Mathematics GU4051 Topology
Assignment # 5

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Problem 1.

Let $f_1 : V_1 \to Y$ and $f_2 : V_2 \to Y$ be functions on sets V_1 and V_2 with $V_1 \cup V_2 = X$. Also, let $f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2}$ so that the function, $f : X \to Y$ given by,

$$f(x) = \begin{cases} f_1(x) & x \in V_1 \\ f_2(x) & x \in V_2 \end{cases}$$

is well defined. The following fact will be of use:

$$f^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U)$$

This equality hold because:

$$x \in f^{-1}(U) \iff f(x) \in U \iff f_1(x) \in U \text{ or } f_2(x) \in U \iff x \in f_1^{-1}(U) \cup f_2^{-1}(U)$$

Suppose that f is continuous on any (not necessarily open or closed) sets V_1 and V_2. Then for any open $U \subset Y$ the set $f^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U)$ is open in X. Thus, $f^{-1}(U) \cap V_1$ is open in V_1. However,

$$f_2^{-1}(U) \cap V_1 \subset f_1^{-1}(U) \cap V_1$$

because if $x \in f_2^{-1}(U) \cap V_1$ then $x \in V_1 \cap V_2$ so $f_1(x) = f_2(x) \in U$ so $x \in f_1^{-1}(U)$. Thus,

$$f^{-1}(U) \cap V_1 = (f_1^{-1}(U) \cap V_1) \cup (f_2^{-1}(U) \cap V_1) = f_1^{-1}(U) \cap V_1 = f_1^{-1}(U)$$

because $f_1^{-1}(U) \subset V_1$ and thus $f_1^{-1}(U)$ is open in V_1. Thus, f_1 is continuous. The continuity of f_2 follows identically. Now, we will prove the converse in the cases that V_1 and V_2 are both closed or both open.

(a). Suppose that V_1 and V_2 are open and that $f_1 : V_1 \to Y$ and $f_2 : V_2 \to Y$ are continuous. For an open $U \subset Y$, consider

$$f^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U)$$

However, by continuity, $f_1^{-1}(U)$ is open in V_1 and $f_2^{-1}(U)$ is open in V_2. Thus, there are sets $S_1, S_2 \subset X$ which are open in X s.t. $f_1^{-1}(U) = S_1 \cap V_1$ and $f_2^{-1}(U) = S_2 \cap V_2$. Therefore, these sets are open in X because V_1 and V_2 are open. Thus,

$$f^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U)$$

is open in X so f is continuous.

(b). Suppose that C_1 and C_2 are closed and that $f_1 : V_1 \to Y$ and $f_2 : V_2 \to Y$ are continuous. For a closed $D \subset Y$, consider

$$f^{-1}(D) = f_1^{-1}(D) \cup f_2^{-1}(D)$$

However, by continuity, $f_1^{-1}(D)$ is closed in C_1 and $f_2^{-1}(D)$ is closed in C_2. Thus, there are sets $S_1, S_2 \subset X$ which are closed in X s.t. $f_1^{-1}(D) = S_1 \cap C_1$ and $f_2^{-1}(D) = S_2 \cap C_2$. Therefore, these sets are closed in X because C_1 and C_2 are closed. Thus,

$$f^{-1}(D) = f_1^{-1}(D) \cup f_2^{-1}(D)$$

is closed in X so f is continuous.

Problem 2.

