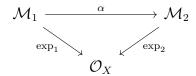
1 Definitions

Definition 1.0.1. A pre-log structure on X is a sheaf of commutative monoids \mathcal{M} on $X_{\text{\'et}}$ along with a morphism of sheaves of monoids,

$$\exp: \mathcal{M} \to (\mathcal{O}_X, \times)$$

A morphism of pre-log structues $\alpha: (\mathcal{M}_1, \exp_1) \to (\mathcal{M}_2, \exp_2)$ is a morphism of sheaves of monoids $\alpha: \mathcal{M}_1 \to \mathcal{M}_2$ such that the diagram,



commutes.

Remark. From now on, when we write \mathcal{O}_X in the category of monoids we mean (\mathcal{O}_X, \times) .

Example 1.0.2. Some examples,

- (a) $\mathcal{O}_X^{\times} \hookrightarrow \mathcal{O}_X$
- (b) $\mathcal{O}_X \to \mathcal{O}_X$

Example 1.0.3. Suppose that P is a commutative monoid and $\exp: P \to \Gamma(W, \mathcal{O}_W)$ is a map then we get a pre-log structure $\exp: \underline{P} \to \mathcal{O}_W$ by adjunction.

For a monoid P let $X_P = \operatorname{Spec}(k[P])$ then we see,

$$\operatorname{Hom}(W, X_P) = \{ \text{pre-log structures } \underline{P} \to \mathcal{O}_W \}$$

Definition 1.0.4. A pre-log structure exp : $\mathcal{M} \to \mathcal{O}_X$ is called a *log structure* if $\exp^{-1}(\mathcal{O}_X^{\times}) \to \mathcal{O}_X^{\times}$ is an isomorphism.

Remark. In particular, there is a unique morphism $\alpha: \mathcal{O}_X^{\times} \to \mathcal{M}$ making the diagram,

$$\mathcal{O}_X^{\times} \xrightarrow{\alpha} \mathcal{M}$$

$$\downarrow^{\exp}$$
 \mathcal{O}_X

commute so all log-structures "lie between" \mathcal{O}_X^{\times} and \mathcal{O}_X in the sense that \mathcal{O}_X^{\times} is the initial object and \mathcal{O}_X the terminal object of the categories of log structures.

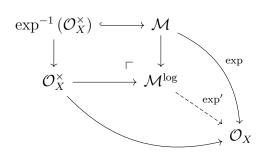
Definition 1.0.5. A log scheme (X, \mathcal{M}) is a scheme X equipped with a log structure \mathcal{M} .

Proposition 1.0.6. Let $Z \subset X$ be a closed subset. Let $\mathcal{M}_Z \subset \mathcal{O}_X$ be the subsheaf of functions invertible on $U = X \setminus Z$. Then \mathcal{M}_Z is a log structure.

Proof. This just says that if $f \in \mathcal{O}_X(V)$ then it is invertible on U which is obvious by restriction. \square

1.1 Logification

Proposition 1.1.1. There is a left-adjoint $\mathcal{M} \mapsto \mathcal{M}^{\log}$ to the forgetful functor $\{\log_X\} \to \{\text{pre-log}_X\}$ given by,

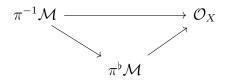


Proof. (DO THIS!!!)

Definition 1.1.2. Given a morphism $\pi: X \to Y$ and a pre-log structure $\exp: \mathcal{M} \to \mathcal{O}_Y$ on Y. Then the pullback is $\pi^{-1}\mathcal{M} \to \pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. If \mathcal{M} is a log structure on Y then the pullback is,

$$\pi^{\flat}\mathcal{M} = (\pi^{-1}\mathcal{M})^{\log}$$

By the universal property there is a map,



Remark. There is a tautological log structure $\underline{P}^{\log} \to \mathcal{M}_{X_P}$. In the toric case $P = \sigma^{\vee} \cap M$ we can define $Z = X_{\sigma} \setminus T$ where $T = \operatorname{Spec}(k[M])$ is the torus then I claim that $\underline{P}^{\log} = \mathcal{M}_Z$. (PROVE THIS!!)

Remark. Consider a morphism $W \to X_P$ then the log structure $\underline{P}^{\log} \to \mathcal{O}_W$ is $\pi^{\flat} \mathcal{M}_P$. (PROVE TIS!!)

Proposition 1.1.3. Let $\pi: X \to Y$ be a morphism. Then the following diagram commutes,

$$\begin{cases} \operatorname{pre-log}_Y \rbrace & \xrightarrow{\pi^{-1}} \left\{ \operatorname{pre-log}_X \right\} \\ & \downarrow^{\log} & \downarrow^{\log} \\ \left\{ \log_Y \right\} & \xrightarrow{\pi^{\flat}} & \left\{ \log_X \right\} \end{cases}$$

Proof. IS THIS ACTUALLY TRUE??

Definition 1.1.4. A morphism $f:(X,\mathcal{M})\to (Y,\mathcal{N})$ of log schemes is a morphism of schemes $f:X\to Y$ and a morphism of sheaves of monoids α such that,

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\alpha} & \pi_* \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{O}_V & \longrightarrow & \pi_* \mathcal{O}_V
\end{array}$$

commutes. Equivalently this is a morphism of log structures $\alpha : \pi^{\flat} \mathcal{N} \to \mathcal{M}$ because then automatically the diagram,

$$\begin{array}{cccc}
\pi^{-1}\mathcal{N} & \longrightarrow & \pi^{\flat}\mathcal{N} & \longrightarrow & \mathcal{M} \\
\downarrow & & & \downarrow \\
\pi^{-1}\mathcal{O}_{Y} & \longrightarrow & \mathcal{O}_{X}
\end{array}$$

commutes.