1 Tensor Densities

Definition 1.0.1. For any vector bundle $\pi : E \to M$ there is an associated frame bundle $F(E)$. For $s \in \mathbb{R}$, consider the homomorphism $\rho_s : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^\times$ via $A \mapsto |\det A|^{-s}$ which gives a 1-dimensional representation. Then the *bundle of E densities* is the line bundle,

$$D^s(E) = F(E) \times_{\rho_s} \mathbb{R}$$

Because ρ has image inside $\mathbb{R}^+$ and therefore ρ gives an action on $\mathbb{R}^+$ by multiplication. Therefore, there is a canonical principal $\mathbb{R}^+$-bundle inside $D^s(E)$,

$$D^s_+(E) = F(E) \times_{\rho_s} \mathbb{R}^+ \hookrightarrow F(E) \times_{\rho_s} \mathbb{R} = D^s(E)$$

corresponding to the positive densities. This is a restriction of the structue group of $D^s(E)$ from $\mathbb{R}^\times$ to $\mathbb{R}^+$ which corresponds to putting an orientation on $D^s(E)$ and thus a notion of positivity.

Proposition 1.0.2. The line bundle $D^s(E)$ is always trivial.

Proof. The subbundle $D^s_+(E) \hookrightarrow D^s(E)$ gives an orientation of $D^s(E)$ and thus $D^s(E)$ is trivial. Furthermore, all principal $\mathbb{R}^+$-bundles are trivial and thus $D^s_+(E)$ has a section which gives a nonvanishing section of $D^s(E)$ that trivializes it. □

Remark. Notice: there does not exist a canonical trivialization of $D^s(E)$. Here we recall some cohomological arguments for the structure of the torsors in question. There is a morphism of exact sequences of abelian sheaves on M where the second row is $\exp : \mathcal{O}_M \to \mathcal{O}_M^\times$ replaced by its image,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_M \arrow[r, "\exp"] \arrow[d, "\exp"'] & \mathcal{O}_M^\times \arrow[r] \arrow[d, equal] & \underline{\{\pm 1\}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_M^+ \arrow[r] & \mathcal{O}_M \arrow[r] & \underline{\{\pm 1\}} \arrow[r] & 0
\end{tikzcd}$$

Then we get a morphism of long exact sequence of cohomology,

$$\begin{tikzcd}
H^1(M, \mathcal{O}_M) \arrow[r] \arrow[d, "\exp"] & H^1(M, \mathcal{O}_M^\times) \arrow[r] \arrow[d, equal] & H^1(M, \{\pm 1\}) \arrow[r] \arrow[d, equal] & H^2(M, \mathcal{O}_M) \arrow[d, "\exp"] \\
H^1(M, \mathcal{O}_M^+) \arrow[r] & H^1(M, \mathcal{O}_M^\times) \arrow[r] & H^1(M, \{\pm 1\}) \arrow[r] & H^2(M, \mathcal{O}_M)
\end{tikzcd}$$

but $\mathcal{O}_M$ is soft so $H^i(M, \mathcal{O}_M) = 0$ for all $i > 0$ and thus $w_1 : H^1(M, \mathcal{O}_M^\times) \to H^1(M, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism classifying line bundles by their first Steifel-Whitney class. Furthemore, since exp is an isomorphism we see that $H^i(M, \mathcal{O}_M^+) = 0$ so in particular, every prinicpal $\mathbb{R}^+$-bundle is trivial. A more prosaic reason is that principal $\mathbb{R}^+$-bundles are reduction of structure groups of a line bundle $\mathcal{L}$ thus giving the strucutre of an orientation on $\mathcal{L}$ but an oriented line bundle is trivial.

