# Contents

# 1 Chapter 1.1

## 1.1 1.2

Consider the following conditions on a ring $R$,

(I) $R$ satisfies the IBP (if $R^n \cong R^m$ then $n = m$).

(II) For all $m, n$ and $P$ if $R^m \cong R^n \oplus P$ then $m \geq n$.

(III) For all $n$ and $P$ if $R^n \cong R^n \oplus P$ then $P = 0$

We will show (III) $\implies$ (II) $\implies$ (I). First suppose $R$ satisfies (III) and consider the situation that $R^m \cong R^n \oplus P$ and $m < n$. We can add $R^{n-m}$ to each side to get,

$$R^n \cong R^n \oplus (P \oplus R^{n-m})$$

then applying (III) we find $P \oplus R^{n-m} = 0$ a contradiction proving (II).

Now assume property (II) and suppose that $R^m \cong R^n$. By applying (II) in the case $P = 0$ we find $m \geq n$ and $n \geq m$ and thus $m = n$ proving the IBP i.e. property (I).

## 1.2 1.3

We need to show that the following conditions on a ring $R$ are equivalent,

(a). For all $n$, every surjection $R^n \to R^n$ is an isomorphism.

(b). For all $n$, and $f, g \in M_n(R)$ if $fg = \text{id}$ then $gf = \text{id}$ and $g \in \text{GL}_n(R)$.

(c). For all $n$ and $P$ if $R^n \cong R^n \oplus P$ then $P = 0$.

First suppose property (a) and let $fg = \text{id}$ for $f, g \in M_n(R) = \text{End}(R^n)$. Since $fg = \text{id}$ the map $g : R^n \to R^n$ is surjective and thus an isomorphism by property (a). so we find that $g \in \text{GL}_n(R)$ and there is some $h \in \text{GL}_n(R)$ such that $gh = hg = \text{id}$. However,

$$fgh = (fg)h = h = f(gh) = f$$

so $h = f$ and thus $fg = gf = \text{id}$ proving (b).

Now suppose (b) holds and suppose we have the situation $R^n \cong R^n \oplus P$. Then consider the maps $\iota : R^n \to R^n \oplus P$ and $\pi : R^n \oplus P$ which satisfiy $\pi \circ \iota = \text{id}$. Now let $f : R^n \to R^n \oplus P$ be the given isomorphism then define $\tilde{\iota} = f^{-1} \circ \iota : R^n \to R^n$ and $\tilde{\pi} = \pi \circ f : R^n \to R^n$ and thus $\tilde{\pi} \circ \tilde{\iota} = \text{id}$ and $\tilde{\pi}, \tilde{\iota} \in \text{End}(R^n) = M_n(R)$. Thus by (b), $\tilde{\iota} \circ \tilde{\pi} = \text{id}$ so $\tilde{\iota} = f^{-1} \circ \iota$ is an isomorphism which implies that $\iota : R^n \to R^n \oplus P$ is an isomorphism (since $f^{-1}$ is) and thus $P = 0$ proving (c).

Finally, suppose (c) and suppose that $f : R^n \to R^n$ is a surjection. Then consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \longrightarrow R^n \longrightarrow 0$$

Then $R^n$ is free and thus projective so the sequence is split,

$$R^n \cong R^n \oplus \ker f$$

so by (c) we have $\ker f = 0$ and thus $f$ is an isomorphism proving (a).

Finally suppose that $R$ is commutative and $f, g \in M_n(R)$ with $fg = \text{id}$. Then $\det fg = \det f \det g = 1$ so $f, g \in \text{GL}_n(R)$ and thus there exists a matrix (coefactors) $h$ such that $gh = \text{id}$ then $f = h$ by a previous argument. Therefore commutative rings satisfy all the above properties.

### 1.3 1.7

(NO IDEA)

### 1.4 1.8

(NO IDEA)

### 1.5 1.9

(NO IDEA)

## 2 Chapter 1.2

*Remark.* In this section $R$ is a commutative (unital) ring.

