1 Remedial Curve Theory

1.1 Geometric Irreducibility of Generic Fibers

Lemma 1.1.1 (Tag 0553)**.** Let $f : X \to Y$ be a morphism of schemes. Assume,

(a) Y is irreducible with generic point η,

(b) X_η is geometrically irreducible

(c) f is of finite type

then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibers.

Lemma 1.1.2. Let $f : X \to Y$ be a morphism of schemes. Suppose that,

(a) X and Y are integral

(b) X is normal

(c) the fibers of f are geometrically connected (e.g. $f_* \mathcal{O}_X = \mathcal{O}_Y$)

then the generic fiber $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically irreducible.

Proof. $X_\eta / \kappa(\eta)$ is geometrically irreducible iff $\kappa(\eta)$ is separable closed in $\kappa(\xi)$. This follows from Tag 054Q and Tag 0G33. Let $\alpha \in \kappa(\xi)$ be separably algebraic over $\kappa(\eta)$ i.e. a root of a separable polynomial $p \in \kappa(\eta)[x]$. There is a coordinate ring A of Y where all the denominators of p are invertible. We claim that $A[\alpha] \subset B$ where B is any coordinate ring of X containing A. Indeed, α is integral over A and hence over B so by normality $\alpha \in B$ so we get morphisms,

$$X_A \to \mathrm{Spec}\,(A[\alpha]) \to \mathrm{Spec}\,(A)$$

but the fibers of $X_A \to \mathrm{Spec}\,(A)$ are geometrically connected so we must have $\alpha \in A$ since otherwise the fibers of $\mathrm{Spec}\,(A[\alpha]) \to \mathrm{Spec}\,(A)$ and hence $X_A \to \mathrm{Spec}\,(A)$ are not geometrically irreducible. $\square$

Remark. If we only assumed that X/k is geometrically irreducible (which is weaker than X being normal) the result would not follow. Indeed, consider,

$$X = \mathrm{Proj}\left(k[t][X,Y,Z]/(X^2 - tY^2)\right) \to \mathrm{Spec}\,(k[t]) = Y$$

where k is algebraically closed. Then X and Y are geomtrically integral since they are integral. Indeed, we need to check that the polynomials on the charts,

$$\left(\frac{X}{Z}\right)^2 - t\left(\frac{X}{Y}\right)^2 \qquad \left(\frac{X}{Y}\right)^2 - t \qquad 1 - t\left(\frac{Y}{X}\right)^2$$

are irreducible. They are since t does not admit a square root. However, the generic fiber is,

$$X = \mathrm{Proj}\left(k(t)[X,Y,Z]/(X^2 - tY^2)\right) \to \mathrm{Spec}\,(k(t))$$

is not geometrically irreducible since after the extension $k(t^{\frac{1}{2}})/k(t)$ we can split the polynomial. However, X is not normal since $t^{\frac{1}{2}}$ is in the fraction ield (look at the second chart) but not in every chart since $H^0(X, \mathcal{O}_X) = k[t]$ and this does not contain $t^{\frac{1}{2}}$. The normalization of X is $\mathbb{P}^1 \times \mathrm{Spec}\left(k[t^{\frac{1}{2}}]\right)$ with the map,

$$[T_0 : T_1] \to [t^{\frac{1}{2}} T_0 : T_0 : T_1]$$

This "hits both branches" since $t^{\frac{1}{2}}$ "remembers which branch of the suqare root it is on" while still making $\widetilde{X}$ an integral scheme as it must be since it is the normalization of an integral schemes.

Remark. When the base has $\dim Y = 1$ and is over a perfect field then we can also ensure that the generic fiber is geometrically integral.

Proposition 1.1.3. Let $f : X \to Y$ be a proper morphism of schemes. Let X, Y be integral and finite type over a perfect field k. If X is normal and $\dim Y = 1$ then the following are equivalent,

(a) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically integral

(b) $\kappa(\eta)$ is algebraically closed in $\kappa(\xi)$

(c) $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism.

Proof. Lemma 7.2 of Badescu. □

Example 1.1.4. If the base has dimension > 1 this is false. For example,

$$X = \mathrm{Proj}\,(\mathbb{F}_p[s,t][X,Y,Z]/(X^p + sY^p + tZ^p)) \to \mathrm{Spec}\,(\mathbb{F}_p[s,t]) = Y$$

satisfies $f_* \mathcal{O}_X = \mathcal{O}_Y$ and X is normal but the generic fiber,

$$X = \mathrm{Proj}\,(\mathbb{F}_p(s,t)[X,Y,Z]/(X^p + sY^p + tZ^p)) \to \mathrm{Spec}\,(\mathbb{F}_p(s,t))$$

is not geometrically reduced. Indeed, alhough $\mathbb{F}_p(s,t)$ is algebraically closed in,

$$\mathrm{Frac}\,(\mathbb{F}_p(s,t)[x,y]/(x^p + sy^p + t))$$

it is not separable since separability implies reducedness fo the base change by the field extension $\mathbb{F}_p(s^{\frac{1}{p}}, t^{\frac{1}{p}})$.

