1 Appendix

1.1 Curves and Genera

Lemma 1.1.1. Let X be a integral scheme proper over k then $K = H^0(X, \mathcal{O}_X)$ is a finite field extension of k and for any coherent $\mathcal{O}_X$-module $\mathscr{F}$, the cohomology $H^p(X, \mathscr{F})$ is a finite-dimensional $H^0(X, \mathcal{O}_X)$-module.

Proof. Since $\mathcal{O}_X$ is coherent, and X is proper over k so $K = H^0(X, \mathcal{O}_X)$ is a finite k-module. However, since X is integral $H^0(X, \mathcal{O}_X)$ is a domain but a finite k-algebra domain is a field and we see K/k is a finite extension of fields. Furthermore, the $\mathcal{O}_X(X)$-module structure on $H^p(X, \mathscr{F})$ gives it a K-module structure. Since X is proper over k then $H^p(X, \mathscr{F})$ is a finite k-module and thus finite as a K-module. □

Remark. Unfortunately, when k is not algebraically closed then we may not have $H^0(X, \mathcal{O}_X) = k$ even for smooth projective varieties. Therefore, some caution must be taken in defining numerical invariants of the curve such as genus. However, by [?, Tag 0BUG], whenever X is proper geometrically integral then indeed $H^0(X, \mathcal{O}_X) = k$. Furthermore, for proper X if $H^0(X, \mathcal{O}_X) \neq k$ then X cannot be geometrically connected by [?, Tag 0FD1].

Definition 1.1.2. Let C be a smooth proper curve over k with $H^0(C, \mathcal{O}_C) = K$. Then we define $g(C) := \dim_K H^0(X, \Omega_{C/k})$. If C is any curve over k then there is a unique smooth proper curve S over k which is k-birational to C. Then we define $g(C) := g(S)$.

Remark. By definition, the genus of a curve is clearly a birational invariant since there is a unique smooth complete curve in every birational equivalence class of curves.

Remark. There is a slight subtlety in this definition in the case of a non-perfect base field. It it always true that we can find a proper *regular* curve C in each birational equivalence class however when k is non-perfect the curve C may not be smooth. However, under a finite purely separable extension K/k, we can ensure that C_K admits a smooth proper model. Then we define $g(C) := g(C_K)$ in the case that C_K is a curve. The only thing that can go wrong is when C is not geometrically irreducible since then C_K will not be integral.

Definition 1.1.3. The *arithmetic genus* $g_a(C)$ of a proper curve C over k with $H^0(C, \mathcal{O}_C) = K$ is,

$$g_a(C) := \dim_K H^1(X, \mathcal{O}_C)$$

By Serre duality, if C is smooth then $H^0(C, \Omega_C) = H^1(C, \mathcal{O}_X)^\vee$ meaning that $g_a(C) = g(C)$.

Remark. The arithmetic genus depends on the projective compactification and singularities meaning it will not be a birational invariant unlike the (geometric) genus.

Example 1.1.4. Let $k = \mathbb{F}_p(t)$ for an odd prime $p = 2k + 1$ and consider the curve,

$$C = \operatorname{Spec}\left(k[x, y]/(y^2 - x^p - t)\right)$$

which is regular but not smooth at $P = (y, x^p - t)$. Consider the purely inseperable extension $K = \mathbb{F}(t^{1/p})$. Then $C_K = \operatorname{Spec}\left(K[x, y]/(y^2 - (x - t^{1/p})^p)\right) \cong \operatorname{Spec}\left(K[x, y]/(y^2 - x^p)\right)$. Taking the normalization of C_K gives $\mathbb{A}^1_K \to C_K$ via $t \mapsto (t^p, t^2)$. This is birational since the following ring map is an isomorphism,

$$(K[x, y]/(y^2 - x^p))_x \to K[t]_t$$

sending $x \mapsto t^2$ and $y \mapsto t^p$ which has an inverse $t \mapsto y/x^k$ since $x \mapsto t^2 \mapsto y^2/x^{2k} = x$ and $y \mapsto t^p \mapsto y^p/x^{kp} = y(y^{2k}/x^{pk}) = y$ and $t \mapsto y/x^k \mapsto t^{p-2k} = t$.

Therefore, $C_K \overset{\sim}{\dashrightarrow} \mathbb{P}^1_K$ so $g(C) = g(C_K) = 0$. However, consider the projective closure,

$$\overline{C} = \text{Proj}\left(k[X,Y,Z]/(Y^2 Z^{p-2} - X^p - tZ^p)\right)$$

then $\overline{C} \hookrightarrow \mathbb{P}^2_k$ is a Cartier divisor (since $\mathbb{P}^2_k$ is locally factorial) so we find that $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = k$ and $\dim_k H^1(\overline{C}, \mathcal{O}_{\overline{C}}) = \frac{1}{2}(p-1)(p-2) = k(2k-1)$ since its sheaf of ideals is $\mathcal{O}_{\mathbb{P}^2_k}(-p)$. Then $p = 3$ we expect this to be an elliptic curve and we do see $g_a(\overline{C}) = 1$. However, $g(\overline{C}) = 0$ and correspondingly C is not smooth due to the positive characteristic phenomenon.

Lemma 1.1.5. Suppose that $f : X \to Y$ is a finite birational morphism of n-dimensional irreducible Noetherian schemes. Then $H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(X, \mathcal{O}_X)$ is surjective.

