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(a) Measure theory notes

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# 2 Important Notes to Self

## 2.1 Common Mistakes

(a) Reflexive seperable Banach spaces need not admit a Schauder basis!! However, separable Hilbert spaces always admit an othogonal Schauder basis and thus are all isomorphic to $\ell^2$.

# 3 Measure Theory Definitions and Theorems

**Definition 3.0.1.** A measure space $(X, \mathcal{F}, \mu)$ is called $\sigma$-finite if there exists a countable cover $\{A_i\}$ of $X$ by measurable sets $A_i \in \mathcal{F}$ with $\mu(A_i) < \infty$.

## 3.1 Integration

*Remark.* Fix a measure space $(\Omega, \mathcal{F}, \mu)$. A function on $X \in \mathcal{F}$ means a map $f : X \to \hat{\mathbb{R}}$ where $\hat{\mathbb{R}} = [-\infty, \infty]$ is the extended real numbers. We say that $f$ is measurable if it is measurable with respect to the Borel $\sigma$-algebra on $\hat{\mathbb{R}}$.

*Remark.* For functions $f : \mathbb{R} \to \mathbb{R}$ the term "measurable" is a priori ambigious because we have not specified the measure space on the domain (although our conventions do prescribe the Borel $\sigma$-algebra on the codomain). We take the standard convention that measurable means with respect to the Borel $\sigma$-algebra i.e. $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ is measurable. Then we say that $f$ is Lebesgue measurable if $f : (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{B})$ is measureable where $\mathcal{L}$ is the $\sigma$-algebra of Lebesgue measurable sets. Notice that measurable implies Lebesgue measurable. However, neither implies that the function $f : (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{L})$ is measurable, a condition which has the distasteful property of not holding for all continuous functions (of course continiuous functions are (Borel) measurable and Lebesgue measurable).

**Definition 3.1.1.** A measurable function $s : X \to \hat{\mathbb{R}}$ is *simple* if it takes on finitely many values. Clearly, any simple function can be writen as,

$$s = \sum_{i=1}^{n} c_i \chi_{A_i}$$

for $c_i \in \mathbb{R}$. For a nonnegative simple function (i.e. $c_i \geq 0$) we define,

$$\int_X s \, \mathrm{d}\mu = I(s) = \sum_{i=1}^{n} c_i \mu(A_i)$$

*Remark.* This is well-defined even when $\mu(A_i) = \infty$ because $c_i \geq 0$. Clearly $I(s)$ is independent of the sum representation of $s$.

**Definition 3.1.2.** For a nonnegative function $f \geq 0$ we define,

$$\int_X f \, \mathrm{d}\mu = \sup\{I(s) \mid 0 \leq s \leq f \text{ where } s \text{ is simple}\}$$

**Definition 3.1.3.** We say that a measurable function $f$ is *integrable* if,

$$\int_X |f| \, \mathrm{d}\mu < \infty$$

Notice that the integral of $f$ may be *definable* even if $f$ is not integrable where the integral is definable if when writing $f = f^+ - f^-$ for nonegative $f^+ = \max\{(f, 0)\}$ and $f^- = \max\{(-f, 0)\}$ with one of $f^+$ or $f^-$ integrable in which case,

$$\int_X f \, \mathrm{d}\mu = \int_X f^+ \, \mathrm{d}\mu - \int_X f^- \, \mathrm{d}\mu$$

which of course may be infinite so we still do not say that $f$ is integrable unless both integrals are finite in which case,

$$\int_X |f| \, \mathrm{d}\mu = \int_X f^+ \, \mathrm{d}\mu + \int_X f^- \, \mathrm{d}\mu < \infty$$

is finite so indeed $f$ is integrable in the previous sense.

**Proposition 3.1.4.** The following are basic properties of the Lebesgue integral,

(a) if $f \overset{\text{a.e.}}{=} g$ then,

$$\int_X f \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu$$

(b) if $f \le g$ a.e. then,

$$\int_X f \, \mathrm{d}\mu \le \int_X g \, \mathrm{d}\mu$$

(c) if $f \ge 0$ a.e. then,

$$f \overset{\text{a.e.}}{=} 0 \iff \int_X f \, \mathrm{d}\mu = 0$$

**Theorem 3.1.5.** If $f : X \to \mathbb{R}$ is a nonnegative measurable function then,

$$\int_X f \, \mathrm{d}\mu = \int_0^\infty f^*(t) \, \mathrm{d}t$$

where $f^*(t) = \mu(\{x \in X \mid f(x) > t\})$ is monotonic and thus Riemann integrable.

**Theorem 3.1.6.** If $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable then for the Lebesgue measure $\mu$,

$$\int_{\mathbb{R}} f \, \mathrm{d}\mu = \int_{\mathbb{R}} f \, \mathrm{d}x$$

**Theorem 3.1.7.** Let $C_c(\mathbb{R})$ be the space of compactly supported continuous functions on $\mathbb{R}$ with the following norm,

$$||f|| = \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x$$

which may be computed by the Riemann integral since $f$ is continuous. This always exists because $f$ is supported on a compact (and thus bounded set) and it must thus be bounded. Then $C_c(\mathbb{R})$ is not complete but its completion is isomorphic to $L^1(\mathbb{R})$. Then, $\int : C_c(\mathbb{R}) \to \mathbb{R}$ is a bounded operator and is defined on the dense subspace $C_c(\mathbb{R}) \subset L^1(\mathbb{R})$ so it has a unique extension $\int : L^1(\mathbb{R}) \to \mathbb{R}$ which is exactly the Lebesgue integral on the space of Lebesgue integrable function (modulo $\overset{\text{a.e.}}{=}$).

**Theorem 3.1.8** (Fubini)**.** Let $(X_1, \Sigma_1, \mu_1)$ and $(X_2, \Sigma_2, \mu_2)$ be $\sigma$-finite measure spaces and $f : X_1 \times X_2 \to \hat{\mathbb{R}}$ measurable. Then,

$$\int\limits_{X_1 \times X_2} |f| \, \mathrm{d}\mu = \int_{X_1} \left( \int_{X_2} |f(x, y)| \, \mathrm{d}\mu_2(y) \right) \mathrm{d}\mu_1(x) = \int_{X_2} \left( \int_{X_1} |f(x, y)| \, \mathrm{d}\mu_1(x) \right) \mathrm{d}\mu_2(y)$$

and if this value is finite then $f$ is by definition integrable and furthermore,

$$\int\limits_{X_1 \times X_2} f \, \mathrm{d}\mu = \int_{X_1} \left( \int_{X_2} f(x, y) \, \mathrm{d}\mu_2(y) \right) \mathrm{d}\mu_1(x) = \int_{X_2} \left( \int_{X_1} f(x, y) \, \mathrm{d}\mu_1(x) \right) \mathrm{d}\mu_2(y)$$

## 3.2 Convergence in Measure (WIP)

**Definition 3.2.1.** Let $(X, \mathcal{F}, \mu)$ be a measure space and $f_n$ a sequence of measurable functions and $f$ a measurable function. We say that,

(a) $f_n \to f$ *globally in measure* if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

(b) $f_n \to f$ *locally in measure* if for every finite measure $A \subset X$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\})$$

**Theorem 3.2.2** (Borel-Cantelli)**.** Let $A_n \subset X$ be a sequence of measurable sets such that,

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

Then,

$$\mu\left(\limsup_{n\to\infty} A_n\right) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n\right) = 0$$

**Proposition 3.2.3.** Let $(X, \mathcal{F}, \mu)$ be a measure space and $f_n$ a sequence of measurable functions and $f$ a measurable function. Suppose either,

(a) $f_n \to f$ globally in measure or

(b) $(X, \mathcal{F}, \mu)$ is $\sigma$-finite and $f_n \to f$ locally in measure

then $f_n$ has a subsequence converging pointwise a.e. to $f$.

*Proof.* For each $n > 0$ there exists some $j_n$ such that for any $k \geq j_n$,

$$\mu(\{x \in X \mid |f_k(x) - f(x)| \geq \tfrac{1}{n}\}) < \tfrac{1}{2^n}$$

Let,

$$E_n = \{x \in X \mid |f_{j_n}(x) - f(x)| \geq \tfrac{1}{n}\}$$

I claim that the subsequence $f_{j_n}$ converges to $f$ on $U = X \setminus \limsup\limits_{n\to\infty} E_n$ and that $\mu\left(\limsup\limits_{n\to\infty} E_n\right) = 0$.

Indeed, for $x \in U$ there is some $m$ such that for $n > m$ we have $x \notin E_n$. Thus for any $\epsilon > 0$ choose $N > m$ such that $\frac{1}{N} < \epsilon$ then for $n > N$ we have $\frac{1}{n} < \epsilon$ so,

$$|f_{j_n}(x) - f(x)| < \tfrac{1}{n} < \epsilon$$

because $x \notin E_n$. Furthermore, by construction,

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Therefore by Borel-Cantelli,

$$\mu\left(\limsup_{n\to\infty} E_n\right) = 0$$

as required. $\square$

**Proposition 3.2.4.** If $f_n \to f$ pointwise locally almost everywhere then $f_n \to f$ locally in measure.

*Proof.* For any $A \in \mathcal{F}$ with $\mu(A) < \infty$ we know that,

$$\mu(\{x \in A \mid \lim_{n\to\infty} f_n(x) \neq f(x)\}) = \mu\left(\bigcup_{k=1}^{\infty}\{x \in A \mid \forall N : \exists n > N : |f_n(x) - f(x)| \geq \tfrac{1}{k}\}\right) = 0$$

Therefore, for each $\epsilon > 0$ we have,

$$\mu(\{x \in A \mid \forall N : \exists n > N : |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

Furthermore,

$$\begin{aligned}\{x \in A \mid \forall N : \exists n > N : |f_n(x) - f(x)| \geq \epsilon\} &= \bigcap_{N=1}^{\infty}\bigcup_{n=N+1}^{\infty}\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\} \\ &= \limsup_{n\to\infty}\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}\end{aligned}$$

Now because $(A, \mathcal{F}|_A, \mu|_A)$ is a finite measure space $\mu$ is continuous with respect to descending limits of sets and thus,

$$\limsup_{n\to\infty} \mu(\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}) \geq \mu(\limsup_{n\to\infty}\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

which implies (since the sequences are nonegative) that,

$$\lim_{n\to\infty} \mu(\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

□

*Remark.* Even given $f_n \to f$ pointwise a.e. globally we cannot in general conclude that $f_n \to f$ globally in measure unless the measure space is finite. Indeed, consider, $f_n = \chi_{[n,\infty)}$ then $f_n \to 0$ pointwise everywhere. However, $f_n$ does not converge to 0 in measure.

**Proposition 3.2.5.** Let $(X, \mathcal{F}, \mu)$ be a measure space functions $f_n, f \in L^p(X)$ such that $f_n \to f$ in the $L^p$-norm. Then $f_n \to f$ globally in measure.

*Proof.* First consider $1 \leq p < \infty$. Since we have $f_n \to f$ in $L^p(X)$ we know that $||f_n - f||_p \to 0$ in $L^p(X)$. Let $g_n = |f_n - f|^p$. Then we know, for any $\epsilon > 0$ and $\eta > 0$ there is some $N$ such that for $n > N$,

$$\int_X g_n \,\mathrm{d}\mu < \epsilon\eta^p$$

Then,

$$\mu(\{x \in X \mid g_n(x) \geq M\}) \leq \frac{1}{M}\int_X g_n \,\mathrm{d}\mu$$

taking $M = \eta^p$ we see that,

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \eta\}) = \mu(\{x \in X \mid g_n(x) \geq \eta^p\}) \leq \epsilon$$

and therefore,

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \eta\}) = 0$$

proving that $f_n \to f$ in measure. Finally, if $p = \infty$ then convergence in $L^\infty(X)$ implies unifom and thus pointwise convergence and therefore convergence in measure. □

*Remark.* A good counter example for the converse is the sequence $f_n = n\chi_{[0,\frac{1}{n}]}$ which converges to 0 almost everywhere and globally in measure but not in $L^p$-norm for any $p \geq 1$.

**Corollary 3.2.6.** If $f_n \to f$ converges in $L^p(X)$ then there is a subsequence converging pointwise a.e. to $f$.

**Corollary 3.2.7.** If $f_n \in L^p(X) \cap L^{p'}(X)$ and $f_n \to f$ in the $L^p$-norm and $f_n \to f'$ in the $L^{p'}$-norm. Then $f \overset{\text{a.e.}}{=} f'$ so we see that $f_n \to f$ also in the $L^{p'}$-norm and $f_n \to f'$ also in the $L^p$-norm.

*Proof.* Since $f_n \to f$ in $L^p$ and $f_n \to f'$ in $L^{p'}$ we know that there are subsequences $\{f_{n_j}\}$ and $\{f_{m_j}\}$ such that poinwise a.e. $f_{n_j} \to f$ and $f_{m_j} \to f'$. However, $f_{m_j} \to f$ in $L^p$ because it is a subsequence of a $L^p$ convergent sequence. Then let $g_j = |f_{m_j} - f|^p$ and by Fatou's lemma,

$$\int_X \liminf_{j \to \infty} g_j \, \mathrm{d}\mu \leq \liminf_{j \to \infty} \int_X g_j \, \mathrm{d}\mu = 0$$

because $f_{m_j} \to f$ in $L^p$. However,

$$\liminf_{j \to \infty} g_j = \liminf_{j \to \infty} |f_{m_j} - f|^p \overset{\text{a.e.}}{=} |f' - f|^p$$

Therefore,

$$\int_X |f' - f|^p \, \mathrm{d}\mu = 0$$

which implies that $f \overset{\text{a.e.}}{=} f'$. □

## 3.3 Lebesgue-Radon-Nikodym

**Definition 3.3.1.** Let $(\Omega, \Sigma)$ be a measurable space and $\mu, \nu$ be $\Sigma$-measures. We say that,

(a) $\nu$ is *absolutely continuous* with respect to $\mu$ ($\nu \ll \mu$) if $\mu(A) = 0 \implies \nu(A) = 0$ for all $A \in \Sigma$

(b) $\mu$ and $\nu$ are *mutually singular* ($\mu \perp \nu$) if there is a decomposition $X = A \cup B$ such that $A \cap B = \varnothing$ and for any measurable $E \subset A$ we have $\nu(E) = 0$ and for any measurable $E \subset B$ we have $\mu(E) = 0$.

**Theorem 3.3.2** (Lebesgue)**.** Let $\mu$ and $\nu$ be $\sigma$-finite measures. Then there is a unique decomposition $\nu = \nu_c + \nu_s$ where $\nu_c, \nu_s$ are $\sigma$-finite measures such that $\nu_c \ll \mu$ and $\nu_s \perp \mu$.

**Theorem 3.3.3** (Radon-Nikodym)**.** Suppose that $\mu, \nu$ are $\sigma$-finite measures with $\nu \ll \mu$. Then there exists a nonegative measurable function $f : X \to [0, \infty)$ such that for any $A \in \Sigma$,

$$\nu(A) = \int_A f \, \mathrm{d}\mu$$

*Remark.* The above function is uniquely determined $\mu$-a.e. and therefore we write suggestively,

$$f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$$

to suggest that $f$ is the derivative.

**Corollary 3.3.4.** If $\nu \ll \mu$ and $g$ is a $\nu$-integrable function then,

$$\int_A g \, \mathrm{d}\nu = \int_A g \cdot \left( \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right) \, \mathrm{d}\mu$$

*Proof.* By splitting $g$, it suffices to consider the case that $g \geq 0$. Then applying Fubini,

$$\begin{aligned} \int_A g \, \mathrm{d}\nu &= \int_0^\infty \nu(\{x \in A \mid g(x) > t\}) \, \mathrm{d}t = \int_0^\infty \left( \int_{g^{-1}((-\infty,t))} \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu \right) \mathrm{d}t \\ &= \int_0^\infty \int_X \chi_{g^{-1}((-\infty,t))} \cdot \left( \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right) \mathrm{d}\mu \, \mathrm{d}t = \int_X \left( \int_0^\infty \chi_{g^{-1}((-\infty,t))} \, \mathrm{d}t \right) \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu \\ &= \int_X g \cdot \left( \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right) \mathrm{d}\mu \end{aligned}$$

because,

$$\chi_{g^{-1}((-\infty,t))}(x) = \chi_{[0,g(x))}(t)$$

and $\mu_{\mathcal{L}}([0, g(x))) = g(x)$. □

## 3.4 Lebesgue Differentiation

*Remark.* Here let $X = \mathbb{R}^n$ with the Lebesgue measure or more generally a Riemannian manifold with a $\sigma$-finite Borel measure.

**Theorem 3.4.1.** Let $f : X \to \hat{\mathbb{R}}$ be a locally integrable function. Then, the limit,

$$\lim_{\epsilon \to 0} \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} f \, \mathrm{d}\mu \overset{\text{a.e.}}{=} f(x)$$

exists and almost everywhere equals $f(x)$. In fact,

$$\lim_{\epsilon \to 0} \left( \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} |f(y) - f(x)| \, \mathrm{d}\mu(y) \right) = 0$$

almost everywhere.

