Revisiting Tardos's Framework for Linear Programming

Faster Exact Solutions using Approximate Solvers

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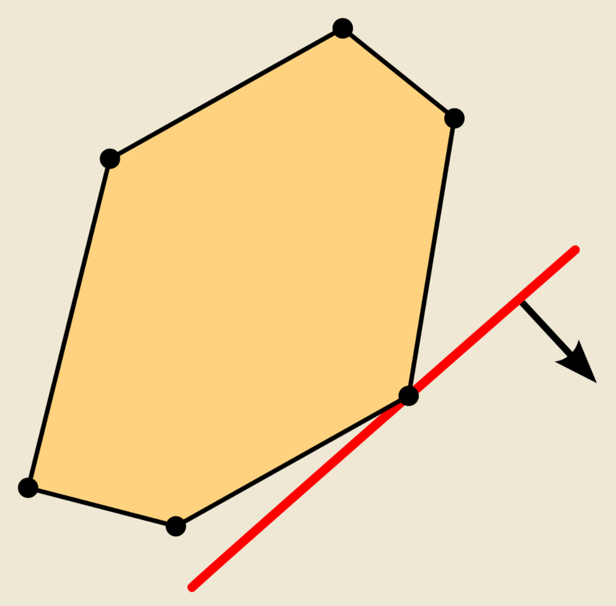
Linear Programming

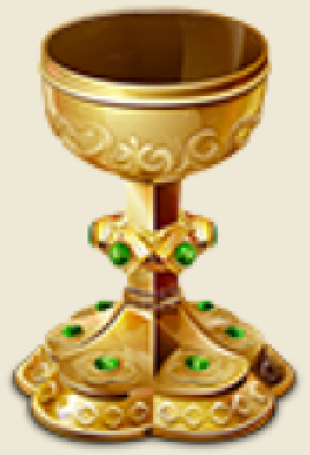
In standard form for $A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$,

$$
\begin{aligned}
\min\ &\langle c, x\rangle \\
Ax &= b \\
x &\geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\max\ &\langle y, b\rangle \\
A^\top y + s &= c \\
s &\geq 0
\end{aligned}
$$



Timeline

?	**Strongly polynomial algorithm for LP** Smale's 9th question
1980s	**Interior point methods** Karmarkar
1970s	**Ellipsoid Method** Khachiyan
1940s	**Simplex Method** Dantzig
1820s	**Origins** Fourier



Weakly vs Strongly Polynomial Algorithms for LP

LP with n variables, m constraints
L: encoding length of the input.

weakly polynomial

- $\text{poly}(m, n, L)$ basic arithmetic operations.
- Standard variants of Ellipsoid and interior point methods: running time bound heavily relies on L.

strongly polynomial

- $\text{poly}(m, n)$ basic arithmetic operations.
- PSPACE: all numbers occurring in the algorithm must remain polynomially bounded in input size.

Fast Weakly Polynomial Algorithms for LP

ε-approximate solution

- Approximately optimal: $\langle c, x\rangle \leq \mathrm{OPT} + \varepsilon\|c\|R$
- Approximately feasible: $\|Ax - b\| \leq \varepsilon(\|A\|_F R + \|b\|)$
- $\log(1/\varepsilon)$ dependence $\Rightarrow$ exact algorithm with L dependence.

Recent progress

- Randomized $O((\mathrm{nnz}(A) + m^2)\sqrt{m}\log^{O(1)}(n)\log(n/\varepsilon))$ Lee, Sidford '13–'19
- Randomized $O(n^\omega \log^{O(1)}(n)\log(n/\varepsilon))$ Cohen, Lee, Sidford '19
- Deterministic $O(n^\omega \log^2(n)\log(n/\varepsilon))$ van den Brand '20
- Randomized $O((mn + m^3)\log^{O(1)}(n)\log(n/\varepsilon))$ van den Brand, Lee, Sidford, Song '20

Strongly Polynomial Algorithms for LP

Network flow problems

- Maximum flow: Edmonds–Karp–Dinitz '70-72
- Min-cost flow: Tardos '85

Special classes of LP

- Feasibility of 2-variable-per-inequality systems: Megiddo '83
- Discounted Markov Decision Processes: Ye '05, Ye '11
- Maximum generalized flow problem: V. '17, Olver– V. '20
- ...

Dependence on the constraint matrix only

$$\min\langle c, x\rangle \; Ax = b, \; x \geq 0$$

Running time dependent **only** on constraint matrix A, but **not** on b and c.

