1 Week 1 Reading

2 Week 1 Meeting Notes

Proposition 2.0.1. Any such A/S is commuative.

Proof. Step -1 reduce to the case S is locally noetherian by spreading out. $\square$

2.1 Duality

Given some abelian scheme A/S there is a dual $A^\vee/S$ of the same dimension. If $S = \mathrm{Spec}\,(\mathcal{C})$ then $H^1(A, \Omega_A)/H^1(A, \mathbb{Z})$ gives a dual which we think of as Line bundles with trivial Chern class (degree zero). This gives a complex manifold and it turns out to be algebraic. We can make sense of this using the exponential sequence,

$$0 \longrightarrow 2\pi\mathbb{Z} \longrightarrow \mathcal{O}_A \xrightarrow{\exp} \mathcal{O}_A^\times \longrightarrow 0$$

giving the map $H^1(A, \mathbb{Z}) \to H^1(A, \mathcal{O}_A)$. We could also use,

$$H^0(X, \Omega_A)^\vee / H_1(A, \mathbb{Z})$$

with the map given by integrating along a homology cycle. We can geometrise the moduli space of line bundles with Picard Schemes.

Apparently if A is a complex torus and A and $A^\vee$ are isogenous then A is algebraic.

2.2 Defining Pic

We consider the functor,

$$(T \to S) \mapsto \mathrm{Pic}\,(X_T)\,/\mathrm{Pic}\,(T)$$

this defines the set of maps,

$$T \to \mathrm{Pic}\,(X/S)$$

Remark. $\mathrm{Pic}\,(T) \to \mathrm{Pic}\,(X_T)$ is injective for $X = A$ because there is a section but also because $\pi_* \mathcal{O}_X = \mathcal{O}_T$.

Theorem 2.2.1 (Grothendieck)**.** If A is zariski locally projective then $\mathrm{Pic}\,(A/S)$ is representable by a scheme locally of finite type.

Remark. According to Sean even if $A \to S$ is not locally projective $\mathrm{Pic}\,(A/S)$ is represented by an algebraic space but a theorem of Raynaud tells you that if an algebraic space is an abelian space then it is actually a scheme.

Proposition 2.2.2. If $S = \mathrm{Spec}\,(k)$ then $T_e\mathrm{Pic}\,(A/S) = H^1(A, \mathcal{O}_A) = \mathrm{Ext}^1_{\mathcal{O}_A}\,(\mathcal{O}_A, \mathcal{O}_A)$.

Remark. It should be true that $e^*\Omega^1_{\mathrm{Pic}(A)/S} = (R^1\pi_*\mathcal{O}_A)^\vee$.

Proposition 2.2.3. $\mathrm{Pic}\,(A/S)$ is smooth.

Proof. Use formal smoothness and deformation theory. Have to use the group structure. Let B be an Artin local ring over $\mathcal{O}_{S,s}$ and an extension $B \hookrightarrow B'$ with square-zero kernel I (probably actually want $\mathfrak{m}I = 0$). Given a line bundle $\mathcal{L}$ on A_B we must show it lifts to $A_{B'}$. There is an obstruction element,

$$H^2(A_s, \mathcal{O}_{A_S}) \otimes_k I$$

$\square$

3 April 15

Let A/k be an abelian variety over a field k. Let $\mathcal{L}$ be a line bundle on A. Define,

$$\Lambda(\mathcal{L}) = \mu^*\mathcal{L} \otimes p_1^*\mathcal{L}^{\otimes -1} \otimes p_2^*\mathcal{L}^{\otimes -1}$$

is a line bundle on $A \times A$ and therefore defines $\Phi_{\mathcal{L}} : A \to \mathrm{Pic}\,(A)$. We need to check that $0 \mapsto 0$ then by rigidity it is automatically a group map.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A \times A \arrow[r] \arrow[d, "\pi_1"] & \mathrm{Pic}\,(A) \times A \arrow[d] \\
e \arrow[r] & A \arrow[r] & \mathrm{Pic}\,(A)
\end{tikzcd}$$

then a direct calculation shows that $P \mapsto \Lambda(\mathcal{L}) \mapsto \mathcal{O}_A$ so it sends e to the trivial bundle in $\mathrm{Pic}\,(A)$.

Theorem 3.0.1. If $\mathcal{L}$ is ample then $\Phi_{\mathcal{L}}$ is finite flat (hence surjective) so an isogeny.

