# 1 Appendix

## 1.1 Curves and Genera

**Lemma 1.1.** Let $X$ be a integral scheme proper over $k$ then $K = H^0(X, \mathcal{O}_X)$ is a finite field extension of $k$ and for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, the cohomology $H^p(X, \mathcal{F})$ is a finite-dimensional $H^0(X, \mathcal{O}_X)$-module.

*Proof.* Since $\mathcal{O}_X$ is coherent, and $X$ is proper over $k$ so $K = H^0(X, \mathcal{O}_X)$ is a finite $k$-module. However, since $X$ is integral $H^0(X, \mathcal{O}_X)$ is a domain but a finite $k$-algebra domain is a field and we see $K/k$ is a finite extension of fields. Furthermore, the $\mathcal{O}_X(X)$-module structure on $H^p(X, \mathcal{F})$ gives it a $K$-module structure. Since $X$ is proper over $k$ then $H^p(X, \mathcal{F})$ is a finite $k$-module and thus finite as a $K$-module. $\square$

Unfortunately, when $k$ is not algebraically closed then we may not have $H^0(X, \mathcal{O}_X) = k$ even for smooth projective varieties. Therefore, some caution must be taken in defining numerical invariants of the curve such as genus. However, by [**?**, Tag 0BUG], whenever $X$ is proper geometrically integral then indeed $H^0(X, \mathcal{O}_X) = k$. Furthermore, for proper $X$ if $H^0(X, \mathcal{O}_X) \neq k$ then $X$ cannot be geometrically connected by [**?**, Tag 0FD1].

Let $C$ be a smooth proper curve over $k$ with $H^0(C, \mathcal{O}_C) = K$. Then we define $g(C) := \dim_K H^0(X, \Omega_{C/k})$. If $C$ is any curve over $k$ then there is a unique smooth proper curve $S$ over $k$ which is $k$-birational to $C$. Then we define $g(C) := g(S)$.

By definition, the genus of a curve is clearly a birational invariant since there is a unique smooth complete curve in every birational equivalence class of curves.

There is a slight subtlety in this definition in the case of a non-perfect base field. It it always true that we can find a proper *regular* curve $C$ in each birational equivalence class however when $k$ is non-perfect the curve $C$ may not be smooth. However, under a finite purely separable extension $K/k$, we can ensure that $C_K$ admits a smooth proper model. Then we define $g(C) := g(C_K)$ in the case that $C_K$ is a curve. The only thing that can go wrong is when $C$ is not geometrically irreducible since then $C_K$ will not be integral.

The *arithmetic genus* $g_a(C)$ of a proper curve $C$ over $k$ with $H^0(C, \mathcal{O}_C) = K$ is,

$$g_a(C) := \dim_K H^1(X, \mathcal{O}_C)$$

By Serre duality, if $C$ is smooth then $H^0(C, \Omega_C) = H^1(C, \mathcal{O}_X)^\vee$ meaning that $g_a(C) = g(C)$.

The arithmetic genus depends on the projective compactification and singularities meaning it will not be a birational invariant unlike the (geometric) genus.

**Example 1.2.** Let $k =_p (t)$ for an odd prime $p = 2k + 1$ and consider the curve,

$$C = k[x, y]/(y^2 - x^p - t)$$

which is regular but not smooth at $P = (y, x^p - t)$. Consider the purely inseperable extension $K =_p (t^{1/p})$. Then $C_K = K[x, y]/(y^2 - (x - t^{1/p})^p) \cong K[x, y]/(y^2 - x^p)$.

Taking the normalization of $C_K$ gives ${}^1_K \to C_K$ via $t \mapsto (t^p, t^2)$. This is birational since the following ring map is an isomorphism,

$$(K[x,y]/(y^2 - x^p))_x \to K[t]_t$$

sending $x \mapsto t^2$ and $y \mapsto t^p$ which has an inverse $t \mapsto y/x^k$ since $x \mapsto t^2 \mapsto y^2/x^{2k} = x$ and $y \mapsto t^p \mapsto y^p/x^{kp} = y(y^{2k}/x^{pk}) = y$ and $t \mapsto y/x^k \mapsto t^{p-2k} = t$.

Therefore, $C_K \mathbb{P}^1_K$ so $g(C) = g(C_K) = 0$. However, consider the projective closure,

$$\overline{C} = k[X,Y,Z]/(Y^2 Z^{p-2} - X^p - tZ^p)$$

then $\overline{C} \hookrightarrow \mathbb{P}^2_k$ is a Cartier divisor (since $\mathbb{P}^2_k$ is locally factorial) so we find that $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = k$ and $\dim_k H^1(\overline{C}, \mathcal{O}_{\overline{C}}) = \frac{1}{2}(p-1)(p-2) = k(2k-1)$ since its sheaf of ideals is $\mathcal{O}_{\mathbb{P}^2_k}(-p)$. Then $p = 3$ we expect this to be an elliptic curve and we do see $g_a(\overline{C}) = 1$. However, $g(\overline{C}) = 0$ and correspondingly $C$ is not smooth due to the positive characteristic phenomenon.