Let $\{A_n \mid n \in \mathbb{N}\}$ be a sequence of connected subsets of X s.t. for every $n \in \mathbb{N} : A_n \cap A_{n+1} \neq \varnothing$. Consider the sequence of sets, $C_n = \bigcup\limits_{i=0}^{n} A_i$. By induction, I will show that these sets are connected. $C_0 = A_0$ which is by hypothesis connected. Suppose that C_n is connected then $C_{n+1} = C_n \cup A_{n+1}$ and $A_n \subset C_n$ but $A_n \cap A_{n+1} \neq \varnothing$ so $C_n \cap A_{n+1} \neq \varnothing$. Thus, C_{n+1} is the union of two intersecting connected sets and is therefore connected. Since $A_n \subset C_n$ and $C_n \subset \bigcup\limits_{i=0}^{\infty} A_i$ we have,

$$\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} C_n$$

and $A_0 \subset C_n$ so $A_0 \subset \bigcap\limits_{n=0}^{\infty} C_n$ which is therefore not empty because $A_0 \cap A_1 \neq \varnothing$ so A_0 is not empty. Thus, because every C_n is connected and the total intersection in nonempty, the union is also connected.

Problem 3.

I claim that every proper nonempty subset $A \subset X$ has $\partial A \neq \varnothing$ if and only if X is connected. Note, I claim "yes" to both the question and its converse and I use $\partial A = \mathrm{Bd} A$.

Proof. Suppose that some $A \subset X$ has $\bar{A} \backslash A^\circ = \varnothing$. Thus, $x \in \bar{A} \implies x \in A^\circ$ so $\bar{A} \subset A^\circ$ but $A^\circ \subset A \subset \bar{A}$ so $A^\circ = A = \bar{A}$. Furthermore, A° is open and $\bar{A}$ is closed so A is clopen. Thus, if X is connected then A must be nonproper or empty. Converseley, if X is disconnected then there exists a proper nonempty clopen set $U \subset X$ then $U = U^\circ = \bar{U}$ because both U and $X \backslash U$ are closed thus $\partial U = \bar{U} \backslash U^\circ = \varnothing$. $\square$

Problem 4.

(a). Suppose that $f : [0,1] \to (0,1)$ is a homeomorphism. Take $A = [0,1] \backslash \{1\} = [0,1)$ then by bijectivity, $f(A) = (0,1) \backslash \{f(1)\}$. However, $0 < f(1) < 1$ so $f(A)$ is not an interval and thus disconnected. However, $A = [0,1)$ is connected and by assumption f is continuous so $f(A)$ must be connected which is a contradiction.

Suppose that $f : (0,1] \to (0,1)$ is a homeomorphism. Take $A = (0,1]\backslash\{1\} = (0,1)$ then by bijectivity, $f(A) = (0,1)\backslash\{f(1)\}$. However, $0 < f(1) < 1$ so $f(A)$ is not an interval and thus disconnected. However, $A = (0,1)$ is connected and by assumption f is continuous so $f(A)$ must be connected which is a contradiction.

Suppose that $f : [0,1] \to (0,1]$ is a homeomorphism. Take $A = [0,1]\backslash\{0\} = (0,1]$ then by bijectivity, $f(A) = (0,1]\backslash\{f(0)\}$. However, $0 < f(0) \leq 1$. In the case $f(0) < 1$, we proceed as above, since $f(A)$ is not an interval and thus disconnected. However, $A = [0,1)$ is connected and by assumption f is continuous so $f(A)$ must be connected which is a contradiction. In the case $f(0) = 1$, we have $A = (0,1]$ and $f(A) = (0,1)$ which we allready know are not homeomorphic contradicting Lemma **??**.

(b). Suppose that $f : \mathbb{R} \to \mathbb{R}$ for $n > 1$ is a homeomorphism. Take $A = \mathbb{R}\backslash\{0\}$ then $f(A) = \mathbb{R}^n\backslash\{f(0)\}$ but by Lemma **??**, $\mathbb{R}^n\backslash\{f(0)\}$ is connected. However, $\mathbb{R}\backslash\{0\}$ is not an interval so it is diconnected. However, by Lemma **??**, A and $f(A)$ are homeomorphic which is a contradiction because connectedness is preserved by homeomorphism.

Problem 5.

