VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \qquad d\tau = dx\,dy\,dz$

Gradient:
$$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence:
$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:
$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian:
$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \qquad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient:
$$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence:
$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian:
$$\boxed{\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \qquad d\tau = s\,ds\,d\phi\,dz$

Gradient:
$$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence:
$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:
$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian:
$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Satisfying the normalization condition (9.139), we find

$$\langle \theta, \phi | l, l \rangle = Y_{l,l}(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)!}{4\pi}} e^{il\phi} \sin^l \theta \tag{9.146}$$

• We now apply the lowering operator to determine the remaining spherical harmonics. From Chapter 3 we know that

$$\hat{L}_- |l, m\rangle = \sqrt{l(l+1) - m(m-1)}\, \hbar |l, m-1\rangle \tag{9.147}$$

Combining (9.146) and (9.147), we find (see Problem 9.18) for $m \geq 0$

$$Y_{l,m}(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(l+m)!}{4\pi (l-m)!}} e^{im\phi} \frac{1}{\sin^m \theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} \sin^{2l} \theta \tag{9.148}$$

The choice of the phase factor $(-1)^l$ is taken to ensure that $Y_{l,0}(\theta, \phi)$, which is independent of ϕ, has a real positive value for $\theta = 0$. In fact,

$$Y_{l,0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \tag{9.149}$$

where $P_l(\cos\theta)$ is the standard **Legendre polynomial**. The spherical harmonics for $m < 0$ are given by

$$Y_{l,-m}(\theta, \phi) = (-1)^m [Y_{l,m}(\theta, \phi)]^* \tag{9.150}$$

It is useful to list the spherical harmonics with $l = 0$, 1, and 2:

$$Y_{0,0}(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \tag{9.151}$$

$$Y_{1,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} e^{\pm i\phi} \sin\theta \tag{9.152a}$$

$$Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \tag{9.152b}$$

$$Y_{2,\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\phi} \sin^2\theta \tag{9.153a}$$

$$Y_{2,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} e^{\pm i\phi} \sin\theta \cos\theta \tag{9.153b}$$

$$Y_{2,0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \tag{9.153c}$$

Figure 9.11 shows plots of $|Y_{l,m}(\theta, \phi)|^2$ as a function of θ and ϕ. Since the spherical harmonics depend on ϕ through $e^{im\phi}$, these plots are all independent of ϕ. The $l = 0$ state, often called an s state, is spherically symmetric. Thus if a

Table 4.1: Some associated Legendre functions, $P_l^m(\cos\theta)$.

$P_1^1 = \sin\theta$	$P_3^3 = 15\sin\theta(1-\cos^2\theta)$
$P_1^0 = \cos\theta$	$P_3^2 = 15\sin^2\theta\cos\theta$
$P_2^2 = 3\sin^2\theta$	$P_3^1 = \frac{3}{2}\sin\theta(5\cos^2\theta - 1)$
$P_2^1 = 3\sin\theta\cos\theta$	$P_3^0 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$
$P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$	

Notice that l must be a nonnegative integer for the Rodrigues formula to make any sense; moreover, if $|m| > l$, then Equation 4.27 says $P_l^m = 0$. For any given l, then, there are $(2l+1)$ possible values of m:

$$l = 0, 1, 2, \ldots; \quad m = -l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l. \qquad [4.29]$$

But wait! Equation 4.25 is a second-order differential equation: It should have *two* linearly independent solutions, for *any old* values of l and m. Where are all the *other* solutions? *Answer*: They *exist*, of course, as mathematical solutions to the equation, but they are *physically* unacceptable because they blow up at $\theta = 0$ and/or $\theta = \pi$, and do not yield normalizable wave functions (see Problem 4.4).

Now, the volume element in spherical coordinates[7] is

$$d^3\mathbf{r} = r^2 \sin\theta\, dr\, d\theta\, d\phi, \qquad [4.30]$$

so the normalization condition (Equation 4.6) becomes

$$\int |\psi|^2 r^2 \sin\theta\, dr\, d\theta\, d\phi = \int |R|^2 r^2\, dr \int |Y|^2 \sin\theta\, d\theta\, d\phi = 1.$$

It is convenient to normalize R and Y individually:

$$\int_0^\infty |R|^2 r^2\, dr = 1 \quad \text{and} \quad \int_0^{2\pi}\int_0^\pi |Y|^2 \sin\theta\, d\theta\, d\phi = 1. \qquad [4.31]$$

The normalized angular wave functions[8] are called **spherical harmonics**:

[7] See, for instance, Boas, (footnote 2), Chapter 5, Section 4.

[8] The normalization factor is derived in Problem 4.47. The ϵ factor is chosen for consistency with the notation we will be using in the theory of angular momentum; it is reasonably standard, though some older books use other conventions. Notice that

$$Y_l^{-m} = (-1)^m Y_l^{m*}.$$