1 Chapter 2

2 2.2

Given an exact sequence of vector bundles,

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0$$

consider the exact sequence,

3 2.5

Let L, L^* be holomorphic line bundles on a compact complex manifold X. Suppose that L and L^* admit nonzero global holomorphic sections s, s'. Then consider $s \otimes s'$ a global section of $L \otimes L^* \cong \mathcal{O}_X$. However, all nonzero sections of $\mathcal{O}_X$ are nonvanishing because X is compact and thus $H^0(X, \mathcal{O}_X) = \mathbb{C}$. Therefore, s and s' are nonvanishing meaning that $L \cong L^* \cong \mathcal{O}_X$.

4 2.6

4.0.1 1

I think f is holomorphic iff $\mathrm{d}f(Iv) = i\mathrm{d}f(v)$

4.0.2 2

4.0.3 3

4.0.4 4

Let $f : X \to Y$ be a surjective holomorphic map between connected xomplex manifolds. We want to look at the smooth locus of f.

I claim the following is true: for a morphism of vector budles (not necessarily constant rank) $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ then ϕ has full rank $k = \min\{m, n\}$ iff the morphism $\phi' : \bigwedge^k \mathcal{E}_1 \to \bigwedge^k \mathcal{E}_2$ is nonzero (is this true).

Therefore, the locus where ϕ is not full rank is the vanishing the section

$$\phi' \in \mathcal{HOM}_{\mathcal{O}_X}\left(\bigwedge^k \mathcal{E}_, \bigwedge^k \mathcal{E}_2 \right)$$

Now apply this to the map $f^* \Omega_Y \to \Omega_X$ to get the nonsmooth locus.

4.0.5 6

The cousins' problem has a solution because $H^1(X, \mathcal{O}_X) = 0$. Question: why is every hypersurface defined by a $H^0(K^\times / \mathcal{O}_X^\times)$. Question: how are we supposed to use the poincare lemma.

4.0.6 7

We define,

$$H_{\mathrm{BC}}^{p,q}(X) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) \mid \mathrm{d}\alpha = 0\}}{\partial\bar{\partial}\mathcal{A}^{p-1,q-1}(X)}$$

This makes sense because if $\alpha = \partial\bar{\partial}\gamma$ then

$$\mathrm{d}\alpha = \partial^2\bar{\partial}\gamma - \bar{\partial}^2\partial\gamma = 0$$

Now, the inclusion of d-closed forms into $\bar{\partial}$-closed forms induces a map,

$$H_{\mathrm{BC}}^{p,q}(X) \to H^{p,q}(X)$$

which is well-defined because if $\alpha = \partial\bar{\partial}\gamma$ then $\alpha = -\bar{\partial}\partial\gamma$ and is thus $\bar{\partial}$-exact. If X is furthermore compact Kahler then by the $\partial\bar{\partial}$-lemma we see if α maps to zero i.e. $\alpha = \partial\bar{\partial}\beta$ and $\mathrm{d}\alpha = 0$ then $\alpha = \mathrm{d}\gamma$ so the map is injective. Furthermore, by the Hodge decomposition, $H^{p,q}(X)$ can be represented by Harmonic forms which are d-closed and thus this map is surjective as well.

4.0.7 8

Is this just because we can take $M \to M$ via complex conjugation.

4.1 3.2

4.1.1 3.2.4

What does this really mean?? Ask Ron.

4.1.2 3.2.6

Let X be a compact Kähler manifold. Then,

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

Furthermore, $H^{q,p} = \overline{H^{p,q}}$. Therefore,

$$b_{2k+1} = \sum_{p+q=2k+1} h^{p,q} = \sum_{i=0}^{k} (h^{2k+1-i,i} + h^{i,2k+1-i}) = 2\sum_{i=0}^{k} h^{2k+1-i,i}$$

is even.

