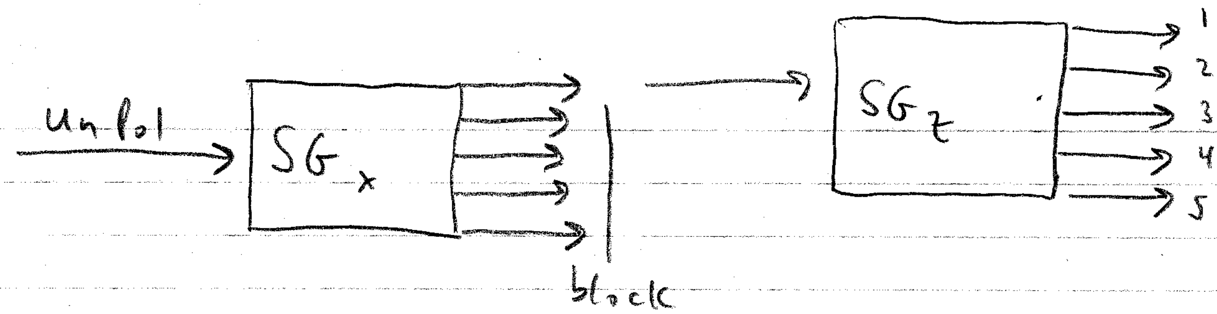
1. 

A) If we see 5 beams exit the 1st SG, we have a spin $s=2$ particle, spin states $|2,-2\rangle_x$, $|2,-1\rangle_x$, $|2,0\rangle_x$, $|2,1\rangle_x$ are blocked and $|2,2\rangle_x$ moves on, for a total of 5 beams from $s=5$.

B) The states exiting through ports 1-5 are

$1 \to |2,2\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$ $\quad 2 \to |2,1\rangle \longrightarrow \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$

$3 \to |2,0\rangle \longrightarrow \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$ $\quad 4 \to |2,-1\rangle \longrightarrow \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$

$5 \to |2,-2\rangle \longrightarrow \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$

We need to compute $|\langle 2,m|2,2\rangle_x|^2$ for $m \in \{2,1,0,-1,2\}$

Now I need $|2,2\rangle_x$ in the z basis. To do this, I'll bring in the single, general result

$$\hat{S}_+|s,m\rangle = \sqrt{s(s+1)-m(m+1)}\,\hbar\,|s,m+1\rangle$$

From this rule, we can see:

$$\hat{S}_+ |2,1\rangle = \sqrt{2(2+1) - 1(1+1)}\, \hbar\, |2,2\rangle$$
$$= \sqrt{6-2}\, \hbar\, |2,2\rangle$$
$$= 2\hbar |2,2\rangle$$

$$\hat{S}_+ |2,0\rangle = \sqrt{6 - 0}\, \hbar\, |2,1\rangle$$
$$= \sqrt{6}\, \hbar\, |2,1\rangle$$

$$\hat{S}_+ |2,-1\rangle = \sqrt{6 + 1(-1+1)}\, \hbar\, |2,0\rangle$$
$$= \sqrt{6}\, |2,0\rangle$$

$$\hat{S}_+ |2,-2\rangle = \sqrt{6 + 2(-2+1)}\, \hbar\, |2,-1\rangle$$
$$= 2\hbar |2,-1\rangle$$

Note that all of these states are eigen states of $\hat{S}_z$. If I work in the z basis, I need a matrix representation of $\hat{S}_+$ that operates on the states listed at the start of part B that generates the eigenvalues listed above.

$$\hat{S}_+ \rightarrow \hbar \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

You can get this by analogy with ex 3.4 and check it by letting it act on the states

Again $\hat{S}_-$ is the transpose

$$\hat{S}_- \rightarrow \hbar \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Again, you can check it by acting on states.

So again $\hat{S}_x$ in the z basis is provided by

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Now we need the eigenvectors of this thing. For that, I'll turn to mathematica. See the attached notebook.

Having extracted the right state, we can now find the 5 probabilities.

