Hello,

My name is Ben Church and I am applying for the mathematics Ph.D. program at NYU Courant. For a long time I've known that I wanted to be an academic and to devote myself to doing mathematics. I want to be an academic not just to do research, teaching is also very important to me. In fact, Courant helped instill my admiration for excellent teaching since I was deeply inspired by attending New York math circle hosted by Courant. More recently, delving into the research problem I am about to describe, I began to read about Prof. Bogomolov's techniques and results. Having been inspired in these two critical junctures, I hope to attend Courant institute for my graduate studies.

Now I would like to describe a piece of math I find particularly fascinating. It concerns a particular sequence: given a variety X with integer coefficients, for simplicity we assume it is smooth and proper over $\mathrm{Spec}\,(\mathbb{Z})$ minus finitely many primes. Now for a prime power $q = p^k$, consider the sequence $a_n = \#X(\mathbb{F}_{q^n})$ counting solutions over successively larger finite fields. We would be interested in knowing the asymptotics of this series. To do this we might consider the generating function,

$$f(t) := \sum_{n=1}^{\infty} a_n t^{n-1}$$

Since we expect a_n to grow exponentially, we would like there to exist some constants c_i and r_i controlling the growth,

$$a_n \sim \sum_{i=1}^{g} c_i r_i^n$$

This would imply that,

$$f(t) \sim \sum_{i=1}^{g} c_i (r_i + r_i^2 t + r_i^3 t^2 + \cdots) = \sum_{i=1}^{g} \frac{c_i r_i}{1 - r_i t}$$

However, notice that,

$$\frac{c_i r_i}{1 - c_i r_i} = \frac{\mathrm{d}}{\mathrm{d}t} \log{(1 - r_i t)^{c_i}}$$

Therefore, we would want to show that,

$$f(t) := \sum_{n=1}^{\infty} a_n t^{n-1} \sim \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{g} \log{(1 - r_i t)^{c_i}} = \frac{\mathrm{d}}{\mathrm{d}t} \log \left(\prod_{i=1}^{g} (1 - r_i t)^{c_i} \right)$$

This requirement is much simplier if we introduce a new function $\zeta_X(t)$ such that,

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \zeta_X(t)$$

Explicitly,

$$\zeta_X(t) := \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} t^n \right)$$

Then our hypothesis about asymototics becomes,

$$\zeta_X(t) \sim \prod_{i=1}^{g} (1 - r_i t)^{c_i}$$

so we are looking for a rational function asymtotic to $\zeta_X(t)$. The amazing fact is that $\zeta_X(t)$ exactly equals a rational function! Weil conjectured that ζ_X has the percise form,

$$\zeta_X(t) = \prod_{i=0}^{2d} P_i(t)^{(-1)^{i+1}} = \frac{P_1(t) \cdots P_{2d-1}(t)}{P_0(t) \cdots P_{2d}(t)}$$

where $P_i(t)$ is an *integer* polynomial of degree b_i factoring as $(1 - \alpha_{i1} t) \cdots (1 - \alpha_{i b_i} t)$ with $P_0(t) = 1 - t$ and $P_{2d}(t) = 1 - q^d t$. In analogy with the Riemann zeta function, Weil conjectured two additional properties,

(a) The functional equation: there should exist an integer χ such that,

$$\zeta_X(q^{-d}t^{-1}) = \pm q^{\frac{\chi d}{2}} t^{\chi} \zeta_X(t)$$

(b) The Riemann hypothesis: $|\alpha_{ij}| = q^{i/2}$

Therefore we get not an asymtotic but an exact formula,

$$a_n = \sum_{i=1}^{2d} (-1)^i \sum_{j=1}^{b_i} \alpha_{ij}^n$$

and the Riemann hypothesis precisely controlls this growth. For example, if X is an elliptic curve then,

$$\zeta_X(t) = \frac{1 - a_p t + pt^2}{(1-t)(1-pt)}$$

and $1 - a_p t + pt^2 = (1 - \alpha t)(1 - \beta t)$ such that $|\alpha_i| = \sqrt{p}$. Then,

