Mathematics GU4044 Representations of Finite Groups
Assignment # 3

Benjamin Church

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Problem 1.

(i) Take an element of $O(2)$ given in matrix form by,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then, we know that $\det M = \pm 1$ so $ad - cb = \pm 1$. Furthermore, $MM^\top = I$ so,

$$MM^\top = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, $a^2 + b^2 = 1$ and $c^2 + d^2$ so (a, b) and (c, d) are points on the unit circle. Therefore, $a = \cos\theta$ and $b = \sin\theta$ and $c = \cos\theta'$ and $d = \sin\theta'$ for some $\theta, \theta' \in [0, 2\pi)$. However, $ad - bc = \pm 1$ so $\cos\theta \sin\theta' - \sin\theta \cos\theta' = \sin(\theta' - \theta) = \pm 1$. Thus, $\theta' - \theta = (2n + 1)\pi$ for $n \in \mathbb{Z}$. Then, $\sin\theta' = \sin(\theta + (2n + 1)\pi) = \pm \sin\theta$ and $\cos\theta' = \cos(\theta + (2n + 1)\pi) = \mp 1$. Therefore,

$$M = \begin{pmatrix} \cos\theta & \pm \sin\theta \\ \sin\theta & \mp \cos\theta \end{pmatrix} = B(\theta) \text{ or } A(\theta)$$

(ii) First,

$$\begin{aligned} A_{\theta_1} A_{\theta_2} &= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & -\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \\ \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} = A_{\theta_1 + \theta_2} \end{aligned}$$

Therefore, $A_\theta^2 = A_{2\theta}$ and $A_\theta A_{-\theta} = A_0 = I$ so $A_\theta^{-1} = A_{-\theta}$. Furthermore, define the matrix,

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then clearly $R^2 = I$ and,

$$A_\theta R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = B_\theta$$

Next,

$$B_{\theta_1}B_{\theta_2} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta_1-\theta_2) & -\sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) \end{pmatrix} = A_{\theta_1-\theta_2}$$

so $B_\theta^2 = A_0 = I$ and thus $B_\theta^{-1} = B_\theta$. Now, $R^{-1}A_\theta R = RA_\theta R = RB_\theta$ and,

$$RB_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A_{-\theta}$$

Therefore, $R^{-1}A_\theta R = RB_\theta = A_{-\theta}$. Thus, $B_\theta B_\theta = A_\theta R A_\theta R = A_\theta A_{-\theta} = I$ since $R = R^{-1}$.

(iii) As calculated above, $A_{\theta_1}A_{\theta_2} = A_{\theta_1+\theta_2}$ and $B_{\theta_1}B_{\theta_2} = A_{\theta_1-\theta_2}$. Then,

$$A_{\theta_1}B_{\theta_2} = A_{\theta_1}A_{\theta_2}R = A_{\theta_1+\theta_2}R = B_{\theta_1+\theta_2}$$

Likewise,

$$B_{\theta_1}A_{\theta_2} = A_{\theta_1}RA_{\theta_2} = A_{\theta_1}A_{-\theta_2}R = A_{\theta_1-\theta_2}R = B_{\theta_1-\theta_2}$$

Finally,

$$A_\theta R A_\theta^{-1} = A_\theta R A_{-\theta} = A_\theta A_\theta R = A_{2\theta}R = B_{2\theta}$$

(iv) Let $\mathbf{u}_1 = A_{\theta/2}\mathbf{e}_1$ and $\mathbf{u}_2 = A_{\theta/2}\mathbf{e}_2$. Since $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis and $A_\theta^{-1} = A_{-\theta} = A_\theta^\top$ so A is an orthogonal matrix and thus $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis. Then,

$$B_\theta\mathbf{u}_1 = A_{\theta/2}RA_{\theta/2}^{-1}\mathbf{u}_1 = A_{\theta/2}R\mathbf{e}_1 = A_{\theta/2}e_1 = \mathbf{u}_1$$

