1 Complex Singularities on Hypersurfaces (Milnor, Brieskorn)

Let $A \cong \mathbb{C}\{z_1, \dots, z_n\}/(y^2 - x^n)$ where $\mathbb{C}\{x_1, \dots, z_n\}$ is the algebra of convergent power series.

Definition 1.0.1. A *germ* is an element of the opposite category of these algebras.

Definition 1.0.2. A *deformation* of a germ X_0 is a flat mop of germs $f : X \to S$ where S has a distinguished point $0 \in S$ sauch that,

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & S \end{tikzcd}$$

(is Cartesian probably?) There is an equivalence relation under,

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

Example 1.0.3. Milnor considered deformations of the form,

$$\begin{tikzcd} \mathbb{C}\{x, y\} \arrow[r] & \frac{\mathbb{C}\{x,y,s\}}{(y^2-x^n+s)} \\ \mathbb{C} \arrow[r] \arrow[u] & \mathbb{C}\{s\} \arrow[u] \end{tikzcd}$$

(remember the arrows go the other way for germs).

1.1 The Topology

Let $f(z_0, z_1, \dots, z_n)$ be a nonconst polynomial in $n + 1 \geq 2$ complex vars. Then,

$$V = \{z \mid f(z) = 0\}$$

is a complex hypersurface of dimension n. Consider,

$$V \cap S_\epsilon(z^0) = K$$

where $z^\bullet$ is a singular point and $S_\epsilon(z^0)$ is the $(2n+1)$-sphere of radius $\epsilon > 0$ about z^0 in the ambiant $\mathbb{C}^n$-space.

If z^0 is regular point, V smooth dim $2n$ then K is a smooth $(2n - 1)$-dim manifold diffeomorphic to $(2n - 1)$-sphere. Then $K \hookrightarrow S_\epsilon$ is a knotted sphere.

Example 1.1.1. If $f(z_1, z_2) = z_1^p + z_2^q$ with $(p, q) = 1$ and $p, q \geq 2$ so there is a singularity at the origin.

Proposition 1.1.2 (Brauner)**.** $f^{-1}(0) \cap S_\epsilon$ is a (p, q)-torus knot.

Remark. We want to consider cases where K is homeomorphic to S^{2n-1}. However, it is possible that K has a nonstandard smooth structure from the induced submanifold structure!

Introduce a fibration for describing $K \hookrightarrow S_\epsilon$.

Theorem 1.1.3 (Milnor)**.** If $z^0 \in V$ and $\epsilon > 0$ sufficiently small and $S_\epsilon(z^0)$ is the sphere of radius ϵ at z^0. Then let $\phi : S_\epsilon \setminus K \to S^1$ defined by,

$$\phi(z) = \frac{f(z)}{|f(z)|}$$

viewing S^1 as the complex unit circle. Then ϕ forms a smooth fiber bundle with each fiber $F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon \setminus K$ is a paralllelizable $2n$-dimensional manifold.

Furthermore, if z^0 is an isolated singularity then F_θ is homotopy equivalent to $S^n \vee \cdots \vee S^n$.

Remark. The proof of this theorem goes through Morse theory.

Remark. Is $K = f^{-1}(0) \cap S_\epsilon$ a topological sphere? Any two n-dimensional homology classes α, β of F_θ have a geometric intersection number $s(\alpha, \beta)$

Lemma 1.1.4. *K is a $\mathbb{Z}$-homology sphere iff $s : H_n(F_\theta) \otimes H_n(F_\theta) \to \mathbb{Z}$ has determinant ± 1.*

Remark. Given a fiber bundle $\phi : E \to S^1$ there is an action of $1 \in \pi(S^1)$ on homology of the fiber. It is given by an automorphism $h_* : H_*(F_\theta) \to H_*(F_\theta)$. We call h the characteristic homeomorphism of $F_1 = \phi^{-1}(1)$. Define,

$$\Delta(t) = \det\left(tI_* - h_* | H_*(F_\theta)\right)$$

is related to the alexander polynomial of K.

Theorem 1.1.5. *If $n \neq 2$ then K is a topological sphere if and only if $\Delta(1) = \det\left(I_* - h_*\right) = \pm 1$.*

2 Brieskorn Varieties

For $a_1, \dots, a_{n+1} \geq 2$ coprime integers then let,

$$f(z_1, \dots, z_n) = (z_1)^{a_1} + \cdots + (z_{n+1})^{a_{n+1}}$$

Then let $V = f^{-1}(0)$ be the hypersurface defined by f. The origin is the unique singular point of V. Then let,

$$K = f^{-1}(0) \cap S_\epsilon$$

is smooth of dimension $2n - 1$. Consider fibration,

$$\phi : S_\epsilon \setminus K \to S^1$$

the fibers are F_θ.

Theorem 2.0.1 (Brieskorn-Pham)**.** *$H_n(F_\theta)$ is free of rank,*

$$M = (a_1 - 1) \cdots (a_{n+1} - 1)$$

and the roots of $\Delta(t)$ are exactly the set of all products,

$$\omega_1 \cdots \omega_{n+1}$$

where ω_i range over the a_i^{th}-roots of unity besides 1 (of which there are $a_i - 1$) for each i. Thus,

$$\Delta(t) = \prod_\omega (t - \omega_1 \cdots \omega_{n+1})$$

Example 2.0.2. For $a_1 = \cdots = a_n = 2$ and $a_{n+1} = 3$ we call this the "generalized trefoil" (because $a_1 = 2$ and $a_2 = 3$ gives exactly a trefoil knot in S_3). Then,

$$\omega_1 = \cdots \omega_n = -1 \quad \text{and} \quad \omega_{n+1} = \frac{-1 \pm i\sqrt{3}}{2}$$

Therefore,

$$\Delta(t) = (t - (-1)^n \zeta_3) \left(t - (-1)^n \bar{\zeta_3}\right) = t^2 + (-1)^n t + 1$$

Therefore K is a topological sphere for dimension $1, 5, 9, 13, \ldots$ and for dimension 9 you get Kervera's exotic sphere.