Physics C2801, Fall 2013 Prof. Brian Cole

Issued: **Nov. 20**	Problem Set # 9	Due: **Nov. 27**

Problem 1. Lorentz transformations, Lorentz boost factors, non-relativistic limit

All this semester I have been discussing relationships between results of special relativity and Newtonian mechanics. Newtonian mechanics can be viewed as an approximation to (more correct) special relativity that is valid when velocities of particles are much less than the speed of light and when considering inertial frames whose relative velocities are much less than the speed of light. To this end we will explore the Lorentz transformation for small boost velocities and see that we (mostly) recover the Galilean transformation.

a. First, I want you to calculate the Lorentz boost factor $\gamma_B \equiv \frac{1}{\sqrt{1-\beta_B^2}}$ for $\beta_B = 0.1, 0.2, 0.4, 0.6, 0.8$ and $1 - \beta_B = 1 \times 10^{-n}$ for $n = 1, 2, 3, 4, 6, 8$ and 10.

b. Writing out the Lorentz transformation (LT) in physical units, show what happens when $v_B/c << 1$ keeping terms to first order (power) in v_B/c. You should recover the Galilean transformation with one modification. Explain how that modification might be sufficiently small that it could have little practical impact on Newtonian mechanics when applied on Terrestrial distance scales (see part e below).

c. Now evaluate the form of the Lorentz transformation in the limit $v_B/c << 1$ keeping terms to second power in v_B/c. How is the Galilean transformation modified? Evaluate the resulting terms for $v_B = 28,000$ km/hour, approximately the velocity of the space shuttle in orbit.

d. To help develop your intuition for times in natural units, consider that for a physical time interval Δt, $c\Delta t$ represents the distance that light will propagate over that time interval. Calculate $c\Delta t$, or equivalently Δt in natural units, for $\Delta t = 1 \times 10^{-n}$ s with $n = 0, 3, 6, 9, 15, 21$. Express your results in meters. What physical times intervals would correspond to natural time intervals of 1 Å(typical atom size), 1×10^{-14} m (typical nucleus size), 1.5×10^{11} m (1 astronomical unit or the distance from the earth to the sun), 3×10^{16} m (approx. 1 parsec), and 3×10^4 parsec (size of the milky way).

e. Consider the LT for the time coordinate between two inertial frames where the primed frame is moving at $\beta_B = 0.99$ relative to the unprimed frame. Suppose $t = 1$ s. For what value of x does the inclusion of the x term change the the transformed time (t') by 1% compared to the result that would be obtained if the x term were neglected?

f. Natural units are often "natural" when consider the motion and decay of unstable particles. Suppose that in some frame, a particle with lifetime τ (*note:* lifetimes of particles are always expressed as proper times) moves with velocity β. Suppose that the particle is created with that velocity and then decays in a proper time τ. Show that when accounting for the effects of time dilation, the particle will travel a distance $\beta\gamma\tau$ between its production and decay positions.

g. Particles called muons are created in collisions of high-energy cosmic rays with (the nuclei of) gas atoms in the atmosphere. The muon has a lifetime $\tau = 2.2\ \mu$s in physical units. Calculate its lifetime in natural units. Then, evaluate the typical distance that muons will propagate when they are created with $\beta = 0.999$. Relativistic time dilation is the reason that most muons survive to reach the earths surface. Nearly all "cosmic rays" reaching the earths surface are not actually true cosmic rays but are actually muons generated by cosmic ray interactions in the upper atmosphere.

Problem 2. Relativistic transformation of velocities

a. Using the LT, derive the expression for the transformation of velocities parallel to the boost direction. More explicitly, consider a particle that in some inertial frame has velocity in natural units, $\vec{\beta} = \beta\hat{i} \equiv \frac{v}{c}\hat{i}$. Now evaluate the velocity, β' as observed in an inertial frame moving with respect to the first with $\vec{\beta}_B = \beta_B\hat{i}$. You can do this by considering the motion of the particle in the original frame over a finite time interval Δt.

b. Continuing from part a, consider the situation when $\beta << 1$ and $\beta_B << 1$. Show that (converting back to physical units) the transformation of velocities reduces to the usual, Galilean result when keeping terms to first order in small quantities. Evaluate the transformation keeping terms to the next (non-zero) power in small quantities.

c. Show that if $\beta = \pm 1$, then $\beta' = \beta$, i.e. the speed of light is the same in all frames.

