1 Early Developments

Definition 1.0.1. The Riemann zeta function is,

$$\zeta_(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Proposition 1.0.2. The summation form of ζ converges absolutely and is analytic for $\mathrm{Re}\,(s) > 1$.

Proof. Let $s = a + ib$ then,

$$\sum_{n=1}^{\infty} \frac{1}{|n^s|} = \sum_{n=1}^{\infty} \frac{1}{n^a}$$

converges when $a > 1$. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s) = \sum_{n=1}^{\infty} \frac{\ln n}{n^s}$$

and additionally,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^a}$$

converges for $a > 1$. $\square$

Proposition 1.0.3. For all $\mathrm{Re}\,(s) > 1$ there is a convergent Euler product representation,

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

Corollary 1.0.4. There are infinitely many primes.

Proof. The limit $\lim_{s \to 1^+} \zeta(s) = \infty$ and therefore,

$$\prod_p \frac{1}{1 - p^{-s}}$$

cannot be a finite product else it would converge in the limt $s \to 1^{+1}$ to,

$$\prod_p \frac{1}{1 - p^{-1}}$$

$\square$

1.1 Linear Characters

Definition 1.1.1. Let G be an abelian group. Then $\widehat{G} = \mathrm{Hom}\,(G, \mathbb{C}^\times)$ is the character group where $(\varphi_1 \cdot \varphi_2)(g) = \varphi_1(g)\varphi_2(g)$.

Proposition 1.1.2. Let χ_1, χ_2 be characters,

$$\sum_{g \in G} \overline{\chi_1(g)} \chi_2(g) = \begin{cases} |G| & \chi_1 = \chi_2 \\ 0 & \chi_1 \neq \chi_2 \end{cases}$$

Let $g, h \in G$ be group elements,

$$\sum_{\chi \in \widehat{G}} \overline{\chi(g)} \chi(h) = \begin{cases} |\widehat{G}| & g = h \\ 0 & g \neq h \end{cases}$$

Proof. We know,

$$\overline{\chi_1(g)} \left(\sum_{h \in G} \overline{\chi_1(h)} \chi_2(h) \right) \chi_2(g) = \sum_{h \in G} \overline{\chi_1(gh)} \chi_2(gh) = \sum_{h' \in G} \overline{\chi_1(h')} \chi_2(h')$$

Therefore, either $\chi_1(g) = \chi_2(g)$ for all g or the sum is zero. The second is similar. □

1.2 Dirichlet L-Functions

Definition 1.2.1. Let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a character. This extends to a multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}^\times$ by setting $\chi(a) = 0$ for $\gcd(a, m) > 1$ and $\chi_0(a) = 1$. Then the Dirichlet L-function,

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Remark. We have $L(s, \chi_0) = \zeta(s)$.

Proposition 1.2.2. For $\chi \neq \chi_0$ the L-function $L(s, \chi)$ is well-defined in the limit $s \to 1^+$ and $L(1, \chi) \neq 0$.

Proof. □

Proposition 1.2.3. The L function has an Euler product for $\mathrm{Re}\,(s) > 1$,

$$L(s, \chi) = \prod_p \frac{1}{1 - \chi(p) p^{-s}}$$

Proposition 1.2.4. The density of primes in the arithmetic progression $an + b$ is $\frac{1}{\varphi(a)}$.

Proof. Consider □