Lecture 3

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1 Meromorphic Functions: a reprise

There was some confusion about meromorphic functions. Here I hope to clear up some of the confusion. The definition we have previously was:

Definition: We say a function $f : \Omega \to \mathbb{C}$ for $\Omega \subset \mathbb{C}$ is meromorphic if there is a set of isolated poles $P \subset \Omega$ such that f is holomorphic on $\Omega \setminus P$ and f has a pole at each point $p \in P$.

where having a pole means the quite specific statement defined as follows.

Definition: We say that a function $f : \mathbb{C} \to \mathbb{C}$ has a *pole* of order n at z_0 if near z_0 we can write,

$$f(z) = (z - z_0)^{-n} u(z)$$

where $u(z)$ is a nonvanishing holomorphic function.

Definition: We say a function $f : \Omega \to \mathbb{C}$ for $\Omega \subset \mathbb{C}$ is *meromorphic* if there is a set of isolated poles $P \subset \Omega$ such that f is holomorphic on $\Omega \setminus P$ and f has a pole at each point $p \in P$.

We see that a function with a jump discontinuity cannot be meromorphic because, in that case, near the jump, the function would not blow up like $\sim \frac{1}{z^n}$. We can give an alternative definition which may be more intuitive.

Definition: A function f is *meromorphic* if it is a ratio of two holomorphic functions g, h i.e.

$$f(z) = \frac{g(z)}{h(z)}$$

where h is not identically zero i.e. $h \neq 0$ (h is not the constant zero function).

Perhaps a more sophisticated definition of a meromorphic functions involves the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which is the complex plane closed into a sphere with the point at infinity. We will think about this space much more when we talk about projective space. Then we could make the following definition.

Definition: A *meromorphic* function on $\mathbb{C}$ is a holomorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$.

This corresponds to our previous notion that whenever f is undefined (i.e. is sent to ∞) then f must "blow up" nearby. This corresponds to the fact that for a continuous function $f : \mathbb{C} \to \hat{\mathbb{C}}$ to hit ∞ it must, nearby, pass through arbitrarily large values in $\mathbb{C} \subset \hat{\mathbb{C}}$. Thus we can't have a function which is nice and bounded and suddenly jumps up to ∞ since this corresponds to a discontinuity viewed as a map $\mathbb{C} \to \hat{\mathbb{C}}$. Now let's try to prove that these definitions agree.

Theorem 1.1. Let $f : \mathbb{C} \setminus P \to \mathbb{C}$ be a holomorphic function where $P \subset \mathbb{C}$ is an isolated set of points. Then the following are equivalent:

1. near each $p \in P$ we can write $f(z) = (z-p)^{-n} u(z)$ where $u(z)$ is holomorphic and nonvanishing near p.
2. $f = \frac{g}{h}$ where $g, h : \mathbb{C} \to \mathbb{C}$ are entire functions and h is nonvanishing outside of P.
3. there exists a holomorphic map $\tilde{f} : \mathbb{C} \to \hat{\mathbb{C}}$ extending f i.e. $\tilde{f}$ restricts to f when $\tilde{f}(z) \neq \infty$,

$$\tilde{f}|_{\mathbb{C} \setminus P} = f \text{ viewed as a map } \mathbb{C} \setminus P \to \mathbb{C} = \hat{\mathbb{C}} \setminus \{\infty\} \subset \hat{\mathbb{C}}$$

2 The Weierstrass $\wp$ Function

Our plan to find a holomorphic doubly periodic (elliptic) function has been thwarted as soon as it was devised. Since we cannot find interesting holomorphic examples of elliptic functions we now ask for the next best thing. We want to consider elliptic meromorphic functions. That is, meromorphic functions $f : \mathbb{C}/\Lambda \to \mathbb{C}$. It turns out that now were are in luck. Notice the following fact. If $f : \mathbb{C} \to \mathbb{C}$ is any function then,

$$g(z) = \sum_{\omega \in \Lambda} f(z + \omega)$$

is invariant under shifts by any $\omega \in \Lambda$ since this simply reorders the summation. However, in general there is no reason such a function should converge. In fact. using Liouville's theorem, we know that if f is any holomorphic function that g cannot be a convergent holomorphic function. Thus we should try with a simple meromorphic f instead. In order for the sum to converge, we need that f goes to zero sufficiently quickly (faster that $1/z$ because harmonic series do not converge). Therefore, we might try the meromorphic function is z^{-2} and try,

$$f(z) = \sum_{\omega \in \Lambda} \frac{1}{(z + \omega)^2}$$

which is doubly periodic because if $\omega \in \Lambda$ then,

$$f(z + \omega) = \sum_{\omega' \in \Lambda} \frac{1}{(z + \omega + \omega')^2} = \sum_{\omega'' \in \Lambda} \frac{1}{(z + \omega'')^2} = f(z)$$

However, this definition has one major flaw. That sum still does not converge! This is because,

$$f(z) = \sum_{n,m \in \mathbb{Z}} \frac{1}{(z + n\omega_1 + m\omega_2)^2} \sim \sum_{r \in \mathbb{Z}^+} \frac{2\pi r}{(z + r)^2} \sim 2\pi \sum_{r \in \mathbb{Z}^+} \frac{r}{r^2} \to \infty$$

where I have reordered the sum to sum over circles with radius r in the $\mathbb{Z}^2$ plane each circle having approximately $2\pi r$ lattice points. To fix this, we use a clever subtraction to remove the divergent part of the sum and arrive at the following definition by Weierstrass.

