1 CM Types

2 Algebraic de Rham Cohomology

Let X be a smooth quasi-projective variety over K a characteristic zero field. Let $\dim X = n$. The algebraic de Rham complex is,

$$\Omega_X^\bullet := [0 \to \mathcal{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots]$$

where $\Omega_X^p := \wedge^p \Omega_{X/K}$ is the sheaf of Kahler differentials and differential is defined as,

$$\mathrm{d}(f_0 \, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_r) = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_r$$

Then we define the algebraic de Rham cohomology,

$$H^i_{\mathrm{dR}}(X/K) = \mathbb{H}^i(X, \Omega_X^\bullet)$$

There is a filtration on $\Omega_X^\bullet$ denoted,

$$\Omega_X^{\bullet \ge p} := [0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots]$$

The graded parts are,

$$\Omega_X^{\bullet \ge p} / \Omega_X^{\bullet \ge p+1} = \Omega_X^p[-p]$$

This gives a filtration on $H^i_{\mathrm{dR}}(X/K)$ by,

$$F^p H^i_{\mathrm{dR}}(X/K) := \mathrm{im}\,(\mathbb{H}(X, \Omega_X^{\bullet \ge p}) \to \mathbb{H}(X, \Omega_X^\bullet))$$

Then we get the Hodge-de Rham spectral sequence,

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies \mathbb{H}^{p+q}(X, \Omega_X^\bullet)$$

If $X/\mathbb{C}$ has structure of a complex manifold then we can consider the analogous construction in the analytic topology:

$$\Omega^\bullet_{X^{\mathrm{an}}} := [0 \to \mathcal{O}_{X^{\mathrm{an}}} \to \Omega^1_{X^{\mathrm{an}}} \to \Omega^2_{X^{\mathrm{an}}}]$$

Then GAGA implies (in the proper case),

$$H^i_{\mathrm{dR}}(X^{\mathrm{an}}/\mathbb{C}) := \mathbb{H}^i(X, \Omega_X^{\mathrm{an}}) = H^i_{\mathrm{dR}}(X/\mathbb{C})$$

Note that we need to use the compatibility of the algebraic and analtyic Hodge-de Rham spectral sequences since the de Rham complex is not a complex of $\mathcal{O}_X$-modules (differentials are nonlinear) but we can compare the Hodge cohomology directly from GAGA). This is true in more generality (FIND CITATION)

Proposition 2.0.1 (Grothendieck)**.** Let X be smooth over $\mathbb{C}$. Then,

$$H^i_{\mathrm{dR}}(X/\mathbb{C}) \cong H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$$

Proof. In the analytic topology there is a quasi-isomorphism from the Poincare lemma,

$$\mathbb{C}[0] \to \Omega^\bullet_{X^{\mathrm{an}}}$$

And therefore,

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \cong H^i_{\mathrm{dR}}(X^{\mathrm{an}}/\mathbb{C})$$

since this is an isomorphism on hypercohomology. $\square$

Moreover, if X is a compact Kahler manifold then degeneration of Hodge-de Rham spectral sequence comes down to the Hodge decomposition in terms of harmonic forms. Therefore,

$$H^n(X, \mathbb{C}) = H^n_{\mathrm{dR}}(X/\mathbb{C}) \cong \bigoplus_{p+q=n} H^q(X, \Omega^p_{X^{\mathrm{an}}})$$

Then H^n_{dR} is an example of a $\mathbb{Q}$-Hodge structure of pure weight n.

Definition 2.0.2. Let $Q \subset \mathbb{R}$ be a subring. A $\mathbb{Q}$-Hodge structure of pure weight k is a projective Q-module V with bigrading of $V_{\mathbb{C}} = V \otimes_Q \mathbb{C}$ meaning,