Proof. Suffices to prove that $K_{\mathcal{L}} = \ker \Phi_{\mathcal{L}}$ is quasi-finite (miracle flatness and dimension theory). Let $B = (K_{\mathcal{L}}^{\mathrm{red}})^\circ$ observe that $M = \mathcal{L} \otimes [-1]^*\mathcal{L}$ is ample and $M|_B$ is trivial therefore B is finite. $\square$

Remark. Group surjective maps of smooth groups are flat. Either use generic flatness and translate or use miracle flatness since all fibers isomorphic to kernel so constant dimension.

Definition 3.0.2. Let A/S be an abelian scheme, then a polarization is a group map $\lambda : A \to A^\vee$ such that for all geometric points $\bar{s} \to S$ we have $\lambda_{\bar{s}} = \Phi_{\mathcal{L}}$ for some ample $\mathcal{L}_{\bar{s}} \in \mathrm{Pic}\,(A_{\bar{S}})$.

Remark. Even over a field k we can have polarizations which do not arise from λ because the Λ might live over some field extension. I think étale locally every polarization comes from Λ.

Remark. Fibral flatness implies that φ is finite flat and rigidity says that λ is a group map and hence $A^\vee \cong A/\ker\lambda$.

Remark. Let P be the universal bundle on $A \times A^\vee$ then we get $M = \lambda^* P \in \mathrm{Pic}\,(A \times A)$. Then $\Delta^* M$ defines a line bundle. Now $\Delta^*\Phi_\Lambda^* P = \mathcal{L}^{\otimes -1}$.

Remark. Polarization is the same as a section of the section of the Neron-Severi scheme which is étale. Then the map $\mathrm{Pic}\,(A) \to \mathrm{NS}_A$ étale-locally admits sections so we étale-locally do indeed get a line bundle étale-locally.

Definition 3.0.3. The stack $\mathcal{A}_{g,1}/\mathrm{Spec}\,(\mathbb{Z})$ is the category fivered in groupoids of,

$$(S, \mathcal{A}/S, \lambda : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^\vee)$$

where $\lambda : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^\vee$ is a principal polarization (it is an isomorphism) whose morphisms are Cartesian diagram,

$$\begin{tikzcd}
\mathcal{A}_S \arrow[r] \arrow[d] & \mathcal{A}'_{S'} \arrow[d] \\
S \arrow[r] & S'
\end{tikzcd}$$

therefore,

$$\mathcal{A}_{g,1}(S) = \{(\mathcal{A}, S, \lambda)\}$$

with morphisms $f : A' \to A$ such that,

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "\lambda'"] & A \arrow[d, "\lambda"] \\ A'^\vee & A^\vee \arrow[l, "f^\vee"] \end{tikzcd}$$

commutes. Then we can also define $\mathcal{A}_{g,d,n}/\mathrm{Spec}\,(\mathbb{Z}[1/n])$ whose objects have degree $\sqrt{d}$-polarizations and $\eta : (\mathbb{Z}/n\mathbb{Z})^{\oplus 2g} \xrightarrow{\sim} A[n]$ which is compatible with the Weil pairing up to a scale.

3.1 Deformation Theory

Need to show that $\mathrm{Def}(A, m, e, i)$ is formally smooth. Then $\mathrm{Def}(A, m, e, i) \subset \mathrm{Def}(A)$ is actually an equality then the latter is formally smooth by a trick.

Now we need to deform $m : A \times A \to A$ this is the same as deforming $\Gamma_m \subset A \times A \times A$. Deforming Γ_m has a tangent obstruction theory,

$$H^{i-1}(\Gamma_m, N_{\Gamma_m})$$

Upshot: given (A_0, m_0, e_0, ι_0) over R_0 and given a fixed (A, e) over R we can lift m and i uniquely to get a group structure of R so the deformation theory is formally smooth. Given (A, e) and (A, e') then $(A, e', m', i') \cong (A, e, m, i)$ by rigidity. Thus the defomational theorey is formally smooth and $\mathrm{Def}(A_0)$ and the tangent space is $H^1(A, T_A) \otimes I$.

