1 Sep. 21

Interested in studying perverse sheaves in the Euclidean and étale topologies for studying Poincare duality for singular varieties. We studied:

(a) tiangulated categories

(b) t-structures

(c) gluing t-structues $\mathcal{D}_Z \to \mathcal{D} \to \mathcal{D}_U$ with nice functors between these triangulated categories

(d) perverse sheaves over $\mathcal{C}$

We are now going to develop the theory of perverse sheaves in the étale setting for étale sheaves and étale cohomology. The main goal for the first two weeks is to define the derived categories for $\overline{\mathbb{Q}}_\ell$-sheaves and six functor formalism. Then we study the theory of weights due to Deligne (Weil II) for varieties over $\mathbb{F}_q$.

1.1 Etale Sheaves

Assume all schemes are Noetherian. Then $X_{\text{ét}}$ is the small étale site with objects: schemes étale over X and covers and jointly surjective families of étale maps.

1.1.1 Constructibility

Let Λ be a finite "coefficient" ring. We say that an étale sheaf of sets $\mathscr{F}$ on $X_{\text{ét}}$ is locally constant constructible (lcc) if it is represented by an object of $\text{FEt}(X)$ i.e. there is a finite étale map $X' \to X$ such that,

$$\mathscr{F}(U) = \text{Hom}_X(U, X')$$

We say that a sheaf $\mathscr{F}$ is Λ-modules is *constructible* if each $\mathscr{F}_x$ is a finite Λ-module and there exists a stratification,

$$X = \bigcup X_i$$

such that $X_i \subset X$ is locally closed s.t. $\mathscr{F}|_{X_I}$ is lcc. Furthermore, $\mathscr{F} \in D^b(X, \Lambda)$ is constructible if each $H^i(\mathscr{F})$ is constructible. This gives the triangulated subcategory $D^i_c(X, \Lambda)$.

Now given a map of Noetherian schemes $f : X \to Y$ we get a six functor formalism. We have $f_* : \text{Ab}(X) \to \text{Ab}(Y)$ and $Rf_* : D^+(x, \Lambda) \to D^+(Y, \Lambda)$ and $f^* : \text{Ab}(Y) \to \text{Ab}(X)$ which is exact. Then there is an adjointness,

$$\text{Hom}_X(f^*\mathscr{F}, \mathscr{G}) = \text{Hom}_Y(\mathscr{F}, f_*\mathscr{G})$$

When $\iota : Z \hookrightarrow X$ is a closed immersion and $j : U \hookrightarrow X$ is the complementary open immersion we can define the "exceptional" functors. We define,

$$j_! : \text{Ab}(Y) \to \text{Ab}(X)$$

which is exact because it preserves stalks. Then ι_* is also exact because it preserves stalks (and extends by zero on U). Furthermore,

$$\iota^! : \text{Ab}(X) \to \text{Ab}(Z)$$

is defined by,

$$\iota^! \mathscr{F} = \iota^* \ker(\mathscr{F} \to j_* j^* \mathscr{F})$$

which is the subsheaf of sections supported on Z.

However, $Rf_!$ is not the naive derived functor of $f_!$ unfortunately. Assume that $f : X \to Y$ is separated of finite type. Then by Nagata, there is a compactification,

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

Then we define $Rf_! = D^+(X, \Lambda) \to D^+(Y, \Lambda)$ by $Rf_! = (R\bar{f}_*) \circ j_!$.

Remark. It does not work to define $Rf_! = R(f_!)$ where $f_! = \bar{f}_* \circ j_!$. For example for a curve X over a field k and $f : X \to \mathrm{Spec}\,(k)$. Then,

$$\Gamma_c(X, \Lambda) = \bigoplus_{x \in X} \Gamma_x(X, \mathscr{F})$$

where Γ_x is local cohomology. But this derived functor can be too large.

Then we can also define $Rf^! : D^+(Y, \Lambda) \to D^+(X, \Lambda)$ to be the right adjoint of $Rf_!$. For example if $f = \iota$ is a closed immersion then $Rf^! = R(\iota^!)$ is actually a derived functor. If f is a smooth map of relative dimension d (and $n\Lambda = 0$ for $n \in \mathcal{O}_Y^\times$) then $Rf^! = f^*(\alpha)[2d]$

Theorem 1.1.1. Let $f : X \to Y$ be finite tpye over a field k. Then the six functors presserve $D_c^+(X, \Lambda)$ and we have biduality when $n\Lambda = 0$ for $n \in k^\times$ and Λ is an injective Λ-module. Let $f : X \to \mathrm{Spec}\,(k)$ and set $K_X = Rf^! \Lambda$ and $DL := \mathrm{RHom}\,(L, K_X) \in D^b(X, \Lambda)$. Then $L \xrightarrow{\sim} DDL$ in $D_c^b(X, \Lambda)$.

