1 Mar. 30

Remark. Here is a reason that a -1-category should be the initial (empty) category or the terminal category (single object and single morphism). We want the Hom spaces of a 0-category (a set) to be -1-categories but these are singletons or empty. Therefore, we can say that a -1-sheaf (-2-stack) should be a function because its a category fibered over X in -1-categories so just a single element over open compatibly with restriction.

Given a (pre)sheaf on a topological space X we can "glue the fibers together" to get a fibered category over $\text{Open}(U)$ where the fibers are the sets $\mathscr{F}(U)$ over some open. The morphisms in this category are exactly $f|_U \to f$ where the morphism represents the restriction of the function f over $U \hookrightarrow V$. To make this a sheaf we need axioms involving the topology.

We do the same thing for stacks.

Definition 1.0.1. A 2-presheaf over a topological space X is a functor $f : \mathcal{C} \to \text{Open}(X)$ such that

(a) pullbacks exist

(b) every morphism in $\mathcal{C}$ is a pullback

Remark. Some exercises:

(a) the fibers of a 2-presheaf are groupoids.

Definition 1.0.2. Let $\mathcal{C}$ be a 2-presheaf then $\mathcal{C}$ is a stack if

(a) for $a, b \in \mathcal{C}(U)$ the functor $\text{Isom}(a, b)$ is a sheaf on U

(b) objects glue.

Definition 1.0.3. A category of geometric spaces is a category $\mathscr{G}$ such that there is a distinguished class of "open immersions" which is

(a) closed under composition

(b) local in nature

(c) preserved by pullbacks (fibered products)

Proposition 1.0.4 (Yonega)**.** Consider the category $\text{PSh}_{\mathcal{C}}$ which is the contravariant functors from $\mathcal{C}$ to Set. Then $X \mapsto h^X = \text{Hom}_{\mathcal{C}}(-, X)$ gives a fully faithfully embedding $\mathcal{C} \hookrightarrow \text{PSh}_{\mathcal{C}}$.

2 Geometry on PSh

What is a vectorbundle on a presheaf? If we are going to give it geometry we should know an answer to this question.

Example, the Hodge bundle. For any $S \to \mathcal{M}_g$ there is a family of curves $\pi : \mathcal{C} \to S$ and thus we get the Hodge bundle $\pi_* \Omega_{\mathcal{C}/S}$ which is a rank g vector bundle on S. These vector bundles are compatible (by cohomology and base change) with pullbacks $S' \to S \to \mathcal{M}_g$.

We call this data a vector bundle on $\mathcal{M}_g$.

Definition 2.0.1. A vector bundle $\mathcal{E}$ on $\mathscr{F} \in \mathrm{PSh}_{\mathscr{G}}$ is a vector bundle $\mathcal{E}(S)$ on each $S \in \mathscr{G}$ along with isomorphisms (DO THIS)

Exercise 2.0.2. Let $\mathscr{G}$ be the category of open balls in $\mathcal{C}^n$ and holomorphic maps between them. Then $\mathrm{Man}_{\mathbb{C}} \to \mathrm{PSh}_{\mathscr{G}}$ is a fully faithful embedding.

2.1 Fiber Products

$\mathscr{G}$ may not have fiber products because. For example if $\mathscr{G}$ is the category of smooth manifolds and smooth maps then fiber products of non submersions is not a smooth manifold.

However, $\mathrm{PSh}_{\mathscr{G}}$ does have fiber products. Indeed we construct fiber products point-wise.

Exercise 2.1.1. Any fiber product in $\mathscr{G}$ agrees with the corresponding fiber product in $\mathrm{PSh}_{\mathscr{G}}$ (the Yoneda embedding preserves fiber products).

The Yoneda functor preserves fiber products basically by definition because

$$h^{A \times_B C}(X) = \mathrm{Hom}_{\mathscr{G}}(X, A \times_B C) = \mathrm{Hom}_{\mathscr{G}}(X, A) \times_{\mathrm{Hom}_{\mathscr{G}}(X,B)} \mathrm{Hom}_{\mathscr{G}}(X, C)$$

Definition 2.1.2. A morphism $f : F \to G$ in $\mathrm{PSh}_{\mathscr{G}}$ is representable when for any map $S \to G$ from $S \in \mathscr{G}$ then $F \times_G S$ is representable.

Remark. If $\mathscr{G}$ has fiber products then every morphism between representable functors is representable.

Exercise 2.1.3. Representable morphisms are preserved by base change.

