

Most taken for the section in Jarod Alper's notes on Quasi-Coherent Sheaves.

1 Quasi-Coherent Sheaves

Recall that for a DM-stack we defined the small étale site:

Definition 1.0.1. Let \mathcal{X} be a DM-stack. Then the *small étale site* $\mathcal{X}_{\text{ét}}$ of \mathcal{X} is the category of schemes equipped with an étale map $U \rightarrow \mathcal{X}$. A covering is $\{U_i \rightarrow U\}$ over \mathcal{X} such that $\sqcup_i U_i \rightarrow U$ is surjective.

Then for a sheaf \mathcal{F} on $\mathcal{X}_{\text{ét}}$ we defined its global sections,

$$\Gamma(\mathcal{X}, \mathcal{F}) := \text{Hom}_{\mathfrak{Sh}(\mathcal{X}_{\text{ét}})}(1, \mathcal{F})$$

where 1 is the terminal sheaf (the sheafification of $U \mapsto *$).

Remark. This definition works nicely for \mathcal{X} DM and naturally generalizes the étale site $X_{\text{ét}}$ of a scheme. However, there is a glaring flaw if we attempt to extend this definition to Artin stacks there is a catastrophic failure: $\mathcal{X}_{\text{ét}}$ could be empty! For example, $(B\mathbb{G}_m)_{\text{ét}}$ is empty. Indeed, DM-stacks are exactly those stacks with schemes as étale neighborhoods. To remedy this we could take the smooth site of \mathcal{X} . To stay in the world of étale cohomology we consider a hybrid site where the schemes are smooth over \mathcal{X} but the covers are all étale.

Definition 1.0.2. Let \mathcal{X} be an algebraic stack. Then the *lisse-étale site* $\mathcal{X}_{\ell\text{-ét}}$ is the category of schemes smooth over \mathcal{X} with *arbitrary* maps of schemes over \mathcal{X} . A covering $\{U_i \rightarrow U\}$ is a collection of morphisms such that $\sqcup_i U_i \rightarrow U$ is surjective or étale.

Definition 1.0.3. Let \mathcal{F} be a sheaf on $\mathcal{X}_{\ell\text{-ét}}$ then,

$$\Gamma(\mathcal{U}, \mathcal{F}) = \text{Hom}_{\mathfrak{Sh}(\mathcal{U}_{\ell\text{-ét}})}(1_{\mathcal{U}}, \mathcal{F}|_{\mathcal{U}_{\ell\text{-ét}}})$$

where $1_{\mathcal{U}}$ is the *indicator sheaf* of the smooth \mathcal{X} -stack $\mathcal{U} \rightarrow \mathcal{X}$ the sheafification of the constant sheaf $*$. This is the terminal object of $\mathcal{U}_{\ell\text{-ét}}$. This can be computed by choosing a smooth presentation,

$$R \rightrightarrows U \rightarrow \mathcal{U}$$

and setting,

$$\Gamma(\mathcal{U}, \mathcal{F}) = \text{eq}(\mathcal{F}(U) \rightrightarrows \mathcal{F}(R))$$

Definition 1.0.4. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is defined via,

$$\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$$

is a ring object in the abelian category $\mathbf{Ab}(\mathcal{X}_{\ell\text{-ét}})$. We therefore define the abelian category $\mathbf{Mod}_{\mathcal{O}_{\mathcal{X}}}$. Given a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of algebraic stacks there are adjoint pairs of functors¹,

$$\begin{array}{ccc} \mathbf{Ab}(\mathcal{X}_{\ell\text{-ét}}) & \xrightleftharpoons[f^*]{f_*} & \mathbf{Ab}(\mathcal{Y}_{\ell\text{-ét}}) \\ \mathbf{Mod}_{\mathcal{O}_{\mathcal{X}}} & \xrightleftharpoons[f^*]{f_*} & \mathbf{Mod}_{\mathcal{O}_{\mathcal{Y}}} \end{array}$$

¹I think these do NOT arise from morphisms of topoi on the topoi of sheaves of sets, in the sense that f^{-1} is not left exact in general. This is why people say that the lisse-étale site is NOT functorial. However, f^* is left exact on abelian sheaves see [Tag 00XS](#)

Given two $\mathcal{O}_{\mathcal{X}}$ -modules \mathcal{F} and \mathcal{G} we define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$ as the sheafification of,

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)} \mathcal{G}(U)$$

and the Hom sheaf $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})$ as the sheaf,

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

where $\mathcal{F}|_U$ means the restriction to the site $U_{\ell\text{-}\acute{e}t}$ (note this is much more data than the restriction to U_{Zar}).

1.1 Quasi-Coherent Sheaves

As above we denote by $\mathcal{F}|_U$ the restriction of \mathcal{F} to $U_{\ell\text{-}\acute{e}t}$ and $\mathcal{F}|_{U_{\mathrm{Zar}}}$ the restriction to U_{Zar} . Then we define,

Definition 1.1.1. Let \mathcal{X} be an algebraic stack. A $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{F} is *quasi-coherent* if:

- (a) for every smooth $U \rightarrow \mathcal{X}$ the restriction $\mathcal{F}|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$ -module
- (b) for every morphism $f : V \rightarrow U$ of smooth \mathcal{X} -schemes, the induced morphism,

$$f^*(\mathcal{F}|_{U_{\mathrm{Zar}}}) \rightarrow \mathcal{F}|_{V_{\mathrm{Zar}}}$$

is an isomorphism.

Remark. The above definition can be made in any site which refines the Zariski topology on each of its opens. However, in this generality such an object is usually called a *crystal in quasi-coherent sheaves* and the term *quasi-coherent* in an arbitrary site is reserved for the notion developed below. However, in most sites the two notions agree.

Definition 1.1.2. In an arbitrary ringed site $(\mathcal{C}, \mathcal{O})$ (or even an arbitrary ringed topos) a \mathcal{O} -module \mathcal{F} is *quasi-coherent* if for each object $U \in \mathcal{C}$ there exists a cover $\{U_i \rightarrow U\}$ such that $\mathcal{F}|_{\mathcal{C}/U_i}$ is a *presentable* \mathcal{O} -module meaning there exists a presentation,

$$\bigoplus_J \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus_I \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \mathcal{F}|_{\mathcal{C}/U_i} \longrightarrow 0$$

We call the abelian subcategory of such sheaves $\mathrm{QCoh}(\mathcal{C}) \subset \mathbf{Mod}_{\mathcal{O}_{\mathcal{C}}}$.

Definition 1.1.3. Let S be a scheme and $\mathcal{C} \subset \mathbf{Sch}_S$ a subcategory. Consider the presheaf of rings,

$$\begin{aligned} \mathcal{O} : \mathcal{C}^{\mathrm{op}} &\rightarrow \mathrm{Ring} \\ (T \rightarrow S) &\mapsto \Gamma(T, \mathcal{O}_T) \end{aligned}$$

This is a sheaf for the fpqc topology. Furthermore, for any sheaf \mathcal{F} on S_{Zar} there is a presheaf,

$$\begin{aligned} \mathcal{O} : \mathcal{C}^{\mathrm{op}} &\rightarrow \mathrm{Ab} \\ (f : T \rightarrow S) &\mapsto \Gamma(T, f^* \mathcal{F}) \end{aligned}$$

which is a \mathcal{O} -module. Furthermore, if \mathcal{F} is quasi-coherent then \mathcal{F}^a is a fpqc sheaf by descent.

Theorem 1.1.4 ([Tag 03OJ](#)). Let S be a scheme. Let \mathcal{C} be a site such that,

- (a) \mathcal{C} is a full subcategory of \mathbf{Sch}_S
- (b) any Zariski covering of $T \in \mathcal{C}$ can be refined by a covering of \mathcal{C}
- (c) $\text{id} : S \rightarrow S$ is an object of \mathcal{C} (so in particular \mathcal{C} has a terminal object)
- (d) every covering of \mathcal{C} is an fpqc covering of schemes

Then the presheaf \mathcal{O} is a sheaf on \mathcal{C} and there is an equivalence of categories,

$$\begin{aligned} \text{QCoh}(S) &\xrightarrow{\sim} \text{QCoh}(\mathcal{C}) \\ \mathcal{F} &\mapsto \mathcal{F}^a \end{aligned}$$

Proof. This is basically a rephrasing of fpqc descent. □

Proposition 1.1.5. Let \mathcal{F} be a $\mathcal{O}_{\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}}$ -module. Then the following are equivalent,

- (a) \mathcal{F} is quasi-coherent in the general sense
- (b) \mathcal{F} is quasi-coherent in the crystal sense.

