1 Definitions

Definition 1.0.1. An *algebraic space* is a functor $X : (\mathbf{Sch}_S)^{\text{op}}_{\text{fppf}} \to \text{Set}$ such that,

(a) F is a sheaf in the fppf topology

(b) the diagonal $\Delta_{X/S} : X \to X \times_S X$ is representable by schemes

(c) there is a scheme U and an étale surjection $U \twoheadrightarrow X$.

Definition 1.0.2. An *algebraic stack* is a category fibered in groupoids $\mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$ such that,

(a) $\mathcal{X}$ is a stack in the fppf topology

(b) $\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces

(c) there is an algebraic space U and an étale surjection $U \twoheadrightarrow \mathcal{X}$.

Remark. The map $U \to \mathcal{X}$ is only necessarily representable by algebraic spaces so to express the property of being an étale surjection consider any map from a scheme $T \to \mathcal{X}$ and an étale cover from a scheme $V \to U \times_{\mathcal{X}} T$ in the diagram,

$$\begin{tikzcd}
V \arrow[r] \arrow[rr, bend left, "\text{ét surj}"] & U \times_{\mathcal{X}} T \arrow[r] \arrow[d] & T \arrow[d] \\
& U \arrow[r] & \mathcal{X}
\end{tikzcd}$$

This property is independent of the choice of étale cover $V \to U \times_{\mathcal{X}} T$ by étale descent for étale surjective morphisms.

Remark. Why do we only require that $\mathcal{X}$ be smooth locally an algebraic space and its diagonal be representable by only algebraic spaces? The diagonal is closely related to the automorphism groups of objects $\mathcal{X}$ parametrizes.

(PRODUCTS OF STACKS)
(INERTIA)
(STABILIZERS)

2 Presentations

Proposition 2.0.1. Let X be an algebraic space over S and $f : U \twoheadrightarrow X$ an étale surjection from a scheme U. Set $R = U \times_X U$ in the pullback diagram,

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & U \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

then we have,

(a) $j : R \to U \times_S U$ is a monomorphism and $R(T) \subset U(T) \times U(T)$ is an equivalence relation for all $T \to S$

(b) the projections $s, t : R \to U$ are étale

(c) the diagram,

$$R \overset{s}{\underset{t}{\rightrightarrows}} U \longrightarrow X$$

is a coequalizer in $\mathrm{Sh}((\mathbf{Sch}_S)_{\mathrm{fppf}})$.

Proof. The first two are immediate

The last follows in any category of sheaves given that $U \to X$ is surjective. ☐

(PRESENTATIONS OF STACKS)

3 Examples

(BG) (MG) (AG)

4 Infinitesimal Deformation Theory

Remark. First we recall how to apply infinitessimal deformation theory in the relative setting. In the basic case, we want to probe properties of a morphisms of schemes $f : X \to S$ near a finite type point $x : \mathrm{Spec}\,(k) \to S$. There is some affine open $\mathrm{Spec}\,(\Lambda) \subset X$ containing x. Then we need to consider Artinian local rings A and diagrams,

$$\begin{tikzcd} & & & X \arrow[d, "f"] \\ \mathrm{Spec}\,(k) \arrow[r] & \mathrm{Spec}\,(A) \arrow[r] \arrow[urr, dashed] & \mathrm{Spec}\,(\Lambda) \arrow[r, hook] & S \end{tikzcd}$$

and consider the set of dashed arrows. This means our base category should be the category of local Artinian Λ-algebras with residue field k.

Definition 4.0.1. Let Λ be a Noetherian ring and $\Lambda \to k$ a finite ring map with k a field. Let $\mathcal{C}_\Lambda$ be the category of,

(a) (A, φ) where A is an Artinian local Λ-algebra and $\varphi : A/\mathfrak{m}_A \to k$ a Λ-algebra isomorphism

(b) morphisms $f : (B, \psi) \to (A, \varphi)$ are local Λ-algebra maps such that $\varphi \circ (f \bmod \mathfrak{m}_A) = \psi$

Remark. As in the absolute case (which corresponds to $\Lambda = k$) we can factor any extension $B \twoheadrightarrow A$ into *small* extensions $\varphi : B' \twoheadrightarrow A$ where $\ker \varphi$ is principal and annihilated by $\mathfrak{m}_B$.

