1 Sep. 20

1.1 History

(a) 19th century

    (a) $Z(f_1, \ldots, f_n) \subset \mathbb{C}^n$ using analytic tools

    (b) Riemann's idea of moduli of algebraic curves (1857)

(b) 20th century

    (a) $Z(f_1, \ldots, f_n) \subset \mathbb{C}^n$ using algebraic tools (commutative algebra)

    (b) replace $\mathbb{C}$ with algebraically closed field k

    (c) number theory: want $k = \mathbb{F}_p$ or $\mathbb{Q}$ not algebraically closed. Examples: Fermat's Last Theorem:
$$u^n + v^n = 1$$

want geometry for,
$$\mathbb{Q}[u, v]/(u^2 + v^2 - 1)$$

but this only has finitely many points so how is there a "geometry".

1.1.1 Question: for any field k is there a "geometry" for $k[X_1, \ldots, X_n]/I$?

First attempt (Weil and Zariski 1930s - 1940s) use Galois theory with algebraic sets in $\overline{k}^n$ for ideals $I \subset k[X_1, \ldots, X_n]$. This only works for perfect fields (not $\mathscr{F}_p(t)$ which we want to consider generic families of equations over $\mathbb{Z}$). Weil's Foundations of Algebraic Geometry.

1.1.2 The Weil Conjectures (1948)

For $f_1, \ldots, f_r \in \mathbb{F}_p[x_1, \ldots, x_n]$ then define,

$$N_m = \{x \in \mathbb{F}_{p^m}^r \mid f_1(x) = \cdots = f_r(x) = 0\}$$

Then we define a Zeta function,

$$\zeta(s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} p^{-sm}\right)$$

which should control the behavior of N_m as $m \to \infty$. Furthermore,

$$Z_{\mathbb{C}}(F_1, \ldots, F_r) \subset \mathbb{C}^n \quad \text{and} \quad Z_p = \mathbb{Z}(F_1, \ldots, F_r) \subset \overline{\mathbb{F}_p}^n$$

are closely related where algebraic topology invariants of $Z_{\mathbb{C}}$ gives formulas for counts of Z_p.

1.1.3 1950's: Chaos (Kähler, Shimura, Nagata, etc.)

Proposal for algebraic geometry over Dedekind domains but chaotic and confusing. Then Schemes resolve all of these problems to give "geometry over any commutative ring".

1.2 Affine Algebraic Sets

Let k be an algebraically closed field. Let $\mathbb{A}^n = k^n$ and define a subset $Z \subset \mathbb{A}^n$ to be *algebraic* if $Z = Z(\Sigma)$ where $\Sigma \subset k[X_1, \dots, X_n]$ is a set of polynomials. Then $Z(\Sigma) = Z(I)$ where I is the ideal generated by Σ.

Theorem 1.2.1. The algebraic sets form (the complement of) a topology on $\mathbb{A}^n$.

Remark. We call this the Zariski topology.

There is a base of open sets given by,

$$U_f = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$$

1.2.1 Examples

The Zariski topology on $\mathbb{A}^1$ has the cofinite topology. However, $\mathbb{A}^2 \neq \mathbb{A}^1 \times \mathbb{A}^1$ as a topological space.

Remark. Some weird properties of the Zariski topology,

(a) In $\mathbb{C}^n$ any nonzero open ball is Zariski dense.

(b) $Z(f) = Z(f^n)$ and $Z(I) = Z(\sqrt{I})$.

Definition 1.2.2. For any $Y \subset \mathbb{A}^n$ define,

$$I(Y) = \{f \in k[X_1, \dots, X_n] \mid \forall y \in Y : f(y) = 0\}$$

which is a radical ideal.

Proposition 1.2.3 (Nullstellensatz)**.** For a field k and $\mathfrak{m} \subset K[X_1, \dots, X_n]$ a maximal ideal. Then $K[X_1, \dots, X_n]/\mathfrak{m}$ is a finite dimensional K-vector space.

Proof. 210B [Mat, Thm 5.3] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Corollary 1.2.4. If K is algebraically closed then $K \to K[X_1, \dots, X_n]/\mathfrak{m}$ is an isomorphism so we have $a_i \mapsto X_i$ and thus $X_i - a_i = 0$ in the quotient so,

$$\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$$

is the kernel of the map $K[X_1, \dots, X_n] \to K[X_1, \dots, X_n]/\mathfrak{m}$. Therefore, points in $Z(J)$ correspond to $\mathfrak{m} \supset J$. Therefore,

$$I(Z(J)) = \bigcap_{\mathfrak{m} \supset J} \mathfrak{m}$$

Theorem 1.2.5. The following hold,

(a) $I_1 \subset I_2 \implies Z(I_1) \supset Z(I_2)$

(b) $Y_1 \subset Y_2 \implies I(Y_1) \supset I(Y_2)$

(c) $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(d) $Z(I_1 \cap I_2) = Z(Y_1) \cup Z(Y_2)$

(e) $Z(I(Y)) = \overline{Y}$

(f) $I(Z(J)) = \sqrt{J}$ (Hilbert's Nullstellensatz).

Proof. The first three follow directly from definitions. Suppose that $x \notin Z(I_1)$ and $x \notin Z(I_2)$ then there is some $f_i \in I_i$ such that $f_i(x) \neq 0$ so $f_1(x)f_2(x) \neq 0$ but $f_1 f_2 \in I_1 \cap I_2$ so $x \notin Z(I_1 \cap I_2)$.

Now $Y \subset Z(I(Y))$ so $\overline{Y} \subset Z(I(Y))$. Pick $x \notin \overline{Y}$ so there is some $x \in U_f$ such that $U_f \cap \overline{Y} = \varnothing$. Therefore, $U_f \cap Y = \varnothing$ so $f|_Y = 0$ since if $x \in Y$ then $x \notin U_f$. Thus $f \in I(Y)$ so $x \notin Z(I(Y))$ proving that $Z(I(Y)) \subset \overline{Y}$.

Since $J \subset I(Z(J))$ is radical we see that $\sqrt{J} \subset I(Z(J))$. The key is to apply the Nullstellensatz. $\qquad \square$

2   Sep. 22

Proposition 2.0.1. Let K be a field and A a finitely generated K-algebra, $J \subset A$ an ideal. Then,

$$\sqrt{J} = \bigcap_{\mathfrak{m} \supset J} \mathfrak{m} = \mathrm{Jac}\,(J)$$

Proof. Replace A by $A/\sqrt{J}$ such that $J = (0)$ and $\mathrm{nilrad}\,(A) = (0)$. Choose $f \neq 0$ then f is not nilpotent so A_f is nonzero and $A_f = A[x]/(xf-1)$ is a finitely generated K-algebra. Thus A_f has a maximal ideal $\mathfrak{m} \subset A_f$. Now under $\varphi : A \to A_f$ we see that $\varphi^{-1}(\mathfrak{m}) \subset A$ is a prime. However, $A/\varphi^{-1}(\mathfrak{m}) \hookrightarrow A_f/\mathfrak{m}$ but $A_f/\mathfrak{m}$ is a finite field extension of K so $A/\varphi^{-1}(\mathfrak{m})$ is a finite dimensional K-algebra and a domain so its a field and thus $\varphi^{-1}(\mathfrak{m})$ is maximal and $f \notin \varphi^{-1}(\mathfrak{m})$ and thus $f \notin \mathrm{Jac}\,(A)$. $\qquad \square$

Remark. Any domain D that is a finite dimensional K-algebra is a field because if $r \in D$ is nonzero then $D \xrightarrow{\times r} D$ is injective and thus surjective so $xr = 1$ for some $x \in D$ so D is a field.

Remark. Usually difficult to compute $\sqrt{J}$ given generators of J.

Definition 2.0.2. Say that $f \in k[x_1, \ldots, x_n]$ is *radical* if $f \in k$ and no repeated irreducible factors. The hypersurface $Z(f)$ for non-constant f is radical.

Definition 2.0.3. A topological space Y is *irreducible* if $Y \neq \varnothing$ and $Y \neq Y_1 \cup Y_2$ for closed $Y_1, Y_2 \subsetneq Y$. Otherwise, Y is *reducible*.

Remark. If Y is irreducible then every nonempty open $U \subset Y$ is dense. This is because $Y = (Y \setminus U) \cup \overline{U}$ but if U is nonempty then $Y \setminus U$ is a proper subset so $\overline{U} = Y$.

Definition 2.0.4. A topological space Y is *Noetherian* if it satisfies the DCC for closed sets meaning if,

$$Z_1 \supset Z_2 \supset Z_3 \supset \cdots$$

is a descending chain then it stabilizes meaning $Z_n = Z_{n+1}$ for all sufficiently large n.

Example 2.0.5. $\mathbb{A}^n$ is Noetherian because closed sets correspond to ideals and $k[x_1, \ldots, x_n]$ is Noetherian.

Proposition 2.0.6. Let $Z \subset \mathbb{A}^n$ be an algebraic set. Then Z is irreducible if and only if $I(Z)$ is prime.

Proof. Irreducible are nonempty and prime ideals I are proper subsets. Thus consider the case that $I(Z) \neq (1)$ equivalently that Z is nonempty. We see that,

$$Z = Z_1 \cup Z_2 \iff I(Z) = I(Z_1) \cap I(Z_2)$$

and $Z_i \subsetneq Z$ iff $I(Z_1) \supsetneq I(Z)$. Therefore, irreducibility of Z is equivalent to the condition that if $I(Z) = I_1 \cap I_2$ with I_1 and I_2 radical then $I_1 = I(Z)$ or $I_2 = I(Z)$ which is equivalent to in $A = k[x_1, \ldots, x_n]/I(Z)$ the property that if $(0) = J_1 \cap J_2$ then either $J_1 = (0)$ or $J_2 = (0)$. Therefore, we reduce to showing the following: if A is a nonzero reduced ring, then A is a domain iff $J_1 \cap J_2 = (0)$ for radical ideals J_1, J_2 then either $J_1 = (0)$ or $J_2 = (0)$.

If A is a domain then $J_1 J_2 \subset J_1 \cap J_2 = (0)$ so if $a_i \in J_i$ are nonzero then $a_1 a_2 \in J_1 J_2$ so $a_1 a_2 = 0$ contradicting the fact that A is a domain. Now suppose that A has this property. Choose $f, g \in A$ such that $fg = 0$ then let $Q = \sqrt{(f)} \cap \sqrt{(g)}$. If $a \in Q$ then $a^n = pf$ and $a^m = qg$ so $a^{n+m} = pqfg = 0$ and thus $a \in \operatorname{nilrad}(A)$ but A is reduced so $Q = (0)$ and thus either $f = 0$ or $g = 0$ by the assumption. $\square$

Corollary 2.0.7. If f is radical then $Z(f)$ is irreducible iff f is irreducible.

Proof. Both are equivalent to (f) being prime. $\square$

Theorem 2.0.8. Every Noetherian topological space is a finite union of irreducible closed sets,

$$Y = Y_1 \cup \cdots \cup Y_r$$

which is unique if we require the irredundancy,

$$Y_i \not\subset \bigcup_{j \neq i} Y_i$$

Furthermore, in the irredundant case, the Y_i are exactly the maximal irreducible subsets (i.e. irreducible components).

Corollary 2.0.9. Every algebraic set Z is a finite union of irreducible closed subsets.

Definition 2.0.10. An *affine variety* is a irreducible algebraic set.

3 Dimension and Regular Functions

Lemma 3.0.1. If $Y \subset X$ is irreducible in the subspace topology and $Y \subset Z_1 \cup Z_2$ for closed $Z_j \subset X$ then $Z \subset Z_1$ or $Y \subset Z_2$.

Proof. Then $Y = (Y \cap Z_1) \cap (Y \cap Z_2)$. $\square$

Remark. This is why for an irredundant decomposition,

$$X = Z_1 \cup \cdots \cup Z_n$$

into its irreducible components then every irreducible $Y \subset X$ lies inside some Z_i. Therefore, the Z_i are indeed maximal irreducible subsets.

Definition 3.0.2. The (combinatorial) *dimension* of a topological space X is,

$$\dim(X) = \sup\{n \geq 0 \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X \text{ of irreducible closed } Z_j \subset X\}$$

Furthermore we set $\dim(\varnothing) = -\infty$.

Remark. We may have $\dim(X) = \infty$.

Definition 3.0.3. For a commutative ring A,

$$\dim A = \sup\{n \geq 0 \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset A \text{ for prime } \mathfrak{p}_j \subset A\}$$

and we set $\dim(0) = -\infty$.

Definition 3.0.4. For $Y \subset \mathbb{A}^n$ an algebraic set, we define the coordinate ring $k[Y] := k[x_1, \ldots, x_n]/I(Y)$. Notice this depends on the embedding into affine space not necessarily the intrinsic structure of Y.

Remark. We see that there are inclusion reversing equivalences,

$$\{\text{Radical ideals of } k[Y]\} \iff \{\text{closed subsets } Z \subset Y\}$$

and likewise,

$$\{\text{Prime ideals of } k[Y]\} \iff \{\text{irreducible closed subsets } Z \subset Y\}$$

Therefore,
$$\dim Y = \dim k[Y]$$

Remark. For irreducible closed $Z \subset X$ where X is an affine algebraic set, does there exist a maximal length chain,
$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d \subset X$$
with some $Z_j = Z$? If X is reducible then the answer is no because we can have irreducible components of different dimensions. However, for irreducible algebraic sets the answer is yes.

Theorem 3.0.5. Let B be a domain finitely generated over a field k. Then,

(a) $\dim B = \operatorname{trdeg}_k(\operatorname{Frac}(B))$ which is, in particular, finite

(b) For any prime $\mathfrak{p} \subset B$,
$$\dim B = \dim B/\mathfrak{p} + \dim B_\mathfrak{p}$$

Remark. We interpret the second part of this theorem as follows. The primes of $B/\mathfrak{p}$ are exactly the primes containing $\mathfrak{p}$ and thus we consider maximal chains,

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

and the primes of $B_\mathfrak{p}$ are exactly the primes contained in $\mathfrak{p}$ and thus we consider maximal chains,

$$\mathfrak{p} = \mathfrak{q}_m \supsetneq \cdots \subsetneq \mathfrak{q}_0$$

and thus splicing them together, by the theorem, gives a maximal length chain in B containing $\mathfrak{p}$.

Proof. For (a) [Mat. Thm. 5.6] for (b) [Mat, Ex. 5.1] (the solution is in the back of the book and uses part (a) and induction for non-algebraically closed fields even if you only care about the case of algebraically closed fields). □

Theorem 3.0.6 (Krull)**.** For all local Noetherian rings, $\dim A < \infty$.

