Notes on Conformal Field Theory

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1 Introduction

2 The Basics of QFT

Theorem 2.1. Consider a QFT coupled to a background metric g. Correlators are given produced by a path-integral inserion,

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \int \mathscr{D}\phi \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_1) e^{iS[g,\phi]}$$

The stress-energy tensor insersion is given by,

$$\langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \frac{2}{i\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g$$

Proof. Simply computing the right hand side gives,

$$\frac{2}{i\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \frac{2}{\sqrt{g}} \int \mathscr{D}\phi \frac{\delta S}{\delta g_{\mu\nu}} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{iS[g,\phi]} = \langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g$$

Since the Einstein-Hilbert action implies that,

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

Furthermore, suppose that S is diffeomorphism invariant. Consider an infinitessimal change of variables, $x \mapsto x - \epsilon(x)$ under which $\phi(x) \mapsto \phi(x) + \epsilon^\mu(x)\partial_\mu \phi(x)$. Operator $\mathcal{O}(x)$ with a spin structure will transform in a representation of the Lorentz group as $\mathcal{O}(x) \mapsto (1 + R(\epsilon)) \cdot \mathcal{O}(x) + \epsilon^\mu(x) \partial_\mu \mathcal{O}(x)$. Since S is a diffeomorphism invariant, the path-integral is invariant under this coodinate transformation which we can view as a change of variables of the fields. Furthermore, if we have initially flat space then the perturbation to the metric is,

$$\delta g_{\mu\nu} = \partial_\mu \epsilon^\alpha \eta_{\alpha\nu} + \partial_\nu \epsilon^\beta \eta_{\mu\beta}$$

Under the given transformation of the fields, the path-integral must be invariant. Therefore,

$$\begin{aligned}
\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle &= \int \mathscr{D}\phi' \, \mathcal{O}'_1(x_1) \cdots \mathcal{O}'_n(x_n) e^{iS[g,\phi']} \\
&= \int \mathscr{D}\phi \, (\mathcal{O}_1(x_1) + \epsilon^\mu(x_1)\partial_\mu \mathcal{O}_1) \cdots (\mathcal{O}_n(x_n) + \epsilon^\mu(x_n)\partial_\mu \mathcal{O}_n) e^{iS[g,\phi]} \exp\left(i \int \mathrm{d}^4 x \, \frac{\delta S}{\delta \phi} \delta\phi(x) \right)
\end{aligned}$$

Therefore, to first-order in ϵ,

$$\int \mathscr{D}\phi \left(\frac{\delta S}{\delta\phi}\delta\phi\right) \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{iS[g,\phi]} = i\epsilon^\mu(x_1)\left\langle \partial_\mu\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle + \cdots + i\epsilon^\mu(x_n)\left\langle \mathcal{O}_1(x_1)\cdots\partial_\mu\mathcal{O}_n(x_n)\right\rangle$$

□

However, since S is a diffeomorphism invariant and shifting both the fields and the metric as above is equivalent to a coodinate transformation. Thus,

$$\delta S = \frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu} + \frac{\delta S}{\delta\phi}\phi = \int \mathrm{d}^4x \, \left(\frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu}(x) + \frac{\delta S}{\delta\phi}\delta\phi(x)\right) = 0$$

Therefore,

$$\left\langle \left(\frac{\delta S}{\delta\phi}\delta\phi(x)\right)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle = -\left\langle \frac{\sqrt{g}}{2} T^{\mu\nu}(x)\delta g_{\mu\nu}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle$$

Expanding about flat Minkowski-space,

$$T^{\mu\nu}\delta g_{\mu\nu} = T^{\mu\nu}\left[\partial_\mu\epsilon^\alpha\eta_{\alpha\nu} + \partial_\nu\epsilon^\beta\eta_{\mu\beta}\right] = T^{\mu\nu}\left[\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right] = 2T^{\mu\nu}\partial_\mu\epsilon_\nu$$

and thus,

$$\left\langle \left(\frac{\delta S}{\delta\phi}\delta\phi(x)\right)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle = -(\partial_\mu\epsilon_\nu)\left\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle$$