Problem $S \to \{A/S\}$ does not have effective deformation rings meaning we can lift all the way to a formal scheme but cannot algebrize it. What about $\mathrm{Def}(A, \lambda)$. Consider $I \to R \to R_0$ and (A_0, λ_0) over R_0. We can lift A to R and ask does λ_0 lift,

$$\begin{tikzcd} A \arrow[r, dashed, "\lambda"] \arrow[d] & A^\vee \arrow[d] \\ A_0 \arrow[r, "\lambda_0"] & A_0^\vee \end{tikzcd}$$

If a lift exists, then by rigidity it is unique. Therefore $\mathrm{Def}(A_0, \lambda_0) \subset \mathrm{Def}(A_0)$ but this is a strict containment. There is an obstruction,

$$H^1(\mathcal{A}, T) \otimes I \to H^2(\mathcal{A}, \mathcal{O}_A) \otimes I$$

it turns out this is linear and there is a diagram,

$$\begin{tikzcd} H^1(A, T_A) \otimes I \arrow[r] \arrow[d, "(1\, c(\mathcal{L}))"] & H^2(A, \mathcal{O}_A) \otimes I \\ H^1(A, T_A) \otimes H^1(A, \Omega^1) \otimes I \arrow[r, "\smile"] & H^2(A, T \otimes \Omega^1) \otimes I \arrow[u] \end{tikzcd}$$

where c is the Chern class. This gives $\dim = g(g+1)/2$ and formal deformations are effective so $\mathcal{A}_{g,1}$ is a smooth algebraic stack over $\mathrm{Spec}\,(\mathbb{Z})$ of relative dimension $g(g+1)/2$. There is a universal family $B \to \mathcal{A}_{g,1}$. Over $\mathbb{Z}$ we have $B[n]$ is proper and flat (only étale over $\mathbb{Z}[1/n]$) so its dimension can be checked over $\mathbb{C}$

Remark.
$$\mathrm{Isom}((A,\lambda),(B,\mu))$$
is finite étale over k. It is étale because of rigidity so there is unique lifting of maps. Then we show for $n \ge 3$,
$$\mathrm{Isom}((A,\lambda),(B,\mu)) \hookrightarrow \mathrm{Isom}(A[n],B[n])$$
which is finite because these are finite group schemes.

Remark. Is there a way to do level structure over all of $\mathbb{Z}$? de Jong defined $\mathcal{A}_{g,\Gamma_0(p)}$ over $\mathrm{Spec}\,(\mathbb{Z})$ where we fix a flag in $A[p]$ isotropic for the Weil pairing. But this has complicated geometry.

4 April 28

Let $\mathscr{F} = \varprojlim \mathscr{F}_n$ then,
$$\Omega_{\mathscr{F}/S} = \varprojlim \Omega_{\mathscr{F}_n/S}$$
but the maps,
$$\begin{tikzcd} \mathscr{F}_{n+m} \arrow[r] \arrow[d, "p^m"] & \mathscr{F}_n \arrow[d] \\ \mathscr{F}_{n+m} \arrow[r, equal] & \mathscr{F}_{n+m} \end{tikzcd}$$

4.1 Calculating Extensions

Let E be a supersingular elliptic curve and $H \subset E \times E \times \mathbb{P}^1$ where H is $\alpha_p \times \mathbb{P}^1$ embedded via,
$$\alpha_p \times \mathbb{P}^1 \hookrightarrow E \times E \times \mathbb{P}^1 \quad \text{via} \quad (x,[a:b]) \mapsto (\tfrac{a}{b}x, x, [a:b])$$
There is a unique $\alpha_p \subset E$ kernel of Frobenius. The claim,
$$\dim_k \mathrm{Hom}\,(\alpha_p, (E\times E)/(a,b)\alpha_p) = \begin{cases} 2 & \frac{a}{b} \in \mathbb{F}_{p^2} \\ 1 & \text{else} \end{cases}$$
Therefore, $\ker F_{(E\times E)/H}$ is $\alpha_p \times \alpha_p$ if $\frac{a}{b} \in \mathbb{F}_{p^2}$ and W_2 otherwise. Consider the exact sequences,
$$\begin{tikzcd} 0 \arrow[r] & \alpha_p \arrow[r, "a"] \arrow[d, "\frac{a}{b}"] & E \arrow[r, "F"] \arrow[d, equal] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \alpha_p \arrow[r, "b"] & E \arrow[r, "F"] & E \arrow[r] & 0 \end{tikzcd}$$
Applying $\mathrm{Hom}\,(\alpha_p, -)$ gives a diagram of exact sequences,
$$\begin{tikzcd} \mathrm{Hom}\,(\alpha_p,E) \arrow[r, "\delta"] \arrow[d, equal] & \mathrm{Ext}^1\,(\alpha_p,\alpha_p) \arrow[r] \arrow[d, "\frac{a}{b}\cdot"] & \mathrm{Ext}^1\,(\alpha_p,E) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1\,(\alpha_p,E) \arrow[d, equal] \\ \mathrm{Hom}\,(\alpha_p,E) \arrow[r, "\partial"] & \mathrm{Ext}^1\,(\alpha_p,\alpha_p) \arrow[r] & \mathrm{Ext}^1\,(\alpha_p,E) \arrow[r] & \mathrm{Ext}^1\,(\alpha_p,E) \end{tikzcd}$$

