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1.1 Line Bundles

There exists a map,

$$\Gamma(X, \mathcal{L}^{\otimes a}) \otimes \Gamma(X, \mathcal{L}^{\otimes b}) \rightarrow \Gamma(X, \mathcal{L}^{\otimes ab})$$

since we have an isomorphism $\mathcal{L}^{\otimes a} \otimes \mathcal{L}^{\otimes b} = \mathcal{L}^{\otimes ab}$. Furthermore, since \mathcal{L} is rank 1 this map is commutative since $s \times s' = s' \otimes s$ since they only differ by a section of \mathcal{O}_X . This allows us to define the following graded ring structure.

Definition 1.1. Let \mathcal{L} be an invertible \mathcal{O}_X -module, \mathcal{F} any \mathcal{O}_X -module and $s \in \mathcal{L}(X)$ a global section. Then we define the following graded ring.

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

and then the following module,

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which is a graded $\Gamma_*(X, \mathcal{L})$ -module. Furthermore, there is a map,

$$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \rightarrow \mathcal{F}(X_s) = \Gamma(X_s, \mathcal{F})$$

sending $\frac{t}{s^n} \mapsto t|_{X_s} \otimes (s|_{X_s})^{\otimes -n}$.

Proposition 1.2. Let X be a quasi-compact, quasi-separated scheme and \mathcal{F} be quasi-coherent. Then the above map is an isomorphism.

Proof. Tag OB5K. (Compare with that Hartshorne Exercise 2.16). □

Example 1.3. Let A be a graded ring such that A is generated by A_1 as a A_0 -algebra (e.g. $A = k[X_0, \dots, X_n]$). Let $X = \text{Proj}(A)$ and consider the graded module $M = A(n)$ which is the graded module $M_k = A_{k+n}$. Then we can construct the Serre twists,

$$\mathcal{O}_X(n) = \widetilde{M} = \widetilde{A(n)}$$

which is an invertible \mathcal{O}_X -module. Furthermore,

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$$

Remark. This will not be invertible and these maps will not be isomorphisms in general when A does not satisfy the required conditions.

Proof. We can decompose,

$$X = \bigcup_{f \in A_1} D_+(f) = \bigcup_{f \in A_1} \text{Spec}(A_{(f)})$$

via the given assumptions. We know that,

$$\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{A(n)}|_{D_+(f)} = A[f^{-1}]_n$$

However it is clear that $A[f^{-1}]_n = A[f^{-1}]_0 \cdot f^n$ so this sheaf is free of rank 1. □

Remark. For $n = 1$ any element $f \in A_1$ gives a global section $f \in \Gamma(X, \mathcal{O}_X(1))$ such that $D_+(f) = X_s$ and hence,

$$\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(1)|_{X_s}$$

Corollary 1.4. In the setting above, further assume that A is generated by finitely many $f \in A_1$ as an A_0 -algebra. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} if we set,

$$M = \Gamma_*(X, \mathcal{O}_X(1), \mathcal{F})$$

as a graded A -module via the map,

$$A \rightarrow \Gamma_*(X, \mathcal{O}_X(1)) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

Then we get, $\mathcal{F} = \widetilde{M}$.

Proof. Tag □

2 Feb. 13

Definition 2.1. Let X be a scheme and \mathcal{L} an invertible \mathcal{O}_X -module. We say \mathcal{L} is *ample* if X is quasi-compact and $\forall x \in X : \exists n > 0 : s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $x \in X_s$.

Example 2.2. Let $X = \text{Proj}(A)$ where A is generated by A_1 as a A_0 -algebra and $A_1 = f_1 A_0 + \dots + f_r A_0$. Then $\mathcal{O}_X(1)$ is invertible and X is covered by $D_+(f_i)$ and is quasi-compact, and $D_+(f_i) = X_{s_i}$ where $s_i \in \Gamma(X, \mathcal{O}_X(1))$ is a section corresponding to f_i .

Proposition 2.3. Let X be quasi-compact and quasi-separated for $\mathcal{L} \in \text{Pic}(X)$ the following are equivalent,

- (a). \mathcal{L} is ample
- (b). for all \mathcal{O}_X -modules \mathcal{F} locally of finite type there exists $n > 0$ s.t. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections.

Proof. TAG 01Q3. □

Lemma 2.4. \mathcal{L} is ample iff $\mathcal{L}^{\otimes n}$ is ample for any $n > 0$.

Lemma 2.5. If X is affine, and \mathcal{L} is invertible, and $s \in \Gamma(X, \mathcal{L})$ then X_s is affine.

Definition 2.6. A scheme is noetherian if it has a finite open cover by spectra of noetherian rings.

Remark. It is equivalent to require that X is quasi-compact and $\mathcal{O}_X(U)$ is noetherian for each affine open.

Lemma 2.7. A locally noetherian scheme is quasi-separated.

Proof. If U, V are affines then $U \cap V$ is quasi-compact since every subspace of a noetherian space is quasi-compact. □

Definition 2.8. Let X be a noetherian scheme. An \mathcal{O}_X -module \mathcal{F} is *coherent* if it is quasi-coherent and locally of finite type.

Remark. It is equivalent to require that locally on affine opens $\mathcal{F}|_U = \widetilde{M}$ for a finitely-generated module M .

Remark. The inclusion functors,

$$\mathcal{Coh}(\mathcal{O}_X) \subset \mathcal{QCoh}(\mathcal{O}_X) \subset \mathcal{Mod}(\mathcal{O}_X)$$

are exact and preserved under extensions i.e. given a short exact sequence,

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

if $\mathcal{F}_1, \mathcal{F}_2$ are (quasi)-coherent then \mathcal{F}_3 is also (quasi)-coherent.

Lemma 2.9. A scheme of finite type over a noetherian scheme is noetherian.

Proof. Since $f : X \rightarrow Y$ is finite type f is quasi-compact but Y is quasi-compact open so its preimage X is also quasi-compact. Furthermore, for any affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(B) = V \subset Y$ such that $f(U) \subset V$ we get a ring map $B \rightarrow A$ of finite type so $B[x_1, \dots, x_n] \rightarrow A$ and since B is noetherian we see that A is noetherian so X is quasi-compact and covered by $\text{Spec}(A)$ for noetherian rings A . \square

Remark. We want to prove the following theorem. Let R be a noetherian ring, X a projective (or proper) scheme over R (then X is noetherian), and \mathcal{F} a coherent sheaf on X , then,

$$H^i(X, \mathcal{F})$$

is a finite R -module for any i and $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.

3 Feb 18

Definition 3.1. An immersion $j : X \rightarrow Y$ is a morphism which may be factored as $X \rightarrow U \rightarrow Y$ where $X \rightarrow U$ is a closed immersion and $U \rightarrow Y$ is an open immersion.

Definition 3.2. Let R be a ring, and X a scheme over R . We say X is *quasi-projective over R* iff there exists a quasi-compact immersion $j : X \rightarrow \mathbb{P}_R^n$ over R .

