PHYS 2801 PROBLEM SET 4 SOLUTIONS

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1. K&K 3.4

The first realization to make is that the projectiles take the same amount of time to reach the peak of their trajectory as they do to fall to earth again. This reduces it to a one dimensional conservation of momentum problem. We will call the total mass M, so that the smaller projectile has a mass of $M/4$ and the larger one has mass $3M/4$. To land back at the launching station, the velocity of the smaller piece must be the negative of the velocity V of the pair just before the explosion. We can then apply conservation of momentum to find V', the velocity of the larger piece.

$$p_{tot} = MV = -\frac{M}{4}V + \frac{3M}{4}V'$$

Solving for V' gives $V' = \frac{5}{3}V$, meaning the large fragment will travel $\frac{5}{3}$ as far in the second half of its flight. Thus the total distance X travelled by the larger piece is

$$X = \frac{8}{3}L$$

2. K&K 3.7

This problem is split into two parts. First, block 2 accelerates in the x direction until it gets to the equilibrium position. At this point there will no longer be a force holding m_1 against the stop, and the system will be carried forward by the momentum gained by m_2. After this time there will be oscillation between the two masses but the center of mass will move at constant velocity since there are no external forces on the system along the x axis.

If x_2 is the coordinate of m_2 from its equilibrium position, then Hooke's law gives us $m_2\ddot{x}_2 = -kx_2$. We have seen that the solution to this differential equation is $x_2 = A\cos\omega t + B\sin\omega t$ where $\omega \equiv \sqrt{\frac{k}{m_2}}$. Since $x_2(0) = -l/2$ and $\dot{x}_2(0) = 0$, we can set A and B to get

$$x_2(t) = -\frac{l}{2}\cos\omega t$$

until m_2 reaches the equilibrium position, when $x_2 = 0$. This occurs when $\cos\omega t = 0$, i.e. when $t = \frac{\pi}{2\omega}$. The total momentum at this point is $m_2\dot{x}_2 = \frac{1}{2}m_2 l\omega$.

The position of the center of mass relative to the wall is

$$x_{cm} = \frac{m_1x_1 + m_2(l + x_2)}{m_1 + m_2}$$

Date: September 2013.

Note that the position of the 2nd mass is at $l + x_2$, since x_2 indicates the position of the 2nd block relative to the equilibrium position, which is l to the right of the wall. Thus

$$x_{cm}(t) = \frac{m_2 l}{m_1 + m_2}\left(1 - \frac{1}{2}\cos\omega t\right) \quad \text{for } t \leq \frac{\pi}{2\omega}$$

Once $t \geq \frac{\pi}{2\omega}$, the system will be moving with the momentum we found earlier, $\frac{1}{2}m_2 l\omega$, so its velocity $\dot{x}_{cm} = \frac{m_2 l\omega}{2(m_1+m_2)}$. Now we can write x_{cm} in the form of $x(t) = x_0 + v_0(t - t_0)$:

$$\begin{aligned} x_{cm}(t) &= \frac{m_2 l}{m_1 + m_2} + \frac{m_2 l\omega}{2(m_1 + m_2)}\left(t - \frac{\pi}{2\omega}\right) \\ &= \frac{m_2 l}{m_1 + m_2}\left(1 + \frac{1}{2}\omega t - \frac{\pi}{4}\right) \quad \text{for } t \geq \frac{\pi}{2\omega} \end{aligned}$$

3. K&K 3.12

We will use $M(t)$ and $v(t)$ to refer to the car's mass and velocity respectively as functions of time, $u = 5\,\mathrm{m/s}$ as the relative velocity of the sand to the car, and $\mu \equiv \frac{dm}{dt} = 10\,\mathrm{kg/s}$ as the rate of flow of mass. Since the rate of flow of mass is constant, $M(t) = M_0 + \mu t$, where M_0 is given to be 2000 kg. At time t, a small mass of sand Δm with velocity $v + u$ collides with the car of mass $M(T)$ and velocity $v(t)$. A small time Δt later, the combinded system has momentum $M(t + \Delta t)v(t + \Delta t)$. Keeping only 1st order terms and taking the $\Delta t \to 0$ limit when appropriate, we have by conservation of momentum