**Corollary 3.4.2.** Let $A \subset X$ be measurable. Then,

$$\lim_{\epsilon \to 0} \frac{\mu(A \cap B_\epsilon(x))}{\mu(B_\epsilon(x))} = 1$$

for almost all $x \in A$.

*Proof.* Apply Lebesgue differentiation to the function $\chi_A$ which is locally integrable. Then, for almost every $x \in A$,

$$\lim_{\epsilon \to 0} \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} \chi_A \, \mathrm{d}\mu = \lim_{\epsilon \to 0} \frac{\mu(A \cap B_\epsilon(x))}{\mu(B_\epsilon(x))} \overset{\text{a.e.}}{=} \chi_A(x) = 1$$

□

### 3.5 Egorov's and Lusin's Theorems (WIP)

**Theorem 3.5.1** (Egorov)**.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $M$ a separable metric space $f_n : X \to M$ a sequence of measurable functions (for the Borel $\sigma$-algebra on $M$) such that $f_n \to f$ pointwise a.e. then for any $\epsilon > 0$ there exists a measurable subset $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $X \setminus E$.

*Remark.* Finiteness of the measure space is essential. For example, consider $X = \mathbb{R}$ with the Lebesgue measure and $M = \mathbb{R}$. Then let $f_n = \chi_{[n,n+1]}$. Pointwise $f_n \to 0$ but this convergence is not uniform on the complement of any finite measure open set (notice that every finite measure set is contained in a finite measure open set by regularity).

### 3.6 •

# 4 Convergence Theorems for Lebesgue Integrals

## 4.1 A Few Remarks on Measurable Limits

**Lemma 4.1.1.** Let $X \in \mathcal{F}$ be measurable, and $\{f_n\}$ a sequence of measurable functions on $X$. Then define,

(a) $(\liminf f_n)(x) = \liminf\limits_{n \to \infty} f_n(x)$

(b) $(\limsup f_n)(x) = \limsup\limits_{n \to \infty} f_n(x)$

(c) $(\inf f_n)(x) = \inf\limits_{n \geq 1} f_n(x)$

(d) $(\sup f_n)(x) = \sup\limits_{n \geq 1} f_n(x)$

which always exist as functions $X \to \hat{\mathbb{R}}$. Then $\liminf f_n, \limsup f_n, \inf f_n, \sup f_n$ are all measurable.

*Proof.* Rudin RCA Theorem 1.14. □

**Corollary 4.1.2.** If $f_n \to f$ pointwise and $f_n$ is a sequence of measurable functions then $f$ is measurable.

**Lemma 4.1.3.** Assume that $(\Omega, \mathcal{F}, \mu)$ is a complete measure space. Then if $f \overset{\text{a.e.}}{=} g$ and $f$ is measurable then $g$ is measurable.

*Proof.* Let $N = \{x \in X \mid f(x) \neq g(x)\}$ and we know $\mu(N) = 0$. It suffices to show that $g^{-1}(Y) \in \mathcal{F}$ for all $Y \in \mathcal{B}$. However,

$$g^{-1}(Y) = [g^{-1}(Y) \cap N] \cup [g^{-1}(Y) \cap N^C] = [g^{-1}(Y) \cap N] \cup [f^{-1}(Y) \cap N^C]$$

where the last equality holds because,

$$\begin{aligned} x \in g^{-1}(-\infty, c) \cap N^C &\iff x \in N \text{ and } g(x) \in (Y) \iff x \in N^C \text{ and } f(x) \in (Y) \\ &\iff x \in f^{-1}(Y) \cap N^C \end{aligned}$$

since $f(x) = g(x)$ for $x \in N^C$. Now because $\mu(N) = 0$ and $g^{-1}(Y) \cap N \subset N$ we have $g^{-1}(Y) \cap N \in \mathcal{F}$ because the measure space is complete. Therefore $g^{-1}(Y) \in \mathcal{F}$ so $g$ is measurable. □

**Corollary 4.1.4.** Let $f_n$ be a sequence of measureable functions on $X$ and $f$ a function such that $f_n \to f$ pointwise a.e. If $(\Omega, \mathcal{F}, \mu)$ is complete then $f$ is measurable.

*Proof.* We know that $\liminf f_n$ exists and that $\liminf f_n = f$ almost everywhere since wherever $f_n(x) \to f(x)$ converges we have $(\liminf f_n)(x) = \lim_{n\to\infty} f_n(x) = f(x)$. We know that $\liminf f_n$ is measurable so we find that $f$ is measurable as well because $(\Omega, \mathcal{F}, \mu)$ is complete. □

## 4.2 The Main Lemma

**Lemma 4.2.1** (Fatou)**.** Let $X \in \mathcal{F}$ be measurable, and $\{f_n\}$ a sequence of measurable functions which are nonegative almost everywhere. Define $f : X \to \hat{\mathbb{R}}$ via,

$$f(x) = \liminf_{n\to\infty} f_n(x)$$

Then $f$ is measurable and,

$$\int_X f \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$$

*Proof.* The proof from first principles is annoying. □

**Corollary 4.2.2** (Reverse Fatou)**.** Now suppose that $\{f_n\}$ is a sequence of measurable functions on $X$ such that $|f_n| \le g$ almost everywhere for an integrable function $g$ on $X$. Then,

$$\limsup_{n\to\infty} \int_X f_n \,\mathrm{d}\mu \le \int_X \limsup_{n\to\infty} f_n \mathrm{d}\mu$$

*Proof.* Consider $f'_n = g - f_n$. Then $f'_n$ is measurable and a.e. nonegative. Applying Fatou's lemma we see that,

$$f' = \liminf f'_n = g - \limsup f_n$$

is measurable and,

$$\int_X (g - \limsup f_n) \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X (g - f_n) \,\mathrm{d}\mu$$

However, $g$ is integrable so we can pull it out of the integrals,

$$\int_X (g - \limsup f_n) \,\mathrm{d}\mu = \int_X g \,\mathrm{d}\mu - \int_X \limsup f_n \,\mathrm{d}\mu$$
$$\liminf_{n\to\infty} \int_X (g - f_n) \,\mathrm{d}\mu = \int_X g \,\mathrm{d}\mu - \limsup_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$$

Therefore,

$$\limsup_{n\to\infty} \int_X f_n \,\mathrm{d}\mu \le \int_X \limsup_{n\to\infty} f_n \mathrm{d}\mu$$

□

## 4.3 The Main Theorems

**Theorem 4.3.1** (Monotone Convergence)**.** Let $\{f_n\}$ be a pointwise a.e. non-decreasing sequence of a.e. non-negative measurable functions on $X$, explicitly this means

$$0 \le f_n(x) \le f_{n+1}(x)$$

for all $n$ almost everywhere. Then let $f = \liminf_{n \to \infty} f_n$. Then $f$ is measurable and,

$$\lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$$

*Proof.* Notice that $f_n$ is nondecreasing a.e. and thus $\int f_n$ is nondecreasing. Then applying Fatou's lemma, we see that $f$ is measurable and,

$$\int_X f \, \mathrm{d}\mu \le \liminf_{n \to \infty} \int_X f_n \, \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu$$

Furthermore, $f_n \le f$ a.e. so we find that,

$$\int_X f_n \, \mathrm{d}\mu \le \int_X f \, \mathrm{d}\mu$$

proving the equality. □

*Remark.* Notice that there is no confusion about if the hypothis means that,

$$\mu(\{x \in X \mid \forall n : 0 \le f_n(x) \le f_{n+1}(x)\}^C) = 0 \quad \text{or} \quad \forall n : \mu(\{x \in X \mid 0 \le f_n(x) \le f_{n+1}(x)\}^C) = 0$$

because these are equivalent since the first set is the countable union of the second sequence of sets the measure is countably additive.

**Theorem 4.3.2** (Dominated Convergence)**.** Let $\{f_n\}$ be a sequence of measurable functions on $X$ dominated almost everywhere by an integrable function $g$ meaning that $|f_n| \le g$ almost everywhere. Suppose that $f_n \to f$ pointwise almost everywhere where $f$ is measurable. Then $f_n$ and $f$ are integrable and,

$$\lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$$

*Proof.* First, because $f_n$ is measurable and $|f_n| \le g$ a.e. where $g$ is integrable we see that $f_n$ is integrable. Furthermore, almost everywhere we have,

$$|f - f_n| \le |f| + |f_n| \le 2g$$

Therefore, we may apply reverse Fatou to the sequence $\{|f - f_n|\}$ bounded by the integrable function $2g$ to find that,

$$\limsup_{n \to \infty} \int_X |f - f_n| \, \mathrm{d}\mu \le \int_X \limsup |f - f_n| \, \mathrm{d}\mu$$

but $\limsup |f - f_n| \overset{\text{a.e.}}{=} 0$ because $f \to f_n$ a.e. and thus,

$$\limsup_{n \to \infty} \int_X |f - f_n| \, \mathrm{d}\mu \le 0$$

but $|f - f_n| \geq 0$ so the limit exists because the liminf is bounded below by zero as well so,

$$\lim_{n\to\infty} \int_X |f - f_n| \,\mathrm{d}\mu = 0$$

and therefore,

$$\left| \int_X f_n \,\mathrm{d}\mu - \int_X f \,\mathrm{d}\mu \right| \leq \int_X |f_n - f| \,\mathrm{d}\mu \to 0$$

meaning that,

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu$$

□

*Remark.* If the measure space is complete or $f_n \to f$ everywhere then $f$ is automatically measurable.

## 4.4 Scheffé's Lemma

**Lemma 4.4.1** (Scheffé)**.** Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with $f$ integrable. Then,

$$\lim_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0 \iff \lim_{n\to\infty} \int |f_n| \,\mathrm{d}\mu = \int |f| \,\mathrm{d}\mu$$

*Proof.* One direction is obvious. If,

$$\lim_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0$$

then because $||f_n| - |f|| \leq |f_n - f|$ we know that,

$$\left| \int_X |f_n| \,\mathrm{d}\mu - \int_X |f| \,\mathrm{d}\mu \right| \leq \int_X ||f_n| - |f|| \,\mathrm{d}\mu \leq \int_X |f_n - f| \,\mathrm{d}\mu \to 0$$

and therefore,

$$\lim_{n\to\infty} \int |f_n| \,\mathrm{d}\mu = \int |f| \,\mathrm{d}\mu$$

Conversely, let $g_n = |f_n| - |f_n - f|$ then,

$$|g_n| \leq ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f|$$

so because $f$ is integrable we can apply the dominated convergence theorem to conclude that,

$$\lim_{n\to\infty} \int g_n \,\mathrm{d}\mu = \int |f| \,\mathrm{d}\mu$$

because $g_n \to |f|$ almost everywhere. However,

$$\lim_{n\to\infty} \int g_n \,\mathrm{d}\mu = \lim_{n\to\infty} \left( \int |f_n| \,\mathrm{d}\mu - \int |f - f_n| \,\mathrm{d}\mu \right)$$

and we assume that,

$$\lim_{n\to\infty} \int |f_n| \,\mathrm{d}\mu = \int |f| \,\mathrm{d}\mu$$

so combining the two equations we find that,

$$\lim_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0$$

$\square$

**Corollary 4.4.2.** Under the hypothses of the lemma, if

$$\lim_{n\to\infty} \int |f_n| \, \mathrm{d}\mu = \int |f| \, \mathrm{d}\mu$$

then

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$$

*Proof.* By the lemma, our assumption implies that,

$$\lim_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0$$

and then we conclude from

$$\left| \int_X f_n \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu \right| \le \int_X |f_n - f| \, \mathrm{d}\mu \to 0$$

meaning that,

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$$

$\square$

*Remark.* However, the converse of this corollary does *not* hold in general (it does say when $f_n$ are nonnegative and it becomes trivial). For example, let

$$f_n = \frac{1}{2\sqrt{\pi}} \left( e^{-(x-n)^2} - e^{-(x+n)^2} \right)$$

Then $f_n \to 0$ everywhere pointwise. And likewise,

$$\int_X f_n \, \mathrm{d}\mu = 0$$

for each $n$ so we indeed have,

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu = 0$$

trivially where $f = 0$. However,

$$\int_X |f_n| \, \mathrm{d}\mu \to 1 \quad \text{where as} \quad \int_X |f| \, \mathrm{d}\mu = 0$$

# 5 Functional Analysis Theorems

## 5.1 Banach-Steinhaus

**Theorem 5.1.1** (Banach-Steinhaus)**.** Let $X$ be a Banach space and $Y$ a normed space. Then for any subset $\mathcal{M} \subset \mathcal{L}(X, Y)$ we have,

$$\mathcal{M} \text{ is pointwise bounded} \iff \mathcal{M} \text{ is uniformly bounded}$$

**Corollary 5.1.2.** Let $X, Y$ be normed spaces with $X$ Banach and $T_n \in \mathcal{L}(X, Y)$ a sequence such that $\lim_{n\to\infty} T_n x$ exists for all $x \in X$. Then there exists $T \in \mathcal{L}(X, Y)$ defined by $Tx = \lim_{n\to\infty} T_n x$.

*Proof.* It is clear that $T$ is linear and thus suffices to prove that $T$ is bounded. Consider,

$$\mathcal{M} = \{T_n \mid n \in \mathbb{N}\}$$

For any $x \in X$ the sequence $T_n x$ is bounded because it has a limit. Thus $\mathcal{M}$ is pointwise bounded and therefore uniformly bounded by Banach-Steinhaus. In particular, there exists some $C > 0$ such that $||T_n|| \leq C$ for all $n$. Therefore,

$$||T|| = \sup_{||x||=1} ||Tx|| = \sup_{||x||=} \lim_{n\to\infty} ||T_n x|| \leq \lim_{n\to\infty} ||T_n|| \leq C$$

so $T$ is bounded. $\square$

## 5.2 Hahn-Banach

**Theorem 5.2.1** (Hahn-Banach)**.** Let $X$ be a normed space and $U \subset X$ a linear subspace. Let $\ell : U \to \mathbb{R}$ be a bounded linear functional. Then there exists a linear functional $\tilde{\ell} : X \to K$ extending $\ell$ that satisfies $||\tilde{\ell}||_{X^*} = ||\ell||_{U^*}$.

*Proof.* First, we show that given $z \in X \setminus U$ it is possible to extend $\ell$ to $U' = U + \text{span}\{z\}$. Indeed,

$$\ell'(u + \alpha z) = \ell(u) + \alpha a$$

is well-defined since $z \notin U$ implies the decomposition is unique and is clearly linear. Thus we need to show that $a$ can be choosen to fix the norm. Because $U \subset U'$ we only need to show that $\ell'$ is bounded above by $||\ell||_U$ (bounded below then follows immediately by linearity) since the norm of any extension is automatically larger as it is a supremum over a superset. We need to choose $a$ such that,

$$\ell'(u + \alpha z) = \ell(u) + \alpha a \leq ||\ell||_U \cdot ||u + \alpha z||$$

If $\alpha = 0$ this says nothing. If $\alpha > 0$ then we require,

$$\ell(u) + a \leq ||\ell||_U \cdot ||u + z||$$

for all $u \in U$ by pulling out $\alpha$ and rescaling $u$. Similarly, for $\alpha < 0$ we require for all $u \in U$ that,

$$\ell(u) - a \leq ||\ell||_U \cdot ||u - z||$$

However,

$$\ell(u_1) + \ell(u_2) = \ell(u_1 + u_2) \leq ||\ell||_U \cdot ||u_1 + u_2|| \leq ||\ell||_U \cdot (||u_1 + z|| + ||u_2 - z||)$$

and therefore,

$$\ell(u_2) - ||\ell||_U \cdot ||u_2 - z|| \leq ||\ell||_U \cdot ||u_1 + z|| - \ell(u_1)$$

for all $u_1, u_2 \in U$ and therefore the supremum of the left is less than the infimum of the right so there exists an $a$ fitting between i.e. satisfiying,

$$a \leq ||\ell||_U \cdot ||u + z|| - \ell(u) \quad \text{and} \quad a \geq \ell(u) - ||\ell||_U \cdot ||u - z||$$

Therefore, such an extension $\ell'$ exists.

Now we complete the proof via Zorn's lemma applied to pairs $(U', \ell')$ where $U' \subset X$ is a linear subspace and $\ell' : U' \to \mathbb{R}$ is an extension such that $||\ell'||_{U'} = ||\ell||_U$. This is a poset under inclusion $U_1 \subset U_2$ with the restriction that $\ell_2|_{U_1} = \ell_1$. Therefore, chains give compatible bounded linear functionals on their union and thus give an upper bound. By Zorn's lemma there is a maximal element but by the above construction a maximal pair $(U', \ell')$ cannot have any $z \in X \setminus U'$ or it could be extended to $U' + \text{span}\{z\}$ so $U' = X$ proving the theorem. □

**Corollary 5.2.2.** Let $X$ be a normed space and $x \in X$ nonzero. Then there exists $\ell \in X^*$ such that $\ell(x) = ||x||$ and $||\ell|| = 1$.