General LP

- 'Combinatorial LPs'
  If A integral and $\Delta_A := \max\{|\det(B)| : \; B \text{ is a square submatrix of } A\}$, then LP solvable in $\mathrm{poly}(m, n, \log \Delta_A)$ arithmetic operations: Tardos '86
- 'Layered-least-squares (LLS) Interior Point Method'
  LP solvable in $O(n^{3.5} \log \bar{\chi}_A)$ linear system solves: Vavasis–Ye '96
- 'Scaling invariant Layered-least-squares (LLS) Interior Point Method'
  LP solvable in $O(n^{2.5} \log n \log \bar{\chi}^*_A)$ linear system solves: Dadush–Huiberts–N.–Végh '20

Dependence on the constraint matrix only

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'Layered-least-squares (LLS) Interior Point Method'
LP solvable in $O(n^{3.5} \log \bar{\chi}_A)$ linear system solves: Vavasis–Ye '96

Condition number $\bar{\chi}_A$

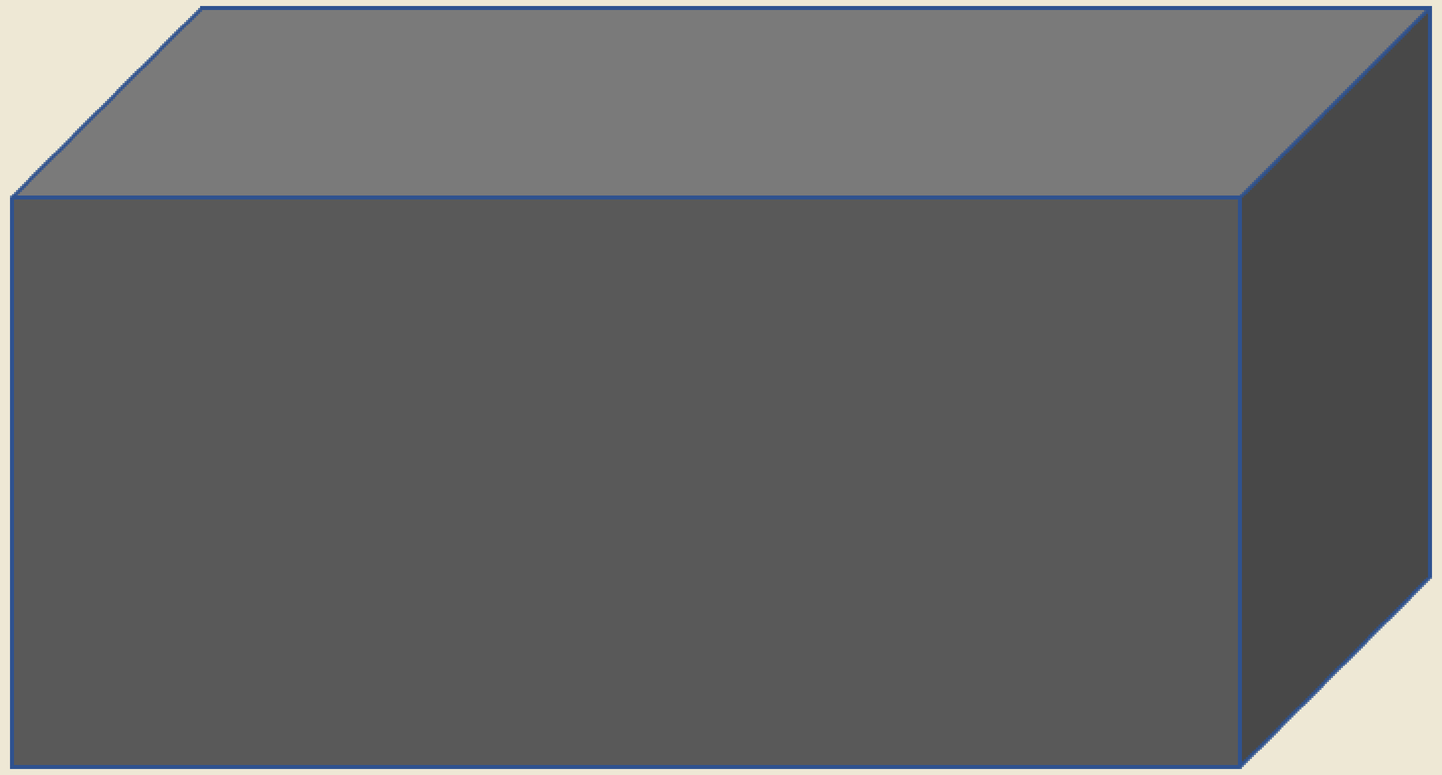
- $\bar{\chi}_A = O(2^{L_A})$.
- Depends **only** on the subspace $\ker(A)$.
- NP-hard to approximate within a factor $2^{\mathrm{poly}(\mathrm{rank}(A))}$: Tunçel '99

Can an exact LP algorithm also be fast?

Layered Least Squares IPMs

- Require computationally expensive special step directions
- Extending them to weighted central paths seems difficult

Black box approach



- Use fast approximate solver in black box manner
- Learn information about the support of the optimal solution
- Relies on proximity results on LP solutions

Tardos's framework: variable fixing

$$\min\langle c, x\rangle \; Ax = b,\; x \geq 0, A \in \mathbb{Z}^{m\times n}$$

Running time dependent **only** on constraint matrix A, but **not** on b and c.
Key idea for the first strongly polynomial algorithm for minimum cost flows.