Remark. If X is proper over R (or just universally closed) then j is automatically a closed immersion since $\mathbb{P}_R^n \rightarrow \text{Spec}(R)$ is separated and $X \rightarrow \text{Spec}(R)$ is universally closed implies that $j : X \rightarrow \mathbb{P}_R^n$ is universally closed and in particular topologically closed and thus closed as an immersion. This gives the following lemma.

Lemma 3.3. X is projective over R iff X is quasi-projective and proper over R .

Theorem 3.4. Let R be a ring and X a scheme over R . The TFAE,

- (a). X is quasi-projective over R
- (b). X is of finite type over R and X has an ample invertible module \mathcal{L}
- (c). there exists a quasi-compact open immersion $X \hookrightarrow X'$ with X' projective over R .

Lemma 3.5. Let $j : X \rightarrow Y$ be a quasi-compact immersion and \mathcal{L} an ample line bundle on Y . Then $j^*\mathcal{L}$ is an ample line bundle on X .

Proof. (DO THIS!!) □

Lemma 3.6. Let $j : X \rightarrow Y$ be a quasi-compact immersion and X' is scheme-theoretic image. Then $j : X \rightarrow X'$ is an open immersion.

Proof. Since j is qc and qs (immersions are separated) then $j_*\mathcal{O}_X$ is quasi-coherent and thus $\mathcal{I} = \ker(\mathcal{O}_Y \rightarrow \mathcal{O}_X)$ is quasi-coherent so we find $X' = V(\mathcal{I})$ (FINSIH THIS) □

Example 3.7. $\text{Spec}(k[[x]]) \rightarrow \text{Spec}(k[x])$ has scheme theoretic image $\text{Spec}(k[x])$ since its image contains the generic point. However, its set theoretic image is two points.

Proof. of Theorem (2) \implies (1). Choose $r \geq 0$ and $n \geq 1$ and $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.t.

$$X = \bigcup_{i=0}^r X_{s_i}$$

and X_{s_i} affine. Write $X_{s_i} = \text{Spec}(A_i)$. Now R is finite type over R so A_i is finite type over R so we may take $a_{i1}, \dots, a_{iN_i} \in A_i$ which generate A_i as an R -algebra. Choose $m \geq 1$ and $s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ such that $a_{ij} = s_{ij} \cdot s_i^{\otimes -m}|_{X_{s_i}}$. Therefore, $s_0^{\otimes m}, \dots, s_r^{\otimes m}, s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ generate $\mathcal{L}^{\otimes mn}$ and therefore define a morphism $\varphi : X \rightarrow \mathbb{P}_R^{r+\sum N_i}$. It suffices to check that $X_{s_i} \rightarrow D_+(T_i)$ are a closed immersion. This holds because it is given by the ring map,

$$R[\frac{T_0}{T_1}, \dots, \frac{T_r}{T_i}, \frac{T_{ij}}{T_i}] \rightarrow A_i = \mathcal{O}_X(X_{s_i})$$

given by $\frac{T_{ij}}{T_i} \rightarrow a_{ij}$ which is clearly surjective so $X_{s_i} \rightarrow D_+(T_i)$ is a closed immersion. □

Remark. If we had checked that $X_{s_{ij}} \rightarrow D(T_{ij})$ we also a closed immersion with $X_{s_{ij}}$ affine then $\varphi : X \rightarrow \mathbb{P}_R^N$ would be a *closed* immersion. We checked only that it is locally a closed immersion on X

3.1 Functorial Characterization of \mathbb{P}_R^n

Consider the functor, $F : \mathfrak{Sch}_R \rightarrow \mathfrak{Set}$ via,

$$T \mapsto \{(\mathcal{L}, s_0, \dots, s_n) \mid \mathcal{L} \in \text{Pic}(T) \xrightarrow{\mathcal{O}_T^{n+1} \xrightarrow{(s_0, \dots, s_n)}} \mathcal{L} \text{ i.e. } s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \text{ generate}\} / \cong$$

where $(\mathcal{L}, s_0, \dots, s_n) \cong (\mathcal{L}', s'_0, \dots, s'_n)$ if there is an isomorphism $\alpha : \mathcal{L} \rightarrow \mathcal{L}'$ with $\alpha(s_i) = s'_i$.

Theorem 3.8. \mathbb{P}_R^n represents this functor, $\text{Hom}_{\mathfrak{Sch}_R}(T, \mathbb{P}_R^n) = F(T)$.

Proof. Given $\varphi : T \rightarrow \mathbb{P}_R^n$ we get $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}_R^n}(1)$ and $s_i = \varphi^*(T_i)$.

Conversely, given $(\mathcal{L}, s_0, \dots, s_n)$ and $U \subset T$ and □

Theorem 3.9. If R is Noetherian and X is proper over R and \mathcal{L} is ample on X then,

$$X \cong \text{Proj}(\Gamma_*(X, \mathcal{L}))$$

and $\Gamma_*(X, \mathcal{L})$ is a finitely-generated graded R -algebra whose degree zero part is a finite R -module.

Remark. We will prove this using cohomology.

4 Cohomology

Theorem 4.1. $\mathbf{Mod}_{\mathcal{O}_X}$ is a Grothendieck abelian category so there are enough injectives.

Definition 4.2. Therefore, we can produce the right-derived functors $H^i(X, -)$ of the global sections functor,

$$\Gamma(X, -) : \mathbf{Mod}_{\mathcal{O}_X} \rightarrow \mathbf{Mod}_{\Gamma(X, \mathcal{O}_X)}$$

where (X, \mathcal{O}_X) is a ringed space. Since this is right-exact we find $H^0(X, -) = \Gamma(X, -)$.

Definition 4.3. Furthermore, given a morphism $f : X \rightarrow Y$ we can produce $R^i f_* : \mathbf{Mod}_{\mathcal{O}_X} \rightarrow \mathbf{Mod}_{\mathcal{O}_Y}$ the right-derived functors of $f_* : \mathbf{Mod}_{\mathcal{O}_X} \rightarrow \mathbf{Mod}_{\mathcal{O}_Y}$.

Remark. $\mathbf{Ab}(X) = \mathbf{Mod}_{\mathbb{Z}}$ so we may apply the theory of cohomology of \mathcal{O}_X -modules to the ringed space (X, \mathbb{Z}) to get a cohomology theory for abelian sheaves.

Lemma 4.4 (locality of cohomology). Given $\xi \in H^p(X, \mathcal{F})$ with $p > 0$ there exists an open covering,

$$X = \bigcup_{i \in I} U_i$$

s.t. $\xi|_{U_i} = 0$ for each $i \in I$.

Proof. □

Remark. The pullback is defined as follows,

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5.1 Čech Cohomology

For any open covering \mathfrak{U} of a space X and a sheaf \mathcal{F} there is a simplicial abelian group,

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0})$$

Then $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})$ is the complex associated to the cosimplicial object.

Example 5.1. Given an exact sequence of sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

An obstruction to lifting a section $s \in \Gamma(X, \mathcal{H})$ is a cocycle in $\check{C}^1(\mathfrak{U}, \mathcal{F})$.

Lemma 5.2. Čech cohomology vanishes on injective objects in the category of presheaves.

