Mathematics GU4044 Representations of Finite Groups
Assignment # 2

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Problem 1.

Let k be a field of characteristic zero. Define the subspaces,

$$W_1 = \{(t, \cdots, t) \mid t \in k\}$$
$$W_2 = \{(t_1, \cdots, t_n) \mid \sum_{i=1}^{n} t_i = 0\}$$

Define the map $p_1 : k^n \to W_1$ by $(t_1, \cdots, t_n) \mapsto (a(v), \cdots, a(v))$ where $a(v) = \frac{1}{n}\sum_{i=1}^{n} t_n$. Clearly, p_1 is linear and for $(t, \cdots, t) \in W_1$ we have $a = \frac{1}{n}(nt) = t$ so $p_1(w) = (t, \cdots t)$. Finally, given any $t \in k^n$ take $v = (tn, 0, \cdots, 0) \in k^n$ then $p_1(v) = (t, \cdots, t)$ so $W_1 \subset \text{Im}(p_1)$ but clearly $\text{Im}(p_1) \subset W_1$ so $\text{Im}(p_1) = W_1$. Therefore, p_1 is a projection map. Furthermore, $v \in \ker p_1 \iff a(v) = 0 \iff \sum_{i=1}^{n} t_i = 0$ so $\ker p_1 = W_2$. Thus, $k^n = W_1 \oplus W_2$.

Similarly, let $p_2 : k^n \to W_2$ be given by, $p_2 = \text{id}_{k^n} - p_1$ so $p_2(t_1, \cdots, t_n) = (t_1 - a(v), \cdots, t_n - a(v))$. As we have seen on the previous homework, p_2 has image W_2 and kernel W_1.

Problem 2.

Let $v_1 \in \mathbb{R}^2$ be the vector $(1, -3)$ and let $L_1 = \text{span}\{v_1\}$.

(a). Let $v_2 \in \mathbb{R}^2 \backslash L_1$ and $L_2 = \text{span}\{v_2\}$. Then, because $v_2 \notin L_1$ the set $\{v_1, v_2\}$ is independent which implies that $L_1 \cap L_2 = \varnothing$. Furthermore, $\dim \mathbb{R}^2 = 2$ so $\{v_1, v_2\}$ being independent is also a basis. Therefore, $L_1 + L_2 = \mathbb{R}^2$ so $\mathbb{R}^2 = L_1 \oplus L_2$.

(b). Let $v_2 = (-4, 9)$. Define $p : \mathbb{R}^2 \to \mathbb{R}^2$ such that $p(v_1) = v_1$ and $p(v_2) = 0$. The kernel of p is nontrivial (since $v_2 \in \ker p$) but not full (because $v_1 \notin \ker p$). Thus $\dim \ker p = 1$. Thus, the kernel is spanned by any nonzero element. In particular, $\ker p = \text{span}\{v_2\} = L_2$. Similarly, if $v \in L_1$ then $v = cv_1$ for $c \in \mathbb{R}$ so $p(v) = p(cv_1) = cp(v_1) = cv_1 = v$ so $p(v) = v$ on L_1. This shows that $L_1 \subset \text{Im}(p)$. However, by rank-nullity, $\dim \text{Im}(p) = 1$ and $\dim L_1 = 1$ so $\text{Im}(p) = L_1$.

(c). In the basis $\{v_1, v_2\}$ the matrix of p satisfying $p(v_1) = v_1$ and $p(v_2) = 0$ is given by,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Now, define the change of basis matrix C such that $C(e_1) = v_1$ and $C(e_2) = v_2$ i.e.

$$C = \begin{pmatrix} 1 & -4 \\ -3 & 9 \end{pmatrix} \quad \text{with inverse} \quad C^{-1} = \begin{pmatrix} -3 & -4/3 \\ -1 & -1/3 \end{pmatrix}$$

Therefore, in the standard basis, p is given by the matrix,

$$A' = CAC^{-1} = \begin{pmatrix} -3 & -4/3 \\ 9 & 4 \end{pmatrix}$$

(d). $\mathrm{Tr}\, A' = -3 + 4 = 1$ and likewise $\mathrm{Tr}\, A = 1 + 0 = 1$.

Problem 3.