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

such that,

$$\overline{H}^{p,q} = H^{q,p}$$

Let $X/\mathbb{C}$ an consider $f : X \to Y$ a family of varities over $\mathbb{C}$ with f smooth proper. Then there are cohomology sheaves,

$$\rfloor\mathcal{H}^q_{\mathrm{dR}}(X/Y) := R^q f_* \Omega^\bullet_{X/Y}$$

finite locally free over Y which is filtered by submodules,

$$H^p \rfloor\mathcal{H}^q = R^q f_* \Omega^{\bullet \geq p}_{X/Y}$$

These are called Hodge bundles and define a variation of Hodge structures. Spectral sequences applied to $\Omega^\bullet_{X/\mathbb{C}}$ has,

$$E_1^{p,q} = \Omega^p_Y \otimes_{\mathcal{O}_Y} \rfloor\mathcal{H}^q_{\mathrm{dR}}(X/Y) \implies H^{p+q}_{\mathrm{dR}}(X/\mathbb{C})$$

2.1 Gauss-Manin Connection

$\Omega^p_{Y/\mathbb{C}} \otimes_{\mathcal{O}_Y} \rfloor\mathcal{H}^q_{\mathrm{dR}} \implies R^{p+q} f_* \Omega^\bullet_X$ differentials on E_1-page give,

$$\mathrm{d}_1^{p,q} : \Omega^p_Y \otimes_{\mathcal{O}_Y} \rfloor\mathcal{H}^q_{\mathrm{dR}} \to \Omega^{p+1}_Y \otimes_{\mathcal{O}_Y} \rfloor\mathcal{H}^q_{\mathrm{dR}}$$

For $p = 0$ this is the Gauss-Manin connection ∇. This is an algebraic definition so if $f : X \to Y$ is efined over $K \subset \mathbb{C}$ then ∇ is defined over K and $\rfloor\mathcal{H}^q_{\mathrm{dR}}$ is defined over K.

Cycle class map on de Rham or Betti cohomology,

$$H^i(X^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X/K) \otimes_K \mathbb{C}$$

rational structures on both sides not the same (even if $K = \mathbb{Q}$) there are at least factors of $2\pi i$. Then we define cycle class maps, for a line bundle $\mathcal{L}$ we get,

$$c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}(1))$$

Indeed from the exponential sequence,

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

then we get,

$$H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}\,(X) \longrightarrow H^2(X, \mathbb{Z}(1)) \longrightarrow H^2(X, \mathcal{O}_X)$$

from the long exact sequence. Alternatively, we get a map,

$$\mathrm{dlog} : \mathcal{O}_X^\times \to \Omega^1_{X/K}$$

and this gives,

$$H^1(X, \mathcal{O}_X^\times) \to H^1(X, \Omega^1_{X/K})$$

However we want a class in de Rham cohomology. To do this, notice that dlog lands inside closed forms so we get a map of complexes,

$$\mathrm{dlog} : \mathcal{O}_X^\times[-1] \to \Omega^\bullet_X$$

and taking $\mathbb{H}^2(X, -)$ gives,

$$c_1 : \mathrm{Pic}\,(X) = H^1(X, \mathcal{O}_X^\times) \to H^2_{\mathrm{dR}}(X/K)$$

In general for a vector bundle $\mathcal{E}$ we want to define,

$$c_p(\mathcal{E}) \in F^p H^{2p}_{\mathrm{dR}}(X/K)$$

and compatibly get a class,

$$c_p(\mathcal{E}) \in H^{2p}(X^{\mathrm{an}}, \mathbb{Z}(p))$$

We define these inductively from c_1 using,

$$H^{2p}(\mathbb{P}(\mathcal{E}), \mathbb{Z}(p)) = \bigoplus_{i=0}^{r-1} \xi^i \cdot \pi^* H^{2p-2i}(X^{\mathrm{an}}, \mathbb{Z}(p-i))$$

where $\xi := c_1(\mathcal{O}(-1))$ and then,

$$\xi^r + \pi^*(c_1)\xi^{r-1} + \cdots + \pi^*(c_p) = 0$$

where we define $c_r(\mathcal{E}) := c_r$. This definition extends from vector bundle to Grothendieck group of coherent sheaves,

$$K_0(X) = G_0(X)$$

since X is smooth. Then for some subvariety $Z \subset X$ we get $[\mathcal{O}_Z] \in G_0(X) = K_0(X)$. Then we define,

$$[Z] = \frac{(-1)^{p-1}}{(p-1)!} c_p([\mathcal{O}_Z]) \in H^{2p}(X^{\mathrm{an}}, \mathbb{Z}(p))$$

where p is the codimension of Z.