Applying integration by parts,

$$\begin{aligned}
\int \mathscr{D}\phi \, \left(\frac{\delta S}{\delta\phi}\delta\phi\right) \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{iS[g,\phi]} &= \int \mathrm{d}^4x \, \left\langle \left(\frac{\delta S}{\delta\phi}\delta\phi(x)\right)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle \\
&= -\int \mathrm{d}^4x \, (\partial_\mu\epsilon_\nu)\left\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle \\
&= \int \mathrm{d}^4x \, \epsilon_\nu(x)\partial_\mu\left\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle
\end{aligned}$$

Applying the invariance under field change of the path-integral,

$$\int \mathrm{d}^4x\epsilon_\nu(x)\partial_\mu\left\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle = i\epsilon^\mu(x_1)\left\langle \partial_\mu\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle + \cdots + i\epsilon^\mu(x_n)\left\langle \mathcal{O}_1(x_1)\cdots\partial_\mu\mathcal{O}_n(x_n)\right\rangle$$

Since this holds for all ϵ^μ, by the definition of the Dirac distribution,

$$\partial_\mu\left\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle = i\delta(x-x_1)\left\langle \partial^\nu\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\right\rangle + \cdots + i\delta(x-x_n)\left\langle \mathcal{O}_1(x_1)\cdots\partial^\nu\mathcal{O}_n(x_n)\right\rangle$$

2.1 Operators With Spin

3 Conformal Invariance

Theorem 3.1. Scale-invariant theories have traceless Stress-Energy tensors.

Proof. Suppose that we have a scale-invariant theory. In particular, whenever $\delta g_{\mu\nu} = \omega(x)g_{\mu\nu}$ then,

$$\frac{\delta S}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta\omega} = T^{\mu\nu}g_{\mu\nu} = 0$$

which implies that $T^\mu_\nu = 0$ the Stress-Energy tensor is traceless. □

The traceless condition implies a weaker conformal killing equation. For a vector field $\epsilon^\mu(x)$ we want to consider when the charge

$$Q_\epsilon(\Sigma) = -\int_\Sigma \mathrm{d}S_\mu \, \epsilon_\nu(x) T^{\mu\nu}(x)$$

is conserved. Using the divergence theorem, for two space-like slices,

$$Q_\epsilon(\Sigma_2) - Q_\epsilon(\Sigma_1) = \int_{\Sigma_1} \mathrm{d}S_\mu \, \epsilon_\nu(x) T^{\mu\nu}(x) - \int_{\Sigma_2} \mathrm{d}S_\mu \, \epsilon_\nu(x) T^{\mu\nu}(x) = \int_V \mathrm{d}^4 x \, \partial_\mu \left(\epsilon_\nu(x) T^{\mu\nu}(x) \right)$$

Therefore, Q_ϵ is conserved over all space-like slices exactly if,

$$\partial_\mu \left(\epsilon_\nu(x) T^{\mu\nu}(x) \right) = 0$$

For arbitrary symmetric divergence-free $T^{\mu\nu}$ we have,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0$$

and thus

$$\partial_\mu \epsilon_\nu T^{\mu\nu} + \epsilon_\nu \partial_\mu T^{\mu\nu} = 0$$

However, $\partial_\mu T^{\mu\nu} = 0$ and T is symmetric so this equation is equivalent to,

$$\partial_\mu \epsilon_\nu T^{\mu\nu} = \tfrac{1}{2} \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \right) T^{\mu\nu} = 0$$

Distinguished solutions to the strict Killing equation in flat space are,

$$\begin{aligned} p_\mu &= \partial_\mu && \text{(translations)} \\ m_{\mu\nu} &= x_\nu \partial_\mu - x_\mu \partial_\nu && \text{(rotations)} \end{aligned}$$

which have Hermitian generators P_μ and $M_{\mu\nu}$ respectively. However, for a conformal equation, $T^{\mu\nu}$ is traceless and symmetric this equation implies that,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(x) \eta_{\mu\nu}$$

