Mathematics GU4044 Representations of Finite Groups
Assignment # 7

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Problem 1.

Let G be a finite group and V and W be irreducible representations of G which characters χ_V and χ_W.

(i) Let,

$$F_{V,\chi_W} = \sum_{h \in G} \chi_W(h) \rho_V(h)$$

I claim that F is a G-morphism,

$$\rho_V(g)^-1 \circ F_{V,\chi_W} \circ \rho_V(g) = \sum_{h \in G} \chi_W(h) \rho_V(g^{-1}hg) = \sum_{h' \in G} \chi_W(gh'g^{-1}) \rho_V(h') = F_{V,\chi_W}$$

because χ_W is a class function. Therefore, by Schur's lemma, $F_{V,\chi_W} = t \cdot \mathrm{id}$ because V is irreducible. Taking the trace,

$$t \cdot \dim V = \mathrm{Tr}\, F_{V,\chi_W} = \sum_{g \in G} \chi_W(g) \mathrm{Tr}\, \rho_V(g) = \sum_{g \in G} \chi_W(g) \chi_V(g) = \#(G) \left\langle \chi_W, \overline{\chi_V} \right\rangle$$

and thus

$$F_{V,\chi_W} = \frac{\#(G) \left\langle \chi_W, \overline{\chi_V} \right\rangle}{\dim V} \mathrm{id}$$

Define,

$$e_W = \frac{\dim W}{\#(G)}$$

and then,

$$F_{V,e_W} = \sum_{g \in G} \frac{\dim W}{\#(G)} \chi_W(g) \rho_V(g) = \frac{\dim W}{\#(G)} \cdot \frac{\#(G) \left\langle \chi_W, \overline{\chi_V} \right\rangle}{\dim V} \mathrm{id} = \frac{\dim W}{\dim V} \left\langle \chi_W, \overline{\chi_V} \right\rangle \cdot \mathrm{id}$$

However, if $V \cong W^*$ then $\dim V = \dim W^* = \dim W$ and we know $\left\langle \chi_W, \overline{chi_V} \right\rangle = 1$ so $F_{V,e_W} = \mathrm{id}$. If $V \ncong W^*$ then $\left\langle \chi_W, \overline{\chi_V} \right\rangle = 0$ so $F_{V,e_W} = 0$. Thus,

$$F_{V,e_W} = \begin{cases} \mathrm{id} & V \cong W^* \\ 0 & V \ncong W^* \end{cases}$$

(ii) From the above,

$$F_{V,\overline{\chi_W}} = \sum_{g \in G} \overline{\chi_W}(g)\rho_V(g) = \sum_{h \in G} \chi_W(h)\rho_V(h^{-1}) = \frac{\#(G)\left\langle \overline{\chi_W}, \overline{\chi_V} \right\rangle}{\dim V} \cdot \mathrm{id}$$

Multipying by $\rho_V(g)$,

$$\sum_{h \in G} \chi_W(h)\rho_V(h^{-1}g) = \frac{\#(G)\left\langle \overline{\chi_W}, \overline{\chi_V} \right\rangle}{\dim V} \cdot \rho_V(g)$$

then taking the trace,

$$(\chi_W * \chi_V)(g) = \sum_{h \in G} \chi_W(h)\chi_V(h^{-1}g) = \frac{\#(G)\left\langle \overline{\chi_W}, \overline{\chi_V} \right\rangle}{\dim V} \cdot \chi_V(g) = \frac{\#(G)\left\langle \chi_W, \chi_V \right\rangle}{\dim V} \cdot \chi_V(g)$$

(iii) Let $V_1, \cdots, V_h$ be the irreducible representations of G up to isomorphism with $dimV_i = d_i$. For each i, we define $e_i \in L^2(G)$ by,

$$e_i = e_{V_i^*} = \frac{d_i}{\#(G)}\chi_{V_i^*} = \frac{d_i}{\#(G)}\overline{\chi_{V_i}}$$

