Math GR6262 Algebraic Geometry
Assignment # 3

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1 Problem 1

Let A be a Noetherian domain such that $\dim A = 1$ with maximal ideal $\mathfrak{p} \subset A$. Let $K = \mathrm{Frac}\,(A)$ and take any $f \in \mathrm{Frac}\,(K)$ such that $f \notin A_{\mathfrak{p}}$ (e.g. p^{-1} for any $p \in \mathfrak{p}$). Consider the ideal

$$I = (A : f) = \{x \in A \mid xf \in A\}$$

Then if $x \in I$ we have $xf \in A$ so if $x \in A \setminus \mathfrak{p}$ then $f = \frac{xf}{x} \in A_{\mathfrak{p}}$. Since $f \notin A_{\mathfrak{p}}$ we must have $I \subset \mathfrak{p}$. Since A is Noetherian and I is proper it has a primary decomposition,

$$I = \mathfrak{q}_0 \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

such that $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Therefore,

$$\sqrt{I} = \sqrt{\mathfrak{q}_0} \cap \cdots \cap \sqrt{\mathfrak{q}_n} = \mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_n$$

which implies that $\mathfrak{p}_0 \ldots, \mathfrak{p}_n \in V(I)$. Furthermore, $\dim A = 1$ so each prime $\mathfrak{p}_i$ is maximal and thus $V(I) = \{\mathfrak{p}_0, \ldots, \mathfrak{p}_n\}$ since if some prime $\mathfrak{q} \supset I$ then $\mathfrak{q} \supset \sqrt{I}$ and thus $\mathfrak{q} \supset \mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_n$ but $\mathfrak{q}$ is prime so $\mathfrak{q} \supset \mathfrak{p}_i$ for some but $\mathfrak{p}_i$ is maximal so $\mathfrak{q} = \mathfrak{p}_i$. In particular there are a finite number of primes above I and since $\mathfrak{p} \in V(I)$ we can take $\mathfrak{p}_0 = \mathfrak{p}$ WLOG.

By prime avoidance $\mathfrak{p}_i \not\subset \bigcup_{j \neq i} \mathfrak{p}_j$ and thus there exist elements, $a_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$. Then let $\tilde{a} = \prod_{i=1}^n a_i$ and thus $a_0 \tilde{a} \in \mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_n = \sqrt{I}$ so $(a_0 \tilde{a})^N \in I$ for some positive integer N. Consider $I' = (A : \tilde{a}^N f) \supset I$. Since $a_0^N \tilde{a}^N \in I$ we know that $a_0^N (\tilde{a}^N f) \in A$ so $a_0^N \in I'$. However, $a_0 \notin \mathfrak{p}_i$ for $i > 0$ and thus neither is a_0^N so $I' \not\subset \mathfrak{p}_i$ for $i > 0$. But since $I' \supset I$ we have $V(I') \subset V(I)$ so $V(I') = \{\mathfrak{p}\}$. Furthermore,

$$g \in A_{\mathfrak{q}} \iff \exists s \in A \setminus \mathfrak{q} : sg \in A \iff (A : g) \not\subset \mathfrak{q}$$

Therefore $a_0^N f \notin A_{\mathfrak{p}}$ but $a_0^N f \in A_{\mathfrak{q}}$ for each prime $\mathfrak{q} \neq \mathfrak{p}$.

2 Problem 2

Let A be a domain an M a torsion-free finite A-module. Take $K = \mathrm{Frac}\,(A)$ and consider the sequence,

$$0 \longrightarrow A \longrightarrow K \longrightarrow K/A \longrightarrow 0$$

Tensoring with $(-) \otimes_A M$ gives a long exact sequence,

$$\mathrm{Tor}_1^R(K, M) \longrightarrow \mathrm{Tor}_1^R(K/A, M) \longrightarrow A \otimes_A M \longrightarrow K \otimes_A M \longrightarrow K/A \otimes_A M \longrightarrow 0$$

However, $\mathrm{Tor}_1^R(K, M) = 0$ because K is flat. Thus we have,

$$0 \longrightarrow \mathrm{Tor}_1^R(K/A, M) \longrightarrow M \longrightarrow K \otimes_A M$$

However, $\mathrm{Tor}_1^R(K/A, M)$ is the torsion of M and thus vanishes since M is torsion free. Thus the map $M \to K \otimes_A M$ is an injection. Furthermore, $K \otimes_A M$ is a K-module and therefore free (since it is a vectorspace) as a K-module. Thus if $m_1, \dots, m_n$ generate the image of M in $K \otimes_A M$ then each m_i can be expressed in terms of a basis $b_1, \dots, b_k$ of $K \otimes_A M$. Choosing d large enough to clear all denominators we can write,

$$M \hookrightarrow d^{-1}(b_1 R \oplus \cdots \oplus b_k R) \subset K \otimes_A M$$

which is an inclusion into a free R-module.