4.1.3 3.2.7

No! (PROVE IT)

4.1.4 3.2.8

Let X be a compact Kähler manifold. Let $\omega \in H^0(X, \Omega_X^p)$. Clearly, $\bar{\partial}\omega = 0$ since ω is a holomorphic $(p,0)$-form. Furthermore,

$$\bar{\partial}^*\omega = -(\bar{\star} \circ \bar{\partial} \circ \bar{\star})\,\omega$$

but $\bar{\star}\omega$ is a $(n-p,n)$-form and thus $\bar{\partial}\bar{\star}\omega = 0$. Therefore, $\bar{\partial}\omega = 0$ and $\bar{\partial}^*\omega = 0$ and thus $\Delta_{\bar{\partial}}\,\omega = 0$.

4.1.5 3.2.13

Let X be a complex Kähler manifold and $\alpha \in \mathcal{A}^k(X)$ which is d-closed and d^c-exact where $\mathrm{d}^c = i(\bar{\partial} - \partial)$. Notice that $\mathrm{dd}^c = 2i\partial\bar{\partial}$. Write $\alpha = \alpha^{k,0} + \cdots + \alpha^{0,k}$. (FINISH!!)

4.1.6 3.2.14

DO!

4.1.7 3.2.15

DO!

4.1.8 3.2.16

Let X be a compact Kähler manifold. Let ω and ω' be Kähler forms such that $[\omega] = [\omega'] \in H^2(X, \mathbb{R})$. Then $\eta = \omega - \omega' = \mathrm{d}\alpha$ for some real 1-form α. Thus η is a closed real $(1,1)$-form which is d-exact and thus by the $\partial\bar{\partial}$-lemma $\eta = i\partial\bar{\partial}f$ for some $f \in \mathcal{A}^{0,0}$. Notice,

$$\bar{\eta} = -i\bar{\partial}\partial\bar{f} = i\partial\bar{\partial}\bar{f}$$

however η is real so $\bar{\eta} = \eta$ and thus $\bar{f} = f$ so $f \in \mathcal{A}^0_{\mathbb{R}}$ is a real function and,

$$\omega = \omega' + i\partial\bar{\partial}f$$

5 Extra Questions for Ron

5.0.1 1

Kodaira embedding says that every positive line bundle is ample in the sense of having some power very ample. Does the algebraic geometry definition work here? I.e. L is ample iff for each bundle Q we have $Q \otimes L^n$ generated by global sections for $n \gg 0$. Do we need Q to be arbitrary coherent sheaf.
Yes, in fact we only need this for vector bundles because it then follows by resolution for all coherent sheaves.

5.0.2 2

If we have a big line bundle $H^0(X, L^{\otimes m}) \sim m^n$ then does it follow there is an ample line bundle i.e. X is projective. I am guessing not. This is similar to asking if there are non algebraic examples of compact Moishezon manifolds $a(X) = \dim X$.

5.0.3 3.3.1

5.0.4 3.3.2

5.0.5 3.3.3

6 Chapter 4

6.1 Section 4.3

6.1.1 4.3.1

6.1.2 4.3.2

6.1.3 4.3.3

6.1.4 4.3.4

6.1.5 4.3.5

Let X be complex manifold. Let L be a holomorphic line bundle with a hermitian structure h whose Chern connection has positive curvature. Then $F_\nabla \in \mathcal{A}^{1,1}(X)$ is an imaginary $(1,1)$-form. Furthermore, note that $F_\nabla = \bar{\partial}\partial \log h$ and thus,

$$\mathrm{d}F_\nabla = (\partial + \bar{\partial})\bar{\partial}\partial \log h = 0$$

because $\bar{\partial}^2 = 0$ and $\partial\bar{\partial}\partial = -\partial^2\bar{\partial} = 0$. Since $\omega = iF_\nabla$ is positive, it is a Kähler form. Furthermore if X is compact then,

$$\int_X A(L)^n = \int_X F_\nabla^n = \int_X \omega^n = n! \int_X \mathrm{vol}_\omega > 0$$

(CHECK THIS! FACTORS OF I)