Then using the above result,

$$e_i * e_j = \frac{d_i d_j}{\#(G)^2}\chi_{V_i^*} * \chi_{V_j^*} = \frac{d_i d_j}{\#(G)^2} \cdot \frac{\#(G)\left\langle \chi_{V_i}, \chi_{V_j} \right\rangle}{\dim V_i} \cdot \chi_V = \frac{d_j}{\#(G)}\left\langle \chi_{V_i}, \chi_{V_j} \right\rangle \cdot \chi_{V_j}$$

If $i \neq j$ then by definition $V_i \ncong V_j$ so $\left\langle \chi_{V_i}, \chi_{V_j} \right\rangle = 0$ and thus $e_i * e_j = 0$. Furthermore, if $i = j$ then since V_i is irreducible, $\left\langle \chi_{V_i}, \chi_{V_i} \right\rangle = 1$. Thus,

$$e_i * e_i = \frac{d_i}{\#(G)}\chi_{V_i} = e_i$$

so in summary,

$$e_i * e_j = \begin{cases} e_i & i = j \\ 0 & i \neq j \end{cases}$$

Furthermore, since the regular representation contains every irreducible G-representation with multiplicty d_i, we know that,

$$\chi_{reg} = d_1 \cdot \chi_{V_1} + \cdots + d_h \cdot \chi_{V_h} = \#(G)\left(e_1 + \cdots + e_h\right)$$

However,

$$\chi_{reg}(g) = \begin{cases} \#(G) & g = e \\ 0 & g \neq e \end{cases}$$

and therefore, $e_1 + \cdots + e_h = \delta_e$.

Problem 2.

(a) Let G_1 and G_2 be finite abelian groups and $f : G_1 \to G_2$ is a homomorphism. Let $\chi \in \hat{G}_2$ then χ is a homomorphism so $\chi \circ f$ is a homomorphism since it is the composition of homomorphisms. Therefore, $f^*(\chi) = \chi \circ f \in \hat{G}_1$. Furthermore, let $\chi_1, \chi_2 \in \hat{G}_2$ then for $g \in G_1$ we have,

$$f^*(\chi_1 \cdot \chi_2)(g) = (\chi_1 \cdot \chi_2) \circ f(g) = \chi_1(f(g))\chi_2(f(g)) = (f^*(\chi_1)(g))(f^*(\chi_2)(g)) = (f^*(\chi_1) \cdot f^*(\chi_2))(g)$$

Therefore,

$$f^*(\chi_1 \cdot \chi_2) = f^*(\chi_1) \cdot f^*(\chi_2)$$

so f^* is a homomorphism.

(b) Let G_1, G_2, G_3 be three finite abelian groups. Let $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ be homomorphisms. Consider the map $(f_2 \circ f_1)^* : \hat{G}_3 \to \hat{G}_1$,

$$(f_2 \circ f_1)^*(\chi) = \chi \circ (f_2 \circ f_1) = (\chi \circ f_2) \circ f_1 = (f_2^*(\chi)) \circ f_1 = f_1^*(f_2^*(\chi)) = (f_1^* \circ f_2^*)(\chi)$$

Thus,

$$(f_2 \circ f_1)^* = f_1^* \circ f_2^*$$

In summary, the map $G \mapsto \hat{G}$ and $f \to f^*$ is a contravariant endofunctor on the category of finite abelian groups. This endofunctor is a special case of the contravariant hom functor $\mathrm{Hom}\,(-, \mathbb{C})$. Explicitly, $\hat{G} = \mathrm{Hom}\,(G, \mathbb{C})$ and given a map $f : G_1 \to G_2$ we have a map $f^* : \mathrm{Hom}\,(G_2, \mathbb{C}) \to \mathrm{Hom}\,(G_1, \mathbb{C})$ given by its action of a map $h : G_2 \to \mathbb{C}$ by $f^*(h) = h \circ f : C_1 \to \mathbb{C}$.