Remark. Warning! In topology, the singular cohomology group $H^1(M, G)$ only classifies G-bundles for a discrete groups G. To classify G-bundles for a topoolgical group we need to use the sheaf $\mathscr{F}_G$ of continuous maps to G. Then sheaf cohomology $H^1(M, \mathscr{F}_G)$ classifies G-torsors on M. Warning $\mathscr{F}_G$ is NOT the constant sheaf $\underline{G}$ unless G is discrete and thus,

$$H^1_{\mathrm{sing}}(M, G) = H^1(M, \underline{G}) \neq H^1(M, \mathscr{F}_G) = \{\text{principal } G\text{-bundles on } M\}$$

For example, principal $\mathbb{R}$-bundles are all trivial but $H^1(M, \mathbb{R})$ need not be zero.

2 Conformal Geometry

Definition 2.0.1. Let M be a smooth manifold. Two Riemannian metrics g_1, g_2 are *conformally equivalent* if there exists a positive smooth function $\lambda : M \to \mathbb{R}$ such that $g_1 = \lambda^2 g_2$.

Definition 2.0.2. A smooth map $f : M \to N$ of Riemannian manifolds is *conformal* if g_M and $f^* g_N$ are conformally equivalent.

Definition 2.0.3. A *conformal manifold* is a pair $(M, [g])$ where M is a smooth manifold and $[g]$ is a conformal class of Riemannian metrics on M. A conformal map $f : (M, [g]) \to (N, [h])$ is a smooth map $f : M \to N$ such that $f^*[h] = [g]$.

Definition 2.0.4. We say that an almost complex structure I is *compatible* with a metric g if

$$g(I(v), I(u)) = g(v, u)$$

that is $I \in \mathrm{O}(TM)$

Proposition 2.0.5. Let (X, g, I) be a Riemannian manifold with a compatible almost complex structue. Then $\omega(-,-) = g(I(-),-)$ is a 2-form called the fundamental form.

Proof. We need to show that ω is antisymmetric. Note,

$$\omega(v, u) = g(I(v), u) = g(I^2(v), I(u)) = -g(v, I(u)) = -g(I(u), v) = -\omega(u, v)$$

$\square$

Proposition 2.0.6. If I is compatible with g then

Proposition 2.0.7. Let (X, I) be a (paracompact) smooth manifold with an almost complex structure. Then there exists a Riemannian metric g on X compatible with I.

Proof. First we show that there exsits a metric g' on X. Choose charts $\{(U_i, \varphi_i)\}$ which we refine such that it is locally finite and choose a subordinate partition of unity χ_i. The standard metric gives a metric g_i on U_i. Now consider,

$$g = \sum \chi_i g_i$$

then g is symmetric and positive definite since $\chi_i \geq 0$ with at least one positive at each point and $g_i(v, v) > 0$ for $v \neq 0$ so $g(v, v) > 0$.

Now consider,

$$g(v, w) = g'(v, w) + g'(I(v), I(w))$$

It is clear that g is symmetric and positive definite because $g(v, v) = g'(v, v) + g'(I(v), I(v))$ is positive unless $v = 0$ so g is a metric. Furthermore,

$$g(I(v), I(u)) = g'(I(v), I(u)) + g'(I^2(v), I^2(u)) = g'(v, u) + g'(I(v), I(u)) = g(v, u)$$

$\square$

Definition 2.0.8. A *pseudo-holomorphic* map $f : (X, I) \to (X', I')$ is a smooth map $f : X \to X'$ such that $\mathrm{d}f \circ I = I' \circ \mathrm{d}f$.

Remark. When (X, I) and (X', I') are integrable almost complex structues (i.e. are induced by complex structues on X and X') then pseudo-holomorphic maps are exactly holomorphic maps $f : X \to X'$.

2.1 The Two Dimensional Case

Lemma 2.1.1. Let V be a oriented 2-dimensional $\mathbb{R}$-vectorspace. Let $J : V \to V$ be an endomorphism such that $v, J(v)$ is a positively ordered basis for each $v \neq 0$. Then $J \in \mathrm{GL}^+(V)$.

