1 Group Theory

1.1 Semi-Direct Products

Proposition 1.1.1. If $N, M \subset G$ are normal subgroups such that $N \cap M = \{e\}$ and $NM = G$ then $G \cong N \times M$.

Proof. Consider the map $\varphi : N \times M \to G$ via $\varphi(n, m) = nm$. First, we need to show that this is a homomorphism. It suffices to show that if $n \in N$ and $m \in M$ then $nm = mn$. Indeed, $nmn^{-1}m^{-1} \in N \cap M$ because both are normal (so $nmn^{-1} \in M$ and thus so is $nmn^{-1}m^{-1}$ and ditto for N). However, $N \cap M = \{e\}$ thus $nm = mn$. Because $NM = G$ the map φ is surjective. Finally, if $nm = e$ then $n = m^{-1}$ so $n \in N \cap M$ and thus $n = m = e$ so $\ker \varphi = \{e\}$ and thus φ is an isomorphim. $\square$

Remark. The semidirect product $N \rtimes_\varphi H$ for the action $\varphi : H \to \mathrm{Aut}(N)$ is defined by $(n, h) \cdot (n', h') = (n\varphi(h) \cdot n', hh')$. Then notice that $(n, h)^{-1} = (\varphi(h^{-1}) \cdot n^{-1}, h^{-1})$ because,

$$(n, h) \cdot (\varphi(h^{-1}) \cdot n^{-1}, h^{-1}) = (n\varphi(h)\varphi(h^{-1})n^{-1}, hh^{-1}) = (e, e)$$

and likewise,

$$(\varphi(h^{-1}) \cdot n^{-1}, h^{-1}) \cdot (n, h) = (\varphi(h^{-1})n^{-1}\varphi(h^{-1}) \cdot n, h^{-1}h) = (\varphi(h^{-1})(n^{-1}n), e) = (e, e)$$

Then notice,

$$(e, h) \cdot (n, e) \cdot (e, h^{-1}) = (\varphi(h) \cdot n, h) \cdot (e, h^{-1}) = (\varphi(h) \cdot n, e)$$

so we say that $H \circlearrowright N$ through φ via conjugation inside $G = N \rtimes H$.

Proposition 1.1.2. Fix two groups N and H. Isomorphism classes of semi-direct products $N \rtimes_\varphi H$ correspond to classes of homomorphisms $\varphi : H \to \mathrm{Aut}(N)$ up to inner automorphism.

Proof. Suppose that $\varphi, \psi : H \to \mathrm{Aut}(N)$ are homomorphisms such that $\psi(h) \cdot n = q(\varphi(h) \cdot n)q^{-1}$ for $q \in N$. Then consider the bijection $f_q : N \rtimes_\varphi H \to N \rtimes_\psi H$ via $(n, h) \mapsto (qnq^{-1}, h)$. Then,

$$\begin{aligned} f_q((n, h) \cdot_\varphi (n', h')) = f_q((n\varphi(h') \cdot n', hh')) &= (qn(\varphi(h') \cdot n')q^{-1}, hh') = (qnq^{-1}q(\psi(h') \cdot n')q^{-1}, hh') \\ &= (qnq^{-1}, h) \cdot_\psi (qn'q^{-1}, h') = f_q(n, h) \cdot_\psi f_q(n', h') \end{aligned}$$

Therefore $f_q : N \rtimes_\varphi H \xrightarrow{\sim} N \rtimes_\psi H$ is an isomorphism.

Conversely, suppose that $f : N \rtimes_\varphi H \xrightarrow{\sim} N \rtimes_\psi H$ is an isomorphism. Then, consider,

$$f((e, h) \cdot_\varphi (n, e) \cdot_\varphi (e, h^{-1})) = f((\varphi(h) \cdot n, e))$$

(WAIT IS THIS TRUE!!!) $\square$

Remark. Let H be any group then $\varphi : H \to \mathrm{Aut}(H)$ sending $h \mapsto \varphi_h$ where φ_h is the inner automorphism $\varphi_h : x \mapsto hxh^{-1}$ is a crossed module. Indeed,

$$\varphi(\psi \cdot h) = \psi \circ \varphi_h \circ \psi^{-1}$$

because,

$$\varphi(\psi \cdot h)(x) = (\psi \cdot h)x(\psi \cdot h^{-1}) = \psi(h\psi^{-1}(x)h^{-1}) = (\psi \circ \varphi_h \circ \psi^{-1})(x)$$

and furthermore,

$$\varphi(h) \cdot h' = hh'h^{-1}$$

by definition.

