5 Integrals

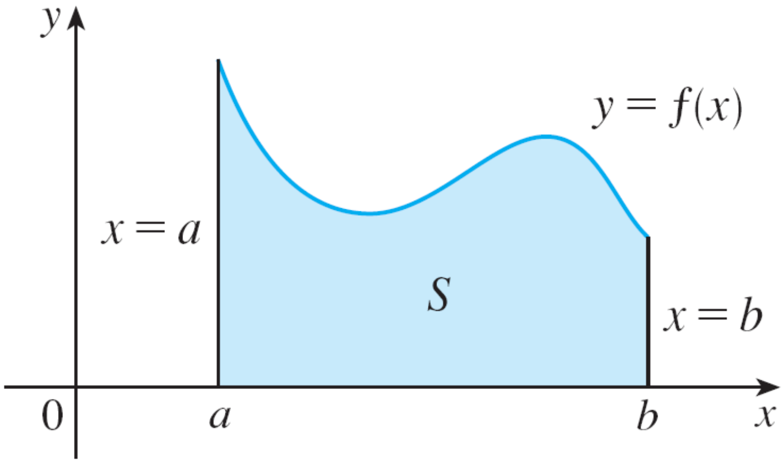
5.1 Areas and Distances

The Area Problem

The Area Problems

We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.



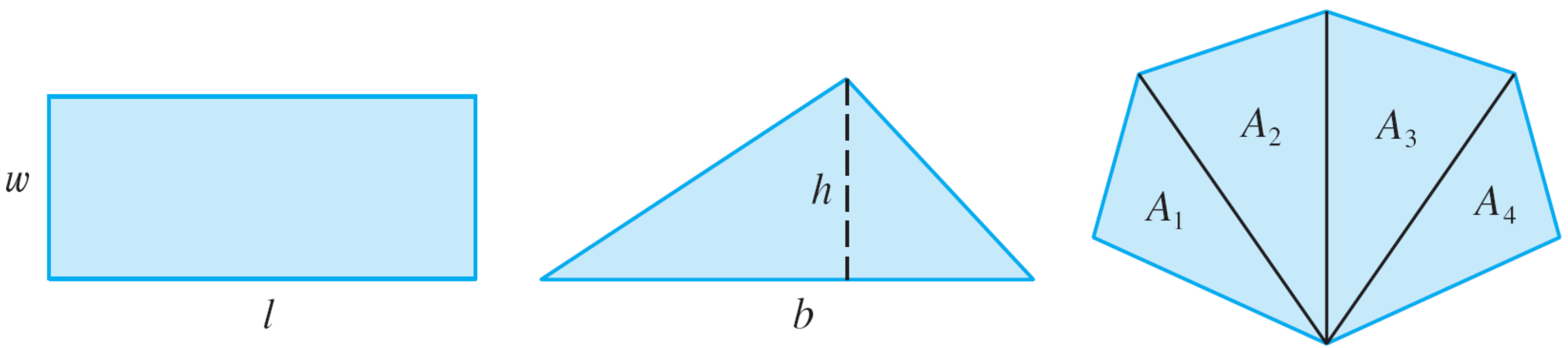
$S = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$

Figure 1

The Area Problems

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height.

The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.



