1. a) $\langle \psi | \hat{P}_+ | \phi \rangle$
$= \langle \psi | +z \rangle \langle +z | \phi \rangle$
$= \langle +z | \phi \rangle \langle \psi | +z \rangle$ can write in any order
$= \langle \phi | +z \rangle^* \langle +z | \psi \rangle^*$
$= \langle \phi | \hat{P}_+ | \psi \rangle^*$
Thus $\hat{P}_+$ is Hermitian

b) $\hat{P}_+ |\lambda\rangle = \lambda |\lambda\rangle$
$\hat{P}_+^2 |\lambda\rangle = \hat{P}_+ \hat{P}_+ |\lambda\rangle = \lambda^2 |\lambda\rangle$

but now since $\hat{P}_+ = \hat{P}_+^2$, we have
$\lambda |\lambda\rangle = \lambda^2 |\lambda\rangle$, which is true only for
$\lambda = 1, 0$.

c) Recall unitary operators have the property
$\hat{U} \hat{U}^\dagger = \hat{U}^\dagger \hat{U} = \mathbb{1}$.

An orthonormal basis satisfies
$\langle a_m | a_n \rangle = \delta_{mn}$

I'll insert $\hat{U}^\dagger U$ between my $|a\rangle$ states
since it's an identity

$\delta_{mn} = \langle a_m | a_n \rangle$
$= \langle a_m | \hat{U}^\dagger \hat{U} | a_n \rangle$
$= \langle b_m | b_n \rangle$
where $|b_n\rangle = \hat{U} |a_n\rangle$

So the $|b_n\rangle = \hat{U} |a_n\rangle$ are also
orthonormal.

We also need to show that the $|b_n\rangle$
are complete, ie a basis

$$\mathbb{1} = \sum_n |a_n\rangle\langle a_n|$$

$$= \sum_n \hat{U}|a_n\rangle\langle a_n|\hat{U}^\dagger$$ multiply both sides from the right by $\hat{U}^\dagger$ and from the left by $\hat{U}$. Since $\hat{U}\hat{U}^\dagger = \mathbb{1}$ the LHS is not changed

$$= \sum_n |b_n\rangle\langle b_n|$$

So the $|b_n\rangle = \hat{U}|a_n\rangle$ are also complete.

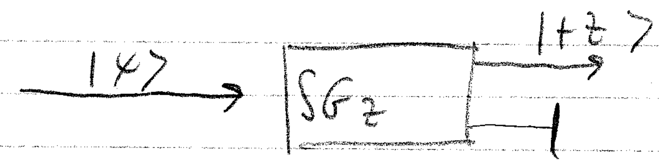
D) $\langle\lambda|\hat{U}^\dagger\hat{U}|\lambda\rangle = \langle\lambda|\lambda^*\lambda|\lambda\rangle$ $*1$

$*2$ $\langle\lambda|\lambda\rangle = \lambda^*\lambda\langle\lambda|\lambda\rangle$

$\lambda^*\lambda \Rightarrow \lambda = e^{i\theta}$

My Eq $*1$ is true since λ is the eigen value of $\hat{U}$ and $\langle\lambda|\hat{U}^\dagger = (\lambda|\lambda\rangle)^\dagger = \langle\lambda|\lambda^*$

My Eq $*2$ is true because $\hat{U}^\dagger\hat{U}$ is an identity.

E) 

This "projects" out the $|+z\rangle$ component of $|\psi\rangle$. The probability that $|\psi\rangle$ yields $|+z\rangle$ is $|\langle +z|\psi\rangle|^2 = cc^*$ if we write $|\psi\rangle = c|+z\rangle + b|-z\rangle$. projection operator expectation also yields $\langle\psi|P_{+z}|\psi\rangle = \langle\psi|c|+z\rangle = c\langle\psi|+z\rangle = cc^*$

So our expectation value is the probability to observe $|\psi\rangle$ in the $|+z\rangle$ state.