Remark. Note that if X is any of,

(a) reduced

(b) integral

(c) normal

(d) regular

then the same is true of X_η for any map $f : X \to Y$ by localization. However, unlike the case for irreducibiliy above, the corresponding geometric versions do *not* hold as the following and previous examples show.

Example 1.1.5. Quasi-elliptic fibrations $\mathrm{Bl}\mathbb{P}^2 \to \mathbb{P}^1$ have fibers which are not geometrically normal or regular.

Theorem 1.1.6 (Fujita, 1982)**.** Let $f : X \to Y$ be a proper dominant morphism of integral locally noetherian schemes. Consider the following properties,

(a) $\kappa(\xi_Y)$ is algebraically closed in $\kappa(\xi_X)$

(b) $\text{rank}_Y(f_* \mathcal{O}_X) = 1$

(c) the general fiber satisfies $h^0(X_y, \mathcal{O}_{X_y}) = 1$

(d) $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism.

Then the following implications hold,

$$\begin{tikzcd}
(a) \arrow[r] & (b) \arrow[l, bend right, "X \text{ normal}"'] \arrow[r, bend left, "Y \text{ normal}"] \arrow[d, leftrightarrow] & (d) \arrow[l] \\
& (c) &
\end{tikzcd}$$

Proof. DO IT!!! $\square$

Example 1.1.7. Consider,

$$X = \text{Proj}\,(k[t][X, Y, Z]/(X^p + cY^p + tZ^p)) \to \text{Spec}\,(k[t])$$

where $c \in k$ is not a p^{th}-power. Then X_η is a smooth genus $\frac{(p-1)(p-2)}{2}$ curve but X_0 is integral and $H^0(X_0, \mathcal{O}_{X_0}) = k$ but X_0 is not geometrically reduced. The arithmetic genus is still constant but the geometric genus drops to zero.

1.2 Genera of Curves

Definition 1.2.1. A *curve* C over k is a separated finite type scheme over k of pure dimension 1.

Definition 1.2.2. Let X be a proper curve over k. The *arithmetic genus* of X is,

$$p_a(X/k) := \dim_k H^1(X, \mathcal{O}_X)$$

If $H^0(X, \mathcal{O}_X) = K$ is a field then we write,

$$p_a(X) := \dim_K H^1(X, \mathcal{O}_X)$$

Remark. The arithmetic genus is stable under field extension by flat base change. However, if X admits $X \to \text{Spec}\,(k') \to \text{Spec}\,(k)$ then the arithmetic genus of X viewed over k is $[k' : k]$ times the arithmetic genus of X viewed over k'. The point of the second definition is that when it it applies the base field is unambigious.

Definition 1.2.3. Let X be a curve which is a disjoint union of finitely many smooth proper curves over an algebraically closed field k. Then the *geometric genus* (or just *genus*) of X is,

$$g(X) := p_a(X/k) = \sum_{i=1}^{n} p_a(C_i/k)$$

Definition 1.2.4. Let X be a proper curve over a field k. Consider $\widetilde{X}$ which is the normalization of $(X_{\bar{k}})_{\text{red}}$. This is a disjoint union of finitely many smooth proepr curves C_i over $\bar{k}$. Thus we can define,

$$g(X/k) := g(\widetilde{X})$$

If $H^0(X, \mathcal{O}_X) = K$ is a field then we set,

$$g(X) := g(X/k)$$

Remark. The geometric genus is stable under field extension by definition. However, notice that $g(X/k)$ does depend on the base field. If X admits $X \to \operatorname{Spec}(k') \to \operatorname{Spec}(k)$ then the geometric genus of X viewed over k is $[k' : k]$ times the geometric genus of X viewed over k'. The point of the second definition is that when it it applies the base field is unambigious.

Proposition 1.2.5. The geometric genus is a birational invarian of proper curves over k.

