Name: ________________

PHYS 335 HW 3

Due 1/28 at 5:00PM

The usual instructions, plus:
1. A note about **party tricks**....
When I call something a party trick, I mean that it's a rather stand-alone derivation of something. Often when we do derivations in class or in a textbook reading or when we solve homework problems, we appeal to a number of separately derived results from various equations in a textbook. At the end of those calculations, I'm sometimes left wondering if I understand the big picture. When I call something a party trick, I mean that I start with something rather fundamental and end with some interesting result without pulling in "anything" beyond ideas that one could imagine being foundational ideas and mathematical results that one could "just know", as though one were answering a question from a friend at a party. I often then stretch the definition of "just know" a little to things that with some more practice maybe I could "just know". I find these kinds of calculations really satisfying when I don't stretch the rules too much.
2. If you use Mathematica, include a print out or write down the command you used and what the output was as a part of your solution.

0. (2 pts) Estimate the time it took to do this problem set. Include any comments about the homework that you'd like to share with Jay.

1. *Party trick 1.*

After Jay does his party trick (posted on moodle as class7_notes), sassy party goer Sacha Paine-Díaz says, "Oh yeah, you think you're so special? I can do the same calculation with 5 beams exiting the Stern-Gerlachs." (Said with a friendly smile on their face, of course.) Acting as Sacha, "one-up" Jay by working out the party trick. Take the setup as identical to Jay's party trick, just add one more beam coming out of each SG. You can assume that you have a laptop running Mathematica with you at the party. Note that Sacha's work will in some sense be more of a party trick than Jay's because she will need to work though the analogue of Example 3.4 rather than just appealing to the result. (She'll still need to bring in the general result

$$\hat{S}_+ \left| s, m \right\rangle = \sqrt{s(s+1) - m(m+1)}\hbar \left| s, m+1 \right\rangle , \tag{1}$$

but I find this to be within the spirit of the party trick since (1) it's a rather simple and general result that might be reasonable for Sacha to have memorized, and (2) if she wanted to, she could bring the derivation of this result into her trick by working through the steps at the start of section 3.4.)

2. *Party trick 2.*

After Sacha completed their trick, Sue Pernice says, "Wow, those are really great tricks. I love how Jay set up the basic approach and Sacha showed that it can be done with particles of any spin. Great job! You pholks both used Mathematica in your party tricks, and that's fine, since your calculations would take a lot of time by hand. But I know that our next adventure together will be a survival experience in which we will be dropped off in the wilderness without technology to find our way back to civilization. I'd like to make sure we are able to have fun by the campfire in the evenings without Mathematica. I do plan to memorize a few more trig identities before the trip, but for tonight, I'm going to do this trick using Mathematica or a math handbook only for trig identities. I'll do the matrix and linear algebra manipulations by hand. To illustrate these techniques as well as work out some features of the spin 1/2 problem with the quantization axis in an arbitrary direction, I'll show that when we deal with a quantization axis pointing in an arbitrary direction

$$\vec{n} = \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k}, \tag{2}$$

we get eigenstates described by

$$\begin{aligned} |{+n}\rangle &= \cos\tfrac{\theta}{2}\,|{+z}\rangle + \sin\tfrac{\theta}{2}e^{i\phi}\,|{-z}\rangle & (3)\\ |{-n}\rangle &= \sin\tfrac{\theta}{2}\,|{+z}\rangle - \cos\tfrac{\theta}{2}e^{i\phi}\,|{-z}\rangle & (4) \end{aligned}$$

as we used in the first problem set." Acting as Sue, write up the party trick by filling in the steps below.

a. The spin operator along the $\vec{n}$ direction is $\hat{\vec{S}} \cdot \vec{n}$. Write this as

$$\hat{\vec{S}} \cdot \vec{n} = \hat{S}_x \sin\theta\cos\phi + \ldots \tag{5}$$

b. Now write $\hat{\vec{S}} \cdot \vec{n}$ as a matrix in the s_z basis by filling in the relevant matrices for $\hat{S}_x$ etc. You could look them up in chapter 3. If you feel like that violates the spirit of the party trick, these matrices are used so often in the field that many people come to know them off hand. You could also work them out as you did for similar matrices in the last problem.

c. So it's the eigenvectors of $\hat{\vec{S}} \cdot \vec{n}$ that we seek. We could write the eigen equation as

$$\hat{\vec{S}} \cdot \vec{n}\,|\psi\rangle = \tfrac{\hbar}{2}\lambda\,|\psi\rangle\,, \tag{6}$$

where $\lambda = \pm 1$. Explain how we know these eigenvalues.

