1 Jan 12

1.1 Summary

Developed the basics of ∞-categories and spaces.

classical	∞-math
sets	spaces / ∞-groupoids
categories	∞-categories
sheaves	∞-sheaves
abelian categoires / triangulated	stable ∞-categories

the advantage is that stability for ∞-categories is a *property* not a structure. Likewise, for algebraic gadgets,

classical	∞-math
monoids	E_1-spaces (equivalently A_∞)
groups	(grouplike) E_1-spaces
abelian groups	E_∞-spaces (equivalently connective spectra)
associative rings	(connective) E_1-ring spectra
commutative ring	(connective) E_∞-ring spectra
schemes	derived / spectral schemes

1.2 Homotopy theories of simplicial sets

Given a simplical set S we can think of S as either,

(a) modeling space $|S|$ = Quillen model structure

(b) modeling an ∞-category = Joyal model structure

and these two representations come with different notions of equivalence.

Definition 1.2.1. $S \to T$ is a

(a) *weak equivalence* when $|S| \to |T|$ is a homotopy equivalence

(b) *categorical equivalence* when $|\mathcal{C}[S]| \to |\mathcal{C}[T]|$ is an equivalence of

Proposition 1.2.2. If $\mathcal{C}$ is an ∞-category and all morphisms are equivalences (or equivalently every morphism is invertible in $h\mathcal{C}$) then $\mathcal{C}$ Kan complex.

2 Fibrations

(a) trivial Kan fibrations: have right lifting property w.r.t. $\partial\Delta^n \to \Delta^n$ this means that all fibers are contractible Kan complexes. These are both weak and categorical equivalences.

(b) Kan fibrations: have right lifting property w.r.t Horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ this means all fibers are Kan complexes and if $(s \to t)$ is an edge of S then we get a map $X_s \to X_t$ defined up to coherent homotopies and these are thus homotopy equivalences.

(c) left (right) fibrations have right lifting properties w.r.t $\Lambda^n_k \hookrightarrow \Delta^n$ for $0 \le k < n$ ($0 < k \le n$). This has all fibers Kan complexes (hard) and for each edge $(s \to t) \in S$ we have a map $X_s \to X_t$ (for a right fibration we get a map $X_t \to X_s$).

(d) inner fibrations have right lifting property w.r.t $\Lambda^n_k \hookrightarrow \Delta^n$ for $0 < k < n$. These have all fibers ∞-categories and for each edge $(s \to t) \in S$ there is a correspondence between X_s and X_t given by a functor $X_s^{\mathrm{op}} \times X_t \to \mathcal{S}$ indeed this is given by filling the degenerate Horn given by $(s \to t)$. Given a triangle $(t, s, r) \in S$ there is a morphism,

$$M_{t \to s} \circ M_{s \to r} \to M_{t\ tor}$$

this is "lax functor" $S \to (\infty, 1)$ the $(\infty, 2)$-category of $(\infty, 1)$-categories

(e) cocartesian fibrations (the thing that should correspond to $\mathrm{Fun}(\mathcal{C}, \mathrm{QCat})$. Indeed, functors are special correspondences: given $F : \mathcal{C} \to \mathcal{D}$ consider the correspondence $M(C, D) = \mathrm{Hom}_D(F(C), D)$. So coCartesian fibrations are inner fibrations satisfiny the properties making their correspondences arise from functors.

Lemma 2.0.1 (Joyal)**.** For K a simplicial set and $\iota : \Lambda^n_k \times K \hookrightarrow \Delta^n \times K$ for $0 \le k < n$ this is left-anodyne menaing left fibrations have the right lifting property with respect to ι^n_k.

Theorem 2.0.2 (Straightening-Unstraightening)**.** If $\mathcal{C}$ is an ∞-category then there is an equivalene of ∞-categories,

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}) \xrightarrow{\sim} \mathrm{LFib}(\mathcal{C})$$

and likewise,

$$\mathrm{Fun}(\mathcal{C}, \mathcal{S}^{\simeq}) \xrightarrow{\sim} \mathrm{KanFib}(\mathcal{C})$$

where $\mathcal{S}^{\simeq}$ is the subcategory of equivalences.

Definition 2.0.3. Let $p : X \to S$ be an inner fibration. An edge $f : x \to y$ in X is *cocartesian* if $X_{f/} \to X_{x/} \times_{S_{p(x)/}} S_{f/}$ (which is always a left fibration is a trivial Kan fibration

Proposition 2.0.4. Let $p : X \to S$ be an inner fibration. If S is an ∞-category, so is X. Then $f : x \to y$ is cocarteisan if and only if for every 2-simplex $p(z) \to p(x) \to p(y)$ in S with a lfit $z \in X$ the map,

$$\mathrm{Hom}_X(z, X) \to \mathrm{Hom}_X(z, y)$$

is a weak equivalence where we restrict to maps lying over the triangle in S.

Definition 2.0.5. A *cocartesian fibration* is an inner fibration $p : X \to S$ s.t. for every edge $x \to y$ in S and lift $\widetilde{x}$ of x there exists a cocartesian lift $\widetilde{f} : \widetilde{x} \to \widetilde{y}$ of f.