Proof. This is almost by definition. Let $f : X \dashrightarrow Y$ be a birational map of curves meaning there is a dense open on which it becomes an isomorphism. Then by functoriality this gives a birational map $f : \widetilde{X} \dashrightarrow \widetilde{Y}$ which is an isomorphism since both sides are collections of regular curves over $\bar{k}$. Hence $g(X) = g(Y)$. $\square$

Lemma 1.2.6. Let $f : X \to Y$ be a nonconstant map of proper regular curves over an algebrcially closed field k. Then $g(X) \geq g(Y)$.

Proof. Riemann-Hurwitz and Frobenius tricks [Hartshorne, Chapter IV] $\square$

Proposition 1.2.7. Let $f : X \to Y$ be a dominant map of proper curves over a field k. Then $g(X/k) \geq g(Y/k)$.

Proof. By definition, we set $\widetilde{X}$ to be the normalization of $(X_{\bar{k}})_{\text{red}}$ and then $g(X/k) = g(\widetilde{X})$. Then the induced map $f : \widetilde{X} \to \widetilde{Y}$ is also surjective since it is dominant (because this is preserved by base change and reduction and normalization) and proper. Therefore, each component of $\widetilde{Y}$ is hit by some component of $\widetilde{X}$ so we reduce to the previous lemma and conclude,

$$g(X/k) \geq g(Y/k)$$

$\square$

Example 1.2.8. Say $E = \operatorname{Proj}(\mathbb{R}[X, Y, Z]/(Y^2Z - X^3 - xZ^2))$ is an elliptic curve over $\mathbb{R}$. It is important that we consider the genus of $E_{\mathbb{C}}$ *as a curve over* $\mathbb{R}$ as 2 and not 1 because,

$$X = \operatorname{Proj}\left(\mathbb{R}[X, Y, Z]/((Y^2Z - X^3)^2 + (XZ^2)^2)\right)$$

has normalization $E_{\mathbb{C}}$. However, X has genus 2 since $H^0(X, \mathcal{O}_X) = \mathbb{R}$ so we must view it over $\mathbb{R}$ and to compute its genus we base change to $X_{\mathbb{C}}$ then our definition will give genus 2. If we want the map $E_{\mathbb{C}} \to X$ to satisfy the above lemma we must have $g(E_{\mathbb{C}}/\mathbb{R}) = 2$.

Corollary 1.2.9. Let $f : X \to Y$ be a dominant map of proper curves over k with,

$$k \to H^0(Y, \mathcal{O}_Y) \to H(X, \mathcal{O}_X)$$

isomorphisms. Then $g(X) \geq g(Y)$.

Remark. The above example shows that the assumption on the fields is necessary.

1.3 Birational Maps of Curves and the Relationship Between Genera

Lemma 1.3.1. Let C, S be proper integral curves over k which are birational over k. Suppose that S is regular. Let $k_C = H^0(C, \mathcal{O}_C)$ and $k_S = H^0(S, \mathcal{O}_S)$. Then the genera satisfy,

(a) $g(C) = g(S)$

(b) $p_a(C) \ge p_a(S)$

and if one of the following holds,

(a) $p_a(C) = p_a(S)$ with $p_a(S) > 0$

(b) $p_a(C) = p_a(S) = 0$ and $k_C = k_S$

then $C \cong S$ so C is regular.

Proof. We have already seen that g is a birational invariant for all curves. Now focus on p_a. Given a birational map $S \overset{\sim}{\dashrightarrow} C$ we can extend it to a birational morphism $S \to C$ since S is regular. The morphism $f : S \to C$ is automatically finite since it is a non-constant map of proper curves. In particular, f is affine so for each $y \in C$ we may choose an affine open $y \in V \subset C$ whose preimage $U = f^{-1}(V)$ is also affine. On sheaves, this gives a map of domains $\mathcal{O}_C(V) \to \mathcal{O}_S(U)$ which localizes to an isomorphism on the fraction fields. However, the localization map of a domain is injective so $\mathcal{O}_C(V) \hookrightarrow \mathcal{O}_S(U)$ is an injection. This shows that $\mathcal{O}_C \to f_*\mathcal{O}_S$ is an injection of sheaves which is generically an isomorphism. Extending to an exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_*\mathcal{O}_S \longrightarrow \mathscr{C} \longrightarrow 0$$

where $\dim \mathrm{Supp}\,(\mathscr{C}) = 0$ and hence $H^1(C, \mathscr{C}) = 0$. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, \mathscr{C}) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow 0$$