**Lemma 1.3.** Suppose that $f : X \to Y$ is a finite birational morphism of $n$-dimensional irreducible Noetherian schemes. Then $H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(X, \mathcal{O}_X)$ is surjective.

*Proof.* The map $f$ must restrict on some open subset $U \subset X$ to an isomorphism $f|_U : U \to V$. Thus, the sheaf map $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ restricts on $V$ to an isomorphism $\mathcal{O}_Y|_V \xrightarrow{\sim} (f_*\mathcal{O}_X)|_V$. We factor this map into two exact sequences,

$$0 \longrightarrow \quad \longrightarrow \mathcal{O}_Y \longrightarrow \quad \longrightarrow 0$$

$$0 \longrightarrow \quad \longrightarrow f_*\mathcal{O}_X \longrightarrow \quad \longrightarrow 0$$

with $= \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $= \mathrm{coker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $= \mathrm{Im}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. Taking cohomology and using that it vanishes in degree above $n$ we get,

$$H^{n-1}(Y, ) \longrightarrow H^n(Y, ) \longrightarrow H^n(Y, \mathcal{O}_Y) \longrightarrow\!\!\!\!\!\to H^n(Y, ) \longrightarrow 0$$

$$H^{n-1}(Y, ) \longrightarrow H^n(Y, ) \longrightarrow H^n(X, \mathcal{O}_X) \longrightarrow\!\!\!\!\!\to H^n(X, ) \longrightarrow 0$$

where we have used that $f : X \to Y$ is affine to conclude that $H^p(Y, f_*\mathcal{F}) = H^p(Y, \mathcal{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Furthermore, $|_V = 0$ so $\mathcal{O}_Y \subset X \setminus V$ but is coherent so the support is closed. Since $V$ is dense open, is supported in positive codimension so $H^n(Y, ) = 0$ (since $H^n(S, )$ vanishes due to dimension on the closed subscheme $S = \mathcal{O}_X$ on which is supported). Thus we have,

$$H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(Y, ) \twoheadrightarrow H^n(Y, ) \twoheadrightarrow H^n(X, \mathcal{O}_X)$$

proving the proposition. $\square$

Let $S$ and $C$ be proper curves over $k$ where $S$ is smooth which are birationally equivalent and $H^0(S, \mathcal{O}_S) \cong H^0(C, \mathcal{O}_C)$. Then the genera satisfy,

1. $g_a(C) \geq g_a(S)$

2. $g(C) = g(S)$

3. $g(C) \leq g_a(C)$ with equality if and only if $C$ is smooth.

*Proof.* Given a birational map $SC$ we can extend it to a birational morphism $S \to C$ since $S$ is regular. The morphism $S \to C$ is automatically finite since it is a non-constant map of proper curves. Then the previous lemma implies that $g_a(S) \leq g_a(C)$. (b). follows from the definition of $g(C)$. The third follows from the fact that $g(S) = g_a(S)$ because of Serre duality,

$$H^1(S, \mathcal{O}_S) \cong H^0(S, \Omega_{S/k})^\vee$$

using that $S$ is smooth. Then we see that $g(C) = g(S) = g_a(S) \leq g_a(C)$ proving the inequality part of (c). Finally, if $C$ is smooth we see by Serre duality that $g(C) = g_a(C)$. Conversely, suppose that $g(C) = g_a(C)$ then $g_a(C) = g(C) = g(S) = g_a(S)$ and consider the map $f : S \to C$ which is finite birational map of integral schemes over $k$. In particular, $f$ is affine so for each $y \in C$ we may choose an affine open $y \in V \subset C$ whose preimage $U = f^{-1}(V)$ is also affine. On sheaves, this gives a map of domains $\mathcal{O}_C(V) \to \mathcal{O}_S(U)$ which localizes to an isomorphism on the fraction fields. However, the localization map of a domain is injective so $\mathcal{O}_C(V) \hookrightarrow \mathcal{O}_S(U)$ is an injection. This shows that $\mathcal{O}_C \to f_* \mathcal{O}_S$ is an injection of sheaves which we extend to an exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_* \mathcal{O}_S \longrightarrow \quad \longrightarrow 0$$

Note that $f : S \to C$ induces an isomorphism $H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$ since it is a map of fields with the same (finite) dimension over $k$. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, ) \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, ) = 0$$

