Name: ________________

PHYS 335 HW 2

Due 1/21 at 5:00PM

Before starting this homework, you should make sure you have read the homework section of the syllabus and the homework rubric from the homework folder on Moodle. To keep the accounting simple, all problems are worth 10 pts unless otherwise noted. In some cases a lot may be required for your 10 points, in other cases a little. Be sure to fill out the academic honesty statement at the end.

0. (2 pts) Estimate the time it took to do this problem set. Include any comments about the homework that you'd like to share with Jay.

1. *Hermitian, Unitary, Orthonormal, Oh My!*

a. Use Dirac notation directly (properties of kets, inner products, etc. without explicitly using matrix representations), to establish that the projection operator $\hat{P}_{+z} = |+z\rangle \langle +z|$ is Hermitian, taking $\langle\psi| \hat{A} |\phi\rangle = \langle\phi| \hat{A} |\psi\rangle^*$ as the definition of Hermitian.

b. Show that since $\hat{P}^2_{+z} = \hat{P}_{+z}$, the eigenvalues of the projection operator are either 0 or 1. I suggest that you start by considering the action of the projection operator on an eigenstate $\hat{P}_{+z} |\lambda\rangle = \lambda |\lambda\rangle$, do other similar manipulations, then argue that λ is 0 or 1.

c. For a set of vectors $|a_i\rangle$ to be an orthonormal basis, they must be both orthonormal $\langle a_m|a_n\rangle = \delta_{mn}$ and a complete set $\sum_i |a_i\rangle \langle a_i| = \mathbb{I}$. Show that if the states $|a_i\rangle$ form an orthonormal basis, then the states $\hat{U} |a_i\rangle$ do too. Here $\hat{U}$ is some unitary operator.

d. Show that the eigenvalues of a unitary operator can be written as $e^{i\theta}$ for some real θ. I suggest considering the following expression where $|\lambda\rangle$ is an eigenstate of the unitary operator $\hat{U}$ with eigenvalue λ:

$$\langle\lambda| \lambda^*\lambda |\lambda\rangle = \langle\lambda| \hat{U}^\dagger\hat{U} |\lambda\rangle \,, \tag{1}$$

after you explain why it's true.

e. Since the projection operator is Hermitian, $\langle\psi| \hat{P}_{+z} |\psi\rangle$ should correspond to the expectation value of some observable. What is the observable and how could you use a Stern Gerlach experiment to observe it?

2. *Why not the y basis?*
Find the column vectors representing the states $|\pm x\rangle$ using the $|\pm y\rangle$ basis. Recall

$$\begin{aligned} |\pm x\rangle &= \tfrac{1}{\sqrt{2}} |+z\rangle \pm \tfrac{1}{\sqrt{2}} |-z\rangle \,, & (2)\\ |\pm y\rangle &= \tfrac{1}{\sqrt{2}} |+z\rangle \pm \tfrac{i}{\sqrt{2}} |-z\rangle \,. & (3) \end{aligned}$$

Write your answers in the form

$$|+x\rangle \;\rightarrow\; re^{i\theta} \begin{bmatrix} 1 \\ i \end{bmatrix}, \tag{4}$$

$$|-x\rangle \;\rightarrow\; re^{i\theta} \begin{bmatrix} i \\ 1 \end{bmatrix} \tag{5}$$

for some real r, θ.

3. *Base your x on z.*

a. Express the $\hat{J}_x$ operator in the S_z basis. You can do it however you want, but I suggest finding it first in the S_x basis, then use the $|\pm x\rangle$ states (shown above) to find the $\mathbb{S}$ matrix suitable for transforming it to the S_z basis.

b. Find $\langle S_x \rangle$ for the state $|\psi\rangle$, which can be expressed in the S_z basis as follows

$$|\psi\rangle \to \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} . \tag{6}$$

Do so by doing matrix manipulations in the S_z basis using the things you worked out in part (a).

4. *Doofenshmirtz Evil Inc.*

a. A key communications satellite can be modeled as a uniform disk of mass 1 kg and radius 1 m. Dr. Doofenshmirtz knows that photons carry angular momentum, and he plans to take out communications by shining a bright light on the satellite with his 'Rotatinator' such that it will be spun up to a high angular speed and eventually tare itself apart. Assume that the satellite is oriented such that the face of the disk is normal to Dr. D's beam, which travels in the z direction. Further assume that the surface of the satellite is perfectly absorbing (photons are absorbed rather than being reflected). Dr. D's beam carries N photons per second, and each photon is in the state

$$|\psi\rangle = \sqrt{\frac{2}{3}}\,|x\rangle + \frac{i}{\sqrt{3}}\,|y\rangle . \tag{7}$$

Find the torque exerted on the disk by the incident photons. (Hints: What fractions of his beam are left and right circularly polarized? What angular momentum does each carry? How is angular momentum related to torque?)

b. The laser that was bounced off of the moon in the 1980s to measure how far away it is had $N = 10,180$ photons per second. They now have a bigger one, but that's a bright laser never the less. Assuming Dr. D uses this laser and all of the photons shot hit the disk. How long would it take to reach an angular speed of 1 rad/s? Is Dr. D's device practical for this application? Can one in principle rotate a macroscopic object by letting it absorb quantum spin angular momentum? Could he do it (in principle) by shooting electrons instead?

c. If an ideal polarizer is inserted into the beam with it's transmission axis oriented in the y direction, what is the probability that a photon will make it through?

d. Would the addition of this polarizer enhance or degrade the effectiveness of the Rotatinator? Consider both the effect on the number of photons and the effect of the polarization that the transmitted photons now have.

5. *Enlighten a representation*
Example 2.8 introduces 2 representations for the $\hat{J}_z$ operator. Use each, in turn, to find the expectation value of the angular momentum for the state $a\,|R\rangle + b\,|L\rangle$, where a and b are arbitrary complex numbers.

6. *Would operators in any other order smell as sweet?*
Consider a set of operators $\hat{A}, \hat{B}, \hat{C}$.

a. Show that $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$

b. Show that $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$

c. Show that $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

d. Show that if $\hat{A}$ and $\hat{B}$ are Hermitian, and $[\hat{A}, \hat{B}] = i\hat{C}$, then $\hat{C}$ is Hermitian. You can do this by showing $\hat{C}^\dagger = \hat{C}$.

Academic Honesty Statement

____I did not receive help on this problem set. I did all the work on my own.
____I received on this problem set from:
 ____Jay on problems ______________
 ____Student Assistants on problems ______________
 ____Other Students (please name ___________________) on problems ______________
 ____Other Instructor (please name ___________________) on problems ______________
 ____A resource other than our text (please name ___________________)
  on problems ______________

By way of acknowledgements, these problems were adapted from Townsend.