Mathematics GU4044 Representations of Finite Groups
Assignment # 5

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Problem 1.

We know that the character of the dual representation satisfies $\chi_{V*} = \overline{\chi_V}$. Therefore,

$$\langle \chi_{V*}, \chi_{V*} \rangle = \langle \overline{\chi_V}, \overline{\chi_V} \rangle = \langle \chi_V, \chi_V \rangle$$

However, a representation is irreducible if and only if $\langle \chi_W, \chi_W \rangle = 1$. Therefore, V^* is irreducible if and only if V is irreducible.

Problem 2.

Suppose G acts on X doubly transitively with $\#(X) = n$. Now consider the action of G on P, the set of ordered distinct pairs of elements in X. By definition, this action must be transitive and thus there is one orbit of size $\#(P) = n^2 - n$. By orbit-stabilizer, $\#(G) = \#(\text{Orb}(x))\#(\text{Stab}(x))$ and thus $\#(P) \mid \#(G)$. Therefore, $n^2 - n \mid \#(G)$. However, the order of G must be positive so $\#(G) \geq n^2 - n$.

Problem 3.

Let G be a nonabelian group of order 6. We know that the number of conjugacy classes is equal to the number of irreducible representation of G. Furthermore, $\sum\limits_{i=1}^{h} d_i^2 = \#(G) = 6$ where d_i is the dimension of the i^{th} irreducible representation. Since G is nonabelian, we cannot have $d_i = 1$ for all i. Therefore, at least one $d_i > 1$. However, $3^2 > 6$ so there must be exactly one 2-dimensional representation. Thus, up to order,$d_1 = 2$ which forces $d_2 = 1$ and $d_3 = 1$ so that $d_1^2 + d_2^2 + d_3^2 = 6$. Thus, there are three irreducible representations and thus three conjugacy classes.

Let G be a nonabelian group of order 8. We know that the number of conjugacy classes is equal to the number of irreducible representation of G. Furthermore, $\sum\limits_{i=1}^{h} d_i^2 = \#(G) = 8$ where d_i is the dimension of the i^{th} irreducible representation. Since G is nonabelian, we cannot have $d_i = 1$ for all i. Therefore, at least one $d_i > 1$. However, $3^2 > 8$ so there must be exactly one 2-dimensional representation. Thus, up to order,$d_1 = 2$. However, the trivial representation is always irredcible so take $d_2 = 1$. Since $8 - d_1^2 - d_2^2 = 3$ the rest of the sum is forced to be $d_3 = 1$, $d_4 = 1$ and $d_5 = 1$ so that $d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8$. Thus, there are five irreducible representations and thus three conjugacy classes.

Lemmas

Lemma 0.1.

Proof. ∎