Therefore,

$$p_a(C/k) \ge p_a(S/k)$$

and the fields $k_C = H^0(C, \mathcal{O}_C)$ and $k_S = H^0(S, \mathcal{O}_S)$ satisfy $k_C \hookrightarrow k_S$. Therefore, dividing by the respective degrees of the extensions gives,

$$p_a(C/k_C) \ge p_a(S/k_S)$$

since $[k_C : k] \le [k_S : k]$. Now if $p_a(C) = p_a(S)$ and are nonzero then the two constituent inequalities are equalities meaning $p_a(C/k) = p_a(S/k)$ and $k_C = k_S$. Indeed whenever this holds, the extact sequence shows that $H^0(C, \mathscr{C}) = 0$ so $\mathscr{C} = 0$ since it is supported on an affine scheme. Therefore $f : S \to C$ is an isomorphism since it is affine and $\mathcal{O}_C \xrightarrow{\sim} f_*\mathcal{O}_S$ is an isomorphism. ☐

Example 1.3.2. The normalization map,

$$\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Proj}\left(\mathbb{R}[X, Y, Z]/(X^2 + Y^2)\right)$$

gives an example where $p_a(C) = p_a(S) = 0$ but the map is not an isomorphism since $k_C \hookrightarrow k_S$ is not an isomorphism.

Proposition 1.3.3. Let C be a proper integral curve over k. Then $g(C) \le p_a(C)$. If C is smooth, this is an equality. If equality holds and C is geometrically reduced then C is smooth.

Proof. Change the field such that $k = H^0(C, \mathcal{O}_C)$. Then $p_a(C) = p_a(C_{\bar{k}})$ by flat base change. Then $\widetilde{C} \to (C_{\bar{k}})_{\text{red}}$ satisfies the above hypotheses so,

$$g(C) = p_a(\widetilde{C}/\bar{k}) \le p_a((C_{\bar{k}})_{\text{red}}/\bar{k}) \le p_a(C_{\bar{k}}/\bar{k}) = p_a(C)$$

Now if C is smooth then so is $C_{\bar{k}}$ so the above are equalities. Now if $g(C) = p_a(C)$ the above are equalities. This implies that $\widetilde{C} \xrightarrow{\sim} (C_{\bar{k}})_{\text{red}}$ is an isomorphism so if C is geometrically reduced then $C_{\bar{k}} \cong \widetilde{C}$ and hence C is smooth. $\square$

Example 1.3.4. If C is not geometrically reduced then we may have $g(C) = p_a(C)$ but C not smooth. For example,

$$C = \operatorname{Proj}\left(\mathbb{F}_p(s,t)[X,Y,Z]/(X^p + sY^p + tZ^p)\right)$$

with $p = 2$ then $p_a(C) = 0$ since it is a conic in $\mathbb{P}^2$ and hence $g(C) = 0$ but $C_{\bar{k}}$ is a nonreduced double line.

Remark. There is a lot more that can be said about curves satisfying $g(C) = p_a(C)$ without extra assumptions for example,

(a) Ji and Waldron

(b) Schröer

1.4 Degenerations of Curves

Notation: let $(R, \mathfrak{m}, \kappa)$ be a DVR with fraction field $K = \operatorname{Frac}(R)$. Let $S = \operatorname{Spec}(R)$. For $X \to S$ let $X_\eta = X_K$ be the generic fiber and let $X_s = X_\kappa$ the special fiber.

Definition 1.4.1. A *degeneration of curves* is a proper flat family $X \to S = \operatorname{Spec}(R)$ over a DVR R where X_η is an integral normal projective curve over $K = \operatorname{Frac}(R)$. If X is normal we say that X is a *model* of X_η over R.

Lemma 1.4.2. The total space X of a degeneration of curves is integral.

Proof. We need to show that every affine open $\operatorname{Spec}(A) = U \subset X$ has A a domain. Indeed, $R \to A$ is flat so $A \hookrightarrow A_K$ is injective but A_K is an affine open of X_K which in integral so A_K and hence A is a domain. $\square$

Lemma 1.4.3. Let $f : X \to Y$ be a proper flat map of integral schemes with Y normal. Then the following are equivalent,

(a) $f_* \mathcal{O}_Y = \mathcal{O}_Y$

(b) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = \kappa(\eta)$

Proof. Indeed, $f_*\mathcal{O}_X$ is a finite $\mathcal{O}_Y$-algebra and since X is integral it is a sheaf of domains. We need to show that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism which is a local question so we reduce to $\operatorname{Spec}(A) \subset Y$ and $\operatorname{Spec}(B) \subset X$ such that $A \to B$. Then we have maps $A \to (f_*\mathcal{O}_X)(A) \to B$ and $A \to B$ is flat hence injective since they are domains. Hence $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective. Furthermore, by flat base change,

$$H^0(X_\eta, \mathcal{O}_{X_\eta}) = (f_*\mathcal{O}_X)_\eta$$

so if (b) holds then $(f_*\mathcal{O}_X)_\eta = \kappa(\eta)$. Since $\mathcal{O}_Y$ is normal and $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_Y$ we see that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism since it is contained in the fraction field. □

