1. a) $$2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 = \overbrace{2\hat{S}_{1x}\hat{S}_{2x} + 2\hat{S}_{1y}\hat{S}_{2y}}^{**} + 2\hat{S}_{1z}\hat{S}_{2z}$$

$$\hat{S}_{1+}\hat{S}_{2-} = (\hat{S}_{1x} + i\hat{S}_{1y})(\hat{S}_{2x} - i\hat{S}_{2y})$$
$$= \hat{S}_{1x}\hat{S}_{2x} - \cancel{i\hat{S}_{1x}\hat{S}_{2y}} + \cancel{i\hat{S}_{1y}\hat{S}_{2x}} + \hat{S}_{1y}\hat{S}_{2y}$$

$$\hat{S}_{1-}\hat{S}_{2+} = (\hat{S}_{1x} - i\hat{S}_{1y})(\hat{S}_{2x} + i\hat{S}_{2y})$$
$$+ \quad = \hat{S}_{1x}\hat{S}_{2x} - \cancel{i\hat{S}_{1y}\hat{S}_{2x}} + \cancel{i\hat{S}_{1x}\hat{S}_{2y}} + \hat{S}_{1y}\hat{S}_{2y}$$

$$* \quad \hat{S}_{1+}\hat{S}_{2+} + \hat{S}_{1-}\hat{S}_{2-} = 2\hat{S}_{1x}\hat{S}_{2x} + 2\hat{S}_{1y}\hat{S}_{2y}$$

Subing * at ** we get

$$*** \quad 2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 = \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z}$$

b) $$\langle 3|\hat{H}|3\rangle = \frac{A}{\hbar^2}\langle -z, +z|2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2|-z+z\rangle$$

Acting with a raising or lowering operator after plugging in *** will yield 0 since it will raise or lower the ket making it different from the bra. This is why writing the x + y operators as + + − is so helpful. You know what they do to the z states whereas you don't easily know what $\hat{S}_x$ does.

This leaves

operates, pulls out eigenvalue $\frac{\hbar}{2}$

$$= \frac{A}{\hbar^2}\langle -z, +z|2\hat{S}_{1z}\hat{S}_{2z}|-z, +z\rangle$$

$$= \frac{A}{\hbar^2}\, 2\left(\frac{-\hbar}{2}\right)\left(\frac{\hbar}{2}\right)$$

$$= -\frac{A}{2} \checkmark$$

c) Lets name the eigenstates of $\hat{H}$ as follows and add

$$\frac{1}{\sqrt{2}}\left(|1,0\rangle = \frac{1}{\sqrt{2}}|+z,-z\rangle + \frac{1}{\sqrt{2}}|-z,+z\rangle\right)$$

$$-\frac{1}{\sqrt{2}}\left(|0,0\rangle = \frac{1}{\sqrt{2}}|+z-z\rangle - \frac{1}{\sqrt{2}}|-z,+z\rangle\right)$$

$$\frac{1}{\sqrt{2}}|1,0\rangle - \frac{1}{\sqrt{2}}|0,0\rangle = |-z,+z\rangle = |3\rangle$$

So now we have the $|3\rangle$ state in terms of the eigenstates of $\hat{H}$ so we now how $\hat{H}$ acts on $|3\rangle$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|3\rangle$$

$$= \frac{1}{\sqrt{2}}e^{-iAt/2\hbar}|1,0\rangle - \frac{1}{\sqrt{2}}e^{3iAt/2\hbar}|0,0\rangle$$

$$= \frac{1}{2}\left(e^{-iAt/2\hbar}(|2\rangle + |3\rangle) - e^{3iAt/2\hbar}(|2\rangle - |3\rangle)\right)$$

$$= \frac{1}{2}\left(e^{-iAt/2\hbar} - e^{3iAt/2\hbar}\right)|2\rangle$$
$$+ \frac{1}{2}\left(e^{-iAt/2\hbar} + e^{3iAt/2\hbar}\right)|3\rangle$$

$$= e^{iAt/2\hbar}\left[\left(\frac{e^{-iAt/\hbar} - e^{iAt/\hbar}}{2}\right)|2\rangle + \left(\frac{e^{-iAt/\hbar} + e^{iAt/\hbar}}{2}\right)|3\rangle\right]$$

$$= e^{iAt/2\hbar}\left(i\sin At/\hbar\, |2\rangle + \cos At/\hbar\right)|3\rangle$$

So at $t=0$ it's the $|3\rangle$ state (up to phase) and at $t = \pi\hbar/A$ it's $|2\rangle$, then at $2\pi\hbar/A$, it's back to $|3\rangle$. The freq of oscillations is $\omega = \frac{A}{\hbar}$ or $\nu = \frac{A}{h}$.