Proof. The map f must restrict on some open subset $U \subset X$ to an isomorphism $f|_U : U \to V$. Thus, the sheaf map $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ restricts on V to an isomorphism $\mathcal{O}_Y|_V \xrightarrow{\sim} (f_*\mathcal{O}_X)|_V$. We factor this map into two exact sequences,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_Y \longrightarrow \mathscr{I} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{I} \longrightarrow f_*\mathcal{O}_X \longrightarrow \mathscr{C} \longrightarrow 0$$

with $\mathscr{K} = \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{C} = \text{coker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{I} = \text{Im}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. Taking cohomology and using that it vanishes in degree above n we get,

$$H^{n-1}(Y, \mathscr{I}) \longrightarrow H^n(Y, \mathscr{K}) \longrightarrow H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(Y, \mathscr{I}) \longrightarrow 0$$

$$H^{n-1}(Y, \mathscr{C}) \longrightarrow H^n(Y, \mathscr{I}) \longrightarrow H^n(X, \mathcal{O}_X) \twoheadrightarrow H^n(X, \mathscr{C}) \longrightarrow 0$$

where we have used that $f : X \to Y$ is affine to conclude that $H^p(Y, f_*\mathscr{F}) = H^p(Y, \mathscr{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$. Furthermore, $\mathscr{C}|_V = 0$ so $\text{Supp}_{\mathcal{O}_Y}(\mathscr{C}) \subset X \setminus V$ but $\mathscr{C}$ is coherent so the support is closed. Since V is dense open, $\mathscr{C}$ is supported in positive codimension so $H^n(Y, \mathscr{C}) = 0$ (since $H^n(S, \mathscr{C})$ vanishes due to dimension on the closed subscheme $S = \text{Supp}_{\mathcal{O}_X}(\mathscr{C})$ on which $\mathscr{C}$ is supported). Thus we have,

$$H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(X, \mathcal{O}_X)$$

proving the proposition. □

Corollary 1.1.6. Let S and C be proper curves over k where S is smooth which are birationally equivalent and $H^0(S, \mathcal{O}_S) \cong H^0(C, \mathcal{O}_C)$. Then the genera satisfy,

(a) $g_a(C) \geq g_a(S)$

(b) $g(C) = g(S)$

(c) $g(C) \leq g_a(C)$ with equality if and only if C is smooth.

Proof. Given a birational map $S \overset{\sim}{\dashrightarrow} C$ we can extend it to a birational morphism $S \to C$ since S is regular. The morphism $S \to C$ is automatically finite since it is a non-constant map of proper curves. Then the previous lemma implies that $g_a(S) \le g_a(C)$. (b). follows from the definition of $g(C)$. The third follows from the fact that $g(S) = g_a(S)$ because of Serre duality,

$$H^1(S, \mathcal{O}_S) \cong H^0(S, \Omega_{S/k})^\vee$$

using that S is smooth. Then we see that $g(C) = g(S) = g_a(S) \le g_a(C)$ proving the inequality part of (c). Finally, if C is smooth we see by Serre duality that $g(C) = g_a(C)$. Conversely, suppose that $g(C) = g_a(C)$ then $g_a(C) = g(C) = g(S) = g_a(S)$ and consider the map $f : S \to C$ which is finite birational map of integral schemes over k. In particular, f is affine so for each $y \in C$ we may choose an affine open $y \in V \subset C$ whose preimage $U = f^{-1}(V)$ is also affine. On sheaves, this gives a map of domains $\mathcal{O}_C(V) \to \mathcal{O}_S(U)$ which localizes to an isomorphism on the fraction fields. However, the localization map of a domain is injective so $\mathcal{O}_C(V) \hookrightarrow \mathcal{O}_S(U)$ is an injection. This shows that $\mathcal{O}_C \to f_* \mathcal{O}_S$ is an injection of sheaves which we extend to an exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_* \mathcal{O}_S \longrightarrow \mathscr{C} \longrightarrow 0$$

Note that $f : S \to C$ induces an isomorphism $H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$ since it is a map of fields with the same (finite) dimension over k. Then the long exact sequence of cohomology gives,

$$0 \to H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S) \to H^0(X, \mathscr{C}) \to H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S) \to H^1(S, \mathscr{C}) = 0$$

I claim that $H^1(S, \mathscr{C}) = 0$. Since f is birational, $\mathscr{C}$ is supported in codimension one. Thus, the map $H^1(C, \mathcal{O}_C) \twoheadrightarrow H^1(S, \mathcal{O}_S)$ is surjective but $g_a(C) = g_a(S)$ so these vectorspaces have the same dimension so $H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S)$ is an isomorphism. Thus, from the exact sequence we have $H^0(X, \mathscr{C}) = 0$. However, $\mathrm{Supp}_{\mathcal{O}_C}(\mathscr{C})$ is a closed ($\mathscr{C}$ is coherent) dimension zero subset i.e. finitely many discrete closed points. However, a sheaf supported on a discrete set of points is zero iff it has no global sections. Therefore, $\mathscr{C} = 0$ so $\mathcal{O}_C \xrightarrow{\sim} f_* \mathcal{O}_S$. In particular $\mathcal{O}_C(V) \xrightarrow{\sim} \mathcal{O}_S(U)$ is an isomorphism which implies that the map of affine schemes $f|_U : U \to V$ is an isomorphism. Since the affine opens V cover C we see that $f : S \to C$ is an isomorphism. In particular, C is smooth. ☐

