### 1 The Yoneda Embedding

**Lemma 1.0.1.** Let  $\eta : \text{Hom } (A, -) \to \text{Hom } (B, -)$  be a natural transformation. Then  $\eta$  is uniquely determined by  $\eta_A(\text{id}_A)$  via  $\eta_X(f) = f \circ \eta_A(\text{id}_A)$  for any  $f \in \text{Hom } (A, X)$ .

*Proof.* Let  $f: A \to X$  be some map. Consider the naturality diagram,

$$\operatorname{Hom}(A, A) \xrightarrow{f_*} \operatorname{Hom}(A, X) 
\downarrow^{\eta_A} \qquad \downarrow^{\eta_X} 
\operatorname{Hom}(B, A) \xrightarrow{f_*} \operatorname{Hom}(B, X)$$

Consider the element  $\mathrm{id}_A \in \mathrm{Hom}\,(A,A)$  which, under the upper path, maps to  $\eta_X(f_*(\mathrm{id}_A)) = \eta_X(f \circ \mathrm{id}_A) = \eta_X(f)$  and, under the lower path,  $f_*(\eta_A(\mathrm{id}_A)) = f \circ \eta_A(\mathrm{id}_A)$ . Therefore,

$$\eta_X(f) = f \circ \eta_A(\mathrm{id}_A)$$

Corollary 1.0.2. Natural transformations  $\eta : \text{Hom } (A, -) \to \text{Hom } (B, -)$  are in one-to-one correspondence with functions Hom (B, A). We say  $f^*$  is the natural transformation  $f_X^*(g) = g \circ f$  for any  $g \in \text{Hom } (A, X)$ .

**Theorem 1.0.3.** Let  $\mathcal{C}$  be any category. The functor  $Y: \mathcal{C}^{op} \to \mathbf{Set}^{\mathcal{C}}$  sending  $A \mapsto h^A$  where  $h^A = \mathrm{Hom}(A, -)$  and  $f \mapsto f^*$  described above is fully faithful.

*Proof.* Clearly  $(\mathrm{id}_A)^* = \mathrm{id}_{h^A}$  since  $(\mathrm{id}_A)^*(f) = f \circ \mathrm{id}_A = f$  and for  $f \in \mathrm{Hom}(B,A)$  and  $g \in \mathrm{Hom}(C,B)$  then  $(f \circ g)^* = g^* \circ f^*$  since for any  $g \in \mathrm{Hom}(A,X)$  we send,

$$(f \circ g)^*(q) = q \circ (f \circ g) = (q \circ f) \circ g = g^*(f^*(q))$$

The above corollary proves that Y is fully faithful.

**Lemma 1.0.4.** Let  $F: \mathcal{C} \to \mathcal{D}$  be fully faithful then  $X \cong Y \iff F(X) \cong F(Y)$ .

*Proof.* If  $F(X) \cong F(Y)$  then there are morphisms  $f \in \text{Hom }(F(X), F(Y))$  and  $g \in \text{Hom }(F(Y), F(X))$  which are inverses. However, since F is full there exist morphisms  $\tilde{f}$ : Hom (X, Y) and  $g \in \text{Hom }(Y, X)$  such that  $F(\tilde{f}) = f$  and  $F(\tilde{g}) = g$ . Then,

$$F(\tilde{f}\circ\tilde{g})=F(\tilde{f})\circ F(\tilde{g})=f\circ g=\mathrm{id}_{F(Y)}\quad\text{and}\quad F(\tilde{g}\circ\tilde{f})=F(\tilde{g})\circ F(\tilde{f})=g\circ f=\mathrm{id}_{F(X)}$$

However, since F is faithful then,

$$\tilde{f} \circ \tilde{g} = \mathrm{id}_Y$$
 and  $\tilde{g} \circ \tilde{f} = \mathrm{id}_X$ 

proving that  $X \cong Y$ .

**Definition 1.0.5.** We say a functor  $F: \mathcal{C} \to \mathbf{Set}$  is representable if  $F \cong h^A$  for some  $A \in \mathcal{C}$ .