Proof. See [Mat, Thm. 13.5] and [AM, Cor. 11.11]. □

Corollary 3.0.7. $\dim \mathbb{A}^n = \dim k[x_1, \ldots, x_n] = \operatorname{trdeg}_k(k(x_1, \ldots, x_n)) = n$

Corollary 3.0.8. Let $Z \subset \mathbb{Z}^n$ be irreducible and closed. Then,

(a) For nonempty open $U \subset Z$ (so $\overline{U} = Z$ because Z is irreducible) then $\dim U = \dim Z$

(b) If $Z = Z(f)$ for irreducible f then $\dim Z = n - 1$

(c) For each $x \in Z$ we have $\dim k[Z]_{\mathfrak{m}_z} = \dim Z$.

Remark. $\dim k[Z]_{\mathfrak{m}_z}$ corresponds to chains of irreducibles beginning at $Z_0 = \{z\}$.

Proof. (c) is immediate. Now we do (a). We see that U is irreducible because $\overline{U} = Z$ since $Y \subset U$ is closed then $\overline{Y} \cap U = Y$. Suppose that,

$$Y_0 \subsetneq \cdots \subsetneq Y_n \subset U$$

is a chain of closed irreducible subsets of U then,

$$\overline{Y}_0 \subsetneq \cdots \subsetneq \overline{Y}_n \subset Z$$

is a chain of closed irreducible subsets of Z since $\overline{Y}_i \cap U = Y_i$ so the containments are proper. Therefore, $\dim U \leq \dim Z$. Now, by the previous theorem, we can choose a maximal chain such that $Z_0 = \{z\}$ with $z \in U$ (the point can be chosen arbitrarily) and get,

$$Z_0 \subsetneq \cdots \subsetneq Z_n = Z$$

so take $Y_j = Z_j \cap U$ which is clearly closed in U and irreducible. Since $z \in U \cap Z_j$ we see that Y_j is nonempty but open in Z_j and thus $\overline{Y}_j = Z_j$ and thus the containments must be proper since $Z_j \subsetneq Z_{j+1}$.

Finally to show (b) we apply the dimension formula,

$$\dim Z(f) + \mathbf{ht}\,((f)) = n$$

so it suffices to prove that $(0) \subsetneq (f)$ is a minimal nonzero prime. However, for any nonzero $\mathfrak{p} \subset (f)$ take an irreducible element $g \in \mathfrak{p}$ (factor any element and by primality its irreducible factors are inside $\mathfrak{p}$) and thus $(g) \subset \mathfrak{p} \subset (f)$ so $g = fr$ but g is irreducible and f is not a unit so r is a unit and thus $(g) = (f)$ so $\mathfrak{p} = (f)$. □

Proposition 3.0.9. Let A be a Noetherian domain and suppose that $f \in A$ is nonzero and (f) is prime. Then (f) is a minimal nonzero prime.

Proof. then take $x \in \mathfrak{p}$ so $x = fr$ but $f \notin \mathfrak{p}$ so $r \in \mathfrak{p}$ and thus $f\mathfrak{p} = \mathfrak{p}$. Thus if $\mathfrak{p}$ is finitely generated (it is because we are in a Noetherian ring) then there is $r \in (f)$ such that $(r-1)\mathfrak{p} = 0$ by Nakayama but in a domain this implies $\mathfrak{p} = 0$ because $r - 1 \neq 0$. □

Remark. The closed sets $Z \subset \mathbb{A}^n$ whose irreducible components are all of dimension $n-1$ are *exactly* $Z = Z(f)$ for nonconstant $f \in k[x_1, \ldots, x_n]$ (look at irreducible components $Z_j = Z(f_j)$).

3.0.1 "Nice" functions on algebraic sets

We have $k[Z] = k[x_1, \ldots, x_n]/I(Z) \hookrightarrow \mathrm{Func}(Z, k)$ by sending $g \mapsto (z \mapsto g(z))$ because these functions by definition do not care about polynomials that vanish on Z. Consider $U = Z_f \subset Z$ then we get $\alpha_f : k[Z]_f \to \mathrm{Func}(Z_f, k)$ because f is non-vanishing on U and thus f^{-1} makes sense as a function.

Definition 3.0.10. For any open $U \subset Z$ nonempty we define,

$$\mathcal{O}_Z(U) = \{\varphi : U \to k \mid \forall u \in U : \exists u \in V \subset U : \varphi|_V = \tfrac{g}{h} \text{ for } g, h \in k[Z] \text{ and } h|_V \text{ nonvanishing }\}$$

Proposition 3.0.11. The map $\alpha_f : k[Z]_f \xrightarrow{\sim} \mathcal{O}_Z(Y)$ is an isomorphism.

4 Sep. 27

4.1 Regular Functions and Coordinate Rings

Let $Z \subset \mathbb{A}^n$ be an algebraic set. For nonempty open $U \subset Z$,

$$\mathcal{O}_Z(U) = \{\varphi : U \to k \mid \text{locally on } Z : \varphi \in \mathrm{Frac}\,(k[Z])\}$$

furthermore $\mathcal{O}_Z(\varnothing) = (0)$. Last time: for $f \in k[Z]$ we said that $\alpha_f : k[Z]_f \to \mathcal{O}_Z(Z_f)$ via $\frac{g}{f^n} \mapsto \frac{g|_{Z_f}}{(f|_{Z_f})^r}$. We want to show that α_f is an isomorphism.

Remark. $f = 0$ in $k[Z]$ iff $k[Z]_f = 0$ iff $Z_f = \varnothing$

Example 4.1.1.

Let $Z = \{xy = zw\} \subset \mathbb{A}^4$ then $k[Z] = k[x, y, z, w]/(xy - zw)$ is a domain because $xy - zw$ is irreducible. Therefore,
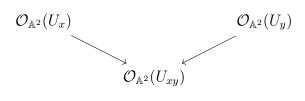
$$\frac{x}{w} = \frac{z}{y} \in \mathrm{Frac}\,(k[Z])$$

On Z_w we have the regular function $\frac{x}{w}$ and on Z_y we have the regular function $\frac{z}{y}$. Furthermore, they agree on $Z_w \cap Z_y = Z_{wy}$. Therefore, they glue to a regular function on $Z_w \cup Z_y$. However, we don't see $g, h \in k[Z]$ with $h|_U$ nonvanishing and $\varphi = \frac{g}{h}$.
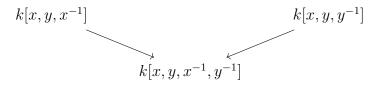
Example 4.1.2. Consider $U = \mathbb{A}^2 \setminus \{(0, 0)\}$ which is Zariski open. We want to determine $\mathcal{O}_{\mathbb{A}^2}(U)$. However, $U = U_x \cup U_y$ where $U_f = Z_f^c$. Furthermore, $U_x \cap U_y = U_{xy}$. Granting the theorem of today, then $\mathcal{O}_{\mathbb{A}^2}(U_x) = k[x, y]_x$ and $\mathcal{O}_{\mathbb{A}^2}(U_y) = k[x, y]_y$ and $\mathcal{O}_{\mathbb{A}^2}(U_{xy}) = k[x, y]_{xy}$. These are domains so there are inclusions $k[x, y]_x \to k[x, y]_{xy}$ and $k[x, y]_y \to k[x, y]_y$. Therefore, inside $k[x, y]_{xy}$ we need to compute the intersection,

$$k[x, y]_x \cap k[x, y]_y = k[x, y]$$

because $k[x, y]$ is a UFD with units $k^\times$. Explicitly,

$$\mathcal{O}_{\mathbb{A}^2}(U_x) \qquad\qquad\qquad \mathcal{O}_{\mathbb{A}^2}(U_y)$$

$$\mathcal{O}_{\mathbb{A}^2}(U_{xy})$$

is the diagram,

$$k[x, y, x^{-1}] \qquad\qquad\qquad\qquad k[x, y, y^{-1}]$$

$$k[x, y, x^{-1}, y^{-1}]$$

Equivalently, we can intersect inside the function field since $k[x, y, x^{-1}, y^{-1}] \hookrightarrow k(x, y)$. Therefore, $\mathcal{O}_{\mathbb{A}^2}(U) = k[x, y]$.

This is the algebraic version of Harthog's theorem from several complex variables.

Proposition 4.1.3. For $\varphi \in \mathcal{O}_Z(U)$ the map $\varphi : U \to \mathbb{A}^1$ is continuous.

Proof. The only interesting closed sets in $\mathbb{A}^1$ are finite collections of points. Thus we only need to check that the preimage of a point is closed. Being closed is local (on all of Z not just on the set that would be "locally closed") so we can choose an open cover U_i such that $\varphi|_{U_i} = \frac{f}{g}$ with g nonvanishing. Then,

$$\varphi^{-1}(y) \cap U_i = \{x \in U_i \mid f(x) - yg(x) = 0\} = Z(f - yg)$$

is Zariski closed. Thus $\varphi^{-1}(y)$ is Zariski closed. $\qquad\square$

Proposition 4.1.4. The map $\alpha_f : k[Z]_f \to \mathcal{O}_Z(Z_f)$ is an isomorphism.

Proof. We can assume that $Z_f = \varnothing$ because in that case $f = 0$ and this is obvious. Now we reduce to the case $f = 1$ by the Rabinawitz trick. Let $Z \subset \mathbb{A}^n$ cut out by $I \subset k[x_1, \ldots, x_n]$. Then consider,

$$Y = \{(\underline{x}, t) \in \mathbb{A}^{n+1} \mid \forall h \in I(Z) \; f(\underline{x})t = 1\}$$

I claim that $Z_f \cong Y$ and Y is cut out by the ideal $I + (Tf - 1) \subset k[x_1, \ldots, x_n, T]$. To see this, consider,

$$k[x_1, \ldots, x_n, T]/(I(Z), Tf - 1) = k[Z][T]/(Tf - 1) = k[T]_f$$

which is reduced and thus $I + (Tf - 1)$ is radical.

Furthermore, there is a bijection ξ between these sets,

$$(x \in Z_f) \mapsto (\underline{x}, \tfrac{1}{f(x)})$$
$$\underline{x} \leftmapsto (\underline{x}, t)$$

This transforms the problem for $Z_f \subset Z$ into the problem for Y with "$f = 1$". We need to check that,

$$\mathcal{O}_Z(Z_f) \xrightarrow{\sim} \mathcal{O}_Y(Y)$$

First we should check that ξ is a homeomorphism. We need to check that $\xi^{-1}(Z)$ is closed where Z is closed. Let $Z = (g_1, \ldots, g_n)$ for $g_i \in k[\underline{x}, T]$. It suffices to show that $\xi^{-1}(Z(g))$ for one polynomial. Thus we need to show that,

$$\xi^{-1}(Z(g)) = \{x \in Z_f \mid g(\underline{x}, f(\underline{x})^{-1}) = 0\}$$

is closed. However, there is a largest power of f^{-1} in this expression so,

$$g(\underline{x}, f(\underline{x})^{-1}) = \frac{p(\underline{x})}{f(\underline{x})^n}$$

but the left hand size is a polynomial $p \in k[x_1, \ldots, x_n]$. Thus because f is nonvanishing on Z_f we see that $\xi^{-1}(Z(g)) = Z(p) \cap Z_f$ and thus is closed in Z_f. Now we need to show that ξ is open. Now for $q \in k[x_1, \ldots, x_n]$ we need to show that $\xi(Z(q) \cap Z_f)$ is closed but,

$$\xi(Z(q) \cap Z_f) = \{(\underline{x}, t) \in Y \mid q(\underline{x}) = 0\}$$

which is closed. It remains to prove that the following diagram commutes,

$$
\begin{array}{ccc}
k[Z]_f & \longrightarrow & \mathcal{O}_Z(Z_f) \\
{\scriptstyle g \mapsto g \circ f^{-1}} \uparrow & & \| \xi \\
k[Y] & \longrightarrow & \mathcal{O}_Y(Y)
\end{array}
$$

Therefore, we have reduced to showing that $\alpha : k[Y] \to \mathcal{O}_Y(Y)$ is an isomorphism.

Therefore we do the case $f = 1$. Pick $\varphi \in \mathcal{O}_Z(Z)$ we want $\varphi \in k[Z]$. For all $P \in Z$ there is an open $V_P \subset Z$ such that $\varphi|_{V_p} = \frac{g_P}{h_P}$, for $g_P, h_P \in k[Z]$ and $h_P|_{V_P}$ is non-vanishing. Shrink V_P to some $P \in Z_{H_P} = \{H_p \neq 0\}$ in Z. Shrink further to $\{h_P H_P \neq 0\}$ and φ on here is,

$$\varphi|_{Z_{H_P}} = \frac{g_P H_P}{h_P H_P}$$

which we rename to $\frac{g_P}{h_P}$. Thus Z is covered by Z_{h_P} with $\varphi|_{Z_{h_P}} = \frac{g_P}{h_P}$. By HW1, a Noetherian space is quasi-compact , so Z is covered by finitely many,

$$Z_{h_1}, \ldots, Z_{h_m}$$

with $\varphi|_{Z_{h_j}} = \frac{g_j}{h_j}$ for $g_j, h_j \in k[Z]$. That these cover say that $\{h_j\}$ have no common zero in Z, so $(h_1, \ldots, h_m) = (1)$ in $k[Z]$ else it would contain some maximal ideal and thus by the Nullstellensatz it would vanish. Then we apply the following ring theory fact. If M is an R module and $(f_1, \ldots, f_n) = (1)$ then the following sequence is exact,

$$0 \longrightarrow M \longrightarrow \prod_{i=1}^{n} M_{f_i} \longrightarrow \prod_{i,j} M_{f_i f_j}$$

$\square$

5 Sept. 29

5.1 Morphisms of Algebraic Sets

For C^∞-manifolds M, N a C^∞-morphism $\varphi : M \to N$ satisfies,

(a) φ is continuous

(b) $\varphi^* : C^\infty(V) \to C^\infty(\varphi^{-1}(V))$ for all open $V \subset N$.

Proposition 5.1.1. Conversely, if $\varphi : M \to N$ satisfies (a) and (b) then it is C^∞.