$A = lw$ $\qquad$ $A = \frac{1}{2}bh$ $\qquad$ $A = A_1 + A_2 + A_3 + A_4$

Figure 2

The Area Problems

However, it isn't so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

Example 1

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

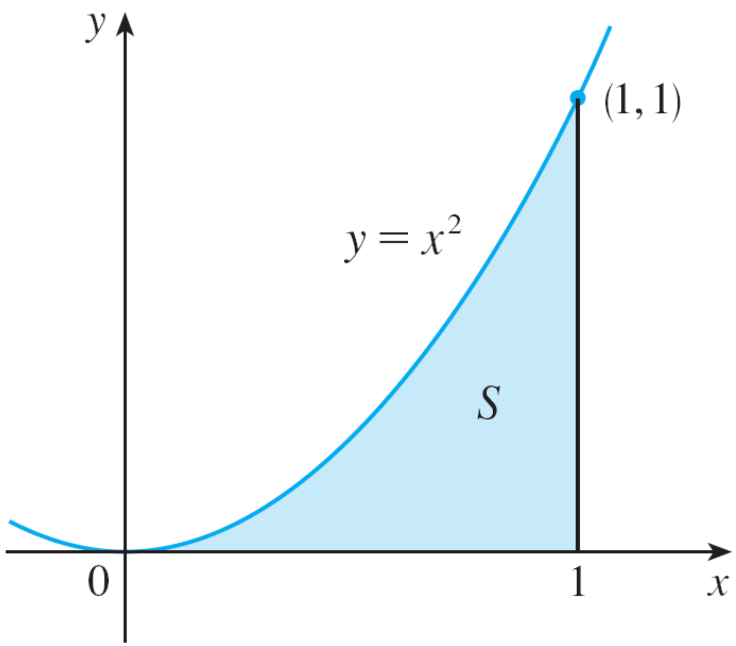


Figure 3

Example 1 – *Solution*

We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that.

Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a).

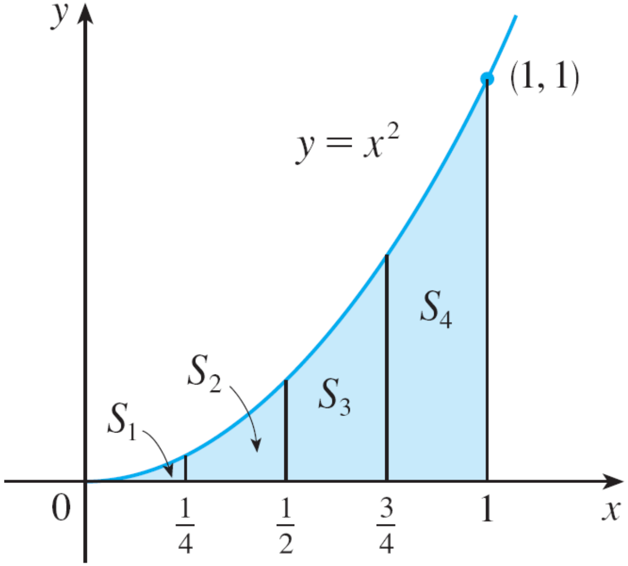


Figure 4(a)

Example 1 – *Solution*

cont'd

We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)].

In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the *right* endpoints of the subintervals $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

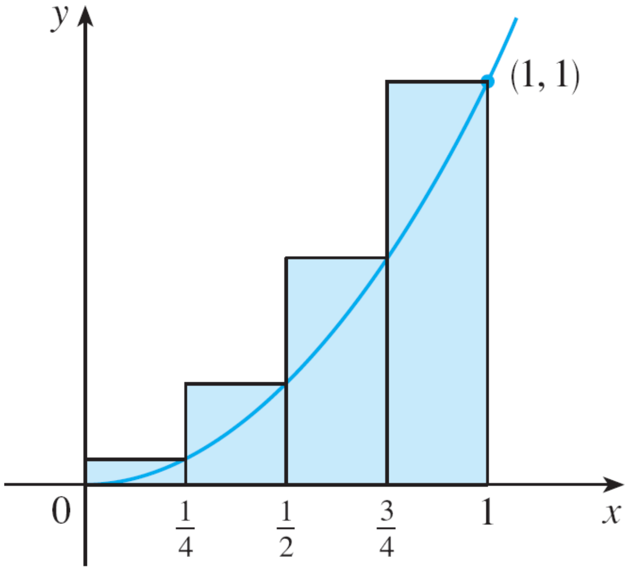


Figure 4(b)

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2$, and 1^2.

Example 1 – *Solution*

cont'd

If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2$$

$$= \frac{15}{32}$$

$$= 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Example 1 – *Solution*

cont'd

Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of *f* at the *left* endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.)

