The Behaviour of a Forced Damped Pendulum

B Taylor
School of Physics and Astronomy
University of Southampton

May 5, 2016

Contents

Abstract

This report examines the behaviour of a forced damped pendulum and attempts to discern when different types of behaviour, such as chaotic or periodic, occur. We find that, while is not possible to get a concrete pattern, it is possible to predict the conditions under which chaotic and different types of periodic are most likely.

1 Introduction

The pendulum is a mechanic widely used to show the properties of simple harmonic motion. Harmonic motion is one where the restoring force is proportional to the displacement of the pendulum, and this can be described by the equation $x(t) = A\sin(\omega t + \varphi)$ where A is the amplitude, t is the time, x is the horizontal displacement, ω is the frequency and φ is the phase difference.

We can then introduce a damping term to this equation, which if there is no force propelling the pendulum, will eventually cause the pendulum to come to rest. This damping term will be proportional to the velocity of the pendulum. With a real life example this term will usually represent the friction at the top of the pendulum string and air resistance, although it could also represent the viscosity of the fluid the pendulum is travelling through.

Finally, we can introduce a harmonic force to this equation, proportional to $\cos(\Omega t)$ where Ω is the frequency of the harmonic force. When looking at the motion of the forced damped pendulum we can vary this value of Ω and examine the different behaviours.

In this report the behaviour of a forced damped pendulum will be examined and, in particular the behaviour that occurs at large amplitudes.

2 Theory

In order to examine the motion of the forced damped pendulum we must first look at its differential equation

$$mL^2 \frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + mgL\sin\theta = FL\cos(\Omega t) \quad (1)$$

where m is the mass of the pendulum, L is its length and g is gravitation acceleration. It is more convenient to measure time relative to the period of small amplitude oscillations, so we write $t = \tau(L/g)^{1/2}$ and rewrite the equation in terms of this.

$$\frac{d^2\theta}{d\tau^2}+\frac{k}{mL\sqrt{gL}}\frac{d\theta}{d\tau}+\sin\theta=\frac{F}{mg}\cos\left(\Omega\tau\sqrt{\frac{L}{g}}\right) \quad (2)$$

The interesting behaviours of the pendulum occur when Ω is slightly less than the natural frequency for small oscillations, because as the amplitude increases the frequency of the oscillations decreases slightly. It is therefore convenient to write Ω as $\Omega = (1 - \eta)(g/L)^{1/2}$.

$$\frac{d^2\theta}{d\tau^2}+\frac{k}{mL\sqrt{gL}}\frac{d\theta}{d\tau}+\sin\theta=\frac{F}{mg}\cos(\tau(1-\eta)) \quad (3)$$

In order to solve this equation we then split it into two first order differential equations

$$v=\frac{d\theta}{d\tau} \quad (4)$$

$$\frac{dv}{d\tau}=\frac{F}{mg}\cos(\tau(1-\eta))-\frac{kv}{mL\sqrt{gL}}-\sin\theta \quad (5)$$

where v is the angular velocity.

3 The Computer Simulation

In order to simulate the pendulum the Runge–Kutta method was used to find a numerical solution to (5). The mass of the pendulum was assumed to be 1kg and the length was assumed to be 1m in order to keep the equation as simple as possible. When using the Runge–Kutta method step sizes of 0.01 were used, which was determined to be low enough to prevent any errors, by testing different step sizes.

In order to show the results of each simulation a phase plane plot was used. This was obtained by plotting the angular velocity on the y-axis and the angular displacement on the x-axis. A different coloured line was used to plot the last portion of iterations, making it easier to see whether there was any periodic motion.

Below are some examples of common pendulum motions shown on a phase plane diagram.

