Adaptive Control for a Class of Multi-Input Multi-Output Plants with Arbitrary Relative Degree

Zheng Qu, *Member, IEEE*, Benjamin T. Thomsen, and Anuradha M. Annaswamy, *Fellow, IEEE*

***Abstract*—In this paper, a new adaptive output-feedback controller for a class of multi-input multi-output linear plant models with relative degree three or higher is developed. The adaptive controller includes a baseline design based on observers and parameter adaptation based on a modified closed-loop reference model. The overall design guarantees stability and tracking performance in the presence of large parametric uncertainties that are common in aircraft models with actuator dynamics. The controller is applied to the longitudinal dynamics of a very-flexible aircraft model with non-negligible second-order actuator dynamics.**

***Index Terms*—Adaptive control, arbitrary relative degree, computational simplicity, multi-input multi-output, very flexible aircraft.**

I. Introduction

Output-feedback control designs have been studied extensively because of their ability to control a plant with incomplete state measurements. One strategy is to use an observer to generate state estimates, and use the estimates to perform state-feedback-like control [1]. Observer-based controllers have been widely employed for aircraft control and their performance is quite satisfactory for a nominal plant model [2], [3]. Some amount of model uncertainties can be tolerated when these linear controllers are designed with adequate stability margins [1], [4]. Adaptive control has been investigated as an improvement over these linear controllers due to its guarantees of stability and performance in the presence of parametric uncertainties [5].

The classical approach to multi-input multi-output (MIMO) adaptive controllers (see [5, Chapter 10] and [6, Chapter 9]) is based on the underlying plant transfer function matrix. Such a design typically requires knowledge of the plant's Hermite form [7], [8] and uses a non-minimal observer along with a reference model, resulting in the use of a large number of integrators. In contrast to the classical method, recent literature propose a new approach based on a state-space representation, which uses a minimal observer to generate the underlying state estimates [9, Chapter 14]. The state estimates are then used for both feedback and parameter adaptation. The minimal observer is used simultaneously as a reference model, by appealing to the notion of a closed-loop reference model (CRM), which is a promising development in adaptive control due to improved transients [10]–[12]. The use of CRM allows for a substantial reduction in the number of integrators compared to classical adaptive output-feedback controllers, and is therefore an attractive alternative in the case of MIMO plant models (see [9], [13]–[16]). The controllers proposed in these references, however, are based on a restrictive assumption that the underlying relative degree (see Definition 2) of the plant is unity. In practice, this implies that any actuator dynamics that may be present must be sufficiently fast, and that the number of integrations of each sensor measurement may not exceed one. Recent research relaxed the relative degree one assumption by allowing first-order actuators in the plant models, and proposed a new adaptive control solution for actuator aging which becomes an important consideration in long-endurance operation [17], [18].

In this paper, we further relax the relative degree assumption by explicitly including second- or higher-order actuator dynamics in the plant dynamics, resulting in a relative degree three or higher plant model. We propose a new adaptive control design which uses a modified observer/CRM, and a new adaptive law which uses high-order tuners instead of standard first-order differential equations for parameter adaptation. This model can accommodate parametric uncertainties in both the actuator model and the original plant dynamics. Adaptive controllers for these types of uncertainties have been addressed in single-input single-output (SISO) plants [5], [12] and for square MIMO plants in [5], [6]. The adaptive controller that we propose in this paper addresses nonsquare MIMO plants and has significantly lower computational complexity compared to those in [5], [6], [12], and therefore can be implemented with relative ease in most practical applications.

This paper is organized as follows. Section II introduces mathematical preliminaries necessary for the design and analysis of the proposed controller. Section III motivates the control design using a SISO plant with a relative degree three transfer function as an example. The main problem addressed in this paper is presented in Section IV where a MIMO plant model with arbitrary relative degree, motivated by flight control for very flexible aircraft (VFA), is considered. The SISO control design is then extended in Section V to address the solution for MIMO relative degree three plant models. The relative degree assumption is further relaxed in Section VI to accommodate plant models with arbitrary relative degree. Section VII demonstrates the adaptive controller on a nonlinear VFA model navigating through multiple equilibrium flight conditions.

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Z. Qu, B. T. Thomsen and A. M. Annaswamy are with the Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139. {mzqu, thomsen, aanna}@mit.edu

II. Preliminaries

This section presents a number of definitions, lemmas, and propositions that will be used throughout the paper. Throughout this paper, the ith derivative of χ is denoted by $\chi^{(i)}$, and the ith power of χ is denoted by χ^i or $\chi^{\{i\}}$ if necessary. Specific variables are denoted by $\chi^{[i]}$ and χ_n^i. The outputs of filters are denoted by $\overline{\chi}^{[i]}$. The notation χ^* represents a true value, or a variable constructed from true values, while its counterpart χ represents either an estimate or nominal value of χ^*. The difference $(\chi - \chi^*)$ is denoted using $\widetilde{\chi}$. The differential operator $\frac{d}{dt}$ is denoted by s, and $\pi_r^i(s)$ denotes a polynomial in s with a_r^i as its coefficients, i.e.

$$\pi_r^i(s) = (s+\alpha_1)(s+\alpha_2)\cdots(s+\alpha_i) = \sum_{j=0}^{i} a_r^{r-j} s^{i-j} \quad (1)$$

for $i = 1, 2, \cdots, r$, and $\alpha_i \in \mathbb{R}$. Note that $s^{(-1)}[\cdot] := \int [\cdot] dt$. The coefficients of the expansion $s^r(xy)$ are $d_r^i = \frac{r!}{i!(r-i)!}$ with $0 \leq i \leq r$, per the general Leibniz rule of differentiation. The following property, derived from (1),

$$\pi_r^{i+1}(s) = \pi_r^i(s) \cdot s^{r-i} + \sum_{j=0}^{r-i-1} a_r^j s^j \quad (2)$$

will be used in the control design.

Consider a state space representation (A, B, C) with m inputs and p outputs. Square systems have $m = p$ while nonsquare systems have $m \neq p$. Curly brace notation $\{A, B, C\}$ denotes the transfer function matrix of the model, i.e. $\{A, B, C\} := C(sI - A)^{-1}B$.

Definition 1. [19] For a non-degenerate m-input, p-output and nth order linear system with minimal realization (A, B, C), the transmission zeros (referred as "zeros" hereafter) are defined as the finite values of z such that $rank[R(z)] < n + \min[m, p]$, with the Rosenbrock system matrix given by

$$R(z) = \begin{bmatrix} zI - A & -B \\ C & 0 \end{bmatrix}. \quad (3)$$

The input relative degree of the plant model is defined as follows, with the ith column of B denoted b_i and corresponding to input u_i.

Definition 2. A linear square plant model $\{A, B, C\}$ has

a) input relative degree $\mathfrak{r} = [r_1, r_2, \cdots, r_m]^T \in \mathbb{N}^{+(m\times 1)}$ if and only if

$$\begin{aligned} i) \quad & \forall j \in (1, \cdots, m),\ \forall k \in (0, \cdots, r_j - 2) \ : \\ & CA^k b_j = 0_{m\times 1}, \quad \text{and} & (4) \\ ii) \quad & rank[CA^{r_1-1}b_1, \cdots, CA^{r_m-1}b_m] = m; & (5) \end{aligned}$$

b) uniform input relative degree $r \in \mathbb{N}^+$ if and only if it has input relative degree $\mathfrak{r}$ and $r_1 = r_2 = \cdots = r_m = r$.

Remark 1. If $\{A, B, C\}$ has u_i of input relative degree r_i, u_i needs to be differentiated r_i times to ensure that u_i appears in the minimal realization of y equation, i.e.

$$\begin{aligned} C(sI-A)^{-1}b_i s &= C(sI-A)^{-1}Ab_i \\ &\vdots \\ C(sI-A)^{-1}b_i s^{r_i-1} &= C(sI-A)^{-1}A^{r_i-1}b_i \\ C(sI-A)^{-1}b_i s^{r_i} &= C(sI-A)^{-1}A^{r_i}b_i + CA^{r_i-1}b_i \end{aligned} \quad (6)$$

The term "relative degree" refers to input relative degree in this paper, and b_i is named as a "relative degree r_i input path". The relative degree of a plant model relates to its transmission zeros in the following Lemma (see [20, Corollary 2.6] for proof).

Lemma 1. *For a square system $\{A, B, C\}$ with uniform input relative degree r, define controllability and observability matrices*

$$\begin{aligned} \mathcal{B} &= \begin{bmatrix} B & AB & \cdots & A^{r-1}B \end{bmatrix} \in \mathbb{R}^{n\times mr}, \\ \mathcal{C} &= \begin{bmatrix} C^T & (CA)^T & \cdots & (CA^{r-1})^T \end{bmatrix}^T \in \mathbb{R}^{mr\times n}, \end{aligned} \quad (7)$$

along with $\mathcal{M} \in \mathbb{R}^{n\times(n-mr)}$ such that $\mathcal{C}\mathcal{M} = 0$, and $\mathcal{N} \in \mathbb{R}^{(n-mr)\times n}$ such that $\mathcal{N}\mathcal{B} = 0$ and $\mathcal{N}\mathcal{M} = I$; Then $\mathcal{C}\mathcal{B}$ is full rank, and the eigenvalues of $(\mathcal{N}A\mathcal{M})$ are the transmission zeros of $\{A, B, C\}$.

The following Lemma states two equivalent realizations of linear system when differentiators s are added to inputs. Its proof directly follows (6) and is therefore omitted here.

Lemma 2. *Given a linear system $\{A, B, C\}$ with uniform input relative degree r, the following realizations are equivalent for $i = 1, 2, \cdots, r-1$:*

$$\begin{aligned} i) \quad & \dot{x} = Ax + B\pi_{r-1}^{r-i}(s)[u], \qquad y = Cx & (8) \\ ii) \quad & \dot{x}' = Ax' + B_i^a u, \qquad y = Cx' & (9) \end{aligned}$$

where $\{A, B_i^a, C\}$ has relative degree i with

$$B_i^a = \sum_{j=0}^{r-i} A^{r-j} B a_{r-1}^{r-i-j} = \pi_{r-1}^{r-i}(A)B. \quad (10)$$

Remark 2. It is noted that the implementation of representation (8) includes differentiation of input u, while the implementation of (9) does not require differentiations of u. State coordinates in (8) and (9) are related by

$$x' = x - \sum_{j=i+1}^{r} (B_j^a s^{j-i-1})[u]. \quad (11)$$

$\pi_{r-1}^{r-i}(s)$ in (8) adds $(r-i)$ transmission zeros to $\{A, B, C\}$, as stated in the following Proposition, whose proof uses Definition 1 directly and therefore is omitted here.

Proposition 1. *With $\{A, B, C\}$ and B_i^a given in Lemma 2, $\boldsymbol{Z}\{A, B_i^a, C\} = \boldsymbol{Z}\{A, B, C\} \cup \boldsymbol{Z}\{\pi_{r-1}^{r-i}(s)\}$, where $\boldsymbol{Z}\{\}$ denotes the set of transmission zeros of a transfer function or the set of roots of a polynomial.*

III. Adaptive Output-Feedback Control of SISO Relative Degree Three Plant

To illustrate the main idea behind the proposed controller, this section presents the control design when applied to a SISO relative degree three plant. Section III-A describes the SISO control problem relevant to flight control. The control design solution is derived in Section III-B along with analysis of error dynamics. A law for parameter adaptation is described in Section III-C, along with proof of stability.

A. Plant Model Description

A SISO first-order dynamical system driven by an actuator with second-order dynamics is considered. The response of aircraft roll rate x_p with respect to aileron deflection u_p can be described by the first-order model

$$\dot{x}_p = (-a_p - b_p\theta_p^*)x_p + b_p\lambda_p^* u_p, \tag{12}$$

where a_p is a nominal pole location and b_p is a nominal high-frequency gain. θ_p^* is the uncertainty in the aircraft dynamics and λ_p^* is the uncertainty in aircraft control surface. The aileron deflection u_p is driven by an actuator modeled as

$$\ddot{u}_p + (2\zeta\omega_n + \omega_n^2\theta_{\zeta\omega}^*)\dot{u}_p + \omega_n^2(1+\theta_\omega^*)u_p = \omega_n^2\lambda_a^* u \tag{13}$$

where u is the command to the actuator, ζ is the nominal damping ratio and ω_n is the nominal natural frequency, and $\theta_{\zeta\omega}^*$ and θ_ω^* are their corresponding uncertainties. Define $w_u = \lambda_p^* u_p$. Then the overall plant model can be written as

$$\begin{aligned}\begin{bmatrix}\dot{x}_p\\ \dot{w}_u\\ \ddot{w}_u\end{bmatrix} &= \underbrace{\begin{bmatrix}-a_p & b_p & 0\\ 0 & 0 & I\\ 0 & -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_p\\ w_u\\ \dot{w}_u\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}b_p\\ 0\\ 0\end{bmatrix}}_{b_1}\underbrace{\begin{bmatrix}-\theta_p^* & 0 & 0\end{bmatrix}}_{\psi_1^{*T}}x \\ &+ \underbrace{\begin{bmatrix}0\\ 0\\ \omega_n^2\end{bmatrix}}_{b_3}\underbrace{\begin{bmatrix}0 & -\theta_\omega^* & -\theta_{\zeta\omega}^*\end{bmatrix}}_{\psi_3^{*T}}x + \underbrace{\begin{bmatrix}0\\ 0\\ \omega_n^2\end{bmatrix}}_{b_3}\underbrace{\lambda_p^*\lambda_a^*}_{\lambda^*}u \\ y &= \underbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}_{c^T}x\end{aligned} \tag{14}$$

which can be rewritten concisely as

$$\dot{x} = A^*x + b_3\lambda^*u, \qquad y = c^Tx \tag{15}$$

where $A^* = (A + b_1\psi_1^{*T} + b_3\psi_3^{*T})$, and A, b_1, b_3 and c are fully known, while ψ_1^*, ψ_3^* and λ^* contain unknowns. It is assumed that there exist known bounds $\lambda_{max} \geq |\lambda^*|$ and $\psi_{max} \geq \|\psi_i^*\|$ for $i = 1, 3$. The uncertain plant model in (15) is a SISO plant with relative degree three. It is noted that $\psi_1^{*T}b_3 = \psi_1^{*T}A^*b_3 = 0$ as a result of the structure of ψ_1^*. b_3 is a "relative degree three input path" (see Remark 1) and satisfies $c^Tb_3 = c^TA^*b_3 = 0$ and $c^TA^{*2}b_3 = b_p\omega_n^2$. b_1 is a "relative degree one input path" with $c^Tb_1 = b_p$. The following Proposition specifies the relation between b_1 and b_3.

Proposition 2. *The vector b_1 belongs to the linear subspace spanned by $(b_3, A^*b_3, A^{*2}b_3)$.*

Proposition 2 is a result of the fact that the matrix $\begin{bmatrix}b_3, & A^*b_3, & A^{*2}b_3\end{bmatrix}$ is full rank. It is noted that

$$c^T(sI - A^*)^{-1}b_1 = \frac{b_p\omega_n^2}{p_1^*(s)}, \qquad c^T(sI - A^*)^{-1}b_3 = \frac{b_p\omega_n^2}{p_3^*(s)} \tag{16}$$

where $p_1^*(s) = s + (a_p + b_p\theta_p^*)$ and $p_3^*(s) = p_1^*(s)\big[s^2 + (2\zeta\omega_n + \omega_n^2\theta_{\zeta\omega}^*)s + \omega_n^2(1+\theta_\omega^*)\big]$.

B. Control Design and Analysis of Error Dynamics

The control goal is to design u such that y will track a reference trajectory y_m despite the presence of uncertainties. The main challenge here is to arrive at an error transfer function of relative degree one, starting from the transfer function at hand, which has relative degree three. This is necessary to guarantee strictly positive real (SPR) error dynamics, a ubiquitous requirement in adaptive control [5]. For the plant model (15), a modified closed-loop reference model (CRM) [10], [12], [21] is designed as

$$\begin{aligned}\dot{x}_m &= A_mx_m + \ell e_y + f_\ell(e_y) + f_{x_m}(e_y)\\ y_m &= c^Tx_m\end{aligned} \tag{17}$$

where $(A_m = A - b_3k^T)$ is chosen using the linear quadratic regulator (LQR) technique, for example, and is Hurwitz. $f_\ell(e_y)$ and $f_{x_m}(e_y)$ are functions to be designed later. y_m is a reference output trajectory for y, $(e_y = y - y_m)$ is the tracking error, and ℓ is the CRM error feedback gain. From (15) and (17), the state error dynamics is derived as

$$\begin{aligned}\dot{e}_x = &\left(A^* - \ell c^T\right)e_x + b_1\psi_1^{*T}x_m + b_3\psi_3^{'*T}x_m\\ &+ b_3\lambda^*u - f_\ell(e_y) - f_{x_m}(e_y), \quad e_y = c^Te_x,\end{aligned} \tag{18}$$

where $e_x = x - x_m$ and $\psi_3^{'*} = \psi_3^* + k$. We now define a SPR error model, and then describe the controller which will satisfy the requirement of SPR error dynamics using ℓ, $f_\ell(e_y)$, $f_{x_m}(e_y)$, and u.