*Proof.* Let $U = \text{span}\{x\}$ then define $\ell : U \to \mathbb{R}$ via $\lambda x \mapsto \lambda ||x||$. This is clearly linear and furthermore,

$$||\ell|| = \sup_{||\lambda x|| = 1} ||\ell(\lambda x)|| = \sup_{||\lambda x|| = 1} ||\lambda x|| = 1$$

By Hahn-Banach there is an extension $\ell' : X \to \mathbb{R}$ such that $||\ell'|| = ||\ell|| = 1$ and $\ell'(x) = ||x||$. □

**Corollary 5.2.3.** Let $X$ be a normed space. Then $X^*$ separates points.

*Proof.* If $x_1, x_2 \in X$ are not equal then there exists $\ell \in X^*$ such that $||\ell(x_1 - x_2)|| = ||x_1 - x_2|| \neq 0$ but $\ell(x_1 - x_2) = \ell(x_1) - \ell(x_2)$ so $\ell(x_1) \neq \ell(x_2)$. □

**Corollary 5.2.4.** For each $x \in X$,

$$||x||_X = \sup_{||\ell|| = 1} |\ell(x)|$$

*Proof.* By definition,

$$||\ell|| \geq \frac{|\ell(x)|}{||x||}$$

Therefore,

$$||x|| \geq \sup_{||\ell|| = 1} |\ell(x)|$$

However, by above there exists $\ell \in X^*$ such that $||\ell|| = 1$ and $\ell(x) = ||x||$ and therefore,

$$||x|| \leq \sup_{||\ell|| = 1} |\ell(x)|$$

proving the claim. □

**Corollary 5.2.5.** Let $U \subset X$ be a closed linear subspace, $x \in X$ with $x \neq U$. Then there exists $\ell \in X^*$ with $\ell|_U = 0$ and $\ell(x) = ||x||_X$.

*Proof.* Consider $X/U$ which inherts a normed structure via,

$$||[x]||_{X/U} = \inf_{u \in U} ||x+u||_X$$

Then, there exits a bounded linear functional $\ell : X/U \to K$ such that $\ell([x]) = ||[x]||_{X/U}$. Take $\tilde{\ell} = \ell \circ \pi$ where $\pi : X \to X/U$ is the projection which is bounded by 1. Thus, $\tilde{\ell} \in X^*$ and $\ell|_U = 0$. Furthermore,

$$\tilde{\ell}(x) = \ell([x]) = ||[x]||_{X/U} \neq 0$$

because $x \notin U$ so $[x] \neq 0$ so we can scale $\tilde{\ell}$ to give the required result. $\square$

## 5.3 Open Mapping Theorem

**Lemma 5.3.1.** Let $X$ be a normed space. A linear subspace $U \subset X$ is open iff $U = X$.

*Proof.* If $U$ is open then $0 \in U$ there must exist some open ball $B_\epsilon(0) \subset U$ and thus for any $x \in X$ we have $x = \lambda x'$ for $x' \in B_\epsilon(0) \subset U$ so $U = X$ because it is closed under scaling. $\square$

**Corollary 5.3.2.** If $T : X \to Y$ is a linear map of topological vector spaces and $Y$ is normed. Then if $T$ has open image then $T$ is surjective.

*Proof.* By linearity, the image $T(X) \subset Y$ is a linear subspace. Furthermore, $T(X)$ is open by assumption so by the lemma $T(X) = Y$. $\square$

**Theorem 5.3.3** (Open Mapping)**.** Let $X, Y$ be Banach spaces. For $T \in \mathcal{L}(X, Y)$ then,

$$T \text{ is surjective} \iff T \text{ is open}$$

*Remark.* By above, the direction $T$ is open $\implies$ $T$ is surjective is trivial.

**Theorem 5.3.4** (Bounded Inverse)**.** Let $X, Y$ be Banach spaces and $T \in \mathcal{L}(X, Y)$. If $T$ is bijective then $T^{-1} \in \mathcal{L}(X, Y)$.

*Proof.* It is obvious that $T^{-1}$ is linear so it suffices to check that $T^{-1}$ is bounded. Because $T$ is bijective we have,

$$||T^{-1}|| = \sup_{y \neq 0} \frac{||T^{-1}y||}{||y||} = \sup_{x \neq 0} \frac{||x||}{||Tx||} = \left[ \inf_{x \neq 0} \frac{||Tx||}{||x||} \right]^{-1}$$

Therefore, it suffices to show that $T$ is positively bounded below in addition to being bounded above. By the Open Mapping theorem, $T$ is open and thus a homeomorphism. Let,

$$S(X) = \{x \in X \mid ||x|| = 1\}$$

is closed so $T(S(X))$ is also closed and $0 \notin T(S(X))$ because $T$ is injective. Thus $d(0, T(S(X))) > 0$ because $T(S(X))$ is a closed subset of a metric space. Therefore,

$$\inf_{||x||=1} ||Tx|| > 0$$

and thus $T$ is bounded below proving the theorem. $\square$

**Lemma 5.3.5.** A linear operator $T : X \to Y$ between normed spaces is continuous iff it is bounded.

*Proof.* If $T$ is bounded then,

$$||Tx - Ty|| = ||T(x-y)|| \leq ||T|| \cdot ||x-y||$$

so indeed $T$ is even Lipschitz. If $T$ is continuous then consider,

$$S(Y) = \{y \in Y \mid ||y|| = 1\}$$

is closed so $A = T^{-1}(S(Y))$ is closed. Furthermore, $0 \notin A$ because $T(0) = 0 \notin S(Y)$. Therefore, because $A$ is closed in a metric space and $0 \notin A$ so $d(0, A) > 0$ and therefore,

$$||T|| = \sup_{x \in A} \frac{1}{||x||} = \left[\inf_{x \in A} ||x||\right]^{-1}$$

exists because $\inf\limits_{x \in A} ||x||$ is positive. Alternatively, $T^{-1}(B_1(0))$ is open so there is a ball $B_\delta(0) \subset T^{-1}(B_1(0))$ and thus,

$$||Tx|| = \frac{||x||}{\delta} ||T(x\delta/||x||)|| \leq \frac{||x||}{\delta}$$

so $T$ is bounded. $\square$

*Remark.* We can thus give a slick proof of the bounded operator theorem. Since $T$ is surjective it is open and thus $T^{-1}$ is continuous and thus bounded.

**Example 5.3.6.** If $X, Y$ are not both complete then the above may fail. For example, let $X = C([0,1])$ and $Y = C^1([0,1])$ with supremum norms and take,

$$(Tf)(t) = \int_0^t f(s) \mathrm{d}s$$

which is linear and bounded because,

$$||Tf||_\infty = \sup_{t \in [0,1]} \left| \int_0^t f(s) \mathrm{d}s \right| \leq ||f||_\infty$$

because $f$ is bounded. Thus $||T|| \leq 1$ and is injective by the fundamental theorem of calculus. Therefore it is bijective onto its image. However $T$ is not bounded below because we can take a sequence of spikier functions with vanishing area but the same supremum norm. However, if we give $Y$ the "corrrect" $C^1$ norm $||f||_{C^1} = ||f||_\infty + ||f'||_\infty$ then the operator is invertible because,

$$||Tf||_{C^1} = \sup_{t \in [0,1]} \left| \int_0^t f(s) \mathrm{d}s \right| + \sup_{t \in [0,1]} |(Tf)'(t)| \leq 2||f||_\infty$$

and also,

$$||Tf||_{C^1} \geq ||(Tf)'||_\infty = ||f||_\infty$$

so we see that $||f||_\infty \leq ||Tf||_{C^1} \leq 2||f||_\infty$ so $T$ is also bounded below and thus has closed image which is thus a Banach space (the image is the subspace of $C^1([0,1])$ vanishing at 0).

# 6 Closed Image Theorem

**Lemma 6.0.1.** Let $T : X \to Y$ be a continuous map of Banach spaces. Then $T$ is injective and $\operatorname{im} T$ is closed if and only if $T$ is bounded below.

*Proof.* Suppose there is $C > 0$ such that,

$$||x||_X \le C||Tx||_Y$$

for all $x \in X$. In particular, if $Tx = 0$ then $x = 0$ so $T$ is injective. Furthermore, if $x_n \in X$ is a sequence such that $Tx_n$ is Cauchy (any Cauchy sequence in $\operatorname{im} T$ is of this form) then,

$$||x_n - x_m|| \le C||T(x_n - x_m)|| \le C||Tx_n - Tx_m|| < C\epsilon$$

for $n, m > N$. Thus, $\{x_n\}$ is Cauchy and therefore since $X$ is Banach the sequence is convergent,

$$\lim_{n\to\infty} x_n = x$$

Furthermore, $T$ is continuous so,

$$\lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx$$

and thus $x_n$ is convergent and $x \in \operatorname{im} T$ so $\operatorname{im} T$ is closed.

Conversely, since $T$ is injective $T : X \to \operatorname{im} T$ is bijective and $\operatorname{im} T \subset Y$ is closed so $\operatorname{im} T$ is Banach. Therefore, by the bounded inverse theorem $T : X \to \operatorname{im} T$ is invertible by a bounded operator $T^{-1} \in \mathcal{L}(\operatorname{im} T, X)$. Thus,

$$||x||_X = ||T^{-1}(Tx)||_X \le ||T^{-1}|| \cdot ||Tx||_Y$$

so $||T^{-1}||$ gives the required constant. $\square$

**Theorem 6.0.2** (Closed Graph)**.** Let $T : X \to Y$ be a linear operator between Banach spaces. Then $T$ is continuous if and only if its graph is closed.

## 6.1 Closed Range Theorem

**Definition 6.1.1.** Let $V \subset X$ be a subset then define $V^\perp \subset X^*$ via,

$$V^\perp = \{\ell \in X^* \mid \forall v \in V : \ell(v) = 0\} = \bigcap_{v \in V} \ker \operatorname{ev}_v$$

Clearly this is a weak-$*$ closed subspace and thus norm closed. Let $W \subset X^*$ be a subset then define $W^\perp \subset X$ via,

$$W^\perp = \{x \in X \mid \forall \ell \in W : \ell(x) = 0\} = \bigcap_{\ell \in W} \ker \ell$$

which is clearly a weakly closed subspace and thus norm closed.

**Lemma 6.1.2.** If $V \subset X$ is a subspace then $V^{\perp\perp} = \overline{V}$.

*Proof.* If $v \in V$ then for any $\ell \in V^\perp$ by definition $\ell(v) = 0$ so $v \in V^{\perp\perp}$ so $V \subset V^{\perp\perp}$ and thus $\overline{V} \subset V^{\perp\perp}$. Let $x \notin \overline{V}$. Since $\overline{V}$ is a subspace, by Hahn-Banach there is some $\ell \in X$ such that $\ell|_{\overline{V}} = 0$ but $\ell(x) = 1$ so $\ell \in V^\perp$ and thus $x \notin V^{\perp\perp}$. Thus $V^{\perp\perp} = V$. $\square$

*Remark.* Weakly closed and closed subspaces coincide so this is also the weak closure of $V$.

**Lemma 6.1.3.** If $W \subset X^*$ is a subspace then $W^{\perp\perp}$ is the weak-$*$ closure of $W$.

*Proof.* If $\ell \in W$ then for any $x \in W^\perp$ by definition $\ell(x) = 0$ so $\ell \in W^{\perp\perp}$ so $W \subset W^{\perp\perp}$ and thus if $Y$ is the weak-$*$ closure of $W$ then $Y \subset W^{\perp\perp}$. Now suppose that $\ell \in W^{\perp\perp}$ then $\ell|_{W^\perp} = 0$ and we need to show that $\ell \in Y$. Now $\ell$ descends to $\ell \in (X/W^\perp)^*$ For any finite dimensional subspace $V \subset X/W^\perp$ we can find $\ell_V \in W$ such that $\ell|_V = \ell_V$. This is becaues if $\eta(v) = 0$ for all $\eta \in W$ then $v \in W^\perp$ so $[v] = 0$. Thus $W \to X^* \to (X/W^\perp)^* \to V^*$ is surjective because its dual $v \mapsto (\eta \mapsto \eta(v))$ is injective and $V$ is finite dimensional (see this answer). Therefore there is such a $\ell_V$. Now we see that $\ell_V \to \ell$ pointwise i.e. in the weak-$*$ topology as $V$ ranges over the finite dimensional subspaces of $(X/W^\perp)$. Therefore $\ell \in Y$ since $Y$ is weak-$*$ closed and $\ell_V \in W \subset Y$. ☐

*Remark.* This also shows that $Y \cong (X/W^\perp)^*$.

**Proposition 6.1.4.** Let $T : X \to Y$ be a continuous linear operator. Then,

(a) $\ker T = (\operatorname{im} T^*)^\perp$

(b) $\ker T^* = (\operatorname{im} T)^\perp$

*Proof.* If $Tx = 0$ then if $\ell = T^*\eta$ we have $\ell(x) = \eta(Tx) = 0$ so $\ker T \subset (\operatorname{im} T^*)^\perp$. If $x \in (\operatorname{im} T^*)^\perp$ then for any $\ell \in Y^*$ we see that $\ell(Tx) = (T^*\ell)(x) = 0$ so $Tx = 0$ by Hahn-Banach.

If $T^*\ell = 0$ then if $y = Tx$ we see that $\ell(y) = (T^*\ell)(x) = 0$ so $\ell \in (\operatorname{im} T)^\perp$. If $\ell \in (\operatorname{im} T)^\perp$ then for all $x \in X$ we have $(T^*\ell)(x) = \ell(T(x)) = 0$ so $T^*\ell = 0$ and thus $\ell \in \ker T^*$. ☐

**Corollary 6.1.5.** We see that,

$$\overline{\operatorname{im} T} = (\ker T^*)^\perp$$

and that the weak-$*$ closure of $\operatorname{im} T^*$ is $(\ker T)^\perp$.

**Corollary 6.1.6.** Let $T : X \to Y$ be a continuous linear operator. Then,

(a) $\operatorname{im} T$ is dense iff $T^*$ is injective

(b) $\operatorname{im} T^*$ is weak-$*$ dense iff $T$ is injective.

*Proof.* We see that,

$$\operatorname{im} T \text{ is dense} \iff \overline{\operatorname{im} T} = Y \iff (\ker T^*)^\perp = Y \iff \ker T^* = (0) \iff T^* \text{ is injective}$$

using that $\ker T^*$ is weak-$*$ closed and

$$\operatorname{im} T^* \text{ is weak-* dense} \iff \overline{\operatorname{im} T*} = X^* \iff (\ker T)^\perp = X^* \iff \ker T = (0) \iff T \text{ is injective}$$

using that $\ker T$ is closed. ☐

**Theorem 6.1.7** (Closed Range)**.** Let $T : X \to Y$ be a continuous map of Banach spaces and $T^* : Y^* \to X^*$ its dual. Then the following are equivalent,

(a) $\operatorname{im} T$ is closed

(b) $\operatorname{im} T^*$ is closed

(c) $\operatorname{im} T = (\ker T^*)^\perp = \{y \in Y \mid \forall \ell \in \ker T^* : \langle \ell, y \rangle = 0\}$

(d) $\operatorname{im} T^* = (\ker T)^\perp = \{\psi \in Y^* \mid \forall x \in \ker T : \langle \psi, x \rangle = 0\}$

*Remark.* This result can be extended to densely defined operators as long as they have closed graph. Notice that everywhere defined continuous operators between Banach spaces automatically have closed graph.

# 7 Examples of Banach Spaces

**Example 7.0.1.** Let $K$ be a compact metric space and $C(K)$ the space of continuous functions equipped with the sup-norm $||\bullet||_\infty$. This is well-defined because continuous functions on a compact space are bounded and if $||f||_\infty = 0$ then $f = 0$ because then $|f(x)| = 0$ so $f(x) = 0$ for all $x$. Finally, convergence in the sup-norm is what is usually called uniform convergence which preserves continuity so $C(K)$ is complete.

**Example 7.0.2.** However, $C^k([0,1])$ for $k \geq 1$ is not a Banach space because the uniform limit of differentiable functions is not differentiable.

**Proposition 7.0.3.** Let $X, Y$ be normed spaced and $\mathcal{L}(X,Y)$ the space of bounded linear operators endowed with the operator norm. Then $\mathcal{L}(X,Y)$ is a normed space and is Banach if $Y$ is Banach.