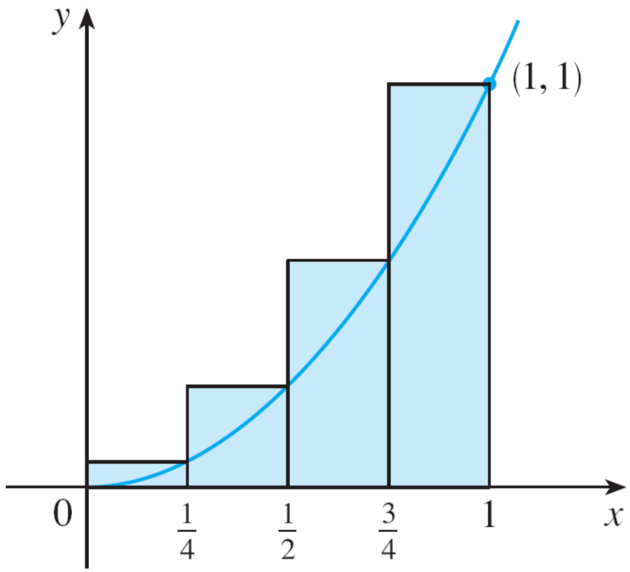


Figure 4(b)

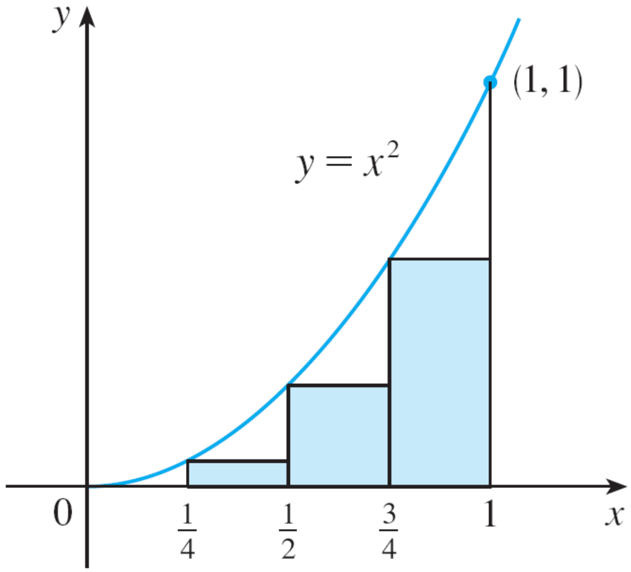


Figure 5

Example 1 – *Solution*

cont'd

The sum of the areas of these approximating rectangles is

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2$$

$$= \frac{7}{32}$$

$$= 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

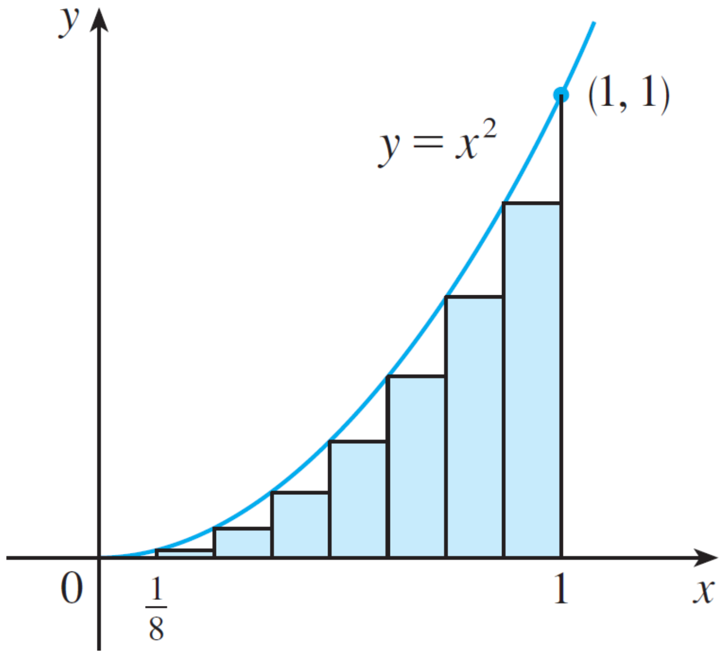
$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips.