Definition 3. A strictly positive real error model is defined as

$$e_y(t) = W_1^*(s)\widetilde{\Omega}(t)\xi(t) \tag{19}$$

where $\widetilde{\Omega}(t) = \Omega(t) - \Omega^*$ is the parameter error between a vector of adjustable parameters $\Omega(t)$ and actual values Ω^*, $\xi(t)$ is a known regressor, and $W_1^*(s)$ is a SPR transfer function.

In Section III-B1) and III-B2), we will show that by suitably designing ℓ, along with $f_\ell(e_y)$, $f_{x_m}(e_y)$, and u, a known regressor $\xi(t)$ can be found so that (18) can be expressed in the form of (19) with a corresponding $W_1^*(s)$ that is SPR. Since the plant is of relative degree three, the basic challenge is to introduce two zeros without any explicit differentiation. This is accomplished through suitable designs of $f_\ell(e_y)$ and $f_{x_m}(e_y)$ in combination with high-order tuner adaptive laws and the product rule of derivatives for implementation. $f_\ell(e_y)$, along with ℓ, will be addressed in Section III-B1), while $f_{x_m}(e_y)$ will be addressed in Section III-B2). In both sections, we will use Lemma 2 and Remark 2 extensively to design u and CRM that are free of inaccessible time derivatives, and show their equivalent zero additions in analysis.

1) Design ℓ and $f_\ell(e_y)$: Zero addition in the CRM can be realized by utilizing a new state coordinate along with new b vectors, as shown in Lemma 2. These vectors are given by

$$b_1^{a*} = A^{*2}b_3 + 2A^*b_3 + b_3, \qquad b_2^{a*} = A^*b_3 + 2b_3. \tag{20}$$

These b_i^{a*} yield zeros in (18), since

$$\begin{aligned}c^T(sI - A^*)^{-1}b_1^{a*} &= \frac{b_p\omega_n^2}{p_3^*(s)}\left(s^2 + 2s + 1\right)\\ c^T(sI - A^*)^{-1}b_2^{a*} &= \frac{b_p\omega_n^2}{p_3^*(s)}\left(s + 2\right)\end{aligned} \tag{21}$$

where $p_3^*(s)$ is defined in (16). Without loss of generality, we chose coefficients in (20) so that in (21) two repeated zeros at $s = -1$ are generated. It is noted that $c^T(sI - A^*)^{-1}b_1^{a*}$

has relative degree one and $c^T(sI-A^*)^{-1}b_2^{a*}$ has relative degree two. The following Lemma (whose proof can be found in [16, Lemma 5]) states that a ℓ^* can be designed such that an underlying SPR property can be guaranteed.

Lemma 3. *There exists a ℓ^* defined as*

$$\ell^* = \varepsilon b_1^{a*} b_p \omega_n^2, \tag{22}$$

and a finite scalar constant ε^* such that for $\forall \varepsilon > \varepsilon^* > 0$, the transfer function

$$W_1^*(s) = c^T(sI - A_{\ell^*}^*)^{-1} b_1^{a*} = \frac{b_p\omega_n^2(s+1)^2}{p_3^*(s) + \varepsilon(b_p\omega_n^2)^2(s+1)^2} \tag{23}$$

is SPR, where $A_{\ell^*}^* = A^* - \ell^* c^T$.

The first point to note is that the SPR transfer function in (23) depends on unknown parameters ℓ^* and b_1^{a*}. To accommodate this, we choose ℓ in the CRM in (17) as

$$\ell = \varepsilon b_1^a b_p \omega_n^2, \tag{24}$$

where b_i^a are known versions of (20) chosen as

$$b_1^a = A^2 b_3 + 2Ab_3 + b_3, \quad b_2^a = Ab_3 + 2b_3. \tag{25}$$

Constant ε^* is a function of A^* and b_1^{a*} and therefore is unknown. One can find a known upper bound $\bar{\varepsilon} = \bar{\varepsilon}(A, b_1^a, c, \lambda_{max}, \psi_{max})$ such that $\varepsilon > \bar{\varepsilon} > \varepsilon^*$ for which (22) is satisfied. This fact is explicitly proved for the MIMO case in Section V and is omitted here. The difference between b_i^a and b_i^{a*} is addressed in the following Lemma.

Lemma 4 (Recursive Properties of b_i^a)**.** *Input matrices b_i^a as in (25) and b_i^{a*} as in (20) satisfy*

$$(b_1^{a*} - b_1^a) = b_2^{a*}\psi_3^{1*} + b_3\psi_3^{2*} \tag{26}$$
$$(b_2^{a*} - b_2^a) = b_3\psi_3^{1*} \tag{27}$$

where $\psi_3^{1*} = -\omega_n^2\theta_{\zeta\omega}^*$ and $\psi_3^{2*} = \omega_n^2(\theta_\omega^* + 2\zeta\omega_n\theta_{\zeta\omega}^*)$.

It should be noted that Lemma 4 has the label "recursive" since $(b_i^{a*} - b_i^a)$ lies in the range of b_{i+1}^{a*}.

Using (21), (23), (24), and (26), the dynamics in (18) can be rewritten as

$$\begin{aligned} e_y = c^T(sI - A_{\ell^*}^*)^{-1}\big[&b_3\lambda^* u - f_\ell(e_y) - f_{x_m}(e_y) \\ &+ b_1\psi_1^{*T}x_m + b_3\psi_3^{'*T}x_m \\ &+ \underbrace{b_3(s+2)\psi_3^{1*}e_{sy} + b_3\psi_3^{2*}e_{sy}}_{\text{Use } f_\ell \text{ to make SPR}}\big] \end{aligned} \tag{28}$$

with scaled error $e_{sy} = \varepsilon b_p\omega_n^2 e_y$ and $A_{\ell^*}^*$ defined in Lemma 3. It is noted that several terms in this dynamics enter through relative degree three input path b_3. To design the controller which will lead to SPR error dynamics, we introduce filtered error quantities

$$\begin{aligned} \bar{e}_{\psi y}^{[1]} &= \frac{s+2}{s^2+2s+1}e_{sy}, \qquad \bar{e}_{\psi y}^{[2]} = \frac{1}{s^2+2s+1}e_{sy} \\ \bar{e}_{\psi y}^{[1][2]} &= \frac{s}{s^2+2s+1}\big[\psi_3^1\bar{e}_{\psi y}^{[1]}\big] \end{aligned} \tag{29}$$

where the superscript $(\cdot)^{[i]}$ indicates that its ith derivative is available for control, and $\psi_3^i(t)$ is the estimate of unknown parameter ψ_3^{i*}. Note that (29) allows us to make the following substitution in the error dynamics (28)

$$\begin{aligned} &b_3(s+2)\psi_3^{1*}e_{sy} + b_3\psi_3^{2*}e_{sy} \\ &\qquad = b_3(s+1)^2(\psi_3^{1*}\bar{e}_{\psi y}^{[1]} + \psi_3^{2*}\bar{e}_{\psi y}^{[2]}). \end{aligned} \tag{30}$$

In what follows, we design the term $f_\ell(e_y)$ to give a portion of the error dynamics the desired SPR property. Defining

$$\begin{aligned} f_\ell(e_y) = \big(b_2^a s + b_3\big)\big[\psi_3^1\bar{e}_{\psi y}^{[1]}\big] + b_3(s+1)^2\big[\psi_3^1\bar{e}_{\psi y}^{[1][2]}\big] \\ + b_3(s+1)^2\big[\psi_3^2\bar{e}_{\psi y}^{[2]}\big] \end{aligned} \tag{31}$$

and making use of Lemma 4 and Equations (16) and (21), the error dynamics becomes

$$\begin{aligned} e_y = c^T(sI - A_{\ell^*}^*)^{-1}\big[&b_3\lambda^* u - f_{x_m}(e_y) \\ &+ \underbrace{b_1\psi_1^{*T}x_m + b_3\psi_3^{'*T}x_m}_{\text{Use } f_{x_m} \text{ to make SPR}} \\ &- b_1^{a*}(\widetilde{\psi}_3^1\bar{e}_{\psi y}^{[1]} + \widetilde{\psi}_3^1\bar{e}_{\psi y}^{[1][2]} + \widetilde{\psi}_3^2\bar{e}_{\psi y}^{[2]})\big]. \end{aligned} \tag{32}$$

where $\widetilde{\psi}_3^i(t) = \psi_3^i(t) - \psi_3^{i*}$. Note that we have managed to transform a portion of the error dynamics to have the SPR transfer function $W_1^*(s) = c^T(sI - A_{\ell^*}^*)^{-1}b_1^{a*}$. Implementation of $f_\ell(e_y)$ using the product rule of derivatives in (31) requires the first and second derivative of parameter estimates $\psi_3^1(t)$ and $\psi_3^2(t)$. The use of a high-order tuner for the adaptive law, described later in Section III-C, ensures that the derivatives of the parameter estimates are available. f_ℓ also includes the first and second derivatives of the errors $\bar{e}_{\psi y}^{[1]}$, $\bar{e}_{\psi y}^{[1][2]}$, and $\bar{e}_{\psi y}^{[2]}$, which have been constructed to ensure these derivatives are available.

2) Design u and $f_{x_m}(e_y)$: Here we define control input u and the remaining undefined term in the reference model, $f_{x_m}(e_y)$, to complete the SPR error model. To begin, we reparameterize the uncertainties $b_1\psi_1^{*T} + b_3\psi_3^{'*T}$ in (32) in Proposition 3. Its proof is straightforward by noting the lower triangular structure of the matrix $\big[b_1^{a*}, \quad b_2^{a*}, \quad b_3\big]$.

Proposition 3. *There exist $\underline{\psi}_1^*$, $\underline{\psi}_2^*$ and $\underline{\psi}_3^*$ such that*

$$b_1\psi_1^{*T} + b_3\psi_3^{'*T} = b_1^{a*}\lambda^*\underline{\psi}_1^{*T} + b_2^{a*}\lambda^*\underline{\psi}_2^{*T} + b_3\lambda^*\underline{\psi}_3^{*T} \tag{33}$$

where b_i^{a*} is defined as in (20) and $\underline{\psi}_1^* = \begin{bmatrix} \times & 0 & 0 \end{bmatrix}^T$.

Following Proposition 3, and making use of the filtered reference model states (note that the ith derivative of $\bar{x}_m^{[i]}$ is available for control)

$$\bar{x}_m^{[1]} = \frac{s+2}{s^2+2s+1}x_m, \quad \bar{x}_m^{[2]} = \frac{1}{s^2+2s+1}x_m \tag{34}$$

we can make the following substitution in the error dynamics

$$\begin{aligned} &b_1\psi_1^{*T}x_m + b_3\psi_3^{'*T}x_m \\ &\qquad = b_3(s+1)^2[\underline{\psi}_1^{*T}x_m + \underline{\psi}_2^{*T}\bar{x}_m^{[1]} + \underline{\psi}_3^{*T}\bar{x}_m^{[2]}] \end{aligned} \tag{35}$$

The equation (35) suggests that we choose control law u that includes $(s+1)^2[\underline{\psi}_1^T x_m]$, where $\underline{\psi}_1(t)$ is the estimate of $\underline{\psi}_1^*$, however such an input is not implementable since $\ddot{x}_m$ is not available. Instead, introducing scaled integral error signal

$$e_{y_1^0}(t) = \int_0^t l e_y(\tau) d\tau \tag{36}$$

and synthetic variables (the accessible parts of $x_m^{(i)}$)

$$v_m^{[2]} = (A^2 x_m - A\ell e_y),\ v_m^{[1]} = (Ax_m),\ v_m^{[0]} = (x_m) \tag{37}$$

we define the control law $(u = u_1 + u_2)$ as

$$\begin{aligned} u_1 =& -(s+1)^2 \left[\underline{\psi}_2^T \overline{x}_m^{[1]} + \underline{\psi}_3^T \overline{x}_m^{[2]}\right] \\ u_2 =& -(s^2+2s)\left[\underline{\psi}_1^T\right] e_{y_1^0} \\ & - \left[(\ddot{\underline{\psi}}_1^T v_m^{[0]} + 2\dot{\underline{\psi}}_1^T v_m^{[1]} + \underline{\psi}_1^T v_m^{[2]}) \right. \\ & \left. + 2(\dot{\underline{\psi}}_1^T v_m^{[0]} + \underline{\psi}_1^T v_m^{[1]}) + \underline{\psi}_1^T v_m^{[0]}\right] \end{aligned} \tag{38}$$

where $\underline{\psi}_i(t)$ is the estimate of $\underline{\psi}_i^*$, whose derivatives will be realized in Section III-C using high-order tuners. Using (35) and substituting u_1 from (38), (32) becomes

$$\begin{aligned} e_y = c^T(sI - A_{\ell^*}^*)^{-1}&\left[b_3\lambda^*(u_2 + (s+1)^2[\underline{\psi}_1^{*T} x_m])\right. \\ & + b_1^{a*}\lambda^*(\widetilde{\underline{\psi}}_2^T \overline{x}_m^{[1]} + \widetilde{\underline{\psi}}_3^T \overline{x}_m^{[2]}) - f_{x_m}(e_y) \\ & \left. - b_1^{a*}(\widetilde{\psi}_3^1 \overline{e}_{\psi y}^{[1]} + \widetilde{\psi}_3^1 \overline{e}_{\psi y}^{[1][2]} + \widetilde{\psi}_3^2 \overline{e}_{\psi y}^{[2]})\right] \end{aligned} \tag{39}$$

with $\widetilde{\underline{\psi}}_i(t) = \underline{\psi}_i(t) - \underline{\psi}_i^*$. To deal with the inaccessible signal $\ddot{x}_m$ using u_2 and f_{x_m}, we proceed to introduce filtered signals

$$\begin{aligned} \overline{e}_{\Psi_1 y_1^0}^{[1]} &= \frac{s^2+2s}{s^2+2s+1}\left[\underline{\psi}_1^T e_{y_1^0}\right] \\ \overline{e}_{\Psi_1 y_1^0}^{[1][2]} &= \frac{s}{s^2+2s+1}\left[\lambda \overline{e}_{\Psi_1 y_1^0}^{[1]}\right] \end{aligned} \tag{40}$$

where $\lambda(t)$ is the estimate of λ^*, and then define

$$\begin{aligned} f_{x_m}(e_y) =& \\ & -(b_2^a s + b_3)\left[\lambda \overline{e}_{\Psi_1 y_1^0}^{[1]}\right] - b_3(s+1)^2\left[\psi_3^1 \overline{e}_{\Psi_1 y_1^0}^{[1][2]}\right]. \end{aligned} \tag{41}$$

Lemma 5, whose proof is in the Appendix, shows that u_2 and $f_{x_m}(e_y)$, chosen as in (38) and (41), accommodate the inaccessible parts of $x_m^{(i)}$.

Lemma 5. *With u_2 as in (38) and $f_{x_m}(e_y)$ as in (41), it follows that*

$$\begin{aligned} b_3\lambda^*(u_2 + (s+1)^2[\underline{\psi}_1^{*T} x_m]) - f_{x_m}(e_y)& \\ = b_1^{a*}[-\widetilde{\underline{\psi}}_1^T x_m + \widetilde{\lambda}\overline{e}_{\Psi_1 y_1^0}^{[1]} + \widetilde{\psi}_3^1 \overline{e}_{\Psi_1 y_1^0}^{[1][2]}]& \end{aligned} \tag{42}$$

with $\widetilde{\lambda}(t) = \lambda(t) - \lambda^*$.

