1 Regularity

Definition A ring map $\varphi : A \to B$ is *flat* if it makes B a flat A-module.

Definition A morphism of schemes $f : X \to Y$ is *flat* if for each $x \in X$ the stalk map $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat ring map.

Definition A scheme X is regular at $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular i.e. $\dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$.

Definition A scheme X is regular if it is regular at each $x \in X$.

Lemma 1.1. The localization of a local regular ring is regular.

Corollary 1.2. A noetherian scheme X is regular iff it is regular at each closed point.

Proof. On a noetherian scheme every point x specializes to a closed point y and thus $\mathcal{O}_{X,y}$ localizes to $\mathcal{O}_{X,x}$ so $\mathcal{O}_{X,x}$ is regular. $\square$

Definition We say that a k-scheme X is *geometrically regular* if $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}\left(\overline{k}\right)$ is regular where $\overline{k}$ is the algebraic closure of k. We say that X is geometrically regular at $x \in X$ if $\mathcal{O}_{X,x} \otimes_k \overline{k}$ is regular.

1.1 Normal Rings and Schemes

Definition A ring A is normal if each of its localizations $A_\mathfrak{p}$ is a local integrally closed domain.

Lemma 1.3. Let A be a domain then inside $K = \mathrm{Frac}(A)$,

$$A = \bigcap_{\mathfrak{m} \in \mathrm{mSpec}(A)} A_\mathfrak{m}$$

Proof. Suppose that $z \in K \setminus A$ then the ideal define $I = \{a \in A \mid az \in A\}$. Since $1 \notin I$ it is proper and thus there exists a maximal ideal $I \subset \mathfrak{m}$. If $z \in A_\mathfrak{m}$ then there must exist $s \notin \mathfrak{m}$ such that $sz \in A$ but then $s \in I \subset \mathfrak{m}$ a contradiction so $z \notin A_\mathfrak{m}$. $\square$

Lemma 1.4. A domain A is normal iff it is integrally closed.

Proof. If A is an integrally closed domain then $A_\mathfrak{p}$ is an integrally closed domain so A is normal. Conversely, suppose that $A_\mathfrak{p}$ is an integrally closed domain for each prime $\mathfrak{p} \subset A$. Since A is a domain, inside $K = \mathrm{Frac}(A)$,

$$A = \bigcap_{\mathfrak{m} \in \mathrm{mSpec}(A)} A_\mathfrak{m}$$

Thus if $a \in K$ is integral over A then it is integral over $A_\mathfrak{m}$ and thus $a \in A_\mathfrak{m}$ for each $\mathfrak{m}$ since we assume that each $A_\mathfrak{m}$ is integrally closed. Thus $a \in A$ so A is integrally closed. $\square$

Proposition 1.5. If A is a UFD then A is normal.

Proof. Since A is a domain it suffices to show that A is integrally closed. Consider a monic $f \in A[X]$,

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$$

and suppose that $\frac{\alpha}{\beta} \in K$ is a root of f. Then,

$$\alpha^n + a_{n-1}\alpha^{n-1}\beta + \cdots + a_0\beta^n = 0$$

Therefore, $\beta \mid \alpha^n$. Since A is a UFD then $\beta \mid \alpha$ so $\frac{\alpha}{\beta} \in A$. □

Definition A scheme X is normal if for each $x \in X$ the local ring $x \in \mathcal{O}_{X,x}$ is a local integrally closed domain.

Proposition 1.6. A scheme X is normal iff $\mathcal{O}_X(U)$ is normal for each $U \subset X$.

Proof. The ring $\mathcal{O}_X(U)$ is normal iff its localization at each prime $\mathcal{O}_{X,x}$ for $x \in U$ is a normal domain by definition. □

2 Smooth Morphims

Definition A ring map $\phi : A \to B$ is of *finite presentation* if B is a f.g A-algebra and $\ker(A[x_1, \dots, x_n] \to B)$ is finitely generated. This is equivalent to asking that,

$$B \cong A[x_1, \dots, x_n]/(f_1, \dots, f_k)$$

for finitely may polynomials in finitely many variables.

Definition A morphism of schemes $f : X \to Y$ is *locally of finite presentation* at $x \in X$ if there exist affine neighbrohoods $x \in U = \mathrm{Spec}\,(B)$ and $f(x) \in V = \mathrm{Spec}\,(A)$ with $f : U \to V$ such that $A \to B$ is of finite presentation.

Remark. If $\phi : A \to B$ is of finite type and A is Noetherian (so B is also Noetherian) then ϕ is automatically of finite presentation. This gives the following which will be the generic case we work under.

Proposition 2.1. Let $f : X \to Y$ be locally of finite-type and X be locally Noetherian. Then f is locally of finite presentation.

Definition $f : X \to Y$ is smooth at x if,

(a). f is flat at x

(b). f is locally of finite presentation at x

(c). f has geometrically regular fibers at x i.e. $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \overline{\kappa(f(x))}$ is regular.

Definition

The relative dimension of $f : X \to Y$ at x is $\dim_x(f) = \dim X_{f(x)}$.

Proposition 2.2. A morphism $f : X \to Y$ is smooth at x iff,

(a). f is flat at x

(b). f is locally of finite presentation at x

(c). $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module in a neighbrohood of U of rank $\dim_x f$.

Remark. Therefore, quantifying over all $x \in X$, we get.

Proposition 2.3. A morphism $f : X \to Y$ is smooth of relative dimension n iff,

(a). f is flat

(b). f is locally of finite presentation

(c). $\Omega_{X/Y}$ is locally free of rank n

(d). f has constant relative dimension $\dim_x f = n$.

Definition We say a scheme X over S is smooth if the structure morphism $X \to S$ is smooth.

Definition Let X be a scheme of finite type over k. Then X is smooth iff $\Omega_{X/k}$ is locally free of rank $n = \dim X$.

Proof. Any finite type map $X \to \mathrm{Spec}(k)$ is automatically flat and locally of finite presentation. Furthermore $X_{f(x)} = X$. Therefore, smoothness is equivalent having $\Omega_{X/Y}$ be locally free of rank $n = \dim X_{f(x)} = \dim X$. $\square$

3 Etale Morphims

Definition A morphism $f : X \to Y$ is *étale* if it is smooth of relative dimension zero.

Lemma 3.1. Let A be a K-algebra. Then $\mathrm{Spec}(A) \to \mathrm{Spec}(K)$ is étale iff A is a finite product of finite seperable extensions L_i/K,

$$A = \prod_{i=1}^{n} L_i$$

Proposition 3.2. A morphism $X \to \mathrm{Spec}(K)$ is étale iff X is $\mathrm{Spec}(L_1 \otimes \cdots \otimes L_n)$ where L_i/K is a finite seperable extension.

Corollary 3.3. Irreducible étale covers of $\mathrm{Spec}(K)$ correspond exactly to finite seperable extensions L/K.

Corollary 3.4. Let $f : X \to Y$ be étale then the fibre $X_y \to \mathrm{Spec}(\kappa(y))$ is étale and thus $X_y = \mathrm{Spec}(L_1 \otimes \cdots \otimes L_n)$ where $L_i/\kappa(y)$ is a finite seperable extension.

4 Sites

5 The Étale Site

6 ℓ-adic Cohomology

7 The Crystalline and Infinitessimal Sites

8 Crystalline Cohomology