Definition 2.1.4. Given a property $\mathcal{P}$ of morphisms in $\mathscr{G}$. Then we say a representable morphism $f : F \to G$ in $\mathrm{PSh}_{\mathscr{G}}$ has property $\mathcal{P}$ if for every $S \to G$ with $S \in \mathscr{G}$ the morphism $F \times_G S \to S$ (which is a $\mathscr{G}$-morphism) has property $\mathcal{P}$.

Remark. For this to make sense, we need $\mathcal{P}$ to be a property preserved under base change so that $X \to Y$ has property $\mathcal{P}$ if and only if $X_{Y'} \to Y'$ has property $\mathcal{P}$.

Definition 2.1.5. We can now define an open cover in $\mathrm{PSh}_{\mathscr{G}}$. A representable morphism is open in the above sense.

3 April 4

Remark. Notice that every representable presheaf on $\mathscr{G}$ is a sheaf when restricted to each object $X \in \mathscr{G}$.

Definition 3.0.1. A presheaf $F \in \mathrm{PSh}_{\mathscr{G}}$ is a *sheaf* if for each $X \in \mathscr{G}$ the presheaf $X|_{\mathscr{G}}$ (restriction to the open subsets of $\mathscr{G}$) is a sheaf.

Remark. This will be a sheaf for the topology on $\mathscr{G}$ induced by open embeddings.

Definition 3.0.2. Let $\mathscr{G}$ be a category (not necessarily with fiber products). A topology on $\mathscr{G}$ is a connection of morphisms $\mathscr{G}^\circ \subset \mathscr{G}$ (the "open immersions") satisfying the following properties:

(a) and isomorphism $f : X \to Y$ is in $\mathscr{G}^\circ$ (for example id_X because $\mathscr{G}^\circ$ is a subcategory)

(b) openness is preserved under composition ($\mathscr{G}$ is a subcategory)

(c) pullbacks of morphisms in $\mathscr{G}^\circ$ by morphisms in $\mathscr{G}$ exist and are in $\mathscr{G}^\circ$.

(d) the fiber product of $U_1 \to X$ and $U_2 \to X$ gives $U_1 \times_X U_2 \to X$ is open (this is implied by composition and preservation under fiber products).

Along with the data of distinguished collections of morphisms in $\mathscr{G}^\circ$ called covering families such that

(a) every isomorphism $f : X \to Y$ is a covering family

(b) given a covering on Y and a morphism $f : X \to Y$ then the base change is a cover of X

(c) a cover of a cover is a cover meaning if $\{X_\alpha \to X\}$ is a covering family and $\{X_{\beta\alpha} \to X_\alpha\}$ are covering families then $\{X_{\beta\alpha} \to X\}$ is a covering family.

Definition 3.0.3. The category of sheaves $\mathfrak{Sh}_{\mathscr{G}} \subset \mathrm{PSh}_{\mathscr{G}}$ is the full subcategory of objects "determined locally on covers" i.e. satisfying the usual sheaf axiom.

Exercise 3.0.4. $\mathfrak{Sh}_{\mathscr{G}}$ has all fiber products and they agree with fiber products in $\mathscr{G}$ (when they exist) and an in $\mathrm{PSh}_{\mathscr{G}}$ under the fully faithfully embeddings,

$$\mathscr{G} \hookrightarrow \mathfrak{Sh}_{\mathscr{G}} \hookrightarrow \mathrm{PSh}_{\mathscr{G}}$$

Definition 3.0.5. If $X \in \mathscr{G}$ define $\mathscr{G}_X$ the slice category of morphisms $f : Y \to X$.

Definition 3.0.6. A sheaf $F \in \mathfrak{Sh}_{\mathscr{G}}$ is *locally representable* if there is an open cover by representable sheaves. Explicitly there are representable sheaves and representable morphisms $U_i \to F$ such that for every such diagram,

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ U_i \arrow[r] & F \end{tikzcd}$$

for $X \in \mathscr{G}$ we have $\{X_i \to X\}$ is a covering family in $\mathscr{G}$.

Remark. Applying this construction we get:

(a) for affine schemes and open immersions get all schemes

(b) for varieties and open immersions get pre-varieties (no separatedness or quasi-compactness)

(c) for open balls in $\mathcal{C}^n$ with open holomorphic embeddings get pre-manifolds (no Hausdorffness or second countability).

4 April 6

Reminder about $\mathscr{G}$: maybe it doesn't have fiber products (e.g. manifolds). We require our topology is *subcanonical* meaning for any $Y \in \mathscr{G}$ the functor h^Y is a sheaf.

Our category $\mathscr{G}$ is often a subcategory of locally ringed spaces. In most cases we can recover the sheaf of rings via maps to a ring object $\mathbb{A}^1 \in \mathscr{G}$.

