1 Borel-Weil-Bott

Let G/k be a split connected reductive group. Let $T \subset \overline{B} \subset G$ be a split maximal torus and $\overline{B}$ a borel containing it. Let $X^*(T) = \mathrm{Hom}\,(T, \mathbb{G}_m)$. Let $X^*(T)^+$ be the cone of dominant characters which is

$$\{\lambda : \langle \lambda, \alpha^\vee \rangle \geq 0\}$$

where $\alpha^\vee$ are coroots for the Borel B opposite to $\overline{B}$ (for example this means that $P_\alpha = B$ for the dynamical thing). Write $B = TN$ where N is the unipotent radical of B.

Theorem 1.0.1 (Highest Weight)**.** Suppose k has characteristic zero. Let W be an irreducible finite dimensional algebraic representation of G. Then $\dim_k W^N = 1$ and $T \circlearrowright W^N$ by some $\chi_W \in X^\bullet(T)^+$

(a) the association $W \mapsto \chi_W$ induces a bijection,

$$\{\text{irred. fin dim reps}\}/\text{isom} \xrightarrow{\sim} X^\bullet(T)^+$$

(b) The category of algebraic finite dimensional reps of G is semisimple.

Borel-Weil-Bott gives an explicit way to construct these highest-weight representations algebraically.

Let $X = G/\overline{B}$ be the flag variety. A $\overline{B}$-representation E induces a vector bundle $\mathcal{L}_E$ on X where,

$$\Gamma(U, \mathcal{L}_E) = \{f : \pi^{-1}(U) \to E \mid \forall g \in \overline{B} : f(gb) = b^{-1} f(g)\}$$

Theorem 1.0.2 (Borel-Weil)**.** For any $\lambda \in X^\bullet(T)$ consider $\mathcal{L}_\lambda$ by induction to $\overline{B}$ $H^0(\lambda) := H^0(G/B, \mathcal{L}_\lambda)$ is either:

(a) the irreducible of heighest weight λ if $\lambda \in X^\bullet(T)^+$

(b) 0 otherwise.

Theorem 1.0.3 (Bott)**.** First for $w \in W := N(T)/T$ consider the action on $X^\bullet(T)$ by $w \bullet \lambda = w(\lambda + \rho) - \rho$ where,

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

Then $H^{i+\ell(w)}(G/B, \mathcal{L}_{w \bullet \lambda}) \cong H^i(G/B, \mathcal{L}_\lambda)$ where $\ell(w)$ is the number of simple reflections needed to write w. In particular $H^i(G/B, \mathcal{L}_\ell) = 0$ unless $i = i(\lambda)$ where $i(\lambda)$ is the minimal length of a w such that $\lambda = w \bullet \mu$ for $\mu \in X^\bullet(T)^+$.

Remark. $\mathcal{L}_{-2\rho} = \omega_{G/B}$.

1.1 Proof of Borel-Weil

First recall the Bruhat decomposition,

$$G = \bigcup_{w \in W} \overline{N} w B$$

and $\mathrm{codim}\left(\overline{N} w B\right) = \ell(w)$. Therefore, $\overline{N} B$ is an open subset which is rational.

Work over $\mathbb{C}$.

Consider $H^0(\lambda) := H^0(G/B, \mathcal{L}_\lambda) = \{f : G \to \mathbb{C} \mid f(gb) = \lambda(b)^{-1} f(b)\}$. If $B = TN$ so $\overline{B} = T\overline{N}$ then,

$$H^0(\lambda)^N = \{f : G \to \mathbb{C} \mid f(gb) = \lambda(b^{-1}) f(g) \text{ and } f(ab) = f(b)\}$$

For any $f \in H^0(\lambda)$ it is determined on the open cell $\overline{N}B \subset G$ of the Bruhat decomposition via $f(nb) = \lambda(b)^{-1} f(1)$ therefore $\dim_{\mathbb{C}} H^0(\lambda)^N \le 1$ determined by f. If $g \in H^0(\lambda)^N$ such that $g(1) = 1$ note that if $t \in T$ then $(t \cdot f)(1) = f(t^{-1}) = \lambda(t) f(1) = \lambda(t)$ so $T \circlearrowright H^0(\lambda)^N$ by the character χ_W. Therefore, if $H^0(\lambda)^N \neq 0$ then $\lambda \in X(T)^+$ by the theorem of highest weight. For this we used that G/B is proper so $H^0(\lambda)$ is a finite-dimensional algebraic representation. Moreover, we need to show that if λ is dominant then $H^0(\lambda)^N \neq 0$.