Proof. Choose an open cover $\{W_\alpha\}$ on W such that $\varphi(W_\alpha) \subset V_\alpha$ which is a chart of V_α (using that φ is continuous to choose open $W_\alpha \subset \varphi^{-1}(V_\alpha)$ around each point). If $V_\alpha \subset \mathbb{R}^n$ is open. Choose coordinates $x_1, \ldots, x_n$ then, $W_\alpha \xrightarrow{\varphi} V_\alpha \subset \mathbb{R}^n$ is $w \mapsto (\varphi_1(w), \ldots, \varphi_2(w))$ where $\varphi_j = x_j \circ \varphi = \varphi^*(x_j)$ for the local coordinate functions $x_j : V_\alpha \to \mathbb{R}$. By (b) we know that $\varphi_j = x_j \circ \varphi = \varphi^*(x_j)$ is a smooth function on V_α. Therefore, φ is C^∞ because $W_\alpha \to V_\alpha \subset \mathbb{R}^n$ is smooth. $\square$

Definition 5.1.2. For algebraic sets $Z_1 \subset \mathbb{A}^n$ and $\mathbb{Z}_2 \subset \mathbb{A}^m$ and open $U_j \subset Z_j$, a *morphism* $\varphi : U_1 \to U_2$ is a continuous map such that for any open $V \subset U_2$ and f a regular function on V then $\varphi^* f = f \circ \varphi|_V$ is a regular function on $\varphi^{-1}(V)$.

Remark. Given a morphism $\varphi : U_1 \to U_2$ we get a map of k-algebras $\varphi^* : \mathcal{O}_{Z_2}(V) \to \mathcal{O}_{Z_1}(\varphi^{-1}(V))$ for each open $V \subset U_2$. This notion is Zariski-local meaning "morphisms glue". Furthermore, it is immediate that morphisms compose.

Remark. There appears to be a defect in this definition: it seems to depend on an ambient $Z_j \supset U_j$ and how Z_j are embedded in affine space.

Example 5.1.3. Given a map $\alpha : k[Z_2] \to k[Z_1]$ of coordinate rings consider coordinates,

$$
\begin{array}{ccc}
k[Y_1, \ldots, Y_m] & \dashrightarrow^{\Phi} & k[X_1, \ldots, X_n] \\
\downarrow & & \downarrow \\
k[Z_2] & \xrightarrow{\;\;\alpha\;\;} & k[Z_1]
\end{array}
$$

corresponding to the embeddings $Z_1 \hookrightarrow \mathbb{A}^n$ and $Z_2 \hookrightarrow \mathbb{A}^m$. These maps have ideals $J_i = I(Z_i)$. Consider $\alpha : \bar{Y_j} \to \overline{h_j(X)}$. Define Φ by $Y_j \mapsto h_j(X)$ which is not unique but it is unique up to J_1. Then we get a corresponding diagram of algebraic sets,

$$
\begin{array}{ccc}
\mathbb{A}^m & \xleftarrow{\tilde{\Phi}} & \mathbb{A}^n \\
\uparrow & & \uparrow \\
Z_2 & \xleftarrow{\tilde{\alpha}} & Z_1
\end{array}
$$

be defining,
$$ \tilde{\Phi}(x_1, \ldots, x_n) = (h_1(x), \ldots, h_m(x)) $$

Then I claim that $\tilde{\Phi}(Z_1) \subset Z_2$ and $Z_1 \to Z_2$ depends only on α. Ambiguity in h_j is $I(Z_1)$ which vanishes on Z_1 so $\tilde{\Phi}|_{Z_1}$ is independent of choices. Since $Z_2 = Z(J_2)$ it is enough to show for $g \in J_2$ that $g \circ \tilde{\Phi} = 0$. But $g \circ \tilde{\Phi} = \Phi(g)$, it is enough to check this on generators of the polynomial algebra $g \in k[Y_1, \ldots, Y_m]$ but $Y_j \circ \tilde{\Phi} = h_j = \Phi(Y_j)$ by definition. Then for $g \in J_2$ we have $g \circ \tilde{\Phi} = \Phi(g) \in \Phi(J_2) \subset J_1$ by the diagram.

Now we show that $\tilde{\alpha}$ is a morphism. To check that $\tilde{\alpha}$ is continuous, it suffices to show that $\tilde{\Phi}$ is continuous because the Z_j have the subspace topology. However,

$$ \tilde{\Phi}(Z(I)) = \{x \in \mathbb{A}^n \mid \tilde{\Phi}(x) \in Z(I)\} = \{x \in \mathbb{A}^n \mid \forall g \in I : g(\tilde{\Phi}(x)) = 0\} = \{x \in \mathbb{A}^n \mid \forall g \in I : \Phi(g)(x) = 0\} $$
$$ = Z(\Phi(I)) $$

is closed. Regular functions on Z_1 are "locally restrictions of reg. functions on $\mathbb{A}^n$" and similarly for Z_2, so it is enough to show $\tilde{\Phi}$ pulls back regular functions to regular functions.

For open $V_2 \subset \mathbb{A}^m$ and $f \in \mathcal{O}_{\mathbb{A}^m}(V_2)$ want $f \circ \tilde{\Phi}$. This is local on V_2 because $\tilde{\Phi}$ is continuous. We can shrink V_2 so WLOG $V_2 = U_h = \{h \neq 0\}$ for some $h \in k[Y]$. We can choose sufficiently small U_h (depending on f) such that $f|_{U_h} = \frac{g}{h}$ for $g, h \in k[Y]$. But, we computed that

$$ \tilde{\Phi}(Z(h)) = Z(\Phi(h)) $$

10

and thus by taking complements $\tilde{\Phi}^{-1}(U_h) = U_{\Phi(h)}$. Furthermore,

$$\frac{g}{h} \circ \tilde{\Phi} = \frac{g \circ \tilde{\Phi}}{h \circ \tilde{\Phi}} = \frac{\Phi(g)}{\Phi(h)}$$

which is manifestly a regular function on $U_{\Phi(h)}$. Therefore $\tilde{\alpha}$ is a morphism.

Observe that,

$$
\begin{array}{ccc}
\mathcal{O}_{Z_2}(Z_2) & \xrightarrow{\ \tilde{\alpha}^*\ } & \mathcal{O}_{Z_1}(Z_1) \\
\| & & \| \\
k[Z_2] & \xrightarrow{\ \ \alpha\ \ } & k[Z_1]
\end{array}
$$

commutes so $\tilde{\alpha}$ induces α. Therefore we see that,

$$\mathrm{Hom}\,(\mathrm{alg}, Z_1)\, Z_2 \to \mathrm{Hom}\,(k\text{-alg}, \mathcal{O}_{Z_2}(Z_2))\, \mathcal{O}_{Z_1}(Z_1)$$

via $\varphi \mapsto (f \mapsto f \circ \varphi)$. We showed that this is surjective by given α constructing $\tilde{\alpha}$. We constructed a map,

$$\mathrm{Hom}\,(k\text{-alg}, \mathcal{O}_{Z_2}(Z_2))\, \mathcal{O}_{Z_1}(Z_1) \to \mathrm{Hom}\,(\mathrm{alg}, Z_1)\, Z_2$$

such that $\alpha \mapsto \tilde{\alpha} \mapsto \alpha$ and thus a section. We next need to check that the map $\alpha \mapsto \tilde{\alpha}$ is injective or equivalently that $\tilde{\alpha} \mapsto \alpha \mapsto \tilde{\alpha}$.

Remark. If $A \to B$ is a map and $I \subset A$ and ideal then $(\varphi^*)^{-1}(V(I)) = V(\varphi(I))$ because,

$$\varphi^{-1}(\mathfrak{p}) \supset I \iff \mathfrak{p} \supset \varphi(I)$$

Indeed, if $\varphi^{-1}(\mathfrak{p}) \supset I$ then $\mathfrak{p} \supset \varphi(\varphi^{-1}(\mathfrak{p})) \supset \varphi(I)$ and if $\mathfrak{p} \supset \varphi(I)$ then $\varphi^{-1}(\mathfrak{p}) \supset \varphi^{-1}(\varphi(I)) \supset I$. This was really an important aspect of the above proof.

Remark. $\tilde{\Phi} : \mathbb{A}^n \to \mathbb{A}^m$ then $\Phi^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{\tilde{\Phi}(P)}$ because $\Phi(f)(P) = f(\tilde{\Phi}(P))$ and thus,

$$f \in \Phi^{-1}(\mathfrak{m}_P) \iff \Phi(f)(P) = 0 \iff f(\tilde{\Phi}(P)) = 0 \iff f \in \mathfrak{m}_{\Phi(P)}$$

Therefore, this describes $\tilde{\Phi}$ in terms of Φ intrinsically. Consequently, $\tilde{\alpha} : Z_1 \to Z_2$ satisfies $\alpha^{-1}(\mathfrak{m}_{Q_1}) = \mathfrak{m}_{\tilde{\alpha}(Q_1)}$.

Example 5.1.4. Let $Z_1 = Z(x - y^2)$ and $Z_2 = \mathbb{A}^1$. Consider $(x, y) \mapsto x$. Then $k[Z_1] = k[x, y]/(y^2 - x) \cong k[y]$ and $k[Z_2] = k[x]$ and the map $k[Z_2] \to k[Z_1]$ via $x \mapsto x = y^2$. The map $k[x] \to k[y]$ is finite free of rank 2.

Example 5.1.5. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ be given by $t \mapsto (t^3, t^4, t^5)$ then $k[t] \xleftarrow{\alpha} k[x, y, z]$ via $x \mapsto t^3$ and $y \mapsto t^4$ and $y \mapsto t^5$. Let $J = \ker \alpha$ which is prime since $k[x, y, z]/J \hookrightarrow k[t]$ is thus a domain. The image of φ is clearly contained in $Z(J)$ but $k[x, y, z]/J \hookrightarrow k[t]$ is a finite extension and thus surjective on maxspec so φ is surjective onto $Z(J)$.

6 Oct. 1

Theorem 6.0.1. For algebraic sets $Z_1 \subset \mathbb{A}^n$ and $Z_2 \subset \mathbb{A}^m$ the map,

$$\mathrm{Hom}\,(\text{algset}, Z_1)\, Z_2 = \mathrm{Hom}\,(k\text{-alg}, \mathcal{O}_{Z_2}(Z_2))\, \mathcal{O}_{Z_1}(Z_1)$$

via $(\varphi : Z_1 \to Z_2) \mapsto (f \mapsto f \circ \varphi)$ is bijective.

Proof. Last time, we proved surjectivity by, given a k-algebra map,

$$\alpha : k[Z_2] \to k[Z_1]$$

we build $\tilde{\alpha} : Z_1 \to Z_2$ such that $f \circ \tilde{\alpha} = \alpha(f)$. For injectivity, given $\varphi, \psi : Z_1 \to Z_2$ that induce the same map $k[Z_2] \to k[Z_1]$ then we want to show that $\varphi = \psi$. Composing,

$$Z_1 \to Z_2 \hookrightarrow \mathbb{A}^m$$

then the compositions also act identically on coordinate functions. Therefore, we may assume we have maps $\varphi, \psi : Z_1 \to \mathbb{A}^m$. However, these are the maps $x \mapsto (h_1(x), \ldots, h_m(x))$ where

$$h_j = Y_j \circ \varphi = Y_j \circ \psi$$

and therefore $\varphi = \psi$. $\qquad\square$

Remark. For Z an affine variety and $U, V \subset Z$ non-empty open then,

$$k[Z] \hookrightarrow \mathcal{O}_Z(U), \mathcal{O}_Z(V) \subset \mathrm{Frac}\,(k[Z]) = k(Z)$$

since $k[Z]$ is a domain and,
$$\mathcal{O}_Z(U) \cap \mathcal{O}_Z(V) = \mathcal{O}_Z(U \cup V)$$

Corollary 6.0.2. For $U \subset Z \hookrightarrow \mathbb{A}^n$ with U open in Z and $Z \hookrightarrow \mathbb{A}^n$ closed. Then,

$$\mathrm{Hom}\,(, U)\, \mathbb{A}^m \xrightarrow{\dagger} \mathcal{O}_Z(U)^{\oplus m}$$

given by $\varphi \mapsto (Y_1 \circ \varphi, \ldots, Y_m \circ \varphi)$ is bijective.

Proof. If $U = Z_f$ is a basic affine open then by Rabinowitz trick we have that U is an algebraic set so we use the previous result to get,

$$\mathrm{Hom}\,(, Z_f)\, \mathbb{A}^m = \mathrm{Hom}\,(k, k[Y_1, \ldots, Y_m])\, k[U_f] = k[U_f]^{\oplus m}$$

Then, a general U is covered by finitely many principal opens $Z_{f_1}, \ldots, Z_{f_r}$ with $Z_{f_i} \cap Z_{f_j} = Z_{f_i f_j}$ so we can "glue" using the functoriality of $\dagger$. $\qquad\square$

Remark. The corollary says that,

$$\mathrm{Hom}\,(, -)\, \mathbb{A}^m = \mathrm{Hom}\,(, -)\, \mathbb{A}^{1 \oplus m}$$

so we begin to see that $\mathbb{A}^m$ is the m-fold product of $\mathbb{A}^1$ in the category of algebraic sets.

6.1 Local Rings and Smoothness

For algebraic set $Z \subset \mathbb{A}^n$ and $P \in Z$, "local" properties of Z at P are encoded in the "local ring" at P.

Definition 6.1.1. $\mathcal{O}_{Z,P} = \varinjlim_{P \in U} \mathcal{O}_Z(U)$ where this is a directed system of k-algebras via $U_1 \geq U_2$ if $U_1 \subset U_2$ then $\mathcal{O}_Z(U_1) \leftarrow \mathcal{O}_Z(U_2)$.

Remark. For C^∞ or C^ω manifolds there are corresponding analytic notions called "germs" furthermore there is a diagram,

$$
\begin{array}{ccc}
\mathcal{C}^\omega_{\mathbb{R}^n,0} & \hookrightarrow & \mathcal{C}^\infty_{\mathbb{R}^n,0} \\
 & \searrow \quad \swarrow & \\
 & \mathbb{R}[[x_1,\ldots,x_n]] &
\end{array}
$$

where $\mathcal{C}^\infty_{\mathbb{R}^n,0} \twoheadrightarrow RR[[x_1,\ldots,x_n]]$ a theorem of Borel.

Example 6.1.2. $\mathcal{O}_{\mathbb{A}^n,a} = \left\{ \frac{f(x)}{g(x)} \big| g(a) \neq 0 \right\} \subset k(x_1,\ldots,x_n) = k(\mathbb{A}^n)$

Theorem 6.1.3. If $\mathfrak{m} \subset k[Z]$ is a maximal ideal corresponding to $P \in Z_1$ then,

$$
\begin{array}{ccc}
k[Z] & \longrightarrow & \mathcal{O}_{Z,P} \\
 & \searrow & \downarrow \\
 & & k[Z]_{\mathfrak{m}}
\end{array}
$$

there is a unique isomorphism making this diagram commute. Moreover, for open U containing P then the map $\mathcal{O}_Z(U) \to \mathcal{O}_{Z,P} \twoheadrightarrow k$ is evaluation $f \mapsto f(P)$.

