1 Basic Definitions

Definition 1.0.1. We say a k-group G is *algebraic* if it is finite type over k.

Proposition 1.0.2. A connected algebraic k-group G is geometrically connected and geometrically irreducible.

Proof. See HW 1 for proof. $\square$

Remark. Over an imperfect field k a reduced connected algebraic k-group need not be geometrically reduced (and thus not smooth). For example see W1 Exercise 4(iii).

Proposition 1.0.3. Any k-group scheme is automatically separated.

Proof. The map $e : \mathrm{Spec}\,(k) \to G$ is a closed immersion. Notice that $\Delta_{G/k} : G \to G \times_k G$ is the base change of e via $f = m \circ (\mathrm{id}, \iota) : G \times_k G \to G$ because,

$$\begin{tikzcd}
G \arrow[dr, dashed] \arrow[drr, bend left, "\Delta_{G/k}"] \arrow[ddr, bend right] & & \\
& X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \times_k G \arrow[d, "f"] \\
& \mathrm{Spec}\,(k) \arrow[r, "e"] & G
\end{tikzcd}$$

commutes giving an arrow $G \to X$. However, $X \to G \times_k G$ gives a pair of maps $f, g : X \to G$ such that $m \circ (f, \iota \circ g) = e$ then by the group axioms,

$$g = m \circ (e, g) = m \circ (m \circ (f, \iota \circ g), g) = m \circ (f, m(\iota \circ g, g)) = m \circ (f, e) = f$$

so $f = g$ and thus $(f, g) = \Delta_{G/k} \circ f$ so the universal map factors uniquely through $\Delta_{G/k}$ and thus $G \to X$ is an isomorphism. $\square$

2 Smoothness of Algebraic Groups

Lemma 2.0.1. Let A be a local Noetherian k-algebra with residue field k. Let k'/k be a field extension $\mathfrak{m}' = \ker (A \otimes_k k' \to k')$. Assume that $A' = (A_{k'})_{\mathfrak{m}'}$ is Noetherian. Then A is regular if and only if A' is regular.

Proof. The map $A \to A'$ is a local map of local Noetherian ring. Furthermore, $k \to k'$ is flat so $A \to A_{k'}$ is flat and thus $A \to A_{k'} \to A'$ is flat because localization is flat. Thus, by the dimension formula,

$$\dim A' = \dim A + \dim A'/\mathfrak{m}A'$$

but $\mathfrak{m}A' = \mathfrak{m}'$ so $\dim A'/\mathfrak{m}A' = \dim k' = 0$. Thus,

$$\dim A' = \dim A$$

Furthermore, by flatness of k'/k we see that $k' \otimes_k \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathfrak{m}'/\mathfrak{m}'^2$ canonically. Thus,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_{k'} \mathfrak{m}'/\mathfrak{m}'^2$$

Therefore,

$$\dim A' = \dim_{k'} \mathfrak{m}'/\mathfrak{m}'^2 \iff \dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$$

□

Remark. This fails for $A = \mathbb{Q}[[x]]$ and $k' = \mathbb{C}$ because $\mathfrak{m}' = \ker(\mathbb{Q}[[x]] \otimes_{\mathbb{Q}} \mathbb{C} \twoheadrightarrow \mathbb{C})$ is exactly the ideal (X). Notice that if $\alpha \in \mathbb{C}$ is transcendental then a power series like,

$$1 - \alpha x$$

is not in $\mathfrak{m}'$ but has no inverse because its inverse,

$$1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots$$

is not in $A_{\mathbb{C}}$ because it has infinitely many $\mathbb{Q}$-independent coefficients. Thus A' is larger than $A_{\mathbb{C}}$ and is not Noetherian.

Remark. Consider $k = \mathbb{F}_p(t)$ and $k' = \mathbb{F}_p(t^{\frac{1}{p}})$ then let $A = \mathbb{F}_p[t]/(t^p - 1) \cong \mu_p \cong k'$. Then $A \otimes_k k' = k'[t]/(t^{\frac{1}{p}} - 1)^p$ is non-reduced but local and Noetherian. This fails because the residue field of A is k'. It is the rationality of the point $\mathfrak{m}$ that saves the day.

Proposition 2.0.2. Let G be an algebraic k-group. Then the following are equivalent,

(a) G is smooth

(b) $G_{\bar{k}}$ is regular

(c) $\mathcal{O}_{G_{\bar{k}}, e_{\bar{k}}}$ is regular

(d) $\mathcal{O}_{G,e}$ is regular

Proof. The equivalence of (a) and (b) is general. The equivalence of (b) and (c) comes from translation by $G(\bar{k})$ acting transitively on $G_{\bar{k}}$ and thus if one point is regular then every point is regular since translation is an automorphism. Equivalence of (c) and (d) follow from the lemma. □

Proposition 2.0.3. An algebraic k-group is smooth if and only if it is geometrically reduced.

Proof. Since $G_{\bar{k}}$ is a reduced finite type $\bar{k}$-scheme, by generic smoothness, it has a smooth point. Therefore, every point is smooth by translation. See the proof of Lemma 3.2.3. for more details. □

Remark. Therefore, we can add the condition that $\mathcal{O}_{G,e}$ is geometrically reduced to the above list.

Corollary 2.0.4. Over a perfect field k, an algebraic k-group is smooth if and only it it is reduced.

Proof. Over a perfect field, any reduced k-algebra is geometrically reduced by Tag 030U. □