Remark. Unless otherwise stated, all rings are commutative and unital.

1 Definitions

Definition 1.0.1. An element $p \in A$ is prime if (p) is a prime ideal. Equivalently p is prime if whenever $p \mid xy$ either $p \mid x$ or $p \mid y$.

Definition 1.0.2. An element $r \in A$ which is nonzero and not a unit is irreducible if whenever $r = xy$ either $x \in A^\times$ or $y \in A^\times$.

2 Domains

Definition 2.0.1. A ring A is a domain if A has no zero divisors i.e. if $ab = 0$ then $a = 0$ or $b = 0$.

Proposition 2.0.2. Let A be a domain then any nonzero prime element is irreducible.

Proof. Let $p \in A$ be a prime. Now suppose that $p = xy$ for $x, y \in A$. Thus, $p \mid xy$ so (WLOG) we have $p \mid x$ so $x = pz$ and thus $p = pzy$. However, p is nonzero and A is a domain so $zy = 1$ and thus $y \in A^\times$ proving that p is irreducible. □

3 Principal Ideal Domains

Definition 3.0.1. A principal ideal domain (PID) is a domain A such that every ideal is principal.

Lemma 3.0.2. If A is a PID then A is Noetherian.

Proof. Every ideal is principal and thus finitely generated. □

Lemma 3.0.3. Let A be a PID and $r \in A$ irreducible then (r) is maximal and thus r is prime.

Proof. Consider an intermediate ideal $(r) \subset J \subset A$ then since A is a PID we have $J = (a)$ so $r \in (a)$ and thus $r = ac$ so either $a \in A^\times$ in which case $J = A$ or $c \in A^\times$ in which case $J = (r)$ so (r) is maximal and thus a prime ideal. □

Theorem 3.0.4. Let A be a PID and not a field then $\dim A = 1$.

Proof. Any prime ideal $\mathfrak{p} \subset A$ is principal so $\mathfrak{p} = (p)$ and p is prime. Either $p = 0$ which is prime since A is a domain or p is irreducible and so we have shown (p) is maximal. So every prime ideal is zero or maximal and thus $\dim A \le 1$. If $\dim A = 0$ then (0) is maximal so A is local and any nonzero element is thus invertible so A is a field. □

Theorem 3.0.5 (Kaplansky)**.** Let A be Noetherian then A is a principal ideal ring iff every maximal ideal is prime.

Theorem 3.0.6 (Cohen)**.** A ring A is Noetherian iff every prime ideal is finitely generated.

Corollary 3.0.7. A ring A is a principal ideal ring iff every prime ideal is principal.

4 Unique Factorization Domains

Definition 4.0.1. A domain A is a unique factorization domain (UFD) if every nonzero element has a unique factorization into irreducible elements.

Definition 4.0.2. A factorization ring A is a ring such that every nonzero element has a factorization into irreducible elements.

Lemma 4.0.3. If A is a Noetherian domain then it is a factorization domain.

Proof. Take $a_0 \in A$. If a is irreducible, zero, or a unit then we are done. Then we can write, $a = a_1^{(1)} a_2^{(1)}$ for $a_1, b_1 \notin A^\times$. Continuing in this manner we get,

$$(a) \subsetneq (a_1^{(1)}, a_2^{(1)}) \subsetneq (a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, a_4^{(3)}) \subsetneq \cdots$$

(CHECK THIS) This sequence is proper since if $a = bc$ and $b \in (a)$ then $a = arc$ so $rc = 1$ and thus $c \in A^\times$ contradicting our construction. However, A is Noetherian then the sequence must terminate so at some point the factorization must become irreducible. $\square$

Theorem 4.0.4. Let A be a factorization domain. Then A is a UFD iff every irreducible is prime.

Proof. If A is a UFD and p an irreducible. Let $x, y \in A$ and $p \mid xy$ then p is in the factorization of xy and thus, by uniqueness must be in the factorization of either x or y so $p \mid x$ or $p \mid y$.

Conversely, if A is a factorization domain and every irreducible is prime then given two factorizations of x each irreducible must, by primality, divide an irreducible in the other factorization so they are equal. (DO THIS BETTER) $\square$

Corollary 4.0.5. If A is a PID then A is a UFD.

Proof. If A is a PID then it is Noetherian and thus a factorization domain. Furthermore, every irreducible is prime so A is a UFD. $\square$

4.1 Height One Prime Ideals

Proposition 4.1.1. Let A be Noetherian. Then any principal prime ideal has height at most one.

Proof. Let $\mathfrak{p} = (p) \subset A$ be a principal prime ideal. Then consider the localization which is $A_{(p)}$ Noetherian and the unique maximal ideal $pA_{(p)}$ is principal. Take $N = \text{nilrad}\left(A_{(p)}\right)$ then,

$$\dim A_{(p)}/N = \dim A_{(p)} = \mathbf{ht}\,(\mathfrak{p})$$

but $A_{(p)}/N$ is a Noetherian domain and the unique maximal ideal $pA_{(p)}$ is principal so $A_{(p)}/N$ is a PID and thus $\dim A_{(p)}/N \leq 1$. $\square$

Proposition 4.1.2. If A is a UFD then every prime ideal of height one is principal.

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal with $\mathbf{ht}\,(\mathfrak{p}) = 1$. Take any nonzero element $x \in \mathfrak{p}$ and consider its factorization into irreducibles. Since $\mathfrak{p}$ is prime some irreducible factor $p \mid x$ must be in $\mathfrak{p}$ so $(p) \subset \mathfrak{p}$. Since A is a UFD all irreducibles are prime so $(p) \subset \mathfrak{p}$ is prime. However $\mathbf{ht}\,(\mathfrak{p}) = 1$ and $(p) \neq (0)$ so $(p) = \mathfrak{p}$ and thus $\mathfrak{p}$ is principal. $\square$

Theorem 4.1.3. Let A be a Noetherian domain. Then A is a UFD iff every height one prime ideal is principal.

Proof. We showed one direction above. Conversely, suppose every height one prime ideal is principal. Since A is a Noetherian domain, it suffices to show that each irreducible is prime. Let r be irreducible and consider a minimal prime $\mathfrak{p} \supset (r)$. Then by Krull's Hauptidealsatz, $\mathfrak{p}$ has height one so by our assumption $\mathfrak{p} = (p)$ is principal. However, $(r) \subset (p)$ so $p \mid r$ but r is irreducible so we must have $(r) = (p) = \mathfrak{p}$ and thus r is prime. $\square$

Theorem 4.1.4 (Krull's Hauptidealsatz)**.** Let $I \subset A$ be an ideal in a Noetherian ring A with n generators then any minimal prime ideal $\mathfrak{p} \supset I$ has height at most n.

5 Simple Modules

Definition 5.0.1. A nonzero R-module is *simple* if it has no nontrivial submodules.

Proposition 5.0.2. Let R be a ring and M an R-module. Then the following are equivalent,

(a) M is simple

(b) $\ell_R(M) = 1$

(c) $M = R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.

