Q1 I pick Y_{11}

If we project in to position space we have
$\hat{L}_z |E l m\rangle = m\hbar |E l m\rangle$

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi} Y_{11} = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right)$$

$$= \hbar\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right)$$

$= \hbar Y_{11}$ as needed for $m = 1$ ✓

Q2 I'll again pick Y_{11}
$\hat{L}^2 |E l m\rangle = l(l+1)\hbar^2 |E l m\rangle$

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{11}$$

$$= -\hbar^2\left[\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sin\theta}{\partial\theta}\right)}\left(-\sqrt{\frac{3}{8\pi}}e^{i\phi}\right) + \cancel{\sqrt{\frac{3}{8\pi}}\frac{e^{i\phi}}{\sin\theta}}\right]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\cos\theta = -\sin\theta + \frac{\cos^2\theta}{\sin\theta}$$

$$= -\sin\theta + \cancel{\frac{1}{\sin\theta}} - \frac{\sin^2\theta}{\sin\theta}$$

$$= 2\hbar^2\left(-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta\right)$$

$= 2\hbar^2 Y_{11}$, the right thing for $l = 1$ ✓

Q3 They are the probability distribution in θ & ϕ for where the electron could be found. These are then multiplied by the radial distribution to form a full probability distribution in 3D.

Q4 $l(l+1)$

Q5 $$-\cancel{\hbar^2}\left[\frac{\Phi}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Theta}{\partial\Theta}\right)+\frac{\Theta}{\sin^2\Theta}\frac{\partial^2\Phi}{\partial\phi^2}\right]=l(l+1)\cancel{\hbar^2}\,\Theta\,\Phi$$

pull in Θ + Φ

Now divide by Θ + Φ, multiply by $\sin^2\Theta$

$$\underbrace{-\frac{\sin\Theta}{\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Theta}{\partial\Theta}\right)}\underbrace{-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}}_{\text{add this}}=\underbrace{l(l+1)\sin^2\Theta}_{\text{subtract this}}$$

to get (7)

Q6 The equality must hold for any Θ, independent of ϕ and visa versa.

Q7 The wave f'n has to be smooth @ $\phi = 0$. It means m is an integer.

Q8 $$P_1(\cos\Theta)=\frac{1}{2}\frac{d}{d\cos\Theta}(\cos^2\Theta-1)=\cos\Theta$$

$$P_{11}(\cos\Theta)=\underbrace{(1-\cos^2\Theta)}_{\sin^2\Theta}{}^{1/2}\underbrace{\left(\frac{d}{d\cos\Theta}\right)\cos\Theta}_{1}$$

$$=\sin\Theta \checkmark$$

Q9 The particle needs to be somewhere in $\Theta + \phi$ so the probabilities must "add up" to 1.

$$\int_0^\pi \sin\Theta\, d\Theta \int_0^{2\pi} d\phi$$

$$= -\cos\Theta\Big|_0^\pi \quad \phi\Big|_0^{2\pi} =$$

$$= 2 \cdot 2\pi = 4\pi$$

So $\frac{1}{\sqrt{4\pi}}$ is the right normalization for Y_{00} ie $A_{00} = 1/\sqrt{4\pi}$

Q10 $(x^2-1)^l$ is a polynomial of degree $2l$. We take l derivatives in (13) and m more in (12).