ASTR GR6001 Radiative Processes
Assignment # 8

Benjamin Church

November 25, 2019

1 Problem 1

Consider mono-energetic electrons of a very large γ_0 are injected at a constant rate into a region with uniform B field. Assume that there is sufficient time for these electrons to radiate away all their energy so we reach a steady state.

We have shown that the Lorentz factor for an electron emitting synchrotron radiation will decrease as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -A\gamma^2$$

where,

$$A = \frac{2e^4}{3m^2c^5}B_\perp^2$$

Let $n(\gamma)$ be the number of electrons at a given energy (and thus Lorentz factor γ). There is a "current" $n(\gamma)\dot{\gamma}$ of electrons of energy γ flowing downwards in energy. For steady state to be reached, this current must be constant else the electron density changes at some energy. Therefore,

$$\frac{\mathrm{d}n}{\mathrm{d}\gamma}\dot{\gamma} + n(\gamma)\frac{\partial \dot{\gamma}}{\partial \gamma} = 0$$

However, we know that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -A\gamma^2$$

and thus,

$$\frac{\partial \dot{\gamma}}{\partial \gamma} = -2A\gamma$$

This implies that,

$$(-A\gamma^2)\frac{\mathrm{d}n}{\mathrm{d}\gamma} - 2A\gamma\, n(\gamma) = 0$$

Thus,

$$\frac{\mathrm{d}n}{\mathrm{d}\gamma} = -2\gamma^{-1}n(\gamma)$$

and thus,

$$\frac{\mathrm{d}\log n}{\mathrm{d}\log\gamma} = -2$$

which implies that,

$$n(\gamma) = n_0 \left(\frac{\gamma}{\gamma_0}\right)^{-2}$$

so the power law of electron energies has index $p = 2$. Now we need to compute the spectrum of synchrotron radiation emitted by this distribution. This is,

$$P(\omega) = \frac{\sqrt{3}}{2\pi} \cdot \frac{e^3 B \sin\alpha}{mc^2} \int_1^{\gamma_0} n(\gamma) F(\omega/\omega_c)\, \mathrm{d}\gamma$$

where,

$$\omega_c = \tfrac{3}{2}\gamma^3 \left(\frac{eB}{\gamma mc}\right) \sin\alpha$$

We define,

$$\omega_0 = \frac{3\gamma_0^2 eB}{2mc} \sin\alpha$$

and thus,

$$\omega_c = \left(\frac{\gamma}{\gamma_0}\right)^2 \omega_0$$

Therefore, we can write this integral in terms of,

$$x = \frac{\omega}{\omega_c} = \frac{\omega}{\omega_0}\left(\frac{\gamma}{\gamma_0}\right)^{-2}$$

Now,

$$\mathrm{d}x = -\frac{\omega}{\omega_0} \cdot \frac{2}{\gamma_0}\left(\frac{\gamma}{\gamma_0}\right)^{-3} \mathrm{d}\gamma$$

and thus,

$$\mathrm{d}\gamma = -\frac{\gamma_0}{2} \cdot \left(\frac{\omega}{\omega_0}\right)^{\frac{1}{2}} \cdot \frac{\mathrm{d}x}{x^{\frac{3}{2}}}$$

Therefore,

$$\begin{aligned}
P(\omega) &= \frac{\sqrt{3}}{2\pi} \cdot \frac{n_0 e^3 B \sin\alpha}{mc^2} \int_1^{\gamma_0} \left(\frac{\gamma}{\gamma_0}\right)^{-p} F(\omega/\omega_c)\, \mathrm{d}\gamma \\
&= \frac{\sqrt{3}}{2\pi} \cdot \frac{n_0 e^3 B \sin\alpha}{mc^2} \left(\frac{\omega}{\omega_0}\right)^{-\frac{p}{2}} \int_1^{\gamma_0} x^{\frac{p}{2}} F(x)\, \mathrm{d}\gamma \\
&= \frac{\sqrt{3}}{4\pi} \cdot \frac{n_0 \gamma_0 e^3 B \sin\alpha}{mc^2} \left(\frac{\omega}{\omega_0}\right)^{-\frac{p-1}{2}} \int_{x_1}^{x_2} x^{\frac{p-3}{2}} F(x)\, \mathrm{d}x
\end{aligned}$$