Proximity

Use **exact** solvers to find optimal solution x to ε-rounded (perturbed) problems. Proximity yields that an optimal solution x^* to the original problem is within $\mathrm{poly}(n)\Delta_A \cdot \varepsilon$ of the rounded problem.

Variable Fixing

If the proximity is better than $\|x\|_\infty$, then we learn $x_i^* > 0$ for a variable and so the corresponding slack variable is $s_i^* = 0$.
$\rightsquigarrow$ **delete** variable and **recurse** on smaller problem.

Our contributions: Dadush–N.–Végh '20

Generalizing Tardos' result to real matrices

We give a blackbox algorithm that can handle any **real** matrix $A \in \mathbb{R}^{m\times n}$ and dependence $\log \bar{\chi}_A$ instead of $\log \Delta_A$.

Usage of approximate solvers

We only require any **approximate** LP solver, and can directly leverage the fast approximate LP algorithms. $O(mn^{\omega+1+o(1)} \log \bar{\chi}_A)$ exact deterministic LP algorithm using van den Brand '20.

Certificates for infeasibility and large condition numbers

If primal or dual linear programming are infeasible we provide a Farkas certificate. In case that the condition number is larger than our guess, we are able to provide a certificate.

$\bar{\chi}$ is hard to estimate. Iterative guesses $M \to \max\{M^2, \text{certified lower bound at failure}\}$.

The mysterious $\bar{\chi}_A$

through a matroidal lens

The condition number $\bar{\chi}_A$

Definition.

$$\bar{\chi}_A := \sup\left\{\|A^\top\left(ADA^\top\right)^{-1}AD\| \,:\, D \in \mathbf{D}\right\}$$

- Introduced by Dikin '67, Stewart '89, Todd '90, ...
- Bounds norm of oblique projections.
- Depends only on the subspace $\ker(A)$.
- Plays key role in certain interior point methods.

The circuit imbalance measure

...the "combinatorial" sister of $\bar{\chi}_A$

Definition. A **circuit** of A is a minimal linearly dependent subset of columns $C \subseteq [n]$. Let $\mathcal{C}$ denote the set of all circuits.

Definition. The **circuit imbalance measure** of A is

$$\kappa_A := \max\left\{ \left|\frac{g_j}{g_i}\right| : Ag = 0,\ \mathrm{supp}(g) \in \mathcal{C}, i, j \in \mathrm{supp}(g), \right\}$$

Lemma. *If A is a TU-matrix, then $\kappa_A = 1$. More generally, if A is integer, then $\kappa_A \leq \Delta_A$.*

Proof. For a TU-matrix, $Ax = 0, -1 \leq x \leq 1, x_J = 0$ is an integral polytope for all $J \subseteq n$. The second part follows by Cramer's rule.

Theorem. *[DHNV20]* $\sqrt{1+\kappa_A^2} \leq \bar{\chi}_A \leq n\kappa_A$. *Thus,* $\log(n+\kappa_A) = \Theta(\log(n+\bar{\chi}_A))$.