(c) Let G be a finite abelian group and let $H \subset G$ be a subgroup with the quotient map $\pi : G \to G/H$. Consider the map $\pi^* : \widehat{G/H} \to \hat{G}$. Suppose that $\pi^*(\chi) = 1$ then $\chi \circ \pi = 1$. Thus, for any $g \in G$ we have $\chi \circ \pi(g) = 1$ so $\chi(gH) = 1$ for any g. However gH enumerates every element of G/H so $\chi = 1$. Thus, π^* is an injection. Next, we consider $\mathrm{Im}(\pi^*)$. If $\chi = \pi^*(\chi') = \chi' \circ \pi$ then for any $h \in H$ we have $\chi(h) = \chi' \circ \pi(h) = \chi'(e_{G/H}) = 1$. Conversely, if $\chi(h) = 1$ for any $h \in H$ then if g_1 and g_2 lie in the same coset i.e. $g_1 H = g_2 H$ so $g_1 = g_2 h$ and thus

$$\chi(g_1) = \chi(g_2) \cdot \chi(h) \implies \chi(g_1) = \chi(g_2)$$

Thus, χ is constant on cosets so it decends to a map χ' on the quotient G/H such that $\chi = \chi' \circ \pi$. Therefore, the image π^* is equivalent to the set of characters which are trivial on H.

(d) Let $\iota : H \to G$ be the inclusion map. Suppose that $\chi \in \ker \iota^*$ then $\iota^*(\chi) = \chi \circ \iota = 1$. Thus, for any $h \in H$ we have $\chi \circ \iota(h) = \chi(h) = \iota^*(\chi)(h) = 1$. Conversely, suppose that $\chi(h) = 1$ for every $h \in H$ then $\iota^*(\chi)(h) = \chi \circ \iota(h) = \chi(h) = 1$. Thus, $\iota^*(\chi)$ is the trivial character in $\hat{H}$ so $\chi \in \ker \iota^*$. Thus, $\ker \iota^*$ is the set of characters which are trivial on H and thus $\ker \iota^* = \mathrm{Im}(\pi^*)$. Since π^* is injective we know that π^* is a bijection onto its image so,

$$\#(\widehat{G/H}) = \#(\mathrm{Im}(\pi^*))$$

and therefore,

$$\#(\ker \iota^*) = \#(\mathrm{Im}(\pi^*)) = \#(\widehat{G/H}) = \#(G/H) = \#(G)/\#(H)$$

because $\#(\hat{K}) = \#(K)$.

(e) Since $\iota^* : \hat{G} \to \hat{H}$ is a homomorphism, we know that $\text{Im}(\iota^*) \cong \hat{G}/\ker \iota^*$ and therefore,

$$\#(\text{Im}(\iota^*)) = \#(\hat{G}/\ker \iota^*) = \#(\hat{G})/\#(\ker \iota^*) = \#(G)\frac{\#(H)}{\#(G)} = \#(H) = \#(\hat{H})$$

and therefore ι is a surjection since $\#(\text{Im}(\iota)) = \#(\hat{H})$ but $\text{Im}(\iota) \subset \hat{H}$ so $\text{Im}(\iota) = \hat{H}$.

Problem 3.

Let G be abelian and define the map $ev : G \to \hat{\hat{G}}$ given by,

$$\text{ev}(g)(\chi) = \chi(g)$$

Let $g, h \in G$ then,

$$\text{ev}(gh)(\chi) = \chi(gh) = \chi(g) \cdot \chi(h) = \text{ev}(g)(\chi) \cdot \text{ev}(h)(\chi) = (\text{ev}(g) \cdot \text{ev}(h))(\chi)$$

so ev is a homomorphism. Furthermore, suppose $\text{ev}(g) = \hat{\hat{e}}$, that is, $\text{ev}(g)(\chi) = 1$ for every character χ. Then, $\chi(g) = 1$ for each character χ. If every character is trivial on g then it is also triival on $\langle g \rangle$ by multiplicativity. Therefore, $\chi(g)$ decends to the quotient $G/\langle g \rangle$ so $\hat{G} \cong \widehat{G/\langle g \rangle}$ which contradicts the fact that

$$\#(G) = \#(\hat{G}) = \#(\widehat{G/\langle g \rangle}) = \#(G/\langle g \rangle) = \#(G)/\#(\langle g \rangle)$$

unless $\#(\langle g \rangle) = 1$. Thus, $g = e$ which implies that ev is injective. However, $\#(G) = \#(\hat{G}) = \#(\hat{\hat{G}})$ so ev must also be a surjection. Thus, $\text{ev} : G \to \hat{\hat{G}}$ is a isomorphism.