1.2 The Locus on Which Morphisms Agree

Lemma 1.2.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then for schemes X there is a natural bijection,

$$\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(R), X) \cong \{x \in X \text{ and local map } \mathcal{O}_{X,x} \to R\}$$

Proof. Given $\mathrm{Spec}(R) \to X$ we automatically get $\mathfrak{m} \mapsto x$ and $\mathcal{O}_{X,x} \to R_\mathfrak{m} = R$. Now, note that taking any affine open neighborhood $x \in \mathrm{Spec}(A) \subset X$ and then $A \to A_\mathfrak{p} = \mathcal{O}_{X,x}$ to give $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(A) \to X$. Clearly, this map sends $\mathfrak{m}_x \mapsto x$ and at $\mathfrak{m}_x$ has stalk map $\mathrm{id} : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ since it is the localization at $\mathfrak{p}$ of $A \to A_\mathfrak{p}$.

Thus we get an inverse as follows. Given a point $x \in X$ and a local map $\phi : \mathcal{O}_{X,x} \to R$ then take,

$$\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$$

This is inverse since $\mathfrak{m} \mapsto \mathfrak{m}_x$ (because $\mathcal{O}_{X,x} \to \mathfrak{m}_x$ is local) and $\mathfrak{m}_x \mapsto x$ and the stalk at $\mathfrak{m}$ gives $\mathcal{O}_{X,x} \xrightarrow{\mathrm{id}} \mathcal{O}_{X,x} \xrightarrow{\phi} R$.

Finally, I claim that any $f : \mathrm{Spec}\,(R) \to X$ factors through $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}) \to X$ and thus is reconstructed from $x \in X$ and $\mathcal{O}_{X,x} \to R$. Choose an affine open neighborhood $x \in \mathrm{Spec}\,(A) \subset X$ then consider $f^{-1}(\mathrm{Spec}\,(A))$ which is open in $\mathrm{Spec}\,(R)$ and contains the unique closed point $\mathfrak{m} \in \mathrm{Spec}\,(R)$ so there is some $f \in R$ s.t. $\mathfrak{m} \in D(f) \subset f^{-1}(\mathrm{Spec}\,(A))$ so $f \notin \mathfrak{m}$ so $f \in R^\times$ and thus $D(f) = \mathrm{Spec}\,(R)$. Therefore, we get a map $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(A)$ and thus $\phi : A \to R$ where $\phi^{-1}(\mathfrak{m}) = \mathfrak{p} = x$ so $A \setminus \mathfrak{p}$ is mapped inside $R^\times$ so this map factors through $A \to A_{\mathfrak{p}} \to R$ giving the desired factorization $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}) \to \mathrm{Spec}\,(A) \to X$. □

Definition 1.2.2. The locus Z on which two maps $f, g : X \to Y$ over S agree is given as the pullback,

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

with $F = (f, g)$. This is the equalizer of $f, g : X \to Y$. Furthermore $Z \to X$ is an immersion since it is the base change of $\Delta_{Y/S}$ which is an immersion.

Lemma 1.2.3. Topologically, the locus on which S-morphisms $f, g : X \to Y$ agree is,

$$Z = \{x \in X \mid f(x) = g(x) \text{ and } f_x = g_x : \kappa(f(x)) \to \kappa(x)\}$$

Proof. On some S-subscheme $G \subset X$, the maps $f|_G = g|_G$ agree iff there exists $G \to Y$ such that,

$$\begin{tikzcd} G \arrow[r, dashed] \arrow[d, hook] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

commutes. In particular, for any point $x \in X$ consider $\iota : \mathrm{Spec}\,(\kappa(x)) \to X$ then $f \circ \iota = g \circ \iota$ iff $f(x) = g(x)$ and $f_x = g_x : \kappa(f(x)) \to \kappa(x)$. Consider a point $z \in Z$ and $\mathrm{Spec}\,(\kappa(z)) \to Z$, such a point is equivalent to giving a diagram,

$$\begin{tikzcd} \mathrm{Spec}\,(\kappa(z)) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ & X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

However, $\iota : Z \to X$ is an immersion so $\iota_x : \kappa(\iota(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism. Therefore, points $\mathrm{Spec}\,(\kappa(z)) \to Z$, are exactly points of X for which a lift $\mathrm{Spec}\,(\kappa(x)) \to Y$ exists i.e. points such that f and g agree in the required way. □

Lemma 1.2.4. If $f : X \to Y$ is an immersion then $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective for each $x \in X$ and $f_x : \kappa(f(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism.

Proof. For closed immersions, $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective by definition. Thus we get a surjection $f_x : \mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_{f(x)}$. Furthermore, topologically, $f : X \to Y$ is a homomorphism onto its image so for any open $U \subset X$ there exists an open $V \subset Y$ s.t. $U = f^{-1}(V)$ showing that,

$$(f_* \mathcal{O}_X)_{f(x)} = \varinjlim_{f(x) \in V} \mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}$$

Furthermore, for an open immersion, $f^\flat : f^{-1} \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism so $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. Thus the composition, $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective. Furthermore, f_x is local we get $f_x : \kappa(f(x)) \twoheadrightarrow \kappa(x)$ which is a surjection of fields and thus an isomorphism. $\square$

Lemma 1.2.5. If $Y \to S$ is separated then the locus on which $f, g : X \to Y$ over S agree is closed.