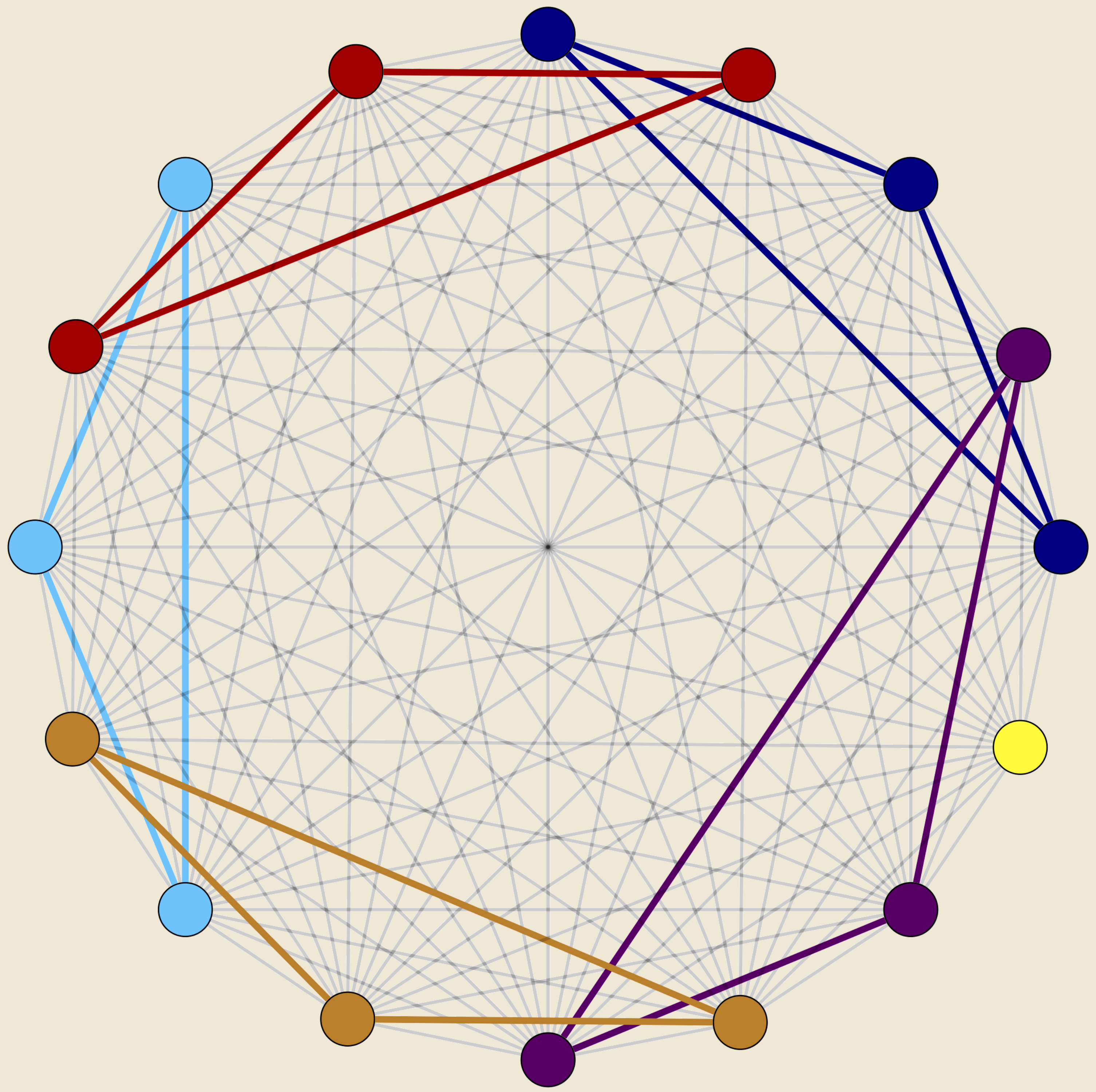
Δ vs κ

- In general $\kappa_A \leq n\Delta_A$.
- For complete undirected graph:

$$\kappa_A = 2, \text{ but } \Delta_A \geq 2^{\lfloor n/3 \rfloor}$$

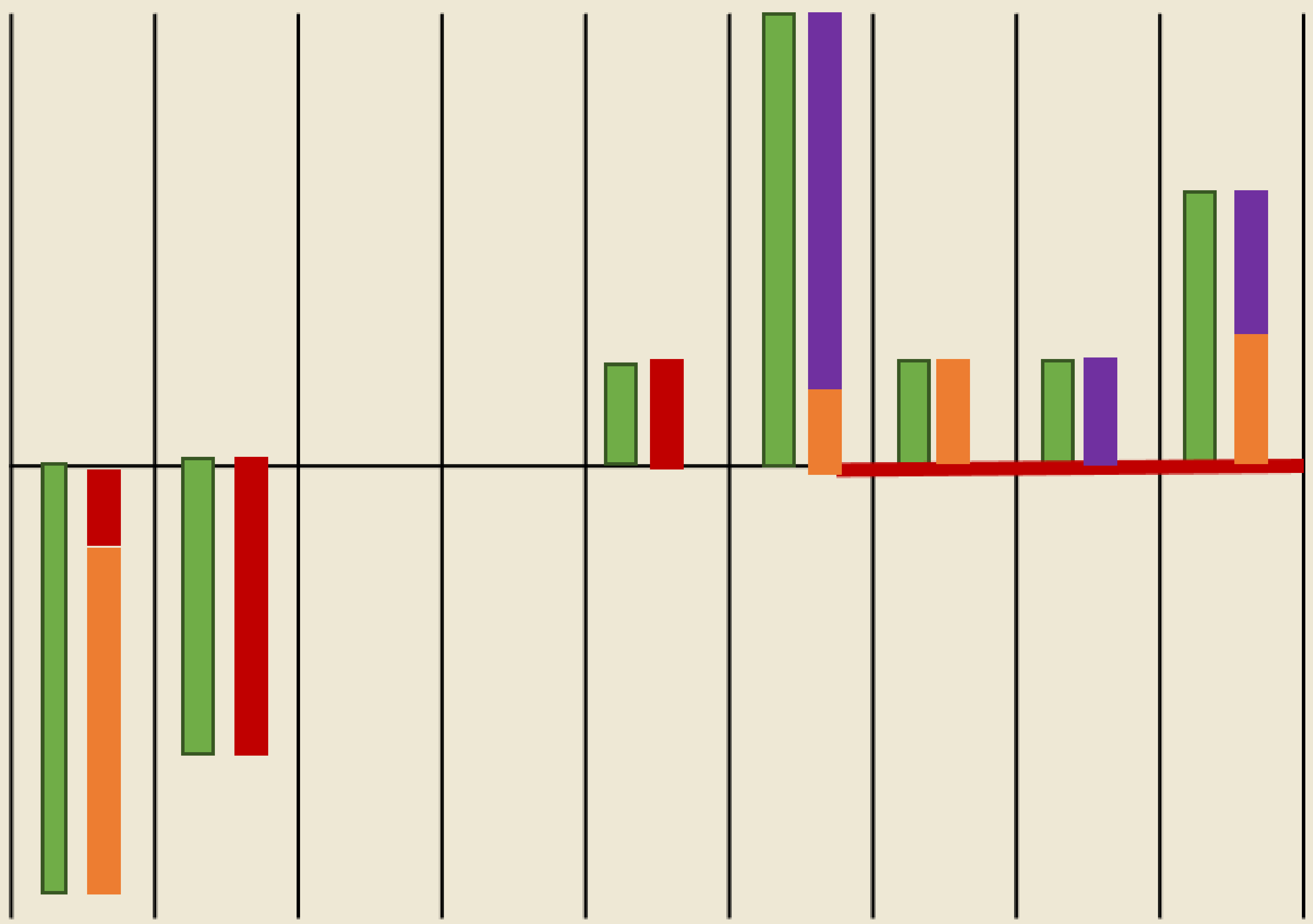
as

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2$$



Near-optimal rescaling

Proximity theorems for κ_A



Linear Programming in subspace view

...a change of perspective

In standard form for $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$,

$$\begin{aligned} \min \ \langle c, x \rangle \quad & \quad \max \ \langle y, b \rangle \\ Ax = b \quad & \quad A^\top y + s = c \\ x \geq 0 \quad & \quad s \geq 0 \end{aligned}$$

In subspace view for $W = \ker(A), d \in \mathbb{Q}^n$, s.t. $Ad = b$,

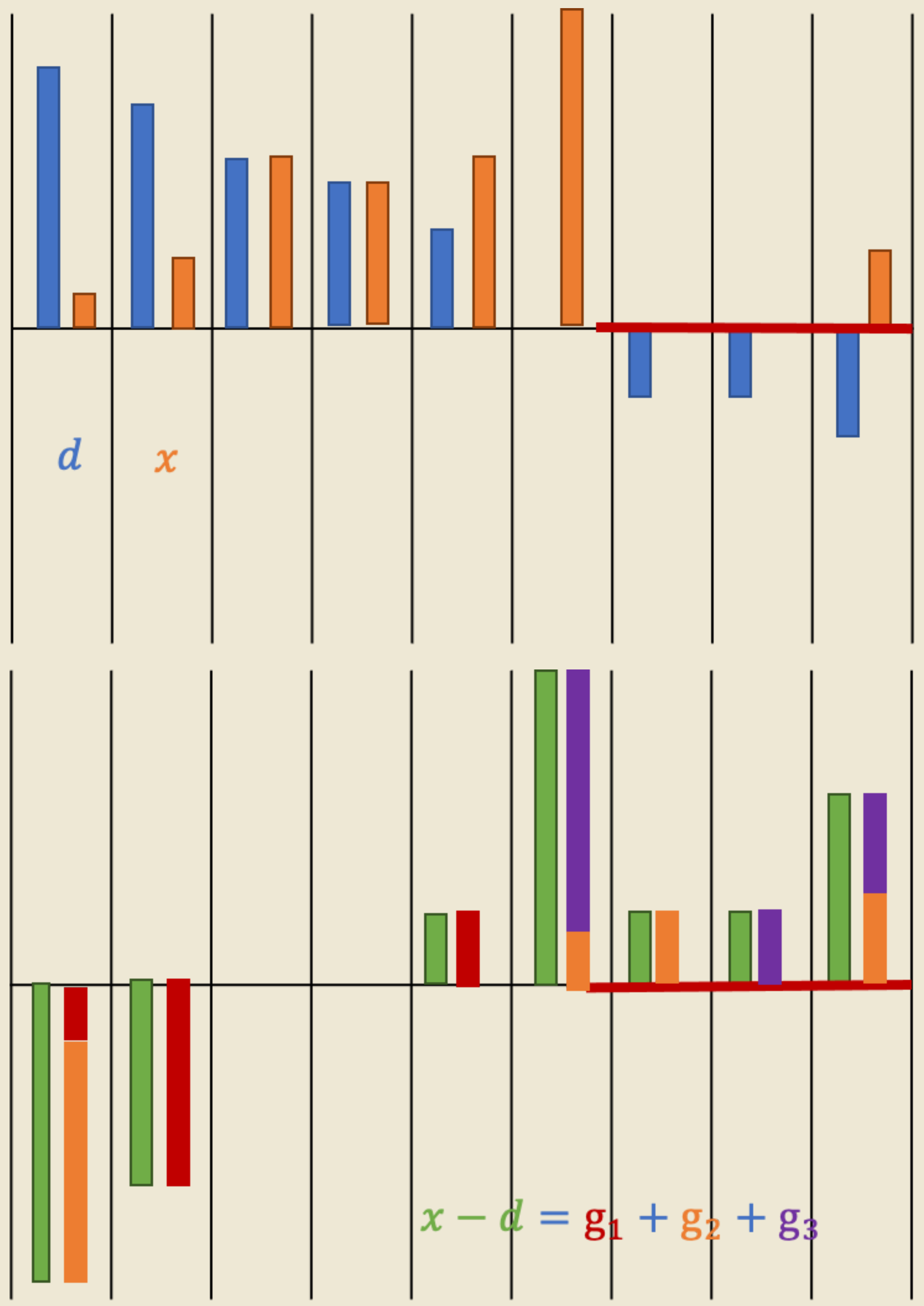
$$\begin{aligned} \min \ \langle c, x \rangle \quad & \quad \max \ \langle c - s, d \rangle \\ x \in W + d \quad & \quad s \in W^\perp + c \\ x \geq 0 \quad & \quad s \geq 0 \end{aligned}$$

