1 Convext Sets

2 Convex Optimization

2.1 Motivation

Definition: $\mathbb{T}^n = (\mathbb{G}_m)^n = (\mathbb{C}^\times)^n$

Remark 1. This torus is noncompact (i.e. not proper) so we compactify it to recover a toric variety. The compactification $X_\Sigma = \overline{\mathbb{T}}^\Sigma$ is given by a fan $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ where $N = \operatorname{Hom}(\mathbb{G}_m, \mathbb{T}) \cong \mathbb{Z}^n$ is a lattice.

Remark 2. Consider an ample line bundle $L \to X_\Sigma$ which is compatible with the toric structure. We can associate each line bundle L with a convex polytope Δ_{L,X_Σ} in the vectorspace $N^\vee \otimes_{\mathbb{Z}} \mathbb{R} = (N \otimes_{\mathbb{Z}} \mathbb{R})^*$. Furthermore,

$$\langle L \cdot L \cdots L \rangle = \operatorname{Vol}(L) = \deg_L X_\Sigma = n! \operatorname{Vol}_n(\Delta_{L,X_\Sigma})$$

Remark 3. Given line bundles $L_i \to X_\Sigma$ over a toric variety then we have many polytopes $\Delta_i \subset N^\vee \otimes_{\mathbb{Z}} \mathbb{R}^n$. Consider the mixed intersection numbers,

$$\langle L_{i_1} \cdot L_{i_2} \cdots L_{i_n} \rangle$$

give mixed volumes of the polytopes Δ_i. Look for inequalities between mixed intersection numbers. Thus we need to study inequalities for mixed volumes of convext polytopes.

2.2 Convex Functions

Definition: A function $f : V \to \mathbb{R}$ is convex if it satisfies Jensen's inequality:

$$\forall x, y \in V : \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

In general, if $K \subset V$ is convex then we may define a function $f : K \to \mathbb{R}$ to be convex.

Remark 4. We can relate convext functions and sets to eachother.

On a convex set $K \subset V$, a function $f : K \to \mathbb{R}$ is convex iff

$$\text{epigraph}(f) = \{(x, r) \mid x \in K \text{ and } r \ge f(x)$$

is convex in $V \times \mathbb{R}$.

Futhermore, given a convex function $f : K \to \mathbb{R}$ then the set,

$$L(f,t) = \{x \in K \mid f(x) \le t\} \subset K$$

is convex in V.

Let $K \subset V$ be convex then the function,

$$-\log(1_K) = \begin{cases} +\infty & V \setminus K \\ 0 & K \end{cases}$$

is convex.

Definition: A nonegative function $f : K \to \mathbb{R}$ is log-convex (concave) if $\log f$ is convex (concave). Equivalently,

$$\forall x, y \in K : \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda)y) \leq f(x)^{\lambda} \cdot f(y)^{1-\lambda}$$

Definition: A supporting hyperplane for $f : K \to \mathbb{R}$ given by $\xi \in V^*$ is such that,

$$\forall y \in K : f(y) - f(x) \geq \langle \xi, y - x \rangle$$

Lemma 2.1. The function f is convex iff $\forall x \in K : f$ has a supporting hyperplane at x.

Lemma 2.2 (Operations on Convex Functions)**.** Let $f_1, f_2 : V \to \mathbb{R}$ be convex. Then the following functions are convex,

1. $\max\{f_1, f_2\}$
2. $f_1 + f_2$
3. $g(f_1)$ for g any increasing function
4. $f_1 \boxplus f_2(z) = \sup\{\lambda f_1(x) + (1-\lambda) f_2(y) \mid \lambda x + (1-\lambda)y = z\}$ here,

$$\text{epigraph}\,(f_1 \boxplus f_2) = \text{epigraph}\,(f_1) + \text{epigraph}\,(f_2)$$

2.3 Legendre - Fenchel Transform

Definition: Let $f : V \to \mathbb{R}$ be any function. Then we define the convex dual $f^{\vee} : V^* \to \mathbb{R}$ via,

$$f^{\vee}(\xi) = \sup_x \left(\langle \xi, x \rangle - f(x)\right)$$

Remark 5. If f is differentiable at x then x_ξ is such that,

$$\nabla_x \left(\langle \xi, x \rangle - f(x)\right)\Big|_{x = x_\xi} = 0$$

meaning that,

$$\nabla f(x_\xi) = \xi$$

in V^* where we have defined the gradient as a dual vector via,

$$x \mapsto \nabla f(x) : v \mapsto \langle \nabla f(x), v \rangle = \nabla_v f(x)$$

Lemma 2.3. Suppose that $f : K \to \mathbb{R}$ is lower-semicontinuous on a compact convex set K. Then $f^\vee : V^* \to \mathbb{R}$ is convex and lower-semicontinuous.

Proof. For $\lambda \in [0,1]$. $\square$

Corollary 2.4. The double dual $f^{\vee\vee}$ is convex.

Proposition 2.5. In general, $f^{\vee\vee}(x) \leq f(x)$ and $f^{\vee\vee}$ is the largest convex function below f i.e. the convex envelope of f.

Proof. $\square$

Theorem 2.6. $f^{\vee\vee}(x) = f(x) \iff f$ has a supporting hyperplane at x.

Proof. f has a supporting hyperplane at x given by ξ iff $f^\vee$ has a supporting hyperplane at ξ given by x since,

$$\begin{aligned} f^\vee(\eta) = \sup_y \left(\langle \eta, y\rangle - f(y)\right) \geq \langle \eta, x_\xi\rangle - f(x_\xi) \\ &= \langle \eta - \xi, x_\xi\rangle + \langle \eta, x_\xi\rangle - f(x_\xi) \\ &= \langle \eta - \xi, x_\xi\rangle + f^\vee(\xi) \end{aligned}$$

Therefore,

$$f^\vee(\eta) - f^\vee(\xi) \geq \langle \eta - \xi, x_\xi\rangle$$

$\square$

Corollary 2.7. $f^{\vee\vee} = f \iff f$ is convex.

Remark 6. The involution $f^{\vee\vee} = f$ encodes the convexity of f in the convexity of $f^\vee$. If f is strictly convex and differentiable then the map $\xi \to x_\xi$ is bijective.

Remark 7. Otherwise then corners of f are sent to affine parts of $f^\vee$ i.e. dualizing replaces failure to be differentiable with failure to be *strictly* convex.

Proposition 2.8. $\forall x \in V, \xi \in V^* : \langle \xi, x\rangle \leq f(x) + f^\vee(\xi)$

2.4 Some Ineqaulities

Theorem 2.9 (Pakopa - Lindler Inequality)**.** Consider positive $f, g, h : V \to \mathbb{R}$ such that,

$$\forall x, y \in V : h(\lambda x + (1-\lambda)y) \geq f(x)^\lambda \cdot g(y)^{1-\lambda}$$

for a fixed $\lambda \in [0,1]$. Then,

$$\int_V h(z)\mathrm{d}z \geq \left(\int_V f(x)\mathrm{d}x\right)^\lambda \cdot \left(\int_V g(y)\mathrm{d}y\right)^{1-\lambda}$$

Theorem 2.10 (Brunn-Minkowski Inequality I)**.** Let $X, Y \subset V$ be convex bodys then,

$$\mathrm{Vol}_n\left(\lambda X + (1-\lambda)Y\right) \geq \mathrm{Vol}_n\left(X\right)^\lambda \cdot \mathrm{Vol}_n\left(Y\right)^{1-\lambda}$$

Theorem 2.11 (Brunn-Minkowski Inequality II)**.** Let $X, Y \subset V$ be convex bodys then,

$$\mathrm{Vol}_n\left(\lambda X + (1-\lambda)Y\right)^{1/n} \geq \lambda \mathrm{Vol}_n\left(X\right)^{1/n} + (1-\lambda)\mathrm{Vol}_n\left(Y\right)^{1/n}$$

Remark 8. We will prove these innequalities in the following way,

$$\text{1D BM} \implies \text{1D PL} \implies \text{nD PL} \implies \text{nD BM}$$

Proof of 1D BM. Consider the $n = 1$ case. We need to prove that,

$$\mathrm{Vol}_n\left(\lambda X + (1-\lambda)Y\right) \geq \lambda \mathrm{Vol}_n\left(X\right) + (1-\lambda)Y$$

Suppose that X and Y are compact then $X + Y$ is compact. We may translate without changing the volumes such that $X \subset \mathbb{R}_{\leq 0}$ and $Y \subset \mathbb{R}_{\geq 0}$ and $X \cap Y = \{0\}$. Now,

$$\lambda X + (1-\lambda)Y \supset \lambda X \cup (1-\lambda)Y$$

which implies that,

$$\mathrm{Vol}_n\left(\lambda X + (1-\lambda)Y\right) \geq \mathrm{Vol}_n\left(\lambda X\right) + \mathrm{Vol}_n\left((1-\lambda)Y\right) = \lambda \mathrm{Vol}_n\left(X\right) + (1-\lambda)\mathrm{Vol}_n\left(Y\right)$$

□

Proof of 1D PL. Via Lebesgue integration,

$$\int_{\mathbb{R}} h(z)\mathrm{d}z = \int_0^\infty \mu(\{z \in \mathbb{R} \mid h(z) \geq t\})\mathrm{d}t$$

However, □

Proof of nD PL. Assume for induction that PL is true in $\mathbb{R}^n$. Now, $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. Define $h_c(z) = h(c, z)$ for $c \in \mathbb{R}$ and $z \in \mathbb{R}^n$. Then,

$$h(\lambda(a, x) + (1-\lambda)(b, y)) = h_{\lambda a + (1-\lambda)b}(\lambda x + (1-\lambda)y)$$

However, by assumption,

$$h(\lambda(a, x) + (1-\lambda)(b, y)) \geq f(a, x)^\lambda \cdot g(b, y)^{1-\lambda} = f_a(x)^\lambda \cdot g_b(y)^{1-\lambda}$$

By nD PL we have,

$$\int_{\mathbb{R}^n} h_{\lambda a + (1-\lambda)b}(z)\mathrm{d}z \geq \left(\int_{\mathbb{R}^n} f_a(x)\mathrm{d}x\right)^\lambda \cdot \left(\int_{\mathbb{R}^n} g_b(y)\mathrm{d}y\right)^{1-\lambda}$$

Denote,

$$H(c) = \int_{\mathbb{R}^n} h_c(z)\mathrm{d}z \qquad F(a) = \int_{\mathbb{R}^n} f_a(z)\mathrm{d}z \qquad G(b) = \int_{\mathbb{R}^n} g_b(y)\mathrm{d}y$$

Thus we have shown that,

$$H(\lambda a + (1-\lambda)b) \geq F(a)^\lambda \cdot G(b)^{1-\lambda}$$

Therefore, by 1D PL we have,

$$\int_{\mathbb{R}} H(c)\mathrm{d}c \geq \left(\int_{\mathbb{R}} F(a)\mathrm{d}a\right)^{\lambda} \cdot \left(\int_{\mathbb{R}} G(b)\mathrm{d}b\right)^{1-\lambda}$$

Then, reparametrizing the integrals by Fubini's theorem,

$$\int_{\mathbb{R}^{n+1}} h(z)\mathrm{d}z \geq \left(\int_{\mathbb{R}^{n+1}} f(x)\mathrm{d}z\right)^{\lambda} \cdot \left(\int_{\mathbb{R}^{n+1}} g(y)\mathrm{d}y\right)^{1-\lambda}$$

□

Proof of nD BM. □

Theorem 2.12 (Isoperimetric Inequality)**.**

$$\frac{1}{n} \lim_{\epsilon \to 0} \frac{\mathrm{Vol}_n\left(X + \epsilon Y\right) - \mathrm{Vol}_n\left(X\right)}{\epsilon} \geq \mathrm{Vol}_n\left(X\right)^{\frac{n-1}{n}} \cdot \mathrm{Vol}_n\left(Y\right)^{\frac{1}{n}}$$

3 Mixed Volumes

3.1 Scaling Volumes

Consider a measureable set $S \subset V$ with measure $\mathrm{Vol}_n(S)$. Given $\lambda \in \mathbb{R}_{\geq 0}$ then $\mathrm{Vol}_n(\lambda \cdot S) = \lambda^n \mathrm{Vol}_n(S)$. We want to understand the volume of the Minkowski sum of two shapes. We define,

$$\mathrm{Vol}_n(S, T) = \tfrac{1}{2}\left[\mathrm{Vol}_n(S+T) - \mathrm{Vol}_n(S) - \mathrm{Vol}_n(T)\right]$$

In general, $\mathrm{Vol}_n(\lambda S + \mu T)$ is a homogeneous polynomial of degree n.

Theorem 3.1. Let $S_1, \dots, S_r \subset V$ be compact convex measureable subsets and $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$. Then, $\mathrm{Vol}_n(\lambda_1 S_1 + \cdots + \lambda_r S_n)$ is a homogeneous polynomial of degree n i.e.

$$\mathrm{Vol}_n(\lambda_1 S_1 + \cdots + \lambda_r S_r) = \sum_{i_1, \dots, i_n = 1}^{r} \mathrm{Vol}_n(S_i, \dots, S_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}$$

Proof. Assume for now that $S_1, \dots, S_r$ are polytopes and by shifting that each contains the origin in its interior. Set $K_\lambda = \lambda_1 S_1 + \cdots + \lambda_r S_r$. We proceed by induction on $\dim V = n$. First, for $n = 1$ polytopes are intervals $S_i = [a_i, b_i]$ for $a_i \leq 0 \leq b_i$. Then, $K_\lambda = [\lambda_1 a_1 + \cdots + \lambda_r a_r, \lambda_1 b_1 + \cdots + \lambda_r b_r]$ so we find,

$$\mathrm{Vol}_1(K_\lambda) = (\lambda_1 b_1 + \cdots + \lambda_r b_r) - (\lambda_1 a_1 + \cdots + \lambda_r b_r) = \sum_{i=1}^{r} (b_i - a_i)\lambda_r$$

proving the theorem in the case $n = 1$.

For $n > 1$, we write the boundary of K_λ as the union of the facets. Let $\mathrm{pyr}_0(F_i)$ be the convex hull of $F_i \cup \{0\}$ which is the pyramid with base F_i and apex 0. Thus,

$$K_\lambda = \bigcup_{i=1}^{s} \mathrm{pyr}_0(F_i)$$

Furthermore, the pyramids intersect in lower dimensional faces and thus their intersections have zero Lebesgue measure. Therefore,

$$\mathrm{Vol}_n\left(K_\lambda\right) = \sum_{i=1}^{s} \frac{1}{n}\, h_i \,\mathrm{Vol}_{n-1}\left(F_i\right)$$

Now, if $K = S_1 + \cdots + S_r$ then the faces of K are $F = F_1 + \cdots + F_r$ are sums of faces $F_i \subset S_i$. Furthermore, the heights from $0 \in K$ to F decomposes as $h = h_1 + \cdots + h_r$ where h_i is the height from $0 \in S_i$ to F_i. Therefore,

$$\mathrm{Vol}_n\left(K_\lambda\right) = \sum_{i=1}^{s} \frac{1}{n}\, h_i \,\mathrm{Vol}_{n-1}\left(F_i\right) = \frac{1}{n}\sum_{i=1}^{n}(\lambda_1 h_{i_1} + \cdots + \lambda_r h_{i_r})\mathrm{Vol}_{n-1}\left(\lambda_1 F_{i_1} + \cdots \lambda_r F_{i_r}\right)$$

By the induction hypothesis, $\mathrm{Vol}_{n-1}\left(\lambda_1 F_{i_1} + \cdots \lambda_r F_{i_r}\right)$ is a homogeneous polynomial of degree $n-1$. Therefore, $\mathrm{Vol}_n\left(K_\lambda\right)$ is a homogeneous polynomial of degree n. $\square$

Example 3.2. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$ s.t. $\lambda_1 + \cdots + \lambda_r = 1$ then,

$$\begin{aligned}\mathrm{Vol}_n\left(S\right) = \mathrm{Vol}_n\left(\lambda_1 S + \cdots + \lambda_r S\right) &= \sum_{i_1,\ldots,i_n=1}^{r} \mathrm{mVol}\left(S, \ldots, S\right) \lambda_{i_1} \cdots \lambda_{i_n} \\ &= \mathrm{mVol}\left(S, \ldots, S\right)\left(\lambda_1 + \cdots + \lambda_r\right)^n = \mathrm{mVol}\left(S, \ldots, S\right)\end{aligned}$$

Therefore, $\mathrm{mVol}\left(S, \ldots, S\right) = \mathrm{Vol}_n\left(S\right)$.

Proposition 3.3. Properties of Mixed Volumes:

1. $\mathrm{mVol}\left(S, \ldots, S\right) = \mathrm{Vol}_n\left(S\right)$
2. Symmetric, $\mathrm{mVol}\left(S_1, \ldots, S_n\right) = \mathrm{mVol}\left(S_{\pi(1)}, \ldots, S_{\pi(n)}\right)$
3. Multilinear: $\mathrm{mVol}\left(\lambda S + \lambda' S', S_2, \cdots, S_n\right) = \lambda \mathrm{mVol}\left(S, S_2, \ldots, S_n\right) + \lambda' \mathrm{mVol}\left(S', S_2, \ldots, S_n\right)$
4. Nonegative: $\mathrm{mVol}\left(S_1, \ldots, S_n\right) \geq 0$
5. Monotonic: if $S \subset S'$ then $\mathrm{mVol}\left(S, S_2, \ldots, S_n\right) \leq \mathrm{mVol}\left(S', S_2, \ldots, S_n\right)$

Proof. To prove (3), consider, $K_\lambda = \lambda_1(\lambda S + \lambda' S') + \lambda_2 S_2 + \cdots + \lambda_r S_n$. Thus we find,

$$\mathrm{Vol}_n\left(K_\lambda\right) = \sum_{i_1,\ldots,i_n=1}^{n} \mathrm{mVol}\left(S_{i_1}, \ldots, S_{i_n}\right) \lambda_{i_1} \cdots \lambda_{i_n}$$

Consider the $\lambda_1 \ldots \lambda_n$ term which has coefficient $n!\,\mathrm{mVol}\left(S_1, \ldots, S_n\right)$. Furthermore,

$$\mathrm{Vol}_n\left(K_\lambda\right) = \sum_{i_1,\ldots,i_n=0}^{n} \mathrm{mVol}\left(S_{i_1}, \ldots, S_{i_n}\right) \alpha_{i_1} \cdots \alpha_{i_n}$$

where now $\alpha_0 = \lambda_1 \lambda$ and $\alpha_1 = \lambda_1 \lambda'$ and for $i > 2$, $\alpha_i = \lambda_i$. $\square$

Theorem 3.4 (Alexandrov-Fenchel Inequality)**.** Let $A, B, S_3, \ldots, S_n \subset V$ be compact convex measurable sets. Then,

$$\mathrm{mVol}\,(A, B, S_3, \ldots, S_n)^2 \geq \mathrm{mVol}\,(A, A, S_3, \ldots, S_n) \cdot \mathrm{mVol}\,(B, B, S_3, \cdots, S_n)$$

Definition: A sequence $\{a_n\}$ is log-concave iff $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i > 0$.

Lemma 3.5. Fix $1 \leq m \leq n$ and take some compact convex measurable sets $A, B, S_{m+1}, \ldots, S_n \subset V$. Then, let $a_i = \mathrm{mVol}\,(A^{m-i}B^i, S_\bullet)$ for $i = 0, 1, \ldots, m$. Then the sequence $\{a_n\}$ is log-concave.

Proof. By AF,

$$\begin{aligned} a_i^2 &= \mathrm{mVol}\left(A^{m-i}B^i, S_\bullet\right)^2 = \mathrm{mVol}\left(A, B, A^{m-i-1}B^{i-1}, S_\bullet\right)^2 \\ &\geq \mathrm{mVol}\left(A^{m-i+1}, B^{i-1}, S_\bullet\right) \cdot \mathrm{mVol}\left(A^{m-i-1}, B^{i+1}, S_\bullet\right) = a_{i-1} \cdot a_{i+1} \end{aligned}$$

□

Theorem 3.6 (Generalized Brunn–Minkowski inequality)**.** Fix $1 \leq m \leq n$ and convex compact measureable bodies $A, B, S_{m+1}, \ldots, S_n \subset V$. Set $S_\lambda = (1-\lambda)A + \lambda B$ for $\lambda \in [0, 1]$ and consider the function $f : [0, 1] \to \mathbb{R}$ given via,

$$f(\lambda) = \mathrm{mVol}\,(S_\lambda^m, S_\bullet)^{\frac{1}{m}}$$

then f is concave on $[0, 1]$.

Proof. Consider,

$$\begin{aligned} f''(0) &= (m-1)\,\mathrm{mVol}\left(B, B, B^{m-2}, S_\bullet\right)^{\frac{1}{m}-2} \\ &\quad \cdot \left(\mathrm{mVol}\left(A, A, B^{m-2}, S_\bullet\right)\mathrm{mVol}\left(B, B, B^{m-2}, S_\bullet\right) - \mathrm{mVol}\left(A, B, B^{m-2}, S_\bullet\right)^2\right) \\ &= (m-1)\, a_0^{\frac{1}{m}-2}\left(a_0a_2 - a_1^2\right) \leq 0 \end{aligned}$$

via the previous lemma. Thus $f'' \leq 0$ so f is concave. □

Corollary 3.7. $\mathrm{Vol}_n\,(A+B)^{\frac{1}{n}} \geq \mathrm{Vol}_n\,(A)^{\frac{1}{n}} + \mathrm{Vol}_n\,(B)^{\frac{1}{n}}$

Definition: Let $S \subset V$ be a compact convex set. The *support function* of S is $h_S : V^* \to \mathbb{R}$ via $u \mapsto \sup_{x \in S} \langle u, x \rangle$.

Remark 9. For each $u \in V^*$ the set $\{v \in V \mid \langle u, v \rangle = h_S(u)\}$ is a supporting hyperplane for S and all such supporting hyperplanes arise this way.

Lemma 3.8. $h_{A+B} = h_A + h_B$.