Exercise 4.0.1. $\mathrm{LRep}_{\mathrm{AffSch}} \subset \mathbf{Sch}$.

Theorem 4.0.2. If $\mathscr{G}$ contains all fiber products (and a terminal object) then every $M \in \mathrm{LRep}_{\mathscr{G}}$ has $\Delta : M \to M \times M$ representable.

Remark. In the example $\mathscr{G} = \mathrm{AffSch}$ we see that $\mathbb{A}^2$ with the doubled origin is not in $\mathrm{LRep}_{\mathscr{G}}$ because its diagonal is not affine and hence not representable.

Lemma 4.0.3. If $\mathscr{G}$ has products and fiber products. Then all maps $X \to F$ for $X \in \mathscr{G}$ and $F \in \mathrm{PSh}_{\mathscr{G}}$ are representable if and only if $\Delta : F \to F \times F$ is representable.

Proof. First, assume that Δ is representable. Then,

$$\begin{tikzcd} X \times_F Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r] & F \end{tikzcd}$$

The following diagram is also cartesian,

$$\begin{tikzcd} X \times_F Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times Y \arrow[d] \\ F \arrow[r] & F \times F \end{tikzcd}$$

and $X \times Y \in \mathscr{G}$ so $X \times_F Y \in \mathscr{G}$ proving the claim. Next, suppose that $X \to F$ is always representable. For $U \in \mathscr{G}$ and $U \to F \times F$ we want to show that $U \times_{F \times F} F \in \mathscr{G}$.

$$\begin{tikzcd} U \times_{F \times F} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d] \\ U \times_F U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \times U \arrow[d] \\ F \arrow[r] & F \times F \end{tikzcd}$$

We see that $U \times_F U$ is representable because $U \to F$ is representable and thus the top square is all in $\mathscr{G}$ and hence because $\mathscr{G}$ has fiber products we conclude. $\square$

Remark. Given a diagram,

$$\begin{tikzcd} \mathrm{Isom} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d] \\ F \arrow[r] & F \times F \end{tikzcd}$$

then the pullback will classify isomorphisms between the objects over U represented by F under the two maps $U \to F$.

Proof of Theorem. We need to show that $F \to F \times F$ is representable. By the lemma, it is equivalent to ask if for every diagram with $X, Y \in \mathscr{G}$ we have,

$$\begin{tikzcd} X \times_F Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & F \end{tikzcd}$$

we want $X \times_F Y$ is representable. Choose a cover $U_i \to F$ by representable U_i. Then we pullback to get □

4.1 Complex Analytic Spaces

4.2 Sheafification

5 April 8

Given the data $U_i \in \mathscr{G}$ along with opens immersions $U_{ij} \hookrightarrow U_i$ and $U_{ij} \to U_j$ such that $U_{ij} \times_{U_i} U_{ik} = U_{kj} \times_{U_j} U_{ij}$ MAKE THIS WORK!!!

Then we get that,

$$M^-(X) = \mathrm{Hom}_{\mathscr{G}}\left(X, \coprod U_i\right) / \mathrm{Hom}_{\mathscr{G}}\left(X, \coprod U_{ij}\right)$$

automatically $M^-(X) \in \mathrm{PSh}^+_{\mathscr{G}}$. This cannot be the right presheaf however, for example id_M doesn't make sense because we are not stratifying X. To do this we exactly take the sheafification. Therefore we define $M = (M^-)^+$.

The claim is that $U_i \to M$ is open and make M be locally representable.

Remark. Suppose that $\Delta : M \to M \times M$ is representable. Then for any cover $U_i \to M$ is

6 April —

7 April 13

Question: given $\pi : U \to X$ in $\mathscr{G}$ does F satisfy the sheaf condition for π meaning is,

$$F(X) \longrightarrow F(U) \rightrightarrows F(U \times_X U)$$

Theorem 7.0.1. If there is $\sigma : X \to U$ such that $\pi \circ \sigma = \mathrm{id}$ then any presheaf F satisfies the sheaf condition for $\pi : U \to X$.