Theorem 2.1. Let $\Lambda \subset \mathbb{C}$ be a Lattice. The Weierstrass $\wp$-function, defined as,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left[\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right]$$

is a well-defined meromorphic function $\wp : \mathbb{C}/\Lambda \to \mathbb{C}$ with double poles on Λ.

Proof. We can write,

$$\frac{1}{(z + \omega)^2} = -\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{z + \omega} \right) = -\frac{1}{\omega} \frac{\mathrm{d}}{\mathrm{d}z} \left[\sum_{n=0}^{\infty} \left(-\frac{z}{\omega} \right)^n \right] = -\frac{1}{\omega} \sum_{n=0}^{\infty} \left[\frac{(-1)^n n z^{n-1}}{\omega^n} \right]$$
$$= \frac{1}{\omega^2} \left[\sum_{n=0}^{\infty} (-1)^n (n+1) \frac{z^n}{\omega^n} \right]$$

and this series is uniformly convergent for $|z| < |\omega|$. Thus,

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2} \sum_{n=1}^{\infty} (-1)^n (n+1) \frac{z^n}{\omega^n}$$

and therefore,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

converges uniformly. Furthermore,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\omega \in \Lambda^\times} \left(-\frac{2}{(z+\omega)^3} \right) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^3}$$

Thus, $\wp'(z)$ is doubly periodic with periods ω_1 and ω_2. Thus, $\wp(z+\omega_1) = \wp(z) + c_1$ so $\wp(-\frac{1}{2}z_1) = \wp(\frac{1}{2}z) + c_1$ but $\wp$ is even so $c_1 = 0$. The same for ω_2. Thus, $\wp(z+\omega_1) = \wp(z)$ and $\wp(z+\omega_2) = \wp(z)$. $\square$

2.1 The Defining Differential Equation for $\wp$

We need to work out he expansion for $\wp(z)$ near 0. We have,

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2} \cdot \frac{1}{(1-\frac{z}{\omega})^2} = \frac{1}{\omega^2} \sum_{n=0}^{\infty} (m+1) \left(\frac{z}{\omega} \right)^m$$

Therefore,

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \sum_{m=1}^{\infty} (m+1) \frac{z^m}{\omega^{m+2}} \\
&= \frac{1}{z^2} + \sum_{m=1}^{\infty} (m+1) z^m \left(\sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{m+2}} \right) \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) z^{2k} G_{k+1}(\Lambda)
\end{aligned}$$

where we have defined,

$$G_k(\Lambda) = \sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{2k}}$$

The odd terms vanish because if we sum an odd function over the lattice then we get zero. Explicitly,

$$\wp(z) = \frac{1}{z^2} + 3G_2(\Lambda) z^2 + 5G_3(\Lambda) z^4 + O(z^6)$$

Next,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty} (2k+1)(2k) G_{k+1}(\Lambda) z^{2k-1}$$

which we sum as,

$$\wp'(z) = -\frac{2}{z^3} + 6G_2(\Lambda) z + 20 G_3(\Lambda) z^3 + O(z^5)$$

Thus, compute,

$$\begin{aligned}
\wp'(z)^2 &= \left(-\frac{2}{z^3} + 6G_2(\Lambda) z + 20G_3(\Lambda) z^3 + O(z^5) \right)^2 \\
&= \frac{4}{z^6} - 24 G_2(\Lambda) \frac{1}{z^2} - 80 G_3(\Lambda) + O(z^2)
\end{aligned}$$

Similarly, compute,

$$\begin{aligned}\wp(z)^3 &= \left(\frac{1}{z^2} + 3G_2(\Lambda)z^2 + 5G_3(\Lambda)z^4 + O(z^6)\right)^3 \\ &= \frac{1}{z^6} + 9G_2(\Lambda)\frac{1}{z^2} + 15G_3(\Lambda) + O(z^2)\end{aligned}$$

Therefore,

$$\begin{aligned}\wp'(z)^2 - 4\wp(z)^3 &= -24G_2(\Lambda)\frac{1}{z^2} - 36G_2(\Lambda)\frac{1}{z^2} - 80G_3(\Lambda) - 60G_3(\Lambda) + O(z^2) \\ &= -60G_2(\Lambda)\frac{1}{z^2} - 140G_3(\Lambda) + O(z^2)\end{aligned}$$

Which implies that,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda) = O(z^2)$$

Therefore the function,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda)$$

is holomorphic on $\mathbb{C}$ and is doubly periodic and thus constant. However, it vanishes at $z = 0$ and thus must be the zero function. Therefore, we have the differential equation,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda) = 0$$

We can interpret this differential equation as telling us that the image of the elliptic Weierstrass $\wp$-function lies on a special type of object called an elliptic curve.