6.1.6 4.3.6

6.1.7 4.3.7

6.1.8 4.3.8

6.1.9 4.3.9

Let X be a compact Kähler manifold with $b_1(X) = 0$. Suppose that ∇ is a flat connection on $\mathcal{O}_X$ with $\nabla^{0,1} = \bar{\partial}$. Then $\nabla = \mathrm{d} + \omega$ where $\omega : \mathcal{A}^0(X) \to \mathcal{A}^1(X)$ is $\mathcal{A}^0(X)$-linear and thus $\omega \in \mathcal{A}^1(X)$. Futhermore, $\nabla^{0,1} = \bar{\partial}$ so ω is a smooth $(1,0)$-form. Now consider the curvature,

$$F_\nabla = \nabla \circ \nabla(1) = \nabla(\omega \otimes 1) = \mathrm{d}\omega \otimes 1 - \omega \wedge \nabla(1) = \mathrm{d}\omega \otimes 1 - \omega \wedge \omega \otimes 1 = \mathrm{d}\omega$$

Since ∇ is flat we must have $\mathrm{d}\omega = 0$. Thus ω defines a de Rham cohomology class $[\omega] \in H^1(X, \mathbb{C})$ but $b_1(X) = 0$ so ω is exact. Take $\omega = \mathrm{d}f$ for some smooth function f. However, ω is a $(1,0)$-form so f is holomorphic. But X is compact so f is constant and thus $\omega = 0$ showing that $\nabla = \mathrm{d}$.

Now suppose that L is a line bundle on X with $c_1(L) = 0$. From the exponential sequence,

$$H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}\,(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

and thus $\ker c_1 = \mathrm{Im}\,(H^1(X, \mathcal{O}_X) \to \mathrm{Pic}\,(X))$. However, $b_1(X) = 0$ so by the Kähler decomposition, $H^1(X, \mathcal{O}_X) = 0$. Therefore, $\ker c_1$ is trivial so $L = \mathcal{O}_X$.

6.1.10 4.3.10

Let ∇ be a connection on a complex vector bundle E. We want to show that E locally has parallel frames iff $F_\nabla = 0$.

Suppose that E has a local frame $e_1, \dots, e_n$ of parallel sections over U i.e. $\nabla e_i = 0$ and these are independent on each fiber. Since the curvature form $\omega_\nabla(s) = \nabla_1 \circ \nabla(s)$ is $\mathcal{O}_X$-linear, writing $s = f_i e_i$ we get,

$$\omega_\nabla(f_i e_i) = f_i \omega_\nabla(e_i) = f_i \nabla_1 \circ \nabla e_i = 0$$

Therefore, $\omega_\nabla = 0$ so ∇ must be flat.

Locally write $E|_U \cong \mathcal{O}_U^{\oplus n}$ write e_i for a local frame of $E|_U$. Now write $\nabla e_j = \omega_{ij} \otimes e_i$ thus we see,

$$\nabla(f_j e_j) = \mathrm{d}f_j \otimes e_j + \omega_{ij} f_j \otimes e_i = (\mathrm{d}f_i + \omega_{ij} f_j) \otimes e_i$$

Now, applying $\nabla_1 : \Omega^1_X \otimes E \to \Omega^2_X \otimes E$ we get,

$$\begin{aligned}
\nabla_1 \circ \nabla(f_j e_j) &= \nabla_1(\mathrm{d}f_i + \omega_{ij} f_j) \otimes e_i = \mathrm{d}\mathrm{d}f_i \otimes e_i + \mathrm{d}(\omega_{ij} f_j) \otimes e_i - (\mathrm{d}f_i + \omega_{ij} f_j) \wedge \nabla e_i \\
&= (\mathrm{d}\omega_{ij}\, f_j - \omega_{ij} \wedge \mathrm{d}f_j) \otimes e_i - (\mathrm{d}f_i + \omega_{ij} f_j) \wedge \omega_{ki} \otimes e_k \\
&= \mathrm{d}\omega_{ij} f_j \otimes e_i + \mathrm{d}f_j \wedge \omega_{ij} \otimes e_i - \mathrm{d}f_i \wedge \omega_{ki} \otimes e_k + \omega_{ki} \wedge \omega_{ij} f_j \otimes e_k \\
&= (\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj}) f_j \otimes e_i
\end{aligned}$$