Proof. The first two are equivalent by definition. Clearly if $\mathfrak{m} \subset R$ is maximal then $R/\mathfrak{m}$ is simple. Now suppose that M is simple and take a nonzero $x \in M$. Then $(x) = M$ by simplicity so consider $I = \ker(R \xrightarrow{x} M) = \mathrm{Ann}_A(x) = \{r \in R \mid rx = 0\}$. Since $M = Rx$ we know that $M \cong R/I$. However, by the lattice isomorphism theorem, submodules of R/I correspond to ideals above I so since M is simple we must have I maximal. $\square$

6 Artinian Modules

Definition 6.0.1. An R-module M is *noetherian/artinian* if it satisfies the ascending/descending chain condition on submodules.

Theorem 6.0.2. An R-module M has finite length iff it is both noetherian and artinian.

Proof. If M has finite length then clearly it is noetherian and artinian since chains of submodules are bounded in length. Alternatively, simple modules are noetherian and artinian so given a composition series we see that M is noetherian and artinian by repeated extension. Now, conversely, assume that M is noetherian and artinian. By the artinian property we can take a minimal nonzero submodule $M_1 \subset M$. Then M_1 is simple. Either M/M_1 is simple or we may repeat to get $M_2 \supset M_1$ and M_2/M_1 is simple. Thus we get an ascending chain $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$ with M_{i+1}/M_i simple. Since M is Noetherian, this must terminate at $M_n = M$ so we get a finite length composition series showing that M has finite length. $\square$

7 Artinian Rings

Definition 7.0.1. A ring A is *artinian* if it satisfies the descending chain condition on ideals: given a chain of ideals,

$$I_0 \supset I_1 \supset I_2 \supset \cdots$$

the chain stabilizes $I_{n+i} = I_n$.

Remark. A is artinian iff it is artinian as a module over itself.

Proposition 7.0.2. An artinian ring has finitely many maximal ideals.

Proof. Let $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \dots$ be a list of maximal ideals. Then consider the chain,

$$\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supset \cdots$$

By the artinian condition, we must have $\mathfrak{m}_1 \cdots \mathfrak{m}_n = \mathfrak{m}_1 \cdots \mathfrak{m}_n\mathfrak{m}_{n+1}$ for some n. But then by prime avoidence $\mathfrak{m}_{n+1}$ must be one of $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ since $\mathfrak{m}_{n+1} \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n$ so $\mathfrak{m}_{n+1} \supset \mathfrak{m}_i$ and $\mathfrak{m}_i$ is maximal. ☐

Proposition 7.0.3. Let A be an artinian ring. Then every prime ideal is maximal so $\dim A = 0$.

Proof. Let $\mathfrak{p}$ be prime and $x \notin \mathfrak{p}$. Consider the chain,

$$(x) \supset (x^2) \supset (x^3) \supset \cdots$$

By the artinian condition $(x^n) = (x^{n+1})$ for some n so $x^n = rx^{n+1}$ for some $r \in A$. Thus $x^n(rx-1) = 0$. However, $x^n \notin \mathfrak{p}$ so $rx - 1 \in \mathfrak{p}$ and thus $x \in A/\mathfrak{p}$ is invertible so $A/\mathfrak{p}$ is a field and thus $\mathfrak{p}$ is maximal. ☐

Proposition 7.0.4. Let A be artinian. Then $\mathrm{nilrad}\,(A)$ is a nilpotent ideal.

Proof. Let $I = \mathrm{nilrad}\,(A)$. Consider the chain of ideals,

$$I \supset I^2 \supset I^3 \supset \cdots$$

By the artinian condition, $I^{n+1} = I^n$ for some n.

Consider $J = \{x \in A \mid xI^n = 0\}$. If $J \neq R$ we can choose $J' \supsetneq J$ minimal (using the artinian property). Then take $y \in J'$ so by minimality $J' = J + (y)$. Suppose $J + I(y) = J'$ then, since $J \subset \mathrm{Jac}\,(A)$ and (y) is finitely generated, by Nakayama, $J' = J + I(y) = J$ which is false so $J \subset J + I(y) \subsetneq J'$ and thus $J = J + I(y)$ by minimality so $I(y) \in J$. Therefore, $y \cdot I^{n+1} = 0$ but $I^{n+1} = I^n$ so $y \cdot I^n = 0$ and thus $y \in J$ contradicting our situation so $J = R$ and thus $I^n = 0$. ☐

Proposition 7.0.5. Every artinian ring is a product of local artinian rings: $A_{\mathfrak{m}_i} = A/\mathfrak{m}_i^n$.

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals. Then we know that $\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r} = 0$ for some integers $n_1, \dots, n_r \in \mathbb{Z}$. Therefore, by the Chinese remainder theorem,

$$A = A/(\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r}) = \prod_{i=1}^{r} A/\mathfrak{m}_i^{n_i}$$

Furthermore, $A/\mathfrak{m}_i^{n_i}$ is local because $\mathfrak{m}_i$ is the only maximal ideal above $\mathfrak{m}_i^{n_i}$. Furthermore,

$$A_{\mathfrak{m}_i} = (A/\mathfrak{m}_i^{n_i})_{\mathfrak{m}_i} = A/\mathfrak{m}_i^{n_i}$$

since $A \setminus \mathfrak{m}_i$ is not contained in any maximal ideal of $A/\mathfrak{m}_i^{n_i}$ and thus is invertible. ☐

Proposition 7.0.6. A ring A is artinian iff it has finite length as a module over itself.

Proof. If A has finite length as an A-module then it satisfies both the ascending and descending chain conditions on A-submodules i.e. ideals thus A is both noetherian and artinian. Conversely, let A be artinian. Since A is a finite product of local artinian rings we may reduce to the case that A is local artinian with maximal ideal $\mathfrak{m}$. Since $\text{nilrad}(A) = \mathfrak{m}$ then $\mathfrak{m}^n = 0$ for some n so we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

Then $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a $A/\mathfrak{m}$-module and,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

which must be finite since $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is an artinian module and thus must have finite dimension else there would be a nonterminating descending chains. Thus from the series A has finite length. $\square$

Theorem 7.0.7. A ring A is artinian iff A is noetherian and $\dim A = 0$.

Proof. If A is artinian then it has finite length over itself and thus is noetherian. Also every prime is maximal so $\dim A = 0$. Conversely, suppose that A is noetherian and $\dim A = 0$. Then $\text{Spec}(A)$ is a noetherian topological space which has finitely many irreducible componets so A has finitely many minimal primes which are also maximal since $\dim A = 0$. Thus A has finitely many primes all of which are maximal. Since $\dim A = 0$ we have $I = \text{Jac}(A) = \text{nilrad}(A)$ so any $f \in I$ is nilpotent so I is nilpotent because A is noetherian so I is finitely generated. Thus by the Chines remainder theorem A is a finite product of local rings so we reduce to the case that A is local with maximal ideal $\mathfrak{m}$. Then we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

but $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite $A/\mathfrak{m}$-module since A is noetherian so,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is finite and thus $\ell_A(A)$ is finite from the series showing that A is artinian. $\square$

Proposition 7.0.8. Let A be an artinian ring. Then,

$$\ell_A(A) = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i})$$

Proof. We can write, $A = A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r}$ and thus the formula immediately follows. $\square$

Proposition 7.0.9. Any finite dimensional k-algebra is artinian.

