# 1 Newton-Okounkov Bodies

Let $X/k$ be a variety, with $k$ arbitrary (is not equipped with a valuation). We consider valuations $\nu : K(X)^\times \to \mathbb{R}^d$ with $\mathbb{R}^d$ taking the lexicographic order where $d = \dim X$. There is a valuation with full rank for $d$ the dimension meaning its image is a lattice.

We use this valuation to measure the size of divisors.

**Definition 1.0.1.** The *Newton-Okounkov Body* of a divisor $D$ wrt $\nu$ is the set,

$$\Delta_\nu(D) = \overline{\bigcup_{n \geq 1} \{ \tfrac{1}{n}\nu(f) \mid f \in H^0(X, nD) \}} \subset \mathbb{R}^d$$

**Theorem 1.0.2** (Lazarsfeld-Mustata)**.** $\mathrm{Vol}(\Delta_\nu(D)) = \frac{1}{d!}\mathrm{Vol}(D) \cdot \mathrm{Vol}(\mathrm{Im}\,(\nu))$ where the volume of a divisor $D$ is $\mathrm{Vol}(D) = d! \deg D^d$ is given by intersection theory as is equal (by Riemann-Roch) to,

$$\mathrm{Vol}(D) = \lim_{n \to \infty} \frac{h^0(X, nD)}{n^d}$$

*Remark.* The usual setting is where we take $\nu$ to be a flag valuation. Consider a flag of subvarities,

$$* = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_{d-1} \subsetneq Y_d = X$$

where the valuation is defined by the order of vanishing of a function proceeding down the flag.