1. Dotting in from Mma with the $|2,2\rangle$ gives $|\langle 2,2|2,2\rangle_x|^2 = \left(\frac{1}{4}\right)^2 = 1/16$
2. $|\langle 2,1|2,2\rangle_x|^2 = \left(\frac{1}{2}\right)^2 = 1/4$
3. $|\langle 2,0|2,2\rangle_x|^2 = \left(\frac{1}{2}\sqrt{\frac{3}{2}}\right)^2 = 3/8$
4. by symmetry $1/4$
5. $1/16$

As we've seen before, higher probability near the middle.

Here I' ll imput the sx operator

In[3]:=
```
sx = 1 / 2 {{0, 2, 0, 0, 0}, {2, 0, √6, 0, 0},
    {0, √6, 0, √6, 0}, {0, 0, √6, 0, 2}, {0, 0, 0, 2, 0}}
```

Out[3]= $\left\{\{0, 1, 0, 0, 0\}, \left\{1, 0, \sqrt{\frac{3}{2}}, 0, 0\right\}, \left\{0, \sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}, 0\right\}, \left\{0, 0, \sqrt{\frac{3}{2}}, 0, 1\right\}, \{0, 0, 0, 1, 0\}\right\}$

If I show it in matrix form, it will just help me make sure that it looks right

In[4]:=
```
MatrixForm[sx]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This checks that I get the right eigenvalues, which I do

In[5]:=
```
Eigenvalues[sx]
```

Out[5]= $\{-2, 2, -1, 1, 0\}$

In[6]:=
```
Eigenvectors[sx]
```

Out[6]= $\left\{\left\{1, -2, \sqrt{6}, -2, 1\right\}, \left\{1, 2, \sqrt{6}, 2, 1\right\}, \{-1, 1, 0, -1, 1\}, \{-1, -1, 0, 1, 1\}, \left\{1, 0, -\sqrt{\frac{2}{3}}, 0, 1\right\}\right\}$

I' m guessing that the second vector above corresponds to +2 since this is the second Eigen-value listed above. Now I'll normalize it.

In[7]:=
```
n = Normalize[{1, 2, √6, 2, 1}]
```

Out[7]= $\left\{\frac{1}{4}, \frac{1}{2}, \frac{\sqrt{\frac{3}{2}}}{2}, \frac{1}{2}, \frac{1}{4}\right\}$

Let' s check if we' re using the right vector

In[8]:=
```
Dot[sx, n]
```

Out[8]= $\left\{\frac{1}{2}, 1, \sqrt{\frac{3}{2}}, 1, \frac{1}{2}\right\}$

This looks like 2 times the normalized vector, so that confirms that we're dealing with the right Eigenvector.

2. We're putting the quantization axis in an arbitrary direction in space described by the unit vector

$$\vec{n} = \sin\theta\cos\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k}$$

We need to find the z basis representation of $\hat{\vec{S}}\cdot\vec{n}$, the spin operator along the $\vec{n}$ direction

a) $\hat{\vec{S}}\cdot\vec{n} = \hat{S}_x\sin\theta\cos\phi + \hat{S}_y\sin\theta\sin\phi + \hat{S}_z\cos\theta$

b) Now I'll plug in the z representations for the $\hat{S}_j$ Townsend 3.88, 3.89 and easy $\hat{S}_z$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

The eigen eqn is

$$\hat{\vec{S}}\cdot\vec{n}\,|\psi\rangle = \frac{\hbar}{2}\lambda\,|\psi\rangle$$

$$\text{or}\quad \left(\hat{\vec{S}}\cdot\vec{n} - \frac{\hbar}{2}\lambda\,\mathbb{1}\right)|\psi\rangle = 0$$

c) We know λ will be ± 1 since the spin eigenvalues will be $\pm\hbar/2$ No matter the quantization direction.

d) So in matrix notation

$$\begin{pmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - \lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

Working with $\lambda = 1$

$$(\cos\theta - 1)a + (\sin\theta\, e^{-i\phi})b = 0$$

$$(*) \quad \frac{a}{b} = \frac{\sin\theta\, e^{-i\phi}}{1 - \cos\theta}$$

So I could write the vector
$|+n\rangle = c(\sin\theta\, e^{-i\phi}\,|+z\rangle + (\cos\theta - 1)\,|-z\rangle)$

where c is a normalization constant but it will look more "normal" if I use