Theorem 2.0.6 (Straightening-Unstraightening)**.** If $\mathcal{C}$ is an ∞-category there is an equivalence,

$$\mathrm{Fun}(\mathcal{C}, \mathrm{QCat}) \xrightarrow{\sim} \mathrm{Cocart}(\mathcal{C})$$

3 Misc

$A = k[x, y]/(x^2)$. Then $\Omega_A = (A\mathrm{d}x \oplus A\mathrm{d}y)/(2x\mathrm{d}x)$. We have $J/J^2 \to \Omega_P \otimes_P A$ given by $J/J^2 \cong A$ and the map sends $1 \mapsto 2x\mathrm{d}x$. Then we consider,

$$\mathrm{Hom}_A(\Omega_P \otimes A, A) \to \mathrm{Hom}_A\left(J/J^2, A\right)$$

sends a map $\varphi \mapsto \varphi(2x\mathrm{d}x) = 2x\varphi(\mathrm{d}x)$. The $\varphi(\mathrm{d}x)$ can be arbitrary since $\Omega_P \otimes A$ is free. Therefore, the cokernel is $T^1(A) = A/(2x) = k[y]$. This is enormous. We need to do this for $\mathbb{P}^1_{k[\epsilon]}$. Likewise the kernel $T^0(A) = A\frac{\partial}{\partial y}$ since we send $\varphi \mapsto 2x\varphi(\mathrm{d}x)$ for this to vanish we need to it be zero on $\mathrm{d}x$. Now we globalize these computations.

We do have an exact sequence, for any divisor $X \subset \mathbb{P}^2$ of degree d,

$$0 \longrightarrow \mathscr{O}_X(-d) \longrightarrow \Omega_{\mathbb{P}^2}|_X \longrightarrow \Omega_X \longrightarrow 0$$

therefore we get,

$$0 \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\Omega_X, \mathscr{O}_X) \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\Omega_{\mathbb{P}^2}, \mathscr{O}_X) \longrightarrow \mathscr{O}_X(d) \longrightarrow \mathscr{E}\!xt^1_{\mathscr{O}_X}(\Omega_X, \mathscr{O}_X) \longrightarrow 0$$

3.1 Cohomology of Moduli Space of Curves

Projective space is the moduli space for line sub bundles in the trivial $\mathbb{A}^{n+1}$-bundle (or dually line bundle generated by $n+1$ sections). Then the cohomology of a moduli space gives the characteristic classes of the objects in the moduli problem. We have,

$$H^*(\mathbb{P}^n) = \mathbb{Q}[x]/(x^{n+1})$$

generated by the universal Chern class. Likewise the Grasmannian $G(k, n)$ has two universal bundles, the universal quotient $\mathbb{Q}$ and the universal sub $\mathscr{E}$. Then,

$$H^*(\mathrm{Gr}(k, n)) = \mathbb{Q}[x_1, x_2, \dots, x_k]/\sim$$

where $x_i = c_i(\mathscr{E})$ and there is the relation $c(\mathscr{E}) \cdot c(\mathbb{Q}) = 1$

These stabilize meaning the relations vanish as I go to higher degree.

We want to do the same for $\mathcal{M}_g$ the moduli space of algebraic curves of genus g. For a family $\pi : \mathcal{C} \to \mathcal{M}_g$ we have $\mathscr{L}$ the relative cotangent bundle. Then $c_1(\mathscr{L})$ gives a universal class. Then the κ classes are,

$$\kappa_i = \pi_*(c_1(\mathscr{L})^{i+1})$$

Conjecture 3.1.1 (Mumford)**.** $H^*(\mathcal{M}_g)$ stabilizes to $\mathbb{Q}[\kappa_1, \kappa_2, \dots]$

This is now a theorem of Macbeson and Weiss.

Theorem 3.1.2 (Haertagie)**.**

$$(-1)^{g+1}\chi(\mathcal{M}_g) \sim \sqrt{\frac{\pi}{g}}\left(\frac{G}{\pi e}\right)^{2g}$$

Theorem 3.1.3 (Haer)**.** $\mathrm{vcd}(\mathcal{M}_g) = 4g - 5$ and $H^{4g-5}(\mathcal{M}_g) = 0$.

4 Jan 19

4.1 Review of t-structures

Remark. Here we use the ungodly homological grading.

First lets recall what a t-structure is on a triangulated category.