Proof. By dimensionality arguments every descending chain stabilizes. $\square$

Proposition 7.0.10. Let $A \to B$ be a local map and M an B-module of finite length. Then,

$$\ell_A(M) = \ell_B(M) \cdot [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

and in particular $\ell_A(M)$ is finite if $\kappa(\mathfrak{m}_B)$ is a finite extension of $\kappa(\mathfrak{m}_A)$.

Proof. Consider a composition series,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

Then M_i/M_{i-1} is a simple A-module so $M_i/M_{i-1} \cong B/\mathfrak{m}_B = \kappa(\mathfrak{m}_B)$ since B is local. Therefore,

$$\ell_A(M) = \sum_{i=1}^{n} \ell_A(M_i/M_{i-1}) = \sum_{i=1}^{n} \ell_A(\kappa(\mathfrak{m}_B)) = n \cdot [\kappa(B_\mathfrak{m}) : \kappa(A_\mathfrak{m})]$$

where $\ell_A(\kappa(\mathfrak{m}_B)) = \ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$ because $A \to B$ is local and,

$$\ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = \dim_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

□

Corollary 7.0.11. If A is a local artinian finite type k-algebra. Then,

$$\dim_k A = \ell_A(A) \cdot \dim_k (A/\mathfrak{m})$$

in particular A is a finite k-module.

Proof. Viewing A as a module over itself we know it has finite length since A is artinian. Furthermore, $A/\mathfrak{m}$ is a field finitely generated over k and thus a finite extension of k by the Nullstellensatz. Then applying the previous result we conclude. □

Corollary 7.0.12. Let A be an artinian finite type k-algebra. Then,

$$\dim_k A = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}) \cdot \dim_k (A/\mathfrak{m}_i)$$

Proof. Since A is artinian we can write,

$$A = \prod_{i=1}^{r} A_{\mathfrak{m}_i}$$

where $A_{\mathfrak{m}_i}$ are the local artinian factors associated to the finitely many prime ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$. The result follows from above by additivity of the dimensions. □

8 Cohen-Macaulay Rings

8.1 Dimension

Proposition 8.1.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. Then,

$$\dim A/(f) \geq \dim A - 1$$

with equality iff f is a nonzero divisor.

Proof. https://math.stackexchange.com/questions/2085779/the-dimension-modulo-a-principal-ideal-in-a-noetherian-local-ring □

8.2 Depth

8.3 Properties

Proposition 8.3.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$ a nonzero divisor. Then A is Cohen-Macaulay iff $A/(f)$ is Cohen-Macaulay.

Proof. We have $\text{depth}\,(A/(f)) = \text{depth}\,(A) - 1$ and $\dim A/(f) = \dim A - 1$. $\square$

9 Weakly Associated Points

9.1 Weakly Associated Primes

Definition 9.1.1. Let A be a ring and M an A-module. Then a prime $\mathfrak{p} \subset A$ is *weakly associated* to M if $\mathfrak{p}$ is minimal over $\text{Ann}_A(m)$ for some $m \in M$. We denote these primes $\text{WAss}_A(M)$.

Lemma 9.1.2. Let M be an A module then the natural map,

$$M \to \prod_{\mathfrak{p} \in \text{WAss}_A(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Suppose that $m \in M$ maps to zero. Then $\mathfrak{p} \not\subset \text{Ass}_A(m)$ for each $\mathfrak{p} \in \text{WAss}_A(M)$ which implies $\text{Ass}_A(m) = A$ since otherwise some associated prime will be minimal over $\text{Ann}_A(m)$. Thus $m = 0$. $\square$

Lemma 9.1.3. Let M be an A-module. Then,

$$M = (0) \iff \text{WAss}_A(M) = \varnothing$$

Proof. If $M = (0)$ then this is clear. Otherwise, by the previous lemma $M \hookrightarrow (0)$ is injective so $M = (0)$. $\square$

Lemma 9.1.4. Let A be a ring and M an A-module. Then,

$$\mathfrak{p} \in \text{WAss}_A(M) \iff \mathfrak{p}A_{\mathfrak{p}} \in \text{WAss}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

Proof. Consider the exact sequence for each $m \in M$,

$$0 \longrightarrow \text{Ann}_A(m) \longrightarrow A \xrightarrow{m} M \longrightarrow 0$$

Since localization is exact, we get an exact sequence,

$$0 \longrightarrow (\text{Ann}_A(m))_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}} \xrightarrow{m} M_{\mathfrak{p}} \longrightarrow 0$$

Therefore, $\text{Ann}_{A_{\mathfrak{p}}}(m) = (\text{Ann}_A(m))_{\mathfrak{p}}$. If $\mathfrak{p} \supset \text{Ann}_A(m)$ is minimal then $\mathfrak{p}A_{\mathfrak{p}} \subset (\text{Ann}_A(m))_{\mathfrak{p}} = \text{Ann}_{A_{\mathfrak{p}}}(m)$ is minimal. Conversely, if $\mathfrak{p}A_{\mathfrak{p}} \supset \text{Ann}_{A_{\mathfrak{p}}}(m/s)$ is minimal then,

$$\text{Ann}_{A_{\mathfrak{p}}}(m/s) = \text{Ann}_{A_{\mathfrak{p}}}(m) = (\text{Ann}_A(m))_{\mathfrak{p}}$$

which implies that $\mathfrak{p} \supset \text{Ann}_A(m)$ is minimal because if $x \in \text{Ann}_A(m)$ and $x \notin \mathfrak{p}$ then $(\text{Ann}_A(m))_{\mathfrak{p}} = A_{\mathfrak{p}}$ and any prime $\mathfrak{q}$ such that $\mathfrak{p} \subset \mathfrak{q} \subset \text{Ann}_A(m)$ implies that $\mathfrak{q}A_{\mathfrak{p}}$ is intermediate. $\square$

Lemma 9.1.5. Let A be a ring and M an A-module. Then $\text{WAss}_A(M) \subset \text{Supp}_A(M)$ furthermore any minimal element of $\text{Supp}_A(M)$ is an element of $\text{WAss}_A(M)$.

Proof. Since $\mathfrak{p} \subset \text{Ann}_A(m)$ we know $M_\mathfrak{p} \neq 0$ since m is nonzero in $M_\mathfrak{p}$. Furthermore, suppose that $\mathfrak{p} \in \text{Supp}_A(M)$ is minimal. Then $\text{Supp}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{p}A_\mathfrak{p}\}$ and $M_\mathfrak{p} \neq 0$ so $\text{WAss}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \{A_\mathfrak{p}\}$ and thus $\mathfrak{p} \in \text{WAss}_A(M)$. $\square$

Lemma 9.1.6. Let A be a ring and M an A-module and $S \subset A$ a multiplicative subset. Then.