Proof. C.f. [06WK](#). Let $\mathcal{C} = \mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}$. Suppose that \mathcal{F} satisfies (a). Then the restriction of \mathcal{F} is quasi-coherent on \mathcal{C}/U and thus by the previous lemma $\mathcal{F}|_{\mathcal{C}} = (\mathcal{F}|_{U_{\text{Zar}}})^a$ and therefore satisfies (b). Given (b) take any $U \rightarrow \mathcal{X}$ smooth. Then we know $\mathcal{F}|_{U_{\text{Zar}}}$ is quasi-coherent so there is an affine Zariski open cover $\{U_i \rightarrow U\}$ such that $\mathcal{F}|_{(U_i)_{\text{Zar}}}$ is presented. Then the claim is that $\mathcal{F}|_{\mathcal{C}/U_i}$ is also presented. Indeed, the comparison map induced by $f : V \rightarrow U$ is an isomorphism the presentation pulls back to give a presentation of $\mathcal{F}|_{\mathcal{C}/U_i}$. □

1.2 Descent Data

Definition 1.2.1. Let (U, R, s, t, c, e) be a groupoid scheme over S where $s, t : R \rightrightarrows U$ are the source and target maps and $c : R \times_{s, U, t} R \rightarrow R$ is the composition and $e : U \rightarrow R$ is the identity. Then the category of *descent data* consists of the category of pairs (\mathcal{F}, φ) where \mathcal{F} is a sheaf on U and φ is an isomorphism,

$$\varphi : t^* \mathcal{F} \xrightarrow{\sim} s^* \mathcal{F}$$

such that $e^* \varphi = \text{id}$ and satisfying the cocycle condition,

$$c^* \varphi = \pi_2^* \varphi \circ \pi_1^* \varphi$$

as morphisms of sheaves on $R \times_{s, U, t} R$.

Example 1.2.2. For any cover $U \rightarrow X$ we can form the ‘‘Cech groupoid’’ $U \times_X U \rightrightarrows U$ whose composition is given by projection,

$$(U \times_X U) \times_{\pi_1, U, \pi_2} (U \times_X U) = U \times_X U \times_X U \rightarrow U \times_X U \quad ((a, b), (c, a)) \mapsto (c, a, b) \mapsto (c, b)$$

For this we recover the ordinary notion of a descent datum.

Example 1.2.3. Let $G \curvearrowright X$ be an action of an algebraic group on a scheme. Then there is a groupoid $G \times X \rightrightarrows X$ whose composition $G \times G \times X \rightarrow G \times X$ is given by multiplication in the group.

For this we will recover the notion of G -equivariance.

Proposition 1.2.4. Let $R \rightrightarrows U$ be a smooth presentation of an algebraic stack \mathcal{X} by schemes. There is an equivalence of categories,

$$\mathrm{QCoh}(\mathcal{X}) \rightarrow \mathrm{DD}_{\mathrm{QCoh}}(U/R) \quad \mathcal{F} \mapsto (\mathcal{F}|_{U_{\mathrm{Zar}}}, \varphi)$$

where $\mathrm{DD}_{\mathrm{QCoh}}(U/R)$ is the category of descent data for quasi-coherent sheaves along the groupoid $R \rightrightarrows U$.

Proof. For any smooth map $V \rightarrow \mathcal{X}$ there is a further smooth refinement $V' \rightarrow V$ such that $V' \rightarrow \mathcal{X}$ factors through $U \rightarrow \mathcal{X}$. Hence, applying descent to $V' \rightarrow V$, any quasi-coherent sheaf \mathcal{F} on $\mathcal{X}_{\ell\text{-}\acute{\mathrm{e}}\mathrm{t}}$ is determined by its descent data over $R \rightrightarrows U$. \square

Definition 1.2.5. Let $G \curvearrowright X$ be an action of a group scheme on a scheme (or algebraic space). The category of G -equivariant sheaves is defined as the category of descent data for the groupoid $G \times X \rightrightarrows X$.

Remark. Some standard diagram chasing shows that this is formally the same as the usual definition of a G -equivariant sheaf in [Stacks]. In the case that G is a finite constant group it is easy to check that this agrees with the naive notion in terms of compatible isomorphisms between the pullbacks along the action by elements $g \in G$.

Proposition 1.2.6. There is an equivalence of categories,

$$\mathrm{QCoh}([X/G]) \rightarrow \mathrm{QCoh}_G(X)$$

Proof. This is a special case of the previous proposition. \square

1.3 Examples

Example 1.3.1. Let $\mathcal{X} \rightarrow S$ be a DM-stack. Then the sheaf,

$$\Omega_{\mathcal{X}/S} : (U \rightarrow \mathcal{X}) \mapsto \Gamma(U, \Omega_{U/S})$$

is quasi-coherent since any morphism $f : V \rightarrow U$ in $\mathcal{X}_{\acute{\mathrm{e}}\mathrm{t}}$ is étale so the map,

$$f^* \Omega_{U/S} \xrightarrow{\sim} \Omega_{V/S}$$

is an isomorphism. However, if $\mathcal{X} \rightarrow S$ is not DM we don't have access to $\mathcal{X}_{\acute{\mathrm{e}}\mathrm{t}}$ nor can we define $(\Omega_{X/S})^a$ on X_{fppf} as we can for a scheme since there is no Zariski or étale site to define this sheaf over for a bootstrap. There is still a sheaf of $\mathcal{O}_{\mathcal{X}_{\ell\text{-}\acute{\mathrm{e}}\mathrm{t}}}$ -modules,

$$\Omega_{\mathcal{X}/S} : (U \rightarrow \mathcal{X}) \mapsto \Gamma(U, \Omega_{U/S})$$

but it is not quasi-coherent. This is the sort of sheaf the stacks project calls *locally quasi-coherent* meaning that it is quasi-coherent when restricted to $U_{\acute{\mathrm{e}}\mathrm{t}}$ for any $U \rightarrow \mathcal{X}$.

Remark. Indeed, it is not clear that an Artin stack $\mathcal{X} \rightarrow S$ should have any good notion of a cotangent bundle $\Omega_{\mathcal{X}/S}$. For example, consider $\mathcal{X} = \mathbf{B}\mathbb{G}_m$ which is smooth of relative dimension -1 so what should $\Omega_{\mathcal{X}/S}$ even be? It can't be a vector bundle of rank -1 can it! To fix this conundrum, we either work with $\Omega_{\mathcal{X}/S}$ as defined above which is not quasi-coherent and hence does not have a well-defined rank or we define the cotangent complex $\mathbb{L}_{\mathcal{X}/S} \in D_{\mathrm{QCoh}}^{\leq 1}(\mathcal{X})$ (technically it's an ind-object in this generality) [Champs Algebriques, Chapter 17] which encodes the deformation theory of \mathcal{X} . Note that unlike for a scheme, $\mathbb{L}_{\mathcal{X}/S}$ can be supported in degree 1. In fact, the following are equivalent,

- (a) $\mathcal{X} \rightarrow S$ is DM
- (b) $\mathcal{H}^1(\mathbb{L}_{\mathcal{X}/S}) = 0$

Proof: [Champs Algebriques, Cor. 17.9.2].

1.4 Picard Groups

Let \mathcal{X} be an algebraic stack. Then $\text{Pic}(\mathcal{X})$ denotes the set of isomorphism classes of line bundles on \mathcal{X} which becomes an abelian group under \otimes .

Example 1.4.1. If G is an affine algebraic k -group then $\text{Pic}(\mathbf{B}G) = \text{Hom}_{\text{gp}}(G, \mathbb{G}_m)$ is the group of characters. For example,

- (a) $\text{Pic}(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$
- (b) $\text{Pic}(\mathbf{B}\text{GL}_n) = \mathbb{Z}$
- (c) $\text{Pic}(\mathbf{B}\text{PGL}_n) = \{0\}$.

