1 Two Capacitors

Suppose we hook up two capacitors with capacitances C_1, C_2 one charged with charge Q the other uncharged. The total internal resistance is R.

First, we can actually calculate the equilibrium is when the charge is distributed to equalize the voltage, i.e. $q_1 - q_2 = Q$ because Let $q_1(0) = Q$ and $q_2(0) = 0$ and $I = \dot{q}_1 = \dot{q}_2$ so $q = q_1 - q_2$ is constant so $q_1 = Q + q_2$. Now,

$$R\dot{q}_1 + \frac{q_1}{C_1} + \frac{q_2}{C_2} = 0$$

Therefore,

$$\dot{q}_1 = \left(\frac{Q}{RC_2} - q_1 \left(\frac{1}{RC_1} + \frac{1}{RC_2} \right) \right) = \frac{1}{RC_S} \left(Q\frac{C_S}{C_2} - q_1 \right) = \frac{1}{RC_S} \left(\frac{C_1}{C_1 + C_2} Q - q_1 \right)$$

Therefore,

$$\dot{q}_1 = \frac{1}{RC_2} \left(Q - q_1 \frac{C_2}{C_S} \right)$$

where,

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2}$$

viewing the capacitors in series. In particular, we have equilibrium when,

$$q_1 = Q\frac{C_S}{C_2} = \frac{C_1}{C_1 + C_2} Q$$

and thus at equilibrium,

$$q_1 = \frac{C_1}{C_1 + C_2} Q \quad \text{and} \quad q_2 = -\frac{C_2}{C_1 + C_2} Q$$

At the begining there is energy,

$$E_0 = \frac{Q^2}{2C_1}$$

consider the energy afterwards,

$$E_1 = \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} = \frac{Q^2}{2} \left(\frac{C_1}{(C_1 + C_2)^2} + \frac{C_2}{(C_1 + C_2)^2} \right) = \frac{Q^2}{2(C_1 + C_2)} = \frac{Q^2}{2C_P}$$

where $C_P = C_1 + C_2$ viewing the capacitors in parallel. Therefore the energy loss is,

$$\Delta E = E_0 - E_1 = \frac{Q^2}{2C_1} \left(1 - \frac{C_1}{C_1 + C_2} \right) = \frac{Q^2}{2C_1} \cdot \frac{C_2}{C_1 + C_2} = E_0 \left(\frac{C_2}{C_1 + C_2} \right)$$

Now likewise, we can solve the differential equation,

$$q_1(t) = Q \left(\frac{C_1}{C_1 + C_2} + \frac{C_2}{C_1 + C_2} e^{-\frac{t}{RC_S}} \right)$$

Furthermore, we can consider,

$$\begin{aligned}\Delta E &= \int_0^\infty P\,\mathrm{d}t = \int_0^\infty I^2 R\,\mathrm{d}t = \int_0^\infty \dot{q}_1^2 R\,\mathrm{d}t = \frac{1}{RC_S^2}\int_0^\infty \left(\frac{C_1}{C_1+C_2}Q - q_1\right)^2 \mathrm{d}t \\ &= \frac{Q^2}{RC_S^2}\left(\frac{C_2}{C_1+C_2}\right)^2 \int_0^\infty e^{-\frac{2t}{RC_S}}\,\mathrm{d}t = \frac{Q^2}{RC_S^2}\left(\frac{C_2}{C_1+C_2}\right)^2 \frac{RC_S}{2} = \frac{Q^2}{2C_S}\left(\frac{C_2}{C_1+C_2}\right)^2 \\ &= \frac{Q^2}{2C_1}\frac{C_2}{C_1+C_2}\end{aligned}$$

agreeing with our previous result where I used,

$$\frac{1}{C_S}\left(\frac{C_2}{C_1+C_2}\right)^2 = \frac{C_1+C_2}{C_1C_2}\cdot\frac{C_2^2}{(C_1+C_2)^2} = \frac{1}{C_1}\cdot\frac{C_2}{C_1+C_2}$$

2 Moving a Chain

I claim that a chain (or string) in some smooth curve through space (without gravity) will follow its curve in the absense of gravity.