How to define $D_c^b(X, \overline{\mathbb{Q}}_\ell)$? Let k be a finite field or a separably closed field $\ell \in k^\times$. Let X be seperated of finite type over k. Fact 1 for $E/\mathbb{Q}_\ell$ finite extension then $\mathcal{O}_E$ have triangulated cat $D_c^b(X, \mathcal{O}_E)$ and standard nontrivial t-structure. For $r \geq 1$ let $\mathcal{O}_r = \mathcal{O}_E/\lambda^r$ where λ is a uniformizer. Then $\mathcal{O}_r$ is constructible.

Let $D_{ctf}^b(X, \mathcal{O}_r) \subset D_c^b(X, \mathcal{O}_r)$ be defined by objects isomorphic to bounded complexes of flat $\mathcal{O}_r$-module that give $D_c^b(X, \mathcal{O}_E)$ is defined as $K = (K_r)_r$ where $K_r \in D_{ctf}^b(X, \mathcal{O}_r)$ and $K_{r+1} \otimes^{\mathbb{L}} \mathcal{O}_r \cong K_r$. Moreover,

$$\mathrm{Hom}_{D_c^b(X, \mathcal{O}_E)}(K, L) = \varprojlim_{V} \mathrm{Hom}_{D_c^b(X, \mathcal{O}_r)}(K_r, L_r)$$

Furthermore, $D_c^b(X, \mathcal{O}_E)$ is a triangulated category. Furthermore, there is a standard t-structure on $D_c^b(X, \mathcal{O}_E)$. The problem is that $\tau_{\leq n}$ and $\tau_{\geq n}$ dont preserve $D_{cft}^b(X, \mathcal{O}_r)$ but Deligne defined them in such a way to preserve everything you want.

Remark. There are some serious problems with trying to first define an abelian category of $\mathbb{Z}_\ell$-sheaves and then take derived categories. For example, if $X = \mathrm{Spec}\,(k)$ then sheaves of $\mathbb{Z}/\ell^n$-modules is the same as $\mathbb{Z}/\ell^n[\mathrm{Gal}\,(\bar{k}/k)]$-modules but the limit of these categories gives the category of continuous $\mathrm{Gal}\,(\bar{k}/k)$-representations over $\mathbb{Z}_\ell$ which is not an abelian category. Look at Bhatt-Scholtze and condensed mathematics.

1.2 Theory of Weights

Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$ and let $X = X_0 \otimes \overline{\mathbb{F}_q}$. Then $F : X \to X$ is the geometric Frobenius i.e. $F = \mathrm{id} \otimes \mathrm{Spec}\,(F)$ where $F : \mathbb{F}_q \to \mathbb{F}_q$ is the inverse of $x \mapsto x^p$. Let $|X_0|$ denote the closed points and $x \in |X_0|$ we have $\#\kappa(x) = N(x) = q^{d(x)}$. For X_0 geometrically connected over $\mathbb{F}_q$ we have an exact sequence,

$$1 \longrightarrow \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X_0, \bar{x}) \longrightarrow \mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \longrightarrow 1$$

and $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) = \hat{\mathbb{Z}}$ generated by geometric Frobenius. Therefore, we have an exact squence,

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(X, \bar{x}) \arrow[r] & \pi_1^{\text{ét}}(X_0, \bar{x}) \arrow[r] & \mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(X, \bar{x}) \arrow[r] \arrow[u, equal] & W(X_0, \bar{x}) \arrow[r] \arrow[u, hook] & \mathbb{Z} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

Definition 1.2.1. A Weil sheaf on X_0 is a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}$ on X with an isomorphism $\varphi : F^*\mathscr{F} \to \mathscr{F}$.

Remark. Let $g : X \to X_0$ be the canonical map. Then $g \circ F = g$ so there is a natural isomorphism $\eta : F^* \circ g^* \to g^*$ and thus for any $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}_0$ on X_0 we have an natural isomorphism $\eta : F^* g^* \mathscr{F}_0 \xrightarrow{\sim} g^*\mathscr{F}_0$ and thus $\mathscr{F} := g^*\mathscr{F}_0$ and $\eta : F^*\mathscr{F} \to \mathscr{F}$ gives a Weil sheaf.