Proof. The orientation induces a notion of positivity on $\bigwedge^2 V$. The form $q(v) = v \wedge J(v)$ is positive definite. Choosing a basis we write J in a matrix form,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $q(x) = x^\top Bx$ where,

$$B = \tfrac{1}{2}(SA - A^\top S)$$

where,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then,

$$B = \begin{pmatrix} -c & \frac{a-d}{2} \\ \frac{a-d}{2} & b \end{pmatrix}$$

Since q is positive definite,

$$\det B = -cb - \left(\frac{a-d}{2}\right)^2 > 0$$

Therefore,

$$ad - bc > ad + \left(\frac{a-d}{2}\right)^2 = \left(\frac{a+d}{2}\right)^2 > 0$$

and thus $J \in \mathrm{GL}^+(V)$. $\square$

Lemma 2.1.2. Let V be a 2-dimensional $\mathbb{R}$-vectorspace with an inner product $\langle -, - \rangle$. Suppose that $J : V \to V$ is an endomorphism such that $\langle v, J(v) \rangle = 0$ for all $v \in V$. Then $J^2 = -\lambda^2 \mathrm{id}$.

Proof. Choose an orthonormal basis $\{e_i\}$ then we write I in a matrix form,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $\langle v, J(v) \rangle = x^\top Ax$ where $v = x^i e_i$. Thus if the form is zero we must have $A^\top = -A$ and thus,

$$A = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

meaning that $A^2 = -\lambda^2 I$ and thus $J^2 = -\lambda^2 \mathrm{id}$. $\square$

Lemma 2.1.3. Let V be a 2-dimension $\mathbb{R}$-vectorspace and $J : V \to V$ an endomorphism such that $J^2 = -\mathrm{id}$. Then $J \in \mathrm{SL}(V)$.

Proof. We can choose a basis $v, J(v)$ for any $v \neq 0$ since if $J(v) = \lambda v$ then $\lambda^2 = -1$ which is not possible over $\mathbb{R}$. Thus $J(v \wedge J(v)) = J(v) \wedge J^2(v) = -J(v) \wedge v = v \wedge J(v)$ and thus J preserves $\det V$ so $\det J = 1$ and thus $J \in \mathrm{SL}(V)$. $\square$

Proposition 2.1.4. Let X be an oriented 2-manifold. Then the following data are equivalent,

(a) an almost complex structure (X, I) compatible with the orientation

(b) a conformal structure $(X, [g])$

The graph of this correspondence is the set of compatible pairs $(I, [g])$.

Proof. Given a conformal class $[g]$ choose a representative g. For any vector field σ consider the line bundle $L = \ker g(\sigma, -) \subset TX$ giving an exact sequence,

$$0 \longrightarrow L \longrightarrow TX \xrightarrow{\iota_\sigma g} \mathbb{R} \longrightarrow 0$$

Therefore, we find,

$$\det TX \cong L$$

An orientation of X induces an orientation on $\det TX$ and thus on L so L has nonvanishing global sections. Then the fixed norm bundle $U_\sigma(L) = \{(x, v) \in L \mid g(v, v) = g(\sigma, \sigma)\}$ has a unique positive section $\sigma^\perp \in \Gamma(X, U(L)) \subset \Gamma(X, TX)$ under the induced orientation. Then $\sigma \mapsto \sigma^\perp$ is $\mathcal{C}^\infty$-linear giving an endomorphism $I : TX \to TX$. Furthermore, $g(\sigma, I(\sigma)) = 0$ so, applying the previous lemmas, $I^2 = -\lambda^2 \mathrm{id}$ but $g(I(\sigma), I(\sigma)) = g(\sigma, \sigma)$ and thus $\lambda^2 = 1$ so $I^2 = -\mathrm{id}$. Finally, since $v \wedge I(v)$ is positive, we see I induces a compatible orientation on X. Furthermore, clearly I is the unique almost complex structure compatible with g and orientation since $U_\sigma(L)$ has only two sections the other of which is oppositely oriented.

