1 Group Cohomology

Let A be an abelian group and G a group equiped with an action $G \circlearrowright A$ via group automorphisms. Then consider an extension,

$$0 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 0$$

Then we see that,

$$\text{Aut}\,(E_\alpha) \cong Z^1(G, Z) = \{\beta; G \to A \mid \beta(g_1 g_2) = \beta(g_1)^{g_1}\beta(g_2)\}$$

1.1 Functoriality

Suppose we are given $(h, f) : (G, A) \to (G', A')$ such that $f({}^g a) = {}^{h(g)} f(a)$. Then suppose we have cocycles $\alpha \in Z^2(G, A)$ and $\alpha' \in Z^2(G', A')$ then there is a diagram,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & E_\alpha \arrow[r] \arrow[d, dashed, "F"] & G \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & E_{\alpha'} \arrow[r] & G' \arrow[r] & 0
\end{tikzcd}$$

1.2 The Category

Consider the category $\mathbb{C}$ with objects (G, A, α) with $G \circlearrowright A$ and $\alpha \in Z^2(G, A)$. Then the morphisms $(G, A, \alpha) \to (G', A', \alpha')$ given by (h, f, β) where,

$$\begin{aligned} h &: G \to G' \\ f &: A \to A' \\ \beta &: G \to A' \end{aligned}$$

such that ${}^\beta f_* \alpha = h^* \alpha' \in Z^2(G, A')$. Then composition is given by,

$$(h', f', \beta') \circ (h, f, \beta) = (h' \circ h, f' \circ f, \beta'(f' * \beta))$$

There is a functor $\mathbb{C} \to \text{Ext}(G, A)$ which is fully faithful and essentially surjective but there is no canonical quasi-inverse.

Furthermore, (G, A, α) and $h : G' \to G$ then get $h^* \alpha \in Z^2(G', A)$ then $E_{h^*\alpha} = E_\alpha \times_{G,h} G'$.

Likewise given a G-linear map $f : A \to A'$ then there is a cocycle $f_* \alpha \in Z^2(G, A')$ then we get an extension $E_{f_*\alpha} = (A' \ltimes E_\alpha)/A$ where we map in A via $a \mapsto (f(a)^{-1}, a)$. The image is a normal subgroup (CHECK)

1.3 Transfer Maps

Given $H \subset G$ of finite index there is a map $V_{G,H} : G^{\text{ab}} \to H^{\text{ab}}$ defined as follows. Write as a disjoint union,

$$G = \bigsqcup_{i \in I} H g_i$$

Then define,

$$V_{G,H} : g \mapsto \prod_{i \in I} g_i g g_{i'}^{-1}$$

where $Hg_i g = Hg_{i'}$. Thus $g' g g_{i'}^{-1} \in H$ so this is a map. I claim this is well-defined up to choosing coset representations. Indeed if we replace $g_i \mapsto h_i g_i$ then we get,

$$g \mapsto \prod_{i \in I} h_i g_i g g_{i'} h_{i'}^{-1} = \left(\prod_i h_i \right) \left(\prod_{i \in I} g_i g g_{i'}^{-1} \right) \left(\prod_{i'} h_{i'}^{-1} \right) = prod_{i \in I} g_i g g_{i'}^{-1}$$

using that the image is in H^{ab} so we can commute elements. Then $V_{G,H}$ is a homomorphism also by using that the target is abelian.

If $H \triangleleft G$ then $\operatorname{im} V_{G,H} \subset (H^{\mathrm{ab}})^{G/H}$ since,

$$h \left(\prod_i g_i g g_{i'}^{-1} \right) k^{-1} = \prod_i (k g_i) g (k g_{i'})^{-1}$$

which is a new set of coset representatives and hence gives the same transer map.