Proof. Since $X \to S$ is separated, $\Delta_{Y/S} : Y \to Y \times_S Y$ is a closed immersion. So $Z \to X$ is the base change of a closed immersion and thus a closed immersion. $\square$

Lemma 1.2.6. Let X be a reduced and Y be a separated scheme over S and $f, g : X \to Y$ be morphism over S. If $f \circ j = g \circ j$ agree on a dense subscheme $j : G \hookrightarrow X$ then $f = g$.

Proof. Consider $F = (f, g) : X \to Y \times_S Y$. Since $\Delta : Y \to Y \times_S Y$ is a closed immersion (by separateness). Then $F^{-1}(\Delta)$ is the locus on which $f = g$ which is closed because $\Delta : Y \to Y \times_S Y$ is a closed immersion. Since $f|_G = g|_G$ we get a diagram,

$$\begin{tikzcd}
G \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, hook, bend right] & & \\
& Z \arrow[r, "\tilde{F}"] \arrow[d, hook, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\
& X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}$$

Since $\iota : Z \hookrightarrow X$ is a closed immersion with dense image, $Z \hookrightarrow X$ is surjective. By the following, $\iota : Z \to X$ is an isomorphism. Thus, $F = F \circ \iota \circ \iota^{-1} = \Delta_Y \circ \tilde{F} \circ \iota^{-1}$. By the universal property of maps $X \to Y \times_S Y$ this implies that $f = g = \tilde{F} \circ \iota^{-1}$. $\square$

Lemma 1.2.7. Let X be a scheme and consider an exact sequence of quasi-coherent $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A} \longrightarrow 0$$

and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebra. Suppose that $\mathscr{F}_x \neq 0$ for each $x \in X$. Then $\mathscr{I} \hookrightarrow \mathcal{N}$ where $\mathcal{N}$ is the sheaf of nilpotent.

Proof. Take an affine open $U = \mathrm{Spec}\,(R) \subset X$ such that $\mathcal{A}|_U = \widetilde{A}$. Then we have an surjection of rings $R \twoheadrightarrow A$ giving $R/I = A$ for $I = \ker(R \to A)$. Now, for each $\mathfrak{p} \in \mathrm{Spec}\,(R)$ we know $R_\mathfrak{p} = \mathcal{O}_{X,\mathfrak{p}} \neq 0$. However, if $\mathfrak{p} \not\supset I$ then $(R/I)_\mathfrak{p} = A_\mathfrak{p} = 0$ so we must have $\mathfrak{p} \supset I$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ i.e. $I \subset \mathrm{nilrad}\,(R)$. Therefore, $\mathscr{I}|_U \hookrightarrow \mathcal{N}|_U$ for any affine open $U \subset X$ showing that $\mathscr{I}$ is comprised of nilpotents. $\square$

Corollary 1.2.8. If X is reduced and $\iota : Z \hookrightarrow X$ is a surjective closed immersion then $\iota : Z \xrightarrow{\sim} X$ is an isomorphism.

Proof. Since $\iota : Z \hookrightarrow X$ is a homeomorphism onto its image X it suffices to show that the map of sheaves $\iota^{\#} : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. Since $\iota : Z \to X$ is a closed immersion $\iota^{\#} : \mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Z$ is a surjection and $\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras giving an exact sequence,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Furthermore,

$$\mathrm{Supp}_{\mathcal{O}_X}(\iota_* \mathcal{O}_Z) = \mathrm{Im}(\iota) = X$$

since $(\iota_* \mathcal{O}_Z)_x = \mathcal{O}_{Z,x}$ when $x \in \mathrm{Im}(\iota)$ (and zero elsewhere). by the above, $\mathscr{I} \hookrightarrow \mathcal{N} = 0$ since X is reduced to $\iota^{\#} : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. ☐

Lemma 1.2.9. A rational S-map $f : X \dashrightarrow Y$ with X reduced and $Y \to S$ separated is equivalent to a morphism $f : \mathrm{Dom}(f) \to Y$.

Proof. For any (U, f_U) and (V, f_V) representing f there must be a dense (in X) open $W \subset U \cap V$ on which $f_U|_W = f_V|_W$ and thus $f_U|_{U \cap V} = f_V|_{U \cap V}$ since $f_U, f_V : U \cap V \to Y$ are morphisms from reduced to irreducible schemes. Now $\mathrm{Dom}(f)$ has an open cover (U_i, f_i) for which $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ so these morphisms glue to give $f : \mathrm{Dom}(f) \to Y$ ($\mathrm{Hom}_S(-, Y)$ is a sheaf on the Zariski site). ☐

1.3 Extending Rational Maps

Lemma 1.3.1. Regular local rings of dimension 1 exactly correspond to DVRs.

Proof. Any DVR R has a uniformizer $\varpi \in R$ then $\dim R = 1$ and $\mathfrak{m}/\mathfrak{m}^2 = (\varpi)/(\varpi^2) = \varpi\kappa$ which also has $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 1$ so R is regular. Conversely, if R is a regular local ring of dimension $\dim R = 1$ then, by regularity, R is a normal Noetherian domain so by $\dim R = 1$ then R is Dedekind but also local and thus is a DVR. ☐

Proposition 1.3.2. Let X be a Noetherian S-scheme and $Z \subset X$ a closed irreducible codimension 1 generically nonsingular subset (with generic point $\eta \in Z$ such that $\mathcal{O}_{X,\eta}$ is regular). Let $f : X \dashrightarrow Y$ be a rational map with Y proper over S. Then $Z \cap \mathrm{Dom}(f)$ is a dense open of Z.