If we take the trace of both sides,

$$c(x) = \frac{2}{d} \partial_\mu \epsilon^\mu$$

This alows two new types of transformations, dilations,

$$d = x^\mu \partial_\nu \text{ corresponding to the vector field } \epsilon^\mu(x) = x^\mu$$

which have Hermitian generator D and special conformal transformations,

$$k_\alpha = 2x_\alpha (x^\nu \partial_\nu) - x^\nu x_\nu \partial_\alpha \text{ corresponding to the vector field } \epsilon^\mu_\alpha(x) = 2x_\alpha x^\mu - x_\nu x^\nu \delta^\mu_\alpha$$

with Hermitian generator K_α. These together with the Poincare algebra (in Euclidean signature) satisfy,

$$\begin{aligned} [M_{\mu\nu}, P_\alpha] &= \delta_{\nu\alpha} P_\mu - \delta_{\mu\alpha} P_\nu \\ [M_{\mu\nu}, K_\alpha] &= \delta_{\nu\alpha} K_\mu - \delta_{\mu\alpha} K_\nu \\ [M_{\mu\nu}, M_{\alpha\beta}] &= \delta_{\nu\alpha} M_{\mu\beta} - \delta_{\mu\alpha} M_{\nu\beta} + \delta_{\nu\beta} M_{\alpha\mu} - \delta_{\mu\beta} M_{\alpha\nu} \\ [D, P_\mu] &= P_\mu \\ [D, K_\mu] &= -K_\mu \\ [K_\mu, P_\nu] &= 2\delta_{\mu\nu} D - 2M_{\mu\nu} \end{aligned}$$

and all other commutators vanish.

Theorem 3.2. A conformal field theory has only massless states in its spectrum.

Proof. Suppose that $|\Psi\rangle$ is a state with mass m. Therefore,

$$P_\mu P^\mu |\Psi\rangle = m^2 |\Psi\rangle$$

However,

$$[D, P_\mu P^\mu] = DP^\mu P_\mu - P^\mu P_\mu D = P^\mu D P_\mu + iP^\mu P_\mu - P^\mu P_\mu D = 2iP^\mu P_\mu$$

Thus,

$$\langle\Psi| [D, P^\mu P_\mu] |\Psi\rangle = \langle\Psi| 2iP^\mu P_\mu |\Psi\rangle = 2im^2$$

However,

$$\langle\Psi| [D, P^\mu P_\mu] |\Psi\rangle = \langle\Psi| DP^\mu P_\mu |\Psi\rangle - \langle\Psi| P^\mu P_\mu D |\Psi\rangle = m^2 [\langle\Psi| D |\Psi\rangle - \langle\Psi| D |\Psi\rangle] = 0$$

since $P^\mu P_\mu$ is a Hermitian operator. Thus, we must have $m = 0$. □

3.1 Finite Conformal Representation

Consider an infinitesimal transformation $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x)$. If ϵ^μ satisfies the confromal Killing equation, then

$$\frac{\partial x'^\mu}{\partial x^\mu} = \delta^\mu_\nu + \partial_\nu \epsilon^\mu = \left(1 + \frac{1}{d}(\partial \cdot \epsilon)\right)\left(\delta^\mu_\nu + \frac{1}{2}(\partial_\nu \epsilon^\mu - \epsilon^\mu \epsilon_\nu)\right)$$

This is an infinitesimal rescaling times an infinitesimal rotation. Exponentiating gives a coordinate transformation $x \mapsto x'$ such that

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) R^\mu_\nu(x) \qquad R^\top R = I$$

where $\Omega(x)$ and $R^\mu_\nu(x)$ are finite position-dependent rescalings and rotations. Equivalently, the transformation $x \mapsto x'$ rescales the metric by a scale factor,

$$\delta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial \beta} = \Omega(x)^2 \delta_{\alpha\beta}$$

Such transformations are called *conformal* which comprise the conforml group.

3.1.1 Reflections

3.2 Charge Representation

To find these charges i.e. Hermitian transformation generators, we need to search for charges associated to a vector field. We have already defined the charge $Q_\epsilon(\Sigma)$ for a vector vield $\epsilon = \epsilon^\mu \partial_\mu$ via,

$$Q_\epsilon(\Sigma) = -\int_\Sigma \mathrm{d}S_\mu \, \epsilon_\nu(x) T^{\mu\nu}(x)$$

Theorem 3.3. When $d \geq 3$,

$$[Q_\epsilon, T^{\mu\nu}] = (\epsilon \cdot \partial) T^{\mu\nu} + (\partial \cdot \epsilon) T^{\mu\nu} - \partial_\rho \epsilon^\mu T^{\rho\nu} + \partial^\nu \epsilon_\rho T^{\rho\mu}$$

Proof. □

Theorem 3.4. The chrages Q_ϵ form a representation of the conformal algebra via,

$$[Q_{\epsilon_1}, Q_{\epsilon_2}] = Q_{-[\epsilon_1,\epsilon_2]}$$

Proof. □

Proposition 3.5. For $d = 2$ there exists an additional term, ...

