Physics C2801, Fall 2013 Prof. Brian Cole

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Problem 1. Kleppner and Kolenkow problem 7.1

Problem 2. Kleppner and Kolenkow problem 7.2

Problem 3. Kleppner and Kolenkow problem 7.3

Problem 4. Kleppner and Kolenkow problem 7.6

Problem 5. Kleppner and Kolenkow problem 7.7

Problem 6. Rotations and rotation matrices

In this problem, we will look more into the properties of rotations as transformations and their implementation via matrices in both 2 and 3 dimensions and start to develop some experience working with matrices. This problem will also illustrate some of the characteristics of transformations that we will see in Lorentz transformations in special relativity.

Suppose the position vector $\vec{r} = x\hat{i} + y\hat{j}$ represents the position of some particle or physical object. Then suppose that the particle or object is rotated around the origin by an angle θ about the z axis. The new position vector will be written, $\vec{r}' = x'\hat{i} + y'\hat{j}$. Now, in problem set 2, you determined the transformation properties of a vector when the coordinate axes are rotated by an angle θ. Such a rotation is called a "passive" rotation because the position of a particle is kept fixed while the coordinate axes are rotated. In contrast, an "active" rotation keeps the coordinate axes fixed in position, but rotates the particle or, equivalently, the position vector of the object. Convince yourself that active and passive rotations will differ only by a change in the sign of the angle: $\theta_{active} = -\theta_{passive}$. For the remainder of this problem, all rotations will be active. So, for example, the rotation in the x–y plane by an angle θ will take the form:

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}$$

In class I also introduced matrices as a way to re-state such a relationship in a compact way. Namely, we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

Using the language of linear algebra, $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ are termed "column vectors." The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the rotation matrix in two dimensions. It is also a rank-2 tensor in that it transforms one vector into another vector.

If you are not familiar with matrix multiplication, the product on the right hand side of Equation 1 evaluates to:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

In words, the product is evaluated by multiplying the elements of a row against the elements of a column and summing. That sum yields one of the elements of the resulting matrix. For the product in Equation 1 there are two rows in the square matrix with which we can perform this operation, so there will be two rows in the resulting column vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$. See the nice Wikipedia entry *http://en.wikipedia.org/wiki/Matrix_multiplication* for more details and a nice picture of how matrix multiplication works.

Sometimes it is useful to represent a product such as that in Equation 1 symbolically, e.g. $X' = R(\theta)\,X$ where X and X' represent the column vectors and R represents the rotation matrix. We write $R(\theta)$ to make explicit the dependence of the rotation matrix on the angle θ. Alternatively, we can represent elements of vectors and matrices algebraically with subscript(s) that run over the dimensions of the matrix. For example, if X represents the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then $X_1 \equiv x$ and $X_2 \equiv y$,

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

with R_{ij} representing the different elements of the rotation matrix given above (i.e. $R_{11} = \cos\theta$, $R_{12} = -\sin\theta$ etc.) It is customary for the first index in the element of the matrix to represent the row and the second index to represent the column. This different notation for the elements of the vector and the elements of the matrix allows us to re-write the matrix multiplication rule algebraically

$$X_i' = \sum_{j=1}^{2} R_{ij} X_j$$

a.) Show that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the "identity" matrix in two dimensions, namely that it is a matrix that when multiplied against another matrix leaves that matrix unchanged More specifically, show that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show that $R(0)$ is the identity matrix – i.e. a rotation by zero angle leaves the vector unchanged.

b.) Show that for the rotation matrix in Equation 1 evaluated at special angles $\theta = \pi/2, \pi, 3\pi/2, 2\pi$ you obtain sensible results that correspond to interchanging the x and y axes or inverting the x or y coordinates (make sure to show why the results make sense).