Proof. Choose some representative (U, f_U) for $f : X \dashrightarrow Y$. Note that $\mathcal{O}_{X,\eta}$ is a regular dimension one (see Lemma 1.4.3) ring and thus a DVR. Consider the generic point $\xi \in X$ of X then, by localizing, we get an inclusion of the generic point $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ and $\mathcal{O}_{X,\xi} = K(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta})$. Furthermore, the inclusion of the generic point gives $\mathrm{Spec}(K(X)) \to U \xrightarrow{f_U} Y$ and thus we get a diagram,

$$\begin{tikzcd} \mathrm{Spec}(K(X)) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{X,\eta}) \arrow[r] \arrow[ur, dashed, "\ell"] & \mathrm{Spec}(k) \end{tikzcd}$$

and a lift $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to Y$ by the valuative criterion for properness applied to $Y \to \mathrm{Spec}(k)$ since $\mathcal{O}_{X,\eta}$ is a DVR. Choose an affine open $\mathrm{Spec}(R) \subset Y$ containing the image of $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to Y$ (i.e. choose a neighborhood of the image of η which automatically contains $f(\xi)$ since the map factors $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to \mathrm{Spec}\left(\mathcal{O}_{Y,f(\eta)}\right) \to \mathrm{Spec}(R) \to Y$) and let $\eta \in V = \mathrm{Spec}(A) \subset X$ be an affine open neighborhood of ξ mapping onto $\mathrm{Spec}(R)$. By Lemma 1.4.7, since $\mathcal{O}_{X,\eta}$ is a domain, we may shrink V so that A is a domain. Since X is irreducible $U \cap V$ is a dense open. Note that if $\eta \in U$ then $\eta \in \mathrm{Dom}(f)$ and thus $Z \cap \mathrm{Dom}(f)$ is a nonempty open of the irreducible space Z

and therefore a dense open so we are done. Otherwise, let $\mathfrak{p} \in \mathrm{Spec}\,(A)$ correspond to $\eta \in Z$ then $A_\mathfrak{p} = \mathcal{O}_{X,\eta}$ is a DVR. Take some principal affine open $D(f) \subset U \cap V$ for $f \in A$ so $f \in \mathfrak{p}$ since $\mathfrak{p} \notin D(f) \subset U \cap V$. Since $A_\mathfrak{p}$ is a DVR we may choose a uniformizer $\varpi \in \mathfrak{p}$ so the map $A \to \mathfrak{p}$ via $1 \mapsto \varpi$ is as isomorphism when localized at $\mathfrak{p}$. Since A is Noetherian both are f.g. A-modules so there must be some $s \in A \setminus \mathfrak{p}$ such that $A_s \to \mathfrak{p}_s$ is an isomorphism. Replacing A by A_s we may assume $\mathfrak{p} = (\varpi) \subset A$ is principal. Since $f \in \mathfrak{p}$ we can write $f = t\varpi^k$ for some $a \in A \setminus \mathfrak{p}$ (see Lemma 1.4.1). Then consider $\tilde{V} = \mathrm{Spec}\,(A_t)$. Since $t \notin \mathfrak{p}$ then $\eta \in \tilde{V}$ and since $f = t\varpi^k$ we have $D(f) \subset D(t) = \tilde{V}$. Now we get the following diagram,

$$\begin{tikzcd}
& & \mathrm{Spec}\,(R) \\
\mathrm{Spec}\,(A_\mathfrak{p}) \arrow[r] \arrow[urr, bend left, "\ell"] & \mathrm{Spec}\,(A_t) \arrow[ur, dashed, "f_V"] & \\
\mathrm{Spec}\,(\mathrm{Frac}\,(A)) \arrow[r] \arrow[u] & \mathrm{Spec}\,(A_f) \arrow[u] \arrow[uur, bend right, "f_U"'] &
\end{tikzcd}$$

I claim the square is a pushout in the category of affine schemes because maps $R \to A_\mathfrak{p}$ and $R \to A_f$ which agree under the inclusion to $\mathrm{Frac}\,(A)$ gives a map $R \to A_\mathfrak{p} \cap A_f \subset \mathrm{Frac}\,(A)$. However, consider,

$$x \in A_\mathfrak{p} \cap A_t \implies x = \frac{u\varpi^r}{s} = \frac{a}{f^n}$$

for $u, s, t \in A \setminus \mathfrak{p}$ and $a \in A$. Thus we get,

$$ut^n \varpi^{r+nk} = sa$$

so $a \in \mathfrak{p}^{r+nk} \setminus \mathfrak{p}^{r+nk+1}$ ($s \notin \mathfrak{p}$ which is prime) and thus $a = u'\varpi^{r+nk}$ for $u' \in A \setminus \mathfrak{p}$. Therefore,

$$x = \frac{u'\varpi^{r+nk}}{t^n \varpi^{nk}} = \frac{u'\varpi^r}{t^n} \in A_t$$

Thus, $A_\mathfrak{p} \cap A_f \subset A_f$ so we get a map $R \to A_t$. Therefore we get a map $f_{\tilde{V}} : \tilde{V} \to Y$ such that $(f|_{\tilde{V}})|_{D(f)} = (f_U)|_{D(f)}$ which implies that $\eta \in \tilde{V} \subset \mathrm{Dom}\,(f)$ so $Z \cap \mathrm{Dom}\,(f)$ is a dense open of Z. $\square$

Proposition 1.3.3. Let $C \to S$ be a proper regular Noetherian scheme with $\dim C = 1$ and $f : C \dashrightarrow Y$ a rational S-map with $Y \to S$ proper. Then f extends uniquely to a morphism $f : C \to Y$.