### 2.1 2.2

### 2.2 2.4

Consider a continuous function $f : \text{Spec}\,(R) \to \mathbb{Z}$. First, $\text{Spec}\,(R)$ is quasi-compact. This is easily shown since every affine cover $U_i$ can be refined to a cover by principal opens $D(f_i)$ then,

$$\text{Spec}\,(R) = \bigcup_{i=1}^{\infty} D(f_i) = D(\langle f_i \rangle)$$

(since $f_i \notin \mathfrak{p}$ for some $f_i$ iff $\langle f_i \rangle \not\supset \mathfrak{p}$) and thus $\langle f_i \rangle = R$ (otherwise it would be contained in a maximal ideal) but then $1 = r_1 f_1 + \cdots + r_n f_n$ for finitely many so,

$$\text{Spec}\,(R) = D(\langle f_1, \ldots, f_n \rangle) = \bigcup_{i=1}^{n} D(f_i)$$

so there is a finite subcover of $U_i$.

Therefore, $f(\text{Spec}\,(R)) \subset \mathbb{Z}$ is compact and thus finite so it must take finitely many values $n_1, \ldots, n_c$. Then $V_i = f^{-1}(n_i)$ is a closed subset of $\text{Spec}\,(R)$ since $\mathbb{Z}$ is discrete.

If $R$ is not reduced then consider $R_{\text{red}} = R/\text{nilrad}\,(R)$ and $\text{Spec}\,(R) \cong \text{Spec}\,(R_{\text{red}})$ naturally so we may assume that $R$ is reduced and we may use idempotent lifitng (2.2).

Since $V_i$ is closed $V_i = V(I_i)$ for some ideal $I_i \subset R$. Furthermore,

$$\mathrm{Spec}\,(R) = \bigcup_{i=1}^{n} V_i = \bigcup_{i=1}^{n} V(I_i) = \bigcup_{i=1}^{n} V(I_n) = V(I_1 \cdots I_n)$$

Thus $\sqrt{I_1 \cdots I_n} = \mathrm{nilrad}\,(R) = (0)$ so $I_1 \cdots I_n = (0)$. Furthermore, the $V_i$ are disjoint so,

$$\varnothing = V_i \cap V_j = V(I_i) \cap V(I_j) = V(I_i + I_j)$$

and thus $I_i + I_j = R$ so the ideals $I_i$ and $I_j$ are coprime. Therefore, by CRT,

$$R = R/(0) = R/(I_1 \cdots I_n) = (R/I_1) \times \cdots \times (R/I_n)$$

since these ideals are pairwise coprime. (Note, there is an error in the text, it has these two conditions backwards).

## 2.3 2.5

Conisder the following properties,

(a). $\mathrm{Spec}\,(R)$ is connected.

(b). Every finitely generated projective $R$-module has constant rank.

(c). $R$ has no idempotent elements except 0 and 1.

I claim that these are equivalent.

See the background material in Appendix A, but for any finitely-generated projective module If $\mathrm{Spec}\,(A)$ is connected then since $\mathrm{rank}(P)$ is continuous (see Appendix) then then its image must be connected in $\mathbb{Z}$ and thus constant.

Suppose $e \in R$ were a nontrivial idepotent. Then consider the module $P = (e)$ which I claim is f.g. (obvious) and projective. It suffices to show that $P$ is free on some open cover. On the open set $D(e)$ we have $P_e \cong R_e$ so $P$ is free on $D(e)$ of rank 1. Furthermore, on the open set $D(1-e)$ we have $P_{1-e} = (e)_{1-e} = (0)$ since $e^2 = e$ and thus $P$ is free of rank 0. Since $e + (1-e) = 1$ these open sets cover $\mathrm{Spec}\,(R)$. Therefore $P$ is f.g. projective but does not have finite rank. Thus (b) $\implies$ (c).

Finally, if $\mathrm{Spec}\,(R)$ is not connected then we can write $\mathrm{Spec}\,(R) = V(I) \cup V(J)$ for two nontrivial disjoint closed sets in which case $IJ = (0)$ and $I + J = R$. Thus by CRT, $R = (R/I) \times (R/J)$. However, the element $(1,0)$ in this product is a nontrivial idempotent in the ring. Thus (c) $\implies$ (a).

## 2.4 2.8

## 2.5 2.10

Let $P, Q$ be $R$-modules and $P \otimes_R Q \cong R^n$ for $n > 0$. Then $P$ and $Q$ are f.g. projective $R$-modules.

*Proof.* (DO) $\square$

## 2.6 2.11

Let $M$ be a finitely generated module over a commuative ring $R$. I claim that the following are equivalent for every $n$,

(a). $M$ is f.g. projective of constant rank $n$

(b). $M_\mathfrak{p} \cong R_\mathfrak{p}^n$ for every prime ideal $\mathfrak{p}$ of $R$.