*Proof.* It is clear that $\mathcal{L}(X,Y)$ is a vector space. We need to check that $||\bullet||$ is indeed a norm on $\mathcal{L}(X,Y)$. It is obviously homogeneous and nonegative so we need to check positivity and the triangle inequality. First, suppose that $||T|| = 0$ then by definition $||Tx|| = 0$ for all $x \in X$ so $Tx = 0$ meaning $T = 0$. If $T, S \in \mathcal{L}(X,Y)$ then,

$$||X + Y|| = \sup_{||x||=1} ||(X+Y)x|| \leq \sup_{||x||=1} (||Tx|| + ||Sx||) \leq ||T|| + ||S||$$

Now suppose that $Y$ is Banach. Given a Cauchy sequence $T_n \in \mathcal{L}(X,Y)$ we know that,

$$||T_n x - T_m x|| \leq ||T_n - T_m|| \cdot ||x||$$

and thus for each fixed $x \in X$ the sequence $T_n x$ is Cauchy and thus converges to a limit in $Y$ because $Y$ is complete. It is clear that

$$Tx = \lim_{n \to \infty} T_n x$$

defines a linear operator. We need to show that it is bounded. Notice that $||T_n||$ is bounded by $C$ because $T_n$ is Cauchy and therefore,

$$\begin{aligned} ||T|| &= \sup_{||x||=1} ||Tx|| = \sup_{||x||=1} \lim_{n \to \infty} ||T_n x|| \\ &= \sup_{||x||=1} \limsup_{n \to \infty} ||T_n x|| \leq \sup_{||x||=1} \limsup_{n \to \infty} ||T_n|| \cdot ||x|| = \limsup_{n \to \infty} ||T_n|| \leq C \end{aligned}$$

is bounded. Thus it suffices to show that $T_n \to T$ in operator norm. Recall that,

$$||(T_n - T_m)x|| \leq ||T_n - T_m|| \cdot ||x||$$

for any $x$ and $T_n$ is Cauchy so for any $\epsilon > 0$ we can choose $N$ large enough such that $||T_n - T_m|| < \epsilon$ for any $n, m > N$. By the continuity of the norm on $Y$ we find that,

$$||(T - T_n)x|| = \lim_{m \to \infty} ||(T_m - T_n)x|| < \epsilon ||x||$$

for $n > N$ and thus,

$$||T - T_n|| = \sup_{||x||=1} ||(T - T_n)x|| < \epsilon$$

for all $n > N$ so $||T - T_n|| \to 0$. $\square$

*Remark.* A nice write up can be found here.

**Example 7.0.4.** We will prove that for any locally compact Hausdorff space, $C_0(X)$ equiped with the uniform norm is a Banach space. However, here we consider $C_c(X)$ the space of compactly supported functions. Although $(C_c(X), || \bullet ||_\infty)$ is clearly a normed space, it is *not* complete.

Here we provide an example. Let $X = \mathbb{R}$ and and consider a smooth bump function $\chi(x)$ which satisfies $\chi(x) = 1$ if $|x| < 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Therefore, $\chi$ is compactly supported and continuous. Then, consider,

$$f_n(x) = \frac{\chi(x/n)}{1 + x^2}$$

which is clearly continuous and compactly supported since $\text{Supp}\,(f_n) \subset \overline{B_{2n}(0)}$. Furthermore,

$$||f_n - f_m||_\infty \leq \frac{2}{1 + \min{(n,m)}^2}$$

because for $x < \min{(n,m)}$ we have $f_n(x) - f_m(x) = 0$ and for $x \geq \min{(n,m)}$ each function is bounded by $(1 + \min{(n,m)}^2)^{-1}$. Therefore, $f_n$ is a Cauchy sequence in the uniform norm and therefore converges in $C_0(X)$. Clearly $f_n \to f$ where $f(x) = \frac{1}{1+x^2}$ which is not compactly supported and since $C_c(X) \subset C_0(X)$ (and normed spaces being Haudorff have unique limits) we see that $C_c(X)$ cannot be complete.

However, we can give $C_c(X)$ a better topology such that it is complete. Indeed,

$$C_c(X) = \varinjlim_{\substack{K \supset X \\ \text{compact}}} C(X, K)$$

where $C(X, K)$ is the space of continuous functions supported on $K$. This is Banach because obviously $C(X, K) \cong C(K)$ which when given the uniform norm is Banach. Therefore, we equip $C_c(X)$ with the locally convex direct limit topology. Then the sup norm $|| \bullet ||_\infty$ is continuous on $C_c(X)$ because it is on each $C(X, K)$ but the topology is finer than the norm topology e.g. there are sequences which coverge in the uniform norm which are not Cauchy in $C_c(X)$ for example the sequence $f_n$ defined above.

# 8 Locally Convex Analysis

**Definition 8.0.1.** A subset $X$ of a $\mathbb{R}$-vector space $V$ is called *convex* if for all $x, y \in X$ and $\lambda \in [0, 1]$ the point $\lambda x + (1 - \lambda) y \in X$.

**Definition 8.0.2.** A topological vector space is called *locally convex* if the origin has a neighborhood basis of covex sets.

*Remark.* Because translation is a homeomorphism, every point of a locally convex topological vector space (LCTVS) has a convex neighborhood basis.

**Definition 8.0.3.** A *semi-norm* on a $\mathbb{R}$-vector space $V$ is a function $p : V \to \mathbb{R}$ such that,

(a) $p(x) \geq 0$ for all $x \in X$

(b) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$ and $x \in X$

(c) $p(x+y) \le p(x) + p(y)$ for all $x, y \in V$.

Notice that a semi-norm satisfying $p(x) = 0 \implies x = 0$ is a norm.

**Definition 8.0.4.** Given a family of semi-norms $\mathcal{P}$ on a vector space $V$, the *initial topology* $\mathcal{T}_{\mathcal{P}}$ on $X$ is the coarsest topology for which each $p \in \mathcal{P}$ is continuous.

**Proposition 8.0.5.** The initial topology makes $V$ a locally convex topological vector space.

*Proof.* A function $f : X \to V$ is continuous if and only if $p \circ f : X \to \mathbb{R}$ is continuous for each $p \in \mathcal{P}$. First, for $f : \mathbb{R} \times V \to V$ via $(\lambda, v) \mapsto \lambda v$ we have $p \circ f(\lambda, v) = |\lambda| p(v)$ so $p \circ f$ is the same as $m \circ (| \bullet | \times p)$ which is continuous because $p$ is continuous and the other functions we know are continuous on $\mathbb{R}$. Now for $f : V \times V \to V$ we know that $p \circ f(x, y) = p(x+y) \le p(x) + p(y)$.

Furthermore, the open sets $\{U_{\epsilon,p} = p^{-1}(B_\epsilon(0))\}_{\epsilon > 0, p \in \mathcal{P}}$ form a neighborhood basis of the origin and if $x, y \in U_{\epsilon,p}$ then $p(x) < \epsilon$ and $p(y) < \epsilon$ so for any $\lambda \in [0,1]$ we have,

$$p(\lambda x + (1-\lambda) y) \le \lambda p(x) + (1-\lambda) p(y) < \lambda \epsilon + (1-\lambda)\epsilon = \epsilon$$

so $\lambda x + (1-\lambda) y \in U_{\epsilon,p}$ and thus $U_{\epsilon,p}$ is convex. □

# 9 Stone-Weierstrass Theorem

**Definition 9.0.1.** Let $X$ be a locally compact Hausdorff space. Then $C_0(X, \mathbb{R})$ is the space of continuous functions that vanish at infinity meaning that for each $f \in C_0(X, \mathbb{R})$ any any $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|f|_{X \setminus K}| < \epsilon$.

*Remark.* If $X$ is compact then $C_0(X, \mathbb{R}) = C(X, \mathbb{R})$.

**Proposition 9.0.2.** When $C_0(X, \mathbb{R})$ is equipped with the supremum norm it is a Banach space.

*Proof.* Because the greater than 1 support of $f$ is contained in a compact set $K$ we know that $f$ is bounded because $f|_K$ is bounded since it is continuous on a compact space. Thus $||f||_\infty$ exists. Furthermore, it is trivial to show this is a normed space. Now suppose that $\{f_n\}$ is Cauchy sequence. Then $|f_n(x) - f_m(x)| < ||f_n - f_m||$ so $\{f_n(x)\}$ is Cauchy for each $x \in X$. Therefore, we can define a function,

$$f(x) = \lim_{n \to \infty} f_n(x)$$

which exists because $\mathbb{R}$ is complete. I claim that $f_n \to f$ in the sup norm. Indeed, for any $\epsilon > 0$ we have $||f_n - f_m|| < \epsilon$ for $n, m > N$ with some $N$ so,

$$|f_n(x) - f_m(x)| < \epsilon$$

Taking the limit as $m \to \infty$ we find,

$$|f_n(x) - f(x)| < \epsilon$$

and since this is a uniform bound for all $x \in X$ then $||f_n - f|| < \epsilon$ so $f_n \to f$. Now we need to show that $f \in C_0(X, \mathbb{R})$. For continuity, it suffices to show that $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$ has an open neighborhood for all $\epsilon > 0$ and $x_0 \in X$. Notice that we can choose $n$ large enough such that,

$||f_n - f|| < \frac{\epsilon}{3}$ and by continuity of $f_n$ there is an open $x_0 \in U \subset f_n^{-1}(B_{\frac{\epsilon}{3}}(f_n(x_0)))$. We need to show that $f(U) \subset B_\epsilon(f(x_0))$. for $x \in U$,

$$|f(x) - f(x_0)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon$$

and therefore $U \subset f^{-1}(B_\epsilon(f(x_0)))$ and $x_0 \in U$ so we find that $f$ is continuous. Finally, we need to show that $f$ vanishes at infinity. For any $\epsilon > 0$ we can find some $f_n$ such that $||f - f_n|| < \frac{\epsilon}{2}$ and because $f_n \in C_0(X, \mathbb{R})$ also a compact $K \subset X$ such that $||f_n|_{X \setminus K}||_\infty < \frac{\epsilon}{2}$. Then,

$$||f|_{X \setminus K}|| \le ||f - f_n|| + ||f_n|_{X \setminus K}|| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

so $f$ vanishes at infinity. □

*Remark.* This proof is showing that uniform limits of continuous functions are continuous.

**Theorem 9.0.3** (Stone-Weierstrass)**.** Let $X$ be a locally compact Hausforff and $A \subset C_0(X, \mathbb{R})$ be a subalgebra such that,

(a) $A$ vanishes nowhere meaning for each $x \in X$ there is $f \in A$ with $f(x) \neq 0$

(b) $A$ separates points meaning for each $x, y \in X$ with $x \neq y$ there is $f \in A$ with $f(x) \neq f(y)$

then $A$ is dense in $C_0(X, \mathbb{R})$.

**Corollary 9.0.4.** Polynomials with rational coefficients are dense in $C([a, b], \mathbb{R})$ and thus $C([a, b], \mathbb{R})$ is separable.

# 10 Duals and Weak-$*$ Topology

## 10.1 Definitions

*Remark.* Here let $K$ be a complete normed field.

**Definition 10.1.1.** Let $X$ be a topological $K$-vector space. Then the *algebraic dual space* $X^\#$ is the space of linear functionals and the *continuous dual space* $X'$ is the space of continuous (equivalently bounded if $X$ is normed) linear functionals.

*Remark.* Notice that if $X$ is a normed space then $X^* = \mathcal{L}(X, K)$ inherits the operator norm and is a Banach space with respect to this norm because $\mathcal{K}$ is complete.

**Definition 10.1.2.** Let $X$ be a normed space. Consider the canonical map $\varphi : X \to X^{**}$ via sending $x \mapsto (\ell \mapsto \ell(x))$. First, notice that $\varphi(x)$ is a bounded operator because,

$$|\varphi(x)(\ell)| = |\ell(x)| \le ||\ell|| \cdot ||x||$$

and thus,

$$||\varphi(x)||_{X^{**}} \le ||x||$$

Therefore, $\varphi$ is a bounded operator $\varphi \in \mathcal{L}(X, X^{**})$ and a short map $||\varphi|| \le 1$. We say that $X$ is *reflexive* if $\varphi$ is an isomorphism.

**Proposition 10.1.3.** Let $X$ be a normed space. Then $\varphi$ is an isometry and is thus injective.

*Proof.* For each $x \in X$, by Hahn-Banach, there exists a linear functional $\ell_x : X \to K$ such that $\ell_x(x) = ||x||$ and $||\ell_x|| = 1$. Therefore,

$$||\varphi(x)|| \geq |\varphi(x)(\ell_x)| = ||x||$$

but we already showed that $||\varphi(x)|| \leq ||x||$ so indeed $||\varphi(x)|| = ||x||$. Injectivity follows from being an isometry because if $\varphi(x) = 0$ then $||x|| = ||\varphi(x)|| = 0$ so $x = 0$. We alternatively know that $X^*$ separates points again by the Hahn-Banach theorem so if $x \neq y$ then there is a linear functional $\ell$ such that $\ell(x) \neq \ell(y)$ and thus $\varphi(x)(\ell) \neq \varphi(y)(\ell)$ so $\varphi(x) \neq \varphi(y)$. $\square$

**Theorem 10.1.4.** Let $X$ be a normed space. If $X^*$ is separable then $X$ is separable.

**Corollary 10.1.5.** If $X$ is a reflexive Banach space then $X$ is separable if and only if $X^*$ is separable.

## 10.2 Reflexive Spaces

**Example 10.2.1.** If $X$ is finite dimensional then $X$ is reflexive.

**Example 10.2.2.** On a measure space $(\Omega, \mathcal{F}, \mu)$ for $1 < p < \infty$ we have $L^p(\Omega)^* \cong L^q(\Omega)$ where $p^{-1} + q^{-1} = 1$. Therefore $L^p(\Omega)$ is reflexive. However, for $p = 1, \infty$ this does not hold.

**Example 10.2.3.** Indeed, here we consider the $p = \infty$ case. Consider $X = C([-1, 1], \mathbb{R})$ with the sup-norm. By Hahn-Banach, for any $\ell \in X^*$ there is some $\varphi \in X^{**}$ such that $||\varphi|| = 1$ and $\varphi(\ell) = ||\ell||$. If we assume that $X$ is reflexive then $\varphi$ is in the image and thus $\varphi(\ell) = \ell(x)$ for some $x \in X$. However, consider,

$$\ell(g) = \int_{-1}^{0} g(t)\mathrm{d}t - \int_{0}^{1} g(t)\mathrm{d}t$$

Then $||\ell|| \leq 2$ so there exists some $f \in X$ with $||f|| = 1$ and $\ell(f) = 2$. However, $|\ell(f)| < 2$ for all $f$ with $||f|| = 1$ giving a contradiction.

**Theorem 10.2.4** (Riesz)**.** Let $H$ be a Hilbert space. Then, $\langle -, - \rangle$ defines an isomorphism $H \to \overline{H}^*$ via $x \mapsto \langle x, - \rangle$ where $\overline{H}^*$ is the conjugate dual space or equivalently the space of continuous anti-linear functionals.

**Corollary 10.2.5.** Any Hilbert space is reflexive.

*Proof.* Since $\overline{H}^*$ is a Hilbert space there are canonical isomorphisms by Riesz $\varphi_1 : H \to \overline{H}^*$ and $\varphi_2 : \overline{H}^* \to H^{**}$. It suffices to show that $\varphi_2 \circ \varphi_1 = \varphi$ where $\varphi : H \to H^{**}$ is the canonical map considered above. Indeed, for any $x \in H$ and $\ell \in H^*$ consider,

$$(\varphi_2 \circ \varphi_1)(x)(\ell) = \varphi_2(\langle x, - \rangle)(\ell) = \langle \langle x, - \rangle, \ell \rangle_{H^*}$$

However, by Riesz we know that $\ell = \langle y, - \rangle$ for some $y \in H$ and that $\varphi_1$ is an isometry (although anti-linear so it reverses the inner product) so,

$$(\varphi_2 \circ \varphi_1)(x)(\ell) = \langle y, x \rangle = \ell(x)$$

$\square$

**Proposition 10.2.6.** If $X$ is a normed space and $V \subset X$ is a closed linear subspace. If $X$ is reflexive then so is $V$.

*Proof.* We need to prove that if $\psi \in Y^{**}$ then there exists $y \in Y$ such that $\varphi_Y(y) = \psi$. Under the canonical map $Y^{**} \to X^{**}$ we see that any $\psi \in Y^{**}$ maps to the image of $\varphi_X : X \to X^{**}$ so there is some $x \in X$ such that the image $\psi'$ of $\psi$ under $Y^{**} \to X^{**}$ satisfies,

$$\forall \ell \in X^* : \psi'(\ell) = \psi(\ell|_Y) = \ell(x)$$

Therefore it suffices to show that $x \in Y$ since by Hahn-Banach we can extend any linear functional on $Y$ to $X$ such that its restriction is unchanged. Suppose not, there there exists some $m \in X^*$ such that $m|_Y = 0$ and $m(x) \neq 0$ by Hahn-Banach. Now, clearly,

$$\psi'(m) = \psi(m|_Y) = 0$$

However, $\psi'(m) = m(z) \neq 0$ giving a contradiction so $z \in Y$ and thus $Y$ is reflexive. □

## 10.3 Weak Convergence

**Definition 10.3.1.** Let $X$ be a topological $K$-vector space. We say a sequence $x_n \in X$ *converges weakly* to $x \in X$ or $x_n \to x$ weakly if the sequence converges in the weak topology $\sigma(X, X^*)$ on $X$. Explicilty, $x_n \to x$ weakly if for every $\ell \in X^*$ we have,

$$\lim_{n \to \infty} \ell(x_n) = \ell(x)$$

*Remark.* From here on let $X$ be a normed space.