Proof.

$$\begin{aligned} h_{A+B}(u) &= \sup_{\substack{x \in A \\ y \in B}} \langle u, x+y \rangle \\ &= \sup_{x \in A} \langle u, x \rangle + \sup_{y \in B} \langle u, y \rangle = h_A(u) + h_B(u) \end{aligned}$$

□

4 Sheaves

4.1 Categories

Definition: A class of of objects $\mathbb{C}$ and for each $X, Y \in \mathcal{C}$ a set of morphisms $\mathcal{C}(X, Y)$ and $\forall X, Y, Z \in \mathcal{C}^3$ a map $\mathbb{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ via $(f, g) \mapsto g \circ f$ such that,

1. $\forall X \in \mathcal{C} \exists 1_X \in \mathbb{C}(X, X)$ such that,
$$\forall f \in \mathcal{C}(X, Y) : f \circ 1_X = f \quad \text{and} \quad \forall g \in \mathcal{C}(Y, X) : 1_X \circ g = g$$

2. for $f \in \mathcal{C}(X, Y), g \in \mathbb{C}(Y, Z), h \in \mathcal{C}(Z, W) : h \circ (g \circ f) = (h \circ g) \circ f$

5 Scheme Theory

Theorem 5.1. If X is a locally ringed space and A a ring then,
$$\mathrm{Hom}_{\mathbf{LRS}}(X, \mathrm{Spec}(A)) \cong \mathrm{Hom}_{\mathbf{Ring}}(A, \mathcal{O}_X(X))$$

Corollary 5.2. In particular, for $X = \mathrm{Spec}(B)$ we have,
$$\mathrm{Hom}_{\mathbf{LRS}}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \cong \mathrm{Hom}_{\mathbf{Ring}}(A, B)$$
and thus the Spec functor is fully faithful giving an antiequivalence of functors between the category of rings and the category of affine schemes.

5.1 Geometric Realization of Functors

Definition: Let X be a locally ringed sapce. We define a functor,
$$\begin{aligned} \mathfrak{S}_X : \mathbf{Ring} &\to \mathbf{Set} \\ A &\mapsto \mathrm{Hom}_{\mathbf{LRS}}(\mathrm{Spec}(A), X) \end{aligned}$$

Remark 10. By Yoneda's lemma $\mathfrak{S}_X(A) = \mathrm{Hom}_{\mathbf{Set}^{\mathbf{Ring}}}(h_A, \mathfrak{S}_X)$

Theorem 5.3. The functor $X \mapsto \mathfrak{S}_X$ admits a left adjoint $\mathrm{Rg} : \mathbf{Set}^{\mathbf{Ring}} \to \mathbf{LRS}$ such that,
$$\mathrm{Hom}_{\mathbf{LRS}}(\mathrm{Rg}(F), X) \cong \mathrm{Hom}_{\mathbf{Set}^{\mathbf{Ring}}}(F, \mathfrak{S}_X)$$
Furthermore, $\mathrm{Rg}(h_A) = \mathrm{Spec}(A)$.

Proof. Define,
$$I_F = \{(A, \rho) \mid A \in \mathbf{Ring} \text{ and } \rho \in F(A)\}$$
Furthermore, let,
$$I_F((A_1, \rho_1), (A_2, \rho_2)) = \{f : A_1 \to A_2 \mid F(f)(\rho_1) = \rho_2\}$$

Now,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Set}^{\mathbf{Ring}}}(F, \mathfrak{S}_X) &= \{\alpha_A : F(A) \to \mathfrak{S}_X(A) \mid \alpha_{A'} = \mathfrak{S}_X(f)(\alpha_A(\rho)) \text{ for } f : (A,\rho) \to (A',\rho')\} \\
&= \varprojlim_{(A,\rho)\in I_F} \mathrm{Hom}_{\mathbf{Set}^{\mathbf{Ring}}}(h_A, \mathfrak{S}_X) = \varprojlim_{(A,\rho)\in I_F} \mathfrak{S}_X(A) \\
&= \varprojlim_{(A,\rho)\in I_F} \mathrm{Hom}_{\mathbf{LRS}}(\mathrm{Spec}\,(A), X) = \mathrm{Hom}_{\mathbf{LRS}}\left(\varinjlim_{(A,\rho)\in I_F^{\mathrm{op}}} \mathrm{Spec}\,(A), X\right)
\end{aligned}$$

Thus the functor $\mathrm{Hom}_{\mathbf{Set}^{\mathbf{Ring}}}(F, \mathfrak{S}_-)$ is representable and thus $\mathfrak{S}_-$ admits a left adjoint. □

5.2 Schemes

Definition: We call an *affine scheme* any locally ringed space isomorphic to the spectrum of a ring. We call a *scheme* any locally ringed space which admits an open cover by affine schemes i.e. $\forall x \in X$ there exists an open neighborhood U of x such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. Finally, denote the category of schemes with morphisms of locally ringed spaces by **Sch**.

Proposition 5.4. If X is a scheme then $\mathrm{Rg}(\mathfrak{S}_X) = X$.

Corollary 5.5. The functor,

$$\begin{aligned}
&\mathbf{Sch} \to \mathbf{Set}^{\mathbf{Ring}} \\
&X \mapsto \mathfrak{S}_X
\end{aligned}$$

is fully faithfull because,

$$\mathrm{Hom}_{\mathbf{Sch}}(X, Y) \cong \mathrm{Hom}_{\mathbf{Set}^{\mathbf{Ring}}}(\mathfrak{S}_X, \mathfrak{S}_Y)$$

Definition: If a functor $F : \mathbf{Ring} \to \mathbf{Set}$ is isomorphic to $\mathfrak{S}_X$ for some scheme X then we say that X *represents* F or F is *represented* by X.

Remark 11. The functor F above is not necessarily representable in the category of rings unless X is an affine scheme.

5.3 Toric Varieties

Let N be a fixed free $\mathbb{Z}$-module of finite rank n and $M = N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Now, consider the rational polyhedral cone,

$$\sigma = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_s$$

for $v_1, \ldots, v_s \in \mathbb{N}$. Then,

$$\dim \sigma_r = \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(\sigma))$$

and we may define the dual cone,

$$\sigma^\vee = \{\alpha \in M_\mathbb{R} = N_\mathbb{R}^\vee \mid \forall x \in \sigma \quad \alpha(x) \geq 0\}$$

A face is,

$$\tau = \sigma \cap \alpha^\perp = \{x \in \sigma \mid \alpha(x) = 0$$

for $\alpha \in \sigma^\vee$.

Lemma 5.6 (Gordon)**.** $S_\sigma = \sigma^\vee \cap M$ is a finitely-generated moniod.

Definition: Let k be a ring and σ a strongly convex rational polyhedral cone. Let $X_\sigma \to \mathrm{Spec}\,(k)$ be a scheme over $\mathrm{Spec}\,(k)$ such that,

$$\begin{aligned}\mathfrak{S}_{X_\sigma} &: \mathbf{Alg}_k \to \mathbf{Set} \\ &(k \to A) \mapsto \mathrm{Hom}_k\,(k[S_\sigma], A) = \mathrm{Hom}_{\mathbf{Mon}}\,((S_\sigma, +), (A, \times))\end{aligned}$$

where the functor is represented by,

$$k[S_\sigma] = \bigoplus_{m \in \sigma} k \cdot x^m$$

Then $X_\sigma = \mathrm{Spec}\,(k[S_\sigma]) \to \mathrm{Spec}\,(k)$ corresponds to $k \to k[S_\sigma]$. We call X_σ an *affine toric variety.*

Remark 12. If τ is a face of σ then $S_\tau \supset S_\sigma$ induces $k[S_\sigma] \to k[S_\tau]$ and thus a morphism $X_\tau \to X_\sigma$ which is an open embedding because it at the level of rings it is injective.

Remark 13. The smallest face $\{0\}$ has $\{0\}^\vee = M$ has,

$$X_{\{0\}} = \mathrm{Spec}\,(k[M]) \cong \mathrm{Spec}\,(k[\mathbb{Z}^n]) = \mathbb{G}_{m,k}^n$$

which is a torus.

Remark 14. If σ and τ intersect in a common face $S_{\sigma\cap\tau} = S_\sigma + S_\tau$ then the embeddings $X_{\sigma\cap\tau} \to X_\sigma, X_\tau$ allow gluing.

Definition: A *fan* is a collection Σ of strongly convex rational polyhedral cones in $N_\mathbb{R}$ such that,

1. $\forall \sigma \in \Sigma$ and any face τ of σ then $\tau \in \Sigma$
2. $\forall \sigma, \tau \in \Sigma$ the intersection $\sigma \cap \tau$ is a common face of σ and τ and $\sigma \cap \tau \in \Sigma$.

Definition: Given a fan Σ we define the toric variety X_Σ via gluing X_σ for each $\sigma \in \Sigma$. To see that this gluing works, consider the functor,

$$\begin{aligned}\mathfrak{S}_{X_\Sigma \to \mathrm{Spec}(k)} &: \mathbf{Alg}_k \to \mathbf{Set} \\ &A \mapsto \left\{ \bigcup_{\sigma \in \Sigma} S_\sigma \to A \;\middle|\; \forall \sigma \in \Sigma : f|_{S_\sigma} \to (A, \times) \text{ is a morphism of monoids} \right\}\end{aligned}$$

This functor is represented by the scheme X_Σ.

5.4 Divisors

5.5 Toric Divisors

Let $X_\Sigma \supset X_{\{0\}} = \mathrm{Spec}\,(k[M])$ be a toric variety. Then,

$$\mathrm{Rat}\,(X_\Sigma) = \mathrm{Frac}\,(k[M])$$

Definition: We call toric Cartier divisors of X_Σ andy Cartier divisor D which is locally defined by some $\chi^m \in \mathrm{Frac}\,(k[M])$ for $m \in M$.

5.5.1 Combinatorial Interpretation

First consider the support of the fan,

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$$

We define the notion of a vertual support function $\psi : |\Sigma| \to \mathbb{R}$ such that $\psi|_\sigma$ identifies with the restriction of some $m_\sigma \in M$ for any $\sigma \in \Sigma$.

Remark 15. m_σ is unique up to an element of $\sigma^\perp$.

Definition: $D_\psi = \{(X_\sigma, \chi^{-m_\sigma})\}$

Theorem 5.7. $\psi \mapsto D_\psi$ defines a bijection between toric Cartier divisors on X_Σ and vertual support functions on Σ.

6 The Open Problem

Let $X \to \mathrm{Spec}\,(k)$ be a projective scheme of dimension d over k. Given a line bundle L on X we may construct the Okounkov body $\Delta_L \subset \mathbb{R}^d$ a convex body. Then,

$$\int_{\Delta_d} 1 \,\mathrm{d}x = \lim_{n \to \infty} \frac{\dim_k H^0(X, L^{\otimes n})}{n^d}$$

In particular, given a scheme $\mathcal{X}$ over a projective curve C over k with a line bundle $\mathcal{L}$ on L then condier the diagram,

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\
X = \mathcal{X}_\eta \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\eta \arrow[r] & C \arrow[d] \\
 & \mathrm{Spec}\,(k)
\end{tikzcd}$$

with $L = \mathcal{L}|_X$ and η is the generic point of the curve C. Then there is a function on the Okounvof bundle $G_{\mathcal{L}} : \Delta_L \to \mathbb{R}$ such that,

$$\int_{\Delta_L} \max\{(G_{\mathcal{L}}(x), 0)\} = \lim_{n\to\infty} \frac{\dim_k H^0(\mathcal{X}, \mathcal{L}^{\otimes n})}{n^{d+1}}$$

For two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ form $\mathcal{L}_1 \otimes \mathcal{L}_2$. We have,

$$\Delta_{L_1} + \Delta_{L_2} \subset \Delta_{L_1 \otimes L_@}$$

and also

$$\forall (x, y) \in \Delta_{L_1} \times \Delta_{L_2} : G_{\mathcal{L}_1 \otimes \mathcal{L}_2}(x + y) \geq G_{\mathcal{L}_1}(x) + \mathcal{G}_{\mathcal{L}_2}(y)$$

7 Line Bundles

Definition: Let R be a ring and M an R-module. We say that M is invertible if there exists an R-module N such that $M \otimes_R N \cong R$. This is equivalent to the statement that the functor $M \otimes_R (-)$ is an equivalence of categories.

Definition: Let X be a locally ringed space and $\mathcal{L}$ an $\mathcal{O}_X$-module. We say that $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module if there exists an $\mathcal{O}_X$-module $\mathcal{L}'$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}' \cong \mathcal{O}_X$ which is equivalent to the statment that the functor $\mathcal{L} \otimes_{\mathcal{O}_X} (-)$ is an equivalence of cateogries.

Proposition 7.1. Let $\mathcal{L}$ be an invertable $\mathcal{O}_X$-module. Then,

1. $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$ where $\mathcal{L}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is the dual sheaf
2. given a morphism of locally ringed space $f : X \to Y$ and $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module then $f^*\mathcal{L}$ is an inertible $\mathcal{O}_X$-module
3. $\mathcal{L}$ is locally-free of rank 1 i.e. there exits an open cover of X such that on each open set $\mathcal{L}|_U \cong \mathcal{O}_X|_U$
4. on stalks, $\mathcal{L}_x \cong \mathcal{O}_{X,x}$

Proof. □

Definition: A *line bundle* on a locally ringed space is an invertible sheaf on X.

Proposition 7.2. Given two invertible sheaves of $\mathcal{O}_X$-modules $\mathcal{L}$ and $\mathcal{L}'$ on X their tensor product $\mathcal{L} \otimes_{\mathcal{O}_X} L'$ is an invertible $\mathcal{O}_X$-module.

Corollary 7.3. Isomorphism classes of line bundles on X form a group under tensor product which we denote $\mathrm{Pic}\,(X)$.

7.1 Divisors and Line Bundles

Given a Cartier divisor on an integral scheme X which we realize as $D = \{(U_i, f_i)\}$ where $\{U_i\}$ is an open cover of X and $f_i \in \mathrm{Rat}\,(X)$ such that $f_i f_j^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$. Then we may form an invertible sheaf $\mathcal{L}(D)$ as the subsheaf of $K^\times/\mathcal{O}_X^\times$ generated by $\{(U_i, f_i)\}$.

Proposition 7.4. In the previous situation,

1. $\mathcal{L}(D)$ is an invertible sheaf
2. $D_1 \sim D_2 \iff \mathcal{L}(D_1) \cong \mathcal{L}(D_2)$
3. $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}\,(X)$ as groups
4. $\mathrm{Cl}(X) \cong \mathrm{Pic}\,(X)$ when X is integral

Remark 16. Consider the exact sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{K}^\times \longrightarrow K^\times/\mathcal{O}_X^\times \longrightarrow 0$$

then taking cohomology,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^0(X, \mathcal{O}_X^\times) & \longrightarrow & H^0(X, \mathcal{K}_X^\times) & \longrightarrow & H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) & \longrightarrow & H^1(X, \mathcal{O}_X^\times) & \longrightarrow & H^1(X, \mathcal{K}_X^\times) \\ & & \| & & \| & & \| & & \| & & \| \\ 0 & \longrightarrow & \mathcal{O}_X(X)^\times & \longrightarrow & K(X)^\times & \longrightarrow & \mathrm{Ca}(X) & \longrightarrow & \mathrm{Pic}\,(X) & \longrightarrow & 0 \end{array}$$

Therefore, we get an isomorphism,

$$\mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Pic}\,(X)$$

where $\mathrm{CaCl}(X) = \mathrm{coker}\,(K(X)^\times \to H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. To see why $H^0(X, \mathcal{K}_X^\times) = 0$, recall that on an integral scheme (actualy any irreducible topological space) all open sets are connected and thus all constant sheaves are flasque and thus have trivial higher cohomology.

7.2 Examples

Consider the toric variety X_Σ then the picard group of Σ is given by virtual support functions modulo global linear functions. Let h be a virtual support function represented by elements $\{m_\sigma\}_{\sigma \in \Sigma}$ then we may construct $\mathcal{L}(h)$ an $\mathcal{O}_X$-module satisfying,

1. $\mathcal{L}(\varphi) = \{0\}$
2. $\mathcal{L}(U) = z^{m_\tau}\mathcal{O}_X(U)$ for $U \subset X_{\tau^\vee}$
3. $\mathcal{L}(U_1 \cup \cdots \cup U_s) = z^{m_1}\mathcal{O}_X(U_1) \cap \cdots \cap z^{m_s}\mathcal{O}_X(U_s)$

Theorem 7.5. We have,

1. $\mathcal{L}_h \cong \mathcal{L}_{h'} \iff h - h'$ is global linear
2. every invertible sheaf of X_Σ is isomorphic to $\mathcal{L}_h$ for some virtual support function h.

Corollary 7.6. For a toric variety X_Σ we have,

$$\mathrm{Pic}\,(X) = \mathrm{Pic}\,(\Sigma) = \frac{\mathrm{VS}(\Sigma)}{\Sigma} = \mathrm{CaCl}(X)$$

7.3 The Case of Noetherian Domains

Remark 17. In the affine case $X = \mathrm{Spec}\,(R)$ then any invertible sheaf is coherent since it is locally $\widetilde{R}$. Therefore it must be $\widetilde{M}$ for some R-module M which must be invertible as an R-module. Therefore, $\mathrm{Pic}\,(X) = \mathrm{Pic}\,(R)$ where $\mathrm{Pic}\,(R)$ is the group of invertible R-modules under tensor product.

Remark 18. Now restrict to the case of R a Noetherian domain.

Definition: A fractional ideal of R is is a f.g. submodule of $\mathrm{Frac}\,(R)$.

Theorem 7.7. Every invertible module is isomorphic to some fractional ideal. The invertable fractional ideals is free abelian group generated by height 1 prime ideas.

8 Cohomology of Sheaves

8.1 Derived Category of an Abelian Category

Definition: Let $\mathcal{A}$ be an abelian category and $\mathbf{Ch}\,(A)$ denote the category of cocomplexes in $\mathcal{A}$ i.e. diagrams,

$$\cdots \longrightarrow A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \cdots$$

such that $d^n \circ d^{n-1} = 0$. A morphism $f : A \to B$ in the category $\mathbf{Ch}\,(A)$ is a commutative diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f_{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "f_n"] & A^{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

i.e. such that $f_{n+1} \circ d_A^n = d_B^n \circ f_n$.

Definition: For any cocomplex A we may shift by m to form the complex $A[m]$ such that $A[m]^n = A^{n-m}$ and $d^n_{A[m]} = (-1)^m d_A^n$.

Definition: There exists a cohomology functor $H^n : \mathbf{Ch}(A) \to \mathcal{A}$ defined as follows. Since $d^{n+1} \circ d^n = 0$ we have that d^n factors through the kernel of d^{n+1} to give a map $d^n : A^n \to \ker d^{n+1}$. Then we define $H^n(A) = \operatorname{coker} d^n : (A^n \to \ker d^{n+1})$.

Definition: The homotopy category $\mathbf{K}(\mathcal{A})$ is the quotient of $\mathbf{Ch}(\mathcal{A})$ by chain homotopy where chain maps $f, g : A \to B$ are chain homotopic via a homotopy s if there exists a diagram,

$$
\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "\Delta_{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "\Delta_n"] \arrow[dl, "s_n"'] & A^{n+1} \arrow[r] \arrow[d, "\Delta_{n+1}"] \arrow[dl, "s_{n+1}"'] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}
$$

such that $s_{n+1} \circ d_A^n + d_B^{n-1} \circ s_n = \Delta_n = f_n - g_n$.

Proposition 8.1. The cohomology functor $H^n : \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$ factors through the quotient functor $\mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$.

Definition: We define the categories $\mathbf{Ch}^+(\mathcal{A})$ of bounded below cocomplexes i.e. cocomplexes A such that $A^n = 0$ for all sufficiently small n and the category $\mathbf{Ch}^b(\mathcal{A})$ of bounded cocomplexes i.e. cocomplexes A such that $A^n = 0$ for all but finitely many n.

Remark 19. These chain categories are not abelian. We want to fix this issue.

Definition: Let $A, B \in \mathbf{Ch}(\mathcal{A})$ and $f : A \to B$. We construct the cone $c(f) \in \mathbf{Ch}(\mathcal{A})$ via $c(f)^n = A^{n+1} \oplus B^n$ with a boundary map,

$$d_c^n = (-d_A^{n+1}, -f_{n+1} + d_B^n)$$

Then we may check,

$$d_c^{n+1} \circ d_c^n = (d_A^{n+1} \circ d_A^n, f_{n+1} \circ d_A^{n+1} - d_A^{n+2} \circ f_{n+1} + d_B^{n+1} \circ d_B^n) = 0$$

Proposition 8.2. There exists an exact sequence,

$$0 \longrightarrow B \xrightarrow{g} c(f) \xrightarrow{\delta} A[-1] \longrightarrow 0$$

where $g = (0, \mathrm{id})$ and $\delta = -\mathrm{pr}_1$. Such an exact sequence induces a long exact sequenc of cohomology,

$$\cdots \longrightarrow H^{n-1}(c(f)) \longrightarrow H^{n-1}(A[-1]) \longrightarrow H^n(B) \longrightarrow H^n(c(f)) \longrightarrow H^{n+1}(A) \longrightarrow \cdots$$

however, $H^{n-1}(A[-1]) = H^n(A)$ and the map $H^{n-1}(A[-1]) \to H^n(B)$ is simply the map induced by $f : A \to B$. Thus we find,

$$\cdots \longrightarrow H^{n-1}(c(f)) \longrightarrow H^n(A) \xrightarrow{f} H^n(B) \longrightarrow H^n(c(f)) \longrightarrow H^{n+1}(A) \longrightarrow \cdots$$

Corollary 8.3. We say that f is a quasi-isomorphism if $H^n(f)$ is an isomorphism for each n iff $c(f)$ is exact i.e. acyclic.

Definition: In general, a diagram,

$$A' \xrightarrow{u} B' \xrightarrow{v} C' \xrightarrow{w} A'[-1]$$

is called an *exact triangle* if there exists $f : A \to B$ in $\mathbf{Ch}(\mathcal{A})$ and α, β, γ isomorphisms in $\mathcal{K}(\mathcal{A})$ such that the following diagram commutes in $\mathcal{K}(\mathcal{A})$,

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "\alpha"] & B' \arrow[r, "v"] \arrow[d, "\beta"] & C' \arrow[r, "w"] \arrow[d, "\gamma"] & A'[-1] \arrow[d, "\alpha"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & c(f) \arrow[r, "\delta"] & A[-1]
\end{tikzcd}$$

Definition: The derived category $\mathfrak{D}(\mathcal{A})$ is the category obtained from $\mathcal{K}(\mathcal{A})$ by inverting all quasi-isomorphisms.