(a) $\text{WAss}_A(S^{-1}M) = \text{WAss}_{S^{-1}A}(S^{-1}M)$

(b) $\text{WAss}_A(M) \cap \text{Spec}(S^{-1}A) = \text{WAss}_A(S^{-1}M)$.

Proof. We have,

$$\mathfrak{p} \in \text{WAss}_A\left(S^{-1}M\right) \iff \mathfrak{p}A_\mathfrak{p} \in \text{WAss}_{A_\mathfrak{p}}\left(S^{-1}M_\mathfrak{p}\right)$$

For $\mathfrak{p} \in \text{Spec}(S^{-1}A)$ (i.e. $S \subset A \setminus \mathfrak{p}$) we have $S^{-1}M_\mathfrak{p} = M_\mathfrak{p}$ and $(S^{-1}A)_\mathfrak{p} = A_\mathfrak{p}$ so both equalities hold. Otherwise, $\mathfrak{p}A_\mathfrak{p}$ containes an element of S so $\mathfrak{p}A_\mathfrak{p}$ has some nonzero divisor on $S^{-1}M_\mathfrak{p}$ and thus $\mathfrak{p} \notin \text{WAss}_A(S^{-1}M)$. $\square$

Proposition 9.1.7. Let A be a ring M an A-module then $\mathfrak{p} \in \text{Supp}_A(M)$ if and only if there exists $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \in \text{WAss}_A(M)$. Therefore,

$$\bigcap_{\mathfrak{p} \in \text{Supp}_A(M)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \text{WAss}_A(M)} \mathfrak{p}$$

Proof. Take $\mathfrak{p} \in \text{Supp}_A(M)$ so $M_\mathfrak{p} \neq 0$ and then $\text{Ass}_A(M_\mathfrak{p}) \neq \varnothing$. Using the previous lemma, there exists $\mathfrak{q} \in \text{Ass}_A(M_\mathfrak{p}) = \text{Ass}_A(M) \cap \{\mathfrak{q} \mid \mathfrak{q} \subset \mathfrak{p}\}$. Furthermore, the support is an upward set (if $\mathfrak{q} \subset \mathfrak{p}$ and $M_\mathfrak{q} \neq 0$ then $M_\mathfrak{p} \neq 0$ since $M_\mathfrak{p} \to M_\mathfrak{q}$ is localization). Thus, if we have $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \in \text{Ass}_A(M) \subset \text{Supp}_A(M)$ then $\mathfrak{p} \in \text{Supp}_A(M)$. $\square$

Lemma 9.1.8. Let $M \hookrightarrow N$ be an injection of A-modules. Then $\text{WAss}_A(M) \subset \text{WAss}_A(N)$.

Proof. This follows because the set of annihilators of elements of M is a subset of the set of annihilators of elements of N. $\square$

Lemma 9.1.9. Consider an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

then,

$$\text{WAss}_A(M_2) \subset \text{WAss}_A(M_1) \cup \text{WAss}_A(M_3)$$

Proof. Let $\mathfrak{p} \in \text{WAss}_A(M_2)$ and $\mathfrak{p} \notin \text{WAss}_A(M_1)$. Using the previous lemma it suffices to consider the case that A is local with maximal ideal $\mathfrak{p}$ (since we may localize the exact sequnence at $\mathfrak{p}$). Then $\mathfrak{p}$ is minimal over $\text{Ann}_A(m)$ for some $m \in M_2$ not in the image of $M_1 \to M_2$ (else $\mathfrak{p} \in \text{WAss}_A(M_1)$). Therefore $\bar{m} \in M_3$ is nonzero and $\text{Ann}_A(\bar{m}) \supset \text{Ann}_A(m)$ but $\text{Ann}_A(\bar{m})$ is proper since $\bar{m}$ is nonzero and thus contained in $\mathfrak{p}$. Since $\mathfrak{p}$ is minimal over $\text{Ann}_A(m)$ it must also be minimal over $\text{Ann}_A(\bar{m})$ and thus we conclude that $\mathfrak{p} \in \text{WAss}_A(M_3)$. $\square$

Lemma 9.1.10. Let A be a ring and M and A-module. Then,

$$\bigcup_{\mathfrak{p} \in \text{WAss}_A(M)} = \{\text{zero divisors on } M\}$$

Proof. Let $m \in M$ have zero divisors then there is exists a minimal prime (by Zorn's Lemma) above $\mathrm{Ann}_A(m)$ which must be associated. Conversely, if $f \in \mathfrak{p} \in \mathrm{WAss}_A(M)$ then $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A(m)$ for some $m \in M$. Then $R = (A/\mathrm{Ann}_A(m))_{\mathfrak{p}}$ has a unique minimal prime $\mathfrak{p}$ so $\mathfrak{p} = \mathrm{nilrad}(R)$ and thus $gf^n \in \mathrm{Ann}_A(m)$ for some least $n > 0$ and $g \notin \mathfrak{p}$. Thus $gf^n m = 0$ so $f(gf^{n-1}m) = 0$ but $gf^{n-1}m \neq 0$ because n is minimal so f is a zero divisor. $\square$

Proposition 9.1.11. Let $(A, \mathfrak{m})$ be a local ring then $\mathfrak{m} \in \mathrm{WAss}_A(A)$ iff $\mathfrak{m} = \{\text{zero divisors}\}$.

Proof. Immediate from the above since zero divisors are not units and thus contained in $\mathfrak{m}$. $\square$

Corollary 9.1.12. Given a prime $\mathfrak{p} \in \mathrm{Spec}(A)$ and an A-module M we have,

$$\mathfrak{p} \in \mathrm{WAss}_A(A) \iff \mathfrak{p}A_{\mathfrak{p}} = \{\text{zero divisors of } A_{\mathfrak{p}}\}$$

Proposition 9.1.13. Let A be reduced then $\mathrm{WAss}_A(A)$ are exactly the minimal primes of A.