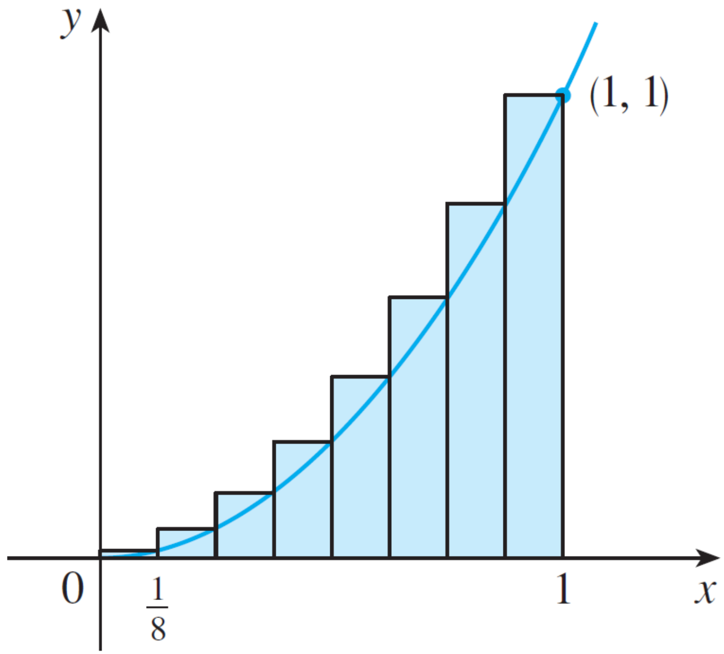
Example 1 – *Solution*

cont'd

Figure 6 shows what happens when we divide the region *S* into eight strips of equal width.



(a) Using left endpoints



(b) Using right endpoints

Approximating *S* with eight rectangles

Figure 6

Example 1 – *Solution*

cont'd

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips.

Example 1 – *Solution*

cont'd

The table at the right shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n).

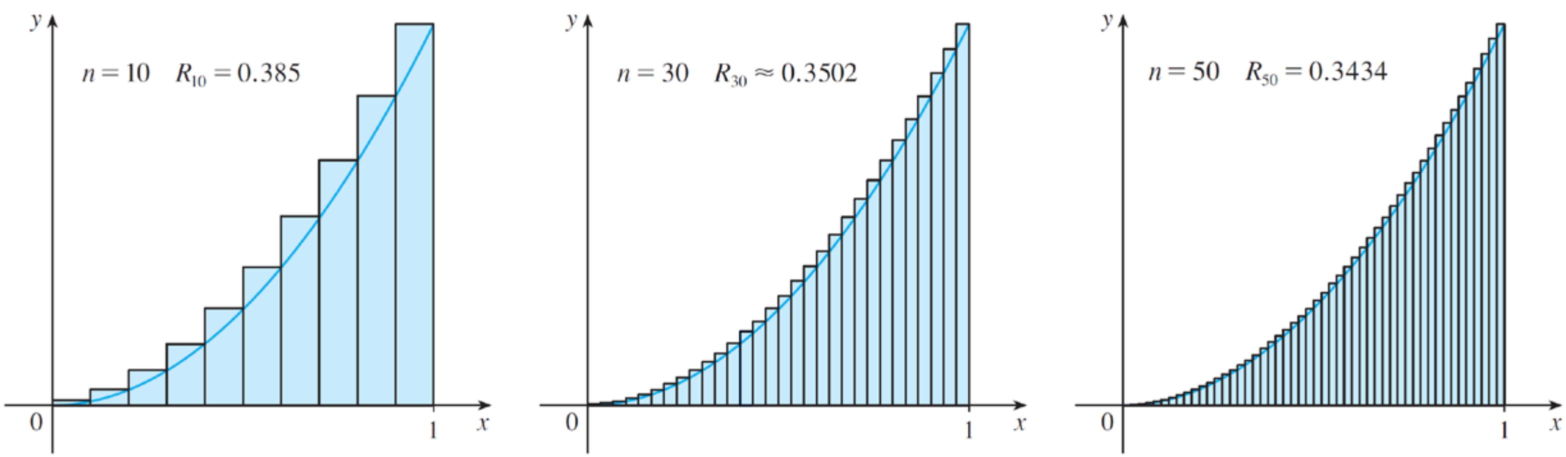
n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335.