Therefore,

$$\omega_\nabla(f_j e_j) = (\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj}) f_j \otimes e_i$$

is linear as it should be. Now assume ∇ is flat i.e. $\omega_\nabla = 0$. Thus,

$$\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj} = 0$$

First, in the case $n = 1$ the connection is given by a 1-form ω. Then $\omega_\nabla = 0 \iff \mathrm{d}\omega = 0$ in which case locally $\omega = -\mathrm{d}f$ and thus $\nabla(fe) = \mathrm{d}f \otimes e + \omega \otimes e = 0$ so we get a frame of parallel sections.

Now we proceed by induction for the general case. First, using a $\mathrm{GL}_((n), \mathbb{C})$ transformation we can
Assume we can find a frame $e_1, \dots, e_{n-1}, s$ such that $\nabla e_i = 0$.

6.2 Section 4.4

6.2.1 4.4.2

Let X be a compact complex manifold and L a basepoint-free line bundle. Then L defines a map $f : X \to \mathbb{P}^N$ such that $f^* \mathcal{O}_{\mathbb{P}^N}(1) = L$. Let h be the standard hermitian structure on $\mathcal{O}_{\mathbb{P}^N}(1)$ so $f^* h$ gives a hermitian structure on L. Taking the Chern connections $\nabla_{f^* h} = f^* \nabla_h$ and thus,

$$F(L, f^* h) = F(f^* \mathcal{O}_{\mathbb{P}^N}(1), f^* h) = f^* F(\mathcal{O}_{\mathbb{P}^N}(1), h) = f^* \omega_{\mathrm{FS}}$$

which is a positive form. Therefore,

$$c_1(L) = f^*[\omega_{\mathrm{FS}}]$$

so we see that,

$$\int_X c_1(L)^n = \int_X (f^* \omega_{\mathrm{FS}})^n = \int_X f^* \omega_{\mathrm{FS}}^n \geq 0$$

6.2.2 4.4.4

Ask Ron about interpretation!!

6.2.3 4.4.9

Note that $\text{End}\,(E) \cong E^* \otimes E$ then,

$$c_k(\text{End}\,(E)) = \sum_{i+j=k} c_i(E^*) \cdot c_j(E) = \sum_{i+j} (-1)^i c_i(E) \cdot c_j(E)$$

In particular,

$$c_1(\text{End}\,(E)) = c_0(E) \cdot c_1(E) - c_1(E) \cdot c_0(E) = 0$$

and likewise,

$$c_2(\text{End}\,(E)) = c_0(E) \cdot c_2(E) - c_1(E) \cdot c_1(E) + c_2(E) \cdot c_0(E) = 2c_2(E) - c_1(E)^2$$

Then if $E = L \oplus L$ where L is a line bundle we have,

$$c(L) = 1 + c_1(L)$$

and thus,

$$c_1(E) = 2c_1(L) \quad \text{and} \quad c_2(E) = c_1(L)^2$$

Therefore, we see that,

$$(4c_2 - c_1^2)(E) = 4c_1(E)^2 - (4c_1(E))^2 = 0$$

Furthermore, if $E \cong E^*$ then $c_{2k+1}(E) = c_{2k+1}(E^*) = (-1)^{2k+1} c_{2k+1}(E) = -c_{2k+1}(E)$ and thus $c_{2k+1}(E) = 0$.

6.2.4 4.4.10

Let L be a holomorphic line bundle on X a compact Kähler manifold. Suppose that $c_1(L) = [\alpha]$ where α is closed a real $(1,1)$-form. Let h_0 be a Hermitian structure on L then,

$$c_1(L, h_0) = \frac{i}{2\pi} \bar{\partial} \partial \log h_0$$

Now consider,

$$\eta = \alpha - c_1(L, h_0)$$

is a real $(1,1)$-form and since $[\alpha] = [c_1(L, h_0)]$ also η is d-exact. Thus, by the $\partial\bar{\partial}$-lemma, we know,

$$\eta = -\frac{i}{2\pi} \partial \bar{\partial} f$$

for $f \in \mathcal{A}^{0,0}_{\mathbb{R}}(X)$ i.e. f is a real smooth function. Therefore,

$$\alpha = \frac{i}{2\pi} \bar{\partial} \partial \left[f + \log h_0 \right] = \frac{i}{2\pi} \bar{\partial} \partial \log e^f h_0$$

Therefore, let $h = e^f h_0$ be annother Hermitian structure (since f is real) then we see $c_1(L, h) = \alpha$.