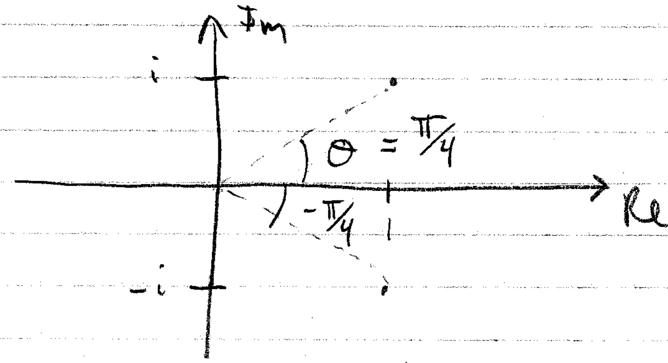
2. $$|\pm x\rangle = \begin{pmatrix} \langle +y | \pm x\rangle \\ \langle -y | \pm x\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left(\langle -z| \frac{-i}{\sqrt{2}} + \langle +z| \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|+z\rangle \pm \frac{1}{\sqrt{2}}|-z\rangle\right) \\ \left(\langle -z| \frac{i}{\sqrt{2}} + \langle +z \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|+z\rangle \pm \frac{1}{\sqrt{2}}|-z\rangle\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \mp \frac{1}{2} i \\ \frac{1}{2} \pm \frac{1}{2} i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \mp i \\ 1 \pm i \end{pmatrix}$$

$$|1-i| = \sqrt{(1-i)(1+i)} = \sqrt{1+1} = \sqrt{2}$$
$$|1+i| = \sqrt{(1+i)(1-i)} = \sqrt{2}$$



So $1-i = \sqrt{2}\, e^{-i\pi/4}$ and $1+i = \sqrt{2}\, e^{i\pi/4}$

$$|+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi/4} \\ e^{i\pi/4} \end{pmatrix} = \frac{e^{-i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|-x\rangle = \frac{1}{\sqrt{2}}\, e^{-i\pi/4}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

3a) $$\hat{J}_x \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle +z|\hat{J}_x|+z\rangle & \langle +z|\hat{J}_x|-z\rangle \\ \langle -z|\hat{J}_x|+z\rangle & \langle -z|\hat{J}_x|-z\rangle \end{pmatrix}$$

One could use the $|\pm z\rangle$ states in the $|\pm x\rangle$ basis to work this out directly or (It's really the same calculations in a different order) do the following

$$= S^+ \begin{pmatrix} \langle +x|\hat{J}_x|+x\rangle & \langle +x|\hat{J}_x|-x\rangle \\ \langle -x|\hat{J}_x|+x\rangle & \langle -x|\hat{J}_x|-x\rangle \end{pmatrix} S$$

The $S = \begin{pmatrix} \langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle \end{pmatrix}$

This would take a ket from z to x, so, as used above it takes $\hat{J}_x$ from the x representation to the z rep.

We can work this out from the $|\pm x\rangle$ states in the z basis

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

So now we have

$$\hat{J}_x \rightarrow \frac{\hbar}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we can check this result with $|+x\rangle \xrightarrow[S_z \text{ basis}]{} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\langle +x|\hat{S}_x|+x\rangle$

$\xrightarrow[S_z \text{ basis}]{} \frac{\hbar}{4}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$

$= \frac{\hbar}{4}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$

$= \frac{\hbar}{2}$ ✓

b) $\langle S_x\rangle = \langle\psi|S_x|\psi\rangle$

$= \frac{\hbar}{3\cdot 2}\begin{pmatrix}1 & \sqrt{2}\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$

$= \frac{\hbar}{6}\begin{pmatrix}1 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2}\\1\end{pmatrix}$