Definition 4.0.2. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. Define the category $\widehat{\mathcal{C}}_\Lambda$ of,

(a) pairs (R, φ) where R is a Noetherian complete local Λ-algebra and $\varphi : R/\mathfrak{m}_R \to k$ is a Λ-algebra isomorphism,

(b) morphisms $f : (S, \psi) \to (R, \varphi)$ are local Λ-algebra map such that $\varphi \circ (f \bmod \mathfrak{m}_S) = \psi$.

Remark. Then $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is naturally a full subcategory.

4.1 Deformation Functors

Definition 4.1.1. A *predeformation functor* is a functor $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ such that $F(k) = \{*\}$.

Remark. The condition $F(k) = \{*\}$ corresponds to choosing a fixed base object for the deformations.

Definition 4.1.2. Given a predeformation functor $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ we extend it to $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to \mathrm{Set}$ via,

$$\widehat{F}(R) = \varprojlim_n F(R/\mathfrak{m}_R^n)$$

A functor F is *pro-representable* if $\widehat{F}$ is representable.

Definition 4.1.3. Let $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ be a predeformation functor. A *hull*[1] for F is a pair (R, η) where $R \in \widehat{\mathcal{C}}_\Lambda$ and $\eta \in \widehat{F}(R)$ such that $h_R \to F$ is formally smooth and bijective on tangent spaces.

Remark. Let $k[\epsilon]$ be the ring $k[\epsilon]/(\epsilon^2)$ with the trivial Λ-algebra structure.

Definition 4.1.4. Let $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ be a predeformation functor. If $A' \to A$ and $A'' \to A$ are morphisms in $\mathcal{C}_\Lambda$ there is a natural map,

$$(*) \quad F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

Then Schlessinger's conditions are as follows,

(H1) if $A'' \twoheadrightarrow A$ is a small thickening then $(*)$ is surjective

(H2) if $A = k$ and $A'' = k[\epsilon]$ then $(*)$ is bijective

(H3) $T_F = F(k[\epsilon])$ is finite-dimensional

(H4) if $A'' = A'$ and $A' \to A$ is a small thickening, then $(*)$ is bijective.

Definition 4.1.5. A predeformation functor $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ is a *deformation functor* if it satisfies (H1) and (H2).

Theorem 4.1.6 (Schlessinger)**.** Let $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ be a deformation functor. Then,

(a) F admits a hull if and only if it satisfies (H3)

(b) F is prorepresentable if and only if it satisfies (H3) and (H4).

Example 4.1.7. Let X be a scheme, the functor,

$$\mathrm{Def}_X : A \mapsto \{(X', \varphi) \mid X' \text{ flat } A\text{-scheme with } \varphi : X' \otimes_A k' \xrightarrow{\sim} X\}/\cong$$

is a deformation functor.

Example 4.1.8. Let $X = \mathrm{Spec}\,(k[x,y]/(xy))$ and $F = \mathrm{Def}_X$. If A is a finite type k-algebra and $P \twoheadrightarrow A$ is a presentation from a polynomial ring with kernel K then [H, Ex. 9.8] shows that,

[1]Some authors use the terminology *miniversal* formal object. However, in the deformation category setting, a minimal versal object may not induce an isomorphism of the tangent space so we reserve the term *miniversal* for a minimal versal object see Tag 06T0.

$$\mathrm{Hom}_A\left(\Omega_{P/k} \otimes_k A, A\right) \longrightarrow \mathrm{Hom}_A\left(J/J^2, A\right) \longrightarrow T_{\mathrm{Def}_A} \longrightarrow 0$$

arising from the conormal exact sequence,

$$J/J^2 \longrightarrow \Omega_{P/k} \otimes_P A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