Proof. For any point $x \notin \mathrm{Dom}\,(f)$ let $Z = \overline{\{x\}} \subset D$ for $D = C \setminus \mathrm{Dom}\,(f)$. Since $\mathrm{Dom}\,(f)$ is a dense open, by lemma 1.4.2, we have $\mathrm{codim}\,(Z, C) \geq \mathrm{codim}\,(D, C) \geq 1$ but $\dim C = 1$ so $\mathrm{codim}\,(Z, C) = 1$. Furthermore, since C is regular $\mathcal{O}_{C,x}$ is regular and thus, by the previous proposition, $Z \cap \mathrm{Dom}\,(f)$ is a dense open and in particular $x \in \mathrm{Dom}\,(f)$ meaning that $\mathrm{Dom}\,(f) = C$ so we get a morphism $C \to Y$. This is unique because C is reduced (it is regular) and Y is separated (it is proper over S) so morphisms $C \to Y$ are uniquely determined on a dense open which any representative for $f : C \dashrightarrow Y$ is defined on. $\square$

Corollary 1.3.4. Rational maps between normal proper curves are morphisms.

Corollary 1.3.5. Birational maps between normal proper curves are isomorphisms.

Proof. Let $f : C_1 \dashrightarrow C_2$ and $g : C_2 \dashrightarrow C_1$ be birational inverses of smooth proper curves. Then we know that these extend to morphisms $f : C_1 \to C_2$ and $g : C_2 \to C_1$. Furthermore, the maps $g \circ f : C_1 \to C_1$ must extend the identity on some dense open. However, since curves are separated and reduced there is a unique extension of this map so $g \circ f = \mathrm{id}_{C_1}$ and likewise $f \circ g = \mathrm{id}_{C_2}$. □

Theorem 1.3.6. If k is perfect then there exists a unique normal curve in each birational equivalence class of curves.

Proof. It suffices to show existence. Given a curve X, we consider the projective closure $X \hookrightarrow \overline{X}$ which is birational and $\overline{X} \to \mathrm{Spec}\,(k)$ is proper. Then take the normalization $\overline{X}^{\nu} \to \overline{X}$ which remains proper over $\mathrm{Spec}\,(k)$ and is birational. Then $\overline{X}^{\nu}$ is regular and thus smooth over k since k is perfect and $\overline{X}^{\nu} \to X$ is birational. □

1.4 Lemmas

Lemma 1.4.1. Let A be a Noetherian domain and $\mathfrak{p} = (\varpi)$ a principal prime. Then any $f \in \mathfrak{p}$ can be written as $f = t\varpi^k$ for $f \in A \setminus \mathfrak{p}$.

Proof. From Krull intersection,

$$\bigcap_{n \ge 0}^{\infty} \mathfrak{p}^n = (0)$$

so there is some n such that $f \in \mathfrak{p}^n \setminus \mathfrak{p}^{n+1}$. Thus $f = t\varpi^n$ for some $f \in A$ but if $t \in \mathfrak{p}$ then $f \in \mathfrak{p}^{n+1}$ so the result follows. □

Lemma 1.4.2. Consider a closed subset $Y \subset X$ and an open $U \subset X$ with $U \cap Z \neq \varnothing$. Then $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Y \cap U, U)$.

Proof. Consider a chain of irreducible $Z_i \supsetneq Z_{i+1}$ with $Z_0 \subset Y$. I claim that $Z_i \mapsto Z_i \cap U$ and $Z_i \mapsto \overline{Z_i}$ are inverse functions giving a bijection between closed irreducible chains in X with final terms contained in Y and closed irreducible chains in U with final term contained in $Y \cap U$. Note, if $Z_i \subset Y \cap U$ then $\overline{Z_i} \subset Y$ since Y is closed in X.

First, $\overline{Z_i \cap U} \subset Z_i$ and is closed in X. Then $\overline{Z_i \cap U} \cup U^C \supset Z_i$ so because Z_i is irreducible $\overline{Z_i \cap U} = Z_i$ since by assumption $Z_i \not\subset U^C$. Conversely, if $Z_i \subset U$ is a closed irreducible subset then $\overline{Z_i}$ is closed and irreducible in X and $Z_i \subset \overline{Z_i} \cap U$ but $Z_i = C \cap U$ for closed $C \subset X$ so $Z_i \subset C$ and thus $\overline{Z_i} \subset C$ so $\overline{Z_i} \cap U \subset C \cap U = Z_i$ meaning $Z_i = \overline{Z_i} \cap U$. Thus we have shown these operations are inverse to each other.

Finally, if $Z_i \cap U - Z_{i+1} \cap U$ then $\overline{Z_i \cap U} = \overline{Z_i \cap U}$ so $Z_i = Z_{i+1}$ so the chain does not degenerate. Likewise, if $\overline{Z_i} = \overline{Z_{i+1}}$ then $\overline{Z_i} \cap U = \overline{Z_{i+1}} \cap U$ so $Z_i = Z_{i+1}$. Therefore, we get a length-preserving bijection between the chains defining $\mathrm{codim}\,(Y, X)$ and $\mathrm{codim}\,(Y \cap U, U)$. □

Lemma 1.4.3. Let $Z \subset X$ be a closed irreducible subset with generic point $\eta \in Z$. Then $\mathrm{codim}\,(Z, X) = \dim \mathcal{O}_{X,\eta}$.