Proof. To compute $\mathcal{O}_{Z,P}$ it is enough to use the cofinal poset of principal opens $U = Z_h$ for $h \in k[Z]$ with $P \in Z_h$. Then we have,

$$\mathcal{O}_Z(Z_h) = k[Z]_h$$

Now $Z_h \subset Z_{h'} \iff h' \mid h^n$ for some n because $Z_h \subset Z_{h'}$ iff $Z(h) \supset Z(h')$ iff $\sqrt{(h)} \subset \sqrt{(h')}$ iff $h \in \sqrt{(h')}$ iff $h^n \in (h')$. Therefore,

$$\mathcal{O}_{Z,P} = \varinjlim_{h, h(P) \neq 0} k[Z]_h$$

with $h \geq h'$ iff $h' \mid h^n$ for some n. Therefore, we are localizing with increasing subsets of $k[Z] \setminus \mathfrak{m}$ and thus,

$$\mathcal{O}_{Z,P} = \varinjlim_{h, h(P) \neq 0} k[Z]_h = k[Z]_{\mathfrak{m}}$$

To see this,

$$
\begin{array}{ccc}
k[Z]_{h'} & \longrightarrow & k[Z]_{\mathfrak{m}} \\
\downarrow & \nearrow & \\
k[Z]_h & &
\end{array}
$$

where h' is a unit in $k[Z]_h$ because $h' \mid h^n$ giving the existence of the downward map. $\qquad\square$

Remark. Prime of $\mathcal{O}_{Z,P} = k[Z]_{\mathfrak{m}}$ are the primes $\mathfrak{p}$ of $k[Z]$ inside $\mathfrak{m}$ equivalent to the irreducible closed $Y \subset Z$ through P. Therefore minimal primes of $\mathcal{O}_{Z,P}$ are the irreducible components passing through P. Now,

$$\dim \mathcal{O}_{Z,P} = \max_{P \in Z} \dim Z$$

Definition 6.1.4. If $I(Z0 = (f_1, \ldots, f_r) \subset k[X_1, \ldots, X_n]$ for $Z \hookrightarrow \mathbb{A}^n$ say Z is *smooth* at $P \in Z$ if,

$$\text{rank} \left(\frac{\partial f_i}{\partial x_j}(P) \right) = n - \dim \mathcal{O}_{Z,P}$$

7 Oct. 4

7.1 Tangent Spaces and Presheaves

For $P \in Z$ an algebraic set, say P is *smooth* on Z if the inequality,

$$\text{rank}\, (J(P)) \leq n - \dim \mathcal{O}_{Z,P}$$

is an equality. This is equivalent to $\mathcal{O}_{Z,P}$ being an equality.

Definition 7.1.1. The Zariski tangent space is defined by,

$$T_p(Z) = \text{Hom}\left(k, \mathfrak{m}_P/\mathfrak{m}_P^2\right) k$$

where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{Z,P}$.

Remark. To give some motivation, consider a C^∞-manifold M, and $a \in M$ then,

$$T_a(M) = \text{Hom}\left(\mathbb{R}, \mathfrak{m}_a/\mathfrak{m}_a^2\right) \mathbb{R}$$

for the maximal ideal $\mathfrak{m}_a$ of the local ring $\mathcal{O}_{M,a}$ of gems of C^∞-functors near a. Then $\mathcal{O}_{M,a} \cong \mathbb{R} \oplus \mathfrak{m}_a$ and $T_a(M)$ is the space of point derivations at a on $\mathcal{O}_{M,a}$.

Now we apply the Hadamard lemma: any $f \in \mathcal{O}_{M,a}$ can be written in the following form,

$$f(x) = f(a) + \sum_{i=1}^{n} (x - a)_i g_i(x)$$

for some smooth functions g_i.

Example 7.1.2. Let $Z \hookrightarrow \mathbb{A}^n$ corresponding to $k[\mathbb{A}^n] \twoheadrightarrow k[Z]$ then $M - (x_j - x_j(P))_j \twoheadrightarrow \mathfrak{m}_P$ and thus $M/M^2 \twoheadrightarrow \mathfrak{m}_P/\mathfrak{m}_P^@$ so we get a map,

$$T_P(Z) \hookrightarrow T_P(\mathbb{A}^n) = \bigoplus_{i=1}^{n} k(x_j - x_j(P)) \cong k^n$$

Then if $I(Z) = (f_1, \ldots, f_r)$ then,

$$T_P(Z) = \ker J(P) = \left\{ t \in k^n \,\middle|\, \forall i : \sum_j \frac{\partial f_i}{\partial x_j}(P) t_j = 0 \right\}$$

14

To see this,
$$k[Z] = k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$$
so $\mathfrak{m}_P = (X_1 - X_1(P), \ldots, X_n - X_n(P))/(f_1, \ldots, f_r)$ where this makes sense because all f_i vanish at P. Therefore,
$$\mathfrak{m}_P/\mathfrak{m}_P^2 = (X_j - X_j(P))_j/((X_i - X_i(P))(X_j - X_j(P)), f_1, \ldots, f_r)$$
$$= \frac{\bigoplus_{j=1}^r (X_j - X_j(P))}{\left\langle \sum_j \frac{\partial f_i}{\partial x_j}(P)(X_j - X_j(P)) \right\rangle} \cong \frac{k^n}{\mathrm{Im}\,(\nabla f(P))}$$

Then,
$$\mathrm{Hom}\left(k, \mathfrak{m}_P/\mathfrak{m}_P^2\right) k \cong \ker \nabla f(P)$$

Example 7.1.3. If $Z = Z(f)$ for radical f nonconstant so $\dim Z = n - 1$ and $\dim \mathcal{O}_{Z,P} = n - 1$ (so all irreducible components of Z have dimension $n - 1$). Then,
$$T_P(Z) = (\Delta f(P))^\perp \subset k^n$$
so Z is *smooth* at P iff $(\Delta f)(P) \neq 0$ iff *some*,
$$\frac{\partial f}{\partial x_j}(P) \neq 0$$

For example, if,
$$f = \sum X_j^2$$
in characteristic not 2 and $n \geq 2$ then $Z(f)$ is smooth except at the origin.

Example 7.1.4. For $n = 2$, radical $f \in k[x, y]$ non-constant then $C = Z(f)$ is a curve in $\mathbb{A}^2$. Now C is smooth at P iff $\frac{\partial f}{\partial x}(P)$ or $\frac{\partial f}{\partial y}(P)$ is nonzero. At a smooth point, $\mathcal{O}_{C,P}$ is a regular 1-dimensional local Noetherian domain and thus a DVR [Mat, Thm.11.2].

We might ask how to detect when $h \in \mathfrak{m}_P$ (e.g. $h \in k[x, y]$ with $h(P) = 0$) is a uniformizer. The valuation at a smooth point P corresponds to the order of tangency of C with the curve defined by h. Therefore, h gives a uniformizer if it is *not* tangent to C i.e. its linear part is not a scalar multiple of $(\nabla f)(P)$. To see this,
$$\mathfrak{m}_P/\mathfrak{m}_P^2 = \frac{k^2}{k(\nabla f)(P)}$$
so we need that $\bar{h} \in \mathfrak{m}_P/\mathfrak{m}_P^2$ is nonzero and thus its linear part should not be in the subspace $k(\nabla f)(P) \subset k^2$. Therefore, the line,
$$\ell = V(c_1(x - a) + c_2(y - b))$$
gives a uniformizer when it cuts the curve at P if and only if $(c_1, c_2) \notin k\left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\right)$.

7.2   Schemes

To build up schemes we want,

(a) to allow any field k

(b) to remove the ambient $\mathbb{A}^n$

(c) allow rings not containing a field (e.g. "geometry over $\mathbb{Z}$").

For manifolds there are two points of view:

(a) charts and atlases

(b) sheaves of rings on a topological space.

Definition 7.2.1. Let X be a topological space, $\mathcal{C}$ a category with a final object. Then a <u>presheaf</u> on X valued in $\mathcal{C}$ is an assignment,
$$\mathscr{F} : U \mapsto \mathscr{F}(U)$$
for open subsets $U \subset X$ and for each containment $V \subset U$ a restriction map $\mathrm{res}_{V,U} : \mathscr{F}(U) \to \mathscr{F}(V)$ such that given $W \subset V \subset U$ we have $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$ and $\mathrm{res}_{U,U} = \mathrm{id}$.

Example 7.2.2.   (a) Let X be a topological space. Take,
$$\mathscr{F}(U) = \{\varphi : U \to \mathbb{R} \mid \varphi \text{ is continuous}\}$$

(b) Let X be a C^∞-manifold. Take,
$$\mathcal{O}_X : U \mapsto C^\infty(U)$$
or,
$$\Omega_X^3 : U \mapsto \{\text{smooth 3-forms on } U\}$$
or also,
$$\mathrm{Vec}_X : U \mapsto \{C^\infty \text{ vector fields on } U\}$$
and also,
$$\mathcal{O}_X^\times : U \mapsto \{\varphi \in C^\infty(U) \mid \forall x \in U : \varphi(x) \neq 0\}$$

(c) Let Z be an algebraic set, with the Zariski topology then,
$$\mathcal{O}_Z : U \mapsto \{\text{ regular functions on } U\}$$
is a presheaf of k-algebras.

(d) If A is an abelian group then I can form two presheaves,
$$\mathscr{F}_A : U \mapsto A \text{ and } \mathrm{res}_{V,U} = \mathrm{id}$$
or you can form,
$$\underline{A} : U \to \{\varphi : U \to A \mid \varphi \text{ is continuous where } A \text{ has the discrete topology}\}$$
(Notice that $\underline{A}(\varnothing) = 0$ while $\mathscr{F}_A(\varnothing) = A$).

Definition 7.2.3. A <u>morphism</u> $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves is a collection $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ which are compatible in the sense that,

$$
\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\
{\scriptstyle \mathrm{res}^{\mathscr{F}}_{V,U}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}^{\mathscr{G}}_{V,U}} \\
\mathscr{F}(V) & \xrightarrow{f_V} & \mathscr{G}(V)
\end{array}
$$

If all f_U are monic we say that $\mathscr{F}$ is a <u>subpresheaf</u> of $\mathscr{G}$.

Example 7.2.4. The map $\mathrm{d} : \Omega^2_X \to \Omega^3_X$ is a map of presheaves on a C^∞-manifold X. Furthermore, there is a sequence of presheaves,

$$
\mathcal{O}_X \overset{\exp}{\longhookrightarrow} \mathcal{O}_X^\times \overset{\mathrm{sign}}{\longrightarrow} \underline{\{\pm 1\}}
$$

When X is a complex manifold then $\mathcal{O}_X$ is the sheaf of holomorphic functions the exponential map has a kernel. There is a sequence,

$$
\underline{2\pi i \mathbb{Z}} \longrightarrow \mathcal{O}_X \overset{\exp}{\longrightarrow} \mathcal{O}_X^\times
$$

but exp is not surjective as presheaves because the global logarithm does not exist. Therefore we need to upgrade our notion to sheaves so capture this local data.

8 Sheaves and Ringed Spaces

Let $\mathcal{C}$ be a category with final object: (e.g. **Ab**, **Ring**, **Set**). Assume $\mathcal{C}$ is a subcategory of **Set** compatibly with,

(a) $\prod_{i \in I}$

(b) $\varinjlim$

(c) and the final object

Definition 8.0.1. A $\mathcal{C}$-valued sheaf on a topological space X is a $\mathcal{C}$-valued presheaf $\mathscr{F}$ on X such that, for any open $U \subset X$ and any open cover $\{U_\alpha\}_{\alpha \in A}$ of U, the diagram,

$$
\mathscr{F}(U) \longrightarrow \prod_\alpha \mathscr{F}(\alpha) \rightrightarrows \prod_{(\beta,\gamma)} \mathscr{F}(U_\alpha \cap U_\beta)
$$

given by $s \mapsto (s|_{U_\alpha})$ and $(s_\alpha) \mapsto (s_\beta|_{U_{\beta\gamma}})$ or $(s_\gamma|_{U_{\beta\gamma}})$ via the two maps is an equalizer diagram.

Remark. The says that for any $(s_\alpha)_{\alpha \in A}$ agreeing on overlaps, there exists a unique $s \in \mathscr{F}(U)$ s.t. $s|_{U_\alpha} = s_\alpha$.

Remark. Some remarks on the definition,

(a) If $\mathcal{C} = \mathbf{Ab}$, we can instead use the "difference" map,

$$
(s_\alpha) \mapsto (s_\beta|_{U_{\beta\gamma}} - s_\gamma|_{U_{\beta\gamma}})
$$

(b) Often $U_{\beta\gamma} = \varnothing$

(c) For any sheaf, $\mathscr{F}(\varnothing)$ is the final object. We could just insist on this but it is a consequence of the definition because the empty product is the final object.

Example 8.0.2. For any abelian group B, we see that $\underline{B}$ is a sheaf but the "very constant" presheaf $\mathscr{F}_B$ is not a sheaf because it does not glue property on a disconnected cover.

Example 8.0.3. Sheaves are the same notion of morphisms in the sense that they can be locally specified and glue.

Example 8.0.4. Non-example: let M be a C^∞-manifold,

$$\mathscr{F}_k : U \mapsto \bigwedge_{\mathcal{O}_M(U)}^{k} (\Omega_M^1(U))$$

is not a sheaf because locally wedges of 1-forms glue to a global k-form but this need not be globally a wedge of global 1-forms. However, Ω_M^k is the "wedge product of sheaves" but this only means that locally k-forms are wedges of 1-forms.

Definition 8.0.5. The stalk of a presheaf $\mathscr{F}$ at $x \in X$ is,

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$$

Example 8.0.6. Consider the following examples,

(a) If X is a manifold or an algebraic set, we have,

$$\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U)$$

are the "germs of [nice] functions at x".

(b) For any topological space, $A \xrightarrow{\sim} (\underline{A})_x$ by sending $a \mapsto$ constant function a. However, we also have $A \xrightarrow{\sim} (\mathscr{F}_A)_x$. So even though $\underline{A}$ and $\mathscr{F}_A$ have the same stalks one is a sheaf and the other is not.

(c) Consider a covering space $\pi : E \to X$. Introduce $\Sigma_{E/X}$ the sheaf of sections of π. Meaning,

$$\Sigma_{E/X}(U) = \{s : U \to E \mid \pi \circ s = \mathrm{id}_U\}$$

is the set of continuous sections. This is a sheaf of sets because continuous maps on opens glue. Now,

$$(\Sigma_{E/X})_x \xrightarrow{\sim} \pi^{-1}(x)$$

by evaluating the section at x. This is because $\pi : E \to X$ is locally a product $U \times \pi^{-1}(x)$ and the section condition completely determines $s : U \to U \times \pi^{-1}(x) \to U$ (the map to the first factor) to be the identity. For example, if $E = A \times X$ with A discrete then $\underline{A} \cong \Sigma_{E/X}$ as sheaves of sets.