Definition 4.1.1. Let $\mathscr{D}$ be a triangulated category. A *t-structure* is a pair of full subcategories $(\mathscr{D}_{\ge 0}, \mathscr{D}_{\le 0})$ satisfying the following properties. Set,

$$\mathscr{D}_{\ge n} = \mathscr{D}_{\ge 0}[n]$$
$$\mathscr{D}_{\le n} = \mathscr{D}_{\le 0}[n]$$

Then we require,

(a) if $X \in \mathscr{D}_{\ge 1}$ and $Y \in \mathscr{D}_{\le 0}$ then $\mathrm{Hom}_{\mathscr{D}}(X, Y) = 0$ (we write $\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}_{\ge 1}, \mathscr{D}_{\le 0}) = 0$)

(b) $\mathscr{D}_{\ge 1} \subset \mathscr{D}_{\ge 0}$ and $\mathscr{D}_{\le 0} \subset \mathscr{D}_{\le 1}$ meaning $\mathscr{D}_{\ge 0}$ is invariant under positive shifts and $\mathscr{D}_{\le 0}$ is invariant under negative shifts

(c) if $X \in \mathscr{D}$ then there exists a distinguished triangle,

$$X' \to X \to X'' \to X[1]$$

with $X \in \mathscr{D}_{\ge 1}$ and $Y \in \mathscr{D}_{\le 0}$.

Definition 4.1.2. Given a t-structue $(\mathscr{D}_{\ge 0}, \mathscr{D}_{\le 0})$ on a triangulated category $\mathscr{D}$. Then the *heart* is the full subcategory $\mathscr{D}^{\heartsuit} = \mathscr{D}_{\ge 0} \cap \mathscr{D}_{\le 0}$.

Remark. The reason we care about hearts is the following property.

Proposition 4.1.3. For any t-structure on a triangulated category $\mathscr{D}^{\heartsuit}$ is an abelian category.

Example 4.1.4. The motivating example of a t-structure is on the derived category $\mathscr{D} = \mathscr{D}^b(\mathscr{A})$ of bounded complexes of objects in an abelian category $\mathscr{A}$. Then we define,

$$X \in \mathscr{D}_{\ge 0} \iff \forall i < 0 : H_i(X) = 0$$
$$X \in \mathscr{D}_{\le 0} \iff \forall i > 0 : H_i(X) = 0$$

which are the complexes which are acyclic in negative and in positive degrees respectively. Then by shifting this implies,

$$X \in \mathscr{D}_{\ge n} \iff \forall i < n : H_i(X) = 0$$
$$X \in \mathscr{D}_{\le n} \iff \forall i > n : H_i(X) = 0$$

thus for $n \ge 0$ we have $\mathscr{D}_{\ge n} \subset \mathscr{D}_{\ge 0}$ and $\mathscr{D}_{\le 0} \subset \mathscr{D}_{\ge n}$.

If $X \in \mathscr{D}_{\ge 1}$ and $Y \in \mathscr{D}_{\le 0}$ then we can choose and injective resolution of Y which are bounded above by 0. Then $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(X, I) = 0$ becuase the support of these complexes is not overlapping and hence $\mathrm{Hom}_{\mathscr{D}}(X, Y) = 0$ for the derived category. This proves that $\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}_{\ge 1}, \mathscr{D}_{\le 0}) = 0$. Notice that the order ($\mathscr{D}_{\ge 1}$ mapping to $\mathscr{D}_{\le 0}$) is important. If we reverse it say take the target $Y \in \mathscr{D}_{\ge 1}$ then we still have nonoverlapping cohomology. However, we cannot choose an injective resolution supported in positive degrees only (you build injective resolutions in the direction of the maps and can't stop at any fixed point). Indeed, in $\mathcal{D}(\mathbb{Z})$ there are nonzero maps,

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z})}\left(\mathbb{Z}/2\mathbb{Z}[0], \mathbb{Z}[1]\right) = \mathrm{Ext}^1_{\mathbb{Z}}\left(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}\right) = \mathbb{Z}/2\mathbb{Z}$$

For the last property, we want to factor a complex into bounded above and bounded below parts. This is accomplished with the truncation functors: $\tau_{\ge n} A \in \mathcal{D}_{\ge n}$ and $\tau_{\le n} A \in \mathcal{D}_{\le n}$ giving an exact triangle,

$$\tau_{\ge 1} A \to A \to \tau_{\le 0} A \to (\tau_{\ge 1} A)[1]$$

The truncations are given by,

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
A_2 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker \mathrm{d}^1 \arrow[d] \\
\ker \mathrm{d}^1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & \operatorname{coker} \mathrm{d}^1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_{-1} \arrow[r] \arrow[d] & A_{-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

which makes $H_i(\tau_{\ge n} X) \to H_i(X)$ an isomorphism for $i \ge n$ and zero for $i < n$ and likewise $H_i(X) \to H_i(\tau_{\le n} X)$ is an isomorphism for $i \le n$ and zero for $i > n$. Notice here that this decomposition is functorial. Moreover, for any X we have,

$$\tau_{\ge 0}\tau_{\le 0} X = \tau_{\le 0}\tau_{\ge 0} X = H_0(X)[0] \in \mathcal{D}^{\heartsuit}$$

Furthermore, this gives every element of the heart because,

$$\mathcal{D}^{\heartsuit} = \{X \in \mathcal{D} \mid \forall i \neq 0 : H_i(X) = 0\} \cong \mathcal{A}$$

because the maps $\tau_{\le 0}\tau_{\ge 0} X \to \tau_{\ge 0} X \leftarrow X$ are quasi-isomorphisms if $X \in \mathcal{D}^{\heartsuit}$ and thus isomorphisms in $\mathcal{D}$. This, in this case we see that $\mathcal{D} = \mathcal{D}^b(\mathcal{D}^{\heartsuit})$. However, there exist examples of triangulated categories which t-structures that do not arrise as the derived category of any abelian category. However, we will see that functorial truncation does exist generally and thus we will get a "cohomology" functor $\tau_{\ge 0}\tau_{\le 0}$ into the heart.