The main result of our design of ℓ, $f_\ell(e_y)$, u, and $f_{x_m}(e_y)$ is that we can rewrite (39) as

$$e_y(t) = W_1^*(s)\lambda^* \widetilde{\Omega}^T \xi + W_1^*(s)\widetilde{\Phi}^T \nu \tag{43}$$

where $\widetilde{\Omega}(t) = \Omega(t) - \Omega^*$ and $\widetilde{\Phi}(t) = \Phi(t) - \Phi^*$ with

$$\begin{aligned} \Omega(t) &= \begin{bmatrix}\underline{\psi}_1^T & \underline{\psi}_2^T & \underline{\psi}_3^T\end{bmatrix}^T, \quad \Omega^* = \begin{bmatrix}\underline{\psi}_1^{*T} & \underline{\psi}_2^{*T} & \underline{\psi}_3^{*T}\end{bmatrix}^T \\ \Phi(t) &= \begin{bmatrix}\psi_3^{1T} & \psi_3^{2T} & \lambda^T\end{bmatrix}^T, \quad \Phi^* = \begin{bmatrix}\psi_3^{1*T} & \psi_3^{2*T} & \lambda^{*T}\end{bmatrix}^T \\ \xi &= \begin{bmatrix}-x_m^T, & -\overline{x}_m^{[1]T}, & -\overline{x}_m^{[2]T}\end{bmatrix}^T \end{aligned} \tag{44}$$

$$\nu = \begin{bmatrix}(\overline{e}_{\Psi_1 y_1^0}^{[1][2]} - \overline{e}_{\psi y}^{[1]} - \overline{e}_{\psi y}^{[1][2]}), & -\overline{e}_{\psi y}^{[2]}, & \overline{e}_{\Psi_1 y_1^0}^{[1]}\end{bmatrix}^T$$

with $\Omega(t)$ and $\Phi(t)$ as estimates of corresponding true parameter values Ω^* and Φ^*. It is noted that (43) represents an SPR error model of the form of (19), which is formally stated in Lemma 6.

Lemma 6. *For the plant model (14), the CRM (17) with ℓ in (24), $f_\ell(e_y)$ in (31), u in (38), and $f_{x_m}(e_y)$ in (41), the error equation (43) is an SPR error model and has a minimal state-space realization:*

$$\dot{e}_{mx} = A_{\ell^*}^* e_{mx} + b_1^{a*}\lambda^* \widetilde{\Omega}^T \xi + b_1^{a*}\widetilde{\Phi}^T \nu, \quad e_y = c^T e_{mx} \tag{45}$$

In the above, it has been assumed that second derivatives of all parameter estimates are realizable. We proceed to define an adaptive law for $\Omega(t)$ and $\Phi(t)$ which makes this possible using high-order tuners.

C. Adaptive Law with High-Order Tuners

We adapt the parameter estimates $\Omega(t)$ and $\Phi(t)$ using high-order tuners, similar to the method in [22]. This is described for each element of $\Omega(t) = [\Omega_1, \cdots, \Omega_{3n}]^T$ by

$$\begin{aligned} \dot{x}_{h,k} &= \left(A_h x_h + b_h \Omega_k^{'T}(t)\right) g(\xi_k) \\ g(\xi_k) &= \left(1 + \mu(\xi_k)^2\right), \qquad \dot{\Omega}_k^{'}(t) = -\xi_k e_y \\ \left[\Omega_k^T(t),\ \ \dot{\Omega}_k^T(t),\ \ \ddot{\Omega}_k^T(t)\right] &= \left[c_h^T x_{h,k},\ \ c_{h,1}^T x_{h,k},\ \ c_{h,2}^T x_{h,k}\right] \end{aligned} \tag{46}$$

where $k = 1, 2, \cdots, 3n$, $\xi(t) = [\xi_1, \cdots, \xi_{3n}]^T$, and

$$\begin{aligned} A_h = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix},\ b_h = \begin{bmatrix}0\\1\end{bmatrix},\ c_h = \begin{bmatrix}1 & 0\end{bmatrix}^T, \\ c_{h,1} = \begin{bmatrix}0 & 1\end{bmatrix}^T,\ c_{h,2} = \begin{bmatrix}-1 & -2\end{bmatrix}^T. \end{aligned} \tag{47}$$

The system (46) consists of a second-order filter and a time-varying gain g, and yields an output $\Omega(t)$. High-order tuners for $\Phi(t)$ are also constructed but omitted here due to high similarity. The definition of gain μ is given later in MIMO case, and therefore omitted here. The structure of these second-order filters implies that up to the second derivatives of their outputs are available for implementation. The following theorem, whose proof can be found in the Appendix, guarantees the overall stability of the closed-loop system.

Theorem 1. *For plant model (14) and the CRM (17), if u is designed as in (38) with parameters adjusted as in (46), i) the closed-loop system has a bounded solution and, ii) $e_y(t) \to 0$ as $t \to \infty$.*

As $e_y(t) \to 0$, $f_\ell(e_y) \to 0$ and $f_{x_m}(e_y) \to 0$ as $t \to \infty$, and therefore CRM (17) recovers its open-loop reference model form as $t \to \infty$. As a result, the tracking goal is achieved.

IV. MIMO Plants with Arbitrary Relative Degree: Problem Statement

We now state the main problem of interest in this paper, which is the adaptive control of a MIMO plant with a uniform input relative degree $r \in \mathbb{N}^+$. Our starting point is a nonlinear plant model, an example of which is a mechanical system that includes rigid body dynamics and flexible dynamics [23].

Following [23] and [18], the nonlinear model can be linearized around a single equilibrium point, leading to a linear plant model

$$\begin{aligned}\dot{x}_p &= A_p x_p + B_p \Theta_p^{*T} x_p + B_p \Lambda^* u_p \\ y_p = C_p x_p, &\qquad z = C_{pz} x_p + D_z \Theta_p^{*T} x_p + D_z \Lambda^* u_p,\end{aligned} \tag{48}$$

similar to (12). In (48) $x_p \in \mathbb{R}^{n_p}$ are states, $u_p \in \mathbb{R}^m$ are control inputs, $y_p \in \mathbb{R}^{p_p}$ are measurement outputs and $z \in \mathbb{R}^d$ are tracking outputs. The control goal is to design u such that z tracks a trajectory z_m from a reference model despite the presence of uncertainties. It is assumed that $p_p + d > m$, thus the plant model is nonsquare. Matrices $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, $C_p \in \mathbb{R}^{p_p \times n_p}$, $B_{pz} \in \mathbb{R}^{n_p \times d}$, $C_{pz} \in \mathbb{R}^{d \times n_p}$ and $D_z \in \mathbb{R}^{d \times m}$ are known, and $\Lambda^* \in \mathbb{R}^{m \times m}$ and $\Theta_p^* \in \mathbb{R}^{n_p \times m}$ are unknown. Λ^* is a diagonal matrix of actuator effectiveness, and Θ_p^* represents state-dependent uncertainties (such as flexible effects). It is assumed that $C_p B_p$ is full rank and therefore the plant model has relative degree one. Suppose in addition, there is additional k^{th}-order dynamics present between the input u_p and an actual available input u, which may be either due to actuator dynamics or other unmodeled effects. This is assumed to be of the form

$$s^k u_p + \sum_{i=1}^{k} (D_i + \Theta_i^{*T}) s^{i-1} u_p = D_1 u \tag{49}$$

where D_i are diagonal matrices representing nominal actuator parameters and $\Theta_i^* \in \mathbb{R}^{m \times m}$ are their uncertainties. When $\Theta_i^* = 0$, this additional dynamics can be written as

$$\underbrace{\begin{bmatrix} s w_u \\ \vdots \\ s^k w_u \end{bmatrix}}_{\dot{x}_{act}} = \underbrace{\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -D_1 & \cdots & -D_k \end{bmatrix}}_{A_{act}} \underbrace{\begin{bmatrix} w_u \\ \vdots \\ s^{k-1} w_u \end{bmatrix}}_{x_{act}} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ D_1 \end{bmatrix}}_{B_{act}} \Lambda^* u \tag{50}$$

where $w_u = \Lambda^* u_p$. For command tracking performance, we introduce integral error states $w_z = \int (z - z_{cmd}) dt$. An augmented plant model with $x = \begin{bmatrix} x_p^T & x_{act}^T & w_z^T \end{bmatrix}^T$ can written compactly as

$$\begin{aligned}\dot{x} &= \left(A + B_1 \Psi_1^{*T} + B_r \Psi_r^{*T}\right) x + B_r \Lambda^* u + B_z z_{cmd} \\ y &= Cx, \qquad z = C_z x + D_z \Psi_1^{*T} x\end{aligned} \tag{51}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are redefined states, inputs and outputs, respectively. It can be seen that (51) has an arbitrary relative degree $r = k+1$, and is a MIMO linear plant with parametric uncertainties Ψ_1^*, Ψ_r^*, and Λ^*. This augmented model is defined by nominal state, input, and output matrices

$$\begin{aligned} A &= \begin{bmatrix} A_p & \begin{bmatrix} B_p & 0 \end{bmatrix} & 0 \\ 0 & A_{act} & 0 \\ C_{pz} & \begin{bmatrix} D_z & 0 \end{bmatrix} & 0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} B_p \\ 0 \\ D_z \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ B_{act} \\ 0 \end{bmatrix}, \quad B_z = \begin{bmatrix} 0 \\ 0 \\ -I \end{bmatrix} \\ C &= \begin{bmatrix} C_p & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad C_z = \begin{bmatrix} C_{pz} & \begin{bmatrix} D_z & 0 \end{bmatrix} & 0 \end{bmatrix} \end{aligned} \tag{52}$$

as well as uncertainty matrices given by

$$\Psi_1^* = \begin{bmatrix} \Theta_p \\ 0 \\ 0 \end{bmatrix}, \quad \Psi_r^* = -\begin{bmatrix} 0 \\ \begin{bmatrix} \Theta_1^{*T} & \cdots & \Theta_k^{*T} \end{bmatrix}^T \\ 0 \end{bmatrix} (D_1)^{-1} \tag{53}$$

The matrices $A \in \mathbb{R}^{n \times n}$, $B_1, B_r \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $B_z = \mathbb{R}^{n \times d}$, $C_z \in \mathbb{R}^{d \times n}$ and $D_z \in \mathbb{R}^{d \times m}$ are known, while $\Psi_1^*, \Psi_r^* \in \mathbb{R}^{n \times m}$ are unknown. The uncertainties $B_1 \Psi_1^{*T}$ are from the original plant model with the relative degree one input path B_1, and $B_r \Psi_r^{*T}$ from the actuator dynamics with the relative degree r input path B_r. Although not shown in (51), the considered plant model can have other uncertainties $B_i \Psi_i^{*T}$ for $i = 2, \cdots, k$ with relative degree i input path B_i.

We will present the control design for (51) with second-order additional dynamics in Section V and then extend the design to higher-order dynamics models in Section VI. The adaptive controller that we will present requires the following assumptions regarding the plant model (51):

Assumption 1. *(A, B_r, C) is a minimal realization;*

Assumption 2. *$\{A, B_r, C\}$ has stable transmission zeros;*

Assumption 3. *$\{A, B_i, C\}$ has uniform input relative degree i, for $i = 1, 2, \cdots, r$;*

Assumption 4. *B_i can be spanned by a linear combination of $[B_r, AB_r, \cdots, A^{r-i} B_r]$ for $i = 1, 2, \cdots, (r-1)$;*

Assumption 5. *For $i = 1, 2, \cdots, (r-1)$, Ψ_i^* satisfies*

$$\Psi_i^{*T} \begin{bmatrix} B_r & AB_r & \cdots & A^{r-2} B_r \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix};$$

Assumption 6. *$\|\Psi_i^*\|$ are bounded by a known value, respectively, i.e. $\|\Psi_i^*\| < \Psi_{max}$ for $i = 1, 2, \cdots, r$;*

Assumption 7. *Λ^* is diagonal and $\|\Lambda^*\|$ is bounded by a known value, $\|\Lambda^*\| < \Lambda_{max}$ and $sign(\Lambda^*)$ is known.*

Assumption 1 is standard. Assumption 2 is ubiquitous for adaptive control. It is noted that plant models with stable transmission zeros do not require zeros of each individual transfer functions to be stable. For nominal MIMO plant models satisfying Assumptions 1 and 2, a baseline observer-based controller (such as LQG/LTR [1]) can be designed to achieve a satisfactory tracking performance with adequate stability margins. For adaptive control, additional assumptions on the plant model are needed. Assumption 3 implies $\begin{bmatrix} (C_p B_p)^T & D_z^T \end{bmatrix}$ is full rank (see [16] for justifications). Assumption 4 and 5 are always satisfied if the plant model has the structure as in (51) (see Proposition 2 for derivations). Assumption 6 implies that an upper bound on the magnitude of parametric uncertainties can be established, which is reasonable in most applications. Assumption 7 implies that actuator anomalies are physically bounded and are independent of each other. In Section V, we show that under the above assumptions, the control goal mentioned above can be achieved when relative degree $r = 3$. Extension to the case when $r > 3$ is addressed in Section VI.

V. Adaptive Output-Feedback Control of MIMO Relative Degree Three Plant

This section extends the adaptive controller presented in Section III to the relative degree three MIMO plant model. A complete adaptive control design is summarized in Section V-A. An error model is derived with guaranteed SPR properties in Section V-B. High-order tuners for parameter adaptation, along with a stability proof, are presented in Section

V-C. For simplicity, we first design controller assuming (51) is square, i.e. $p = m$. Extension to nonsquare cases is discussed in Section V-D.

A. Control Design

The first step in designing the adaptive control input is to choose a reference model. This is accomplished using a modified closed-loop reference model (CRM) [10], [12], [21] given by

$$\begin{aligned}\dot{x}_m &= A_m x_m + B_z z_{cmd} + L e_y + F_L(e_y) + F_{x_m}(e_y)\\ y_m &= C x_m, \qquad z_m = C_z x_m\end{aligned} \tag{54}$$

where $e_y = y - y_m$, $A_m = A - BK^T$ and $K \in \mathbb{R}^{n\times m}$ is a nominal feedback gain matrix designed for the system without uncertainty, using the linear quadratic regulator (LQR) technique, for example. Similar to (24), L is designed as

$$L = B_1^a R^{-1} S \tag{55}$$

which utilizes

$$S = (CB_1^a)^T, \qquad \overline{C} = SC, \tag{56}$$

$$R^{-1} = (\overline{C}B_1^a)^{-1}[\overline{C}AB_1^a + (\overline{C}AB_1^a)^T](\overline{C}B_1^a)^{-1} + \varepsilon I. \tag{57}$$

$\varepsilon > \overline{\varepsilon} > \varepsilon^*$ is chosen with $\overline{\varepsilon}$ defined as

$$\overline{\varepsilon} = \max[\varepsilon_1, \varepsilon_2] \tag{58}$$

with definitions of ε_1 and ε_2 given in the Appendix, so that an underlying SPR condition is guaranteed in Section V-B. Similar to (31) and (41), $F_L(e_y)$ and $F_{x_m}(e_y)$ are designed as

$$\begin{aligned}F_L(e_y) = F_2(\psi_3^{1T}(t)\overline{e}_{\psi y}^{[1]}) &+ F_3(\psi_3^{1T}(t)\overline{e}_{\psi y}^{[1][2]})\\ &+ F_3(\psi_3^{2T}(t)\overline{e}_{\psi y}^{[2]}),\end{aligned} \tag{59}$$

$$F_{x_m}(e_y) = -F_2(\Lambda^T(t)\overline{e}_{\Psi_1 y_1^0}^{[1]}) - F_3(\psi_3^{1T}(t)\overline{e}_{\Psi_1 y_1^0}^{[1][2]}). \tag{60}$$

where $F_i(w(t))$ are defined as

$$\begin{aligned}F_2(w(t)) &= \left(B_2^a s + B_3^a a_2^0\right)[w(t)]\\ F_3(w(t)) &= B_3^a \pi_2^2(s)[w(t)]\end{aligned} \tag{61}$$

and B_i^a, similar to (25), are defined as

$$\begin{aligned}B_1^a &= A^2 B_3 a_2^2 + AB_3 a_2^1 + B_3 a_2^0\\ B_2^a &= AB_3 a_2^2 + B_3 a_2^1, \qquad B_3^a = B_3 a_2^2\end{aligned} \tag{62}$$

which are relative degree i input paths for $\{A, B_i^a, C\}$. $\psi_3^1(t)$ and $\psi_3^2(t)$ are estimates of $\psi_3^{1*} \in \mathbb{R}^{m\times m}$ and $\psi_3^{2*} \in \mathbb{R}^{m\times m}$, which are elements of a transformed Ψ_3^*. $\Lambda(t)$ is the estimate of Λ^*. $\pi_2^i(s)$ are defined in (1). We will define high-order tuners for parameter adaptation in Section V-C so that signals $\ddot{\Lambda}(t)$, $\ddot{\psi}_3^1(t)$ and $\ddot{\psi}_3^2(t)$ are accessible. We proceed to define filtered error signals $\overline{e}_{[]}^{[]}$. Utilizing the scaled error

$$e_{sy} = R^{-1} S e_y \tag{63}$$

we then define $\overline{e}_{\psi y}^{[i][j]}(t)$, similar to (29), as filter outputs of

$$\begin{aligned}&\overline{e}_{\psi y}^{[1]}(t) = \frac{\pi_2^1(s)}{\pi_2^2(s)} e_{sy}(t), \qquad \overline{e}_{\psi y}^{[2]}(t) = \frac{\pi_2^0(s)}{\pi_2^2(s)} e_{sy}(t)\\ &\overline{e}_{\psi y}^{[1][2]}(t) = \frac{s\cdot\pi_2^0(s)}{\pi_2^2(s)}\left[\psi_3^{1T}(t)\overline{e}_{\psi y}^{[1]}(t)\right],\end{aligned} \tag{64}$$

where S and R^{-1} are define in (56) and (57), respectively. Using scaled integral error signal

$$e_{y_1^0}(t) = \int_0^t L e_y(\tau) d\tau \tag{65}$$

we define $\overline{e}_{\Psi_l y_i^{i-l}}^{[i][j]}$, similar to (40), as filter outputs of

$$\begin{aligned}\overline{e}_{\Psi_1 y_1^0}^{[1]}(t) &= \frac{s\cdot\pi_2^1(s)}{\pi_2^2(s)}\left[\underline{\Psi}_1^T(t) e_{y_1^0}(t)\right]\\ \overline{e}_{\Psi_1 y_1^0}^{[1][2]}(t) &= \frac{s\cdot\pi_2^0(s)}{\pi_2^2(s)}\left[\Lambda^T(t)\overline{e}_{\Psi_1 y_1^0}^{[1]}(t)\right].\end{aligned} \tag{66}$$

where $\Lambda(t)$ and $\underline{\Psi}_1(t)$ are estimates of parameters Λ^* and $\underline{\Psi}_1^*$ (defined in Proposition 4). It is noted that jth derivative of $(\cdot)^{[i][j]}$ is available for implementation.

