Math 56: Proofs and Modern Mathematics
Homework 3 Solutions

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Problem 1 (Axler 2.B.4). (i) Let U be the subspace of $\mathbb{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(ii) Extend the basis in (i) to a basis of $\mathbb{C}^5$.

(iii) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

Solution. (a) Let $(z_1, z_2, z_3, z_4, z_5)$ be an element in U. The equations defining U give us the relations $z_2 = 6z_1$, $z_3 = -2z_4 - 3z_5$, so we can rewrite this as

$$(z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5) = z_1(1, 6, 0, 0, 0) + z_4(0, 0, -2, 1, 0) + z_5(0, 0, -3, 0, 1).$$

This shows that the vectors $u_1 = (1, 6, 0, 0, 0), u_2 = (0, 0, -2, 1, 0), u_3 = (0, 0, -3, 0, 1)$ span U. Suppose that $a_1u_1 + a_2u_2 + a_3u_3 = 0$. This gives us the equations

$$\begin{aligned} a_1 &= 0 \\ 6a_1 &= 0 \\ -2a_2 - 3a_3 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \end{aligned}$$

The first, fourth, and fifth equations give us $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. Hence u_1, u_2, u_3 are linearly independent, and so form a basis for U.

(b) To extend u_1, u_2, u_3 to a basis for $\mathbb{C}^5$, we need to add two additional vectors to the set while keeping it linearly independent. From Problem 1, if we add a vector w_1 that is not in $\text{span}(u_1, u_2, u_3) = U$, then u_1, u_2, u_3, w_1 is still linearly independent. Let us take $w_1 = (1, 0, 0, 0, 0)$; this is not in U since $z_2 \neq 6z_1$, so u_1, u_2, u_3, w_1 is still linearly independent. However, w_1 does satisfy the second equation $z_3 + 2z_4 + 3z_5 = 0$, so every vector in $\text{span}(u_1, u_2, u_3, w_1)$ must satisfy $z_3 + 2z_4 + 3z_5 = 0$. With this in mind, we take $w_2 = (0, 0, 1, 0, 0)$, which does not satisfy this equation, so $w_2 \notin \text{span}(u_1, u_2, u_3, w_1)$, hence u_1, u_2, u_3, w_1, w_2 are linearly independent and form a basis for $\mathbb{C}^5$.

(c) We define $W = \text{span}(w_1, w_2)$. First, we note that U, W are subspaces, since they are spans of a set of vectors. Second, given an arbitrary vector $v \in \mathbb{C}^5$, we have

$$v = a_1 u_a + a_2 u_2 + a_3 u_3 + b_1 w_1 + b_2 w_2 = u + w,$$

where $u = a_1 u_a + a_2 u_2 + a_3 u_3$ is in U and $w = b_1 w_1 + b_2 w_2$ is in W. Hence $\mathbb{C}^5 = U + W$. Finally, suppose that $v \in U \cap W$. Using the bases for U and W, this means that we have scalars a_1, a_2, a_3, b_1, b_2 such that

$$v = a_1 u_1 + a_2 u_2 + a_3 u_3 = b_1 w_1 + b_2 w_2,$$

which is equivalent to the equation

$$a_1 u_1 + a_2 u_2 + a_3 u_3 - b_1 w_1 - b_2 w_2 = 0.$$

By linear independence of the basis, we have $a_1, a_2, a_3, b_1, b_2 = 0$. Hence $v = 0$, so $U \cap W = 0$, and $\mathbb{C}^5 = U \oplus W$.

Problem 2 (Axler 2.B.6)**.** Suppose that v_1, v_2, v_3, v_4 is a basis of V. Prove that $v_1 + v2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of V.

Solution. We need to prove that $v_1 + v2, v_2 + v_3, v_3 + v_4, v_4$ are linearly independent and span V.

Linearly independent: Suppose that we have

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4 = 0.$$

Rearranging, this gives

$$a_1 v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_{=}0.$$

Since v_1, v_2, v_3, v_4 are linearly independent, all these coefficients must be 0, so we have the equations

$$\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_2 + a_3 &= 0 \\ a_3 + a_4 &= 0 \end{aligned}$$

The first equation gives us $a_1 = 0$. Plugging that into the second equation gives us $a_2 = 0$. Continuing to substitute, we get $a_3 = 0$ and $a_4 = 0$. Hence $v_1 + v2, v_2 + v_3, v_3 + v_4, v_4$ are linearly independent.

Span V: Let v be any vector in V. Since v_1, v_2, v_3, v_4 is a basis, we have $v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$. We can rewrite this as

$$v = a_1v_1+a_2v_2+a_3v_3+a_4v_4 = a_1(v_1+v_2)+(a_2-a_1)(v_2+v_3)+(a_3-a_2+a_1)(v_3+v_4)+(a_4-a_3+a_2-a_1)v_4.$$

Hence every $v \in V$ can be written as a linear combination of $v_1 + v2, v_2 + v_3, v_3 + v_4, v_4$, so $v_1 + v2, v_2 + v_3, v_3 + v_4, v_4$ span V.