Proposition 4.1.5. Let $(\mathcal{D}_{\ge 0}, \mathcal{D}_{\le 0})$ be a t-structure.

(a) the inclusion $\mathcal{D}_{\ge n} \to \mathcal{D}$ has a right adjoint $\tau_{\ge n} : \mathcal{D} \to \mathcal{D}_{\ge n}$ in particular there are universal canonical maps $\tau_{\ge n} X \to X$

(b) the inclusion $\mathcal{D}_{\le n} \to \mathcal{D}$ has a right adjoint $\tau_{\le n} : \mathcal{D} \to \mathcal{D}_{\le n}$ in particular there are universal canonical maps $X \to \tau_{\le n} X$

(c) there is a unique natrual transformation $\delta : \tau_{\leq n}(-) \to \tau_{\geq n+1}(-)[1]$ such that for each $X \in \mathcal{D}$,

$$\tau_{\geq n+1}X \to X \to \tau_{\leq n}X \xrightarrow{\delta} (\tau_{\geq n+1}X)[1]$$

is a distinguished triangle

(d) $\tau_{\geq n} = [n] \circ \tau_{\geq 0} \circ [-n]$ and $\tau_{\leq n} = [n] \circ \tau_{\leq 0} \circ [-n]$.

Remark. These give canonical distinguished triangles which can be often useful. Notice also that a t-structure is defined by only one of the subcategories $\mathcal{D}_{\geq 0}$ or $\mathcal{D}_{\leq 0}$ since we can recover the other as,

$$\mathcal{D}_{\geq 1} = \{X \in \mathcal{D} \mid \forall Y \in \mathcal{D}_{\leq 0} : \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0\}$$

Proposition 4.1.6. $H_n : \mathcal{D} \to \mathcal{D}^{\heartsuit}$ defined as,

$$H_n = [-n] \circ \tau_{\geq n}\tau_{\leq n} = [-n] \circ \tau_{\leq n}\tau_{\geq n} = \tau_{\geq 0}\tau_{\leq 0} \circ [-n]$$

is a homological functor: it sends distinguished triangles in $\mathcal{D}$ to long exact sequences in $\mathcal{A}$.

4.2 t-Structures on Stable ∞-Categories

Definition 4.2.1. Let $\mathcal{C}$ be a stable ∞-category. A *t-structure* on $\mathcal{C}$ is a t-structure on $h\mathcal{C}$. If $\mathcal{C}$ is equipped with a t-structure, we let $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ denote the full subcategories spanned by those objects which belong to $(h\mathcal{C})_{\geq n}$ and $(h\mathcal{C})_{\leq n}$ respectively.

Proposition 4.2.2. Let $\mathcal{C}$ be a stavle ∞-category equipped with a t-structure. For each n then subcategory $\mathcal{C}_{\leq n}$ is a localization of $\mathcal{C}$.

Proof. It suffices to prove this for $n = 0$. We need to show that for each $X \in \mathcal{C}$ there is a map $f : X \to X''$ where $X'' \in \mathcal{C}_{\leq 0}$ and for any other $Y \in \mathcal{C}_{\leq 0}$ the map,

$$\mathrm{Map}_{\mathcal{C}}(X'', Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak homotopy equivalence. We can choose a distinguished triangle,

$$X' \to X \to X'' \to X'[1]$$

with $X' \in \mathcal{C}_{\geq 1}$ in $h\mathcal{C}$. Recall that,

$$\begin{aligned}\pi_n \mathrm{Map}_{\mathcal{C}}(X, Y) = \pi_0(\Omega^n \mathrm{Map}_{\mathcal{C}}(X, Y)) = \pi_0(\mathrm{Map}_{\mathcal{C}}(X, \Omega^n Y)) = \mathrm{Hom}_{h\mathcal{C}}(X, \Omega^n Y) \\ = \mathrm{Hom}_{h\mathcal{C}}(X, Y[-n]) = \mathrm{Ext}^{-n}_{h\mathcal{C}}(X, Y)\end{aligned}$$

where $\Omega^n = [-n]$ for the trangulated structure on $h\mathcal{C}$ arising from the stable shifts in $\mathcal{C}$. Therefore, by Whitehead's theorem, it suffices to prove that for all $i \leq 0$,

$$\mathrm{Ext}^i_{h\mathcal{C}}(X'', Y) \to \mathrm{Ext}^i_{h\mathcal{C}}(X, Y)$$

is an isomorphism. This now becomes an exercise in trianguled categories which follows from $\mathrm{Hom}_{h\mathcal{C}}(-, Y)$ being homological. From the triangle,

$$X' \to X \to X'' \to X'[1]$$

We get a long exact sequence,

$$\mathrm{Hom}_{h\mathcal{C}}(X'[1], Y[i]) \longrightarrow \mathrm{Hom}_{h\mathcal{C}}(X'', Y[i]) \longrightarrow \mathrm{Hom}_{h\mathcal{C}}(X, Y[i]) \longrightarrow \mathrm{Hom}_{h\mathcal{C}}(X', Y[i])$$