Likewise,

$$B_\theta\mathbf{u}_2 = A_{\theta/2}RA_{\theta/2}^{-1}\mathbf{u}_2 = A_{\theta/2}R\mathbf{e}_2 = -A_{\theta/2}e_2 = -\mathbf{u}_2$$

Therefore, B_θ represents a reflection. Consider the eigenvalues of A_θ which must satisfy,

$$\det(I\lambda - A_\theta) = \det\begin{pmatrix} \lambda-\cos\theta & \sin\theta \\ -\sin\theta & \lambda-\cos\theta \end{pmatrix} = (\lambda-\cos\theta)^2 + \sin^2\theta = \lambda^2 - (2\cos\theta)\lambda + 1 = 0$$

Therefore,

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

If λ is to be real then $\cos^2\theta - 1 \geq 0$ so $\cos\theta = \pm 1$. Therefore, only rotations by $\theta = 0, \pi$ have real eigenvectors. This corresponds to either the identity transformation which fixes every vector or the transformation which reflects through the origin. In the second case, any vector v is taken to $-v$ so every vector is an eigenvector.

Problem 2.

Let $A \in \mathbb{M}_n(\mathbb{C})$. Then $(A^*)_{ij} = \bar{A}_{ji}$. Therefore,

$$\text{Tr}\, A^* = \sum_{i=1}^n (A^*)_{ii} = \sum_{i=1}^n \bar{A}_{ii} = \overline{\sum_{i=1}^n A_{ii}} = \overline{\text{Tr}\, A}$$

Problem 3.

Given any $\alpha \in \mathbb{C}$ such that $\alpha\bar{\alpha} = 1$. Then take $z = \alpha^{1/n}$. In particular, since $\alpha \in S^1$ we can take $\alpha = e^{i\theta}$ then take $z = e^{i\theta/n}$. Now, consider the diagonal matrix, $U = \text{diag}(z, \cdots, z)$. This matrix is unitary because,

$$U^* = \text{diag}(\bar{z}, \cdots, \bar{z}) = \text{diag}(\tfrac{1}{z}, \cdots, \tfrac{1}{z}) = U^{-1}$$

However, $\det U = z^n = (e^{i\theta/n})^n = e^{i\theta} = z$.

Problem 4.

Take any $U \in SU(2)$. For $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ we can write,

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \text{and thus} \qquad U^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

Using the identity, $UU^* = I$, we have,

$$UU^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} + \beta\bar{\beta} & \alpha\bar{\gamma} + \beta\bar{\delta} \\ \gamma\bar{\alpha} + \delta\bar{\beta} & \gamma\bar{\gamma} + \delta\bar{\delta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ and $\gamma\bar{\gamma} + \delta\bar{\delta} = 1$ and $\alpha\bar{\gamma} + \beta\bar{\delta} = 0$ and the last equality follows from the conugate of the previous. If $\alpha \neq 0$ then write $\delta = \bar{\alpha}r$ so $\alpha\bar{\delta} + \beta\bar{\alpha}\bar{r} = 0$ then $\gamma = -\bar{\beta}r$. However, $U \in SU(2)$ so $\det U = 1$ and therefore, $\alpha\delta - \beta\gamma = |\alpha|^2 r + |\beta|^2 r = 1$ but $|\alpha|^2 + |\beta|^2 = 1$ so $r = 1$. Thus, $\delta = \bar{\alpha}$ and $\gamma = -\bar{\beta}$.

If $\alpha = 0$ then $|\beta| = 1$ and $\beta\bar{\delta} = 0$ so $\delta = 0$. However, $\det U = 1$ so $-\beta\gamma = 1$ and therefore $\gamma = -\bar{\beta}$. In either case,

$$U = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

where $|z|^2 + |w|^2 = 1$.

Problem 5.