Corollary 5.3. As a functor ON THE CATEGORY OF PRESHEAVES $\check{H}^i(\mathfrak{U}, -)$ are the right-derived functors of $\check{H}^0(\mathfrak{U}, -)$.

Lemma 5.4. Given a ringed space, (X, \mathcal{O}_X) and B is a basis of top and Cov a set of coverings s.t.

- (a). \mathfrak{U} in cov implies that its union and all finite intersections are in B
- (b). for U basis the coverings of U in Cov are cofinal

If $\mathcal{F} \in \mathcal{Mod}(\mathcal{O}_X)$ and

$$(*) \forall \mathfrak{U} \in \text{Cov} : \check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$$

Then $H^p(\mathfrak{U}, \mathcal{F}) = 0$ for any U in the basis.

6 Feb 25

Lemma 6.1. Let \mathfrak{U} be an open covering of X and $\mathcal{F} \in \mathcal{Mod}(\mathcal{O}_X)$ s.t. $H^p(U_{i_1} \cap \cdots \cap U_{i_n}, \mathcal{F}) = 0$ for all finite intersections. Then $H^p(X, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$ for all $p \geq 0$.

Proof. See proof in Hartshorne Ex. It goes as follows,

(a). Use an exact sequence,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

with \mathcal{I} injective.

(b). Show for any sheaf $\check{H}^0(X, \mathcal{F}) = H^0(X, \mathcal{F})$ just by the sheaf property.

(c). By the assumptions, there is an exact sequence on check complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{I}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{G}) \longrightarrow 0$$

(d). this gives a long exact sequence of Cech cohomology

(e). use this exact sequence plus $\check{H}^p(\mathfrak{U}, \mathcal{I}) = 0$ for $p > 0$ (since flasque) to show that $\check{H}^p(\mathfrak{U}, \mathcal{G}) = \check{H}^{p+1}(\mathfrak{U}, \mathcal{F})$ and $\check{H}^1(\mathfrak{U}, \mathcal{F}) = \text{coker } \check{H}^0(\mathfrak{U}, \mathcal{I}) \rightarrow \check{H}^0(\mathfrak{U}, \mathcal{G})$

(f). use long exact sequence of $H^p(U_{i_0, \dots, i_n}, -)$ to show that \mathcal{G} also satisfies the hypotheses.

(g). use long exact sequence of $H^p(X, -)$ to show that the above hold for usual cohomology.

(h). then by induction we get $\check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) = \check{H}^p(\mathfrak{U}, \mathcal{G}) = H^p(X, \mathcal{G}) = H^{p+1}(X, \mathcal{F})$ and the base case holds since they are both kernels.

□

Corollary 6.2. Let X be a scheme whose diagonal is affine (for example a separated scheme). Let \mathfrak{U} be a covering of affine opens and \mathcal{F} a quasi-coherent \mathcal{O}_X -module. Then,

$$H^p(X, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$$

Remark. There is a Cech to cohomology spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \underline{H}^q(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F})$$

Corollary 6.3. Let $f : X \rightarrow Y$ be a quasi-compact quasi-separated morphism of schemes. Then $R^i f_*$ sends quasi-coherent modules to quasi-coherent modules.

Lemma 6.4. Let $f : X \rightarrow Y$, $\mathcal{F} \in \mathcal{Mod}(\mathcal{O}_X)$ then $R^p f_* \mathcal{F}$ is the sheaf associated to the presheaf,

$$V \mapsto H^i(f^{-1}(V), \mathcal{F})$$

Proposition 6.5. We define the following modifications to the Čech complex,

$\check{C}_{\text{alt}}^\bullet$ is elements of the form $(s_{i_0 \dots i_p})$ which are antisymmetric and vanish if any two indices agree and the ordered Čech complex for a total order $<$ on I ,

$$\check{C}_{\text{ord}}^p = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p})$$

There are the following relations between Čech complexes,

$$\check{C}_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{include}} \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{project}} \check{C}_{\text{ord}}^\bullet(\mathcal{U}, \mathcal{F})$$

the curved arrow is an isomorphism of complexes and the horizontal arrows are homotopy equivalences.

7 Feb. 27

Proposition 7.1. Let R be a Noetherian ring and \mathcal{F} a coherent sheaf on \mathbb{P}_R^n . Then,

- (a). $\exists r \geq 0 : \exists m \in \mathbb{Z}$ and a surjection $\mathcal{O}_X(m)^{\oplus r} \twoheadrightarrow \mathcal{F}$
- (b). $H^i(\mathbb{P}_R^n, \mathcal{F}) = 0$ for $i \notin [0, n]$
- (c). $H^i(\mathbb{P}_R^n, \mathcal{F})$ is a finite R -module
- (d). for $i > 0$, $H^i(\mathbb{P}_R^n, \mathcal{F}(d)) = 0$ for any $d \geq d_0(\mathcal{F})$
- (e). $\bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{F}(d))$ is a finite $P = R[T_0, \dots, T_n]$ -module.

Proof. Recall that $\mathcal{O}_X(1)$ is ample so $\mathcal{F} \otimes \mathcal{O}_X(d)$ is generated by global sections for sufficiently large d and thus we get $\mathcal{O}_X^{\oplus r} \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ and thus $\mathcal{O}_X(-d)^{\oplus r} \twoheadrightarrow \mathcal{F}$.

Note that $\mathbb{P}_R^n = \bigcup_i D_+(T_i)$ which is an open cover of $n+1$ affines so by Čech cohomology, cohomology vanishes above n .

Now we apply descending induction since $H^{n+1}(\mathbb{P}_R^n, \mathcal{F}) = 0$. Now we assume (3) and (4) for degree $k+1$. For a coherent sheaf \mathcal{F} consider the exact sequence,

$$0 \longrightarrow \mathcal{G}(d) \longrightarrow \mathcal{O}_X(m+d)^{\oplus n} \longrightarrow \mathcal{F}(d) \longrightarrow 0$$

then, from the LES we get,

$$H^k(\mathbb{P}_R^n, \mathcal{O}_X(m+d)^{\oplus n}) \longrightarrow H^k(\mathbb{P}_R^n, \mathcal{F}(d)) \longrightarrow H^{k+1}(\mathbb{P}_R^n, \mathcal{G}(d))$$

For the case $d = 0$ we assume that $H^{k+1}(\mathbb{P}_R^n, \mathcal{G})$ is a finite R -module and, by computation, so is $H^k(\mathbb{P}_R^n, \mathcal{O}_X(m)^{\oplus n})$ and thus $H^k(\mathbb{P}_R^n, \mathcal{F})$ is a finite R -module. For $d \gg 0$ then we assume that $H^{k+1}(\mathbb{P}_R^n, \mathcal{G}(d)) = 0$ for sufficiently large d . Furthermore, for $k > 0$ we computed that $H^k(\mathbb{P}_R^n, \mathcal{O}_X(m)^{\oplus n}) = 0$ for $d \geq m$ and thus we see that $H^k(\mathbb{P}_R^n, \mathcal{F}(d)) = 0$ for sufficiently large d proving (3) and (4).