Proposition 1.4.4. Let $X \to S$ be a degeneration of curves. Consider the following properties,

(a) $X_\eta \to \operatorname{Spec}(\kappa(\eta))$ is geometrically integral

(b) $X_\eta \to \operatorname{Spec}(\kappa(\eta))$ is geometrically irreducible

(c) $X_\eta \to \operatorname{Spec}(\kappa(\eta))$ is geometrically connected

(d) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = \kappa(\eta)$

(e) $f_*\mathcal{O}_X = \mathcal{O}_S$

then the following implications hold,

$$\begin{tikzcd}
(a) \arrow[r] \arrow[d] & (d) \arrow[d, leftrightarrow] \\
(b) \arrow[u, bend left, "X_\eta \text{ geom.red.}"] \arrow[d] & (e) \arrow[dl] \\
(c) \arrow[u, bend left, "X \text{ normal}"] &
\end{tikzcd}$$

In particular, if X is normal and X_η is geometrically reduced all the properties are equivalent.

Proof. The only nontrivial implications are:

- $(a) \implies (d)$ is Tag 0BUG (8)
- $(d) \implies (e)$ is exactly Lemma 1.4.3
- $(c) \implies (b)$ is Lemma 1.1.2 and the fact that geometric connectedness of fibers can be checked generically in universally open (e.g. flat finitely presented) families [EGA IV, Cor. 15.5.4].

□

Remark. Even if $f_*\mathcal{O}_X = \mathcal{O}_S$ we don't necessarily have that X_η is geometrically reduced e.g. Example 1.1.7.

1.5 Controlling the Arithmetic Genus in Families

1.6 Graph Theory

Definition 1.6.1. By a *graph* we mean an undirected graph with muliple and self-edges.

Definition 1.6.2. Let G be a graph. The *genus* $g(G)$ of G is the dimension of $H^1(G, \mathbb{Q})$ with G viewed as a 1-truncated CW complex.

Proposition 1.6.3. Let $G = (V, E)$ be a graph. Then,

$$g(G) = \#E - \#V + 1$$

Proof. We prove this by induction on $\#V$. Choose an edge $e \in E$ and consider the contraction of this edge $G/\{e\}$. We see that the map $G \to G/\{e\}$ is a homotopy equivalence so,

$$g(G) = g(G/\{e\}) = (\#E - 1) - (\#V - 1) + 1 = \#E - \#V + 1$$

and the base case with $\#V = 1$ and $\#E$ self-loops is obvious. □

1.6.1 Setup

Let $X \to S$ be a normal degeneration of curves. Then consider the following data. Let $\Gamma_i \subset X_s$ be the (reduced) irreducible componetns of the special fiber and the following κ-algebras,

(a) $A = H^0(X_s, \mathcal{O}_{X_s})$

(b) $\kappa' = H^0((X_s)_{\text{red}}, \mathcal{O}_{(X_s)_{\text{red}}})$

(c) $\kappa_i = H^0(\Gamma_i, \mathcal{O}_{\Gamma_i})$

where A is an Artin local κ-algebra and κ' and κ_i are finite field extensions of κ by Tag 0BUG (1) since these schemes are connected and the second two are reduced.

1.6.2 Inequalities

The jumping off point is constancy of the Euler characteristic,

$$\chi(X_K, \mathcal{O}_{X_K}) = \chi(X_s, \mathcal{O}_{X_s})$$

Therefore the following are equivalent,

$$p_a(X_s/\kappa) \le p_a(X_K/K) \iff p_a(X_s/\kappa) = p_a(X_K/K) \iff A = \kappa$$