I claim that $H^1(S, ) = 0$. Since $f$ is birational, is supported in codimension one. Thus, the map $H^1(C, \mathcal{O}_C) \twoheadrightarrow H^1(S, \mathcal{O}_S)$ is surjective but $g_a(C) = g_a(S)$ so these vectorspaces have the same dimension so $H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S)$ is an isomorphism. Thus, from the exact sequence we have $H^0(X, ) = 0$. However, $\mathcal{O}_C$ is a closed ( is coherent) dimension zero subset i.e. finitely many discrete closed points. However, a sheaf supported on a discrete set of points is zero iff it has no global sections. Therefore, $= 0$ so $\mathcal{O}_C \xrightarrow{\sim} f_* \mathcal{O}_S$. In particular $\mathcal{O}_C(V) \xrightarrow{\sim} \mathcal{O}_S(U)$ is an isomorphism which implies that the map of affine schemes $f|_U : U \to V$ is an isomorphism. Since the affine opens $V$ cover $C$ we see that $f : S \to C$ is an isomorphism. In particular, $C$ is smooth. $\square$

## 1.2 The Locus on Which Morphisms Agree

**Lemma 1.4.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then for schemes $X$ there is a natural bijection,

$$\text{Hom}\,(R, X) \cong \{x \in X \text{ and local map } \mathcal{O}_{X,x} \to R\}$$

*Proof.* Given $R \to X$ we automatically get $\mathfrak{m} \mapsto x$ and $\mathcal{O}_{X,x} \to R_{\mathfrak{m}} = R$. Now, note that taking any affine open neighborhood $x \in A \subset X$ and then $A \to A_{=}\mathcal{O}_{X,x}$ to give $\mathcal{O}_{X,x} \to A \to X$. Clearly, this map sends $\mathfrak{m}_x \mapsto x$ and at $\mathfrak{m}_x$ has stalk map $\text{id} : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ since it is the localization at of $A \to A$.

Thus we get an inverse as follows. Given a point $x \in X$ and a local map $\phi : \mathcal{O}_{X,x} \to R$ then take,

$$R \to \mathcal{O}_{X,x} \to X$$

This is inverse since $\mathfrak{m} \mapsto \mathfrak{m}_x$ (because $\mathcal{O}_{X,x} \to \mathfrak{m}_x$ is local) and $\mathfrak{m}_x \mapsto x$ and the stalk at $\mathfrak{m}$ gives $\mathcal{O}_{X,x} \xrightarrow{\text{id}} \mathcal{O}_{X,x} \xrightarrow{\phi} R$.

Finally, I claim that any $f : R \to X$ factors through $R \to \mathcal{O}_{X,x} \to X$ and thus is reconstructed from $x \in X$ and $\mathcal{O}_{X,x} \to R$. Choose an affine open neighborhood $x \in A \subset X$ then consider $f^{-1}(A)$ which is open in $R$ and contains the unique closed point $\mathfrak{m} \in R$ so there is some $f \in R$ s.t. $\mathfrak{m} \in D(f) \subset f^{-1}(A)$ so $f \notin \mathfrak{m}$ so $f \in R^\times$ and thus $D(f) = R$. Therefore, we get a map $R \to A$ and thus $\phi : A \to R$ where $\phi^{-1}(\mathfrak{m}) == x$ so $A\backslash$ is mapped inside $R^\times$ so this map factors through $A \to A_{\to}R$ giving the desired factorization $R \to \mathcal{O}_{X,x} \to A \to X$. $\square$

**Definition:** The locus $Z$ on which two maps $f, g : X \to Y$ over $S$ agree is given as the pullback,

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_Y"] \\
X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}$$

with $F = (f, g)$. This is the equalizer of $f, g : X \to Y$. Furthermore $Z \to X$ is an immersion since it is the base change of $\Delta_{Y/S}$ which is an immersion.

**Lemma 1.5.** Topologically, the locus on which $S$-morphisms $f, g : X \to Y$ agree is,

$$Z = \{x \in X \mid f(x) = g(x) \text{ and } f_x = g_x : \kappa(f(x)) \to \kappa(x)\}$$

*Proof.* On some $S$-subscheme $G \subset X$, the maps $f|_G = g|_G$ agree iff there exists $G \to Y$ such that,

$$\begin{tikzcd}
G \arrow[r, dashed] \arrow[d, hook] & Y \arrow[d, "\Delta"] \\
X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}$$

commutes. In particular, for any point $x \in X$ consider $\iota : \kappa(x) \to X$ then $f \circ \iota = g \circ \iota$ iff $f(x) = g(x)$ and $f_x = g_x : \kappa(f(x)) \to \kappa(x)$. Consider a point $z \in Z$ and $\kappa(z) \to Z$, such a point is equivalent to giving a diagram,

$$
\begin{tikzcd}
\kappa(z) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_Y"] \\
& X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}
$$

However, $\iota : Z \to X$ is an immersion so $\iota_x : \kappa(\iota(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism. Therefore, points $\kappa(z) \to Z$, are exactly points of $X$ for which a lift $\kappa(x) \to Y$ exists i.e. points such that $f$ and $g$ agree in the required way. $\square$

**Lemma 1.6.** If $f : X \to Y$ is an immersion then $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective for each $x \in X$ and $f_x : \kappa(f(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism.