Proposition 1.1.3. If $N \subset G$ is normal and $H \subset G$ is any subgroup such that $N \cap H = \{e\}$ and $NH = G$ then $G \cong N \rtimes H$ for some action $\varphi : H \to \mathrm{Aut}(N)$.

1.2 The Isomorphism Theorem

Theorem 1.2.1 (Second Isomorphism Theorem)**.** Let $N \subset G$ be a normal subgroup and $H \subset G$ any subgroup. Then $N \cap H$ is normal in H and $HN \subset G$ is a subgroup and $H/H \cap N \cong HN/N$.

Proof. First, suppose that $h \in H$ and $n \in H \cap N$. Then consider hnh^{-1}. Because $N \subset G$ is normal then $hnh^{-1} \in N$ but $n \in H$ so $hnh^{-1} \in H$ and thus $hnh^{-1} \in H \cap N$ so $H \cap N$ is normal in H. Consider the map $\varphi : H \to HN/N$ via $h \mapsto [h \cdot 1]$. Consider $hn, h'n' \in HN$ then notice that $hn \cdot h'n' = hnh'n' = hh'n''n' \in HN$ for $n'' = h'^{-1}nh' \in N$ because N is normal. Furthermore, $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}n' \in HN$ where $n' = hn^{-1}h^{-1} \in N$ because N is normal. Thus $NH \subset G$ is a subgroup. Now, for any $hn \in HN$ clearly, $[h \cdot 1] = [hn]$ in HN/N so φ is surjective. Furthermore, clearly $\ker \varphi = H \cap N$ so the result follows. $\square$

1.3 Groups of Order pq

Let G be a group of order $n = pq$ for distinct primes p, q. Let $P, Q \subset G$ be the Sylow p and q subgroups. From the Sylow theorems,

$$n_P = pk_P + 1 \mid q \quad \text{and} \quad n_Q = qk_Q + 1 \mid p$$

Without loss of generality, let $p < q$ then we must have $n_Q = 1$ so $Q \subset G$ is normal. By the second isomorphism theorem,

$$PQ/Q \cong P/P \cap Q$$

However, $P \cap Q$ is a subgroup of both P and Q which must be trivial by Lagrange since they have coprime orders. Thus, $P \cong PQ/Q$ so $|PQ| = |P||Q| = pq$ and thus $PQ = G$. Therefore, we conclude that,

$$G \cong Q \rtimes P$$

for some action $P \to \text{Aut}\,(Q)$. Furthermore, since P and Q have prime orders they must be cyclic. Thus $P \to \text{Aut}\,(Q)$ is a map $C_p \to \text{Aut}\,(C_q) \cong C_{q-1}$. Such a map is given by sending a generator x to y^k for some generator $y \in C_{q-1}$ where $q - 1 \mid pk$.

1.4 Exercises

Exercise 1.4.1. Let G be a finite group and $N \subset G$ normal such that $|N|$ and $[G : N]$ are coprime. Then prove that $H \subset G$ is the unique subgroup of order $|N|$.

Suppose that $H \subset G$ is a subgroup of order $|N|$. By the second isomorphism theorem,

$$HN/N \cong H/H \cap N$$

Write $n = |G|$ as $n = ab$ where $a = |N|$ and $b = [G : N]$. Then, HN/N must divide b because $|HN|$ divides ab but is divisible by a (it contains N) so $|HN/N| = |NH|/a$ divides b. However, $H/H \cap N$ divides a so both sides must be 1 since a and b are coprime. Thus $H \cap N = N$ so $H = N$ since they have the same number of elements.