8.2 Injective Objects

Definition: In an abeian cateogory $\mathcal{A}$ we call an object I injective if the contravariant hom functor $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.

Remark 20. This condition is equivalent to the following. For any monomorphism $f : A \to B$ in $\mathcal{A}$ and a morphism $\alpha : A \to I$ there exists a unique extension $\beta : B \to I$ such that $\beta \circ f = \alpha$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\beta"] \\
 & I &
\end{tikzcd}$$

Definition: We say that $\mathcal{A}$ has *enough injectives* if for any $A \in \mathcal{A}$ there exists an injective object $I \in \mathcal{A}$ and a monomorphism $\alpha : A \to I$.

Theorem 8.4. Let $A, B \in \mathcal{A}$ be objects and $f : A \to B$ a morphism. Consider two complexes in $\mathcal{K}(A)$,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & \mathbf{M}^{\bullet} \arrow[d, dashed] \\
0 \arrow[r] & B \arrow[r] & \mathbf{I}^{\bullet}
\end{tikzcd}$$

such that the first is exact and $\mathbf{I}^{\bullet}$ is injective. Then there exists a unique morphism in $\mathbf{K}(\mathcal{A})$ between these complexes.

Corollary 8.5. Let $I \in \mathbf{Ch}^{+}(A)$ consist of injectives. If $f : I \to M$ is a quasi-isomorpihsm then f admits an inverse of the left in the category $\mathbf{K}(\mathcal{A})$. Furthermore, $\mathrm{Hom}_{\mathfrak{D}(\mathcal{A})}(M, I) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M, I)$.

8.3 Derived Functors

Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathbf{K}^+(\mathfrak{I})$ be the full subcategory of $\mathbf{K}^+(\mathcal{A})$ consisting of complexes of injective objects.

Theorem 8.6. There is an equivalence of categories $\mathfrak{D}^+(\mathcal{A}) \cong \mathbf{K}^+(\mathfrak{I})$.

Proof. Consider the inclusion functor $\mathbf{K}^+(\mathfrak{I}) \to \mathfrak{D}^+(\mathcal{A})$ which is fully faithfull because $\mathrm{Hom}_{\mathfrak{D}^+(\mathcal{A})}(I, J) = \mathrm{Hom}_{\mathbf{K}^+(\mathfrak{I})}(I, J)$ when $I, J \in \mathbf{K}^+(\mathfrak{I})$ are injective complexes. Thus it suffices to show that the functor is essentially surjective i.e. that for any complex $M \in \mathfrak{D}^+(\mathcal{A})$ there exists an injective resolution $I \in \mathfrak{D}^+(\mathcal{A})$ with a quasi-isomorphism $q : M \to I$. This is true when $\mathcal{A}$ has enough injectives. $\square$

Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ a functor preserving exact triangles. A right derived funtion of F is a functor $RF : \mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(\mathcal{B})$ and a morphism of functors ξ from $\mathbf{K}(\mathcal{A}) \xrightarrow{F} \mathbf{K}(\mathcal{B}) \to \mathfrak{D}(\mathcal{B})$ to $\mathbf{K}(\mathcal{A}) \to \mathfrak{D}(\mathcal{A}) \xrightarrow{RF} \mathfrak{D}(\mathcal{B})$ which satisfies the following universal property. If $G : \mathfrak{D}(\mathcal{A}) \to \mathfrak{D}(B)$ is another functor preserving exact triangles with a morphism of functors $\zeta : q \circ F \to G \circ q$ then there exists a unique morphism of functors $\eta : RF \to G$ such that $\zeta = \eta \circ \xi$.

Theorem 8.7. If $\mathcal{A}$ has enough injectives then $R^+F : \mathfrak{D}^+(\mathcal{A}) \to \mathfrak{D}(B)$ exists. Moreover, for any comlex consisting of injective objects $R^+F(I) = q \circ F(I) \in \mathfrak{D}(B)$.

Definition: We say that $A \in \mathbf{K}(\mathcal{A})$ is F-acyclic if $F(A)$ is exact.

Example 8.8. Let X be a topological space and $\mathcal{A} = \mathbf{Ab}(X)$ the category of sheaves of abelian groups on X. If $\mathcal{F} \in \mathcal{A}$ is a sheaf then consider $\mathcal{F}_x$

9 δ-functors

Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A (cohomological) δ-functor is a sequence of additive functors $T^n : \mathcal{A} \to \mathcal{B}$ and associated to each short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ a family of morphisms $\delta^n : T^n(C) \to T^{n+1}(A)$ such that there is a long exact sequnce,

$$\begin{tikzcd}
0 \arrow[r] & T^0(A) \arrow[r] & T^0(B) \arrow[r] & T^0(C) \arrow[r, "\delta^0"] & T^1(A) \arrow[r] & T^1(B) \arrow[r] & T^1(C) \arrow[dllllll, rounded corners, "\delta^1"'] & \\
& T^2(A) \arrow[r] & T^2(B) \arrow[r] & T^2(C) \arrow[r, "\delta^2"] & T^3(A) \arrow[r] & T^3(B) \arrow[r] & T^3(C) \arrow[r] & \cdots
\end{tikzcd}$$

Furthermore, associated to each morphism of short exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

the induced squares,

$$\begin{array}{ccc}
T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\
\downarrow & & \downarrow \\
T^n(C') & \xrightarrow{\delta^n} & T^{n+1}(A')
\end{array}$$

commute such that there is a morphism of long exact sequences,

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & T^0(A) & \longrightarrow & T^0(B) & \longrightarrow & T^0(C) & \xrightarrow{\delta^0} & T^1(A) & \longrightarrow & T^1(B) & \longrightarrow & T^1(C) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^0(A') & \longrightarrow & T^0(B') & \longrightarrow & T^0(C') & \xrightarrow{\delta^0} & T^1(A') & \longrightarrow & T^1(B') & \longrightarrow & T^1(C') & \longrightarrow & \cdots
\end{array}$$

Definition: A morphism of δ-functors $f : S \to T$ is a sequence of natural transformations $f^n : S^n \to T^n$ which, for each short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

commutes with the connecting maps as follows,

$$\begin{array}{ccc}
S^n(C) & \xrightarrow{\delta^n} & S^{n+1}(A) \\
\downarrow{\scriptstyle f_C^n} & & \downarrow{\scriptstyle f_A^{n+1}} \\
T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A)
\end{array}$$

such that there is a morphism of long exact sequences,

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & S^0(A) & \longrightarrow & S^0(B) & \longrightarrow & S^0(C) & \xrightarrow{\delta^0} & S^1(A) & \longrightarrow & S^1(B) & \longrightarrow & S^1(C) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_A^0} & & \downarrow{\scriptstyle f_B^0} & & \downarrow{\scriptstyle f_C^0} & & \downarrow{\scriptstyle f_A^1} & & \downarrow{\scriptstyle f_B^1} & & \downarrow{\scriptstyle f_C^1} & & \\
0 & \longrightarrow & T^0(A) & \longrightarrow & T^0(B) & \longrightarrow & T^0(C) & \xrightarrow{\delta^0} & T^1(A) & \longrightarrow & T^1(B) & \longrightarrow & T^1(C) & \longrightarrow & \cdots
\end{array}$$

Remark 21. Note that, by definition, if $T : \mathcal{A} \to \mathcal{B}$ is a cohomological δ-functor than $T^0 : \mathcal{A} \to \mathcal{B}$ is a left-exact additive functor. A homological δ-functor would give a left-exact additive functor $T^0 : \mathcal{A} \to \mathcal{B}$.

Definition: We call a δ-functor $S : \mathcal{A} \to \mathcal{B}$ universal if for any other δ-functor $T : \mathcal{A} \to \mathcal{B}$ with a natural transformation $\alpha : S^0 \to T^0$ it extends to a unique morphism of δ-functors $f : S \to T$ with $f^0 = \alpha$.

Proposition 9.1. Universal δ-functors with a given fixed initial additive functor $S^0 : \mathcal{A} \to \mathcal{B}$ are unique up to unique isomorphism.

Proof. Let S and T be two universal δ-functors with a natural isomorphism $\alpha^0 : S^0 \to T^0$. Applying the universal properties of S and T give morphism of δ-functors $f : S \to T$ and $g : T \to S$ such that $f^0 = \alpha$ and $g^0 = \alpha^{-1}$. Thus $g \circ f : S \to S$ is a morphism of δ-functors lifting $\mathrm{id}_{S^0} : S^0 \to S^0$ and thus $g \circ f = \mathrm{id}_S$ by the uniqueness of lifts in the universal property for S. Similarly, $f \circ g : T \to T$ is a morphism of δ-functors lifting $\mathrm{id}_{T^0} : T^0 \to T^0$ and thus $f \circ g = \mathrm{id}_T$ applying the uniqueness in the universal property for T. $\square$

Definition: Given a left-exact additive functor $F : \mathcal{A} \to \mathcal{B}$ if there exists a universal δ-functor $S : \mathcal{A} \to \mathcal{B}$ such that $S^0 = F$ then we call S^n the right-sattelite functors associated to F.

Definition: An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called *effaceable* if for each $A \in \mathcal{A}$ there exists a monomorphism $a : A \to M$ for some $M \in \mathcal{A}$ such that $F(a) = 0$. In particular, this is satisfied if $F(M) = 0$.

Theorem 9.2. Let $S : \mathcal{A} \to \mathcal{B}$ be a δ-functor. If S^n is effaceable for all $n \geq 1$, then S is universal.

Proof. Suppose that $T : \mathcal{A} \to \mathcal{B}$ is a δ-functor and $\alpha : S^0 \to T^0$ is a natural transformation. We construct the morphism of δ-functors $f : S \to T$ by induction. Such a natural transformation is given for $n = 0$ so assume we have constructed $f^n : S^n \to T^n$. Now for any $A \in \mathcal{A}$ since S^{n+1} is effaceable we may choose a monomorphism $a : A \hookrightarrow M$ such that $S^{n+1}(a) = 0$. Now consider the short exact sequence,

$$0 \longrightarrow A \xrightarrow{a} M \longrightarrow K \longrightarrow 0$$

which gives rise to long exact sequences,

$$\begin{tikzcd}
S^n(A) \arrow[r, "S^n(a)"] \arrow[d, "f^n_A"] & S^n(M) \arrow[r] \arrow[d, "f^n_M"] & S^n(K) \arrow[r, "\delta^n"] \arrow[d, "f^n_K"] & S^{n+1}(A) \arrow[r, "S^{n+1}(a)"] \arrow[d, dashed, "f^{n+1}_A"] & S^{n+1}(M) \\
T^n(A) \arrow[r, "T^n(a)"] & T^n(B) \arrow[r] & T^n(K) \arrow[r, "\delta^n"] & T^{n+1}(A) \arrow[r, "T^{n+1}(a)"] & T^{n+1}(B)
\end{tikzcd}$$

Since the morphism $S^{n+1}(a) : S^{n+1}(A) \to S^{n+1}(M)$ is zero then $S^{n+1}(A)$ is the cokernel of the morphism $S^n(M) \to S^n(K)$. By commutativity and exactness of the lower sequence, the morphism $S^n(M) \to S^n(K) \to T^n(K) \to T^{n+1}(A)$ is zero and thus factors uniquely through $f_A^{n+1} : S^{n+1}(A) \to T^{n+1}(A)$ this defines a morphism $f^{n+1} : S^{n+1} \to T^{n+1}$. It suffices to prove that f^{n+1} is natural and well-defined. (SHOW THIS) $\square$

Corollary 9.3. Let $S, T : \mathcal{A} \to \mathcal{B}$ be effaceable δ-functors which agree in degree zero i.e. $S^0 \cong T^0$ naturally. Then $S \cong T$ by a unique isomorphism lifting $S^0 \cong T^0$.

Theorem 9.4. Let $\mathcal{A}$ be an abelian category with enough injectives and $F : \mathcal{A} \to \mathcal{B}$ an additive functor. Then the right-derived functors $R^i F : \mathcal{A} \to \mathcal{B}$ form a universal δ-functor.

Proof. We have already proven that given an additive functor $F : \mathcal{A} \to \mathcal{B}$ on an abelian category $\mathcal{A}$ having enough injectives, the derived functors form a δ-functor. For each $A \in \mathcal{A}$ because $\mathcal{A}$ has enough injectives there exists an injective object $I \in \mathcal{A}$ and a monomorphism $a : A \to I$. Since I is injective $R^i F(I) = 0$ for any $i \geq 1$ and thus $R^i F(a) : R^i F(A) \to R^i F(I)$ is the zero morphism. Thus for each $i \geq 1$, the derived functor $R^i F$ is effaceable. Therefore, right-derived functors $R^i F : \mathcal{A} \to \mathcal{B}$ form a universal δ-functor. $\square$

Corollary 9.5. Let $\mathcal{A}$ be an abelian category with enough injectives and $F : \mathcal{A} \to \mathcal{B}$ a left-exact additive functor. Then the right-satellite functors of F exist and are canonically isomorphic to the right-derived functors $R^i F : \mathcal{A} \to \mathcal{B}$.

Corollary 9.6. Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{B}$ be an abelian category. Suppose that $S, T : \mathcal{A} \to \mathcal{B}$ are δ-functors such that $F = S^0 \cong T^0$ naturally and for each injective $I \in \mathcal{A}$ we have $S^n(I) = T^n(I) = 0$ for all $n \geq 1$. Then there are canonical isomorphism of δ-functors, $S \cong T \cong RF$.

Definition: Let $T : \mathcal{A} \to \mathcal{B}$ be a δ-functor. We say that $A \in \mathcal{A}$ is T-acyclic if $T^n(A) = 0$ for all $n \geq 1$.

Proposition 9.7. Let $T : \mathcal{A} \to \mathcal{B}$ be a δ-functor and,

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow K \longrightarrow 0$$

be an exact sequence in which the C^i are T-acyclic. Then we have,

$$\forall i > n+1 : T^i(A) = T^{i-n}(K) \qquad T^{n+1}(A) = \text{coker}\,(T^0(C^n) \to T^0(K))$$

Proof. We proceed by induction. First, consider the case $n = 0$ in which we have an exact sequence,

$$0 \longrightarrow A \longrightarrow C \longrightarrow K \longrightarrow 0$$

where C is T-acyclic. This short exact sequence gives a long exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & T^0(A) \arrow[r] & T^0(C) \arrow[r] & T^0(K) \arrow[r] & T^1(A) \arrow[r] & 0 \arrow[r] & T^1(K) \arrow[dllllll, rounded corners] \\ & T^2(A) \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & T^i(K) \arrow[r] & T^{i+1}(A) \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Thus we find isomorphism $T^{i+1}(A) = T^i(K)$ for $i \geq 1$ and furthermore,

$$T^1(A) = \text{coker}\,(T^0(C) \to T^0(K))$$

Now assume the statement holds for fixed n and consider the case $n+1$. We may split the exact sequence,

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow K \longrightarrow 0$$

into a pair of exact sequences,

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow N \longrightarrow 0$$

and

$$0 \longrightarrow N \longrightarrow C^{n+1} \longrightarrow K \longrightarrow 0$$

where $N = \ker(C^{n+1} \to K)$. By the induction hypothesis applied to the first sequence we have,

$$\forall i > n+1 : T^i(A) = T^{i-n}(N) \qquad T^{n+1}(A) = \text{coker}\,(T^0(C^n) \to T^0(N))$$

and from the $n = 0$ case applied to the second short exact sequence we find,

$$\forall i > 1 : T^i(N) = T^{i-1}(K) \qquad T^1(N) = \text{coker}\,(T^0(C^{n+1}) \to T^0(K))$$

Therefore, for $i > n+2$ applying the first and then second result we find,

$$T^i(A) = T^{i-n}(N) = T^{i-n-1}(K)$$

which holds since $i - n > 1$. Furthermore, setting $i = n+2$ we find,

$$T^{n+2}(A) = T^1(N) = \text{coker}\,(T^0(C^{n+1}) \to T^0(K))$$

proving the result by induction. □

Proposition 9.8. Let $T : \mathcal{A} \to \mathcal{B}$ be δ-functor and for $A \in \mathcal{A}$ let,

$$0 \longrightarrow A \longrightarrow \mathbf{C}_A^\bullet$$

be an T-acylic resultion of A i.e. an exact complex such that C^i is T-acylic for each C^i. Then for all $n \geq 0$ we may compute the satellite functors as the cohomology,

$$T^n(A) = H^n(T^0(\mathbf{C}_A^\bullet))$$

Proof. First, since $A = \ker(C^0 \to C^1)$ and since T^0 is left-exact it preserves kernels so $T^0(A) = \ker(T^0(C^0) \to T^0(C^1)) = H^0(T^0(\mathbf{C}_A^\bullet))$. For $n \geq 1$ we may terminate the acyclic resoluton at C^{n-1} by adding $K = \text{coker}\,(C^{n-2} \to C^{n-1})$ to form an exact sequence,

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^{n-1} \longrightarrow K \longrightarrow 0$$

By the previous proposition $T^n(A) = \text{coker}\,(T^0(C^{n-1}) \to T^0(K))$. However, by the exactnes of the resolution $K = \ker(C^n \to C^{n+1})$ and thus, again by left-exactness, $T^0(K) = \ker(T^0(C^n) \to T^0(C^{n+1}))$. Therefore,

$$T^n(A) = \text{coker}\,(T^0(C^{n-1}) \to \ker(T^0(C^n) \to T^0(C^{n+1})) = H^n(T^0(\mathbf{C}_A^\bullet)$$

□

Remark 22. Now we return to the situation of an abelian category $\mathcal{A}$ with enough injectives and an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories. We say that $C \in \mathcal{A}$ is F-acylic if $R^i F(C) = 0$ for all $i > 0$ i.e. if "all higher-derived vanish".

Corollary 9.9. The derived functors of F can be computed as the cohomology of F applied to any F-acyclic resolution i.e. if

$$0 \longrightarrow A \longrightarrow \mathbf{C}_A^\bullet$$

is an exact sequence such that C^n is F-acyclic for each $n \geq 0$ then, for all $n \geq 0$,

$$R^n F(A) = H^n(F(\mathbf{C}_A^\bullet))$$

Theorem 9.10 (de Rham)**.** Let M be a smooth manifold then for each $n \geq 0$ there is a natural isomorphism,

$$H^n_{\text{sing}}(M; \mathbb{R}) = H^n_{\text{dR}}(M)$$

Theorem 9.11. Consider the complex of sheaves on M,

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \Omega^0 \xrightarrow{\mathrm{d}^0} \Omega^1 \xrightarrow{\mathrm{d}^1} \Omega^2 \longrightarrow \cdots$$

where Ω^k is the sheaf of differential k-forms on M. The above is clearly a complex because $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$ by the definition of the exterior derivative. Furthermore, by the Poincare lemma this complex of sheaves is exact since every k-form is locally exact. Furthermore, by taking partitions of unity, we may extend any locally defined k-form to a globally defined k-form meaning that the restriction maps are surjective. Thus the sheaves Ω^k are flasque and thus Γ-acylic so the sheaves of differential forms form a Γ-acyclic resolution of $\underline{\mathbb{R}}$. Therefore, by the above propositions,

$$H^n_{\text{sing}}(X; \mathbb{R}) = H^n_{\text{sheaf}}(X, \underline{\mathbb{R}}) = R^n\Gamma(X, \underline{\mathbb{R}}) = H^n(\Gamma(X, \Omega^\bullet)) = H^n(\Omega^\bullet(M)) = H^n_{\text{dR}}(M)$$

10 Cartier Divisors

Definition: Let X be a locally ringed space and S_X the sheaf on X defined by,

$$S_X(U) = \{s \in \mathcal{O}_X(U) \mid \mathcal{O}_X|_U \xrightarrow{s} \mathcal{O}_X|_U \text{ is a monomorphism}\}$$

Then let $\mathcal{K}_X = (U \mapsto S_X(U)^{-1}\mathcal{O}_X(U))^{++}$ be the sheafification.

Definition: The sheaf of divisors is defined as the $\mathcal{O}_X$-module,

$$\mathfrak{Div}_X = \mathcal{K}_X^\times / \mathcal{O}_X^\times$$

Then the Cartier divisors on X are the group $\mathrm{Ca}(X) = \mathrm{Div}_X(X) = \Gamma(X, \mathfrak{Div}_X)$. Furthermore, we define the Cartier divisor class group to be the quotient of Cartier divisors by global invertible rational sections i.e. sheaf map $\mathcal{K}_X^\times \to \mathcal{K}_X^\times / \mathcal{O}_X^\times$ giving a map on global sections gives a cokernel sequence,

$$\Gamma(X, \mathcal{K}_X^\times) \longrightarrow \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \longrightarrow \mathrm{CaCl}(X) \longrightarrow 0$$

Proposition 10.1. There is a monomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}\,(X)$ which is an isomorphism whenever $H^1(X, \mathcal{K}_X^\times) = 0$.

Proof. Consider the exact sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{K}_X^\times \longrightarrow \mathfrak{Div}_X \longrightarrow 0$$

Taking the long exact sequence of cohomology we find,

$$\begin{array}{ccccccccc} 1 \longrightarrow H^0(X, \mathcal{O}_X^\times) & \longrightarrow & H^0(X, \mathcal{K}_X^\times) & \longrightarrow & H^0(X, \mathfrak{Div}_X) & \longrightarrow & H^1(X, \mathcal{O}_X^\times) & \longrightarrow & H^1(X, \mathcal{K}_X^\times) \\ & & & & & & \| & & \\ & & & & & & \mathrm{Pic}\,(X) & & \end{array}$$

By exactness,

$$\ker\,(H^1(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{K}_X^\times)) = \mathrm{coker}\,(H^0(X, \mathcal{K}_X^\times) \to H^0(X, \mathfrak{Div}_X)) = \mathrm{CaCl}(X)$$

so we have an exact sequence,

$$1 \longrightarrow \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}\,(X) \longrightarrow H^1(X, K_X^\times)$$

□

Proposition 10.2. Let X is a reduced scheme with finitely many irreducible components then $H^1(X, \mathcal{K}_X^\times) = 0$.