If ω_0 is very large then x_1 and x_2 can be taken to be approximately constant in ω and thus the integral is constant. Therefore, P satisfies a power law,

$$P(\omega) \propto \left(\frac{\omega}{\omega_0}\right)^{-\frac{p-1}{2}}$$

so the spectral index of the emitted synchrotron radiation is $s = \frac{1}{2}(p-1) = \frac{1}{2}$ since $p = 2$.

2 Problem 2

(a)

Relativistic electrons satisfy the Lorentz force law,

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\vec{v}) = e\left(E + \frac{\vec{v}}{c} \times B\right)$$

Then, for circular motion,

$$a_\perp = \frac{v_\perp^2}{r_g} = \omega_B v_\perp$$

For the case $E = 0$ then the force and the acceleration are perpendicular so,

$$a_\perp = \frac{F_\perp}{\gamma m} = \frac{ev_\perp B}{\gamma mc}$$

and $a_\parallel = 0$. Therefore,

$$\omega_B = \frac{eB}{\gamma mc}$$

and

$$r_g = \frac{\gamma mcv_\perp}{eB}$$

Now note that for a pitch angle α we have $v_\perp = v\sin\alpha$. We must have $L \gg \hbar$ for the classical approximation to be valid. The orbital angular momentum is,

$$L = \gamma mv_\perp r_g = \frac{\gamma^2 m^2 cv_\perp^2}{eB}$$

Furthermore, if the electron energy becomes too great then the electron will radiate energy at a rate much faster than the orbital period which makes synchrotron radiation possible. The relativistic Larmor formula gives,

$$P = \frac{2}{3}\frac{e^2}{c^3}\gamma^4(a_\perp^2 + \gamma^2 a_\parallel^2) = \frac{2e^2}{3c^3}\gamma^4\left(\frac{ev_\perp B}{\gamma mc}\right)^2 = \frac{2e^2}{3c^3}\gamma^2 v_\perp^2\left(\frac{eB}{mc}\right)^2$$

We must have $P \ll \omega_B E$ where $E = \gamma mc^2$ is the energy of the orbiting electron and,

$$\omega_B = \frac{eB}{\gamma mc}$$

and thus,

$$\omega_B E = \left(\frac{eB}{\gamma mc}\right)\gamma mc^2 = eBc$$

is the orbital period. Therefore,

$$\frac{2e^2}{3c^3}\gamma^2 v_\perp^2\left(\frac{eB}{mc}\right)^2 \ll eBc$$

which implies that,

$$\frac{3m^2c^6}{2e^3B} \gg \gamma^2 v_\perp^2$$

Furthermore,

$$\gamma^2 v_\perp^2 = \frac{eBL}{m^2 c}$$

and thus,

$$\frac{3m^2c^6}{2e^3B} \gg \frac{eBL}{m^2c}$$

which implies that,

$$B^2 \ll \frac{3m^4c^7}{2e^4L}$$

However, we also require $L \gg \hbar$ and thus,

$$B \ll B_{\max} = \left(\frac{3m^4c^7}{2e^4\hbar}\right)^{\frac{1}{2}} = \left(\frac{3m^2c^3}{2r_0^2\hbar}\right)^{\frac{1}{2}} = 6.32 \cdot 10^{14}\ \mathrm{G}$$

For magnetic fields greater than than maximum field, synchrotron radiation is not possible. Notice that this limit does not depend on the electron pitch angle.