But the freq of osillations of <u>probability</u> is $2\frac{A}{\hbar}$ since,

$$|\langle 2|\psi(t)\rangle|^2 = \sin^2 At/\hbar = \tfrac{1}{2}(1-\cos 2At/\hbar)$$

2 a) When $B_x = 0$, 4.38 is diagonal making the basis states $|\pm z\rangle$ eigen states of $\hat{H}$.

b) Looking at what we did to get the Rabi formula

$$\frac{\omega_1^2/4}{(\omega_0 - \omega)^2 + \omega_1^2/4} \sin^2 \frac{\sqrt{(\omega_0-\omega)^2 + \omega_1^2/4}}{2} t$$

$\uparrow$

difference between energy states before the oscillating field is turned on over $\hbar$. For NH_3) $\frac{1}{\hbar}|E_0 - A - E_0 - A| = 2A/\hbar$

ω is the drive frequency in both cases

$\frac{\hbar}{2}\omega_1 \to \mu_e|\vec{E}_0|$ so $\omega_1 \to \frac{2\mu_e}{\hbar}|\vec{E}_0|$

So the analogue of Rabi for NH_3 is

$$\frac{\mu_e^2|\vec{E}_0|^2}{(2A - \omega\hbar)^2 + \mu_e^2|E_0|^2} \sin^2\left[t\frac{\sqrt{(2A-\omega\hbar)^2 + \mu_e^2|\vec{E}_0|^2}}{2\hbar}\right]$$

c) These are transitions between the $|I\rangle$ and $|II\rangle$ states, the eigenstates of the NH_3 molecule when $\vec{E} = 0$. If we start in state $|II\rangle$, the answer in (b) is the probability to be found in $|I\rangle$.

d) The basis used in 4.57, $|1\rangle, |2\rangle$ correspond to N above and N below the H_3 plane respectively, fig 4.9.

e) $\begin{pmatrix} \langle I|1\rangle & \langle I|2\rangle \\ \langle II|1\rangle & \langle II|2\rangle \end{pmatrix}\begin{pmatrix} \langle 1|\hat{H}|1\rangle & \langle 1|\hat{H}|2\rangle \\ \langle 2|\hat{H}|1\rangle & \langle 2|\hat{H}|2\rangle \end{pmatrix}\begin{pmatrix} \langle 1|I\rangle & \langle 1|II\rangle \\ \langle 2|I\rangle & \langle 2|II\rangle \end{pmatrix}$

$$= \begin{pmatrix} \langle I|1\rangle & \langle I|2\rangle \\ \langle II|1\rangle & \langle II|2\rangle \end{pmatrix}\begin{pmatrix} \langle 1|\hat{H}|1\rangle\langle 1|I\rangle + \langle 1|\hat{H}|2\rangle\langle 2|I\rangle & \langle 1|\hat{H}(|1\rangle\langle 1| + |2\rangle\langle 2|)|II\rangle \\ \langle 2|\hat{H}\underbrace{(|1\rangle\langle 1| + |2\rangle\langle 2|)}|I\rangle & \cdots \end{pmatrix}$$

In each case we have a complete set of states which is $\mathbb{1}$

$$= \begin{pmatrix} \langle I|1\rangle & \langle I|2\rangle \\ \langle II|1\rangle & \langle II|2\rangle \end{pmatrix}\begin{pmatrix} \langle 1|\hat{H}|I\rangle & \langle 1|\hat{H}|II\rangle \\ \langle 2|\hat{H}|I\rangle & \langle 2|\hat{H}|II\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle I|(|1\rangle\langle 1| + |2\rangle\langle 2|)\hat{H}|I\rangle & \langle I|\mathbb{1}\hat{H}|II\rangle \\ \langle II|\mathbb{1}\hat{H}|I\rangle & \langle II|\mathbb{1}\hat{H}|II\rangle \end{pmatrix}$$

once again, we get complete sets of states that we can turn into identies.