Finally, we also use descending induction and consider the exact sequence,

$$\bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{O}_X(m+d)^{\oplus r}) \longrightarrow \bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{F}(d)) \longrightarrow \bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{G}(d))$$

By computation, the first term is a submodule of a finite P -module and the last term is zero is sufficiently large degrees. Thus the middle term M has a f.g. P -submodule M' such that M/M' is finite as an R -module so M is a f.g. P -module. \square

Lemma 7.2. Let $f : X \rightarrow Y$ be an affine morphism of schemes. Then $H^p(X, \mathcal{F}) = H^p(Y, f_* \mathcal{F})$ for \mathcal{F} quasi-coherent.

Proof. We use the Grothendieck spectral sequence and not that for $f : X \rightarrow Y$ affine and \mathcal{F} quasi-coherent we have $R^p f_* \mathcal{F} = 0$ for $p > 0$ since quasi-coherent higher cohomology vanishes on affine schemes. \square

Example 7.3. If X is a projective scheme over a Noetherian ring R . For closed immersion $X \hookrightarrow \mathbb{P}_R^n$,

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}_R^n, j_* \mathcal{F})$$

for quasi-coherent \mathcal{O}_X -modules.

Lemma 7.4. If $\mathcal{F} : X \rightarrow Y$ is finite and X and Y are Noetherian then f_* preserves coherent sheaves.

Proof. Since f is affine it preserves quasi-coherent modules. Since the morphism is additionally finite on rings so it changes finite modules to finite modules on the affine open level. \square

Corollary 7.5. For any coherent \mathcal{F} on a scheme X projective over Noetherian R then the above proposition holds with $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ where \mathcal{L} is an ample line bundle.

Remark. Let X be Noetherian over Noetherian R then let $n = \max\{\dim X_s \mid s \in \text{Spec}(R)\}$ then $H^i(X, \mathcal{F}) = 0$ for $i > n$. Warning, this is not true for quasi-projective X over a Noetherian ring. For example, consider $X = \mathbb{A}_{\mathbb{Q}}^2 \setminus \{0\} \rightarrow \mathbb{A}_{\mathbb{Q}}^2$ is quasi-projective over $R = \mathbb{Q}[x, y]$ but X does not have finitely generated cohomology.

Lemma 7.6. Let X be projective over a field k then X has an open cover by $\dim X + 1$ affines.

Proof. Choose $X \hookrightarrow \mathbb{P}_k^n$ show that we can find $F \in k[T_0, \dots, T_n]_d$ s.t. $\dim(X \cap V(F)) < \dim X$. Namely, choose F not vanishing at the generic points of X by graded prime avoidance. Then we can repeat to get,

$$X \cap V(F_1) \cap \dots \cap V(F_{\dim X + 1}) = \emptyset$$

and thus,

$$X = (X \cap D_+(F)) \cup \dots \cup (X \cap D_+(F_{\dim X + 1}))$$

where these factors are affine. \square

Corollary 7.7. $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$ for \mathcal{F} quasi-coherent on X projective over a field.

Theorem 7.8 (Grothendieck). If (X, \mathcal{O}_X) is a Noetherian ringed space then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$ and any \mathcal{O}_X -module \mathcal{F} .

Remark. Since we can always choose $\mathcal{O}_X = \mathbb{Z}$ in the above theorem applies to all abelian sheaves.

Lemma 7.9. If X is qc and qs then for \mathcal{F}_i quasi-coherent and I an *arbitrary* index set,

$$H^p(X, \bigoplus_{i \in I} \mathcal{F}_i) = \bigoplus_{i \in I} H^p(X, \mathcal{F}_i)$$

Remark. The above is always true in general for *finite* I since biproducts preserve exact sequences and injectives.

Proof. It is enough to show the above for Čech cohomology for finite affine open covers. Thus, it is enough to show that,

$$\left(\bigoplus_{i \in I} \mathcal{F}_i \right) (U) = \bigoplus_{i \in I} \mathcal{F}_i(U)$$

If X is affine open in X (WAIT WHAT??) □

7.1 Duality

Lemma 7.10. Let R be a ring, M an R -module, and X qc + sep over R . And some $n \geq 0$ such that $H^{n+1}(X, \mathcal{F}) = 0$ for all \mathcal{F} quasi-coherent. Then, the functor $F : \mathbf{Qcoh}(\mathcal{O}_X) \rightarrow \mathbf{Mod}_R$ via $\mathcal{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathcal{F}), M)$ is representable by some $\omega_{X/R, M} \in \mathbf{Qcoh}(\mathcal{O}_X)$. That is,

$$F(-) = \mathrm{Hom}_{\mathcal{O}_X}(-, \omega_{X/R, M})$$

Example 7.11. For $X = \mathrm{Spec}(A)$ then we have $\widetilde{N} \mapsto \mathrm{Hom}_R(N_R, M)$. Then,

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_A(N, \mathrm{Hom}_R(A, M))$$

so we would have $\omega_{A/R, M} = \widetilde{\mathrm{Hom}_R(A, M)}$.

Proof. First note that F acts on direct sums as,

$$F\left(\bigoplus_{i \in I} \mathcal{F}_i\right) = \mathrm{Hom}_R\left(H^n(X, \bigoplus_{i \in I} \mathcal{F}_i), M\right) = \mathrm{Hom}_R\left(\bigoplus_{i \in I} H^n(X, \mathcal{F}_i), M\right) = \prod_{i \in I} \mathrm{Hom}_R(H^n(X, \mathcal{F}_i), M)$$

Furthermore, F takes epis to monos since given an exact sequence,

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

then we get,

$$H^n(X, \mathcal{F}_2) \longrightarrow H^n(X, \mathcal{F}_3) \longrightarrow H^{n+1}(X, \mathcal{F}_1) = 0$$

These together shows that F takes all small colimits to products. Then if F satisfies some mild set-theoretic condition then the adjoint functor theorem gives $\omega_{X/R, M}$ as a functor on M . The ideal goes as follows. We take,

$$\omega_{X/R, M} = \mathrm{colim}_{\mathcal{C}} \mathcal{F}$$

where \mathcal{C} is a category of pairs (\mathcal{F}, α) where \mathcal{F} is a quasi-coherent sheaf and $\alpha \in F(\mathcal{F})$ and $\mathrm{Hom}_{\mathcal{C}}((\mathcal{F}, \alpha), (\mathcal{G}, \beta)) = \varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $\varphi^* \beta = \alpha$. However, this category is big so we cannot take a total colimit over it. We must resolve this set-theoretic issue.

In the case R is Noetherian and X is finite type over R then any quasi-coherent \mathcal{F} can be written as a filtered colimit,

$$\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$$

with \mathcal{F}_i coherent. This means that in the colimit defining $\omega_{X/R, M}$ we can restrict to only coherent \mathcal{F} and there is a set of isomorphism classes of coherent sheaves. □

8 Mar 3

Remark. Here X will be a Noetherian scheme.

Lemma 8.1. Let X be a Noetherian scheme. Any presheaf on $\mathfrak{QCo}\mathfrak{h}(\mathcal{O}_X)$ which transforms colimits into limits is representable.

Lemma 8.2. Any quasi-coherent module \mathcal{F} on X is a filtered colimit of coherent \mathcal{O}_X -modules. (In fact \mathcal{F} is the rising union of its coherent submodules).