$$\begin{aligned} M(t + \Delta t)v(t + \Delta t) &= M(t)v(t) + \Delta m(v(t) + u) \\ (M(t) + \Delta m)(v(t) + \Delta v) &= M(t)v(t) + v(t)\Delta m + u\Delta m \\ M(t)\Delta v &= u\Delta m \\ M(t)\frac{\Delta v}{\Delta t} &= u\frac{\Delta m}{\Delta t} \\ \frac{dv}{dt} &= \frac{u\mu}{M(t)} \\ \int_0^{v(t)} dv' &= u\mu\int_0^t \frac{dt'}{M_0 + \mu t'} \\ v(t) &= u\ln\left(1 + \frac{\mu t}{M_0}\right) \end{aligned}$$

Along the way, we used the fact that $\frac{\Delta m}{\Delta t} \to \frac{dm}{dt} = \mu$ and $M(t) = M_0 + \mu t$. You probably did not have to be this rigorous (with all the Δm, Δv business) to solve this problem, but I wanted to illustrate a general method that can be applied to trickier problems. Plugging in numerically, we find that

$$v(100\,\mathrm{s}) = (5\,\mathrm{m/s})\ln\left(1 + \frac{1}{2}\right) \approx 2.0\,\mathrm{m/s}$$

4. K&K 3.13

First we will evaluate the average force the tow must apply over a single skier's trip to the top. The skier has mass $m = 70\,\mathrm{kg}$ and the slope is at an angle $\theta = 20°$. The tow is active over a distance $L = 100\,\mathrm{m}$ and pulls at a constant velocity of $v = 1.5\,\mathrm{m/s}$. The temptation is to say there is only a force $mg\sin\theta$ needed to maintain constant velocity to the top. However, the wording of the problem implies that the skiier must be accelerated *from rest* but leaves the top of the tow with momentum mv. To clarify the notation: in this problem, "Δ" before a quantity does not imply that it will be taken to an infinitesimal limit - merely that it represents a net change in that quantity. We have in general

$$\Delta\mathbf{p} = \int_{\mathbf{p}_1}^{\mathbf{p}_2} d\mathbf{p} = \int_{t_1}^{t_2} \frac{d\mathbf{p}}{dt}dt = \int_{t_1}^{t_2} \mathbf{F}_{\text{net}}dt = \bar{\mathbf{F}}_{\text{net}}\Delta t$$

where $\Delta t = t_2 - t_1$ is the time period we are averaging over and the bar over $\bar{\mathbf{F}}_{\text{net}}$ indicates that it is the average net force. Working along an axis parallel to and pointing up the slope, we apply this to our problem and find

$$\bar{F}_{\text{net}} = \bar{F}_1 - mg\sin\theta$$

where $\bar{F}_1$ is the average force applied by the tow for a single skier. We do not need to average the gravitational component because it is constant. Now, since $\Delta p = mv$, we have

$$mv = \Delta p = \bar{F}_{\text{net}}\Delta t = (\bar{F}_1 - mg\sin\theta)\Delta t$$

so therefore

$$\bar{F}_1 = mg\sin\theta + \frac{mv}{\Delta t}$$

Note that the time over which the skier accelerates to velocity v does not play a role in our answer, because we are merely interested in the *average* force over the entire trip to the top. The time taken to reach the top, Δt, is given by L/v.

Finally, we can find the average number of skiers N on the tow at any one time by dividing the trip time Δt by the average time it takes a new skier to latch on, $\tau = 5\,\mathrm{s}$. This gives $N = \frac{\Delta t}{\tau} = \frac{L}{v\tau}$, and since the total average force applied by the tow $\bar{F}_{\text{tow}} = N\bar{F}_1$, we obtain

$$\bar{F}_{\text{tow}} = \frac{L}{v\tau}\left(\frac{mv}{\Delta t} + mg\sin\theta\right) = \frac{m}{\tau}\left(v + \frac{gL}{v}\sin\theta\right)$$

After mashing our fingers against our calculators (or more probably, typing into Wolfram Alpha), we arrive at our final answer:

$$\bar{F}_{\text{tow}} = 3150\,\mathrm{N}$$

5. K&K 3.14

Since the problem does not specify, we will assume that the flatcar begins at rest. If it were not at rest, we could always transform to the inertial frame in which it is stationary, and if we cared, add its initial velocity to our answers.