In our case, let $P = k[x, y]$ and $J = (xy)$. Then we have,

$$A\partial_x \oplus A\partial_y \longrightarrow A \longrightarrow T_{\mathrm{Def}_A} \longrightarrow 0$$

and therefore $T_{\mathrm{Def}_A} = A/(x, y) = k$. Thus Def_X satisfies (H1) - (H3) so it should have a hull. Indeed,

$$(k[[t]], \mathrm{Spf}(k[[t]][x, y]/(xy - t)))$$

is a hull (note the formal object is effective). Let's first understand why this hull is not a pro-representing object. For any map, $\varphi : k[[t]] \to A$ the induced object,

$$\varphi_*(\mathrm{Spf}(k[[t]][x, y]/(xy - t))) = \mathrm{Spec}\left(A[x, y]/(xy - \varphi(t))\right)$$

is unchanged (in isomorphism class) if we replace φ buy $\varphi' = u\varphi$ for any unit $u \in A$ since then we can scale x or y to remove u. However, recall that a deformation X' is equipped with a distinguished isomorphism $\varphi : X' \otimes_A k \xrightarrow{\sim} X$ with which isomorphisms of deformations must be compatible. Therefore, $\varphi' = u\varphi$ and φ define the same deformation if $u \in A^\times$ is a unit and $u \equiv 1 \mod \mathfrak{m}_A$. Therefore, the map, $h_R \to \mathrm{Def}_X$ is not injective for general A but is injective for $A = k[\epsilon]$ (since $(1 + a\epsilon) \cdot \epsilon = \epsilon$ so multiplication by such a does nothing) as must be true for a hull.

However Def_X is not pro-representable since it does not satisfy (H4). Indeed, consider $A = k[\epsilon]/(\epsilon^3)$ and consider,

$$\mathrm{Def}_X(A \times_k A) \to \mathrm{Def}_X(A) \times \mathrm{Def}_X(A)$$

I claim this is not injective. Indeed, $t = \epsilon_1 + \epsilon_2$ and $t = \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$ map to the same pair of deformations but I claim they are not related by such a unit. Write,

$$u = 1 + a\epsilon_1 + b\epsilon_2 + O(\epsilon^2)$$

then,

$$u(\epsilon_1 + \epsilon_2) = \epsilon_1 + \epsilon_2 + a\epsilon_1^2 + (a + b)\epsilon_1\epsilon_2 + b\epsilon_2^2 + O(\epsilon^3)$$

and we cannot have $a = b = 0$ but $a + b = 1$.

Remark. The above illustrates why it is necessary to define deformations of a scheme as equipped with a distinguished isomorphism $\varphi : X' \otimes_A l \xrightarrow{\sim} X$ otherwise Def_X will not be a deformation functor.

4.2 Deformation Categories

Definition 4.2.1. A *predeformation category* is a category cofibered in groupoids $\mathcal{F} \to \mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to the trivial category.

Remark. Let $\mathcal{F}$ be a predeformation category and $x_0 \in \mathcal{F}(k)$. Then for any $x \in \mathcal{F}$ over A let $q : A \to k$ then there is a pushforward $x \to q_*x$ and $q_*x \in \mathcal{F}(k)$ so there is a unique isomorphism $q_*x \xrightarrow{\sim} x_0$ and hence there is a canonical morphism $x \to x_0$ in $\mathcal{F}$.

Remark. If $F : \mathcal{C}_\Lambda \to \text{Set}$ is a predeformation functor then the associated cofibered set $\mathcal{F}_F \to \mathcal{C}_\Lambda$ is a predeformation category. Likewise, if $\mathcal{F} \to \mathcal{C}_\Lambda$ is a predeformation category then the functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \text{Set}$ is a predeformation functor.

Definition 4.2.2. Let $\mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The *category of formal objects of* $\widehat{\mathcal{F}}$ is the category of,

(a) formal objects (R, ξ_n, f_n) consists of an object $R \in \widehat{\mathcal{C}}_\Lambda$, and objects $\xi_n \in \mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ over the projection $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$

(b) morphisms $a : (R, \xi_n, f_n) \to (S, \eta_n, g_n)$ consists of a map $a_0 : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \to \eta_n$ of morphisms in $\mathcal{F}$ lying over $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ such that the diagrams,

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"] \arrow[d, "a_{n+1}"] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commute for each $n \in \mathbb{N}$.