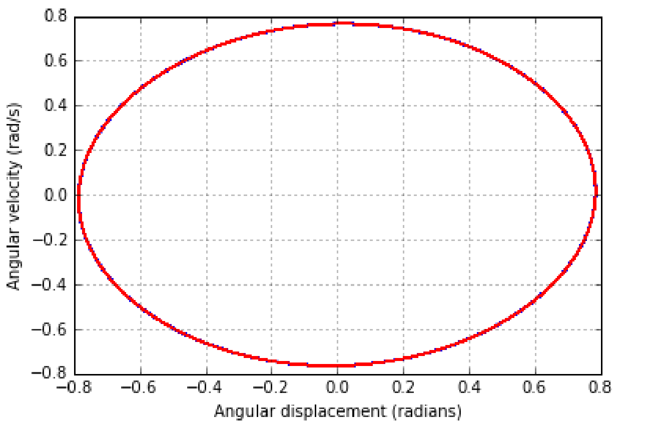

Figure 1 - simple harmonic motion of amplitude π/4

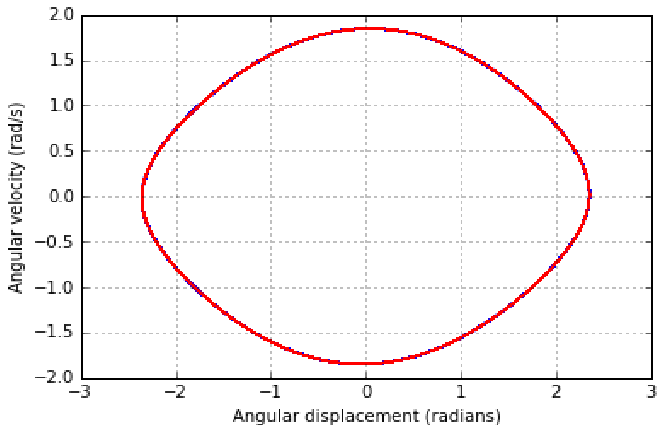

Figure 2 - Pendulum with initial angle of 3π/2

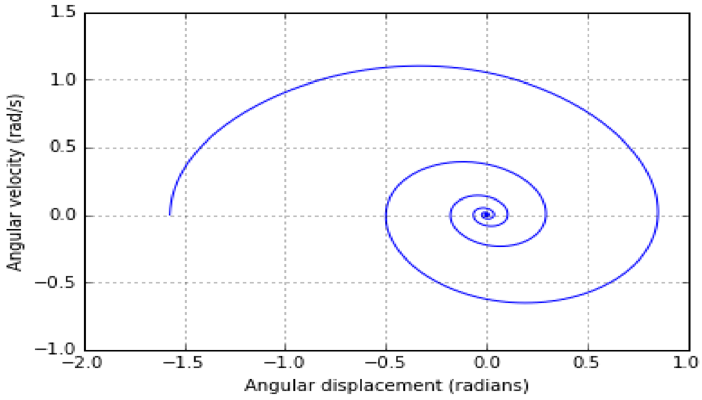

Figure 3 - Damped Harmonic motion starting from π/2

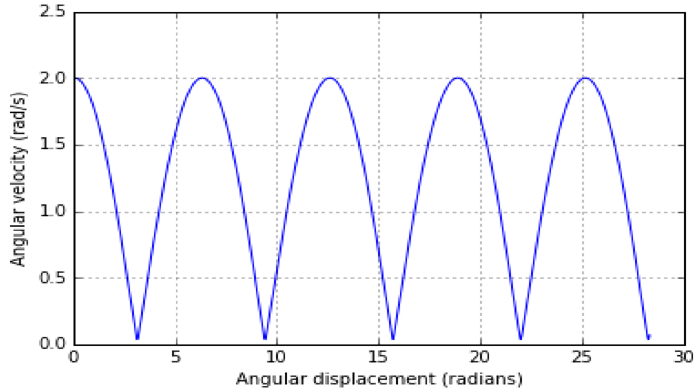

Figure 4 - Pendulum with just enough initial velocity to make a full swing

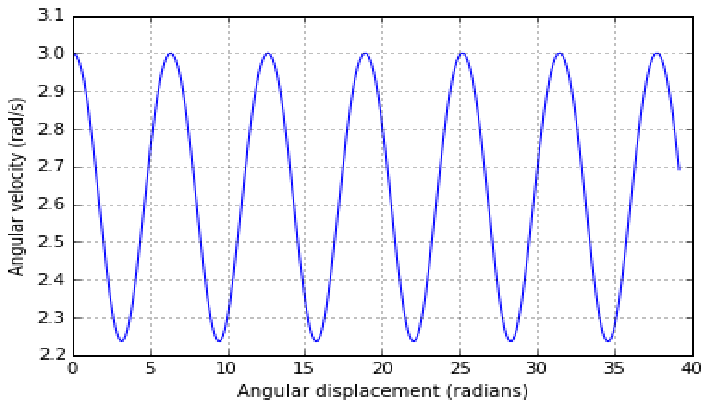

Figure 5 - Pendulum with easily enough initial velocity for a full swing

As can be seen from figures 1 and 2, a closed loop indicates periodic motion, and as can be seen from figures 4 and 5, the line moving in only the positive or negative x-direction indicates full swings of the pendulum.