$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
$1-\cos\theta = 2\sin^2\frac{\theta}{2}$

Then I can write

$$|+n\rangle = \cos\frac{\theta}{2}\,|+z\rangle + \sin\frac{\theta}{2}\, e^{i\phi}\,|-z\rangle$$

Where I've flipped the $e^{i\phi}$ to the bottom at $(*)$ by multiplying the top and bottom by $e^{i\phi}$.

Since $\langle +n|+n\rangle = 1$, this is already normalized.

the $|-n\rangle$ case has

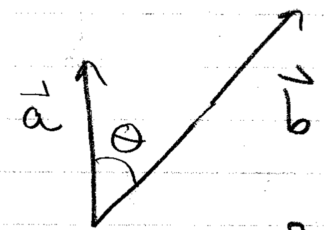
$$\frac{a}{b} = \frac{\sin\theta \, e^{i\phi}}{-(1+\cos\theta)}$$

and I'll use $1+\cos\theta = 2\cos^2\theta/2$

so $$\frac{a}{b} = \frac{2\sin\theta/2 \cos\theta/2}{-2\cos^2\theta/2 \; e^{i\phi}}$$

or $|-n\rangle = \sin\theta/2 \, |+z\rangle - \cos\frac{\theta}{2} e^{i\phi} |-z\rangle$

3 a)



$$(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) \overset{?}{\geq} (a\,b\cos\theta)^2$$
$$a^2\,b^2 \geq a^2 b^2 \cos^2\theta$$

The identity is demonstrated to be true since $\cos^2\theta \leq 1$

You could also do this part graphically by noting that the projection of $\vec{b}$ on $\vec{a}$ is always less than $|\vec{b}|$.

b) A complex number times it's complex cojugate is real and positive as follows

$$(r\,e^{i\phi})(r\,e^{i\phi})^* = r^2 e^{i\phi} e^{-i\phi} = r^2$$

Since a complex vector doted with its complex cojugate is a sum of complex components times their complex conjugate as follows,

$$(r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, \dots)\begin{pmatrix} r_1 e^{-i\phi_1} \\ r_2 e^{-i\phi_2} \\ \vdots \end{pmatrix}$$

$$= r_1^2 + r_2^2 + \dots$$

$$\langle \psi | \psi \rangle \geq 0$$

So now we have

$$0 \le \langle \psi | \psi \rangle = (\langle \alpha | + \lambda^* \langle \beta |)(|\alpha\rangle + \lambda |\beta\rangle)$$

$$* \quad 0 \le \langle \alpha | \alpha \rangle + \lambda^* \langle \beta | \alpha \rangle + \lambda \langle \alpha | \beta \rangle + \lambda^* \lambda \langle \beta | \beta \rangle$$

Now pick a value of λ that minimizes the RHS

$$\frac{d}{d\lambda}\left(\langle \alpha | \alpha \rangle + \lambda^* \langle \beta | \alpha \rangle + \lambda \langle \alpha | \beta \rangle + \lambda^* \lambda \langle \beta | \beta \rangle\right) = 0$$

Since the real and imaginary parts of λ are arbitrary at this stage, we can treat λ and λ^* as independent

$$\langle \alpha | \beta \rangle + \lambda^* \langle \beta | \beta \rangle = 0 \quad \Rightarrow \lambda^* = -\frac{\langle \alpha | \beta \rangle}{\langle \beta | \beta \rangle} \Rightarrow \lambda = \frac{\langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle}$$

plugging back into $*$

$$0 \le \langle \alpha | \alpha \rangle \cancel{- \frac{\langle \alpha | \beta \rangle \langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle}} \cancel{+ \frac{\langle \alpha | \beta \rangle \langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle}} - \frac{\langle \alpha | \beta \rangle \langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle}$$

$$0 \le \langle \beta | \beta \rangle \langle \alpha | \alpha \rangle - \langle \alpha | \beta \rangle \langle \beta | \alpha \rangle$$

$$|\langle \alpha | \beta \rangle|^2 \le \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \quad \checkmark$$

There are a lot of ways to organize these steps.

4. I'll start w/ the schwarz
