1 Topological Groups

Definition: A topological group is a group object in **Top**.

Theorem 1.1. Let X be a topological group then $\pi_1(X)$ is an abelian group.

Proof. The functor $\pi_1 : \mathbf{Top}_\bullet \to \mathbf{Grp}$ preserves products and thus preserves group objects. Thus $\pi_1(X)$ is a group object in **Grp** which is an abelian group. □

Proposition 1.2. Let G be a connected topological group and $K \subset G$ a discrete normal subgroup. Then $K \subset Z(G)$.

Proof. Consider the continuous map $G \times K \to K$ given by $(g, k) \mapsto gkg^{-1}$ which is well-defined by normality $K \triangleleft G$. For each fixed $k \in K$ consider the map $G \to K$ via $g \mapsto gkg^{-1}$. Since G is connected its image is also connected in K and thus is a point since K is discrete. However, $1 \mapsto k$ meaning that $gkg^{-1} = k$ for all $g \in G$ and each fixed $k \in K$. Thus $K \subset Z(G)$. □

Proposition 1.3. Let $H \triangleleft G$ be topological groups then the quotient $\pi : G \to G/H$ is an open homeomorphism.

Proof. A set $U \subset G/H$ is open iff $\pi^{-1}(U)$ is open. Furthermore, for any open $U \subset G$ consider,

$$\pi^{-1}(\pi(U)) = H \cdot U = \bigcup_{h \in H} h \cdot U$$

which is a union of open sets and thus open since $h\cdot$ is a homeomorphism and thus $h \cdot U$ is open. □

Proposition 1.4. Let $H \subset G$ be topological groups. If H is open then H is closed. If H is closed of finite index then H is open.

Proof. Because the cosets form a disjoint cover, we may write,

$$G \setminus H = \bigcup_{gH \in (G/H) \setminus H} gH$$

If H is open then gH is open because multiplication by g is open (it is a homeomorphism) so $G \setminus H$ is a union of open sets and thus open i.e. H is closed. If H is closed then gH is closed since g is a closed map and if furthermore $[G : H]$ is finite then $G \setminus H$ is also closed since it is a finite union of closed sets and thus H is open. □

Proposition 1.5. Let G be a compact topological group and $H \subset G$ an open subgroup. Then G/H is finite.

Proof. The open sets $\{gH \mid g \in G\}$ form a cover of G which has a finite subcover because G is compact. However, cosets are equivalence classes and thus disjoint so there must be a finite number of cosets. Thus $[G : H]$ is finite so G/H is finite. □

Proposition 1.6. A topological group G is Hausdorff iff $1 \in G$ is a closed point.

Proof. If G is Hausdorff then G is T_1 so, in particular, $1 \in G$ is closed. Conversely, assume that $1 \in G$ is closed. Consider the continuous map $G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$. The preimage of $\{1\} \subset G$ under this map is the diagonal $\Delta \subset G \times G$ which is then closed. Therefore G is Haudorff. □

Proposition 1.7. Let $H \triangleleft G$ be topological groups then,

$$G/H \text{ is Hausdorff} \iff H \subset G \text{ is closed}$$

Proof. G/H is Hausdorff $\iff$ $1 \in G/H$ is a closed point $\iff$ $H \subset G$ is closed. $\square$

Proposition 1.8. Let $H \triangleleft G$ be topological groups then,

$$G/H \text{ is discrete} \iff H \subset G \text{ is open}$$

Proof. G/H is discrete $\iff$ $1 \in G/H$ is an open point $\iff$ $H \subset G$ is open. $\square$

Proposition 1.9. Let $H \subset G$ be a subgroup of a topological group then $\overline{H} \subset G$ is a closed subgroup. Furthermore if $H \triangleleft G$ is normal then $\overline{H} \triangleleft G$ is normal.

Proof. Let $a, b \in \overline{H}$ and consider the continuous map $f : G \times G \to G$ given by $f(x, y) = xy^{-1}$. Let U be any neighborhood of ab^{-1} then $f^{-1}(U)$ is open with $(a, b) \in f^{-1}(U)$ so there exist open sets $A, B \subset G$ such that $(a, b) \in A \times B \subset f^{-1}(U)$. However, since $a, b \in \overline{H}$ and A, B are neighborhoods of a, b repectivly, since a and b are closure points of H then $\exists x \in A \cap H$ and $y \in B \cap H$. Thus $(x, y) \in f^{-1}(U)$ so $xy^{-1} \in H \cap U$ since H is a subgroup. Since U is arbitrary containing ab^{-1} then $ab^{-1} \in \overline{H}$. Therefore $\overline{H}$ is a subgroup.