*Proof.* For closed immersions, $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective by definition. Thus we get a surjection $f_x : \mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_{f(x)}$. Furthermore, topologically, $f : X \to Y$ is a homomorphism onto its image so for any open $U \subset X$ there exists an open $V \subset Y$ s.t. $U = f^{-1}(V)$ showing that,

$$(f_* \mathcal{O}_X)_{f(x)} = \varinjlim_{f(x) \in V} \mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}$$

Furthermore, for an open immersion, $f^\flat : f^{-1}\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism so $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. Thus the composition, $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective. Furthermore, $f_x$ is local we get $f_x : \kappa(f(x)) \twoheadrightarrow \kappa(x)$ which is a surjection of fields and thus an isomorphism. $\square$

**Lemma 1.7.** If $Y \to S$ is separated then the locus on which $f, g : X \to Y$ over $S$ agree is closed.

*Proof.* Since $X \to S$ is separated, $\Delta_{Y/S} : Y \to Y \times_S Y$ is a closed immersion. So $Z \to X$ is the base change of a closed immersion and thus a closed immersion. $\square$

**Lemma 1.8.** Let $X$ be a reduced and $Y$ be a separated scheme over $S$ and $f, g : X \to Y$ be morphism over $S$. If $f \circ j = g \circ j$ agree on a dense subscheme $j : G \hookrightarrow X$ then $f = g$.

*Proof.* Consider $F = (f, g) : X \to Y \times_S Y$. Since $\Delta : Y \to Y \times_S Y$ is a closed immersion (by separateness). Then $F^{-1}(\Delta)$ is the locus on which $f = g$ which is closed because $\Delta : Y \to Y \times_S Y$ is a closed immersion. Since $f|_G = g|_G$ we get a diagram,

$$\begin{tikzcd}
G \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, hook, bend right] & & \\
& Z \arrow[r, "\tilde{F}"] \arrow[d, hook, "\iota"'] & Y \arrow[d, "\Delta_Y"] \\
& X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}$$

Since $\iota : Z \hookrightarrow X$ is a closed immersion with dense image, $Z \hookrightarrow X$ is surjective. By the following, $\iota : Z \to X$ is an isomorphism. Thus, $F = F \circ \iota \circ \iota^{-1} = \Delta_Y \circ \tilde{F} \circ \iota^{-1}$. By the universal property of maps $X \to Y \times_S Y$ this implies that $f = g = \tilde{F} \circ \iota^{-1}$. $\square$

**Lemma 1.9.** Let $X$ be a scheme and consider an exact sequence of quasi-coherent $\mathcal{O}_X$-modules,

$$0 \longrightarrow \quad \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A} \longrightarrow 0$$

and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebra. Suppose that $\mathcal{F}_x \neq 0$ for each $x \in X$. Then $\hookrightarrow \mathcal{N}$ where $\mathcal{N}$ is the sheaf of nilpotent.

*Proof.* Take an affine open $U = R \subset X$ such that $\mathcal{A}|_U = A$. Then we have an surjection of rings $R \twoheadrightarrow A$ giving $R/I = A$ for $I = \ker(R \to A)$. Now, for each $\in R$ we know $R_{=}\mathcal{O}_{X,} \neq 0$. However, if $\not\supset I$ then $(R/I)_{=}A_{=}0$ so we must have $\supset I$ for all $\in R$ i.e. $I \subset R$. Therefore, $|_U \hookrightarrow \mathcal{N}|_U$ for any affine open $U \subset X$ showing that is comprised of nilpotents. $\square$

**Corollary 1.10.** If $X$ is reduced and $\iota : Z \hookrightarrow X$ is a surjective closed immersion then $\iota : Z \xrightarrow{\sim} X$ is an isomorphism.

*Proof.* Since $\iota : Z \hookrightarrow X$ is a homeomorphism onto its image $X$ it suffices to show that the map of sheaves $\iota^\# : \mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is an isomorphism. Since $\iota : Z \to X$ is a closed immersion $\iota^\# : \mathcal{O}_X \twoheadrightarrow \iota_*\mathcal{O}_Z$ is a surjection and $\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras giving an exact sequence,

$$0 \longrightarrow \quad \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Z \longrightarrow 0$$

Furthermore,

$$\mathcal{O}_X \iota_*\mathcal{O}_Z = \mathrm{Im}(\iota) = X$$

since $(\iota_*\mathcal{O}_Z)_x = \mathcal{O}_{Z,x}$ when $x \in \mathrm{Im}(\iota)$ (and zero elsewhere). by the above, $\hookrightarrow \mathcal{N} = 0$ since $X$ is reduced to $\iota^\# : \mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is an isomorphism. $\square$

**Lemma 1.11.** A rational $S$-map $f : XY$ with $X$ reduced and $Y \to S$ separated is equivalent to a morphism $f : f \to Y$.

