Angular Wave F'n

Last time, we showed that the Schrödinger Eqn in position space could be written

(1) $$\left[\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{l(l+1)\hbar^2}{2\mu r}+V(r)\right]\langle\vec{r}|Elm\rangle = E\langle\vec{r}|Elm\rangle,$$

for the 2-body central force problem, and we argued that the states $|Elm\rangle$ are the simultaneous eigenstates of $\hat{H}$, $\hat{L}^2$, $\hat{L}_z$, with the following eigenvalues

(2) $\hat{H}|Elm\rangle = E|Elm\rangle$

(3) $\hat{L}^2|Elm\rangle = l(l+1)\hbar^2|Elm\rangle$

(4) $\hat{L}_z|Elm\rangle = m\hbar|Elm\rangle$

$\langle\vec{r}|Elm\rangle$ is the projection of these eigenstates onto position space. In spherical coordinates, they will be a f'n of the 3 position space variables r, θ, ϕ. The structure of (1) means that we can assume a "seperable sol'n", where $\langle\vec{r}|Elm\rangle$ takes the following form

$$\langle\vec{r}|Elm\rangle = R(r)\,Y(\theta)(\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi).$$

$R(r)$ can only be found once we pick what kind of potential $V(r)$ we are dealing with. The angular functions are general. Before we solve, lets look at the answers and check them. Some of the angular eigenfunctions are listed on the next page. Noting that the position space representation of $\hat{L}_z$ is

Q1 $\hat{L}_z \to \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, pick a Y_{lm} and show that it satisfies Eq. (4).

Particle Data Book

(2)

Townsend writes lower m index. There is no difference in meaning. It's just notation

42. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

$Y_0^0 = \sqrt{\frac{1}{4\pi}}$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
⋮ ⋮			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

Here is how you get −m

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1 m_2	$J=1$, $M=+1$
+1/2 +1/2	1

m_1 m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=1$, $M=-1$
−1/2 −1/2	1

2×1/2

m_1 m_2	$J=5/2$, $M=+5/2$
+2 +1/2	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2, −5/2
−2 −1/2	1

1×1/2

m_1 m_2	3/2, +3/2
+1 +1/2	1

m_1 m_2	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2, −3/2
−1 −1/2	1

3/2×1/2

m_1 m_2	2, +2
+3/2 +1/2	1

m_1 m_2	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2, −2
−3/2 −1/2	1

2×1

m_1 m_2	3, +3
+2 +1	1

m_1 m_2	3, +2	2, +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3, −1	2, −1	1, −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3, −2	2, −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3, −3
−2 −1	1

3/2×1

m_1 m_2	5/2, +5/2
+3/2 +1	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2, −3/2	3/2, −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2, −5/2
−3/2 −1	1

1×1

m_1 m_2	2, +2
+1 +1	1

m_1 m_2	2, +1	1, +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2, −1	1, −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2, −2
−1 −1	1

3/2×3/2

m_1 m_2	3, +3
+3/2 +3/2	1

m_1 m_2	3, +2	2, +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	3, +1	2, +1	1, +1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	3, 0	2, 0	1, 0	0, 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	3, −1	2, −1	1, −1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	3, −2	2, −2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	3, −3
−3/2 −3/2	1

2×3/2

m_1 m_2	7/2, +7/2
+2 +3/2	1

m_1 m_2	7/2, +5/2	5/2, +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	7/2, −5/2	5/2, −5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	7/2, −7/2
−2 −3/2	1

2×2

m_1 m_2	4, +4
+2 +2	1

m_1 m_2	4, +3	3, +3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	4, +2	3, +2	2, +2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	4, +1	3, +1	2, +1	1, +1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	4, −1	3, −1	2, −1	1, −1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	4, −2	3, −2	2, −2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	4, −3	3, −3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	4, −4
−2 −2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 42.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

Since $\hat{L} \to \vec{r} \times \frac{\hbar}{i} \vec{\nabla}$ one can show that

(5) $$\hat{L}^2 \to -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

when represented in spherical coords in position space. You can do it by looking up $\vec{\nabla}$ in spherical coords and multiplying it out. It takes a little time, so don't do it now.