For $i \le 0$ we have $Y[i] \in \mathcal{D}_{\le i} \subset \mathcal{D}_{\le 0}$ and $X' \in \mathcal{D}_{\ge 1}$ so,

$$\mathrm{Hom}_{h\mathcal{C}}(X', Y[i]) = 0 \quad \text{and} \quad \mathrm{Hom}_{h\mathcal{C}}(X'[1], Y[i]) = \mathrm{Hom}_{h\mathcal{C}}(X', Y[i-1]) = 0$$

proving the required isomorphism from the long exact sequence. □

Corollary 4.2.3. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then $\mathcal{C}_{\le 0}$ are stable under all limts which exist in $\mathcal{C}$ and $\mathcal{C}_{\ge 0}$ are stable unider all colimits which exist in $\mathcal{C}$.

Remark. Since we have localizations, we denote by $\tau_{\ge n} : \mathcal{C} \to \mathcal{C}_{\ge n}$ the right adjoint to the inclusion and $\tau_{\le n} : \mathcal{C} \to \mathcal{C}_{\le n}$ the left adjoint to the inclusion.

Proposition 4.2.4. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then the natural transformation,

$$\theta : \tau_{\le m} \circ \tau_{\ge n} \to \tau_{\ge n} \circ \tau_{\le m}$$

is an equivalence.

Proof. This is true for t-structures on triangulated categories. Therefore, this map induces an isomorphism in $h\mathcal{C}$ meaning it is an equivalence.

(GIVE THE FULL PROOF) □

Definition 4.2.5. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. The *heart* $\mathcal{C}^{\heartsuit}$ of $\mathcal{C}$ is the full subcategory,

$$\mathcal{C}^{\heartsuit} = \mathcal{C}_{\ge 0} \cap \mathcal{C}_{\le 0}$$

For define $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ by $\tau_{\le 0} \circ \tau_{\ge 0} \circ [-n]$.

Remark. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $X, Y \in \mathcal{C}^{\heartsuit}$ then the homotopy hroup $\pi_n \mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Ext}^{-n}_{\mathcal{C}}(X, Y)$ vanishes for $n > 0$. It follows that $\mathcal{C}^{\heartsuit}$ is equivalent to its homotopy category $h\mathcal{C}^{\heartsuit}$. The category $h\mathcal{C}^{\heartsuit}$ is abelian by the corresponding result for triangulated categories.

Definition 4.2.6. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Let,

$$\mathcal{C}^+ = \bigcup \mathcal{C}_{\le n} \quad \text{and} \quad \mathcal{C}^- = \bigcup \mathcal{C}_{\ge -n} \quad \text{and} \quad \mathcal{C}^b = \mathcal{C}^+ \cap \mathcal{C}^-$$

Remark. This is in direct analogy with the bounded, bounded below, and bounded above derived categories.

Remark. We see that the subcategories $\mathcal{C}^+, \mathcal{C}^-, \mathcal{C}^b$ are stable. We say that $\mathcal{C}$ is *bounded below* (left bounded) if $\mathcal{C} = \mathcal{C}^+$ and *bounded above* (right bounded) if $\mathcal{C} = \mathcal{C}^-$ and *bounded* if $\mathcal{C} = \mathcal{C}^b$.

Definition 4.2.7. If $\mathcal{C}$ is a stable ∞-category with a t-structure then the *left completion* $\widehat{\mathcal{C}}$ of $\mathcal{C}$ is the homotopy limit of the tower,

$$\cdots \to \mathcal{C}_{\le 2} \xrightarrow{\tau_{\le 1}} \mathcal{C}_{\le 1} \xrightarrow{\tau_{\le 0}} \mathcal{C}_{\le 0} \to \cdots$$

Remark. Explicitly, $\widehat{\mathcal{C}}$ is equivalent to the full subcategory of $\mathrm{Fun}(N(\mathbb{Z}), \mathcal{C})$ spanned by functors $F : N(\mathbb{Z}) \to \mathcal{C}$ with the properties that,

(a) $F(n) \in \mathcal{C}_{\le -n}$ for all n

(b) $F(m) \to F(n)$ induces an equivalence $\tau_{\leq -n} F(m) \to F(n)$ for all $m \leq n$.

Proposition 4.2.8. Let $\mathcal{C}$ be a stable ∞-category equipped with a t structure. Then,

(a) the left completion $\widehat{\mathcal{C}}$ is stable

(b) let $\widehat{\mathcal{C}}_{\leq 0}$ and $\widehat{\mathcal{C}}_{\geq 0}$ be the full subcategoires of $\widehat{\mathcal{C}}$ spanned by those functors which factor though $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$ respectively. These give a t-structure on $\widehat{\mathcal{C}}$.

