1 Dimension Theory

1.1 Introduction

Definition 1.1.1. Let X be a topological space. The *Krull dimension* or *combinatorial dimension* of X is the maximal length of chains of irreducible closed subsets,

$$\dim(X) = \max\{n \in \mathbb{Z} \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ is a chain of closed irreducible subsets } Z_i \subset X\}$$

and $\dim X = \infty$ if there is no maximum and $\dim X = -\infty$ if X is empty.

Definition 1.1.2. For $x \in X$ we define the dimension at x as,

$$\dim_x(X) = \inf_{x \in U} \dim(U)$$

taken over open neighborhoods U of x.

Remark. For any subset $S \subset X$, if $Z \subset S$ is closed irreducible then $\overline{Z} \subset X$ is closed irreducible so we get an inclusion of chains in S to chains in X. Thus,

$$\dim S \leq \dim X$$

Definition 1.1.3. Let $Z \subset X$ be a closed irreducible subset. Then,

$$\operatorname{codim}(Z, X) = \{n \in \mathbb{Z} \mid Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ is a chain of closed irreducible subsets } Z_i \subset X\}$$

and for any $Y \subset X$ we define,

$$\operatorname{codim}(Y, X) = \inf_{Z \subset Y} \operatorname{codim}(Z, X)$$

over $Z \subset Y \subset X$ closed irreducible subsets in X.

Lemma 1.1.4.