Q2 Instead, pick a $Y_{\ell m}$ and show that (3) is satisfied given (5). Pick an easy one like Y_{11}.

The combination of Q1 and Q2 should convince you that the $Y_{\ell m}$ are the eigenfunctions of $\hat{L}_z$ and $\hat{L}^2$. The figure on the next page plots $|Y_{\ell m}(\theta, \phi)|^2$.

Q3 If you imagine a specific problem, like the hydrogen atom, what do these plots mean?
(an important one!)

As a side note, since the $Y_{\ell m}(\theta, \phi)$ form a complete set of functions in $\theta + \phi$, <u>any well behaved f'n</u> on a sphere can be expanded in spherical harmonics, hence they come up <u>a lot</u>!

Now lets solve for the $Y_{\ell m}$ directly instead of just checking them. Projecting (3) in position space we have

(6) $$-\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \Theta(\theta) \Phi(\phi)$$
$$= \underline{\qquad} \hbar^2 \, \Theta \, \Phi(\phi)$$

Q4 Fill in the blank in (6) given (3) and (5).

A little algebra on (6) allows us to separate the variables, ie put the ϕs on the right and the θs on the left.

Townsend

(4)

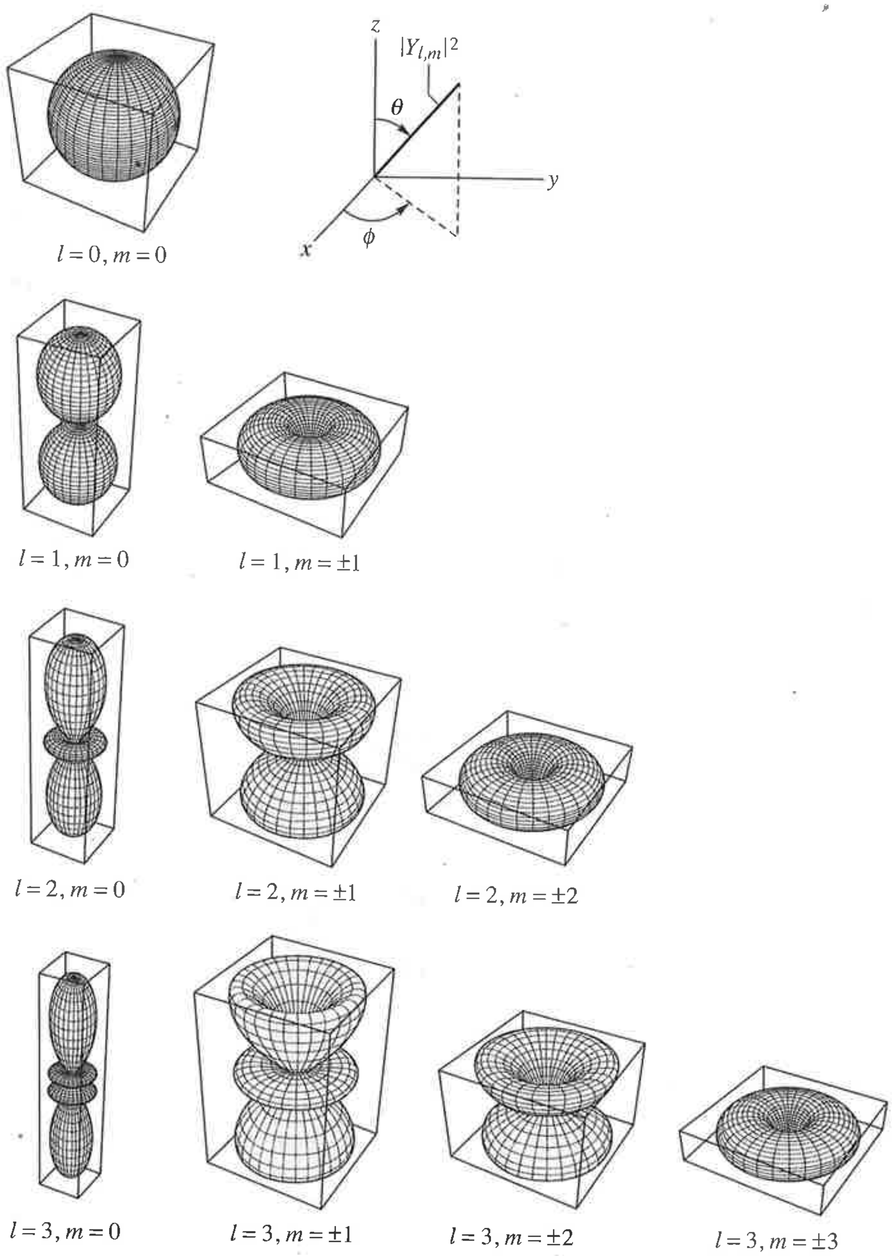

Figure 9.11 Plots of $|Y_{l,m}(\theta, \phi)|^2$ for $l = 0, 1, 2,$ and 3.

rotator, such as the diatomic molecule discussed in Section 9.6, is in an s state, a measurement of the orientation of the rotator is equally likely to find it oriented in any direction. The $l = 1$ states are known as p states. The states with $m = \pm 1$ have a probability density that tends to reside in the x-y plane, which is just the sort of behavior that you might expect for an object rotating around the z axis with nonzero

After separation, I find

(7) $$\frac{\sin\Theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+l(l+1)\sin^2\theta=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$$

Q5 If you have time, show (7) from (6). Else do it later.

Q6. Convince your self that each side of (7) equals a constant. Lets call that constant m^2, so we now have