Clearly (a) $\implies$ (b) so we assume that $M_\mathfrak{p} \cong R_\mathfrak{p}^n$ at each prime $\mathfrak{p}$. By Lemma 2.4 it suffices to show that $M$ is finitely presented since then freeness of the stalks implies projectivity and $M$ is automatically of constant rank $n$ by definition.

Lift the basis map $R_\mathfrak{p}^n \to M_\mathfrak{p}^n$ to a map $f : R^n \to M$ by clearing denominators. Now consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \longrightarrow M \longrightarrow \operatorname{coker} f \longrightarrow 0$$

Since $M$ is finitely generated then so is $\operatorname{coker} f$. Furthermore, when we localize at $\mathfrak{p}$ we get,

$$0 \longrightarrow (\ker f)_\mathfrak{p} \longrightarrow R_\mathfrak{p}^n \longrightarrow M_\mathfrak{p} \longrightarrow (\operatorname{coker} f)_\mathfrak{p} \longrightarrow 0$$

but we know $R_\mathfrak{p}^n \to M_\mathfrak{p}$ is an isomorphism so $(\operatorname{coker} f)_\mathfrak{p} = (\ker f)_\mathfrak{p} = 0$. Since $\operatorname{coker} f$ is f.g. there exists $g \in R$ such that $(\operatorname{coker} f)_g = 0$. Then localizing at $g$ instead we find,

$$0 \longrightarrow (\ker f)_g \longrightarrow R_g^n \longrightarrow M_g \longrightarrow 0$$

Then for any prime $\mathfrak{q} \in D(g)$ we may localize again to find,

$$0 \longrightarrow (\ker f)_\mathfrak{q} \longrightarrow R_\mathfrak{q}^n \longrightarrow M_\mathfrak{q} \longrightarrow 0$$

so $R_\mathfrak{q}^n \to M_\mathfrak{q}$ is a surjection. However, my assumeption $M_\mathfrak{q}$ is free of rank $n$ and $R$ is commutative so by 1.3 property (a). we know $R_\mathfrak{q}^n \to M_\mathfrak{q}$ is an isomorphism and thus $\ker f_\mathfrak{q} = 0$. Therefore $(\ker f)_g$ is an $A_g$-module with empty support so $\ker f_g = 0$. Therefore, $M_g \cong R_g^n$ so $M$ is locally free and thus projective.

Therefore, suppose that $M$ is finitely generated free at each stalk with $\text{rank}(M)$ continuous. Then $\text{Spec}\,(R)$ has a finite open cover $U_i = (\text{rank}(M))^{-1}(n_i)$ such that $M|_{U_i}$ is f.g. with $M_\mathfrak{p} = R_\mathfrak{p}^{n_i}$ for fixed $n_i$ on each $U_i$. Thus we have shown that $M$ is locally free on $U_i$ and thus locally free on $\text{Spec}\,(R)$ and thus projective. Conversely if $M$ is f.g. projective then we know (by Lemma 2.4) that $M$ is locally free and thus $M_\mathfrak{p} \cong R_\mathfrak{p}^{n_\mathfrak{p}}$ and has continuous rank funciton.

## 2.7 2.12

Let $\phi : R \to S$ be a morphism of rings then let $f = \phi^{-1} : \text{Spec}\,(S) \to \text{Spec}\,(R)$ be the associated morphism of affine schemes. Now there is a functor,

$$f^* : \mathfrak{QCoh}\,(\text{Spec}\,(R)) \to \mathfrak{QCoh}\,(\text{Spec}\,(S))$$

given explicitly by $M \mapsto M \otimes_R S$. I claim that if $P$ is f.g. projective then $f^*P$ is f.g. projective. This is clear using the following property and noting that $(-) \otimes_R S$ is left adjoint to restriction of an $S$ module to an $R$ module which is clearly exact.

**Lemma 2.1.** If a functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$ between abelian categories and $G$ is exact then $F$ preserves projectives.