Conversely, given a complex structure (X, I) compatible with the orientation consider the set $c(I)$ of metrics on X compatible with I. Suppose g and g' are two such metrics. At a point $x \in X$ choose some $v \neq 0$. We know $v, I(v)$ forms a basis $T_x X$ so g_x and g'_x are determined by their values on $v, I(v)$. Choose $\lambda_x > 0$ such that $g'_x(v, v) = \lambda_x^2 g_x(v, v)$. Furthermore,

$$g'_x(I(v), I(v)) = g'_x(v, v) = \lambda_x^2 g(v, v) = \lambda_x^2 g_x(I(v), I(v))$$

and $g'_x(v, I(v)) = 0$ and $g_x(v, I(v))$ so $g'_x = \lambda_x^2 g_x$. Then because g and g' are smooth and nonzero tensors we see that λ is a smooth function so g and g' are conformally equivalent. ☐

Proposition 2.1.5. Let $(X, I, [g])$ and $(X', I', [g'])$ be 2-manifolds with compatible conformal and almost complex structures. Let $f : X \to X'$ be a smooth map. Then the following are equivalent,

(a) f is conformal and orientation preserving

(b) f is a pseudo-holomorphic local diffeomorphism.

Proof. If $f : X \to X'$ is conformal then choosing a representative, g' for the conformal class on X' we see that $f^* g'$ is a metric i.e. positive definite. In particular, $(f^* g')(v, v) = g'(\mathrm{d}f(v), \mathrm{d}f(v))$ but if $\mathrm{d}f(v) = 0$ then $v = 0$ since $(f^* g')(v, v) = 0$. Thus f is a local diffeomorphism and since $f^* g' \in [g]$, we see that I is compatible with with $f^* g'$. Furthermore, $\tilde{I} = \mathrm{d}f^{-1} \circ I' \circ \mathrm{d}f$ is an almost complex structure on X compatible with $f^* g'$ because,

$$\begin{aligned}(f^* g')(\tilde{I}(v), \tilde{I}(u)) &= g'(\mathrm{d}f(I(v)), \mathrm{d}f(I(u))) = g'(I'(\mathrm{d}f(v)), I'(\mathrm{d}f(u))) \\ &= g'(\mathrm{d}f(v), \mathrm{d}f(u)) = (f^* g')(v, u)\end{aligned}$$

Furthermore, if f is orientation preserving then $\tilde{I}$ is compatible with the orientation. To see this, note that $\mathrm{d}f(v) \wedge I' \circ \mathrm{d}f(v)$ is positive (because I' is compatible with the orientation on X) so applying $\mathrm{d}f^{-1}$ we see that $v \wedge \tilde{I}(v)$ is positive meaning that $\tilde{I}$ is compatible with the orientation on X. Thus $\tilde{I} = I$ because there is a unique almost complex structure compatible with the orientation and conformal structure so f is pseudo-holomorphic.

Conversely, suppose that f is pseudo-holomorphic and a local diffeomorphism. Choose a representative g' for the conformal class. Then consider,

$$(f^*g')(I(v), I(u)) = g'(\mathrm{d}f \circ I(v), \mathrm{d}f \circ I(u)) = g'(I' \circ \mathrm{d}f(v), I' \circ \mathrm{d}f(u)) = g'(\mathrm{d}f(v), \mathrm{d}f(u)) = (f^*g')(v, u)$$

because I' is compatible with g'. Thus, f^*g' is compatible with I. Because f is a local diffeomorphism f^*g' is a metric on X and thus $f^*g' \in [g]$ because f^*g' is compatible with I and thus defines the same conformal class. Therefore, $f : (X, [g]) \to (X', [g'])$ is conformal. Furthermore, $v \wedge I(v)$ positive but $\mathrm{d}f(I(v)) = I'(\mathrm{d}f(v))$ so $\mathrm{d}f(v \wedge I(v)) = \mathrm{d}f(v) \wedge I'(\mathrm{d}f(v))$ which is positive because I' is compatible with the orientation on X'. $\square$