The F_L term in the CRM, together with L, accommodates the uncertainties introduced by forming an SPR error model (see Section V-B). F_{x_m}, on the other hand, accommodates the inaccessibility of $\ddot{x}_m$, which is required to address $B_1\Psi_1^{*T}x$ in the plant model. We define synthetic variables $v_m^{[i]}$ (the accessible parts of $x_m^{(i)}$) as

$$\begin{aligned}v_m^{[2]} &= (A^2 x_m + AB_z z_{cmd} + B_z \dot{z}_{cmd} - ALe_y)\\ v_m^{[1]} &= (Ax_m + B_z z_{cmd}), \qquad v_m^{[0]} = (x_m).\end{aligned} \tag{67}$$

Similar to the SISO case, we introduce filtered CRM states

$$\overline{x}_m^{[1]} = \frac{\pi_2^1(s)}{\pi_2^2(s)} x_m, \qquad \overline{x}_m^{[2]} = \frac{\pi_2^0(s)}{\pi_2^2(s)} x_m. \tag{68}$$

and note that ith derivatives of $\overline{x}_m^{[i]}$ are available for control. The complete control law, similar to (38), is defined as

$$\begin{aligned}u = &-\sum_{i=0}^{2}\sum_{k=0}^{i} a_2^i d_i^k \underline{\Psi}_1^{(i-k)T} v_m^{[k]} - \sum_{i=1}^{2} a_2^i d_i^i \underline{\Psi}_1^{(i)T}(t) e_{y_1^0}\\ &+ \pi_2^2(s)\left[-\underline{\Lambda}^T(t)K^T\overline{x}_m^{[2]} - \underline{\Psi}_2^T(t)\overline{x}_m^{[1]} - \underline{\Psi}_3^T(t)\overline{x}_m^{[2]}\right]\end{aligned} \tag{69}$$

The adjustable parameters $\underline{\Lambda}(t)$ and $\underline{\Psi}_i(t)$ are estimates of Λ^{*-1} and $\underline{\Psi}_i^*$, respectively, which will be defined in Section V-C. The complete adaptive controller is specified by Eqs. (54)-(69), with only the parameter adaptation rules remaining to be defined in Section V-C. It should be noted that provided the higher-order derivatives of the parameters $\underline{\Psi}_i(t)$, $\psi_3^i(t)$, and $\underline{\Lambda}(t)$ are realizable in (59), (60) and (69), this controller is implementable.

B. Analysis of Error Dynamics

The following paragraphs show that the control design in the previous section yields an underlying SPR error model. The error equation is derived by subtracting the CRM (54) from the plant model (51), which yields (similar to (18))

$$\begin{aligned}\dot{e}_x = &(A^* - LC)e_x - F_L(e_y) - F_{x_m}(e_y)\\ &+ B_1\Psi_1^{*T}x_m + B_3(\Psi_3^* + K)^T x_m + B_3\Lambda^* u\end{aligned} \tag{70}$$

where $A^* = (A + B_1\Psi_1^{*T} + B_3\Psi_3^{*T})$. The relative degree i input paths for (70) are B_i^{a*}, which, similar to (20), are defined as

$$\begin{aligned}B_1^{a*} &= A^{*2} B_3 a_2^2 + A^* B_3 a_2^1 + B_3 a_2^0\\ B_2^{a*} &= A^* B_3 a_2^2 + B_3 a_2^1, \qquad B_3^{a*} = B_3 a_2^2,\end{aligned} \tag{71}$$

are the unknown counterparts of (62). Define $A_L^* = A^* - LC$. It is noted that B_i^{a*} are also the relative degree i input paths for $\{A_L^*, B_i^{a*}, C\}$ for any $L \in \mathbb{R}^{n\times m}$.

The error model analysis will be a direct extension of its SISO case in Section III-B, and therefore only pertinent Lemmas will be provided in the following. The following Lemma (proof given in Appendix), which is an extension of Lemma 3, guarantees an SPR transfer function.

Lemma 7. *Choosing R^{-1} as in (55) and $\varepsilon > \overline{\varepsilon}$ as in (58), $W_1^*(s) = \overline{C}(sI - A^* + L^*C)^{-1}B_1^{a*}$ is SPR, where $L^* = B_1^{a*}R^{-1}S$.*

The goal of the rest of the section is to demonstrate how the control design leads to an SPR error model which include $W_1^*(s)$ and has the form of Definition 3. To realize L^* in $W_1^*(s)$ while using L in the CRM, we first address the difference between B_i^{a*} and B_i^a in the following Lemma (an extension of Lemma 4). The proof can be found in the Appendix.

Lemma 8 (Recursive Properties of B_i^a)**.** *B_i^a as in (62) and B_i^{a*} as in (71) satisfies*

$$\begin{aligned}
&(B_1^{a*} - B_1^a) = B_2^{a*}\psi_3^{1*T} + B_3^{a*}\psi_3^{2*T} \\
&(B_2^{a*} - B_2^a) = B_3^{a*}\psi_3^{1*T}, \qquad (B_3^{a*} - B_3^a) = 0
\end{aligned} \tag{72}$$

where unknown terms ψ_3^{i*} are elements of a transformed Ψ_3^*.

The difference $(B_1^{a*} - B_1^a)$ is addressed in $F_L(e_y)$ as in (59). $F_{x_m}(e_y)$ in (60), together with u, addresses $B_i\Psi_i^{*T}x_m$ for $i = 1, 3$ that comes with the plant model (51). For analysis, $B_i\Psi_i^{*T}x_m$ is rewritten in the range of B_i^{a*} using the following Proposition (an extension of Proposition 3). The proof is straightforward by following the proof of Lemma 8.

Proposition 4. *For plant (A, B_3, C) satisfying Assumptions 3 and 4, there exist $\underline{\Psi}_1^*$, $\underline{\Psi}_2^*$ and $\underline{\Psi}_3^*$ such that*

$$\begin{aligned}
&B_1\Psi_1^{*T} + B_3\Psi_3^{*T} \\
&\qquad = B_1^{a*}\Lambda^*\underline{\Psi}_1^{*T} + B_2^{a*}\Lambda^*\underline{\Psi}_2^{*T} + B_3\Lambda^*\underline{\Psi}_3^{*T}
\end{aligned} \tag{73}$$

where B_i^{a*} is defined in (72) and $\underline{\Psi}_1^* = \begin{bmatrix} \times & 0 & 0 & 0 \end{bmatrix}^T$.

Both the error terms in (72) and (73) are in the range of B_i^{a*} and can be addressed by $F_L(e_y)$ and $F_{x_m}(e_y)$ in the error model analysis together with the following Lemma, whose proof is in the Appendix.

Lemma 9 (Recursive Adaptation)**.** *Suppose an error model*

$$\dot{e}_x = A_L^* e_x + B_2^{a*}\phi^{*T}\omega(t) - F(t), \quad e_y = Ce_x \tag{74}$$

where $A_L^* = A^* - LC$ with any $L \in \mathbb{R}^{n\times m}$, B_2^{a*} defined in (71), $\{A, B_3, C\}$ satisfies Assumptions 3 to 5, ϕ^* is unknown but constant, and $\omega(t)$ is a known regressor with $\dot{\omega}(t)$ inaccessible, then $F(t)$ can be chosen as

$$F(t) = F_2(\phi^T(t)\overline{\omega}^{[1]}(t)) + F_3(\psi_3^{1T}(t)\overline{\omega}^{[1][2]}(t)) \tag{75}$$

where F_i are defined in (61), $\phi(t)$ is an estimate of ϕ^* with $\ddot{\phi}(t)$ accessible, $\psi_3^1(t)$ is an estimate of ψ_3^{1*} as in (72) with $\ddot{\psi}_3^1(t)$ accessible, and $\overline{\omega}^{[1]}(t)$ and $\overline{\omega}^{[1][2]}(t)$ are defined as

$$\begin{aligned}
\overline{\omega}^{[1]}(t) &= \frac{\pi_2^1(s)}{\pi_2^2(s)}\omega(t) \\
\overline{\omega}^{[1][2]}(t) &= \frac{s\cdot\pi_2^0(s)}{\pi_2^2(s)}\left[\phi^T(t)\overline{\omega}^{[1]}(t)\right]
\end{aligned} \tag{76}$$

such that the error model can be transformed into

$$\begin{aligned}
\dot{e}'_x &= A^*e'_x - B_1^{a*}\widetilde{\phi}^T(t)\overline{\omega}^{[1]}(t) - B_1^{a*}\widetilde{\psi}_3^{1T}\overline{\omega}^{[1][2]}(t) \\
e_y &= Ce'_x
\end{aligned} \tag{77}$$

where $\widetilde{\phi}(t) = \phi(t) - \phi^*$, $\widetilde{\psi}_3^1(t) = \psi_3^1(t) - \psi_3^{1*}$ and

$$\begin{aligned}
e'_x = &e_x + B_3^{a*}\phi^{*T}\omega(t) - B_3^{a*}\phi^T(t)\omega(t) \\
&+ \left[B_2^{a*} + B_3^{a*}s\right]\left[\widetilde{\phi}^T(t)\overline{\omega}^{[1]}(t) + \widetilde{\psi}_3^{1T}\overline{\omega}^{[1][2]}(t)\right].
\end{aligned} \tag{78}$$

Remark 3. If in (51) $\Psi_3^* = 0$, then $\underline{\Psi}_3^* = 0$ and $\psi_3^{1*} = 0$. As a result, we can choose $\psi_3^1(t) \equiv 0$ and all terms with $\overline{\omega}^{[1][2]}$ will be zero in $F(t)$.

The design of $F_L(e_y)$ (59) and $F_{x_m}(e_y)$ (60) follows Lemma 9 and leads to SPR error dynamics as shown in the following Lemma (an extension of Lemma 6), whose proof can be found in the Appendix.

Lemma 10. *For the plant model (51) satisfying Assumptions 1 to 7, with the CRM in (54) and u in (69), the error model (70) is strictly positive real and has a minimal state-space realization*

$$\begin{aligned}
\dot{e}_{mx} &= A_{L^*}^* e_{mx} + B_1^{a*}\Lambda^*\widetilde{\Omega}^T\xi + B_1^{a*}\widetilde{\Phi}^T\nu \\
e_y &= Ce_{mx}
\end{aligned} \tag{79}$$

where $A_{L^*}^* = A^* - L^*C$, B_1^{a*} is defined in (71) and

$$\begin{aligned}
\xi &= \begin{bmatrix} -K^T\overline{x}_m^{[2]} \\ -x_m \\ -\overline{x}_m^{[1]} \\ -\overline{x}_m^{[2]} \end{bmatrix}, \quad \nu = \begin{bmatrix} -\overline{e}_{\psi y}^{[1]} - \overline{e}_{\psi y}^{[1][2]} + \overline{e}_{\Psi_1 y_1^0}^{[1][2]} \\ -\overline{e}_{\psi y}^{[2]} \\ \overline{e}_{\Psi_1 y_1^0}^{[1]} \end{bmatrix} \\
\Omega^* &= \begin{bmatrix} \Lambda^{*-T} & \underline{\Psi}_1^{*T} & \underline{\Psi}_2^{*T} & \underline{\Psi}_3^{*T} \end{bmatrix}^T \\
\Omega(t) &= \begin{bmatrix} \underline{\Lambda}^T(t) & \underline{\Psi}_1^T(t) & \underline{\Psi}_2^T(t) & \underline{\Psi}_3^T(t) \end{bmatrix}^T \\
\Phi^* &= \begin{bmatrix} \psi_3^{1*T} & \psi_3^{2*T} & \Lambda^{*T} \end{bmatrix}^T \\
\Phi(t) &= \begin{bmatrix} \psi_3^{1T}(t) & \psi_3^{2T}(t) & \Lambda^T(t) \end{bmatrix}^T
\end{aligned} \tag{80}$$

with $\widetilde{\Omega}(t) = \Omega(t) - \Omega^*$ and $\widetilde{\Phi}(t) = \Phi(t) - \Phi^*$.

(79) is an SPR error model and will be used in the stability analysis which follows the description of high-order tuners for online adaptation of $\Omega(t)$ and $\Phi(t)$.

C. Adaptive Law with High-Order Tuners

In this section, a high-order tuner adaptive law [22] is used for estimates $\Omega(t)$ and $\Phi(t)$ as in (80), such that $\ddot{\Omega}(t)$ and $\ddot{\Phi}(t)$ are generated and the control input (69) and CRM (54) are realizable. The design is very similar to (46) except that the equations presented here are in matrix form. The adjustable parameters $\Omega(t) : \mathbb{R} \to \mathbb{R}^{(3n)\times m}$ and $\Phi(t) : \mathbb{R} \to \mathbb{R}^{3m\times m}$ are outputs of tuners given by

$$\begin{aligned}\dot{X}_\Omega &= \big(A_H X_\Omega + B_H \Omega^{'T}\big) g(\xi;\mu_\xi), \quad \Omega^T(t) = C_H X_\Omega \\ \dot{X}_\Phi &= \big(A_H X_\Phi + B_H \Phi^{'T}\big) g(\nu;\mu_\nu), \quad \Phi^T(t) = C_H X_\Phi\end{aligned} \tag{81}$$

where $X_\Omega(t) : \mathbb{R} \to \mathbb{R}^{2m\times(m+3n)}$ and $X_\Phi(t) : \mathbb{R} \to \mathbb{R}^{2m\times 3m}$ are augmented tuner state matrices. $\Omega^{'}(t)$ and $\Phi^{'}(t)$ are adjusted using

$$\begin{aligned}\dot{\Omega}^{'}(t) &= -\Gamma \xi e_y^T S^T sign(\Lambda^*) \\ \dot{\Phi}^{'}(t) &= -\Gamma \nu e_y^T S^T\end{aligned} \tag{82}$$

where $\Gamma = \gamma I > 0$ are adaptation gains. $A_H \in \mathbb{R}^{2m\times 2m}$, $B_H \in \mathbb{R}^{2m\times m}$ and $C_H \in \mathbb{R}^{m\times 2m}$ are block diagonal matrices with A_h, b_h and c_h as their diagonal elements, respectively. (A_h, b_h, c_h), similar to (47), is chosen as

$$A_h = \begin{bmatrix} 0 & 1 \\ -a_2^0/a_2^2 & -a_2^1/a_2^2 \end{bmatrix}, \; b_h = \begin{bmatrix} 0 \\ a_2^0/a_2^2 \end{bmatrix}, \; c_h = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{83}$$

such that $c_h(sI - A_h)^{-1} b_h = \frac{\pi_2^0(s)}{\pi_2^2(s)}$. Then the derivatives of $\Omega(t)$ and $\Phi(t)$ can be realized by

$$\begin{aligned}\dot{\Omega}^T(t) &= C_H^1 X_\Omega, \qquad \ddot{\Omega}^T(t) = C_H^2 X_\Omega \\ \dot{\Phi}^T(t) &= C_H^1 X_\Phi, \qquad \ddot{\Phi}^T(t) = C_H^2 X_\Phi\end{aligned} \tag{84}$$

where $C_H^1 \in \mathbb{R}^{m\times 2m}$ and $C_H^2 \in \mathbb{R}^{m\times 2m}$ are block diagonal matrices with $c_h^1 = [0, 1]$ and $c_h^2 = \frac{-1}{a_2^2}[a_2^0, a_2^1]$ as their diagonal block elements, respectively. The time-varying gain $g(x;\mu)$ in (81) is given by

$$g(x;\mu) = 1 + \mu x^T x \tag{85}$$

with scalar gain μ defined in the Appendix for brevity in the text. With the high-order tuner (81) and the tracking error defined as $e_z(t) = (z - z_m)$, the global stability and asymptotic tracking of the adaptive system can be guaranteed in the following theorem, whose proof can be found in the Appendix.