a. This is the obvious case. By conservation of momentum, since N men with mass m jump off simultaneously with speed u, the final velocity V of the cart of mass M must be

$$V = \frac{Nmu}{M}$$

b. When each of the men jump off individually, we must be more careful in our analysis. Suppose at any time there are n men on the cart. When one man jumps off, conservation of momentum tells us

$$(M + (n-1)m)\, \Delta V_n = mu$$

where we are using ΔV_n to indicate the change in velocity from the jump from n to $n-1$ men. To find the final velocity, we must sum over every jump's contribution, from $n = N$ to $n = 1$. Of course, the order we take the sum in does not matter, so we can write it out as

$$V = \sum_{n=1}^{N} \Delta V_n = \sum_{n=1}^{N} \frac{mu}{M + (n-1)m} = u \sum_{n=0}^{N-1} \frac{1}{\frac{M}{m} + n}$$

If we want to show off (and of course we do) we can find a closed form for our answer. The trick lies in noticing that

$$\frac{1}{x+n} = \frac{d}{dx} \ln(x+n)$$

Using this along with properties of logarithms, we can write it as

$$\sum_{n=0}^{N-1} \frac{1}{x+n} = \frac{d}{dx} \sum_{n=0}^{N-1} \ln(x+n) = \frac{d}{dx} \ln \left[\prod_{n=0}^{N-1} (x+n) \right]$$

In case you haven't seen this notation before, the big Π indicates the product over all the terms in the parentheses, from $n = 0$ to $n = N - 1$. It is much like the big Σ except the operation is multiplication instead of addition.

If x were an integer, we could write this product as

$$\prod_{n=0}^{N-1} (x+n) = x(x+1)(x+2)\cdots(x+N-2)(x+N-1) = \frac{(x+N-1)!}{(x-1)!}$$

But what if x is not an integer? This motivates the definition of the *gamma function*,

$$\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} dx$$

You should check for yourself that $\Gamma(n+1) = n\Gamma(n)$ (HINT: integrate by parts). This along with the observation that $\Gamma(1) = 1$ allows us to see that $\Gamma(n+1) = n!$ for integer n. The gamma function allows us to generalize the factorial function to real numbers (in fact, it is defined in the entire complex plane, but that does not concern our application here). Now we are able to write our product for any real x:

$$\prod_{n=0}^{N-1} (x+n) = x(x+1)\cdots(x+N-1) = \frac{\Gamma(x+N)}{\Gamma(x)}$$

Plugging back into our expression for the original sum gives

$$\sum_{n=0}^{N-1} \frac{1}{x+n} = \frac{d}{dx} \ln\left(\frac{\Gamma(x+n)}{\Gamma(x)}\right) = \frac{d}{dx} \ln(\Gamma(x+n) - \frac{d}{dx} \ln(\Gamma(x))$$

At this point we might as well introduce the *digamma function*:

$$\psi(x) \equiv \frac{d}{dx} \ln(\Gamma(x))$$

The consequence of all this machinery is that if we still remember what we were doing here in the first place we can write our expression for the final velocity V of the flatcar as

$$V = u\left[\psi\left({}^{M}\!/\!_{m} + N\right) - \psi\left({}^{M}\!/\!_{m}\right)\right]$$

Look how pretty that is!

c. The first case should result in a higher final velocity for the cart. The additional inertia from the men remaining on the cart when one man jumps off causes each jump to be less influential to the final velocity than if they had all jumped at once.

6. K&K 3.15

a. If $x(t)$ denotes the length of rope hanging through the hole at time t, let us call $d(t) = l - x(t)$ the length of rope lying on the table. Denote the mass of a length x of rope by $m(x) = \frac{x}{l}M$ where M and l are the mass and length of the entire rope. The rope hanging through the hole experiences a downwards gravitational pull of $F = m(x(t))g$ and an upwards tension force F_T. The rope above the table is pulled through the hole by F_T. We then have

$$F_{\text{x,net}} = m(x)\ddot{x} = m(x)g - F_T$$

for the rope beneath the table and

$$F_{\text{d,net}} = m(d)\ddot{d} = -m(d)\ddot{x} = -F_T$$

Combining these two yields

$$\begin{aligned} m(x)\ddot{x} &= m(x)g - m(d)\ddot{x} \\ (m(x) + m(d))\ddot{x} &= \frac{x}{l}Mg \\ \ddot{x} &= \gamma^2 x \end{aligned}$$

with $\gamma^2 \equiv \frac{g}{l}$. We know this differential equation has exponential solutions

$$x(t) = Ae^{\gamma t} + Be^{-\gamma t}$$

where the constants A and B are arbitray without defining initial conditions.