(d) Consider C^∞-map $\varphi : X \to Y$ for C^∞-manifolds X, Y we have maps,

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$

but NOT a map $\mathcal{O}_Y \to \mathcal{O}_X$. To make sense of a map of sheaves between two spaces we need to introduce a way of moving a sheaf on one space to another.

Definition 8.0.7. For a continuous map $f : X \to Y$ of topological spaces, $\mathscr{F}$ a presheaf on X, the *pushforward* $f_*\mathscr{F}$ on Y is,

$$(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$$

Remark. Easy to check that if $\mathscr{F}$ is a sheaf then $f_*\mathscr{F}$ is also a sheaf.

Remark. In the preceding example: have "pullback on functions" $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remark. Given $\alpha : \mathscr{F} \to \mathscr{G}$ on X then we get $f_*\alpha : f_*\mathscr{F} \to f_*\mathscr{G}$ via,

$$\alpha_{f^{-1}(U)} : \mathscr{F}(f^{-1}(U)) \to \mathscr{G}(f^{-1}(U))$$

Suppose we have continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and a sheaf $\mathscr{F}$ on X then $g_*(f_*\mathscr{F}) \xrightarrow{\sim} (g \circ f)_*\mathscr{F}$.

Example 8.0.8. Let $f : X \to Y$ be a map of ringed spaces and $y = f(x)$. Then the sheaf map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ gives a map on stalks,

$$\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y = \varinjlim_{U \ni y} \mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_{X,x}$$

Example 8.0.9. Let $\iota : Y \hookrightarrow X$ be a closed submanifold. Then,

$$(\iota_*\mathcal{O}_Y)_x = \begin{cases} \mathcal{O}_{Y,x} & x \in Y \\ 0 & x \notin Y \end{cases}$$

Definition 8.0.10. Assume $\mathcal{C}$ is a category of rings (e.g. k-algebras, A-algebras, all CRings). A *$\mathcal{C}$-ringed space* is a pair $(X, \mathcal{O}_X)$ for a $\mathcal{C}$-valued sheaf of rings $\mathcal{O}_X$. A *morphism* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where,

(a) $f : X \to Y$ is continuous

(b) $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings

Remark. NOTE: $f^\#$ may not be determined by f.

Remark. Composition is given as follows,

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$$

has composition $g \circ f : X \to Y$ and map of sheaves,

$$
\begin{array}{ccc}
\mathcal{O}_Z & \dashrightarrow & (g \circ f_*\mathcal{O}_X \\
\downarrow{\scriptstyle g^\#} & & \| \\
g_*\mathcal{O}_Y & \xrightarrow{g_* f^\#} & g_*(f_*\mathcal{O}_X)
\end{array}
$$

9 Oct. 8

9.1 Locally Ringed Spaces

For ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a *morphism* $\varphi : (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that,

(a) f is continuous

(b) $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$

We want a more refined notion of "ringed space" to make connection between $\mathcal{O}_X$ and "functions" on X and to relate $f^{\#}$ more directly to to f. Let's see how $\mathcal{O}_X x$ is functorial in $(X, \mathcal{O}_X, x)$.

Suppose we have,

$$(X, \mathcal{O}_X) \xrightarrow{\varphi = (f, f^{\#})} (Y, \mathcal{O}_Y) \xrightarrow{\psi = (g, g^{\#})} (Z, \mathcal{O}_Z)$$

then we have maps,

$$\varphi_x : \mathcal{O}_{Y, f(x)} \xrightarrow{f^{\#}_{f(x)}} (f_* \mathcal{O}_X)_{f(x)} = \varinjlim_{U \ni y} \mathcal{O}_X(f^{-1}(U)) \to \varinjlim_{V \ni x} \mathcal{O}_X(V) = \mathcal{O}_{X,x}$$

where the second map exists by the universal property of the direct limit. This is compatible with composition: if $x \mapsto y \mapsto z$ then the following diagram commutes,

$$
\begin{array}{ccccccc}
\mathcal{O}_{Z,z} & \xrightarrow{g_z^{\#}} & (g_* \mathcal{O}_Y)_z & \xrightarrow{g_* f^{\#}} & (g_*(f_* \mathcal{O}_X))_z & = & ((g \circ f)_* \mathcal{O}_X)_z \\
& \searrow{\psi_y} & \downarrow & & & & \downarrow \\
& & \mathcal{O}_{Y,y} & & \xrightarrow{\quad\varphi_x\quad} & & \mathcal{O}_{X,x}
\end{array}
$$

and therefore $\varphi_x \circ \psi_y = (\psi \circ \varphi)_x$

9.2 Examples of Ringed Spaces

(a) $X = \varnothing$ and $\mathcal{O}_X(\varnothing) = 0$

(b) Smooth manifolds (over $\mathbb{R}, \mathbb{C}$) and usual notion of morphism (smooth or holomorphic map) and define $f^{\#} = (-) \circ f$ by pulling back smooth function. These are ringed spaces of $\mathbb{R}$-algebras or $\mathbb{C}$-algebras.

(c) Algebraic sets and usual morphisms (ring spaces of k-algebras) and again $f^{\#} = (-) \circ f$.

(d) $\mathrm{Spec}\,(\mathbb{C})$ is the ringed space where $X = *$ and $\mathcal{O}_X(*) = \mathbb{C}$. We can view this in three ways:

    (a) as a ringed space of $\mathbb{C}$-algebras and thus having a trivial automorphism group

    (b) as a ringed space of $\mathbb{R}$-algebras and thus having a automorphism group $\{\mathrm{id}, \sigma\}$

    (c) as a ringed space (of $\mathbb{Z}$-algebras) and thus having a massive automorphism group $\mathrm{Aut}\,(\mathbb{C})$.

(e) $\mathrm{Spec}\,(k[t]/(t^2))$ is the ringed space where $X = *$ and $\mathcal{O}_X(*) = k[t]/(t^2)$ has a nonzero nilpotent.

(f) $\mathrm{Spec}\,(\mathbb{Z})$ is the ringed space $X = \{\text{primes}\} \cup \{0\}$ where $\{0\}$ is dense. A nonempty open U is $X \setminus \{p_1, \ldots, p_r\}$ some finite set of closed points. Then define,

$$\mathcal{O}_X(U) = \mathbb{Z}[\tfrac{1}{p_1 \cdots p_r}] = \{q \in \mathbb{Q} \mid q = \tfrac{a}{b} \text{ where } p \mid b \implies p = p_i \text{ for some } i\}$$

with the obvious restriction maps $V \subset U$ gives $\mathcal{O}_X(U) \subset \mathcal{O}_X(V) \subset \mathbb{Q}$. Furthermore,

$$\mathcal{O}_{X,(p)} = \varinjlim_{U \ni (p)} \mathcal{O}_X(U) = \bigcup_{U \ni (p)} \mathcal{O}_X(U) = \mathbb{Z}_{(p)}$$

and likewise,

$$\mathcal{O}_{X,(0)} = \varinjlim_U \mathcal{O}_X(U) = \bigcup_U \mathcal{O}_X(U) = \mathbb{Q} = \mathbb{Z}_{(0)}$$

Therefore, $\mathcal{O}_{X,x} = \mathbb{Z}_x$ for each $x \in X$.

20

Remark. In the case of $\mathrm{Spec}(\mathbb{Z})$ and also for manifolds, we saw that the stalks $\mathcal{O}_{X,x}$ are all local rings. This is the phenomenon we want to capture.

9.3 Locally Ringed Spaces

Definition 9.3.1. A *locally $\mathcal{C}$-ringed space* is a $\mathcal{C}$-ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,x}$ is local $\forall x \in X$. A *morphism* of locally $\mathcal{C}$-ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of $\mathcal{C}$-ringed spaces φ such that $\varphi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local $\forall x \in X$ meaning that $\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ (or equivalently $\varphi_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$).

Example 9.3.2. Some examples of locally ringed spaces.

(a) Manifolds (over $\mathbb{R}$ or $\mathbb{C}$) then $\mathfrak{m}_x$ is the germs of functions vanishing at x so $\varphi_x(\mathfrak{m}_{f(x)}) = \mathfrak{m}_x$ because $f^\# = (-) \circ f$ so if g vanishes at $f(x)$ then $g \circ f$ vanishes at x.

(b) Let $\mathcal{C}$ be k-algebras with $k = \bar{k}$. Then algebraic sets $(Z, \mathcal{O}_Z)$ are examples. Then $\mathcal{O}_{Z,z}$ is a local k-algebra with $\mathcal{O}_Z z/\mathfrak{m}_z = k$ (germs of regular functions j near z with $h(z) = 0$). We've seen for k-alg map $\alpha : k[Z_2] \to k[Z_1]$ and $\tilde{\alpha} : Z_1 \to Z_2$ that $\mathfrak{m}_{\tilde{\alpha}(P)} = \alpha^{-1}(\mathfrak{m}_P)$ and thus,

$$
\begin{array}{ccc}
k[Z_2] & \xrightarrow{\ \alpha\ } & k[Z_1] \\
\downarrow & & \downarrow \\
k[Z_2]_{\mathfrak{m}_{\tilde{\alpha}(P)}} & \xrightarrow{\ \tilde{\alpha}_P\ } & k[Z_1]_{\mathfrak{m}_P}
\end{array}
$$

Exercise 9.3.3. Let $\mathcal{C}$ be k-algebras for a field k. Consider locally $\mathcal{C}$-ringed spaces $(X, \mathcal{O}_X)$ s.t. $k \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}_x$ is an isomorphism $\forall : x \in X$. Then morphisms as $\mathcal{C}$-ringed spaces is automatically "local" meaning is actually a morphism of $\mathcal{C}$-locally ringed spaces.

Remark. We call $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$. Call $\mathcal{O}_X$ the *structure sheaf*.

Example 9.3.4. Non-example: X is any topological space then $(X, \mathbb{Z})$ is a ringed space but not a locally ringed space.

9.4 Evaluation

For $U \ni x$ and $s \in \mathcal{O}_X(U)$ I get a germ $s_x \in \mathcal{O}_{X,x}$ and thus $s(x) \in \kappa(x)$ by taking its class in the quotient,

$$
\mathcal{O}_X(Y) \to \varinjlim_{V \ni} \mathcal{O}_X(V) = \mathcal{O}_{X,x} \twoheadrightarrow \kappa(x)
$$

We call $s(x)$ the "value" of s at x.

Remark. Notice the field $\kappa(x)$ may vary from point to point so $s(x) \in \kappa(x)$ cannot be glued together to give a usual map to a field.

Example 9.4.1. In familiar cases however, $s(x)$ does recover the "value" of s at a point. For example,

(a) For manifolds and algebraic sets under the isomorphism $k \xrightarrow{\sim} \kappa(x)$ indeed $s(x) \in k$ is the usual value of the function.

(b) In general if $\mathcal{C}$ is k-algebras and $k \to \mathcal{O}_{X,x} \twoheadrightarrow \kappa(x)$ is an isomorphism $\forall x \in X$ then get a map of sheaves of k-algebras,

$$\mathcal{O}_X \to \mathcal{A}_{X,k}$$

where $\mathcal{A}_{X,k}$ is the sheaf of k-valued functions on X. This sends $s \mapsto (x \mapsto s(x))$. In the previous example this is the subsheaf of smooth functions inside all functions. However, in general this is not injective: for example a variety over a non-algebraically closed field or a nonreduced scheme over an algebraically closed field. Lets give an example. For $X = \mathrm{Spec}\,(k[t]/(t^2))$ and $t \in \mathcal{O}_X(X)$ gives the zero function on X because $t \in \mathfrak{m}_x$ at the unique point (indeed any nilpotent must be sent to zero in a field under a ring homomorphism and therefore any nilpotent $s \in \mathcal{O}_X(U)$ has $\forall x \in X : s(x) = 0$.

(c) Let $X = \mathrm{Spec}\,(\mathbb{Z})$ and $s = \frac{5}{3} \in \mathcal{O}_X(U)$ where $U = X \setminus \{(3)\}$. Then, $x \in U$ is either (p) for $p \neq 3$ or (0). Then,

$$\kappa(x) = \begin{cases} \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{F}_p & p \neq 0 \\ \mathbb{Q} & p = 0 \end{cases}$$

and $s(p) = 5 \cdot 3^{-1} \in \mathbb{F}_p$ and $s(0) = \frac{5}{3} \in \mathbb{Q}$.
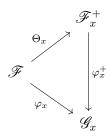
9.5 Functoriality of Evaluation

If $\varphi = (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $x \mapsto f(x)$. Then there is a commutative diagram,

$$\begin{array}{ccc} \mathcal{O}_Y y & \longrightarrow & \mathcal{O}_X x \\ \downarrow & & \downarrow \\ \kappa(y) & \longrightarrow & \kappa(x) \end{array}$$

where the bottom map sends $s(y) \mapsto (f^\# s)(x)$ which must be the identity map if these are maps of k-algebras and $\kappa(x)$ is identified with k because its a k-algebra map. Therefore on values, functions are pulled back as expected.

10 Oct. 11

By earlier result today, enough $\varphi_x^+ : \mathscr{F}_x^+ \to \mathscr{G}_x$ is an isometry for all x. Then the diagram,

Need to construct $(\Phi, \mathscr{F}^+)$ with this universal property. Here we deviate from [H], for method applicable to Grothendieck topologies. There are two possible problems to fix,

(a) $s \in \mathscr{F}(U)$ may not be determined by local data

(b) compatible local data may not glue.

Now we propose an iterated "completion" that fixes (a) in the first step and (b) in the second step.

First we get $\Theta_0 : \mathscr{F} \to \mathscr{F}_0$ where $\mathscr{F}_0$ is a separated presheaf (sections are determined by local data) and $\mathscr{F} \xrightarrow{\Theta_0} \mathscr{F}_0 \xrightarrow{(\Theta_0)_0} (\mathscr{F}_0)_0$. We are going to build, $\mathscr{F}_0(U)$ as $\varinjlim$ over open covers of U in a manner similar to Cauchy sequences for metric space completion. Let,

$$\Sigma_U = \{\mathfrak{V} = \{V_i\}_{i \in I} \mid \text{ open cover }\}$$

is a poset under refinement where $\mathfrak{V}' \geq \mathfrak{V}$ if there exists a map $\tau : I' \to I$ such that $V_j' \subset V_{\tau j}$ for each $j \in I'$. Furthermore, Σ_U is a directed poset because we can construct,

$$\mathfrak{V} \cap \mathfrak{V}' = \{V_i \cap V_j'\}_{(i,j) \in I \times I'}$$

Then we define,

$$\mathscr{F}_0(U) = \varinjlim_{\mathfrak{V} \in \Sigma_U} \{(s_i) \in \prod \mathscr{F}(V_i) \mid \forall i, j \in I : s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}\}$$

Furthermore, for $U' \subset U$ we get $\Sigma_U \to \Sigma_{U'}$ given by $\mathfrak{V} \mapsto \mathfrak{V} \cap U'$ giving a map $\mathscr{F}_0(U) \to \mathscr{F}_0(U')$ so $\mathscr{F}_0$ is a presheaf. It is easy to check that $\mathscr{F}_0$ is a separated presheaf. Furthermore, $\{U\} \in \Sigma_U$ and this gives a map $\mathscr{F}(U) \xrightarrow{\Theta_0} \mathscr{F}_0(U)$.