Theorem 3.6. The conformal charges satisfy the commutation relations, ...

Proof. □

3.3 Conformal Angular Momentum Representation

Consider the definitions,

$$\begin{aligned}
L_{\alpha\beta} &= M_{\alpha\beta} \\
L_{-1,0} &= D \\
L_{0,\mu} &= \tfrac{1}{2}(P_\mu + K_\mu) \\
L_{-1,\mu} &= \tfrac{1}{2}(P_\mu - K_\mu)
\end{aligned}$$

where $L_{ab} = -L_{ba}$. From above, it follows that L_{ab} for $a, b \in \{1, \dots, d\}$ satisfy the commutation relations of the Lie algebra $\mathfrak{so}(d)$. We need to show that the entire object satisfies the Lie algebra of $\mathfrak{so}(1, d+1)$. First we consider the rotation part, L_{ab} for $a, b \in \{0, 1, \dots, d\}$. We need to show that,

$$[L_{ab}, L_{cd}] = \delta_{bc}L_{ad} - \delta_{ac}L_{bd} + \delta_{bd}L_{ca} - \delta_{ad}L_{bc}$$

We have already shown this when all $a, b, c, d > 0$. Furthermore, this expression is antisymmetric in a, b and c, d. First, let $a = 0$ and $b, c, d > 0$. Then we have,

$$[L_{0,b}, L_{cd}] = \delta_{cb}L_{0,d} - \delta_{bd}L_{0,c}$$

because $L_{0,b} = \frac{1}{2}(P_b + K_b)$ is a vector under $\mathrm{SO}(0, d)$. This satisfies the condition since $\delta_{ac} = \delta_{ad} = 0$. An indentical argument holds any one of a, b, c, d zero. Now take the case $a = c = 0$. Then we have,

$$[L_{0,b}, L_{0,d}] = \tfrac{1}{4}[P_b+K_b, P_d+K_d] = \tfrac{1}{4}\left([K_b, P_d] + [P_b, K_d]\right) = \tfrac{1}{2}(\delta_{bd}D - M_{bd} - \delta_{db}D + M_{db}) = -M_{bd} = -L_{bd}$$

satisfying the commutation relations because $\delta_{bc} = \delta_{ad} = 0$ and $L_{ca} = 0$ and $\delta_{ac} = 1$. If any three variables are zero then one generator must vanish by antisymmetry so we are done checking the rotational part.

4 Primary Operators

4.1 Scaling Dimension and Correlators

Consider operators diagonalized at the origin such that,

$$[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0)$$

where the eigenvalue Δ is the *scaling dimension* of the operator $\mathcal{O}$. Now consider the scaling action away from the origin,

$$[D, \mathcal{O}(x)] = [D, e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}] = e^{x\cdot P}(e^{-x\cdot P}De^{x\cdot P}\mathcal{O}(0) - \mathcal{O}(0)e^{-x\cdot P}De^{x\cdot P})e^{-x\cdot P}$$

By the Hausdorff formula,

$$e^A B e^{-A} = e^{[A,\cdot]}B = B + [A,B] + \tfrac{1}{2!}[A,[A,B]] + \cdots$$

Therefore,

$$e^{x\cdot P}De^{-x\cdot P} = e^{[\cdot, x\cdot P]}D = D + [D, x\cdot P] + \frac{1}{2!}[[D, x\cdot P], x\cdot P] + \cdots$$