Proposition 4.2.3. The formal objects forms a category cofibered in groupoids $\hat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$.

Definition 4.2.4. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We say that $\mathcal{F}$ satisfies the *Rim-Schlessinger (RS) condition* if for all $A_1 \to A$ and $A_2 \to A$ in $\mathcal{C}_\Lambda$ with $A_2 \twoheadrightarrow A$ surjective,

$$\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence. A *deformation category* is a predeformation category $\mathcal{F}$ satisfying (RS).

Lemma 4.2.5. The RS condition is equivalent to: for every diagram in $\mathcal{F}$,

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, there exists a fiber product $x_1 \times_x x_2$ in $\mathcal{F}$ such that the diagram,

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

Lemma 4.2.6. Let $F : \mathcal{C}_\Lambda \to \text{Set}$ be a predeformation functor then if the associated predeformation category $\mathcal{F}_F$ satsifies (RS) then F satisfies (H1), (H2), and (H4).

Remark. IS IT ACTUALLY EQUIVALENT e.g. https://stacks.math.columbia.edu/tag/06J1

Lemma 4.2.7. If $\mathcal{X} \to S$ is an algebraic stack then for any $\text{Spec}\,(k) \to S$ and $x_0 \in \mathcal{X}(k)$ the deformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (RS).

Remark. By Schlessinger's theorem, this is telling us that a deformation functor $F = D_{X,x_0}$ represented by some pointed algebraic space $x_0 \in X$ is pro-representable. So even though X does not have a canonical local ring it does have a formal local ring $\widehat{\mathcal{O}_{X,x_0}}$. We can calculate it from the formal local ring of any étale cover $U \to X$. This is well-defined because for two étale covers $U_1 \to X$ and $U_2 \to X$ we have $U_1 \times_X U_2$ is an étale cover of both and these maps identify the formal local rings.

4.3 Versality

Remark. A versal object is a universal object without the "uni" i.e. without the uniqueness.

Definition 4.3.1. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of categories cofibered in groupoids over $\mathcal{C}_\Lambda$ is *smooth* if for every extension $B \twoheadrightarrow A$ in $\mathcal{C}_\Lambda$ the map,

$$\mathcal{F}(B) \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{F}(B)$$

is essentially surjective.

Remark. This is basically the formal lifting criterion for formal smoothness. Indeed, if these deformation categories are induced by the representable functors for a morphism of schemes $f : X \to Y$ then we get that,

$$X(B) \to X(A) \times_{Y(A)} Y(B)$$

is surjective which is equivalent to there existing a dashed arrow in each lifting diagram,

$$\begin{tikzcd} \mathrm{Spec}\,(A) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathrm{Spec}\,(B) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Lemma 4.3.2. Smoothness of $\varphi : \mathcal{F} \to \mathcal{G}$ is equivalent to the following explicit condition. For every surjection $B \twoheadrightarrow A$ in $\mathcal{C}_\Lambda$ and $y \in \mathcal{G}(B)$ and $x \in \mathcal{F}(A)$ equipped with a map $y \to \varphi(x)$ over $B \twoheadrightarrow A$ there is $x' \in \mathcal{F}(B)$ and a morphism $x' \to x$ over $B \twoheadrightarrow A$ and a morphism $\varphi(x') \to y$ over $\mathrm{id} : B \to V$ such that,

$$\begin{tikzcd} \varphi(x') \arrow[r] \arrow[dr] & y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

Definition 4.3.3. Let $R \in \widehat{\mathcal{C}}_\Lambda$. We say $\xi \in \widehat{\mathcal{F}}(R)$ is *versal* if the morphism $\xi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ defined by ξ is smooth.

