1 Lifting and Extensions

1.1 Smoothness

Definition 1.1.1. We say that a map $T \to T'$ is an *order n infinitesimal thickening (or extension)* if it is a closed immersion whose defining ideal $\mathscr{I}$ satisfies $\mathscr{I}^{n+1} = 0$.

Remark. Notice that a zeroth order infinitesimal thickening is an isomorphism. Furthermore, in the affine case, this corresponds to $A = A'/I$ for an ideal $I \subset A'$ with $I^{n+1} = 0$.

Definition 1.1.2. Let $f : X \to Y$ be a morphisms of schemes. If for any diagram,

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $T \to T'$ a first-order infinitesimal thickening of *affine schemes* we say that f is

(a) *formally smooth* if there exists at least one dashed arrow

(b) *formally unramified* if there exists at most one dashed arrow

(c) *formally étale* if there exists exactly one dashed arrow.

Furthermore, we say that f is smooth (resp. unramified, resp. étale) if f is formally smooth (resp. unramified, resp. étale) and locally of finite presentation.

Remark. Notice that any order-n infinitesimal thickening $T \to T'$ may be factored as,

$$T = T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n = T'$$

where T_i is the closed subscheme of T' cut out by I^{i+1}. Therefore, $T_i \to T_{i+1}$ is a closed immersion cut out by I^{i+1}/I^{i+2} which has zero square and thus is a first-order infinitesimal thickening. Therefore, by repeatedly applying the lifting criteria, we may replace "first-order" in the definition by n^{th}-order.

Remark. The definition given above appears in the Stacks project. The definition in our text refers to diagrams of (possibly not affine) infinitesimal thickenings,

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

and asks only about liftings $T' \to X$ *Zariski-locally* on T'. Therefore, it is clear that Stacks project formal smoothness implies [I] formal smoothness. In fact, both definitions for all three properties agree. Indeed, if f is formally étale then the uniqueness of the lift on affines implies gluing so there exists a unique *global* map $T' \to X$ for any infinitesimal thickening $T \to T'$. This contrasts the smooth case for which we will construct a global obstruction to the existence of a global lift $T' \to X$.

1.2 Extensions

Definition 1.2.1. Let $f : X \to S$ be an S-scheme and $\mathscr{I}$ a quasi-coherent $\mathcal{O}_X$-module. A *S-extension of* X by $\mathscr{I}$ is a S-morphism $\iota : X \to X'$ which is a first-order infinitesimal thickening by an ideal isomorphic to $\mathscr{I}$ via the data of an $\mathcal{O}_{X'}$-module map $\varphi : \iota_* \mathscr{I} \to \mathcal{O}_{X'}$.

Remark. If $\mathcal{J} \subset \mathcal{O}_{X'}$ is the sheaf of ideals corresponding to the thickening $\iota : X \to X'$ then $\mathcal{J}^2 = 0$ so $\mathcal{J}$ is naturally a $\mathcal{O}_X = \mathcal{O}_{X'}/\mathcal{J}$-module.

Remark. The situation to have in mind is a R-algebra A and an A-module I. Then a first-order R-extension of A by I is a map of R-algebras $A' \twoheadrightarrow A$ such that,

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

is exact such that the image of $I \to A'$ is an ideal with $I^2 = 0$ in A'.

Remark. We now need a notion of when two extensions are equivalent or more generally the concept of a morphism between them.

Definition 1.2.2. A morphism between two S-extensions of X by $\mathscr{I}$, namely $\iota_1 : X \to X_1'$ and $\iota_2 : X \to X_2'$ is an X-morphism $g : X_1' \to X_2'$ meaning that

$$\begin{tikzcd} X_1' \arrow[rr, "g"] & & X_2' \\ & X \arrow[ul, "\iota_1"] \arrow[ur, "\iota_2"'] & \end{tikzcd}$$

commutes and such that,

$$\begin{tikzcd} \iota_1^{-1}\mathcal{O}_{X_1'} & & \iota_2^{-1}\mathcal{O}_{X_2'} \arrow[ll, "g^\#"'] \\ & \mathscr{I} \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

commutes as a diagram of $f^{-1}\mathcal{O}_S$-modules (notice that ι is a homeomorphism so we may apply ι_* and ι^{-1} freely as inverses to get the sheaves on the correct spaces).

