1 Rational Curves on K3 Surfaces

Definition 1.0.1. A K3 surface X/k is a smooth projective surface such that $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$.

Remark. The condition $H^1(X, \mathcal{O}_X) = 0$ is used to rule out abelian surfaces. Equivalently we could require $\pi_1(X) = 0$.

1.1 Basics over $\mathbb{C}$

The Hodge diamond has $h^{2,0} = 1$ and $h^{1,1} = 20$ and all other (not obviously nonzero by symmetry) are zero.

From the exponential sequence,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

by the long exact sequence,

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X, \mathbb{Z})$$

but $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathbb{Z}) = \mathbb{Z}^{22}$ thus $\mathrm{Pic}\,(X)$ is free of rank $\rho \le 22$. In fact since $\mathrm{Pic}\,(X) \to H^{1,1}(X)$ is injective we have,

$$\rho := \mathrm{rank}\,\mathrm{Pic}\,(X) \le 20$$

Theorem 1.1.1 (Mori-Mukai '83)**.** Every K3 surface over $\mathbb{C}$ has a rational curve. Furthermore, a very general K3 has infinitely many.

Definition 1.1.2. A *polarization* H on a K3 X is an ample line bundle H which is primitive (not an integral multiple of another class).

Let $\mathscr{K}_g$ be the moduli stack of polarized K3 surfaces with $H^2 = g$. Fact: H^2 is always even. Indeed, by adjunction,

$$2g(H) - 2 = H \cdot (H + K_X) = H^2$$

Conjecture 1.1.3 (Bogomolov)**.** For any K3 surface X over $\mathbb{C}$ has infinitely many rational curves. Or if X/K with K a number field then any K-point has a rational curve defined over $\overline{\mathbb{Q}}$ passing through it.

Theorem 1.1.4 (Bogomolov-Hasset-Tschinkel)**.** If $\rho = 1$ and $g = 2$ then there exist infinitely many rational curves.

Theorem 1.1.5 (Li-Liedtke)**.** The same is true for any odd ρ and any degree g.

1.2 Proof Strategy

Reduction mod p.

(a) Deformation theory reduces to X/F for some number field F

(b) Find many good primes p to reduce at

(c) Compare Picard groups -> show there exists a rational curve not lifting

(d) Exhibit a sum of rational curves which does lift

1.3 K3 surfaces over finite fields

Theorem 1.3.1. If X is a K3 surface over a field k of characteristic p and $k = \bar{k}$ then there exists $T/W(k)$ finite and some $\mathscr{X} \to \mathrm{Spec}\,(T)$ smooth projective lifting X and generically a K3. Let S be the generic fiber. Then by smooth base change,

$$H^2_{\text{ét}}(X, \mu_{\ell^n}) \cong H^2(S, \mathbb{Z}/\ell^n)$$

Pitfall, X in characteristic p can be supersingular. Recall that we have étale cohomology groups $H^{2i}_{\text{ét}}(X, \mathbb{Z}_\ell(i))$ and there is a cycle class map,

$$\mathrm{Pic}\,(X) \to H^2(X, \mathbb{Z}_\ell(1))$$

given by the limit of the connecting maps in the sequence,

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

The tate conjecture predicts that,

$$\mathrm{Pic}\,(X)_{\mathbb{Q}} \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))$$

surjects onto the Frob_q-fixed part.

Theorem 1.3.2 (Charles)**.** Over k finite with characteristic ≥ 5 and X/k a K3 surface then the Tate conjecture holds.

Definition 1.3.3. A K3 is *supersingular* if $\mathrm{Frob} \circlearrowright H^2(X, \mathbb{Z}_\ell(1))$ is trivial.

Under the tate conjecture, supersingularity is equvalent to $\rho = 22$. Another pathology: K3 can be unirational but then they are supersingular.

Proposition 1.3.4. If X is a K3 and the Tate conjecture holds, then ρ is even over $\bar{k}$.

Proof. Let $\alpha_1, \dots, \alpha_{22}$ be the eigenvalues of $\mathrm{Frob} \circlearrowright H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))$. This representation is semisimple. Poincare duality identifies $\alpha_i \mapsto \alpha^{-1}$. Consider classes of eigenvalues,

(a) not roots of unity: even cardinality by Poincare pairing

(b) roots of unity: also even since must add up to 22.

The second class must be algebraic cycles over $\bar{k}$ by Tate. □

1.4 Reduction to Number Fields

Let S/k be a K3 with k a field of characteristic zero. WLOG $k = \mathrm{Frac}\,(B)$ where B/F smooth variety over F with F a number field. Then S spreads out to $\mathcal{S} \to B$ smooth projective with fibers K3 surfaces. We want to spread out the property of having infinitely many rational curves.

1.5 Comparison of Picard Groups

Theorem 1.5.1. Let S/F be a number field and $\mathfrak{p}$ is some prime. There is a specialization map $\text{Pic}\,(S_{\bar{F}}) \to \text{Pic}\,(S_{\bar{\kappa}})$ which is injective away from characteristic of $\mathfrak{p}$.

Proposition 1.5.2. If $p \geq 5$ then there exists $\mathcal{L}_p$ in $\text{Pic}\,(S_{\bar{\mathfrak{p}}})$ not lifting to $S_{\bar{\mathbb{Q}}}$.

Theorem 1.5.3 (Bogomolov-Tschinkel)**.** If X is a K3 over $k = \bar{k}$ then any effective divisor has a representative by an effective sum of rational curves.

Corollary 1.5.4. There exists a rational curve C_p not lifting to $S_{\bar{K}}$.

Because H is ample, there is some large N_p such that $N_pH - C_p$ is effective so applying the theorem again we get,

$$C_p + \sum_i n_i R_{p,i} \in |N_pH|$$

where the $R_{p,i}$ are also rational curves. But H lifts by definition.

Proposition 1.5.5. Assume that S_p is smooth, not supersingular, and assume $C_1 + \cdots + C_r$ all distinct rational curves lifts as a divisor class but no subset does. Then there exists a rational curve $C \subset S_{\bar{K}}$ whose divisor class specializes to the divisor class $C_1 + \cdots + C_r$.

Proof. Consider the moduli space of stable maps $\overline{\mathcal{M}}_0 \to_{S_p/W(\bar{k})}$. Since S_p is not uniruled, the fibers have dimension 0. Fact: this map has relative dimension -1 and the image is the deformation space compatible with the polarization. Therefore, there is a generic lift to a stable map $T \to S$ and minimality means that $\operatorname{im} T$ is irreducible. $\square$

Varying the prime we can ensure the minimality for different values of N_p