## 2 Additive Categories

**Definition 2.0.1.** A category  $\mathcal{C}$  is pre-additive if its hom sets have the structure of an abelian group and composition of maps distributes over addition. Explicitly, for  $X, Y, Z \in \mathcal{C}$ , there exits a binary operation,

$$+: \operatorname{Hom}(X,Y) \times \operatorname{Hom}(X,Y) \to \operatorname{Hom}(X,Y)$$

such that (Hom (X,Y),+) is an abelian group and, for  $f,g:X\to Y$  and  $h,k:Y\to Z$  we have  $h\circ (f+g)=h\circ f+h\circ g$  and  $(h+k)\circ f=h\circ f+k\circ f$ . This is equivalent to the requirement that hom is a functor,

$$\operatorname{Hom} (-,-): \mathcal{C}^{\operatorname{op}} \times \mathcal{C} \to \mathbf{Ab}$$

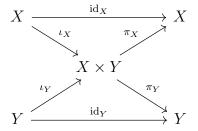
**Lemma 2.0.2.** In a pre-additive cateogory, there exists an identity element  $0 \in \text{Hom } (X,Y)$  such that 0 + f = f + 0 = f for  $f \in \text{Hom } (X,Y)$  and  $f \circ 0 = 0$  for  $f \in \text{Hom } (Y,Z)$  and  $0 \circ f = 0$  for  $f \in \text{Hom } (Z,X)$ .

Proof. The hom sets are abelian groups by definiton and thus must have unique identity elements satisfying f + 0 = 0 + f = f for all  $f \in \text{Hom}(X, Y)$ . Furthermore, for  $f \in \text{Hom}(Y, Z)$  we have  $f \circ 0 = f \circ (0 + 0) = f \circ 0 + f \circ 0$  and thus  $f \circ 0 = 0_{XZ}$ . Furthermore for  $f \in \text{Hom}(Z, X)$  we know that  $0 \circ f = (0 + 0) \circ f = 0 \circ f + 0 \circ f$  so  $0 \circ f = 0_{ZY}$ .

**Definition 2.0.3.** A biproduct of an indexed set  $\{X_i\}_I$  is an object  $X = \bigoplus_I X_i$  along with projection maps  $\pi_i : X \to X_i$  and inclusion maps  $\iota_i : X_i \to X$  such that  $(X, \{\pi_i\}_I)$  is the product of  $\{X_i\}_I$  and  $(X, \{\iota_i\}_I)$  is the coproduct of  $\{X_i\}_I$ .

**Proposition 2.0.4.** Let  $\mathcal{C}$  be a pre-additive category. Every finite product and finite coproduct is a biproduct. In particular, finite products and coproducts are equal.

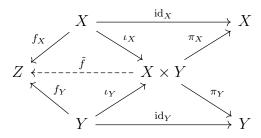
*Proof.* Let  $X \times Y$  be the product of X and Y. Consider the diagram,



where the maps  $\iota_X: X \to X \times Y$  and  $\iota_Y: Y \to X \times Y$  are defined via the universal property of the product applied to  $(\mathrm{id}_X, 0)$  and  $(0, \mathrm{id}_Y)$  respectively where  $0 \in \mathrm{Hom}(X, Y)$  is the identity element of the abelian group. The universal property gives,

$$\pi_X \circ \iota_X = \mathrm{id}_X \quad \pi_Y \circ \iota_X = 0$$
  
 $\pi_X \circ \iota_Y = 0 \quad \pi_Y \circ \iota_Y = \mathrm{id}_Y$ 

so the diagram commutes. We need to show that  $X \times Y$  is universal with respect to the maps  $\iota_X$  and  $\iota_Y$ . Suppose we have maps  $f_X : Z \to X$  and  $f_Y : Z \to Y$  then define  $\tilde{f} = f_X \circ \pi_X + f_Y \circ \pi_Y$ .