This is because line bundles on $\mathbf{B}G$ are the same as line bundles on $\text{Spec}(k)$ along with descent data i.e. a G -action. This is the same as a 1-dimensional G -representation.

Example 1.4.2. Consider the action, $\mathbb{G}_m \curvearrowright \mathbb{A}^n$ with weights d_1, \dots, d_n . Let the *weighted projective stack* be the DM-stack (at least if $p \nmid d_i$),

$$\mathcal{P}(d_1, \dots, d_n) = [(\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m]$$

Here let k be a field of characteristic not dividing any d_i or 2 or 3.

- (a) The map $\text{Pic}(\mathbf{B}\mathbb{G}_m) \rightarrow \text{Pic}(\mathcal{P}(d_1, \dots, d_n))$ induced by the canonical \mathbb{G}_m -bundle is an isomorphism. Indeed, this reduces to classifying \mathbb{G}_m -actions on $\mathcal{O}_{\mathbb{A}^n \setminus \{0\}}$. By Hartogs' these correspond to \mathbb{G}_m -actions on $\mathcal{O}_{\mathbb{A}^n}$ and thus to different grading of $A = k[x_1, \dots, x_n]$ as an A -module with x_i given weight d_i . These are just overall shifts $A(d)$ i.e. putting 1 in degree d . This is the same as the pullback of the bundle over $\mathbf{B}\mathbb{G}_m$ corresponding to $\mathbb{G}_m \xrightarrow{z^d} \mathbb{G}_m$.
- (b) Using Weierstrass models we get an isomorphism,

$$\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4, 6)$$

Therefore, $\text{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}\omega_{\mathcal{C}/\mathcal{M}}$

- (c) Then it turns out that,

$$\text{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z}\omega_{\mathcal{C}/\mathcal{M}}$$

This is because the discriminant Δ is a section of $\mathcal{O}(12)$ which is nowhere vanishing for smooth families.

1.5 Global Quotients and the Resolution Property

Definition 1.5.1. An algebraic stack \mathcal{X} is a *global quotient stack* if there is an isomorphism $\mathcal{X} \cong [U/\mathrm{GL}_n]$ where U is an algebraic space. This is equivalent to asking for the existence of a GL_n -bundle $U \rightarrow \mathcal{X}$ where U is an algebraic space. By definition this is the same as a representable morphism $\mathcal{X} \rightarrow \mathbf{BGL}_n$.

Proposition 1.5.2. Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a surjective, flat, and projective morphism of noetherian algebraic stacks. If \mathcal{X} is a quotient stack then \mathcal{Y} is a quotient stack.

Definition 1.5.3. A noetherian algebraic stack has the *resolution property* if every coherent sheaf is a quotient of a vector bundle.

A smooth or quasi-projective scheme over a field has the resolution property. Any noetherian normal \mathbb{Q} -factorial scheme with affine diagonal also has the resolution property.

Proposition 1.5.4. Let G be an affine algebraic k -group with an action $G \curvearrowright U$ on a quasi-projective k -scheme U . Assume that there is an ample line bundle \mathcal{L} with a G -action. Then $[\mathrm{Spec}(A)/G]$ has the resolution property.

Remark. While not every line bundle \mathcal{L} on a normal k -scheme admits a G -action, it turns out there is always some positive power such that $\mathcal{L}^{\otimes n}$ has a G -action.

Proof. The G -line bundle \mathcal{L} corresponds to a line bundle on $[U/G]$ which is ample with respect to the morphism $p : [U/G] \rightarrow \mathbf{BG}$ since relative ampleness can be checked after smooth covers (it can be reduced to a fiberwise condition). For a coherent sheaf \mathcal{F} on $[U/G]$ the natural map,

$$\mathcal{L}^{-\otimes N} \otimes p^*p_*(\mathcal{L}^{\otimes N} \otimes \mathcal{F}) \rightarrow \mathcal{F}$$

is surjective for $N \gg 0$ since relative ampleness implies global generation of $\mathcal{L}^{\otimes N} \otimes \mathcal{F}$. The pushforward $p_*(\mathcal{L}^{\otimes N} \otimes \mathcal{F})$ is quasi-coherent on \mathbf{BG} hence a G -representation. We can hence write it as an increasing union of finite-dimensional G -representations V_i and obtain,

$$\mathrm{colim}_i (\mathcal{L}^{-\otimes N} \otimes p^*V_i) \rightarrow \mathcal{F}$$

since \mathcal{F} is coherent, this stabilizes at some stage meaning,

$$\mathcal{L}^{-\otimes N} \otimes p^*V_i \rightarrow \mathcal{F}$$

at some finite stage i . □

Theorem 1.5.5 (Totaro-Gross). Let \mathcal{X} be a quasi-separated normal algebraic stack of finite type over k . Assume that the stabilizer group at every closed point is smooth and affine. Then the following are equivalent:

- (a) \mathcal{X} has the resolution property
- (b) $\mathcal{X} \cong [U/\mathrm{GL}_n]$ with U quasi-affine
- (c) $\mathcal{X} \cong [\mathrm{Spec}(A)/G]$ with G an affine algebraic group.

In particular, \mathcal{X} has affine diagonal.

Remark. The normality hypothesis on \mathcal{X} and smoothness hypothesis on the stabilizers are unnecessary. However, the affineness hypothesis on the stabilizers is necessary. For example, \mathbf{BE} the classifying stack of an elliptic curve has the resolution property.

1.6 Sheaf Cohomology

Lemma 1.6.1. If \mathcal{X} is an algebraic stack, the categories $\mathbf{Ab}(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}})$ and $\mathbf{Mod}_{\mathcal{X}}$ have enough injective. If \mathcal{X} is quasi-separated then $\mathbf{QCoh}(\mathcal{X})$ has enough injectives.

Definition 1.6.2. Let \mathcal{X} be an algebraic stack and \mathcal{F} a sheaf on $\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}$. The *cohomology groups* $H^i(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F})$ are the derived functors of,

$$\Gamma(\mathcal{X}, -) : \mathbf{Ab}(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}) \rightarrow \mathbf{Ab}$$

applied to \mathcal{F} ,

$$H^i(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F}) = R^i\Gamma(\mathcal{X}, \mathcal{F})$$

Definition 1.6.3. Given a smooth covering $\mathfrak{U} = \{\mathcal{U}_i \rightarrow \mathcal{X}\}_{i \in I}$ of algebraic stacks and an abelian presheaf \mathcal{F} on $\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}$ the *Cech complex* $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})$ of \mathfrak{U} with respect to \mathfrak{U} is,

$$\check{C}^n(\mathfrak{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_n) \in I^{n+1}} \mathcal{F}(\mathcal{U}_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{U}_{i_n})$$

with differential,

$$d^n : \check{C}^n(\mathfrak{U}, \mathcal{F}) \rightarrow \check{C}^{n+1}(\mathfrak{U}, \mathcal{F}) \quad (s_{i_0, \dots, i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_k^* s_{i_0, \dots, \hat{i}_k, \dots, i_n} \right)_{i_0, \dots, i_{n+1}}$$

where the projection p_k forgets the k^{th} coordinate. The *Cech cohomology* of \mathcal{F} with respect to \mathfrak{U} is,

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) := H^i(\check{C}^\bullet(\mathfrak{U}, \mathcal{F}))$$

Theorem 1.6.4. Let \mathcal{X} be an algebraic stack and \mathcal{F} a quasi-coherent sheaf on $\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}$. Then for any cover $\mathfrak{U} = \{\mathcal{U}_i \rightarrow \mathcal{X}\}_{i \in I}$ there exists a spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, H^q(-, \mathcal{F})) \implies H^{p+q}(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F})$$

where $H^q(-, \mathcal{F})$ is the presheaf $U \mapsto H^q(U_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F})$.

