1. $\hat{J}_z \hat{J}_x |+x\rangle = \frac{\hbar}{2} \hat{J}_z |+x\rangle$

$$= \frac{\hbar}{2} \hat{J}_z \left(\frac{1}{\sqrt{2}} |+z\rangle + \frac{1}{\sqrt{2}} |-z\rangle \right)$$

$$= \frac{\hbar^2}{4} \left(\frac{1}{\sqrt{2}} |+z\rangle - \frac{1}{\sqrt{2}} |-z\rangle \right)$$

$$= \frac{\hbar^2}{4} |-x\rangle$$

$\hat{J}_x \hat{J}_z |+x\rangle = \frac{\hbar}{2} \hat{J}_x |-x\rangle$

$$= -\frac{\hbar^2}{4} |-x\rangle$$

4. $i\hbar J_y |+x\rangle = i\frac{\hbar^2}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar^2}{2\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \frac{\hbar^2}{2} |-x\rangle$$

So $[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$

2. Jay's answer (may be others)

Lets call the expectation value of the observable a and the (Hermitian) operator $\hat{A}$
$a = \langle \lambda | \hat{A} | \lambda \rangle$. Normally the expectation value is also the eigen value so

$a = a \langle \lambda | \lambda \rangle$ so $\langle \lambda | \lambda \rangle = 1$

Eg Let $\hat{A} = \hat{J}_z$ and $|\lambda\rangle = |+z\rangle$

$\frac{\hbar}{2} = \langle +z | \hat{J}_z | +z \rangle = \frac{\hbar}{2} \langle +z | +z \rangle \Rightarrow \langle +z | +z \rangle = 1$

We also normally use the eigenvectors as an orthonormal basis

3. Note

$$\langle j, m' | \hat{J}_\pm | j, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,\hbar \times \langle j, m' | j, m+1\rangle$$

Now we need to work out

$$\begin{pmatrix} \langle \tfrac{1}{2}\,\tfrac{1}{2} | \hat{J}_+ | \tfrac{1}{2}\,\tfrac{1}{2}\rangle \to ①\ 0 & \langle \tfrac{1}{2}\,\tfrac{1}{2} | \hat{J}_+ | \tfrac{1}{2}\,\tfrac{-1}{2}\rangle \\ \langle \tfrac{1}{2}\,\tfrac{-1}{2} | \hat{J}_+ | \tfrac{1}{2}\,\tfrac{1}{2}\rangle \to ① & \langle \tfrac{1}{2}\,\tfrac{-1}{2} | \hat{J}_+ | \tfrac{1}{2}\,\tfrac{-1}{2}\rangle \to 0 \end{pmatrix}$$

① b/c can't raise $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$

② If we raise $|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$ to $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$ it will be orthogonal to the bra on the left

$$\langle \tfrac{1}{2}\,\tfrac{1}{2} | \hat{J}_+ | \tfrac{1}{2}\,\tfrac{-1}{2}\rangle = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) + \tfrac{1}{2}(\tfrac{-1}{2}+1)}\,\hbar \langle \tfrac{1}{2}\,\tfrac{1}{2} | \tfrac{1}{2}\,\tfrac{1}{2}\rangle$$
$$= \sqrt{3/4 + \tfrac{1}{4}}\,\hbar$$
$$= \hbar$$

So $\hat{J}_+ \xrightarrow[z\text{ basis}]{} \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Check $\hat{J}_+ |-z\rangle = \hat{J}_+ |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$

$$\to \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ |+z\rangle\ ✓$$

$$\hat{J}_+ |+z\rangle = \hat{J}_+ |\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$\longrightarrow \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\ ✓$$