Now, we can write the result of two rotations symbolically $X'' = R'(\theta')X' = R'(\theta')R(\theta)X$, or in matrix form,

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (2)$$

Matrix multiplication is **not** commutative but it is associative. So we can multiply the two square matrices on the right-hand side of Equation 2 first and then multiply the result against the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. But, the product of the two matrices is yet another rotation matrix, which we will represent as R''. We can write the matrix multiplication rule for the two square matrices

$$R''_{ij} = \sum_{k=1}^{2} R'_{ik} R_{kj}$$

c.) Evaluate the elements of R'' and show that R'' is just an ordinary rotation matrix for a (combined) rotation angle of $\theta + \theta'$. This is as we should expect: namely, in one dimension if we rotation by θ and then by θ' we obtain a result equivalent with a single rotation by an angle $\theta + \theta'$.

d.) The "inverse" of a rotation matrix R is that rotation (matrix) that undoes the effect of R. Namely, if R^{-1} is the inverse rotation, $R^{-1}R = 1$. Here "1" means the identity matrix (see part a above). Show by explicit multiplication that $R^{-1}(\theta) = R(-\theta)$, i.e. that $R(-\theta)R(\theta) = 1$. This result is also consistent with part b, since if $\theta' = -\theta$, $\theta'' = 0$. Clearly, the fact that the inverse rotation works out as it should is due to the symmetry of the rotation matrix – i.e. how it depends on the sign of the angle – and the fact that the components of the rotation matrix are $\sin\theta$ and $\cos\theta$.

e.) The transpose of a matrix is obtained by switching the rows and columns. For example, the transpose of the rotation matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. The transpose of a matrix A is usually written A^T. Show that $R^T = R^{-1}$.

Now, the transpose of the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is a "row vector" $\begin{bmatrix} x & y \end{bmatrix}$. The matrix product of a row vector times a column vector is a scalar (number) – the matrix equivalent of the inner product. For example,

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2.$$

An interesting question is how does the row vector $\begin{bmatrix} x & y \end{bmatrix}$ transform under rotations by an angle θ? We would expect

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta \end{bmatrix}$$

But, we can think of $\begin{bmatrix} x' & y' \end{bmatrix}$ as the transpose of $\begin{bmatrix} x' \\ y' \end{bmatrix}$ or, equivalently, as the transpose of the product $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. An important result from linear algebra is that the transpose of the product of two matrices, is equal to the inverted product of the transposes, i.e. $(A\,B)^T = B^T\,A^T$. So,

$$\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)^T = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \qquad (3)$$

f.) Show that the final product in Equation 3 gives the correct result for $\begin{bmatrix} x' & y' \end{bmatrix}$. You have, thus, shown that $X'^T = X^T R^T = X^T R^{-1}$.

g.) Now, use the result from part f to show that rotations preserve the length (squared) of the position vector in two dimensions. Namely, that $X'^T X' = X^T X$.

Now, if we extend the analysis of rotations to three dimensions, the vectors have three elements and the rotation matrices will be 3×3 square matrices. The rotation matrix for rotations about the z axis by angle θ_z will be $R_z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

h.) Work out the form of the rotation matrix, $R_y(\theta_y)$, for active rotations around the y axis by an angle θ_y, and $R_x(\theta_x)$ for rotations about the x axis by an angle θ_x. Be careful with the signs and make sure you preserve a right-handed coordinate system.

i.) Show that rotations about the z axis and rotations about the y axis do not commute. Namely that $R_y(\theta_y)\,R_z(\theta_z) \neq R_z(\theta_z)\,R_y(\theta_y)$. Explicitly calculate the two products of the rotation matrices and show that they are not equal.