d. You can now make the eigen equation look like the "usual lin-al form" by writing

$$(\hat{\vec{S}} \cdot \vec{n} - \tfrac{\hbar}{2}\lambda\mathbb{I})\,|\psi\rangle = 0 \tag{7}$$

as a matrix. Using the known values of λ, find the eigenvectors and show that they are the ones we expect above. When I do it, I initially find that the ratio of the components of one of the eigenvectors is

$$\frac{\sin\theta e^{-i\phi}}{1-\cos\theta}, \tag{8}$$

and for the other, I generate a similar thing. These can be turned into the normalized eigenvectors we seek using the following trig identities:

$$\begin{aligned} \sin\theta &= 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} & (9)\\ 1-\cos\theta &= 2\sin^2\tfrac{\theta}{2} & (10)\\ 1+\cos\theta &= 2\cos^2\tfrac{\theta}{2}, & (11) \end{aligned}$$

along with a manipulation of the complex exponential. Finish out the calculation.

3. *Party trick 3.*

After finishing her trick, Sue looks to Ben Waiting, who has been eagerly but quietly watching the fun, and says, "Ben, would you like to share a trick?" Ben responds, "maybe, but it's more of a derivation of basics. I'm not sure if it will be as much fun as the spiffy applications you pholks have been doing." To which Sacha says, "nonsense Waiting! Let's see it!" Ben proceeds to explain that his interest is in the foundational nature of the commutation relations and how they lead to the uncertainty relations. For his party trick, he presents a derivation that starts with a general commutation relation like Townsend Eq. 3.64 and proceeds party-trick style (presenting a stand-alone argument working through the details) through the derivation of the uncertainty relation, Townsend Eq. 3.74. Set up Ben's party trick as follows in this problem, and the next:

a. Ben is going to need the Schwarz identity. But to use it in a party trick, he wants to first convince the audience that it makes sense conceptually, then prove with certainty that it's true for our complex space. In this part, draw some pictures and/or do a couple lines of algebra, and/or write some words that would convince Ben's audience that the Schwarz identity would apply to a pair of vectors $\vec{a}$ and $\vec{b}$ in a real 2 dimensional vector space.

b. Prove the Schwarz identity via the following steps:
(i) Show that $\langle\psi|\psi\rangle$ is real and positive for an arbitrary state $|\psi\rangle$.
(ii) Write $|\psi\rangle = |\alpha\rangle + \lambda\,|\beta\rangle$, where λ is a complex number. (For any state $|\psi\rangle$, I could divide it up like this in an infinite number of ways.) Use this broken up version of $|\psi\rangle$ to expand the equation

$$\langle\psi|\psi\rangle \geq 0 \tag{12}$$

(iii) Minimize the left side of your equation from ii with respect to λ. When you do this, you can treat λ and λ^* as independent.
(iv) Plug your expression for λ back into Eq. (12) and rearrange to finish the proof.

4. *Party trick 4.*
Having filled in the details behind the Schwarz identity in part 3, proceed to finish the party trick showing that 3.64 implies 3.74. This will partly involve "copying" stuff from a page in Townsend, but it will be more than that because there are steps that will not be obvious to you or your audience without additional steps and/or comments, such as the first equality in 3.71. You can also feel free to organize the steps in whatever way you want to make them clearer to you.

5. *Some general Hamiltonian practice.*
Townsend 4.13.
Address the following in your solution:
(i) The states $|1\rangle, |2\rangle, |3\rangle$ are not all eigenstates of the Hamiltonian. Say how you know just by looking at the Hamiltonian in this basis.
(ii) You'll need to find the eigenstates of this Hamiltonian such that you can express the states $|\psi(0)\rangle$ in terms of the eigenstates of the Hamiltonian. Why do you need to do this in order to find $|\psi(t)\rangle$?
(iii) For each of parts a and b, say whether the state has changed in a "meaningful" way.

Academic Honesty Statement

___I did not receive help on this problem set. I did all the work on my own.
___I received on this problem set from:
___Jay on problems __________
___Student Assistants on problems __________
___Other Students (please name __________) on problems __________
___Other Instructor (please name __________) on problems __________
___A resource other than our text (please name __________)
on problems __________

By way of acknowledgements, these problems were adapted from Townsend.