Remark. In the affine case, this corresponds exactly to an R-algebra map $g : A_1' \to A_2'$ giving a morphism of exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & A_1' \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & A_2' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Notice, by the 5-lemma, g is an isomorphism so a morphism of lifts is always an isomorphism.

Definition 1.2.3. We say that an extension $\iota : X \to X'$ is *split* if there exists a section $s : X' \to X$ such that $s \circ \iota = \mathrm{id}_X$. In this case, the exact sequence of $\iota^{-1}\mathcal{O}_{X'}$-modules,

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \arrow[r] & \iota^{-1}\mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend right, "s^\#"'] & 0 \end{tikzcd}$$

is split meaning that,

$$\iota^{-1}\mathcal{O}_{X'} \cong \mathcal{O}_X \oplus \mathscr{I}$$

with the unique $\mathcal{O}_X$-algebra structure such that $\mathscr{I}^2 = 0$. Therefore, there is a unique split extension up to isomorphism.

Remark. The map $\iota^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X$ is surjective because ι is a closed immersion and a homeomorphism so ι^{-1} and ι_* are inverse functors.

Remark. The split extension in the affine case is given by $A' = A \oplus I$ with the unique A-algebra structure such that $I^2 = 0$.

Definition 1.2.4. We denote the set of isomorphism classes of S-extensions of X by $\mathscr{I}$ as,

$$\mathrm{Ext}_S\left(X, \mathscr{I}\right)$$

This is a group under "Bayer sum" with the split extension as the identity as we shall soon see.

Example 1.2.5. Let $X \to X_1 \to X \times_S X$ be the first infinitesimal neighborhood of the diagonal i.e. if $\Delta_{X/S} : X \to X \times_S X$ is cut out by $\mathscr{I}$ then X_1 is cut out by $\mathscr{I}^2$. Then $\Delta_1 : X \to X_1$ is a first-order infinitesimal thickening with ideal $\mathscr{I}/\mathscr{I}^2 \cong \Omega^1_{X/S}$. Then the two projections $p_1, p_2 : X_1 \to X$ split the extension giving two splittings of,

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{X/S} \arrow[r] & \mathcal{P}_{X/S} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend right, "j_1"'] \arrow[l, bend left, "j_2"] & 0 \end{tikzcd}$$

where $\mathcal{P}_{X/S} = \iota^{-1}\mathcal{O}_{X_1}$ is the sheaf of first principal parts. The two splittings correspond to two $\mathcal{O}_X$-module structures on $\mathcal{P}$ and $\mathrm{d}_{X/S} = j_2 - j_1$ is the universal derivation $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$.

Proposition 1.2.6. Let $\iota : X \to X'$ be an S-extension by $\mathscr{I}$. Then the automorphisms of X' as a lift are naturally isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}\left(\Omega_{X/S}, \mathscr{I}\right)$.

Proof. Since ι is a homeomorphism any S-automorphism $g : X' \to X'$ over X must topologically be the identity. Therefore, we just need to classify sheaf maps $g^\# : \varphi : \iota^{-1}\mathcal{O}_{X'} \to \iota^{-1}\mathcal{O}_{X'}$ over $\varphi : \mathscr{I} \to \iota^{-1}\mathcal{O}_{X'}$. Thus we consider diagrams,

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \arrow[r] \arrow[d, equal] & \iota^{-1}\mathcal{O}_{X'} \arrow[r] \arrow[d, "g^\#"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{I} \arrow[r] & \iota^{-1}\mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

Then $\theta = g^\# - \mathrm{id}$ is a map $\iota^{-1}\mathcal{O}_{X'} \to \mathscr{I}$ but $\mathscr{I}^2 = 0$ so θ factors through an S-linear derivation $\mathcal{O}_X \to \mathscr{I}$ giving an element $\theta \in \mathrm{Der}_{\mathcal{O}_S}\left(\mathcal{O}_X, \mathscr{I}\right) = \mathrm{Hom}_{\mathcal{O}_X}\left(\Omega_{X/S}, \mathscr{I}\right)$.