Theorem 2. *For the plant (51) that satisfies Assumptions 1 to 7, and for any $z_{cmd}(t)$ that is piecewise continuous, the adaptive controller in (54)-(85), guarantees that i) the closed-loop system has bounded solutions, ii) $e_y(t) \to 0$ as $t \to \infty$, and iii) $e_z(t) \to 0$ as $t \to \infty$.*

D. Extension to Nonsquare Plant Models

This section extends the control design in Section V to a plant model whose number of outputs exceeds that of inputs, i.e. $p > m$. The design for plants whose number of inputs exceeds that of outputs, however, is similarly derived through duality. Define n_z as the number of transmission zeros in the plant model (51). The control design (see [18] for details) incorporates a procedure which adds fictitious inputs to the system (the squaring-up procedure), requiring an additional assumption on the plant model:

Assumption 8. *The dimensions of $B_3 \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ satisfy that $(n - 3m - n_z) \geq (p - m)$.*

The following Lemma shows that a $\overline{B}_3 \in \mathbb{R}^{n\times p}$ can be found by squaring-up B_3. The proof is similar to [24] and therefore is omitted here.

Lemma 11. *For plant models satisfying Assumptions 1 to 3 and 8, there exists a $B_{s1} \in \mathbb{R}^{n\times m_s}$ such that $\{A, \overline{B}_3, C\}$, where $\overline{B}_3 = [B_3, B_{s1}]$, has stable transmission zeros and nonuniform relative degree with $r_i = 3$ for $i = 1, 2, \cdots, m$ and $r_i = 1$ for $i = m+1, m+2, \cdots, p$.*

All equations outlined in the control design in Section V-A can be adopted with a few modifications. The CRM error feedback gain L will use (55) through (58) modified to replace B_1^a with $\overline{B}_1^a$, where

$$\overline{B}_1^a = [B_1^a, B_{s1}], \; B_1^a = A^2 B_3 a_2^2 + A B_3 a_2^1 + B_3 a_2^0. \tag{86}$$

Then L guarantees the SPR property of $\{A_{L^*}^*, \overline{B}_1^{a*}, \overline{C}\}$ where $\overline{B}_1^{a*} = [B_1^{a*}, B_{s1}]$, and $B_1^{a*} = A^{*2} B_3 a_2^2 + A^* B_3 a_2^1 + B_3 a_2^0$. The partition $S = [S_3^T, S_{s1}^T]^T$ with $S_3 \in \mathbb{R}^{m\times p}$ leads to the SPR properties $\{A_{L^*}^*, B_1^{a*}, S_3 C\}$ using the Kalman–Yakubovich–Popov (KYP) Lemma. As a result, Lemma 7 holds. Also, Lemma 8 holds with $\overline{B}_1^a$ and $\overline{B}_1^{a*}$ (instead of B_1^a and B_1^{a*}). Since L is designed using $\overline{B}_1^a$, $F_L(e_y)$ in (59) is modified to use $\overline{B}_3$ and $\overline{B}_2^a$ (in instead of B_3 and B_2^a). Since u enters the plant through B_3, F_{x_m} in (60) is unchanged. In (82), $\Omega^{'}(t)$ is adjusted using S_3 instead of S. The rest of the design follows Section V-B and Section V-C.

VI. Extension to Higher-Order Actuator Models

This section extends the MIMO adaptive output-feedback control design in Section V to the plant models (51) in Section IV with arbitrary relative degree r. The design in this section will be a direct extension of the design in Section V for relative degree three cases and therefore only key steps are presented.

L and S in (55) and u in (69) can be extended using

$$B_i^a = \sum_{j=0}^{r-i} a_{r-1}^{r-1-j} A^{r-i-j} B_r, \; i = 1, 2, \cdots r, \tag{87}$$

(an extension of (62)) to guarantee the SPR property of $\{A_{L^*}^*, B_1^{a*}, SC\}$ with $A^* = (A + B_1 \Psi_1^{*T} + B_r \Psi_r^{*T})$ and

$$B_i^{a*} = \sum_{j=0}^{r-i} a_{r-1}^{r-1-j} A^{*\{r-i-j\}} B_r, \; i = 1, 2, \cdots r, \tag{88}$$

(an extension of (71)). The design of F_L and F_{x_m} in the CRM, on the other hand, requires two key lemmas, i.e. Lemma 8 and Lemma 9, to be extended to relative degree r cases. We first present the extension of Lemma 8, which specifies the difference between B_i^a and B_i^{a*} and is used to formulate $L^* = B_1^{a*} R^{-1} S$. Its proof is omitted due to high similarity.

Lemma 12 (Recursive Properties of B_i^a)**.** *B_i^a as in (87) and B_i^{a*} (88) satisfy*

$$B_i^{a*} - B_i^a = B_{i+1}^{a*} \psi_r^{1*T} + \cdots + B_r^{a*} \psi_r^{[r-i]*T} \tag{89}$$

for $i = 1, 2, \cdots, r$ with ψ_r^{i*} being elements of a transformed Ψ_r^*.

It's noted that in (89) and the remaining part of this section that terms with index $^{[]} > r$ and $^{[]} \leq 0$ are zeros. Define

$$\pi_{r-1}^{i}(s) = \sum_{j=0}^{i} a_{r-1}^{r-1-j} s^{i-j}, \quad \underline{\pi}_{r-1}^{i}(s) = \sum_{j=0}^{i} a_{r-1}^{j} s^{j} \tag{90}$$

such that the property in (2) holds as $\pi_{r-1}^{i}(s) \cdot s^{r-i-1} + \underline{\pi}_{r-1}^{r-i-1}(s) = \pi_{r-1}^{r-1}(s)$. Similar to (61), define

$$F_i(x) = B_i^a s^{i-1}[x] + B_r \underline{\pi}_{r-1}^{i-2}(s)[x]. \tag{91}$$

Proposition 4 can be extended similarly. Then all uncertainty terms can be rewritten in the range of B_i^{a*} and addressed in the following Lemma (an extension of Lemma 9).

Lemma 13 (Recursive Adaptation)**.** *Using the same notation and under the same assumptions as in Lemma 9, for an error equation*

$$\dot{e}_x = A_L^* e_x + B_i^{a*} \phi^{*T} \omega(t) - F(t), \qquad e_y = C e_x \tag{92}$$

with $i = 1, 2, \cdots, r$, $F(t)$ can be chosen as

$$\begin{aligned} F(t) = & F_i(\phi^T(t)\overline{\omega}^{[i-1]}(t)) \\ & + \sum_{j=1}^{r-i} \Big[F_{i+j}(\psi_r^{jT}(t)\overline{\omega}^{[i-1][i-1+j]}(t)) \\ & + \sum_{k=1}^{r-i-j} F_{i+j+k}(\psi_r^{kT}(t)\overline{\omega}^{[i-1][i-1+j][i-1+j+k]}(t)) + \cdots \Big] \end{aligned} \tag{93}$$

where the sequence "$\cdots$" stops when the last index of $\overline{\omega}^{[]\cdots[]}$ reaches $[r-1]$. F_i is defined as in (91), $\overline{\omega}^{[i]}(t)$ is chosen as

$$\overline{\omega}^{[i]}(t) = \frac{\pi_{r-1}^{r-1-i}(s)}{\pi_{r-1}^{r-1}(s)} \omega(t), \tag{94}$$

and $\overline{\omega}^{[i]\cdots[j][k]}(t)$ for $(j+1) \leq k \leq (r-1)$ is chosen as

$$\overline{\omega}^{[i]\cdots[j][k]}(t) = \frac{\pi_{r-1}^{r-1-k}(s)}{\pi_{r-1}^{r-1}(s)} \cdot s^{\{j\}} \left[\phi^T(t) \overline{\omega}^{[i]\cdots[j]}(t) \right], \tag{95}$$

such that the error model can be transformed into

$$\begin{aligned} \dot{e}_x' = & A_L^* e_x' + B_1^{a*} \widetilde{\phi}^T(t) \overline{\omega}^{[i-1]}(t) \\ & + B_1^{a*} \sum_{j=1}^{r-i} \big[\widetilde{\psi}_r^{jT} \overline{\omega}^{[i-1][i-1+j]}(t) \\ & + \sum_{k=1}^{r-i-j} \widetilde{\psi}_r^{kT}(t) \overline{\omega}^{[i-1][i-1+j][i-1+j+k]}(t) + \cdots \big] \\ e_y = & C e_x' \end{aligned} \tag{96}$$

where the sequence "$\cdots$" stops when the last index of $\overline{\omega}^{[]\cdots[]}$ reaches $[r-1]$.

Remark 4. The relation between e_x' and e_x can be found using Lemma 2 (see Lemma 9 for example).

Remark 5. Lemma 9 is a special case of Lemma 13 when the input path $i = 2$, i.e. $B_i^{a*} = B_2^{a*}$ in (92) and $r = 3$.

Remark 6. While in (93) all $\overline{\omega}^{[i]\cdots[j][k]}(t)$ are accessible due to the filters designs in (95), the term $F(t)$ is designed assuming all the required derivatives of ϕ^T and $\psi_r(t)$ are accessible, which is achieved using a high-order tuner similar to (81).

Remark 7. It's noted that the summation sequences in (93) are nested due to the recursive properties of B_i^{a*} as in (89, which determines that the number of integrators used is in the order of $O((n+m)r^2 + mr(n+mr))$. ($F(t)$ requires at most $O((n+m)r^2)$ and high order tuners at most $O(mr(m+nr))$). For comparison, the number of integrators in the classical adaptive control approach [5] is in the order of $O(m^2n^2r^2)$.

Following the design in Section III-B, Lemma 13 can be used to design $F_L(e_y)$ (similar to 59), which addresses $B_i^{a*}\psi_r^{i*T} x_m$ in (89), and $F_{x_m}(e_y)$ (similar to 60), which addresses $B_i^{a*}\Lambda^*\underline{\Psi}_i^{*T} x_m$ that comes from (51). The analysis will lead to an SPR error model similar with that in Lemma 10. The analysis, along with the stability proof, is highly similar to Theorem 2 and therefore is omitted here.

VII. Applications to VFA

This section applies the relative degree three adaptive controller to a nonlinear 3-wing very-flexible aircraft (VFA) model. The aircraft features three rigid wings hinged side-by-side [25], and the outer wings can rotate with respect to the center wing about the longitudinal axis (i.e. dihedral angle). The platform captures essential flexible wing effects and can be viewed as building blocks of large VFA. A 6-state nonlinear model has been developed in [25, Eq.s (45) and (46)] for the aircraft's pitch mode and dihedral dynamics. This model has states V as airspeed, α as the angle of attack, θ as pitch angle, q as pitch rate and η as the dihedral angle. Linearization of the model around an equilibrium yields a MIMO plant model as shown in (48) with uncertainties caused by flexible effects [18]. Measurements are vehicle vertical acceleration A_z, dihedral angle η and pitch rate q. Other states, α and $\dot{\eta}$, are unmeasurable and can not be used for control. The goal is to use center elevators δ_e and outer ailerons δ_a to track A_z commands and simultaneously regulate η.

The model is linearized around each of 25 trim points defined by $V_0 = 30$ ft/sec, $\alpha_0 = 0°$, $\theta_0 = 0°$, $q = 0°$/sec, $\eta_0 \in [10, 12]°$ with a step of 0.5, and $\dot{\eta}_0 \in [-0.2, 0.2]°$/sec with a step of 0.1. Numerical values for the linearized model are given in (97) for $\eta_0 = 10°$ and $\dot{\eta}_0 = 0°$/sec. It is verified that (48) holds for all trims in the range $\eta_0 \in [10, 12]°$, with $\{A, B_3, C\}$ fixed on the trim of $\eta_0 = 10°$ and Θ_p^* varying for different η_0. For example, the linearized model for $\eta_0 = 11°$ can be approximated using

$$\Theta_p^* = \begin{bmatrix} 0.6 & -4.52 & 0 & 0.05 & 0.41 & 1.47 \\ 0.1 & 1.83 & 0 & -0.02 & -0.35 & -0.59 \end{bmatrix}, \Lambda^* = \begin{bmatrix} 0.91 & 0.52 \\ 0.52 & 0.79 \end{bmatrix}$$

The pitch mode of the VFA when $\eta \geq 11°$ is unstable. The model (97) used for design includes second-order actuator dynamics with nominal natural frequency $\omega_n = 1$ rad/sec and nominal damping ratio $\zeta = 0.7$. u_e are elevator commands and u_a are aileron commands.

For control design, we derived control parameters for the trim of (97), using the squaring-up method for the nonsquare plant. L and S_3 are found using $a_2^2 = 1$, $a_2^1 = 2$, $a_2^0 = 1$, $\varepsilon = 30$, $\Lambda_{max} = 2$ and $\Psi_{max} = 30$. For the baseline controller (without adaptation), use of the LQR method results in an observer-based linear controller. The simulation results with the nonlinear VFA model are shown in Figure 1 for

$$
\begin{bmatrix} V \\ \dot{\alpha} \\ \dot{\theta} \\ \dot{q} \\ \dot{\eta} \\ \ddot{\eta} \\ \dot{\delta}_e \\ \dot{\delta}_a \\ \ddot{\delta}_e \\ \ddot{\delta}_a \\ \dot{w}_\eta \\ \dot{w}_{A_z} \end{bmatrix} = \underbrace{\begin{bmatrix} -0.279 & 3.476 & -32.2 & -0.015 & 0.514 & 0.525 & 0 & 0 & -2.57 & -6.47 & 0 & 0 \\ -0.070 & -4.104 & 0 & 1.013 & 0.193 & 0.100 & 0 & 0 & -0.795 & -0.079 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -54.04 & 0 & 0.255 & 1.845 & 21.41 & 0 & 0 & 5.991 & -6.363 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.002 & 0.044 & 0 & 0.819 & -0.075 & -6.518 & 0 & 0 & 0.195 & -0.034 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1.4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -123.12 & 0 & 0 & 0 & 0 & 0 & 0 & -23.84 & -2.376 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} V \\ \alpha \\ \theta \\ q \\ \eta \\ \dot{\eta} \\ \delta_e \\ \delta_a \\ \dot{\delta}_e \\ \dot{\delta}_a \\ w_\eta \\ w_{A_z} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{B_3} \underbrace{\begin{bmatrix} u_e \\ u_a \end{bmatrix}}_{u} \tag{97}
$$

$$
y = \begin{bmatrix} q \\ w_\eta \\ w_{A_z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} x
$$

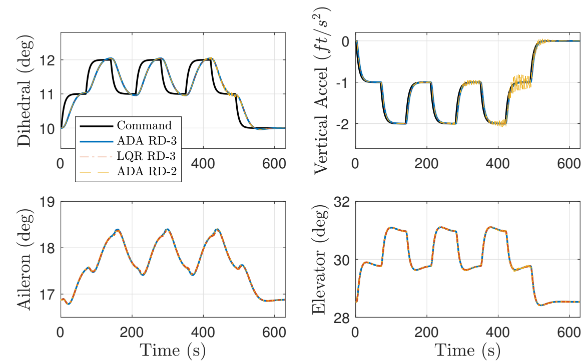
Baseline relative degree three	Adaptive relative degree two	Adaptive relative degree three	Classical adaptive controller [5]
6	48	169	1328

Table I: Total number of integrators for each controller

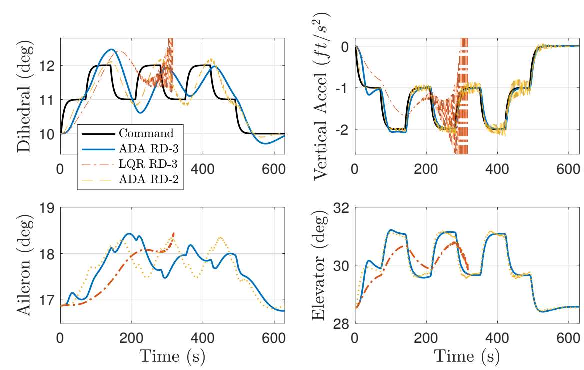
three actuator models and and three control designs. The simulated actuators are the nominal model with $\omega_n = 1$ rad/sec and $\zeta = 0.7$, a fast model with $\omega_n = 2$ rad/sec and $\zeta = 0.5$, and a slow model with $\omega_n = 0.5$ rad/sec and $\zeta = 2$. The controllers simulated are the baseline LQR controller, one adaptive controller designed assuming a first-order actuator model is sufficient ("relative degree two" design as developed in [18]), and the relative degree three controller described in Section V of this work, based on a nominal second-order actuator model in (97). The number of integrators required for each controller is listed in Table I, demonstrating that the adaptive controllers proposed in this paper require an order of magnitude fewer integrations compared to the classical adaptive controller, and thus is able to reduce the computational complexity required for control.