11 Properties of Sheafification

Restriction to open subsets: for presheaf $\mathscr{F}$ on X, open $U \subset X$ define $\mathscr{F}|_U : (V \subset U) \mapsto \mathscr{F}(V)$.

Proof. Naturally, $(\mathscr{F}^+)|_U \cong (\mathscr{F}|_U)^+$. $\qquad\square$

Proof. It would be bogus to say "they are equal because they have the same stalks" because you need a global map that *induces* an isomorphism on stalks. However, there is $\theta : \mathscr{F} \to \mathscr{F}^+$ then restricting gives,

$$
\begin{array}{ccc}
& & (\mathscr{F}|_U)^+ \\
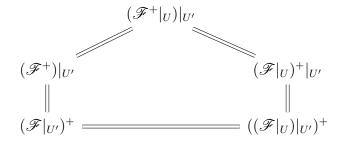& \nearrow & \downarrow{\scriptstyle\alpha} \\
\mathscr{F}|_U & \longrightarrow & (\mathscr{F}^+)|_U
\end{array}
$$

there exists a unique $\alpha : (\mathscr{F}|_U)^+ \to (\mathscr{F}^+)|_U$. The map α is an isomorphism because it is an isomorphism on stalks because the diagram commutes and the other two maps are both isomorphisms on stalks. $\qquad\square$

Example 11.0.1. If X is a C^∞-manifold and $U \subset X$ then $\mathcal{C}_X^\infty|_U \cong \mathcal{C}_U^\infty$ and likewise $\Omega_X^1|_U \cong \Omega_U^1$.

"How much more pedantic than this can one get? Watch this!"

Remark. For open $U' \subset U$ then the restrictions are compatible,

$$(\mathscr{F}^+|_U)|_{U'}$$

$$(\mathscr{F}^+)|_{U'} \qquad\qquad (\mathscr{F}|_U)^+|_{U'}$$

$$\|\qquad\qquad\qquad\qquad\qquad\quad\|$$

$$(\mathscr{F}|_{U'})^+ =\!=\!=\!=\!=\!=\!=\!= ((\mathscr{F}|_U)|_{U'})^+$$

You can check this commutes by looking at the diagram of stalk maps and noting that it commutes using the following lemma.

Lemma 11.0.2. If $\varphi_1, \varphi_2 : \mathscr{F} \to \mathscr{G}$ are maps of sheaves on X then

$$\varphi_1 = \varphi_2 \iff \forall x \in X : (\varphi_1)_x = (\varphi_2)_x$$

Definition 11.0.3. $\mathbf{Ab}(X)$ is the category of abelian sheaves (sheaves of abelian groups) on X. For example $\mathcal{O}_X$ additively, $\mathcal{O}_X^\times$ multiplicatively.

Definition 11.0.4. For $\varphi : \mathscr{F} \to \mathscr{G}$ is a map in $\mathbf{Ab}(X)$. Define the following sheaves,

(a) $\ker \varphi : U \mapsto \ker \varphi|_U$ is a subsheaf of $\mathscr{F}$

(b) $\operatorname{im} \varphi$ is the sheafification of the presheaf $U \mapsto \operatorname{im} \varphi|_U$

(c) $\operatorname{coker} \varphi$ is the sheafification of the presheaf $U \mapsto \operatorname{coker} \varphi|_U$.

Remark. If we have subpresheaf $j : \mathscr{H}_1 \hookrightarrow \mathscr{H}_2$ of a presheaf $\mathscr{H}_2$. Then,

$$j^+ : \mathscr{H}_1^+ \to \mathscr{H}_2^+$$

is a subsheaf meaning $(j^+)_U$ is injective for every open U (in particular $\operatorname{im} \varphi \to \mathscr{G} = \mathscr{G}^+$ is a subsheaf). We can prove this in two ways,

(a) look at the stalks (for sheaves, subsheaf iff injective on stalks and if j is injective then its injective on stalks and j^+ and j have the same stalk maps).

(b) look at the construction which exhibits $\mathscr{H}_1^+$ as a subsheaf of $\mathscr{H}_2^+$ using exactness of $\varinjlim$.

Remark. Explicitly,

$$(\operatorname{im} \varphi)(U) = \{s \in \mathscr{G}(U) \mid s \text{ is locally in the image of } \varphi\}$$
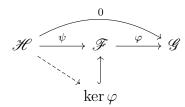
since for subsheaf $\mathscr{H}_1 \subset \mathscr{H}_2$ and $s \in \mathscr{H}_2(U)$ then $s \in \mathscr{H}_1(U)$ iff $s_x \in (\mathscr{H}_1)_x \subset (\mathscr{H}_2)_x$ for all $x \in U$. This is because an element locally is in the image iff its stalks are in the image because

Example 11.0.5. Consider $\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ for X a complex manifold. Then $\operatorname{im} \exp = \mathcal{O}_X^\times$ as a *sheaf* since non-vanishing holomorphic functions *locally* has a continuous logarithm. However, $\operatorname{im} \varphi_U$ is often strictly contained in $\mathcal{O}_X^\times(U)$ because global logarithms do not exist. Here, $\operatorname{coker} \varphi = \{1\}$ but the presheaf cokernel is nontrivial.

Remark. In general, $\operatorname{coker} \varphi = 0 \iff \forall x \in X : \varphi_x : \mathscr{F}_x \twoheadrightarrow \mathscr{G}_x$ NOT $\varphi_U : \mathscr{F}(U) \twoheadrightarrow \mathscr{G}(U)$ for "small" open $U \subset X$. It means that sections are locally in the image but how locally you have to look depends on the section so there is not necessarily a cover on which it is surjective.

Theorem 11.0.6. Consider $\varphi : \mathscr{F} \to \mathscr{G}$ in $\mathrm{Ab}(X)$ then,

(a) $\ker \varphi \subset \mathscr{F}$ and $\operatorname{im} \varphi \subset \mathscr{G}$ are categorical kernel and image respectively. Explicitly, this means that if $\psi : \mathscr{H} \to \mathscr{G}$ composes with φ to give zero then there exists a unique $\mathscr{H} \to \ker \varphi \to \mathscr{F}$,

$$
\begin{array}{ccc}
 & 0 & \\
\mathscr{H} \xrightarrow{\ \psi\ } & \mathscr{F} \xrightarrow{\ \varphi\ } & \mathscr{G} \\
 & \uparrow & \\
 & \ker \varphi &
\end{array}
$$

and likewise,

$$
\mathscr{F} \xrightarrow{\ \varphi\ } \mathscr{G} \xrightarrow{\ \alpha\ } \mathscr{H}
$$
$$
\underset{0}{\phantom{\mathscr{F}}}
$$

if and only if $\alpha|_{\operatorname{im}\alpha} = 0$. (IS THIS RIGHT??)

(b) $\operatorname{im} \varphi = \ker (\mathscr{G} \to \operatorname{coker} \varphi)$ as subsheaves of $\mathscr{G}$

(c) $\varphi : \mathscr{F} \to \mathscr{G}$ factors through a subsheaf $\mathscr{G}' \subset \mathscr{G}$ iff $\operatorname{im} \varphi \subset \mathscr{G}'$ as subsheaves meaning that $\operatorname{im} \varphi$ is the smallest subsheaf through which φ factors.

Proof. For (c) just check on stalks since for subsheaves,

$$
\mathscr{H}_1, \mathscr{H}_2 \subset \mathscr{H}
$$

and $\mathscr{H}$ is a sheaf then $\mathscr{H}_1 \subset \mathscr{H}_2 \iff (\mathscr{H}_1)_x \subset (\mathscr{H}_2)_x$ inside $\mathscr{H}_x$ for all $x \in X$.

Likewise for (b) also you can compare stalks inside $\mathscr{G}_x$ because these are both subsheaves so its enough to check that they have the same stalks inside $\mathscr{G}_x$.

For (a), it is easy to check the kernel property on U sections since there is no sheafification involved. For the image, you use the fact that a map is zero iff it is zero on stalks so we again reduce to the case of abelian groups. $\qquad\square$

Definition 11.0.7. Say $\varphi : \mathscr{F} \to \mathscr{G}$ in $\mathbf{Ab}(X)$ is *surjective* if $\operatorname{im} \varphi = \mathscr{G}$ or equivalently $\forall x \in X :$ $\operatorname{im} \varphi_x = (\operatorname{im} \varphi)_x = \mathscr{G}_x$ iff $\forall x \in X : \varphi_x$ is surjective.

Definition 11.0.8. We say the $\varphi : \mathscr{F} \to \mathscr{G}$ in $\mathbf{Ab}(X)$ is *injective* if $\ker \varphi = 0$. This is equivalent to $\ker \varphi_x = (\ker \varphi)_x = 0$ if and only if φ_x is injective $\forall x \in X$.

Proposition 11.0.9. φ is injective and surjective iff φ_x is an isomorphism for all $x \in X$ iff φ is an isomorphism.

Remark. Notice that $\mathscr{G} \twoheadrightarrow \operatorname{coker} \varphi$ is always surjective and $\ker \varphi \hookrightarrow \mathscr{F}$ is always injective.

Proposition 11.0.10. Given a map of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$,

(a) φ is injective iff $\mathscr{F} \xrightarrow{\sim} \operatorname{im} \varphi$ is an isomorphism

(b) φ is surjective iff $\operatorname{im} \varphi \xrightarrow{\sim} \mathscr{G}$ is an isomorphism.

Definition 11.0.11. A complex in $\mathbf{Ab}(X)$,

$$
\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\ \varphi^{i-1}\ } \mathscr{F}^{i} \xrightarrow{\ \varphi^{i}\ } \mathscr{F}^{i+1} \longrightarrow \cdots
$$

is *exact* if $\operatorname{im} \varphi^{i-1} = \ker \varphi^{i}$ as subsheaves of $\mathscr{F}^{i}$.

Proposition 11.0.12. A sequence is exact iff it is exact of stalks for all $x \in X$.

25

12 Oct. 15

12.1 Some Loose Ends on Exactness

Example 12.1.1. The exponential sequences.

(a) for $\mathbb{R}$-manifolds,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\text{exp}} \mathcal{O}_X^\times \xrightarrow{\text{sign}} \underline{\{\pm 1\}} \longrightarrow 0$$

The long exact sequence shows that the first Stifel-Whitney class is an isomorphism,

$$\text{Pic}\,(X) = H^1(X, \mathcal{O}_X^\times) \xrightarrow{\sim} H^1(X, \mathbb{Z}/2\mathbb{Z})$$

(recall that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ because $\mathcal{O}_X$ is soft).

(b) for $\mathbb{C}$-manifolds,

$$0 \longrightarrow \underline{2\pi i \mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\text{exp}} \mathcal{O}_X^\times \longrightarrow 0$$

Example 12.1.2. For a subsheaf $j : \mathscr{F} \hookrightarrow \mathscr{F}$ write $\mathscr{G}/\mathscr{H} = \text{coker}\,j$ so $\mathscr{G} \twoheadrightarrow \mathscr{G}/\mathscr{F}$ induces $\mathscr{G}_x \twoheadrightarrow (\mathscr{G}/\mathscr{F})_x$ with kernel $\mathscr{F}_x$ and thus $(\mathscr{G}/\mathscr{F})_x = \mathscr{G}_x/\mathscr{F}_x$.

Example 12.1.3. Closed submanifolds of X relate to quotients of $\mathcal{O}_X$ by (certain) "ideal sheaves" (subsheaves of $\mathcal{O}_X$ which are ideals on sections or equivalently ideals on stalks).

Say $\iota : Y \hookrightarrow X$ is a closed C^∞-submanifold. Then topologically, ι is a closed embedding. Furthermore, on the sheaves of rings $\iota^\# : \mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Y$ is surjective because locally charts are subvectorspaces of the ambient chart (using the Immersion Theorem).

Exercise 12.1.4. Check that any topological closed embedding with $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Y$ surjective then ι is a closed submanifold.

Use the fact that if $\iota^\# : \mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Y$ is surjective then $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{Y,y}$ is surjective where $x = \iota(y)$ and local and thus the map $\mathfrak{m}_x/\mathfrak{m}_x^2 \twoheadrightarrow \mathfrak{m}_y/\mathfrak{m}_y^2$ is surjective (because $\iota_x^{-1}(\mathfrak{m}_y) = \mathfrak{m}_x$) meaning that its dual $T_y \to T_x$ is injective proving that ι is an immersion. Furthermore, since ι is a topological closed embedding this shows that ι makes Y a (closed) embedded submanifold.

12.2 Sheaf Pullback

For $h : M \to N$ of C^∞-manifolds, for open $V \subset N$ and $\omega \in \Omega_N^k(V)$ get $h^*\omega \in \Omega_M^k(h^{-1}V)$ giving a pullback map $h^* : \Omega_N^k \to h_* \Omega_M^k$.

Let $f : X \to Y$ be continuous then we defined the functor. $f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$. Will will now define the functor $f^{-1} : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$. Our goal is to convert $\mathscr{F} \to f_* \mathscr{G}$ to a map $f^{-1}\mathscr{F} \to \mathscr{G}$.

Loose analogy: $\alpha : A \to B$ a ring map. Then $\mathfrak{p} \mapsto \alpha^{-1}(\mathfrak{p})$ gives $\tilde{\alpha} : \text{Spec}\,(B) \to \text{Spec}\,(A)$. There is

a forgetful map $\tilde{\alpha}_* : \mathbf{Mod}_B \to \mathbf{Mod}_A$ sending $M \mapsto_A M$. There is a functor going the other way $N \mapsto B \otimes_A N$. Furthermore,

$$\mathrm{Hom}\,(A, N)\,_A M = \mathrm{Hom}\,(B, B \otimes_A N)\, M$$

This is supposed to be the analogue of,

$$\mathrm{Hom}\,(Y, \mathscr{F})\, f_* \mathscr{G} = \mathrm{Hom}\,\left(X, f^{-1}\mathscr{F}\right) \mathscr{G}$$

Definition 12.2.1. Let $f : X \to Y$ be a continuous map. For any presheaf of sets $\mathscr{F}$ on Y define,

$$f_{\mathrm{pre}}^{-1}\mathscr{F} : U \mapsto \varinjlim_{U \subset f^{-1}(V)} \mathscr{F}(V)$$

For a sheaf of sets $\mathscr{F}$ on X then we define $f^{-1}\mathscr{F}$ to be the sheafification of $f_{\mathrm{pre}}^{-1}\mathscr{F}$. For $\mathscr{F} \in \mathbf{Ab}(Y)$ then $f^{-1}\mathscr{F} \in \mathbf{Ab}(X)$.

