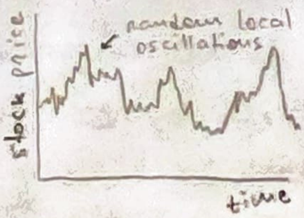
- Geometric Brownian motion and the efficient market hypothesis

1.1 Stock prices as a random walk

- Stock prices seem to follow a local behavior very similar to random walks. Thus, Monte Carlo simulation methods are the appropriate tool for simulating them.

- The data of the stock prices of the 500 biggest companies in the world (S&P-500) is a good description of the market as a whole.

- Future stock price, while random, nevertheless obeys constraints — there is more chance of ending at some values and less at other values. Thus, it follows a probability distribution.



1.2 Brownian motion

- In 1827 Robert Brown observed, using a microscope, that polen grains seemed to "move by themselves" in water. Now we know that this is because of random impacts of the grains by the water molecules.

- In 1900 the random motion of stock prices recalled the mathematician Louis Bachelier of Brown's observations.

- In 1905 Eintein showed that Brownian motion particles must satisfy the partial differential eq.

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2} \quad \text{(diffusion eq.)}$$

where $p(x,t)$ is the distibution of particles over space and time and D is a (diffusion) constant.

- The sequence of movements a particle experiences over a period of time is called a <u>world state</u>, or <u>scenario</u>. There are infinitely-possible future scenarios for stock prices.
- A crude approximation to 1-D Brownian motion can be obtained via a simple coin-toss analysis. Remarkably, accurate Brownian motion can be obtained by varying the conditions of the approximation.

- Assume time is divided into discrete periods Δt and in each such period a particle, starting at $x_0 = 0$, moves a step right or left by Δx. This is a <u>random walk</u>.

- After n steps the particle will be between $-n\Delta x$ and $n\Delta x$.

- Let $p(x,t)$ denote the probability that the particle is at $x = m\Delta x$ after n steps (i.e. at $t = n\Delta t$).

- The random walk can be viewed as the tossing n-coins: if the coin lands heads (tails) the particle moves right (left).

- If $r \equiv$ # steps to the right and $l \equiv$ # steps to the left, of

Course:

$$m = r - l \quad \text{and} \quad n = r + l$$

or even,

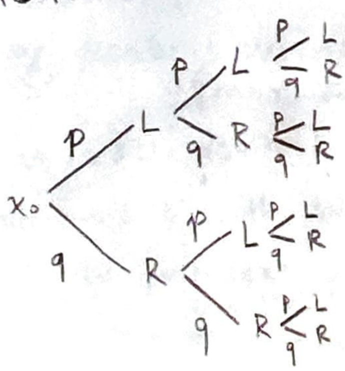
$$r = \frac{n+m}{2} \quad \text{and} \quad l = \frac{n-m}{2}$$

- Now note that, fixing m (and of course n) the particle always lands at $m\Delta x$ after the same ammount of steps left or right. Ex: for $n=4$ and $m=2$, $r=3$ and $l=1$. (Note that certain values are impossible. For ex., it is impossible to arrive at $m=3$ with only $n=4$ steps.)

- Moreover, the number of possible n-steps random walks with r-steps to the right is

$$C(n,r) = \frac{n!}{r!\,(n-r)!} \equiv \binom{n}{r}$$

- If the probability of stepping left was p and of stepping right was $q \equiv 1-p$, $p(x,t)$ would follow a binomial distribution.



...etc. $\quad p(x,t) = C(n,r)\, p^{n-r} q^{r}$

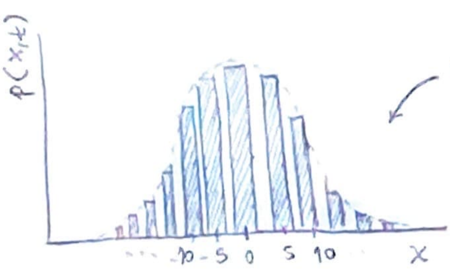
Independent steps

# of was to obtain each (mutually-exclusive) trajectory with r steps right and l steps left.

- However, assuming $p = q = 1/2$ (coin toss) we have

$$p(x,t) = \frac{C(n,r)}{2^n} \quad , \quad r = \frac{n+m}{2}$$

- If we plot a histogram of $p(x,t)$ for, say, $n = 24$, $\Delta x = 1$, we get something like this:



- The expectation value (mean) of a random walk is then

$$E(x,t) = \sum_{n \text{ times}} \left[\frac{1}{2}\Delta x + \frac{1}{2}(-\Delta x) \right]$$
$$= 0 \; //$$

- And the standard deviation (std) is

$$\sigma = \sqrt{E(x^2;t) - \underbrace{E^2(x,t)}_{=0}}$$
$$= \sqrt{\sum_{n \text{ times}} \left[\frac{1}{2}\Delta x^2 + \frac{1}{2}\Delta x^2 \right]}$$
$$= \sqrt{n}\,\Delta x \sim \sqrt{t} \quad (\text{as } t = n\Delta t)$$

- However, computing $C(n,r)$ can be very expensive when n is very large. But, in this limit, the Central Limit Theorem (CLT) states that

$$p(x,t) \simeq \frac{1}{\sqrt{2\pi n \Delta x^2}} e^{-x^2/2n\Delta x^2}$$

Normal distribution $\left(\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right)$ with mean $\mu = 0$ and variance $\sigma^2 = n\Delta x^2$.

- Defining then $D = \Delta x^2 / 2\Delta t$, we can write

$$p(x,t) \simeq \frac{1}{\sqrt{4\pi t D}} e^{-x^2/4tD}$$

Solution to the diffusion eq. proposed by Einstein //

- Therefore, we can conclude that the probability of finding a Brownian particle between $X = a$ and $X = b$ is given by the cumulative distribution function

$$Pr(a < X < b) = \int_a^b dx \frac{1}{\sqrt{4\pi t D}} e^{-\frac{x^2}{4tD}} \quad //.$$

1.3 Wiener processes

- The mathematical theory of Brownian motion was developed by Norbert Wiener, and is also called a Wiener process.

- Let W_t, $t \geq 0$, denote the position of a Brownian particle at time t with $W_0 = 0$.

- Axioms of a Wiener process:

1. Every increment $W_{t+h} - W_t$ is normally distributed with mean 0 and variance $\sigma^2 h$, where σ = fixed parameter.

2. For every pair of disjoint time intervals $[t_1, t_2]$, $[t_3, t_4]$, the increments $W_{t_2} - W_{t_1}$, $W_{t_4} - W_{t_3}$ are disjoint and distributed as in (1).

3. W_t is continuous at $t = 0$.

- Standard Brownian motion: Wiener process with $\sigma = 1$.

- Consequence of axiom 2: W_{t+h} only depends on W_t (Markov property).

- Consequence of axiom 1: $W_{t+h} = W_t + \Delta W \Rightarrow E(W_{t+h} | W_t) =$
$= \underset{\substack{\nearrow \\ \text{fixed value}}}{E(W_t)} + \underbrace{E(\Delta W)}_{=0} = W_t$. That is, the future expectation of the process equals the present value W_t (Martingale property).

- Remarkable: the axioms hold for infinitesimal increments:

$$dW_t = W_{t+dt} - W_t \quad //$$

- Axiom 1 also implies that we can change variables, so that

$$W_{0+t} - \underbrace{W_0}_{=0} = W_t \xrightarrow{h=t} \frac{W_t}{\sqrt{\sigma^2 t}} \equiv Z \quad \therefore \quad W_t = \sigma \sqrt{t}\, Z \quad //$$

where $Z = N(0,1)$ = standardized normal distribution.

- A consequence of this last eq. is that as the increments $h=t$ between the jumps tend to zero, also does the size of the jump. Thus, W_t is <u>continuous</u>. Also that within every visible jumps there are an infinite number of much smaller jumps of much smaller size.

- Note: although continuous, Brownian paths are <u>nowhere differentiable</u>. The proof is beyond our scope.

1.3.1 Simulating Brownian motion end points

- Simulating the end point W_T of a Wiener process is just a matter of generating standard normal samples $Z \sim N(0,1)$

- If $Z \sim N(0,1)$ is a sample from the standard normal, then $X = \mu T + \sigma \sqrt{T} Z$ is a sample from $N(\mu T, \sigma^2 T)$.

1.3.2 Simulating Brownian motion paths

- We can simulate the Brownian motion leading to W_T considering a sequence of <u>discrete</u> times. For ex., at $\Delta t, 2\Delta t, \ldots, n\Delta t \equiv T$ we have

$$W_0 = 0, \quad W_{i\Delta t} = W_{(i-1)\Delta t} + \sigma \sqrt{\Delta t}\, \underbrace{Z_i}_{\sim N(0,1)}, \quad i = 1, \ldots, n$$

1.3.3 Wiener processes with drift

- The Wiener process described above has mean displacement zero. This is because for every path allowed, its negative is also an admissible path occuring with the same probability. This fact can also be seen by axiom 1.