Furthermore,

$$[D, x\cdot P] = x^\mu [D, P_\mu] = x^\mu P_\mu = x\cdot P$$

and therefore, the higher-order commutators are all zero. Thus,

$$e^{x\cdot P}De^{-x\cdot P} = D + x\cdot P$$

This implies that,

$$\begin{aligned}
[D, \mathcal{O}(x)] &= [D, e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}] = e^{x\cdot P}(e^{-x\cdot P}De^{x\cdot P}\mathcal{O}(0) - \mathcal{O}(0)e^{-x\cdot P}De^{x\cdot P})e^{-x\cdot P} \\
&= e^{x\cdot P}([D, \mathcal{O}(0)] + [x\cdot P, \mathcal{O}(0)])e^{-x\cdot P} = e^{x\cdot P}(\Delta\mathcal{O}(0) + [x\cdot P, \mathcal{O}(0)])e^{-x\cdot P} \\
&= (x^\mu\partial_\mu + \Delta)e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P} = (x^\mu\partial_\mu + \Delta)\mathcal{O}(x)
\end{aligned}$$

because,

$$\partial_\mu \mathcal{O}(x) = \partial_\mu e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P} = P_\mu e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P} - e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}P_\mu = [P_\mu, \mathcal{O}(x)]$$

is the Hiesenberg equation of motion. Therefore, we find the result,

$$[D, \mathcal{O}(x)] = (x^\mu\partial_\mu + \Delta)\mathcal{O}(x)$$

This result is strict enough to fix the form of $\mathcal{O}$-two-point correlation functions. By invariance under the Poincare group we can write,

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = f(|x-y|)$$

In a scale invariant theory, we must have,

$$D\,|\Omega\rangle = 0$$

otherwise if the vacuum had nonzero scaling charge then it would change under a scale transformation. Thus,

$$\langle [D, \mathcal{O}_1(x)\mathcal{O}_2(y)]\rangle = \langle\Omega|\, D\mathcal{O}_1(x)\mathcal{O}_2(y)\,|\Omega\rangle - \langle\Omega|\,\mathcal{O}_1(x)\mathcal{O}_2(y)D\,|\Omega\rangle = 0$$

However,

$$\begin{aligned}
[D, \mathcal{O}_1(x)\mathcal{O}_2(y)] &= [D, \mathcal{O}_1(x)]\mathcal{O}_2(y) + \mathcal{O}_1(x)[D, \mathcal{O}_2(y)] = (x^\mu\partial_{x^\mu} + \Delta_1)\mathcal{O}_1(x)\mathcal{O}_2(y) + \mathcal{O}_1(x)(y^\mu\partial_{y^\mu} + \Delta_2)\mathcal{O}_2(x) \\
&= (x^\mu\partial_\mu + \Delta_1 + y^\mu\partial_\mu + \Delta_2)\,\mathcal{O}_1(x)\mathcal{O}_2(y)
\end{aligned}$$

Therefore,

$$\langle [D, \mathcal{O}_1(x)\mathcal{O}_2(y)] \rangle = \langle (x^\mu \partial_\mu + \Delta_1 + y^\mu \partial_\mu + \Delta_2)\, \mathcal{O}_1(x)\mathcal{O}_2(y) \rangle = (x^\mu \partial_\mu + \Delta_1 + y^\mu \partial_\mu + \Delta_2)\, \langle \mathcal{O}_1(x)\mathcal{O}_2(y) \rangle = 0$$

Which implies that,

$$(x^\mu \partial_\mu + \Delta_1 + y^\mu \partial_\mu + \Delta_2)\, f(|x-y|) = 0$$

This differential equation forces,

$$f(|x-y|) = \frac{C}{|x-y|^{\Delta_1 + \Delta_2}}$$

For the correlation functions to satisfy the cluster decomposition, we require the correlators to decrease with distance so the scaling dimensions Δ of all operators must be positive.

4.2 Conformal Representations

Here we will use the notation $Q \cdot \mathcal{O} = [Q, \mathcal{O}]$ which is associative since $\mathcal{O}$ transforms in the adjoint representation i.e. by the Jacobi identity,

$$\begin{aligned}(Q_1 \cdot Q_2) \cdot \mathcal{O} &= ([Q_1, Q_2]) \cdot \mathcal{O} = [[Q_1, Q_2], \mathcal{O}] = [Q_1, [Q_2, \mathcal{O}]] + [Q_2, [\mathcal{O}, Q_2]] \\ &= Q_1 \cdot (Q_2 \cdot \mathcal{O}) - Q_2 \cdot (Q_1 \cdot \mathcal{O}) = [Q_1 \cdot, Q_2 \cdot] \mathcal{O}\end{aligned}$$

I will now drop the $\cdot$ to denote the adjoint action.