Proximity theorem

Theorem. *Assume that the system $x \in W + d, x \geq 0$ is feasible. Then there exists a feasible solution such that $\|x - d\|_\infty \leq \kappa_W \|d^-\|_1$.*

Proof sketch.

- Take any feasible $x \in W + d, x \geq 0$. Thus, $x - d \in W$.
- We decompose $x - d = g_1 + g_2 + \ldots + g_t$ into sign-consistent circuits $g_i \in W$ by Carathéodory's theorem.
- Delete circuits that do not intersect $\mathrm{supp}(d^-)$.
- For all other circuits g and indices j, $|g_j| \leq \kappa_A |g_k|$ for some $k \in \mathrm{supp}(d^-)$.

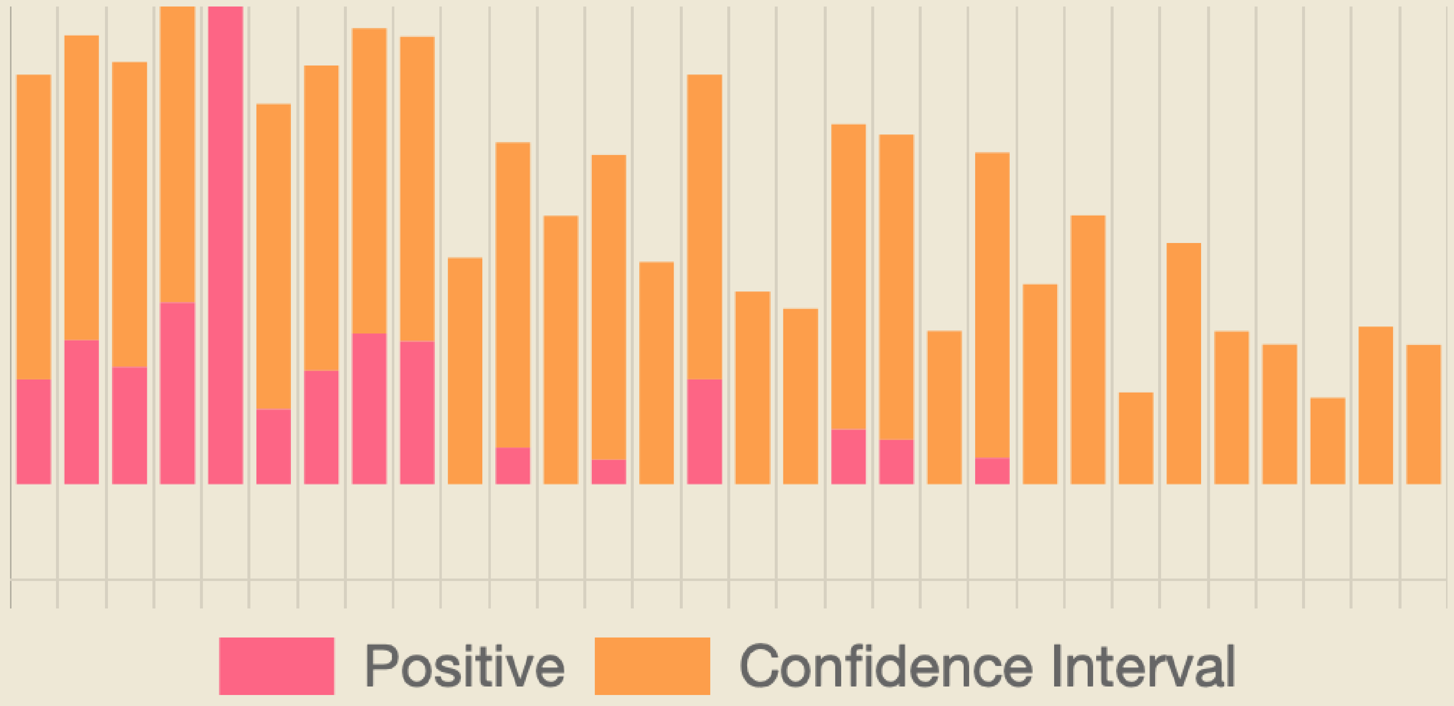


Variable fixing for feasibility

Theorem. *Assume that the system* $x \in W + d, x \geq 0$ *is feasible. Then there exists a feasible solution such that* $\|x - d\|_\infty \leq \kappa_W \|d^-\|_1$.