Exercise 1.4.2. Let G be a group of order $30 = 2 \cdot 3 \cdot 5$ then,

(a) show that G has a subgroup of order 15

(b) show that every group of order 15 is cyclic

(c) show that G is a semi-direct product $C_{15} \rtimes C_2$

(d) exhibit three nonisomorphic (with proof) groups of order 30.

Let P, Q be the Sylow 3 and 5 subgroups. By the Sylow theorems,

$$n_P = 3k_P + 1 \mid 2 \cdot 5 \quad \text{and} \quad n_Q = 5k_Q + 1 \mid 2 \cdot 3$$

So $n_P = 1$ or $n_P = 10$ and $n_Q = 1$ or $n_Q = 6$. If niether one is normal then $n_P = 10$ and $n_Q = 6$ which would mean there are $10 \cdot 2$ elements of order 3 (these groups are prime order so they must be disjoint except at e) and $6 \cdot 4$ elements of order 4 but $10 \cdot 2 + 6 \cdot 4 + 1 = 45$ which is bigger than 30 so one must be normal. Let N be the normal one and H the other subgroup. Then $N \cap H = \{e\}$ because they have coprime order so by the second isomorphism theorem,

$$NH/N \cong H/H \cap N = H$$

meaning that $|NH| = |N| \cdot |H| = 3 \cdot 5 = 15$ so NH is a subgroup of order 15.

This follows from Sylow arguments. Groups of order 15 are type pq which are all semi-direct products $C_q \rtimes C_p$ if $p < q$ and there are no nontrivial maps $C_3 \to \mathrm{Aut}\,(C_5) = C_4$ so this semi-direct product is direct. Thus $C_3 \times C_5 = C_{15}$ is the only group of order 15.

Let R be the Sylow 2 subgroup and let H be the cyclic subgroup of order 15. Because $[G : H] = 2$ we know H is normal so by the second isomorphism theorem,

$$RH/H \cong R/R \cap H$$

but $R \cap H = \{e\}$ because they have coprime orders so $|RH| = |H| \cdot |R| = 30$ and thus $RH = G$. Therefore, $G \cong H \rtimes R$ but we know that $H \cong C_{15}$ and $R \cong C_2$ so we find $G \cong C_{15} \rtimes C_2$.

Semi-direct products $C_{15} \rtimes C_2$ are (almost) classified by conjugation types of homomorphisms

$$C_2 \to \mathrm{Aut}\,(C_{15}) \cong C_2 \times C_4$$

Since C_{15} is abelian there are no inner automorphisms. Consider three maps, the trivial group φ_0 the map $\varphi_1 : C_2 \hookrightarrow C_2 \times C_4$ into the first factor and the map $\varphi_2 : C_2 \hookrightarrow C_2 \times C_4$ sending $C_2 \to C_4$ the unique subgroup of order 2. Then let $G_i = C_{15} \rtimes_{\varphi_i} C_2$. Clearly G_0 is abelian but G_1 and G_2 are not so it suffices to show that G_1 and G_2 are not isomorphic. Just write down the table. I don't want to but we could also just consider D_{15} and $C_5 \times S_3$. Indeed $Z(D_{15})$ is trivial but $Z(C_5 \times S_3)$ is not.

Exercise 1.4.3. Let G be a group of order $105 = 3 \cdot 5 \cdot 7$ and let P, Q, R be the corresponding Sylow subgroups. Prove that,

(a) one of Q or R is normal in G

(b) G has a cyclic subgroup of order 35

(c) both Q and R are normal in G

(d) if P is normal in G then G is cyclic

By the Sylow theorems,

$$n_Q = 5k_Q + 1 \mid 3 \cdot 7 \quad \text{and} \quad n_R = 7k_P + 1 \mid 3 \cdot 5$$

then either $n_Q = 1$ or $n_Q = 21$ and $n_R = 1$ or $n_R = 15$. Because these subgroups have prime order the conjugates but be disjoint (except for e). Each n_Q contains 4 elements of order 5 and thus if $n_Q = 21$ there are $4 \cdot 21$ elements of order 5 and if $n_R = 15$ there must be $6 \cdot 15$ elements of order 7. However, $4 \cdot 21 + 6 \cdot 15 + 1 = 175$ greater than the total number of elements so either $n_Q = 1$ or $n_R = 1$ proving that either P or R is normal.

