Class 12 Clebsch-Gordon Exercises

1. We showed that a system of 2 spin ½ particles forms the following eigenstates of the operators $\hat{\vec{S}}^2 = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2$ and $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$, the total spin and total z component respectively.

$$|s_t, m_t\rangle = \Sigma |s_1\ s_2\ m_1\ m_2\rangle$$

$$|1, 1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$
$$|1, 0\rangle = \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\ \tfrac{1}{2}\rangle$$
$$|0, 0\rangle = \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\rangle - \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\ \tfrac{1}{2}\rangle$$
$$|1, -1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\ \tfrac{-1}{2}\rangle$$

This information is summarized in the $\frac{1}{2} \times \frac{1}{2}$ section of the Clebsch Gordan table. See if you can figure out how it works.

2. A spin $\frac{3}{2}$ and a spin 1 particle form a 2-particle state that we could label with the eigenvalues of $\hat{\vec{S}}^2$ and $\hat{S}_z$ as follows: $|\frac{3}{2}, -\frac{1}{2}\rangle$ use your C.G. table to write this as a superposition of single-particle states.

3. If you operate on $|\frac{3}{2}\ -\frac{1}{2}\rangle$ with $\hat{S}_z$, you get $\hat{S}_z|\frac{3}{2}\ -\frac{1}{2}\rangle = -\frac{\hbar}{2}|\frac{3}{2}\ -\frac{1}{2}\rangle$. If you operate on the answer to number 2 with $\hat{S}_{1z} + \hat{S}_{2z}$, you should get the same eigenvalue. Check.

4. If you operate on $|\frac{3}{2}\ -\frac{1}{2}\rangle$ with $\hat{\vec{S}}^2$, you get $\hat{\vec{S}}^2|\frac{3}{2}\ -\frac{1}{2}\rangle = \frac{3}{2}(\frac{3}{2}+1)\hbar^2|\frac{3}{2}\ -\frac{1}{2}\rangle$. If you operate on the answer to number 2 with

$$\hat{\vec{S}}_1^2 + \hat{\vec{S}}_2^2 + 2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \hat{\vec{S}}_1^2 + \hat{\vec{S}}_2^2 + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z}$$

what do you expect? Think a little about how the calculation would go.