1. It should say around the z axis

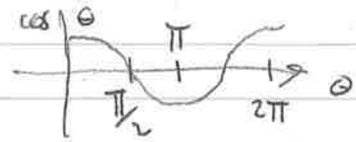
So that's apply $R(\phi k) = e^{-i\hat{J}_z\phi/\hbar}$

$$\hat{R}(\phi k)\,|+x\rangle = \frac{1}{\sqrt{2}}e^{-i\phi/2}\,|+z\rangle + \frac{1}{\sqrt{2}}e^{+i\phi/2}\,|-z\rangle$$

$$|\langle +x|\hat{R}|+x\rangle|^2 = \frac{1}{4}\left|e^{-i\phi/2} + e^{i\phi/2}\right|^2$$

$$\cos\frac{\phi}{2} - i\sin\phi$$
$$\cos\frac{\phi}{2} + i\sin\phi$$

$$= \cos^2\frac{\phi}{2}$$



Checks: $\phi = 0 \Rightarrow P_{+x} = 1$ ✓

$\phi = \pi \Rightarrow P_{+x} = 0$ ✓

$\phi = \pi/2 \Rightarrow P_{+x} = 1/2$ ✓

2. $$\hat{J}_z \xrightarrow[a/b\text{ basis}]{} \begin{pmatrix} \langle a|\hat{J}_z|a\rangle & \langle a|\hat{J}_z|b\rangle \\ \langle b|\hat{J}_z|a\rangle & \langle b|\hat{J}_z|b\rangle \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \langle a|\left(\frac{\hbar}{2}|+z\rangle - \frac{\hbar}{2}|-z\rangle\right) & \langle a|\left(-\frac{\hbar}{2}|+z\rangle - \frac{\hbar}{2}|-z\rangle\right) \\ \langle b|\left(\frac{\hbar}{2}|+z\rangle - \frac{\hbar}{2}|-z\rangle\right) & \langle b|\left(-\frac{\hbar}{2}|+z\rangle - \frac{\hbar}{2}|-z\rangle\right) \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} -1 & -1 \\ +1 & -1 \end{pmatrix} \end{pmatrix}$$

$$= -\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3. No. $|+x\rangle$ is not an eigenvector of $\hat{J}_z$

$$\hat{J}_z|+x\rangle = \hat{J}_z\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right)$$
$$= \frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right)$$
$$= \frac{\hbar}{2}|-x\rangle$$

We don't get the state back so it's not an eigenvector.