Any group of order 35 is cyclic by a Sylow argument, notice there are only trivial maps $C_5 \to \text{Aut}(C_7) \cong C_6$. Therefore it suffices to find a subgroup of G of order 35. Consider QR. Since one is normal, call it N and the other H, by the second isomorphism theorem,

$$HN/N \cong H/H \cap N$$

but H and N have coprime orders so $H \cap N = \{e\}$. Therefore $|HN| = |H| \cdot |N|$ so $|QR| = |Q| \cdot |R| = 5 \cdot 7 = 35$ so the subgroup QR is a subgroup of order 35.

Clearly $Q, R \subset QR$ and since QR is cyclic any subgroup is also cyclic proving that both Q and R are cyclic.

Let H be the cyclic subgroup of order 35. Suppose that P is normal in G. Then by the second isomorphism theorem,

$$HP/P \cong H/H \cap P$$

However, P and H have coprime order so $H \cap P = \{e\}$ and thus we find that $|HP| = |H| \cdot |P| = |G|$ so $HP = G$. Therefore, G is a semi-direct product of P and H but $\text{Aut}(P) \cong C_2$ and there is no nontrivial map $H \to C_2$ because $|H|$ is odd. Thus $G = P \times H$ and since P and H are cyclic of coprime orders we have that G is also cyclic by the Chinese remainder theorem.

Exercise 1.4.4. Let F be a field and E/F an extension. Let $\alpha \in E$ be algebraic of odd degree over F. Then prove that,

(a) $F(\alpha) = F(\alpha^2)$

(b) the element $\alpha^n \in E$ has odd degree over F

Assume that $\alpha \notin F(\alpha^2)$ then $1, \alpha$ is clearly a basis of $F(\alpha)$ over $F(\alpha^2)$ so $[F(\alpha) : F(\alpha^2)] = 2$ but $[F(\alpha) : F] = [F(\alpha) : F(\alpha^2)][F(\alpha^2) : F]$ is even giving a contradiction so $\alpha \in F(\alpha^2)$. Furthermore,

$$[F(\alpha) : F] = [F(\alpha) : F(\alpha^n)][F(\alpha^n) : F]$$

and thus $[F(\alpha^n) : F]$ is odd so the degree of α^n is odd.

2 Analysis

Exercise 2.0.1. Let $f : D^\circ \to \mathfrak{h}$ be a holomorphic function from the open unit disk to the upper half plane. Assume that $f(0) = in$ then find a sharp bound on $|f'(0)|$.

Notice that $g = e^{if/s} : D^\circ \to \mathbb{C}$ is bounded by 1. Then by Cauchy,

$$g'(0) = \frac{1}{2\pi i} \oint_\gamma \frac{g(z)}{z^2} \mathrm{d}z = \int_0^1 \frac{g(re^{2\pi i t})}{re^{2\pi i t}} \mathrm{d}t$$

Therefore,

$$|g'(0)| \le \frac{1}{r}$$

so we can take the limit $r \to 1$ and get $|g'(0)| \le 1$. Then,

$$g'(0) = \frac{if'(0)}{s} e^{if(0)/s}$$

so we find that,

$$|f'(0)| \le |g'(0)| \, s \, e^{n/s}$$

Now we minimize over s. Consider $m(s) = s\, e^{2/s}$ then $m'(s) = (1 - n/s)e^{n/s}$ so the minimum occurs at $s = n$ and thus we find that,

$$|f'(0)| \le ne$$

Actually though, we can do better by using a better transform. Consider,

$$g : \mathfrak{h} \to D^\circ \quad \text{via} \quad g(z) = \frac{z - is}{z + is}$$