*Proof.* For any $(U, f_U)$ and $(V, f_V)$ representing $f$ there must be a dense (in $X$) open $W \subset U \cap V$ on which $f_U|_W = f_V|_W$ and thus $f_U|_{U\cap V} = f_V|_{U\cap V}$ since $f_U, f_V : U \cap V \to Y$ are morphisms from reduced to irreducible schemes. Now $f$ has an open cover $(U_i, f_i)$ for which $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ so these morphisms glue to give $f : f \to Y$ ($\mathrm{Hom}_S(-, Y)$ is a sheaf on the Zariski site). $\square$

## 1.3 Extending Rational Maps

**Lemma 1.12.** Regular local rings of dimension 1 exactly correspond to DVRs.

*Proof.* Any DVR $R$ has a uniformizer $\varpi \in R$ then $\dim R = 1$ and $\mathfrak{m}/\mathfrak{m}^2 = (\varpi)/(\varpi^2) = \varpi\kappa$ which also has $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 1$ so $R$ is regular. Conversely, if $R$ is a regular local ring of dimension $\dim R = 1$ then, by regularity, $R$ is a normal Noetherian domain so by $\dim R = 1$ then $R$ is Dedekind but also local and thus is a DVR. $\square$

**Proposition 1.13.** Let $X$ be a Noetherian $S$-scheme and $Z \subset X$ a closed irreducible codimension 1 generically nonsingular subset (with generic point $\eta \in Z$ such that $\mathcal{O}_{X,\eta}$ is regular). Let $f : XY$ be a rational map with $Y$ proper over $S$. Then $Z \cap f$ is a dense open of $Z$.

*Proof.* Choose some representative $(U, f_U)$ for $f : XY$. Note that $\mathcal{O}_{X,\eta}$ is a regular dimension one (see Lemma 1.16) ring and thus a DVR. Consider the generic point $\xi \in X$ of $X$ then, by localizing, we get an inclusion of the generic point $\mathcal{O}_{X,\xi} \to \mathcal{O}_{X,\eta} \to X$ and $\mathcal{O}_{X,\xi} = K(X) = \mathcal{O}_{X,\eta}$. Furthermore, the inclusion of the generic point gives $K(X) \to U \xrightarrow{f_U} Y$ and thus we get a diagram,

$$\begin{tikzcd} K(X) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \mathcal{O}_{X,\eta} \arrow[r] \arrow[ur, dashed, "\ell"] & k \end{tikzcd}$$

and a lift $\mathcal{O}_{X,\eta} \to Y$ by the valuative criterion for properness applied to $Y \to k$ since $\mathcal{O}_{X,\eta}$ is a DVR. Choose an affine open $R \subset Y$ containing the image of $\mathcal{O}_{X,\eta} \to Y$ (i.e. choose a neighborhood of the image of $\eta$ which automatically contains $f(\xi)$ since the map factors $\mathcal{O}_{X,\eta} \to \mathcal{O}_{Y,f(\eta)} \to R \to Y$) and let $\eta \in V = A \subset X$ be an affine open neighborhood of $\xi$ mapping onto $R$. By Lemma 1.20, since $\mathcal{O}_{X,\eta}$ is a domain, we may shrink $V$ so that $A$ is a domain. Since $X$ is irreducible $U \cap V$ is a dense open. Note that if $\eta \in U$ then $\eta \in f$ and thus $Z \cap f$ is a nonempty open of the irreducible space $Z$ and therefore a dense open so we are done. Otherwise, let $\in A$ correspond to $\eta \in Z$ then $A_{=}\mathcal{O}_{X,\eta}$ is a DVR. Take some principal affine open $D(f) \subset U \cap V$ for $f \in A$ so $f \in$ since $\notin D(f) \subset U \cap V$. Since $A$ is a DVR we may choose a uniformizer $\varpi \in$ so the map $A \to$ via $1 \mapsto \varpi$ is as isomorphism when localized at . Since $A$ is Noetherian both are f.g. $A$-modules so there must be some $s \in A \setminus$ such that $A_s \to_s$ is an isomorphism. Replacing $A$ by $A_s$ we may assume $= (\varpi) \subset A$ is principal. Since $f \in$ we can write $f = t\varpi^k$ for some $a \in A \setminus$ (see Lemma 1.14). Then consider $\tilde{V} = A_t$. Since $t \notin$ then $\eta \in \tilde{V}$ and since $f = t\varpi^k$ we have $D(f) \subset D(t) = \tilde{V}$. Now we get the following diagram,

$$
\begin{tikzcd}
 & & R \\
A \arrow[r] \arrow[urr, bend left, "\ell"] & A_t \arrow[ur, dashed, "f_V"] & \\
A \arrow[u] \arrow[r] & A_f \arrow[u] \arrow[uur, bend right, "f_U"'] &
\end{tikzcd}
$$