Let $S \subset \mathbb{R}^2$ be countable. Now, consider two points $\mathbf{v}, \mathbf{u} \in \mathbb{R}^2\backslash S$. Consider the set of lines passing through a given point:

$$\mathcal{L}(\mathbf{w}) = \{L(\mathbf{w}, \theta) \mid \theta \in [0, \tfrac{\pi}{2}]\} \text{ with } L(\mathbf{w}, \theta) = \{\mathbf{r} \in \mathbb{R}^2 \mid (r_x - w_x)\sin\theta = (r_y - w_x)\sin\theta\}$$

$L(\mathbf{w}, \theta)$ contains $(\cos\theta, \sin\theta) + \mathbf{w}$. Also, no two distinct lines intersect at more than one point so the number of lines about any point is uncountable since it is in bijection with the points on a half circle surrounding $\mathbf{w}$. Thus, every point has a line through it which does not intersect S. If this were false, we could construct a map $f : \mathcal{L}(\mathbf{w}) \to S$ given by mapping a line L to the smallest $s \in S$ (S is in bijection to a set of integers and thus can be well-ordered) that intersects L. This map would be an injection because two distinct lines through the same point cannot intersect but at that point. However, there cannot exist an injection from a uncountable set to a countable set so there must exist some (uncountably many in fact) lines which do not intersect S. Choose $\tilde{L}(\mathbf{v})$ and $\tilde{L}(\mathbf{u})$ to be two such lines which intersect eachother at $\mathbf{r}$ which is always possible because there is exactly one line though $\mathbf{u}$ which is parallel to $\tilde{L}(\mathbf{v})$ so take any other of the uncoutably many options for $\tilde{L}(\mathbf{u})$. Define $\gamma : [0,1] \to \mathbb{R}^2\backslash S$ by

$$\gamma(t) = \begin{cases} \gamma_1(t) = \mathbf{v} + 2t(\mathbf{r} - \mathbf{v}) & x \in [0, \frac{1}{2}] \\ \gamma_2(t) = \mathbf{r} + (2t-1)(\mathbf{u} - \mathbf{r}) & x \in [\frac{1}{2}, 1] \end{cases}$$

Since $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed and intersect only at $\frac{1}{2}$ where $\gamma_1(\frac{1}{2}) = \mathbf{v} + (\mathbf{r} - \mathbf{v}) = \mathbf{r}$ and $\gamma_2(\frac{1}{2}) = \mathbf{r}$ so by the glueing lemma, γ is continuous since γ_1 and γ_2 are continuous. Also, $\gamma(0) = \mathbf{v}$ and $\gamma(1) = \mathbf{r} + (\mathbf{u} - \mathbf{r}) = \mathbf{u}$. Finally, γ is well defined because $\gamma_1(t) \in \tilde{L}(\mathbf{v}) \subset \mathbb{R}^2\backslash S$ and $\gamma_2(t) \in \tilde{L}(\mathbf{u}) \subset \mathbb{R}^2\backslash S$. Thus, $\gamma(t) \in S$ so γ is a path from $\mathbf{u}$ to $\mathbf{v}$ proving that $\mathbb{R}^2\backslash S$ is path connected.

Problem 6.

Let $A \subset \mathbb{R}^n$ be connected and open. Take $\mathbf{x}_0 \in A$ and consider

$$U = \{\mathbf{x} \in A \mid \exists \text{ path from } \mathbf{x}_0 \text{ to } \mathbf{x}\}$$

Consider $\mathbf{z} \in U$, then $\mathbf{z} \in A$ which is open so $\exists \delta > 0$ s.t. $\mathbf{z} \in B_\delta(\mathbf{z}) \subset A$. Also, there exists a continuous map $\gamma : [0,1] \to A$ s.t. $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = \mathbf{z}$. For any $\mathbf{x} \in B_\delta(\mathbf{z})$, take the function $\gamma_G : [0,2] \to A$ given by:

$$\gamma_G(t) = \begin{cases} \gamma(t) & t \in [0,1] \\ r(t) = \mathbf{z} + (t-1)(\mathbf{x} - \mathbf{z}) & t \in [1,2] \end{cases}$$