6.2.5 4.4.11

Let X be compact Kähler and E a vector bundle with a Chern connection ∇. If we let,

$$\sum_{i=0}^{r} \tilde{P}_i(B) = \mathrm{tr}\left(e^B\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{tr}\left(B^n\right)$$

so,

$$\tilde{P}_k(B_1, \dots, B_k) = \frac{1}{k!}\mathrm{tr}\left(B_1 \cdots B_k\right)$$

and then define,

$$\mathrm{ch}_k(E, \nabla) := \tilde{P}_k\left(\frac{i}{2\pi}F_\nabla\right) \in \mathcal{A}^{2k}_{\mathbb{C}}(M)$$

where $\tilde{P}_k$ acts on $\mathrm{End}\,(E)$-valued 2-forms via,

$$\tilde{P}_k(\alpha_1 \otimes \varphi_1, \dots, \alpha_k \otimes \varphi_k) = (\alpha_1 \wedge \cdots \wedge \alpha_k)\,\tilde{P}_k(\varphi_1, \dots, \varphi_k) = (\alpha_1 \wedge \cdots \wedge \alpha_k)\,\frac{1}{k!}\mathrm{tr}\left(\varphi_1 \cdots \varphi_k\right)$$

This is the composition of $(\Omega^2_X)^{\otimes k} \to \Omega^{2k}_X$ via exterior product and $\mathrm{End}\,(E)^{\otimes k} \to \mathrm{End}\,(E)$ via composition and finally taking trace. We see that,

$$\mathrm{ch}_k(E, \nabla) = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \mathrm{tr}\left(F_\nabla^{\otimes k}\right)$$

where $F_\nabla^{\otimes k}$ is the image under $(\Omega^2_X \otimes \mathrm{End}\,(E))^{\otimes k} \to \Omega^{2k}_X \otimes \mathrm{End}\,(E)$. Now taking Dolbeault cohomology classes via $\mathcal{A}^{k,k}_{\mathbb{C}}(\mathrm{End}\,(E)) \to H^k(X, \Omega^k \otimes \mathrm{End}\,(E))$,

$$\mathrm{ch}_k(E) = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \mathrm{tr}\left([F_\nabla]^{\otimes k}\right)$$

where $[F_\nabla]^{\otimes k}$ is the image under the map,

$$H^1(X, \Omega^1_X \otimes \mathrm{End}\,(E)) \times \cdots \times H^1(X, \Omega^1_X \otimes \mathrm{End}\,(E)) \to H^k(X, \Omega^k \otimes \mathrm{End}\,(E))$$

Furthermore $[F_\nabla] = A(E)$ so we get,

$$\mathrm{ch}_k(E) = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \mathrm{tr}\left(A(E)^{\otimes k}\right)$$

as a class under the map $H^k(X, \Omega^k \otimes \mathrm{End}\,(E)) \xrightarrow{\mathrm{tr}} H^k(X, \Omega^k_X) \subset H^{2k}(X, \mathbb{C})$.

6.2.6 4.4.12

Let X be compact Kähler and E a holomorphic vector bundle admitting a holomorphic connection. Then $A(E) = 0$ and therefore $c_k(E) = 0$.

7 Chapter 5

7.1 Section 5.1

7.1.1 5.1.1

7.2 Section 5.2

7.2.1 5.2.1

7.3 Section 5.3

7.3.1 5.3.1

8 Chapter 6

8.1 Section 6.1

8.2 6.1.1

8.3 6.1.2

8.4 6.1.3