**Proposition 10.3.2.** Bounded linear operators $T : X \to Y$ are weakly continuous.

*Proof.* If $T : X \to Y$ is bounded then it is continuous in the norm topology. Thus for any continuous linear functional $\ell \in Y^*$ we know that $T^*\ell = \ell \circ T \in X^*$ is a continuous linear functional and thus continuous in the weak topology on $X$. Therefore, $T : X \to Y$ is continuous in the weak topologies. □

**Proposition 10.3.3.** If $x_n \in X$ is a weakly convergent sequence then $\{x_n\}$ is bounded.

*Proof.* Consider the operators $\mathrm{ev}_{x_n} \in X^{**}$. Because $x_n$ converges weakly, for each $\ell \in X^*$ we know that,

$$\lim_{n \to \infty} \ell(x_n) = \lim_{n \to \infty} \mathrm{ev}_{x_n}(\ell)$$

exists and thus $\{\mathrm{ev}_{x_n}(\ell)\}$ is bounded. Since $X^*$ is a Banach space, we can apply Banach-Steinhaus to the sequence of pointwise bounded operators $\{\mathrm{ev}_{x_n}\} \subset \mathcal{L}(X^*, K)$ to conclude that $\{\mathrm{ev}_{x_n}\}$ is uniformly bounded. Therefore,

$$||x_n|| = \sup_{||\ell||=1} |\ell(x_n)| = \sup_{|\ell|=1} |\mathrm{ev}_{x_n}(\ell)| = ||\mathrm{ev}_{x_n}||$$

is uniformly bounded. □

**Proposition 10.3.4** (Lower semi-continuity of weak limits)**.** Let $x_n \to x$ weakly. Then,

$$||x|| \le \liminf_{n \to \infty} ||x_n||$$

*Proof.* By Hahn-Banach, there exists some $\ell \in X^*$ with $|\ell(x)| = ||x||$ and $||\ell|| = 1$. Therefore,

$$||x|| = |\ell(x)| = \lim_{n\to\infty} |\ell(x_n)| = \liminf_{n\to\infty} |\ell(x_n)| \le \liminf_{n\to\infty} ||x_n|| \cdot ||\ell|| = \liminf_{n\to\infty} ||x_n||$$

□

**Lemma 10.3.5.** Suppose that $C \subset X$ is convex and (norm) closed then $C$ is weakly closed.

*Proof.* Let $C$ be a closed and convex. Then for any $x_0 \notin C$ by convex Hahn-Banach there is some $\ell \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x_0) < \alpha < f(x)$ for all $x \in X$. Then $f^{-1}((-\infty, \alpha))$ is a weakly open neighborhood of $x$ in $C^C$ so $C$ is weakly closed. □

**Proposition 10.3.6.** If $\varphi : X \to \mathbb{R}$ is convex and lower semi-continuous in the norm topology then $\varphi$ is lower semi-continuous in the weak topology on $X$.

*Proof.* It suffices to show that $C_\alpha = \varphi((-\infty, \alpha])$ is weakly closed. Since $\varphi$ is lower semi-cotinuous in the norm topology $C_\alpha$ is norm closed. Furthermore, because $\varphi$ is convex if $a, b \in C_\alpha$ then for any $\lambda \in [0, 1]$,

$$f(\lambda a + (1 - \lambda b)) \le \lambda f(a) + (1 - \lambda) f(b) \le \lambda\alpha + (1 - \lambda)\alpha = \alpha$$

Therefore, $C_\alpha$ is convex so it is weakly closed by above. Therefore $\varphi$ is weakly lower semi-continuous. □

*Remark.* Since $|| \bullet || : X \to \mathbb{R}$ is convex and norm continuous we get another proof of the weak lower semi-continuity.

**Proposition 10.3.7.** If $X$ and $Y$ are Banach spaces and $T : X \to Y$ is linear then $T$ is bounded if and only if $T$ is weakly continuous.

*Proof.* One direction is immediate. Suppose that $T : X \to Y$ is weakly continuous. To show that $T$ is continuous it suffices to show that its graph is closed. Let $x_n \to x$ and $Tx_n \to y$ in norm. Then $x_n \to x$ weakly so $Tx_n \to Tx$ weakly and thus $Tx = y$ because $Tx_n \to y$ weakly and weak limits are unique. □

## 10.4 The Weak-$*$ Topology

*Remark.* When $X$ is a normed space we know that $X^*$ is also normed (in fact it is Banach) so we get a canonical topology on $X^*$ compatible with the vector space structure. However, in general it is not clear how to topologize $X^*$. One option is the weak-$*$ topology which we now define although this will usually not agree with the norm topology when one is available.

**Definition 10.4.1.** The weak-$*$ topology on $X^*$ is the coarsest topology on which the maps $\ell \mapsto \ell(x)$ for $x \in X$ are continuous.

**Proposition 10.4.2.** Using weak-$*$ topology, the canonical map $\varphi : X \to X^{**}$ is continuous.

*Proof.* First we must check that $\varphi$ is actually well-defined. Then $\varphi(x) : \ell \mapsto \ell(x)$ is continuous by hypothesis and thus $\varphi(x) \in X^{**}$. Furthermore, we topologize $X^{**}$ with the weak-$*$ topology induced by the weak-$*$ topology on $X$. Therefore, for any $\ell \in X^*$ we know that $e_\ell : \psi \mapsto \psi(\ell)$ is continuous. In particular, $\varphi$ is continuous iff $e_\ell \circ \varphi$ is continuous for all $\ell \in X^*$. However, $e_\ell \circ \varphi : x \mapsto \ell(x)$ is continuous by the definition of the weak-$*$ topology on $X^*$. □

*Remark.* Sometimes the weak-$*$ topology is called the "topology of pointwise convergence" because a net $f_\bullet$ in $X^*$ converges to $f$ iff $f_\bullet(x)$ coverges to $f(x)$ for each $x \in X$.

**Proposition 10.4.3.** If $X$ is a Banach space and $\ell_n \in X^*$ is a weak-$*$ convergent sequence then $\{\ell_n\}$ is bounded.

*Proof.* For each $x \in X$, notice that,

$$\lim_{n \to \infty} \ell_n(x)$$

exists and thus $\{\ell_n\} \subset \mathcal{L}(X, K)$ is a pointwise bounded sequence of operators. Because $X$ is Banach we can apply Banach-Steinhaus to conclude that $\{\ell_n\}$ is uniformly bounded. $\square$

**Theorem 10.4.4** (Banach-Alaoglu)**.** Let $X$ be a topological vector space and $U \subset X$ a neighborhood of the origin. Then,

$$U^\circ = \{\ell \in X^* \mid \sup_{x \in U} |\ell(x)| \leq 1\}$$

is compact in the weak-$*$ topology.

**Corollary 10.4.5.** Let $X$ be a normed space and $B = \{x \in X \mid ||x|| \leq 1\}$. Then,

$$B^\circ = \{\ell \in X^* \mid \sup_{x \in U} |\ell(x)| \leq 1\} = \{\ell \in X^* \mid ||\ell|| \leq 1\} = B^*$$

is compact in the weak-$*$ topology.

**Proposition 10.4.6.** Let $X$ be a Banach space. Then $B = \{x \in X \mid ||x|| \leq 1\}$ is compact in the weak topology on $X$ if and only if $X$ is reflexive.

*Proof.* If $X$ is reflexive then $X \to X^{**}$ is an isomorphism. Furthermore, the weak-$*$ topology on $X^{**}$ is equal to the weak topology on $X$ under this isomorphism because the weak-$*$ topology is the weakest topology for which $\psi = \mathrm{ev}_x \mapsto \mathrm{ev}_x(\ell) = \ell(x)$ is continuous which is exactly the weak topology on $X$. Furthermore, $X \to X^{**}$ is isometric so the unit balls agree. Therefore, by Banach-Alaoglu, $B$ is compact. Conversely, suppose that $B$ is compact. (HOW TO SHOW THIS!!!) $\square$

**Proposition 10.4.7.** If $X$ is a separable normed space then $B^*$ in the weak topology is metrizable.

*Proof.* We apply Spring 2010 part 2 Q2. Let $\{x_i\} \subset X$ be a dense countable set then $f_i = \mathrm{ev}_{x_i}$ are continuous on $X^*$ in the weak-$*$ topology. Furthermore, for $\ell_1, \ell_2 \in X^*$ if $\ell_1(x_i) = \ell_2(x_i)$ for all $x_i$ then $\ell_1 = \ell_2$ because $\mathcal{C}$ is Hausdorff and $\{x_i\}$ is dense. Therefore $f_j$ separate points. Furthermore, by Banach-Alaoglu, $B^*$ is compact so we apply the problem to conclude that $B^*$ is metrizable in the weak topology. $\square$

**Lemma 10.4.8.** Let $X$ be an infinite dimensional normed space. In the weak topology, $\overline{S} = B$.

*Proof.* Let $||x_0|| \leq 1$. Now any weakly open neighborhood contains an open of the form,

$$U = \bigcap_{i=1}^{n} (f_i^{-1}(B_\epsilon(0)) + x_0)$$

for some $f_i \in X^*$. Then consider the map $f : V \to K^n$ via $x \mapsto (f_1(x), \dots, f_n(x))$. Because $V$ is not finite dimensional, $f$ cannot be injective. Choose some $y_0 \neq 0$ with $f(y_0) = 0$. Then $f_i(ty_0) = 0$ for all $t$ and therefore $x_0 + ty_0 \in U$. Let $g(t) = ||x_0 + ty_0||$. Notice that $g(0) = ||x_0|| \leq 1$ and $g(t) \to \infty$ as $t \to \infty$ because,

$$g(t) \geq ||t|||y_0|| - ||x_0|||$$

and $||y_0|| > 0$. Therefore, there exists some $t \in \mathbb{R}$ such that $g(t) = 1$. Therefore, $x_0 + ty_0 \in U \cap S$ so we see that $x_0$ is a weak limit point of $S$. Therefore, $B \subset \overline{S}$. However, $B$ is weakly closed because it is convex and norm closed. Therefore, $B = \overline{S}$. $\square$

# 11 Measure Theory

**Proposition 11.0.1** (Chebyshev)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f$ a measurable function and $\epsilon > 0$. Then,

$$\mu(\{x \in X \mid |f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f| \, \mathrm{d}\mu$$

*Remark.* The following establishes the continuity of the Lebesgue integral.

**Proposition 11.0.2.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : X \to \hat{\mathbb{R}}$ an integrable function with $X \in \mathcal{F}$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $E \in \mathcal{F}$ with $\mu(E) < \delta$,

$$\left| \int_E f \, \mathrm{d}\mu \right| < \epsilon$$

*Proof.* Because,

$$\left| \int_E f \, \mathrm{d}\mu \right| \leq \int_E |f| \, \mathrm{d}\mu$$

it suffices to assume that $f \geq 0$. Now by definition,

$$\int_X f \, \mathrm{d}\mu = \int_0^\infty f^*(t) \, \mathrm{d}t = \lim_{s \to \infty} \int_0^s f^*(t) \, \mathrm{d}t$$

where $f^*(t) = \mu(\{x \in X \mid f(x) > t\})$ exists. Because the integral is an increasing function bounded by its limit, there exists some $s_0$ such that,

$$\int_{s_0}^\infty f^*(t) \, \mathrm{d}t < \frac{\epsilon}{2}$$

Now set $\delta = \frac{\epsilon}{2s_0}$ and suppose that $\mu(E) < \delta$. Then consider,

$$\left| \int_E f \, \mathrm{d}\mu \right| = \int_0^\infty f_E^*(t) \, \mathrm{d}t$$

where,

$$f_E^*(t) = \mu(\{x \in E \mid f(t) > x\}) = \mu(\{x \in X \mid f(t) > x\} \cap E) \leq \mu(E) < \delta$$

so $f_E^*(t) \leq f^*(t)$ and $f_E^*(t) < \delta$. Therefore,

$$\left| \int_E f \, \mathrm{d}\mu \right| = \int_0^{s_0} f_E^*(t) \, \mathrm{d}t + \int_{s_0}^\infty f_E^*(t) \, \mathrm{d}t < \int_0^{s_0} \delta \, \mathrm{d}t + \int_{s_0}^\infty f^*(t) \, \mathrm{d}t < s_0 \delta + \tfrac{\epsilon}{2} = \epsilon$$

∎

# 12 Spectral Theory

**Definition 12.0.1.** Let $X$ be a complex Banach space and $T : X \to X$ a bounded operator. Define the *spectrum*,

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not invertible}\}$$

and the *resolvent*

$$\rho(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ invertible}\} = \mathbb{C} \setminus \sigma(T)$$

By the bounded inverse theorem $T - \lambda I$ is invertible iff it is bijective. Therefore, we can decompose,

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

where,

$$\sigma_p(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not injective}\}$$
$$\sigma_p(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective and } \overline{\mathrm{Im}\,(T)} = X\}$$
$$\sigma_r(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective and } \overline{\mathrm{Im}\,(T)} \neq X\}$$

## 12.1 Some Lemmas

**Lemma 12.1.1.** If $S : X \to Y$ and $T : Y \to Z$ are bounded operators then $T \circ S$ is bounded and,

$$||T \circ S|| \leq ||T|| \cdot ||S||$$

*Proof.* Since $||Ty|| \leq ||T|| \cdot ||y||$ for any $y \in Y$,

$$||T \circ S|| = \sup_{||x||=1} ||T(S(x))|| \leq \sup_{||x||=1} ||T|| \cdot ||Sx|| = ||T|| \cdot ||S||$$

□

**Lemma 12.1.2.** Let $X, Y, Z$ be normed spaces and $T : Y \to Z$ a bounded operator. Then composition $T \circ - : \mathcal{L}(X, Y) \to \mathcal{L}(X, Z)$ is a bounded operator and thus continuous.

*Proof.* We have shown that for any $S \in \mathcal{L}(X, Y)$ that $T \circ S$ is bounded and,

$$||T \circ S|| \leq ||T|| \cdot ||S||$$

and therefore $||T \circ -||_{\mathcal{L}(X,Y) \to \mathcal{L}(X,Z)} \leq ||T||_{\mathcal{L}(Y,Z)}$ so $T \circ -$ is a bounded operator. □

**Lemma 12.1.3.** Let $X$ be a Banach space. Suppose that $S \in \mathcal{L}(X, X)$ has $||S|| < 1$. Then $(I - S)$ is invertible and in fact,

$$(I - S)^{-1} = \sum_{n=0}^{\infty} S^n$$

*Proof.* First, we need to show that,

$$Q = \lim_{n \to \infty} Q_n \text{ where } Q = \sum_{i=0}^{n} S^i$$

exists. We know that $||S^n|| \leq ||S||^n = r^n$ and thus since $r = ||S|| < 1$ this is a Cauchy sequence because,

$$||Q_{n+k} - Q_n|| = ||S^n Q_k|| \leq ||S^n|| \cdot ||Q_k|| \leq ||Q_k|| \cdot r^n$$

Furthermore, by the triangle inequality, $||Q_k|| \leq 1 + r + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$ and thus,

$$||Q_{n+k} - Q_n|| \leq r^n \cdot \frac{1 - r^{k+1}}{1 - r} < \frac{r^n}{1 - r} \to 0$$

Therefore, the limit exists because $X$ is a Banach space so $\mathcal{L}(X, X)$ is also Banach. Thus $Q$ is well-defined. Now we apply the continuity of composition to see that,

$$(I - S) \circ Q = \lim_{n \to \infty} \sum_{i=0}^{n} (S^i - S^{i+1}) = \lim_{n \to \infty} (I - S^{n+1}) = I$$

because $||(I - S^{n+1}) - I|| = ||S^{n+1}|| \leq r^{n+1} \to 0$. Clearly the operators commute so $q \circ (I - S) = I$ as well. □

## 12.2 Basic Properties

*Remark.* We write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

**Theorem 12.2.1.** Let $X$ be a Banach space and $T \in \mathcal{L}(X)$ a bounded operator. Then,

(a) $\sigma(T) \subset \mathbb{C}$ is closed

(b) for any $\lambda \in \rho(T)$ we have $B_\delta(\lambda) \subset \rho(T)$ where $\delta = ||(T - \lambda I)^{-1}||^{-1} \le d(\lambda, \sigma(T))$

(c) the map $\rho(T) \to \mathcal{L}(X)$ via $\lambda \mapsto (T - \lambda I)^{-1}$ is analytic.