(b)

Now we fix the magnetic field B and consider the maximum frequency,

$$\omega_B = \frac{eB}{\gamma mc}$$

However, recall that the peak of synchrotron radiation occurs not at a frequency of ω_B but rather at approximately,

$$\omega_C = \tfrac{3}{2}\gamma^3\omega_B \sin\alpha = \gamma^2\frac{3eB}{2mc}\sin\alpha$$

Therefore, finding maximum ω_S is equivalent to asking for the maximum value of γ for which our approximations are valid since the other parameters are fixed. We have shown that,

$$\frac{3m^2c^6}{2e^3B} \gg \gamma^2 v_\perp^2$$

Furthermore,

$$v_\perp = v\sin\alpha$$

and thus,

$$\gamma^2 v_\perp^2 = c^2(\gamma^2 - 1)\sin^2\alpha$$

Therefore, we must have,

$$\gamma^2 \ll \frac{3m^2c^4}{2e^3B\sin^2\alpha}$$

and thus,

$$\omega_C \ll \frac{9mc^3}{4e^2\sin\alpha} = \frac{9c}{4r_0\sin\alpha}$$

Therefore, we find an upper limit for the peak emission frequency of synchrotron radiation,

$$\omega_{\max} = \frac{9c}{4r_0\sin\alpha} = \left(2.39 \cdot 10^{23}\ \mathrm{s}^{-1}\right) \cdot (\sin\alpha)^{-1}$$

which notice is independent of the magnetic field B. This peak frequency corresponds to a photon energy of,

$$E_{\max} = \hbar\omega_{\max} = \frac{9\hbar c}{4r_0\sin\alpha} = (157.4\ \mathrm{MeV}) \cdot (\sin\alpha)^{-1}$$

3 Problem 3

The Crab Nebula has a uniform magnetic field of $B = 5 \times 10^{-4}$ G and a volume of 3×10^{56} cm^3.

(a)

From the formula,

$$\omega_c = \tfrac{3}{2}\,\gamma^3 \omega_B \;\sin\alpha$$

where,

$$\omega_B = \frac{eB}{\gamma mc}$$

Assuming the electron velocities are randomly distributed in angle, we can average $\sin\alpha$ over a sphere to find,

$$\langle \sin\alpha \rangle = \frac{1}{4\pi} \int_0^\pi \sin\alpha (2\pi \sin\alpha \; \mathrm{d}\alpha\;) = \frac{\pi}{4}$$

Therefore,

$$\omega_c = \tfrac{3\pi}{8} \gamma^2 \omega_0$$

with,

$$\omega_0 = \frac{eB}{mc} = 8.79 \times 10^3 \text{ s}^{-1}$$

Thus, we find that the average electron must have a γ-factor of,

$$\gamma = \left(\frac{8}{3\pi}\right)^{\frac{1}{2}} \cdot \left(\frac{\omega_c}{\omega_B}\right)^{\frac{1}{2}}$$

Furthermore, the gyroradius,

$$r_g = \frac{\gamma mc v_\perp}{eB} = \frac{\gamma c \beta}{\omega_0}$$

For highly relativistic electrons $\beta \approx 1$ and thus,

$$r_g = \frac{\gamma c}{\omega_0} = \gamma \cdot (3.4 \cdot 10^6 \text{ cm})$$

Furthermore, the synchrotron lifetime is,

$$t_{\frac{1}{2}} = \left(\frac{2e^4 B^2}{3m^3 c^5} \cdot \gamma\right)^{-1}$$

Therefore,

$$t_{\frac{1}{2}} = (5.1 \cdot 10^8 \text{ s}) \cdot \left(\frac{B}{1\,\text{G}}\right)^{-2} \cdot \gamma^{-1}$$

For $B = 5 \cdot 10^{-4}$ G we have,

$$t_{\frac{1}{2}} = (2.0 \cdot 10^{15} \text{ s}) \cdot \gamma^{-1}$$

Now we can calculate the table,

ν (Hz)	γ	r_g (cm)	$t_{\frac{1}{2}}$ (s)
10^8	250	$8.5 \cdot 10^8$	$8.0 \cdot 10^{12}$
10^{14}	2.5×10^5	$8.5 \cdot 10^{13}$	$8.0 \cdot 10^9$
10^{22}	$2.5 \cdot 10^9$	$8.5 \cdot 10^{17}$	$8.0 \cdot 10^5$

Furthermore, the radius of the Nebula,

$$r_N = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = 4.15 \cdot 10^{18}$$

Therefore, the gyroradius for γ-rays (10^{22} Hz) is on the order of the radius of the nebula $r_N \sim 10^{18}$ cm although smaller by a factor of 5.