Proof. We get maps $\sigma_i : U \to U \times_X U$ given by $(\sigma \circ \pi, \mathrm{id})$ and $(\mathrm{id}, \sigma \circ \pi)$. Now we get $s^* \pi^* = \mathrm{id}$ so we see that π^* is injective giving the first part. For gluing, given $s \in F(U)$ such that $\pi_1^* s = \pi_2^* s$. Then take $t = \sigma^* s$ and we need to show that $\pi^* t = s$. Now $\pi_1 \circ \sigma_2 = \sigma \circ \pi$ and thus,

$$\pi^* \sigma^* s = \sigma_2^* \pi_1^* s = \sigma_2^* \pi_2^* s = s$$

because $\pi_1^* s = \pi_2^* s$ and $\pi_2 \circ \sigma_2 = \mathrm{id}$. □

Remark. Here is an alternative proof. We rewrite the above sequence via the Yoneda lemma as,

$$\mathrm{Hom}\left(h^X, F\right) \longrightarrow \mathrm{Hom}\left(h^U, F\right) \rightrightarrows \mathrm{Hom}\left(h^{U\times_X U}, F\right)$$

since $\mathrm{Hom}(-, F)$ takes colimits to limits it suffices to show that,

$$h^{U\times_X U} \rightrightarrows h^U \to h^X$$

is a coequalizer in the category of presheaves. However, limits and colimits are computed pointwise in the category of presheaves so we just need to show that

$$\mathrm{Hom}(T, U \times_X U) \rightrightarrows \mathrm{Hom}(T, U) \longrightarrow \mathrm{Hom}(T, X)$$

is a coequalizer in Set. Since the map to $\mathrm{Hom}(T, X)$ coequalizes and the section $\sigma : U \to X$ shows that $\mathrm{Hom}(T, U) \to \mathrm{Hom}(T, X)$ is sujective. Finally, given two maps $\alpha, \beta : T \to U$ then $\pi\circ\alpha = \pi\circ\beta$ if and only if α, β arise as the projections of a morphism $(\alpha, \beta) : T \to U \times_X U$ so we conclude.

Corollary 7.0.2. Suppose we have a diagram,

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & U \arrow[d] \\ & X \end{tikzcd}$$

Suppose that,

(a) F satisfies the sheaf condition for $V \to X$

(b) $F(U) \to F(V \times_X U)$ is injective

then F satisfies the sheaf condition for $U \to X$.

Proof. Consider,

$$\begin{tikzcd} V \times_X U \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] \arrow[ur] & X \end{tikzcd}$$

We know automatically that F satisfies the sheaf condition for $V \times_X U \to V$ because there is a section $V \to V \times_X U$. By assumption F is a sheaf for $V \to X$ so F is a sheaf for $V \times_X U \to X$. Using that $F(V \times_X U) \to F(U)$ is injective, we reduce to the following lemma. ☐

Lemma 7.0.3. Consider maps $V \to U \to X$ and let F be a presheaf. If F satisfies the sheaf condition for $V \to U \to X$ and $\pi^* : F(U) \to F(V)$ is injective then F satisfies the sheaf condition for $U \to X$.

Proof. Consider the diagram,

$$\begin{tikzcd} 0 \arrow[r] & F(X) \arrow[r] & F(V) \arrow[r, shift left] \arrow[r, shift right] & F(V \times_X V) \\ 0 \arrow[r] & F(X) \arrow[r] \arrow[u, equal] & F(U) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & F(U \times_X U) \arrow[u] \end{tikzcd}$$

where the top row is an equalizer. Then $F(X) \to F(V)$ is injective so $F(X) \to F(U)$ is injective. Suppose $\beta \in F(U)$ has equal pullbacks then $\pi^* \beta \in F(V)$ has equal projections and hence arises from a unique class $\alpha \in F(X)$ so that $\alpha \mapsto \pi^* \beta$. Since $F(U) \to F(V)$ is injective this means that $\alpha \mapsto \beta$ along $F(X) \to F(U)$ proving the claim. ☐

Remark. The condition (b) in the corollary illustrates the utility of having our covers preserved via arbitrary base change in the definition of a Grothendieck topology. Indeed, we will use it essentially in the proof of the following.

Corollary 7.0.4. Let F be a sheaf on a site $\mathcal{C}$. If $U \to X$ is refined by a cover of $\mathcal{C}$ then F satisfies the sheaf condition for $U \to X$.

Proof. Indeed, suppose there is a cover $V \to X$ which factors as $V \to U \to X$. Then by definition, F satisfies the sheaf condition for the covers $V \to X$ and $V \times_X U \to U$ using that covers are preserved under base change. In particular $F(U) \to F(V \times_X U)$ is injective so we can apply our previous result. ☐

Corollary 7.0.5. Let τ and τ' be Grothendieck topologies on $\mathcal{C}$. If F is a sheaf for τ and every morphism in τ' is refined by a τ-cover then F is a τ'-sheaf.

Corollary 7.0.6. Let τ and τ' be Grothendieck topologies on $\mathcal{C}$. Suppose that τ and τ' are cofinal meaning every covering morphism in one is refined by a covering morphism in the other. Then τ and τ' have the same categories of sheaves (as subcategories of $\mathrm{PSh}(\mathcal{C})$).