Proof. Consider the commutative diagram of functors,

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}) & \xleftarrow{a} & \mathbf{PSh}(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}) \\ & \searrow \Gamma & \downarrow \check{H}^0 \\ & & \mathbf{Ab} \end{array}$$

Notice that $\check{C}^\bullet(\mathfrak{U}, -)$ is exact in the category of presheaves which shows that $\check{H}^\bullet(\mathfrak{U}, -)$ forms a δ -functor. In fact, since $\check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$ for $i > 0$ and any injective sheaf (this is a very general fact, see [Tag 03OR](#)) it is a universal δ -functor. Now the inclusion a takes injectives to injectives because sheaves form a reflexive subcategory (maps to a sheaf factors through the sheafification). Therefore, we apply the Grothendieck spectral sequence so it suffices to compute $R^q a(\mathcal{F})$ of a sheaf \mathcal{F} . Since the functor $(-) \mapsto \Gamma(U, -)$ is exact on presheaves we see that,

$$R^q a(\mathcal{F})(U) = R^q \Gamma(U, \mathcal{F}) = H^q(U, \mathcal{F})$$

so we conclude. □

Theorem 1.6.5. If X is an affine scheme and \mathcal{F} is a quasi-coherent $\mathcal{O}_{\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}}$ -module then,

$$H^i(X_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F}) = \begin{cases} \Gamma(X, \mathcal{F}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. We refine to affine coverings $\{\text{Spec}(B) \rightarrow \text{Spec}(A)\}$ then \mathcal{F} is quasi-coherent (in all equivalent notions) and hence arises from some A -module M . To show that $\check{H}^{>0} = 0$ for this covering we show that the Amistur complex,

$$0 \longrightarrow M \longrightarrow M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \otimes_A B \longrightarrow \cdots$$

is exact. Indeed, after applying $B \otimes_A -$ which is faithfully flat this complex obtains a nullhomotopy. Now to conclude, we can either apply Cartan's criterion ([Tag 03F9](#)) or use hypercoverings and the fact that hypercover Cech cohomology computes derived functor cohomology. \square

Proposition 1.6.6. Let \mathcal{X} be an algebraic stack with affine diagonal and \mathcal{F} be a quasi-coherent sheaf. If $\mathfrak{U} = \{U_i \rightarrow \mathcal{X}\}$ is an étale covering with each U_i affine, then $H^i(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F}) = \check{H}^i(\mathfrak{U}, \mathcal{F})$.

Proof. Follows immediately from the Cech-to-derived spectral sequence and the above. \square

Remark. To remove the “affine diagonal” condition we need to use hypercovers. Indeed, if $U_\bullet \rightarrow \mathcal{X}$ is a simplicial hypercover in $\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}$ where each U_\bullet is an affine scheme and \mathcal{F} is quasi-coherent then,

$$H^i(\mathcal{X}, \mathcal{F}) = \check{H}^i(U_\bullet, \mathcal{F})$$

Proposition 1.6.7. Let X be a scheme (or a DM-stack with a sheaf on $\mathcal{X}_{\acute{\text{e}}\text{t}}$) with affine diagonal² and \mathcal{F} a quasi-coherent sheaf. Then,

$$H^i(X, \mathcal{F}) = H^i(X_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F}_{\ell\text{-}\acute{\text{e}}\text{t}})$$

for all i where $\mathcal{F}_{\ell\text{-}\acute{\text{e}}\text{t}}$ is the $\mathcal{O}_{X_{\ell\text{-}\acute{\text{e}}\text{t}}}$ -module defined by,

$$\mathcal{F}_{\ell\text{-}\acute{\text{e}}\text{t}}(U) = \Gamma(U, f^* \mathcal{F})$$

for a smooth map $f : U \rightarrow X$. (In the stack case it is pullback under $f : \mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}} \rightarrow \mathcal{X}_{\acute{\text{e}}\text{t}}$).

Proof. Choose an affine Zariski cover U of X (affine étale cover U of \mathcal{X}) by the assumption on the diagonal we see that,

$$H^i(X_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F}) = \check{H}^i(U, \mathcal{F}) = H^i(X, \mathcal{F})$$

(and similarly for \mathcal{X}). The affine diagonal condition is to ensure that projects in the Cech complex are affine and hence have vanishing $H^{>0}$. However, this condition is not necessary. We can always choose a Zariski hypercover $U_\bullet \rightarrow X$ by affines and similar arguments show that,

$$H^i(X_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F}) = \check{H}^i(U_\bullet, \mathcal{F}) = H^i(X, \mathcal{F})$$

\square

Proposition 1.6.8. Let \mathcal{X} be an algebraic stack.

- (a) \mathcal{F} is an $\mathcal{O}_{\mathcal{X}}$ -module then $H^i(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F})$ agrees with $R^i\Gamma : \mathbf{Mod}_{\mathcal{O}_{\mathcal{X}}} \rightarrow \mathbf{Ab}$ computed in the category of $\mathcal{O}_{\mathcal{X}}$ -modules.

²If we use hypercovers (see the discussion in the proof then we can remove this condition.

- (b) If \mathcal{X} has affine diagonal and \mathcal{F} is a quasi-coherent sheaf on \mathcal{X} , then the cohomology $H^i(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F})$ agrees with $R^i\Gamma : \mathbf{QCoh}(\mathcal{X}) \rightarrow \mathbf{Ab}$ computed in the category of quasi-coherent modules.

For a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of algebraic stacks (resp. quasi-compact morphism of algebraic stacks with affine diagonals) then (a) (resp. (b)) holds also for $R^if_*\mathcal{F}$ of an $\mathcal{O}_{\mathcal{X}}$ -module (resp. quasi-coherent sheaf).

Remark. If \mathcal{X} does not have affine diagonal, then the sheaf cohomology $H^i(\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}, \mathcal{F})$ of a quasi-coherent sheaf may differ from the derived functor $R^i\Gamma(\mathcal{X}, -) : \mathbf{QCoh}(\mathcal{X}) \rightarrow \mathbf{Ab}$.

Proposition 1.6.9. If \mathcal{X} is an algebraic stack and \mathcal{F}_i is a directed system of abelian sheaves in $\mathcal{X}_{\ell\text{-}\acute{\text{e}}\text{t}}$ then $\text{colim}_i H^i(\mathcal{X}, \mathcal{F}_i) \rightarrow H^i(\mathcal{X}, \text{colim}_i \mathcal{F}_i)$ is an isomorphism.

2 July 8 Affine GIT and Good moduli spaces

2.1 Good Moduli Spaces

Definition 2.1.1. A quasi-compact quasi-separated morphism $\pi : \mathcal{X} \rightarrow X$ from an algebraic stack to an algebraic space is a *good moduli space* if,

- (a) $\mathcal{O}_X \xrightarrow{\sim} \pi_*\mathcal{O}_{\mathcal{X}}$
- (b) $\pi_* : \mathbf{QCoh}(\mathcal{X}) \rightarrow \mathbf{QCoh}(X)$ is exact.

Example 2.1.2. Let G be linearly k -reductive (meaning taking invariants of representations is exact) then $G \curvearrowright \text{Spec}(A)$ gives a diagram,

$$\begin{array}{ccc} \text{Spec}(A) & & \\ \downarrow & \searrow & \\ [\text{Spec}(A)/G] & \xrightarrow{\pi} & \text{Spec}(A^G) \end{array}$$

Then π satisfies the properties of a good moduli space morphism. Indeed,

- (a) $\Gamma([\text{Spec}(A)/G], \mathcal{O}_{[\text{Spec}(A)/G]}) = A^G$
- (b) and $\pi_* : \mathbf{QCoh}([\text{Spec}(A)/G]) \rightarrow \mathbf{QCoh}(\text{Spec}(A^G))$ is exact since taking invariants of a G -representation is exact.

Example 2.1.3. $[\mathbb{A}^1/\mathbb{G}_m] \rightarrow \text{Spec}(k)$ is a good moduli space

Example 2.1.4. $[\mathbb{P}^1/\mathbb{G}_m] \rightarrow \text{Spec}(k)$ is NOT a good moduli space. Indeed (2) fails because $\pi_* = H^0$ is not exact for \mathbb{G}_m -equivariant sheaves on \mathbb{P}^1 . Here two closed points specializes to 1 closed point downstairs. We will see this is a problem.

Theorem 2.1.5. Let $\pi : \mathcal{X} \rightarrow X$ be a good moduli space and \mathcal{X} be q-sep over a scheme S .