Corollary 8.3. For any $\mathcal{F} \in \mathfrak{QCo}\mathfrak{h}(\mathcal{O}_X)$ there exists an exact sequence,

$$\bigoplus_{j \in J} \mathcal{G}_j \longrightarrow \bigoplus_{i \in I} \mathcal{F}_i \longrightarrow 0$$

where \mathcal{F}_i and \mathcal{G}_j are coherent.

Lemma 8.4. There is a set of isomorphism classes of coherent \mathcal{O}_X -modules.

Proposition 8.5. Let X be finite type over R Noetherian. Let n be an integer s.t. $H^{n+1}(X, \mathcal{F}) = 0$ for any $\mathcal{F} \in \mathfrak{QCo}\mathfrak{h}(\mathcal{O}_X)$. Then, for any R -module M , the functor,

$$\mathcal{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathcal{F}), M)$$

is representable by $\omega_{X/R, M, n} \in \mathfrak{QCo}\mathfrak{h}(\mathcal{O}_X)$ i.e.

$$\mathrm{Hom}_R(H^n(X, \mathcal{F}), M) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_{X/R, M, n})$$

functorially in $\mathcal{F} \in \mathfrak{QCo}\mathfrak{h}(\mathcal{O}_X)$.

Remark. For any integer p and $\mathcal{F} = \mathrm{colim} \mathcal{F}_i$ is a filtered colimit of \mathcal{O}_X -modules on a Noetherian scheme (or qcqs scheme) we have,

$$H^p(X, \mathcal{F}) = \mathrm{colim} H^p(X, \mathcal{F}_i)$$

Theorem 8.1. If k is a field and $n \geq 0$. Then $\omega_{\mathbb{P}_k^n/k, k, n} = \mathcal{O}_{\mathbb{P}_k^n}(-n-1)$. In particular,

$$H^n(\mathbb{P}_k^n, \mathcal{F})^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_k^n}(-n-1))$$

functorially in $\mathcal{F} \in \mathfrak{QCo}\mathfrak{h}(\mathcal{O}_{\mathbb{P}_k^n})$.

Proof. It suffices to show for \mathcal{F} coherent. Pick a resolution,

$$\bigoplus_{j=1}^s \mathcal{O}_{\mathbb{P}_k^n}(e_j) \longrightarrow \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}_k^n}(d_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

Since $H^n(\mathbb{P}_k^n, -)$ is right exact (by dimension vanishing) we get,

$$\bigoplus_{j=1}^s H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(e_j)) \longrightarrow \bigoplus_{j=1}^r H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d_i)) \longrightarrow H^n(\mathbb{P}_k^n, \mathcal{F}) \longrightarrow 0$$

Then taking k -linear duals,

$$\begin{array}{ccccccc}
\bigoplus_{j=1}^s H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(e_j))^\vee & \longleftarrow & \bigoplus_{j=1}^r H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d_i))^\vee & \longleftarrow & H^n(\mathbb{P}_k^n, \mathcal{F}) & \longleftarrow & 0 \\
\parallel & & \parallel & & \vdots & & \\
\bigoplus_{j=1}^s H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-e_j - n - 1)) & \xleftarrow{t} & \bigoplus_{j=1}^r H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-d_i - n - 1)) & \longleftarrow & \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_k^n}(-n - 1)) & \longleftarrow & 0
\end{array}$$

Note that,

$$\mathcal{O}_{\mathbb{P}_k^n}(-d - n - 1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{\mathbb{P}_k^n}(d), \mathcal{O}_{\mathbb{P}_k^n}(-n - 1))$$

gives the above “transpose” map t above by functoriality in the first argument along with the fact,

$$H^0(\mathbb{P}_k^n, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}, \mathcal{G})) = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

□

9 March 5

9.1 Serre Duality for \mathbb{P}_k^n Continued.

Write $\omega = \mathcal{O}_{\mathbb{P}_k^n}(-n - 1)$ and $t : H^n(\mathbb{P}_k^n, \omega) \rightarrow k$ via the Check class,

$$\frac{1}{T_0 \cdots T_n} \mapsto 1$$

Then we know that ω represents the functor,

$$\mathcal{F} \mapsto H^n(\mathbb{P}_k^n, \mathcal{F})^\vee$$

on $\mathfrak{QCoh}(\mathcal{O}_X)$ with universal object t .

Theorem 9.1. For coherent modules \mathcal{F} , there is an isomorphism,

$$H^{n-i}(\mathbb{P}_k^n, \mathcal{F})^\vee = \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \omega)$$

Proof. Both sides are contravariant δ -functors in \mathcal{F} so it suffices to show that both are universal for which it suffices to show that both are coeffecable. For any coherent sheaf \mathcal{F} we can find,

$$\mathcal{O}_{\mathbb{P}_k^n}(-q) \oplus^r \twoheadrightarrow \mathcal{F}$$

and then for $i > 0$ we know,

$$H^{n-i}(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-q)) = 0 \quad \text{and} \quad \text{Ext}_{\mathcal{O}_X}^i(\mathcal{O}_{\mathbb{P}_k^n}(-q), \omega) = H^i(\mathbb{P}_k^n, \omega(q)) = H^i(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-n-1+q)) = 0$$

for sufficiently large $q \gg 0$ using our Cech calculations. □

Lemma 9.1. Let (X, \mathcal{O}_X) be a ringed space and \mathcal{E} a finite locally free \mathcal{O}_X -module. Then.

$$\text{Ext}_{\mathcal{O}_X}^i(\mathcal{E}, \mathcal{G}) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

Proof. Choose an injective resolution $\mathcal{G} \rightarrow \mathcal{I}^\bullet$ then,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet) = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet)) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)$$

Now I claim that $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$ is an injective resolution over $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}$. To see this, we use,

$$\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} -, \mathcal{I}^\bullet)$$

but \mathcal{I}^\bullet is injective and \mathcal{E} is flat so this is an exact functor. Taking cohomology of the first equality proves the lemma. \square

Remark. We could also just say, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, -) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$ so taking their derived functors gives the same thing. However, $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} -$ is exact so taking derived functors of $\Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$.

Remark. The perfect pairings,

$$\mathrm{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^i(\mathcal{F}, \omega) \times H^{n-i}(\mathbb{P}_k^n, \mathcal{F}) \rightarrow H^n(\mathbb{P}_k^n, \omega) \xrightarrow{t} k$$

factors through $H^n(\mathbb{P}_k^n, \omega)$. The first map can be realized as composition of ext classes or a cup product.

