1 Moduli Spaces of Cubic Threefolds

We can define such a moduli space as follows,

$$M = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(3)))^{\text{smooth}} // \mathrm{SL}_5(\mathcal{C})$$

Then $\dim M = 35 - 1 - 24 = 10$. This is quasi-projective but not projective. We can find a projective model as follows,

$$M^{\text{SIT}} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(3))^{\text{SS}} // \mathrm{SL}_5(\mathcal{C})$$

which has finite quotient singularities. Now,

$$M^{\text{SIT}} = M^K \setminus K$$

where K is the K-stabilit

1.1 Period Maps

Jacobians,

$$IJ(x) = H^{2,1}(X)^*/H_3(X, \mathbb{Z}) \in A_5$$

Clemes - Griffith: X is not rational. Then we get a map $M \to A_f$.

Theorem 1.1 (Clemens-Griffiths, Mumford)**.** Torelli holds and $M \hookrightarrow A_5$.

There is another period map associated to the Boll quooteint via Alloch, Toledo. Suppose we have a cubic threefold,

$$X = \{f_3(x_0, \dots, x_4) = 0\} \subset \mathbb{P}^4$$

Then we associate to it a cubic fourfold,

$$Z = \{z^3 - f_3 = 0\} \subset \mathbb{P}^5$$

If X is smooth then Z is also smooth. There is a 3 to 1 map $Z \to \mathbb{P}^4$ birational along X. Then the Fano variety of lines,

$$F(Z) = \{\ell \subset \mathbb{P}^5 \mid \ell \subset Z\}$$

Then $\omega_{F(Z)} = \mathcal{O}_{F(Z)}$. Then $F(Z)$ is a smooth four-fold and a holomorphic symplectic manifold i.e. a hyperKahler manifold. The Hodge theory on hyper-Kahlers alows $\omega = \omega(F(H)) \in \Omega$ a period point where Ω is period domain $\dim \Omega = 20$. Furthermore, Z has $\mathbb{Z}/32\mathbb{Z}$-action so the period point lie $\omega \in B^{10}$ where B^{10} is a 10-dimensional ball. Then $M \hookrightarrow B^{10}/\Gamma$ where Γ is arithmetic group.

Remark. We have $B^{10}/\Gamma - M \cong B^g/\Gamma_h$ which is the moduli space of hyperelliptic curves of genus 5.