(c) There is a canonical functor $\mathcal{C} \to \widehat{\mathcal{C}}$ which is exact and induces an equivalence $\mathcal{C}_{\leq 0} \to \widehat{\mathcal{C}}_{\leq n}$.

Proof. DO THIS!!! □

4.3 Dold - Kan

Let $\mathcal{A}$ be an abelian category. We work in $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes. This is not triangulated untill we pass to the homotopy category or invert quasi-isomorphisms. However, we get a pre-t-structure given by $\mathrm{Ch}(\mathcal{A})_{\geq n}$ denoting the full subcategory of complexes for which $A_k = 0$ for $k < n$. Similarly, $\mathrm{Ch}(\mathcal{A})_{\leq n}$ is the full subcategory of complexes with $A_k = 0$ for $k > n$. Then the truncation functors we defined earlier $\tau_{\geq n} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq n} \to \mathrm{Ch}(\mathcal{A})$ and $\tau_{\leq n} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\leq n}$ are defined as before and are adjoints to the inclusions.

Definition 4.3.1. Let $A \in \mathrm{Ch}(\mathcal{A})$. We define a simplicial object $\mathrm{DK}(A)$ as follows,

(a) for $n \geq 0$ the object,

$$\mathrm{DK}_n(A) = \bigoplus_{\alpha [n] \twoheadrightarrow [k]} A_k$$

where the sum is taken over surjective maps $\alpha : [n] \to [k]$ in Δ ranging over all k (necessarily $k \leq n$).

(b) Let $\beta : [n'] \to [n]$ be a morphism in Δ then the incued map,

$$\beta^* : \mathrm{DK}_n(A) \to \mathrm{DK}_{n'}(A)$$

is given by the matrix $(f_{\alpha,\alpha'} : A_k \to A_{k'})$ where $f_{\alpha,\alpha'}$ is the identity if $k = k'$ and the diagram,

$$\begin{tikzcd} {[n']} \arrow[r, "\beta"] \arrow[d, "\alpha'"] & {[n]} \arrow[d, "\alpha"] \\ {[k']} \arrow[r, "\mathrm{id}"] & {[k]} \end{tikzcd}$$

commutes and $f_{\alpha,\alpha'}$ is the differential d if $k' = k - 1$ and the diagram,

$$\begin{tikzcd} {[n']} \arrow[r, "\beta"] \arrow[d, "\alpha'"] & {[n]} \arrow[d, "\alpha"] \\ {[k']} \arrow[r, "\delta^0"] & {[k]} \end{tikzcd}$$

commutes where $\delta^0(i) = i + 1$ so it hits everything except 0. Otherwise we set $f_{\alpha,\alpha'}$ to zero.

This procedure determines a functor $A \mapsto \mathrm{DK}_\bullet(A)$ sending $\mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$. We denote this functor by DK.

Example 4.3.2. For each simplical set $K_\bullet$ let $\mathbb{Z}K_\bullet$ denote the free simplicial abelian group generated by $K_\bullet$ (meaning $(\mathbb{Z}K)_n$ is the free abelian group generated by K_n and the maps operate by face and degeneracy maps on the generators). Let $\mathbb{Z}[n]$ denote the chain complex of abelian groups,

$$\mathbb{Z}[n]_k = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

Then there is a canonical isomorphism of simplical abelian groups,

$$\mathrm{DK}_\bullet(\mathbb{Z}[n]) = \mathbb{Z}\Delta^n / \mathbb{Z}\partial\Delta^n$$

This is a hit that $\mathbb{Z}[n]$ is dual to a sphere.

Theorem 4.3.3 (Dold-Kan)**.** Let $\mathcal{A}$ be an additive category. The functor,

$$\mathrm{DK} : \mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$$

is fully faithful. If $\mathcal{A}$ is idempotent complete[1] then DK is an equivalence of categories.

Remark. When $\mathcal{A}$ is an abelain category this becomes easier because of the following construction.

Definition 4.3.4. Let $\mathcal{A}$ be an abelian category, and $A_\bullet$ be a simplicial object of A. For each $n \geq 0$ we let $N_n(A)$ denote,

$$N_n(A) := \ker \left(A_n \to \bigoplus_{1 \leq i \leq n} A_{n-1} \right)$$

defined by the maps $\{d_i\}_{1 \leq i \leq n}$ and when $n > 0$ the map d_0 restricts to a boundary map,

$$d_0 : N_n(A) \to N_{n-1}(A)$$

Then $N_\bullet(A)$ forms a chain complex called the *normalized chain complex.* The construction $A_\bullet \mapsto N_\bullet(A)$ determines a functor $N : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq 0}$ which we will refer to as the normalized chain complex functor.