This map satisfies the required universal property because,

$$\tilde{f} \circ \iota_X = (f_X \circ \pi_X + f_Y \circ \pi_Y) \circ \iota_X = f_X \circ \pi_X \circ \iota_X + f_Y \circ \pi_Y \circ \iota_X = f_X + 0 = f_X$$

and likewse,

$$\tilde{f} \circ \iota_Y = (f_X \circ \pi_X + f_Y \circ \pi_Y) \circ \iota_Y = f_X \circ \pi_X \circ \iota_Y + f_Y \circ \pi_Y \circ \iota_Y = 0 + f_Y = f_Y$$

Lastly, we must show that  $\tilde{f}$  is unique. Suppose there exits a map  $\tilde{f}: X \times Y \to Z$  such that  $\tilde{f} \circ \iota_X = f_X$  and  $\tilde{f} \circ \iota_Y = f_Y$ . Consider the map  $I: X \times Y \to X \times Y$  given by,

$$I = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$$

Therefore,

$$\pi_X \circ I = \pi_X \circ \iota_X \circ \pi_X + \pi_X \circ \iota_Y \circ \pi_Y = \pi_X + 0 = \pi_X$$

and furthermore,

$$\pi_Y \circ I = \pi_Y \circ \iota_X \circ \pi_X + \pi_Y \circ \iota_Y \circ \pi_Y = 0 + \pi_Y = \pi_Y$$

However, by the universal property of the product, there exists a unique map, namely  $\mathrm{id}_{X\times Y}$ , satisfying these properties. Thus,  $I=\mathrm{id}_{X\times Y}$ . Thus,

$$\tilde{f} = \tilde{f} \circ \mathrm{id}_{X \times Y} = \tilde{f} \circ I = \tilde{f} \circ \iota_X \circ \pi_X + \tilde{f} \circ \iota_Y \circ \pi_Y = f_X \circ \pi_X + f_Y \circ \pi_Y$$

so the map we constructed earlier is unique.

Similarly, let  $X \coprod Y$  be the coproduct of X and Y. A similar argument will hold reversing all arrows.

Remark. Additionally, we see that a terminal object T (empty product) is also initial (empty coproduct) because  $\operatorname{Hom}(T,X)$  must have a zero element  $0:T\to X$  and for any map  $f:T\to X$  we know that  $f\circ 0_{TT}=0_{TX}$  but  $0_{TT}=\operatorname{id}_T$  is the unique map  $T\to T$  so  $f=0_{TX}$  and thus T is also initial.

**Definition 2.0.5.** A category is additive if it is pre-additive, has a zero object, and has all finite biproducts. The preceding dicussion implies that it is enough to check that either all finite products or all finite coproducts exit.

**Proposition 2.0.6.** In an additive category, the zero map is the indentity obeject of the **Ab**-enriched hom-sets.

**Definition 2.0.7.** A functor  $T: \mathcal{C} \to \mathcal{D}$  is additive if it preserves finite biproducts.

**Proposition 2.0.8.** A functor  $T: \mathcal{C} \to \mathcal{D}$  is additive iff the map on enriched hom-sets,

$$T_{X,Y}: \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(T(X),T(Y))$$

is a homomorphism in the category of abelian groups.

*Proof.* A biproduct  $X \oplus Y$  with its projections and inclusions is completely characterized by the property  $\mathrm{id}_{X \oplus Y} = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$ . Thus T preserves the biproduct structure iff it preserves addition i.e. iff,

$$\mathrm{id}_{T(X\oplus Y)}=T(\mathrm{id}_{X\oplus Y})=T(\iota_X\circ\pi_X+\iota_Y\circ\pi_Y)=T(\iota_X)\circ T(\pi_X)+T(\iota_Y)\circ T(\pi_Y)$$

DEF-COMPLEX PROP ADD-FUNC PRESERVE COMPLEXES

## 3 Normality

**Definition 3.0.1.** A morphism  $f: X \to Y$  is called,

- (a) split on the left (admits a left inverse) if there exists a map  $g: Y \to X$  such that  $g \circ f = \mathrm{id}_X$
- (b) split of the right (admits a right inverse) if there exists a map  $g: Y \to X$  such that  $f \circ g = \mathrm{id}_Y$
- (c) an isomorphism if there exists a map  $g: Y \to X$  such that  $g \circ f = \mathrm{id}_X$  and  $f \circ g = \mathrm{id}_Y$ .

**Lemma 3.0.2.** The following hold in any category:

- (a) a morphism split on the left is monic
- (b) a morphism split of the right is epic
- (c) a morphism is split both on the left and on the right if and only if it is an isomorphism.