(b)

The spectrum of electron energies can be computed from the synchrotron emission spectrum of the nebula. The emission spectrum is computed in the notes as,

$$\mathcal{E}_\nu = (1.7 \cdot 10^{-21}\,\mathrm{erg\,cm^{-3}\,s^{-1}\,Hz^{-1}}) \left(\frac{n_0}{1\,\mathrm{cm^{-3}}}\right) a(p) \left(\frac{B}{1\,\mathrm{G}}\right)^{s+1} \cdot \left(\frac{\nu}{4 \cdot 10^6\,\mathrm{Hz}}\right)^{-s}$$

where s is the spectral index of the radiation and $a(p)$ is a numerical value depending on the index $p = 2s + 1$. Furthermore, the n_0 which appears above is the specific number density appearing in the power-law distribution for electron energy distribution,

$$n(\gamma) = n_0 \gamma^{-p}$$

and therefore, the energy in emitting electrons is,

$$E_{\mathrm{seg}} = V \int_{\gamma_{\min}}^{\gamma_{\max}} (\gamma mc^2) n(\gamma)\,\mathrm{d}\gamma = V \int_{\gamma_{\min}}^{\gamma_{\max}} (\gamma mc^2) n_0 \gamma^{-p}\,\mathrm{d}\gamma = V \frac{n_0 mc^2}{p-2} \left[\frac{1}{\gamma_{\min}^{p-2}} - \frac{1}{\gamma_{\max}^{p-2}}\right]$$

Therefore, it suffices to find p, n_0, and $\gamma_{\min}$ and $\gamma_{\max}$.

Our available information is the observed specific flux spectrum which is related to $\mathcal{E}_\nu$ via the distance to the cluster and the total volume via,

$$f_\nu = \frac{\mathcal{E}_\nu V}{4\pi d^2}$$

since $\mathcal{E}_\nu V$ is the total power emitted by the nebula. Therefore, we have,

$$n_0 = (1\,\mathrm{cm^{-3}}) \cdot \frac{4\pi d^2 f_\nu}{V} \cdot (1.7 \cdot 10^{-21}\,\mathrm{erg\,cm^{-3}\,s^{-1}\,Hz^{-1}})^{-1} a(p)^{-1} \cdot \left(\frac{B}{1\,\mathrm{G}}\right)^{-(s+1)} \cdot \left(\frac{\nu}{4 \cdot 10^6\,\mathrm{Hz}}\right)^{s}$$

Computing this we will find the total energy in emitting electrons,

$$\begin{aligned} E &= V \sum_{\text{segments}} \frac{n_0 mc^2}{p-2} \left[\frac{1}{\gamma_{\min}^{p-2}} - \frac{1}{\gamma_{\max}^{p-2}}\right] \\ &= (1\,\mathrm{cm^{-3}}) \cdot (4\pi d^2 f_\nu) \cdot (1.7 \cdot 10^{-21}\,\mathrm{erg\,cm^{-3}\,s^{-1}\,Hz^{-1}})^{-1} a(p)^{-1} \cdot \left(\frac{B}{1\,\mathrm{G}}\right)^{-(s+1)} \cdot \left(\frac{\nu}{4 \cdot 10^6\,\mathrm{Hz}}\right)^{s} \\ &\quad \cdot \sum_{\text{segments}} \frac{mc^2}{p-2} \left[\frac{1}{\gamma_{\min}^{p-2}} - \frac{1}{\gamma_{\max}^{p-2}}\right] \end{aligned}$$