Remark. The morphism is defined as follows. For any $A \in \mathcal{C}_\Lambda$ and map $\varphi : R \to A$ it will factor as $\varphi_n : R/\mathfrak{m}^n \to A$ we send $(A, \varphi) \mapsto (\varphi_n)_* \xi_n$. The compatibility isomorphisms of the formal object ξ make this well-defined.

Remark. Let ξ be a formal object of $\mathcal{F}$. Versality of ξ is equivalent to: the existence of a dashed arrow for any diagram,

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur, dashed] & x \end{tikzcd}$$

in $\widehat{\mathcal{F}}$ such that $y \to x$ lies over a surjective map $B \twoheadrightarrow A$ of Artinian rings.

Theorem 4.3.4 (Rim-Schlessinger)**.** A deformation category $\mathcal{F}$ with $T\mathcal{F} = \overline{\mathcal{F}}(k[\epsilon])$ is finite dimensional admits a versal formal object.

(DO SOME EXAMPLES!!!)

Definition 4.3.5. Given a category fibered in groupoids,

$$p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$$

and
https://stacks.math.columbia.edu/tag/07T2
(DEF DEF CAT AND VERSAL)

Definition 4.3.6. Let S be a locally noetherian scheme and $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$ a category fibered in groupoits. We say $\mathcal{X}$ satisfies *opennes of versality* if given a scheme U locally of finite type over S, an open $x \in \mathcal{X}(U)$, and a finite type point $u_0 \in Y$ such that x is versal at u_0 then there is exists an open neiborhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

(EXAMPLES)

4.4 Effectivity

Definition 4.4.1. We say a formal object $\xi = (R, \xi_n, f_n) \in \widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$ is *effective* if it arises from some $\tilde{\xi} \in \mathcal{X}(R)$.

Lemma 4.4.2. If $\mathcal{X}$ is an algebraic stack then every formal object is effective.

Proof. First, if X is a scheme then for all local rings R factoring $\text{Spec}(k) \to X$ the map corresponds to $\text{Spec}(R) \to \text{Spec}(\mathcal{O}_{X,x}) \to X$ so if R is complete,

$$X(R) = \text{Hom}_{\text{loc}}(\mathcal{O}_{X,x}, R) = \varprojlim_n \text{Hom}_{\text{loc}}(\mathcal{O}_{X,x}, R/\mathfrak{m}_R^n) = \varprojlim_n X(R/\mathfrak{m}_R^n)$$

Now in general, choose a smooth cover $\pi : U \to \mathcal{X}$ from a scheme.
https://stacks.math.columbia.edu/tag/07X3 □

5 Artin's Axioms

Theorem 5.0.1 (Artin)**.** Let S be a locally noetherian scheme and $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$ a category fibered in groupoids. Let R be a Noetherian complete local ring with redicue field k with $\text{Spec}(R) \to S$ finite type and $x \in \mathcal{X}(R)$. Let $s \in S$ be the image of $\text{Spec}(k) \to \text{Spec}(R) \to S$. Assume that,

(a) $\mathcal{O}_{S,s}$ is a G-ring

(b) p is limit-preserving on objects.

Then for every $N \geq 1$ there exist,

(a) a finite type S-algebra A

(b) a maximal ideal $\mathfrak{m}_A \subset A$

(c) an object $x_A \in \mathcal{X}(A)$

(d) an S-isomorphism $R/\mathfrak{m}_R^N \xrightarrow{\sim} A/\mathfrak{m}_A^N$

(e) an isomorphism $x|_{R/\mathfrak{m}_R^N} \xrightarrow{\sim} x_A|_{A/\mathfrak{m}_A^N}$ over the previous map

(f) an isomorphism $\mathbf{gr}_{\mathfrak{m}_R}(R) \xrightarrow{\sim} \mathbf{gr}_{\mathfrak{m}_A}(A)$ of graded k-algebras.