Conversely, any S-derivation $\theta : \mathcal{O}_X \to \mathscr{I}$ produces an automorphism $\mathrm{id} + \tilde{\theta} : \iota^{-1}\mathcal{O}_{X'} \to \iota^{-1}\mathcal{O}_{X'}$ where $\tilde{\theta}$ is the composite map $\iota^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X \xrightarrow{\theta} \mathscr{I}$. ∎

Remark. For any extension $(\iota : X \to X', \varphi : \mathscr{I} \to \iota^{-1}\mathcal{O}_{X'})$ the map $\iota : X \to X'$ is a closed immersion. Therefore, there is an exact sequence of $\mathcal{O}_X$-modules,

$$\mathscr{I} \xrightarrow{\varphi} \iota^*\Omega_{X'/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

coming from the second fundamental sequence and the map $\varphi : \mathscr{I} \to \iota^{-1}\mathcal{O}_{X'}$ identifying $\mathscr{I}$ with the defining ideal such that $\mathscr{I}^2 = 0$. If $f : X \to S$ is smooth then the sequence is short exact.

Proposition 1.2.7. If $f : X \to S$ is smooth then the map,

$$\mathrm{Ext}_S\left(X, \mathscr{I}\right) \to \mathrm{Ext}^1_{\mathcal{O}_X}\left(\Omega_{X/S}, \mathscr{I}\right)$$

defined by sending an extension $(\iota : X \to X', \varphi : \mathscr{I} \to \iota^{-1}\mathcal{O}_{X'})$ to the extension class of,

$$0 \longrightarrow \mathscr{I} \xrightarrow{\varphi} \iota^*\Omega_{X'/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

is a bijection sending the split extension to the trivial extension.

1.3 Lifting

Proposition 1.3.1. Let $f : X \to Y$ be smooth and $\iota : T \to T'$ an extension of T by $\mathscr{I}$. Then given a diagram,

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ T' \arrow[r] \arrow[ur, dashed, "g'"] & Y \end{tikzcd}$$

there exists an obstruction class,

$$c(g_0) \in \mathrm{Ext}^1_{\mathcal{O}_T}\left(g^*\Omega_{X/Y}, \mathscr{I}\right)$$

to the existence of a Y-morphism $g : T \to X$ extending g. Furthermore, if $c(g) = 0$ then the set of extensions g' of g is a $\mathrm{Hom}_{\mathcal{O}_T}\left(g^*\Omega_{X/Y}, \mathscr{I}\right)$-torsor.

Proof. Consider the sheaf of sets $\mathscr{F}$ on T of local lifts,

$$U \mapsto \{g' \in \mathrm{Hom}_Y\left(U', X\right) \mid g \circ \iota = g_0\}$$

Since ι is a homeomorphism, opens of T and T' agree but they have different scheme structures. Let $\mathscr{G} = \mathscr{H}\!om_{\mathcal{O}_T}\left(g^*\Omega_{X/Y}, \mathscr{I}\right)$. Given $s \in \mathscr{G}(U)$ and $g' \in \mathscr{F}(U)$ we can form $g' + s \in \mathscr{F}(U)$ as follows. Since ι is a homeomorphism, maps g' are determined by sheaf maps $g^{-1}\mathcal{O}_X \to \iota^{-1}\mathcal{O}_{T'}$. Thus consider,

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \arrow[r] & \iota^{-1}\mathcal{O}_{T'} \arrow[r] & \mathcal{O}_T \arrow[r] & 0 \\ & & & g^{-1}\mathcal{O}_X \arrow[u] \arrow[ul, dashed] & \end{tikzcd}$$