(i) Define the matrix,

$$A = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$$

The characteristic polynomial of A is given by,

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda - \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \lambda - \cos(2\pi/n) \end{pmatrix} \\ &= (\lambda - \cos(2\pi/n))^2 + \sin^2(2\pi/n) = \lambda^2 - (2\cos(2\pi/n))\lambda + 1 \end{aligned}$$

Therefore, the eigenvalues of A are,

$$\lambda = \cos(2\pi/n) \pm \sqrt{\cos^2(2\pi/n) - 1} = \cos(2\pi/n) \pm i\sin(2\pi/n) = e^{\pm i2\pi/n}$$

For these eigenvalues, we can find eigenvectors which span the null spaces of $I\lambda - A$ i.e.

$$(Ie^{i2\pi/n} - A)v = \begin{pmatrix} i\sin(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & i\sin(2\pi/n) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

so we can take $a = i$ and $b = 1$. Likewise,

$$(Ie^{-i2\pi/n} - A)v = \begin{pmatrix} -i\sin(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & -i\sin(2\pi/n) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

so we can take $a = 1$ and $b = i$. Thus, we can take $v_1, v_2 \in \mathbb{C}^2$ given by,

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

which each span a A eigenspace. Define two $\mathbb{Z}/n\mathbb{Z}$-representations, $\rho_0 : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(\mathbb{C}^2)$ given by $\rho(k) = A^k$ and $\rho_1 : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(V_1 \oplus V_2)$ given by $\rho_1(k)(v_1) = e^{2\pi ik/n}v_1$ and $\rho_1(k)(v_2) = e^{-2\pi ik/n}v_2$ where $V_1 = \mathbb{C} \cdot v_1$ and $V_2 = \mathbb{C} \cdot v_2$.

Now, define the $\mathbb{C}$-linear map $F : \mathbb{C}^2 \to V_1 \oplus V_2$ given by $F(v_1) = (v_1, 0)$ and $F(v_2) = (0, v_2)$. Then, for any $v \in \mathbb{C}^2$ we can write $v = c_1v_1 + c_2v_2$ in the basis $\{v_1, v_2\}$. Then,

$$\begin{aligned} F(\rho_0(k)v) &= F(A(c_1v_1 + c_2v_2)) = c_1F(Av_1) + c_2F(Av_2) = c_1F(e^{2\pi ik/n}v_1) + v_2F(e^{-2\pi ik/n}v_2) \\ &= c_1e^{2\pi ik/n}F(v_1) + c_2e^{-2\pi ik/n}F(v_2) = (c_1e^{2\pi ik/n}v_1, 0) + (0, c_2e^{-2\pi ik/n}v_2) \\ &= (c_1e^{2\pi ik/n}v_1, c_2e^{-2\pi ik/n}v_2) \end{aligned}$$

Likewise,

$$\begin{aligned} \rho_1(k)F(v) &= \rho_1(k)(c_1F(v_1) + c_2F(v_2)) = \rho_1(k)(c_1(v_1, 0) + c_2(0, v_2)) \\ &= \rho_1(k)(c_1v_2, c_2v_2) = (c_1e^{2\pi ik/n}v_1, c_2e^{-2\pi ik/n}v_2) \end{aligned}$$

Therefore, $F(\rho_0(k)v) = \rho_1(k)F(v)$ but $F : \mathbb{C}^2 \to V_1 \oplus V_2$ is isomorphic because $V_1 \cap V_2 = \{0\}$ and $V_1 + V_2 = \mathbb{C}^2$. Thus, F is a $\mathbb{Z}/n\mathbb{Z}$-isomorphism.

(ii) For $n > 2$ the eigenvalues $e^{\pm 2\pi i/n}$ are not equal because if $2\pi/n = -2\pi/n + 2\pi k$ for $k \in \mathbb{Z}$ then $k = 2/n$ so $n < 2$. Therefore, the vectors v_1 and v_2 must lie in different eigenspaces (Since A acting on them gives a different value). Furthermore, the eigenspaces must be one-dimensional (because each is nonzero and not the full space which is two-dimensional) so their spanning sets are unique up to a scalar. Therefore, the basis $\{v_1, v_2\}$ of A eigenvectors is unique up to order and scaling. Furthermore, the vectors v_1 and v_2 are not eigenvectors of

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

because $Rv_1 = iv_2$ and $Rv_2 = -iv_1$ but $\{v_1, v_2\}$ are indpendent. Therefore, A and R have no common eigenvectors. However, any nontrivial invariant subspace of the representation of D_n mapping to A and R must be one-dimensional simply because $\mathbb{C}^2$ is two-dimensional. A one-dimensional invariant subspace is exactly equivalent to a common eigenvector which we know A and R do not have. Thus, this representation of D_n is irreducible for $n > 2$.