We split the Crab spectrum into a broken power law with two segments in frequency, 10^7 Hz $-$ 10^{14} Hz with spectral index $s_1 \approx 0.25$ and 10^{14} Hz – 10^{24} Hz. The end points of these segments gives the minimum and maximum γ-factors and their slopes give the values of s and p. We need to do a linear (in log-log) fit to each segment. Define the parameter,

$$g = \left(\frac{f_\nu}{10^{-20}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}}} \right) \cdot \left(\frac{\nu}{4 \cdot 10^6\ \mathrm{Hz}} \right)^s$$

with s chosen to fit a segment. This number is approximately constant on each segment. In terms of the parameters, g and s and plugging in for B and d,

$$N = (2.8 \cdot 10^{45}) \left(5 \cdot 10^{-4}\right)^{-(s+1)} \cdot \frac{g}{a(p)(p-1)} \cdot \left[\frac{1}{\gamma_{\min}^{p-1}} - \frac{1}{\gamma_{\max}^{p-1}} \right]$$

From linear fits to the data (note that the data is given in $\mathrm{W\ m^{-2}\ Hz^{-1}} = 10^{-3}\ \mathrm{erg\ cm^{-2}\ Hz^{-1}}$), I find,

segment	s	p	$a(p)$	g	$\gamma_{\min}$	$\gamma_{\max}$
1	0.25	1.5	0.085	7.07	79	$2.5 \cdot 10^5$
2	1.20	3.4	0.074	$7.54 \cdot 10^7$	$2.5 \cdot 10^5$	$2.5 \cdot 10^{10}$

Therefore, for the two segments, we find,

$$N_1 = 6.9 \cdot 10^{50} \qquad N_2 = 2.4 \cdot 10^{48}$$

Therefore,

$$N_{\text{tot}} = 6.9 \cdot 10^{50}$$

4 Problem 4

Consider a compact extra-galactic radio source at $d = 300$ Mpc, with an angular diameter $\theta = 2 \cdot 10^{-3}$ arcsec with observed flux density $f_s = 1 \cdot 10^{-25}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}}$ at a synchrotron self-absorption frequency of $\nu_s = 10^8$ Hz, and a spectral index $s = 0.75$ in the optically thin regime. We assume the source is spherical and homogeneous.

(a)

At the self-absorption frequency ν_s we can compute the brightness temperature,

$$T_B = \frac{2mc^2}{k_B} \left(\frac{\nu mc}{eB} \right)^{\frac{1}{2}}$$

However, at the self-absorption limit, the emission is given by,

$$B_\nu = 4m\nu^2 \left(\frac{\nu mc}{eB} \right)^{\frac{1}{2}}$$

Therefore,

$$T_B = \frac{B_\nu c^2}{2k_B \nu^2}$$

Now, we can compute $\mathcal{E}_\nu$ from the observed flux f_ν. The total specific power is,

$$P_\nu = 4\pi R^2 (\pi B_\nu)$$

and therefore,

$$f_s = \left(\frac{R}{d}\right)^2 \pi B_\nu = \left(\frac{\theta}{2}\right)^2 \pi B_\nu$$

Thus we have,

$$T_B = \frac{f_s c^2}{2\pi k_B \nu^2} \cdot \left(\frac{\theta}{2}\right)^{-2} = 4.4 \cdot 10^{11} \text{ °K}$$

(b)

We can compute the magnetic field from the above equations since,

$$B = \left(\frac{2mc^2}{k_B T_B}\right)^2 \cdot \frac{\nu mc}{e}$$

Using the previous values,

$$B = 4.1 \cdot 10^{-3} \text{ G}$$

Then the total magnetic energy is,

$$U_{\text{mag}} = V\left(\frac{B^2}{8\pi}\right) = \frac{4\pi R^3}{3}\left(\frac{B^2}{8\pi}\right) = \frac{4\pi d^3}{3} \cdot \left(\frac{\theta}{2}\right)^3 \cdot \left(\frac{B^2}{8\pi}\right) = \left(\frac{\theta}{2}\right)^3 \cdot \left(\frac{B^2 d^3}{6}\right)$$