b. Once we do apply the initial conditions, $x(0) = l_0$ and $\dot{x}(0) = 0$, we get two equations for A and B

$$\begin{aligned} A + B &= l_0 \\ \gamma A - \gamma B &= 0 \end{aligned}$$

and we find that

$$A = B = \frac{l_0}{2}$$

Plugging back into our expression for $x(t)$ we can write

$$x(t) = \frac{l_0}{2} e^{\gamma t} + \frac{l_0}{2} e^{-\gamma t} = l_0 \cosh \gamma t$$

7. K&K 3.18

This problem is analyzing a raindrop that is gathering mass as it falls through a rain cloud. We have how the mass changes with velocity and what the initial mass is. We want to find the velocity of the drop with respect to time and then determine the terminal velocity of the drop. We should start newton's 2nd law for varying mass (change in momentum).

$$
\begin{aligned}
\frac{dp}{dt} &= \sum F \\
p(t) &= mv \\
p(t+dt) &= (m+dm)(v+dv) \\
dp &= mv + dmv + mdv + dmdv - mv = mdv + vdm \\
\frac{dp}{dt} &= \frac{dm}{dt}v + m\frac{dv}{dt} \\
\frac{dm}{dt}v + \frac{dv}{dt}m &= \sum F = mg \\
\frac{dm}{dt} &= kmv \\
kmv^2 + \frac{dv}{dt}m &= mg \\
\frac{dv}{dt} &= g - kv^2 \\
\frac{dv}{dt} &= -k(v^2 - g/k) \\
\frac{dv}{v^2 - g/k} &= -kdt \\
\int_0^v \frac{1}{(v')^2 - g/k}\,dv' &= -\int_0^t k\,dt' \\
\frac{1}{v^2 - g/k} &= \frac{A}{v - \sqrt{g/k}} + \frac{B}{v + \sqrt{g/k}} \\
0 &= A + B \\
1 &= \sqrt{g/k}(A - B) \\
B &= -\sqrt{k/g} \\
\int_0^v \left(\frac{\sqrt{k/g}}{v' - \sqrt{g/k}} - \frac{\sqrt{k/g}}{v' + \sqrt{g/k}}\right) dv' &= -kt \\
u1 &= v - \sqrt{g/k} \\
du1 &= dv \\
u2 &= v + \sqrt{g/k} \\
du2 &= dv \\
\sqrt{k/g}\int_0^v \left(\frac{1}{u1} - \frac{1}{u2}\right) du &= -kt \\
(ln(v' - \sqrt{g/k}) - ln(v' + \sqrt{g/k}))|_0^v &= -kt\sqrt{g/k} \\
ln\left(\frac{v - \sqrt{g/k}}{-\sqrt{g/k}}\right) - ln\left(\frac{v + \sqrt{g/k}}{\sqrt{g/k}}\right) &= -\sqrt{kg}t \\
ln\left(\frac{\sqrt{g/k} - v}{\sqrt{g/k} + v}\right) &= -\sqrt{kg}t \\
\frac{\sqrt{g/k} - v}{\sqrt{g/k} + v} &= e^{-\sqrt{kg}t} \\
(\sqrt{g/k} - v)e^{-\sqrt{kg}t} &= (\sqrt{g/k} + v)e^{-\sqrt{kg}t} \\
-v(1 + e^{-\sqrt{kg}t}) &= \sqrt{g/k}(e^{-\sqrt{kg}t} - 1) \\
v(t) &= \sqrt{g/k}\frac{1 - e^{-\sqrt{kg}t}}{1 + e^{-\sqrt{kg}t}} = \sqrt{g/k}tanh(\sqrt{kg}t) \\
v(t- > \infty) &= \sqrt{g/k} \\
v_t &= \sqrt{g/k}
\end{aligned}
$$

The terminal velocity is in the limit as time approaches infinity. Thus, the velocity reaching the value found above eventually and keeps falling at that speed, similar to what happens on a falling object under air resistance.