8 April 20

8.1 Quasi-Coherent Sheaves on Algebraic Spaces

For every $\mathrm{Spec}\,(A) \to X$ where X is an algebraic space, I want an A-module M such that for $\mathrm{Spec}\,(A') \to \mathrm{Spec}\,(A)$ we have M' is the pullback of M.

Proposition 8.1.1. If $\pi : X \to Y$ is a quasi-compact and quasi-separated morphism of algebraic spaces and $\mathscr{F}$ is quasi-coherent then $\pi_* \mathscr{F}$ is quasi-coherent.

8.2 Cech Cohomology

For the Zariski topology (and also other cohomology) on X an algebraic space with quasi-compact and affine diagonal then Cech cohomology works for quasi-coherent sheaves.

9 Example

Consider $\mathcal{M}_3^a$ the moduli space of genus 3 curves with no nontrivial automorphisms over $\mathbb{C}$. Let $\mathscr{G} = \mathbf{Sch}_{\mathbb{C}}$ then $\mathcal{M}_3^a \in \mathrm{PSh}_{\mathscr{G}}$ is the functor,

$S \mapsto \{\pi : C \to S$ relative dim 1 smooth geometrically intergral fibers of genus 3 with no automorphisms$\}$

Consider $\pi_* \Omega_{C/S}$ is a rank 3 vector bundle (by cohomology and base change) and thus we get a closed embedding,

$$C \hookrightarrow \mathbb{P}_S(\pi_* \Omega_{C/S})$$

over S. These are all embedded as plane quartic curves. Quartic divisors are parametrized by $\mathbb{P}^{14}$ there is an open $U \subset \mathbb{P}^{14}$ where the associated plane quadric is a smooth irreducible curve. Then,

$$\mathcal{M}_3^a = U/\mathrm{PGL}_3$$

To see this, consider,

$$\begin{tikzcd} B_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\ B \arrow[r] & \mathcal{M}_3^a \end{tikzcd}$$

Then B_U is exactly $\mathrm{Isom}\left(\mathbb{P}_B(\pi_*\Omega_{C/S}), \mathbb{P}^3_B\right)$ which is a Zarikis PGL_3-torsor OR MAYBE where $C^{\text{univ}} \to U$ is the universal curve over U. Maybe it's actually correct to take the PGL_3-torsor D over B of sections of $\mathbb{P}_S(\pi_*\Omega_{C/S})$ and tak

Let's check this. The T-points $T \to B_U$ are exactly given by the following data (a, b, γ) where $a : T \to B$ and $b : T \to U$ and an isomorphism $\gamma : C_T \xrightarrow{\sim} C_T^{\text{univ}}$ which is the data of a map

$$T \to \mathrm{Isom}_{B \times U}(C_{B\times U}, C^{\text{univ}}_{B\times U})$$

However, since $C^{\text{text}} \to U$ is universal the maps $b : T \to U$ are exactly classified by isomorphism classes $[C_T]$ as closed subschemes since because these curves are canonically embedded, as $C \hookrightarrow \mathbb{P}_B(\pi_*\Omega_{X/S})$ and $C^{\text{univ}} \hookrightarrow \mathbb{P}^3_U$ the isomorphism $C_T \xrightarrow{\sim} C_T^{\text{univ}}$ induces an isomorphism $\mathbb{P}(\pi_*\Omega_{X/S}) \xrightarrow{\sim} \mathbb{P}^3$. For a fixed such isomorphism there is a unique map $T \to U$ defined by the image of C_T in $\mathbb{P}^3$ therefore,

$$B_U = \mathrm{Isom}_{B\times U}(C_{B\times U}, C^{\text{univ}}_{B\times U}) = \mathrm{Isom}_B(\mathbb{P}_B(\pi_*\Omega_{X/S}), \mathbb{P}^3_B)$$

10 April 25

Definition 10.0.1. A *sieve* over X is a sub-presheav of h_X.

Example 10.0.2. $S_{U\to X}$ for any U

Definition 10.0.3. Given τ, a sieve is called a *covering sieve* if it contains some cover.

Example 10.0.4. If $\mathcal{C}$ is the category of opens of a topological space X then a sieve on X is a collection of open in X stable under subsets.

Theorem 10.0.5. A presheaf $\mathscr{F}$ satisfies,

$$\mathscr{F} \text{ is a sheaf} \iff \mathrm{Hom}\,(S, \mathscr{F}) = \mathrm{Hom}\,(h_X, \mathscr{F}) \text{ for all covering sieves } S \subset h_X$$

Remark. This shows that the category of sheaves recovers the covering sieves because we can take the sieves to be those that satisfy,

$$\mathrm{Hom}\,(S, \mathscr{F}) = \mathrm{Hom}\,(h_X, \mathscr{F})$$

for all sheaves $\mathscr{F}$.