However, the h^0 of the special fiber can jump up. Therefore, we need to study more subtle inequalities. Consider the following inequalities where G is the (IS THIS RIGHT!) geometric dual graph of X_s,

$$p_a((X_s)_{\text{red}}/\kappa) \le p_a(X_K/K) \tag{A}$$

$$p_a((X_s)_{\text{red}}/\kappa') \le p_a(X_K/K) \tag{A'}$$

$$\sum_{i=1}^{r} p_a(\Gamma_i/\kappa) + g(G) \le p_a(X_K/K) \tag{B}$$

$$\sum_{i=1}^{r} p_a(\Gamma/\kappa_i) + [\kappa' : \kappa] g(G) \le p_a(X_K/K) \tag{B'}$$

$$\max_i \{p_a(\Gamma_i/\kappa)\} \le p_a(X_K/K) \tag{C}$$

$$\max_i \{p_a(\Gamma_i/\kappa_i)\} \le p_a(X_K/K) \tag{C'}$$

$$\max_{\Gamma_i \text{ normal}} \{p_a(\Gamma_i/\kappa)\} \le p_a(X_K/K) \tag{D}$$

$$\max_{\Gamma_i \text{ normal}} \{p_a(\Gamma_i/\kappa_i)\} \le p_a(X_K/K) \tag{D'}$$

All the unprimed inequalities hold in κ has characteristic zero but fail in characteristic p. The primed inequalities hold in general if X is *regular* and I conjecture that they all hold for X *normal*. Here I will write down what I can prove.

Lemma 1.6.4. We have the following implications,

$$\begin{tikzcd} (A) \arrow[r] \arrow[d] & (B) \arrow[r] \arrow[d] & (C) \arrow[r] \arrow[d] & (D) \arrow[d] \\ (A') \arrow[r] & (B') \arrow[r] & (C') \arrow[r] & (D') \end{tikzcd}$$

Proof. The downward arrows are all trivial as are all the horizontal arrows except $(A) \implies (B)$ and the primed version. These both follow from identical arguments. Suppose that,

$$p_a((X_s)_{\text{red}}/\kappa) \le p_a(X_K/K)$$

Let $Y = (X_s)_{\text{red}}$ and consider the map $\sqcup \Gamma_i \to Y$ breaking the irreducible components. This gives a sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \prod \mathcal{O}_{\Gamma_i} \longrightarrow \mathscr{C} \longrightarrow 0$$

and therefore,

$$\chi(Y, \mathcal{O}_Y) + \dim_\kappa H^0(Y, \mathscr{C}) = \sum_i \chi(\Gamma_i, \mathcal{O}_{\Gamma_i})$$

If $\kappa' = H^0(Y, \mathcal{O}_Y)$ then this becomes,

$$[\kappa' : \kappa](1 - p_a(Y/\kappa')) + \dim_\kappa H^0(Y, \mathscr{C}) = \sum_i [\kappa_i : \kappa](1 - p_a(\Gamma_i/\kappa_i))$$

This implies that,

$$\sum_i p_a(\Gamma_i/\kappa) + [\kappa' : \kappa](\dim_{\kappa'} H^0(Y, \mathscr{C}) - \sum_i [\kappa_i : \kappa'] + 1) = p_a(Y/\kappa)$$

Now $\mathscr{C}$ is supported at each intersection point (edge of G) and its length at the intersection of i and j is divisible by both $[\kappa_i : \kappa']$ and $[\kappa_j : \kappa']$. Hence,

$$(\dim_{\kappa'} H^0(Y, \mathscr{C}) - \sum_i [\kappa_i : \kappa'] + 1) \ge g(G)$$

where G is the *geometric* dual graph so we conclude (DO THIS BETTER AND CLARIFY THE GEOMETRIC DUAL GRAPH!!) $\square$

Lemma 1.6.5. Let $f : X \to S$ be a degeneration of curves with $h^0(X_\eta, \mathcal{O}_{X_\eta}) = h^1(X_s, \mathcal{O}_{X_s})$ then all the inequalities hold.

Proof. It suffices to prove (A). By assumption and constancy of Euler characteristic we have,

$$p_a(X_s/\kappa) = p_a(X_\eta/K)$$

therefore it suffcies to show that $p_a((X_s)_{\text{red}}/\kappa) \le p_a(X_s/\kappa)$. Indeed, there is a surjection,

$$\mathcal{O}_{X_s} \twoheadrightarrow \mathcal{O}_{(X_s)_{\text{red}}}$$

and $\dim X_s = 1$ so $H^2(X_s, -) = 0$ proving that there is a surjection,

$$H^1(X_s, \mathcal{O}_{X_s}) \twoheadrightarrow H^1(X_s, \mathcal{O}_{(X_s)_{\text{red}}})$$

and hence we conclude. $\square$

1.7 Cohomological Flatness

Suppose that we have a flat proper family $f : X \to S$ with $f_* \mathcal{O}_X = \mathcal{O}_S$. Formation of this pushforward my fail to be compatible with basechange, this is failure of cohomological flatness in degree zero. When this happens we can have jumping up of $h^0(X_s, \mathcal{O}_{X_s})$. Consider the finite $\kappa(s)$-algebra,

$$A = H^0(X_s, \mathcal{O}_{X_s})$$

There are three ways we could imagine A jumping up:

(a) A is a finite separable extension of $\kappa(s)$

(b) A is a finite purely-inseparable extension of $\kappa(s)$

(c) A is nonreduced.