I claim the square is a pushout in the category of affine schemes because maps $R \to A$ and $R \to A_f$ which agree under the inclusion to $A$ gives a map $R \to A_\cap A_f \subset A$. However, consider,

$$x \in A_\cap A_t \implies x = \frac{u\varpi^r}{s} = \frac{a}{f^n}$$

for $u, s, t \in A\backslash$ and $a \in A$. Thus we get,

$$ut^n \varpi^{r+nk} = sa$$

so $a \in^{r+nk} \backslash^{r+nk+1}$ ($s \notin$ which is prime) and thus $a = u'\varpi^{r+nk}$ for $u' \in A\backslash$. Therefore,

$$x = \frac{u'\varpi^{r+nk}}{t^n \varpi^{nk}} = \frac{u'\varpi^r}{t^n} \in A_t$$

Thus, $A_\cap A_f \subset A_f$ so we get a map $R \to A_t$. Therefore we get a map $f_{\tilde{V}} : \tilde{V} \to Y$ such that $(f|_{\tilde{V}})|_{D(f)} = (f_U)|_{D(f)}$ which implies that $\eta \in \tilde{V} \subset f$ so $Z \cap f$ is a dense open of $Z$. $\square$

Let $C \to S$ be a proper regular Noetherian scheme with $\dim C = 1$ and $f : CY$ a rational $S$-map with $Y \to S$ proper. Then $f$ extends uniquely to a morphism $f : C \to Y$.

*Proof.* For any point $x \notin f$ let $Z = \overline{\{x\}} \subset D$ for $D = C \setminus f$. Since $f$ is a dense open, by lemma 1.15, we have $Z, C \geq D, C \geq 1$ but $\dim C = 1$ so $Z, C = 1$. Furthermore, since $C$ is regular $\mathcal{O}_{C,x}$ is regular and thus, by the previous proposition, $Z \cap f$ is a dense open and in particular $x \in f$ meaning that $f = C$ so we get a morphism $C \to Y$. This is unique because $C$ is reduced (it is regular) and $Y$ is separated (it is proper over $S$) so morphisms $C \to Y$ are uniquely determined on a dense open which any representative for $f : CY$ is defined on. $\square$

Rational maps between normal proper curves are morphisms.

Birational maps between normal proper curves are isomorphisms.

*Proof.* Let $f : C_1 C_2$ and $g : C_2 C_1$ be birational inverses of smooth proper curves. Then we know that these extend to morphisms $f : C_1 \to C_2$ and $g : C_2 \to C_1$. Furthermore, the maps $g \circ f : C_1 \to C_1$ must extend the identity on some dense open. However, since curves are separated and reduced there is a unique extension of this map so $g \circ f = \mathrm{id}_{C_1}$ and likewise $f \circ g = \mathrm{id}_{C_2}$. $\square$

If $k$ is perfect then there exists a unique normal curve in each birational equivalence class of curves.

*Proof.* It suffices to show existence. Given a curve $X$, we consider the projective closure $X \hookrightarrow \overline{X}$ which is birational and $\overline{X} \to k$ is proper. Then take the normalization $\overline{X}^{\nu} \to \overline{X}$ which remains proper over $k$ and is birational. Then $\overline{X}^{\nu}$ is regular and thus smooth over $k$ since $k$ is perfect and $\overline{X}^{\nu} \to X$ is birational. $\square$

## 1.4 Lemmas

**Lemma 1.14.** Let $A$ be a Noetherian domain and $= (\varpi)$ a principal prime. Then any $f \in$ can be written as $f = t\varpi^k$ for $f \in A\backslash$.

*Proof.* From Krull intersection,

$$\bigcap_{n \geq 0}^{\infty} {}^n = (0)$$

so there is some $n$ such that $f \in^n \backslash^{n+1}$. Thus $f = t\varpi^n$ for some $f \in A$ but if $t \in$ then $f \in^{n+1}$ so the result follows. $\square$

**Lemma 1.15.** Consider a closed subset $Y \subset X$ and an open $U \subset X$ with $U \cap Z \neq \varnothing$. Then $Y, X = Y \cap U, U$.

*Proof.* Consider a chain of irreducible $Z_i \supsetneq Z_{i+1}$ with $Z_0 \subset Y$. I claim that $Z_i \mapsto Z_i \cap U$ and $Z_i \mapsto \overline{Z_i}$ are inverse functions giving a bijection between closed irreducible chains in $X$ with final terms contained in $Y$ and closed irreducible chains in $U$ with final term contained in $Y \cap U$. Note, if $Z_i \subset Y \cap U$ then $\overline{Z_i} \subset Y$ since $Y$ is closed in $X$.

First, $\overline{Z_i \cap U} \subset Z_i$ and is closed in $X$. Then $\overline{Z_i \cap U} \cup U^C \supset Z_i$ so because $Z_i$ is irreducible $\overline{Z_i \cap U} = Z_i$ since by assumption $Z_i \not\subset U^C$. Conversely, if $Z_i \subset U$ is a closed irreducible subset then $\overline{Z_i}$ is closed and irreducible in $X$ and $Z_i \subset \overline{Z_i} \cap U$ but $Z_i = C \cap U$ for closed $C \subset X$ so $Z_i \subset C$ and thus $\overline{Z_i} \subset C$ so $\overline{Z_i} \cap U \subset C \cap U = Z_i$ meaning $Z_i = \overline{Z_i} \cap U$. Thus we have shown these operations are inverse to each other.