Let V be a k-vectorspace. Consider the map $\Phi : \mathrm{Hom}\,(k, V) \to V$ given by, $\Phi(h) = h(1)$ where $h : k \to V$ is any element of $\mathrm{Hom}\,(k, V)$. We must show that Φ is an isomorphism. First, suppose thatt $h \in \ker \Phi$ then $h(1) = 0$ so for any $r \in k$ we have $h(k) = h(1 \cdot k) = h(1)h(k) = 0$ so h is the zero map. Thus, Φ is injective. Furthermore, for any $v \in V$ consider the map $\phi_v \in \mathrm{Hom}\,(k, V)$ given by $\phi_v(c) = cv$. Clearly, ϕ_v is linear and $\Phi(\phi_v) = \phi_v(1) = v$ so Φ is surjective. Finally, for $c_1, c_2 \in k$ and $h_1, h_2 \in \mathrm{Hom}\,(k, V)$ consider,

$$\Phi(c_1 h_1 + c_2 h_2) = (c_1 h_1 + c_2 h_2)(1) = c_1 h_1(1) + c_2 h_2(1) = c_1 \Phi(h_1) + c_2 \Phi(h_2)$$

so Φ is linear. Therefore, $\mathrm{Hom}\,(k, V) \cong V$.

Furthermore, $\{B : k \times V \to W \mid B \text{ is bilinear}\} \cong \mathrm{Hom}\,(k \otimes V, W)$ via the universal property of the tensor product. However, $k \otimes V \cong \mathrm{Hom}\,(k^*, V) \cong \mathrm{Hom}\,(k, V) \cong V$ were I have used the previous result and the fact that k^* is naturally isomorphic to k because k has a natural choice of basis, namely $\{1\}$. Thus, there is a natural isomorphism,

$$\{B : k \times V \to W \mid B \text{ is bilinear}\} \cong \mathrm{Hom}\,(V, W)$$

Problem 4.

Let V and W be finite-dimensional vector spaces with bases, $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ respectively.

(a). Let $F : V \to W$ be a linear map with matrix A such that $F(v_i) = A_{ji} w_j$. Then, consider the matrix of $F^* : W^* \to V^*$ with respect to the dual bases $v_1^*, \cdots, v_n^*$ and $w_1^*, \cdots, w_n^*$. Consider,

$$(F^*(w_i^*))(v_k) = (w_i^* \circ F)(v_k) = w_i^*(A_{jk} w_j) = A_{jk} \delta_{ij} = A_{ik}$$

Therefore,

$$F^*(w_i^*) = A_{ik} v_k^* = (A^\top)_{ki} v_k^*$$

so the matrix for F^* is $A^\top$.

(b). Let V and W be finite-dimensional k-vectorspaces. Consider the map $\Phi : \mathrm{Hom}\,(V, W) \to \mathrm{Hom}\,(W^*, V^*)$ given by $\Phi : F \mapsto F^*$. First, we show that $F \mapsto F^*$ is an injective linear map. This does not depend on the finite dimensional assumption. Suppose F^* is the zero map. Therefore, for any $\phi \in W^*$ the map $F^*(\phi) = \phi \circ F$ is the zero map. However, there exists a ϕ which is nonzero on any $w \in W \setminus \{0\}$. Thus, F must be the zero map so $\Phi : F \mapsto F^*$ is injective. Furthermore, $\Phi(F+G) = (F+G)^*$ which is a map such that $(F+G)^*(\phi) = \phi \circ (F+G) = \phi \circ F + \phi \circ G = F^*(\phi) + G^*(\phi)$ so $(F+G)^* = F^* + G^*$. Thus, Φ is linear.

Now, we need the fact that V and W are finite-dimensional. We know that $\dim \mathrm{Hom}\,(V,W) = (\dim V)(\dim W)$ and likewise $\dim \mathrm{Hom}\,(W^*, V^*) = (\dim W^*)(\dim V^*) = (\dim V)(\dim W)$. Thus, $\dim \mathrm{Hom}\,(W^*, V^*) = \dim \mathrm{Hom}\,(V,W)$ so because $\Phi : \mathrm{Hom}\,(V,W) \to \mathrm{Hom}\,(W^*,V^*)$ is a linear injection it must also be a surjection and thus an isomorphism.