*Proof.* First, if  $f: X \to Y$  admits a left inverse  $g: Y \to X$  then for any  $a, b: Z \to X$  such that  $f \circ a = f \circ b$  we have  $g \circ f \circ a = g \circ f \circ b$  and thus a = b since  $g \circ f = \mathrm{id}_X$ . Dually, if  $f: X \to Y$  admits a right inverse  $g: Y \to X$  then for any  $a, b: Y \to Z$  such that  $a \circ f = b \circ f$  we have  $a \circ f \circ g = b \circ f \circ g$  and thus a = b since  $f \circ g = \mathrm{id}_Y$ .

Finally, an isomorphism is clearly split on the left and right. To prove the converse, it suffices to show that the left and right inverses agree. Indeed if f has a left inverse  $g_L: B \to A$  and right inverse  $g_R: B \to A$  such that  $g_L \circ f = \mathrm{id}_A$  and  $f \circ g_R = \mathrm{id}_B$  then,

$$g_L = g_L \circ \mathrm{id}_B = g_L \circ (f \circ g_R) = (g_L \circ f) \circ g_R = \mathrm{id}_A \circ g_R = g_R$$

*Remark.* Due to the previous result, we alternatively call a morphisms split on the left a "split mono" and a morphism split on the right a "split epi".

*Remark.* Split monos and epis are important because every functor preserves them unlike usual monos and epis.

**Lemma 3.0.3.** Every equalizer is a monomorphism. Dually, every coequalizer is an epimorphism.

*Proof.* Let  $f, g: X \to Y$  be morphisms and  $e: E \to X$  be the equalizer. Given two maps  $a, b: E \to X$  such that  $q = e \circ a = e \circ b$  clearly we have  $f \circ q = g \circ q$  because  $f \circ e = g \circ e$ . Therefore, q factors uniquely through E meaning that a = b.

Corollary 3.0.4. Every kernel is a monomorphism. Dually, every cokernel is a epimorphism.

**Definition 3.0.5.** Let  $\mathcal{A}$  be a category with zero maps.

- (a) a monomorphism is *normal* if it is the kernel of some morphism
- (b) an epimorphism is *conormal* if it is the cokernel of some morphism
- (c)  $\mathcal{C}$  is normal if every monomorphism is the kernel of some morphism
- (d)  $\mathcal{C}$  is conormal if every epimorphism is the cokernel of some morphism

(e) C is binormal if it is normal and conormal.

**Proposition 3.0.6.** Let  $\mathcal{A}$  be a category with zero maps and  $f: A \to B$  a morphism. Then,

- (a) if f is a monomorphism and also a conormal epimorphism then f splits uniquely on the left
- (b) if f is an epimorphism and also a normal monomorphism then f splits uniquely on the right.
- (c) if f is a normal mono and also a conormal epi then f is an isomorphism.

#### (HERE I THINK ITS ALREADY AN ISOMORPHISM)

*Proof.* If f is a conormal epi, it is the cokernel of  $a: K \to A$ . Thus,  $f \circ a = f \circ 0 = 0$  but f is monic so a = 0. Therefore, we have a diagram,

$$K \xrightarrow{0} A \xrightarrow{f} B$$

$$\downarrow id_A \downarrow \qquad \downarrow g$$

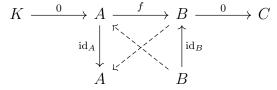
where  $id_A \circ a = 0$  because a = 0 and thus  $id_A$  factors through the cokernel  $f : A \to B$  as  $g \circ f$  for a unique  $g : B \to A$ .

The second is exactly the dual statement and thus is an application of the first in  $\mathcal{A}^{\text{op}}$ . The third follows directly by applying both the previous statements. However, we will spell it out for clarity.

Let  $f: A \to B$  be a normal mono and a conormal epi meaning f must be a kernel and a cokernel of some maps,

$$K \xrightarrow{\quad a\quad} A \xrightarrow{\quad f\quad} B \xrightarrow{\quad b\quad} C$$

where  $f: A \to B$  is the cokernel of  $a: K \to A$  and the kernel of  $b: B \to C$ . Then  $f \circ a = 0 = f \circ 0$  so, since f is monic, a = 0. Furthermore,  $b \circ f = 0 = 0 \circ f$  so, since f is an epic, b = 0. Therefore, consider the diagram,