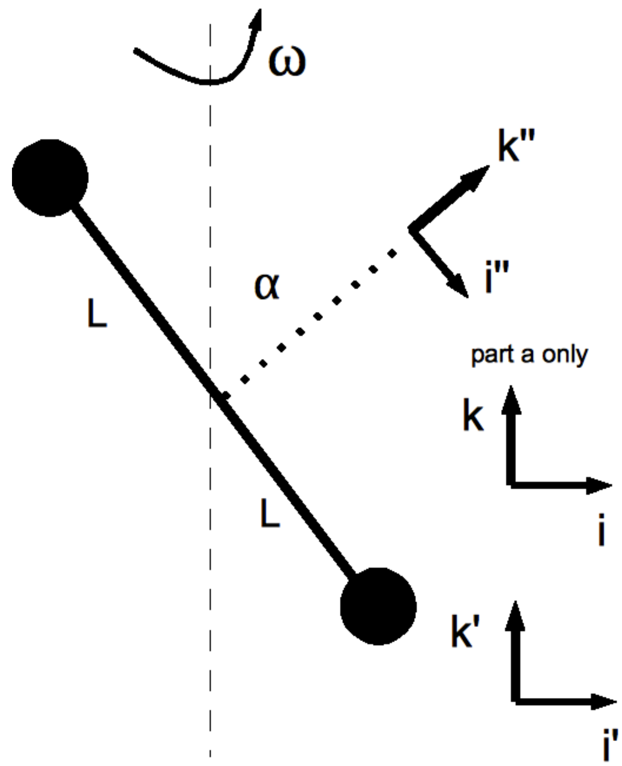
j.) Now show that for $\theta_z << 1$ and for $\theta_y << 1$ keeping terms to first order in small numbers, the successive rotations in part j *do* commute.

k.) Show that the product of three infinitesimal rotations $R_x(d\theta_x)R_y(d\theta_y)R_z(d\theta_z)$ gives the result I motivated in class – namely that $d\vec{r} = \vec{d\theta} \times \vec{r}$ for $\vec{d\theta} = d\theta_x\hat{i} + d\theta_y\hat{j} + d\theta_z\hat{k}$.

Problem 7. Skew rod with non-point masses

We spent significant time in lecture using the "skew rod" (also discussed in your book extensively) as a valuable tool for understanding the complete treatment of angular momentum, the tensor of inertia, and the value of decomposing the angular velocity vector into components using body-centered coordinates. That problem simplified because the masses at the ends of the skew rod were treated as point masses. In this problem, you will analyze the skew rod treating the masses at the ends of the rod as finite-sized spheres with non-zero moments of inertia.

Use the same geometry as we used in class (see the figure). Each half of the rod is of length L. The angle between the z axis and the normal to the rod is α. Take the masses at the end of the rod to be spheres with mass M_s and moments of inertia I_s about an axis through their center. *Note: In principle, the center of the masses is displaced by the radius of the sphere from the end of the rod so that the centers are a distance $L + R$ from the pivot but we will ignore that detail as it just adds unnecessary algebra.* The rod of negligible mass and moment of inertia compared to the spheres is pivoting about the z axis with angular velocity $\vec{\omega} = \omega\hat{k}$ keeping α constant.



a. Evaluate the angular momentum of the rod when it lies in the $x-z$ plane as shown in the figure. Do this using the kinematic expressions for the angular momentum from chapter 6 evaluating the contribution of each mass to the total angular momentum. Express the components of $\vec{L}$ in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ (you may not need all three) when the rod is oriented as shown in the figure. You will obtain a different result from the one obtained in class. Evaluate the angle between $\vec{L}$ and the normal to the rod (specify in terms of the arc tangent of a massless ratio).

b. One problem with the analysis in part a is that we evaluated $\vec{L}$ for a specific orientation of the rod only. If the rod rotates away from the orientation of the figure, your result will no longer hold as expressed in the **fixed** coordinate system ($\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$). But, we can instead use a "body-centered" coordinate system ($\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$) defined so that $\hat{k}'$ and $\hat{\imath}'$ point in the plane of the rod always and with $\hat{\jmath}'$ defined by $\hat{\imath}' \times \hat{\jmath}' = \hat{k}'$. *Note: since we are rotating about the z axis* $\hat{k} = \hat{k}'$. Your results from part a hold for all orientations if we simply replace $\hat{\imath}$ with $\hat{\imath}'$. For use in the rest of the problem, we need to find the relationship beteween the primed and unprimed unit vectors. Suppose $\hat{\imath}$ and $\hat{\imath}'$ are aligned at $t = 0$. Find an expression for $\hat{\imath}'(t)$ and $\hat{\jmath}'(t)$ in terms of $\hat{\imath}$ and $\hat{\jmath}$.