$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

I'll pick $|\alpha\rangle = (\hat{A} - \langle A\rangle)|\psi\rangle$
and $|\beta\rangle = (\hat{B} - \langle B\rangle)|\psi\rangle$
for some normalized state $|\psi\rangle$
where the expectation values of A and B can be found via $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$ and $\langle B\rangle = \langle\psi|\hat{B}|\psi\rangle$ respectively.

To get where we want to go, we expect $\langle\alpha|\alpha\rangle$ to turn into ΔA, or similar.

Lets check out
(i) $\langle\alpha|\alpha\rangle = \langle\psi|(\hat{A}^{+} - \langle A\rangle)(\hat{A} - \langle A\rangle)|\psi\rangle$
since $\hat{A}$ is hermitian, we can write
$= \langle\psi|\hat{A}^2|\psi\rangle - 2\langle\psi|\hat{A}\langle A\rangle|\psi\rangle + \langle A\rangle^2\langle\psi|\psi\rangle$
$= \langle A^2\rangle - 2\langle A\rangle\langle\psi|\hat{A}|\psi\rangle + \langle A\rangle^2$
$= \langle A^2\rangle - 2\langle A\rangle^2 + \langle A\rangle^2$
$= \langle A^2\rangle - \langle A\rangle^2$
This version is familiar to me as the squared uncertainty in A from Eq 1.21.
You could also write
$= \langle\psi|(\hat{A} - \langle A\rangle)^2|\psi\rangle$ at step 2
$= \langle(A - \langle A\rangle)^2\rangle = \Delta A^2$
This version is line 1 at Eq 1.21 and is perhaps more fundamental as the definition of ΔA^2... ie the average of the square deviation from average.

We can of course understand
(ii) $\langle\beta|\beta\rangle = \Delta B^2$ in the same way.

$$(*) \quad \langle\alpha|\beta\rangle = \langle\psi|\overbrace{(\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)}^{\text{Call this operator } \hat{\Theta}}|\psi\rangle$$

We can divide up any operator into a part satisfying $\hat{F} = \hat{F}^+$ and a part satisfying $(i\hat{G})^+ = -i\hat{G}$, Hermitian and "anti Hermitian"

$$\hat{\Theta} = \hat{F} + i\hat{G}$$

or we could pull out a factor of $\frac{1}{2}$ as I does

$$(**) \quad \hat{\Theta} = \tfrac{1}{2}\hat{F} + \tfrac{i}{2}\hat{G}$$

$$\hat{F} = \hat{\Theta} + \hat{\Theta}^+ \quad (\text{clearly } \hat{F}^+ = \hat{F})$$

$$\hat{G} = -i(\hat{\Theta} - \hat{\Theta}^+) \quad (i\hat{G} = \hat{\Theta} - \hat{\Theta}^+$$
$$\text{so } (i\hat{G})^+ = \hat{\Theta}^+ - \hat{\Theta} = -i\hat{G} \,)$$

Note however that $\hat{G}^+ = i(\hat{\Theta}^+ - \hat{\Theta}) = -i(\hat{\Theta} - \hat{\Theta}^+) = \hat{G}$

Further, it's clear that $\hat{F}$ and $\hat{G}$ so written satisfy $(**)$:

$$\tfrac{1}{2}\hat{F} + \tfrac{i}{2}\hat{G} = \frac{\hat{\Theta}}{2} + \frac{\hat{\Theta}^+}{2} + \frac{\hat{\Theta}}{2} - \frac{\hat{\Theta}^+}{2} = \hat{\Theta} \checkmark$$

Replacing Θ in $(*)$ with $\hat{F}$ and $\hat{G}$ gives

$$\langle\alpha|\beta\rangle = \langle\psi|\frac{\hat{F}}{2}|\psi\rangle + i\langle\psi|\frac{\hat{G}}{2}|\psi\rangle$$

since $\hat{F}$ and $\hat{G}$ are Hermitian, the 1st term is real and the 2nd is imaginary

Hence

$$\text{iii} \quad |\langle\alpha|\beta\rangle|^2 = \frac{|\langle\psi|\hat{F}|\psi\rangle|^2}{4} + \frac{|\langle\psi|\hat{G}|\psi\rangle|^2}{4} \geq \frac{|\langle\hat{G}\rangle|^2}{4}$$

Finally $\hat{G} = -i(\hat{A}\hat{B} - \langle A\rangle\hat{B} - \langle B\rangle\hat{A} + \langle A\rangle\langle B\rangle$