Proof. Take affine open neighborhood $\eta \in U = \mathrm{Spec}\,(A) \subset X$. Then for $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponding to η we get $A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$. However, $\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U)$ and $Z \cap U = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Therefore,

$$\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U) = \mathbf{ht}\,(\mathfrak{p}) = \dim A_{\mathfrak{p}} = \dim \mathcal{O}_{X,\eta}$$

□

Lemma 1.4.4. Let X be a Noetherian scheme then the nonreduced locus,

$$Z = \{x \in X \mid \text{nilrad}\left(\mathcal{O}_{X,x}\right) \neq 0\}$$

is closed.

Proof. The subsheaf $\mathcal{N} \subset \mathcal{O}_X$ is coherent since X is Noetherian. Thus $Z = \text{Supp}_{\mathcal{O}_X}(\mathcal{N})$ is closed and $\mathcal{N}_x = \text{nilrad}\left(\mathcal{O}_X x\right)$. Locally, on $U = \text{Spec}\,(A)$ we have $\mathcal{N}|_U = \widetilde{\text{nilrad}\,(\text{A})}$ and $\text{nilrad}\,(A)$ is a f.g. A-module since A is Noetherian so,

$$\text{Supp}_{\mathcal{O}_X}(\mathcal{N}) \cap U = \text{Supp}_A\left(\text{nilrad}\,(A)\right) = V(\text{Ann}_A\left(\text{nilrad}\,(A)\right)$$

is closed in $\text{Spec}\,(A)$. ∎

Lemma 1.4.5. Let X be a Noetherian scheme then X has finitely many irreducible components.

Proof. First let $X = \text{Spec}\,(A)$ for a Noetherian ring A. Then the irreducible components of A correspond to minimal primes $\mathfrak{p} \in \text{Spec}\,(A)$. Then $\dim A_{\mathfrak{p}} = 0$ and $A_{\mathfrak{p}}$ is Noetherian so $A_{\mathfrak{p}}$ is Artinian. $A_{\mathfrak{p}}$ must have some associated prime so $\text{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}\}$. By [**?**, Tag 05BZ], then $\text{Ass}_A(A) \cap \text{Spec}\,(\mathbb{A}_{\mathfrak{p}}) = \text{Ass}_{\mathbb{A}_{\mathfrak{p}}}(\mathbb{A}_{\mathfrak{p}}) = \{\mathfrak{p}\}$ so every minimal prime is an associated prime. However, for A Noetherian then A admits a finite composition series so there are finitely many associated primes.

Now let X be a Noetherian scheme. For any affine open $U \subset X$ we have shown that U has finitely many irreducible components. However, since X is quasi-compact there is a finite cover of affine opens and thus X must have finitely many irreducible components. ∎

Lemma 1.4.6. Let X be a Noetherian scheme and Y is the complement of some dense open U. Then $\text{codim}\,(Y, X) \geq 1$.

Proof. It suffices to show that Y does not contain any irreducible component since then any irreducible contained in Y cannot be maximal. Since X is Noetherian, it has finitely many irreducible components Z_i. Then if $Z_j \subset Y$ for some i we would have $Z_i \cap U = \varnothing$ but then,

$$U = \bigcup_{i \neq j} Z_i$$

which is closed so $\overline{U} \subsetneq X$ contradicting our assumption that U is dense. ∎

Lemma 1.4.7. Let X be a Noetherian scheme and $x \in X$ such that $\mathcal{O}_{X,x}$ is a domain. Then there is an affine open neighborhood $x \in U \subset X$ with $U = \text{Spec}\,(A)$ and A is a domain.

Proof. Take any affine open neighborhood $x \in U \subset X$ with $U = \text{Spec}\,(A)$ and $\mathfrak{p} \in \text{Spec}\,(A)$ corresponding to x. Then $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is a domain. Since X is Noetherian then A is Noetherian so it has finitely many minimal primes $\mathfrak{p}_i$ (corresponding to the generic points of irreducible components of U) with $\mathfrak{p}_0 \subset \mathfrak{p}$. Since $A_{\mathfrak{p}}$ is a domain, it has a unique minimal prime and thus $\mathfrak{p}_0$ is the only minimal prime contained in $\mathfrak{p}$ (geometrically $A_{\mathfrak{p}}$ being a domain corresponds to the fact that $\mathfrak{p}$ is the generic point of a generically reduced irreducible subset which lies in only one irreducible component)

Now for any $i \neq 0$ take $f_i \in \mathfrak{p} \setminus \mathfrak{p}_0$. This is always possible else $\mathfrak{p} \subset \mathfrak{p}_0$ contradicting the minimality of $\mathfrak{p}_0$. If $f \notin \mathfrak{q}$ then $\mathfrak{q} \not\supset \mathfrak{p}_i$ for any $i \neq 0$ so $\mathfrak{q} \supset \mathfrak{p}_0$ since it must lie above some minimal prime. Thus $\text{nilrad}\,(A_f) = \mathfrak{p}_0 A_f$ is prime and $f \notin \mathfrak{p}$ since else $\mathfrak{p} \supset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ which is impossible since $\mathfrak{p} \not\supset \mathfrak{p}_i$ for any i. Now we know that $\text{nilrad}\,(A_\mathfrak{p}) = 0$ and A_f is Noetherian so $\text{nilrad}\,(A_\mathfrak{p})$ is finitely generated. Thus, there is some $g \notin \mathfrak{p}$ such that $\text{nilrad}\,(A_{fg}) = (\text{nilrad}\,(A_f))_g = 0$. Thus A_{fg} is a domain since $\text{nilrad}\,(A_{fg}) = (0)$ and is prime and $\mathfrak{p} \in A_{fg}$ because $fg \notin \mathfrak{p}$. Therefore, $x \in \text{Spec}\,(A_{fg}) \subset U$ is an affine open satisfying the requirements. □

Remark. This does not imply that X is integral if $\mathcal{O}_{X,x}$ is a domain for each $x \in X$ (which is false, consider $\text{Spec}\,(k \times k)$) because it only shows there is an integral cover of X not that $\mathcal{O}_X(U)$ is a domain for each U.