- (a) π is surjective and universally closed

- (b) if $\mathbb{Z}_1, \mathbb{Z}_2 \subset \mathcal{X}$ are closed substacks then $\pi(\mathbb{Z}_1 \cap \mathbb{Z}_2) = \pi(\mathbb{Z}_1) \cap \pi(\mathbb{Z}_2)$ where this denotes scheme-theoretic (or rather stack-theoretic) images and intersections. In particular, for geometric points $x_1, x_2 \in \mathcal{X}(k)$ then $\pi(x_1) = \pi(x_2)$ iff $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$.
- (c) If \mathcal{X} is noetherian then X is also noetherian.
- (d) If \mathcal{X} is of finite type over S and S is noetherian then X is finite type over S and π_* preserves coherent sheaves.
- (e) If \mathcal{X} is noetherian then π is initial for maps to algebraic spaces.

Corollary 2.1.6. Let G be linearly reductive group over $k = \bar{k}$ and $\tilde{\pi} : U = \text{Spec}(A) \rightarrow U//G = \text{Spec}(A^G)$ then,

- (a) $\tilde{\pi}$ is surjective and for any G -invariant closed $Z \subset U$ (meaning corresponding to a closed substack of $[U/G]$) then $\pi(Z) \subset U//G$ is closed. This property remains true after base change
- (b) given G -invariant closed $Z_1, Z_2 \subset U$ then $\pi(Z_1 \cap Z_2) = \pi(Z_1) \cap \pi(Z_2)$ so if $x_1, x_2 \in U(k)$ then $\tilde{\pi}(x_1) = \tilde{\pi}(x_2)$ iff $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$.
- (c) If A is Noetherian so is A^G . If A is f.g. over k then A^G is f.g. over k and for any f.g. A -module M with a G -action, M^G is a f.g. A^G -module
- (d) if A is noetherian then $\tilde{\pi}$ is initial for G -invariant maps to algebraic spaces.

2.2 Cohomologically Affine Morphisms

Definition 2.2.1. A quasi-compact quasi-separated morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is *cohomologically affine* if $f_* : \text{QCoh}(\mathcal{X}) \rightarrow \text{QCoh}(\mathcal{Y})$ is exact. We say \mathcal{X} is *cohomologically affine* if $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$ is.

Example 2.2.2. (a) an affine morphism is cohomologically affine

(b) an affine algebraic group G/k is linearly reductive iff $\mathbf{B}G \rightarrow \text{Spec}(k)$ is cohomologically affine

Lemma 2.2.3. Consider a diagram,

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

- (a) if f is faithfully flat and π' is a GMS then π is a GMS
- (b) if \mathcal{Y} has quasi-affine diagonal and π is a GMS then π' is a GMS

Proof. (a) by flat base change $\pi'_* f^* \cong g^* \pi_*$ and f^* is exact and g^* is faithfully exact so we conclude. For (b), first consider the case that g is quasi-affine. Then factor as,

$$\mathcal{Y}' \hookrightarrow \mathbf{Spec}_{\mathcal{Y}}(f_* \mathcal{O}_{\mathcal{Y}'}) \rightarrow \mathcal{Y}$$

where the second map is affine by definition. If g is affine then g_* is faithfully exact. If g is an open immersion then for $F' \rightarrow G'$ in $\mathrm{QCoh}(\mathcal{X}')$ we conclude that $G := \mathrm{im}(f_*F' \rightarrow f_*G')$ since $f^*f_* = \mathrm{id}$ we get $f^*G = G'$ and π_* is exact so $\pi_*f_*F' \rightarrow \pi_*F$ so pullback via g then,

$$g^*\pi_*f_*F' = \pi'_*f^*f_*F' = \pi'_*F' \rightarrow g^*\pi_*G = \pi'_*f^*G = \pi'_*G$$

the first by flat basechange. Thus we conclude. To do the general case, assume \mathcal{Y} and \mathcal{Y}' are quasi-compact and choose a smooth presentation $Y = \mathrm{Spec}(A) \rightarrow \mathcal{Y}$ which is quasi-affine since \mathcal{Y} is. Then $\mathcal{Y}'_Y \rightarrow \mathcal{Y}$ is faithfully flat so by (a) suffices to show that $\mathcal{X}'_Y \rightarrow \mathcal{Y}'_Y$ is cohomologically affine. By (a) again we can pass to a smooth cover $Y' \rightarrow \mathcal{Y}'_Y$ and reduce to a morphism of affine schemes which is hence affine so we win. \square

Corollary 2.2.4. If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is representable and \mathcal{Y} has quasi-affine diagonal. If f is cohomologically affine then f is affine.

Proof. Suffices to prove this for a map $f : X \rightarrow Y$ of schemes with Y affine. However, this is just Serre's criterion for affineness. \square

2.3 First Properties of GMS

Lemma 2.3.1. Consider a diagram,

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{g} & X \end{array}$$

with X', X quasi-separated algebraic spaces and X has quasi-affine diagonal.

- (a) if g is faithfully flat and π' is a GMS then π is a GMS
- (b) if π is a GMS then π' is a GMS.

Now assume that π is a GMS then,

- (a) there is a projection formula for $F \in \mathrm{QCoh}(\mathcal{X})$ and $G \in \mathrm{QCoh}(X)$ then,

$$\pi_*\mathcal{F} \otimes G \xrightarrow{\sim} \pi_*(F \otimes \pi^*G)$$

In particular,

$$G \xrightarrow{\sim} \pi_*\pi^*G$$

- (b) $F \in \mathrm{QCoh}(\mathcal{X})$ then,

$$g^*\pi_*F \xrightarrow{\sim} \pi'_*f^*F$$

- (c) for any $\mathcal{I} \subset \mathcal{O}_X$ a quasi-coherent ideal sheaf then,

$$\mathcal{I} \xrightarrow{\sim} \pi_*(\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\mathcal{X}})$$

Proof. For (a) and g flat then $g^*(\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\mathcal{X}})$ is just $\mathcal{O}_{X'} \rightarrow \pi'_* \mathcal{O}_{\mathcal{X}'}$ by flat base change. Thus if g is faithfully flat then we get,

$$\mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathcal{X}} \iff \mathcal{O}_{X'} \xrightarrow{\sim} \pi'_* \mathcal{O}_{\mathcal{X}'}$$

Furthermore, the lemma says that cohomological flatness descends under g proving (a). We also showed that (b) holds for flat base change. To prove it for arbitrary base change we first prove the projection formula. Indeed, let $U \rightarrow X$ be an étale presentation with U disjoint union of affine schemes. Then $\pi_U : \mathcal{X}_U \rightarrow U$ is a GMS by flat case. Then the pullback of $\text{id} \rightarrow \pi_* \pi^*$ is $\text{id} \rightarrow \pi_{U*} \pi_U^*$. We can assume that $X = \text{Spec}(A)$ and consider,

$$G_2 \rightarrow G_1 \rightarrow G \rightarrow 0$$

is a free presentation. Then consider,

$$\begin{array}{ccccccc} \pi_* F \otimes G_2 & \longrightarrow & \pi_* F \otimes G & \longrightarrow & \pi_F \otimes G & \longrightarrow & 0 \\ \parallel & & \parallel & & \downarrow & & \\ \pi_*(F \otimes \pi^* G_2) & \longrightarrow & \pi_*(F \otimes \pi^* G_1) & \longrightarrow & \pi_*(F \otimes \pi^* G) & \longrightarrow & 0 \end{array}$$

using that π_* is exact and the locally free form of the projection formula. Then we conclude the projection formula by the 5 lemma. (DO THE REST) \square

Lemma 2.3.2. Let $\pi : \mathcal{X} \rightarrow X$ be a GMS and X is quasi-separated. Then,

- (a) for $\mathcal{A} \in \text{QCoh}(\mathcal{X})$ then $\mathbf{Spec}_{\mathcal{X}}(\mathcal{A}) \rightarrow \mathbf{Spec}_X(\pi_* \mathcal{A})$ is a GMS
- (b) $\mathbb{Z} \subset \mathcal{X}$ is a closed substack defined by \mathcal{I} then $Z = \pi(\mathbb{Z}) \subset X$ satisfies that $\mathbb{Z} \rightarrow Z$ is a GMS.