Recursive algorithm

- Use approximate solver to get near feasible $z \in W + d$ with $\|z^-\|_1$ "small".
- $I := \{i \in [n] : z_i > \kappa_W \|z^-\|_1\}$.
- $J := \{i \in [n] : z_i \leq \kappa_W \|z^-\|_1\}$.
- By proximity, there exists a feasible solution with $x_I > 0$.
- Recurse on the subspace $W' = \mathrm{proj}_J(W)$ with $d' = d_J$.
- If $W = \ker(A)$, then we obtain $W' = \ker(A')$ by eliminating the variables in I.



$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & A' \\ 0 & 0 & \\ 0 & 0 & \end{array}\right)$$

Variable fixing for feasibility

Recursive algorithm

- Use approximate solver to get near feasible $z \in W + d$ with $\|z^-\|_1$ "small".
- $I := \{i \in [n] : z_i > \kappa_W \|z^-\|_1\}$.
- $J := \{i \in [n] : z_i \leq \kappa_W \|z^-\|_1\}$.
- Recurse on the subspace $W' = \text{proj}_J(W)$ with $d' = d_J$.

Questions

- How do we guarantee that $I \neq \emptyset$?
- How can we construct a feasible solution? Given $x' \in \text{proj}_J(W) + d_J, x' \geq 0$, how do we recover $x \in W + d, x \geq 0$?

The lifting operation

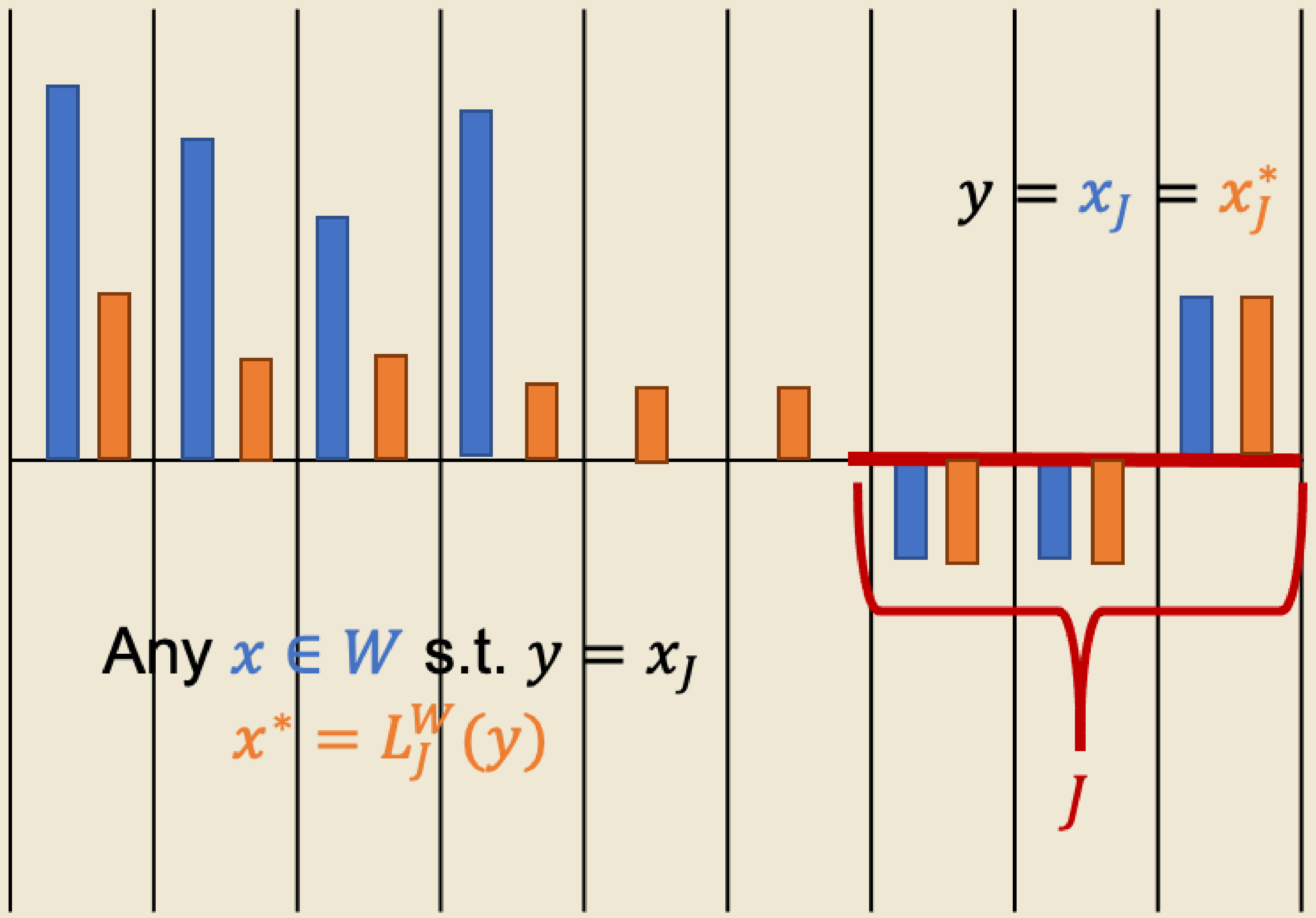
- $W \subseteq \mathbb{R}^n$ subspace, $J \subseteq [n]$
- $y \in \mathrm{proj}_J(W)$, i.e. $\exists x \in W, x_J = y$.
- The lifting of y to W is defined as

$$L_J^W(y) := \arg\min_x \{\|x\|_2 : x \in W, x_J = y\}$$

- Can be computed using a projection matrix.