Remark. If $\mathcal{A} \in \mathrm{Ch}(\mathcal{A}_{\geq 0}$ and $\mathrm{DK}(\mathcal{A})$ then for any $\alpha : [n] \twoheadrightarrow [k]$ surjective there exists $[n-1] \xrightarrow{\delta^i} [n] \to [k]$ with $i > 0$ so that the composition is surjective so the maps. Therefore, the only part not mapped isomorphicly via some d_i for $1 \leq i \leq n$ is the component corresponding to $\mathrm{id} : [n] \to [n]$ which is mapped to zero by all $f_{\alpha,\alpha'}$ for δ^i. Therefore,

$$N_n(\mathrm{DK}(A_\bullet)) = A_n$$

as a summand. This is compatible with the differential by design so,

$$A \xrightarrow{\sim} N_\bullet(\mathrm{DK}(A))$$

Proposition 4.3.5. Let $\mathcal{A}$ be an abelian category. The isomorphism of functors,

$$\eta : \mathrm{id}_{\mathrm{Ch}(\mathcal{A})_{\geq 0}} \xrightarrow{\sim} N_* \circ \mathrm{DK}$$

the the unit of an adjunction, $\mathrm{DK} \dashv N_*$.

[1] idempotent maps have kernels and cokernels

Proof. Let $A \in \mathrm{Ch}(\mathcal{A})$. We want to show that the canonical map,

$$\theta : \mathrm{Hom}\left(\mathrm{DK}(A), B\right) \to \mathrm{Hom}\left(N_*(\mathrm{DK}(A)), N_*(B)\right) \xrightarrow{u} \mathrm{Hom}\left(A, N_*(B)\right)$$

is an isomorphism. We just present an inverse. Given $\phi : A \to N_*(B)$ define $\Phi : \mathrm{DK}(A) \to B$ to be the sum,

$$\Phi_n : \bigoplus_{\alpha:[n]\to[k]} A_k \to B_n$$

of the maps $f_\alpha : A_k \to B_k \to B_n$ where $B_k \to B_n$ is the map induced by α in the simplical object B. This gives a map $\Phi : \mathrm{DK}(A) \to B$ and is the unique preimage under θ of ϕ. ☐

Lemma 4.3.6. Let **Ab** be the category of abelian groups. Then,

$$\mathrm{DK} : \mathrm{Ch}(\mathbf{Ab})_{\geq 0} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Ab})$$

is an equivalence of categories.

Proof. CALCULATION SHOWING THE OPPOSITE COMPOSITION IS AN ISOMORPHISM. ☐

Proof of Dold-Kan. The idea is to work in $\mathcal{A}' = \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ the category of presheaves of abelian groups on $\mathcal{A}$ which is an abelian category. Then Yoneda,

$$j : \mathcal{A} \to \mathcal{A}' \quad \text{via} \quad A \mapsto \mathrm{Hom}_{\mathcal{A}}\left(-, A\right)$$

is a fully-faithfull embedding. This makes sense because $\mathcal{A}$ is an additive category. Since the construction of DK is completely formally functorial the diagram,

$$\begin{tikzcd} \mathrm{Ch}(\mathcal{A})_{\geq 0} \arrow[r, "\mathrm{DK}"] \arrow[d] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \arrow[d] \\ \mathrm{Ch}(\mathcal{A}')_{\geq 0} \arrow[r, "\mathrm{DK}"] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}') \end{tikzcd}$$

commutes up to unique natural isomorphism. The vertical maps are fully-faithful embeddings and we know the bottom map is an equivalence of categories. Therefore DK is a fully-faithful embedding on $\mathcal{A}$ and its essential image is the simplicial objects A such that $N_\bullet(j(A))$ is representable (in the essential image of j). By construction, $N_n(j(A))$ is a direct summand of $\mathrm{DK}_n(N_*(j(A)) \cong j(A_n)$. Therefore, if $\mathcal{A}$ is idempotent-complete this means that $N_n(j(A))$ is in the essential image of j proving that $N_*(j(A))$ is in the essential image of j so DK is essentially surjective. ☐

4.4 Dold - Kan for ∞-Categories

If $X_\bullet$ is a simplicial object of a stable ∞-category $\mathcal{C}$, then $X_\bullet$ determines a simplicial object in $h\mathcal{C}$. The homotopy category $h\mathcal{C}$ is not abelian but it is additive and satisifes,

if $\iota : X \to Y$ is a morphism in $h\mathcal{C}$ admitting a left inverse then there is an isomorphism $Y \cong X \oplus X'$ such that ι is identifies with the map $(\mathrm{id}, 0)$.

This condition suffices to construct Dold-Kan for $h\mathcal{C}$ giving a chain complex in $h\mathcal{C}$.

There is another construction of a chain complex in $h\mathcal{C}$. Namely a $\mathbb{Z}_{\geq 0}$-filtered object $n \mapsto Y_n$ determines a chain complex $C_\bullet$ in $h\mathcal{C}$ defined by $C_n = \mathrm{cofib}(Y_{n-1} \to Y_n)[-n]$.