(8) $$\frac{d^2\Phi(\phi)}{d\phi^2}=-m^2\Phi \xrightarrow{\text{Sol'n}} \Phi=e^{im\phi}$$

and

(9) $$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+l(l+1)\sin^2\theta=m^2$$

$\xrightarrow{\text{Sol'n}}$??! (Later)

The process that happens from (6) to (8)+(9) is a big #$@! deal. It shows up all over physics.

Before we deal with (9), lets have fun with (8). Note that

(10) $\Phi(\phi+2\pi)=\Phi(\phi)$.

Q7. Explain why (10) is so, and say what constraint it places on m, ie, what m's are allowed? Compare your answers to the ϕ part of the spherical harmonics. Does it make sence?

(As an aside, we can't have a superposition of multiple ms for Φ or it wouldn't be an eigenstate of $\hat{L}_z$)

While (9) looks gnarly, specific sol'ns for small m and ℓ are not. If $\ell = m = 1$,

(11) $\Theta = A \sin\theta$ is a sol'n. By inspection of the spherical harmonics, you can see that the sol'ns for Θ are combinations of $\sin\theta$ and $\cos\theta$. The general sol'n can be written in terms of associated Legendre functions

(12) $$P_{\ell m}(x) = (1-x^2)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} P_\ell(x),$$

where $P_\ell(x)$ is a Legendre polynomial

(13) $$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx}\right)^{\ell} (x^2-1)^\ell ,$$

This thing's called the Rodrigues formula

as

(14) $\Theta = A_{\ell m} P_{\ell m}(\cos\theta)$ where $A_{\ell m}$ is an integration constant. So the $P_{\ell m}$ provide a way of expressing the tower of sin and cos that form the $Y_{\ell m}$ in an organized way.

Q 8 Check that (12) - (14) succeed in generating the θ part of a $Y_{\ell m}$, like maybe a simple case like (11).

All that remains of finding the $Y_{\ell m}$ is the normalization. We want

(15) $$\int |Y_{\ell m}|^2 d\Omega = \int_0^{2\pi}\int_0^{\pi} |Y_{\ell m}|^2 \sin\theta \, d\theta \, d\phi = 1$$

Hi! My name is solid angle. Nice to meet you!

Q 9 Convince yourself of (15) and show it works for Y_{00}.

Q 10 " " $P_{\ell m} = 0$ for $|m| > \ell$.