With nominal actuators, all three controllers have the same ideal performance, as shown in Figure 1a. It is noted that the VFA navigates through $\eta = [11, 12]^\circ$ where the pitch mode is unstable. The unstable zeros in the individual transfer function from δ_e to A_z are accommodated by our design since Assumption 2 only asks for stable transmission zeros. With fast actuators, both adaptive controllers were able to achieve tracking goals while the baseline controller failed, as shown in Figure 1b. When actuator dynamics were slow as shown in Figure 1c, only the adaptive relative degree three controller introduced in this paper can achieve stable command tracking. The parameter trajectories of this controller are shown in Figure 2.

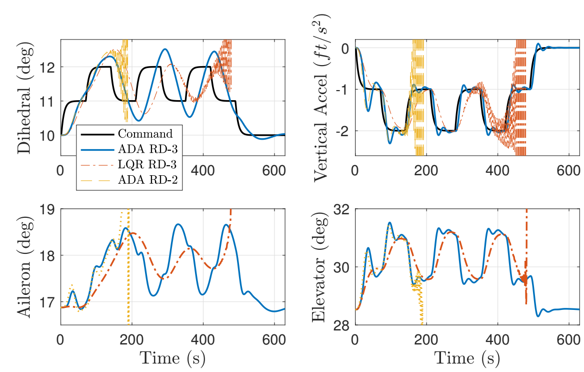
Using the parameter values at the end of simulation, the closed-loop system consists of a linear time-invariant plant and a linear observer-based controller. The robustness of this closed-loop systems is examined in the gang-of-six frequency domain (see [9, Chapter 5] and [26, Chapter 5]) in Figure 3. The comparison shows that at $t = 0$ sec the uncertainties reduce the gain margin from the nominal value (i.e. the base-line controller without uncertainties) of $[-14.8,\ 13.2]dB$ to $[-2.2, 1.8]dB$, and phase margin from $\pm 47.9°$ to $\pm 9.6°$; The closed-loop system for the adaptive relative degree three con-



(a) Nominal actuators with $\omega_n = 1$ rad/sec and $\zeta = 0.7$



(b) Fast actuators with $\omega_n = 2$ rad/sec and $\zeta = 0.5$



(c) Slow actuators with $\omega_n = 0.5$ rad/sec and $\zeta = 2$

Figure 1: The tracking of η and A_z using the relative degree three adaptive controller on the nonlinear VFA model with uncertainty

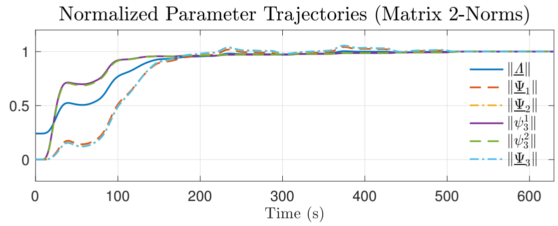


Figure 2: The parameter trajectories of the relative degree three adaptive controller in the simulation shown in Figure 1c

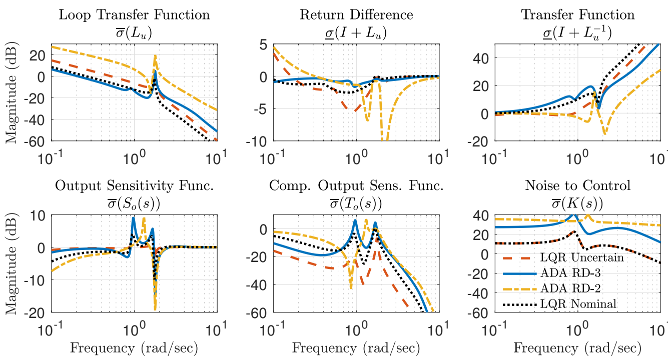


Figure 3: The frequency domain analysis shows that adaptation mitigates the effects of model uncertainties on robustness

troller at $t = 600$ sec recovers these margins to $[-24, 14.2]dB$ and to $\pm 55.0°$, respectively, while the adaptive relative degree two controller has deteriorated stability margins. The trade-off is that the output sensitivity of the systems (with parameter values at $t = 600$) increases to more than 9dB after adaptation, implying increased sensitivities to measurement noise. The loop transfer function also has a spike at around 1.5 rad/sec at the end of adaptation, which implies increased sensitivity to input disturbances.

VIII. Conclusions

This paper develops a new adaptive output-feedback controller for relative degree three or higher MIMO plant models with parametric uncertainties. In order to accommodate the high relative degree, additional filters and adjustable parameters are judiciously introduced so as to generate synthetic zeros and therefore an underlying SPR transfer function matrix. This in turn leads to global stability and asymptotic tracking. The overall design is validated using simulation results on a VFA model with second-order actuators.

IX. Acknowledgment

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References

[1] J. Doyle and G. Stein, “Multivariable feedback design: Concepts for a classical/modern synthesis,” *IEEE Transactions on Automatic Control*, vol. 26, no. 1, pp. 4–16, feb 1981.

[2] E. Rynaski, “Flight control synthesis using robust output observers,” in *AIAA Guidance, Navigation, and Control Conference*, 1982, pp. 10.2514/6.1982–1575.

[3] C. Thompson, E. Coleman, and J. Blight, “Integral LQG controller design for a fighter aircraft,” in *AIAA Guidance, Navigation, and Control Conference*, 1987, pp. 10.2514/6.1987–2452.

[4] K. Wise and E. Lavretsky, “Asymptotic properties of LQG/LTR controllers in flight control problems,” in *AIAA Guidance, Navigation, and Control Conference*, 2012, pp. 10.2514/6.2012–4889.

[5] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Dover Publications, 2004.

[6] G. Tao, *Adaptive Control Design and Analysis*. Wiley, 2003.

[7] A. S. Morse, “Parameterizations for multivariable adaptive control,” in *Proceedings of the 20th IEEE Conference on Decision and Control*. New York, NY, USA: IEEE, 1981, pp. 970–972, 10.1109/CDC.1981.269 358.

[8] R. P. Singh and K. S. Narendra, “Prior information in the design of multivariable adaptive controllers,” *IEEE Transactions on Automatic Control*, vol. AC-29, no. 12, p. 10.1109/TAC.1984.1103441, 1984.

[9] E. Lavretsky and K. A. Wise, *Robust and Adaptive Control with Aerospace Applications*. Springer, 2013.

[10] T. Gibson, A. Annaswamy, and E. Lavretsky, “Improved transient response in adaptive control using projection algorithms and closed loop reference models,” in *AIAA Guidance, Navigation, and Control Conference*, 2012, pp. 10.2514/6.2012–4775.

[11] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, “On adaptive control with closed-loop reference models: Transients, oscillations, and peaking,” *Access, IEEE*, vol. 1, pp. 703–717, 10.1109/ACCESS.2013.2 284 005, 2013.

[12] T. E. Gibson, A. M. Annaswamy, E., and Lavretsky, “Closed-loop reference models for output-feedback adaptive systems,” in *European Control Conference*, 2013, pp. 365–370, 10.1109/TAC.2015.2 405 295.

[13] Z. Qu, E. Lavretsky, and A. M. Annaswamy, “An adaptive controller for very flexible aircraft,” in *AIAA Guidance, Navigation, and Control Conference*, 2013, pp. 10.2514/6.2013–4854.

[14] T. E. Gibson, Z. Qu, A. M. Annaswamy, and E. Lavretsky, “Adaptive output feedback based on closed-loop reference models,” *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2728 – 2733, 2015.

[15] D. P. Wiese, A. M. Annaswamy, J. A. Muse, M. A. Bolender, and E. Lavretsky, “Adaptive output feedback based on closed-loop reference models for hypersonic vehicles,” in *AIAA Guidance, Navigation, and Control Conference*, 2015, pp. 10.2514/6.2015–1755.

[16] Z. Qu and A. M. Annaswamy, “Adaptive output-feedback control with closed-loop reference models for very flexible aircraft,” *AIAA Journal of Guidance, Control, and Dynamics*, p. 10.2514/1.G001282, 2015.

[17] Z. Qu, A. M. Annaswamy, and E. Lavretsky, “Adaptive output-feedback control for relative degree two systems based on closed-loop reference models,” in *IEEE Conference on Decision and Control*, Osaka, Japan, 2015, p. 10.1109/CDC.2015.7403420.

[18] ——, “Adaptive output-feedback control for a class of multi-input-multi-output plants with applications to very flexible aircraft,” in *IEEE, 2016 American Control Conference*, 2016, p. 10.1109/ACC.2016.7525147.

[19] A. G. J. MacFarlane and N. Karcanias, “Poles and zeros of linear multivariable systems: a survey of the algebraic, geometric and complex-variable theory,” *International Journal of Control*, vol. 24, no. 1, pp. 33–74, 10.1080/00 207 177 608 932 805, 1976.

[20] M. Mueller, “Normal form for linear systems with respect to its vector relative degree,” *Linear Algebra and Its Applications*, vol. 430, no. 4, pp. 1292–1312, 10.1016/j.laa.2008.10.014, 2009.

[21] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, “Closed-loop reference models in adaptive control: Stability, robustness and performance,” in *ArXiv: 1201.4897*, 2012.

[22] S. Evesque, A. M. Annaswamy, S. Niculescu, and A. P. Dowling, “Adaptive control of a class of time-delay systems,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 125, no. 2, pp. 186–193, 10.1115/1.1 567 755, Jun 2003.

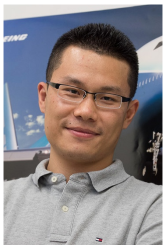
[23] W. Su, “Coupled nonlinear aeroelasticity and flight dynamics of fully flexible aircraft,” Ph.D, University of Michigan, Ann Arbor, 2008.

[24] Z. Qu, A. M. Annaswamy, and E. Lavretsky, “Squaring-up method for relative degree two plants,” in *MIT dSpace Archive*, 2015. [Online]. Available: http://hdl.handle.net/1721.1/98917

[25] T. Gibson, A. Annaswamy, and E. Lavretsky, “Modeling for control of very flexible aircraft,” in *AIAA Guidance, Navigation, and Control Conference*, 2011, pp. 10.2514/6.2011–6202.

[26] K. J. Astrom and R. M. Murray, *Feedback Systems: an Introduction for Scientists and Engineers*. Princeton University Press, 2008.

[27] N. Truhar, "The perturbation bound for the solution of the Lyapunov equation," *Mathematical Communications*, vol. 12, pp. 83–94, 2007.



Dr. Zheng Qu received the B.S. and M.S. degree in Mechanical Engineering from University of Michigan, Ann Arbor, MI, USA in 2009 and 2010, respectively, and received the Ph.D degree in Mechanical Engineering from the Massachusetts Institute of Technology, Cambridge, MA, USA. He is a research scientist with Zero Zero Robotics Inc. and has developed flight control functions, SLAM methods, obstacle avoidance schemes and flight simulation platforms for different models of Hover Camera. He is a young professional IEEE member.

Benjamin Thomsen received his B.Eng. in Mechanical Engineering from McGill University in 2015. He is a graduate student in the Department of Mechanical Engineering at the Massachusetts Institute of Technology, where he is a researcher in the Active Adaptive Control Laboratory. His current research is on adaptive control for unmanned aerial systems, and his research interests lie in the areas of motion planning and control for autonomous systems.

Dr. Anuradha Annaswamy received her Ph.D. in Electrical Engineering from Yale University in 1985. She has been a member of the faculty at Yale, Boston University, and MIT where currently she is the director of the Active-Adaptive Control Laboratory and a Senior Research Scientist in the Department of Mechanical Engineering. Her research interests pertain to adaptive control theory and applications to aerospace, automotive, and propulsion systems, cyber physical systems science, and CPS applications to Smart Grids, Smart Cities, and Smart Infrastructures. She is the author of a hundred journal publications and numerous conference publications, co-author of a graduate textbook on adaptive control (2004), co-editor of two reports, IEEE Vision for Smart Grid Control: 2030 and Beyond, and Impact of Control Technology (1st ed. & 2nd ed.). Dr. Annaswamy has received several awards including the George Axelby and Control Systems Magazine best paper awards from the IEEE Control Systems Society (CSS), the Presidential Young Investigator award from NSF, the Hans Fisher Senior Fellowship from the Institute for Advanced Study at the Technische Universität München, the Donald Groen Julius Prize from the Institute of Mechanical Engineers, a Distinguished Member Award, and a Distinguished Lecturer Award from IEEE CSS. Dr. Annaswamy is a Fellow of the IEEE and IFAC. She has served as the Vice President for Conference Activities (2014-15), and is currently serving as the VP for Technical Activities (2017-18) in the Executive Committee of the IEEE CSS.

Proof of Lemma 5

Proof: With (37), u_2 in (38) can be rewritten as

$$\begin{aligned} u_2 = &-\underline{\psi}_1^T(t)\big[A^2 x_m - A\ell e_y\big] - \big(2\dot{\underline{\psi}}_1^T(t) + 2\underline{\psi}_1^T(t)\big)\big[A x_m\big] \\ &- \big(\ddot{\underline{\psi}}_1^T(t) + 2\dot{\underline{\psi}}_1^T(t) + \underline{\psi}_1^T(t)\big)\big[x_m\big] \\ &- \big(s^2+2s\big)\big[\underline{\psi}_1^T(t)\big]e_{y_1^0} \end{aligned} \tag{98}$$

Since we design $\underline{\psi}_1(t) = \begin{bmatrix} \times & 0 & 0 \end{bmatrix}^T$, it can be shown that $\underline{\psi}_1^T(t) b_3 = 0$ and $\underline{\psi}_1^T(t) b_2^a = 0$, which leads to

$$\underline{\psi}_1^T(t)\dot{x}_m = \underline{\psi}_1^T(t)\big[A x_m - \ell e_y\big], \tag{99}$$

$$\underline{\psi}_1^T(t)\ddot{x}_m = \underline{\psi}_1^T(t)\big[A^2 x_m - A\ell e_y - \ell\dot{e}_y\big]. \tag{100}$$

(99) and (100), together with (36), transform (98) into

$$\begin{aligned} u_2 = &-(s+1)^2\big[\underline{\psi}_1^T(t)x_m\big] - \big[2\dot{\underline{\psi}}_1^T(t) + 2\underline{\psi}_1^T(t)\big]\dot{e}_{y_1^0} \\ &- \underline{\psi}_1^T(t)\ell\ddot{e}_{y_1^0} - \big((s^2+2s)\big[\underline{\psi}_1^T(t)\big]\big)e_{y_1^0}. \end{aligned} \tag{101}$$

Applying the product rule of derivative yields

$$u_2 = -(s+1)^2\big[\underline{\psi}_1^T(t)x_m\big] - (s^2+2s)\big[\underline{\psi}_1^T(t)e_{y_1^0}\big]. \tag{102}$$

Applying (40) yields

$$u_2 = -(s+1)^2\big[\underline{\psi}_1^T(t)x_m + \bar{e}^{[1]}_{\Psi_1 y_1^0}\big] \tag{103}$$

which leads to

$$\begin{aligned} &b_3\big[(s+1)^2(\underline{\psi}_1^{*T}x_m) + u_2\big] - f_{x_m}(e_y) \\ &\quad = b_1^{*a}\lambda^*\big[-\tilde{\underline{\psi}}_1^T x_m - \bar{e}^{[1]}_{\Psi_1 y_1^0}\big] - f_{x_m}(e_y). \end{aligned} \tag{104}$$