Finally, if $Z_i \cap U - Z_{i+1} \cap U$ then $\overline{Z_i \cap U} = \overline{Z_i \cap U}$ so $Z_i = Z_{i+1}$ so the chain does not degenerate. Likewise, if $\overline{Z_i} = \overline{Z_{i+1}}$ then $\overline{Z_i} \cap U = \overline{Z_{i+1}} \cap U$ so $Z_i = Z_{i+1}$. Therefore, we get a length-preserving bijection between the chains defining $Y, X$ and $Y \cap U, U$. $\square$

**Lemma 1.16.** Let $Z \subset X$ be a closed irreducible subset with generic point $\eta \in Z$. Then $Z, X = \dim \mathcal{O}_{X,\eta}$.

*Proof.* Take affine open neighborhood $\eta \in U = A \subset X$. Then for $\in A$ corresponding to $\eta$ we get $A_= \mathcal{O}_{X,\eta}$. However, $Z, X = Z \cap U, U$ and $Z \cap U = \overline{\{\}} = V()$. Therefore,

$$Z, X = Z \cap U, U = \; = \dim A = \dim \mathcal{O}_{X,\eta}$$

$\square$

**Lemma 1.17.** Let $X$ be a Noetherian scheme then the nonreduced locus,

$$Z = \{x \in X \mid \mathcal{O}_{X,x} \neq 0\}$$

is closed.

*Proof.* The subsheaf $\mathcal{N} \subset \mathcal{O}_X$ is coherent since $X$ is Noetherian. Thus $Z = \mathcal{O}_X\mathcal{N}$ is closed and $\mathcal{N}_x = \mathcal{O}_X x$. Locally, on $U = A$ we have $\mathcal{N}|_U = A$ and $A$ is a f.g. $A$-module since $A$ is Noetherian so,

$$\mathcal{O}_X\mathcal{N} \cap U = AA = V(AA$$

is closed in $A$. □

**Lemma 1.18.** Let $X$ be a Noetherian scheme then $X$ has finitely many irreducible components.

*Proof.* First let $X = A$ for a Noetherian ring $A$. Then the irreducible components of $A$ correspond to minimal primes $\in A$. Then $\dim A = 0$ and $A$ is Noetherian so $A$ is Artinian. $A$ must have some associated prime so $AA = \{A_{\}}$. By [**?**, Tag 05BZ], then $AA \cap = = \{\}$ so every minimal prime is an associated prime. However, for $A$ Noetherian then $A$ admits a finite composition series so there are finitely many associated primes.

Now let $X$ be a Noetherian scheme. For any affine open $U \subset X$ we have shown that $U$ has finitely many irreducible components. However, since $X$ is quasi-compact there is a finite cover of affine opens and thus $X$ must have finitely many irreducible components. □

**Lemma 1.19.** Let $X$ be a Noetherian scheme and $Y$ is the complement of some dense open $U$. Then $Y, X \geq 1$.

*Proof.* It suffices to show that $Y$ does not contain any irreducible component since then any irreducible contained in $Y$ cannot be maximal. Since $X$ is Noetherian, it has finitely many irreducible components $Z_i$. Then if $Z_j \subset Y$ for some $i$ we would have $Z_i \cap U = \varnothing$ but then,

$$U = \bigcup_{i \neq j} Z_i$$

which is closed so $\overline{U} \subsetneq X$ contradicting our assumption that $U$ is dense. □

**Lemma 1.20.** Let $X$ be a Noetherian scheme and $x \in X$ such that $\mathcal{O}_{X,x}$ is a domain. Then there is an affine open neighborhood $x \in U \subset X$ with $U = A$ and $A$ is a domain.

*Proof.* Take any affine open neighborhood $x \in U \subset X$ with $U = A$ and $\in A$ corresponding to $x$. Then $A_{=}\mathcal{O}_{X,x}$ is a domain. Since $X$ is Noetherian then $A$ is Noetherian so it has finitely many minimal primes $_i$ (corresponding to the generic points of irreducible components of $U$) with $_0 \subset$. Since $A$ is a domain, it has a unique minimal prime and thus $_0$ is the only minimal prime contained in (geometrically $A$ being a domain corresponds to the fact that  is the generic point of a generically reduced irreducible subset which lies in only one irreducible component)

Now for any $i \neq 0$ take $f_i \in \setminus_0$. This is always possible else $\subset_0$ contradicting the minimality of $_0$. If $f \notin$ then $\not\supset_i$ for any $i \neq 0$ so $\supset_0$ since it must lie above some minimal prime. Thus $A_f =_0 A_f$ is prime and $f \notin$ since else $\supset_1 \cap \cdots \cap_n$ which is impossible since $\not\supset_i$ for any $i$. Now we know that $A = 0$ and $A_f$ is Noetherian so $A$ is finitely generated. Thus, there is some $g \notin$ such that $A_{fg} = (A_f)_g = 0$. Thus $A_{fg}$ is a domain since $A_{fg} = (0)$ and is prime and $\in A_{fg}$ because $fg \notin$. Therefore, $x \in A_{fg} \subset U$ is an affine open satisfying the requirements. $\square$

This does not imply that $X$ is integral if $\mathcal{O}_{X,x}$ is a domain for each $x \in X$ (which is false, consider $k \times k$) because it only shows there is an integral cover of $X$ not that $\mathcal{O}_X(U)$ is a domain for each $U$.