Proof. Indeed (b) is the special case of (a) where $\mathcal{A} = \mathcal{O}_{\mathcal{X}}/\mathcal{I}$. For (a) we see that,

$$\mathbf{Spec}_{\mathcal{X}}(\mathcal{A}) \rightarrow \mathcal{X} \times_X \mathbf{Spec}_X(\pi_* \mathcal{A}) \rightarrow \mathbf{Spec}_X(\pi_* \mathcal{A})$$

the first is affine and the second is cohomologically affine by base change. \square

I SHOULD HAVE BEEN CAREFUL ABOUT π vs im since I MEAN SCHEME THEORETIC IMAGE

Proof of Theorem. (a) if \mathcal{X} is quasi-sep then so is X . Then for all $x \in X(k)$ use Lemma 6.3.20 (b) $\mathcal{X} \times_X \text{Spec}(k) \rightarrow \text{Spec}(k)$ is a GMS. Then $\Gamma(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}) = k$ so $\mathcal{X}_x \neq \emptyset$ and hence π is surjective. To prove universal closedness consider $\mathbb{Z} \subset \mathcal{X}$ a closed substack use Lemma 6.3.22 (b) then $\mathbb{Z} \rightarrow \pi(\mathbb{Z})$ is a GMS and hence surjective hence $\pi(\mathbb{Z})$ the scheme theoretic image is just equal to the image and hence the image is closed. Then use preservation under base change to get universally closed. (b) if $\mathbb{Z}_i \subset \mathcal{X}$ are defined by \mathcal{I}_i then apply π_* to the sequence,

$$0 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{I}_1 + \mathcal{I}_2 \longrightarrow \mathcal{I}_2/(\mathcal{I}_1 \cap \mathcal{I}_2) \longrightarrow 0$$

so we get,

$$\pi_* \mathcal{I}_1 + \pi_* \mathcal{I}_2 \cong \pi^*(\mathcal{I}_1 + \mathcal{I}_2)$$

and hence,

$$\text{im}(\mathbb{Z}_1) \cap \text{im}(\mathbb{Z}_2) = \text{im}(\mathbb{Z}_1 \cap \mathbb{Z}_2)$$

\square

2.4 Finite Typeness of GMS

Definition 2.4.1. A morphism $f : X \rightarrow Y$ of schemes is *universally submersive* if it is surjective and Y has the quotient topology ($U \subset Y$ open iff $f^{-1}(U)$ is open) and this is true after any base change.

Lemma 2.4.2. Valuative criterion:

$$\begin{array}{ccc} \mathrm{Spec}(R') & \dashrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & Y \end{array}$$

if $X \rightarrow Y$ is universally submersive then there exists R'/R extensions of DVRs lifting.

Example 2.4.3. The following maps are universally submersive,

- (a) fppf covering
- (b) good moduli space morphism

Proof.

□

3 Determinantal Line Bundles

Recall that k is an algebraically closed field of characteristic zero.

Let X be a smooth, projective and connected curve over k . Let \mathcal{S} be an algebraic stack over k . Then consider the diagram,

$$\begin{array}{ccc} & X \times \mathcal{S} & \\ q \swarrow & & \searrow p \\ X & & \mathcal{S} \end{array}$$

Since $X \times \mathcal{S} \rightarrow \mathcal{S}$ is representable by schemes we can use cohomology and base change theorem. If \mathcal{E} is a vector bundle on $X \times \mathcal{S}$ then $\mathbf{R}p_*\mathcal{E}$ is a reflexive complex on \mathcal{S} with amplitude in $[0, 1]$ since this is true for the projection map $X \times T \rightarrow T$ from a test scheme. **what does amplitude mean**

Proposition 3.0.1. Let k be a field, S a k -scheme of finite type and $f : X \rightarrow S$ a smooth projective morphism of relative dimension n . If F is a flat coherent sheaf on X then there is a locally free resolution

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow F \rightarrow 0$$

such that $R^n f_* F_\nu$ is locally free for $\nu = 0, \dots, n$ and $R^i f_* F_\nu = 0$ for $i \neq n$ and $\nu = 0, \dots, n$. Moreover, there is a natural quasi-isomorphism

$$[R^n f_* F_n \rightarrow R^n f_* F_{n-1} \rightarrow \cdots \rightarrow R^n f_* F_0] \rightarrow (\mathbf{R}f_* F)[n]$$

Proof. Let $\mathcal{O}_X(1)$ be an f -very ample line bundle on X . Since f is a Cohen-Macaulay morphism (it is even smooth) we have a Serre duality pairing,

$$R^i f_* \mathcal{F}(-m) \times R^{n-i} f_* (\mathcal{F}^\vee(m) \otimes \omega_{X/Y}) \rightarrow R^n f_* \omega_{X/Y} = \mathcal{O}_X$$

which is perfect. Using the mumford complex, there is some m_0 such that for $m \geq m_0$ we have $R^{n-i} f_* (\mathcal{F}^\vee(m) \otimes \omega_{X/Y}) = 0$ for all $i < n$ and hence $R^i f_* \mathcal{F}(-m) = 0$. Define S -flat sheaves K_ν and G_ν inductively as follows. Let $K_0 := F$. Assume that K_ν has been constructed. Since this is a bounded family, for sufficiently large $m \gg m_0$ all fibers $(K_\nu)_s$ are m -regular. Hence, by cohomology and base change $f_* K_\nu(m)$ is locally free (since its higher cohomology vanishes) and there is a natural surjection $G_\nu := f^*(f_* K_\nu(m))(-m) \rightarrow K_\nu$ (surjective on fibers by m -regularity). Then G_ν is locally free and

$$R^i f_* G_\nu = f_* K_\nu(m) \otimes R^i f_* \mathcal{O}_X(-m)$$

by the projection formula. In particular, $R^n f_* G_\nu$ is locally free and the other direct image sheaves vanish. Finally, define

$$K_{\nu+1} := \ker(G_\nu \rightarrow K_\nu)$$

Therefore we get an infinite locally free resolution $G_\bullet \rightarrow F$. Since for each $s \in S$

$$(K_n)_s = \ker((G_{n-1})_s \rightarrow (G_{n-2})_s)$$

and the fiber X_s is regular of dimension n it has global dimension n and therefore $(K_n)_s$ is locally free since it is the n^{th} syzygy module in the locally free resolution of the locally free sheaf F_s . Hence K_n is itself locally free by Nakayama³. Therefore, we can truncate to get the locally free resolution F_\bullet . The last statement follows from the first. Indeed, the quasi-isomorphism,

$$[F_n \rightarrow \cdots \rightarrow F_0] \rightarrow F[0]$$

gives the desired result after applying $\mathbf{R}f_*$ and showing that the natural map **WHERE DOES IT COME FROM?**

$$\mathbf{R}f_*[F_n \rightarrow \cdots \rightarrow F_0] \rightarrow [R^n f_* F_n \rightarrow \cdots \rightarrow R^n f_* F_0][-n]$$

is a quasi-isomorphism. Indeed, the derived functor spectral sequence shows that,

$$E_2^{p,q} = \mathcal{H}^p(R^q f_* F_\bullet) \implies \mathcal{H}^{p+q}(\mathbf{R}f_* F_\bullet)$$

but the only nonzero part of the E_2 page is the column (p, n) and thus

$$E^{p,n} = \mathcal{H}^p(R^n f_* F_\bullet) = \mathcal{H}^{p+n}(\mathbf{R}f_* F_\bullet)$$

showing that the natural map is a quasi-isomorphism. □

WHY CAN WE APPLY THIS TO A REPRESENTABLE MAP OF STACKS

Let \mathcal{E} be a vector bundle on $X \times \mathcal{S}$ then there exists a short exact sequence,

³Suppose that $f : X \rightarrow S$ is a morphism and F is a coherent \mathcal{O}_X -module. If F_x is flat over $\mathcal{O}_{S,s}$ then F is locally free at x if and only if F_s is locally free at $x \in X_s$. Recall that F is locally free at x if and only if F_x is free over $\mathcal{O}_{X,x}$ since F is coherent. Then if $(F_s)_x = F_x/\mathfrak{m}_s F_x$ is free we lift the basis to

$$0 \rightarrow K \rightarrow \mathcal{O}_X x^{\oplus r} \rightarrow F_x \rightarrow 0$$

but F_x is flat over $\mathcal{O}_{S,s}$ so this stays exact when applying $-\otimes \mathcal{O}_{S,s}/\mathfrak{m}_s$ so we see that $K/\mathfrak{m}_s K = 0$ and hence $K/\mathfrak{m}_x K = 0$ so by Nakayama's lemma $K = 0$.