For two dashed maps g_1, g_2 the difference $D = g_2 - g_1 : g^{-1}\mathcal{O}_X \to \mathscr{I}$ is a $g^{-1}f^{-1}\mathcal{O}_Y$-derivation because $\mathscr{I}^2 = 0$ and likewise to any g we may add an $\mathscr{I}$-valued derivation and retain an algebra morphism. Therefore $\mathscr{F}$ is a pseudo-torsor (possibly non-split) over,

$$\begin{aligned} \mathscr{D}\!er_{g^{-1}f^{-1}\mathcal{O}_Y}\left(g^{-1}\mathcal{O}_X, \mathscr{I}\right) &= \mathscr{H}\!om_{g^{-1}\mathcal{O}_X}\left(\Omega_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_Y}, \mathscr{I}\right) \\ &= \mathscr{H}\!om_{g^{-1}\mathcal{O}_X}\left(g^{-1}\Omega_{X/Y}, \mathscr{I}\right) = \mathscr{H}\!om_{\mathcal{O}_T}\left(g^*\Omega_{X/Y}, \mathscr{I}\right) = \mathscr{G} \end{aligned}$$

(See Tag 08RR if this makes you uncomfortable). Since f is smooth, lifts exist Zariski locally so $\mathscr{F}$ is a $\mathscr{G}$-torsor and thus it corresponds to a class,

$$c(g) \in H^1(T, \mathscr{G}) = H^1(T, \mathscr{H}om_{\mathcal{O}_T}(g^*\Omega_{X/Y}, \mathscr{I})) = \mathrm{Ext}^1_{\mathcal{O}_T}\left(g^*\Omega_{X/Y}, \mathscr{I}\right)$$

which is zero iff $\mathscr{F}$ is trivial iff $\mathscr{F}$ has a global section. Furthermore, if $\mathscr{F}$ is trivial then $\Gamma(T, \mathscr{F})$ is an affine space over $\Gamma(X, \mathscr{G}) = \mathrm{Hom}_{\mathcal{O}_T}\left(g^*\Omega_{X/Y}, \mathscr{I}\right)$. $\square$

Definition 1.3.2. Let $\iota : Y \to Y'$ be a first-order infinitesimal thickening and $f : X \to Y$ a Y-scheme. A *lift* of X over Y' is a Y'-scheme $f' : X' \to Y'$ and an isomorphism $\varphi : X' \times_{Y'} Y \xrightarrow{\sim} X$. A morphism between lifts $f'_1 : X'_1 \to Y'$ and $f'_2 : X'_2 \to Y$ is a Y'-morphism $g : X'_1 \to X'_2$ such that,

$$\begin{tikzcd} X'_1 \times_{Y'} Y \arrow[rr, "g \times \mathrm{id}"] & & X'_2 \times_{Y'} Y \\ & X \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

commutes.

Proposition 1.3.3. Assume that $f : X \to Y$ is smooth and $\iota : Y \to Y'$ has ideal $\mathscr{I}$. Then,

(a) there exists an obstruction,

$$\omega(f) \in \mathrm{Ext}^2_{\mathcal{O}_X}\left(\Omega_{X/Y}, f^*\mathscr{I}\right)$$

to the existence of a smooth lift of X over Y'

(b) If $\omega(f) = 0$ then the set of isomorphism classes of smooth lifts is an affine space over,

$$\mathrm{Ext}^1_{\mathcal{O}_X}\left(\Omega_{X/Y}, f^*\mathscr{I}\right)$$

(c) If $f' : X' \to Y'$ is a smooth lift of X then the group of automorphisms of f' is naturally,

$$\mathrm{Hom}_{\mathcal{O}_X}\left(\Omega_{X/Y}, f^*\mathscr{I}\right)$$

Remark. Since $f : X \to Y$ is smooth, then $\Omega_{X/Y}$ is locally free and therefore,

$$\mathrm{Ext}^i_{\mathcal{O}_X}\left(\Omega_{X/Y}, f^*\mathscr{I}\right) = H^i(X, \mathcal{T}_{X/Y} \otimes_{\mathcal{O}_X} f^*\mathscr{I})$$