- However, a directional bias can be introduced by considering the variable

$$X_t = \mu t + W_t$$

instead of W_t.

- μ is a constant called the <u>drift</u>. Its effect is to shift the mean position of the Brownian particle from 0 to μt at time t.

- Note that

$$\rightarrow E(X_t) = E(\mu t + W_t) = \underbrace{E(\mu t)}_{=\mu t} + \underbrace{E(W_t)}_{=0} = \mu t$$

$$\rightarrow Var(X_t) = Var(\mu t + W_t) = \underbrace{Var(\mu t)}_{=0} + \underbrace{Var(W_t)}_{=0} = 0$$

- If $W_t \sim N(0, \sigma^2 t)$, then $X_t \sim N(\mu t, \sigma^2 t)$.

1.4 Arithmetical random walk

- We define an arithmetical random walk (ARW) as the simulation $\{X_0, X_{\Delta t}, X_{2\Delta t}, \dots, X_{n\Delta t}\}$ of Brownian motion with drift.

- The term "arithmetical" refers to the fact that all steps have the same size (Δx), the same mean ($\mu \Delta t$) and the same std. $\sigma\sqrt{\Delta t}$.

- General algorithm for an ARW:

```
inputs: X0, T, Δt, μ, σ
  n = T/Δt  # number of Δt steps in time T
  for t = 1, ..., n
      Z_t ~ N(0,1)  # Sample of N(0,1)
      ΔX_t = μΔt + σ√Δt Z_t
      X_t = X_{t-1} + ΔX_t
  endfor
  # The last X_t is X_T
```

- The output of the algorithm is a possible path that can be taken by the particle, out of infinitely many possible paths. A particular realization of a path is called an <u>instance</u>.

1.5 Geometric random walk

- We cannot simulate stock prices directly as an ARW because of two shortcomings:
    - Even if it starts at $X_0 = 0$, the walk can reach negative values
    - Observations show that stocks selling at small prices tend to have small increments, whereas stocks selling at high prices tend to have much larger increments.

- In 1965 the MIT economist Paul Samuelson fixed both problems with a same solution: the stock prices increments should be proportional to the present price:

$$dS = S_t(\mu dt + dW_t)$$

- Note that now stock prices cannot go below zero because when $S_t = 0$ the increment vanishes (recall that a Wiener process is continuous).

1.5.1 Price volatility

- By definition, a Wiener process W_t has a parameter σ and variance $\sigma^2 t$, showing tha σ controls the degree of dispersement of the problem.

- In the scenario of stock prices, σ is called volatility. It measures the tendency of stock prices to oscillate.

- Each stock has its own volatility, which can be estimated using its recent price data. This is called the stock's historical or statistical volatility.

- Henceforth we shall use W_t to refer to a standard Wiener process and σW_t to the process with parameter σ. We have then

$$dS \rightarrow \frac{dS}{S_t} = \mu dt + \sigma dW_t$$
$$= \mu dt + \sigma \sqrt{dt}\, \underbrace{Z_t}_{\sim N(0,1)}$$

- This is called the geometric Brownian motion (GBM), or drift-diffusion model, for stock prices.

- Note that in the GBM model, two parameters characterize any given stock: its drift, μ, and its volatility, σ.

- We can estimate μ and σ like so:

  - Given a sample of recent prices $\{S_0, S_{\Delta t}, S_{2\Delta t}, \dots, S_{n\Delta t}\}$, calculate the sequence of returns:

$$c_i = \frac{\Delta S_i}{S_i} \equiv \frac{S_{i+1} - S_i}{S_i}, \quad i = 1, \dots, n$$

  - But $E(\Delta S_i/S_i) = E(\mu \Delta t) + \sigma \overbrace{E(\Delta W_{\Delta t})}^{=0} = \mu \Delta t$ and $Var(\Delta S_i/S_i) = \underbrace{Var(\mu \Delta t)}_{=0} + Var(\sigma \Delta W_{\Delta t}) = \sigma^2 \Delta t$, so that:

by the new notation

$\rightarrow \quad \mu \simeq \frac{1}{n\Delta t} \sum_{i=0}^{n-1} c_i$ //

$\rightarrow \quad \sigma^2 \simeq \frac{1}{(n-1)\Delta t} \sum_{i=0}^{n-1} (c_i - \mu \Delta t)^2$ //

Not "n" because this is the sample variance (in contrast with the population variance). The exact reason comes from a more complex statistical analysis.