10.1 Example

We want to show that $\mathcal{M}_g^a$ is an algebraic space. First we need to show it is a sheaf in the étale (or smooth) topology. In fact, this will work in the fpqcK topology.

Theorem 10.1.1. The data of $(X, \mathcal{L})$ where $\mathcal{L}$ is ample descends because we can descend the algebra,

$$\bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$$

which determines X via embedding into projective space.

11 April 27

Suppose $G \circlearrowright X$ where G is a geometric group. We want to define X/G as an algebraic space.

Example 11.0.1. We want,

(a) $X \to Y$ is a G-bundle then $X/G = Y$

(b) $\mathbb{Z}/2 \circlearrowright \mathbb{A}^1$ via $x \mapsto -x$ is not a G-bundle and we want $\mathbb{A}^1/(\mathbb{Z}/2)$ to be the GIT quotient.

What is the definition of X/G as an algebraic space? There should be a G-invariant map $X \to X/G$ such that any G-invariant map $X \to Y$ factors uniquely through $X \to X/G \to Y$,

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X/G \arrow[d, dashed] \\ & Y \end{tikzcd}$$

We call this the categorical quotient. There are some problems with this definition,

(a) it might not exist

(b) $X \to X/G$ might not be a G-bundle e.g. $\mathbb{A}^1 \to \mathbb{A}^1$ via $x \mapsto x^2$ is the quotient $\mathbb{A}^1/(\mathbb{Z}/2)$ but this is not a $\mathbb{Z}/2$-bundle (ramified over the origin).

Some possible answers,

(a) define X/G as the categorical quotient

(b) define X/G as the categorical quotient by only when $X \to X/G$ is a G-bundle

(c) defined it as the presheaf (or the sheafification)

$$Y \mapsto X(Y)/G(Y)$$

(d) define it as the presheaf sending Y to equivariant maps,

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

where $P \to Y$ is a principal G-bundle. We would have to take this up to isomorphism and this will be bad if there are nontrivial automorphisms of the map $P \to X$.

(e) Consider the presheaf h^X/h^G and take the sheafification to get X/G. If $G \circlearrowright X$ freely then h^X/h^G is a separated presheaf.

Example 11.0.2. If $Y = *$ and P is the trivial $\mathbb{Z}/2$-bundle over Y then the two maps $P \to \mathbb{A}^1$ whose image is ± 1 have no automorphisms but the map $P \to \mathbb{A}^1$ whose image is 0 does have an automorphism because the action is not free.

Definition 11.0.3. The action $G \circlearrowright X$ is free if it is free on T-points $G(T) \circlearrowright X(T)$.

Example 11.0.4. An elliptic curve is $E = \mathbb{C}/\Lambda$ analytically and indeed $h^E = (h^{\mathbb{C}}/h^{\Lambda})^{++}$. Algebraically we have $\Lambda \circlearrowright \mathrm{Spec}\,(\mathbb{C}[t]) = \mathbb{A}^1_{\mathbb{C}}$ where Λ is a discrete group viewed as a scheme. And we can define $\mathbb{A}^1_{\mathbb{C}}/\Lambda$ which is an algebraic space but not isomorphic to an elliptic curve! However $(\mathbb{A}^1_{\mathbb{C}}/\Lambda)^{\mathrm{an}} \cong \mathbb{C}/\Lambda$ so $(\mathbb{A}^1_{\mathbb{C}}/\Lambda)^{\mathrm{an}} \cong E^{\mathrm{an}}$ but the isomorphism (the Weierstrass $\wp$-function) is not algebraic.

12 May 6

Question: how many times do you need to plus h^U/h^R to get a stack?

12.1 How do we tell if a stack is DM

Proposition 12.1.1. If $\mathcal{M}$ is DM then $\mathscr{I}_{\mathcal{M}} \to \mathcal{M}$ is unramified.

Proof. Consider,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \times U \arrow[d] \\ \mathcal{M} \arrow[r] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

where the donward maps are étale. However, $R \to U$ is étale and hence unramified so $R \to U \times U$ is unramified. ☐

Proposition 12.1.2. If $\mathcal{M}$ is an algebraic stack and $\mathscr{I}_{\mathcal{M}} \to \mathcal{M}$ is unramified then $\mathcal{M}$ is DM.

Proof. Given a smooth cover $U \to \mathcal{M}$. For each point $p \in U$ where the relative dimension $n > 0$ we want to slice to find an étale neighborhood. ☐

13 May. 9

Let $\mathcal{M}$ be an algebraic stack (meaning a locally representable stack in the smooth topology on schemes).