4 Method

1. Start with $F = mg$, $k = 3$ and $\eta = 0.1$ so the damping is sufficient to stop large oscillations
2. Reduce the damping in increments of 0.1 until interesting behaviour is observed, and continue to record these observations as the damping gets smaller
3. Repeat step 2 for η = 0.05, 0.2, 0.3, 0.4 and record any differences in behaviour
4. Repeat steps 2 and 3 for $F = 1.5mg$ and record any differences in behaviour

Throughout the simulations behaviour such as periodic motion, chaotic behaviour and anything resembling the behaviours in figures 1-5 will be recorded.

On all simulations the starting angles and velocities were taken as zero.

5 Results and Discussion

For the simulations with high damping most of the results were very similar and looked like harmonic motion of varying shapes and sizes, of which one is shown below. However at both F values of η = 0.4 and η = 0.3, *F = 1.5mg* the diagram was slightly lopsided, which was most pronounced at η = 0.4, *F = 1.5mg* shown below.

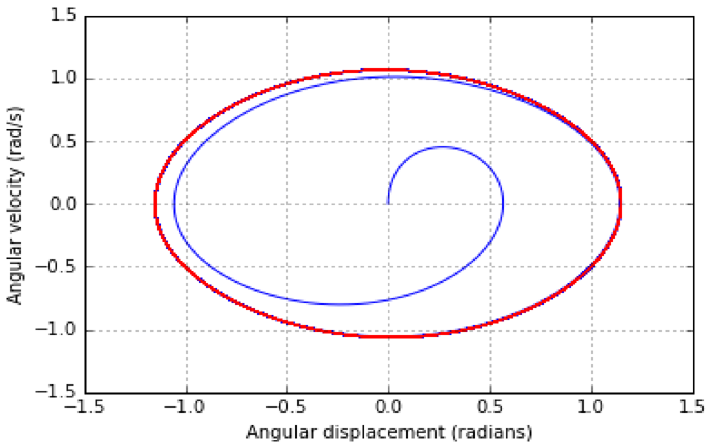

Figure 6 - F = mg, η = 0.1, k = 3. Forced Harmonic motion.

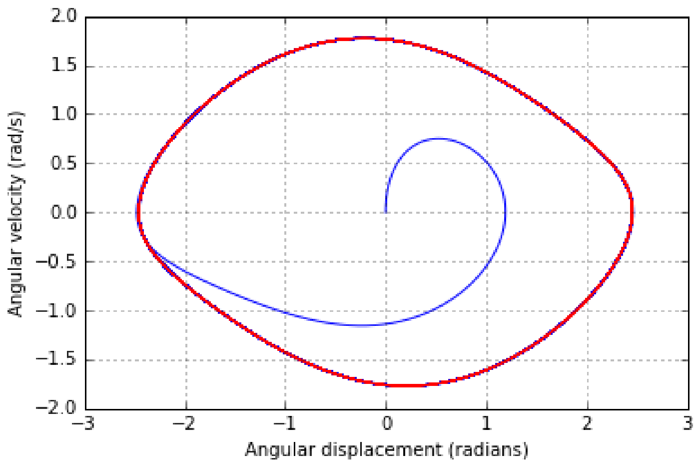

Figure 7 - F = 1.5mg, η = 0.4, k = 3

The shape of figure 7 indicates that the maximum angular velocity of the pendulum is reached just before the bottom of its swing. This is most likely due to the frequency of the force being out of sync with the natural frequency of the pendulum, and probably happens to some extent on all of the high damping results. However, they are only visible when the force and frequency difference is increased, making the effect more pronounced.

The results are shown in the tables below. Any results not shown up to k=3 are because they were all simple closed loops.

	Value of η				
K values	0.05	0.1	0.2	0.3	0.4
1.6					
1.5					
1.4					
1.3					
1.2					
1.1					
1.0					
0.9					
0.8					
0.7					
0.6					
0.5					
0.4					
0.3					
0.2					
0.1					
0					

Table 1 - Results for F = mg. Red - chaotic. Yellow - Periodic without repeating oscillations (ie. no closed loop). Blue - Complex closed loop. Green - simple closed loop.