Proof. First, we consider shrinking X until it is affine and it maps to affines so we have $Y = \mathrm{Spec}\,(B)$ and $X = \mathrm{Spec}\,(A)$ and $Y' = \mathrm{Spec}\,(B')$ where $B = B'/I$ and $I^2 = 0$. We need to show that there exists a unique lift over A and that the group of automorphisms of this lift is $\mathrm{Hom}_A\left(\Omega_{A/B}, I \otimes_B A\right)$.

Suppose that A' is a smooth lift meaning $A' \otimes_{B'} B = A$. Applying $- \otimes_{B'} A'$ to the sequence,

$$0 \longrightarrow I \longrightarrow B' \longrightarrow B \longrightarrow 0$$

we get an exact sequence because A' is flat over B',

$$0 \longrightarrow I \otimes_{B'} A' \longrightarrow A' \longrightarrow A' \otimes_{B'} B \longrightarrow 0$$

then using that $A' \otimes_{B'} B = A$ and that I is naturally a B-module because $I^2 = 0$ we get,

$$
\begin{tikzcd}
0 \arrow[r] & I \otimes_B A \arrow[r] & A' \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u] & B' \arrow[r] \arrow[u, "f'"] & B \arrow[r] \arrow[u, "f"] & 0
\end{tikzcd}
$$

identifying $I' = \ker(A' \to A) = I \otimes_B A$ which is great because it is fixed by the given data. The only unknown is the A' that fill in the diagram.

First we consider automorphisms of A' preserving the diagram. If $\varphi : A' \to A'$ is a ring automorphism then $\varphi - \mathrm{id} : A' \to I'$ is a B'-module map because $\varphi - \mathrm{id}$ projects to zero in A. Because $I^2 = 0$ it is easy to check that $\tilde{D} = \varphi - \mathrm{id}$ is a derivation. Moreover, any B'-derivation kills I' because $I' = IA'$ and $D(ia) = iD(a) \in I^2 A = 0$ so it factors through a B-derivation $D : A \to I$ (since $A'/I' = A$). Conversely, given a B-derivation $D : A \to I'$ we produce a B'-map,

$$
\tilde{D} : A' \to A \xrightarrow{D} I' \to A'
$$

and a direct calculation shows that $\varphi = \mathrm{id} + \tilde{D}$ is a B-algebra automorphism making the diagram commute (because D lands in $I' = \ker(A' \to A)$). Therefore,

$$
\mathrm{Aut}_B(A'/A) = \mathrm{Der}_B(A, I') = \mathrm{Hom}_A\left(\Omega_{A/B}, I \otimes_B A\right)
$$

Next we show the uniqueness of lifts. Suppose that A_1' and A_2' are two smooth lifts of A. Then,

$$
\begin{tikzcd}
 & 0 \arrow[rr] & & I \otimes_B A \arrow[rr] & & A_2' \arrow[rr] & & A \arrow[rr] & & 0 \\
0 \arrow[rr] & & I \otimes_B A \arrow[ur, equal] \arrow[rr] & & A_1' \arrow[ur, dashed] \arrow[rr] & & A \arrow[ur, equal] \arrow[rr] & & 0 & \\
 & 0 \arrow[rr] & & I \arrow[rr] \arrow[uu] & & B' \arrow[rr] \arrow[uu] & & B \arrow[rr] \arrow[uu] & & 0 \\
0 \arrow[rr] & & I \arrow[ur, equal] \arrow[rr] \arrow[uu] & & B' \arrow[ur, equal] \arrow[rr] \arrow[uu] & & B \arrow[ur, equal] \arrow[rr] \arrow[uu] & & 0 &
\end{tikzcd}
$$

commutes giving a commutative square,

$$
\begin{tikzcd}
A & A_2' \arrow[l] \arrow[dl, dashed] \\
A_1' \arrow[u] & B' \arrow[l] \arrow[u]
\end{tikzcd}
$$

where the lift $A_1' \to A_2'$ exists because $B' \to A_2'$ is smooth and $A_1' \to A$ is an infinitesimal extension by I'. Applying the 5-lemma to the top of the preceding diagram we see that $A_1' \to A_2'$ is an isomorphism proving uniqueness of the lift.