*Proof.* $F(P)$ is projective iff $\mathrm{Hom}_{\mathcal{C}}(F(P), -)$ is exact but,

$$\mathrm{Hom}_{\mathcal{D}}(F(P), -) \cong \mathrm{Hom}_{\mathcal{C}}(P, G(-))$$

which is exact since $G$ and $\mathrm{Hom}_{\mathcal{C}}(P, -)$ are for projective $P$. □

Now I claim that $\mathrm{rank}(f^* P) = \mathrm{rank}(P) \circ f$. This is because,

$$(f^* P) \otimes_{S_{\mathfrak{p}}} \kappa(\mathfrak{p}) = P \otimes_R S \otimes_{S_{\mathfrak{p}}} \kappa(\mathfrak{p}) = P \otimes_R \kappa(\mathfrak{p})$$

Via the map $R \to S \to \kappa(\mathfrak{p})$. Now we get an inclusion of fields, $\kappa(f(\mathfrak{p})) \to \kappa(\mathfrak{p})$ which $R \to \kappa(\mathfrak{p})$ factors through. Thus,

$$P \otimes_R \kappa(\mathfrak{p}) = P \otimes_R \kappa(f(\mathfrak{p})) \otimes_{\kappa(f(\mathfrak{p}))} \kappa(\mathfrak{p})$$

In particular, these vectorspaces have equal rank i.e.

$$\begin{aligned} \mathrm{rank}_{\mathfrak{p}}(f^* P) &= \dim_{\kappa(\mathfrak{p})}(P \otimes_R \kappa(\mathfrak{p})) = \dim_{\kappa(\mathfrak{p})}(P \otimes_R \kappa(f(\mathfrak{p})) \otimes_{\kappa(f(\mathfrak{p}))} \kappa(\mathfrak{p})) \\ &= \dim_{\kappa(f(\mathfrak{p}))}(P \otimes_R \kappa(f(\mathfrak{p})) = \mathrm{rank}_{f(\mathfrak{p})}(P) \end{aligned}$$

### 2.8 2.16

Fix a small category of rings $\mathcal{R}$. A big projective $R$-module is a choice of a finitely generated projective $S$-module $P_S$ for each $S$ over $R$ in $\mathcal{R}$ equiped with an isomrophism $P_S \otimes_S T \to P_T$ for every $S \to Y$ over $R$ which satisfies the following properties,

(a). the identity $\mathrm{id} : S \to S$ induces $\mathrm{id} : P_S \to P_S$

(b). to each commutative triangle of $R$-algebras we have a commutative triangle of modules.

Now let $\mathbb{P}'(R)$ denote the category of big $R$-modules and $\mathbb{P}'(R) \to \mathbb{P}(R)$ be the forgetful functor sending $P$ to $P_R$. (FINISH THIS)

# 3 Chapter 1.3

*Remark.* Here $R$ is a commutative (unital) ring.

### 3.1 3.1

We need to show that the following are equivalent properties of an $R$-module $L$,

(a). there is an $R$-module $M$ such that $L \otimes M \cong R$

(b). $L$ is an algebraic line bundle (a f.g. projective module of constant rank 1)

(c). $L$ is a finitely generated $R$-module and $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ for each prime $\mathfrak{p}$.

*Proof.* Assuming (a) then by 2.10 we have $L$ and $M$ are finitely generated projective. Thus $L_\mathfrak{p} \cong R_\mathfrak{p}^n$ and $M_\mathfrak{p} \cong R_\mathfrak{p}^m$ for some $n, m$ but then $L_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p} \cong R_\mathfrak{p}^{mn}$ so $m = n = 1$ proving (b).

(b) $\implies$ (c) is a trivial consequence of Lemma 2.4.

Finally assume (c) then I claim that $L \otimes_R L^\vee \cong R$ where $L^\vee = \mathrm{Hom}_R(L, R)$. First, not there is a natural map $L \otimes L^\vee \to R$ by evaluation. We may check this map is an isomorphism locally on stalks,

$$L_\mathfrak{p} \otimes \mathrm{Hom}_{R_\mathfrak{p}}(L_\mathfrak{p}, R_\mathfrak{p}) \to R_\mathfrak{p}$$

(note that $(\mathrm{Hom}_R(L, R))_\mathfrak{p} = \mathrm{Hom}_{R_\mathfrak{p}}(L_\mathfrak{p}, R_\mathfrak{p})$ holds since $L$ is finitely presented which holds because it is f.g. projective using criterion (4) proved in 2.11). However, $L_\mathfrak{p} \cong R_\mathfrak{p}$ so this above map is clearly an isomorphism with $1 \otimes \mathrm{id} \mapsto 1$. □