	Value of η				
Value of k	0.05	0.1	0.2	0.3	0.4
2.5	Green	Green	Green	Green	Red
2.4	Green	Green	Green	Green	Yellow
2.3	Green	Green	Green	Green	Yellow
2.2	Green	Green	Green	Green	Yellow
2.1	Green	Green	Green	Green	Yellow
2.0	Green	Green	Green	Yellow	Yellow
1.9	Green	Green	Green	Yellow	Red
1.8	Green	Green	Green	Yellow	Blue
1.7	Green	Green	Green	Yellow	Blue
1.6	Green	Green	Yellow	Yellow	Red
1.5	Green	Green	Yellow	Yellow	Yellow
1.4	Green	Blue	Yellow	Yellow	Red
1.3	Green	Green	Yellow	Red	Blue
1.2	Blue	Green	Yellow	Blue	Red
1.1	Green	Yellow	Yellow	Blue	Red
1.0	Green	Yellow	Yellow	Blue	Yellow
0.9	Green	Yellow	Yellow	Blue	Yellow
0.8	Green	Yellow	Yellow	Red	Blue
0.7	Green	Yellow	Red	Blue	Blue
0.6	Green	Yellow	Red	Red	Blue
0.5	Yellow	Yellow	Red	Red	Blue
0.4	Green	Yellow	Red	Red	Blue
0.3	Yellow	Yellow	Yellow	Yellow	Red
0.2	Yellow	Yellow	Red	Yellow	Blue
0.1	Red	Yellow	Red	Red	Red
0	Red	Red	Red	Red	Red

Table 2 - Results for F = 1.5mg. Red - chaotic. Yellow - Periodic without repeating oscillations (ie. no closed loop). Blue - Complex closed loop. Green - simple closed loop.

To clarify the difference between each of the colours in tables 1 and 2, below are examples of phase plane diagrams for each colour.

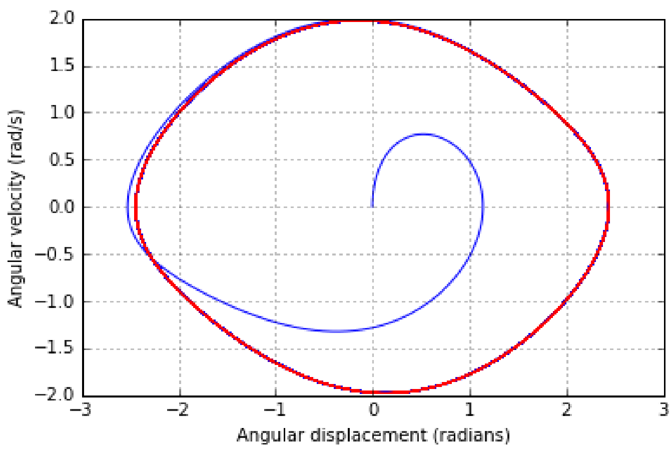

Figure 8 - Closed simple loop (green)

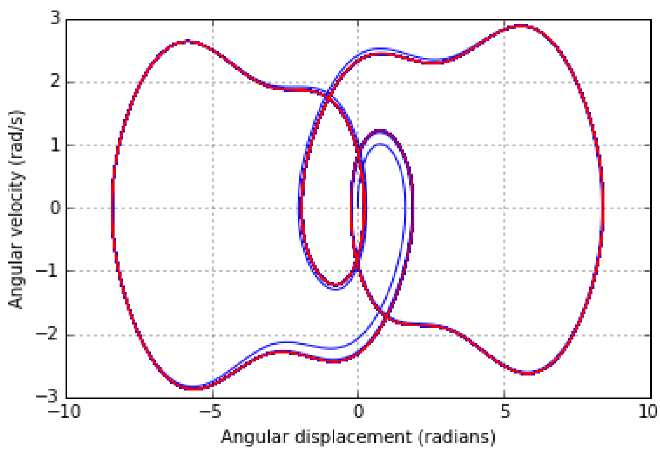

Figure 9 - Closed complex loop (blue)