Remark 4.1. The identity is more clearly expressed under the adjiont map:

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g})$$

on some Lie algbra $\mathfrak{g}$ where $\mathrm{ad}_x(y) = [x, y]$. The above computation shows that,

$$\mathrm{ad}_{[x,y]}(z) = [\mathrm{ad}_x, \mathrm{ad}_y](z)$$

and thus ad is a Lie algebra representation.

Note that K_μ is a lowering operator for D since,

$$D K_\mu \mathcal{O}(0) = ([D, K_\mu] + K_\mu D)\mathcal{O}(0) = K_\mu (D - 1)\mathcal{O}(0) = (\Delta - 1) K_\mu \mathcal{O}(0)$$

A more formal computation gives,

$$\begin{aligned}\mathrm{ad}_D \mathrm{ad}_{K_\mu} \mathcal{O} &= ([\mathrm{ad}_D, \mathrm{ad}_{K_\mu}] + \mathrm{ad}_{K_\mu}\mathrm{ad}_D)\mathcal{O} = (\mathrm{ad}_{[D,K_\mu]} + \mathrm{ad}_{K_\mu}\mathrm{ad}_D)\mathcal{O} = (-\mathrm{ad}_{K_\mu} + \mathrm{ad}_{K_\mu}\mathrm{ad}_D)\mathcal{O} \\ &= \mathrm{ad}_{K_\mu}(\mathrm{ad}_D - 1)\mathcal{O}\end{aligned}$$

However, $\mathcal{O}$ is an eigenvector of ad_D such that $\mathrm{ad}_D \mathcal{O} = \Delta \mathcal{O}$ and thus,

$$\mathrm{ad}_D \mathrm{ad}_{K_\mu} \mathcal{O} = (\Delta - 1)\mathrm{ad}_{K_\mu} \mathcal{O}$$

so $\mathrm{ad}_{K_\mu} \mathcal{O}$ is also an eigenvector of ad_D with eigenvalue $\Delta - 1$.

definition 4.1. In a phyiscally sensible theory, the scaling dimensions are bounded below and thus the lowering process must terminate at some operator $\mathcal{O}$ such that,

$$\mathrm{ad}_{K_\mu} \mathcal{O}(0) = [K_\mu, \mathcal{O}(0)] = 0$$

Such an operator is called *primary*.

Furthermore, we may consider the actions of P_μ on such operators which are scaling eigenvectors. In adjoint notation, we have,

$$DP_\mu \mathcal{O}(0) = ([D, P_\mu] + P_\mu D)\mathcal{O}(0) = (P_\mu + P_\mu D)\mathcal{O}(0) = P_\mu(D+1)\mathcal{O}(0) = (\Delta+1)P_\mu \mathcal{O}(0)$$

Therefore, P_μ (or more accurately ad_{P_μ}) acts as the rasing operator. Applying this process to a primary operator, such operators of higher dimension are called descendents. For example, $\mathcal{O}(x) = e^{x\cdot P}\mathcal{O}(0)$ is an infinite series of descendent operators.

Theorem 4.2. Let $\mathcal{O}(0)$ be a primary operator with rotation representation matrices $\mathcal{S}_{\mu\nu}$ and scaling dimension Δ. Then,

$$[K_\mu, \mathcal{O}(x)] = (k_\mu + 2\Delta x_\mu - 2x^\nu \mathcal{S}_{\mu\nu})\, \mathcal{O}(x)$$

where k_μ is the conformal Kiling vector,

$$k_\mu = 2x_\mu(x\cdot\partial) - x^2\partial_\mu$$

Proof. First consider the commutator,

$$[U, \mathcal{O}(x)] = [U, e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}] = e^{x\cdot P}[e^{-x\cdot P}Ue^{x\cdot P}, \mathcal{O}(0)]e^{-x\cdot P}$$