Having proven that $v_1 + v2, v_2 + v_3, v_3 + v_4, v_4$ are linearly independent and span V, we conclude that $v_1 + v2, v_2 + v_3, v_3 + v_4, v_4$ is a basis.

Note: if we make use of the fact that $\dim V = 4$, we only need to prove one of the above properties.

Problem 3 (Axler 2.B.5)**.** Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Solution. This is TRUE. One way we can find such a basis is using the previous problem. The standard basis for $\mathcal{P}_4(\mathbb{F})$ is $1, x, x^2, x^3$. Let $v_1 = x^2, v_2 = x^3, v_3 = 1, v_4 = x$. By the previous problem, $x^2 + x^3, x^3 + 1, 1 + x, x$ is also a basis for $\mathcal{P}_3(\mathbb{F})$, and none of these has degree 2.

Note: this is not the only possible basis where none of the polynomials has degree 2, e.g. another possibility is $1, x, x^2 + x^3, x^3$.

Problem 4 (Axler 2.B.8)**.** Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis of V.

Solution. We need to prove that $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent and spans V.

Linearly independent: Suppose that we have the equation

$$a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n = 0.$$

We can rearrange this to get the equation

$$a_1u_1 + \cdots + a_mu_m = -b_1w_1 - \cdots - b_nw_n.$$

The left-hand side of this equation is in U and the right-hand side is in W. Since $V = U \oplus W$, we have $U \cap W = 0$, so both sides of the equation, being in both U and W, must be 0. This gives us the equations

$$\begin{aligned} a_1u_1 + \cdots + a_mu_m &= 0 \\ b_1w_1 + \cdots + b_nw_n &= 0 \end{aligned}$$

Since $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W, both sets are linearly independent, so all the coefficients must be 0. Hence $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent.

Spans V: Let v be an arbitrary element of V. Since $V = U \oplus W$, there exist $u \in U$ and $w \in W$ such that $v = u + w$. Since $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W, we have $u = a_1u_1 + \cdots + a_mu_m$, $w = b_1w_1 + \cdots + b_nw_n$ for some scalars $a_1, \ldots, a_m$, $b_1, \ldots, b_n$. Hence $v = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n$, so that $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V.

Hence $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis for V.

Problem 5 (Axler 2.C.10)**.** Suppose $p_0, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ are such that each p_j has degree j. Prove that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Solution. We need to prove that $p_0, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ is linearly independent and spans $\mathcal{P}_m(\mathbb{F})$.

To make things easier, we'll define some notation at the start: let $p_j(x) = p_{j0} + p_{j1}x + p_{j2}x^2 + \cdots + p_{jj}x^j$, where $p_{jj} \neq 0$ since p_j has degree j..

Linearly independent: Suppose that

$$a_0p_0 + \cdots + a_mp_m = 0.$$

Consider the highest-degree term on the left-hand side of this equation: it's $a_mp_{mm}x^m$. This has to be zero, and $p_{mm} \neq 0$, so $a_m = 0$. (Recall that if $xy = 0$ in a field, either $x = 0$ or $y = 0$.) The new highest degree term is $a_{m-1}p_{m-1\quad m-1}x^{m-1}$, and in the same way, we must have $a_{m-1} = 0$. Applying this reasoning repeatedly to remove the highest degree term, we ultimately find that $a_j = 0$ for all j. Hence $p_0, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ is linearly independent.

Spans $\mathcal{P}_m(\mathbb{F})$: Given an arbitrary polynomial $f \in \mathcal{P}_m(\mathbb{F})$, we want to show that $f = a_0 p_0 + \cdots + a_m p_m$ for some scalars $a_0, \dots, a_m \in \mathbb{F}$. We can write $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_m x^m$, so we want to solve

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_m x^m = a_0 p_0 + \cdots + a_m p_m.$$

As above, we first look at the term of highest degree: on the left-hand side, we have $f_m x^m$, and on the right-hand side, we have $a_m p_{mm} x^m$. Since $p_{mm} \neq 0$, we can divide to get $a_m = f_m / p_{mm}$. This means that the x^m terms match on both sides, so if we subtract $a_m p_{mm}$ from both sides, we have

$$f - a_m p_{mm} = a_0 p_0 + \cdots + a_{m-1} p_{m-1} \quad {}_{m-1}.$$

We now do the same thing, but with the x^{m-1} term, which is now the leading term on both sides, and this will give us a_{m-1}. Applying this repeatedly, we get each of $a_{m-2}, \dots, a_0$ in turn. Hence $f = a_0 p_0 + \cdots + a_m p_m$, so $p_0, \dots, p_m \in \mathcal{P}(\mathbb{F})$ spans $\mathcal{P}_m(\mathbb{F})$.

Note: Arguments based on dimension will allow you to prove only one of the above properties.

Problem 6. Prove that if X is a finite dimensional vector space and if V, W are subspaces of X with $V \subset W$ and $\dim V = \dim W$, then $V = W$.

Solution. Let $n = \dim V = \dim W$. Suppose we have a basis of V; by definition of dimension, this basis will have $\dim V = n$ elements in it. We can extend this to a basis of W, but it already has the correct number $n = \dim W$ of elements, and so the basis for V must be the same as the basis for W. Hence W is the span of a basis of V, so $W = V$.