For $n = 2$, the eigenvalues $e^{\pm 2\pi i/2} = -1$ are equal. For $n = 2$, we have $A = -I$ so any vector is an eigenvector of A. Thus, A and R have a two common eigenvectors e_1 and e_2 so the representation of D_2 is reducible.

Problem 6.

Consider the representation of $D_3 \cong S_3$ generated by the matrices,

$$A = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Furthermore, consider the permutation representation over the subspace $W \subset \mathbb{C}^3$

$$W = \{(t_1, t_2, t_3) \mid t_1 + t_2 + t_3 = 0\}$$

with the group S_3 generated by $\sigma = (123)$ and $\tau = (23)$. Let $w_1 = e_1 - e_2 \in W$ and $w_2 = e_2 - e_3 \in W$ then $w_1' = e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 \in W$ is an eigenvector of $\rho(\tau)$ because,

$$\rho(\tau)(e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3) = e_1 - \tfrac{1}{2}e_3 - \tfrac{1}{2}e_2 = e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3$$

Furthermore, $e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 \notin \text{span}\{e_2 - e_3\}$ so the vectors w_2 and w_1' are independend and therefore form a basis because $\dim W = 2$. However, $\rho(\tau)(w_2) = \rho(\tau)(e_2 - e_3) = e_3 - e_2 = -w_2$. Furthermore,

$$\rho(\sigma)(w_1') = \rho(\sigma)(e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3) = e_2 - \tfrac{1}{2}e_3 - \tfrac{1}{2}e_1 = -\tfrac{1}{2}(e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3) + \tfrac{3}{4}(e_2 - e_3) = -\tfrac{1}{2}w_1' + \tfrac{3}{4}w_2$$

Similarly,

$$\rho(\sigma)(w_2) = \rho(\sigma)(e_2 - e_3) = e_3 - e_1 = -(e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3) - \tfrac{1}{2}(e_2 - e_3) = -w_1' - \tfrac{1}{2}w_2$$

Let $w_2' = \frac{\sqrt{3}}{2}w_2$ then $\rho(\sigma)(w_1') = -\frac{1}{2}w_1' + \frac{\sqrt{3}}{2}w_2'$ and $\rho(\sigma)(w_2') = -\frac{\sqrt{3}}{2}w_1' - \frac{1}{2}w_2'$. Therefore, $\rho(\sigma)$ in the basis $\{w_1', w_2'\}$ is given by the matrix,

$$A = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}2 & -1/2 \end{pmatrix}$$

and likewise, $\rho(\tau)(w_1') = w_1'$ and $\rho(\tau)(w_2') = -w_2'$ so $\rho(\tau)$ is given by the matrix,

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Problem 7.

Let V and W be G-representations and let $F : V \to W$ be a G-morphism. Take any $g \in G$. Take, $v \in \ker F$. Then, $F(v) = 0$ and thus, $\rho_W(g)(F(v)) = F(\rho_V(g)(v)) = 0$ so $\rho_V(g)(v) \in \ker F$. Therefore, $\ker F$ is invariant under the action of $\rho_V(g)$ for any $g \in G$. Therefore, $\ker K$ is a G-invariant subspace of V. Similarly, take $w \in \text{Im}(F)$. Then there exists $v \in V$ such that $F(v) = w$. Therefore, $\rho_W(g)(w) = \rho_W(g)(F(v)) = F(\rho_V(g)(v)) \in \text{Im}(F)$. Therefore, $\rho_V(g)(\text{Im}(K)) \subset \text{Im}(F)$ so $\text{Im}(F)$ is a G-invariant subspace of W.