First we illustrate this with a chain in a circular arc. Suppose it has tension T. Then on a chain element, $\mathrm{d}F = T\mathrm{d}\theta$ inwards. Furthermore, the acceleration needed to maintain circular motion is,

$$a = \frac{v^2}{R}$$

where R is the radius of curvature. Finally, the chain element has mass $\mathrm{d}m = \lambda R \mathrm{d}\theta$ where λ is the linear mass density and thus,

$$\mathrm{d}F = T\mathrm{d}\theta = a\,\mathrm{d}m = \mu v^2 \mathrm{d}\theta \iff T = \lambda v^2$$

independent of the radius of curvature. Since infinitessimally every curve is a circular arc with some radius of curvature (that is we can fit it to second order and thus the second derivatives agree) we see that a constant tension gives exactly the required forces to mantain the chain's shape.

What happens if we add gravity?

What happens if we add constant friction? Let f_s be the linear friction force density opposing the tension. The tension must increase throughout the length as,

$$\mathrm{d}T = f_s \mathrm{d}\ell$$

which means we cannot have the constant tension necessary for maintaining curves. Thus,

$$\mathrm{d}F_\perp^{\text{eff}} = a\,\mathrm{d}m - T\mathrm{d}\theta = (\lambda v^2 - T)\mathrm{d}\theta$$

Therefore,

$$a_\perp^{\text{eff}} = \frac{\lambda v^2 - T}{\lambda R}$$

3 Schwarzian Derivative

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and $z \in \Omega$ then consider the limit,

$$\lim_{w \to z} \left[\frac{f'(z)f'(w)}{(f(z)-f(w))^2} - \frac{1}{(z-w)^2} \right]$$

For sufficiently small $|z-w|$ we can write,

$$f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}(w-z)^n$$

and then,

$$\frac{1}{(f(z)-f(w))^2} = \frac{1}{f'(z)^2(z-w)^2} \cdot \left(\frac{1}{1 + \sum_{n=2}^{\infty} \frac{f^{(n)}(z)}{f'(z)n!}(w-z)^{n-1}} \right)^2$$

Furthermore,

$$f'(w) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(z)}{n!}(w-z)^n$$

First we conmute the second-order leading terms of,

$$\begin{aligned}
\left(\frac{1}{1 + \sum_{n=2}^{\infty} \frac{f^{(n)}(z)}{f'(z)n!}(w-z)^{n-1}} \right)^2 &= \left[1 - \left(\frac{1}{2}\frac{f''(z)}{f'(z)}(w-z) + \frac{1}{6}\frac{f'''(z)}{f'(z)}(w-z)^2 \right) + \left(\frac{1}{2}\frac{f''(z)}{f'(z)}(w-z) \right)^2 \right]^2 \\
&= \left[1 - \frac{1}{2}\frac{f''(z)}{f'(z)}(w-z) + \left(\frac{1}{2}\frac{f''(z)^2}{f'(z)^2} - \frac{1}{6}\frac{f'''(z)}{f'(z)} \right)(w-z)^2 \right]^2 \\
&= 1 - \frac{f''(z)}{f'(z)}(w-z) + \left(\frac{5}{4}\frac{f''(z)^2}{f'(z)^2} - \frac{1}{3}\frac{f'''(z)}{f'(z)} \right)(w-z)^2
\end{aligned}$$

and therefore the leading terms are

$$\begin{aligned}
\frac{f'(z)f'(w)}{(f(z)-f(w))^2} &= \frac{f'(z) + f''(z)(w-z) + \frac{1}{2}f'''(z)(w-z)^2}{f'(z)(z-w)^2} \\
&\quad \cdot \left[1 - \frac{f''(z)}{f'(z)}(w-z) + \left(\frac{5}{4}\frac{f''(z)^2}{f'(z)^2} - \frac{1}{3}\frac{f'''(z)}{f'(z)} \right)(w-z)^2 \right] \\
&= \frac{1}{(z-w)^2} \left[1 + \left(\frac{1}{6}\frac{f'''(z)}{f'(z)} - \frac{1}{4}\left(\frac{f''(z)}{f'(z)} \right)^2 \right)(w-z)^2 \right] \\
&= \frac{1}{(z-w)^2} + \left[\frac{1}{6}\frac{f'''(z)}{f'(z)} - \frac{1}{4}\left(\frac{f''(z)}{f'(z)} \right)^2 \right]
\end{aligned}$$

Therefore,

$$\lim_{w \to z} \left[\frac{f'(z)f'(w)}{(f(z)-f(w))^2} - \frac{1}{(z-w)^2} \right] = \frac{1}{6}\frac{f'''(z)}{f'(z)} - \frac{1}{4}\left(\frac{f''(z)}{f'(z)} \right)^2$$

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