Proof. □

Corollary 10.3. On a reduced scheme X with finitely many irreducible components, the natural monomorphism $\mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Pic}\,(X)$ is an isomorphism.

Definition: On a locally ringed space X we define the sheaf of *effective divisors* by the "positive subsheaf" $\mathfrak{Div}_X^+ = S_X/\mathcal{O}_X^\times$ of $\mathfrak{Div}_X$. Furthermore, we define the *effective Cartier divisors* $\mathrm{Ca}^+(X) = H^0(X, \mathfrak{Div}_X^+)$.

Remark 23. A Cartier divisor D is effective $\implies$ $\mathcal{O}_X(D)$ admits a global nonzero section. If X is an integral scheme and $\mathcal{O}_X(D)$ has a non-zero global section then D is equivalent to an effective divisor. There is a non-canonical bijection,

$$H^0(X, \mathcal{O}_X(D)) = \{f \in \mathrm{Rat}\,(X)^\times \mid \mathrm{div}(f) + D \text{ is effective}\} \cup \{0\}$$

10.1 Dimension and Length

Definition: Let X be a noetherian scheme. For any $k \in \mathbb{N}$ define,

$$X^{(k)} = \{x \in X \mid \dim \mathcal{O}_{X,x} = k\}$$

and let $Z^k(X)$ be the free abelian group generated by $X^{(k)}$.

Definition: Let A be a ring and M an A-module. Then $\text{length}_A(M)$ is the largest n such that ther exists a proper chain of submodules,

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \text{Spec}(A)$ which we call a composition sequence.

Proposition 10.4. If A is Noetherian and M is finite type then M admits a composition sequence.

Proposition 10.5. If there is an exact sequence,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

if M has finite length then so does M' and M'' and furthermore,

$$\text{length}_A(M) = \text{length}_A(M') + \text{length}_A(M'')$$

Proposition 10.6. If A is noetherian and M is finitely generated then the following are equivalent,

1. M is of finite length
2. M is artinian
3. the associate prime ideals of M are maximal
4. the spectrum of M consists of maximal prime ideals

Corollary 10.7. Let A be a noetherian domain with $\dim A = 1$. For any nonzero $a \in A$ then any prime containing a is maximal so $A/(a)$ has finite length.

Remark 24. The above obervation allows us to define.

Definition: $\text{ord}_A(a) = \text{length}_A(A/(a))$

Proposition 10.8. For any $a, b \in (A \setminus \{0\})^2$ then $\text{ord}_A(ab) = \text{ord}_A(a) + \text{ord}_A(b)$.

Proof. Consider the exact sequence,

$$0 \longrightarrow (b)/(ab) \longrightarrow A/(ab) \longrightarrow A/(b) \longrightarrow 0$$

Furthermore, since A is a domian, $(b)/(ab) \cong A/(a)$ as A-modules so we have an exact sequence of A-modules,

$$0 \longrightarrow A/(a) \longrightarrow A/(ab) \longrightarrow A/(b) \longrightarrow 0$$

which proves that,

$$\mathrm{ord}_A(ab) = \mathrm{length}_A\left(A/(ab)\right) = \mathrm{length}_A\left(A/(a)\right) + \mathrm{length}_A\left(A/(b)\right) = \mathrm{ord}_A(a) + \mathrm{ord}_B(b)$$

□

Definition: We extend $\mathrm{ord}_A(\cdot)$ to $\mathrm{Frac}\left(A\right)^\times \to \mathbb{Z}$ via $\mathrm{ord}_A(\frac{a}{b}) = \mathrm{ord}_A(a) - \mathrm{ord}_A(b)$.

Definition: Let X be a noetherian integral scheme and $\mathrm{Rat}\left(X\right) = \mathrm{Frac}\left(\mathcal{O}_{X,x}\right)$ the rational functions on X for any $x \in X$. If $x \in X^{(1)}$ is a codimension 1 point then $\mathcal{O}_X x$ is a Noetherian integral scheme with $\dim \mathcal{O}_{X,x} = 1$. Therefore, there is a valuation $\mathrm{ord}_x : \mathrm{Rat}\left(X\right)^\times \to \mathbb{Z}$ via $f \mapsto f_x \mapsto \mathrm{ord}_{\mathcal{O}_{X,x}}(f)$ since $f_x \in \mathrm{Frac}\left(\mathcal{O}_{X,x}\right)$.

Proposition 10.9. If D is a Cartier divisor on X then locally on $x \in X^{(1)}$ the divisor D is defined by a rational function $f_{D,x} \in \mathrm{Rat}\left(X\right)^\times$ up to a section $s \in \mathcal{O}_X^\times$. However, $s_x \in \mathcal{O}_{X,x}^\times$ so $\mathrm{ord}_x(s f_{D,x}) = \mathrm{ord}_x(s) + \mathrm{ord}_x(f_{D,x}) = \mathrm{ord}_x(f_{D,x})$ since s_x is invertible. Therefore, $\mathrm{ord}_x(D)$ is well-defined.

Definition: Each Cartier divisor D defines a cycle in Z^1,

$$[D] = \sum_{x \in X^{(1)}} \mathrm{ord}_x(f_{D,x}) \cdot [x]$$

10.2 General Intersection

Let X be a Noetherian scheme $k \in \mathbb{Z}^+$ and $y \in X^{(k-1)}$. If $x \in X^{(k)}$ such that $x \in \overline{\{y\}} = Y$ then $x \in Y^{(1)}$. Here we consisder Y as an integral closed subscheme of X. For any $f \in \mathrm{Rat}\left(Y\right)^\times$ we define a k-cycle on X as follows,

$$\mathrm{div}(f) = \sum_{x \in X^{(k)} \cap Y} \mathrm{ord}_x(f) \cdot [x] \in Z^k(X)$$

Then we define,

$$R^k(X) = \sum_{y \in X^{(k-1)}} \mathrm{Im}((\mathrm{Rat}\left(\overline{\{y\}}\right)^\times \to Z^k(X)))$$

And funally, we define the Chow group of codimension k,

$$CH^k(X) = Z^k(X)/R^k(X)$$

Proposition 10.10. Let X be a noetherian integral scheme. Then there is a canonical map $\mathrm{CaCl}(X) \xrightarrow{\sim} CH^1(X)$.

Proof. Consider the map $\mathrm{Ca}X \to Z^1(X)$ via $D \mapsto [D]$. Then $\forall f \in \mathrm{Rat}\,(X)^\times$ we have $[f] \in R^1(X)$. Thus the map factors through the quotient $\mathrm{CaCl}(X) \to CH^1(X)$. $\square$

Proposition 10.11. Assume $\forall x \in X$ that $\mathcal{O}_X x$ is integrally closed. Then the map $[\cdot] : \mathrm{Pic}\,(X) \to CH^1(X)$ is injective and so is the map $[\cdot] : \mathrm{Ca}(X) \to Z^1(X)$.

Proposition 10.12. Assume $\forall x \in X$ then $\mathcal{O}_X x$ is a UFD. Then the map $[\cdot] : \mathrm{Pic}\,(X) \to CH^1(X)$ an isomorphism and so is the map $[\cdot] : \mathrm{Ca}(X) \to Z^1(X)$.

10.3 Relative Constructions

Let X, Y be noetherian schemes and $f : X \to Y$ proper. Then we may define,

$$f_* : Z(X) = \bigoplus_{k \in \mathbb{N}} Z^k(X) \to Z(Y)$$

via $f_*([x]) = \deg(x/f(x)) \cdot [f(x)]$ where,

$$\deg(x/f(x)) = \begin{cases} [\kappa(x) : \kappa(f(x))] & \text{finite} \\ 0 & \text{otherwise} \end{cases}$$

Remark 25. Assume that X and Y are integral, $f : X \to Y$ is proper and surjective, and $r \in \mathrm{Rat}\,(X)^\times$ then $d = [R(X) : R(Y)]$. If d is infinite then $f_*(\mathrm{div}(r)) = 0$. If d is finite then $f_*(\mathrm{div}(r)) = \mathrm{div} N_{R(X)/R(Y)}(r)$.

Corollary 10.13. If $f : X \to Y$ is proper and surjective then f_* induces a morphism $f_* : CH(X) \to CH(Y)$.

Corollary 10.14. Let k be a field and $\pi : X \to \mathrm{Spec}\,(k)$ be a proper scheme over k then we get a degree map,

$$\deg = \pi_* : Z(X) \to Z(\mathrm{Spec}\,(k)) = \mathbb{Z}$$

11 Intersections

Remark 26. Let X be a Noetherian integral scheme and D a Cartier divisor on X. Take $V \in Z^k(X)$ and $x \in X^{(k)}$. Locally, D is defined by $f \in \mathrm{Rat}\,(X)^\times$. When $f \in \mathcal{O}_{X,x}^\times$ then f defines an elment $\bar{f}$ of $\kappa(x)$ so let $Y = \overline{\{x\}}$ be an integral closed subvariety. Take $\mathrm{div}(\bar{f})$ as $D \cdot x$. In general, this does not work unless we pass to the Chow group.

Definition: Let $\mathcal{L}$ be a line bundle on X representing the class $[D]$ in $\mathrm{Pic}\,(X)$. Let $\mathcal{L}|_Y$ be its restriction as a line bundle on Y. Take a nonzero section $s \in \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{K}(Y)$ then,

$$\mathrm{div}(s) = \sum_{y \in Y^{(1)}} \mathrm{ord}_r(s) \cdot [y]$$

This divisor class is independent of the choice of section s in the Chow group. Thus we define $D \cdot [x] = [\mathrm{div}(s)]$ giving a map $D \cdot : CH(X) \to CH(X)$ given by,

$$[v] = \left[\sum_{x \in X} \alpha_{V,x} x \right] \mapsto \sum_{x \in X} \alpha_{V,x} D \cdot [x]$$

11.1 The Toric Case

Let Σ be a fan i.e. a collection of strictly convext rational polyhedral cones such that $\sigma, \tau \in \Sigma \implies \sigma \cap \tau \in \Sigma$ and is a face of each. Let k be a field. Then we may construct the toric variety X_Σ. Now,

$$\mathrm{Rat}\left(X_\Sigma\right) = \mathrm{Frac}\left(k[M]\right) \cong k(T_1, \dots, T_n)$$

We have the notion of a toric divisor, D_ψ where ψ is a vertual support functo $\psi : |\Sigma| \to \mathbb{R}$. For $\sigma \in \Sigma$ we have $\psi|_\sigma = m_\sigma \in M_\sigma$. On X_σ the divisor D_ψ is defined by χ^{-m_σ}. Then we may construct a line bundle $L_\psi = \mathcal{O}_X(D)$ and furthermore, $D_{\psi_1} \sim D_{\psi_2} \iff \psi_1 - \psi_2 = m|_{|\Sigma|}$ for $m \in M$.

Now consider,

$$\Sigma^{(k)} = \{\sigma \in \Sigma \mid \sigma \text{ has dimension } k\}$$

For each $\tau \in \Sigma^{(1)}$ we can choose a vector $v_\tau \in N$ which generates $\tau \subset N_\mathbb{R}$ and is of minimal length. Consider $V(\tau) = X_\Sigma \setminus X_\tau \subset X_\Sigma$ is irreducible and closed so take x_τ its generic point. There is a map,

$$\begin{aligned} \mathrm{Div}(X_\Sigma) &\to Z^1(X_\Sigma) \\ D_\psi &\mapsto \sum_{\tau \in \Sigma} -\psi(v_\tau) \cdot x_\tau \end{aligned}$$

For $\sigma \in \Sigma$ we choose $m_\sigma \in M$ such that $\psi|_\sigma = m_\sigma|_\sigma$. Take $\psi' = \psi - m_\sigma|_{|\Sigma|}$ then clearly $D_\psi \sim D_{\psi'}$ furthermore for any face $\tau \in \Sigma^{(1)}$ we have $\psi'|_\sigma = 0$ and thus $\psi'(v_\tau) = 0$.

11.2 NEEDS Work

Theorem 11.1. $\dim_k H^0(D_\psi) = \#\left(\Delta_\psi \cap M\right)$

Corollary 11.2. For higher twists,

$$\dim_k H^0(n\Delta_\psi) = \#\left(n\Delta_\psi \cap M\right) = \#\left(\Delta_\psi \cap \frac{1}{n}M\right) \to n^d \mathrm{Vol}_d\left(\Delta_\psi\right)$$

Theorem 11.3. We have the following properties. If $d = \mathrm{rk}_\mathbb{Z}(N)$ then,

1. For higher tensor powers,

$$\lim_{n\to\infty} \frac{\dim_k(H^0(nD_\psi))}{n^d} = \mathrm{Vol}_d\left(\Delta_\psi\right)$$

2. The bundle $\mathcal{O}_{X_\Sigma}(D_\psi)$ is generated by global sections iff ψ is convave.

3. The bundle $\mathcal{O}_{X_\Sigma}(D_\psi)$ is ample iff ψ is strictly concave.

4. By Riemann-Roch,

$$\mathrm{Vol}_d\left(\Delta_\psi\right) = \frac{1}{d!} \deg\left(D_\psi^d[X_\Sigma]\right)$$

11.3 Absolute Value

Definition: Let K be a field. A *absolute value* on K is a map $|\bullet| : K \to \mathbb{R}_{\geq 0}$ such that,

1. $|x| = 0 \iff x = 0$
2. $|x \cdot y| = |x| \cdot |y|$
3. there exists $c > 0$ s.t. $|1 + x| \leq c$ for all $|x| \leq 1$.

Remark 27. If $|\bullet| : K \to \mathbb{R}_{\geq 0}$ is an absolute value then so is $|\bullet|^\alpha$ for any positive real number $\alpha \in \mathbb{R}_+$.

Example 11.4. The following are absolute values,

1. Trivial: $|0| = 0$ and $|x| = 1$ for $x \in K^\times$.
2. Real $K = \mathbb{R}$ we have,
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$
3. p-adic for $K = \mathbb{Q}$ write,
$$x = \prod_{p \in \mathrm{Spec}(\mathbb{Z})} p^{v_p(x)}$$
with $v_p(x) \in \mathbb{Z}$. Then $v_p(x)$ has finite support on $\mathrm{Spec}(\mathbb{Z})$ (it is the closed set $V((x))$ which is always finite) then take,
$$|x|_p = p^{-v_p(x)}$$

Remark 28. By definition $|1| = |1 \cdot 1| = |1| \cdots |1|$ so $|1| = 1$. Let $x \in K^\times$ such that $x^n = 1$ then $|x|^n = |x^n| = |1| = 1$. In particular, there is only the trivial absolute value on finite fields.

Definition: The pair $(K, |\bullet|)$ is called a valued field. If $k \subset K$ is a subfield then we may restric the absolute value to $(k, |\bullet|)$ to make k a valued field.

Definition: Let $|\bullet|_1$ and $|\bullet|_2$ be absolute values on K. We say these absolute values are equivalent if there exists a positive real constant $c \in \mathbb{R}_+$ such that,
$$|\bullet|_1 = |\bullet|_2$$

Definition: Let $(K, |\bullet|)$ be a valued field. The norm of $|\bullet|$ is,
$$N(|\bullet|) = \sup_{|x| \leq 1} |1 + x|$$

Note that for $x = 0$ we fine $|1 + x| = 1$ so $N(|\bullet|) \geq 1$.

Proposition 11.5. Let a map $|\bullet| : K \to \mathbb{R}_+$ verify (1) and (2) in the definition. Then following are equivalent,

1. $\exists c \in \mathbb{R}_+ : \forall |x| \leq 1 : |1+x| \leq c$
2. $\forall x, y \in K : |x+y| \leq c \max\{|x|, |y|\}$

In particular, if $|\bullet|$ is an absolute value then,

$$\forall x, y \in K : |x+y| \leq N(|\bullet|) \max\{|x|, |y|\}$$

Definition: A map $|\bullet| : K \to \mathbb{R}_+$ satisfies the triangle innequality if,

$$\forall x, y \in K : |x+y| \leq |x| + |y|$$

Remark 29. If $|\bullet| : K \to \mathbb{R}_+$ satisfing (1) an (2) satisfies the triangle innequality then $|\bullet|$ is an absolute value.

Proposition 11.6. Let $(K, |\bullet|)$ be a valued field. Then $N(|\bullet|) \leq 2 \iff |\bullet|$ satisfies the triangle innequality.

Corollary 11.7. Every absolute value on K is equivalent to one which satisfies the trangle inequality.

11.4 Topology on Valued Fields

Definition: Let $(K, |\bullet|)$ be a valued field. Then $d(x,y) = |x-y|$ is a metric on K which thus induces the metric (in this case value) topology. A basis for this topology is the set of open balls,

$$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_+ \text{ and } x \in K\}$$

where,

$$B_\epsilon(x) = \{y \in K : d(x,y) < \epsilon\}$$

Under this topology, the maps,

$$\begin{aligned} (x,y) &\mapsto x+y \\ (x,y) &\mapsto xy \\ x &\mapsto x^{-1} \\ x &\mapsto -x \\ x &\mapsto |x| \end{aligned}$$

are continuous.

11.5 Archemedean and Non-Archemedean Absolute Values

Definition: Let M_K be the equivalence classes of absolute values on K each class is called a *place* of K. In each class choose a representative for the place which satisfies the triangle inequality.

Definition: An absolute value $|\bullet|$ is nonarchemedean if it satisfies the ultrametric inequality,

$$\forall x, y : |x + y| \leq \max\{|x|, |y|\}$$

This is equivalent to $N(|\bullet|) = 1$. Otherwise, if $N(|\bullet|) > 1$ then we say that $|\bullet|$ is archemedean.

Remark 30. If $|\bullet|$ is nonarchemedean then we have,

$$|x + y| = \max\{|x|, |y|\}$$

Example 11.8. For the following absolute values,

1. $(\mathbb{R}, |\bullet|)$ is archemedean with $N(|\bullet|) = 2$
2. $(K, |\bullet|_{\text{triv}})$ is nonarchemedean.
3. $(\mathbb{Q}, |\bullet|_p)$ is nonarchemedean.

Definition: Let $(K, |\bullet|)$ be a valued nonarchemedean field. Then define,

$$\begin{aligned} A &= \{x \in K \mid |x| \leq 1\} \\ \mathfrak{m} &= \{x \in L \mid |x| < 1\} \\ U &= \{x \in K \mid |x| = 1\} \end{aligned}$$

Proposition 11.9. $A = U \cup \mathfrak{m}$ is a subring of K with $K = \text{Frac}(A)$ called the valuation ring which is local with maximal ideal $\mathfrak{m}$. Thus $A^\times = U$ and finally, A is integrally closed. Finally define the residue field $k = A/\mathfrak{m}$.

11.6 Valuations

Definition: Let A be a commutative ring. Then a valuation on A is a map $v : A \to \mathbb{R} \cup \{\infty\}$ satisfing,

1. $v(x) = \infty \iff x = 0$
2. $v(xy) = v(x) + v(y)$
3. $v(x + y) \geq \min\{v(x), v(y)\}$

Remark 31. $v(1) = 0$ and we may extend $v : \text{Frac}(A) \to \mathbb{R}$ via $v(x/y) = v(x) - v(y)$.

Proposition 11.10. Let K be a field. Then there is bijection between valuations on K and nonarchemedean absolute values on K by the mappings,

$$v \mapsto \exp \circ (-v) \qquad\qquad |\bullet| \mapsto -\log \circ |\bullet|$$

Remark 32. Given a nonarchemedean valued field $(K, |\bullet|)$ with corresponding valuation v then we have,

$$\begin{aligned} A &= \{x \in K \mid v(x) \geq 0\} \\ \mathfrak{m} &= \{x \in L \mid v(x) > 0\} \\ U &= \{x \in K \mid v(x) = 0\} \end{aligned}$$

Remark 33. We have a submonoid $v(K^\times) \subset (\mathbb{R}, +)$. If this submonoid is discrete then we may normalize such that $v(K^\times) = \mathbb{Z}$.

Definition: Let $(K, |\bullet|)$ be a nonarchemedean valued field. Then, $|\bullet|$ is discrete iff $\mathfrak{m}$ is a principal ideal. In that case $\mathfrak{m} = (\varpi)$ and thus is A is a discrete valuation ring.

Theorem 11.11. *A is a discrete valuation ring iff A is a local Dedekind domain.*

11.7 Completion of Valued Fields

Theorem 11.12. *Let M be a metric space. Then there exists a completion $\hat{M}$ and a continuous isometric embedding $M \hookrightarrow \hat{M}$ such that $\hat{M}$ is a complete metric space and $M \hookrightarrow \hat{M}$ is dense.*

Definition: Let $(K, |\bullet|)$ be a valued field. Then there exists a complete valued field $(\hat{K}, |\bullet|)$ containing it isometrically. If $v \in M_v$ then we denote this completion as K_v.

Definition: We say that a valued field K is a local field if it is a locally compact topological fiel.

Theorem 11.13. *Every local field is complete.*

Theorem 11.14. *Let $(k, |\bullet|)$ be a local field and $[K : k]$ is finite then there exists a unique absolute value on K which extends $|\bullet|$ defined by,*

$$|x|_K = |N_{K/k}(x)|^{1/[K:k]}$$

12 Isodida's Theorem and the Todd genus

12.1 Polyhedral Laurent Series

Definition: Let A be a unital commutative ring and M a free $\mathbb{Z}$-module of rank r. Let $N = M^\vee$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Also denote $A[M]$ to be the A-algebra generated by M as a monoid and $A(M)$ its qutient ring. Finally, we define the laurent series $A[[M]] = \mathrm{Hom}_{\mathrm{Mod}_A}(A[M], A)$.

Definition: Let σ be a polyhedral cone in $M_{\mathbb{R}}$. Then σ is nonsingular if $\sigma = \mathbb{R}_{+}m_1 + \cdots + \mathbb{R}_{+}m_r$ such that $\{m_1, \cdots m_r\}$ is a masis for M. Then $\mathrm{gen}\,(\sigma) = \{m_1, \ldots, m_r\}$.