*Proof.* Clearly (b) implies (a) since $\sigma(T) = \rho(T)^C$. Choose $\lambda_0 \in \rho(T)$ and set $C = ||(T - \lambda_0 I)^{-1}||$ which exists by the definition of $\rho(T)$. Let $\delta = C^{-1}$ and consider $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \delta$. Then compute,

$$(T - \lambda I) = (T - \lambda_0 I) - (\lambda - \lambda_0) I = (I - \lambda_0 I) \circ [I - (\lambda - \lambda_0) \cdot (T - \lambda_0 I)^{-1}]$$

Now let $S = (\lambda - \lambda_0) \cdot (T - \lambda_0 I)^{-1}$ and notice that,

$$||S|| = |\lambda - \lambda_0| \cdot ||(T - \lambda_0 I)^{-1}|| < \delta \cdot C = 1$$

Therefore, $I - S$ is invertible so $(T - \lambda I) = (I - \lambda_0 I) \circ (I - S)$ is the composition of invertible bounded operators and thus is invertible itself (as a bounded operator meaning its inverse is also in $\mathcal{L}(X)$). Therefore, $\lambda \in \rho(T)$ showing that $B_\delta(\lambda_0) \subset \rho(T)$ so $\rho(T)$ is open proving (b). Furthermore, we know the explict inverse of $I - S$ is,

$$(I - S)^{-1} = \sum_{n=0}^{\infty} S^n$$

Therefore, by the continuity of the composition,

$$(T - \lambda I)^{-1} = (I - S)^{-1} \circ (T - \lambda_0 I)^{-1} = \sum_{n=0}^{\infty} S^k \circ (T - \lambda_0 I)^{-1} = \sum_{n=0}^{\infty} (T - \lambda_0 I)^{-(n+1)} \cdot (\lambda - \lambda_0)^n$$

whenever $|\lambda - \lambda_0| < \delta$. Therefore, the map $\lambda \mapsto (T - \lambda I)^{-1}$ is locally computed by an operator-valued power series in $(\lambda - \lambda_0)$ and thus is analytic. Finally, if $\lambda \in \sigma(T)$ then $\lambda \notin \rho(T)$ so $|\lambda - \lambda_0| \ge \delta$ and thus,

$$d(\lambda_0, \sigma(T)) = \inf_{\lambda \in \sigma(T)} |\lambda - \lambda_0| \ge \delta = C^{-1} = ||(T - \lambda_0 I)^{-1}||^{-1}$$

□

## 12.3 Spectral Radius

**Definition 12.3.1.** For an operator $T : X \to X$ define the *spectral radius*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

**Proposition 12.3.2.** When $|\lambda| > ||T||$ the following series converges,

$$(T - \lambda I)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \left( \frac{T}{\lambda} \right)^n$$

*Proof.* Let $S = \frac{T}{\lambda}$. Since $||T|| < |\lambda|$ then $||S|| < 1$ so by the previous lemma the sum exists and,

$$\sum_{n=0}^{\infty} \left(\frac{T}{\lambda}\right)^n = (I - S)^{-1} = -\lambda(T - \lambda I)^{-1}$$

□

**Corollary 12.3.3.** For any $|\lambda| > ||T||$ we see that $\lambda \in \rho(T)$ and thus $r(T) \le ||T||$.

**Proposition 12.3.4.** We have the following formula,

$$r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}$$

*Proof.* DO THIS!!!! □

*Remark.* Notice that $||T^n|| \le ||T||^n$ and therefore we again see that $r(T) \le ||T||$.

*Remark.* We always know that $1 = ||\text{id}|| \le ||T|| \cdot ||T^{-1}||$. However, the following tells us exactly when this inequality is strict.

**Proposition 12.3.5.** An invertible operator with $||T^{-1}|| = ||T||^{-1}$ has $\sigma(T) \subset S^1 \cdot ||T||$.

*Proof.* Assume that $||T^{-1}||^{-1} = ||T||$. We know that $r(T) \le ||T||$ so it suffices to show that if $\lambda \in \sigma(T)$ then $|\lambda| \ge ||T||$. Indeed, since by assumption $0 \in \rho(T)$ we see that $B_\delta(0) \subset \rho(T)$ where $\delta = ||T^{-1}||^{-1} = ||T||$ and thus if $\lambda \in \sigma(T)$ then $\lambda \notin B_\delta(0)$ so $|\lambda| > ||T||$ proving the result. □

*Remark.* In general, if $T$ is invertible then $\sigma(T)$ is contained in the annulus,

$$\sigma(T) \subset \{\lambda \in \mathbb{C} \mid ||T^{-1}||^{-1} \le |\lambda| \le ||T||\}$$

## 12.4 Some More Properties of the Spectrum

**Proposition 12.4.1.** Let $T : X \to X$ be a bounded operator on a Banach space. Then,

(a) $\sigma(T) \subset \mathbb{C}$ is compact

(b) if $X \neq \{0\}$ then $\sigma(T)$ is nonempty

*Proof.* We know that $\sigma(T) \subset \mathbb{C}$ is closed and also

$$\sup_{\lambda\in\sigma(T)} |\lambda| = r(T) \le ||T||$$

so $\sigma(T)$ is bounded and thus compact. Now suppose that $\sigma(T)$ is empty. For any $\ell \in \mathcal{L}(X)^*$ we see that the function $g : \mathbb{C} \to \mathbb{C}$ given by $\lambda \mapsto \ell((T - \lambda I)^{-1})$ is entire because $\lambda \mapsto (T - \lambda I)^{-1}$ is analytic everywhere and $\ell$ is continuous and linear so,

$$g(\lambda) = \ell((T - \lambda I)^{-1}) = \ell\left(\sum_{n=0}^{\infty}(T - \lambda_0 I)^{-(n+1)} \cdot (\lambda - \lambda_0)^n\right) = \sum_{n=0}^{\infty} \ell((T - \lambda_0 I)^{-(n+1)}) \cdot (\lambda - \lambda_0)^n$$

for $|\lambda - \lambda_0| < \delta$ and thus $g$ is holomorphic about $\lambda_0$ for each $\lambda_0 \in \rho(T) = \mathbb{C}$. Since $g$ is continuous on $\overline{B_{2||T||}(0)}$ it is bounded. Furthermore, for $|\lambda| > ||T||$ we know that the following series is convergent,

$$(T - \lambda I)^{-1} = -\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{T}{\lambda}\right)^n$$

and therefore,

$$g(\lambda) = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{\ell(T^n)}{\lambda^n}$$

is convergent but $|\ell(T^n)| \le ||\ell|| \cdot ||T^n|| \le ||\ell|| \cdot ||T||^n$ and therefore for $|\lambda| > 2||T||$,

$$|g(\lambda)| \le -\frac{1}{|\lambda|} \sum_{n=0}^{\infty} ||\ell|| \left( \frac{||T||}{|\lambda|} \right)^n = \frac{||\ell||}{|\lambda| - ||T||} < \frac{||\ell||}{||T||}$$

is bounded. Therefore $g$ is entire and bounded and thus must be constant. Furthermore, the above calculation shows that $g(\lambda) \to 0$ as $|\lambda| \to \infty$ so $g = 0$ and thus $(T - \lambda I)^{-1} = 0$ which is impossible unless $X = \{0\}$ in which case the $I = 0$ is invertible. Thus if $X \neq \{0\}$ then $\sigma(T)$ is nonempty. □

# 13 Compact Operators on Normed Spaces

## 13.1 Definition and Basic Properties

*Remark.* First we recall the following generalization of Heine-Borel to any complete metric space.

**Proposition 13.1.1.** Let $X$ be a complete metric space. Then $S \subset X$ is compact if and only if $S$ is closed and totally bounded meaning that for any $\epsilon > 0$ there exists a finite open cover of $S$ by balls of radius $\epsilon$.

**Definition 13.1.2.** A linear operator $T : X \to Y$ between normed spaces is compact if for any bounded $B \subset X$ then $\overline{T(B)} \subset Y$ is compact.

**Example 13.1.3.** If $Y$ is finite dimensional then every closed and bounded subset is compact. Therefore, bounded linear operators with finite dimensional codimans are compact.

**Proposition 13.1.4.** Compact operators are bounded and thus continuous.

*Proof.* Consider the bounded subset $B = \{x \in X \mid ||x|| \le 1\}$. Then, $\overline{T(B)}$ is compact and in particular $T(B) \subset \overline{T(B)}$ is bounded. Therefore,

$$\sup_{||x|| \le 1} ||Tx|| = \sup_{y \in T(B)} ||y|| < \infty$$

□

**Proposition 13.1.5.** Let $T_n : X \to Y$ be a sequence of compact linear operators with $Y$ Banach. Suppose that $T_n \to T$ in the norm topology on $\mathcal{L}(X, Y)$. Then $T$ is compact.

*Proof.* It suffices to show that for any bounded $B \subset X$ that $T(B)$ is totally bounded. Let $B$ be bounded from zero by $M$. For each $\epsilon > 0$ we can find an $n$ such that $||T - T_n|| < \frac{\epsilon}{2M}$ and a finite cover $\{B_{\frac{\epsilon}{2}}(y_i)\}_{i \in I}$ of $T_n(B)$. For $x \in B$ we know that,

$$||(T - T_n)x|| \le ||T - T_n|| \cdot ||x|| < \frac{\epsilon}{2}$$

But $T_n x \in T_n(B)$ so there is some $y_i$ such that $T_n x \in B_{\frac{\epsilon}{2}}(y_i)$ thus $Tx \in B_\epsilon(y_i)$ because,

$$||y - Tx|| \le ||y - T_n x|| + ||Tx - T_n x|| < \epsilon$$

Therefore, $\{B_\epsilon(y_i)\}$ provides a finite cover of $T(B)$ by $\epsilon$-balls so $T(B)$ is precompact. □

*Remark.* We did not assume that $X$ is Banach in the above proposition. However, if $Y$ is

**Corollary 13.1.6.** Let $Y$ be a Banach space. The space of compact operators $\mathcal{K}(X,Y) \subset \mathcal{L}(X,Y)$ is closed in the norm topoolgy on $\mathcal{L}(X,Y)$.

*Proof.* Since $\mathcal{L}(X,Y)$ is a normed space it is sequential. We have shown that, when $Y$ is Banach, limits of compact operators are compact and thus $\mathcal{K}(X,Y) \subset \mathcal{L}(X,Y)$ is sequentially closed and thus closed. $\square$

**Corollary 13.1.7.** Let $X$ be a normed space and $Y$ a Banach space and $T_n \in \mathcal{L}(X,Y)$ a sequence of finite rank operators converging to $T \in \mathcal{L}(X,Y)$. Then $T$ is compact.

*Proof.* Finite rank bounded operators are compact and therefore $T_n \to T$ implies that $T$ is compact. $\square$

**Proposition 13.1.8.** Let $T : X \to Y$ be compact. If $x_n \to x$ weakly then $Tx_n \to Tx$ in norm.

*Proof.* By replacing $x_n$ by $x_n - x$ we can assume that $x_n \to 0$ weakly. Because bounded operators are continuous in the weak topology so $Tx_n \to 0$ weakly. Assume that $Tx_n$ does not converge in norm. Then there is some $\epsilon > 0$ such that $||Tx_n|| \ge \epsilon$ infinitely often. Passing to a subsequence, we may assume that $||Tx_n|| \ge \epsilon$ for all $n$ and $x_n \to 0$ weakly.

Furthermore, since $X$ is Banach, weakly convergent sequences are bounded so for some bounded set $B$ we have $\{x_n\} \subset B$ and thus $\{Tx_n\} \subset \overline{T(B)}$ with $\overline{T(B)}$ compact. Because $Y$ is a metric space, $\overline{T(B)}$ is sequentially compact and thus $\{Tx_n\}$ has a convergent subsequence. Therefore, $Tx_{n_j} \to y$ in norm and,

$$||y|| = \lim_{j \to \infty} ||Tx_{n_j}|| \ge \epsilon$$

so $y \neq 0$. However, then $Tx_{n_j} \to y$ weakly contradicting the fact that $Tx_{n_j} \to 0$ weakly because weak limits are unique and any subsequence of a convergent subsequence is also convergent to the same limit. $\square$

**Corollary 13.1.9.** Let $X$ be a separable Hilbert space and $T : X \to Y$ a compact operator. Choose any orthonormal basis $\{e_i\}$ of $X$. Then $Te_i \to 0$ in norm.

*Proof.* By above, it suffices to show that $e_i \to 0$ weakly. For any $x \in X$,

$$||x|| = \sum_{i=1}^{\infty} |\langle e_i, x \rangle|^2 < \infty \implies \lim_{i \to \infty} \langle e_i, x \rangle = 0$$

proving that $e_i \to 0$ weakly. $\square$

## 13.2 Examples and Ascoli's Theorem

**Theorem 13.2.1** (Ascoli)**.** Let $X$ be a compact Hausdorff space. Then a subset $B \subset C(X)$ is precompact if and only if it is (pointwise) equicontinuous and pointwise bounded meaning that,

(a) for all $\epsilon > 0$ there is a neighrborhood $U_x$ of each $x \in X$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in B$ and $y \in U_x$

(b) $\sup_{f \in B} |f(x)| < \infty$ for each $x \in X$.

**Proposition 13.2.2.** Let $X, Y$ be metric spaces with $Y$ compact and $X$ equiped with a Borel measure. Let $K : X \times Y \to \mathbb{R}$ be a function with $K(-, y)$ integrable for each $y \in Y$ and such that $\forall \epsilon > 0$ there exists $g \in L^1(X)$ with $||g||_1 < \epsilon$ and $\delta > 0$ such that $|K(-, y_1) - K(-.y_2)| < |g|$ almost everywhere whenever $|y_1 - y_2| < \delta$. Then the linear operator $T : C_0(X) \to C(Y)$ defined by,

$$(Tf)(y) = \int_X K(x, y) f(x) \, \mathrm{d}\mu_X$$

is compact.

*Proof.* First we need to show that $Tf$ is well-defined. For each $y \in Y$ the function $K(x, y) f(x)$ is integrable because,

$$|(Tf)(y)| = \left| \int_X K(x, y) f(x) \, \mathrm{d}\mu_X \right| \leq \int_X |K(x, y) f(x)| \, \mathrm{d}\mu_X \leq \left( \int_X |K(x, y)| \, \mathrm{d}\mu_X \right) ||f||_\infty$$

and $K(-, y)$ is integrable. Furthermore, this shows that,

$$||Tf||_\infty \leq \sup_{y \in Y} \left( \int_X |K(x, y)| \, \mathrm{d}\mu_X \right) \cdot ||f||_\infty$$

However, I claim that the integral is continuous. Indeed, for any $\epsilon > 0$ there exists $\delta > 0$ and $g \in L^1(X)$ so that whenever $|y_1 - y_2| < \delta$ we have,

$$\left| \int_X |K(x, y_1)| \, \mathrm{d}\mu_X - \int_X |K(x, y_2)| \, \mathrm{d}\mu_X \right| \leq \int_X |K(x, y_1) - K(x, y_2)| \, \mathrm{d}\mu_X \leq \int_X |g| \, \mathrm{d}\mu_X < \epsilon$$

Thus the integral is absolutely continuous and $Y$ is compact so the supremum exists,

$$||T|| \leq \sup \int_X |K(x, y)| \, \mathrm{d}\mu_X$$

Furthermore, if $|y_1 - y_2| < \delta$ then,

$$|(Tf)(y_1) - (Tf)(y_2)| \leq \int_X |K(x, y_1) - K(x, y_2)| \cdot |f(x)| \, \mathrm{d}\mu_X \leq \left( \int_X |g| \, \mathrm{d}\mu_X \right) ||f||_\infty \leq ||f||_\infty \epsilon$$

so we see that $Tf$ is uniformly continuous. Let $B \subset C_0(X)$ be bounded so suppose it is bounded in norm by a constant $M$. We have seen that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|(Tf)(y_1) - (Tf)(y_2)| < \frac{\epsilon}{M} \cdot ||f||_\infty$ whenever $|y_1 - y_2| < \delta$. Therefore, for all $f \in B$,

$$|(Tf)(y_1) - (Tf)(y_2)| < \tfrac{\epsilon}{M} ||f||_\infty \leq \epsilon$$

because $||f||_\infty \leq M$. Thus $T(B)$ is equicontinuous and uniformly bounded because $T$ is bounded and $B$ is bounded so thus pointwise bounded. Indeed,

$$|(Tf)(y)| \leq \left( \int_X |K(x, y)| \, \mathrm{d}\mu_X \right) ||f||_\infty \leq \left( \int_X |K(x, y)| \, \mathrm{d}\mu_X \right) M$$

is a bound over all $f \in B$. Therefore by Ascoli's theorem, $T(B)$ is precompact so $T$ is a compact operator. □

*Remark.* If $X$ is compact and $\mu_X$ is a finite Borel measure then the conditions on $K$ follow immediately from $K$ being continuous. Integrability follows from,

$$\int_X |K(-,y)| \,\mathrm{d}\mu_X \leq \int_X ||K(-,y)||_\infty \,\mathrm{d}\mu_X = \mu(X) \cdot ||K(-,y)||_\infty < \infty$$

Furthermore, because $X$ is compact, $X \times Y$ is compact so $K$ is automatically uniformly continuous. Therefore, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|y_1 - y_2| < \delta$ implies that

$$|K(x, y_1) - K(x, y_1)| < \frac{\epsilon}{2\mu(X)}$$

(here I'm actually using something a bit weaker than uniform continuity on $X \times Y$ I'm using that $K(x, -)$ is uniformly continuous on $Y$ *uniformly* in $x \in X$ meaning the $\delta$ can be choosen independently of $x \in X$). Furthermore, the constant function $\frac{\epsilon}{\mu(X)}$ is integrable and $\int_X \frac{\epsilon}{\mu(X)} \,\mathrm{d}\mu_X < \epsilon$ by definition since $\mu(X)$ is finite.

## 13.3 Spectrum of Compact Operators

**Proposition 13.3.1.** Let $T : X \to X$ be a compact operator on a normed space. If $\dim X = \infty$ then $0 \in \sigma(T)$.