Lemma 5.0.2. Let S be a locally noetherian scheme and $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\mathrm{fppf}}$ a category fibered in groupoids. Let ξ be a formal object of $\mathcal{X}$ with $x_0 = \xi_1$ lying over $\mathrm{Spec}\,(k) \to S$ with image $s \in S$ such that,

(a) ξ is versal

(b) ξ is effective

(c) $\mathcal{O}_{S,s}$ is a G-ring

(d) $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\mathrm{fppf}}$ is limit-preserving

then there exists a finite type morphism $U \to S$, a finite type point $u_0 \in U$ with residue field k and $x \in \mathcal{X}(U)$ such that $x : U \to \mathcal{X}$ is versal at u_0 and $x|_{\mathrm{Spec}\left(\mathcal{O}_{U,u_0}\right)}$ induces ξ.

Proof. Choose an object $x_R \in \mathcal{X}(R)$ whose completion is ξ. Apply Artin approximation with $N = 2$ to obtain $A, \mathfrak{m}_A, x_A \in \mathcal{X}(A)$ apprxoimating ξ. Let η be the formal object completing $x_A|_{\mathrm{Spec}\left(\hat{A}\right)}$ (the completion of A at $\mathfrak{m}_A$). Then a lift for the diagram in $\widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$,

$$\begin{tikzcd} & \eta \arrow[d] \\ \xi \arrow[r] \arrow[ur, dashed] & \xi_2 = \eta_2 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & \hat{A} \arrow[d] \\ R \arrow[r] \arrow[ur, dashed] & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{tikzcd}$$

exists because ξ is versal. Since the map $R \to \hat{A}$ induces an isomorphism on tangent spaces and by construction $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ we conclude that $R \to \hat{A}$ is an isomorphism. Hence $\eta \cong \xi$ is versal so the map $x_A : \mathrm{Spec}\,(A) \to \mathcal{X}$ is versal at $\widehat{x_A|_{\mathrm{Spec}\left(\hat{A}\right)}} = \eta$. □

Theorem 5.0.3. Let S be a locally Noetherian base scheme and consider a category cofibered in groupoids,

$$p : \mathcal{X} \to (\mathbf{Sch}_S)_{\mathrm{fppf}}$$

For each finite type morphism $\mathrm{Spec}\,(k) \to S$ with k a field and $x_0 \in \mathcal{X}(\mathrm{Spec}\,(k))$ assume that,

(a) $\mathcal{X}$ is a stack for the étale topology

(b) $\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces

(c) $\mathcal{X}$ is limit preserving (preserves filtered colimits)

(d) $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS)

(e) $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}\mathcal{F}_{\mathcal{X},k,x_0}$ are finite dimensional for all k and all $x_0 \in \mathcal{F}(k)$

(f) every formal object of $\mathcal{X}$ is effective

(g) $\mathcal{X}$ satisfies opennes of versality

(h) $\mathcal{O}_{S,s}$ is a G-ring for all finite type points $s \in S$

(i) a set theoretic condition

then $\mathcal{X}$ is an algebraic stack.

Proof. It suffices to show that for each finite type $\mathrm{Spec}\,(k) \to S$ and $x_0 \in \mathcal{X}(k)$ there is a finite type morphism $U \to S$ and a smooth map $U \to \mathcal{X}$ such that there is a finite type point $u_0 : \mathrm{Spec}\,(k) \to U$ such that $x|_{u_0} \cong x_0$.

By Rim-Schelssinger $\mathcal{F}_{\mathcal{X},k,x_0}$ admits a versal formal object ξ which is then effective. Artin approximation allows us to approximate an effective formal object by a finite type object $U \to \mathcal{X}$ which is versal at $u_0 \in U$. By openness of versality, we can shrink U such that $U \to \mathcal{X}$ is versal at every finite type point.

Finally, prove that a representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of limit preserving categories fibered in groupoids which is smooth on deformation categories is smooth (Tag 07XX. Indeed, for $T \to \mathcal{Y}$ the condition says that $f : \mathcal{X}_T \to T$ is a formally smooth map of algebraic spaces[2] and the limit-preserving condition gives finitely presented. $\square$

(WHERE NEED INF(X) bounded?)
(EXAMPLES)

[2]There is a subtily here with changing fields that requires the full strength of (RS) where as proving that a versal object exists only requires (S1) and (S2) and finite-dimensionality of tangent spaces