Finally, we need to consider existence. It turns out it will be easiest to consider what seems like a more complication problem: lifting closed subschemes inside an ambient space that is endowed with a fixed lift. In our case, the ambient space will be affine space over B (since A is a finitely presented B-algebra) which is easy to lift (since $B'[x_1, \dots, x_n]$ is obviously a lift of $B[x_1, \dots, x_n]$) so we only need to show that we can lift smooth subschemes of affine space. Hartshorne's deformation theory chapter 2 considers this problem in detail and proves existence in much more generality. I will sketch Hartshorne's argument [H, Thm. 9.2] which applies for local complete intersections. (LOOK AT SEAN'S REFERENCE HERE!!)

Let $P \twoheadrightarrow A$ be the ambient embedding (in our case $P = B[x_1, \dots, x_n]$) and P' a fixed lift of P over B' (in our case $P' = B'[x_1, \dots, x_n]$). Then we need to find A' that fit into a diagram,

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & I \otimes_B J \arrow[r] \arrow[d] & J' \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \otimes_B P \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \otimes_B A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

where the bottom two rows are exact because A' and P' are flat over B' and the leftmost column is exact because A is flat over B and the other two columns are exact by definition so the top row is also exact by the 9-lemma (c.f. [H, Thm. 6.2]). Let $J = (f_1, \dots, f_r)$ and we define J' (and thus A') by lifting $\tilde{f}_i \in P'$ to give an ideal $J' = (f'_1, \dots, f'_r)$. Because A is a local complete intersection in P the Kozul complex $K_\bullet(P; f_1, \dots, f_r)$ is exact and forms a free resolution of A. Since P' is flat over B', tensoring the complex of free P'-modules $K_\bullet(P'; f'_1, \dots, f'_r)$ over the sequence,

$$0 \longrightarrow I \longrightarrow B' \longrightarrow B \longrightarrow 0$$

we get an exact sequences of complexes,

$$0 \longrightarrow I \otimes_B K_\bullet(P; f_1, \dots, f_r) \longrightarrow K_\bullet(P'; f'_1, \dots, f'_r) \longrightarrow K_\bullet(P; f_1, \dots, f_n) \longrightarrow 0$$

again using that I is a B-module. Furthermore, $K_\bullet(P; f_1, \dots, f_r) \otimes_B I$ remains exact because P is flat over B. Therefore, taking the long exact sequence on cohomology shows that $K_\bullet(P'; f'_1, \dots, f'_r)$ is exact in positive degrees and that their quotients form an exact sequence,

$$0 \longrightarrow I \otimes_B A \longrightarrow A' \longrightarrow A \longrightarrow 0$$

so A' fits into the above diagram and moreover is flat over B' (use [H Prop. 2.2]) and therefore smooth because its fibers over B' are equal to the fibers of A over B (the ideal I is nilpotent so $B = B'/I$ has the same points and residue fields) which are smooth. Therefore, we have produced a lift of A inside the lifted ambient space P' giving our desired lift of A.