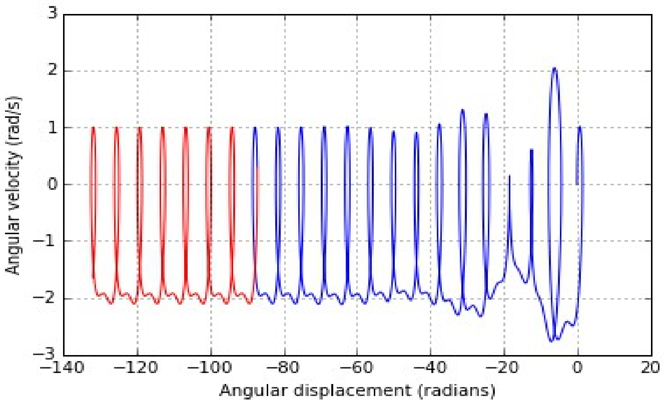

Figure 10 - Periodic motion without a loop (yellow)

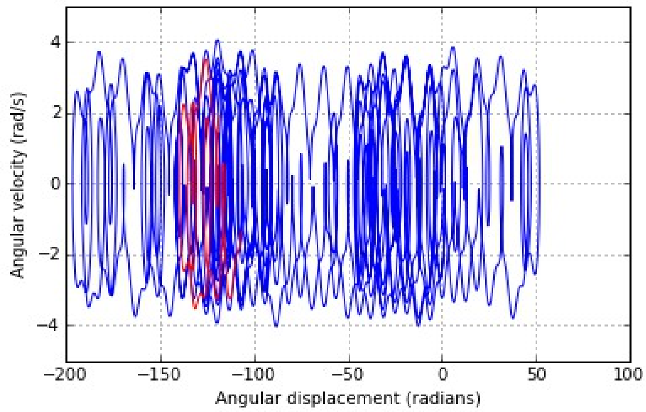

Figure 11 - Chaotic behaviour (red)

In figure 8, the motion is a simple periodic motion, i.e. the pendulum is going back and forth with no full swings. In figure 9, there is still periodic motion but a little more complex. In this particular diagram the pendulum is making a full swing clockwise, then anticlockwise, then swinging back and forth once in the centre, before doing 2 more full swings anticlockwise then clockwise and repeating itself. In figure 10, there is periodic motion, but instead of the anticlockwise and clockwise swings being the same number, it is continuing in a single direction. Figure 11 has chaotic behaviour with no discernible pattern.

As you can see from the tables, there doesn't seem to be any concrete patterns that could help us or predict the motion of the forced damped pendulum, however it is possible to discern correlations more generally as the damping, the force and the frequency of the force change.

Firstly, it is easy to see, by looking at the tables, that chaotic behaviour increases as damping is reduced, η is increased and the force is increased. This correlation between chaotic behaviour and the damping and amplitude of the force is likely due to the amplitude of the pendulum swings becoming large enough to make full swings, which is where chaotic behaviour occurs. The relation to η could be because the frequency of the force is further from the natural frequency of the pendulum, so it is less likely to resonate and find a repeating motion and more likely to continue chaotically.

When the force is increased, the chaotic results are much more spread out on the table than in the lower force table. This is due to the greater force being able to overcome the damping and make full swings, and there is generally less chaotic behaviour because the pendulum has more energy, due to the higher force, and is more likely to be able to keep making full swings, and acquiring a periodic motion. This is also the reason there are much more yellow results (periodic with no loops) in the higher force table.

Finally, the more complex loops (blue) are created because there is either enough force or little enough damping that the pendulum has enough energy to make full swings, and if the frequency of the force resonates with the natural frequency of the pendulum creating periodic motion.

6 Conclusions

In conclusion, it is not possible to discern any concrete pattern in the conditions under which a forced damped pendulum behaves chaotically. However, it is possible to get a general idea of which conditions are most likely to produce chaotic behaviour, and those which are most likely to produce other, periodic, behaviour.

We can conclude numerically that, if $F \sim mg$, chaotic behaviour is unlikely to happen at $\eta < 0.1$, and increases in likeliness at $\eta = 0.2$ up to $\eta = 0.4$ and possibly beyond, however it is impossible to say without further testing. Chaotic behaviour also increases in likelihood if k is lower, with chaotic behaviour being almost certain at $k = 0$ (further testing would be required to determine whether behaviour is always chaotic when $k = 0$ and $\eta \neq 0$). When The value of F is increased, chaotic behaviour is more likely to occur at higher levels of damping, however it becomes harder to predict which values of k will be chaotic. Finally, we can conclude that complex periodic behaviour becomes more likely as F increases.