Notice that,

$$\left| \frac{z - is}{z + is} \right|^2 = \frac{x^2 + (y - s)^2}{x^2 + (y + s)^2} \le 1$$

because $y > 0$ and $s > 0$. Then $g \circ f$ is a self-map of the disk and thus is bounded by 1. Therefore, by the Cauchy integral formula,

$$|(g \circ f)'(0)| \le 1$$

however,

$$(g \circ f)'(z) = \frac{2isf'(z)}{(f(z) + is)^2}$$

Therefore,

$$|f'(0)| = \frac{1}{2s} \cdot (n + s)^2 \cdot |(g \circ f)'(0)|$$

Now we minimize with respect to s. Consider $m(s) = \frac{(n+s)^2}{2s}$ then

$$m'(s) = \frac{n + s}{s} - \frac{(n + s)^2}{2s^2} = \frac{n + s}{2s^2} \cdot (2s - (n + s))$$

and therefore $s = n$ so we find that,

$$|f'(0)| \le 2n$$

To show that this bound is sharp, consider,

$$f(z) = in \cdot \frac{1+z}{1-z}$$

It is easy to show that $\text{Im}\,(f(z)) > 0$ and $f(0) = in$. Furthermore,

$$f'(0) = 2in$$

Exercise 2.0.2. Suppose we have Lebesgue integrable functions $f, g : \mathbb{R} \to \mathbb{R}$ then show that,

$$\lim_{n\to\infty} ||f + g_n||_1 = ||f||_1 + ||g||_1$$

where $g_n(x) = g(x-n)$.

Consider the functions,

$$F(x) = \int_{-\infty}^{x} |f(t)|\mathrm{d}t \quad \text{and} \quad G(x) = \int_{x}^{\infty} |g(t)|\mathrm{d}t$$

Then for any $\epsilon > 0$ we can find x_1 and x_2 such that $F(x_1) > ||f||_1 - \epsilon$ and $G(x_2) > ||g||_1 - \epsilon$ because the limits of each are $||f||_1$ and $||g||_1$ respectively. Notice that F is increasing and G is decreasing. Then choose n large enough such that $x_1 < x_2 + n$. Then consider,

$$||f+g_n||_1 = \int_{-\infty}^{\infty} |f(t)+g(t-n)||\mathrm{d}t = \int_{-\infty}^{x_1} |f(t)+g(t-n)|\mathrm{d}t + \int_{x_1}^{x_2+n} |f(t)+g(t-n)|\mathrm{d}t + \int_{x_2+n}^{\infty} |f(t)+g(t-n)|\mathrm{d}t$$

Each term is nonnegative so we can throw away the middle term and use,

$$||f+g_n||_1 \geq= \int_{-\infty}^{x_1} |f(t)+g(t-n)|\mathrm{d}t + \int_{x_2+n}^{\infty} |f(t)+g(t-n)|\mathrm{d}t \geq F(x_1) + G(x_2) > ||f||_1 + ||g||_1 - 2\epsilon$$

proving that the limit converges,

$$\lim_{n\to\infty} ||f + g_n||_1 = ||f||_1 + ||g_n||_1$$

since of course $||f + g_n||_1 \leq ||f||_1 + ||g||_1$ using that $||g_n||_1 = ||g||_1$.

Exercise 2.0.3. Suppose that $f_n \to f$ almost everywhere and $\int |f_n| \to \int |f|$. Then prove that $\int f_n \to f$.

Consider $g_n = |f_n| - |f_n - f|$ which are measurable and notice that,

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f|$$

and therefore are uniformly bounded by the integrable function $|f|$. Therefore by the dominated convergence theorem we find that,

$$\int g_n \to \int |f|$$

since $g_n \to |f|$ almost eveywhere. However,

$$\int |f_n - f| = \int |f_n| - g_n = \int |f_n| - \int g_n \to \int |f| - \int |f| = 0$$

because we assumed that $\int |f_n| \to \int |f|$. Therefore,

$$\lim_{n\to\infty} \left| \int f - \int f_n \right| \leq \lim_{n\to\infty} \int |f_n - f| = 0$$

meaning that $\int f_n \to \int f$.

Exercise 2.0.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continous function which is zero outside of a finite interval. Then show that,

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt}\mathrm{d}t$$

is entire.