$$0 \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E} \longrightarrow 0$$

such that $R^0 p_* \mathcal{E}^{j-1} = 0$ and $K^j = \mathbf{R}^1 \mathcal{E}^{j-1}$ is locally free for $j = 0, 1$ (Note: I think there is a typo in the indices in the notes) therefore there is a quasi-isomorphism,

$$[K^0 \rightarrow K^1] \rightarrow \mathbf{R}p_* \mathcal{E}$$

Definition 3.0.2. If \mathcal{E} is a vector bundle on $X \times \mathcal{S}$ and $[K^0 \rightarrow K^1]$ is a two-term complex of locally free sheaves quasi-isomorphic to $\mathbf{R}p_* \mathcal{E}$ then we define the line bundle,

$$\det \mathbf{R}p_* \mathcal{E} := \det K^0 \otimes (\det K^1)^\vee$$

We could also make this definition for any perfect complex on \mathcal{S} . The rank of a perfect complex is defined as the alternating sum of the ranks of a representative. This is independent of the choice of representative because localizing at the generic point reduces to the case of finite dimensional vectorspaces for which Euler characteristic of a bounded complex coincides with the alternating sum of dimensions. If $\text{rank}(\mathbf{R}p_* \mathcal{E}) = 0$ then by definition $\text{rank} K^0 = \text{rank} K^1$ so the dual $(\det \mathbf{R}p_* \mathcal{E})^\vee$ is then equipped with a section given by the determinant of the map $K^0 \rightarrow K^1$.

Lemma 3.0.3. The definition of $\det \mathbf{R}p_* \mathcal{E}$ is independent of the choice of representative perfect complex.

Lemma 3.0.4. Given an exact sequence,

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

then,

$$\det \mathbf{R}p_* \mathcal{E} = (\det \mathbf{R}p_* \mathcal{E}') \otimes (\det \mathbf{R}p_* \mathcal{E}'')$$

Proof. We find a complex $\mathcal{E}^\bullet = [\mathcal{E}^{-1} \rightarrow \mathcal{E}^0]$ and $\mathcal{E}^\bullet \rightarrow \mathcal{E}$ a quasi-isomorphism as above. Choose \mathcal{E}'^\bullet and \mathcal{E}''^\bullet similarly. In fact, it follows from the construction that these resolutions may be chosen compatibly so that there is a short exact sequence of complexes,

$$0 \longrightarrow \mathcal{E}'^\bullet \longrightarrow \mathcal{E}^\bullet \longrightarrow \mathcal{E}''^\bullet \longrightarrow 0$$

compatible with the quasi-isomorphisms to the previous sequence. Taking cohomology, we find a short exact sequence

$$0 \longrightarrow K'^\bullet \longrightarrow K^\bullet \longrightarrow K''^\bullet \longrightarrow 0$$

of complex of locally free sheaves on \mathcal{S} . The result follows from the multiplicativity of determinants in short exact sequences of locally free sheaves,

$$\det K^0 = \det K'^0 \otimes \det K''^0 \quad \det K^1 = \det K'^1 \otimes \det K''^1$$

and therefore,

$$\det K^\bullet := \det K^0 \rightarrow (\det K^1)^\vee = \det K'^0 \rightarrow \det K''^0 \otimes (\det K'^1 \otimes \det K''^1)^\vee = \det K'^\bullet \otimes (\det K''^\bullet)^\vee$$

□

We will apply this construction to the case $\mathcal{S} = \mathcal{M}_X(r, d)$ and $\mathcal{E} = \mathcal{E}_{\text{univ}} \otimes q^* V$ where V is a vector bundle on X .

Definition 3.0.5. For a vector bundle V on X we define the *determinantal line bundle*

$$\mathcal{L}_V := (\det \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V))^\vee$$

on $\mathcal{M}_X(r, d)$ associated to V . If $\chi(X, E \otimes V) = 0$ for all $[E] \in \mathcal{M}_X(r, d)$ then the rank of $\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V)$ is zero and we define the section

$$s_V \in \Gamma(\mathcal{M}_X(r, d), \mathcal{L}_V)$$

Remark. Since $\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V)$ is perfect, its construction commutes with base change. In particular, its restriction to a k -point $[E] \in \mathcal{M}_X(r, d)$ is identified with the two-term complex $\mathbf{R}\Gamma(X, \mathcal{E} \otimes V)$. If moreover

$$\chi(X, E \otimes V) = h^0(X, E \otimes V) - h^1(X, E \otimes V) = 0$$

we see that indeed $\text{rank } \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V) = 0$.

Remark. Note that,

$$\deg E \otimes V = (\deg E)(\text{rank } V) + (\text{rank } E)(\deg V)$$

and therefore by Riemann-Roch $\chi(X, E \otimes V) = 0$ if and only if,

$$d \text{rank } V + r \deg V + (1 - g)r \text{rank } V = 0$$

or equivalently,

$$(*) \quad \mu(E) \text{rank } V + \deg V + (1 - g) \text{rank } V = 0$$

Notice that this is actually just a numerical condition on V that only depends on the slope of E . Therefore, if $(*)$ holds then $\chi(X, E \otimes V) = 0$ for all vector bundles E with fixed slope $\mu(E)$.

The construction of the determinantal line bundle defines a morphism,

$$\det : \mathcal{M}_X(r, d) \rightarrow \mathcal{P}ic_X^d$$

such that,

$$(\text{id}_X \times \det)^* \mathcal{P} = p^* \det \mathcal{E}_{\text{univ}}$$

where \mathcal{P} is the Poincare bundle on $\mathcal{P}ic_X^d \times X$.

Proposition 3.0.6. The following hold:

- (a) the assignment $V \mapsto \mathcal{L}_V$ induces a group homomorphism,

$$K_0(X) \rightarrow \text{Pic}(\mathcal{M}_X(r, d))$$

meaning the isomorphism class of \mathcal{L}_V depends only on $\text{rank } V$ and $\det V$

- (b) if V and W are vector bundles of the same rank and degree then there exists a line bundle \mathcal{N} on $\mathcal{P}ic_X^d$ such that,

$$\mathcal{L}_W \cong \mathcal{L}_V \otimes \det^* \mathcal{N}$$

where \det^* is the pullback along the map $\det : \mathcal{M}_X(r, d) \rightarrow \mathcal{P}ic_X^d$.

Proof. For an exact sequence of vector bundles on X ,

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

we get an exact sequence using flatness of q and the fact that $\mathcal{E}_{\text{univ}}$ is a vector bundle,

$$0 \longrightarrow q^*V_1 \otimes \mathcal{E}_{\text{univ}} \longrightarrow q^*V_2 \otimes \mathcal{E}_{\text{univ}} \longrightarrow q^*V_3 \otimes \mathcal{E}_{\text{univ}} \longrightarrow 0$$

which thus induces an isomorphism,

$$\mathcal{L}_{V_2} \cong \mathcal{L}_{V_1} \otimes \mathcal{L}_{V_3}$$

proving the factorization of the map of monoids through $K_0(X)$,

$$\begin{array}{ccc} \text{Vect}_X & \xrightarrow{\quad} & \text{Pic}(\mathcal{M}_X(r, d)) \\ & \searrow & \nearrow \text{dashed} \\ & K_0(X) & \end{array}$$

Then the consequence follows from the isomorphism $K_0(X) = \mathbb{Z} \oplus \text{Pic}(X)$ given by $V \mapsto (\text{rank } V, \det V)$.