Proof. The minimal primes are in $\mathrm{WAss}_A(A)$ by Lemma 9.1.5. Because $\mathfrak{p} \in \mathrm{WAss}_A(A) \iff \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{WAss}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ is suffices to consider the case of a reduced local ring $(R, \mathfrak{m})$ and $\mathfrak{m} \in \mathrm{WAss}_R(R)$. Then $\mathfrak{m}$ is minimal over $\mathrm{Ann}_R(x)$ for some $x \in \mathfrak{m}$ so $\mathfrak{m} = \sqrt{\mathrm{Ann}_R(x)}$. Thus $x^n \in \mathrm{Ann}_R(x)$ so $x^{n+1} = x \cdot x^n = 0$ so $x = 0$ because R is reduced a contradiction unless $\mathfrak{m} = 0$ so R is a field so $\mathfrak{m}$ is minimal showing that $\mathfrak{p}A_{\mathfrak{p}} \subset \mathbb{A}_{\mathfrak{p}}$ and thus $\mathfrak{p} \subset A$ are minimal primes and that $A_{\mathfrak{p}}$ is a field. $\square$

Lemma 9.1.14. Let A be a ring and $\mathfrak{p} \subset A$ a prime then $\mathrm{WAss}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

Proof. For nonzero $a \in A/\mathfrak{p}$ (i.e. $a \notin \mathfrak{p}$) the set $\mathrm{Ann}_A(a) = \{r \in A \mid ra \in \mathfrak{p}\} = \mathfrak{p}$ since $\mathfrak{p}$ is prime and therefore therefore $\mathfrak{p}$ is the unique minimal prime over an annihilator. $\square$

Proposition 9.1.15. Let A be a ring and M a Noetherian A-module. Then,

(a) there exists a finite filtration,

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

such that each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_A(M)$

(b) for any such filtration, $\mathrm{WAss}_A(M) \subset \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$

(c) $\mathrm{WAss}_A(M)$ is finite.

Proof. Since $M \neq (0)$ there is some $\mathfrak{p} \in \mathrm{WAss}_A(M)$ so we have an injection $A/\mathfrak{p} \to M$ let $M_1 \subset M$ be the image of this map so $M_1/M_0 \cong A/\mathfrak{p}_1$. Now take M/M_1 and $\mathfrak{p}_2 \in \mathrm{WAss}_A(M/M_1)$ then we have an injection $A/\mathfrak{p}_2 \to M/M_1$ so take $\bar{M}_2$ to be the image inside M/M_1 and M_2 its preimage in M. Then $M_2/M_1 \cong A/\mathfrak{p}_2$ and continuing by induction we construct a sequence,

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$$

with $M_i/M_{i-1} = A/\mathfrak{p}_i$ and

$$\mathfrak{p}_i \in \mathrm{WAss}_A(M/M_{i-1}) \subset \mathrm{Supp}_A(M/M_{i-1}) \subset \mathrm{Supp}_A(M)$$

However, M is Noetherian so this sequence must stabilize but it is striclty increasing when $M_i \subset M$ is proper. Thus, $M_n = M$ for some n.

For any such filtration we get exact sequences,

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow A/\mathfrak{p}_{i+1} \longrightarrow 0$$

Assume for induction that $\mathrm{Ass}_A(M_i) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_i\}$ then, by Lemma 9.1.9,

$$\mathrm{WAss}_A(M_{i+1}) \subset \mathrm{WAss}_A(M_i) \cup \mathrm{WAss}_A(A/\mathfrak{p}_{i+1}) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{i+1}\}$$

proving (b) by induction. (c) follows directly from (a) and (b). $\square$

9.2 Associated Primes

Definition 9.2.1. Let A be a ring and M an A-module. We say that $\mathfrak{p} \subset A$ is an *associated prime* of M if $\mathfrak{p} = \text{Ann}_A(m)$ for some $m \in M$. We write $\text{Ass}_A(M)$ for the set of associated primes of M.

Remark. Note $\mathfrak{p} = \text{Ann}_A(m) \iff A/\mathfrak{p} \hookrightarrow M$ via $a \mapsto a \cdot m$.

Remark. Clearly $\text{Ass}_A(M) \subset \text{WAss}_A(M)$. We will see equality holds when A is Noetherian.

Lemma 9.2.2. Given an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

we have,

$$\text{Ass}_A(M_2) \subset \text{Ass}_A(M_1) \cup \text{Ass}_A(M_3)$$

Proof. If $\mathfrak{p} \in \text{Ass}_A(M)$ then we have an embedding

$$A/\mathfrak{p} \hookrightarrow M_2$$

which is injective and $\iota(A/\mathfrak{p}) \cap N_1 = (0)$ then we get an injective map $A/\mathfrak{p} \to M_3$ so $\mathfrak{p} \in \text{Ass}_A(M_3)$. If $\iota(A/\mathfrak{p}) \cap M_1 \neq (0)$ then take nonzero $n \in \iota(A/\mathfrak{p}) \cap M_1$. Then $\text{Ann}_A(n) = \text{Ann}_A(\iota(x))$ for $x \in A/\mathfrak{p}$ nonzero. However, if $a \cdot \iota(x) = 0$ then $\iota(a \cdot x) = 0$ but ι is injective so $a \cdot x = 0$ and thus $\text{Ann}_A(\iota(x)) = \text{Ann}_A(x) = \mathfrak{p}$ because if $a \cdot x \in \mathfrak{p}$ for $x \notin \mathfrak{p}$ then $a \in \mathfrak{p}$. □

Lemma 9.2.3. Let $S_{M,\mathfrak{p}} = \{\text{Ann}_A(m) \subset \mathfrak{p} \mid m \in M \setminus \{0\}\}$ then any maximal element in $S_{M,\mathfrak{p}}$ is a prime ideal.

Proof. Let $\mathfrak{q} \in S_{M,\mathfrak{p}}$ be maximal with $\mathfrak{q} = \text{Ann}_A(m)$ for $m \neq 0$. Suppose $ab \in \mathfrak{q}$ and $a, b \notin \mathfrak{q}$. Then $\mathfrak{q} \subsetneq \text{Ann}_A(am)$ since $b \in \text{Ann}_A(am) \setminus \text{Ann}_A(m)$ so by maximality $\text{Ann}_A(am) \not\subset \mathfrak{p}$. Choose $s \in \text{Ann}_A(am) \setminus \mathfrak{p}$. Then $a \in \text{Ann}_A(sm)$ so $\text{Ann}_A(m) \subsetneq \text{Ann}_A(sm)$ and thus by maximality we can choose $t \in \text{Ann}_A(sm) \setminus \mathfrak{p}$ so $st \in \text{Ann}_A(m) \subset \mathfrak{p}$ but $s, t \notin \mathfrak{p}$ contradicting the primality of $\mathfrak{p}$. Thus $\mathfrak{q}$ is prime. □

Proposition 9.2.4. Let A be Noetherian and A-module M. Then,

$$\text{Ass}_A(M) = \text{WAss}_A(M)$$

In particular, $\text{Ass}_A(M) \neq \varnothing$ and all other properties of $\text{WAss}_A(M)$ apply to $\text{Ass}_A(M)$.

Proof. The first inclusion is obvious. If $\mathfrak{p} \in \text{WAss}_A(M)$ then $\mathfrak{p} \supset \text{Ann}_A(m)$ for some $m \in M$ and thus m is nonzero in $M_\mathfrak{p}$ so $\mathfrak{p} \in \text{Supp}_A(M)$. Let A be Noetherian then ascending chains in $S_{M,\mathfrak{p}}$ stabilize and thus by Zorn's Lemma every annhilator $\text{Ann}_A(m) \subset \mathfrak{p}$ is contained in some maximal $\text{Ann}_A(m') \subset \mathfrak{p}$. Thus, if $\mathfrak{p} \in \text{WAss}_A(M)$ then $\mathfrak{p}$ is a minimal prime over some $\text{Ann}_A(m)$ so $\mathfrak{p} = \text{Ann}_A(m')$ since $\text{Ann}_A(m')$ is prime and $\text{Ann}_A(m) \subset \text{Ann}_A(m') \subset \mathfrak{p}$. □

Lemma 9.2.5. Let A be a ring and M an A-module and $S \subset A$ a multiplicative subset. Then.

