1 del Pezzo Surfaces

Definition 1.0.1. A surface is *del Pezzo* if it has cyclic quotient singularities and $-K_X$ is ample (note that X need not be Gorenstein so K_X is just $\mathbb{Q}$-Cartier).

Remark. We can to classify these up to qG-deformations. The idea of a qG-deformation is a family of del Pezzo surfaces $X \to S$ such that $\deg(-K_{X_s})^2$ and $h^0(X_s, -K_{X_s})$ are locally constant on the base (the intersection numbers are automatically constant for flat families).

1.1 Mirror Symmetry for Fanos

Let X be a smooth Famo and p_β is some count of stable maps of class β to X though a fixed generic point. More precisely,

$$P_\beta = (d_\beta)! \int_{M_{0,1}(X,\beta)} \mathrm{ev}^*[*] \cdot \psi^{d_\beta - 2}$$

where $d\beta = -\deg(K_X \cdot \beta)$ is an anti-canonical degree. These are stable maps with $f_*[C] = \beta$ where $[C]$ is the fundamental class.

We fit these together into a generating function,

$$\hat{G}_X(z) := \sum_{\beta \in \mathrm{NE}(X)} P_\beta \cdot z^\beta \in \mathbb{Q}[[\mathrm{NE}(X)]]$$

We can replace $z^\beta \mapsto t^{d_\beta}$ which gives a power series,

$$\hat{G}_X(t) \in \mathbb{Q}[[t]]$$

This is one side of the mirror symmetry story. On the other side we have,

$$f \in \mathbb{Q}[x^{\pm}, y^{\pm}]$$

then consider,

$$\pi_f = \int_{|z_1|=|z_2|=1} \frac{\Omega}{1 - tf} = \sum_{d \geq 0} c_0(f^d) t^d \in \mathbb{Q}[[t]]$$

where $c_0(g)$ is the constant coeffiicent as a Laruent polynomial. We say that f is mirror to X if,

$$\pi_f(t) = \hat{G}_X(t)$$

Note that f is not unique if it exists.

1.2 Conjectures

Definition 1.2.1. A Fano variety if of class TG if there is a qG-deformation to a toric variety.

Conjecture 1.2.2. There exists a bijection,

$$\{X \text{ Fano and of class } TG\}/\{\text{qG-defomration} \iff \{\text{Fano polytopes}\}/\text{mutations}$$

given by $P \mapsto [X_P]$.

1.3 Evidence

Smooth del Pezzos there are $\mathrm{Bl}_{k\mathrm{points}}\mathbb{P}^2$ for $0 \leq k \leq 8$ and $\mathbb{P}^1 \times \mathbb{P}^1$ then the qG-deformations gives 10 families (allthough many of these families have a unique member).

Theorem 1.3.1. The map $\{\text{Fano Poly}\}/\{\text{mutation}\} \to \{X \text{ Fano TG}\}/\sim$ is well defined.

1.4 Gross - Siebert Program

Let (X, D) be a smooth del Pezzo pair D is an anticanonical divisor (snc and a cycle of lines maybe). The Gross-Siebert program produces an algebra,

$$A = \bigoplus_{P \in B(\mathbb{Z})} \theta_p \mathbb{C}[\mathrm{NE}(X)]$$

where we define,

$$\theta_p \cdot \theta_q = \sum_{r \in B(\mathbb{Z})} \alpha_{pqr} \theta_r$$

with $\alpha_{pqr} \in \mathcal{C}[\mathrm{NE}(X)]$ is defined as some Gromov-Witten invariant. Then we get,

$$Y = \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathbb{C}[\mathrm{NE}(X)])$$