- **Note**: Care must be taken when using the above formulae. If we take Δt too small, the drift, $\mu \Delta t$, decreases in the exact proportion. But the std., $\sigma\sqrt{\Delta t}$, which is an estimator for the error in the drift, decreases less. Thus, the error in the drift may become larger than the drift itself, and our predictions may lie all around $\mu \Delta t \pm \underbrace{\sigma\sqrt{\Delta t}}_{> \mu \Delta t \ (??)}$. This phenomenon is known as statistical blur.

- GBM algorithm:

```
inputs: S0, T, μ, σ
  Δt = 1/365 # 1 day time increment in years
  n = T/Δt # number of Δt steps in time T

for t = 1, ..., n:
  Z_t ~ N(0,1)
  ΔS_t = S_{t-1} (μΔt + σ√Δt Z_t)
  S_t = S_{t-1} + ΔS_t
endfor
```

← This is an individual realization of a GBM.

1.5.2 Geometric Brownian motion end point distribution

- The above algorithm works for building individual realizations of stock prices' trajectory. However, it is more important to obtain information about the distribution of the ending price, S_T, over all possible realizations (for ex. its mean and its variance).

- The distribution of S_T is called the maturity distribution.

- We can get a sense of that by running a large number of simulations and graphing the results. It can be checked, however, that, in general, S_T does not follow a normal distribution (see the texbook, pg. 13).

- To determine the maturity distribution we, again, divide $[0,T]$ into n subdivisions of equal lenght, Δt, so that $T = n\Delta t$. Then we have

$$S_i = S_{i-1}\left(1 + \mu \Delta t + \sigma \sqrt{\Delta t}\, \underset{\substack{\uparrow \\ \text{i-th sample} \\ \text{of } N(0,1)}}{Z_i}\right).$$

But, by recurssion,

$$S_n = \underset{\substack{\nearrow \\ \text{starting} \\ \text{price}}}{S_0} \prod_{i=1}^{n} \left(1 + \mu \Delta t + \sigma \sqrt{\Delta t}\, Z_i\right).$$

$$\Rightarrow \log\left(\frac{S_n}{S_0}\right) = \sum_{i=1}^{n} \log\left(1 + \mu \Delta t + \sigma \sqrt{\Delta t}\, Z_i\right)$$

- Using then the Taylor expansion of the logarithm

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

we arrive at

$$\log\left(\frac{S_n}{S_0}\right) = \sum_{i=1}^{n}\Big[(\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_i) - \frac{1}{2}(\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_i)^2 + \frac{1}{3}(\mu\Delta t + \sigma\sqrt{\Delta t}\, Z_i)^3 + \dots\Big]$$

- However, using the central limit theorem, it can be shown that, in the limit $n \to \infty$, we get

$$\log\left(\frac{S_T}{S_0}\right) \sim \mu T - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\, Z \quad //$$

(see the textbook, pg. 15).

- This shows that $\log(S_T/S_0)$ is normally distributed, or, in other words, that S_T/S_0 is lognormally ("LN") distributed.

- As $\log S_0$ is a constant we can rewrite the above formula more conveniently, and obtain its mean and its variance:

$$\log S_T \sim \log S_0 + \mu T - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\, Z \quad //$$

$$\to E(\log S_T) = \log S_0 + \mu T - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\, \underbrace{E(Z)}_{=0} \quad //$$

$$\to Var(\log S_T) = Var(\sigma\sqrt{T}\, Z) = \sigma^2 T \quad //$$

- Moreover, knowing that $\log S_T$ is normally distributed allows us to find the distribution of S_T itself. We omit the details here (see the textbook, pg. 16), but the results are

$$E(S_T) = S_0 e^{\mu T}$$

$$Var(S_T) = S_0^2 (e^{\sigma^2 T} - 1) e^{2\mu T}$$

- A final remark is that $\text{median}(S_T) = S_0 e^{(\mu - \sigma^2/2)T}$, which is <u>less</u> than the mean value. This is due to the fact that S_T is not normally distributed (i.e. its graph contains <u>skewenesses</u>).