Note that $\operatorname{Hom}(\alpha_p, \alpha_p) = k$ and thus we get a k-structure on $\operatorname{Ext}^1(\alpha_p, \alpha_p)$ but in two different ways the first factor gives the right structure and the second the left structure. What is $\operatorname{Ext}^1(\alpha_p, \alpha_p)$. There are 4-isomorphism classes of groups in the extension (NOT isomorphism classes of extension) these are,

(a) $\alpha_p \times \alpha_p$

(b) α_{p^2}

(c) W_2

(d) $E[p]$

Can argue that the second two span with extensions,

$$0 \longrightarrow \alpha_p \xrightarrow{i} \alpha_{p^2} \xrightarrow{F} \alpha_p \longrightarrow 0$$

and likewise,

$$0 \longrightarrow \alpha_p \xrightarrow{p} W_2 \longrightarrow \alpha_p \longrightarrow 0$$

Acting on the right by a^p and on the left by a amounts to,

$$\begin{tikzcd}
0 \arrow[r] & \alpha_p \arrow[r, "i"] \arrow[d, equal] & \alpha_{p^2} \arrow[r, "a^{-1}F"] \arrow[d, equal] & \alpha_p \arrow[r] \arrow[d, "a^p"] & 0 \\
0 \arrow[r] & \alpha_p \arrow[r, "i"] \arrow[d, "a"] & \alpha_{p^2} \arrow[r, "F"] \arrow[d, equal] & \alpha_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \alpha_p \arrow[r, "ia^{-1}"] & \alpha_{p^2} \arrow[r, "F"] & \alpha_p \arrow[r] & 0
\end{tikzcd}$$

these are not isomorphisms of extensions. But I claim the outside two extensions are isomorphic by,

$$\begin{tikzcd}
0 \arrow[r] & \alpha_p \arrow[r, "i"] \arrow[d, equal] & \alpha_{p^2} \arrow[r, "Fa^{-1}"] \arrow[d, "a^{-1}"] & \alpha_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \alpha_p \arrow[r, "ia^{-1}"] & \alpha_{p^2} \arrow[r, "F"] & \alpha_p \arrow[r] & 0
\end{tikzcd}$$

Therefore,

$$a \cdot [\alpha_{p^2}] = [\alpha_{p^2}] \cdot a^p$$
$$a^p \cdot [W_2] = [W_2] \cdot a$$

Then write in terms of the basis,

$$\delta(f) = \beta \cdot [\alpha_{p^2}] + \gamma \cdot [W_2]$$

and thus by commutativity and moving the left $\frac{a}{b} \cdot -$ action to the right action,

$$\partial(f) = \beta \cdot [\alpha_{p^2}] \cdot \left(\frac{a}{b}\right)^p + \gamma \cdot [W_2] \cdot \left(\frac{a}{b}\right)^{\frac{1}{p}}$$

Therefore,

$$\operatorname{im} \delta \cap \operatorname{im} \partial = \{0\} \iff \frac{a}{b} \notin \mathbb{F}_{p^2}$$

Now consider,

$$0 \longrightarrow \alpha_p \xrightarrow{a\,b} E \times E \longrightarrow (E \times E)/H \longrightarrow 0$$

Then applying $\mathrm{Hom}\,(\alpha_p, -)$ we get,

$$0 \longrightarrow \mathrm{Hom}\,(\alpha_p, \alpha_p) \longrightarrow \mathrm{Hom}\,(\alpha_p, E \times E) \longrightarrow \mathrm{Hom}\,(\alpha_p, (E \times E)/H) \longrightarrow \mathrm{Ext}^1\,(\alpha_p, \alpha_p) \longrightarrow \mathrm{Ext}^1\,(\alpha_p, E \times E)$$

