1 Uniform Continuity

Definition 1.0.1. A continuous map $f : X \to Y$ of metric spaces is *uniformly continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that,

$$\forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Lemma 1.0.2. Let $f : X \to Y$ be a continous map of metric spaces with X compact. Then f is uniformly continuous.

Proof. Take $\varepsilon > 0$. Then for each $c \in X$ there exists $\delta_c > 0$ such that $f(B_{\delta_c}(c)) \subset B_{\frac{\varepsilon}{2}}(f(c))$. Now the open balls $\{B_{\frac{\delta_c}{2}}(c)\}$ form an open cover of X. By compactness, there is a finite subcover $\{c_i\}$ then take $\delta = \frac{1}{2}\min\{\delta_{c_i}\}$ by finiteness $\delta > 0$. Now suppose that $d_X(x, y) < \delta$ then since the balls form a cover, $\exists c_x, c_y$ s.t. $x \in B_{\frac{\delta_{c_x}}{2}}(c_x)$ and $y \in B_{\frac{\delta_{c_y}}{2}}(c_y)$. However,

$$\mathrm{d}_X(c_x, y) \le d_X(c_x, x) + d_X(x, y) < \delta + \tfrac{1}{2}\delta_{C_x} \le \delta_{c_x}$$

Therefore $x, y \in B_{\delta_{c_x}}(c_x)$. We know that $f(x), f(y) \in f(B_{\delta_{c_x}}(c_x)) \subset B_{\frac{\varepsilon}{2}}(f(c_x))$. Therefore,

$$d_Y(f(x), f(y)) \le d_Y(f(x), f(c_x)) + d_Y(f(c_x), f(y)) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

so we find,

$$\forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

which shows uniform continuity. □

2 Group Actions

Lemma 2.0.1 (Burnside)**.** Let $G \circlearrowright S$ then the number of orbits is,

$$|S/G| = \frac{1}{|G|} \sum_{g \in G} |\{x \in S \mid g \cdot x = x\}|$$

Proof. Consider the set of fixed pairs,

$$P = \{(g, x) \mid g \in G \quad x \in S \quad g \cdot x = x\}$$

Then we can compute the size of P in two ways,

$$|P| = \sum_{g \in G} |\{x \in S \mid g \cdot x = x\}|$$

Furthermore,

$$|P| = \sum_{x \in S} |\{g \in G \mid g \cdot x = x\}| = \sum_{x \in S} |\mathrm{Stab}(x)|$$

However, by Orbit-Stabilizer,

$$|\mathrm{Stab}(x)| = \frac{|G|}{|\mathrm{Orb}(x)|}$$

Furthermore, since the orbits partition S,

$$|P| = \sum_{x \in S} |\mathrm{Stab}(x)| = |G| \sum_{x \in S} \frac{1}{|\mathrm{Orb}(x)|} = |G| \sum_{O \in |S/G|} \sum_{x \in O} \frac{1}{|O|}$$

□