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Consider the ring $A = k[x, y]/(y^2 - f(x))$ where k is a field with characteristic not 2 and,

$$f(x) = (x - t_1) \cdots (x - t_n)$$

with $t_1, \dots, t_n \in k$ distict and $n \geq 3$ an odd integer. Take the ideal $I = (y, x - t_1) \subset A$. I claim that I is not a free A-module of rank 1. First, if I is not of rank 1 it cannot free because given a generating set $f_1, \dots, f_n$ then $f_2 \cdot f_1 - f_2 \cdot f_1 = 0$ is a nontrivial A-linear combination of the generators that gives zero so it cannot be an A-basis. I will complete the proof of this claim at the end.

We have $\dim k[x, y] = 2$ since k is a field. Then,

$$\dim A = \dim k[x, y] - \mathbf{ht}\left((y^2 - f(x)\right)$$

since these rings are f.g. k-algebras. However, there are strict inclusions,

$$(y, x - t_1) \supsetneq (y^2 - f(x)) \supsetneq (0)$$

so $\mathbf{ht}\left((y^2 - f(x)\right) = 1$ since its height cannot be 2 because it is not maximal and it cannot be 0 because it is not minimal. Therefore $\dim A = 1$ so any nonzero $\mathfrak{p} \in \mathrm{Spec}\,(A)$ must then be maximal. Therefore, every $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponds to a closed point $\mathfrak{p} = (x - a, y - b)$ on the curve.

Now if $\mathfrak{p} = (x - a, y - b)$ with $b \neq 0$ then $y \notin \mathfrak{p}$. Thus, by Lemma 5.0.1, $I_y = A_y$ because $y \in I$. Furthermore, if $\mathfrak{p} = (x - a, y)$ then since $\mathfrak{p}$ is a prime of A then $\mathfrak{p}$ viewed as a prime of $k[x, y]$ must lie above $(y^2 - f(x))$. Thus, $f(a) = 0$ so $a = t_i$ for some i. If $i \neq 1$ then take $g = (x - t_1) \notin \mathfrak{p}$. Since $g \in I$ then by Lemma 5.0.1 we have $I_g = A_g$. Finally, for $\mathfrak{p} = (x - t_1, y) = I$ we may take $g = (x - t_2) \cdots (x - t_n)$. Now consider the map,

$$\frac{x - t_1}{y} A_g \to I_g$$

given by sending,

$$\frac{x-t_1}{y} \to \frac{y}{g}$$

Since $g \notin \mathfrak{p}$ this map is clearly injective. We need to show that this map is surjective i.e. that yA_g and $(x-t_1)A_g$ are in the image. This is easily demonstated via noticing that,

$$g \cdot \frac{x-t_1}{y} \mapsto y$$
$$g \cdot \left(\frac{x-t_1}{y}\right)^2 \mapsto g\frac{y^2}{g^2} = \frac{y^2}{g} = \frac{(x-t_1)\cdots(x-t_n)}{(x-t_2)\cdots(x-t_n)} = x - t_1$$

so the map hits the generators of I_g and thus surjects.

Therefore, we have shown that I is locally free of rank 1 i.e. I is an invertable A-module. Thus, it suffices to show that I is not free of rank 1 and thus represents a nontrivial class of the Picard group. By using the relations in the ring A, we may write an arbitrary element as $\alpha + \beta y$ with $\alpha, \beta \in k[x]$. Consider the norm map, $N : \mathrm{Frac}\,(A) \to k(x)$ which is the multiplicative map given by sending,

$$\alpha + \beta y \mapsto (\alpha + \beta y)(\alpha - \beta y) = \alpha^2 - \beta^2 y^2 = \alpha^2 - \beta^2 f \in k(x)$$

The restriction of this map to A gives a map to $k[x]$. Suppose that $I = (\pi)$ some generator written as $\pi = \alpha + \beta y$. Since $(\pi) = (y, x - t_1)$ we must have $\pi \mid x - t_1$ and $\pi \mid y$ whcih implies, via the multiplicativity of the norm that,

$$N(\pi) \mid N(x-t_1) \implies \alpha^2 - \beta^2 f \mid (x-t_1)^2$$
$$N(\pi) \mid N(y) \implies \alpha^2 - \beta^2 f \mid f$$