$$= \begin{pmatrix} \langle I|\hat{H}|I\rangle & \langle I|\hat{H}|II\rangle \\ \langle II|\hat{H}|I\rangle & \langle II|\hat{H}|II\rangle \end{pmatrix}$$

f) $\begin{pmatrix} \langle 1|I\rangle & \langle 1|II\rangle \\ \langle 2|I\rangle & \langle 2|II\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} E_0 + \mu|\vec{E}| & -A \\ -A & E_0 - \mu|\vec{E}| \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_0 + \mu|\vec{E}| - A & E_0 + \mu|\vec{E}| + A \\ E_0 - \mu|\vec{E}| - A & -(E_0 - \mu|\vec{E}| + A) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2(E_0 - A) & 2\mu|E| \\ 2\mu|\vec{E}| & 2(E_0 + A) \end{pmatrix}$$ ✓

3 a) $[\hat{P}_{+z}, \hat{H}] = \omega_0(\hat{P}_{+z}\hat{S}_x - \hat{S}_x\hat{P}_{+z})$

$$= \frac{\hbar\omega_0}{2}\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]$$

$$= \frac{\hbar\omega_0}{2}\left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]$$

$$= \frac{\hbar\omega_0}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \neq 0 \quad ✓$$

b) For any commutator

$$\Delta P_{+z}\,\Delta E \geq \frac{1}{2}\left|\langle\psi|[\hat{P}_{+z}, \hat{H}]|\psi\rangle\right|$$

↑ uncertainty in the probability to find the particle in the $|+z\rangle$ state. Recall we showed that the observable corresponding to $\hat{P}_{+z}$ was this probability back in HW 1.

(*) $$\Delta P_{+z}\,\Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle\hat{P}_{+z}\rangle}{dt}\right|$$

$$\frac{\Delta P_{+z}}{|d\langle\hat{P}_{+z}\rangle/dt|}\,\Delta E \geq \frac{\hbar}{2}$$

↑ Δt Time for the expectation value of $\hat{P}_{+z}$ to change by an amount equal to the uncertainty in this quantity.

c) For $|+x\rangle$, ΔE will be 0 since it's an eigen state of H. ΔP_{+z} = finite since we have a 50/50 chance of +z. $d\langle\hat{P}_{+z}\rangle/dt = 0$ since it's a stationary state. So we have →

So we have at line ⊛
finite $\cdot\, 0 \geq \hbar/2 \cdot 0$ ✓

If you want to work out ΔP_{+z}

$$\Delta P_{+z}^2 = \langle +x | \hat{P}_z \hat{P}_z | +x \rangle - \langle P_{+z} \rangle^2$$

50% prob to be +z

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} - \frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$\Delta P_{+z} = \frac{1}{2}$$

4. a) m_1 is an element of $\{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}$
m_2 " " " $\{\frac{1}{2}, -\frac{1}{2}\}$

b) The total angular momentum will be 2, or 1 so the states are
$|2,2\rangle$ $|2,1\rangle$ $|2,0\rangle$ $|2\,-1\rangle$ $|2\,-2\rangle$
$|1,1\rangle$ $|1,0\rangle$ $|1,-1\rangle$

These are circled on the CG table on the next page

c) $|1\,-1\rangle = \frac{1}{2}|\frac{3}{2}\,\frac{1}{2}\,-\frac{1}{2}\,-\frac{1}{2}\rangle - \sqrt{\frac{3}{4}}\,|\frac{3}{2}\,\frac{1}{2}\,-\frac{3}{2}\,\frac{1}{2}\rangle$

Look under the circled -1 column to find the $\frac{1}{4} \to \frac{1}{2}$ and $-\frac{3}{4} \to -\sqrt{\frac{3}{4}}$ due to the root convention stated at the top of the CG.

d) $\hat{S}_{1z} + \hat{S}_{1z}\left(\frac{1}{2}|\frac{3}{2}\,\frac{1}{2}\,-\frac{1}{2}\,-\frac{1}{2}\rangle - \sqrt{\frac{3}{4}}\,|\frac{3}{2}\,\frac{1}{2}\,-\frac{3}{2}\,\frac{1}{2}\rangle\right)$

$$= \left(-\frac{\hbar}{2} + -\frac{\hbar}{2}\right)\frac{1}{2}|\frac{3}{2}\,\frac{1}{2}\,-\frac{1}{2}\,-\frac{1}{2}\rangle$$
$$-\left(-\frac{3\hbar}{2} + \frac{\hbar}{2}\right)\sqrt{3/4}\,|\frac{3}{2}\,\frac{1}{2}\,-\frac{3}{2}\,\frac{1}{2}\rangle$$

$$= -\hbar\left(\frac{1}{2}|\frac{3}{2}\,\frac{1}{2}\,-\frac{1}{2}\,-\frac{1}{2}\rangle - \sqrt{\frac{3}{4}}\,|\frac{3}{2}\,\frac{1}{2}\,-\frac{3}{2}\,\frac{1}{2}\rangle\right)$$