*Proof.* Otherwise, $T$ would have a bounded inverse $T^{-1} : X \to X$ but then $TT^{-1} = I$ would be compact. Furthermore, on a normed space, $I$ is compact iff the unit ball is precompact iff $\dim X < \infty$ giving a contradiction. ☐

**Proposition 13.3.2.** Let $T : X \to X$ be a compact operator on a normed space. For any $\lambda \neq 0$,

$$\dim \ker (T - \lambda I) < \infty$$

*Proof.* This kernel is the $\lambda$-eigenspace, $V_\lambda = \ker (T - \lambda I)$. Notice that $T|_{V_\lambda} = \lambda I_{V_\lambda}$ by definition and therefore $I_{V_\lambda} = \lambda^{-1} T|_{V_\lambda}$ is compact. However, on a normed space, the identity is compact iff the unit ball is precompact iff $\dim V_\lambda < \infty$. ☐

**Lemma 13.3.3.** Let $T : X \to X$ be a compact operator on a Banach space and $\lambda \neq 0$. If $(T - \lambda I)$ is surjective then $\ker (T - \lambda I) = \{0\}$.

*Proof.* Let $T_\lambda = (T - \lambda I)$. Define $V_\lambda^{(i)} = \ker T_\lambda^{i+1}$. Since $\lambda \neq 0$, by the previous lemma $V_\lambda^{(1)} = V_\lambda = \ker T_\lambda$ is finite dimensional. Then clearly $V_\lambda^{(i)} \subset V_\lambda^{(i+1)}$. Furthermore, $T_\lambda : V_\lambda^{(i+1)} \to V_\lambda^{(i)}$ is surjective (because $T_\lambda$ is surjective) and

$$\ker T_\lambda|_{V_\lambda^{(i+1)}} = \ker T_\lambda = V_\lambda^{(1)}$$

because $\ker T_\lambda \subset V_\lambda^{(i+1)}$. Thus by rank-nullty,

$$\dim V_\lambda^{(i+1)} = \dim V_\lambda^{(i)} + \dim V_\lambda^{(1)}$$

Therefore, $\dim V_\lambda^{(i)} = i \cdot \dim V_\lambda$. Assume that $\dim V_\lambda \neq 0$ then this is an increasing sequence of closed finite-dimensional subspaces.

Choose $x_i \in V_\lambda^{(i)}$ such that $||x_i|| = 2$ and $||x_i - u|| \geq 1$ for all $u \in V_\lambda^{(i-1)}$ by taking complements. Then consider,

$$||Tx_{i+1} - Tx_i|| = ||T_\lambda x_{i+1} + \lambda x_{i+1} - T_\lambda x_i - \lambda x_i||$$

However, $T_\lambda x_{i+1} \in V_\lambda^{(i)}$ and $x_i, T_\lambda x_i \in V_\lambda^{(i)}$ so therefore,

$$||Tx_{i+1} - Tx_i|| = |\lambda| \cdot ||x_{i+1} - \lambda^{-1}(T_\lambda x_i + \lambda x_i - T_\lambda x_{i+1})|| \geq |\lambda|$$

Since $|\lambda| > 0$, no subsequence of $\{Tx_i\}$ can converge but $\{x_i\}$ is bounded contradicting the compactness of $T$. Therefore, $V_\lambda = \{0\}$. $\square$

**Lemma 13.3.4.** Let $T : X \to X$ be a compact operator on a Banach space and $\lambda \neq 0$. Then $\operatorname{im}(T - \lambda I)$ is closed.

*Proof.* Since $V_\lambda = \ker(T - \lambda I)$ is finite dimensional it has a closed complement $X = V_\lambda \oplus E_\lambda$. Thus it suffices to show the theorem for $T|_{E_\lambda}$ since $\operatorname{im}(T - \lambda I) = \operatorname{im}(T - \lambda I)|_{E_\lambda}$ because it is zero on $V_\lambda$. Thus we may assume that $V_\lambda = \{0\}$.

Let $T_\lambda = T - \lambda I$. Suppose that $\operatorname{im} T_\lambda$ is not closed. Then $T_\lambda$ is not bounded below so there exists a sequence $x_i$ with $||x_i|| = 1$ and $T_\lambda x_i \to 0$. Therefore, by compactness, there exists a convergent subsequence of $\{Tx_i\}$ say $\{Tx_j\}$ then $Tx_j \to y$. Therefore,

$$(T - \lambda I)y = \lim_{j \to \infty} (T - \lambda I)x_j = 0$$

because a subsequence of a convergent sequence converges to the same limit. Therefore,

$$\lim_{j \to \infty} Tx_j = \lambda \lim_{j \to \infty} x_j$$

and since $Tx_j \to y$ we see that $x_j \to \lambda^{-1} y$. Thus, $||y|| = |\lambda| \cdot \lim_{j \to \infty} ||x_j|| = |\lambda| > 0$ and,

$$Ty = T\left(\lim_{j \to \infty} Tx_j\right) = T \lim_{j \to \infty} \lambda x_j = \lambda \lim_{j \to \infty} Tx_j = \lambda y$$

Since $V_\lambda = \{0\}$ we see that $y = 0$ contradicting $||y|| > 0$. Therefore, $\operatorname{im} T_\lambda$ is closed. $\square$

**Theorem 13.3.5.** Let $T : X \to X$ be a compact operator on an infinite dimensional Banach space. Then the following hold,

(a) $0 \in \sigma(T)$

(b) the eigenspaces $V_\lambda$ for $\lambda \neq 0$ are finite dimensional and $\operatorname{im}(T - \lambda I)$ is closed

(c) $\sigma(T) \setminus \{0\}$ consists only of eigenvalues

(d) $\sigma(T) \setminus \{0\}$ is countable so it consists of a sequence of eigenvalues $\{\lambda_i\}$

(e) for any $r > 0$ there are finitely many $\lambda \in \sigma(T)$ with $|\lambda| \geq r$ so $\lim_{i \to \infty} \lambda_i = 0$.

*Proof.* We have already shown (a) and (b). To show (c) we need to prove that if $\lambda \neq 0$ and $V_\lambda = \{0\}$ then $T_\lambda = (T - \lambda I)$ is invertible meaing that $\lambda \notin \sigma(T)$. Indeed, we know that $\operatorname{im} T_\lambda$ is closed so $\operatorname{im} T_\lambda = (\ker T^*_\lambda)^\perp$. However, $T^*$ is compact so $T^*_\lambda = T^* - \lambda I$ and we may apply our previous results. In particular, $\operatorname{im} T^*_\lambda$ is closed so $\operatorname{im} T^*_\lambda = (\ker T_\lambda)^\perp = X^*$ because $V_\lambda = \ker T_\lambda = \{0\}$. By a lemma above, $\ker T^*_\lambda = \{0\}$ because $T^*_\lambda$ is surjective. Therefore, $\operatorname{im} T_\lambda = X$ and $T_\lambda$ is injective so by the bounded inverse theorem $T_\lambda$ is invertible.

Notice that (e) implies (d) because,

$$\sigma(T) \setminus \{0\} = \bigcup_{n=1}^{\infty} \{\lambda \in \sigma(T) \mid |\lambda| \geq \tfrac{1}{n}\}$$

is a countable increasing union of finite sets and thus is countable. Then we can enumerate $\sigma(T) \setminus \{0\}$ via a sequence $\{\lambda_i\}$ and it follows that $\lambda_i \to 0$ because all but finitely many have $|\lambda_i| < \epsilon$ for any $\epsilon > 0$. Therefore it suffices to prove the main statement of (e).

Suppose there is an infinite sequence $\lambda_i \in \sigma(T) \setminus \{0\}$ with $|\lambda_i| \geq r$. Then we know $V_i = V_{\lambda_i} \neq \{0\}$ by (c). Let $E_n = \operatorname{span}\{V_1, \dots, V_n\}$ is closed since it is finite dimensional. Since eigenspaces are disjoint these eigenspaces are nonempty $E_n \subsetneq E_{n+1}$ so there exists a sequence $x_n \in E_n$ with $||x_n|| = 2$ and $||x_n - u|| \geq 1$ for all $u \in E_{n-1}$. Furthermore,f $n > m$ then,

$$||Tx_n - Tx_m|| = ||\lambda_n x_n - \lambda_m x_m|| \geq |\lambda_n| \geq r$$

because $\lambda_n^{-1} \lambda_m x_m \in E_m \subset E_{n-1}$. Therefore, no subsequence of $\{Tx_n\}$ can possibly be Cauchy contradicting the compactness of $T$. Thus, there can only be finitely many such eigenvalues. □

**Corollary 13.3.6** (Fredholm Alternative)**.** Let $T : X \to X$ be a compact operator on a Banach space and $\lambda \neq 0$. Then either $Tu - \lambda u = v$ has a unique solution for each $v \in X$ or there are finitely many linearly independent solutions to $Tu - \lambda u = 0$ and then $Tu - \lambda u = v$ has a solution if and only if,

$$v \in (\ker T^*)^\perp = \{v \in X \mid \forall \ell \in X^* : \ell \circ T = 0 \implies \ell(v) = 0\}$$

in which case the space of all such solutions is $u + u_0$ where $Tu_0 - \lambda u_0 = 0$ which is a finite dimensional affine space.

*Proof.* Since $\lambda \not 0$ we see that either $\lambda$ is an eigenvalue or $(T - \lambda I)$ is invertible. In the latter case $Tu - \lambda u = v$ has a unique solution for all $v$. In the former case the space $V_\lambda = \ker (T - \lambda I)$ of solutions to $Tu - \lambda u = 0$ is finite dimensional. Furthermore, $\operatorname{im} (T - \lambda I)$ is closed so we know that,

$$\operatorname{im} (T - \lambda I) = (\ker T^*)^\perp = \{v \in X \mid \forall \ell \in X^* : \ell \circ T = 0 \implies \ell(v) = 0\}$$

by the closed range theorem and therefore $Tu - \lambda u = v$ has a solution if and only if $v \in (\ker T^*)^\perp$. Finally, if $u, u' \in X$ are such that $T_\lambda u = T_\lambda u' = v$ then $T_\lambda(u' - u) = 0$ so $u' - u \in \ker T_\lambda = V_\lambda$ and thus $u' = u + u_0$ where $u_0 \in V_\lambda$ and thus is a solutino to $Tu - \lambda u = 0$. Therefore, $T_\lambda^{-1}(v) = u_p + V_\lambda$ for any particular solution $u_p$ with $T_\lambda u_p = v$. □

## 13.4 The Approximation Property

**Definition 13.4.1.** Let $X, Y$ be normed spaces. We say that $\mathcal{L}(X, Y)$ has the approximation property if the finite rank operators are dense in the space of compact operators $\mathcal{K}(X, Y)$ with the norm topology. Because $\mathcal{L}(X, Y)$ is a metric space it is sequential and thus this is equivalent to the property that for any compact operator $T \in \mathcal{K}(X, Y)$ there exists a sequence of finite rank operators $T_n \in \mathcal{L}(X, Y)$ such that $T_n \to T$ in norm.

**Definition 13.4.2.** We say that $X$ has the approximation property if $\mathcal{L}(X)$ does.

*Remark.* There are Banach spaces that do not have the approximation property but they are difficult to write down.

**Proposition 13.4.3.** If $Y$ is a Hilbert space then $\mathcal{L}(X,Y)$ has the approximation property.

*Proof.* Let $T : X \to Y$ be compact. Then for any $\epsilon > 0$ there exists a finite cover of $T(B)$ by $\epsilon$-balls $\{B_\epsilon(y_i)\}$. Let $K = \text{span}\{y_1, \dots, y_n\}$ be the span which is finite and thus closed. Therefore, there is an othogonal projection operator $P_\epsilon : Y \to K$. Consider $T_\epsilon = P_\epsilon \circ T$ which has image inside $K \subset Y$ and thus finite rank. Now, for any $x \in B$ there is some $y_i$ with $||Tx - y_i|| < \epsilon$ and thus,

$$||Tx - T_\epsilon x|| \le ||Tx - y_i|| + ||T_\epsilon x - y_i|| < \epsilon + \epsilon$$

because $T_\epsilon x = P_\epsilon T x$ and $y_i \in K$ so $||P_\epsilon T x - y_i|| \le ||Tx - y_i|| < \epsilon$. Therefore,

$$||T - T_\epsilon|| = \sup_{||x|| \le 1} ||Tx - T_\epsilon x|| \le 2\epsilon$$

meaning that $T_\epsilon \to T$ as $\epsilon \to 0$. If we want an actual sequence, let $T_n = T_{\frac{1}{n}}$ and then,

$$||T - T_n|| \le \tfrac{2}{n}$$

and therefore $T_n \to T$ in norm. □

*Remark.* Notice where the proof fails for $Y$ any Banach space. In a general Banach space, it is true that $K$ is complemented (because it is finite dimensional) so there still exists a continuous projection $P_\epsilon : Y \to K$. And still, because $y_i \in K$,

$$||P_\epsilon T x - y_i|| = ||P_\epsilon T x - P_\epsilon|| = ||P_\epsilon(Tx - y_i)|| \le ||P_\epsilon|| \cdot ||Tx - y_i|| < \epsilon ||P_\epsilon||$$

because $P_\epsilon$ is bounded. However, we cannot conclude that this goes to zero as $\epsilon \to 0$ because the bound $||P_\epsilon||$ may not be well-controlled as $\epsilon \to 0$ unlike the Hilbert space case where we can ensure that $||P_\epsilon|| = 1$ uniformly by taking orthogonal projections. Therefore, we can replace Hilbert space by any Banach space which has the property that finite dimensional subspaces have norm-1 projection operators.

**Proposition 13.4.4.** Suppose that $Y$ has a net of operators $P_\alpha \in \mathcal{L}(Y,Y)$ such that $P_\alpha \to I$ strongly and $\{P_\alpha\} \subset \mathcal{L}(Y,Y)$ is pointwise bounded.

# 14 Spectral Theory of Self-Adjoint Operators (WORK IN PROGRESSS!!!)

**Proposition 14.0.1.** Let $T : H \to H$ be self-adjoint and bounded. Then $\sigma(T) \subset \mathbb{R}$.

*Proof.* Let $\lambda$ not be real and consider the imaginary part,

$$\begin{aligned}\mathfrak{Im}\left(\langle (T - \lambda I)x, x\rangle\right) &= \tfrac{1}{2}\left(\langle (T - \lambda I)x, x\rangle - \overline{\langle (T - \lambda I)x, x\rangle}\right) \\ &= \tfrac{1}{2}\left(\langle (T - \lambda I)x, x\rangle - \langle x, (T - \lambda I)x\rangle\right) \\ &= \tfrac{1}{2}\left(\langle Tx, x\rangle - \bar{\lambda}\langle x, x\rangle - \langle x, Tx\rangle + \lambda \langle x, x\rangle\right) = \mathfrak{Im}\left(\lambda\right)||x||^2\end{aligned}$$

because $\langle Tx, x \rangle = \langle x, Tx \rangle$ since $T = T^*$. Therefore,

$$\mathfrak{Im}\,(\lambda)\,||x||^2 = \mathfrak{Im}\,(\langle (T - \lambda I)x, x \rangle) \leq |\,\langle (T - \lambda I)x, x \rangle \leq ||(T - \lambda I)x|| \cdot ||x||$$

Therefore,

$$||x|| \leq \mathfrak{Im}\,(\lambda)^{-1}\,||(T - \lambda I)x||$$

proving that $T - \lambda I$ is injective and has closed image. Furthermore, $H = \ker T^* \oplus \overline{\mathrm{im}\,(T - \lambda I)}$ but $\ker T^* = \ker T = 0$ because $T = T^*$ and thus $T - \lambda I$ has dense image and thus is surjective because we say that its image is closed. Thus by the bounded inverse theorem $T - \lambda I$ is invertible so $\lambda \in \rho(T)$ proving the claim. $\square$

**Corollary 14.0.2.** Let $T : H \to H$ be a compact self-adjoint operator then there is a countable list of eigenvalues $\lambda_i \in \mathbb{R}$ such that $\lambda_i \to 0$ and $\sigma(T) = \{\lambda_i\} \cup \{0\}$.

**Theorem 14.0.3.** Let $H$ be a separable Hilbert space and $T : H \to H$ a compact self-adjoint operator. Then there exists an orthonormal basis of $H$ consisting of eigenvectors of $T$. Explicitly there is an isomorphism,

$$H \cong V_0 \oplus \overline{\bigoplus_{\lambda \in \sigma(T) \setminus \{0\}} V_\lambda}$$

where $V_0$ admits an othonormal basis (by separability) and each $V_\lambda$ admits an orthonormal basis because $\dim V_\lambda < \infty$.

# 15 Some Facts about $L^p$ Space

**Proposition 15.0.1** (Minkowski)**.** $L^p(\Omega)$ is a Banach space. In particular,

$$||f + g||_p \leq ||f||_p + ||g||_p$$

**Proposition 15.0.2.** If $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Then $L^p(\Omega) \subset L^1(\Omega)$ for all $p \geq 1$ and the inclusion is bounded.