Definition: Let σ be a nonsingular cone in $M_{\mathbb{R}}$. Then,

$$q_0(\sigma) = \sum_{m \in \iota(\sigma) \cap M} \chi^m$$

Then $PL_A(M)$ is generated by,

$$\{q_0(\sigma) \mid \sigma \textit{ is a nonsingular cone}\}$$

Then,

$$Q_0(\sigma) = \prod_{m \in \mathrm{gen}(\sigma)} \frac{\chi^m}{1 - \chi^m}$$

Remark 34. Let σ be a nonsingular cone then,

$$\prod_{m \in \mathrm{gen}(\sigma)} (1 - \chi^m) \cdot q_0(\sigma) = \prod_{m \in \mathrm{gen}(\sigma)} \chi^m$$

Theorem 12.1. There exists a unique map $\psi : PL_A(M) \to A(M)$ sendig $q_0(\sigma) \mapsto Q_0(\sigma)$.

Definition: For $S \subset M_{\mathbb{R}}$ and,

$$q(S) = \sum_{m \in M \cap S} \chi^m$$

if $q(S) \in PL_A(M)$ then we defne $Q(S) = \psi(q(S))$

12.2 Brion's Inequality

Let Σ be a fan in $N_{\mathbb{R}}$ and $\Sigma(n) = \{\sigma \in \Sigma \mid \dim \sigma = n\}$. Let ψ be a support function of Σ.

Remark 35. We want to show that $q(\Delta_\psi) \sim q(\Delta_\psi(\sigma))$

Lemma 12.2. Let c be a rational polyhedral cone which is not strongly convex $(\dim\,(c \cap (-c)) > 0)$. Then $Q(c) = 0$.

Proof. Take $m \in M \cap (c \cap (-c))$. Then $m + c = c$ so $q(m + c) = q(c)$. However, $q(m + c) = \chi^m q(c)$. Thus $\chi^m Q(c) = Q(c)$ so $Q(c) = 0$. $\square$

Theorem 12.3. If $\pi \subset \Sigma$ is a rational polyhedral cone of dimension r we define,

$$K(\Sigma, \pi) = \{\sigma \in \Sigma \mid \sigma \cap \pi^\circ \neq \varphi\}$$

Then,

$$\sum_{\sigma \in K(\Sigma, \pi)} (-1)^{\dim \sigma} = (-1)^r$$

Definition: The support function ψ is convex if $\psi(a)+\psi(b) \leq \psi(a+b)$. Furthermore,

$$\Delta_\psi(\sigma) = \{x \in M_{\mathbb{R}} \mid \forall u \in \sigma, x(y) \geq \psi(y)\}$$

Furthermore,

$$\Delta_\psi = \bigcap_{\sigma \in \Sigma} \Delta_\psi(\sigma)$$

Remark 36. $\Delta_\psi(\sigma) = m + \sigma^\vee$ for some $m \in M$.

Lemma 12.4. Suppose that Σ is convex (i.e. $|\Sigma|$ is convex) of dimension r an ψ is a convex support function then,

$$q(\Delta_\psi) = \sum_{\sigma \in K(\Sigma, |\Sigma|)} (-1)^{r - \dim \sigma} q(\Delta_\psi(\sigma))$$

Theorem 12.5. Let Σ be convex and $\dim \Sigma = r$. Let ψ be a convex support function. Then,

$$Q(\Delta_\psi) = \sum_{\sigma \in \Sigma(r)} Q(\Delta_\psi(\sigma))$$

Proof. By the lemma,

$$q(\Delta_\psi) = \sum_{\sigma \in K(\Sigma, |\Sigma|)} (-1)^{r - \dim \sigma} q(\Delta_\psi(\sigma))$$

We need to show thaat if $\dim \sigma < r$ then $\sigma^\vee$ is not strongly convex then $Q(\sigma^\vee) = 0$ and thus, since $\Delta_\psi(\sigma) = m + \sigma^\vee$ so $Q(\Delta_\psi(\sigma)) = \chi^m Q(\sigma^\vee) = 0$. Therefore,

$$Q(\Delta_\psi) = \sum_{\sigma \in \Sigma(r)} Q(\Delta_\psi(\sigma))$$

$\square$

Corollary 12.6. For $\psi = 0$ then,

$$Q(|\Sigma|^\vee) = \sum_{\sigma \in \Sigma(r)} Q(\sigma^\vee)$$

In particular, if Σ is complete then $|\Sigma|^\vee = 0$ thus,

$$Q(|\Sigma|^\vee) = 1 = \sum_{\sigma \in \Sigma(r)} Q(\sigma^\vee)$$

12.3 Ishida's Theorem

Let $A = \mathbb{Q}$ a toric variety above a field k, Σ a finite nonsingular fan in $N_{\mathbb{R}}$. Notation,

$$\Sigma[\rho] = \{\sigma \in \Sigma \mid \sigma \supset \rho\}$$

Definition: Let σ be a nonsingular cone in Σ such that $\dim \sigma = r$. Then consider the map $x(\sigma, -) : \mathrm{gen}\,(\sigma) \to \mathrm{gen}\,(\sigma^{\vee})$ such that $x(\sigma, a)(b) = \delta(a, b)$.

Remark 37. Let ρ be a nonsingular cone in $M_{\mathbb{R}}$ then,

$$Q(\rho) = \prod_{a \in \mathrm{gen}(\rho)} \frac{1}{1 - \chi^a}$$
$$Q_0(\rho) = \prod_{a \in \mathrm{gen}(\rho)} \frac{\chi^a}{1 - \chi^a}$$

Now consider the map $\mathcal{E} : \mathbb{C} \otimes_{\mathbb{Z}} N \to \mathbb{C}^{\times} \otimes_{\mathbb{Z}} N$ via $z \otimes m \mapsto \exp{(-z)} \otimes m$. Then,

$$\mathcal{E}^* Q(\rho) = \prod_{a \in \rho \cap M} \frac{1}{1 - \exp -a}$$
$$\mathcal{E}^* Q_0(\rho) = \prod_{a \in \rho \cap M} \frac{1}{\exp -a - 1}$$

Remark 38.

$$\frac{1}{1 - \exp x} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^{n-1}$$

where B_n is the n^{th} Bernuli number.

Definition: Let $\sigma \in \Sigma$ then define,

$$V(\gamma) = \mathrm{Im}((X_{\Sigma[\gamma]} \to X_{\Sigma}))$$

which is a closed subvariety.

Theorem 12.7. If Σ is a complete fan then,

$$\prod_{\sigma \in \Sigma(1)} \frac{V(\sigma)}{1 - \exp{(-V(\sigma))}} = \prod_{\sigma \in \Sigma(1)} \sum_{n=0}^{\infty} \frac{B_n}{n!} V(\sigma)^n$$

and,

$$\left[\prod_{\sigma \in \Sigma(1)} \frac{V(\sigma)}{1 - \exp{(-V(\sigma))}} \right]_r = 1$$

13 Vanishing of Cohomology

Let N be a lattice and $\Delta \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ be a fan whose cones are generated by lattice points i.e. are rational conces. Let $M = N^\vee$ be the dual lattice. Then $\mathbb{C}[M]$ is the character algebra. Then the dual cone is,

$$\sigma^\vee = \{u \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \forall v \in \sigma : \langle u, v \rangle \geq 0\}$$

Next, the semigroup algebra is,

$$A_\sigma = \mathbb{C}[M \cap \sigma^\vee]$$

and the open set $X_\sigma = \mathrm{Spec}\,(A_\sigma)$ with the torus $T_N = \mathrm{Spec}\,(\mathbb{C}[M])$. Then the toric variety X_Δ is obtained by gluing these open sets X_σ for all $\sigma \in \Delta$.

To each $\rho \in \Delta(1)$ we can associate a torus-invariant Weil divisor $V(\rho) \subset X_\Delta$.

Proposition 13.1. Consider a ray $\rho \in \Delta(1)$ with minimal generator n_ρ in N then,

$$\mathrm{ord}_{V(\rho)}(\chi^u) = \langle u, n_\rho \rangle$$

Proof. If n_ρ is minimap then $\mathbb{Z}[n_\rho]$ is a direct summand of N so n_ρ can be extended to a basis $\{e_1, \dots, e_n\}$ of N with $e_1 = n_\rho$. Take the dual basis $\{e_1^*, \dots, e_n^*\}$. Then $\rho^\vee$ is the half space defined by the line ρ such that the semi-group algebra can be written as,

$$A_\rho = \mathbb{C}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$$

Then $X_\rho = \mathbb{A}^1_{\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}^{n-1}$. Then $V(\rho)$ is the closure of the orbit under $T(N)$ of the distinguished point $x_\rho \in X_\rho$ equal to $(0, 1, \dots, 1)$. Thus, $V(\rho) = \overline{\{x_1 = 0\}}$. For a function $f \in \mathbb{C}(x_1, \dots, x_n)^\times$ then $\mathrm{ord}(f)$ in the local ring $\mathbb{C}[x_1, \dots, x_n]_{(x_1)}$ of $V(\rho)$. Thus,

$$\mathrm{ord}(f) = v \iff f = x_1^v \frac{g}{h} \quad \text{where} \quad g, h \in \mathbb{C}[x_1, \cdots, x_n] \text{ are coprime to } x_1$$

In particular, $\chi^u = x_1^{u_1} \cdots x_n^{u_n}$ and thus,

$$\mathrm{ord}\chi^u = u_1 = \langle u, e_1 \rangle = \langle u, n_\rho \rangle$$

□

Proposition 13.2. Let D be a Weil divisor which is T_N-stable. Consider the action of T_N on $H^0(X, \mathcal{O}_X(D))$ by composition. Then $H^0(X, \mathcal{O}_X(D))$ is T_N-invariant.

Proof. Let $t \in T_N$ and $f \in H^0(X, \mathcal{O}_X(D))$. We need to show that $t \cdot f \in H^0(X, \mathcal{O}_X(D))$. It suffices to prove this holds for each affine open X_σ. Let,

$$D|_{X_\sigma} = \sum_{\rho \in \sigma(1)} -a_\rho V(\rho)$$

We need to show that $t \cdot f$ has vanishing order at least a_ρ on $V(\rho)$. Choose $u \in M$ such that $\mathrm{ord}_{V(\rho}(f) = \frac{1}{\ell}\langle u, n_\rho \rangle$ for some $\ell \in \mathbb{Z}$ because $N_\mathbb{Q}$ and $M_\mathbb{Q}$ are dual as vectorspaces. Then consider the function $\chi^{-u} f^\ell$ which has no zeros nor poles on $V(\rho)$. Thus $t \cdot \chi^{-u} f^\ell$ has no zeros nor poles on $V(\rho)$ becaue the action of t is an automorphism of $V(\rho)$. Thus $\mathrm{ord}_{V(\rho)}(f \cdot \chi^{-u} f^\ell) = 0$. Furthermore, $t \cdot \chi^u(p) = \chi^u(t \cdot p) = \chi^u(t)\chi^u(p)$ which is a scalar multiple and thus has the same order of vanishing on any divisor. Thus,

$$\begin{aligned}\ell \, \mathrm{ord}_{V(\rho)}(t \cdot f) &= \mathrm{ord}_{V(\rho)}(t \cdot f^\ell) = \mathrm{ord}_{V(\rho)}(t \cdot (\chi^u \chi^{-u} f^\ell)) \\ &= \mathrm{ord}_{V(\rho)}(t \cdot \chi^u) + \mathrm{ord}_{V(\rho)}(t \cdot \chi^{-u} f^\ell) = \mathrm{ord}_{V(\rho)}(t \cdot \chi^u) \\ &= \mathrm{ord}_{V(\rho)}(\chi^u) = \langle u, n_\rho \rangle = \ell \, \mathrm{ord}_{V(\rho)}(f)\end{aligned}$$

Thus $\mathrm{ord}_{V(\rho)}(t \cdot f) = \mathrm{ord}_{V(\rho)}(f)$ proving the claim. □

Proposition 13.3. Let D be a $T(N)$-invariant Weil divisor on $X = X_\Delta$. Then we may decompose the T_N-module $H^0(X, \mathcal{O}_X(D))$ as,

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{\chi^u \in H^0(X, \mathcal{O}_X(D))} \mathbb{C} \cdot \chi^u$$

We write formally that $\mathbb{C} \cdot \chi^u = H^0(X, \mathcal{O}_X(D))_u$.

Proposition 13.4. A Cartier divisor D which is invariant by $T(N)$ is equal to $\mathrm{div}(\chi^{-u(\sigma)})$ on each X_σ where $u(\sigma)$ is well-defined up to $M(\sigma) = \sigma^\perp \cap M$.

Definition: A support function is a continuous map $\psi : |\Delta| \to \mathbb{R}$ such that ψ takes rational values on lattice points and is linear on each cone of Δ. Let D be a T_N-invariant Cartier divisor then there is a collection $u(\sigma) \in M/M(\sigma)$ then we get a collection of $\psi_\sigma = \langle u(\sigma), - \rangle$ defined on $|\sigma| \subset |\Delta|$ (well-defined because $M(\sigma) \subset \sigma^\perp$) which agree on the overlaps and thus glue. Indeed, consider the characters $\chi^{-u(\sigma)}|_{X_\tau}$ and $\chi^{-u(\tau)}_{X_\sigma}$ and they conicde up to $M(\sigma \cap \tau) = M(\sigma) = M(\tau)$ thus on $\sigma \cap \tau$ we have $\langle u(\sigma), - \rangle = \langle u(\tau), - \rangle$. Thus these glue to the support function ψ_D. This correspondence is a bijection. The inverse take,

$$\psi \mapsto \sum_{\rho \in \Delta(1)} -\psi(n_\rho) V(\rho)$$

Corollary 13.5. $\chi^u \in H^0(X, \mathcal{O}_X(D)) \iff \langle u, n_\rho \rangle \geq \psi_D(n_\rho)$

Definition: Take a fixed divisor D. For each $u \in M$ then,

$$Z_D(u) = \{v \in |\Delta| \mid \langle u, v \rangle \geq \psi_D(v)\}$$

is a closed cone equal to a hull of cones in Δ.

Corollary 13.6. $\chi^u \in H^0(X, \mathcal{O}_X(D)) \iff Z_D(u) = |\Delta|$

Example 13.7. If $\Delta = \sigma$ then

$$H^0(X_\sigma, \mathcal{O}_{X_\sigma}(D)) = \bigoplus \mathbb{C} \cdot \chi^u$$

where u is such that $Z_D(u) \cap |\sigma| = |\sigma|$.

Definition: Let M be a topological space and $\mathcal{F}$ a sheaf on M. For $Z \subset M$ define the sections over U of $\mathcal{F}$ with support in Z is,

$$H^0_Z(U, \mathcal{F}) = \{s \in H^0(U, \mathcal{F}) \mid \forall V \subset U \cap (M \setminus Z) : s|_V = 0\}$$

If $Z \subset M$ is closed then $H^0_Z(U, \mathcal{F}) = \ker (H^0(U, \mathcal{F}) \to H^0(U \setminus Z, \mathcal{F}))$.

Example 13.8. f $M = |\Delta|$ and $\mathcal{F} = \underline{\mathbb{C}}$ then either,

1. $Z \subsetneq |\Delta|$ then let $s \in H^0(|\Delta|, \underline{\mathbb{C}})$ but since $|\Delta|$ is path-connected (since it is star shaped at zero) so $H^0(|\Delta|, \underline{\mathbb{C}}) = \mathbb{C}$. Thus if $s|_V = 0$ then $s = 0$ for any $V \neq \varnothing$. Thus $H^0_Z(|\Delta|, \underline{\mathbb{C}}) = 0$.
2. $Z = |\Delta|$ in which case $H^0_Z(|\Delta|, \underline{\mathbb{C}}) = H^0(|\Delta|, \underline{\mathbb{C}}) = \mathbb{C}$.

Proposition 13.9. $H^0(X, \mathcal{O}_X(D))_u = H^0_{Z_D(u)}(|\Delta|, \underline{\mathbb{C}})$

Definition: Consider the functor $H^0_Z(U, -)$ which has p^{th}-derived functors $H^p_Z(U, -)$ called cohomology with support in Z.

Theorem 13.10. There is a canonical decomposition,

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^p_{Z_D(u)}(|\Delta|, \underline{\mathbb{C}})$$

we notate, $H^p(X, \mathcal{O}_X(D))_u = H^p_{Z(u)}(|\Delta|, \underline{\mathbb{C}})$.

Corollary 13.11. If $|\psi|$ is concave then $H^i(X, \mathcal{O}_X(D)) = 0$ for all $i > 0$.

Proof. The set $|\Delta| \setminus Z_D(u) = \{v \in |\Delta| \mid \langle u, v \rangle < \psi_D(v)\}$ is convex because $\langle u, - \rangle$ is convex and $-\psi$ is convex. Now apply the long exact sequence noting that $H^i(|\Delta|, \underline{\mathbb{C}}) = 0$ and $H^i(|\Delta| \setminus Z_D(u), \underline{\mathbb{C}}) = 0$ for $i > 0$ since both are contractible and $H^1_Z(|\Delta|, \underline{\mathbb{C}}) = 0$ since the map $H^0(|\Delta|, \underline{\mathbb{C}}) \to H^0(|\Delta| \setminus Z, \underline{\mathbb{C}})$ is surjective since both sets are connected. ☐

Proposition 13.12. $\mathcal{O}_X(D)$ is generated by global sections iff ψ_D is concave and $\mathcal{O}_X(D)$ is ample iff ψ_D is strictly concave

Theorem 13.13 (Demazure)**.** If $\mathcal{O}_X(D)$ is generated by global sections (in particular ample) then,

$$\forall p > 0 : H^p(X, \mathcal{O}_X(D)) = 0$$

14 Cohen's Structure Theorem

Remark 39. All rings are commutative and with identity.

14.1 Topological Rings

Definition: We say a ring A is noetherian if it the satisfies one of the following equivalent conditions,

1. any ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ must satbiliize
2. every nonempty set of ideals has a maximal element (w.r.t. inclusion)
3. every ideal of A is finitely generated as an A-module

Theorem 14.1 (Hilber)**.** If A is noetherian then $A[x]$ is noetherian.

Definition: A ring A is local if it has a unique maximal ideal $\mathfrak{m} \subset A$. We denote the local ring by $(A, \mathfrak{m}, \kappa)$ where $\kappa = A/\mathfrak{m}$.

Theorem 14.2 (Krull Intersection)**.** Let A be noetherian with an ideal $I \subset A$ and M an A-module. Then consider the submodule,

$$N = \bigcap_{n=0}^{\infty} I^n \cdot M$$

Then $I \cdot N = N$.

Corollary 14.3. If $I \subset \text{rad}(A)$ then $N = 0$. Furthermore, in the case $M = A$ and A is a domain then we find,

$$\bigcap_{n=0}^{\infty} I^n = (0)$$

for any proper ideal $I \subset A$ by Nakayama.

Lemma 14.4. Let $M = M_0 \supset M_1 \supset M_2 \supset M_3 \supset \cdots$ be a filtration. Then the sets $\{x + M_n \mid n \in \mathbb{N} \quad x \in M\}$ form the basis for a topology on M.

Proof. Consider $z \in (x + M_n) \cap (y + M_m)$ and $r = \max\{n, m\}$. Then $z - x \in M_n$ and $z - y \in M_m$. Consider $p \in (z + M_r)$ then $p - z \in M_r \subset M_n, M_m$ so $(p - z) - (z - x) = p - x \in M_n$ and $(p - z) - (z - y) \in M_m$ thus $p \in (x + M_n) \cap (x + M_m)$. Therefore, $(z + M_r) \subset (x + M_n) \cap (y + M_m)$. Furthermore, $x \in (x + M_n)$ so the sets clearly cover M prving that they form a basis for a topology. $\square$

Definition: Let A be a ring and M an A-module. We set that a sequence (u_n) of M is *Cauchy* if $\forall n \in \mathbb{N} : \exists N \in \mathbb{N} : \forall i, j > N : u_i - u_j \in M_n$. We say that M is *complete* if every Cauchy sequence is convergent.

Proposition 14.5. The topology induced on M by a filtration is Haudorff iff,

$$\bigcap_{n=0}^{\infty} M_n = (0)$$

in which case we say the filtration is *seperated.*

Proof. $\square$

Remark 40. Given a filtration $M = M_0 \supset M_1 \supset M_2 \supset M_3 \supset \cdots$ then the function $d' : M \times M \to \mathbb{N}$ via $d'(x, y) = \text{argmax} n \in \mathbb{N}(x-y \in M_n)$ defines a pseudo-ultrametric on M via $d(x, y) = 1/N^{d'(x,y)}$ and we set,

$$d(x, y) = 0 \iff x - y \in \bigcap_{n \in \mathbb{N}} M_n$$

whose metric topology coincides with the topology defined above.

1. $d(x, y) = d(y, x)$ since $x - y \in M_n \iff y - x \in M_n$
2. Let $'d(x, z) = n$ and $d'(z, y) = m$ then $(x-z) \in M_n$ and $(z, y) \in M_m$. Therefore, $(x - z) - (z - y) = x - y \in M_{\min\{n,m\}}$ so $d'(x, y) \geq \min\{n, m\}$ which implies that,
$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

Remark 41. This pseudometric is a metric iff the filtration is seperated.

Definition: Given a filtration and the induced topology, the completion of M with respect to this completion is,

$$\hat{M} = \varprojlim_{n \in \mathbb{N}} M/M_n$$

with respect to projection maps $M/M_{n+1} \to M/M_n$. Giving each quotient M/M_n the discrete topology (which is the topology induced by the filtration) makes $\hat{M}$ a topological A-module whose topology agrees with the completion of M with respct to the above metric topology. Furthermore, there is a continuous map $M \to \hat{M}$ with kernel $\bigcap M_n$.

Proof. $\square$

Definition: The completion of A with respect to I is,

$$\hat{A}^I = \varprojlim_{n \in \mathbb{N}} A/I^n$$

This is the completion of A with respect to the I-adic topology defined by the filtration $A \supset I \supset I^2 \supset I^3 \supset \cdots$.

Example 14.6. We may complete the following rings,

1. take $A = k[x_1, \cdots, x_n]$ then with respect to $I = (x_1, \ldots, x_n)$ the completion is $\hat{A}^I = k[[x_1, \ldots, x_n]]$
2. take $A = \mathbb{Z}$ then with respect to $I = (p)$ the competion is $\hat{A}^I = \mathbb{Z}_p$.