The first cannot happen because $f : X \to S$ has geometrically connected fibers but if there is a factorization $X \to \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ with k' separable then it is geometrically disconnected. Therefore, any field inside A must be purely inseparable over k. However both (b) and (c) can happen as we will now see.

Example 1.7.1. RAYNAUD'S EXAMPLES

However, none of these can happen in characteristic zero.

Theorem 1.7.2 (Raynaud)**.** Let $f : X \to S$ be a degeneration of curves with X normal. Suppose that $\mathrm{char}(\kappa(s))$ is coprime to the gcd of the geometric multiplicites of the components of X_s. Then f is cohomologically flat in degree 0.

Corollary 1.7.3. In the above case, all the inequalities hold.

1.8 Results in Positive Characteristic

For simplicitly, we restrict in this section to the case where the curve $C = X_\eta$ over K is smooth and $H^0(C, \mathcal{O}_C) = K$. I think the results should be true without this assumption but I want to quote the stacks project which makes this assumption without caution and use strong desingularization which does use smoothness of C in an important way (see [Liu, Cor. 8.3.51]).

Proposition 1.8.1. For any regular degeneration of curves, inequality (A') holds.

Proof. By Tag 0C68 and Tag 0C69 the effective Cartier divisor,

$$C = \sum_{i=1}^{r} (m_i/d) C_i$$

where m_i is the multiplicity of C_i and $d = \gcd(m_i)$ satisfies,

(a) $H^0(D, \mathcal{O}_D) = \kappa_D$ is a field and

(b) $\chi(X_s, \mathcal{O}_{X_s}) = d\chi(D, \mathcal{O}_D)$

and hence,

$$g - 1 = d[\kappa_D : \kappa](g_D - 1)$$

where $g = p_a(X_K/K)$ and $g_D = p_a(D/\kappa_D)$. Therefore,

$$g_D = \frac{g-1}{d[\kappa_D : \kappa]} + 1 \le g$$

since either $g = 0$ in which case $g_D = 0$ and $d = [\kappa_D : \kappa] = 1$ or $g > 0$ in which case we see that $g_D \le g$. Furthermore, since $(X_s)_{\text{red}} = D_{\text{red}}$ we see that $(X_s)_{\text{red}}$ is a κ_D-scheme and,

$$H^1(X_s, \mathcal{O}_D) \twoheadrightarrow H^1(X_s, \mathcal{O}_{(X_s)_{\text{red}}})$$

so we conclude that,

$$p_a((X_s)_{\text{red}}/\kappa') \le p_a((X_s)_{\text{red}}/\kappa_D) \le p_a(D/\kappa_D) = g_D \le g = p_a(X_K/K)$$

∎

Proposition 1.8.2. For any normal degeneration of curves, inequality (D') holds.

Proof. Let X be a normal degeneration of curves. Then there exists a strong desingularization [Liu, Cor. 8.3.51] $\pi : \widetilde{X} \to X$ meaning it is an isomorphism on the regular locus of X. Then $\widetilde{X} \to S$ is a regular model of $\widetilde{X}_\eta = X_\eta$ and hence verifies (D') by the previous results. However, since $\widetilde{X} \to X$ is an isomorphism away from a finite set of points, for each $\Gamma_i \subset X_s$ there is an irreducible component $\widetilde{\Gamma}_i \subset \widetilde{X}_s$ mapping birationally onto Γ_i. However, Γ_i is a normal curve so the birational map $\widetilde{\Gamma}_i \to \Gamma_i$ is an isomorphism and hence we conclude. ∎

1.8.1 Nonprojective

Definition 1.8.3. Let X be a (nonprojective) curve over k. Then we define the arithmetic genus $p_a(X/k)$ ad the smallest $p_a(\overline{X}/k)$ over all compactifications $X \hookrightarrow \overline{X}$ defined over k. Explicitly, these are proper k-schemes $\overline{X}$ equipped with an open embedding $X \hookrightarrow \overline{X}$.

Remark. We should be able to compute $p_a(X/k)$ via choosing any $X \hookrightarrow \overline{X}$ and taking the partial normalization of $\overline{X}$ along $\overline{X} \backslash X$.

Proposition 1.8.4. If an inequality holds for proper normal/regular models then the corresponding version also holds for nonproper normal/regular models.