A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

The Area Problems

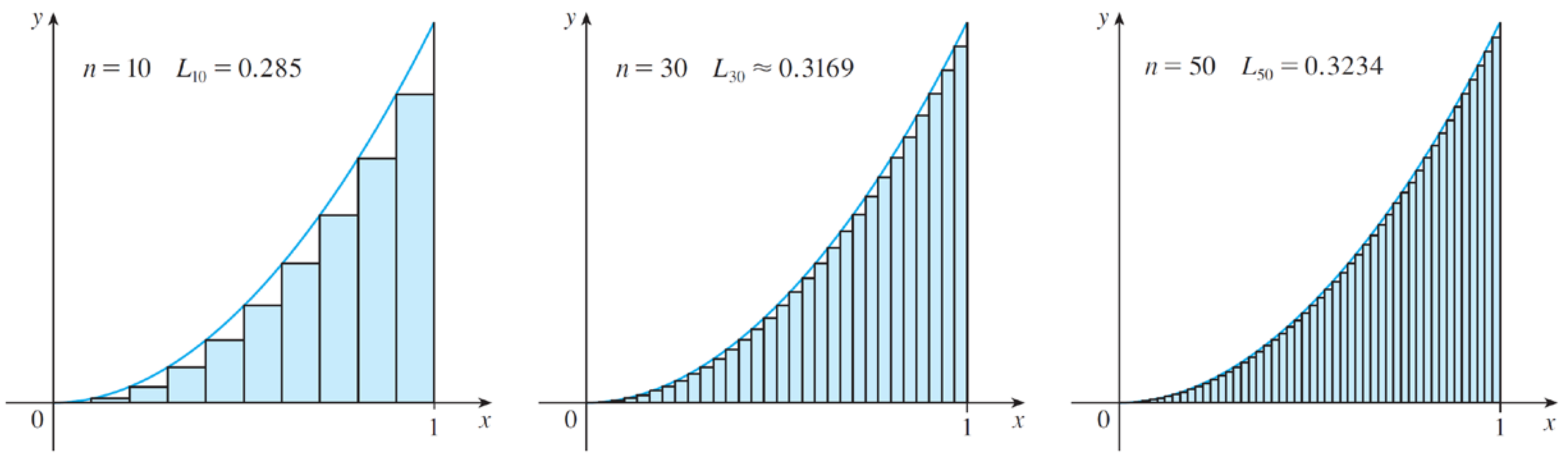
From Figures 8 and 9 it appears that, as *n* increases, both L_n and R_n become better and better approximations to the area of *S*.



Right endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 8

The Area Problems



Left endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 9

Therefore we *define* the area *A* to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

The Area Problems

We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

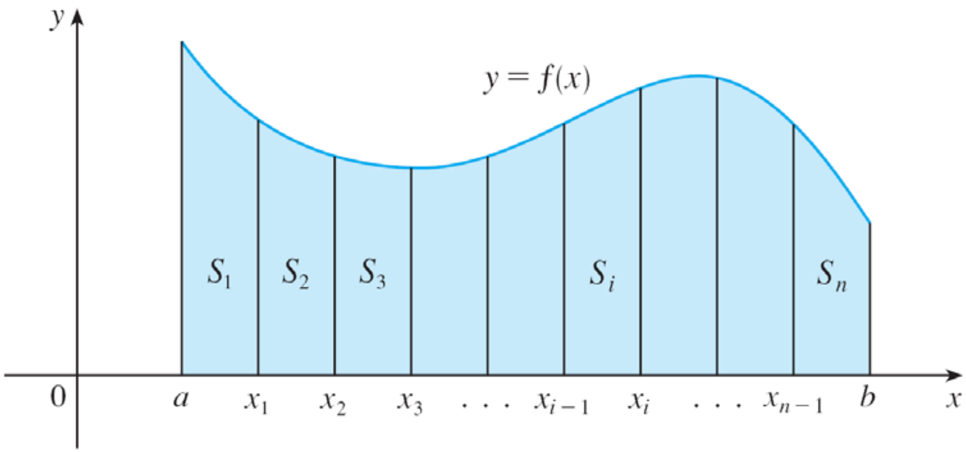


Figure 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

The Area Problems

These strips divide the interval [a, b] into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

The right endpoints of the subintervals are

$$x_1 = a + \Delta x,$$

$$x_2 = a + 2\,\Delta x,$$

$$x_3 = a + 3\,\Delta x,$$

$$\vdots$$

The Area Problems

Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11).