c. Because the body-centered, primed coordinates are rotating with respect to the fixed coordinates, $\vec{L}$ expressed with respect to the primed axes will still have non-zero time derivative. Using the results from part b explicitly calculate $\dot{\vec{L}}$ and express your result in **both** primed and unprimed coordinate systems. Beware the temptation to say that $\dot{\vec{L}} = 0$ in the body-centered coordinate system – this is NOT true. **Note:** since $\dot{\vec{L}} \neq 0$ there must be a torque applied to keep the skew rod rotating with fixed α and fixed ω. That torque would be provided by the pivot/axel on which the skew rod is rotating (not shown in the figure).

d. Now evaluate $\dot{\vec{L}}$ using the general expression derived in class

$$\dot{\vec{L}} = \vec{\Omega} \times \vec{L}_{bc} + \dot{\vec{L}}_{bc}$$

and express the resulting vector in both primed and unprimed coordinates. You should get the same result as part c.

d. Using the relationship between the angular momentum vector and the angular velocity vector, $\vec{L} = [I]\vec{\omega}$, indirectly evaluate the components of the tensor of inertia for the rod+masses system in terms of the body-centered ($\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$) coordinate system (i.e. use $\vec{L}$ and $\vec{\omega}$ to determine $[I]$). You will only be able to determine a subset of the components of $[I]$ – which ones cannot be determined from you result in part a? Now evaluate the tensor of inertia in the fixed coordinate system using the the results from part b to relate the components of $[I]$ between primed and unprimed coordinates. As we saw in lecture, when we express the tensor of inertia for a rotating body in fixed coordinates, the components of the tensor are time-dependent. Show that you obtain the same result for $\dot{\vec{L}}$ as you did in part c if you use the fixed-coordinate tensor of inertia and evaluate $\vec{L} = [I]\vec{\omega}$.

e. We can simplify the tensor of inertia if we introduce a different body-centered coordinate system ($\hat{\imath}''$, $\hat{\jmath}''$, $\hat{k}''$) where $\hat{\imath}''$ is directed along the rod, $\hat{k}''$ is directed normal to the rod and $\hat{\jmath}''$ is determined by $\hat{\imath}'' \times \hat{\jmath}'' = \hat{k}''$. Evaluate directly the diagonal components of the tensor of inertia in this new set of coordinates. Here you may find it useful to use the parallel axis theorem (appropriately since the center of mass motion of the spheres is purely determined by the rotation of the rod.

f. Now let's consider the the off-diagonal components of the tensor of inertia from part e. There's two aspects we should consider. First, let's consider the off-diagonal components of $[I]$ for a sphere centered at the origin. For a sphere all such off-diagonal elements are zero. Explain why. You may use general arguments based on symmetry or you may specifically show how the formula would give zero contribution. Now, for the problem at hand, we need to evaluate the off-diagonal contributions to $[I]$ when the sphere is not centered at the origin but is displaced along the x (actually x'') axis by a distance L. We can find and use an analog of the parallel axis theorem for the components of the tensor of inertia for the displaced mass in terms of the components when the object is centered at the origin. Start with the expression for I^O_{xy} for a sphere centered at the origin and show how to obtain the value I_{xy} when the sphere is displaced by a distance L along the x axis (we are dropping the double-primes here to reduce notation). Now do the same for I_{xz} and I_{yz}. Evaluate all of the off-diagonal components of the tensor of inertia using the body-centered coordinates from part e and show that they are all zero. Thus, in these new coordinates, we have diagonalized the tensor of inertia.

g. Using your results from part e and part f we can now easily evaluate $\vec{L}$. Express $\vec{\omega}$ in terms of ($\hat{\imath}''$, $\hat{\jmath}''$, $\hat{k}''$) and use $\vec{L} = [I]\vec{\omega}$ to obtain $\vec{L}$. Show that you obtain the same result as you obtain from expressing your result from part *a* in components along ($\hat{\imath}''$, $\hat{\jmath}''$, $\hat{k}''$).

h. Now evaluate $\dot{\vec{L}}$ from your expression for $\vec{L}$ in part g using the relationship between time derivatives in body-centered and fixed reference frames (see part d) and show that it is consistent with your result from part c.