Proof. Indeed, we can chosoe a relative compactification of $f : X \to S$ by Nagata to get $\bar{f} : \overline{X} \to S$ with $\overline{X}$ integral and normal by applying the normalization if necessary (which does not change the locus $X \hookrightarrow \overline{X}$ since X is normal). Since S is a DVR then $\bar{f}$ remains flat and proper. Since $X \hookrightarrow \overline{X}$ is an open subscheme the components of the special fiber of $\overline{X}$ are compactifications of the corresponding components of X. Therefore we can apply our restuls to $\overline{X}$ to get an upper bound on the minimal genus of a compactification of each component. □

1.9 Controlling the Geometric Genus in Families (TODO)

2 Application to Specializing Genus of Fibrations

As before let $(R, \mathfrak{m}, \kappa)$ be a DVR with fraction field $K = \mathrm{Frac}\,(R)$ and spectrum $S = \mathrm{Spec}\,(R)$.

Proposition 2.0.1. Let $f : X \to Y$ be a morphism of flat proper S-schemes. Suppose that,

(a) X and Y are integral and normal

(b) X_s decomposes into (reduced) irreducible components $X_1, \dots, X_r$ with multiplicites m_i

(c) Y_s is integral

(d) the map $f_s : X_s \to Y_s$ is dominant

(e) f has relative dimension 1

Let $\xi \in Y$ and $\eta \in Y_s$ be the generic points. Then, the fiber X_η of f decomposes into (reduced) irreducble components $\Gamma_1, \dots, \Gamma_r$ with multiplicities m_i such that Γ_i is the generic fiber of $X_i \to Y_s$. Thus, for each normal X_j we have,

$$p_a(\Gamma_j/\kappa_j) \le p_a(X_\xi/\kappa(\xi))$$

where $\kappa_j = H^0(\Gamma_j, \mathcal{O}_{\Gamma_j})$. and thus the Stein factorization,

$$X_j \to B \to Y_s$$

has $X_j \to B$ a fibration by curves with generic fiber of genus $\le p_a(X_\xi/\kappa(\xi))$.

Proof. Since X is integral, the fiber dimension can only jump on a codim ≥ 2 locus. In fact, the following lemma gives a strengthening of generic flatness. Now $(X_s)_\eta = X_\eta$ and the generic points of Γ_i and of X_i are the same in X and hence they come with the same multiplicity. Indeed, this is a purely local situation. Consider the diagram of ring maps,

$$\begin{tikzcd} A & & B \arrow[ll] \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

with $\mathfrak{m} \subset R$ the maximal ideal. Then $\mathfrak{m}B$ is prime by assumption. Let $S = B \backslash (\mathfrak{m}B)$. Then,

$$(R/\mathfrak{m}) \otimes_R A \otimes_B (S^{-1}B) = [(B/\mathfrak{m})\backslash\{0\}]^{-1}(A/\mathfrak{m}A)$$

has as its local rings those points of the special fiber of X mapping to η. By flatness (see below) each generic point of X_s maps to η so we get the assumed decomposition. Now consider $X_D \to \mathrm{Spec}\,(D)$ where $D = \mathcal{O}_{Y,\eta}$ is a DVR since Y is regular in codimension 1. Since X is irreducible and normal and $X_D \to X$ is a localization then X_D is also irreducible and normal. Finally, $X_R \to \mathrm{Spec}\,(D)$ is proper by basechange and flat because it is dominant by assumption. Therefore, we conclude by an application of Propositione 1.8.2. $\square$

Remark.

(DO I NEED PROPERNESS FOR THIS LEMMA)

Lemma 2.0.2. Let $f : X \to Y$ be a dominant finite type morphism of schemes with Y integral, normal, and noetherian. If X is also integral, then there exists an open $U \subset Y$ containing every codimension 1 point such that $f : X_U \to U$ is flat.

Proof. It (WHY??) suffices to show that for every $y \in Y$ with $\dim \mathcal{O}_{Y,y} = 1$ the pullback over $A = \mathcal{O}_{Y,y}$ is flat. However, in the diagram,

$$\begin{tikzcd} X_A \arrow[r] \arrow[d, "f"] & X \arrow[d, "f"] \\ \mathrm{Spec}\,(A) \arrow[r] & Y \end{tikzcd}$$

the horozontal arrows are localizations and hence X_A is integral. But A is a DVR since Y is normal and X_A is integral so flatness follows from dominance of $\mathrm{Spec}\,(X_A) \to \mathrm{Spec}\,(A)$ which itself follows from dominance of f. $\square$