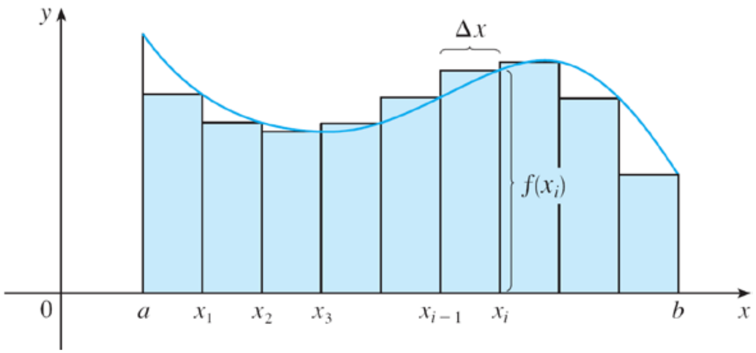


Figure 11

Then the area of the ith rectangle is $f(x_i)\ \Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\ \Delta x + f(x_2)\ \Delta x + \ldots + f(x_n)\ \Delta x$$

The Area Problems

Figure 12 shows this approximation for n = 2, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

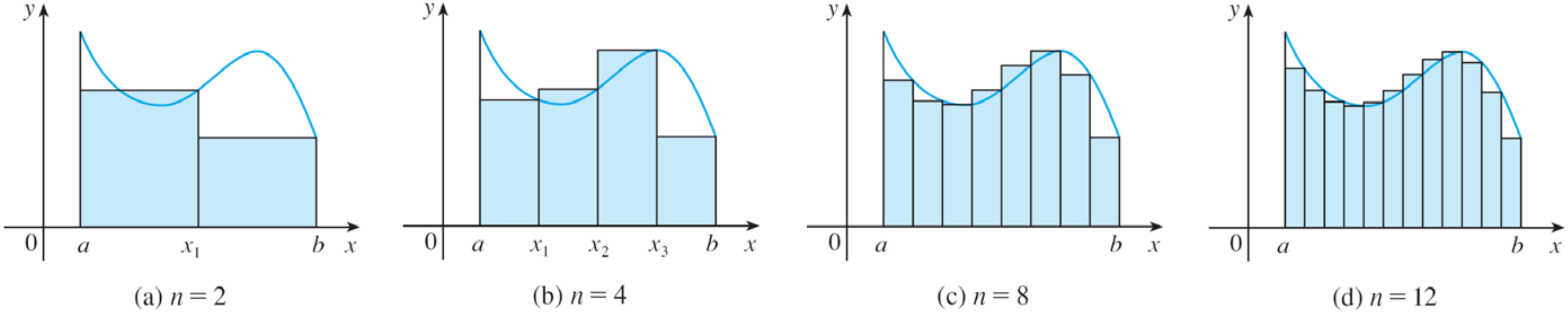


Figure 12

The Area Problems

Therefore we define the area *A* of the region *S* in the following way.

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that *f* is continuous. It can also be shown that we get the same value if we use left endpoints:

3 $$A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

The Area Problems

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1*, x_2*, \ldots, x_n*$ the **sample points**.

Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints.