Given an extension,

$$0 \longrightarrow A \longrightarrow E_\alpha \longrightarrow G \longrightarrow 0$$

Then $V_{E_\alpha, A} : E_\alpha^{\mathrm{ab}} \to A^G$ since A is abelian. This sends,

$$a \mapsto \prod_{g \in G} {}^g a = N_G(a)$$

Then it sends an image of the canonical section to,

$$e_\alpha(g) \mapsto \prod_{h \in G} e_\alpha(h) e_\alpha(g) e_\alpha(hg)^{-1} = \prod_{h \in G} \alpha(h, g)$$

Therefore it gives,

$$V_{E_\alpha, A} : E_\alpha / \langle [E_\alpha, E_\alpha], A \rangle = G^{\mathrm{ab}} \to A^G / N_G(A)$$

since A maps into the norm image. This gives the Nakayama map,

$$H^2(G, A) \to \operatorname{Hom}\left(G^{\mathrm{ab}}, A^G / N_G(A) \right)$$

defined by,

$$\alpha \mapsto \left(g \mapsto \prod_{h \in G} \alpha(h, g) \right)$$

Example 1.3.1. If L/K is a finite extension of p-adic fields and $G = \operatorname{Gal}(L/K)$ and $A = L^\times$ then

$$\operatorname{inv} : H^2(G, A) \xrightarrow{\sim} \frac{1}{[L : K]} \mathbb{Z}/\mathbb{Z}$$

gives a fundamental class $\mathfrak{a}_{L/K} \in H^2(G, A)$ as the preimage of the canonical generator. Then the transfer map,

$$V_{E,A} : G^{\mathrm{ab}} \to K^\times / N_G(K^\times)$$

is the inverse of the Artin map.

2 Local Fields

Let $F/\mathbb{Q}_p$ be finite and $\bar{F}$ the algebraic closure of F.

2.1 Local Class Field Theory

$H^2(I_{\bar{F}/F}, \bar{F}^\times) = 0$.

There is a spectral sequence $H^i(\hat{\mathbb{Z}}, H^j(I_{\bar{F}/F}, M)) \implies H^i(\bar{F}/F, M)$ using that the quotient $\mathrm{Gal}\left(\bar{F}/F\right)/I_{\bar{F}/F} = \mathrm{Gal}\left(F^{\mathrm{ab}}/F\right) = \hat{\mathbb{Z}}$ generated by Frobenius. Then we can compute,

$$\begin{tikzcd}
H^2(F^{\mathrm{nr}}/F, F^{\mathrm{nr},\times}) \arrow[r, "\sim"] \arrow[d] & H^2(\bar{F}/F, \bar{F}^\times) \\
H^2(F^{\mathrm{nr}}/F, \mathbb{Z}) & H^1(F^{\mathrm{nr}}/F, \mathbb{Q}/\mathbb{Z}) \arrow[l, "\sim"'] \arrow[d, equal] \\
 & \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{nr}}/F\right), \mathbb{Q}/\mathbb{Z}\right) \arrow[d, equal] \\
 & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Get the invariant map,

$$\mathrm{inv}_F : H^2(\bar{F}/F, \bar{F}^\times) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

Then from the Kummer sequence we can compute,

$$H^i(\bar{F}/F, \mu_n(\bar{F})) = \begin{cases} \mu_n(F) & i = 0 \\ F^\times/(F^\times)^n & i = 1 \\ \frac{1}{n}\mathbb{Z}/\mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

Then $H^i(\bar{F}/F, M)$ is finite if $\#M < \infty$. Then define $M^* = \mathrm{Hom}\left(M, \mu_\infty(\bar{F}\right)$ and there is a perfect pairing,

$$H^i(\bar{F}/F, M) \times H^{2-i}(\bar{F}/F, M^*) \xrightarrow{\sim} H^2(\bar{F}/F, \mu_\infty(\bar{F})) \xrightarrow{\mathrm{inv}_F} \mathbb{Q}/\mathbb{Z}$$