Definition 13.0.1. The inertia stack,

$$\begin{tikzcd} \mathscr{I}_{\mathcal{M}} \arrow[r] \arrow[d, "\Delta"] & \mathcal{M} \arrow[d, "\Delta"] \\ \mathcal{M} \arrow[r, "\Delta"] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

is defined by the above fiber product.

Proposition 13.0.2. Let $\mathcal{M}$ be an algebraic stack, then $\mathcal{M}$ is as an algebraic space if and only if $\mathscr{I} \to \mathcal{M}$ is an isomorphism.

Proposition 13.0.3. The following are equivalent,

(a) $\mathcal{M}$ is DM

(b) Δ is unramified

(c) $\Omega_{\mathcal{M}/\mathcal{M}\times\mathcal{M}} = 0$

(d) $\Omega_{\mathscr{I}/\mathcal{M}} = 0$.

Theorem 13.0.4. Let $\pi : X \to Y$ be proper flat, whose geometric fibers are reduced and connected with Y locally noetherian. Let $\mathcal{L}$ be a line bundle on $\mathcal{L}$. Then the presheaf sending $Z \to Y$ to data $(\mathcal{M}_Z, \varphi)$ where $\varphi : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}|_Z$ is an isomorphism of line bundles on X_Z for the diagram,

$$\begin{tikzcd} X_Z \arrow[r] \arrow[d, "\pi_Z"] & X \arrow[d, "\pi"] \\ Z \arrow[r] & Y \end{tikzcd}$$

This is represented by a locally closed subscheme of Y. If the fibers of π are integral then a closed subscheme.

Remark. If we assume π is finitely presented then we can drop noetherian assumptions because it is the pullback of a noetherian case and therefore the theorem holds (showing it can be pulled back from a *flat* noetherian case is the tricky part).

Remark. The above theorem works for a schematic morphism $f : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks because it holds after all base changes to schemes.

Proposition 13.0.5. For $g \geq 2$ the stack $\mathcal{M}_g$ sending B to families of smooth genus g curves (flat schematic smooth map of relative dimension 1 with integral curves of genus g fibers) is algebraic. It is a stack in the fpqcK topology.

Proof. Any family of curves $\mathcal{C} \to B$ has $\omega_{\mathcal{C}/B}$ relatively ample and hence we get an embedding into projective space locally. Using cohomology and base change,

$$\mathcal{C} \hookrightarrow \mathbb{P}(\pi_* \omega_{\mathcal{C}/B}^{\otimes 3})$$

Therefore, consider the locus of canonically embedded curves $H \hookrightarrow \mathrm{Hilb}_{\mathbb{P}^n}$ and then $H \to \mathcal{M}_g$ is a PGL_n-torsor. $\square$

Proposition 13.0.6. The stack of dimension n smooth projective varities X where $\det \Omega_X = \omega_X$ is ample is an algebraic stack for the fpqcK topology.

Proof. Let $X \to B$ be such a family. Then there is some $\omega_{X_0}^{\otimes N}$ which is very ample with vanishing higher cohomology. Then we consider the open substack of $\mathcal{M}$ where $h^{>0}(X, \omega_X^{\otimes N}) = 0$ and h^0 is constant which is open by semicontinuity. Then the very ample locus is open (for flat maps the closed embedding locus is open). Then we get $\mathcal{M}_{X_0} \subset \mathcal{M}$ open and $H \to \mathcal{M}_{x_0}$ is a PGL_n-torsor. $\square$

14 May 11

Proposition 14.0.1. Suppose that $\mathcal{M}$ is an algebraic stack. Then the following are equivalent,

(a) $\Omega_{\mathscr{I}/\mathcal{M}} = 0$

(b) $\Omega_\Delta = 0$

(c) $\mathcal{M}$ has a representable étale cover by a scheme so is DM.

Proof. By pullback (b) implies (a) and $\mathscr{I} \to \mathcal{M}$ has a section so (a) implies (b) by pullback as well. We showed previously that if $\mathcal{M}$ is DM then $\Omega_\Delta = 0$ since Δ admits an étale cover by an unramified morphism. Therefore, we just need to show that if Ω_Δ then $\mathcal{M}$ is DM.