## 3.2 3.4

## 3.3 3.15

## 3.4 3.18

Consider the following sequence,

$$1 \longrightarrow \mathrm{Pic}\,(R) \longrightarrow \mathrm{Pic}\,(R[t]) \times \mathrm{Pic}\,(R[t^{-1}]) \longrightarrow \mathrm{Pic}\,(R[t, t^{-1}])$$

the first map is induced by the inclusions $R \to R[t]$ and $R[t^{-1}]$ and the second by the difference of the maps induced by the inclusion $R[t] \to R[t, t^{-1}]$ and $R[t^{-1}] \to R[t, t^{-1}]$. Since $\mathrm{Pic}\,(-)$ is a covariant functor on the category of commutative rings the above sequence is a complex since,

$$R \longrightarrow R[t] \times R[t^{-1}] \longrightarrow R[t, t^{-1}]$$

is exact (this is the computation showing that $\Gamma(\mathbb{P}^1_R, \mathcal{O}_{\mathbb{P}^1_R}) = R$).

Now, given $P \in \mathrm{Pic}\,(R[t])$ and $Q \in \mathrm{Pic}\,(R[t^{-1}])$ suppose that $P \otimes_{R[t]} R[t, t^{-1}]$ and $Q \otimes_{R[t^{-1}]} R[t, t^{-1}]$ are isomorphic as $R[t, t^{-1}]$-modules.

(USE UNITS-PIC sequence and snake lemma)

# 4 Chapter 2.1

## 4.1 2.1

## 4.2 2.2

## 4.3 2.3

# 5 Chapter 2.3

## 5.1 3.3

## 5.2 3.4

## 5.3 3.5

## 5.4 3.7

# 6 Chapter 2.4

## 6.1 4.2

# 7 Chapter 2.5

## 7.1 5.2

## 7.2 5.7

## 7.3 5.8

# 8 Chapter 2.6-8

# 9 Appendix A. Rank Functions

*Remark.* Here $R$ is a commutative (unital) ring.

**Definition 9.1.** Let $M$ be an $R$-module. Then there is a function $\text{rank}(M) : \text{Spec}\,(R) \to \mathbb{Z}$ defined by $x \mapsto \text{rank}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}))$.

**Proposition 9.2.** $\text{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators of $M_p$ as an $R_{\mathfrak{p}}$-module.

*Proof.* If $M_{\mathfrak{p}}$ is generated by $m_1, \dots, m_n$ then $M_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{p}) = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is generated by $\bar{m}_1, \dots, \bar{m}_n$ over $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ since surjectivity of $R_{\mathfrak{p}}^n \to M_{\mathfrak{p}}$ is preserved after applying $(-) \otimes_R \kappa(\mathfrak{p})$. Thus, $\text{rank}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})} M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) \le n$.

Now suppose that $v_1, \dots, v_n$ is a $\kappa(\mathfrak{p})$-basis of $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ then choose a lifts $m_1, \dots, m_n \in M_{\mathfrak{p}}$. I claim that $m_1, \dots, m_n$ generated $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module. Let $N \subset M_{\mathfrak{p}}$ be the $R_{\mathfrak{p}}$-submodule generated by the $m_1, \dots, m_n$ and let $K = M_{\mathfrak{p}}/N$. Then I claim that $\mathfrak{p}K = K$. To see this it suffices to show that $K \subset \mathfrak{p}K$. For any $m \in M_\mathfrak{p}$ we know that its image $\bar{m} \in M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is in the span of the basis $v_1, \dots, v_n$ so,

$$\bar{m} = r_1 v_1 + \cdots r_n v_n$$

for $r_i \in R_\mathfrak{p}$. Thus,

$$m - (r_1 m_1 + \cdots r_n m_n) \in \mathfrak{p}M$$

This implies that in $K$ we have $m \in \mathfrak{p}K$ so $K = \mathfrak{p}K$. Then since $\mathrm{Jac}\,(R_\mathfrak{p}) = \mathfrak{p}$ (because $R_\mathfrak{p}$ is local) by Nakayama $K = 0$ so $M_\mathfrak{p}$ is generated by $m_1, \dots, m_n$. □

**Theorem 9.3.** The following are equivalent:

(a). $M$ is a finitely-generated projective $R$-module

(b). $M$ is a locally-free $R$-module of finite rank $\mathrm{rank}_x(M) < \infty$

(c). $M$ is a finitely-presented $R$-module and for each $\mathfrak{p} \in \mathrm{Spec}\,(R)$, $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module.