(a) $\text{Ass}_A(S^{-1}M) = \text{Ass}_{S^{-1}A}(S^{-1}M)$

(b) $\text{Ass}_A(M) \cap \text{Spec}(S^{-1}A) \subset \text{Ass}_A(S^{-1}M)$ with equality when A is Noetherian.

Proof. Tag 05BZ. □

Proposition 9.2.6. Let A be a Noetherian ring and M a finite A-module. Then,

(a) there exists a finite filtration,

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

such that each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \text{Supp}_A(M)$

(b) for any such filtration, $\text{Ass}_A(M) \subset \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$

(c) $\text{WAss}_A(M)$ is finite.

Proof. M is a Noetherian module so this applies directly from Prop. 9.2.6. ☐

9.3 Primary Decomposition

Remark. In this section we let A be a Noetherian ring.

Definition 9.3.1. An A-module M is called coprimary if $\text{Ass}_A(M) = \{\mathfrak{p}\}$ and if $N \subset M$ we say that N is $\mathfrak{p}$-primary if M/N is coprimary with $\text{Ass}_A(M/N) = \{\mathfrak{p}\}$.

Lemma 9.3.2. M is coprimary iff any zero divisor of M is locally nilpotent i.e. if $a \cdot m = 0$ for some $m \in M \setminus \{0\}$ then $\forall m' \in M : a^n \cdot m' = 0$ for some n.

Proof. Assume that M is coprimary, $\text{Ass}_A(M) = \{\mathfrak{p}\}$. If $x \in M$ is nonzero then Ax is a nonzero submodule of M so $\text{Ass}_A(Ax) = \{\mathfrak{p}\}$ since it is nonempty. Therefore, $\mathfrak{p}$ is a minimal element in $\text{Supp}_A(Ax) = V(\text{Ann}_A(x))$ because $Ax \cong A/\text{Ann}_A(x)$. Thus, $\sqrt{\text{Ann}_A(x)} = \mathfrak{p}$. If a is a zero divisor of M then $a \in \mathfrak{p}$ so $a^n \in \text{Ann}_A(x)$ so a is locally nilpotent. Converely, assume that the set of zero divisors equals the set of locally nilpotent elements. Take $\mathfrak{p}$ to be the ideal of all locally nilpotents. Take $\mathfrak{q} \in \text{Ass}_A(M)$ then $\mathfrak{q} = \text{Ann}_A(x)$ for some x. If $a \in \mathfrak{p}$ then $a^n \cdot x = 0$ for some n implies that $a^n \in \mathfrak{q}$ so $a \in \mathfrak{q}$. so $\mathfrak{p} \subset \mathfrak{q}$. Furthermore,

$$\bigcup_{\mathfrak{q} \in \text{Ass}_A(M)} \mathfrak{q} = \{\text{zero divisors}\} = \mathfrak{p}$$

so for any $\mathfrak{q} \in \text{Ass}_A(M)$ we have $\mathfrak{q} \subset \mathfrak{p}$. Thus, $\mathfrak{p} = \mathfrak{q}$ so $\text{Ass}_A(M)$ constains a unique prime. ☐

Corollary 9.3.3. If $I \subset A$ is an ideal then $\text{Ass}_A(A/I) = \{\mathfrak{p}\}$ if and only if I is a primary ideal and in that case $\sqrt{I} = \mathfrak{p}$.

Proof. Consider $I \subset A$ and A/I is coprimary then take $x, y \in A$ such that $y \notin I$ and $\bar{x} \cdot \bar{y} = 0$ in A/I. Then $\bar{x}$ is a zero divisor of A/I so it is locally nilpotent by the above. Thus, $\bar{x}^n \cdot 1 = 0$ for some n so $x^n \in I$ so $x \in \sqrt{I}$ and thus I is primary. Furthermore,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \text{Supp}_A(A/I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \text{Ass}_A(A/I)} \mathfrak{p} = \mathfrak{p}$$

since $\text{Ass}_A(M)$ is the set of minimal primes of $\text{Supp}_A(M)$ and $\text{Ass}_A(A/I) = \mathfrak{p}$. ☐

Definition 9.3.4. Let M be an A-module and $N \subset M$. We say N has a primary decomposition if,

$$N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

where each Q_i is primary. Moreover, we say that this decomposition is irredundant if

(a) if $i \neq j$ then $\mathrm{Ass}_A(M/Q_i) \neq \mathrm{Ass}_A(M/Q_j)$

(b) we cannot remove any Q_j from the intersection.

Lemma 9.3.5. Let M be an A-module then,

(a) If $Q_1, Q_2 \subset M$ are $\mathfrak{p}$-primary then $Q_1 \cap Q_2$ is $\mathfrak{p}$-primary.

(b) If $N = Q_1 \cap \cdots \cap Q_n$ is a irredundant primary decomposition and for each i, Q_i is $\mathfrak{p}_i$-primary then,

$$\mathrm{Ass}_A(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$$

Proof. Consider the injection,

$$0 \longrightarrow M/Q_1 \cap Q_2 \hookrightarrow M/Q_1 \oplus M/Q_2$$

which implies that,

$$\mathrm{Ass}_A(M/Q_1 \cap Q_2) \subset \mathrm{Ass}_A(M/Q_1 \oplus M/Q_2) = \mathrm{Ass}_A(M/Q_1) \cup \mathrm{Ass}_A(M/Q_2) = \{\mathfrak{p}\}$$

proving the first. For the second, consider the injection,

$$M/N \hookrightarrow M/Q_1 \oplus \cdots \oplus M/Q_n$$

which implies that,

$$\mathrm{Ass}_A(M/N) \subset \mathrm{Ass}_A(M/Q_1) \cup \cdots \cup \mathrm{Ass}_A(M/Q_n) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$$

We need to show that $\mathfrak{p}_i \in \mathrm{Ass}_A(M/N)$ for each i. We have the exact sequence,

$$0 \longrightarrow N \longrightarrow Q_2 \cap \cdots \cap Q_n \longrightarrow M/Q_1$$

and therefore,

$$(Q_2 \cap \cdots \cap Q_n)/N \hookrightarrow M/Q_1$$

which implies that,

$$\mathrm{Ass}_A((Q_2 \cap \cdots \cap Q_n)/N) \subset \mathrm{Ass}_A(M/Q_1) = \{\mathfrak{p}_1\}$$

so since it is nonempy we have,

$$\{\mathfrak{p}_1\} = \mathrm{Ass}_A((Q_2 \cap \cdots \cap Q_n)/N) \subset \mathrm{Ass}_A(M/N)$$

where the inclusion holds via the exact sequence,

$$0 \longrightarrow N \longrightarrow Q_2 \cap \cdots \cap Q_n \longrightarrow M/N$$