8. K&K 3.20

This problem is analyzing the motion of a rocket undergoing a gravitational force and air resistance. The equation for the motion of a rocket was derived in the book so lets use that. We are given the rate that mass is being excelled (γ) and the velocity of the exhaust.

$$\begin{aligned}
m\frac{dv}{dt} - u\frac{dm}{dt} &= \sum F = -mg - mbv \\
\frac{dm}{dt} &= \gamma m \\
m\frac{dv}{dt} - mu\gamma &= -mg - mbv \\
\frac{dv}{dt} &= -g - bv + u\gamma \\
\frac{dv}{dt} &= -b(v + g/b - u\gamma/b) \\
\frac{dv}{v + g/b - u\gamma/b} &= -bdt \\
\int_0^v \frac{1}{v' + g/b - u\gamma/b}\,dv' &= -\int_0^t b\,dt' \\
u &= v' + g/b - u\gamma/b \\
du &= dv' \\
\int_0^v \frac{1}{u}\,du &= -bt \\
ln(v' + g/b - u\gamma/b)|_0^v &= -bt \\
ln(\frac{v + g/b - u\gamma/b}{g/b - u\gamma/b}) &= -bt \\
\frac{v + g/b - u\gamma/b}{g/b - u\gamma/b} &= e^{-bt} \\
v + g/b - u\gamma/b &= (g/b - u\gamma/b)e^{-bt} \\
v(t) &= (u\gamma/b - g/b)(1 - e^{-bt})
\end{aligned}$$

Now check the hint giving is the book for the terminal velocity.

$$v_t = v(t->\infty) = u\gamma/b - g/b$$

9. Center of Mass: Cone

We want to calculate the center of mass of a cone with radius R_c and length R. The equation for the center of mass is given below.

$$\vec{r}_{CM} = \frac{\int \rho\vec{r}\,dV}{\int \rho\,dV}$$

The symmetry of the cone about the y and z axis tells us that $y_{CM} = 0$ and $z_{CM} = 0$. Therefore, we only need to determine the x_{CM}. The differential volume can be written as $dV = A(x)dx$ with A(x) as the area of the face of a slice of the cone. This will depend on the variable y since that is in the radial direction of the cone, but we can use proportions on the cone to determine how y depends on x.

$$\begin{aligned}
x_{CM} &= \frac{\int \rho x A(x)\,dx}{\int \rho A(x)\,dx} \\
A(x) &= \pi y^2 \\
\frac{R_c}{R} &= \frac{y}{x} \\
y &= \frac{R_c}{R}x
\end{aligned}$$

Now we are at the point where we can perform the integral. Start with the integral on the denominator since it is a little easier. The integrands will be x from 0 to R.

$$\begin{aligned}
\int_0^R \rho A(x')\,dx' &= \int_0^R \rho\pi\frac{R_c^2}{R^2}(x')^2\,dx' \\
&= \frac{R_c^2}{R^2}\int_0^R \rho\pi(x')^2\,dx' \\
&= \rho\pi\frac{R_c^2}{R^2}\frac{1}{3}(x')^3|_0^R \\
&= \rho\pi\frac{R_c^2}{3R^2}R^3 = \rho\pi\frac{R_c^2 R}{3}
\end{aligned}$$

Now do the integral in the numerator.

$$
\begin{aligned}
\int_0^R \rho x A(x')\,dx' &= \int_0^{R_c} \rho\pi \frac{R_c^2}{R^2}(x')^3\,dx' \\
&= \rho\pi \frac{R_c^2}{R^2}\frac{1}{4}(x')^4|_0^R \\
&= \rho\pi \frac{R_c^2}{R^2}\frac{1}{4}R^4 = \rho\pi\frac{R_c^2 R^2}{4}
\end{aligned}
$$

Now lets divide them and find the position of the center of mass of the cone in the x direction.

$$
\begin{aligned}
x_{CM} &= \frac{\rho\pi\frac{R_c^2R^2}{4}}{\rho\pi\frac{R_c^2R}{3}} \\
x_{CM} &= \frac{3}{4}R \\
\vec{r}_{CM} &= (\frac{3}{4}R, 0, 0)
\end{aligned}
$$