Now we will show that $f_{x_m}(e_y)$ accommodates $\lambda^*\bar{e}^{[1]}_{\Psi_1 y_1^0}$. Substituting $f_{x_m}(s)$ as in (41) leads to

$$\begin{aligned} -b_1^{a*}\lambda^*\bar{e}^{[1]}_{\Psi_1 y_1^0} - f_{x_m}(e_y) &= b_3(s+1)^2\big[\tilde{\lambda}(t)\bar{e}^{[1]}_{\Psi_1 y_1^0}\big] \\ &\quad - b_3\psi_3^{1*}s\big[\tilde{\lambda}(t)\bar{e}^{[1]}_{\Psi_1 y_1^0}\big] + b_3(s+1)^2\big[\psi_3^1(t)\bar{e}^{[1][2]}_{\Psi_1 y_1^0}\big] \\ &= b_3(s+1)^2\big[\tilde{\lambda}(t)\bar{e}^{[1]}_{\Psi_1 y_1^0} + \tilde{\psi}_3^1(t)\bar{e}^{[1][2]}_{\Psi_1 y_1^0}\big] \end{aligned} \tag{105}$$

where we have applied (40). This leads to the results. ■

Proof of Theorem 1

Proof: Without loss of generality, we will use

$$\dot{e}_{mx} = A^*_{\ell^*}e_{mx} + b_1^{a*}\lambda^*\tilde{\Omega}^T\xi \tag{106}$$

(instead of (45)) for the following stability analysis. For the high-order tuner (46), define an error coordinate z_h as

$$z_{h,k} = x_{h,k} + A_h^{-1}b_h\Omega_k^{\prime T}, \tag{107}$$

which yields

$$c_h^T z_{h,k} = \Omega_k^T(t) - \Omega_k^{\prime T}(t), \tag{108}$$

$$\dot{z}_{h,k}(t) = A_h g(\xi_k)z_{h,k} + A_h^{-1}b_h\dot{\Omega}_k^{\prime T}(t). \tag{109}$$

We propose a Lyapunov function candidate as

$$V = e_{mx}^T P^* e_{mx} + \sum_k \big[(\Omega_k^{\prime} - \Omega_k^*)^2 + \eta z_{h,k}^T P_h z_{h,k}\big] \tag{110}$$

where e_{mx} are the states of error model. P^* satisfies

$$\begin{aligned} P^*(A^* - \ell^* c^T) + (A^* - \ell^* c^T)^T P^* = -Q^* < 0 \\ P^* b_1^{a*} = c, \end{aligned} \tag{111}$$

which are the results of SPR properties of (45), and P_h satisfies $P_h A_h + A_h P_h^T = -I$. Using (45) and (109), $\dot{V}$ is found to be

$$\begin{aligned} \dot{V} = &+2P^* b_1^{a*} e_{mx}^T \lambda^* \big[(\Omega - \Omega') + (\Omega' - \Omega^*)\big]^T \xi \\ &- 2e_y \lambda^* \sum_k (\Omega_k' - \Omega_k^*)\xi_k - \eta \sum_k \big[z_{h,k}^T I z_{h,k}\big] g(\xi_k) \\ &+ 2\eta \sum_k \big[z_{h,k}^T P_h A_{h,k}^{-1} b_h e_y^T \xi_k\big] - e_{mx}^T Q^* e_{mx} \end{aligned} \tag{112}$$

which, combined with (111) and (108), yields

$$\begin{aligned} \dot{V} = &-e_{mx}^T Q^* e_{mx} - \eta \sum_k \big[z_{h,k}^T z_{h,k} + \mu (z_{h,k}\xi_k)^2\big] \\ &+ 2\sum_k \big[e_y \lambda^* c_h^T z_{h,k}\xi_k + \eta z_{h,k}^T P_h A_h^{-1} b_h e_y^T \xi_k\big]. \end{aligned} \tag{113}$$

Then we chose

$$\eta = \frac{\|c_h\| \lambda_{max}}{\|P_h A_h^{-1} b_h\|}, \quad \mu = \frac{2\|c\|^2 \|c_h\|^2 \lambda_{max}^2}{\lambda_{Q^*}\eta}, \tag{114}$$

where λ_{Q^*} is the absolute value of the smallest eigenvalue of Q^* and $|\lambda^*| \le \lambda_{max}$. This leads to

$$\dot{V} \le -\eta \sum_k \big[z_{h,k}^T z_{h,k} - \big(\sqrt{\lambda_{Q^*}}\, \|e_{mx}\| - \sqrt{\eta\mu}\, \|z_{h,k}\xi_k\|\big)^2\big]. \tag{115}$$

As a result, $e_{mx}(t)$, $z_{h,k}(t)$, $\Omega(t)$ are all bounded as $t \to \infty$, and therefore $x_m(t)$ is bounded in the CRM, which proves i). This in turn implies $\dot{z}_{h,k}$ as in (109) is bounded. Applying Barbalat's Lemma yields that $z_{h,k}(t) \to 0$ as $t \to \infty$, which in turns implies that in (115) $e_{mx}(t) \to 0$ as $t \to \infty$. ∎

Definitions of ε_1 and ε_2

$$\begin{aligned} \varepsilon_1 =& \lambda_m\big\{-(\overline{C}B_1^a)^{-1}\big[\overline{C}AB_1^a + (\overline{C}AB_1^a)^T\big](\overline{C}B_1^a)^{-1}\big\} \\ \varepsilon_2 =& \lambda_m\big\{(\overline{C}B_1^a)^{-2}\big[2\|\overline{C}B_1^a\|^2 \big(\overline{\Psi}_{max} + R_{max}\big) + H_{max}^2\big]\big\} \\ H_{max} =& \big[(\|N_1 AB_1^a\| + a_2^2 \overline{\Psi}_{max}^2 R_{max} + a_2^1 \overline{\Psi}_{max} R_{max}) \\ & (1 + 2\overline{\Psi}_{max}\|P_I\|)\|P_I\|\big] + \|\overline{C}AM\| + \|CB_1^a\| \overline{\Psi}_{max}. \end{aligned} \tag{116}$$

In the above equation, $\lambda_m\{\cdot\}$ is the largest real part of the eigenvalues of $\{\cdot\}$, $N_1 = (M^T M)^{-1} M^T \big[I - B_1^a(\overline{C}B_1^a)^{-1}\overline{C}\big]$ is a null space of B_1^a, M is the null space of C, P_I is a solution of $P_I N_1 AM + (N_1 AM)^T P_I = -I$ (see Lemma 1 for their existence), $\overline{\Psi}_{max} = \Psi_{max}\|[\mathcal{B}, \mathcal{M}]\|$, and $R_{max} = \|(\mathcal{CB})^{-1}\mathcal{C}A^2 B_3\|$, where $\mathcal{C}$ and $\mathcal{B}$ (7) are defined for the system (A, B_3, C) and $\mathcal{M}$ is the null space of $\mathcal{C}$.

Proof of Lemma 7

Proof: We will show that R as in (57) guarantees the SPR properties of $\{(A^* - B_1^{a*}R^{-1}SC), B_1^{a*}, C\}$. The proof follows the proof of [17, Lemma 2] using a special state coordinate called "input normal form". For plant models (51) with relative degree three that satisfy Assumption 3, there exists an invertible coordinate transformation matrix

$$T_{in} = \begin{bmatrix} (\mathcal{CB})^{-1}\mathcal{C} \\ \mathcal{N} \end{bmatrix}, \quad T_{in}^{-1} = \begin{bmatrix} \mathcal{B}, & \mathcal{M} \end{bmatrix} \tag{117}$$

where $\mathcal{C}$ and $\mathcal{B}$ (7) are defined for the system (A, B_3, C), and $\mathcal{M}$ is the null space of $\mathcal{C}$ satisfying $\mathcal{CM} = 0$, and $\mathcal{N} = (\mathcal{M}^T\mathcal{M})^{-1}\mathcal{M}^T\big[I_n - \mathcal{B}(\mathcal{CB})^{-1}\mathcal{C}\big]$ satisfying $\mathcal{NB} = 0$ and $\mathcal{NM} = I$ (see Lemma 1), which can transform the plant model (51) into

$$\begin{aligned} \left[\begin{array}{c} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \hline \dot{\eta} \end{array}\right] = & \underbrace{\left[\begin{array}{ccc|c} 0 & 0 & R_1 & V \\ I_m & 0 & R_2 & 0 \\ 0 & I_m & R_3 & 0 \\ \hline 0 & 0 & U & Z \end{array}\right]}_{A_{in}} \underbrace{\left[\begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \\ \hline \eta \end{array}\right]}_{x_{in}} + B_{3,in}\Lambda^* u + B_{in,z} z_{cmd} \\ & + \underbrace{\left[\begin{array}{c} \times \\ \times \\ \times \\ \hline 0 \end{array}\right]}_{B_{1,in}} \underbrace{\begin{bmatrix} 0 & 0 & \psi_1^{3*T} & \psi_1^{[n-3m]*T} \end{bmatrix}}_{\Psi_{1,in}^{*T}} x_{in} \\ & + \underbrace{\left[\begin{array}{c} I_m \\ 0 \\ 0 \\ \hline 0 \end{array}\right]}_{B_{3,in}} \underbrace{\begin{bmatrix} \psi_3^{1*T} & \psi_3^{2*T} & \psi_3^{3*T} & \psi_3^{[n-3m]*T} \end{bmatrix}}_{\Psi_{3,in}^{*T}} x_{in} \\ y = & \underbrace{\left[\begin{array}{ccc|c} 0 & 0 & CA^2B_3 & 0 \end{array}\right]}_{C_{in}} x_{in} \end{aligned} \tag{118}$$

with $x_{in} = T_{in}x$, $A_{in} = T_{in}AT_{in}^{-1}$, $B_{3,in} = T_{in}B_3$, $B_{1,in} = T_{in}B_1$, $B_{in,z} = T_{in}B_z$, $C_{in} = CT_{in}^{-1}$, $\Psi_{1,in}^{*T} = \Psi_1^{*T}T_{in}^{-1}$, $\Psi_{3,in}^{*T} = \Psi_3^{*T}T_{in}^{-1}$ and $[R_1^T, R_2^T, R_3^T]^T = \big((\mathcal{CB})^{-1}\mathcal{C}A^2B\big)^T$ (see [20, Theorem 2.4]). $Z = \mathcal{NAM}$ has eigenvalues that are transmission zeros of $\{A, B_3, C\}$ (see Proposition 1). It is noted that $T_{in}B_1 = \left[\begin{array}{ccc|c} \times & \times & \times & 0 \end{array}\right]^T$ since Assumption 4 holds, and that $\Psi_{1,in}^* = \begin{bmatrix} 0 & 0 & \times & \times \end{bmatrix}^T$ since Assumption 5 holds. All equations in this proof are valid in both coordinates, and matrices in the input norm form coordinate will be denoted with a subscript $(\cdot)_{in}$.

To prove Lemma 7, it is equivalent to show $\{(A_{in}^* - L_{in}^* C_{in}), B_{1,in}^{a*}, SC_{in}\}$ is SPR, where $A_{in}^* = T_{in}A^*T_{in}^{-1}$ and $B_{1,in}^{a*} = T_{in}B_1^{a*}$. Define $\overline{C}_{in} = SC_{in}$,

$$M_{in}^T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{119}$$

which is the null space of C_{in}, and

$$N_{1,in}^* = (M_{in}^T M_{in})^{-1} M_{in}^T \big[I - B_{1,in}^{a*}(\overline{C}_{in}B_{1,in}^{a*})^{-1}\overline{C}_{in}\big],$$

which is the null space of $B_{1,in}^{a*}$ (see Lemma 1). Then propose a P_{in}^* given by

$$\begin{aligned} P_{in}^* = \overline{C}_{in}^T(\overline{C}_{in}B_{1,in}^{a*})^{-1}\overline{C}_{in} + N_{1,in}^{*T}P_I^*N_{1,in}^*, \\ \text{s.t. } P^*B_{1,in}^{a*} = \overline{C}_{in}^T, \end{aligned} \tag{120}$$

where P_I^* is the unique solution of a Lyapunov equation

$$P_I^* N_{1,in}^* A_{in}^* M_{in} + (N_{1,in}^* A_{in}^* M_{in})^T P_I^* = -I, \tag{121}$$

which always exists since $\{A_{in}^*, B_{1,in}^{a*}, C_{in}\}$ has stable transmission zeros (see Proposition 1 and Lemma 1). Define

$$\{\oslash\} := A_{in}^{*T}P_{in}^* + P_{in}^*A_{in}^* - P_{in}^*B_{1,in}^{a*}R^{-1}B_{1,in}^{a*T}P_{in}^* + Q_{in}^* \tag{122}$$

where

$$Q_{in}^* = W_{in}^* + W_{in}^{*T} + \epsilon\overline{C}_{in}^T\overline{C}_{in} + N_{1,in}^{*T}N_{1,in}^*$$

$$
\begin{aligned}
&- \overline{C}_{in}^T (\overline{C}_{in} B_{1,in}^{a*})^{-1} \left[\Delta_{CAB}^* \right. \\
&\left. + \Delta_{CAB}^{*T} \right] (\overline{C}_{in} B_{1,in}^{a*})^{-1} \overline{C}_{in} && (123)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{CAB}^* &= \overline{C}_{in} A_{in}^* B_{1,in}^{a*} - \overline{C}_{in} A_{in} B_{1,in}^{a} \\
&= (C_{in} B_{1,in}^a)^T C_{in} B_{1,in}^a \Psi_R^{*T} && (124)
\end{aligned}
$$

$$
H_{in}^* = M_{in}^T A_{in}^{*T} \overline{C}_{in}^T + P_I^* N_{1,in}^* A_{in}^* B_{1,in}^{a*}. \qquad (125)
$$

and $W_{in}^* = -N_{1,in}^{*T} H_{in}^* (\overline{C}_{in} B_{1,in}^{a*})^{-1} \overline{C}_{in}$, $B_{1,in}^a = T_{in} B_1^a$, and $\Psi_R^{*T} = (\psi_3^{1*} + a_2^1/a_2^2) + (\psi_1^{3*T} + R_3)$. It has been proved in [13], [16] that once ε in L_{in}^* is large enough, $\{\oslash\} = 0$ holds for $P_{in}^* > 0$, $Q_{in}^* > 0$ and $R > 0$ and therefore P_{in}^* guarantees the SPR properties of $\{(A_{in}^* - L_{in}^* C_{in}), B_{1,in}^{a*}, S_{in} C_{in}\}$. We will show that one lower bound of such ε is $\overline{\varepsilon}$ given in (58).