Remark. If \mathcal{F} is locally free then we have a diagram,

$$\begin{array}{ccc} H^i(\mathbb{P}_k^n, \mathcal{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \omega) \times H^{n-i}(\mathbb{P}_k^n, \mathcal{F}) & \longrightarrow & k \\ \downarrow & & \uparrow t \\ H^n(\mathbb{P}_k^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \mathcal{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \omega) & \longrightarrow & H^n(\mathbb{P}_k^n, \omega) \end{array}$$

which gives the same pairing using the unique evaluation pairing,

$$\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \mathcal{F}^\vee = \mathcal{F} \otimes_{\mathbb{P}_k^n} \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_k^n}) \rightarrow \mathcal{O}_{\mathbb{P}_k^n}$$

9.2 Dualizing Sheaves in General

Definition 9.2. Let X be proper over k and $\dim X = n$. A *dualizing sheaf* (ω_X, t) is a pair consisting of a coherent \mathcal{O}_X -module ω_X and a map $t : H^n(X, \omega_X) \rightarrow k$ which represents the functor,

$$\mathcal{F} \mapsto H^n(X, \mathcal{F})^\vee$$

Remark. We have proven, by abstract nonsense, that such a *quasi-coherent* dualizing sheaf exists but now we want to know when such a module is actually *coherent*.

Remark. Consider the case that X is the disjoint union of a curve and a surface. Then $H^2(X, -)$ ignores cohomology on the curve since it vanishes above $H^1(X, -)$. Thus the dualizing sheaf will be zero on the curve. To fix this one looks for a dualizing complex,

$$\omega_X^\bullet \in D^b(\mathfrak{D}\mathfrak{C}\mathfrak{o}\mathfrak{h}(\mathcal{O}_X))$$

such that $H^i(X, \mathcal{F})$ is dual to $\mathrm{Ext}_{\mathcal{O}_X}^{-i}(\mathcal{F}, \omega_X^\bullet)$.

Theorem 9.3. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}_k^n$,

$$\iota_*\omega_X \cong \mathcal{E}xt_{\mathcal{O}_X}^c(\iota_*\mathcal{O}_X, \omega_{\mathbb{P}_k^n})$$

where $c = n - \dim X$ is the codimension.

Lemma 9.4. Let $\iota : X \rightarrow Y$ be a closed immersion of schemes then $\iota_* : \mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_X) \rightarrow \mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_Y)$ defines an equivalence of categories onto its image which is the full subcategory of quasi-coherent \mathcal{O}_Y -modules \mathcal{F} such that $\mathcal{I} \cdot \mathcal{F} = 0$ for $\mathcal{I} = \ker(\mathcal{O}_Y \rightarrow \iota_*\mathcal{O}_X)$.

Remark. If X and Y are Noetherian schemes, then the above holds also for coherent modules.

Remark. If $f : X \rightarrow Y$ is an affine morphism, $\mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_X)$ is the category of pairs (\mathcal{F}, γ) with $\mathcal{F} \in \mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_Y)$ and $\gamma : f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}$ gives \mathcal{F} a $f_*\mathcal{O}_X$ -module structure meaning $f_*\mathcal{O}_X$ is a monoid object and $f_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}$ is an action of a monoid object.

10 Mar. 12

Lemma 10.1. Let A be Noetherian and M, N be finite-presentation A -modules and $X = \text{Spec}(A)$. Then,

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) = \widetilde{\text{Hom}_A(M, N)}$$

Proof. The isomorphism,

$$\text{Hom}_A(M, N)_f = \text{Hom}_{A_f}(M_f, N_f)$$

for finitely-presented modules patch together on the open sets $D(f)$ to give an isomorphism,

$$\widetilde{\text{Hom}_A(M, N)} = \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

□

Lemma 10.2. Let A be Noetherian and M, N be finite A -modules and $X = \text{Spec}(A)$. Then,

$$\mathcal{E}xt_{\mathcal{O}_X}^i(\widetilde{M}, \widetilde{N}) = \widetilde{\text{Ext}_A^i(M, N)}$$

Proof. This holds for $i = 0$ by the above. Then we apply dimension-shifting to prove this in general. Given a

□

Lemma 10.3. For $p < \dim P - \dim X$ we have,

$$\mathcal{E}xt_{\mathcal{O}_X}^p(\iota_*\mathcal{O}_X, \omega_P) = 0$$

Proof. This reduced to the algebra question, given $B = k[x_1, \dots, x_n] \twoheadrightarrow A$ then,

$$\text{Ext}_B^p(A, B) = 0$$

for $p < \dim B - \dim A$. To see this, recall we have $\iota : X \hookrightarrow P = \mathbb{P}_k^n$ then $X \cap D_+(T_i) \subset X$ and $D_+(T_i) = \text{Spec}(B)$. Then, $\omega_P|_{D_+(T_i)} = \mathcal{O}_X|_{D_+(T_i)} = \widetilde{B}$. Furthermore, $\iota : X \hookrightarrow P$ is affine (closed immersion) so $X \cap D_+(T_i) = \text{Spec}(A)$ for $A = B/I$.

Since B is Cohen-Macaulay we have vanishing for,

$$\text{depth}_I(A) \geq \dim B - \dim A$$

□

Proof.

□

Theorem 10.4. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}_k^n$,

$$\iota_*\omega_X \cong \mathcal{E}xt_{\mathcal{O}_X}^c(\iota_*\mathcal{O}_X, \omega_{\mathbb{P}_k^n})$$

where $c = n - \dim X$ is the codimension.

Proposition 10.5. If $\iota : X \rightarrow Y$ is a closed immersion then $\iota^*\iota_*\mathcal{F} = \mathcal{F}$ for any \mathcal{O}_X -module \mathcal{F} and if $\mathcal{I} \cdot \mathcal{G} = 0$ for some \mathcal{O}_Y -module \mathcal{F} then $\mathcal{G} = \iota_*\iota^*\mathcal{G}$ where $\mathcal{I} = \ker(\mathcal{O}_Y \rightarrow \iota_*\mathcal{O}_X)$.

11 Local Property

Definition 11.1. A property P of ring maps is *local* if,

- (a). $P(R \rightarrow A) \implies P(R_f \rightarrow A_f)$ for all $f \in R$
- (b). $P(R_f \rightarrow A)$ for some $f \in R$ then $P(R \rightarrow A_a)$ for any $a \in A$
- (c). if $P(R \rightarrow A_{a_i})$ for $(a_1, \dots, a_r) = A$ then $P(R \rightarrow A)$.

Definition 11.2. We say a morphism of schemes $f : X \rightarrow Y$ is locally P for some local property P if for each $x \in X$ there is an affine open $U = \text{Spec}(A)$ with $x \in U \subset X$ and $V = \text{Spec}(R)$ with $V \subset Y$ with $f(U) \subset V$ such that $P(R \rightarrow A)$.

Lemma 11.3. If $f : X \rightarrow Y$ is locally P then for any affine opens $U \subset X$ and $V \subset Y$ with $f(U) \rightarrow V$ then $P(\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U))$.

Remark.

12 Smooth Maps

Definition 12.1. A ring map $R \rightarrow A$ is *smooth* if it is of finite presentation,

$$A \cong R[x_1, \dots, x_n]/I$$

where I is finitely generated. Then consider,

$$I/I^2 \xrightarrow{d} \bigoplus_{i=1}^n \text{Ad}x_i$$

given by,

$$f \mapsto df = \sum \frac{\partial f}{\partial x_i} dx_i$$

Then d is injective and its cokernel is a projective A -module. Since $K = \text{coker } d$ is projective and finitely generated then it is locally free so it has a rank function. We say that $R \rightarrow A$ is *smooth of relative dimension n* if K is of constant rank n .