γ_G is well defined because $|r(t) - \mathbf{z}| = (t-1)|\mathbf{x} - \mathbf{z}| < \delta$ so $r(t) \in B_\delta(\mathbf{z}) \subset A$. Since γ and $r(t)$ are continuous and $[0,1] \cap [1,2] = \{1\}$ with $\gamma(1) = \mathbf{z} = [\mathbf{z} + (t-1)(\mathbf{x} - \mathbf{z})]_{t=1}$ so by the gluing lemma, γ_G is continuous. Because $f : [0,1] \to [0,2]$ given by $f(x) = 2x$ is a homeomorphism, $\tilde{\gamma} = \gamma_G \circ f : [0,1] \to A$ is a continuous function with $\tilde{\gamma}(0) = \gamma(0) = \mathbf{x}_0$ and $\tilde{\gamma}(1) = \gamma_G(f(1)) = \gamma_G(2) = \mathbf{x}$. Thus, $\tilde{\gamma}$ is a path from $\mathbf{x}_0$ to $\mathbf{x}$ so $\mathbf{x} \in U$. Thus, $B_\delta(\mathbf{z}) \subset U$ so U is open.

Likewise, consider $\mathbf{z} \in A \backslash U$, then $\mathbf{z} \in A$ which is open so $\exists \delta > 0$ s.t. $\mathbf{z} \in B_\delta(\mathbf{z}) \subset A$. Suppose that there exists a continuous map $\gamma : [0,1] \to A$ s.t. $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = \mathbf{x}$ with $\mathbf{x} \in B_\delta(\mathbf{z})$. Then take the function $\gamma_G : [0,2] \to A$ given by:

$$\gamma_G(t) = \begin{cases} \gamma(t) & t \in [0,1] \\ r(t) = \mathbf{x} + (t-1)(\mathbf{z} - \mathbf{x}) & t \in [1,2] \end{cases}$$

γ_G is well defined because $|r(t) - \mathbf{x}| = (t-1)|\mathbf{z} - \mathbf{x}| < \delta$ so $r(t) \in B_\delta(\mathbf{z}) \subset A$. Since γ and $\mathbf{x} + t(\mathbf{z} - \mathbf{x})$ are continuous and $[0,1] \cap [1,2] = \{1\}$ with $\gamma(1) = \mathbf{x} = [\mathbf{x} + (t-1)(\mathbf{z} - \mathbf{x})]_{t=1}$ so by the gluing lemma, γ_G is continuous. Because $f : [0,1] \to [0,2]$ given by $f(x) = 2x$ is a homeomorphism, $\tilde{\gamma} = \gamma_G \circ f : [0,1] \to A$ is a continuous function with $\tilde{\gamma}(0) = \gamma(0) = \mathbf{x}_0$ and $\tilde{\gamma}(1) = \gamma_G(f(1)) = \gamma_G(2) = \mathbf{z}$. Thus, $\tilde{\gamma}$ is a path from $\mathbf{x}_0$ to $\mathbf{z}$ so $\mathbf{z} \in U$ a contradiction. Thus, $\mathbf{x} \notin U$ so $B_\delta(\mathbf{z}) \subset A \backslash U$ so $A \backslash U$ is open. Thus, U is clopen but $\mathbf{x}_0 \in U$ so because A is connected, $U = A$ and therefore A is path-connected.

Lemmas

Lemma 0.1. For any $\mathbf{x}_0 \in \mathbb{R}^n$ with $n > 1$, the set $\mathbb{R}^n \backslash \{\mathbf{x}_0\}$ with the subspace topology is connected.