(c). For a map $F : V \to V$ we know that if the matrix of F is A then the matrix of F^* is $A^\top$. Thus, $\mathrm{Tr}\, F^* = \mathrm{Tr}\, A^\top = \mathrm{Tr}\, A = \mathrm{Tr}\, F$.

Problem 5.

Let $v_1, \cdots, v_n$ be a basis of V_1 and $w_1, \cdots, w_n$ be a basis of V_2 such that $v_i \otimes w_j$ forms a basis of $V_1 \otimes V_2$. Let A be the matrix of $F_1 : V_1 \to V_1$ and B the matrix of $F_2 : V_2 \to V_2$ such that (using summation convention) $F_1(v_i) = A_{ji} v_j$ and $F_2(w_i) = B_{ji}(w_i)$. Then,

$$(F_1 \otimes F_2)(v_i \otimes w_j) = F_1(v_i) \otimes F_2(w_j) = (A_{ai} v_a) \otimes (B_{bj} v_b) = \sum_{a,b} A_{ai} B_{bj} v_a \otimes v_b$$

Therefore, the matrix of $F_1 \otimes F_2$ is $A_{ai} B_{bj}$. Thus,

$$\mathrm{Tr}\, F_1 \otimes F_2 = \sum_{a,b} A_{aa} B_{bb} = \sum_{a=1}^n A_{aa} \sum_{b=1}^n B_{bb} = (\mathrm{Tr}\, F_1)(\mathrm{Tr}\, F_2)$$

Similarly, let $F_1 : V_1 \to V_1$ and $F_2 : V_2 \to V_2$ be linear. Consider the linear map $(F_2)_* \circ (F_1)^* : \mathrm{Hom}\,(V_1, V_2) \to \mathrm{Hom}\,(V_1, V_2)$. A basis for $\mathrm{Hom}\,(V_1, V_2)$ can be written as $v_i^* w_j$ where v_i^* is an element of the dual basis and $(v_i^* w_j)(v) = v_i^*(v) \cdot w_j$. Thus,

$$\begin{aligned} ((F_2)_* \circ (F_1)^*)(v_i^* w_j)(v_l) &= (F_2)_*(v_i^* w_j \circ F_1)(v_l) = F_2 \circ (v_i^* w_j) \circ F_1(v_l) = F_2(v_i^*(A_{al} v_a) w_j) \\ &= F_2(A_{al} v_i^*(v_a) w_j) = A_{al} F_2(\delta_{ia} w_j) = A_{il} B_{rj} w_r \end{aligned}$$

Therefore,

$$((F_2)_* \circ (F_1)^*)(v_i^* w_j) = A_{il} B_{rj} v_l^* w_r$$

so the trace becomes,

$$\mathrm{Tr}\, ((F_2)_* \circ (F_1)^*) = \sum_{i,j} A_{ii} B_{jj} = \sum_{i=1}^n A_{ii} \sum_{j=1}^n B_{jj} = (\mathrm{Tr}\, F_1)(\mathrm{Tr}\, F_2)$$

Problem 6.

Let $A \in GL(n, \mathbb{C})$ be diagonalizable. Suppose that every eigenvalue of A has absolute value 1 then $\lambda^{-1} = \bar{\lambda}$. However, since A is diagonalizable, $\mathrm{Tr}\, A = \sum_{i=1}^n \lambda_i$ counting multiplicity if necessary. However, if $Av = \lambda v$ then $A^{-1} \lambda v = v$ so $A^{-1} v = \frac{1}{\lambda} v$ and visa versa. Thus, the eigenvalues of A^{-1} are exactly one over the eigenvalues of A. Therefore,

$$\mathrm{Tr}\, A^{-1} = \sum_{i=1}^n \frac{1}{\lambda_i} = \sum_{i=1}^n \bar{\lambda}_i = \overline{\mathrm{Tr}\, A}$$

Problem 7.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertivle matrix. Then, consider $(AA^{-1})^\top = (A^{-1})^\top A^\top = I^\top = I$ and $(A^{-1}A)^\top = A^\top(A^{-1})^\top = I^\top = I$. Therefore, $(A^{-1})^\top$ is an inverse of $A^\top$. By the uniqueness of inverses, $(A^{-1})^\top = (A^\top)^{-1}$.

Lemmas