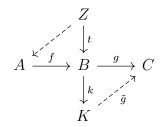
Where  $\mathrm{id}_A \circ a = 0$  and  $b \circ \mathrm{id}_B = 0$  (since a = 0 and b = 0) which implies that  $\mathrm{id}_A$  factors through the cokernel  $f: A \to B$  and  $\mathrm{id}_B$  lifts over the kernel  $f: A \to B$ . Thus f has a left inverse  $g_L: B \to A$  and right inverse  $g_R: B \to A$  such that  $g_L \circ f = \mathrm{id}_A$  and  $f \circ g_R = \mathrm{id}_B$ . Thus f is is both left and right split and thus is an isomorphism. Alternatively, the splittings are unique but notice that  $g_R \circ f = (g_L \circ f) \circ g_R \circ f = g_L \circ f = \mathrm{id}_A$  so  $g_R = g_L$  by uniqueness of the factorization.

Corollary 3.0.7. Let  $\mathcal{A}$  be a binormal category. Then any morphism in  $\mathcal{A}$  which is both monic and epic is an isomorphism.

**Proposition 3.0.8.** Let  $\mathcal{A}$  be a category with zero maps and  $f: A \to B$  a morphism. Then,

- (a) if f is a normal monomorphism then  $f = \ker \operatorname{coker} f$
- (b) if f is a conormal epimorphism then  $f = \operatorname{coker} \ker f$

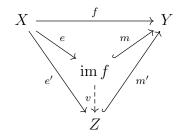
*Proof.* If f is a normal monomorphism then  $f = \ker g$  for some  $g: B \to C$ . Consider the diagram,



Because  $f = \ker g$  we know that  $g \circ f = 0$  and thus g factors through the cokernel of  $f : A \to B$  which is K. Then if  $k \circ t = 0$  we see that  $g \circ t = \tilde{g} \circ k \circ t = 0$  meaning that t factors uniquely through  $f : A \to B$  because  $f = \ker g$  showing that  $f = \ker k$ . The second statement is exactly dual.  $\square$ 

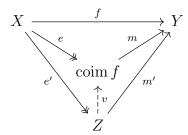
# 4 Images and Coimages (DOOO!!!! COMPARE WITH MAGIC SQUARE)

**Definition 4.0.1.** The image im f of a morphism  $f: X \to Y$  is the smallest subobject of Y such that f factors through im  $f \to Y$ . Explicitly, this is a factorization  $f = m \circ e$  with m monic such that for any other factorization  $f = m' \circ e'$  with m monic as in the diagram,



there is a unique arrow  $v: \operatorname{im} f \to Z$  making the diagram commute.

**Definition 4.0.2.** The coimage coim f of a morphism  $f: X \to Y$  is the image of f in the opposite category or equivalently the largest quotient of X such that f factors through  $X \to \operatorname{coim} f$ . Explicitly, this is a factorization  $f = m \circ e$  with e epic such that for any other factorization  $f = m' \circ e'$  with e epic as in the diagram,



there is a unique arrow  $v: Z \to \text{coim } f$  making the diagram commute.

# 5 Abelian Categories (DO!!!)

DEFINITON OF AB-CAT DEF OF IM AND COIM IM = COIM

**Definition 5.0.1.** We say that a sequence,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a complex if  $g \circ f = 0$  giving a monomorphism  $\operatorname{Im}(f) \to \ker g$ . We say the sequence is *exact* if this morphism is also epic i.e. an isomorphism by the above lemma.

**Proposition 5.0.2.**  $0 \longrightarrow X \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Z$ 

is exact iff  $(X \xrightarrow{f} Y) = \ker g$  and,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact iff  $(Y \xrightarrow{g} Y) = \operatorname{coker} f$ .

*Proof.* DO THIS PROOF

**Definition 5.0.3.** ABELIAN FUNCTOR

**Definition 5.0.4.** Let  $F: \mathcal{A} \to \mathcal{B}$  be an additive functor between abelian categories. Then we say that,

- (a) F is left-exact if F preserves kernels
- (b) F is right-exact if F preserves cokernels
- (c) F is exact if F preserves exact sequences

**Proposition 5.0.5.** F is exact iff F is left and right-exact.

Proof. DO THIS!! □

**Proposition 5.0.6.** Let  $F : A \to B$  and  $G : A \to B$  be an adjoint pair of additive functors between abelian categories. Then F is right-exact and G is left-exact.

*Proof.* Left-adjoints preserve colimits and right-adjoints preserve limits.

# 6 Homology