However, the map,

$$\mathrm{Ext}^1\,(\alpha_p, \alpha_p) \to \mathrm{Ext}^1\,(\alpha_p, E \times E)$$

is the pair of maps after δ and ∂ thus is injective if and only if $\mathrm{im}\,\delta \cap \mathrm{im}\,\partial = \{0\}$ so we see in this case that,

$$\dim \mathrm{Hom}\,(\alpha, (E \times E)/H) = \dim \mathrm{Hom}\,(\alpha_p, E \times E) - \dim \mathrm{Hom}\,(\alpha_p, \alpha_p) = 2 - 1 = 1$$

and otherwise the map to $\mathrm{Ext}^1\,(\alpha_p, \alpha_p)$ is surjective (since it is nonzero and the target is 1-dimensional) so we get,

$$\dim \mathrm{Hom}\,(\alpha, (E \times E)/H) = 2$$

4.2 Goal

Generalize this somehow menaing describe all positive dimensional families of PPAVs with constant isogeny class. Huristically,

$$\text{isogeny classes in } \mathcal{A}_{g,1} \iff \text{families of p-div groups}$$

We have constructed,

$$\mathbb{P}^1 \to \mathcal{A}_{g,1}$$

giving by sending $[a, b] \mapsto (E \times E)/H$ which is constant $\overline{\mathbb{F}}_p$-isogeny class of PPAV (by construction) but with nonconstant p-divisible group.

$$\{\mathrm{BT} X \text{ with } \rho : X \dashrightarrow X_0\} \to \{\mathrm{BT}\}$$

4.3 TB Groups

Let Aff_S be the category of affine schemes over a qcqs scheme S.

Definition 4.3.1. A *Tate-Barsotti* group is a sheaf of abelian groups on Aff_S in the fpqc topology such that,

(a) $[p] : \mathscr{F} \to \mathscr{F}$ is a closed immersion

(b) $\mathscr{F}_n = \mathscr{F}/[p]^n\mathscr{F}$ is a finite flat group scheme over S (taking the fppf quotient)

(c) $\mathscr{F} \xrightarrow{\sim} \varprojlim \mathscr{F}/[p]^n\mathscr{F}$

Proposition 4.3.2. $\mathscr{F}$ is representable because it is the inverse limit of affine morphisms.

Example 4.3.3. Some TB groups,

(a) $\underline{\mathbb{Z}_{p,S}} = \varprojlim_n \underline{\mathbb{Z}/p^n\mathbb{Z}}_S$ which represents continuous maps to $\mathbb{Z}_p$ with the p-adic topology

(b) $T_p\mu_{p^\infty,S} = \varprojlim_n \mu_{p^n}$

(c) If A/S is an abelian scheme then,

$$T_pA = \varprojlim A[p^n]$$

Remark. Notice that,

$$\begin{tikzcd} A[p^{n+m}] \arrow[d, "p^m"] \arrow[dr, two heads] & \\ A[p^{n+m}] \arrow[r] & A[p^n] \end{tikzcd}$$

Proposition 4.3.4. Show there is a short exact sequence,

$$0 \longrightarrow \mathscr{F}_m \longrightarrow \mathscr{F}_{n+m} \longrightarrow \mathscr{F}_n \longrightarrow 0$$

Proof. Notice,

$$\mathscr{F}_{n+m} = \mathscr{F}/p^{n+m}\mathscr{F}$$

Then,

$$\mathscr{F}_{n+m}[p^n] = \frac{p^m\mathscr{F}}{p^{n+m}\mathscr{F}}$$

□

Corollary 4.3.5. We have $\operatorname{rank}\mathscr{F}_1 = p^h$ then $\operatorname{rank}\mathscr{F}_n = p^{hn}$.

Remark. Over $k = \bar{k}$ of characteristic p,

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}^\circ_{n+1} \arrow[r] \arrow[d] & \mathscr{F}_{n+1} \arrow[r] \arrow[d] & \mathscr{F}^{\text{ét}}_{n+2} \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \mathscr{F}^\circ_n \arrow[r] & \mathscr{F}_n \arrow[r] & \mathscr{F}^{\text{ét}}_n \arrow[r] & 0 \end{tikzcd}$$

Theorem 4.3.6 (Raynaud)**.** $\mathscr{F}^\circ_n = \operatorname{Spec}\left(k[x_1, \dots, x_n]/(x_1^{p^{i_1}}, \dots, x_n^{p^{i_n}})\right)$

Remark. We asked if the i_r are locally constant. The degeneration of an ordinary elliptic curve to a supersingular elliptic curve and taking p-torsion gives a counterexample. A better counterexample is the universal extension of α_p by α_p which has fibers α_{p^2} degenerating to α_p^2.

4.4 BT groups

Let $\mathscr{F}$ over S as above. Then,

$$\mathscr{F}[\tfrac{1}{p}] = \operatorname{colim}_{[p]}\mathscr{F} = \operatorname{colim}_n \frac{1}{p^n}\mathscr{F}$$

this is a sheaf (and ind-scheme) but only on quasi-compact test objects. Recall,

$$\mathscr{F}(R) = \varprojlim \mathscr{F}_n(R)$$

is a p-adically complete $\mathbb{Z}_p$-module and hence $\mathscr{F}(R)[\frac{1}{p}]$ is a $\mathbb{Q}_p$-Banach space.