Now suppose that $H \triangleleft G$ is normal. Fix $g \in G$ and consider the continous homomorphism $f_g : G \to G$ given by $f_g(x) = gxg^{-1}$. Because f_g is continuous $f_g(\overline{H}) \subset \overline{f(H)}$. However, since H is normal $f_g(H) = H$ and $f_g(\overline{H}) = g\overline{H}g^{-1}$ so we find $g\overline{H}g^{-1}) \subset \overline{H}$ for each $g \in G$ so $\overline{H} \triangleleft G$ is normal. $\square$

Theorem 1.10. If $f : G \to H$ is an open continuous surjective homomorphism of topological groups then $G/\ker f \cong H$ naturally.

Proof. When $G/\ker f$ is given the quotient topology, the canoncial map on the quotient, $G/\ker f \to H$, is a continuous bijective homomorphism. However, generically it may not be a homeomorphism. However, if $f : G \to H$ is open then consider,

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\pi"'] & & H \\ & G/\ker f \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

If $U \subset G/\ker f$ is open then $U = \pi(\pi^{-1}(U))$ since it is surjective so $\tilde{f}(U) = f(\pi^{-1}(U))$ which is open. Thus $\tilde{f}$ is a open continuous bijective and thus a homeomorphism since $\tilde{f}$ is also a homomorphism it is an isomorphism in the category of topological groups. $\square$

2 Connected Components

Proposition 2.1. The connected components of X are closed and connected. Furthermore, if there are finitely many components then they are open.

Proof. Connectedness is obvious from their maximality in the poset of connected sets and so is closure since if Y is connected then $\overline{Y}$ is also connected so by maximality $Y = \overline{Y}$.

Now, if there are finitely many connected components then the complement of one is a finite union of closed sets (the other components) and thus closed so it is open. $\square$

Remark 1. Finiteness is necessary to ensure that the connected components are open. Accordingly, the space $\mathbb{Q}$ (with the Euclidean topology) has connected connected components $\{q\}$ for $q \in \mathbb{Q}$ which are open but not closed.

Proposition 2.2. Let G be a topological group. Then G_0 the connected component of e is a topological subgroup.

Proof. The map $\ell_g : G \to G$ by left multiplication is continuous. Thus, if $g \in G_0$ then consider $\ell_g(G_0)$ which is connected and contains g so it is contained in G_0. Furthermore, the inversion map $i : G \to G$ is continuous so $i(G_0)$ is connected and contains e so $i(G_0) \subset G_0$. Therefore G_0 is a subgroup. $\square$

Proposition 2.3. Let G be a topological group then there is an exact sequence of topological groups,

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1$$

Proof. First, note that $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ is a functor respecting products and thus preserves group objects. The map $G \to \pi_0(G)$ given by sending g to $[g]$ the unique connected component containing it. This map is a continuous group homomorphism when $\pi_0(G)$ is given the quotient topology. Now $g \in \ker(G \to \pi_0(G))$ iff $[g] = [e]$ iff $g \in G_0 = [e]$ so the sequence is exact. In particular, $G_0 \triangleleft G$ is normal. $\square$

Proposition 2.4. Let G be a topological group and $G_0 \subset G$ the connected component of the identity. Then G_0 is a subgroup of G and the cosets correspond to the connected components via an isomorphism $G/G_0 \cong \pi_0(G)$.

Proof. Take $g \in G_0$ and consider the map $G_0 \to G$ given by $x \mapsto gx^{-1}$. Since this map is continuous and G_0 is connected its image is connected. However, its image contains g since $1 \in G_0$ meaning that the image must lie in the connected component of g which is G_0 since connected components partition G. Thus for $x, y \in G_0$ we have $xy^{-1} \in G_0$ so G_0 is a subgroup.

Furthermore, the set $\pi_0(G)$ is naturally a group. This is because G is a group object in **Top** and π_0 preserves products so $\pi_0(G)$ is a group object in **Set**. Explicitly, multiplication is given by $[x] \cdot [y] = [x \cdot y]$ where $[x]$ is the connected component of $x \in G$. Consider the map $G \to \pi_0(G)$ via $x \mapsto [x]$. Clearly this is surjective with kernel G_0 so $G/G_0 \cong \pi_0(G)$. $\square$

Lemma 2.5. The connected components of any manifold are open.