$$- \hat{B}\hat{A} + \langle A\rangle\hat{B} + \langle B\rangle\hat{A} - \langle A\rangle\langle B\rangle)$$
$$= -i(\hat{A}\hat{B} - \hat{B}\hat{A})$$
$$= -i[\hat{A}, \hat{B}]$$

Plugging i, ii, iii into the Schwarz, we have

$$\Delta A^2 \Delta B^2 \geq |\langle\alpha|\beta\rangle|^2 \geq \frac{|\langle\hat{G}\rangle|^2}{4}$$

So

$$\Delta A^2 \Delta B^2 \geq \frac{|\langle\hat{G}\rangle|^2}{4}$$

or

$$\Delta A \, \Delta B \geq \frac{|\langle\hat{G}\rangle|}{2}$$

or

$$\Delta A \, \Delta B \geq \frac{|[\hat{A},\hat{B}]|}{2}$$

5. Looking at the hamiltonian in the $|1\rangle, |2\rangle, |3\rangle$ basis

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}$$

It's not diagonal, so clearly not all of the states can be eigenstates. State $|2\rangle$ is an eigen state

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = E_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Is an eigen state, but the others are not.

a) From this, we can already answer part a. Since $|2\rangle$ <u>is</u> an eigen state, it "won't change" (stationary state) it will only change by an overall phase

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|2\rangle = e^{-iE_1 t/\hbar}|2\rangle$$

b) Now we need to eigen. Call the eigenvalues of $\hat{H}$ E_a, E_b, E_c

$$\det \begin{vmatrix} E_0 - E & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & E_0 - E \end{vmatrix} = 0$$

$$(E_0 - E)^2 (E_1 - E) - (E_1 - E) A^2 = 0$$

$E = E_1$ should be a sol'n ✓

Now we need 2 more.

I'll "divide off" that one

$$(E_0 - E)^2 - A^2 = 0$$

$$E_0 - E = \pm A \quad \text{(sqrt both sides)}$$

So $E_{a/c} = E_0 \pm A$

Now I need the eigenvectors corresponding to $E_{a/c}$.

$$\begin{pmatrix} -A & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & -A \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$-Aa + Ac = 0 \Rightarrow a = c$

I expect $b = 0$ since the states I want here need to be orthogonal to $|2\rangle$

$Aa - Ac = 0$

For the other vector, only signs flip

$Aa + Ac = 0$ (line 1) $\Rightarrow a = -c$

$Aa + Ac = 0$ (line 2)

So to satisfy these and have normalized vectors is

$$|a\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \qquad |b\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Now we see that

$$|\psi(0)\rangle = |3\rangle = \frac{1}{\sqrt{2}}|a\rangle - \frac{1}{\sqrt{2}}|b\rangle$$

We can check that

$$\hat{H}|a\rangle = E_a|a\rangle \checkmark$$

So now

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|3\rangle$$

$$= e^{-i\hat{H}t/\hbar}\left(\tfrac{1}{\sqrt{2}}|a\rangle - \tfrac{1}{\sqrt{2}}|b\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}} e^{-i(E_0+A)t/\hbar}|a\rangle - \tfrac{1}{\sqrt{2}} e^{-i(E_0-A)t/\hbar}|b\rangle$$

$$= \tfrac{1}{2} e^{-iE_0}\left(e^{-iAt/\hbar}\begin{pmatrix}1\\0\\1\end{pmatrix} - e^{iAt/\hbar}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right)$$

$$= \tfrac{1}{2} e^{-iE_0}\begin{pmatrix} e^{iAt/\hbar} - e^{-iAt/\hbar} \\ 0 \\ e^{-iAt/\hbar} + e^{iAt/\hbar}\end{pmatrix}$$

$$= e^{-iE_0}\begin{pmatrix} i\sin At/\hbar \\ 0 \\ \cos At/\hbar \end{pmatrix}$$

$$= e^{-iE_0}\left(i\sin At/\hbar\,|1\rangle + \cos At/\hbar\,|3\rangle\right)$$

So we have an oscillation between states $|1\rangle$ and $|3\rangle$ with frequency $A/\hbar$.