Theorem 4.4.1 (∞-Categorical Dold-Kan)**.** Let $\mathcal{C}$ be a stable ∞-category. The categories $\mathrm{Fun}(N(\mathbb{Z}_{\geq 0}), \mathcal{C})$ and $\mathrm{Fun}(N(\Delta)^{\mathrm{op}}, \mathcal{C})$ are canonically equivalence.

Remark.

5 Preserve for Posterity the Good Convention

First lets recall what a t-structure is on a triangulated category.

Definition 5.0.1. Let $\mathcal{D}$ be a triangulated category. A *t-structure* is a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the following properties. Set,

$$\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$$
$$\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$$

Then we require,

(a) if $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ then $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ (we write $\mathrm{Hom}_{\mathcal{D}}\left(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}\right) = 0$)

(b) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$ meaning $\mathcal{D}^{\leq 0}$ is invariant under positive shifts and $\mathcal{D}^{\geq 0}$ is invariant under negative shifts

(c) if $A \in \mathcal{D}$ then there exists a distinguished triangle,

$$X \to A \to Y \to X[1]$$

with $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

Definition 5.0.2. Given a t-structue $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category $\mathcal{D}$. Then the *heart* is the full subcategory $\mathcal{D}^{\heartsuit} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Remark. The reason we care about hearts is the following property.

Proposition 5.0.3. For any t-structure on a triangulated category $\mathcal{D}^{\heartsuit}$ is an abelian category.

Example 5.0.4. The motivating example of a t-structure is on the derived category $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ of bounded complexes of objects in an abelian category $\mathcal{A}$. Then we define,

$$X \in \mathcal{D}^{\leq 0} \iff \forall i > 0 : H^i(X) = 0$$
$$X \in \mathcal{D}^{\geq 0} \iff \forall i < 0 : H^i(X) = 0$$

which are the complexes which are acyclic in negative and in positive degrees respectively. Then by shifting this implies,

$$X \in \mathcal{D}^{\leq n} \iff \forall i > n : H^i(X) = 0$$
$$X \in \mathcal{D}^{\geq n} \iff \forall i < n : H^i(X) = 0$$

thus for $n \geq 0$ we have $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\leq n}$.

If $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ then we can choose injective resolutions of X and Y which are bounded above by 0 and below by 1. There are clearly no nonzero maps between these resolutions which proves that $\mathrm{Hom}_{\mathcal{D}}\left(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}\right) = 0$.

For the last property, we want to factor a complex into bounded above and bounded below parts. This is accomplished with the truncation functors: $\tau^{\leq 0}A \in \mathcal{D}^{\leq 0}$ and $\tau^{\geq 1}A \in \mathcal{D}^{\geq 1}$ giving an exact triangle,

$$\tau^{\leq 0}A \to A \to \tau^{\geq 1}A \to (\tau^{\leq 0}A)[1]$$

The truncations are given by,

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
A^{-1} \arrow[r] \arrow[d] & A^{-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker \mathrm{d}^0 \arrow[d] \\
\ker \mathrm{d}^0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \mathrm{coker}\, \mathrm{d}^0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

which makes $H^i(\tau^{\le n}X) \to H^i(X)$ an isomorphism for $i \le n$ and zero for $i > n$ and likewise $H^i(X) \to H^i(\tau^{\ge n}X)$ is an isomorphism for $i \ge n$ and zero for $i < n$. Notice here that this decomposition is functorial. Moreover, for any X we have,

$$\tau^{\le 0}\tau^{\ge 0}X = \tau^{\ge 0}\tau^{\le 0}X = H^0(X)[0] \in \mathscr{D}^\heartsuit$$

Furthermore, this gives every element of the heart because,

$$\mathscr{D}^\heartsuit = \{X \in \mathscr{D} \mid \forall i \neq 0 : H^i(X) = 0\} \cong \mathscr{A}$$

because the maps $\tau^{\ge 0}\tau^{\le 0}X \leftarrow \tau^{\le 0}X \to X$ are quasi-isomorphisms if $X \in \mathscr{D}^\heartsuit$ and thus isomorphisms in $\mathscr{D}$. This, in this case we see that $\mathscr{D} = \mathscr{D}^b(\mathscr{D}^\heartsuit)$. However, there exist examples of triangulated categories which t-structures that do not arrise as the derived category of any abelian category. However, we will see that functorial truncation does exist generally and thus we will get a "cohomology" functor $\tau^{\le 0}\tau^{\ge 0}$ into the heart.

Proposition 5.0.5. For any t-structure, there exist additive functors $\tau^{\le n} : \mathscr{D} \to \mathscr{D}^{\le n}$ and $\tau^{\ge n} : \mathscr{D} \to \mathscr{D}^{\ge n}$ such that for each $X \in \mathscr{D}$,

$$\tau^{\le 0}X \to X \to \tau^{\ge 1}X \to (\tau^{\le 0}X)[1]$$

is a distinguished triangle and we set $\tau^{\le n} = [-n]\tau^{\le 0}[n]$ and $\tau^{\ge n} = [-n]\tau^{\ge 0}[n]$.