Now we do the general case. As before, we would like to consider a "sheaf of lifts" over open of X but this does not make sense because lifts have automorphisms. Indeed, we actually have a stack of lifts $\mathscr{X}$ over X_{Zar}. Explicitly, the objects of $\mathscr{X}$ are smooth lifts U' of an open $U \subset X$ over Y and morphisms are morphisms of Y-schemes $\varphi : U' \to V'$ such that $U' \times_{Y'} Y \to V' \times_{Y'} Y$ is identified with the open inclusion $U \hookrightarrow V$. This is a fibered category over X_{Zar}. An affine local argument shows that every map in $\mathscr{X}$ is an open immersion and that maps over $\text{id} : U \to U$ are isomorphisms so $\mathscr{X}$ is fibered in groupoids. Morphisms glue because an open cover of U will pull back a lift U' of U to an open cover of U'. Furthermore, descent is effective because descent data for a cover $\{U_i \to U\}$ is exactly gluing data on opens for lifts U'_i over each U_i that thus glue to a lift U' over U. Let $\mathscr{G} = \mathscr{H}\!om_{\mathcal{O}_X}\big(\Omega_{X/Y}, f^* \mathscr{I}\big)$. A global version of the automorphisms argument shows that $\mathscr{G}$ acts on the objects of $\mathscr{X}$ precisely giving an action $B\mathscr{G} \times \mathscr{X} \to \mathscr{X}$. What we showed is that on affine opens $U \subset X$, the stack $\mathscr{X}|_U \cong B\mathscr{G}$ because it is connected with automorphism group $\mathscr{G}$ and on affines $\mathscr{G}$-torsors are all trivial (since $\mathscr{G}$ is quasi-coherent).

Therefore $\mathscr{X}$ is a $\mathscr{G}$-gerbe which corresponds to a class $[\mathscr{X}] \in H^2(X, \mathscr{G})$ which is zero if and only if $\mathscr{X}$ is the trivial gerbe if and only if $\mathscr{X}$ "admits a global section" meaning $\mathscr{X}(X)$ is nonempty i.e. there is a global lift. Furthermore, if $\mathscr{X}$ is trivial then $\mathscr{X} \cong B\mathscr{G}$ so the groupoid $\mathscr{X}(X)$ is exactly the category of $\mathscr{G}$-torsors with isomorphisms meaning that $H^1(X, B)$ classifies global lifts[1] and the automorphism group of any global lift is $H^0(X, \mathscr{G})$ as noted earlier.

This can be described more prosaically if Y is separated. Given an open affine cover $\{U_i\}$ of X and lifts U'_i of each U_i then pulling back to the double intersections $U'_{ij} = U'_i|_{U_{ij}}$ (which are affine by separatedness) are lifts over U_{ij}. We showed that any two lifts over an affine scheme are isomorphic so we can fix isomorphism $\varphi_{ij} : U'_{ij} \to U'_{ji}$. Then on triple overlaps U_{ijk} (which are affine by separatedness) there is a cocycle automorphism,

$$u_{ijk} = \varphi_{ik}^{-1} \circ \varphi_{jk} \circ \varphi_{ij}$$

of $U'_i|_{U_{ijk}}$. Then, as previously,

$$c_{ijk} = u_{ijk} - \mathrm{id} \in \mathscr{G}(U_{ijk})$$

and (c_{ijk}) defines a 2-cocyle for $\mathscr{G}$ and therefore defines an obstruction class $[c] \in H^2(X, \mathscr{G})$ to the lifts $\{U'_i\}$ gluing to a global lift X'. Of course, this must be checked to be a 2-cocycle independent of our choices up to the addition of a 2-coboundary and that the isomorphisms φ_{ij} can be modified, possibly on a refinement of the cover $\{U_i\}$, such that the cocycle vanishes and gluing goes through if and only if (c_{ijk}) is a coboundary but we will leave the discussion here. $\square$

Remark. When affine locally flat lifts of subschemes exist inside a lifted ambient space there is an obstruction to global lifting given by an H^1-class of a twisted normal bundle. It seems strange that obstructions to lifting subschemes live in H^1 where as obstructions to lifting schemes without an ambient space live in H^2 until we remember that automorphisms played a central part in the above argument. Notice that subschemes have no automorphisms (as subschemes) and therefore the lifts of subschemes actually form a sheaf (rather than a stack) which is a torsor (rather than a gerbe) over some twisted normal bundle and thus the obstruction to a global lift (corresponding to a global section of the torsor which trivializes it) is an H^1-class corresponding to the torsor.