Proof. Take $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{x}_0\}$. Suppose that $\mathbf{x}_0 - \mathbf{x} \in \text{span}\{\mathbf{y} - \mathbf{x}\}$ then define $\gamma : [0,1] \to \mathbb{R}^n \backslash \{\mathbf{x}_0\}$ to be the map:

$$\gamma(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x}) + t(1-t)\mathbf{b}$$

Where $\mathbf{b}$ is any vector not in the span of $\mathbf{y} - \mathbf{x}$. Such a $\mathbf{b}$ exists because $n > 1$. Now, $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{x} + (\mathbf{y} - \mathbf{x}) = \mathbf{y}$. Also, γ is well defined because if $\gamma(t) = \mathbf{x}_0$ then,

$$\mathbf{b} = \frac{1}{t(1-t)}(\mathbf{x}_0 - \mathbf{x}) - \frac{1}{1-t}(\mathbf{y} - \mathbf{x}) \in \text{span}\{\mathbf{y} - \mathbf{x}\}$$

which contradicts the definition of $\mathbf{b}$. The previous formula is well defined because $t \neq 0$ and $t \neq 1$ since $\gamma(0) = \mathbf{x} \neq \mathbf{x}_0$ and $\gamma(1) = \mathbf{y} \neq \mathbf{x}_0$. Thus, $\mathrm{Im}(\gamma) \subset \mathbb{R}^n \setminus \{\mathbf{x}_0\}$

Otherwise, if $\mathbf{x}_0 - \mathbf{x} \notin \mathrm{span}\{\mathbf{y} - \mathbf{x}\}$ then define $\gamma : [0,1] \to \mathbb{R}^n \setminus \{\mathbf{x}_0\}$ to be the map:

$$\gamma(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$$

Now, $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{x} + (\mathbf{y} - \mathbf{x}) = \mathbf{y}$. Also, γ is well defined because if $\gamma(t) = \mathbf{x}_0$ then $\mathbf{x}_0 - \mathbf{x} = t(\mathbf{y} - \mathbf{x})$ conntradicting the fact that $\mathbf{x}_0 - \mathbf{x} \notin \mathrm{span}\{\mathbf{y} - \mathbf{x}\}$. Thus, $\mathrm{Im}(\gamma) \subset \mathbb{R}^n \setminus \{\mathbf{x}_0\}$. These maps γ are continuous with respect to the Euclidean metric by $\epsilon - \delta$ arguments. Therefore, $\mathbb{R}^n \setminus \{\mathbf{x}_0\}$ is path connected and thus connected. $\square$

Lemma 0.2. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic topological spaces with homeomorphism $f : X \to Y$ then for any $A \subset X$, A is homeomorphic to $f(A)$ with the subspace topologies.

Proof. For $A \subset X$ define $g : A \to f(A)$ by $g : x \mapsto f(x)$ which is trivially a surjection because $\mathrm{Im}(g) = f(A)$. Since f is a bijection, f is injective so $g(x) = g(y) \implies f(x) = f(y) \implies x = y$ so g is a bijection. We must check that g and g^{-1} are continuous. If U is open in $f(A)$ then $\exists V \in \mathcal{T}_Y$ s.t. $U = V \cap f(A)$ then,

$$x \in g^{-1}(U) \iff g(x) \in U \text{ and } x \in A \iff f(x) \in V \cap f(A) \text{ and } x \in A \iff x \in f^{-1}(V) \cap A$$

so $g^{-1}(U) = f^{-1}(V) \cap A$ which is open in A because f is continuous and $V \in \mathcal{T}_Y$. Also, if U is open in A then $U = A \cap V$ with V open in X and consider $(g^{-1})^{-1}(U)$.

$$x \in (g^{-1})^{-1}(U) \iff g^{-1}(x) \in U \iff x \in g(U) = f(U))$$

Thus, $(g^{-1})^{-1}(U) = f(U) = f(A \cap U) = f(A) \cap f(U)$ which is open in $f(A)$. In the last line I have used $f(A \cap B) = f(A) \cap f(B)$ which follows from injectivity. Thus, g^{-1} is a continuous function. $\square$