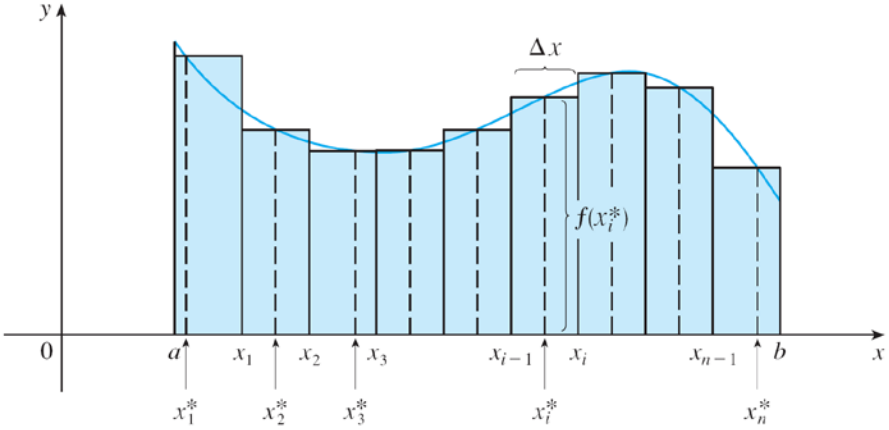


Figure 13

So a more general expression for the area of S is

$$\boxed{4} \quad A = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

The Area Problems

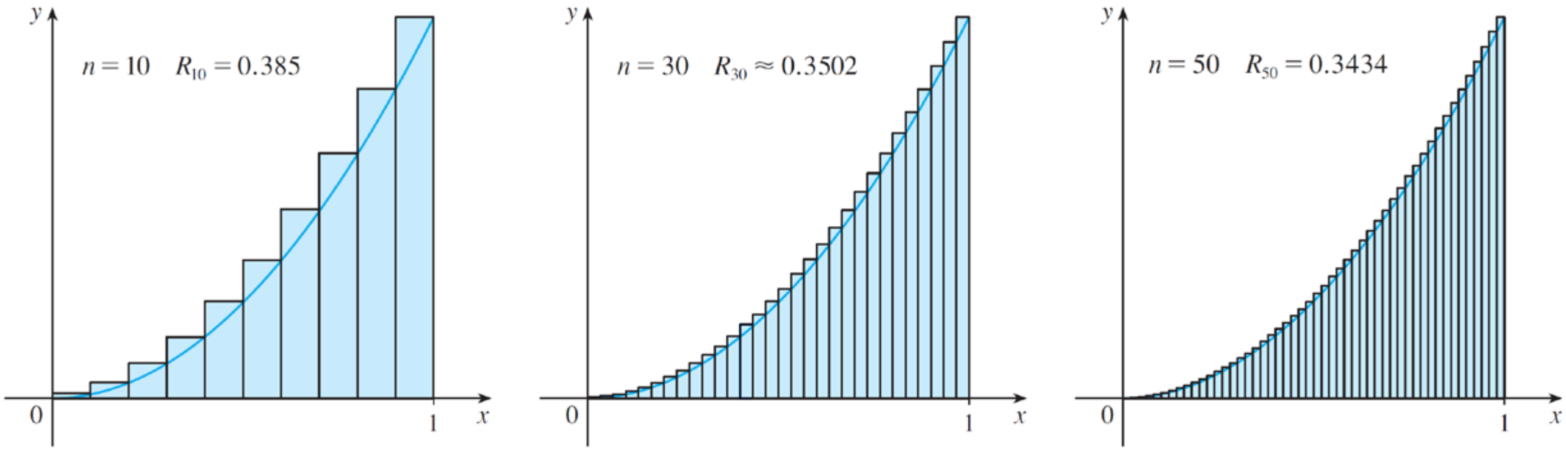
Note:
It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums*.

We saw in Example 1, for instance, that the area $\left(A = \frac{1}{3}\right)$ is trapped between all the left approximating sums L_n and all the right approximating sums R_n.

The function in those examples, $f(x) = x^2$, happens to be increasing on [0, 1] and so the lower sums arise from left endpoints and the upper sums from right endpoints.

The Area Problems

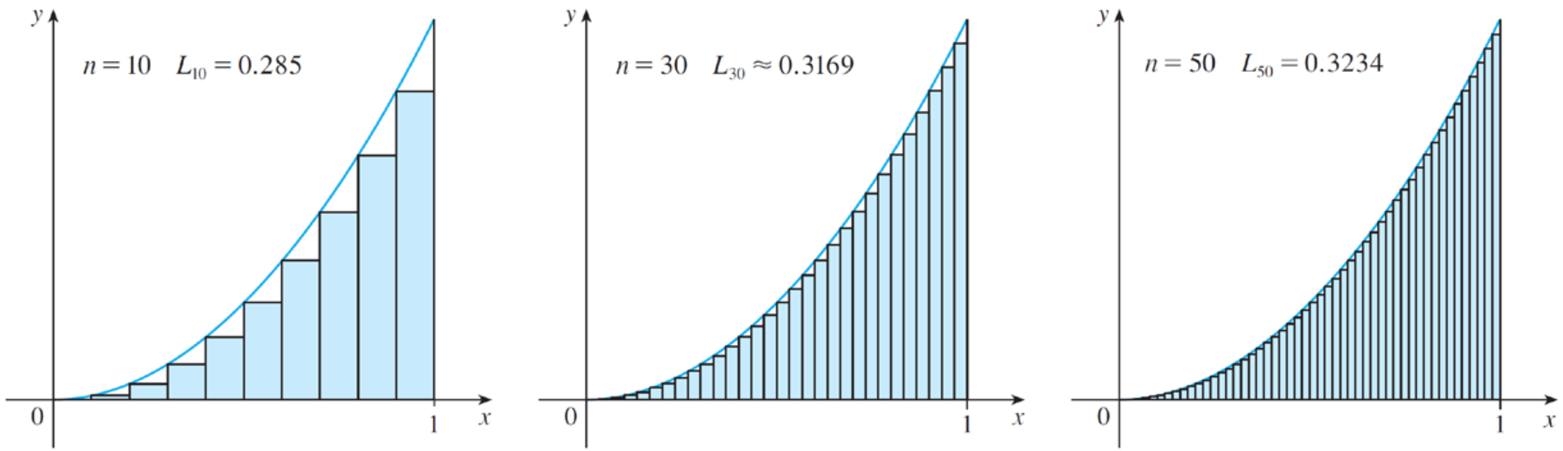
See Figures 8 and 9.