The same argument holds for each i. $\square$

Theorem 9.3.6. Let M be Noetherian. For each $\mathfrak{p} \in \mathrm{Ass}_A(M)$, there exist $Q_{\mathfrak{p}} \subset M$ which are $\mathfrak{p}$-primary such that,

$$\bigcap_{\mathfrak{p} \subset \mathrm{Ass}_A(M)} Q_{\mathfrak{p}} = 0$$

Proof. Fix $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and consider the set $S_\mathfrak{p} = \{Q \subset M \mid \mathfrak{p} \notin \mathrm{Ass}_A(Q)\} \neq \varnothing$ since the zero module is contained in this set. Since M is Noetherian ascending chains stabilize so by Zorn's lemma there exists a maximal element $Q_\mathfrak{p} \in S_\mathfrak{p}$. We know,

$$\mathrm{Ann}_A(M/Q_\mathfrak{p}) \neq \varnothing$$

since we have $M/Q_\mathfrak{p} \neq (0)$. Otherwise, $M = Q_\mathfrak{p}$ which implies $\mathfrak{p} \in \mathrm{Ass}_A(Q_\mathfrak{p})$ but $Q_\mathfrak{p} \in S_\mathfrak{p}$. Let $\mathfrak{p}' \in \mathrm{Ass}_A(M/Q_\mathfrak{p})$ and suppose that $\mathfrak{p}' \neq \mathfrak{p}$ then we have,

$$A/\mathfrak{p}' \hookrightarrow M/Q_\mathfrak{p}$$

The image of this embedding is a submodule, $Q_\mathfrak{p} \subsetneq Q' \subset M$ such that $Q'/Q_\mathfrak{p} \cong A/\mathfrak{p}'$ implying that,

$$\mathrm{Ass}_A(Q'/Q_\mathfrak{p}) = \{\mathfrak{p}'\}$$

Thus we have an exact sequence,

$$0 \longrightarrow Q_\mathfrak{p} \longrightarrow Q' \longrightarrow A/\mathfrak{p} \longrightarrow 0$$

which implies that $\mathrm{Ass}_A(Q') \subset \mathrm{Ass}_A(Q_\mathfrak{p}) \cup \mathrm{Ass}_A(A/\mathfrak{p}') = \mathrm{Ass}_A(Q_\mathfrak{p}) \cup \{\mathfrak{p}'\}$. However, this contradicts the fact that $Q_\mathfrak{p}$ is maximal in $S_\mathfrak{p}$ since $Q' \in S_\mathfrak{p}$ as long as $\mathfrak{p}' \neq \mathfrak{p}$. Therefore, $\mathfrak{p}' = \mathfrak{p}$ so $\mathrm{Ass}_A(A/Q_\mathfrak{p}) = \{\mathfrak{p}\}$. Now consider,

$$\mathrm{Ass}_A\left(\bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(M)} Q_\mathfrak{p}\right) \subset \bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(M)} \mathrm{Ass}_A(Q_\mathfrak{p}) = \varnothing$$

because for any $\mathfrak{p}$ we know $\mathfrak{p} \notin \mathrm{Ass}_A(Q_\mathfrak{p})$. Therefore,

$$\bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(M)} Q_\mathfrak{p} = (0)$$

since it has no associated primes. $\square$

Corollary 9.3.7. If M is a finite A-module then any submodule has a primary decomposition.

Proof. Let $N \subset M$ be a submodule. Apply the theorem to $\bar{M} = M/N$ which has finite type so $\mathrm{Ass}_A(M/N)$ is finite. Write, $\mathrm{Ass}_A(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Therefore, there exist primary ideals Q_i such that,

$$Q_{\mathfrak{p}_1} \cap \cdots \cap Q_{\mathfrak{p}_r} = (0)$$

in M/N. Take Q_i to be the preimage of $Q_{\mathfrak{p}_i}$. Thus,

$$Q_1 \cap \cdots \cap Q_r = N$$

and

$$M/Q_i \cong \bar{M}/Q_{\mathfrak{p}_i} \implies \mathrm{Ass}_A(M/Q_i) = \{\mathfrak{p}_i\}$$

$\square$

9.4 Weakly Associated Points

Definition 9.4.1. Let X be a scheme and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then we define,

(a) $x \in X$ is *weakly associated* to $\mathscr{F}$ if $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is weakly associated to $\mathscr{F}_x$

(b) $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ is the set of weakly associated points of $\mathscr{F}$

(c) the (weakly) associated points of X are $\mathrm{WAss}_{\mathcal{O}_X}(\mathcal{O}_X)$.

Proposition 9.4.2. Let $X = \mathrm{Spec}(A)$ and $\mathscr{F} = \widetilde{M}$ be a quasi-coherent $\mathcal{O}_X$-module then we have,

$$\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F}) = \mathrm{WAss}_A(M)$$

Proof. Immediate consequence of Lemma 9.1.4. □

Proposition 9.4.3. Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then,

$$\mathscr{F} = 0 \iff \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F}) = 0$$

Proof. Choose an affine open cover $U_i = \mathrm{Spec}(A_i)$ such that $\mathscr{F}|_{U_i} = \widetilde{M_i}$. Then $\mathrm{WAss}_A(M_i) = \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F}) \cap U_i = \varnothing$ so $M_i = 0$ and thus $\mathscr{F} = 0$. □

Proposition 9.4.4. Let X be a scheme and $\mathscr{F} \to \mathscr{G}$ a morphism of quasi-coherent $\mathcal{O}_X$-modules. If $\mathscr{F}_x \to \mathscr{G}_x$ is injective for each $x \in \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ then $\mathscr{F} \to \mathscr{G}$ is injective.

Proof. Consider the sequence,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}$$

Since $\mathscr{F}_x \to \mathscr{G}_x$ is an injection $\mathscr{K}_x = 0$ for each $x \in \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$. Furthermore, $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{K}) \subset \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ and thus $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{K}) = \varnothing$ so $\mathscr{K} = 0$. □

9.5 Associated Points: the Noetherian Case

Remark. By analogy, we might define an *associated point* of $\mathscr{F}$ on X to be a point $x \in X$ such that $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is an associated prime of $\mathscr{F}_x$. However, this definition is problematic because, in general, associated primes do not play nicely with localization. In particular $\mathfrak{p} \in \mathrm{Ass}_A(M) \implies \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ but the converse may not hold. Therefore, we may have a scheme X and a quasi-coherent sheaf $\mathscr{F}$ such that on an affine open $U = \mathrm{Spec}(A)$ with $\mathscr{F}|_U = \widetilde{M}$ we have $\mathfrak{p} \in \mathrm{Ass}_A(M)$ but $\mathfrak{p} = x \in X$ is not as associated point of $\mathscr{F}$ on X. To recify this pathology, we only consider associated points on locally noetherian schemes in which case there is no difference between weakly associated points and associated points.