Then we get,

$$\mathrm{Hom}\left(\mathrm{Gal}\left(\bar{F}/F\right), \mathbb{Z}/n\mathbb{Z}\right) \times H^1(\bar{F}/F, \mu_n) \to \mathbb{Q}/\mathbb{Z}$$

which gives an isomorphism,

$$F^\times/(F^\times)^n = H^1(\bar{F}/F, \mu_n) \xrightarrow{\sim} \mathrm{Hom}\left(\mathrm{Hom}\left(\mathrm{Gal}\left(\bar{F}/F\right), \mathbb{Z}/n\mathbb{Z}\right), \mathbb{Q}/\mathbb{Z}\right) \xrightarrow{\sim} \mathrm{Gal}\left(\bar{F}/F\right)^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$

the taking the limit gives the Artin map,

$$\mathrm{Art}_F : \widehat{F^\times} \xrightarrow{\sim} \mathrm{Gal}\left(\bar{F}/F\right)^{\mathrm{ab}}$$

2.2 Functoriality

For E/F finite of degree n consider,

$$\begin{tikzcd}
H^2(\bar{F}/F, \bar{F}^\times) \arrow[r, "\mathrm{inv}_F"] \arrow[d, "\mathrm{Res}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, shift left, "n"] \\
H^2(\bar{E}/E, \bar{E}^\times) \arrow[r, "\mathrm{inv}_E"] & \mathbb{Q}/\mathbb{Z} \arrow[u, shift left, "1"]
\end{tikzcd}$$

Note here $\bar{E} = \bar{F}$ so there is no distiction between writing the other in the second group. If E/F is also Galois then from Inflation-Restriction,

$$\begin{tikzcd}
0 \arrow[r] & H^2(E/F, E^\times) \arrow[r] & H^2(\bar{F}/F, \bar{F}^\times) \arrow[r] \arrow[d] & H^2(\bar{E}/E, \bar{E}^\times) \arrow[d] \\
& & \mathbb{Q}/\mathbb{Z} \arrow[r, "n"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Consider the canonical class $\mathfrak{a}_{E/F} \in H^2(E/F, E^\times)$ given by $\mathrm{inv}_F^{-1}(1)$ in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Forthermore, for $\sigma : \bar{F} \xrightarrow{\sim} \bar{F}'$ with $\sigma F = F'$ we have a diagram,

$$\begin{tikzcd}
H^2(\bar{F}/F, \bar{F}^\times) \arrow[r, "\mathrm{inv}_F"] \arrow[d, "\sigma"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
H^2(\bar{F}'/F', \bar{F}'^\times) \arrow[r, "\mathrm{inv}_{F'}"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

with the map induced by σ on coefficients and conjugation by σ^{-1} on groups and this is an isomorphism.

For the Artin map,

$$\begin{tikzcd}
\widehat{F^\times} \arrow[r, "\sim"] \arrow[d, two heads, "v_F"] & \mathrm{Gal}\left(\bar{F}/F\right)^{\mathrm{ab}} \arrow[d, two heads, "v_F"] \\
\hat{\mathbb{Z}} \arrow[r, equal] & \hat{\mathbb{Z}}
\end{tikzcd}$$

commutes. Therefore $\mathrm{Art}_F : F^\times \xrightarrow{\sim} W_{\bar{F}/F}^{\mathrm{ab}}$ taking the pulback along $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$. Then the functoriality says that, if E/F is finite then we have a diagram,

$$\begin{tikzcd}
\widehat{F^\times} \arrow[r, "\mathrm{Art}_F"] \arrow[d, shift left] & \mathrm{Gal}\left(\bar{F}/F\right)^{\mathrm{ab}} \arrow[d, shift right] \\
\widehat{E^\times} \arrow[r, "\mathrm{Art}_E"] \arrow[u, shift left, "N_{E/F}"] & \mathrm{Gal}\left(\bar{E}/E\right)^{\mathrm{ab}} \arrow[u, shift right, "V"']
\end{tikzcd}$$

If E/F is also Galois then we find,

$$\mathrm{Art}_F : F^\times / N_{E/F}(E^\times) \xrightarrow{\sim} \mathrm{Gal}\left(()\, E/F\right)^{\mathrm{ab}}$$

Lemma 2.2.1 (Nakayama)**.** For E/F finite Galois of degree n,

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z} \cong H^2(E/F, E^\times) \to \mathrm{Hom}\left(\mathrm{Gal}\left(E/F\right)^{\mathrm{ab}}, F^\times / N_{E/F}(E^\times)\right.,$$