Lecture 4

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1 Projective Space

Projective space is the space of lines through the origin in some linear space. For example, real projective space, $\mathbb{RP}^n$, is the space of lines through the origin in $\mathbb{R}^{n+1}$.

2 Projective Elliptic Curves

We homogenize the defining equation of an elliptic curve,

$$y^2 = x^3 + ax + b$$

to get,

$$Y^2 Z = X^3 + aXZ^2 + bZ^3$$

which is a homogeneous polynomial in two variables which therefore naturally lives in $\mathbb{CP}^2$. The elliptic curve should be defined as the projective algebraic set,

$$E = \{[X : Y : Z] \mid Y^2 Z = X^3 + aXZ^2 + bZ^3\} \subset \mathbb{CP}^2$$

We call this a compactification or projectivization of the affine curve,

$$y^2 = x^3 + ax + b$$

which we need to justify. Why is this new curve "the same" as the original affine curve up to completion by adding "points at infinity". Consider the decomposition,

$$\mathbb{CP}^2 = \{Z \neq 0\} \cup \{Z = 0\} = \mathbb{C}^2 \cup \mathbb{CP}^1$$

Now,

$$\begin{aligned} E \cap \mathbb{C}^2 = E \cap \{Z \neq 0\} &= \{[X : Y : Z] \mid Y^2 Z = X^3 + aXZ^2 + bZ^3 \text{ and } Z \neq 0\} \\ &= \{[x : y : 1] \mid y^2 = x^3 + ax + b\} \cong \{(x, y) \mid y^2 = x^3 + ax + b\} \subset \mathbb{C}^2 \end{aligned}$$

which corresponds to setting $Z = 1$ recovering the original equation. This justifies the process of homogenizing our affine equation since when we set $Z = 1$ we recover the original form. Furthermore, the curve E intersects the Riemann sphere at infinity $\{Z = 0\} = \mathbb{CP}^1 \subset \mathbb{CP}^2$ in the following way,

$$E \cap \mathbb{CP}^1 = E \cap \{Z = 0\} = \{[X : Y : 0] \mid X^3 = 0\} = \{[0 : Y : 0]\} = \{[0 : 1 : 0]\}$$

so E intersects infinity in a single point which we call "the point at infinity on the elliptic curve E".

3 Projectivizing the Weierstrass $\wp$ function

Recall we wanted to use the Weierstrass $\wp$ function to create a map,

$$\mathbb{C}/\Lambda \xrightarrow{\Phi} \mathbb{C}^2$$

defined by $z \mapsto (\wp(z), \wp'(z)/2)$ which satisfies the differential equation,

$$(\wp'(z)/2)^2 = \wp(z)^3 - g_2\wp(z) - g_3$$

and therefore the image of Φ is contained in the affine elliptic curve,

$$E = \{(x, y) \mid y^2 = x^3 - g_2x - g_3\}$$

However, there is a problem. At $z = 0$, the $\wp$ function has a pole and therefore the map $\mathbb{C}/\Lambda \to \mathbb{C}^2$ is not well-defined. We want to extend $\mathbb{C}^2$ by adding "points at infinity" to take care of the poles of $\wp$. We do this by extending $\mathbb{C}^2$ to $\mathbb{CP}^2$. Then we can define a map,

$$\mathbb{C}/\Lambda \xrightarrow{\Phi} \mathbb{CP}^2$$

given by $z \mapsto [z^3\wp(z) : z^3\wp'(z)/2 : z^3]$. Notice that when $z \neq 0$,

$$[z^3\wp(z) : z^3\wp'(z)/2 : z^3] = [\wp(z) : \wp'(z)/2 : 1] \in \mathbb{C}^2 \subset \mathbb{CP}^2$$

so this new map agrees with our prior definition hitting the affine patch $\mathbb{C}^2 \subset \mathbb{CP}^2$. However, when $z = 0$, the power of z^3 cancels the poles of $\wp(z)$ and $\wp'(z)$. To see this recall the expansions,

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)z^{2k}G_{k+1}(\Lambda)$$

Therefore,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty}(2k+1)(2k)z^{2k-1}G_{k+1}(\Lambda)$$

which shows that $z^3\wp(z)$ and $z^3\wp'(z)$ are well-defined as $z \to 0$.