14.2 Power Series Rings

Lemma 14.7. Let A be a ring and $a \in A[[X]]$. Then $a \in A[[X]]^\times \iff a_0 \in A^\times$.

Corollary 14.8. The units of $A[[X_1, \dots, X_n]]$ are exactly those whose image in $A = A[[X_1, \cdots, X_n]] \to A[X_1, \dots, X_n]/(X_1, \dots, X_n) = A$ is a unit.

Example 14.9. Power series does not preserve many nice properties,

1. $\mathbb{Z}$ is a PID but $\mathbb{Z}[[x]]$ is not a PID since $(2, x)$ is not principle.

2. $\mathbb{Z}$ is euclidean but $\mathbb{Z}[[x]]$ is not euclidean (since it is not a PID).

14.3 Field of Representatives

Definition: Let $(A, \mathfrak{m}, \kappa)$ be a local ring. Then A is *regular* if $\dim A = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

Example 14.10. The following are regular local rings,

1. any field k since $\dim k = 0$ and $\mathfrak{m} = 0$ so $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = 0$.

2. any DVR R since $\dim R = 1$ and $\mathfrak{m} = (\varpi)$ so $\dim_\kappa(\varpi)/(\varpi^2) = \dim_\kappa A/(\varpi) = 1$

Definition: A local ring $(A, \mathfrak{m}, \kappa)$ is *equicharacteristic* if A and κ have the same characteristic otherwise it has *mixed characteristic*.

Example 14.11. The following local rings satisfy,

1. k is equicharacteristic since $\kappa = k$

2. $\mathbb{Z}_p$ has mixed characteristic since $\kappa = \mathbb{F}_p$

3. $k[[X_1, \dots, X_n]]$ is equicharacteristic since $\kappa = k$

Definition: Let $(A, \mathfrak{m}, \kappa)$ be a local ring and $\pi : A \to A/\mathfrak{m}$ the projection. Suppose that there exists a subring $L \subset A$ s.t. $\pi|_L : L \to A/\mathfrak{m}$ is an isomorphism then L is the *field of representatives* of A.

Theorem 14.12 (Cohen 1)**.** Let $(A, \mathfrak{m}, \kappa)$ be a noetherian, complete, equicharacteristic, local ring then $A \cong \kappa[[X_1, \dots, X_n]]/I$ for some ideal I.

Proof. Since A is equicharacteristic, A is a κ algebra (PROVE THIS). Let $\mathfrak{m} = (a_1, \dots, a_n)$. Consider the map $\kappa[X_1, \dots, X_N] \to A$ via $X_i \mapsto a_i$ which is continuous since it preserves the valuation. Since A is complete this extends to a map on the completion $\kappa[[X_1, \dots, X_N]] \to A$. I claim that this map is surjective proving the theorem. $\square$

Theorem 14.13. In the case above that A is regular then $I = (0)$ so we have $A \cong \kappa[[X_1, \dots, X_n]]$ where $n = \dim A$.

Corollary 14.14. If A is noetherian, complete, equicharacteristic regular local ring, then A is a unique factorization domain.

Theorem 14.15 (Cohen 2)**.** Let $(A, \mathfrak{m}, \kappa)$ be a noetherian, complete, local ring of dimension d. Then there exists a Cohen ring B s.t. $A \cong B[[X_1, \cdots, X_d]]/I$ for some ideal I. In particular, when A is equicharacteristic, then $B = \kappa$. If A is regular then $A = B[[X_1, \dots, X_d]]$.

15 Okunkov Bundles

Remark 42. Let X be a smooth projective variety of dimension d.

15.1 Positivity

Definition: Let $\mathcal{L} \to X$ be a line bundle. We say that,

1. $\mathcal{L}$ is *very ample* if there exists a closed embedding $\iota : X \hookrightarrow \mathbb{P}^N$ such that $\mathcal{L} = \iota^* \mathcal{O}(1)$.
2. $\mathcal{L}$ is *ample* if $\mathcal{L}^{\otimes n}$ is very ample for some $n \in \mathbb{Z}^+$

We also say that a divisor D is (very) ample when $\mathcal{O}_X(D)$ is (very) ample.

Theorem 15.1. A line bundle $\mathcal{L} \to X$ is ample iff for every positive dimension subvariety $V \subset X$ that $V \cdot \mathcal{L}^{\otimes \dim V} = 0$.

Theorem 15.2. A line bundle $\mathcal{L} \to X$ is ample iff for every coherent sheaf $\mathcal{F}$ on X there exists $n \in \mathbb{Z}^+$ s.t. $H^i(X, \mathcal{F} \times \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$.

Definition: Two divisors $D, D' \in \mathrm{Div}X$ are *numerically equivalent* if for every curce $C \subset X$ then $D \cdot C = D' \cdot C$. We say a divisor $D \in \mathrm{Div}X$ is nef (numerically effective) if $C \cdot D \geq 0$ for each curve $C \subset X$. The Neron-Severi group of X is $N^1(X) = \mathrm{Div}X/\{\text{numverically trivial divisors}\}$.

Definition: A divisor D is *big* if there exists $C > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq C \cdot m^d$.

Theorem 15.3. If D is big then there exists an ample divisor A and $m > 0$ and an effective divisor N such that $mD \sim A + N$. The bigness depends only on numerical equivalence class.

Definition: We define K-numverical equivalence classe $N^1(X)_K = N^1(X) \times_{\mathbb{Z}} \mathbb{Q}$. A K-divisor $D \in \mathrm{Div}(X)_K$ is big it it can be written as $\sum a_i D_i$ for $a_i > 0$ and D_i big intergral divisor. Then $\mathrm{Big}(X) \subset N^!(X)_{\mathbb{R}}$ is the convx cone of all big $\mathbb{R}$-divisor classes on X. Furthermore, the effecitve cone,

$$\overline{\mathrm{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$$

is the closure of cone spanned by the classes of effective $\mathbb{R}$-divisors.

Theorem 15.4. $\mathrm{Big}(X) = (\overline{\mathrm{Eff}}(X))^{\circ}$

Definition: Let D be a divisor and $\mathcal{L} = \mathcal{O}_X(D)$. Then the volume of $\mathcal{L}$ is,

$$\mathrm{Vol}_X(L) = \limsup_{m \to \infty} \frac{h^0(X, \mathcal{L}^{\otimes m})}{m^d/d!}$$

Theorem 15.5. Let D be a nef $\mathbb{Q}$-divisor. Then $\mathrm{Vol}_X(D) = D^d$ in the sence of intersection number.

Definition: Let D be a divisor. A *complete linear system* $|D|$ is the set of effective divisors linearly equivalent to D.

Definition: Let $\mathcal{L}$ be a line bundle on X and $s \in H^0(X, \mathcal{L}) \setminus \{0\}$. The *divisor of zeros* $D = (s)_0$ of s is defined as follows: on each open $U \subset X$ such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$ let $(s)_0|_U = \mathrm{div}(s)$ viewing $s|_U \in \mathcal{O}_X(U)$.

Proposition 15.6. Furthermore, there is a bijection $|D| \to (H^0(X, \mathcal{O}_X(D)) \setminus \{0\})/\mathbb{C}^\times$ given by sending a section to its vanish divisor.

Definition: The *base locus* $\mathrm{Bs}(D)$ of a divisor D is defined as,

$$\mathrm{Bs}(D) = \bigcap_{D_{\mathrm{eff}} \in |D|} \mathrm{Supp}(D_{\mathrm{eff}})$$

The *stable base locus* if,

$$\mathrm{B}(D) = \bigcap_{m \geq 1} \mathrm{Bs}(mD)$$

The *augmented base locus* of a $\mathbb{Q}$-divisor is $\mathrm{B}_+(D) = \mathrm{B}(D - A)$ is a sufficiently small ample $\mathbb{Q}$-divisor.

15.2 Construction

Definition: An *admissible flag* $Y_\bullet$ of X is a sequence $X = Y_0 \supset Y_1 \supset \cdots \supset Y_d$ of irreducible closed subvarieties of X s.t. $\mathrm{codim}_X(Y_i) = i$ and each i is smooth at the point Y_d.

Definition: Let X be a locally noetherian integral scheme, $\mathcal{L} \to X$ a line bundle, and $s \in H^0(X, \mathcal{L}) \setminus \{0\}$. Let Z be a prime divisor on X (i.e. a integral irreducible closed subscheme). Then the order of vanishing of s along Z is $\mathrm{ord}_{\mathcal{L},Z}(s) = \mathrm{ord}_{\mathcal{O}_{X,\eta}}(s/s_\eta)$ where $\eta \in Z$ is the generic point of Z and $s_\eta \in \mathcal{L}_\eta$ generates it as a $\mathcal{O}_{X,\eta}$-module.

Remark 43. For any $s \in H^0(X, \mathcal{O}_X(D)) \setminus \{0\}$ then $s/s_\eta \in \mathcal{O}_{X,\eta}$ so $\mathrm{ord}_Z(s) \geq 0$.

Definition: Given an admissible flag and a divisor D we define a valuation $\nu_{Y_\bullet,D} = H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \to \mathbb{Z}^d$ as follows. First let $v_1(s) = \mathrm{ord}_{D,Y_1}(s)$. Choosing a local equition f for Y_1 in X we get a section $\tilde{s}_1 = s/f^{v_1(s)} \in H^0(X, \mathcal{O}_X(D - v_1 Y_1)) \setminus \{0\}$. Then consider $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - v_1 Y_1))$ and $v_2(s) = \mathrm{ord}_{D,Y_2}(s_1)$. Repeating this process we get a sequence $v_1, v_2, \ldots, v_d$.

Remark 44. Generally, for any integres $a_1, \ldots, a_i \geq 0$ we let,

$$\mathcal{O}_{Y_i}(D - (a_1 Y_1 + \cdots + a_i Y_i)) = \mathcal{O}_X(D)|_{Y_i} \otimes \mathcal{O}_X(-a_1 Y_1)|_{Y_i} \otimes \cdots \otimes \mathcal{O}_{Y_{i-1}}(-a_i Y - i)$$

Proposition 15.7. The above valuation satisfies the properties,

1. $v_{Y_\bullet}(s) \in \mathbb{Z}^d_{\geq 0}$
2. Ordering $\mathbb{Z}^d$ lexicographically we have $v_{Y_\bullet}(s_1 + s_2) \geq \min\{v_{Y_\bullet}(s_1), v_{Y_\bullet}(s_2)\}$
3. If nonvero s_1, s_2, s_3 are linearly independent and $s_i \neq cs_j$ (for $i \neq j$) then then smallest two values of $v_{Y_\bullet}(s_i)$ are equal.
4. For each $s \in H^0(X, \mathcal{O}_X(D_1)) \setminus \{0\}$ and $t \in H^0(X, \mathcal{O}_X(D_2)) \setminus \{0\}$ then,

$$v_{Y_\bullet, D_1+D_2}(s \otimes t) = v_{Y_\bullet, D_1}(s) + v_{Y_\bullet, D_2}(t)$$

Example 15.8. Let $X = \mathrm{P}^d$ and $Y_\bullet$ the flag defined as $Y_i = \{X_0 = \cdots = X_i = 0\}$. Then D is a degree m divisor implies that,

$$v_{Y_\bullet}(X_0^{a_1} \cdots X_d^{a_d}) = (a_0, \ldots, a_d)$$

Thus,

$$v_{Y_\bullet}(\sum c_a X^a) = \min\{\{a \mid c_a \neq 0\}\}$$

Lemma 15.9. Let $W \subset H^0(X, \mathcal{O}_X(D))$ be a subspace. Fix $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$. Then let $W_{\geq a} = \{s \in W \mid v_{Y_\bullet}(s) \geq a\}$ and likewise for $W_{>a}$. Then,

$$\dim(W_{\geq a}/W_{>a}) \leq 1$$

In particular, if W is finite dimensional then $\#\left(\mathrm{Im}(W \setminus \{0\}) \xrightarrow{\nu} \mathbb{Z}^d\right)$.

Definition: The *graded semigroup* of D is the subsemigroup,

$$\Gamma_{Y_\bullet}(D) = \{(v_{Y_\bullet}(s), m) \mid s \in H^0(X, \mathcal{O}_X(mD) \text{ and } m \in \mathbb{Z}_{\geq 0}\} \subset N^{\mathrm{d}} \times \mathbb{N} = \mathbb{N}^{d+1}$$

Then let $\Sigma(\Gamma_{Y_\bullet}(D))$ be the closed convex cone of $\Gamma_{Y_\bullet}(D)$ in $\mathbb{N}^{d+1} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^{d+1}_{\geq 0}$.

Definition: The Okounkov body of D is is the conpact convex set,

$$\Delta_{Y_\bullet}(D) = \Sigma(\Gamma_{Y_\bullet}(D)) \cap (\mathbb{R}^d \times \{1\})$$

This is equivalent to the closed covex hull of,

$$\bigcup_{m=1}^{\infty} \tfrac{1}{m}\Gamma_{Y_\bullet}(D)_m \quad \text{where} \quad \Gamma_{Y_\bullet}(D)_m = \mathrm{Im}((H^0(X, \mathcal{O}_X(mD)) \setminus \{0\} \xrightarrow{v_{Y_\bullet}} \mathbb{Z}^d))$$

Remark 45. The Okounkov body $\Delta(D)$ lies in the nongeative orthant of $\mathbb{R}^d$. For a fixed divisor D, for very "general" choices of $Y_\bullet$ the the Okounkov bodies correspond.

Proposition 15.10. The body $\Delta(D)$ is bounded and thus compact.

Proof. It suffices to show that $\exists b > 0$ s.t.

$$\forall i : \forall m > 0 : \forall s \in H^0(X, \mathcal{O}_X(mD)) \setminus \{0\} : v_i(s) < mb$$

Fix an ample divisor H then $Y \cdot H^{d-1} > 0$. Thus, there exists $b_1 > 0$ such that $(D - b_1 Y_1) \cdot H^{d-1} < 0$. Therefore, $v_1(s) < mb_1$. (READ THIS CLAIM) $\square$

Example 15.11. Let $X = \mathbb{P}^d$ and D the hyperplane divisor and $Y_\bullet$ as before. Then,

$$\begin{aligned}\Delta(D) &= \Sigma\left(\bigcup_{m \geq 1} \{\tfrac{1}{m}(a_1, \dots, a_d) \mid a_i \geq 0, a_1 + \cdots + a_d = m\}\right)\\ &= \{(\xi_1, \dots, \xi_d) \in \mathbb{R}^d \mid \xi_i \geq 0, \xi_1 + \cdots + \xi_d \leq 1\}\end{aligned}$$

so $\Delta(D) = \Delta^d$, the standard d-simplex.

15.3 Properties

Let $\Gamma \subset \mathbb{N}^{d+1}$ be a semigroup and $\Sigma = \Sigma(\Gamma)$ the closed cone then

16 Okounkov Bodies in the Toric Case

16.1 Review

We fix d-dimensional lattice $N \cong \mathbb{Z}^d$ and let Σ be a fan in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. That is a set Σ such that,

1. each $\sigma \in \Sigma$ is a strongly convex ($\sigma \cap (-\sigma) = \{0\}$) rational polyhedral cone in $N_\mathbb{R}$
2. if $\tau \subset \sigma \in \Sigma$ is a face then $\tau \in \Sigma$
3. $\forall \sigma, \tau \in \Sigma$ their intersection $\sigma \cap \tau \in \Sigma$ and is a shared face of both σ and τ.

Then set $M = N^\vee = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$.

Recall the following notation,

$$\Sigma(k) = \{\sigma \in \Sigma \mid \dim \sigma = k\}$$

and the affine open sets,

$$U_\sigma = \mathrm{Spec}\,(\mathbb{C}[\sigma^\vee \cap M])$$

glue to form the toric variety X_Σ with the torus $T(N) = N \otimes_\mathbb{Z} \mathbb{C}^\times \subset X_\Sigma$ corresponding to the open set $U_0 = \mathrm{Spec}\,(\mathbb{C}[M])$.

Recall that there is a correspondence between cones $\sigma \in \Sigma$ and T-orbits $O \subset X_\Sigma$ such that,

1. $\sigma \subset \tau$ iff $\overline{O_\tau} \subset \overline{O_\sigma}$

2. $\dim \sigma + \dim O_\sigma = d$

Furthermore, taking the closure $\sigma \mapsto V(\sigma) = \overline{O_\sigma}$ gives a correspondence between cones of dimension i and closed $T(N)$-invariant codimension i, subvarieties (i.e. toric subvarieties of codimension i). For $\sigma \in \Sigma(1)$ then $D_\sigma = V(\sigma)$ gives the set of $T(N)$-invariant prime divisors on X_Σ.

Theorem 16.1. The following hold,

1. X_Σ is normal and Cohen-Macaulay

2. X_Σ is complete (i.e. proper) iff Σ is complete i.e. $|\Sigma| = N_\mathbb{R}$

3. X_Σ is smooth iff Σ is smooth i.e. each cone $\sigma \in \Sigma$ is has gerators which may be extended to

Remark 46. Recall the relationships between various notions of divisors, line bundles, and support funtions.

Definition: Let X be any scheme. Then a *Cartier divisor* on X is a section of the quotient, $\xi \in H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$ which is a set of pairs $\{(U_i, f_i)\}$ where U_i cover X and $f_i \in \mathcal{K}_X(U_i)$ s.t. $f_i / f_j \in \mathcal{O}_X(U_i \cap U_j)^\times$. The epimorphism of sheaves $K_X^\times \to \mathcal{K}_X^\times / \mathcal{O}_X^\times$ defines the Cartier class group as the cokernel on global sections i.e. Cartier divisors modulo global rational functions,

$$H^0(X, \mathcal{K}_X^\times) \longrightarrow H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \longrightarrow \mathrm{CaCl}(X) \longrightarrow 0$$

Proposition 16.2. There is an injective homomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}\,(X)$ wich can be described by sedning a Cartier divisor $D = \{(U_i, f_i)\} \mapsto \mathcal{O}_X(D)$ the invertible subsheaf of $\mathcal{K}_X$ generated on U_i by f_i^{-1} i.e.

$$\mathcal{O}_X|_{U_i} \xrightarrow{f_i^{-1}} \mathcal{O}_X(D)|_{U_i}$$

is an isomorphism. This is well-defined because f_i / f_j is a unit on $U_i \cap U_j$ so they generate the same sheaf. When X is integral, $\mathrm{CaCl}(X) \to \mathrm{Pic}\,(X)$ is an isomorphism.

Definition: Let X be an integral noetherian scheme. Then a *prime divisor* on X is an integral closed subscheme $Y \subset X$ of codimension 1 and a *Weil divisor* is a finite formal sum of prime divisors,

$$D = \sum_{Y \subset X} n_Y \, Y$$

Principal divisors correspond to rational functions $f \in \mathrm{Rat}\,(X)^\times$ where we set,

$$\mathrm{div}(f) = \sum_{Y \subset X} \mathrm{ord}_Y(f) \, Y$$

The map $\mathrm{Rat}\,(X)^\times \to \mathrm{Div}(X)$ defines the Weil class group as the cokernel i.e. divisors modulo principal divisors,

$$\mathrm{Rat}\,(X)^{\times} \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

Proposition 16.3. Let X be an integral noetherian scheme. There is a homomorphism $\mathrm{Ca}(X) \to \mathrm{Div}(X)$ descending to $\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$ given by mapping,

$$\{(U_i, f_i)\} \mapsto \sum_{Y \subset X} \frac{1}{\#\{U_i \cap Y \neq \varnothing\}} \sum_{U_i \cap Y \neq \varnothing} \mathrm{ord}_Y(f_i)\, Y$$

This is well-defined because $f_i/f_j \in \mathcal{O}_X^{\times}(U_i \cap U_j)$ so $\mathrm{ord}_Y(f_i) = \mathrm{ord}_Y(f_j)$ since they differ by a unit.

If X is locally factorial (in particular if X is smooth) then this map is an isomophism.

Proposition 16.4. Given a Weil divisor D there is a corresponding line bundle $\mathcal{O}_X(D)$ via,

$$\mathcal{O}_X(D)(U) = \{f \in \mathrm{Rat}\,(X) \mid (D + \mathrm{div}(f))|_U \geq 0\}$$

Conversely, given a line bundle $\mathcal{L}$ we may assign a Weil divisor $c_1(\mathcal{L})$ to it via,

$$c_1(\mathcal{L}) = \sum_{Y \subset X} \mathrm{ord}_{\mathcal{L},Y}(s)\, Y$$

for some nonzero meromorphic section s of $\mathcal{L}$. This is independent of the choice of section s.

Definition: A *support function* is a continous function $\psi : |\Sigma| \to \mathbb{R}$ such that on each cone $\sigma \in \Sigma$ the restriction $\psi|_\sigma(x) = \langle m_\sigma, x \rangle$ is linear. A global support function is a function of the form $\langle m, - \rangle$ for a global choice of $m \in M$. We define the Picard group of the fan to be the quotient by global support functions $\mathrm{Pic}\,(\Sigma) = SF(\Sigma)/M$.

Proposition 16.5. On a toric variety X_Σ, there is a correspondence between $T(N)$-invariant Cartier divisors D and support functions ψ_D. Given by,

$$D \mapsto \psi_D \text{ such that } \psi|_\sigma = \langle u(\sigma), - \rangle \text{ where } D|_{U_\sigma} = \mathrm{div}(\chi^{-u(\sigma)})$$

and

$$\psi \mapsto \{(U_\sigma, \chi^{-m_\sigma}) \mid \sigma \in \Sigma\}$$

We may furthermore assign a Weil divisor to ψ via the map $\mathrm{Ca}(X) \to \mathrm{Div}(X)$,

$$\psi \mapsto \sum_{\rho \in \Sigma(1)} \mathrm{ord}_Y(\chi^{-m_\rho})\, V(\rho) = \sum_{\rho \in \Sigma(1)} -\langle m_\rho, n_\rho \rangle\, V(\rho) = \sum_{\rho \in \Sigma(1)} -\psi(n_\rho)\, V(\rho)$$

where we recall that $\Sigma(1)$ corresponds to the set of $T(N)$-invariant prime divisors.