$= \frac{\hbar}{6}(2\sqrt{2})$

$= \frac{\sqrt{2}\,\hbar}{3}$

It should be between $\pm\hbar/2$ and $\sqrt{2}/3 = 0.47$ ✓

4. A) We need to find the angular momentum of the beam. I'll use the $|R\rangle$ + $|L\rangle$ from Eq 2.114

$$P_L = |\langle L|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle x| + i\langle y|)\left(\sqrt{\frac{2}{3}}|x\rangle + \frac{i}{\sqrt{3}}|y\rangle\right)\right|^2$$

$$= \left|\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{6}}(\sqrt{2} - 1)\right|^2$$

$$= \frac{1}{6}(2 - 2\sqrt{2} + 1)$$

$$= \frac{1}{2} - \frac{1}{3}\sqrt{2}$$

$$P_R = |\langle R|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle x| - i\langle y|)\left(\sqrt{\frac{2}{3}}|x\rangle + \frac{i}{\sqrt{3}}|y\rangle\right)\right|^2$$

I could work this out as I did above, but since the probabilities of right + left sum to 1, I can just do

$$P_R = |\langle R|\psi\rangle|^2 = 1 - |\langle L|\psi\rangle|^2$$
$$= 1 - \left(\frac{1}{2} - \frac{1}{3}\sqrt{2}\right)$$
$$= \frac{1}{2} + \frac{1}{3}\sqrt{2}$$

So the angular momentum transfer rate to the plate is

$$\frac{dL}{dt} = N(\hbar P_R - \hbar P_L)$$
$$= N\hbar\frac{2\sqrt{2}}{3}$$

Since $\tau = dL/dt$, we have

$$\boxed{\tau = N\hbar\frac{2\sqrt{2}}{3}}$$

B) For a Disk we have $I_D = \frac{1}{2} m r^2$. Here $m = 1\,kg$ and $r = 1\,m$

$$L_D = I\omega$$

$$L_{Df} - \cancel{L_{Di}}^{\,0} = \int_0^t N\hbar \frac{2\sqrt{2}}{3}\, dt$$

$$I_D \omega = N\hbar \frac{2\sqrt{2}}{3} t$$

$$t = \frac{I_D \omega}{N\hbar \frac{2\sqrt{2}}{3}}$$

$$= \frac{(\frac{1}{2}\,kg\,\cancel{m^2})(1\,rad/\cancel{s})}{(10{,}180/s)(1.05\times 10^{-34}\,\cancel{m^2}\,\cancel{kg}/\cancel{s})\,\frac{2\sqrt{2}}{3}}$$

$$\cong 5 \times 10^{29}\, s$$

This is a really long time(!) to reach just 1 rad/s (!) So Dr D's plan is not practical, though in principle the spin angular momentum of light can rotate a macroscopic object.

It's harder with e^-, though the angular momentum would become "a part of the disk." Some issues:

- When e^- are absorbed, depending on the process, they may still be e^-, keeping their angular momentum as spin rather than $\vec{L}$ of the disk.
- The disk would become charged.

C. Probability of transmission is

$$|\langle y|\psi\rangle|^2 = \left|\langle y|\left(\sqrt{\tfrac{2}{3}}\,|x\rangle + \tfrac{i}{\sqrt{3}}|y\rangle\right)\right|^2$$

$$= \left|\tfrac{i}{\sqrt{3}}\right|^2$$

$$= \tfrac{-i}{\sqrt{3}} \cdot \tfrac{i}{\sqrt{3}}$$

$$= \tfrac{1}{3}$$

D. The photons will come out of the polarizer in the $|y\rangle$ state

$$|\langle L|y\rangle|^2 = \left|\tfrac{i}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

So the beam will be a 50/50 mix of $+\hbar$ and $-\hbar$ so $\approx$ no angular momentum will be gained by the disk over a large number of photons.

The intensity is also less since the $|\langle x|\psi\rangle|^2$ fraction of the beam is removed by the polarizer.