5 May 6

Recall: a TB group over S an fpqc sheaf on Aff_S such that,

(a) $[p] : \mathscr{F} \to \mathscr{F}$ is a closed immersion

(b) $\mathscr{F}/p\mathscr{F}$ is finite flat

(c) $\mathscr{F} = \varprojlim \mathscr{F}/p^n\mathscr{F}$.

Consider,

$$\frac{\mathscr{F}[\frac{1}{p}]}{\mathscr{F}} = X_{\mathscr{F}}$$

We showed that $[p] : X_{\mathscr{F}} \to X_{\mathscr{F}}$ is surjective with finite cokernel. Then,

$$X_{\mathscr{F}} = \varinjlim_n X_{\mathscr{F}}[p^n]$$

$$0 \longrightarrow \mathscr{F}/p^n\mathscr{F} \longrightarrow \mathscr{F}/p^{n+m}\mathscr{F} \longrightarrow \mathscr{F}/p^m\mathscr{F} \longrightarrow 0$$

If $p^n = 0$ on S then $X_{\mathscr{F}}$ is formally smooth. The Lie algebra has rank d (locally constant on S). Then (h is the height, locally constnat on S).

Proposition 5.0.1. Show that TB / S and BT / S via,

$$\mathscr{F} \mapsto X_{\mathscr{F}}$$

and

$$X \mapsto T_pX = \mathrm{Hom}\,(\mathbb{Q}_p/\mathbb{Z}_p, X)$$

Proposition 5.0.2. If $S = \mathrm{Spec}\,(k)$ and $\mathscr{F}, \mathscr{G}$ are TB groups over S then,

$$\mathrm{Hom}\,(\mathscr{F}, \mathscr{G})$$

is also a TB group.

Proof. Section 4.1 of Caraiani-Schotze. □

Example 5.0.3. $\mathrm{Hom}\,(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) =$

Definition 5.0.4. An $\mathbb{F}_p$-algebra R is *perfect* if $r \mapsto r^p$ is a bijection on R.

Example 5.0.5. The following are perfect,

(a) $\mathbb{F}_{p^n}$ and $\overline{\mathbb{F}}_p$

(b) $\bigcup_n \mathbb{F}_p[x^{\frac{1}{p^n}}]$.

Definition 5.0.6. A strict p-ring is a ring A that is p-adically complete, p-torsion free with A/p perfect.

Example 5.0.7. The following are strict p-rings,

(a) $\mathbb{Z}_{p^n}, \overline{\mathbb{Z}}_p$

(b) $\left(\bigcup_n \mathbb{Z}_p[x^{\frac{1}{p^n}}]\right)^\wedge_p$.

Theorem 5.0.8. The reduction mod p functor gives an equivalence,

$$\{\text{strict p-rings}\} \to \{\text{perfect } \mathbb{F}_p\text{-algebras}\}$$

given by,

$$A \mapsto A/pA$$

Proof. We construct a multiplicative section of $A \to A/pA$. Indeed, let y_n be a lift of $x^{\frac{1}{p^n}}$ then,

$$[x] = \varinjlim y_n^{p^n}$$

gives a well-defined unique lift. Then,

$$A = \left\{\sum [a_n] p^n\right\}$$

$\square$

Remark. The preimage of a perfect $\mathbb{F}_p$-algebra R is $W(R)$ and $W_n(R) = W(R)/p^n W(R)$.

Remark. $\mathrm{Hom}\left(\mathrm{Spec}\left(B\right), \mathrm{Spf}\left(W(R)\right)\right) = \mathrm{Hom}\left(\mathrm{Spec}\left(B/p\right), R\right)$.

Remark. Claim that $(W(R), (p))$ is Henselian pair and thus gives étale lifting (LOOK UP IN STACKS PROJECT).

Definition 5.0.9. A Dieudonne-module over a perfect ring R is a pair (M, φ_M) where M is a projective $W(R)$-module and an isomorphism

$$\varphi_M : \varphi^* M[1/p] \xrightarrow{\sim} M[1/p]$$

such that,

$$pM \subset \varphi_M(\varphi^* M) \subset M$$

where $\varphi : W(R) \to W(R)$ is the lift of Frobenius on R.