Proposition 12.2.2. $f^{-1} : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$ is a functor (also for $f^{-1} : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$).

Proof. Given $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ on Y then we get,

$$
\begin{array}{ccc}
f_{\mathrm{pre}}^{-1}\mathscr{F}_1 & \longrightarrow & f_{\mathrm{pre}}^{-1}\mathscr{F}_2 \\
\downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_2} \\
f^{-1}\mathscr{F}_1 & \dashrightarrow & f^{-1}\mathscr{F}_2
\end{array}
$$

using universal property of θ_1 there is a unique $f^{-1}\mathscr{F}_1 \to f^{-1}\mathscr{F}_2$ making the diagram commute. $\qquad \square$

Example 12.2.3. For $X = \{y\} \hookrightarrow Y$ then $\iota^{-1}\mathscr{F} = \mathscr{F}_y$ (as a sheaf on the one point space).
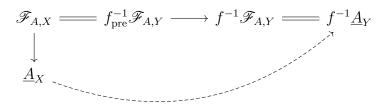
Theorem 12.2.4. For $f : X \to Y$ continuous.

(a) For $x \in X$ then $(f^{-1}\mathscr{F})_x = \mathscr{F}_{f(x)}$ naturally in $\mathscr{F}$.

(b) For $\theta : \mathscr{F} \to \mathscr{F}^+$, then $f^{-1}\theta : f_{\mathrm{pre}}^{-1} \to f^{-1}\mathscr{F}^+$ and therefore $(f_{\mathrm{pre}}^{-1}\mathscr{F})^+ \xrightarrow{\sim} f^{-1}\mathscr{F}^+$ which we could write as saying $f^{-1}\theta : f^{-1}\mathscr{F} \xrightarrow{\sim} f^{-1}\mathscr{F}^+$ (where $f^{-1}\mathscr{F}$ is the sheafification of $f_{\mathrm{pre}}^{-1}\mathscr{F}$).

(c) For $X' \xrightarrow{g} X \xrightarrow{f} Y$ there exists a natural isomorphism,

$$\alpha_{f,g} : g^{-1}(f^{-1}\mathscr{F}) \cong (fg)^{-1}\mathscr{F}$$

which is associative (analogue for modules $B' \otimes_B (B \otimes_A M) \cong B' \otimes_A M$).

Example 12.2.5. For $f : X \to Y$ continuous and $\underline{A}_X = (\mathscr{F}_{A,X})^+$ the constant sheaf on X. We claim that naturally $f^{-1}\underline{A}_Y \xrightarrow{\sim} \underline{A}_X$ (respecting composition in f). This is because,

$$
\begin{array}{ccccccc}
\mathscr{F}_{A,X} & = & f_{\mathrm{pre}}^{-1}\mathscr{F}_{A,Y} & \longrightarrow & f^{-1}\mathscr{F}_{A,Y} & = & f^{-1}\underline{A}_Y \\
\downarrow & & & & & & \\
\underline{A}_X & & & & & &
\end{array}
$$

On x-stalks, check that,

$$
\begin{array}{ccc}
(\underline{A}_X)_X & \longrightarrow & (f^{-1}\underline{A}_Y)_x \\
\| & & \| \\
A & \xrightarrow{\ \sim\ } & (\underline{A}_Y)_{f(x)}
\end{array}
$$

commutes.

Now we prove the theorem.

Proof. For (a) want $\mathscr{F}_{f(x)} \to (f^{-1}\mathscr{F})_x$ and show isomorphism. Now,

$$
\mathscr{F}_{f(x)} = \varinjlim_{V \ni f(x)} \mathscr{F}(Y)
$$

and,

$$
(f^{-1}\mathscr{F})_x = (f_{\mathrm{pre}}^{-1}\mathscr{F})_x = \varinjlim_{U \ni x} f_{\mathrm{pre}}^{-1}(U) = \varinjlim_{U \ni x} \left(\varinjlim_{W \supset f(U)} \mathscr{F}(W) \right)
$$

Notice that all W contain $f(x)$. Each V contains $f(f^{-1}(V))$ so,

$$
\mathscr{F}(V) \to \mathscr{F}(V) \to \varinjlim_{W \supset f(f^{-1}(V))} \mathscr{F}(W) \to (f^{-1}\mathscr{F})_x
$$

because these maps respect inclusion we get,

$$
\mathscr{F}_{f(x)} \to (f^{-1}\mathscr{F})_x
$$

passing to the limit. In reverse,

$$
(f^{-1}\mathscr{F})_x = (f_{\mathrm{pre}}^{-1}\mathscr{F})_x = \varinjlim_{W \ni x} (f_{\mathrm{pre}}^{-1}\mathscr{F})(W) = \varinjlim_{W \ni x} \varinjlim_{V \supset f(W)} \mathscr{F}(V) \to \varinjlim_{V \ni f(x)} \mathscr{F}(V) = \mathscr{F}_{f(x)}
$$

Check these are inverse to each other.

For (b) consider $\theta : \mathscr{F} \to \mathscr{F}^+$ then we get,

$$
\begin{array}{ccc}
f_{\mathrm{pre}}^{-1}\mathscr{F} & \xrightarrow{f_{\mathrm{pre}}^{-1}\theta} & f_{\mathrm{pre}}^{-1}(F^+) \\
\downarrow & & \downarrow \\
f^{-1}\mathscr{F} & \xrightarrow{f^{-1}\theta} & f^{-1}(\mathscr{F}^+)
\end{array}
$$

then this diagram shows that on stalks $f^{-1}\theta$ induces an isomorphism on the stalks.

For (c) build $g^{-1}f^{-1}\mathscr{F} \xrightarrow{\alpha} (fg)^{-1}\mathscr{F}$ using compatibility of g^{-1} and f^{-1} with sheafification (using (b)) and check that $\alpha_{x'}$ respects the identification of both x'-stalks with $\mathscr{F}_{f(g(x'))}$ and thus is an isomorphism. $\qquad\square$

13 Oct. 18

13.1 More sheaf pullback and manifolds without charts

Two instances of sheaf pullback:

$$f : X \to Y \text{ continuous gives } f^{-1}\mathscr{F} = (f_{\text{pre}}^{-1}\mathscr{F})^+$$

(a) for open subspace $j : U \to X$ and $\mathscr{F} \in \mathbf{Shv}(X)$ we get,

$$\mathscr{F}|_U \cong j^{-1}\mathscr{F}$$

and we don't even need to sheafify.

(b) For closed $\iota : Z \hookrightarrow X$ if $\mathscr{G} \in Ab(X)$ then,

$$(\iota_*\mathscr{G})_x = \begin{cases} \mathscr{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

where the second follows from the fact that $X \setminus Z$ is *open* this does not work for a general subset. For $\mathscr{F} \in \mathbf{Ab}(X)$ then $\mathscr{F}|_Z := \iota^{-1}\mathscr{F} \in \mathbf{Ab}(Z)$. If $\mathscr{G} \in \mathbf{Ab}(Z)$ then,

$$\iota^{-1}(\iota_*\mathscr{G}) \cong \mathscr{G}$$

Build dmap $\iota^{-1}(\iota_*\mathscr{G}) \to \mathscr{G}$ using the universal property of sheafification. For open $U \subset Z$ and open $V \subset X$ with $V \cap Z = U$ then $(\iota_*\mathscr{G})(V) = \mathscr{G}(U)$ which we use to build the map and then check on stalks that it is an isomorphism.

Corollary 13.1.1. For a closed $\iota : Z \hookrightarrow X$ then,

$$\mathbf{Ab}(Z) \cong \{\mathscr{F} \in \mathbf{Ab}(X) \mid \iota^{-1}\mathscr{F} = 0\}$$

where the map is given by $\mathscr{G} \mapsto \iota_*\mathscr{G}$ whose inverse is $\mathscr{F} \mapsto \iota^{-1}\mathscr{F}$.

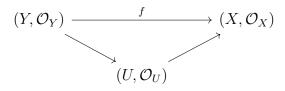**Proposition 13.1.2.** For $f : X \to Y$ continuous then,

$$\text{Hom}\left(X, f^{-1}\mathscr{F}\right)\mathscr{G} = \text{Hom}\left(Y, \mathscr{F}\right)\iota_*\mathscr{G}$$

naturally (as sheaves of sets, then also as abelian sheaves).

Proof. In Homework 4. $\square$

13.2 Universal property of open immersion

Let $(X, \mathcal{O}_X)$ be a $\mathcal{C}$-ringed space, $U \subset X$ an open set. Then $\mathcal{O}_U = \mathcal{O}_X|_U$ makes $(U, \mathcal{O}_U)$ a $\mathcal{C}$-ringed space and have $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ and if $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a map of such with $f(Y) \subset U$ (as sets) then it factors,

because $f^{\#} : \mathcal{O}_X \to f_* \mathcal{O}_Y$ factors.

Definition 13.2.1. Say $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is an open immersion if $f(Y) \subset X$ is open and $(Y, \mathcal{O}_Y) \xrightarrow{\sim} (f(Y), \mathcal{O}_{f(Y)})$.

Remark. I think its better to say that f is a homeomorphism onto an open subspace and the adjoint sheaf map $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is an isomorphism but okay its the same.

Remark. Beware: an open immersion is an open subspace *with the correct sheaf of rings*. For example $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k[t]/(t^2))$ is *not* an open immersion. Similarly, the map $\mathbb{A}^1_{\mathbb{C}} \twoheadrightarrow \{x^3 = y^2\} \subset \mathbb{A}^2_{\mathbb{C}}$ sending $t \mapsto (t^2, t^3)$ given by the pullback map $\mathbb{C}[x, y]/(x^3 - y^2) \to \mathbb{C}[t]$ sending $x \mapsto t^3$ and $y \mapsto t^2$ is *not* an open immersion. Away from the cusp it is an isomorphism but globally it has the wrong sheaf of rings even though it is a homeomorphism.

Definition 13.2.2. A *complex manifold* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ with open cover $\{U_i\}$ where each U_α is isomorphic (as a $\mathbb{C}$-ringed space) to an open subspace of a finite dimensional $\mathbb{C}$-vectorspace V equipped with the sheaf of holomorphic functions.

Remark. These are automatically locally ringed and we saw that maps of $\mathbb{C}$-algebras whose residue fields are all $\mathbb{C}$ are automatically local.

Remark. Convince yourself that this is equivalent to the usual definition (up to topological conditions on the underlying space).

13.3   Gluing of Ringed Spaces

Let $(X_i, \mathcal{O}_{X_i})$ be (locally) $\mathcal{C}$-ringed spaces. Consider the idea of pasting together these spaces along identified open overlaps such that the identifications satisfy some coherence. Given opens $X_{ij} \subset X_i$ for each pair i, j with isomorphisms of (locally) $\mathcal{C}$-ringed spaces,

$$\varphi_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$$

such that,

(a) $X_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}$

(b) $\varphi_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$ for all i, j, k

(c) $\varphi_{jk}|_{X_{ji} \cap X_{jk}} \circ \varphi_{ij}|_{X_{ik} \cap X_{ij}} = \varphi_{ik}|_{X_{ik} \cap X_{ij}}$

Proposition 13.3.1. Given the above setup, there exists a unique (up to unique isomorphism) ringed space X equipped with an open cover $\{U_\alpha\}$ and isomorphisms $\varphi_i : X_i \xrightarrow{\sim} U_i$ such that

(a) $\varphi_i(X_{ij}) = U_i \cap U_j$

(b) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_J} \circ \varphi_i|_{X_{ij}}$

which together have the mapping property: for any (locally) $\mathcal{C}$-ringed space Y,

$$\mathrm{Hom}\,(\mathcal{C}, X)\,Y \xrightarrow{\sim} \left\{ (f_i) \in \prod_i \mathrm{Hom}\,(\mathcal{C}, X_i)\,Y \;\middle|\; f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ \varphi_{ij} \right\}$$

given by sending $f \mapsto (f|_{U_i} \circ \varphi_i)$ is an isomorphism.

Proof. Details in the handout. For topological space, use,

$$X = \left(\coprod X_i \right) / \sim$$

where $x_{ij} \sim \varphi_{ij}(x_{ij})$ for $x_{ij} \in X_{ij} \subset X_i$ and $\varphi_{ij}(x_{[ij}) \in X_j$ with the quotient topology. The cocycle condition gives transitivity of this equivalence relation. Furthermore, we set,

$$\mathcal{O}_X(U_i) := \left\{ (s_i) \in \prod_i \mathcal{O}_{X_i}(U \cap X_i) \,\middle|\, s_i|_{(U \cap X_i) \cap X_j} = s_j|_{(U \cap X_j) \cap X_i} \right\}$$

where for the equality we should stick in some φ_{ij}. $\qquad\square$

14 Oct. 20 Schemes

Let A be a commutative ring. Let $X = \mathrm{Spec}\,(A)$ with the topology whose closed sets are $V(I) = \{ \mathfrak{p} \in \mathrm{Spec}\,(A) \mid \mathfrak{p} \supset I \}$ for ideals $I \subset A$. Notice that many ideals give the same closed subset (e.g. $V(I) = V(\sqrt{I})$). For radical J,

$$J = \bigcap_{\mathfrak{p} \supset J} \mathfrak{p}$$

then A/J is a reduced ring. Therefore, closed sets are in bijective correspondence with radical ideals. This property has some interesting features:

(a) $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ so the closed points are exactly the maximal ideals

(b) as topological spaces,
$$\mathrm{Spec}\,(A/I) = V(I)$$
as *topological spaces*.

(c) $\mathrm{Spec}\,(A_a) = D(a) = X \setminus V(a) = \{ \mathfrak{p} \in \mathrm{Spec}\,(A) \mid a \notin \mathfrak{p} \}$ is also a homeomorphism (call these basic affine opens)

Proposition 14.0.1. The $D(a)$ are a basis of opens in X.

Proof. Let $x \in U = X \setminus V(I)$. Then $x = \mathfrak{p}$ for some prime ideal not containing I. Thus there is some $a \in I$ such that $a \notin \mathfrak{p}$ and thus $\mathfrak{p} \in D(a) \subset U$ because if $a \notin \mathfrak{q}$ then $\mathfrak{q} \not\supset I$ and thus $\mathfrak{q} \in U$. $\qquad\square$

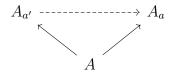*Remark.* $D(a) \cap D(a') = D(aa')$. Call such $D(a)$ basic affine opens.