So we got the state back w/ $-\hbar$ out font ✓

e) $\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\,\hat{S}_1 \cdot \hat{S}_2$
$= \vec{S}_1^2 + \vec{S}_2^2 + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\,\hat{S}_{1z}\hat{S}_{2z}$

That's 5 operators and 2 states, so lets just work out the 10 comos in the chart to follow

Particle Data Book

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42. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J J ... / M M ...
m_1 m_2	
m_1 m_2	Coefficients
. .	

1/2×1/2

m_1 m_2	1, +1
+1/2 +1/2	1

m_1 m_2	1, 0	0, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	1, −1
−1/2 −1/2	1

1×1/2

m_1 m_2	3/2, +3/2
+1 +1/2	1

m_1 m_2	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2, −3/2
−1 −1/2	1

2×1

m_1 m_2	3, +3
+2 +1	1

m_1 m_2	3, +2	2, +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3, −1	2, −1	1, −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3, −2	2, −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3, −3
−2 −1	1

1×1

m_1 m_2	2, +2
+1 +1	1

m_1 m_2	2, +1	1, +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2, −1	1, −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2, −2
−1 −1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

m_1 m_2	5/2, +5/2
+2 +1/2	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2, −5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	2, +2
+3/2 +1/2	1

m_1 m_2	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2, −2
−3/2 −1/2	1

States in part b

Info for part C

3/2×1

m_1 m_2	5/2, +5/2
+3/2 +1	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2, −3/2	3/2, −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2, −5/2
−3/2 −1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

2×3/2

m_1 m_2	7/2, +7/2
+2 +3/2	1

m_1 m_2	7/2, +5/2	5/2, +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	7/2, −5/2	5/2, −5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	7/2, −7/2
−2 −3/2	1

3/2×3/2

m_1 m_2	3, +3
+3/2 +3/2	1

m_1 m_2	3, +2	2, +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	3, +1	2, +1	1, +1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	3, 0	2, 0	1, 0	0, 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	3, −1	2, −1	1, −1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	3, −2	2, −2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	3, −3
−3/2 −3/2	1

2×2

m_1 m_2	4, +4
+2 +2	1

m_1 m_2	4, +3	3, +3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	4, +2	3, +2	2, +2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	4, +1	3, +1	2, +1	1, +1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	4, −1	3, −1	2, −1	1, −1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	4, −2	3, −2	2, −2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	4, −3	3, −3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	4, −4
−2 −2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 42.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

→ acting on

operator ↓	$\lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-1}{2}\, \frac{-1}{2} \rangle$	$\lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-3}{2}\, \frac{1}{2} \rangle$
$\vec{S}_1^{\,2}$	$\frac{3}{2}(\frac{3}{2}+1)\hbar^2$ "	$\frac{3}{2}(\frac{3}{2}+1)\hbar^2$ "
$\vec{S}_2^{\,2}$	$\frac{1}{2}(\frac{1}{2}+1)\hbar^2$ "	$\frac{1}{2}(\frac{1}{2}+1)\hbar^2$ "
$\hat{S}_{1+}\hat{S}_{2-}$	0 "	$\sqrt{3}\,\hbar^2 \lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-1}{2}\, \frac{-1}{2} \rangle$
$\hat{S}_{1-}\hat{S}_{2+}$	$\sqrt{\frac{12}{4}}\,\hbar^2 \lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-3}{2}\, \frac{1}{2} \rangle$	0
$2\hat{S}_{1z}\hat{S}_{2z}$	$\frac{1}{2}\hbar^2 \lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-1}{2}\, \frac{-1}{2} \rangle$	$-\frac{3}{2}\hbar^2 \lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-3}{2}\, \frac{1}{2} \rangle$

Notes:

- Colum 1, row 3 : It's 0 becaus you can't lower $-\frac{1}{2}$ for particle 2.
- Work out colum 1, row 4

$$\hat{S}_{2+} \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-1}{2}\, \tfrac{-1}{2} \rangle = \sqrt{\underset{s}{\tfrac{1}{2}}\,(\underset{s}{\tfrac{1}{2}}+1) - (\underset{m}{\tfrac{-1}{2}})(\underset{m}{\tfrac{-1}{2}}+1)}\;\hbar \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-1}{2}\, \tfrac{1}{2} \rangle$$

$$= \hbar \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-1}{2}\, \tfrac{1}{2} \rangle$$

$$\hat{S}_{1-}\, \hbar \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-1}{2}\, \tfrac{1}{2} \rangle = \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - (\tfrac{-1}{2})(\tfrac{-1}{2}-1)}\;\hbar^2 \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-3}{2}\, \tfrac{1}{2} \rangle$$

$$= \sqrt{\tfrac{15}{4} + \tfrac{3}{4}}\;\hbar^2 \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-3}{2}\, \tfrac{1}{2} \rangle$$

$$\hat{S}_{2-} \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-3}{2}\, \tfrac{1}{2} \rangle = \sqrt{\tfrac{3}{4} - \tfrac{1}{2}(\tfrac{1}{2}-1)}\;\hbar \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-3}{2}\, \tfrac{-1}{2} \rangle$$

$$= \hbar \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-3}{2}\, \tfrac{-1}{2} \rangle$$

$$\hat{S}_{1+}\, \hbar \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-3}{2}\, \tfrac{-1}{2} \rangle = \sqrt{\tfrac{15}{4} - (\tfrac{-3}{2})(\tfrac{-3}{2}+1)}\;\hbar^2 \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-1}{2}\, \tfrac{-1}{2} \rangle$$

$$= \sqrt{\tfrac{15}{4} - \tfrac{3}{4}}\;\hbar^2 \lvert \tfrac{3}{2}\, \tfrac{1}{2}\, \tfrac{-1}{2}\, \tfrac{-1}{2} \rangle$$

Add all coefficients of $\lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-1}{2}\, \frac{-1}{2} \rangle$

$\frac{15}{4}\hbar^2$

$\frac{3}{4}\hbar^2$

$\frac{1}{2}\hbar^2$

$-\sqrt{3}\,\hbar^2 \cdot \sqrt{\frac{3}{4}} \;/\; \frac{1}{2}$

$2\hbar^2 = 1(1+1)\hbar^2$ ✓

Add all coefficiends of $\lvert \frac{3}{2}\, \frac{1}{2}\, \frac{-3}{2}\, \frac{1}{2} \rangle$

$15 + 3 - 6 - 4 = 8$

$\frac{15}{4}\hbar^2$

$\frac{3}{4}\hbar^2$

$-\frac{3}{2}\hbar^2$

$-\sqrt{\frac{12}{4}} \cdot \frac{1}{2} \;/\; \sqrt{\frac{3}{4}}\,\hbar \leftarrow -\hbar$

$2\hbar^2$ ✓

5. (T 5.4)

$$|1,1\rangle = |+z,+z\rangle = |+z\rangle_1 |+z\rangle_2$$
$$= \frac{1}{\sqrt{2}}\left(|+x\rangle_1 + |-x\rangle_1\right)\frac{1}{\sqrt{2}}\left(|+x\rangle_2 + |-x\rangle_2\right)$$
$$= \frac{1}{2}\left(|+x\,+x\rangle + |+x\,-x\rangle + |-x\,+x\rangle + |-x\,-x\rangle\right)$$

We need

$$|1,1\rangle_x = |+x,+x\rangle$$
$$|1,0\rangle_x = \frac{1}{\sqrt{2}}\left(|+x,-x\rangle + |-x,+x\rangle\right)$$
$$|-1,-1\rangle_x = |-x,-x\rangle$$

$$|{}_x\langle 1,1|1,1\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

$$|{}_x\langle 1,0|1,1\rangle|^2 = \left|\frac{1}{2\sqrt{2}}\left(\langle +x\,-x|+x\,-x\rangle + \langle -x\,+x|-x\,+x\rangle\right)\right|^2 = \frac{1}{2}$$

$$|{}_x\langle -1,-1|1,1\rangle|^2 = \frac{1}{4}$$

So the idea was to write the states as eigenstates of $\hat{S}_{1x}$ and $\hat{S}_{2x}$, then work out the projections.