**Proposition 9.4.** If $P$ is a finitely-generated projective module then $\mathrm{rank}(P) : \mathrm{Spec}\,(R) \to \mathbb{Z}$ is continuous.

*Proof.* It suffices to prove for $f = \mathrm{rank}(P)$ that $f^{-1}(n) = V$ is open. For any $\mathfrak{p} \in V$ we know that $P_\mathfrak{p}$ is free of rank $n$. Lift a basis (by clearing demoninators) to a map $f : R^n \to P$ and consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \xrightarrow{f} P \longrightarrow \mathrm{coker}\, f \longrightarrow 0$$

Since $P$ is fintely generated then $\mathrm{coker}\, P$ is also finitely generated. Localizing this exact sequence at $\mathfrak{p}$ we get an exact sequence,

$$0 \longrightarrow (\ker f)_\mathfrak{p} \longrightarrow R^n_\mathfrak{p} \xrightarrow{f} P_\mathfrak{p} \longrightarrow (\mathrm{coker}\, f)_\mathfrak{p} \longrightarrow 0$$

but $f : R^n_\mathfrak{p} \to P_\mathfrak{p}$ is an isomorphism so $(\mathrm{coker}\, f)_\mathfrak{p} = \ker f_\mathfrak{p} = 0$. Since $\mathrm{coker}\, f_\mathfrak{p}$ is finitely generated there is some $g \notin \mathfrak{p}$ such that $\mathrm{coker}\, f_\mathfrak{p} = 0$. Thus we have.

$$0 \longrightarrow (\ker f)_g \longrightarrow R^n_g \longrightarrow P_g \longrightarrow 0$$

We have yet to use projectivity of $P$ so, in particular, we see that $\forall \mathfrak{q} \in D(g) : \mathrm{rank}_\mathfrak{q}(M) \le n$ for any finitely-generated $R$-module $M$. We call this upper-semicontinuity of $\mathrm{rank}(M) : \mathrm{Spec}\,(R) \to \mathbb{Z}$.

Now applying projectivity of $P$ (and thus $P_g$ as a $R_g$-module) the above exact sequence splits to give,

$$R^n \cong P_g \oplus \ker f_g$$

so the projection $R^n \twoheadrightarrow \ker f_g$ shows that $\ker f_g$ is finitely generated and $(\ker f_g)_\mathfrak{p} = 0$ so there is some $h \notin \mathfrak{p}$ such that $\ker f_{gh} = 0$. Then, by exactness of localization we get $R^n_{gh} \xrightarrow{\sim} P_{gh}$ so $P$ is free of rank $n$ on $D(gh)$ and thus $\forall \mathfrak{q} \in D(gh) : \mathrm{rank}_\mathfrak{q}(P) = n$ so $\mathfrak{p} \in D(gh) \subset V$. Therefore, $V$ is open so this function is continuous. □

**Definition 9.5.** Let $X$ be a scheme and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module then there is a function $\mathrm{rank}(\mathscr{F}) : X \to \mathbb{Z}$ defined by $x \mapsto \mathrm{rank}_x(\mathscr{F}) = \dim_{\kappa(x)}(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x))$.

*Remark.* Since $\mathscr{F}$ is coherent then locally $\mathscr{F}|_U = \widetilde{M}$ for some finitely generated $A$-module with $U = \operatorname{Spec}(A)$. (Note that this is necessary for coherence but only sufficient when $X$ is locally noetherian). Thus, $\mathscr{F}_x$ is a finitely-generated $\mathcal{O}_{X,x}$-module and thus $\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is finite dimensional.

**Theorem 9.6.** If $\mathscr{F}$ is a projective coherent $\mathcal{O}_X$-module then $\operatorname{rank}(\mathscr{F}) : X \to \mathbb{Z}$ is continuous.

*Proof.* $\square$

**Proposition 9.7.** Projective coherent $\mathcal{O}_X$-modules on a scheme $X$ are exactly locally-free $\mathcal{O}_X$-modules of finite type. (CHECK).