d. Show that for $|\beta| < 1$, there is always a frame where $\beta' = 0$.

e. Show that for $|\beta| < 1$, $|\beta'| < 1$ for all values of β_B.

f. We will frequently use the Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ when studying the relativistic kinematics of particles. Use the results of part a to evaluate γ' the Lorentz factor in the primed frame for $\beta\hat{i} \equiv \frac{v}{c}\hat{i}$. We will return and examine the transformation of γ more extensively in the next problem set and in lecture.

g. Evaluate the complete transformation of velocities (by components). Specifically, let $\vec{\beta} = \beta_x\hat{i} + \beta_y\hat{j} + \beta_z\hat{k}$ in some (unprimed) inertial frame. Evaluate the three components of the velocity in a frame moving with respect to the first at velocity $\vec{\beta}_B = \beta_B\hat{i}$. Are you surprised by the results for the transformation of the y and z components?

Problem 3. Kleppner and Kolenkow, problem 12.4

Problem 4. Kleppner and Kolenkow, problem 12.6

Problem 5. Kleppner and Kolenkow, problem 12.10

Problem 6. Kleppner and Kolenkow, problem 12.11

Problem 7. Kleppner and Kolenkow, problem 12.12

Problem 8. Lorentz transformations with matrices In problem set 7 we analyzed rotations using matrices. Let's apply a similar analysis to the Lorentz transformations. I have already used this approach in class but here we will work through a complete analysis of Lorentz transformations using a matrix formulation. In fact, the transformation "matrix" is really another rank-2 tensor similar to the tensor of inertia – but one with more interesting properties. We will write the three components of position and time as a four component vector. By convention time is treated as the zeroth (0^{th}) component. Throughout this problem we will work in natural units following Taylor and Wheeler. The Lorentz transformation is described by the matrix $L_x(\beta_B)$ which depends on the boost velocity β_B. For boosts from one (unprimed) inertial frame to another (primed) inertial frame moving with velocity $\vec{\beta}_B = \beta_B \hat{\imath}$ with respect to the first (unprimed) frame,

$$L_x(\beta_B) = \begin{bmatrix} \gamma_B & -\beta_B\gamma_B & 0 & 0 \\ -\beta_B\gamma_B & \gamma_B & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

where $\gamma_B = 1/\sqrt{1-\beta_B^2}$.

a.) Show that if you write

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma_B & -\beta_B\gamma_B & 0 & 0 \\ -\beta_B\gamma_B & \gamma_B & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

you obtain the correct results for the transformation of the time and position coordinates.

b.) Show that $L_x(0)$ is the identity matrix.

c.) Show that $L_x(-\beta_B)$ is the inverse of $L_x(\beta_B)$, i.e. that $L_x(-\beta_B) = L_x^{-1}(\beta_B)$. Do this by explicitly evaluating the product $L_x(-\beta_B)\, L_x(\beta_B)$ and show that you obtain the identity matrix. This result is a mathematical expression of the fact that a boost in the x direction by β_B followed by a boost back by $-\beta_B$ should take us to the original inertial frame. Show by multiplying equation 1 by the inverse transformation matrix on both sides, that the unprimed coordinates can be written in terms of the primed coordinates,

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma_B & \beta_B\gamma_B & 0 & 0 \\ \beta_B\gamma_B & \gamma_B & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}$$

d.) We should always be able to represent two successive boosts in the same direction as a single boost. Namely, if we transform from one (unprimed) frame to a second (primed) frame and then boost yet again to a third (double primed) frame, this should be equivalent to a single boost from the first (unprimed) to the third (double primed) frame. So write

$$\begin{bmatrix} t'' \\ x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \gamma_B{}' & -\beta_B'\gamma_B{}' & 0 & 0 \\ -\beta_B'\gamma_B{}' & \gamma_B{}' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix},$$

and substitute in the expression for $\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}$ from Equation 1. Carry out the resulting matrix product and show that you can express it as a single Lorentz transformation at a "combined" boost velocity $\beta''_B = \dfrac{\beta_b + \beta'_b}{1 + \beta_b\,\beta'_B}$ (i.e. show that $L_x(\beta'_B)\,L_x(\beta_B) = L_x(\beta''_B)$ with β''_B given above). The algebra here will be messy and tedious. Just be careful and watch your signs and you should be able to get it to work out.