Example 1.3.4. If X is affine then,

$$H^i(X, \mathcal{T}_{X/Y} \otimes_{\mathcal{O}_X} f^*\mathscr{I}) = 0$$

for all $i > 0$ and thus smooth lifts always exist and are unique as we demonstrated in the proof.

Example 1.3.5. If $f : X \to Y$ is étale then $\Omega_{X/Y} = 0$. Thus for any first-order thickening $\iota : Y \to Y'$ there is a unique smooth lift X' of X over Y and X' has no nontrivial automorphisms.

Example 1.3.6. If $f : X \to Y$ is a family of smooth curves over a zero dimensional base then,

$$H^2(X, \mathcal{T}_{X/Y} \otimes_{\mathcal{O}_X} f^*\mathscr{I}) = 0$$

[1] Lifts form an affine space over $H^1(X\mathscr{G})$ meaning it only classifies lifts up to choosing a base point or equivalently up to choosing an isomorphism $\mathscr{X} \cong B\mathscr{G}$ which is why the identification is not canonical unlike the case for torsors where 0 corresponds to the trivial torsor (there is no corresponding trivial lift).

because $\dim X = 1$ so curves over infinitessmial schemes always admit liftings. In the case,

$$Y = \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(R) = Y'$$

is an infinitessimal extension for some Artin local ring R with residue field k and maximal ideal $\mathfrak{m} \subset R$ with $\mathfrak{m}^2 = 0$ (e.g. $\mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z}) \to \mathrm{Spec}\,(\mathbb{Z}/p^2\mathbb{Z})$ or $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k[\epsilon])$) then we see that $\mathfrak{m}$ is a k-vectorspace of dimension r so $\mathscr{I} \cong k^{\oplus r}$ and thus,

$$H^1(X, \mathcal{T}_{X/k} \otimes_{\mathcal{O}_X} f^*\mathscr{I}) = H^1(X, \mathcal{T}_{X/Y}^{\oplus r}) = H^0(X, (\omega_{X/Y}^{\otimes 2})^{\oplus r}) \cong \begin{cases} 0 & g = 0 \\ k^{\oplus r} & g = 1 \\ k^{3(g-1)r} & g \geq 1 \end{cases}$$

In particular, for $r = 1$, this gives the expected dimension of the moduli space $\mathcal{M}_g$ of smooth genus g curves: $3(g-1)$. This makes sense because the tangent space $T_{[C]}\mathcal{M}_g$ should correspond to smooth infinitessimal deformations of a curve C/k (i.e. smooth lifts of C/k over $k[\epsilon]$) by the moduli functor description since $\ker(\mathcal{M}_g(k[\epsilon]) \to \mathcal{M}_g(k))$ should classify smooth curves over $k[\epsilon]$ with a fixed pullback to a smooth curve over k on the closed point.

Example 1.3.7. When the extension $\iota : Y \to Y'$ splits then the obstruction class always vanishes $\omega(f) = 0$ because we can form a lift by pulling back along the section $s : Y' \to Y$. This happens, for example, with $D = k[\epsilon]$ and the split extension,

$$0 \longrightarrow k \xrightarrow{\epsilon} D \longrightarrow k \longrightarrow 0$$

Therefore, lifts over D always exist and are classified (in the case that $X \to \mathrm{Spec}\,(k)$ is smooth) by,

$$\mathrm{Ext}^1_{\mathcal{O}_X}\left(\Omega_{X/k}, f^*\underline{k}\right) = H^1(X, \mathcal{T}_{X/Y})$$

Furthermore, the automorphism group over any lift of X over D is naturally isomorphic to,

$$\mathrm{Hom}_{\mathcal{O}_X}\left(\Omega_{X/k}, f^*\underline{k}\right) = H^0(X, \mathcal{T}_{X/k})$$

which identifies tangent fields with "infinitessimal automorphisms of X" meaning $\mathrm{Aut}_X(X \times_k D)$.