Proof. Let $C \subset M$ be a connected component. Then for any $x \in C$ there is a chart (U, φ) containing x. Then $\varphi(U)$ is open in $\mathbb{R}^n$ which is locally connected so there exists an open connected set V containing $\varphi(x)$ which implies that $\tilde{V} = \varphi^{-1}(V)$ is an open connected neighborhood of x so φ is a homeomorphism. Thus, by maximality, $x \in \tilde{V} \subset C$ so C is open. $\square$

Proposition 2.6. Every compact Lie group is a finite extenson of a connected group.

Proof. Let G be a Lie compact group. Then G_0 is open since G is a manifold. Therefore, $\pi_0(G) = G/G_0$ is finite since the cosets form a disjoint open cover. Then the sequence,

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1$$

makes G a finite extension of G_0. $\square$

Remark 2. The requirement that G be a manifold is necessarly. For example $\mathbb{Z}_p$ is a compact topological group but it is totally disconnected and points are not open and it is infinite.

2.1 Covering Groups

3 Manifolds with any Finite Fundamental Group

Remark 3. For loops $\gamma_1, \gamma_2 : I \to X$ we will use the notation $\gamma_1 * \gamma_2$ to denote the loop,

$$h(t) = \begin{cases} \gamma_1(2t) & t \leq \frac{1}{2} \\ \gamma_2(2t-1) & t \geq \frac{1}{2} \end{cases}$$

Definition: An action of of a group G on a topological space X is a homomorphism $A : G \to \text{Homeo}(X)$. Equivalently, one may define a map $\varphi : G \times X \to X$ and let $\varphi_g(x) = \varphi(g, x)$ such that $\varphi_e = \text{id}_X$ and $\varphi_{gh} = \varphi_g \circ \varphi_h$ and φ_g is a continuous map. Because $\varphi_{g^{-1}}$ is also continuous and $\varphi_g \circ \varphi_{g^{-1}} = \varphi_{g^{-1}} \circ \varphi_g = \varphi_e = \text{id}_e$ then each map is a homeomorphism of X to itself so $g \mapsto \varphi_g$ is a homomorphism from G to Homeo(X).

Definition: Let G be a group acting on a topological space X then X/G is the quotient space under the equivalence relation $x \sim y \iff \exists g \in G : g \cdot x = y$.

Remark 4. For $x \in X$, let $[x]_G$ denote the equivalence class under a group action and for $\gamma : I \to X$ let $[\gamma]$ denote the equivalence class under path-homotopy.

Definition: A group G acts freely on a set X if every stabilizer is trivial. Equivalently, if for some $x \in G$ we have $g \cdot x = h \cdot x$ then $(h^{-1}g) \cdot x = x$ so $g = h$.

Definition: A group action on X is properly discontinuous if for any $x \in X$ there exists an open neighborhood $x \in U$ such that $(g \cdot U) \cap U = \varnothing$ for each $g \neq e$.

Lemma 3.1. Let the action of G on X be properly discontinuous, then X is a covering space of X/G with the covering map $\pi : X \to X/G$.

Proof. Take an open set $U \subset X$ and consider $\pi(U)$. Then, $\pi(x) \in \pi(U)$ if and only if $\exists y \in U$ such that $x \sim y \iff \exists g \in G : x = g \cdot y \iff x \in g \cdot U$. Therefore,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

which is open because each g acts as an open map (in fact a homeomorphism). By the definition of X/G then $\pi(U)$ is open so π is an open map. Take a point $x_0 \in X$ and because the action is properly discontinuous, there exists an open $x_0 \in U$ such that $(g \cdot U) \cap U = \varnothing$ for each $g \neq e$. Consider $V = \pi(U) \subset X/G$ which is open. Since for $g \neq h$, we have $(h^{-1}g) \cdot U \cap U = \varnothing$ then $(g \cdot U) \cap (h \cdot U) = \varnothing$ so the slices are disjoint. Finally, take $x, y \in g \cdot U$ then if $\pi(x) = \pi(y)$ we have $[x] = [y]$ so $x = h \cdot y$ for some $g \in G$. But since $y \in g \cdot U$ then $x \in hg \cdot U$ and $x \in g \cdot U$ so $h = e$ and thus $x = y$ because for $g \neq e$ the sets $hg \cdot U$ and $g \cdot U$ are disjoint since $hg \neq g$. Therefore, $\pi|_{g \cdot U}$ is injective but it is trivialy surjective onto $V = \pi(U) = \pi(g \cdot U)$. Furthermore, π is an open continuous map and thus a homeomorphism when restricted to U. Therefore, V is an openly covered neighborhood of $[x]_G$ so π is a covering map of X/G. $\square$

Theorem 3.2. Let X be a simply connected and a let the action of G on X be free and properly discontinuous. Then $\pi_1(X/G, [x_0]_G) \cong G$.