To construct nonzero $f \in H^0(\lambda)^N$ we need to satisfy $f(gb) = \lambda(b)^{-1} f(g)$ and $f(ng) = f(g)$. Using the second equation and $f(1) = 1$ we get a well-defined function on the open cell. We need to show this extends to G. Consider,

$$S = \overline{N}B \cup \overline{N} s_\alpha B = RB$$

where s_α is a simple reflection and R is the root subgroup associated to α and $\overline{N}$. Idea: R has some Levi L, by passing to the universal cover of L and by pulling back, one reduces to the case $G = \mathrm{SL}_2$. On SL_2 we immediately check that for the characters,

$$\lambda_n : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

that the function

$$f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d^n f(1)$$

is a holomorphic function satisfying the required properties.

Matt's suggestion: shouldn't the order of vanishing along some divisor be governed by $\langle \lambda, \alpha^\vee \rangle$ and these are nonnegative by the definition of dominant.

1.2 Bott

The main theorem follows from iterating the following result.

Theorem 1.2.1. If α is a simple root and $\lambda \in X^\bullet(T)$ with $\langle \lambda + \rho, \alpha^\vee \rangle \ge 0$. Then there exists a G-equivariant isomorphism,

$$H^i(G/B, \mathcal{L}_\lambda) = H^{i+1}(G/B, \mathcal{L}_{\alpha \bullet \lambda}$$

Proof. Let P_α be a parabolic contaning $\overline{B}$ and the root subgroup generated by α. Construct a P_α-rep V_α^λ such that as a T-module it is the direct sum of characters: $\lambda, \lambda - \alpha, \cdots, s_\alpha(\lambda)$. One way to do this is $H^0(P_\alpha/B, \mathcal{L}_\alpha)$. If $\langle \lambda, \alpha^\vee \rangle \ge$ then there are two short exact sequence,

$$0 \longrightarrow K \longrightarrow V_\alpha^\lambda \longrightarrow (\lambda) \longrightarrow 0$$

$$0 \longrightarrow s_\alpha(\lambda) \longrightarrow K \longrightarrow V_\alpha^{\lambda \bullet \alpha} \longrightarrow 0$$

Reindexing, we get

$$0 \longrightarrow K \longrightarrow V_\alpha^{\lambda+\rho} \otimes (-\rho) \longrightarrow (\lambda) \longrightarrow 0$$

$$0 \longrightarrow s_\alpha \bullet \lambda \longrightarrow K \otimes (-\rho) \longrightarrow V_\alpha^{\lambda+\rho-\alpha} \otimes (-\rho) \longrightarrow 0$$

Claim,

$$R\Gamma(X, V_\alpha^{\lambda+\rho} \otimes (-\rho)) = R\Gamma(X, K \otimes (-\rho)) = 0$$

Consider the projection:

$$G/B \to G/P_\alpha$$

which is a $\mathbb{P}^1$-bundle. Then $V_\alpha^{\lambda+\rho}$ and V_α^{λ} are P_α-reps so in fact on fibers are trivial. But $\langle -\rho, \alpha^\vee \rangle = -1$ so $V_\alpha^{\lambda+\rho \otimes(-\rho)}$ restricted to any fiber of Π_α vas vanishing cohomology since it restricts to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Then by the Leray spectral sequence,

$$R\Pi_{\alpha *} V_\alpha^{\lambda+\rho} \otimes (-\rho) = 0$$

implies the vanishing. Then taking the two connecting maps,

$$H^i(G/B, \mathcal{L}_\lambda \to H^{i+1}(G/B, K \otimes (-\rho)) \to H^i(G/B, \mathcal{L}_{s_\alpha \bullet \lambda})$$

these maps are isomorphisms by the vanishing in the long exact sequences. The composition of these isomorphisms gives the result.

$\square$