Mathematics GU4053 Algebraic Topology Assignment # 4

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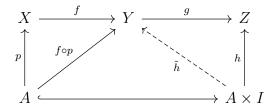
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Note. My order of path concatenation follows Hatcher,

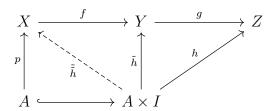
$$\gamma * \delta(x) = \begin{cases} \gamma(2x) & x \le \frac{1}{2} \\ \delta(2x - 1) & x \ge \frac{1}{2} \end{cases}$$

Problem 1.

Let $f: X \to Y$ and $g: Y \to Z$ be fibrations. Given any space A, a map $p: A \to X$ and a homotopy $h: A \times I \to Z$ consider the diagram,

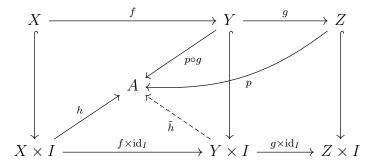


Because g is a fibration, there exists a lift $\tilde{h}: A \times I \to Y$ of h matching $f \circ p$ such that the diagram commutes. Now, rewrite the commutative diagram as,

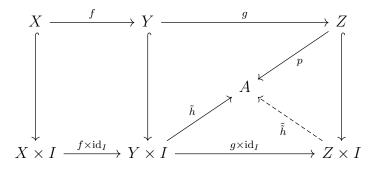


which gives a lift $\tilde{\tilde{h}}$ of \tilde{h} at p because f is a fibration. Therefore, there is a map $\tilde{\tilde{h}}: A \times I \to X$ which makes the diagram commute. That is, $\tilde{\tilde{h}}(a,0) = p(a)$ and $g \circ f \circ \tilde{\tilde{h}} = g \circ \tilde{h} = h$. Thus, $g \circ f$ is a fibration.

Likewise, let $f: X \to Y$ and $g: Y \to Z$ be cofibrations. Given any space A, a map $p: Z \to A$ and a homotopy $h: X \times I \to A$ consider the diagram,



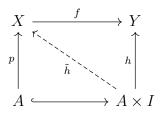
Because f is a cofibration, there exists an extension $\tilde{h}: Y \times I \to A$ of h lifting to $p \circ g$ such that the diagram commutes. Now, rewrite the commutative diagram as,



which gives an extension $\tilde{\tilde{h}}$ of \tilde{h} lifting p because f is a cofibration. Therefore, there is a map $\tilde{\tilde{h}}: Z \times I \to A$ which makes the diagram commute. Thus, $g \circ f$ is a cofibration.

Problem 2.

Let $f: X \to Y = \{*\}$ be a continuous (constant map). Given a map $g: A \to X$ and a homotopy $h: A \times I \to Y$. However, h must be a constant map so we should take $\tilde{h}: A \times I \to X$ given by $\tilde{h}(x,t) = g(x)$. Then, $f \circ \tilde{h} = h$ because both are constant maps. Also, $\tilde{h}(x,0) = g(x)$ by construction. Thus, the following diagram commutes,



which means that f is a fibration.

Problem 3.

Problem 4.

Problem 5.

Problem 6.

Let X be a CW complex equal to an increasing union of subcomplexes,

$$X_1 \subset X_2 \subset X_3 \subset X_4 \subset \cdots$$
 and $X = \bigcup_{i=1}^{\infty} X_i$

such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. I claim that the complex $X = \lim_{\to} X_i$ is the direct limit (colimit) of the system,

$$X_1 \longleftrightarrow X_2 \longleftrightarrow X_3 \longleftrightarrow X_4 \longleftrightarrow \cdots$$

(PROVE THIS)

The projection functor $\pi : \text{Top} \to \text{hTop}$ which maps topological spaces to themselves and maps maps to homotopy classes is cocontinuous. (PROVE THIS) Therefore,

$$\pi(X) = \pi(\lim_{\to} X_i) = \lim_{\to} \pi(X_i)$$

However, each inclusion map ι is nullhomotopic so the map $j_i: X_i \to X$ is also nullhomotopic because $j_i = \iota_i \circ j_{i+1}$ but ι_i is nullhomotopic so j_i is also nullhomotopic. Because X is the limit of this diagram, there must be a unique map $\mathrm{id}_X: X \to X$ which commutes with every cone.

Problem 7.

Consider the pointed space (X, x_0) and the suspension $\Sigma X \cong X \times I/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)$. Consider the homotopy, $X \times I \to \Sigma X$ defined by h(x, t) = (x, t/2). Then, h(x, 0) = (x, 0) but $(x_1, 0) \sim (x_2, 0)$ so h(x, 0) is constant. Also, $h(x, 1) = (x, 1/2) = \iota(x)$. Therefore, $\iota : X \hookrightarrow \Sigma X$ is nullhomotopic.

Problem 8.

Let X be a pointed CW complex. The infinite suspension of X is given by,

$$\Sigma^{\infty}(X) = \bigcup_{i=1}^{\infty} \Sigma^{i}(X)$$

By problem 7, the inclusion maps in the following chain are nullhomotopic,

$$\Sigma X \longrightarrow \Sigma^2 X \longrightarrow \Sigma^3 X \longrightarrow \Sigma^4 X \longrightarrow \Sigma^5 X \longrightarrow \Sigma^6 X \longrightarrow \cdots$$

which, by problem 7, implies that the total complex $\Sigma^{\infty}(X)$ is contractable.