Exercise 1.2.2. For $X_0 = \mathrm{Spec}\,(\mathbb{F}_q)$ we have,

$$\{\text{Weil Sheaves}\} \iff \{\text{continuous reps of } W(X_0) \cong \mathbb{Z} \text{ on finite dimensional } \overline{\mathbb{Q}}_\ell \text{ vector spaces}\}$$

furthermore, the subcategory of $\overline{\mathbb{Q}}_\ell$-sheaves correspond to those representations that extend to $\hat{Z}$.

Exercise 1.2.3. In the rank 1 case let X_0 be a normal geometrically connected / $\mathbb{F}_q$ then,

$$\mathrm{Im}\left(\pi_1(X, \bar{x}) \to W(X_0, \bar{x})^{\text{ét}}\right)$$

is a finite group times a pro-p group.

1.3 Weights

Fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathcal{C}$.

Definition 1.3.1. Let $\mathscr{F}_0$ be a Weil sheaf on X_0 and $\beta \in \mathbb{R}$ then,

(a) $\mathscr{F}_0$ is (pointwise) ι-pure of weight β if $\forall x \in |X_0|$ and α is an eigenvalue of $F_x \circlearrowright (\mathscr{F}_0)_{\bar{x}}$ then $|\iota\alpha| = q^{\frac{\beta d(x)}{2}}$.

(b) $\mathscr{F}_0$ is ι-mixed if there is a filtration,

$$0 = \mathscr{F}_0^{(0)} \subset \cdots \subset \mathscr{F}_0^{(r)} = \mathscr{F}_0$$

by Weil subsheaves s.t. $\mathscr{F}_0^{(i+1)}/\mathscr{F}_0^{(i)}$ is ι-pure of some weight

(c) $\mathscr{F}_0$ is pure of weight β / mixed if it is ι-pure of weight β / ι-mixed for any ι.

Example 1.3.2. The sheaf $\mathscr{F}_0 = \mathbb{Q}(1)$ is pure of weight -2 since $F\zeta_{\ell^n} = \zeta_{\ell^n}^{q^{-1}}$. If X_0 is normal geometrically connected then any rank 1 smooth Weil sheaf on X_0 is ι-pure.

Theorem 1.3.3 (Deligne)**.** For $f_0 : X_0 \to Y_0$ and $\mathscr{F}_0$ on X a ι-mixed of largest weight β then,

$$R^k(f_0)_* \mathscr{F}_0$$

is ι-mixed with weights $\le \beta + k$.

Example 1.3.4. For $f_0 : X_0 \to \mathrm{Spec}\,(\mathbb{F}_q)$ smooth and proper of dimension d and $\mathscr{F}_0$ a smooth sheaf on X_0 that is ι-pure of weight β then by Poincare duality,

$$F \circlearrowright H^k(X_{\text{ét}}, \mathscr{F}) \cong H^{2d-k}(X_{\text{ét}}, \mathscr{F}^\vee(d))^\vee$$

By Deligne's theorem, the left hand side has weights $\le \beta + k$. Furthermore, the sheaf $\mathscr{F}^\vee(d)$ is ι-pure of weight $-\beta - 2d$ and thus by Deligne's theorem the cohomology group has weights $\le -\beta - k$ and thus dualizing the weights are $\ge \beta + k$. Therefore, the weights are all $\beta + k$ because both sides are isomorphic..

Theorem 1.3.5 (semi-continuity)**.** Let $j_0 : U_0 \to X_0$ be a dense open immersion with $S_0 = X_0 \setminus U_0$ with the reduced scheme structure. Let $\mathscr{F}_0$ be a smooth Weil sheaf on X_0. Assume that there is $\beta \in \mathbb{R}$ such that $\forall x \in |U_0|$ and any eigenvalue α of $f_x \circlearrowright \mathscr{F}_{\bar{x}}$ then $|\iota\alpha| \le q^{\frac{\beta d(x)}{2}}$ then $\forall s \in |S_0|$ and any eigenvalue α of $F_s \circlearrowright \mathscr{F}_{\bar{s}}$ then $|\iota\alpha| \le q^{\frac{\beta d(s)}{2}}$.

