

$$1) (x_1, \dots, x_N) \in \mathbb{R}^d \quad x_n \sim N(Wz_n, \sigma^2 I) \quad W \in \mathbb{R}^{d \times k}$$

$$z_n \sim N(0, I)$$

$$p(W) = (2\pi)^{-\frac{k}{2}} \exp\left[-\frac{\lambda}{2} \text{trace}(W^T W)\right]$$

$$p(x_n | W, z_n) = (2\pi\sigma^2 I)^{-d/2} \exp\left[-\frac{1}{2\sigma^2} (x_n - Wz_n)^T (x_n - Wz_n)\right]$$

$$p(z_n) = (2\pi)^{-\frac{k}{2}} \exp\left[-\frac{1}{2} z_n^T z_n\right]$$

goal $\rightarrow W^1 = \underset{W}{\text{argmax}} \ln p(x_1, \dots, x_N, W)$

now, from class

$$\ln p(x_1, \dots, x_N, W) \approx \int q(z_n) \ln \frac{p(x_1, \dots, x_N, W, z_n)}{q(z_n)} dz_n$$

$$q(z) = p(z | W, x) \propto \prod_{n=1}^N p(x_n | W, z_n) p(z_n)$$

$$q(z) \propto \prod_{n=1}^N (2\pi\sigma^2 I)^{-\frac{d}{2}} \exp\left[-\frac{1}{2\sigma^2} (x_n - Wz_n)^T (x_n - Wz_n)\right] (2\pi)^{-\frac{k}{2}} \exp\left[-\frac{1}{2} z_n^T z_n\right]$$

constants
drop out

$$\propto \exp\left[-\frac{1}{2} \sum_{n=1}^N \left[\frac{1}{\sigma^2} z_n^T W^T W z_n - \frac{2}{\sigma^2} x_n^T W z_n + z_n^T z_n \right]\right]$$

$$\propto \exp\left[-\frac{1}{2\sigma^2} \sum_{n=1}^N \left[-2 x_n^T W z_n + z_n^T (\sigma^2 I + W^T W) z_n \right]\right]$$

$$\mu^* = \left(\frac{\sigma^2 I}{\sigma^2 I + W^T W} \right) W^T x_{\text{all}}$$

$$\sigma^2 z_n^T z_n + z_n^T W^T W z_n$$

$$\Sigma^* = \left(\frac{\sigma^2 I}{\sigma^2 I + W^T W} \right)$$

... I think that's it.

$$z_n^T (\sigma^2 I + W^T W) z_n$$

This is a $N(\mu^*, \Sigma^*)$

expectation of joint likelihood

$$\mathcal{L}(W) = E_q[\ln p(z, x, W)]$$

YU

$$p(z_n, x_n, W) \propto p(W) \prod_{i=1}^N p(x_i | z_i, W) p(z_i)$$

$$\ln p(z_n, x_n, W) \propto \ln p(W) + \sum_{i=1}^N \ln p(x_i | z_i, W) + \text{some constant}$$

$$\propto E_q \ln p(W) + E_q \left[\sum_{i=1}^N \left[\ln \left[(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma^2 I} \sum_{i=1}^N (x_i - Wz_i)^T (x_i - Wz_i) \right\} \right] \right] \right]$$

$$\propto \ln p(W) + E_q \left[\sum_{i=1}^N \ln (2\pi\sigma^2 I) + \frac{1}{2\sigma^2 I} \sum_{i=1}^N (x_i - Wz_i)^T (x_i - Wz_i) \right]$$

$$\propto \ln p(W) + E_q \left[\frac{1}{2\sigma^2 I} \sum_{i=1}^N (x_i - Wz_i)^T (x_i - Wz_i) \right]$$

$$\propto \ln p(W) + \frac{1}{2\sigma^2 I} \sum_{i=1}^N E_q \left[x_i^T x_i - 2x_i^T Wz_i + W^T z_i^T z_i W \right]$$

$$\propto \ln p(W) + \frac{1}{2\sigma^2 I} \sum_{i=1}^N \left[E_q(x_i^T x_i) + E_q(z_i^T W^T W z_i) - 2E_q(x_i^T W z_i) \right]$$

$$\propto \ln p(W) + \frac{1}{2\sigma^2 I} \sum_{i=1}^N \left[\overset{\text{data term}}{x_i^T x_i} - 2x_i^T W \mu^* + E_q[z_i^T W^T W z_i] \right]$$

$$= E_q[z_i^T W^T W z_i] + \text{trace}(W^T W \text{cov}(z_i))$$

$$\propto -\frac{\lambda}{2} \text{trace}(W^T W) + \frac{1}{2\sigma^2} \sum_{i=1}^N (\mu^{*T} W^T) (W^* W) + \text{trace}(W^T W \Sigma^*) - 2x_i^T W \mu^*$$

M step

$$\nabla \mathcal{L}_E(W) = -\lambda W + \frac{1}{2\sigma^2} \sum \left[W(\mu^{*T} \mu^* + \Sigma^*) \right] - \frac{2}{2\sigma^2} \sum x_i^T \mu^*$$

$$= 0 = -\lambda W + \frac{1}{2\sigma^2} \sum W(\mu^{*T} \mu^* + \Sigma^*) - \frac{1}{\sigma^2} \sum x_i^T \mu^*$$

$$-\lambda W + \frac{W}{2\sigma^2} (\mu^{*T} \mu^* + \Sigma^*) = \frac{1}{\sigma^2} \sum x_i^T \mu^*$$

$$W_+ = \frac{\sum_{i=1}^N x_i^T \mu^*}{-\lambda \sigma^2 + \frac{1}{2} (\mu^{*T} \mu^* + \Sigma^*)}$$

EM algorithm

1) Initialize W with dimensions

2) for iteration t ,

a) calculate $E_{qt}[z]$, which is $\frac{W^T x_n \mu}{(\sigma^2 I + W^T W)}$ E-step

b) calculate $W_t = \frac{\sum_{i=1}^N x_i \mu^*}{-\lambda \sigma^2 + \frac{1}{2}(N \mu^{*2} + \sum \mu^*)}$

3) once you have this,

calculate $\ln p_t(x_1, \dots, x_N, W)$ from $\ln p(x_1, \dots, x_N, W, z_n)$ until converges \rightarrow
so known W sufficiently small