Start with a smooth cover $U \to \mathcal{M}$ by a scheme U. We need to slice U to make it étale. We can shrink and take disjoint usion so we may take $U = \mathrm{Spec}\,(A)$. Now consider,

$$\begin{tikzcd} R \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & U \arrow[d] \\ U \arrow[r] & \mathcal{M} \end{tikzcd}$$

To slice a smooth morphism we just need that its restriction to the fiber cuts down the dimension of the differentials by one. Consider the sequence for $R \to U \times U \to \mathcal{M}$,

$$\pi_1^* \Omega_{U/\mathcal{M}} \oplus \pi_2^* \Omega_{U/\mathcal{M}} \longrightarrow \Omega_{R/\mathcal{M}} \longrightarrow \Omega_{R/(U \times U)} \longrightarrow 0$$

Since $R \to U \times U$ is unramified we see that $\Omega_{R/U \times U} = 0$ and thus any function $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ for a point $\mathfrak{m} \subset A$ then it has a nonzero differential using the sequence (with structure map $\pi_2 : R \to U$),

$$\pi_2^* \Omega_{U/\mathcal{M}} \longrightarrow \Omega_{R/\mathcal{M}} \longrightarrow \Omega_{R/U} \longrightarrow 0$$

So we see from these sequences that $\pi_1^* \Omega_{U/\mathcal{M}} \twoheadrightarrow \Omega_{R/U}$ therefore for any covector in $(\Omega_{R/U})_p$ arises locally from pullback of some form $\Omega_{U/\mathcal{M}}$ which locally $\mathrm{d}f$ for $f \in \mathfrak{m}$ on $\mathrm{Spec}\,(A) \subset U$. ☐

14.1 Stack of Algebraic Curves

If $g \geq 2$ then $H^0(C, \mathcal{T}_C) = 0$ and therefore there are no infinitessimal automorphisms. This proves that $\Omega_{\mathscr{I}/\mathcal{M}_g} = 0$ so $\mathcal{M}_g$ is DM. Furthermore, because the stabiliers are closed subschemes of PGL_n we see that $\mathcal{M}_g$ has affine diagonal (in particular quasi-compact). However, it is unramified so we see that every genus g curve has finitely many automorphisms.

Consider genus g curves with distinct smooth marked points $p_1, \dots, p_n \in C$ such that,

$$\mathcal{O}_C(p_1 + \cdots + p_n)$$

is very ample with $h^1 = 0$. We claim this is an Artin stack by exactly the same argument: embed into $\mathbb{P}^n$ in families.

15 May 13

Remark. Amazing theorem is that we don't need smooth covers, flat is enough.

Theorem 15.0.1 (04S6)**.** Let F be an fppf sheaf and $f : U \to F$ a representable (by algebraic spaces) morphism which is surjective flat and locally finitely presented. Then F is an algebraic space.

Theorem 15.0.2 (06DC)**.** Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks in the fppf topology. Suppose that $\mathscr{X}$ is an algebraic stack, f is representable by algebraic stacks which is surjective locally of finite presentation and flat. Then $\mathscr{Y}$ is an algebraic space.

Return to $\mathcal{M}_{g,n}^{vp}$ is the moduli stack of families $f : \mathcal{C} \to B$ with f flat finitely presented with n sections whose geometric fibers are 1-dimensional schemes where the sections are distinct smooth points $p_1, \ldots, p_n \in \mathcal{C}_{\bar{s}}$ and $\mathcal{C}_{\bar{s}}$ has arithmetic genus $g = 1 - \chi(C, \mathcal{O}_C)$ and,

$$\mathcal{O}(p_1 + \cdots + p_n)$$

is very ample with vanishing h^1.

Remark. The genus can be weird for example $g(\mathbb{P}^1 \sqcup \mathbb{P}^1) = -1$. But this really is the right notion because χ is locally constant in flat families.

Proposition 15.0.3. Why is $B \hookrightarrow \mathcal{C}$ via the sections σ_i effective Cartier divisors. We need to show $\mathscr{I}_{B/\mathcal{C}}$ is invertible. There is a sequence,

$$0 \longrightarrow \mathscr{I}_{B/\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{O}_B \longrightarrow 0$$

We construct this as follows. Let $U \subset \mathrm{Hilb}(\mathbb{P}^N)$ be the locus where the intersection with a fixed hyperplane H is degree n. Then $V = Z \cap H \to U$ is flat of degree n where Z is the universal familiy over U. Then we take our parameter space $V \times_U \cdots \times_U V \setminus \Delta$ where Δ is the locus where the points are equal. This gives canonical sections of V pulled back to here thus labeling the points. Then we mod out by the PGL_n action fixing H to get our stack.

Then $U \to \mathcal{M}_{g,n}$ where $U \subset \mathcal{M}_{g,N}$ is the smooth geometric fibers locus where $N = n + d$ where d is large enough so that $N > 2g + 2$.

Remark. Consider M_g the stack of nodal genus g curves with connected fibers.