*Proof.* Let $f \in L^p(\Omega)$ we need to show that $||f||_1 < \infty$. Apply Hölder's inequality to $g = \frac{1}{\mu(\Omega)}$ then,

$$||fg||_1 \leq ||f||_p \cdot ||g||_q = ||f||_p$$

However, $||fg||_1 = \frac{1}{\mu(\Omega)}||f||_1$ and thus $||f||_1 \leq ||f||_p \cdot \mu(\Omega)$ so the inclusion is bounded by $\mu(\Omega)$. $\square$

**Corollary 15.0.3.** If $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Then $L^p(\Omega) \subset L^q(\Omega)$ for all $p \geq q$ and the inclusion is bounded.

*Proof.* For $f \in L^p(\Omega)$ then $g = |f|^{\frac{p}{s}} \in L^s(\Omega)$ for any $s \in [1, p]$ and thus by above,

$$||g||_1 = ||f||_{\frac{p}{s}}^{\frac{p}{s}} \leq ||g||_s \cdot \mu(\Omega) = ||f||_{\frac{p}{s}}^{p} \cdot \mu(\Omega)$$

Therefore,

$$||f||_{\frac{p}{s}} \leq ||f||_p \cdot \mu(\Omega)^{\frac{q}{s}}$$

Because $q \in [1, p]$ we see that we can set $q = \frac{p}{s}$ for some $s \in [1, p]$ and therefore,

$$||f||_q \leq ||f||_p \cdot \mu(\Omega)^{\frac{q^2}{p}}$$

showing that $f \in L^1(\Omega)$ and that the inclusion is bounded by $\mu(\Omega)^{\frac{q^2}{p}}$. $\square$

**Proposition 15.0.4** (Extended Hölder)**.** If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $f \in L^a(\Omega)$ and $g \in L^b(\Omega)$ then,

$$||fg||_c \leq ||f||_a \cdot ||h||_b$$

*Proof.* Consider $\tilde{f} = |f|^c$ and $\tilde{g} = |g|^c$ in the usual Hölder inequality with $\frac{c}{a} + \frac{c}{b} = 1$. Then,

$$||\tilde{f}\tilde{g}||_1 \leq ||\tilde{f}||_{\frac{a}{c}} \cdot ||\tilde{g}||_{\frac{b}{c}}$$

Therefore,

$$||fg||_c^c \leq ||f||_a^c \cdot ||g||_b^c$$

and thus,

$$||fg||_c \leq ||f||_a \cdot ||g||_b$$

□

*Remark.* We can get a better bound for $L^p(\Omega) \subset L^q(\Omega)$ as follows. Let $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Then consider, extended Hölder applied to $f$ and $g = 1$,

$$||f||_q = ||fg||_q \leq ||f||_p \cdot ||g||_r = ||f||_p \cdot \mu(\Omega)^{\frac{1}{r}}$$

Therefore,

$$||f||_q \leq ||f||_p \cdot \mu(\Omega)^{\frac{1}{r}}$$

so the inclusion is bounded by $\mu(\Omega)^{\frac{1}{q} - \frac{1}{p}}$. In particular, take $p = \infty$ then,

$$||f||_q \leq ||f||_\infty \cdot \mu(\Omega)^{\frac{1}{q}}$$

# 16 Fourier Series

**Proposition 16.0.1.** Let $\phi \in C(S^1)$ have a continuous extension $f \in C(\overline{\Omega_\delta})$ such that $f|_{\Omega_\delta}$ is holomorphic and $f|_{S^1} = \phi$ where,

$$\Omega_\delta = \{z \in \mathbb{C} \mid 1 - \delta < |z| < 1\}$$

Then the Fourier coefficients,

$$(\mathcal{F}\phi)_n = \int_0^1 e^{-2\pi i n\theta} \phi(\theta) \, \mathrm{d}\theta$$

decay exponentially for $n < 0$ meaning $|(\mathcal{F}\phi)_n| \leq C r^{|n|}$ for some $0 < r < 1$.

*Proof.* Consider the loop $\gamma_{s,n}$ defined by $\gamma_{s,n}(t) = s e^{2\pi i n t}$. Then, the integals,

$$a_{n,s} = \frac{1}{2\pi i} \oint_{\gamma_s} z^{-(n+1)} f(z) \, \mathrm{d}z = \int_0^1 s^{-n} e^{-2\pi i n\theta} f(\gamma_s(\theta)) \, \mathrm{d}\theta$$

for $s \in (1 - \delta, 1)$ do not depend on $s$. Furthermore, because $f$ is continuous on $\overline{\Omega_\delta}$ I claim that,

$$(\mathcal{F}\phi)_n = \lim_{s \to 1} a_{n,s}$$

Indeed, consider,

$$|(\mathcal{F}\phi)_n - a_{n,s}| \leq \int_0^1 |s^{-n} f(\gamma_s(\theta)) - \phi(\theta)| \, \mathrm{d}t \leq \int_0^1 \left[|1 - s^{-n}| \cdot |f(\gamma_s(\theta)| + |f(\gamma_s(\theta)) - \phi(\theta)|\right] \, \mathrm{d}t$$

However, because $\overline{\Omega_\delta}$ is compact we know that $f$ is uniformly continuous so for each $\epsilon > 0$ there is a $\delta' > 0$ such that when $|z - z'| < \delta'$ that $|f(z) - f(z')| < \frac{\epsilon}{2}$. In particular,

$$|f(\gamma_s(\theta)) - \phi(\theta)| = |f(se^{2\pi it}) - f(e^{2\pi it})| < \tfrac{\epsilon}{2}$$

when $|se^{2\pi it} - e^{2\pi it}| = (1 - s) < \delta'$. Therefore, choose $\delta''$ such that $(1 - \delta'')^{-n} - 1 < \frac{\epsilon}{2M}$ where $M = \sup\limits_{z \in \overline{\Omega_\delta}} |f(z)|$ which exists because $f$ is continuous on a compact set. Thus if $|1-s| < \min(\delta', \delta'')$,

$$|(\mathcal{F}\phi)_n - a_{n,s}| \leq \int_0^1 \left[|1 - s^{-n}||f(\gamma_s(\theta)| + |f(\gamma_s(\theta)) - \phi(\theta)|\right] \mathrm{d}t < \tfrac{\epsilon}{2M} \cdot M + \tfrac{\epsilon}{2} = \epsilon$$

meaning that,

$$\lim_{s \to 1} a_{n,s} = (\mathcal{F}\phi)_n$$

Then, by the integral theorem, $a_{n,s}$ is actually constant in $s$. Therefore,

$$(\mathcal{F}\phi)_n = a_{n,\delta'} = s^{-n} \int_0^1 e^{-2\pi in\theta} f(\gamma_{\delta'}(\theta)) \, \mathrm{d}\theta \leq s^{-n} M$$

Therefore, for $n < 0$ we have,

$$|(\mathcal{F}\phi)_n| \leq M s^{|n|}$$

□

*Remark.* If we replace holomorphic by anti-holomorphic or swap the annulus to the outside of $S^1$ then we get exponential decay of the Fourier coefficients for $n > 0$. Thus, we get the following corollary.

**Corollary 16.0.2.** Let $\phi \in C(S^1)$ have a holomorphic extension on some anulus *containing* $S^1$,

$$\Omega_\delta = \{z \in \mathbb{C} \mid 1 - \delta < |z| < 1 + \delta\}$$

Then the Fourier coefficients,

$$(\mathcal{F}\phi)_n = \int_0^1 e^{-2\pi in\theta} \phi(\theta) \, \mathrm{d}\theta$$

decay exponentially meaning $|(\mathcal{F}\phi)_n| \leq C r^{|n|}$ for some $0 < r < 1$.

**Proposition 16.0.3.** Let $\phi \in C^k(S^1)$. Then, there exists a constant $C$ such that for all $n$,

$$|(\mathcal{F}\phi)_n| \leq C(1 + |n|)^{-k}$$

*Proof.* The case of $n = 0$ is trivial so let $n \neq 0$. Applying integration by parts,

$$(\mathcal{F}\phi)_n = \int_0^1 e^{-2\pi int} \phi(t) \, \mathrm{d}t = (2\pi in)^{-k} \int_0^1 e^{-2\pi it} \phi^{(k)}(t) \, \mathrm{d}t$$

Therefore,

$$|(\mathcal{F}\phi)_n| \leq \frac{1}{|n|^k} \cdot \left| \frac{1}{2\pi} \int_0^1 e^{-2\pi it} \phi^{(k)}(d) \, \mathrm{d}t \right| \leq \frac{1}{|n|^k} \left( \frac{1}{2\pi} \int_0^1 |\phi^{(k)}(t)| \, \mathrm{d}t \right) \leq \frac{1}{(1 + |n|)^k} \left( \frac{2^k}{2\pi} ||\phi^{(k)}||_1 \right)$$

□

**Example 16.0.4.** Lacunary series give a good test case. For example,

$$f(z) = \sum_{n=1}^{\infty} n^{-2}\, z^{n^2}$$

Because,

$$|f(z)| \le \sum_{n=1}^{\infty} n^{-2} |z|^{n^2}$$

converges when $|z| \le 1$ we see that $f$ is holomorphic on the open disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and continuous on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Continuity on the circle follows from the $M$-test which shows that,

$$f(e^{2\pi i t}) = \sum_{n=1}^{\infty} n^{-2} e^{i n^2 t}$$

converges uniformly and absolutely since the series of supremums,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

converges. Therefore, $f(t)$ is a uniform limit of continuous functions and thus is itself continuous. However, we can see that $f$ cannot extend holomorphically to any annulus containing $S^1$ for the following reason. Let $\phi(t) = f(e^{2\pi i t})$ which is continuous. Then it is immediate from the definition and Fourier inversion that,

$$a_n = (\mathcal{F}\phi)_n = \begin{cases} n^{-1} & n > 0 \text{ and is a square} \\ 0 & \text{else} \end{cases}$$

Therefore, although $a_n \in \ell^1$ we see that $a_n$ does not decay exponentially in fact does not decay faster than $n^{-1}$ meaning that $\phi$ cannot be $C^k(S^1)$ for $k > 1$. In fact, it is clear that $\psi \notin C^1(S^1)$ therefore there cannot be a holomorphic extension containing $S^1$ else $\phi$ would be smooth. However, of course we see that the $n < 0$ coefficients are all zero and thus do (trivially) decay exponentially.

**Proposition 16.0.5.** A function $\phi : S^1 \to \mathbb{C}$ is real analytic if and only if there exists some annulus $\Omega_\delta = \{z \in \mathbb{C} \mid 1 - \delta < |z| 1 + \delta\}$ and a holomorphic function $f$ on $\Omega_\delta$ such that $f|_{S^1} = \phi$.

*Proof.* If there is such a holomorphic extension $f$ then obviously $\phi$ is real analytic. Conversely, if $\phi$ is real analytic then for each point $z \in S^1$ there exists an open $U_z \subset \mathbb{C}$ and a holomorphic $f_{U_z}$ extension of $\phi$ on $U_z$. Explicitly, we can use the disk $B_{r_z}(z)$ where $r_z$ is the radius of convergence (in $\mathbb{C}$ not along the arc) of the Taylor series at $z$. Therefore, it suffices to check that if $U_z \cap U_{z'} \neq \varnothing$ then $f_{U_z}|_{U_z \cap U_{z'}} = f_{U_{z'}}|_{U_z \cap U_{z'}}$. This holds because both are holomorphic and their difference vanishes on $S^1 \cap U_z \cap U_{z'}$ which has limit points so their difference is identically zero on the connected set $U_z \cap U_{z'}$ (by shrinking the cover we can ensure that the overlaps are connected). Since $S^1$ is compact, there is a finite subcover by sets $U_z$ so there is a holomorphic function glued on,

$$U = \bigcup_{i=1}^{n} U_i$$

and then $S^1$ can be uniformly thickened while still inside $U$ giving the required annulus. ☐

*Remark.* A holomorphic function has no maximal definition set in general. We would like to

# 17 Operator Topologies

**Proposition 17.0.1.** Let $X$ and $Y$ be normed spaces and $T_n \in \mathcal{L}(X, Y)$ a sequence of bounded operators with $||T_n|| \le M$ and $T \in \mathcal{L}(X, Y)$ a bounded operator. Suppose that there is a dense set $D \subset X$ such that $\forall x \in D : T_n x \to Tx$ converges then $T_n \to T$ converges in the strong topology.

*Proof.* We need to show that for any $x \in X : T_n x \to Tx$ converges. Choose a sequence $x_k \in D$ such that $x_k \to x$. Then consider,

$$||T_n x - T_n x_k|| \le ||T_n|| \cdot ||x - x_k|| \le M ||x - x_k||$$

Therefore,

$$||T_n x - Tx|| \le ||T_n x - T_n x_k|| + ||T_n x_k - Tx_k|| + ||Tx_k - Tx|| \le M||x - x_k|| + ||T_n x_k - Tx_k|| + ||T|| \cdot ||x_k - x||$$

However, we know that, for each $k$,

$$\lim_{n \to \infty} ||T_n x_k - Tx_k|| = 0$$

by assumption and thus,

$$\limsup_{n \to \infty} ||T_n x - Tx|| \le (M + ||T||) ||x - x_k||$$

Since $x_k \to x$ taking $k \to \infty$ we see that,

$$\limsup_{n \to \infty} ||T_n x - Tx|| = 0$$

and therefore $T_n x \to Tx$ for each $x \in X$. $\square$

*Remark.* Notice that in the proof we do not get a uniform bound on $||T_n - T||$ tending to zero and therefore $T_n$ may not converge to $T$ in the norm topology only in the strong operator (pointwise convergence) topology.

*Remark.* The crux of the proof is,

$$\lim_{n \to \infty} T_n x = \lim_{n \to \infty} \lim_{k \to \infty} T_n x_k = \lim_{k \to \infty} \lim_{n \to \infty} T_n x_k = \lim_{k \to \infty} T x_k = Tx$$

and justifying the swapping of the limits. This is justified because,

$$\lim_{k \to \infty} T_n x_k = T_n x$$

uniformly in $n$ because $||T_n x - T_n x_k|| \le M ||x - x_k||$. See the following proposition.

**Proposition 17.0.2.** Let $X$ be a metric space and $a_{n,m} \in X$ a double sequence. Suppose that,

$$\lim_{m \to \infty} a_{n,m} = a_n$$

converges uniformly in $n$ and that for all $m$,

$$\lim_{n \to \infty} a_{n,m} = a_m$$

exists and that,

$$\lim_{m \to \infty} a_m = a$$

exists. Then,

$$a = \lim_{m \to \infty} a_m = \lim_{m \to \infty} \lim_{n \to \infty} a_{n,m} = \lim_{n \to \infty} \lim_{n \to \infty} a_{n,m} = \lim_{n \to \infty} a_n$$

*Proof.* For any $\epsilon > 0$ there is some $M$ such that for any $m > M$ and all $n$,

$$d(a_n, a_{n,m}) < \epsilon$$

Now,

$$d(a_n, a) \leq d(a_n, a_{n,m}) + d(a_{n,m}, a_m) + d(a_m, a) < \epsilon + d(a_{n,m}, a_m) + d(a_m, a)$$

Then,

$$\limsup_{n\to\infty} d(a_n, a) < \limsup_{n\to\infty} d(a_{n,m}, a_m) + \epsilon + d(a_m, a) = \epsilon + d(a_m, a)$$

However, $a_m \to a$ and therefore,

$$\limsup_{n\to\infty} d(a_n, a) \leq \epsilon + \limsup_{m\to\infty} d(a_m, a) = \epsilon$$

Since $\epsilon$ is arbitrary we see that,

$$\limsup_{n\to\infty} d(a_n, a) = 0$$

and therefore,

$$\lim_{n\to\infty} a_n = a = \lim_{m\to\infty} a_m$$

proving the proposition. $\square$

# 18 Proving Sets Are Meager

**Definition 18.0.1.** A subset $B \subset V$ of a $K$-vector space is absorbent if for all $v \in V$ there exists a $r > 0$ such that if $|t| > r$ then $v \in tB$.

**Lemma 18.0.2.** Let $X$ be an infinite dimensional normed space and $V \subset X$ a linear subspace and $B_V \subset V$ an absorbent subset. If $B_V$ is precompact then $V$ is meager in $X$.

*Proof.* We see that,

$$V = \bigcup_{n=1}^{\infty} nB_V$$

because for each $v \in V$ there is some $r$ such that for $|t| > r$ we have $v \in tB_V$ and thus for $n > r$ we have $v \in nB_V$. Therefore it suffices to prove that $nB_V$ is nowhere dense for each $n$. Because multiplication by $n$ is a homeomorphism it suffices to show that $B_V$ is nowhere dense. Indeed, $\overline{B_V}$ is compact. So if $x \in \overline{B_V}^{\circ}$ then there is a ball $B_\epsilon(x) \subset \overline{B_V}^{\circ}$ and thus $\overline{B_\epsilon(x)} \subset \overline{B_V}$ so $\overline{B_\epsilon(x)}$ is a closed subset of a compact set and thus compact. However, since $X$ is infinite dimensional, by Riesz lemma, the closed unit ball, which is homeomorphic to $\overline{B_\epsilon(x)}$, is not compact. Thus, $\overline{B_V}^{\circ} = \varnothing$ and therefore $B_V$ is nowhere dense proving that $V$ is of first category. $\square$