If we project out in position space, the Schrödinger Eqn for the SHO can be written

(1) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{m\omega^2 x^2}{2}\psi = E\psi$$

If we again go to dim'less variables we can write this as

(2) $$\frac{d^2\psi}{dx^2} + (\epsilon - x^2)\psi = 0$$

Here $\epsilon = \frac{2E}{\hbar\omega}$ is twice the dim'less energy.

In section 7.9 Townsend solves this directly, getting the same results we saw before from our operator sol'n. The key reasons to look at this second method are:

1. It illustrates how to solve things using a <u>series sol'n</u>, and
2. It illustrates how to use a <u>recursion formula</u>

Both are important methods to meet at this time.

These notes parallel section 7.9 and invite you to work out some of the key steps for yourself.

To get a power series sol'n going a typical first step is to identify the asymptotic (aka large x) behavior of the f'n that we seek. In the large x limit (2) becomes

(3) $$\frac{d^2\psi}{dx^2} - x^2\psi = 0$$

Q1 Why?

Q2 Check that $\psi = Ae^{-x^2/2} + Be^{+x^2/2}$ is a Sol'n to (3).

Q3 $B = 0$. Why?

Given the large x behavior, we'll try a sol'n of the form $\psi = h(x)e^{-x^2/2}$ for the full eqn

If you plug the trial sol'n in at (2) you get

(4) $$\frac{d^2h}{dx^2} - 2x\frac{dh}{dx} + (\epsilon - 1)h = 0.$$

Q4 Show (4).

Since any well behaved f'n $h(x)$ could be expressed as a power series, try $h(x) = \sum_{k=0}^{\infty} a_k x^k$, where a_k are constants.

Let's tee up to plug this sol'n into (4).

Q5 convince your friends that

$x\frac{dh}{dx} = \sum_{k=0}^{\infty} k a_k x^k$. Maybe do it by writing out a few terms of h and differentiating each.

(5) Now $$\frac{d^2h}{dx^2} = \frac{d}{dx}\sum_{k=0}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} k(k-1)a_k x^{k-2}$$

I claim we could start the sum above at $k=2$ instead.

Q6 Why?

So we can write

$$\frac{d^2h}{dx^2} = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$$

Now let $k' = k-2$ or $k = k'+2$ so

(6) $$\frac{d^2h}{dx^2} = \sum_{k'=0}^{\infty} (k'+2)(k'+1)a_{k'+2}x^{k'}$$

Rename k' back to k and plug (5) and (6) into (4)

$$(7) \quad \sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - 2k\,a_k + (\epsilon - 1)a_k \right] x^k = 0$$

Where are we? If we an solve (7) for the a_k coefficients, then we'll know $h(x)$. When we know $h(x)$, we'll know $\psi(x)$ and that's what we want -- the position space eigenfunctions.

I claim that for a polynomial like $A + Bx = 0$ to be 0 for any x, we must have $A = 0$ and $B = 0$. Convince your friends.

Using this idea in (7) the thing in parenthesis must be 0 for each k. This condition can be written

$$(8) \quad a_{k+2} = \frac{2k + 1 - \epsilon}{(k+1)(k+2)} a_k$$

Q7. Check Eq. (8).

If you know a_0 and a_1, you can find all of the other a_k. Hence this is called a recursion formula. The constants a_0 and a_1 are arbitrary at this stage.

Q8. Why is a pair of arbitrary constants expected here?

One can show (and perhaps it's not surprising) that if the sum in our definition of $h(x)$ continues to ∞ we'll again have an un-normalizable wave function. The only way out is if for some k, Eq (8) generates 0 for a_{k+2}. Then all a_k after that will be zero and the sum will end.

Q9 Convince yourself that if $\epsilon = 2n+1$ for some integer n, then the sum will truncate.

This condition implies (since $\epsilon = 2E/\hbar\omega$) that we have allowed energies
$E_n = \hbar\omega(n + \frac{1}{2})$
These are again the equally spaced quantized energy levels! Stand in awe of the power of math! We could stay here and celebrate, but lets get the eigenfunctions too.

Given this quantization, we'll also get quantized eigenfunctions $\psi_n = h_n e^{-x^2/2}$

Since the eigenstates of energy are also parity eigenstates, our eigenstates will either involve all even ks or all odd ks.

Q10 Argue that for $n=0$ a_0 is the only nonzero a_k and the wave function is $\psi_0 = a_0 e^{-x^2/2}$.

Q11 Show that for $n=2$ we have
$\psi_2 = a_0(1-2x^2)e^{-x^2/2}$

The objects appearing between a_0 and e in the ψ_n are proportional to a set of functions called Hermite polynomials. The 1st few are shown below

$H_0 = 1$
$H_1 = 2x$
$H_2 = 4x^2 - 2$
$H_4 = 16x^4 - 48x^2 + 12$

Writing the ψ_n in terms of the H_n and normalizing yeilds

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(x) e^{-x^2/2}$$