$$\#X(\mathbb{F}_{p^n}) = p^n + 1 - \alpha^n - \beta^n$$

so we find,

$$a_p = p + 1 - \#X(\mathbb{F}_{p^n})$$

and for $q = p^n$,

$$|\#X(\mathbb{F}_q) - (q+1)| = |\alpha^n + \beta^n| \le |\alpha|^n + |\beta|^n = 2\sqrt{q}$$

giving Hasse's celebrated theorem. Finally, the complex varitey $X_{\mathbb{C}}$ is smooth and proper giving the complex points $Y = X(\mathbb{C})$ the structure of a compact complex manifold. Amazingly, d is the dimension Y, χ is the Euler characteristic of Y and b_i are the Betti number of Y. Thus ζ_X, defined arithmetically, "knows" about the geometry of $X(\mathbb{C})$.

Even more amazing than these conjectures are the methods that Grothendieck and Deligne used to prove them. What was needed was a cohomology theory like singular cohomology but for varities over $\overline{\mathbb{F}}_q$. Given such a theory, because $X(\mathbb{F}_{q^n})$ are the fixed point of n^{th}-Frobenius $F^n : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$, a Lefschetz trace formula would tell us that,

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}\left(F^n | H^i(X_{\overline{\mathbb{F}}})\right)$$

Therefore, plugging in,

$$\begin{aligned}\zeta_X(t) &= \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right) = \exp\left(\sum_{n=1}^{\infty}\sum_{i=0}^{2d} (-1)^i \mathrm{tr}\left(F^n | H^i(X_{\overline{\mathbb{F}}_q})\right) \frac{t^n}{n}\right) \\ &= \prod_{i=0}^{2d} \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} \mathrm{tr}\left(F^n | H^i(X_{\overline{\mathbb{F}}_q})\right)\right)^{(-1)^i}\end{aligned}$$

However,

$$\sum_{n=1}^{\infty} \frac{t^n}{n} \mathrm{tr}\left(F^n | H^i(X_{\overline{\mathbb{F}}_q})\right) = -\log\det\left(1 - Ft | H^i(X_{\overline{\mathbb{F}}_q})\right)$$

Therefore,

$$\zeta_X(t) = \prod_{i=0}^{2d} \det\left(1 - Ft | H^i(X_{\overline{\mathbb{F}}_q})\right)^{(-1)^{i+1}}$$

proving the formula where,

$$P_i(t) = \det\left(1 - Ft | H^i(X_{\overline{\mathbb{F}}_q})\right)$$

is the characteristic polynomial. Then the functional equation would follow from Poincare duality. Invented by Grothendieck, ℓ-adic étale cohomology $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ provides such a theory proving rationality and the functional equation. Deligne then proved the Riemann hypothesis inductively by building up varities sucessively in Lefschetz pencils.

Finally, we want to know why $\zeta_X(t)$ can compute invariants of $X(\mathbb{C})$. This follows from deep theorems about étale cohomology: smooth and proper base change and Artin comparison. If we have a proper morphism $f : X \to S$ base change alows up to conclude that $(R^i f_* \mathbb{Q}_\ell)_{\bar{s}} \xrightarrow{\sim} H^i_{\text{ét}}(X_{\bar{s}}, \mathbb{Q}_\ell)$ for any geometric point $\bar{s}$ of S. If $f : X \to S$ is additionally smooth (and ℓ is invertible on S) then $f_* \mathbb{Q}_\ell$ is lcc meaning that, for connected S, the stalks of $f_* \mathbb{Q}_\ell$ are constant. We apply this to $X \to U \subset \operatorname{Spec}(\mathbb{Z})$ and conclude that $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. Furthermore, smooth base change and the Artin comparison theorem tell us that,

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \cong H^i_{\text{ét}}(X_{\mathbb{C}}, \mathbb{Q}_\ell) \cong H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}_\ell)$$

proving that b_i and χ give the correct Betti numbers and Euler characteristic.