However, in $k[x]$ the gcd of $(x-t_1)^2$ and f is $(x-t_1)$ since the roots of f are distinct. Therefore, $N(\pi) \mid (x-t_1)$. However, in order for $\alpha^2 - \beta^2 f$ to divide $x - t_1$ we must have $\deg(\alpha^2 - \beta^2 f) \le 1$. But since $\deg f > 0$ either $\beta = 0$, in which case, $\alpha^2 \mid x - t_1$ which is impossible unless $\alpha \in k^\times$ because $x - t_1$ is not a square in $k[x]$. In that case $\pi = \alpha \in k^\times$ which cannot generate I since I is proper. Otherwise, for $\deg(\alpha^2 - \beta^2 f) \le 1$ we must have the leading terms of α^2 and $\beta^2 f$ cancel which implies that they have equal degree. Thus,

$$2\deg\alpha = 2\deg\beta + \deg f$$

However, by hypothesis, $\deg f$ is odd and thus we reach a contradiction so I cannot be principal.

4 02DU

Let A be a ring.

4.1

Suppose that M a is finite locally free A-module and suppose that $\varphi : M \to M$ is an endomorphism. Let $X = \mathrm{Spec}\,(A)$ and consider the induced endomorphism of $\mathcal{O}_X$-modues, $\varphi_* : \tilde{M} \to \tilde{M}$. Because M is finite locally free, at each $\mathfrak{p} \in \mathrm{Spec}\,(A)$ there exists $f \in A$ such that $\mathfrak{p} \in D(f)$ (i.e. $f \notin \mathfrak{p}$) and $\tilde{M}(D(f)) = M_f$ is a free A_f-module. Therefore, $\tilde{\varphi} : \tilde{M}(D(f)) \to \tilde{M}(D(f))$ is a map of free A_f-modules which has a standard trace and determinant in $A_f = \mathcal{O}_X(D(f))$ computed via the matrix representation denoted by $\text{tr}_f(\varphi) \in A_f$ and $\det_f(\varphi) \in A_f$. We need to show that these sections agree on overlaps. Choose a basis $e_1, \dots, e_n$ of M_f as an A_f-module so $M_f = e_1 A_f \oplus \cdots \oplus e_n A_f \cong A_f^{\oplus n}$. We have,

$$\widetilde{M}|_{D(f)} \cong \widetilde{M_f} \cong \widetilde{A_f^{\oplus n}} \cong \mathcal{O}_X|_{D(f)}^{\oplus n}$$

This gives a diagram,

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \xrightarrow{\sim} & \mathcal{O}_X(D(f))^{\oplus n} \\ \downarrow & & \downarrow \\ \widetilde{M}(D(g)) & \xrightarrow{\sim} & \mathcal{O}_X(D(g))^{\oplus n} \end{array}$$

Since the right restriction map sends an A_f basis to an A_g basis, the same must be true of the left restriction map. Then given $D(g) \subset D(f_1) \cap D(f_2)$ then we can write $\varphi(e_i^k) = \sum_{j=1}^n B_{ji}^k e_j^k$ for $k = 1, 2$ and we have $\text{tr}_{f_k} = \sum_{i=1}^n B_{ii}^k$ as an element of A_{k_k}. Under restriction, both $\{e_i^k\}$ for $k = 1, 2$ are sent to a A_g-basis of M_g. Therefore, since we have the diagram,

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \xrightarrow{\varphi} & \widetilde{M}(D(f)) \\ \downarrow & & \downarrow \\ \widetilde{M}(D(g)) & \xrightarrow{\varphi} & \widetilde{M}(D(f)) \end{array}$$

The matrix elements for $\varphi : M_g \to M_g$ in the restriction basis must be the restiction $(A_f \to A_g)$ of the matrix elements of $\varphi : M_f \to M_f$ since,

$$\text{res}(\varphi(e_i^k)) = \text{res}\left(\sum_{j=1}^n B_{ji}^k e_j^k\right) = \sum_{j=1}^n \text{res}_A(B_{ji}^k)\text{res}(e_j^k)$$