Example 1.4.8. Let $X = \text{Spec}\,(k[x,y]/(xy,y^2))$. Then for the bad point $\mathfrak{p} = (x,y)$ we have $\text{nilrad}\,(\mathcal{O}_{X,\mathfrak{p}}) = (y)$. Away from the bad point, say $\mathfrak{p} = (x-1,y)$ we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left(k[x]_{(x-1)}\right)$ so $\text{nilrad}\,(\mathcal{O}_{X,\mathfrak{p}}) = (0)$. Furthermore, at the generic point $\mathfrak{p} = (y)$, we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\,(k(x))$ so $\text{nilrad}\,(\mathcal{O}_{X,\mathfrak{p}}) = (0)$.

Example 1.4.9. Consider $X = \text{Spec}\,(k[x,y,z]/(yz))$ which is the union of the x-y and x-z planes. Consider the generic point of the z-axis $\mathfrak{p} = (x,y)$ then $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left(k[x,z]_{(x)}\right)$ is a domain since the z-axis only lies in one irreducible component. However, at the generic point of the x-axis, $\mathfrak{p} = (y,z)$ we get $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left((k[x,y,z]/(yz))_{(y,z)}\right)$ has zero divisors $yz = 0$ so is not a domain since the x-axis lives in two irreducible components.

1.5 Reflexive Sheaves (WIP)

Definition 1.5.1. Recall the dual of a $\mathcal{O}_X$ module $\mathscr{F}$ is the sheaf $\mathscr{F}^\vee = \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. We say that a coherent $\mathcal{O}_X$-module $\mathscr{F}$ is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism.

Lemma 1.5.2. Let X be an integral locally Noetherian scheme and $\mathscr{F}, \mathscr{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathscr{G}$ is reflexive then $\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is reflexive.

Proof. See [**?**, Tag 0AY4]. □

In particular, since $\mathcal{O}_X$ is clearly reflexive, this lemma shows that for any coherent $\mathcal{O}_X$-module then $\mathscr{F}^\vee$ is a reflexive coherent sheaf. We say the map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ gives the reflexive hull $\mathscr{F}^{\vee\vee}$ of $\mathscr{F}$.

Definition 1.5.3. Let $\mathcal{R}$ be the full subcategory $\mathfrak{Coh}\,(\mathcal{O}_X)$ of coherent reflexive $\mathcal{O}_X$-modules. $\mathcal{R}$ is an additive category and in fact has all kernels and cokernels defined by taking reflexive hulls of the sheaf kernel and cokernel. Furthermore, $\mathcal{R}$ inherits a monoidal structure from the tensor product defined using the reflexive hull as follows,

$$\mathscr{F} \otimes_{\mathcal{R}} \mathscr{G} = (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$$

Finally, we define $\text{RPic}\,(X)$ to be group of constant rank one reflexives induced by the monoidal structure on $\mathcal{R}$. Explicitly, $\text{RPic}\,(X)$ is the group of isomorphism classes of constant rank one reflexive coherent $\mathcal{O}_X$-modules with multiplication $(\mathscr{F}, \mathscr{G}) \mapsto (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$ and inverse $\mathscr{F} \mapsto \mathscr{F}^\vee$.

The importance of reflexive sheaves derives from their correspondence to Weil divisors. Here we let X be a normal integral separated Noetherian scheme.

Proposition 1.5.4. If D is a Weil divisor then $\mathcal{O}_X(D)$ is reflexive of constant rank one.

Proof. (CITE OR DO). □

Theorem 1.5.5. Let X be a normal integral separated Noetherian scheme. There is an isomorphism of groups $\mathrm{Cl}\,(X) \xrightarrow{\sim} \mathrm{RPic}\,(X)$ defined by $D \mapsto \mathcal{O}_X(D)$.

Proof. (DO OR CITE) □

We summarize the important results as follows.

Theorem 1.5.6. Let X be a Noetherian normal integral scheme. Then for any Weil divisors D, E,

(a) $\mathcal{O}_X(D+E) = (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee}$

(b) $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\vee}$

(c) $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(E)) = \mathcal{O}_X(E-D)$

(d) if E is Cartier then $\mathcal{O}_X(D+E) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$

Proof. (DO OR CITE) □

Finally, we have a result which controls when the dualizing sheaf can be expressed in terms of a divisor.

Proposition 1.5.7. Let X be a projective variety over k. Then,

(a) if X is normal then its dualizing sheaf ω_X is reflexive of rank 1 and thus X admits a canonical divisor K_X s.t. $\omega_X = \mathcal{O}_X(K_X)$

(b) if X is Gorenstein then ω_X is an invertible module so K_X is Cartier.

Proof. (FIND CITATION OR DO). □