Proof. Fix $x_0 \in X$, then take $g \in G$ and let $\gamma_g : I \to X$ be a path from x_0 to $g \cdot x_0$. Such a path exists because X is path-connected. Take the projection map $\pi : X \to X/G$ given by $\pi(x) = [x]_G$. These paths project to loops in the quotient space, $\eta_g = \pi \circ \gamma_g$ which is a loop because $\eta_g(0) = \pi(x_0) = [x_0]$ and $\eta_g(1) = \pi(g \cdot x_0) = [g \cdot x_0] = [x_0]$ and action by g is a continuous map.

Define the map $\phi : G \to \pi_1(X/G, [x_0]_G)$ given by $\phi : g \mapsto [\pi \circ \gamma_g]$. Take $g, h \in G$ and consider the path $\delta = \gamma_g * (g \cdot \gamma_h)$ where $(h \cdot \gamma_g)(t) = h \cdot \gamma_g(t)$ with endpoints:

$$\gamma_g * (g \cdot \gamma_h)(0) = \gamma_g(0) = x_0 \text{ and } \gamma_g * (g \cdot \gamma_h)(1) = (g \cdot \gamma_h)(1) = g \cdot (h \cdot x_0) = (gh) \cdot x_0$$

Therefore, because X is simply connected, $\delta \sim \gamma_{gh}$ and thus,

$$\pi \circ \delta = (\pi \circ \gamma_g) * (\pi \circ (g \cdot \gamma_h)) \sim \pi \circ \gamma_{gh} = \eta_{gh}$$

However, $\pi \circ \gamma_g = \eta_g$ and $\pi \circ (g \cdot \gamma_h)(t) = \pi(g \cdot \gamma_h(t)) = [g \cdot \gamma_h(t)]_G = [\gamma_h(t)]_G = \eta_h(t)$ because the orbits are equivalence classses. Thus, $\pi \circ (g \cdot \gamma_h) = \eta_h$ so $\eta_g * \eta_h \sim \eta_{gh}$. Therefore, $\phi(gh) = [\eta_{gh}] = [\eta_g * \eta_h] = [\eta_g][\eta_h] = \phi(g)\phi(h)$ so ϕ is a homomorphism. It remains to show that ϕ is a bijection.

X is the universal cover of X/G so any path $\delta : I \to X/G$ can be lifted to a a unique path $\gamma : I \to X$ up to a choice of initial point. Thus, if δ is a loop at $[x_0]_G$ then there exists a unique path $\gamma : I \to X$ such that $\pi \circ \gamma = \delta$ and $\gamma(0) = x_0$. However, $\pi \circ \gamma(1) = \delta(1) = [x_0]_G$ so $[\gamma(1)]_G = [x_0]_G$ thus $\exists g \in G : \gamma(1) = g \cdot x_0$. Because X is simply connected, $\gamma \sim \gamma_g$ since they share endpoints. Finlly, $\phi(g) = [\pi \circ \gamma_g] = [\pi \circ \gamma] = [\delta]$ so the map ϕ is surjective. Finally, take $g, h \in G$ and suppose that $\phi(g) = \phi(h)$ then $\pi \circ \gamma_g \sim \pi \circ \gamma_h$. By the homotopy lifting lemma, these loops lift to unique path-homotopic paths in X with initial point x_0. However, γ_g and γ_h already satisfy the projection property and therefore must be the unique lifts so $\gamma_g \sim \gamma_h$. In particular, $\gamma_g(1) = \gamma_h(1)$ because they are path homotopic so $g \cdot x_0 = h \cdot x_0$ but because G acts freely on X this implies that $g = h$. Therefore, ϕ is a bijection. ☐

Lemma 3.3. A free action of a finite group on a Hausdorff space is properly discontinuous.

Proof. Take $x \in X$ and, because the action is free, for each $g \neq e$ we have $g \cdot x \neq x$ so because X is Hausdorff, there exist open sets U_g and V_g such that $x \in U_g$ and $g \cdot x \in U_g$ and $U_g \cap V_g$. Now, let,

$$U = \bigcap_{g \in G \setminus \{e\}} (U_g \cap g^{-1} \cdot V_g)$$

which is open because the intersection is finite. Also, for each g, $x \in U_g$ and $g \cdot x \in V_g$ so $x \in g^{-1} \cdot V_g$. Thus, $x \in U$. Now, take any $g \neq e$. We have $U \subset U_g$ and $U \subset g^{-1} \cdot V_g$ so $g \cdot U \subset V_g$. However, U_g and V_g are disjoint so U and $g \cdot U$ are disjoint. ☐

Lemma 3.4. Any quotient of a compact connected space is compact and connected.