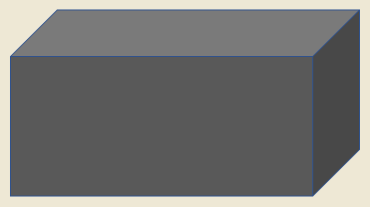
Lemma. $\|L_J^W(y)\|_\infty \leq \kappa_W \|y\|_1$.

Proof. A similar circuit decomposition argument.



The feasibility algorithm

Oracle(ε)



$$z \in W + d$$
$$\|z^-\|_\infty \le \varepsilon \|d^-\|_1$$
$$\|z - d\|_\infty \le C\kappa_W \|d^-\|_1$$

Feasibility(W, d)

$$x \in W + d, \quad x \ge 0$$
$$\|x - d\|_\infty \le C'\kappa_W^2 n \|d^-\|_1$$

Stronger system with proximity constraint useful for "pullback"

- Obtain z by applying the oracle with $\varepsilon = 1/(\kappa \cdot \mathrm{poly}(n))$
- $J := \{i \in [n] : \ z_i < \kappa_W \|z^-\|_1\}$.
- If $J = [n]$ then replace d by the projection d/W.
- Apply the recursive solver to $\mathrm{proj}_J(W)$ and z_J to obtain $\tilde{x} \in \mathrm{proj}_J(W) + z_J, \ \tilde{x} \ge 0$.
- Lift the solution back up to obtain

$$x := z + L_J^W(\tilde{x} - z_J) \ge 0.$$

- Non-negativity and proximity follows from proximity of the recursive solver!

The feasibility algorithm

- As described above, we need $\leq n$ calls to the oracle.
- Can be decreased to $\leq m$ calls (with a little more care.)
- This leads to an $O(mn^{\omega+o(1)}\log(\kappa_W + n))$ feasibility algorithm using van den Brand '20.

Estimating and certifying κ_W

- We maintain a guess M on κ_W.
- If $\|L_J^W(y)\|_\infty \leq M\|y\|_1$ for every lifting call, the algorithm succeeds.
- Otherwise, we can recover a circuit with imbalance $> M$, showing that $\kappa_W > M$.

Proximal optimal solutions

proximity works for optimization as well!

$$\begin{aligned} \min\ & \langle c, x\rangle \\ & x \in W + d \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} \max\ & \langle c - s, d\rangle \\ & s \in W^\perp + c \\ & s \geq 0 \end{aligned}$$

Let $s \geq 0$, $s \in W^\perp + c$ be a feasible dual, but not necessarily optimal solution.

Theorem. *Assuming that the primal is feasible, there exists an* ***optimal*** *solution* $x \in W + d, x \geq 0$ *such that* $\|x - d\|_\infty \leq \kappa_W \big(\|d^-\|_1 + \|d_{\mathrm{supp}(s)}\|_1\big)$.

Optimization algorithm

$$\begin{array}{ll} \min \langle c, x\rangle & \max \langle c-s, d\rangle \\ \quad x \in W + d & \quad s \in W^\perp + c \\ \quad x \geq 0 & \quad s \geq 0 \end{array}$$

Theorem. *Assuming that the primal is feasible, there exists an **optimal** solution $x \in W + d, x \geq 0$ such that $\|x - d\|_\infty \leq \kappa_W(\|d^-\|_1 + \|d_{\mathrm{supp}(s)}\|_1)$.*

- Altogether nm calls to the black box solver.
- We have $\leq n$ **Outer Loops**, each comprising $\leq m$ **Inner Loops**
- Each **Outer Loop** finds $(x, s) \in (W + d) \times (W^\perp + c)$ with $\|x^-\|_1 + \|x_{\mathrm{supp}(s)}\|_1$ "small", and $s \geq 0$ feasible for the dual.
- Using proximity, we can use this to conclude $x_I > 0$ for a certain variable set $I \subseteq [n]$ and recurse.

Open questions

- Feasibility needs m calls—can we make it $\min\{m, n-m\}$ to have the same for primal and dual?
- Optimization takes mn calls—would fewer be enough?
- Can we get better for special cases, such as max flow or min-cost flow?
- Can we get faster (possibly non-deterministic) version of the constructive Hoffman algorithm?
- Can we extend the black box approach to problems with unbounded κ, such as generalized flows?
- κ - theory for more general convex programs e.g. **Convex Quadratic Programs** or **Semidefinite Programs (SDP)**
- κ - theory for **Integer Programming (IP)**