Example 5.0.10. Let $R = \mathbb{F}_p$ then $W(R) = \mathbb{Z}_p$ and $M = \mathbb{Z}_p$ and $\varphi_M = p$ or $\varphi_M = 1$. Furthermore, we can define,

$$M = \bigoplus_{i=1}^{r} e_i \mathbb{Z}_p$$

such that,

$$\varphi_M(e_i) = \begin{cases} e_{i+1} & i \le r - s \\ pe_{i+1} & r - s < i < r \\ pe_1 & i = r \end{cases}$$

Definition 5.0.11. An isocrystal over R is a projective $W(R)[1/p]$-module N with the data,

$$\varphi_N : \varphi^* N \xrightarrow{\sim} N$$

We say that N is of *height* rank N.

Remark. Isocrystals of height n are classified by $\mathrm{GL}_n(W(R)[1/p])/\mathrm{Ad}_\varphi \mathrm{GL}_n(W(R)[1/p])$.

Example 5.0.12. For $\lambda = \frac{s}{r} \in \mathbb{Q}$ with r positive and reduced form,

$$N_\lambda = \mathbb{Q}_p[X]/(X^r - p^s)$$

with $\varphi_{N_\lambda} = X \cdot -$.

Proposition 5.0.13. If $0 \le \lambda \le 1$ then,

$$N_\lambda \cong M_\lambda[1/p]$$

as F-isocrystals.

Proposition 5.0.14. $N_\lambda \otimes N_{\lambda'} = N_{\lambda+\lambda'}^{\gcd(r,r')}$.

Theorem 5.0.15 (Dieudonne-Manin)**.** The category of F-isocrystals over $\bar{k}$ is a $\mathbb{Q}_p$-linear semisimple abelian tensor category with duals. In particular every N decomposes as,

$$N = \bigoplus_\lambda N_\lambda^{c_\lambda}$$

Definition 5.0.16. Fix height h then decompose,

$$N = \bigoplus_\lambda N_\lambda^{c_\lambda}$$

Define a sequence $(\mu_1 \le \cdots \mu_h)$ of $\mu_i \in \mathbb{Q}$. Say that,

$$(\mu_1 \le \cdots \le \mu_n) \le (\mu_1' \le \cdots \le \mu_n') \iff \forall j : \sum_{i=1}^j \mu_i \le \sum_{i=1}^j \mu_i'$$

Remark. Think Bruhat order on $X_*(T)_\mathbb{Q}$.

Definition 5.0.17. Given an isocrystal N over $S = \mathrm{Spec}\,(R)$ then for each geometric point $\bar{s} : \mathrm{Spec}\left(\bar{k}\right) \to S$ get a sequence,

$$N_{\bar{k},\bar{s}} \iff (\mu_{1,\bar{s}} \le \cdots \le \mu_{h,\bar{s}})$$

Theorem 5.0.18 (Grothendieck)**.** The above function is constructible. In fact, for fixed $\mu_1 \le \cdots \le \mu_n$ consider,

$$\{s \in S \mid (\mu_{1,s} \le \cdots \le \mu_{h,s}) \le (\mu_1 \le \cdots \le \mu_h)\}$$

is closed and furthermore,

$$\sum_{i=1}^h \mu_{i,\bar{s}} \text{ is closed}$$

and the denominators are bounded by $h!$.

6 May 13

6.1 Erratum

An isocrystal N over $W(R)[\frac{1}{p}]$ should have a $W(R)$ lattice $M \subset N$ such that $\exists n, m \in \mathbb{Z}$ so that,

$$p^n M \subset \varphi_M(\varphi^* M) \subset p^m M$$

Therefore, N is étale locally in R free over $W(R)[\frac{1}{p}]$.

Remark. Let M be a projective $W(R)$-module such that M/pM is free. Write,

$$M/pM \cong \bigoplus_{i=0}^{n} e_i W(R)$$

Then consider $W(R)^{\oplus n} \to M$ sending $j_i \mapsto \widetilde{e_i}$. Then by Nakayama this is surjective and injective by p-adic completeness checking on $M/p^n M$ for all n.

Exercise 6.1.1. Any rank 1 isocrystal is pro-étale locally on R, isomorphic to an isomcrystal such that $\varphi = p^k$ for some k.