Remark 47. The scheme X_Σ is a variety so, in particular, it is noetherian and integral so Weil divisors are defined and $\mathrm{CaCl}(X_\Sigma) \xrightarrow{\sim} \mathrm{Pic}\,(X_\Sigma)$ is an isomorphism. However, unless X_Σ is locally factorial (in particular when X_Σ is not smooth) then the canonical map $\mathrm{Ca}(X_\Sigma) \to \mathrm{Div}(X_\Sigma)$ may not be surjective i.e. they can be Weil divisors which do not correspond to a Cartier divisor and $T(N)$-invariant Weil divisors which do not correspond to a support function.

Remark 48. Recall the following properties of sections of toric line bundles which we will make repeated use of in the following sections.

Proposition 16.6. Consider a ray $\rho \in \Sigma(1)$ with minimal generator n_ρ in N then,

$$\mathrm{ord}_{V(\rho)}(\chi^u) = \langle u, n_\rho \rangle$$

Proposition 16.7. Let D be a $T(N)$-invariant Weil divisor on $X = X_\Sigma$. Then we may decompose the $T(N)$-module $H^0(X, \mathcal{O}_X(D))$ as,

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{\chi^u \in H^0(X, \mathcal{O}_X(D))} \mathbb{C} \cdot \chi^u$$

We write formally that $\mathbb{C} \cdot \chi^u = H^0(X, \mathcal{O}_X(D))_u$.

Proposition 16.8. $\chi^u \in H^0(X, \mathcal{O}_X(D)) \iff \langle u, n_\rho \rangle \geq \psi_D(n_\rho)$ for each $\rho \in \Sigma(1)$

Proof. The chacters χ^u are invertible rational functions $\chi^u \in \mathrm{Rat}\,(X_\Sigma)^\times$. By definition $\chi^u \in H^0(X, \mathcal{O}_X(D)) \iff \mathrm{div}(\chi^u) + D \geq 0$. However, by above,

$$\mathrm{div}(\chi^u) = \sum_{\rho \in \Sigma(1)} \langle u, n_\rho \rangle \, V(\rho)$$

so by the definition of ψ_D we have,

$$\mathrm{div}(\chi^u) + D = \sum_{\rho \in \Sigma(1)} \langle u, n_\rho \rangle \, V(\rho) + \sum_{\rho \in \Sigma(1)} -\psi_D(n_\rho) V(\rho) \geq 0 \iff \langle u, n_\rho \rangle \geq \psi_D(n_\rho)$$

□

16.2 Construction of the Rational Polytope Corresponding to a Toric Divisor

Definition: Let X_Σ be a toric variety and D a $T(N)$-invariant divisor on X_Σ. Then we construct the set,

$$P_D = \{x \in M_\mathbb{R} \mid \forall \rho \in \Sigma(1) : \langle x, n_\rho \rangle \geq \psi_D(n_\rho)\} = \bigcap_{\rho \in \Sigma(1)} H^+(n_\rho, \psi_D(n_\rho))$$

Since this set is a finite intersection of integral halfspaces, it is clearly a rational polyhedron.

Proposition 16.9. If X_Σ is complete then P_D is bounded and thus a rational polytope.

Proof. X_Σ is complete exactly when $|\Sigma| = N_\mathbb{R}$ in which case,

$$\mathrm{Cone}(\{n_\rho \mid \rho \in \Sigma(1)\}) = N_\mathbb{R}$$

Therefore, the vectors n_ρ span N with positive coefficents impling that P_D is bounded.

□

Proposition 16.10. For a $T(N)$-invariant divisor, the polytopes P_D satisfy the following properties,

1. $P_{D+\mathrm{div}(\chi^u)} = P - u$
2. $P_{nD} = nP_D$
3. $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = \#\,(P \cap M)$

Proof. We know that $\psi_D|_\rho = \langle u(\rho), - \rangle$ where $u(\rho)$ is such that $D|_{U(\rho)} = \mathrm{div}(\chi^{-u(\rho)})$. Let $D' = D + \mathrm{div}(\chi^u)$. Then,

$$D'|_{U(\rho)} = \mathrm{div}(\chi^{-u(\rho)}) + \mathrm{div}(\chi^u) = \mathrm{div}(\chi^{-u(\rho)+u})$$

so $u'(\sigma) = u(\sigma) - u$ meaning that $\psi_{D'}(n_\rho) = \langle u(\sigma) - u, n_\rho \rangle = \psi_D(n_\rho) - \langle u, n_\rho \rangle$. Therefore,

$$\begin{aligned} x \in P_{D'} &\iff \forall \rho \in \Sigma(1) : \langle x, n_\rho \rangle \ge \psi_{D'}(n_\rho) = \psi_D(n_\rho) - \langle u, n_\rho \rangle \\ &\iff \forall \rho \in \Sigma(1) : \langle x + u, n_\rho \rangle \ge \psi_D(n_\rho) \iff x + u \in P_D \end{aligned}$$

Next, consider $\psi_{nD} = n\psi_D$ since on each cone $\psi_D|_{U_\sigma} = \langle nu(\sigma), - \rangle = n\,\langle n(\sigma, - \rangle$ where $D|_{U_\sigma} = \mathrm{div}(\chi^{-nu(\sigma)})$. Therefore,

$$\begin{aligned} x \in P_{nD} &\iff \forall \rho \in \Sigma(1) : \langle x, n_\rho \rangle \ge n\psi_D(n_\rho) \\ &\iff \forall \rho \in \Sigma(1) : \langle x/n, n_\rho \rangle \ge \psi_D(n_\rho) \iff x/n \in P_D \iff x \in nP_D \end{aligned}$$

Now finally, we use the decomposition,

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{\chi^u \in H^0(X, \mathcal{O}_X(D))} \mathbb{C} \cdot \chi^u$$

to show that,

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = \#\{u \in M \mid \chi^u \in H^0(X, \mathcal{O}_X(D))\}$$

However, we have shown that,

$$\chi^u \in H^0(X, \mathcal{O}_X(D)) \iff \forall \rho \in \Sigma(1) : \langle u, n_\rho \rangle \ge \psi_D(n_\rho) \iff u \in P_D$$

Therefore,

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = \#\{u \in M \mid u \in P_D\} = \#(P \cap M)$$

$\square$

16.3 Identification with the Okounkov Body

Remark 49. To consider the Okounkov body we need to fix a abmissible flag on X_Σ. In this section, we assume that X_Σ is smooth.

Definition: Because X_Σ is smooth, we can fix some ordering of $\rho \in \Sigma(1)$ i.e. order the $T(N)$-invariant prime divsors $D_i = V(\rho_i)$ such that the minimal genreators $n_i = n_{\rho_i}$ for $i = 1, \dots, d$ form a bais of N. Then the cones ρ_i generate a maximal cone σ_m. Then we define a flag,

$$Y_i = D_1 \cap \cdots \cap D_i \qquad X = Y_0 \supset Y_1 \supset \cdots \supset Y_d$$

Furthermore, the basis $n_1, \dots, n_d$ defines an isomorphism $N \cong \mathbb{Z}^d$ and a dual isomorphism $\phi : M \to \mathbb{Z}^d$ given by $m \mapsto (\langle m, n_i \rangle)_i$.

Theorem 16.11. Let X_Σ be a smooth projective toric variety and let $\mathcal{L} \to X$ be a big line bundle on X. Let D be the unique $T(N)$-invariant divisor D on X such that $\mathcal{L} \cong \mathcal{O}_{X_\Sigma}(D)$ and $D|_{U_{\sigma_m}} = 0$. Then,

$$\Delta_{Y_\bullet}(D) = \phi(P_D)$$

Proof. Recall that given a section of a line bundle $s \in \Gamma(X, \mathcal{L})$ there is a divisor of zero $(s)_0$ defined as follows. Let U_i be a cover of X such that $\mathcal{O}_X|_{U_i} \xrightarrow{f_i} \mathcal{L}|_{U_i}$ is an isomorphism. Then $\{(U_i, s|_{U_i}/f_i)\}$ is the Cartier divisor $(s)_0$. Now this Cartier divisor defines the Weil divisor,

$$(s)_0 = \sum_{Y \subset X_\Sigma} \mathrm{ord}_Y(s|_{U_i}/f_i)\, Y$$

Then $v_{Y_\bullet}(s) = (a_1, \dots, a_d)$ where $a_i = (s)_0|_{D_i}$. In our case, $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$ then $s \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \subset \mathrm{Rat}\,(X_\Sigma)$. Now,

$$(s)_0 = \sum_{Y \subset X_\Sigma} \mathrm{ord}_Y(s/f_i)\, Y = \sum_{Y \subset X_\Sigma} \mathrm{ord}_Y(s)\, Y - \sum_{Y \subset X_\Sigma} \mathrm{ord}_Y(f_i)\, Y = \mathrm{div}(f) + D$$

since the bundle $\mathcal{O}_{X_\Sigma}(D)$ is generated locally by f_i where $D = \{(U_i, f_i^{-1})\}$. In particular, consider the $T(N)$-invariant sections $\chi^u \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ then,

$$(\chi^u)_0 = D + \sum_{\rho \in \Sigma(1)} \langle u, n_\rho \rangle\, D_\rho$$

However, $D|_{U_{\sigma_m}} = 0$ and $D_i \subset U_\sigma$ for $i = 1, \dots, d$ implying that,

$$v_{Y_\bullet}(\chi^u) = (\langle u, n_i \rangle)_i = \phi(u)$$

Now recall that $\chi^u \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(mD)) \iff u \in mP_D \cap M$ implying that,

$$\Gamma(D)_m = \mathrm{Im}((H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(mD)) \setminus \{0\} \xrightarrow{v_{Y_\bullet}} \mathbb{Z}^d)) \supset \phi(mP_D \cap M)$$

However, because ϕ is injective and $mP_D \cap M$ contains prcisely $h^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(mD))$ lattice points, this inclusion is an equality,

$$\Gamma(D)_m = \phi(mP_D \cap M)$$

Therefore,

$$\Delta_{Y_\bullet}(D) = \Sigma\left(\bigcup_{m\geq 1} \tfrac{1}{m}\Gamma(D)_m\right) = \Sigma\left(\bigcup_{m\geq 1} \tfrac{1}{m}\phi(mP_D \cap M)\right)$$

Let m be any positive integer such that mP_D has all its vertices on lattice points in which case $\phi_{\mathbb{R}}(mP_D)$ is also a lattice polytope because $\phi_{\mathbb{R}} : M_{\mathbb{R}} \to \mathbb{R}^d$ takes lattice points to integer points. Thus the convex hull of $\phi(mP_D \cap M)$ is $\phi_{\mathbb{R}}(mP_D) = m\phi_{\mathbb{R}}(P_D)$ meaning that $\phi_{\mathbb{R}}(P_D)$ is the convex hull of the subset $\frac{1}{m}\phi(mP_D \cap M)$. Furthermore, for any m we have,

$$mP_D \cap M \subset mP_D \implies \tfrac{1}{m}\phi(mP_D \cap M) \subset \phi_{\mathbb{R}}(P_D)$$

Therefore, since it is the convex hull of a subset of the points and contains all of them, $\phi_{\mathbb{R}}(P_D)$ is the smallest closed convex set containing,

$$\bigcup_{m\geq 1} \tfrac{1}{m}\phi(mP_D \cap M)$$

meaning that,

$$\Delta_{Y_\bullet}(D) = \phi(P_D)$$

□

Remark 50. We can think of the condition $D|_{U_{\sigma_m}} = 0$ as centering the body P_B such that it lies in the positive orthant. It corresponds to multipling the Cartier divisor $D = \{(U_i, f_i)\}$ by the global section $f_{\sigma_m}^{-1}$ i.e. subtracting $\mathrm{div}(f_{\sigma_m})$. This corresponds to subtracting a suitable global support function to set a given support funtion equal to zero on the distinguished maximal cone σ_m.

16.4 Construction of a Toric Divisor from a Rational Polytope

Definition: There are a few equivalent characterizations of integral or lattice polytopes. Given a lattice M we say that a lattice polytope $P \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ is one of,

1. the convext hull of a finite subset of M
2. a finite intersection of integral halfspaces,

$$P = \bigcap_F \{m \in M \mid \langle n_F, m\rangle \geq -a_F\}$$

where F are the facets of P and $u_F \in M^\vee$ and $a_F \in \mathbb{Z}$. We may assume that u_F is the minimal inward normal in $M^\vee$.

Definition: Given a lattice polytope $P \subset M_\mathbb{R}$ we define the *normal fan* $\Sigma_P \subset N_\mathbb{R}$ as follows. For each face $A \subset P$ (not necessarily a facet, not including $A = P$ but including $A = \varnothing$) define,

$$\sigma_A = \text{Cone}(\{n_F \mid F \subset P \text{ is a facet s.t. } A \subset F\})$$

Then let $\Sigma_P = \{\sigma_A \mid A \subset P \text{ is a face}\}$.

Proposition 16.12. Given a lattice polytope P, the set Σ_P is a fan in $N_\mathbb{R}$.

Proposition 16.13. There is a duality between P and Σ_P given the inclusion reversing correspondence $A \subset P \leftrightarrow \sigma_A \in \Sigma_P$ satisfying,

1. inclusion reversing, $A \subset B \iff \sigma_B \subset \sigma_A$
2. $\dim A + \dim \sigma_A = \dim P$

Proof. $A \subset B$ implies that if F is a face containing B then F contains A so $\sigma_B \subset \sigma_A$. Furthermore, a face $A \subset P$ is contained in exactly $\dim P - \dim A$ facets giving the second property. □

Definition: Let P be a lattice polytope. Define the proper toric variety $X_P = X_{\Sigma_P}$. Via the above correspondence and the cone - orbit correspondence there is an inclusion preserving correspondence between dimension i faces $A \subset P$ and dimension i torus orbits. In particular,

1. vertices of $P \leftrightarrow$ fixed points of the torus action on X_P
2. facets of $P \leftrightarrow$ T-invariant irreducible divisors in X_P

Remark 51. Therefore we have a construction, given a lattice polytope P, of a proper toric variety $X_P = X_{\Sigma_P}$ of the normal fan. In fact, the following theorem classifies toric varieties arrising from a normal fan.

Theorem 16.14. A toric variety X is projective iff $X = X_P$ for some lattice polytope P i.e. if $X = X_\Sigma$ where $\Sigma = \Sigma_P$ is a normal fan of some lattic polytope P.

Definition: Given a lattice polytope P, we construct a toric variety - toric divisor pair (X_P, D_P) via $X_P = X_{\Sigma_P}$ and summing over the facets $F \subset P$ take,

$$D_P = \sum_{\substack{F \subset P \\ \text{a facet}}} a_F \, V(\sigma_F)$$

Recall that if F is a facet then $\sigma_F \in \Sigma_P(1)$ so the above definition makes sense.

Proposition 16.15. The divisor D_P is an ample Cartier divisor (and thus big) divisor on X_P.

Proof. Let m be a vertex of P and σ_m the corresponding maximal cone. Now I claim that for any facet F,

$$D_F \cap U_{\sigma_m} \neq \varnothing \iff m \in F$$

Indeed,

$$m \in F \iff \sigma_F \subset \sigma_m \iff \sigma_m \in \Sigma[\sigma_F] \iff D_F \cap U_{\sigma_m} \neq \varnothing$$

Therfore,

$$\mathrm{div}(\chi^{-m})|_{U_{\sigma_m}} = \sum_{m \in F} - \langle m, n_F \rangle D_F = \sum_{m \in F} a_F D_F = -D_P|_{U_{\sigma_m}}$$

because $\langle m, n_F \rangle = -a_F$ by the defining representation of P since m is a vertex and F is a facet containing m. Thus, D_P is Cartier since it is principal on the open cover of maximal conces. Therefore, we may consider ψ_D which satisfies $\psi_{D_P}|_{\sigma_m} = \langle m, - \rangle$. Finally, ψ_{D_P} is stricly concave meaning that D_P is ample. $\square$

Theorem 16.16. The polytope associated to the divisor D_P on X_P is $P_{D_P} = P$ therefore the mapping,

$$\{(X, D) \mid \dim X = d\} \to \{\text{integral polytopes of dimension } d\}$$

sending projective toric varieties of dimension d with T-invariant divisors to integral polytopes is surjective.

Proof. Recall that the cones $\rho \in \Sigma_P(1)$ correspond to facets $F \subset P$. The divisor D_P corresponds to the support function ψ_{D_P} with $\psi_{D_P}(n_\rho) = -a_F$. Therefore,

$$P_{D_P} = \bigcap_{\substack{F \subset P \\ \text{a facet}}} H^+(n_F, -a_F) = P$$

$\square$

Remark 52. We can use the theory of toric geometry to give a highly amusing proof of a powerful elementary result in convex geometry.

Theorem 16.17 (Ehrhart Polynomial)**.** Let P be an d-dimensionaly lattice polytope in $M_{\mathbb{R}}$. Then there exists a unique polynomial with rational coefficients $E_P \in \mathbb{Q}[x]$ ssatsfying:

1. For any integer $\nu \in \mathbb{N}$,
$$E_P(\nu) = \#\left((\nu P) \cap M\right)$$

2. The leading coefficient of E_P is $\mathrm{Vol}_M(P)$ i.e. the volume of P normalized to the lattice cell volume of M.

3. There is a reciprocity law for positive integers $\nu > 0$,
$$E_P(-\nu) = (-1)^d \# \left(\nu P^\circ \cap M\right)$$

Proof. Given the lattice polyheron P we have constructed a toric variety X_P with an ample divisor D_P. Furthermore, the lattice polyope of D_P is exactly P. Therefore,

$$\dim_{\mathbb{C}} H^0(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \#\left(\nu P \cap M\right)$$

Recall that the Euler characteristic of the cohernt sheaf $\mathcal{O}_{X_P}(\nu D_P)$ is,

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \sum_{i=0}^{d} (-1)^i \dim_{\mathbb{C}} H^i(X_P, \mathcal{O}_{X_P}(\nu D_P))$$

By the Hirzbruch-Riemann-Roch theorem we have,

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \int_{X_P} \mathrm{ch}(\mathcal{O}_{X_P}(\nu D_P)) \, \mathrm{Td}(\mathcal{T}_{X_P})$$

Recall that the Chern character is,

$$\mathrm{ch}(\mathcal{O}_{X_P}(\nu D_P)) = \exp\left(c_1(\mathcal{O}_{X_P}(\nu D_P))\right) = \sum_{m=0}^{d} \frac{c_1(\mathcal{O}_{X_P}(\nu D_P))^m}{m!}$$

where the sum terminates at $d = \dim X_P$ since higher intersections vanish. Recall that the Chern class c_1 is a homomorphism $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$. Thus, since $\mathcal{O}_{X_P}(\nu D_P) = \mathcal{O}_{X_P}(D_P)^{\otimes \nu}$,

$$\mathrm{ch}(\mathcal{O}_{X_P}(\nu D_P)) = \sum_{m=0}^{d} \frac{c_1(\mathcal{O}_{X_P}(D_P)^{\otimes \nu})^m}{m!} = \sum_{m=0}^{d} c_1(\mathcal{O}_{X_P}(D_P))^m \frac{\nu^m}{m!}$$

Therefore,

$$\begin{aligned} \chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) &= \int_{X_P} \left(\sum_{m=0}^{d} c_1(\mathcal{O}_{X_P}(D_P))^m \frac{\nu^m}{m!} \right) \mathrm{Td}(\mathcal{T}_{X_P}) \\ &= \sum_{m=0}^{d} \frac{\nu^m}{m!} \left(\int_{X_P} c_1(\mathcal{O}_{X_P}(D_P))^m \, \mathrm{Td}(\mathcal{T}_{X_P}) \right) = h(\nu) \end{aligned}$$

is a degree at most d polynomial in ν. Now recall Demazure's theorem on the vanishing of cohomology on toric varieties which states that if $\mathcal{L}$ is ample or generated by global sections then,

$$\forall p > 0 : H^p(X_P, \mathcal{L}) = 0$$

Since $\mathcal{O}_{X_P}$ is generated by global sections and $\mathcal{O}_{X_P}(\nu D_P)$ is ample for $\nu > 0$ we have shown that for $\nu \geq 0$ that,

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \dim_{\mathbb{C}} H^0(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \#\left(\nu P \cap M\right)$$

This implies that for $\nu \in \mathbb{N}$ we have proven there is a polynomial,

$$E_P(\nu) = h(\nu) = \chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \dim_{\mathbb{C}} H^0(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \#\left(\nu P \cap M\right)$$

Furthermore, since D_P is big and $E_P(m)$ counts sections of $\mathcal{O}_{X_P}(mD_P)$, we know that the leading term must be m^d so $\deg E_P = d$. Writing,

$$E_P(x) = a_n x^n + \cdots + a_0$$

we may isolate the leading coefficient as follows,

$$a_n = \lim_{\nu \to \infty} \frac{E_P(\nu)}{\nu^d} = \lim_{\nu \to \infty} \frac{\#\left(\nu P \cap M\right)}{\nu^d} = \mathrm{Vol}_M\left(P\right)$$

Lastly, to prove the duality property, we apply Serre duality. On X_P, the dualizing sheaf is equal to the canonical sheaf,

$$\omega_{X_P} = \mathcal{O}_{X_P}(-\sum_F D_F)$$

where D_F is the divisor $V(\sigma_F)$ for each facet $F \subset P$. Since X_P is a projective Cohen–Macaulay variety (and thus irreducible over k), Serre duality sates that, for any locally free sheaf $\mathcal{F}$ on X_P,

$$H^i(X_P, \mathcal{F}^\vee) = H^{d-i}(X_P, \mathcal{F} \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})^\vee$$

which, by computing dimensions and reordering, implies that,

$$\chi(X_P, \mathcal{F}^\vee) = (-1)^d \chi(X_P, \mathcal{F} \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})$$

In particular, for $\mathcal{F} = \mathcal{O}_{X_P}(\nu D_P)$ we have,

$$E_P(-\nu) = \chi(X_P, \mathcal{O}_{X_P}(-\nu D_P)) = (-1)^d \chi(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})$$

By the Kodaria vanishing theorem, since νD_P is ample for $\nu > 0$,

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P}) = \dim_{\mathbb{C}} H^0(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})$$

Now we consider the invertible sheaf,

$$\mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P} = \mathcal{O}_{X_P}(\nu D_P - \sum_F D_F) = \mathcal{O}_{X_P}(\sum_F (\nu a_F - 1) D_F)$$

which means we should consider the divisor,

$$D' = \sum_F (\nu a_F - 1) D_F$$

which corresponds to the support function $\psi_{D'}$ satisfying $\psi_{D'}(n_F) = -(\nu a_F - 1)$ (recall that cones $\rho \in \Sigma_P(1)$ correspond to facets $F \subset P$). Therefore, the polytope for the divisor D' is,

$$P_{D'} = \bigcap_{\substack{F \subset P \\ \text{a facet}}} H^+(n_F, \psi_{D'}(n_F)) = \bigcap_{\substack{F \subset P \\ \text{a facet}}} H^+(n_F, 1 - \nu a_F)$$

Recall that,

$$\nu P = \bigcap_{\substack{F \subset P \\ \text{a facet}}} H^+(n_F, -a_F) = \bigcap_{\substack{F \subset P \\ \text{a facet}}} \{x \in M_{\mathbb{R}} \mid \forall F : \langle x, n_F \rangle \geq -\nu a_F\}$$

Therefore, the interior is,

$$\nu P^\circ = \bigcap_{\substack{F \subset P \\ \text{a facet}}} \{x \in M_{\mathbb{R}} \mid \forall F : \langle x, n_F \rangle > -\nu a_F\}$$

Therefore, intersecting with the lattice,

$$\nu P^\circ \cap M = \bigcap_{\substack{F \subset P \\ \text{a facet}}} \{m \in M \mid \forall F : \langle m, n_F \rangle \geq -\nu a_F + 1\} = P_{D'} \cap M$$

because the inner product is integer valued on the lattice so,

$$\langle m, n_F \rangle > -\nu a_F \iff \langle n, n_F \rangle \geq -\nu a_F + 1$$

Thus,

$$E_P(-\nu) = \dim_{\mathbb{C}} H^0(X_P, \mathcal{O}_{X_P}(D')) = \#\left(P_{D'} \cap M\right) = \#\left(\nu P^\circ \cap M\right)$$

□

Remark 53. Note that $E_P(0) = \#((0 \cdot P) \cap M) = 1$ so the constant term is 1. Furthermore, in the limit $\nu \to \infty$ if $\dim P = d$ then $E_P(\nu) \in O(\nu^d)$ so $\deg E_P = d$.