To prove the second statement, using the first we may assume that $V = \mathcal{O}_X^{\oplus r_V-1} \oplus \mathcal{O}_X(D)$ for some divisor D on X and similarly for W . Moreover, writing $D = D_1 - D_2$ as a difference of effective divisors we see that,

$$[\mathcal{O}_X(D)] = [\mathcal{O}_X] + [\mathcal{O}_{D_1}] - [\mathcal{O}_{D_2}]$$

in $K_0(X)$. Therefore, the classes of V and W in $K_0(X)$ differ only by the class of a divisor of degree 0 **DOESNT THE PROOF WORK FOR DIVISORS OF ANY DEGREE?**. Thus by the additivity of the derminantal construction, it suffices to prove that,

$$\det \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*\mathcal{O}_x) \cong \det^* \mathcal{N}'$$

for some line bundle \mathcal{N}' on $\mathcal{P}ic_X^d$ where $x \in X$ is a closed point. Viewing \mathcal{E}_x and \mathcal{P}_x as sheaves on $\mathcal{M}_X(r, d)$ and $\mathcal{P}ic_X^d$ respectively as pullback along the sections $\mathcal{S} \rightarrow X \times \mathcal{S}$ defined by $x \in X$ then,

$$\det \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*\mathcal{O}_x) = \det \mathcal{E}_x = \det^* \mathcal{P}_x$$

which proves the claim. \square

Definition 3.0.7. Let E be a vector bundle on X . We say that E is *cohomology-free* if $h^0(X, E) = h^1(X, E) = 0$.

Proposition 3.0.8. If $(*)$ holds for V then the following are equivalent,

- (a) the section $s_V \in \Gamma(\mathcal{M}_X(r, d), \mathcal{L}_V)$ is nonzero at $[E]$
- (b) $E \otimes V$ is cohomology-free

Proof. The morphism $\det j : \det K_0 \rightarrow \det K_1$ of line bundles is nonzero at the point $[E] \in \mathcal{M}_X(r, d)$ if and only if the morphism of vector bundles $j : K - 0 \rightarrow K_1$ is an isomorphism at $[E]$. Since $\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V)$ is quasi-isomorphic to $[K_0 \rightarrow K_1]$ we see that j is an isomorphism if and only if $(\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V))_{[E]} = 0$ if and only if $h^0(X, E \otimes V) = h^1(X, E \otimes V) = 0$ by cohomology and base change. \square

Remark. While \mathcal{L}_V only depends on rank V and $\det V$ the section s_V *does* depend on V itself since it can detect cohomology-freeness as above. We will leverage this fact to produce enough sections of \mathcal{L}_V to establish ampleness. Notice also that, under the assumption that $\chi(X, E \otimes V) = 0$ the vanishing of $H^0(X, E \otimes V)$ is equivalent to the vanishing of $H^1(X, E \otimes V)$.

Now we specialize this discussion to the open substack $\mathcal{M}_X^{ss}(r, d) \subset \mathcal{M}_X(r, d)$. The goal is to prove that the good mouli space $M_X^{ss}(r, d)$ is projective. Our candidate ample line bundle is the descent of some \mathcal{L}_V .

Proposition 3.0.9. The determinantal line bundle \mathcal{L}_V associated to a vector bundle V satisfying $(*)$ descends to $M_X^{ss}(r, d)$ uniquely meaning there exists a unique line bundle $L_V \in \text{Pic}(M_X^{ss}(r, d))$ such that $\mathcal{L}_V \cong \phi^* L_V$.

Proof. By Theorem 3.5(iv), we must show that stabilizers of $\mathcal{M}_X^{ss}(r, d)$ act trivially on the fibers of \mathcal{L}_V . By Theorem 3.12(ii), the closed points of $\mathcal{M}_X^{ss}(r, d)$ correspond to polystable bundles,

$$E = \bigoplus_{j=1}^n E_j^{\oplus m_j}$$

where the E_i are pairwise nonisomorphic stable bundles. Since E is also semistable the slopes of the E_i must all be equal so $\mu(E_i) = \mu(E) = \frac{d}{r}$. Since these are nonisomorphic stable bundles with the same slope there are no nonzero morphisms between them (we have $\text{Hom}(E_i, E_j) = k \cdot \delta_{ij}$). Since $\text{End}(E_j) = k$ we see that,

$$\text{Aut}(E) = \text{GL}_{m_1} \times \cdots \times \text{GL}_{m_n}$$

The fiber of $\mathcal{L}_V|_{[E]}$ is identified with

$$\det \mathbf{R}\Gamma(X, E \otimes V) = \prod_{i=1}^n (\det H^i(X, E \otimes V))^{\otimes (-1)^i}$$

An element $(g_1, \dots, g_n) \in \text{Aut}(E)$ act on,

$$\det H^i(X, E \otimes V) \cong \bigotimes_{j=1}^n (\det H^i(X, E_j \otimes V))^{\otimes m_j}$$

by multiplication with,

$$\prod_{j=1}^n \det(g_j)^{\dim H^i(X, E_j \otimes V)}$$

and thus on $\mathcal{L}_V|_{[E]}$ by multiplication with,

$$\prod_{j=1}^n \det(g_j)^{\chi(X, E_j \otimes V)}$$

but each E_j has slope $\mu(E_j) = \frac{d}{r}$ so by $(*)$ we have $\chi(X, E_j \otimes V) = 0$ since it implies vanishing for all E with $\mu(E) = \frac{d}{r}$ and hence the action is trivial. \square

3.0.1 Moduli of Vector Bundles with Fixed Determinant

Let L be a line bundle on X of degree d corresponding to a closed point of $\mathcal{P}ic_X^d$ represented by a morphism $[L] : \text{Spec}(k) \rightarrow \mathcal{P}ic_X^d$. **IS EVERY k -point closed? in Pic ? I Think so since X/k is SMOOTH so we should have separatedness.** Then consider the diagram,

$$\begin{array}{ccc} \mathcal{M}_X(r, L) & \longrightarrow & \mathcal{M}_X(r, d) \\ \downarrow & \lrcorner & \downarrow \det^* \\ \text{Spec}(k) & \xrightarrow{L} & \mathcal{P}ic_X^d \end{array}$$

Explicitly, $\mathcal{M}_X(r, L)$ is the stack of pairs (\mathcal{E}, φ) where \mathcal{E} is a vector bundle on $X \times S$ of rank r and degree d fiberwise equipped with an isomorphism $\varphi : \det \mathcal{E} \xrightarrow{\sim} L|_{X \times S}$. The condition of constant degree d on fibers is implied by determinant isomorphism which is why d is left out of the notation.

Corollary 3.0.10. For a line bundle L of degree d on X , the restriction of the determinantal line bundle L_V to $M_X^{ss}(r, L)$ only depends on the rank and degree of V .

Proof. We have commuting diagrams,

$$\begin{array}{ccc} \mathcal{M}_X^{ss}(r, L) & \longrightarrow & \mathcal{M}_X^{ss}(r, d) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{L} & \mathrm{Pic}_X^d \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{M}_X^{ss}(r, L) & \hookrightarrow & \mathcal{M}_X^{ss}(r, d) \\ \downarrow & & \downarrow \\ M_X^{ss}(r, L) & \hookrightarrow & M_X^{ss}(r, d) \end{array}$$

where in the second square, both vertical arrows are good moduli space morphisms. If V and W are vector bundles of equal rank and degree on X , both satisfying condition (10) then there exists a line bundle \mathcal{N} on Pic_X^d such that $\mathcal{L}_W \cong \mathcal{L}_V \otimes \det^* \mathcal{N}$. The left diagram shows that, restriction to $\mathcal{M}_X^{ss}(r, L)$, this isomorphism becomes $\mathcal{L}_W \cong \mathcal{L}_V$. The right diagram now shows that the restriction of L_V and L_W to $M_X^{ss}(r, L)$ become isomorphic after pulling back to $\mathcal{M}_X^{ss}(r, L)$, so the restrictions must be isomorphic by the uniqueness of the descent along the good moduli space morphism $\mathcal{M}_X^{ss}(r, L) \rightarrow M_X^{ss}(r, L)$. \square

Theorem 3.0.11 (Drézet-Narasimhan). There exist isomorphisms,

$$\begin{aligned} \mathrm{Pic}(M_X^{ss}(r, d)) &\cong \mathrm{Pic}(\mathrm{Pic}_X^d) \oplus \mathbb{Z} \\ \mathrm{Pic}(M_X^{ss}(r, L)) &\cong \mathbb{Z} \end{aligned}$$

In the first line, \mathbb{Z} is **WHAT IS IT GENERATED BY?** In the second line, \mathbb{Z} is generated by the determinantal line bundle L_V where V is chosen to be of minimal rank,