Wlog let $M = W(R)_{e_1}$ then,

$$\varphi(e_1) = \alpha e_1 \quad \text{ with } \alpha \in W(R)^{\times}[\tfrac{1}{p}]$$

By fudging with denominators we wlog $\alpha \in W(R)^{\times}$. Then we want to pick a basis such that (λe_1) s.t.

$$\varphi(\lambda e_1) = \sigma(\lambda)\alpha e_1$$

Theorem 6.1.2 (Gabber)**.** Let R be a perfect ring, then there is an equivalence of categories,

$$\{\text{TB/Spec}\,(R)\} \to \{\text{Dieudonne-modules over } W(R)\}$$

which we write $\mathscr{F} \mapsto \mathbb{D}(\mathscr{F})$

(a) $\operatorname{rank} \mathscr{F} = \mathbf{ht}\,(\mathscr{F})$

(b) $\operatorname{Lie}(\mathscr{F}/p\mathscr{F}) \cong \frac{\mathbb{D}(\mathscr{F})}{\sigma(\mathbb{D}(\mathscr{F}))}$

Given an abelian scheme $\mathcal{A} \to S$ with S of characteristic p and $\mathcal{A}/S$ is an abelian scheme. Let,

$$S^{\text{perf}} = \varprojlim_{\varphi} S \to S$$

be the terminal perfect scheme mapping to S. Then height $2g$ Dieudonne modules over S^{perf} and thus get isocrystals. Then,

$$S^{\text{perf}} = \bigcup_b S^{\text{perf},b}$$

where b runs over height $2g$ and $\dim = g$ Newton polygons. Apply this to $S = \mathcal{A}_g$, we get,

$$\mathcal{A}_g = \bigcup_b \mathcal{A}_{g,b}$$

with $\mathcal{A}_{g,\text{ss}}$ is closed and $\mathcal{A}_{g,\text{ord}}$ is open.

Remark. The Newton stratification is not functorial it is just a topological stratification which then inherts the reduced induced structures.

6.2 Goal: understand strata

Remark. Gabber tells us that,

$$\overline{\mathcal{A}_{g,b}} \subset \bigcup_{b' \leq b} \mathcal{A}_{g,b}$$

Theorem 6.2.1 (de Jong, Ort)**.** This inclusion is an equality this gives the dimensions of $\mathcal{A}_{g,b}$.

Theorem 6.2.2 (de Jong-Oort)**.** With Honda-Tate theory $\overline{\mathcal{A}_{g,b}} \backslash \mathcal{A}_{g,b}$ has codimension at most one in $\overline{\mathcal{A}_{g,b}}$.

Theorem 6.2.3 (Li-Oort)**.** Compute $\dim \mathcal{A}_{g,\text{ss}}$.

6.3 Ingredients

Given $\mathcal{A}/\overline{\mathbb{F}}_p$ then,

$$\mathrm{Def}(A) \xrightarrow{\sim} \mathrm{Def}(\mathscr{F}_A)$$

Serre-Tate: bijection + deformations of TB groups are effective. We are going to study $\mathcal{A}_g$ or $\mathcal{A}_{g,b}$ via the map,

$$\mathcal{A}_g \to \mathcal{TB}_{2g,g,\mathrm{sym}} = \{\text{stack of height } 2g \text{ dim } g \text{ polarized TB groups}\}$$

and we can fix the Newton polygon on both sides,

$$\widehat{\mathcal{A}_{g,b}} \to \mathcal{TB}_{2g,g,b}$$

We need to take the formal completion to rectify this fact that $\mathcal{A}_{g,b}$ does not have a functorial description. Serre-Tate: this map is formally étale.

Remark. Over the category of perfect schemes these stalks are pro-artin stacks.

Definition 6.3.1. Fix $\mathscr{F}/\mathbb{F}_p$ a TB group. Define $X_{\mathscr{F}}(R)$ as the groupoid,

$$\{\text{TB groups over Spec}\,(R) \text{ with } \alpha : G[\tfrac{1}{p}] \to \mathscr{F}_R[\tfrac{1}{p}]\}$$

The morphisms are,

$$\begin{tikzcd} G \arrow[r, "\sim"] \arrow[d, "\alpha"] & G \arrow[d, "\alpha'"] \\ \mathscr{F}_R[\frac{1}{p}] \arrow[r, equal] & \mathscr{F}_R[\frac{1}{p}] \end{tikzcd}$$

so we see there are no nontrivial automorphisms since,

$$\mathrm{Hom}\,(G, G') \hookrightarrow \mathrm{Hom}\,(G, G')\,[\tfrac{1}{p}]$$

meaning there is no p-torsion. Furthermore, there is an action of the group functor,

$$R \mapsto \mathrm{Aut}\left(\mathscr{F}_R[\tfrac{1}{p}]\right)$$

Caraiani-Scholze prove that this group is a formal algebraic space (but a terrible one!).