1 Pre-talk question.

Let $Y_1(N)/\mathbb{F}_p$ with $(p, N) = 1$ be the level N modular curve which is the moduli space of elliptic curves with level structure.

Recall that an elliptic curve $E/\mathbb{F}_p$ is ordinary if of the equivalent properties hold,

(a) $E[p](\overline{\mathbb{F}_p}) \cong \mathbb{Z}/p\mathbb{Z}$

(b) $E[p^n](\overline{\mathbb{F}_p}) \cong \mathbb{Z}/p^n\mathbb{Z}$

(c) $E_{\overline{\mathbb{F}_p}}[p^\infty] \cong \mathbb{Q}_p/\mathbb{Z}_[\oplus \mu_{p^\infty}$

Otherwise E is supersingular. The locus $Y_1(N)^{\text{ord}} \subset Y_1(N)$ is open. We cannot expect,

$$X = \mathcal{E}[p^\infty]/Y_1(N)^{\text{ord}}$$

to be étale locally isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$. However, there is an exact sequence,

$$0 \longrightarrow X^1 \longrightarrow X \longrightarrow X^0 \longrightarrow 0$$

where X^1 is multiplicative and X^0 is étale (SOMTHING?)

We define the Igusa variety,

$$IG^{\text{ord}}$$

as the moduli space of isomorphisms $X^0 \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_\mathfrak{p}$ and $X^1 \xrightarrow{\sim} \mu_{p^\infty}$. Them

$$IG^{\text{ord}} = \varprojlim IG_p^{\text{ord}}$$

Then IG^{ord} is a $\underline{\mathbb{Z}_p}^\times \times \underline{\mathbb{Z}_p}^\times$ pro-étale torsor.

Remark. Why is this useful? We can lift $Y_1(N)^{\text{ord}}$ to a formal scheme over $\mathbb{Z}_p$ then IG^{ord} lifts to a torsor over the formal scheme $\mathcal{IG}^{\text{ord}}$. Sections of this torsor are used to define p-adic modular forms. Indeed, p-adic modular forms (a la Katz or Hidas) are exactly sections of the lifted Igusa variety.

Theorem 1.0.1 (Igusa)**.** $\pi_0(IG^{\text{ord}}) \cong \mathbb{Z}_p^\times$ equivariantly for the $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ action via $(a, b) \mapsto ab^{-1}$.

Let $\mathcal{A}_{g,N}$ be the moduli space of principally polarized abeilain varieties and let Σ be a fixed polarized p-divisoble group. Then $C_\Sigma \subset \mathcal{A}_{g,N}$ consisting of those $\mathbb{F}_p$-points A such that $A[p^\infty] \cong C_\Sigma$ compatibly wit hthe polarisations up to a $\mathbb{Z}_p^\times$-scalar.

Theorem 1.0.2 (Oort)**.** C_Σ is locally closed, smooth, equidimensional and there is an Igusa variety $IG_\Sigma \to C_\Sigma$ and this is an $\mathrm{Aut}\,(\Sigma)$-torsor.

Theorem 1.0.3 (Chai-Oort)**.** If Σ is not supersingular, then C_Σ is connected and $\pi_0(IG_\Sigma) \cong \mathbb{Z}_p^\times$.

Let (G, X) be a Shimura datum of Hodge type σ and $p > 2$ a prime where $G_{\mathbb{Q}_p}$ is unramified $U^p \subset G(\mathbb{A}_f^p)$ a compact open $U_p \subset G(\mathbb{Q}_p)$ hyperspecial. We let,

$$\mathfrak{Sh}_f/\overline{\mathbb{F}_p}$$

be the special fiver of the canonical integral model (exists by Kisin). Then we also get,

$$C_\Sigma \subset \mathfrak{Sh}_G$$

locally closed, smooth (Kisin) and Igusa varieties,

$$IG_\Sigma \to C_\Sigma$$

Then IG_Σ is a torsor for some profinite group.

Theorem 1.0.4. Assume G^{der} is simply connected, G^{ab} is $\mathbb{Q}$-simple, and that $p \not| W_G$ where W_G is the Weil group. If C_Σ is not contained in the basis locus,

$$\pi_0(IG_\Sigma) \cong \pi_0(\mathfrak{Sh}_G) \times G^{\mathrm{ab}}(\mathbb{Z}_p)$$

Remark. If $C_\Sigma \subset \mathfrak{Sh}_{G_U}$ base U then IG_Σ is 0-dimensional. Some results obtained by Knet-Shing using different methods such as asymtotic point counts.

Remark. $IG_\Sigma \to C_\Sigma$ is a profinite H_Σ-torsor so

$$\pi_0(C_\Sigma) = \pi_0(IG_\Sigma)/H_\Sigma = \pi_0(\mathfrak{Sh}_{G_\sigma}) \times \frac{G^{\mathrm{ab}}(\mathbb{Z}_p)}{H_\Sigma}$$