Plugging in,

$$U_{\text{mag}} = 2.5 \cdot 10^{50} \text{ erg}$$

(c)

The frequency of synchrotron emitting electrons is,

$$\nu = \tfrac{3}{16}\gamma^2\omega_0 \qquad \omega_0 = \frac{eB}{mc}$$

From the previous calculation of B we have,

$$\omega_0 = 7.3 \cdot 10^4 \text{ Hz}$$

At the frequency $\nu_s = 10^8$ Hz we have,

$$\gamma_s = \sqrt{\frac{16}{3}} \cdot \left(\frac{\nu_s}{\omega_0}\right)^{\frac{1}{2}} = 85.5$$

(d)

To compute the energy density in relativistic electrons,

$$\rho = \int_{\gamma_{\min}}^{\gamma_{\max}} n(\gamma)(\gamma mc^2)\,\mathrm{d}\gamma = \int_{\gamma_{\min}}^{\gamma_{\max}} n_0\gamma^{-p}(\gamma mc^2)\,\mathrm{d}\gamma = n_0 mc^2 \int_{\gamma_{\min}}^{\gamma_{\max}} \gamma^{1-p}\,\mathrm{d}\gamma = \frac{n_0 mc^2}{p-2}\left[\frac{1}{\gamma_{\min}^{p-2}} - \frac{1}{\gamma_{\max}^{p-2}}\right]$$

we need to find n_0 and p and the limits of the γ-factor. The total energy is then,

$$E = \rho V = \frac{4\pi R^3}{3}\rho = \frac{N_0 mc^2}{p-2}\left[\frac{1}{\gamma_{\min}^{p-2}} - \frac{1}{\gamma_{\max}^{p-2}}\right]$$

We assume that the emission spectrum of the compact source is a power law with spectral index $s = 0.75$ starting at $\nu_s = 10^8$ Hz and thus $\gamma_{\min} = \gamma_s = 85.5$. We assume that $\gamma_{\max}$ is much larger than $\gamma_{\min}$ so we may drop the second term. The spectral index $s = 0.75$ gives $p = 2s + 1 = 2.5$. Finally, to compute n_0 we use the measured flux and compare it to the emission,

$$\mathcal{E}_\nu = (1.7 \cdot 10^{-21}\ \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}) \left(\frac{n_0}{1\ \mathrm{cm}^{-3}}\right) a(p) \left(\frac{B}{1\ \mathrm{G}}\right)^{s+1} \cdot \left(\frac{\nu}{4 \cdot 10^6\ \mathrm{Hz}}\right)^{-s}$$

Plugging in for B and computing at ν_s,

$$\mathcal{E}_s = (8.58 \cdot 10^{-28}\ \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}) \cdot \left(\frac{n_0}{1\ \mathrm{cm}^{-3}}\right)$$

Furthermore,

$$f_s = \frac{4\pi R^3}{3} \cdot \frac{\mathcal{E}_s}{4\pi d^2} = (4\pi d^2)^{-1} \cdot (8.58 \cdot 10^{-28}\ \mathrm{erg\ s^{-1}\ Hz^{-1}}) \cdot N_0$$

Therefore,

$$N_0 = \frac{4\pi d^2 f_s}{8.58 \cdot 10^{-28}\ \mathrm{erg\ s^{-1}\ Hz}^{-1}} = 1.25 \cdot 10^{57}$$

Finally, we can compute the total energy,

$$E = \frac{N_0 m c^2}{(p-2)\gamma_{\min}^{p-2}} = 2.23 \cdot 10^{50}\ \mathrm{erg}$$

This value is very close to what we computed for the magnetic energy so the energy is in equipartition between the radiating electrons and the magnetic field.