1 Motivation

Theorem 1.0.1. X is smooth quasi-projective over $\mathbb{C}$ there exists a $\mathbb{C}$-local system $L_{\mathbb{C}}$ of rank r such that for all primes ℓ there is L_ℓ an ℓ-adic local system of rank r which is $\overline{\mathbb{Q}}_\ell$-irreducible and $\det L_\ell \cong \mathcal{L}$ and monodromy at ∞ up to ...

Uses tools of companions and hust Langlands.

Remark. Let π be the finitely presented group this gives an obstruction to $\pi = \pi_1(X(\mathbb{C}))$ of a smooth quasi-projective variety X.

Definition 1.0.2. The character variety $\mathbf{Ch}\left(()\,\pi, r\right)$ is *weakly-integral* if,

(a) there is an irreducible $\rho : \pi \to \mathrm{GL}_r(\mathbb{C})$

(b) then for any ℓ there is a $\overline{\mathbb{Q}}_\ell$-irred representation $\rho_\ell : \widehat{\pi} \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ such that $\det \rho_\ell = \rho_{\det}$.

Example 1.0.3. Let $\pi = \mathrm{SL}(2, \mathbb{Z})$. There is an irreducible $\mathbb{C}$-rep but no integral rep at $\ell = 2$.

Theorem 1.0.4. Let Y, X be smooth projective over $\mathbb{C}$ and $f : Y \to X$ and $L_{\mathbb{C}}$ semisimple local system on X then $f^* L_{\mathbb{C}}$ is semisimple.

Remark. This is proved using harmonic geometry. We can prove it again using arithmetic geometry.

Remark. Since $\pi_1(X(\mathbb{C}))$ is finitely generated, there is a finite type $\mathbb{Z}$-algebra $A \subset \mathbb{C}$ such that,

$$\begin{tikzcd}
\pi_1(X(\mathbb{C})) \arrow[r] \arrow[dr] & \mathrm{GL}_r(\mathbb{C}) \\
& \mathrm{GL}_r(A) \arrow[u]
\end{tikzcd}$$

There exists some ℓ so we can map $A \to \mathbb{Z}_\ell$. This will not give all primes however.

1.1 Another Formulation

Consider the moduli space $M_B(X, r, \mathcal{L}, T_i)$ where $\mathcal{L}$ is the determinant and T_i are the monodromy at ∞. This is defines over some number field $K \supset \mathbb{Q}$. In fact it is defined,

$$M_B^{\text{irred}}(X, r, \mathcal{L}, T_i) \to \mathrm{Spec}\left(\mathcal{O}_K\right)$$

Then $\mathcal{L}_{\mathbb{C}}$ maps to $\mathrm{Spec}\left(\mathbb{C}\right) \to \mathrm{Spec}\left(\mathcal{O}_K\right)$. Then the map is dominant so there are at most finitely many primes not in the image. Another version of the theorem is that the map is surjective.

Remark. Irreducible means a representation $\rho : \pi \to \mathrm{GL}_r(A)$ then must be irreducible at each localization (and completion?).

Remark. Given a point over characteristic ℓ such that it is the mod ℓ-reduction of the DVR then I get an ℓ-adic local system. Given $z \in M_B$ closed with $\kappa(z) = \mathbb{F}_{\ell^n}$. Assume there is an R finite over W"$(\mathbb{F}_{\ell^n})$ with residue field $\mathbb{F}_{\ell^n}$ with $\mathrm{Spec}\left(R\right) \to M_B$ specializing to z. Then we get a representation L_R topological local system,

$$\pi_1(X(\mathbb{C})) \to \mathrm{GL}_r(R)$$

but R is profinite so we get a factorization through the completetion.

Sketch of the proof: $(M_B)_{\text{red}} \to \mathrm{Spec}\left(\mathcal{O}_K\right)$ is generically smooth. Therefore, there is an open $U \subset M_B$ where the reduced map is smooth. Choose some closed point $z \in U$. There is a map $D \to \widehat{M_B}_z$ where D is Mazur's deformation space. Its points are $\mathbb{Z}_\ell$-representations which lift z.