Remark 54. To prove the power of this theorem, we can easily derive the classical Pick's theorem as a special case.

Theorem 16.18 (Pick)**.** Let $\dim M = 2$ and $P \subset M_{\mathbb{R}}$ be a lattice polygon. Then,

$$\#(P \cap M) = \mathrm{Vol}_M(P) + \tfrac{1}{2}\#(\partial P \cap M) + 1$$

Proof. Consider the Ehrhart polynomial which takes the form,

$$E_P(x) = \mathrm{Vol}_M(P)\, x^2 + Bx + 1$$

Now we can decompose $P = P^\circ \cup \partial P$ which implies that,

$$E_P(1) = \#\left(P \cap M\right) = \#\left(P^\circ \cap M\right) + \#\left(\partial P \cap M\right)$$

Furthermore, by the reciprocity law,

$$E_P(-1) = \#\left(P^\circ \cap M\right)$$

Putting these together, we find,

$$E_P(1) - E_P(-1) = \#\left(\partial P \cap M\right)$$

However, applying the polynomial form,

$$E_P(1) - E_p(-1) = 2B \implies B = \tfrac{1}{2}\#\left(\partial P \cap M\right)$$

Thus the Ehrhart polynomial is,

$$E_P(x) = \mathrm{Vol}_M\left(P\right)x^2 + \tfrac{1}{2}\#\left(\partial P \cap M\right)x + 1$$

Which, for $x = 1$ we find,

$$E_P(1) = \#\left(P \cap M\right) = \mathrm{Vol}_M\left(P\right) + \tfrac{1}{2}\#\left(\partial P \cap M\right) + 1$$

giving Pick's formula. $\square$

16.5 Examples

16.6 The Picard Group of a Toric Variety

Theorem 16.19. Let X_Σ be a smooth toric variety and $\#(\Sigma(1)) = s$. Then there is an exact sequence,

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Pic}\left(X_\Sigma\right) \longrightarrow 0$$

and $\mathrm{Pic}\left(X_\Sigma\right)$ is torsion free so the sequence splits.

Proof. The map $\mathbb{Z}^s \to \mathrm{Pic}\left(X_\Sigma\right)$ sends $v \mapsto \mathcal{O}_{X_\Sigma}(\sum v_i V(\rho_i))$ where $\phi_i \in \Sigma(1)$ ranges over the rays of Σ. This map is surjective because X_Σ is smooth and integral so $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Pic}\left(X\right)$ is an isomorphism and $\mathrm{Cl}(X)$ is generated by $T(N)$-invariant prime divisors. Furthermore, the kernel $\mathbb{Z}^s \to \mathrm{Div}(X) \to \mathrm{Cl}(X)$ are the $T(N)$-invariant principal divisors i.e. the characters,

$$\mathrm{div}(\chi^u) \in \mathrm{Div}(X)$$

Therefore, this kernel is $\iota : M \to \mathbb{Z}^s$ via,

$$\iota(u) = \mathrm{div}(\chi^u) = \sum_{i=1}^{s} \langle u, n_i \rangle \; D_i$$

since the map $n \mapsto (\langle u, n_i \rangle)_i$ is injective since $\{n_i\}$ forms a basis of N. (FINISH PROOF) $\square$

17 Hodge Index Theorem for Surfaces

Definition: Denote a nonsigular projective variety of dimension two over an algebraically closed field as a *surface* and effective divisor on a surface as a *curve*.

Definition: Let X be a surface and C, C' curves on X. For $p \in X$, choose an open neighborhood of U of p such that C, C' are the vanishing of (f, g) on U. Then consider $A = (f_p, g_p) \subset \mathcal{O}_{X,p}$. I claim that $\mathcal{O}_{X,p}/A$ is finite dimensional. Then we define the intersection multiplicity,

$$\iota(C, C', p) = \dim\left(\mathcal{O}_{X,p}/A\right)$$

We define the intersection number,

$$C \cdot C' = \sum_{p \in X} \iota(C, C', p)$$

Remark 55. Consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

Then there is a sequence,

$$0 \longrightarrow \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C \cap C'} \longrightarrow 0$$

Taking the stalk at p and summing over the two curves gives an exact sequence,

$$0 \longrightarrow A \longrightarrow \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{C \cap C', p} \longrightarrow 0$$

Remark 56. Note that,

$$h^0(\mathcal{O}_{C \cap C'}) = \sum_{p \in X} \iota(C, C', p) = C \cdot C'$$

Since $C \cdot C'$ is zero dimensional its higher cohomology vanishes so,

$$C \cdot C' = \chi(C \cap C', \mathcal{O}_{C \cap C'})$$

Definition: The intersection from $\text{Pic}\,(X) \times \text{Pic}\,(X) \to \mathbb{Z}$ is,

$$\xi \cdot \xi' = \chi(\mathcal{O}_X) - \chi(-\xi) - \chi(-\xi') + \chi(-\xi - \xi')$$

Definition: Let C be a smooth irreducible curve on S. For any line bundle ξ then,

$$\mathcal{O}_X(C) \cdot \xi = \deg(\xi|_U)$$

Theorem 17.1. If $\xi = \mathcal{O}_X(C)$ and $\xi' = \mathcal{O}_X(C')$ then,

$$\xi \cdot \xi' = C \cdot C'$$

Remark 57. Consider the self-intersection $D^2 = D \cdot D$. The self-intersection of $C \subset X$ can be given the following interpretation. Let $N_{X/C}$ be the normal bundle which fits in the exact sequence,

$$0 \longrightarrow T_C \longrightarrow (T_X)|_C \longrightarrow N_{C/X} \longrightarrow 0$$

Then $N_{C/X}$ is the self-intersection of $\xi|_C = \mathcal{O}_C(C)$ i.e. $C^2 = \deg(\xi|_C) = \deg(N_{C/X})$.

Example 17.2. Let C, C' be two plane curves of degree m and n. Take a line ℓ and $C \sim \ell m$ and $C' \sim \ell n$ with $\ell^2 = 1$. Then $C \cdot C' = mn$.

Theorem 17.3 (Hodge Index)**.** Let H be an ample divisor $D \cdot H = 0$ and $D \neq 0$ then $D^2 < 0$.

Definition: Two divisors D, D' are numberically equivalent if for all divisors H we have $D \cdots H = D' \cdots H$. Then $N^1(X)$ is $\mathrm{Pic}\,(X)$ modulo numerical equivalence.

Lemma 17.4. Let H be an ample divisor. For any effective divisor D we have $D \cdot H > 0$.

Lemma 17.5. Let H be an ample divisor on X. Then $\exists m_0 \in N$ s.t. for any D if $DH > m_0$ then $H^2(X, \mathcal{O}_X(D)) = 0$.

Lemma 17.6. Let H be an ample divisor and D such that $D \cdot H > 0$ and $D^2 > 0$. Then for all $m \gg 0$ we have mD is linearly equivalent to an effective divisor.

Proof of Theorem. Suppose not i.e. $D^2 \geq 0$,

First case, $D^2 > 0$. Let $H' = D + mH$ for sufficiently large m. Then H' is ample. Now,

$$H' \cdot D = D^2 + mH \cdot D = D^2 > 0$$

Then mD is effective by previous case. But $md \cdot H > 0$. so $D \cdot H > 0$ which is a contradiction.

Second case, $D^2 = 0$. Since $D \neq 0$ there is a divisor E s.t. $D \cdot E \neq 0$. Let $E' = (H^2)E - (E \cdot H)H$. Then $E' \cdot H = 0$. In addition, $D' = mD + E'$ and $D' \cdot H = mD' + E'$. Then

$$D' \cdot H = mD + E' \cdot H = 0$$

Furthermore,

$$(D')^2 = m^2 D^2 + 2mD \cdot E' + (E')^2 = 2mD \cdot E' + (E')^2$$

Choose m s.t. $(D')^2 > 0$. We apply the first case to D' and get a contradicton. $\square$

18 Alexandrov - Fenchel Inequality

18.1 Review

Fix $n \in \mathbb{Z}^+$ and let κ be the set of convex bodies in $\mathbb{R}^n$ and κ_V the set of integral polytopes. Take scalars $\lambda_1, \lambda_2, \ldots, \lambda_s > 0$ and $\Delta_1, \Delta_2, \ldots, \Delta_s \in \kappa$. Then,

$$\mathrm{Vol}_n\,(\lambda_1 \Delta_1 + \cdots + \lambda_s \Delta_s) = \sum_{i_1,\ldots,i_s=1}^{s} \mathrm{mVol}\,(\Delta_1, \ldots, \Delta_s)\, \lambda_{i_1} \cdots \lambda_{i_n}$$

For any convext sets $\S_1, \ldots, \S_n$ the mixed volume satisfies,

Proposition 18.1. Properties of Mixed Volumes:

1. $\mathrm{mVol}\,(S,\ldots,S) = \mathrm{Vol}_n\,(S)$
2. Symmetric, $\mathrm{mVol}\,(S_1,\ldots,S_n) = \mathrm{mVol}\left(S_{\pi(1)},\ldots,S_{\pi(n)}\right)$
3. Multilinear: $\mathrm{mVol}\,(\lambda S+\lambda' S', S_2,\cdots,S_n) = \lambda\mathrm{mVol}\,(S,S_2,\ldots,S_n)+\lambda'\mathrm{mVol}\,(S',S_2,\ldots,S_n)$
4. Nonegative: $\mathrm{mVol}\,(S_1,\ldots,S_n) \geq 0$
5. Monotonic: if $S \subset S'$ then $\mathrm{mVol}\,(S,S_2,\ldots,S_n) \leq \mathrm{mVol}\,(S',S_2,\ldots,S_n)$

Theorem 18.2 (Alexandrov - Fenchel)**.** For any $\Delta_1,\ldots,\Delta_s \in \kappa$ we have,

$$\mathrm{mVol}\,(\Delta_1,\ldots,\Delta_s) \geq \mathrm{mVol}\,(\Delta_1,\Delta_1,\Delta_3,\ldots,\Delta_s)\cdots\mathrm{mVol}\,(\Delta_2,\Delta_2,\Delta_3,\ldots,\Delta_s)$$

Definition: A bilinear form $B : V\times V \to \mathbb{R}$ is hyperbolic if there exists $v\in V$ s.t. $B(v,v)>0$ but there does not exist a subspace $W\subset V$ s.t. $B|_W \geq 0$ and $\dim W > 1$.

Proposition 18.3. Let $B : V\times V \to \mathbb{R}$ be a hyperbolic form and $v\in V$ s.t. $B(v,v)>0$. Then for any $y\in V$,

$$B(x,y)^2 \geq B(x,x)B(y,y)$$

Theorem 18.4 (Hodge Index)**.** The intersection form $\langle -,-\rangle : \mathrm{Pic}\,(X)\times\mathrm{Pic}\,(X)\to\mathbb{Z}$ is hyperbolic.

18.2 Hausdorff Distance

Definition: Let $B\subset\mathbb{R}^n$ denote the unit ball and $K,L\in\kappa_n$ convex bodies in $\mathbb{R}^n$. Then consider the λ-parallet body $K+\lambda B$. We define the Hausdroff distance,

$$d(K,L) = \inf\{\lambda\geq 0 \mid L\subset K+\lambda B \text{ and } K\subset L+\lambda B\}$$

Lemma 18.5. The Hausdorff distance is a metric.

Remark 58. The Hausdorff distance induces a topology on the space of convex bodies κ_n.

Proposition 18.6. Mixed volumes are continous functions in the Hausdorff topology.

Theorem 18.7. For any covex body $K\in\kappa_n$ there exists an increasing sequence $\Delta_1\subset\Delta_2\subset\Delta_3\subset\cdots$ of polytopes which converge to K in the Hausdorff topology.

18.3 Proof of the Main Theorem

Consider some integral polytopes $\Delta_1, \dots, \Delta_n \in \kappa_n$.

Remark 59. An integral polytope Δ is exactly the convex hull of its vertices $\{v^1, \dots, v^s\}$ which is a finite set. To this set we may associate a Laurent polynomial,

$$p_\Delta(X_1, \dots, X_n) = \sum_{i=1}^{s} X_1^{v_1^i} \cdots X_n^{v_n^i}$$

Theorem 18.8 (Khovanski)**.** If we consider a general system if polynomial equaltions $p_1 = \cdots = p_n = 0$ whose newton polytopes are $\Delta_1, \dots, \Delta_n$ then the number of complex solutions equals $n! \operatorname{mVol}(\Delta_1, \dots, \Delta_n)$.

Remark 60. Now we prove the theorem.

Proof. Consider $\Delta_1, \dots, \Delta_n$ and $f_1, \dots, f_n$ their associated Laurent polynomials. Then let M_Σ be the toric compactification under the fan,

$$\Sigma = \left\{ \sum_{i=1}^{n} \lambda_i \Delta_i \;\middle|\; \lambda_i \geq 0 \right\}$$

We construct a surface F and a family of curves Γ_f on F. First consider the affine surface,

$$F' = \operatorname{Spec}\left(\mathbb{C}[X_1, \dots, X_n]/(f_3, \dots, f_n)\right)$$

Then we let F be its toric closured. Then F is a connected and nonsingular surface so we may apply hodge theory. The curves are constructed via the closure in F of the affine curve,

$$\Gamma_f' = \operatorname{Spec}\left(\mathbb{C}[X_1, \dots, X_n]/(f, f_3, \dots, f_n)\right)$$

If the Newton polytope associated to f is contained in Σ then the curve Γ_f is non-singular.

Proposition 18.9. Let g, h be Laurent polynomials. If Δ_g and Δ_h are non-singular then $\langle \Gamma_g, \Gamma_h \rangle = n! \operatorname{mVol}(\Delta_g, \Delta_h, \Delta_3, \dots, \Delta_n)$.

Proof. All the roots of $g = h = f_3 = \cdots = f_n = 0$ are contained in $\mathbb{C}^\times$ (why?) so we conclude that,

$$\langle \Gamma_g, \Gamma_h \rangle = n! \operatorname{mVol}(\Delta_g, \Delta_h, \Delta_3, \dots, \Delta_n)$$

being the number of intersection points. $\square$

Therefore, for the surface F and the curves Γ_f and Γ_f associated to Δ_1 and Δ_2. Applying the Hodge index theorem,

$$\langle \Gamma_{f_1}, \Gamma_{f_2} \rangle \geq \langle \Gamma_{f_1}, \Gamma_{f_1} \rangle \cdot \langle \Gamma_{f_2}, \Gamma_{f_1} \rangle$$

Therefore we get,

$$\operatorname{mVol}(\Delta_1, \Delta_2, \dots, \Delta_n)^2 \geq \operatorname{mVol}(\Delta_1, \Delta_1, \Delta_3, \dots, \Delta_n) \cdot \operatorname{mVol}(\Delta_2, \Delta_2, \Delta_3, \dots, \Delta_n)$$

Then by using the continuity of the mixed volumes we can apply this to arbitrary convex bodies by approximation via a convergent sequence of polytopes.

$\square$

19 Brenier Maps

Definition: Given two measure spaces (X, Σ_X, μ_X) and (Y, Σ_Y, μ_Y), a transfer problem consider the set of measures,

$$\Pi(\mu_X, \mu_Y) = \{\nu \mid \text{ measure on } X \times Y \text{ s.t. } (\pi_X)_* \nu = \mu_X \text{ and } (\pi_Y)_* \nu = \nu_Y\}$$

on the product measureable space $(X \times Y, \Sigma_X \times \Sigma_Y)$ with marginals μ_X and μ_Y. For example, we might consider the Kantorovich tranport problem which is to attain the infimum,

$$\inf \left\{ \int_{X \times Y} c(x, y) \, \mathrm{d}\nu \;\middle|\; \nu \in \Pi(\mu_X, \mu_Y) \right\}$$

for some cost function $c(x, y)$. Our problem in question is to achieve the infimum,

$$\inf \left\{ \operatorname{ess\,sup} \left(\frac{\mathrm{d}s_*(\nu)}{\mathrm{d}\mu} \right) \;\middle|\; \nu \in \Pi(\mu_X, \mu_Y) \right\}$$

where X and Y are convex bodies and μ_X and μ_Y and μ are the Lebesgue measures on X, Y, and $X + Y$ respectivly and $s : X \times Y \to X + Y$ is the sum map.

Definition: A Monge transport problem is a specialization of the Kantorovich formulation in which we restrict the allowed measures on the product to be diagonal. In particular, we are asked to acheive the infimum,

$$\inf \left\{ \int_X c(x, T(y)) \, \mathrm{d}\mu_X \;\middle|\; T : X \to Y \text{ measureable and } T_*(\mu_X) = \mu_Y \right\}$$

This is equivalent to restricting to measures on $X \times Y$ of the form $\nu = (\mathrm{id} \times T)_*(\mu_X)$.

Remark 61. Given two convex bodies, Δ_1, Δ_2, we are interested in measure-preserving bijections $f : \Delta_1 \to \Delta_2$ which have "nice" extensions $\mathrm{id} + f : \Delta_1 \to \Delta_1 + \Delta_2$. Ideally, such an extension would also be a measure-preserving bijection. In such a case we make take the transer measure $\nu = (\mathrm{id} \times f)_* \mu_1$ have $\rho(\Delta_1, \Delta_2) = 1$.

Proposition 19.1. If $\mathrm{id} + f : \Delta_1 \to \Delta_1 + \Delta_2$ is measure-preserving then

$$\rho(\Delta_1, \Delta_2) = 1$$

is achieved by $\nu = (\mathrm{id} \times f)_*(\mu_X)$.

Proof. Consider the measure $\nu = (\mathrm{id} \times f)_*(\mu_X)$. Then we have $(\pi_X)_* \nu = \mu_X$ since $\pi_X \circ (\mathrm{id} \times f) = \mathrm{id}$ and $(\pi_Y)_* \nu = \mu_Y$ since $\pi_Y \circ (\mathrm{id} \times f) = f$ and $f_*(\mu_X) = \mu_Y$. Finally,

$$s_*(\nu) = (s \circ (\mathrm{id} \times f))_*(\mu_X) = (\mathrm{id} + f)_*(\mu_X) = \mu$$

$\square$

Theorem 19.2 (Knothe)**.** Let Δ_1 and Δ_2 be convex bodies. Then there exists a measure-preserving bijection $f_K : \Delta_1 \to \Delta_2$ s.t. $\det \mathrm{d}f = |\Delta_2| / |\Delta_1|$ is constant everywhere and $\mathrm{d}f$ is upper triangular and $\mathrm{id} + f$ is injective.

Theorem 19.3 (Brenier)**.** Let Δ_1 and Δ_2 be convex bodies and consider the quadratic cost $c(x, y) = |x - y|^2$ via the Euclidean norm. If μ_X is compactly supported and absolutly continuous with respect to the Lebesgue measure then the Monge problem has a solution $T : \Delta_1 \to \Delta_2$ called the Brenier map which is characterized as the unique measure-preserving bijection s.t. there exists a convex function $\phi : \Delta_1 \to \mathbb{R}$ with $T = \nabla\phi$.

Theorem 19.4. Given convex bodies Δ_1 and Δ_2, there exists a measure preserving bijection $\Phi : \Delta_1 \to \Delta_2$ s.t. $\mathrm{id} + \Phi : \Delta_1 \to \Delta_1 + \Delta_2$ is surjective.

Proof. □

Definition: Let $\Omega \subset \mathbb{R}^n$ be open. Then a Monge-Ampere equation is of the form,

$$\det(D^2 u) = f(x, u, \nabla u)$$

for a given function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and some $u : \Omega \to \mathbb{R}$ convex.

Remark 62. Finding the Brenier map $T : \Delta_1 \to \Delta_2$ is equivalent to solving the Monge-Ampere equation,

$$\det(D^2\phi) = \frac{|\Delta_2|}{|\Delta_1|}$$

where $T = \nabla\phi$.

Remark 63. Monge-Ampere theory can thus bound the Jacobian of $\mathrm{id} + \nabla\phi$.