1 Is there a Nice moduli space of Fano variieties

Remark. We work over $\mathbb{C}$. By "nice" we mean separated, proper, projective.

Definition 1.0.1. A variety is Fano if $-K_X$ is $\mathbb{Q}$-Cartier and ample.

Remark. The problem is that there are families $\mathfrak{X} \to C$ of Fanos such that $X_t \cong X$ for all $t \neq t_0$ with fixed X and $\mathfrak{X}_0 \cong X$. This seems to show that the moduli space is not separated.

For example $\mathbb{P}^2$ degenerates to a cone over a conic $\mathbb{P}(1,1,4)$ it can also degenerate to $\mathbb{P}(a^2, b^2, c^2)$ for $a^2 + b^2 + c^2 = 3abc$ (Manetti).

Remark. The same issue persists if we restrict to smooth Fanos.

Remark. The moduli stack of Fanos is "similar" to the moduli stack of vector bundles: very non-separated and has lots of specialization. Therefore we need a stability condition.

1.1 Kähler-Einstein metrics

Metrics with $\mathrm{Ric}\,(g) = g$.

(a) {Fano mfd with KE metric} is a metric space with Gromov-Hausdorff distance between compact metric spaces. Therefore, Moduli of KE Fan var. should be separated.

(b) Donaldson-Sun: there exists a sequence X_k of KE Fano manifolds then it converges X_∞ in GH distance so the moduli space of KE Fano var. should be proper.

1.1.1 The Yau-Tian-Donaldson Conjecture

Let X be a Fano variety. Then there exists a KE metric iff X is K-polystable.

This motivates the following conjecture.

1.1.2 K-moduli Conjecture

There exsits a proper moduli space of K-polystable Fano varieties and all connected components are projective.

Theorem 1.1.1 (Lots of People)**.** (a) First fix $n \in \mathbb{N}^*$ and $v \in \mathbb{Q}_{>0}$. The moduli functor,

$$\mathcal{M}^{\mathrm{Kss}}_{n,v}(S) = \{\text{flat family } \mathfrak{X} \to S \text{ of K-ss Fano var. with } \dim = n \text{ and vol} = (-K)^n = v \text{ satisfying Kolla}$$

is represented by an Artin stack of finite type.

(b) $\mathcal{M}^{\mathrm{Kss}}$ admits a separated good moduli space M^{Kps} which is an algebraic space parametrizing K-polystable Fano varieties meaning there is a map $\mathcal{M}^{\mathrm{Kss}} \to M^{\mathrm{Kps}}$ which étale locally looks like $[\mathrm{Spec}\,(A)\,/G] \to \mathrm{Spec}\left(A^G\right)$.

(c) Every proper subspace of M^{Kps} that parametrizes KE Fanos is projective.

Theorem 1.1.2 (Liu, Xu, -)**.** (a) connected components of M^{Kps} are proper

(b) YTD holds for any Fany variety

1.1.3 Langton's Algorithm

For $\mathcal{M}^{vb}$ moduli space of vector bundles. For $0 \in C$ consider family $\mathcal{E}_t$ of vector bundles. Suppose that $\mathcal{E}_t$ is s.s. for all $t \neq 0$.

However, if $\mathcal{E}_0$ is not s..s then consider the maximal destab. subbundle,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{E}_0 \longrightarrow \mathscr{G}$$

Then let $\mathcal{E}' = \ker(\mathcal{E} \to \mathscr{G})$ then we get,

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathcal{E}'_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

stability improves. After finitely many steps get a semi-stable vector bundle.

1.1.4 K-Stability

Definition 1.1.3. We say that a Fano variety X is K-*semistable* "average divisor in $|-K_X|_{\mathbb{Q}}$ is log canonical". Formally, this means the following.

(a) if E is a divisor on some Y with $\pi : Y \to X$ birational then define the log discrepancy,

$$A_X(E) = 1 + \mathrm{ord}_E(K_{Y/X})$$

(b) A pair (X, D) where $D \subset X$ is a divisor is called log canonical if $A_X(E) \geq \mathrm{ord}_E(\pi^* D)$ for all $E \subset Y$ over X.

(c) the expected vanishing order (order of vanishing of an average divisor) $S_X(E)$ is defined as follows. Let m-basis type divisors,

$$D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{s_i = 0\} \sim_{\mathbb{Q}} -K_X$$

where $s_1, \dots, s_{N_m}$ is a basis of $H^0(-mK_X)$ then let,

$$S_m(E) = \sup\{\mathrm{ord}_E(D) \mid D \text{ m-basis type}\}$$

Then we define,

$$S_X(E) = \lim_{m \to \infty} S_m(E)$$

(d) we say that X is K-*semi-stable* if $A_X(E) \geq S_X(E)$ for all $E \subset Y$ over X

if E is a divisor on some Y with $\pi : Y \to X$ birational then define the log discrepancy,

$$A_X(E) = 1 + \mathrm{ord}_E(K_{Y/X})$$

Then (X, D) where $D \subset X$ is a divisor is called log canonical if $A_X(E) \geq \mathrm{ord}_E(\pi^* D)$ for all $E \subset Y$ over X. Furthermore, the expected vanishing order (order of vanishing of an average divisor) $S_X(E)$ is

Finally, we defn