Proof. Let X be compact and connected. Then, $\pi : X \to X/\sim$ is continuous and surjective. Therefore, $X/\sim$ is the image of a compact and conected set and is thus compact and connected. ☐

Theorem 3.5. For any finite group G, there exists a compact connected manifold with fundamental group G.

Proof. By considering $n \times n$ permutation matrices in $\mathrm{SU}(n)$ we get an embedding of S_n inside $\mathrm{SU}(n)$. However, by Cayley's theorem, any group with order n can be embedded as a subgroup of S_n. Therefore, we have the embeddings,

$$G \hookrightarrow S_n \hookrightarrow \mathrm{SU}(n)$$

Let $\Gamma \subset \mathrm{SU}(n)$ be the embedded copy of G. Γ acts on $\mathrm{SU}(n)$ by left multiplication which is a topological action because $\mathrm{SU}(n)$ is a topological group and thus has continuous multiplcation. Futhermore, if $g \cdot h = h$ then $gh = h$ so $g = e$. Thus, Γ acts freely on $\mathrm{SU}(n)$. However, $\mathrm{SU}(n)$ is a simply connected compact manifold. In particular, $\mathrm{SU}(n)$ is a Hausdorff space and Γ is finite so the action is properly discontinuous. Therefore, since $\mathrm{SU}(n)$ is simply connected, $\pi_1(\mathrm{SU}(n)/\Gamma, [x_0]) \cong \Gamma \cong G$. Furthermore $\mathrm{SU}(n)/\Gamma$ is a compact connected space because it is a quotient of $\mathrm{SU}(n)$ which is compact and connected. Finally, we must show that $\mathrm{SU}(n)/\Gamma$ is a manifold. (WIP) ☐

Theorem 3.6. For any finite cyclic group G, there exists a compact connected 3-manifold with fundamental group G.

Proof. Consider the matrix,

$$M = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} \in \mathrm{SU}(2)$$

Let $\Gamma = \langle M \rangle \subset \mathrm{SU}(2)$. Since M has order n, $\Gamma \cong C_n$. By an identical argument to above, $\pi_1(\mathrm{SU}(2)/\Gamma, [x_0]) \cong \Gamma \cong C_n$ and $\mathrm{SU}(2)/\Gamma$ is compact and connected. It remains to show that $\mathrm{SU}(2)/\Gamma$ is a 3-manifold. (WIP) ☐

4 Lie Groups

Proposition 4.1. Let $f : G \to H$ be a morphism of lie groups with $f_* : \mathfrak{g} \to \mathfrak{h}$ surjective and H connected. Then f is surjective.

Proof. Since $f_* : \mathfrak{g} \to \mathfrak{h}$ is surjective then $\mathrm{d}f : T_g G \to T_{f(g)} H$ is surjective so f is a submersion and thus open. Then $f(G) \subset H$ is an open subgroup and thus closed. However, H is connected and $f(G)$ is nonempty clopen so $f(G) = H$. ☐

Lemma 4.2. Let $f : M \to N$ be a local diffeomorphism. Then the fibres $f^{-1}(y)$ are discrete.

Proof. Let $x \in f^{-1}(y)$ then there exists a neighborhood $U \subset M$ on which $f|_U : U \to f(U)$ is a diffeomorphism and thus $f|_U$ is injective so $U \cap f^{-1}(y) = \{x\}$ which implies that $f^{-1}(y)$ is a discrete set. ☐

Proposition 4.3. Let $f : G \to H$ be a morphism of connected lie groups such that the lie algebra map $f_* \mathfrak{g} \to \mathfrak{h}$ is an isomorphism. Then $\Gamma = \ker f$ is a discrete subgroup $\Gamma \subset Z(G)$ and f induces an isomorphism $f : G/\Gamma \xrightarrow{\sim} H$.

Proof. Since $f_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism we know that $\mathrm{d}f : T_g G \to T_{f(g)} H$ is an isomorphism and thus f is a local diffeomorphism (by the inverse function theorem). Thus, by the lemma, $\Gamma = \ker f = f^{-1}(0)$ is discrete and also normal (since it is a kernel) so by 1.2 we have $\Gamma \subset Z(G)$. Furthermore, local diffeomorphisms are open maps and, by above, f is surjective so by 1.10 the induced map $\tilde{f} : G/\Gamma \xrightarrow{\sim} H$ is a homeomorphism. It suffices to show that $\tilde{f}$ is, in fact, a diffeomorphism (DO THIS). ☐