5. Let's start with the $|R\rangle / |L\rangle$ basis. That will be easy since our state is already in that basis

$$\langle S_z \rangle = (a^*, b^*) \begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= (a^*, b^*) \begin{pmatrix} \hbar a \\ -\hbar b \end{pmatrix}$$

$$= \hbar (a a^* - b b^*)$$

Now to do it in the $|x\rangle / |y\rangle$ basis we first need to transform our state to this basis.

The S matrix to go from R/L to x/y is also worked out in Example 2.8:

$$S = \begin{pmatrix} \langle R | x \rangle & \langle R | y \rangle \\ \langle L | x \rangle & \langle L | y \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

Though it's not hard to work out these inner products yourself.

$$\langle S_z \rangle = \underset{\text{RL basis}}{\langle \psi |} S S^\dagger \underset{\text{RL basis}}{\hat{J}_z} S S^\dagger \underset{\text{RL basis}}{| \psi \rangle}$$

When we write it in this way, it makes it clear that the answer is invariant, but proceeding to matrix notation we get

$$|\psi\rangle \xrightarrow[\text{xy basis}]{} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a+b \\ i(a-b) \end{pmatrix}$$

$$J_z \xrightarrow[xy]{} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$$

$$= \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \Leftarrow \text{ worked out in 2.8 too}$$

$$\langle S_z \rangle = \frac{\hbar}{2}\left(a^*+b^*, -i(a^*-b^*)\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a+b \\ i(a-b) \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(a^*+b^*, -i(a^*-b^*)\right)\begin{pmatrix} a-b \\ i(a+b) \end{pmatrix}$$

$$= \frac{\hbar}{2}\left((a^*+b^*)(a-b) + (a+b)(a^*-b^*)\right)$$

$$= \frac{\hbar}{2}\left(a^*a - \cancel{a^*b} + \cancel{ab^*} - bb^* + aa^* - \cancel{ab^*} + \cancel{a^*b} - bb^*\right)$$

$$= \hbar\left(a\,a^* - b\,b^*\right)$$

Matches what we got in the R/L basis

6, a) $[\hat{A}, \hat{B} + \hat{C}] = \hat{A}(\hat{B} + \hat{C}) - (\hat{B} + \hat{C})\hat{A}$

$= \hat{A}\hat{B} + \hat{A}\hat{C} - \hat{B}\hat{A} - \hat{C}\hat{A}$

$= (\hat{A}\hat{B} - \hat{B}\hat{A}) + (\hat{A}\hat{C} - \hat{C}\hat{A})$

$= [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$

b) $[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C}^{①} - \overset{②}{\hat{B}\hat{C}\hat{A}}$

We need to RHS terms with $\hat{C}$ at the back and 2 with $\hat{B}$ at the front. If we add $0 = \hat{B}\hat{A}\hat{C}^{③} - \hat{B}\hat{A}\hat{C}^{④}$ we can pair up terms ① and ④ as well as terms ② and ③ as follows

$= (\overset{①}{\hat{A}\hat{B}} - \overset{④}{\hat{B}\hat{A}})\hat{C} + \hat{B}(\overset{③}{\hat{A}\hat{C}} - \overset{②}{\hat{C}\hat{A}})$

$= [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

c) You could do a proof just like (b), but you could use (b) as follows

$[\hat{A}\hat{B}, \hat{C}] = -[\hat{C}, \hat{A}\hat{B}]$ Now apply rule (b)

$= -(\hat{A}[\hat{C}, \hat{B}] + [\hat{C}, \hat{A}]\hat{B})$

$= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

D) $-i\hat{C}^\dagger = [\hat{A}, \hat{B}]^\dagger$

$= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger$

$= \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger$ Hermitian

$= \hat{B}\hat{A} - \hat{A}\hat{B}$

$= -[\hat{A}, \hat{B}]$

$= -i\hat{C}$

So $\hat{C} = \hat{C}^\dagger$ ✓