Right endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 8

The Area Problems

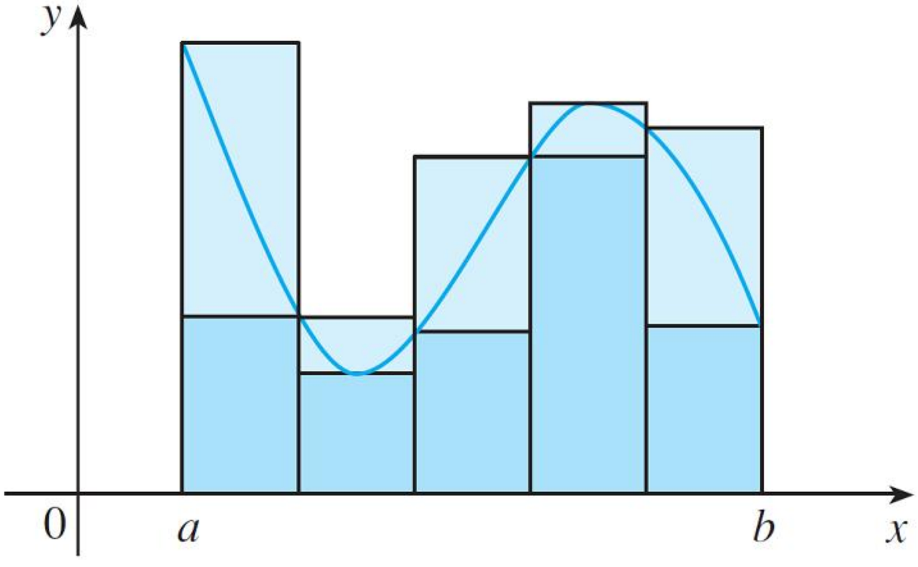


Left endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 9

The Area Problems

In general, we form **lower** (and **upper**) **sums** by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of *f* on the *i*th subinterval. (See Figure 14)



Lower sums (short rectangles) and upper sums (tall rectangles)

Figure 14

The Area Problems

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

The Area Problems

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

We can also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

The Distance Problem

The Distance Problems

Now let's consider the *distance problem:* Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

If the velocity remains constant, then the distance problem is easy to solve by means of the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

But if the velocity varies, it's not so easy to find the distance traveled.

Example 4

Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

Example 4

cont'd

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant.

If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Example 4

cont'd

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when $t = 5$ s.

So our estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

$$31 \text{ ft/s} \times 5 \text{ s} = 155 \text{ ft}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5)$$
$$= 1135 \text{ ft}$$

Example 4

cont'd

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity.

Then our estimate becomes

$$(31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) + (41 \times 5) = 1215 \text{ ft}$$

If we had wanted a more accurate estimate, we could have taken velocity readings every two seconds, or even every second.

The Distance Problems

In general, suppose an object moves with velocity $v = f(t)$, where $a \le t \le b$ and $f(t) \ge 0$ (so the object always moves in the positive direction).

We take velocity readings at times t_0 (= a), t_1, t_2,. . ., t_n (= b) so that the velocity is approximately constant on each subinterval.

If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$.

The Distance Problems

Similarly, the distance traveled during the second time interval is about $f(t_1)\ \Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The Distance Problems

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance *d* traveled is the *limit* of such expressions:

$$\boxed{5} \qquad d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$