In problem set 8 we were able to use the transformation properties of row vectors and column vectors to show that rotations preserved the length of position vectors – and more generally, that rotations preserved the length of all dot products. We can do the same for Lorentz transformations but there's one additional subtlety, namely that the "metric" is non-trivial. You can think of the metric as a matrix that needs to appear between the row and column vector in the definition of the inner product. The metric for Euclidean vectors is the identity matrix so that the inner product of a vector $\vec{V}$ and a vector $\vec{U}$ can be written in matrix form:

$$\vec{V} \cdot \vec{U} = \begin{bmatrix} V_x & V_y & V_z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}$$

Under Lorentz transformations, we have seen that the quantity $t^2 - x^2 - y^2 - z^2$ is left invariant. Therefore, this quantity is what we should interpret as the "length" of a four vector. So, the length of the position four-vector $X = \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$ could be written,

$$X \cdot X = \begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} t \\ -x \\ -y \\ -z \end{bmatrix} = t^2 - x^2 - y^2 - z^2$$

This is a perfectly consistent definition of the "length", but the minus signs in the right-hand "vector", necessary for the correct length definition, seem nonetheless *ad hoc.* However, if we write the length in the following way,

$$X \cdot X = \begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

we obtain the correct "length" and we see that the metric yielding the correct length of four-vectors is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \equiv g.$

You should think of the metric g as more than just a mathematical tool for properly defining the inner product. The metric really defines the properties of the space on which the corresponding vectors are defined. The metric for ordinary three vectors defines a three-dimensional "Euclidean" space The four-dimensional space consisting of the coordinates t, x, y, z is termed "Minkoswki space" though it is often referred to informally as "space-time" and the corresponding metric is called the "Minkowski metric". Clearly the structure of Minkowski space is very different from that of a four-dimensional Euclidean space for which the metric would be the four-dimensional identity matrix.

Now you may wonder about why the signs are the way they are in the Minkowski metric. The metric could have been defined with all of the signs inverted and we would still have a proper description of space-time. The choice of signs shown above is a convention. However, it was chosen because it gives a positive inner product for time-like intervals between events. In other words if $|\Delta\vec{r}| < c\Delta t$ between two events – which means that a physical particle could propagate between the two events (i.e. $|v| \leq c$), then we obtain a positive $\Delta X \cdot \Delta X$ where Δx is the four-vector coordinate difference between two events.

Now, let's apply the analysis similar to what we did in problem set 8 for rotations to show that Lorentz transformations preserve the "length" of any four-vector (we will see several different four-vectors this semester and next). As for rotations in problem set 8, the (four) row vectors transform according to the transpose of L_x But observe that there's an important difference between Lorentz transformations and rotations – namely that that $L_x^T = L_x$.

e.) Suppose we have two four-vectors U and V that transform under Lorentz transformations just like the coordinate four vector (as all proper four vectors must). We can use the index notation introduced in problem 1 to refer to the components of U and V, e.g. $U = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{bmatrix}$ **this way of indexing the components breaks with a nearly universal notation used in more advanced treatments of relativity (and tensor calculus in general). We will come back and "fix it up" next semester. But for now we will use sub-scripts to index the components of four-vectors** Now suppose we write the inner product $U \cdot V$ in one inertial frame (unprimed) and compare to the inner product in a different (primed) inertial frame. Remember that we have to include the metric in the inner product. Show by substituting in the transformations for the column and row vector representations of the four vectors and multiplying the matrices that $U' \cdot V' = U \cdot V$.

Now, the metric above is another rank-2 (Minkowski) tensor. As we have discussed previously tensors are defined by their transformation properties. A vector (Euclidean or Minkowski) is a "rank-1" tensor in that it transforms with one multiplication by the transformation matrix (tensor). A rank-2 tensor transforms with multiplication of the transformation matrix on the left (transpose) and the right. The metric tensor is intimately connected with the Lorentz transformation tensor in that g is the **only** Minkowski tensor left invariant by Lorentz transformations.

f. Show that if we write $g' = L_x^T g L_x$ that the Lorentz transformation leaves the metric tensor unchanged, i.e. that $g' = g$. This result is really just another way of expressing the result from part e since $L_x^T g L_x$ is simply the product of the three matrices appearing in the expression of $U' \cdot V'$ in terms of $U \cdot V$.. However, this way of looking at the invariance of the inner product shows how the form of the Lorentz transformations is connected to the form of the invariant length – which in turn is proscribed by the constancy of the speed of light.