Because f is continuous and supported on a compact set it is bounded, say by M. We need to consider,

$$g'(z) = \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \int_{-\infty}^{\infty} f(t)e^{-izt}\left(\frac{e^{-iht}-1}{h}\right)\mathrm{d}t$$

Now, for all z and t the following series converges absolutly,

$$\left(\frac{e^{-iht}-1}{h}\right)\mathrm{d}t = -it\sum_{n=9}^{\infty}\frac{(-iht)^n}{(n+1)!}$$

On any compact interval for t this power series also converges uniformly by the M-test. Therefore, because f is supported on such a compact interval as is bounded, by the M-test,

$$\sum_{n=0}^{\infty} f(t)e^{-izt}\frac{(-iht)^n}{(n+1)!}$$

is also uniformly convergent on that interval and each term is a continuous function of t with compact support and thus integrable meaning that,

$$g'(z) = \lim_{h\to 0}\sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-izt}\frac{(-it)^{n+1}}{(n+1)!}\mathrm{d}t\right)h^n$$

Since we know this power series converges for any fixed h this implies that its radius of convergence must be infinite (it would have been easier to just use the series expansion for e^{-izt} whoops) and in particular it is a continuous function everywhere the the limit exists,

$$g'(z) = \int_{-\infty}^{\infty} f(t)(-it)e^{-izt}\mathrm{d}t$$

Exercise 2.0.5. Is every complete bounded metric space compact?

No, for example take the closed unit ball in ℓ_2. Explicitly, $B = \{(a_n) \mid \sum_{i=1}^{\infty} a_i^2 = 1\}$. Since B is a closed subset of a complete metric space is it is complete and is bounded by construction. However, B is not compact. To see this, consider the open cover $\{U_i\}$ where U_i is the open subset where $a_i \neq 0$. Then for any finite subset the union is contained in $\bigcup_{i=1}^{k} U_i$ which does not contain,

$$a_i = \begin{cases} 0 & i \le k \\ \frac{1}{2^{i-k}} & \end{cases}$$

and thus there is no finite subcover so B is not compact.

Exercise 2.0.6. Let $(X, \mathscr{F}, \mu)$ be a finite measure space. Let $\{f_n\} \subset \mathcal{L}^1(X,\mu)$ be a sequence of functions and $f \in \mathcal{L}^1(X,\mu)$ such that $f_n \to f$ pointwise a.e. Prove theat for every $\epsilon > 0$ there exists $M > 0$ and a set $E \subset X$, such that $\mu(E) \le \epsilon$ and $|f_n(x)| \le M$ for all $x \in X \setminus E$ and all $n \in \mathbb{N}$.

Let $N \subset X$ be the set on which $f_n(x)$ does not converge to $f(x)$. Then by assumption $\mu(N) = 0$. Now, $f_n \to f$ pointwise on $X \setminus N$ so by Egorov's theorem, for any $\epsilon > 0$ there exists some measurable $E \subset X \setminus N$ such that $f_n \to f$ uniformly on $X \setminus (E \cup N)$ and $\mu(E) < \frac{1}{2}\epsilon$. By unifom convergence, on $X \setminus (N \cup E)$ we have that for $n > N$,

$$|f_n(x) - f(x)| \leq 1$$

Furthermore, $f_n, f \in L^1(X, \mu)$ meaning that,

$$\int_0^\infty \mu(\{x \in X \mid |f(x)| > t\}) \, \mathrm{d}t < \infty$$

so the integrands must tend to zero. Therefore, there is some $M > 0$ such that the sets,

$$E_i = \{x \in X \mid |f_i(x)| > (M-1)\} \quad \text{and} \quad E' = \{x \in X \mid |f(x)| > (M-1)\}$$

for $i = 1, \ldots, N$ have measure less than $\frac{1}{2(N+1)}\epsilon$. Therefore on $X \setminus (N \cup E_1 \cup \cdots \cup E_N \cup E')$ we have,

$$|f_n(x)| \leq M$$

because if $n \leq N$ then this follows since $x \notin E_n$ and if $n > N$ then,

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq 1 + (M-1)$$

because $x \notin (E \cup N)$ and $x \notin E'$.