Remark. Smoothness satisfies the following,

- (a). local

(b). preserved under composition

(c). preserved under base change

Example 12.2. Take $R \rightarrow R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ such that,

$$\det \left(\frac{\partial f_j}{\partial x_i} \right)_{\substack{i=1, \dots, c \\ j=1, \dots, c}}$$

maps to an invertible element of A . Then,

$$\frac{(f_1, \dots, f_c)}{(f_1, \dots, f_c)^2} \rightarrow \bigoplus_{i=1}^n \text{Ad} x_i \rightarrow \text{coker}$$

In differential geometry, we have,

$$\begin{array}{ccc} f^{-1}(\{0\}) & \hookrightarrow & \mathcal{C}^n \\ & & \downarrow f \\ \{0\} & \hookrightarrow & \mathcal{C}^n \end{array}$$

Then $f^{-1}(\{0\})$ is a smooth manifold by the implicit function theorem. We call this situation standard smooth.

Lemma 12.3. A map $R \rightarrow A$ is smooth if and only if there exist a_i s.t. $(a_1, \dots, a_r) = A$ and $R \rightarrow A_{a_i}$ is standard smooth.

Definition 12.4. For a standard smooth ring map, $R \rightarrow A$ we can factor,

$$\begin{array}{ccc} R & \xrightarrow{\quad} & R[x_{n+1}, \dots, x_n] \\ & \searrow & \downarrow \\ & & R[x_1, \dots, x_n]/(f_1, \dots, f_c) \end{array}$$

Then the downward map is étale.

Definition 12.5. A smooth morphism of schemes is a morphism which is locally smooth.

Remark. Using the previous lemma, for any smooth morphism of schemes $X \rightarrow Y$ it is locally standard smooth so we can factor,

$$\begin{array}{ccccc} X & \hookleftarrow & U & & \\ \downarrow f & & \downarrow & \searrow & \\ Y & \hookleftarrow & V & \swarrow & \mathbb{A}^d \end{array}$$

Definition 12.6. A variety X over k is smooth iff $X \rightarrow \text{Spec}(k)$ is smooth.

Definition 12.7. A locally noetherian scheme X is *regular* or *nonsingular* iff $\mathcal{O}_{X,x}$ is regular at each $x \in X$.

Remark. For locally Noetherian schemes it suffices to check regularity on the closed points.

Theorem 12.8. If $X \rightarrow \operatorname{Spec}(k)$ is smooth then X is regular.

Theorem 12.9. If k is perfect then a variety is smooth iff it is regular.

Example 12.10. Let $k = \mathbb{F}_p(t)$ then take $\operatorname{Spec}(k[x]/(y^2 - (x^p - t)))$ which is regular but not smooth. Consider,

$$d(y^2 - x^p + t) = 2ydy + 0$$

and thus we have,

$$(f)/(f^2) \mapsto Adx \oplus Ady$$

which is injective but the cokernel is $A \oplus A/yA$ but A/yA has torsion so cannot be projective and thus not smooth.

However, we just need to check regularity at $(y, x^p - t) = (y) \subset A$ which is a height one ideal and generated by one element so $A_{(y)}$ is regular.

13 Differentials

Definition 13.1. For a ring map $\varphi : R \rightarrow A$ the A -module of differentials $\Omega_{A/R}$ is generated by the symbols da for $a \in A$ such that,

- (a). $d(a_1 + a_2) = da_1 + da_2$
- (b). $da_1 a_2 = da_1 \cdot a_2 + a_1 \cdot da_2$
- (c). $dr = 0$ for $r \in R$

Then $d_{R/A} : R \rightarrow \Omega_{R/A}$ is the universal derivation meaning that $\Omega_{R/A}$ represents the functor $\operatorname{Der}_R(A, -)$ i.e.

$$\operatorname{Hom}_A(\Omega_{A/R}, M) \cong \operatorname{Der}_R(A, M)$$

via $(f : \Omega_{A/R} \rightarrow M) \mapsto f \circ d_{A/R}$

(O8RL, O8RT)

Definition 13.2. Given a morphism of ringed spaces $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ then there is a universal derivation $d_{X/Y} : \mathcal{O}_X \rightarrow \Omega_{X/Y}$ s.t.

$$\operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{F}) \cong \operatorname{Der}_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$$

Where a derivation $\varphi : \mathcal{O}_X \rightarrow \mathcal{F}$ is an abelian map such that $\varphi(fs) = f\varphi(s) + \varphi(f)s$ and under $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ we send $s \in \mathcal{O}_Y(U)$ to $\varphi(s) = 0$.

Lemma 13.3. For a morphism of schemes $f : X \rightarrow Y$ we have,

$$\begin{array}{ccc} X & \longleftarrow & U = \operatorname{Spec}(A) \\ \downarrow f & & \downarrow \\ Y & \longleftarrow & V = \operatorname{Spec}(R) \end{array}$$

Then we have $\Omega_{X/Y}|_U = \widetilde{\Omega_{A/R}}$.

13.1 The Diagonal

(Tag O1R1) Consider $R \rightarrow A$ then consider the map,

$$\Omega_{A/R} \xrightarrow{\sim} J/J^2$$

via $da \mapsto a \otimes 1 - 1 \otimes a$ where $J = \ker(A \otimes_R A \rightarrow A)$ via $a \otimes b \mapsto ab$. This situation generalizes to Schemes in which,

$$\Omega_{X/Y} = \Delta_{X/Y}^*(\mathcal{J})$$

where \mathcal{J} is the sheaf of ideals of $\Delta : X \rightarrow X \times_Y X$ i.e. $\mathcal{J} = \ker(\mathcal{O}_{X \times_Y X} \rightarrow \Delta_{X/Y}^* \mathcal{O}_X)$. This is the conormal sheaf of $\Delta_{X/Y} : X \rightarrow X \times_Y X$.

14 Mar. 31

14.1 Conormal Sheaf

Definition 14.1. Let $\iota : Z \hookrightarrow X$ be a closed immersion with $\mathcal{J} = \ker(\mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z)$. Then the conormal sheaf is $\mathcal{C}_{Z/X} = \iota^* \mathcal{J}$ and thus $\iota_* \mathcal{C}_{Z/X} = \mathcal{J}/\mathcal{J}^2$.

Remark. Affine locally $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. Then we get the A/I -module $I \otimes_A A/I = I/I^2$ pushing forward to the A -module I/I^2 .

Proposition 14.2. Let $\iota : Z \hookrightarrow Z$ is a closed immersion over S then there is an exact sequence,

$$\mathcal{C}_{Z/X} \longrightarrow \iota^* \Omega_{X/S} \longrightarrow \Omega_{Z/S} \longrightarrow 0$$

If $Z \rightarrow S$ is smooth then it is short exact.