However, $\text{res} \circ \varphi = \varphi \circ \text{res}$ and $\text{res}(e_i')$ is also a basis with matrix $B_{ij}'^k$ so we have,

$$\text{res}(\varphi(e_i^k)) = \varphi(\text{res}(e_i^k)) = \sum_{j=1}^n B_{ji}'^k \text{res}(e_j^k)$$

proving the claim. Therefore, we can compute the trace and determinant in either basis $B_{ij}'^k$ which must be equal since they are coordinate independent,

$$\text{tr}_g = \sum_{i=1}^n B_{ii}'^k = \sum_{i=1}^n \text{res}_A(B_{ii}^k) = \text{res}_A\left(\sum_{i=1}^n B_{ii}^k\right) = \text{res}_A(\text{tr}_{f_k})$$

where I simply used the fact that $\text{res}_A : A_{f_k} \to A_g$ is a ring map. Similarly, expressing the determinant in either induced basis we find,

$$\det_g = \det B'^k = \det(\text{res}_A(B^k)) = \text{res}_A(\det B^k) = \text{res}_A(\det_{f_k})$$

Therefore, both the determinant and trace agree when restricted to the overlap. Thus, we may glue to obtain unique global sections $\text{tr}\varphi$ and $\det\varphi$.

Let M be a finite locally-free A module and N a finite locally-free B-module. Consider a ring map $r : A \to B$ and compatible module map $g : M \to N$ and two endomorphisms $\varphi : M \to M$ and $\psi : N \to N$ compatible with the module maps such that,

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "g"'] & M \arrow[d, "g"] \\ N \arrow[r, "\psi"] & N \end{tikzcd}$$

commutes. Viewing N as an A-module, the above commutes as a diagram of A-module maps. I am not sure what being functorial in this triple means for a section such as $\text{tr}\varphi \in A$ since the sections $\text{tr}\varphi$ and $\text{tr}\psi$ are not, in general, equal (consider $M \subset N$ vectorspaces over $A = k$ of different dimension and φ, ψ the corresponding identity maps which clearly make the square commute but have different traces).

4.2

Locally, the trace is computed standardly on maps of free modules. Given maps $\varphi : M \to N$ and $\psi : N \to M$ of finite locally free A-modules, about each point $\mathfrak{p} \in \text{Spec}\,(A)$ choose a neighborhood $D(f)$ such that both M_f and N_f are free. Then the localized maps $\varphi_f : M_f \to N_f$ and $\psi_f : N_f \to M_f$ satisfy $\text{tr}(\varphi_f \circ \psi_f) = \text{tr}(\psi_f \circ \varphi_f)$ and $\det(\varphi_f \circ \psi_f) = \det(\psi_f \circ \varphi_f)$ for standard linear algbra reasons. The global traces and determinants restrict uniquely to these local traces and determinants which forces $\text{tr}(\varphi \circ \psi) = \text{tr}(\psi \circ \varphi)$ and $\det(\varphi \circ \psi) = \det(\psi \circ \varphi)$ since both global sections restrict to the same local sections on some cover.

4.3

Let M be a finite locally-free A-module. Consider the map $\text{tr} : \text{End}_A\,(M) \to A$ defined above. Let $\varphi, \psi : M \to M$ be endomorphisms and $a, b \in A$. Then consider $\text{tr}(a\varphi + b\psi)$. For each point $\mathfrak{p} \in \text{Spec}\,(A)$ there exists an open neigborhood $D(f)$ such that M_f is free. Furthermore, by construction, the trace $\text{tr}(a\varphi + b\psi)$ restricts to $\text{tr}_f(a\varphi + b\psi)$ which is the trace of the map $a\varphi + b\psi : M_f \to M_f$ which satisfies

$$\text{tr}_f(a\varphi + b\psi) = a\,\text{tr}_f\varphi + b\,\text{tr}_f\psi$$

for standard linear algebra reasons on free modules. Thus, $a\,\text{tr}\varphi + b\,\text{tr}\psi$ restricts to the same local sections as $\text{tr}(a\varphi + b\psi)$ on an open cover and thus they must be equal as global sections. The exact same argument shows that $\det(\varphi \circ \psi) = \det(\varphi)\det(\psi)$.

5 Lemmas

Lemma 5.0.1. Let $I \subset A$ be an ideal and $f \in I$ then $I_f = A_f$.

Proof. Consier the exact sequence of A-modules,

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

Since localization is an exact functor we get the exact sequence,

$$0 \longrightarrow I_f \longrightarrow A_f \longrightarrow (A/I)_f \longrightarrow 0$$

However, since $f \in I$ then $[f] = 0$ in A/I which implies that $(A/I)_f = 0$. Therefore the inclusion $I_f \to A_f$ is an isomorphism. $\square$