By the Hausdorff formula,

$$e^A B e^{-A} = e^{[A,\cdot]}B = B + [A,B] + \tfrac{1}{2!}[A,[A,B]] + \cdots$$

Therefore,

$$e^{x\cdot P}K_\mu e^{-x\cdot P} = e^{[\cdot, x\cdot P]}K_\mu = K_\mu + [K_\mu, x\cdot P] + \tfrac{1}{2!}[[K_m u, x\cdot P], x\cdot P] + \cdots$$

Furthermore,

$$[K_\mu, x\cdot P] = x^\nu[K_\mu, P_\nu] = x^\nu(2\delta_{\mu\nu}D - 2M_{\mu\nu}) = 2x_\mu D - 2x^\nu M_{\mu\nu}$$

and therefore we need to check higher-order commutator terms,

$$\begin{aligned}[[K_\mu, x\cdot P], x\cdot P] &= x^\gamma[2x_\mu D - 2x^\nu M_{\mu\nu}, P_\gamma] = 2x^\gamma(x_\mu P_\gamma - x^\nu(\delta_{\nu\gamma}P_\mu - \delta_{\mu\gamma}P_\nu)) \\ &= 4x_\mu(x\cdot P) - 2x^2 P_\mu\end{aligned}$$

which commutes with $x\cdot P$ so we need not investigate any more terms. This implies that,

$$e^{x\cdot P}K_\mu e^{-x\cdot P} = K_\mu + 2x_\mu D - 2x^\nu M_{\mu\nu} + 2x_\mu(x\cdot P) - x^2 P_\mu$$

Therefore,

$$[K_\mu, \mathcal{O}(x)] = e^{x\cdot P}[e^{-x\cdot P}K_\mu e^{x\cdot P}, \mathcal{O}(0)]e^{-x\cdot P} = e^{x\cdot P}[K_\mu + 2x_\mu D - 2x^\nu M_{\mu\nu} + 2x_\mu(x\cdot P) - x^2P_\mu, \mathcal{O}(0)]e^{-x\cdot P}$$

Now we apply the known commutation relations of conformal charges on $\mathcal{O}(0)$. Because $\mathcal{O}(0)$ is a primary operator, $[K_\mu, \mathcal{O}(0)] = 0$. Furtherore, since $\mathcal{O}(0)$ is a scaling eigenvector with scaling dimension Δ, we have $[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0)$. Lastly, the spin representation $\mathcal{S}_{\mu\nu}$ of $\mathcal{O}(0)$ means that,

$$[M_{\mu\nu}, \mathcal{O}(0)] = \mathcal{S}_{\mu\nu}$$

Therefore,

$$[K_\mu, \mathcal{O}(x)] = e^{x\cdot P}(2x_\mu\Delta - 2x^\nu\mathcal{S}_{\mu\nu} + 2(x_\mu x_\nu - \delta_{\mu\nu}x^2)\mathrm{ad}_{P_\nu})\mathcal{O}(0)e^{-x\cdot P}$$

Furthermore,

$$e^{x\cdot P}\mathrm{ad}_{P_\nu}\mathcal{O}(0)e^{-x\cdot P} = e^{x\cdot P}[P_\nu, \mathcal{O}(0)]e^{-x\cdot P} = [e^{x\cdot P}P_\nu e^{-x\cdot P}, \mathcal{O}(x)] = [P_\nu, \mathcal{O}(x)] = \mathrm{ad}_{P_\nu}\mathcal{O}(x)$$

and therefore,

$$[K_\mu, \mathcal{O}(x)] = (2x_\mu\Delta - 2x^\nu\mathcal{S}_{\mu\nu} + 2(x_\mu x_\nu - \delta_{\mu\nu}x^2)\mathrm{ad}_{P_\nu})\mathcal{O}(x)$$

Using the Heisenberg equations of motion,

$$\mathrm{ad}_{P_\nu}\mathcal{O}(x) = [P_\nu, \mathcal{O}(x)] = \partial_\nu\mathcal{O}(x)$$

and therefore,

$$\begin{aligned}[K_\mu, \mathcal{O}(x)] &= (2x_\mu\Delta - 2x^\nu\mathcal{S}_{\mu\nu} + 2(x_\mu x_\nu - \delta_{\mu\nu}x^2)\partial_\nu)\mathcal{O}(x) \\ &= (2x_\mu\Delta - 2x^\nu\mathcal{S}_{\mu\nu} + 2x_\mu(x\cdot\partial) - x^2\partial_\mu)\mathcal{O}(x)\end{aligned}$$