Proof. This follows from the exact sequence, with $R \rightarrow A$ and $B = A/I$,

$$I/I^2 \xrightarrow{\alpha} \Omega_{A/R} \otimes_A B \xrightarrow{\beta} \Omega_{B/R} \longrightarrow 0$$

given by $f \mapsto df \otimes 1$ and $da \otimes a \mapsto d\bar{a}$. The second, β , is clearly surjective. Furthermore, given $B \rightarrow \Omega_{A/R} \otimes_A B/\text{Im}(\alpha)$ via $\bar{a} \mapsto da \otimes 1$ is a well-defined R -derivation and thus factors through $\Omega_{B/R}$ giving the required isomorphism. \square

Proposition 14.3. Given

Remark. The complex,

$$I/I^2 \rightarrow \bigoplus_{i=1}^n \text{Ad} x_i$$

for $A = R[x_1, \dots, x_n]/I$. This is a truncated version of $NL_{A/R}$, the naive cotangent complex of A/R and $\text{coker } d = \Omega_{A/R}$ and $H^{-1}(NL_{A/R}) = H^{-1}(L_{A/R})$.

Remark. In the case $A = R[x_1, \dots, x_n]/I$, we get,

$$I/I^2 \longrightarrow \Omega_{R[x_1, \dots, x_n]/R} \otimes A \longrightarrow \Omega_{A/R} \longrightarrow 0$$

and thus we see $\text{coker } d = \Omega_{A/R}$ as required above.

Definition 14.4. Consider the presentation $R[A] \rightarrow A$ and $J = \ker(R[A] \rightarrow A)$. Then we can define the naive cotangent complex is $NL_{A/R} := (J/J^2 \xrightarrow{d} \Omega_{R[A]/R} \otimes_{R[A]} A)$ with J/J^2 in degree -1 .

Remark. The naive cotangent complex is homotopy equivalent to the complex above for a choice of presentation therefore we see the following.

Proposition 14.5. A ring map $R \rightarrow A$ is smooth if the naive cotangent complex $NL_{A/R}$ is quasi-isomorphic to a projective module in degree zero. Explicitly $H^{-1}(NL_{A/R}) = 0$ and $H^0(NL_{A/R}) = \Omega_{A/R}$ is projective.

15 April 2

Theorem 15.1. Let $f : X \rightarrow S$ be a morphism of schemes then the FAE,

- (a). f is smooth
- (b). f is locally finite presentation, flat, and the fibers $X_s \rightarrow \operatorname{Spec}(\kappa(s))$ are smooth.
- (c). f is locally finitely presentation and f is formally smooth.

Definition 15.2. A morphism $f : X \rightarrow S$ is formally smooth iff every diagram of the form,

$$\begin{array}{ccc} \operatorname{Spec}(A/I) & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow f \\ \operatorname{Spec}(A/I^2) & \longrightarrow & S \end{array}$$

gives a map $\operatorname{Spec}(A/I^2) \rightarrow X$ making it commutes.

15.1 Smoothness over Fields

Let k be a field and $S = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ with $\mathfrak{q} \subset S$ a prime ideal. Let $d = \dim S_{\mathfrak{q}} + \operatorname{trdeg}_k(\kappa(\mathfrak{q}))$.

Proposition 15.3. The following are equivalent,

- (a). S is smooth over k at \mathfrak{q}
- (b). the rank over $\kappa(\mathfrak{q})$ of the matrix,

$$\left(\frac{\partial f}{\partial x_i} \right)_{j=1, \dots, m / i=1, \dots, n} \pmod{\mathfrak{p}}$$

is equal to $n - d$ (it is always at most $n - d$).

Remark. Smoothness at \mathfrak{q} implies that S is regular at \mathfrak{q} but this is not sufficient.

Remark. If S is equidimensional of dimension d then the nonsmooth locus of $\operatorname{Spec}(S)$ is the vanishing locus of all $(n - d) \times (n - d)$ minors of the matrix,

$$\left(\frac{\partial f}{\partial x_i} \right)_{j=1, \dots, m / i=1, \dots, n}$$

Example 15.4. Let $Z = V(F_1, \dots, F_c) \subset \mathbb{P}_k^n = X$ be a complete intersection of type (d_1, \dots, d_c) . We saw that,

$$\omega_Z \cong \mathcal{E}xt_{\mathcal{O}_X}^i(\mathcal{O}_Z, \omega_X) = \omega_X(d_1 + \dots + d_c)|_Z$$

Theorem 15.5. If $Z \subset X = \mathbb{P}_k^n$ is a smooth projective scheme equidimensional of $\dim Z = n - c$ then,

$$\omega_Z = \mathcal{E}xt_{\mathcal{O}_X}^c(\mathcal{O}_Z, \omega_X) \cong \omega_X|_Z \otimes \bigwedge^c \mathcal{C}_{Z/X}^\vee$$

Proof. See HAR [III.7.11 + II.8.17] The exact sequence,

$$0 \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \Omega_{X/k}|_Z \longrightarrow \Omega_{Z/k} \longrightarrow 0 \text{ implies that } \mathcal{C}_{Z/X} \text{ is a vector bundles of rank } c.$$

Then Z is locally a complete intersection so we can conclude using our previous argument. \square

Lemma 15.6. Consider an exact sequence of locally free sheaves,

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow 0$$

then there is a canonical isomorphism,

$$\bigwedge^{\text{top}} \mathcal{F}_1 \otimes_{\mathcal{O}_X} \bigwedge^{\text{top}} \mathcal{F}_3 = \bigwedge^{\text{top}} \mathcal{F}_2$$

Proof. Consider the map \square

Proposition 15.7. For $X = \mathbb{P}_k^n$ we know,

$$\omega_X \cong \mathcal{O}_X(-n-1) \cong \bigwedge^n \Omega_{X/k}$$

Proof. There is a short exact sequence of locally free sheaves,

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \bigoplus_{i=0}^n \mathcal{O}_X(-1) dT_i \longrightarrow \mathcal{O}_X \longrightarrow 0$$

on $D_+(T_i)$ given by,

$$d\left(\frac{T_i}{T_j}\right) = \frac{dT_i}{T_j} - \frac{T_i dT_j}{T_j^2}$$

and,

$$\frac{dT_i}{T_j} \mapsto \frac{T_i}{T_j}$$

Then, taking top exterior powers gives.

$$\bigwedge^{n+1} \left(\bigoplus_{i=0}^n \mathcal{O}_X(-1) \right) = \mathcal{O}_X(-n-1)$$

\square

Lemma 15.8 (Adjunction). Let $Z \subset X$ be a smooth closed subscheme. Then,

$$\bigwedge^{\dim Z} \Omega_{Z/k} \cong \bigwedge^{\dim X} \Omega_{X/k}|_Z \otimes \bigwedge^c \mathcal{C}_{Z/X}^\vee$$

Proof. This is exactly the top exterior power of the sequence,

$$0 \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \Omega_{X/k}|_Z \longrightarrow \Omega_{Z/k} \longrightarrow 0$$

□

Remark. This implies that $K_Z = K_X|_Z + c_1(N_{Z/X})$.

Theorem 15.9. Let Z be a smooth projective variety over k . Then $\omega_Z \cong \bigwedge^{\dim Z} \Omega_{Z/k}$.

Proof. Choose an embedding $Z \hookrightarrow \mathbb{P}_k^n$ then we use adjunction for dualizing sheaves and for canonical sheaves which are the same to conclude. □