Definition 9.5.1. Let X be a locally noetherian scheme and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. We say $x \in X$ is an *associated point* of $\mathscr{F}$ if x is a *weakly associated point.* Likewise we write,

$$\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F}) = \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$$

Remark. Notice this definition is purely notational. In the locally noetherian case we simply will write $\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F})$ for $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ as a reminder that these sets behave as expected for associated points in the case of Noetherian rings.

Proposition 9.5.2. Let X be noetherian and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module. Then $\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F})$ is finite.

Proof. Since X is quasi-compact we may choose a finite open cover $U_i = \mathrm{Spec}(A_i)$ with A_i Noetherian on which $\mathscr{F}|_{U_i} = \widetilde{M_i}$ for finite A_i-modules. Then $\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F}) \cap U = \mathrm{Ass}_{A_i}(M_i)$ each of which is finite since M_i is a Noetherian module. □

10 Pseudomorphisms

Lemma 10.0.1. Let $f : X \to Y$ be a morphism of schemes such that for each weakly associated point $y \in Y$ there exists a point $x \in X$ such that $f(x) = y$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. Then the map on sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective.

Proof. To show that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective, it suffices to show that $\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y$ is injective on each weakly associated point $y \in Y$. Furthermore, we know there exists $x \in X$ with $f(x) = y$ and the composition $\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y \to \mathcal{O}_{X,x}$ is injective and thus $\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y$ is injective. □

Remark. In particular, if $f : X \to Y$ is a dominant map of integral schemes then $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective.

Example 10.0.2. Consider the map $\mathrm{Spec}\,(k[x]) \to \mathrm{Spec}\,(k[x,y]/(xy, y^2))$. Although this map hits the generic point (y), it does not hit the embedded associated point (x, y^2) at the origin and thus $k[x,y]/(xy,y^2) \to k[x]$ is not injective since $y \mapsto 0$.

Definition 10.0.3. We say an immersion $\iota : Y \hookrightarrow X$ is *scheme theoretically dense* if the scheme theoretic image is X.

Lemma 10.0.4. An open immersion $\iota : U \to X$ is scheme theoretically dense iff U contained all weakly associated points of X.

Proof. □

When can we ensure that the coker of $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is supported in codimension one.

10.1 Closed Immersions of Locally Ringed Spaces

Definition 10.1.1. A map $f : X \to Y$ of ringed spaces is a *closed immersion* if,

(a) $f : X \to Y$ is topologically a homeomorphism onto a closed subset

(b) $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective,

Remark. The stacks project requires that $\mathscr{I} = \ker(\mathcal{O}_Y \to f_* \mathcal{O}_X)$ be locally generated by sections such that in the case of schemes $\mathscr{I}$ is quasi-coherent. Therefore, every closed immersion into a scheme comes from a scheme showing that closed subspaces of schemes are automatically schemes. We do not take this definition so schemes may have closed subspaces which are not schemes however the closed subspaces given by quasi-coherent sheaves of ideals are schemes.

Remark. Recall the Zariski tangent space at $x \in X$ is $T_{X,x} = \mathrm{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x))$.

Proposition 10.1.2.

Lemma 10.1.3. Let $\iota : Z \hookrightarrow X$ be a closed immersion of locally ringed spaces. Then there are equivalences of categories,

$$\{\text{abelian sheaves on } Z\} \xrightarrow{\sim} \{\text{abelian sheaves on } X \text{ supported on } \iota(Z)\}$$
$$\mathscr{F} \mapsto \iota_* \mathscr{F}$$
$$\iota^{-1}\mathscr{G} \mapsfrom \mathscr{G}$$

and likewise,

$$\mathbf{Mod}_{\mathcal{O}_X} \xrightarrow{\sim} \{\text{abelian sheaves on } X \text{ supported on } \iota(Z)\}$$
$$\mathscr{F} \mapsto \iota_* \mathscr{F}$$
$$\iota^{-1}\mathscr{G} \mapsfrom \mathscr{G}$$

10.2 Annhiliators

Remark. Here we let X be a scheme. Warning: many of these results do not hold for arbitrary locally ringed spaces. In particular, the kernel of quasi-coherent sheaves need not be quasi-coherent on an arbitrary locally ringed space. However, this holds locally on schemes because kernels and cokerns of sheaves associated to modules are associated to modules.

Definition 10.2.1. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then we define the sheaf of annihilators:

$$\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F}) = \ker\left(\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})\right)$$

Lemma 10.2.2. Let $\mathscr{F}, \mathscr{G}$ be quasi-coherent $\mathcal{O}_X$-modules with $\mathscr{F}$ finitely presented. Then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is quasi-coherent.

Proof. Locally on $U \subset X$ we have a presentation,

$$\bigoplus_{i=1}^m \mathcal{O}_U \longrightarrow \bigoplus_{j=1}^n \mathcal{O}_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Applying the functor $\mathscr{H}om_{\mathcal{O}_U}(-, \mathscr{G})$ gives,

$$0 \longrightarrow \mathscr{H}om_{\mathcal{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U) \longrightarrow \bigoplus_{j=1}^n \mathscr{G}|_U \longrightarrow \bigoplus_{i=1}^m \mathscr{G}|_U$$

Since $\mathscr{G}$ is quasi-coherent and finite sums and kernels of quasi-coherent sheaves are quasi-coherent we see that $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is locally quasi-coherent and thus quasi-coherent. $\square$

Lemma 10.2.3. If $\mathscr{F}$ is finitely presented then $\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ is quasi-coherent.

Proof. From the previous lemma, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})$ is quasi-coherent. Therefore, the kernel,

$$\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F}) = \ker\left(\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})\right)$$

is quasi-coherent. $\square$

Proposition 10.2.4. Let $\mathscr{F}$ be finitely presented. Then $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F})$ is closed and the quasi-coherent sheaf of ideals $\mathrm{Ann}_{\mathcal{O}_X}(\mathscr{F})$ gives a scheme structure on $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F})$. Furthermore, $\mathscr{F}$ is naturally a $\mathcal{O}_X / \mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ - module.

Lemma 10.2.5. Let $f : X \to Y$ be a morphism of schemes. Assume that $\mathcal{O}_Y$ and $f_* \mathcal{O}_X$ are coherent on Y. Furthermore, for each generic point of an irreducible component $\xi \in Y$ assume that there exists some $x \in X$ with $f(x) = \xi$ and $\mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,x}$ surjective. Then $\mathscr{C} = \mathrm{coker}\left(\mathcal{O}_Y \to f_* \mathcal{O}_X\right)$ has $Z = \mathrm{Supp}_{\mathcal{O}_Y}(\mathscr{C})$ in positive codimension.

11 Singularities of Curves

Definition 11.0.1. NORMALIZATION

Proposition 11.0.2. Normalization of a curve exists and is regular.

(CAN WE GET $H^0(O_X)$ is the same?)