**Example 1.21.** Let $X = k[x, y]/(xy, y^2)$. Then for the bad point $= (x, y)$ we have $\mathcal{O}_{X,} = (y)$. Away from the bad point, say $= (x - 1, y)$ we have, $\mathcal{O}_{X,} = k[x]_{(x-1)}$ so $\mathcal{O}_{X,} = (0)$. Furthermore, at the generic point $= (y)$, we have, $\mathcal{O}_{X,} = k(x)$ so $\mathcal{O}_{X,} = (0)$.

**Example 1.22.** Consider $X = k[x, y, z]/(yz)$ which is the union of the $x$-$y$ and $x$-$z$ planes. Consider the generic point of the $z$-axis $= (x, y)$ then $\mathcal{O}_{X,} = k[x, z]_{(x)}$ is a domain since the $z$-axis only lies in one irreducible component. However, at the generic point of the $x$-axis, $= (y, z)$ we get $\mathcal{O}_{X,} = (k[x, y, z]/(yz))_{(y,z)}$ has zero divisors $yz = 0$ so is not a domain since the $x$-axis lives in two irreducible components.

## 1.5 Reflexive Sheaves (WIP)

Recall the dual of a $\mathcal{O}_X$ module $\mathcal{F}$ is the sheaf $\mathcal{F}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. We say that a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

**Lemma 1.23.** Let $X$ be an integral locally Noetherian scheme and $\mathcal{F}, \mathcal{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is reflexive then $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is reflexive.

*Proof.* See [**?**, Tag 0AY4]. $\square$

In particular, since $\mathcal{O}_X$ is clearly reflexive, this lemma shows that for any coherent $\mathcal{O}_X$-module then $\mathcal{F}^\vee$ is a reflexive coherent sheaf. We say the map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ gives the reflexive hull $\mathcal{F}^{\vee\vee}$ of $\mathcal{F}$.

Let $\mathcal{R}$ be the full subcategory $\mathcal{O}_X$ of coherent reflexive $\mathcal{O}_X$-modules. $\mathcal{R}$ is an additive category and in fact has all kernels and cokernels defined by taking reflexive hulls of the sheaf kernel and cokernel. Furthermore, $\mathcal{R}$ inherits a monoidal structure from the tensor product defined using the reflexive hull as follows,

$$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\vee\vee}$$

Finally, we define $\mathrm{RPic}\,(X)$ to be group of constant rank one reflexives induced by the monoidal structure on $\mathcal{R}$. Explicitly, $\mathrm{RPic}\,(X)$ is the group of isomorphism classes of constant rank one reflexive coherent $\mathcal{O}_X$-modules with multiplication $(\mathcal{F}, \mathcal{G}) \mapsto (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\vee\vee}$ and inverse $\mathcal{F} \mapsto \mathcal{F}^\vee$.
The importance of reflexive sheaves derives from their correspondence to Weil divisors. Here we let $X$ be a normal integral separated Noetherian scheme.

If $D$ is a Weil divisor then $\mathcal{O}_X(D)$ is reflexive of constant rank one.

*Proof.* (CITE OR DO). □

**Theorem 1.24.** Let $X$ be a normal integral separated Noetherian scheme. There is an isomorphism of groups $\mathrm{C}\ell X \xrightarrow{\sim} \mathrm{RPic}\,(X)$ defined by $D \mapsto \mathcal{O}_X(D)$.

*Proof.* (DO OR CITE) □

We summarize the important results as follows.

**Theorem 1.25.** Let $X$ be a Noetherian normal integral scheme. Then for any Weil divisors $D, E$,

1. $\mathcal{O}_X(D+E) = (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee}$
2. $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^\vee$
3. $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(E)) = \mathcal{O}_X(E - D)$
4. if $E$ is Cartier then $\mathcal{O}_X(D+E) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$

*Proof.* (DO OR CITE) □

Finally, we have a result which controls when the dualizing sheaf can be expressed in terms of a divisor. Let $X$ be a projective variety over $k$. Then,

1. if $X$ is normal then its dualizing sheaf $\omega_X$ is reflexive of rank 1 and thus $X$ admits a canonical divisor $K_X$ s.t. $\omega_X = \mathcal{O}_X(K_X)$
2. if $X$ is Gorenstein then $\omega_X$ is an invertible module so $K_X$ is Cartier.

*Proof.* (FIND CITATION OR DO). □