With the definition of R in (57), the equality (124) and the transformation matrix $T_B = [M_{in}, B_{1,in}^{a*}]$, it follows that

$$
T_B^T \{\oslash\} T_B = \begin{bmatrix} M_{in}^T \{\oslash\} M_{in} & M_{in}^T \{\oslash\} B_{1,in}^{a*} \\ B_{1,in}^{a*T} \{\oslash\} M_{in} & B_{1,in}^{a*T} \{\oslash\} B_{1,in}^{a*} \end{bmatrix} = 0, \qquad (126)
$$

which implies that $\{\oslash\} = 0$ and that P_{in}^* satisfies

$$
\begin{aligned}
(\overline{A}_{in}^* - L_{in}^* \overline{C}_{in})^T P_{in}^* + P_{in}^* (\overline{A}_{in}^* - L_{in}^* \overline{C}_{in}) &= -\overline{Q}_{in}^* \\
P_{in}^* B_{1,in}^* &= \overline{C}_{in}^T.
\end{aligned}
$$

where $L_{in}^* = B_{1,in}^{a*} R^{-1} S$ and $\overline{Q}_{in}^* = Q_{in}^* + \overline{C}_{in}^T R_{in}^{-1} \overline{C}_{in}$. Now we will show that $R > 0$ and $Q_{in}^* > 0$. Since $\varepsilon > \varepsilon_1$, $R^{-1} > 0$. To show $Q_{in}^* > 0$, equivalently we will show $T_B^T Q_{in}^* T_B > 0$. It is noted that

$$
T_B^T Q_{in}^* T_B = \begin{bmatrix} I & -H_{in}^* \\ -H_{in}^{*T} & \epsilon (\overline{C}_{in} B_{1,in}^{a*})^2 - \left[\Delta_{CAB}^* + \Delta_{CAB}^{*T} \right] \end{bmatrix} \qquad (127)
$$

Since $\overline{C}_{in} B_{1,in}^{a*} = \overline{C}_{in} B_{1,in}^{a} = C B_1^{a*}$ and $C A^* B_1^{a*} = C_{in} A_{in}^* B_{1,in}^{a*}$, it follows $\varepsilon > \varepsilon_2$ that

$$
\begin{aligned}
\epsilon I \quad &\geq \quad (\overline{C}_{in} B_{1,in}^a)^{-1} \left[(C_{in} B_{1,in}^a)^T C_{in} B_{1,in}^a \Psi_R^{*T} \right. \\
&\quad \left. + \Psi_R^* (C_{in} B_{1,in}^a)^T C_{in} B_{1,in}^a \right] (\overline{C}_{in} B_{1,in}^a)^{-1} \\
&\quad + (\overline{C}_{in} B_{1,in}^a)^{-1} H_{max}^2 (\overline{C}_{in} B_{1,in}^a)^{-1} && (128) \\
&\geq \quad (\overline{C}_{in} B_{1,in}^{a*})^{-1} \left[\Delta_{CAB}^* + \Delta_{CAB}^{*T} \right] (\overline{C}_{in} B_{1,in}^{a*})^{-1} \\
&\quad + (\overline{C}_{in} B_{1,in}^{a*})^{-1} H_{in}^{*T} H_{in}^* (\overline{C}_{in} B_{1,in}^{a*})^{-1}. && (129)
\end{aligned}
$$

From (128) to (129), we have used the fact that

$$
\|H_{in}^*\| \leq \|P_I^*\| \left\| N_{in}^{1*} A_{in} B_{1,in}^{a*} \right\| + \left\| \overline{C}_{in} A_{in}^* M_{in} \right\|, \qquad (130)
$$

$$
N_{1,in}^* \overline{A}_{in}^* B_{1,in}^{a*} = N_{1,in} A_{in} B_{1,in}^a + \begin{bmatrix} (a_2^2 R_3 + a_2^1 R_2 + a_2^0 R_1)((\psi_3^{1*})^2 + \psi_3^{2*}) \\ (a_2^2 R_3 + a_2^1 R_2) \psi_3^{1*} \\ 0 \end{bmatrix}, \qquad (131)
$$

$$
C_{in} A_{in}^* M_{in} = C_{in} A_{in} M_{in} + \begin{bmatrix} 0 & 0 & C A^2 B_3 \psi_1^{[n-3m]*T} \end{bmatrix}, \qquad (132)
$$

and the sensitivity of Lyapunov equation solutions [27] as

$$
\|P_I^*\| \leq \left(1 + 2 \overline{\Psi}_{max} \|P_I\| \right) \|P_I\|, \qquad (133)
$$

where P_I is a nominal solution to (121) without uncertainties. By Schur's complement, the fact that Inequality (129) holds implies that $T_B^T Q_{in}^* T_B > 0$ and therefore $Q_{in}^* > 0$. ■

Proof of Lemma 8

Proof: The proof is carried out in the input normal form as in (118) where

$$
\Psi_{3,in}^* = \begin{bmatrix} \psi_3^{1*T} & \psi_3^{2*T} & \psi_3^{3*T} & \psi_3^{[n-3m]*T} \end{bmatrix}^T. \qquad (134)
$$

Similar to the proof of Lemma 4, the definition of B_i^{a*} in (71) implies that

$$
\begin{aligned}
&\begin{bmatrix} B_3^{a*} & | & B_2^{a*} & | & B_1^{a*} \end{bmatrix}_{in} \\
&= \left[\begin{array}{c|c|c} a_2^2 & a_2^2 \psi_3^{1*T} + a_2^1 & a_2^2 \left(\psi_3^{1*2} + \psi_3^{2*T} \right) + a_2^1 \psi_3^{1*T} + a_2^2 \\ 0 & a_2^2 & a_2^2 \psi_3^{1*T} + a_2^1 \\ 0 & 0 & a_2^2 \\ \hline 0 & 0 & 0 \end{array} \right] \qquad (135)
\end{aligned}
$$

and B_i^a in (62), its known version without uncertainties. Then after some algebra, the results will follow. ■

Proof of Lemma 9

Proof: Similar to the SISO case, it follows that

$$
\begin{aligned}
F(t) &= \left[(B_2^a s + B_3^a) \phi^T(t) \overline{\omega}^{[1]} + \pi_2^2(s) \left(\psi_3^1(t) \overline{\omega}^{[1][2]}(t) \right) \right] \\
&= B_3 \pi_2^2(s) \left(\phi^T(t) \overline{\omega}^{[1]} + \widetilde{\psi}_3^1(t) \overline{\omega}^{[1][2]}(t) \right), \qquad (136)
\end{aligned}
$$

where we have used the property of B_2^a in Lemma 8, the equivalent realization of B_2^{a*} in Lemma 2, the recursive property of $\pi_{r-1}^i(s)$ in (2), and the definition of $\overline{\omega}^{[1][2]}(t)$ in (76). It is noted that each time Lemma 2 is used in the error model analysis, (11) has to be used to change the state coordinate, which yields (78). ■

Proof of Lemma 10

Proof: The proof will be a direct extension of Section III-B and therefore only a brief step is shown below. Following Lemma 9, we replace L with

$$
L^* = B_1^{a*} R^{-1} S \qquad (137)
$$

in the error model (70), and use Eq.(72) to account for the difference. Applying Proposition 4, plugging in (61), applying Lemma 9, applying Lemma 2 and substituting (68) yields

$$
\begin{aligned}
\dot{e}_x'' &= A_{L^*}^* e_x'' - B_1^{a*} \widetilde{\psi}_3^{1T} \left[\overline{e}_{\psi y}^{[1]} + \overline{e}_{\psi y}^{[1][2]} \right] - B_1^{a*} \widetilde{\psi}_3^{2T} \overline{e}_{\psi y}^2 \\
&\quad + B_3 \Lambda^* \pi_2^2(s) \left[\underline{\Psi}_1^{*T} x_m + \underline{\Psi}_2^{*T} \overline{x}_m^{[1]} + \underline{\Psi}_3^{*T} \overline{x}_m^{[2]} \right] \\
&\quad + B_3 \pi_2^2(s) K^T \overline{x}_m^{[2]} + B_3 \Lambda^* u - F_{x_m}(e_y). \qquad (138)
\end{aligned}
$$

The rest of the proof follows the proof of Lemma 5. Using the fact that $\underline{\Psi}_1^T(t) F_L(e_y) = 0$ and $\underline{\Psi}_1^T(t) F_{x_m}(e_y) = 0$, expanding the compact form of u in (69) using (67), writing out $\underline{\Psi}_1^T(t) \dot{x}_m$ and $\underline{\Psi}_1^T(t) \ddot{x}_m$ (similar to (99) and (100)), substituting the definition of $e_{y_1^0}$ in (66), and applying the product rule of derivative yields

$$
\begin{aligned}
u &= \pi_2^2(s) \left[-\underline{\Lambda}^T(t) K^T \overline{x}_m^{[2]} - \underline{\Psi}_1^T(s) x_m - \underline{\Psi}_2^T(t) \overline{x}_m^{[1]} \right. \\
&\quad \left. - \underline{\Psi}_3^T(t) \overline{x}_m^{[2]} \right] - \left(a_2^2 s + a_2^1 \right) \cdot s \left[\underline{\Psi}_1^T(t) e_{y_1^0} \right]. \qquad (139)
\end{aligned}
$$

Substituting u (139), $F_{x_m}(e_y)$ (60) in (138), and applying Lemma 9 yields

$$
\begin{aligned}
\dot{e}_{mx} &= A_{L^*}^* e_{mx} - B_1^{a*} \widetilde{\psi}_3^{1T} \left[\overline{e}_{\psi y}^{[1]} + \overline{e}_{\psi y}^{[1][2]} \right] - B_1^{a*} \widetilde{\psi}_3^{2T} \overline{e}_{\psi y}^{[2]} \\
&\quad + B_1^{a*} \Lambda^* \left[-\widetilde{\underline{\Lambda}}^T K^T \overline{x}_m^{[2]} - \widetilde{\underline{\Psi}}_1^T(t) x_m - \widetilde{\underline{\Psi}}_2^T \overline{x}_m^{[1]} - \widetilde{\underline{\Psi}}_3^T \overline{x}_m^{[2]} \right] \\
&\quad + B_1^{a*} \widetilde{\Lambda}^T \overline{e}_{\Psi_1 y_1^0}^{[1]} + B_1^{a*} \widetilde{\psi}_3^{1T} \overline{e}_{\Psi_1 y_1^0}^{[1][2]}. \qquad (140)
\end{aligned}
$$

Grouping terms, and noting that Lemma 7 guarantees that $\{A_{L^*}^*, B_1^{a*}, SC\}$ is SPR, yields the results. ■

DEFINITION OF TUNER GAIN μ

$$\mu_\xi = \frac{4(m+3n)\left\|SC\right\|^2\left\|C_H\right\|^2}{n\lambda_Q\delta_\xi}, \quad \delta_\xi = \frac{\left\|C_H\right\|\Lambda_{max}}{\gamma\left\|P_H A_H^{-1}B_H\right\|}$$
$$\mu_\nu = \frac{4\cdot 3m\left\|SC\right\|^2\left\|C_H\right\|^2}{n\lambda_Q\delta_\nu}, \quad \delta_\nu = \frac{\left\|C_H\right\|}{\gamma\left\|P_H A_H^{-1}B_H\right\|} \tag{141}$$

where $\gamma = \|\Gamma\|$. P_H is a solution matrix for the equation $P_H A_H + A_H^T P_H = -I$ and λ_Q is the absolute value of the smallest eigenvalues of Q, which is defined as $Q = N_1^T H(\overline{C}B_1^a)^{-1}\overline{C}\,\overline{C}^T(\overline{C}B_1^a)^{-1}H^T N_1 + \varepsilon\overline{C}^T\overline{C} + N_1^T N_1$ where $H = M^T A^T \overline{C}^T + P_I N_1 A B_1^a$ and ε is defined in (58).

PROOF OF THEOREM 2

Proof: Without loss of generality, we use the following error model for this proof (instead of (79)).

$$\dot{e}_{mx} = A_{L^*}^* e_{mx} + B_1^{a*}\Lambda^*\widetilde{\Omega}^T\xi, \quad e_y = Ce_{mx}. \tag{142}$$

The proof will be very similar to the proof of Theorem 1 and therefore only crucial steps are listed. First we define a new tuner state as $Z_\Omega = X_\Omega + A_H^{-1}B_H\Omega^{'T}$ which satisfies $\Omega^T - \Omega^{'T} = C_H Z_\Omega$ since $C_H A_H^{-1}B_H = -I$. Let P^* be the matrix that guarantees the SPR properties of $\{A_{L^*}^*, B_1^{a*}, SC\}$:

$$\begin{aligned} P^* A_{L^*}^* + A_{L^*}^{*T}P^* = -Q^* < 0 \\ P^* B_1^{a*} = C^T S^T. \end{aligned} \tag{143}$$

We propose a Lyapunov function defined as

$$\begin{aligned} V = e_{mx}^T P^* e_{mx} + Tr\left[\left(\Omega' - \Omega^*\right)^T\Gamma^{-1}\left(\Omega' - \Omega^*\right)|\Lambda^*|\right] \\ + \delta_\xi \cdot Tr\left[Z_\Omega^T P_H Z_\Omega\right] > 0, \end{aligned} \tag{144}$$

$\dot{V}$ are derived by applying (142), (143) and (141) as

$$\begin{aligned} \dot{V} \le -\sum_{j=1}^{m+3n}\Big[\frac{n\lambda_Q}{m+3n}e_{mx}^2 - 4\left\|SC\right\|\left\|C_H\right\|\left\|e_{mx}\right\|\left\|z_\Omega^j\right\|\left\|\xi^j\right\| \\ + \delta_\xi\mu_\xi z_\Omega^{jT}z_\Omega^j\xi^{j2}\Big] - \delta_\xi\cdot\sum_{j=1}^{m+3n} z_\Omega^{jT}z_\Omega^j, \end{aligned}$$

where z_Ω^j is defined as $Z_\Omega = [\ z_\Omega^1 \quad z_\Omega^2 \quad \cdots \quad z_\Omega^{[m+3n]}\ ]$, ξ^j is defined as $\xi = [\ \xi^1 \quad \xi^2 \quad \cdots \quad \xi^{[m+3n]}\ ]$, and we have used the fact that $Q > Q^*$. Plugging in the definition of δ_ξ and δ_ν in (141) yields

$$\begin{aligned} \dot{V} \le -\sum_{j=1}^{m+3n}\left[\sqrt{\frac{n\lambda_Q}{m+3n}}\left\|e_{mx}\right\| - \sqrt{\delta_\xi\mu_\xi}\left\|z_\Omega^{jT}\right\|\left\|\xi^{jT}\right\|\right]^2 \\ - \delta_\xi\cdot\sum_{j=1}^{m+3n} z_\Omega^{jT}z_\Omega^j \le 0. \end{aligned}$$

As a result, all signals in (144) are bounded and therefore signals in CRM are also bounded, which proves i). The rest of the proof follows that of Theorem 1 and shows that $e_{mx}(t) \to 0$ and $e_y(t) \to 0$ as $t \to \infty$, which proves ii).

Following the derivation of the recursive adaptation as in (78), e_x and e_{mx} is related as

$$e_{mx} = e_x + \left[B_2^{a*} + B_3^{a*}s\right]\Lambda^*\left[\widetilde{\Omega}^T\xi\right] \tag{145}$$

It is noted that since all signals, including z_Ω, in V are bounded, $\left[B_2^{a*} + B_3^{a*}s\right]\widetilde{\Omega}^T\xi$ is also bounded. Since $\int(z - z_{cmd})dt$ is an element of x and $\int\left[(z_m - z_{cmd}) + L_d e_y\right]dt$, where L_d are the last d rows of L, is an element of x_m, it follows $e_{mx}(t) \to 0$ and x_m is bounded that e_x is bounded, and therefore $\int(z - z_m)dt$ is bounded. Also it is noted that $\ddot{x}_m$ is bounded and therefore $(z - z_{cmd})^{(1)}$ is bounded. Applying Barbalat's Lemma proves iii). ■

PROOF OF LEMMA 13

Proof: Lemma 2 and (71) imply that

$$\dot{e}_x = A_L^* e_x + B_i^{a*}\phi^{*T}\omega(t) - F(t), \quad e_y = Ce_x \tag{146}$$

is equivalent to

$$\begin{aligned} \dot{e}_x' = A_L^* e_x' + B_r\phi^{*T}\pi_{r-1}^{r-1-i}(s)\left[\omega(t)\right] - F(t) \\ e_y = Ce_x. \end{aligned} \tag{147}$$

Step 1: Substituting the definition of $\overline{\omega}^{[i-1]}(t)$ yields

$$\dot{e}_x' = A_L^* e_x' + B_r\phi^{*T}\pi_{r-1}^{r-1}(s)\left[\overline{\omega}^{[i-1]}(t)\right] - F(t) \tag{148}$$

Step 2: Substituting the definitions of $F(t)$ and F_i in (91), applying Lemma 8, the recursive property of $\pi_{r-1}^i(s)$ in (2), and applying the product rule of s yields

$$\begin{aligned} \dot{e}_x'' = A_L^* e_x'' - B_r\pi_{r-1}^{r-1}(s)\left[\widetilde{\phi}^T(t)\overline{\omega}^{[i-1]}(t)\right] \\ + \sum_{j=1}^{r-i} B_r\psi_r^{j*}\pi_{r-1}^{r-i-j}(s)\cdot s^{i-1}\left[\phi^T(t)\overline{\omega}^{[i-1]}(t)\right] \\ - \sum_{j=1}^{r-i}\Big[F_{i+j}(\psi_r^{jT}(t)\overline{\omega}^{[i-1][i-1+j]}(t)) + \\ + \sum_{k=1}^{r-i-j} F_{i+j+k}(\psi_r^{kT}(t)\overline{\omega}^{[i-1][i-1+j][i-1+j+k]}(t)) + \cdots\Big]. \end{aligned} \tag{149}$$

The sequence "$\cdots$" stops when the last index of $\overline{\omega}^{[i]\cdots[j][k]}$ reaches $[r-1]$. Then repeating Step 1-2 on $\overline{\omega}^{[i-1][i-1+j]}(t)$, applying the design of $F_{i+j}(t)$ in (91) and so on will yield the results. ■