□

Next we consider comutators of the charge,

$$Q_\epsilon(\Sigma) = -\int_\Sigma \mathrm{d}S_\mu\ \epsilon_\nu(x)T^{\mu\nu}(x)$$

Theorem 4.3. Let ϵ be a conformal Killing vector. Then,

$$[Q_\epsilon, \mathcal{O}(x)] = \left(\epsilon\cdot\partial + \frac{\Delta}{d}(\partial\cdot\epsilon) - \frac{1}{2}(\partial^\mu\epsilon^\nu)\mathcal{S}_{\mu\nu}\right)\mathcal{O}(x)$$

Proof. First, note that,

$$[Q_\epsilon, \mathcal{O}(x)] = [Q_\epsilon, e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}] = e^{x\cdot P}[e^{-x\cdot P}Q_\epsilon e^{x\cdot P}, \mathcal{O}(0)]e^{-x\cdot P}$$

Furthermore, by the Hausdorff formula,

$$e^{-x\cdot P}Q_\epsilon e^{x\cdot P} = Q_\epsilon + [Q_\epsilon, x\cdot P] + \tfrac{1}{2!}[[Q_\epsilon, x\cdot P], x\cdot P] + \cdots$$

However,

$$[Q_\epsilon, P_\mu] = [Q_\epsilon, Q_{p_\mu}] = Q_{-[\epsilon, p_\mu]}$$

Where,

$$-[\epsilon, p_\mu] = p_\mu\epsilon - \epsilon p_\mu = \partial_\mu\epsilon$$

Therefore,

$$[Q_\epsilon, P_\mu] = -\int \mathrm{d}S_\alpha\ (\partial_\mu\epsilon_\beta)T^{\alpha\beta}(x)$$

Furthermore, ϵ satisfies the conformal Killing equation,

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial\cdot\epsilon)\delta_{\mu\nu}$$

Therefore,

$$[Q_\epsilon, P_\mu] = \int \mathrm{d}S_\alpha\ \left(\partial_\beta\epsilon_\mu - \frac{2}{d}(\partial\cdot\epsilon)\delta_{\mu\beta}\right)T^{\alpha\beta}(x) = -\int \mathrm{d}S_\alpha\ \left(\epsilon_\mu\partial_\beta T^{\alpha\beta}(x) + \frac{2}{d}(\partial\cdot\epsilon)T^\alpha_\mu(x)\right)$$

However, $\partial_\beta T^{\alpha\beta}(x) = \partial_\beta T^{\beta\alpha}(x) = 0$. And thus,

$$[Q_\epsilon, P_\mu] = -\int \mathrm{d}S_\alpha\ \frac{2}{d}(\partial\cdot\epsilon)T^\alpha_\mu(x) = Q_{\frac{2}{d}(\partial\cdot\epsilon)\partial_\mu}$$

(Expand ϵ in the basis of conformal vectorfields) □

4.3 Finite Conformal Transformations

An exponential charge $U_\epsilon = e^{Q_\epsilon}$ gives a unitary transformation corresponding to a finite conformal transformation. The corresponding diffeomorphism e^ϵ is denoted $x \mapsto x'(x)$.

Theorem 4.4. Let $\mathcal{O}$ be a primary operator. Then,

$$U_\epsilon \mathcal{O}(x)\mathcal{O}^{-1} = \mathcal{O}(x')^\Delta D(R(x'))\mathcal{O}(x')$$

where,

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x')R^\mu_\nu(x') \qquad R^\mu_\nu(x') \in \mathrm{SO}(d,0)$$

and $D(R)$ is a matrix representing the action of R as a $\mathrm{SO}(d,0)$ representation.

Proof. □

Theorem 4.5. The map $\epsilon \mapsto U_\epsilon$ is a representation of the conformal group. That is,

$$U_{g_1}U_{g_2}\mathcal{O}(x)U_{g_2}^{-1}U_{g_1}^{-1} = U_{g_1g_2}\mathcal{O}(x)U_{g_1g_2}^{-1}$$

Proof. □

5 Conformal Correlators