1 KP Equation

This is a PDE on $u = u(x, y, t)$

$$\frac{\partial}{\partial x}(4u_t - 6u \cdot u_x - u_{xxx}) = 3u_{yy}$$

describes the motion of waves in shallow water. This has a surprising connection ot algebraic curves and abelian varities.

Let C be a smooth projective algebraic curve over $\mathcal{C}$ of genus g. Let $\omega_1, \dots, \omega_g$ be a basis of holomorphic differentials. The well-defined object is the Abel map,

$$C \to \mathbb{C}^g/\Lambda_C$$

given by,

$$p \mapsto \left[\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g\right]$$

Then,

$$\mathrm{Jac}(C) = \mathbb{C}^g/\Lambda_C$$

is the Jacobian of C and is an abelian variety. There is a map,

$$a : C^n \to \mathrm{Jac}\,(C)$$

given by,

$$(p_1, \dots, p_n) \mapsto \left(\sum \int_{p_0}^{p_i} \omega_1, \dots, \sum \int_{p_0}^{p_i} \omega_g\right)$$

Theorem 1.0.1 (Abel)**.** $a(p_1, \dots, p_n) = a(q_1, \dots, q_n) \iff p_1 + \cdots p_n \sim q_1 + \cdots q_n$ as divisors.

Corollary 1.0.2. The dimension of the image of $a : C^{g-1} \to \mathrm{Jac}\,(C)$ is $g - 1$ which is a divisor called Θ_C the Theta divisor of C.

Remark. This divisor pulls back along the holomorphic map $\mathbb{C}^g \to \mathrm{Jac}\,(C)$ to an analytic (not algebraic) hypersurface,

$$\Theta = \{\theta(q_1, \dots, q_g) = 0\}$$

where θ is the θ-function of C.

Theorem 1.0.3 (Krichnever)**.** There exist vectors $U, V, W \in \mathbb{C}^g$ and $c \in \mathbb{C}$ such that,

$$u(x, y, t) = 2\partial_x^2 \log \theta(Ux + Vu + Wt) + c$$

is a solution to the KP equation.

Remark. The vectors U, V, W can be obtained as follows. For $P \in C$ with local coordinate z write $\omega_i = f_i(z)\mathrm{d}z$ then,

$$U = \begin{pmatrix} f_1(P) \\ \vdots \\ f_g(P) \end{pmatrix} \quad V = \begin{pmatrix} f_1'(P) \\ \vdots \\ f_g'(P) \end{pmatrix} \quad W = \begin{pmatrix} f_1''(P) \\ \vdots \\ f_g''(P) \end{pmatrix}$$

The set of such U, V, W is naturally an algebraic variety that we study with Tori and we call it the Dubrovin 3-fold of C.

Remark. Such solutions to the KP equation are called *quasi-periodic* since the θ-function is quasi-periodic. Everything here is explicitly computable. Usually in terms of the Riemman θ-functions,

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} a_n \exp\left(2\pi i (n \cdot z)\right)$$

Remark. Krichev used the theory of integrable systems. In a recent work we take a point of view based on two ingredients:

(a) the Sato Grassmannian: integrable systems

(b) Abel's Theorem.

Note: everything works for singular curves as well.

2 Rational Nodal Curves

Let C be a rational nodal curve.

Example 2.0.1. Let $\pi : \mathbb{P}^1 \to C$ be a nodal curve of genus g construced by gluing together g-pairs of points on $\mathbb{P}^1$.

Let P_0 be a smooth base point on C. Consider $\omega_1, \dots, \omega_g$ basis of canonical differentials on C (meromorphic differentials on $\mathbb{P}^1$ with poles only at the preimages of the singularities and having a residue condition at the pairs).

Example 2.0.2. On the above example for $\mathbb{P}^1$ glued at pairs $(\kappa_1, \kappa_2), \dots, (\kappa_{2g-1}, \kappa_{2g})$ then consider,

$$\omega_i = \left(\frac{1}{x - \kappa_{2i-1}} - \frac{1}{x - \kappa_{2i}} \right) \mathrm{d}x$$

Then we can integrate to get,

$$\int_{p_0}^{n} \omega_i = \log\left(x - \kappa_{2i-1}\right) - \log\left(x - \kappa_{2i}\right) = \log\left(\frac{x - \kappa_{2i-1}}{x - \kappa_{2i}} \right)$$

so we have an Abel map,

$$a : \mathbb{P}^1 \dashrightarrow \mathbb{C}^g \xrightarrow{\exp} \mathbb{C}^g / \mathbb{Z}^g = (\mathcal{C}^\times)^g$$

sending,

$$x \mapsto \left(\left(\frac{x - \kappa_1}{x - \kappa_2} \right), \dots, \left(\frac{x - \kappa_{2g-1}}{x - \kappa_{2g}} \right) \right)$$

We have $(\mathcal{C}^\times)^g$ is the generalized Jacobian. We have again the theta divisor $\Theta = a(C^{g-1})$ gives an analytic hypersurface,

$$\Theta = \{\theta(z_1, \dots, z_g) = 0\}$$

for degenerate θ-functions.

Theorem 2.0.3. This degenerate θ-function is a finite linear combination of exponentials,

$$\theta(z) = \sum_{n \in \mathcal{C}} a_n \exp\left(2\pi i (n \cdot z)\right)$$

where $\mathcal{C} \subset \mathbb{Z}^g$ is finite. We describe the set of $\mathcal{C}$ in terms of the tropical Riemann matrix of C.

Remark. Again we get KP solutions,

$$u = 2\partial_x^2 \log \theta(Ux + Vy + Wt)$$

called *soliton solutions*.

2.1 More Singular Curves

We have seen,

(a) if C is smooth thne θ is an infinite sum of exponentials

(b) if C is nodal then θ is a finite linear combination of exponentials

(c) if C is even more special we can have θ be a polynomial.

Theorem 2.1.1. Let C be an irreducible gorenstein curve then C has a polynomial θ-function if and only if C is rational and has only unibrach singularities (meaning the normalization is $\mathbb{P}^1$ and the map is bijective so all the singularities are higher-order cusps).

Remark. In this case, the θ-polynomial has degree at most $\frac{1}{2}g(g+1)$.

Example 2.1.2. Let C be the image of $\mathbb{P}^1 \to \mathbb{P}^3$ via,

$$[u, t] \mapsto [u^6, t^4u^2, t^5u, t^6]$$

Then C is rational and has one unibrach singularity $Q = [1, 0, 0]$. A basis of differentials is given by,

$$\omega_1 = \mathrm{d}u \quad \omega_2 = u\mathrm{d}u \quad \omega_3 = u^2\mathrm{d}u \quad \omega_4 = u^6\mathrm{d}u$$

Then we can integrate these,

$$\int_0^u \omega_1 = u \quad \int_0^u \omega_2 = \frac{1}{2}u^2 \quad \int_0^u \omega_3 = \frac{1}{3}u^3 \quad \int_0^u \omega_4 = \frac{1}{6}u^7$$

and therefore we get an actually well-defined map,

$$a : C \to \mathbb{C}^g$$

(not needing to mod out by a lattice) and we get a polynomially defined Θ-divisor so an actual algebraic hypersurface not just an analytic one.