Proof. We can always take a chain of curves connecting x and s so we reduce to the case $\dim X_0 = 1$ with X_0 geometrically irreducible and affine. Recall,

$$L(X_0, \mathscr{F}_0, t) = \prod_{x \in |X_0|} \det\,(1 - F_x t^{d(x)} | \mathscr{F}_{\bar{x}})^{-1} = \frac{\det\,(1 - Ft | H^1_c(X, \mathscr{F}))}{\det\,(1 - Ft | H^0_c) \det 1 - Ft | H^2_c)}$$

Then, $H^0_c(X, \mathscr{F}) = 0$ because X is affine and $\mathscr{F}_0$ is smooth so there are no global sections with compact support. Furthermore,

$$H^2_c(X, \mathscr{F}) \cong H^0(X, \mathscr{F}^\vee(1))^\vee$$

by Poincare duality. Fix $x \in |U_0|$ and because $\mathscr{F}_0$ is lisse then it corresponds to some representation V of $W(X_0, \bar{x})$ on $\mathscr{F}_{\bar{x}}$. Then, $H^0(X, \mathscr{F}) = V^{\pi_1(X)}$ and likewise,

$$H^2_c(X, \mathscr{F}) = H^0(X, \mathscr{F}^\vee)^\vee(-1) = V_{\pi_1(X)}(-1)$$

because the dual of invariants are coinvariants. Take α an eigenvalue of F acting on $V_{\pi_1(X)}$ then $\alpha^{d(x)}$ is an eigenvalue of $F^{d(x)} = F_x$ acting on $V_{\pi_1(X)}$ which is a quotient of $\mathscr{F}_{\bar{x}}$ and thus $q|\iota\alpha| \le q^{\frac{\beta d(x)}{2}+1}$ and thus $q\alpha$ is an eigenvalue of $F \circlearrowright H^2_c$ iff $(q\alpha)^{-1}$ is a zero of $\det\,(1 - Ft | H^2_c)$. Therefore, the possible poles of $\iota L(X_0, \mathscr{F}_0, t)$ is $\iota(q\alpha)^{-1}$ for α as above. Therefore $\iota L(X_0, \mathscr{F}_0, f)$ has no poles for $|t| < q^{-\frac{\beta}{2}-1}$. However,

$$L(X_0, \mathscr{F}_0, t) = L(U_0, \mathscr{F}_0|_{U_0}, t) \cdot \prod_{s \in S_0} \det\,(1 - F_s t | \mathscr{F}_{\bar{s}})^{-1}$$

I claim that $\iota L(U_0, \mathscr{F}_0|_{U_0}, t)$ converges and has no zeros for $|t| < q^{-\frac{\beta}{2}-1}$. Then,

$$\iota\left(\frac{L'(U_0)}{L(U_0)}\right) = \iota\left(\sum_{n=1}^{\infty}\left(\sum_{\substack{x\in|U_0|\\ d(x)|n}} d(x)\iota\mathrm{Tr}\left(F_x^{\frac{n}{d(x)}}\right)\right)t^{n-1}\right)$$

Then,

$$|\iota\mathrm{Tr}\left(F_x^{\frac{n}{d(x)}}\right)| \leq r\left(q^{\frac{\beta d(x)}{2}}\right)^{\frac{n}{d(x)}} = rq^{\frac{\beta n}{2}}$$

Therefore,

$$\sum_{\substack{x\in|U_0|\\ d(x)|n}} d(x) = \#U_0(\mathbb{F}_{q^n}) \leq Cq^n$$

and therefore the logarithmic derivative is dominated by,

$$\sum_{n=1}^{\infty} rCq^{n\left(\frac{\beta}{2}+1\right)}t^{n-1}$$

which converges absolutely for $|t| < q^{-\frac{\beta}{2}-1}$ and thus $|\iota\alpha| \leq q^{\beta+1}$.

Now we apply the same argument to $\mathscr{G}_0 = \mathscr{F}_0^{\otimes k}$ so α^k is an eigenvalue of $F_s \circlearrowright \mathscr{G}_{\bar{s}}$ and thus $|\iota\alpha^k| \leq q^{\frac{k\beta}{2}+1}$ which implies that,

$$|\iota\alpha| \leq q^{\frac{\beta}{2}+\frac{1}{k}}$$

and thus taking $k \to \infty$ this goes to $q^{\frac{\beta}{2}}$ so $|\iota\alpha| \leq q^{\frac{\beta}{2}}$. $\square$