Remark. Warning: $D(a)$ does not determine a or even the set $\{a, a^2, a^3, \dots\}$.

14.1 Goal

We want to build a sheaf of rings $\mathcal{O}_A$ on X with $\mathcal{O}_A(D(a)) = A_a$ naturally.

(a) need to specify restriction maps for $D(a') \subset D(a)$

(b) handle general open $U \subset X$

(c) make sure $\mathcal{O}_A(D(a))$ only depends on subset $D(a) \subset X$, not $a \in A$.

If $D(a) \subset D(a')$ then a' is not in every prime of A_a and thus a' is a unit in A_a meaning that by the universal property of localization,
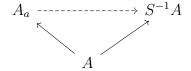
$$A_{a'} \dashrightarrow A_a$$
$$A$$

This map is uniquely determined so we choose it a the restriction map $\mathrm{res}_{D(a),D(a')}$.

Proposition 14.1.1. As A-algebras, there is a unique isomorphism $A_a = S^{-1}A$ for the set

$$S = \{a' \in A \mid \forall \mathfrak{p} \in D(a) : a' \notin \mathfrak{p}\}$$

By definition, S only depends on A and $D(a)$ not on the element a.

Proof. Clearly a is a unit in $S^{-1}A$ so we get a diagram,

$$A_a \dashrightarrow S^{-1}A$$
$$A$$

Because maps over A are unique by the universal property it suffices to produce a map the other way (because then they compose to be the unique automorphism over A which is the identity). By the universal property, a map the other direction simply follows from $S \subset (A_a)^\times$ and $\mathrm{Spec}\,(A_a) = D(a)$ so every $s \in S$ is not contained in any prime in $D(a)$ so not contained in any prime of $\mathrm{Spec}\,(A_a)$ and thus s is a unit in A_a. $\square$

Remark. Because S only depends on the open set $D(a)$ and not on the choice of presenting element $a \in A$ this means that A_a depends up to isomorphism only on the open set $D(a)$ and therefore $\mathcal{O}_A(D(a)) = S^{-1}A \cong A_a$ is well-defined and restriction maps from the universal property. This forms a sheaf of A-algebras. The A-algebra structure is essential to make the restriction maps uniquely defined.

Example 14.1.2. $\mathcal{O}_A(\varnothing) = (0)$ and $\mathcal{O}_A(X) = A$.

Remark. This is not yet a sheaf because we have only defined it on principal affine opens. However, let's first check that it satisfies the sheaf condition when it is defined.

Proposition 14.1.3. If $\{U_i = D(a_i)\}$ form an open cover of $U = D(a)$ then unique gluing holds in the sense that, given $s_i \in A_{a_i} = \mathcal{O}_A(D(a_i))$ such that,

$$A_{a_i} \to A_{a_i a_j} \leftarrow A_{a_j}$$

agree meaning s_i and s_j map to the same element. Then there is a unique $s \in A_a$ so that $s \mapsto s_i$ under $A_a \to A_{a_i}$ for each i.

Proof. First we can set $B = A_a$ and $A_{a_i} = B_{b_i}$ where $b_i = \frac{a_i}{1} \in B$ since $U_i \subset U$ this means that a is a unit in A_{a_i} so you can factor $A \to A_a \to A_{a_i}$. and $A_a \to A_{a_i}$ is a localization. Thus we can replace A by B and forget about the first localization. First we show uniqueness. If $s, t \in A$ agree in all A_{a_i} then want $s = t$. For each $\mathfrak{p}$ we have $A_\mathfrak{p} = (A_{a_i})_{\mathfrak{p}_{a_i}}$ as A-algebras so $s = t$ in $A_\mathfrak{p}$ at each prime and therefore $s = t$. Now for gluing. Given $s_i \in A_{a_i}$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j

seek $s \in A$ so that $s \mapsto s_i$ in A_{a_i} for each i. However, X is quasi-compact, so some finite subcover $\{U_{a_i}, \dots, U_{a_n}\}$ covers and by uniqueness a solution for the finite subcover will work for the whole cover because for each element of the cover the element is locally (in the overlaps with the finite cover) the correct element and thus correct by the uniqueness.

Because $\{U_1, \dots, U_n\}$ covers X is equivalent to $(a_1, \dots, a_n) = (1)$. However, we solved this finite gluing problem earlier when showing that $\mathcal{O}_Z$ is a sheaf. $\qquad\square$

Lemma 14.1.4. Let X be a topological space, B a base of open sets[1], and $\mathscr{F}$ is a B-sheaf then there is a unique (up to unique isomorphism) sheaf $\mathscr{F}'$ on X extending $\mathscr{F}$. This satisfies the mapping property, for sheaves $\mathscr{G}$ on X,

$$\text{Hom}\,(B, \mathscr{F})\,\mathscr{G}|_B = \text{Hom}\,(X, \mathscr{F}')\,\mathscr{G}$$

Proof. For details [3.2.1-3.2.5 in EGA O_I]

$$\mathscr{F}'(U) = \varprojlim_{V \subset U} \mathscr{F}(V) = \left\{ (s_V) \in \prod_{V \subset U} \mathscr{F}(V) \,\middle|\, s_V|_{V \cap V'} = s_{V'}|_{V \cap V'} \right\}$$

$\qquad\square$

Remark. Now we have a sheaf $\mathcal{O}_A$ so we write $\text{Spec}\,(A)$ to mean $(\text{Spec}\,(A), \mathcal{O}_A)$. We want this to be functorial in A, enhancing the topological map meaning that,

$$\varphi : A \to B \rightsquigarrow \tilde{\varphi} : \text{Spec}\,(B) \to \text{Spec}\,(A)$$

where $\tilde{\varphi}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ which is continuous because $\tilde{\varphi}^{-1}(D(a)) = D(\varphi(a))$. However, we want a map of sheaves of rings,

$$\mathcal{O}_A \to \tilde{\varphi}_* \mathcal{O}_B$$

However, we have $A_a \to B_{\varphi(a)}$ and thus,

$$\mathcal{O}_A(D(a)) \to \mathcal{O}_B(D(\varphi(a))) = (\tilde{\varphi}_*(\mathcal{O}_B))(D(a))$$

Lemma 14.1.5. Let $f : X \to Y$ be continuous and bases of opens B' for X and B for Y such that $f^{-1}(U) \in B'$ for $U \in B$ then for B'-sheaf $\mathscr{F}$ on X' then,

$$f_*(\mathscr{F}') = (f_*\mathscr{F})'$$

Corollary 14.1.6. There is a functor $\textbf{Ring} \to \textbf{RingSp}$ via $A \mapsto \text{Spec}\,(A)$ satisfying some properties,

  (a) $\text{Spec}\,(A)$ is quasi-compact and $\text{Spec}\,(A_a)$ gives a basis of ringed spaces because,

$$\mathcal{O}_A|_{D(a)} = \mathcal{O}_{A_a}$$

  (b) $\mathcal{O}_{A,\mathfrak{p}} = \varinjlim_{D(a) \ni x} \mathcal{O}_A(D(a)) = \varinjlim_{a \notin \mathfrak{p}} A_a = A_\mathfrak{p}$ so $\text{Spec}\,(A)$ is a locally ringed space.

Definition 14.1.7. A scheme is a ringed space X covered by *affine* schemes as open subspaces. Thus X is locally ringed.

Remark. Open subspaces of schemes are schemes because $\text{Spec}\,(A)$ has a base of $\text{Spec}\,(A_a)$.

---

[1]Base means closed under finite overlaps here

15 Oct. 22

15.1 Some basic properties of schemes and morphisms

First we start we a few remarks:

(a) A scheme X is q-c iff X is covered by finitely many affine opens.

(b) On schemes, usually for $x \neq y$ there are not isomorphic neighborhoods of x and y (even for smooth varieties and classical closed points).

Last time: given $\varphi : A \to B$ we made $(f, f^{\#}) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ where $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Furthermore, $f^{-1}(D(a)) = D(\varphi(a))$ and $f^{\#} : \mathcal{O}_A(D(a)) \to \mathcal{O}_B(D(\varphi(a)))$ is $\varphi_a : A_a \to B_{\varphi(a)}$.

Proposition 15.1.1. For $y = \{\mathfrak{p}\} \in \operatorname{Spec}(B)$ then $x = f(y) = \{\varphi^{-1}(\mathfrak{p})\} \in \operatorname{Spec}(A)$,

$$
\begin{array}{ccccc}
\mathcal{O}_{A,x} & \xrightarrow{f_x^{\#}} & (f_*\mathcal{O}_B)_x & \longrightarrow & \mathcal{O}_{B,y} \\
\| & & & & \| \\
A_{\mathfrak{p}} & & \longrightarrow & & A_{\varphi^{-1}(\mathfrak{p})}
\end{array}
$$

commutes and is a *local* map.

Proof. Consider,

$$
\begin{array}{ccccccc}
A & = & \Gamma(X, \mathcal{O}_X) & \xrightarrow{f^{\#}} & \Gamma(Y, \mathcal{O}_Y) & = & B \\
& \text{can} \searrow & \downarrow & & \downarrow & \swarrow \text{can} & \\
& & A_{\varphi^{-1}(\mathfrak{p})} & \xrightarrow{f_y^{\#}} & B_{\mathfrak{p}} & &
\end{array}
$$

however, there is a unique map factoring $A \to B_{\mathfrak{p}}$ through $A \to A_{\varphi^{-1}(\mathfrak{p})}$ which is local because it satisfies $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ on pullbacks. □

Definition 15.1.2. A *morphism* of schemes $X \to Y$ is a map as locally ringed spaces.

Theorem 15.1.3. The map,

$$
\operatorname{Hom}(\mathbf{Ring}, A) B \to \operatorname{Hom}(\mathbf{LRS}, \operatorname{Spec}(B)) \operatorname{Spec}(A)
$$

is bijective.

Proof. Given $\varphi : A \to B$ for $f = \operatorname{Spec}(\varphi) : Y \to X$ we've seen that $\Gamma(X, f^{\#}) : A \to B$ is φ so the map is injective.

Let $Y = \operatorname{Spec}(B)$ and $X = \operatorname{Spec}(A)$. Let $f : Y \to X$ be a map of locally ringed spaces. We get a ring map $\varphi : \Gamma(X, f^{\#}) : A \to B$ because $\mathcal{O}_X(X) = A$ and $(f_*\mathcal{O}_Y)(X) = B$. Then we get $g = \operatorname{Spec}(\varphi) : Y \to X$. First we show that f and g agree topologically. For $y \in \{\mathfrak{p}\} \in Y$ we know that $g(y) = \varphi^{-1}(\mathfrak{p})$. Observe,

$$
\begin{array}{ccccccc}
A & = & \Gamma(X, \mathcal{O}_X) & \xrightarrow{f^{\#}} & \Gamma(Y, \mathcal{O}_Y) & = & B \\
\text{can} \downarrow & & \downarrow & & \downarrow & & \downarrow \text{can} \\
A_{f(y)} & = & \mathcal{O}_{X,f(y)} & \xrightarrow{f_y^{\#}} & \mathcal{O}_{Y,y} & = & B_{\mathfrak{p}}
\end{array}
$$

34

However, since $f_y^\#$ is local, the maximal ideal of $A_{f(y)}$ is $(f_y^\#)^{-1}(\mathfrak{p}B_\mathfrak{p})$ and the preimage of this maximal ideal in A is $f(y)$. However, going around the top we get $\mathfrak{p}B_\mathfrak{p}$ pulls back to $\varphi^{-1}(\mathfrak{p}) = g(y)$ so $f(y) = g(y)$.

Now, to check that $f^\#, g^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ (which are now maps between the same sheaves because the topological maps are the same) are the same it suffices to check on the base $D(a)$. Then,

$$
\begin{array}{ccccccc}
A & =\!=\!= & \Gamma(X, \mathcal{O}_X) & \xrightarrow{\;f^\#\;} & \Gamma(Y, \mathcal{O}_Y) & =\!=\!= & B \\
{\scriptstyle\text{can}}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\text{can}} \\
A_a & =\!=\!= & \mathcal{O}_X(D(a)) & \underset{g^\#}{\overset{f^\#}{\rightrightarrows}} & \mathcal{O}_Y(f^{-1}(D(a))) & =\!=\!= & B_{\varphi(a)}
\end{array}
$$

so both $f^\#$ and $g^\#$ both must equal φ_a because they factor $A \to B \to B_{\varphi(a)}$ though $A \to A_a$ which is unique by the universal property when it exists. $\qquad\square$

Remark. For <u>any</u> LRS Y,

$$\operatorname{Hom}(\mathbf{LRS}, Y)\operatorname{Spec}(A) \to \operatorname{Hom}(\mathbf{Ring}, A)\,\mathcal{O}_Y(Y)$$

sending $f \mapsto \Gamma(f)$ is bijective. This is an observation of Tate.

Remark. In general, its very hard to characterize in terms of $\varphi : A \to B$ when $\operatorname{Spec}(\varphi)$ is an open immersion (except when its a principal open i.e. when $A \to B$ is the localization at an element).
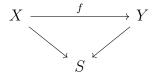
Example 15.1.4. Some schemes:

(a) $\mathbb{A}_R^n = \operatorname{Spec}(R[t_1, \ldots, t_n]) \to \operatorname{Spec}(R)$ has the universal property: for a scheme Y,

$$\operatorname{Hom}(, Y)\,\mathbb{A}_R^n = \operatorname{Hom}(, Y)\operatorname{Spec}(R) \times \{g_1, \ldots, g_n \in \mathcal{O}_Y\}$$

where $g_i = f^\#(t_i)$. In the category of R-schemes (schemes with a map to $\operatorname{Spec}(R)$) then,

$$\operatorname{Hom}(R, Y)\,\mathbb{A}_R^n = \mathcal{O}_Y(Y)^n$$

Definition 15.1.5. For a scheme S, an S-scheme is a morphism $X \to S$. A morphism of S-schemes X, Y is a scheme map $f : X \to Y$ such that,

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & Y \\
& \searrow \quad \swarrow & \\
& S &
\end{array}
$$

For $S = \operatorname{Spec}(A)$ we call these A-schemes (generalization of A-algebras).

Remark. The classical setting is equivalent to studying k-schemes where k is algebraically closed.

Example 15.1.6. $\operatorname{Spec}(\mathbb{C})$ as a $\mathbb{C}$-scheme is "just a point" but $\operatorname{Spec}(\mathbb{C})$ as a $\mathbb{Z}$-scheme is "huge" it covers $\operatorname{Spec}(\mathbb{Z}[\pi, e])$ which is a big scheme.

16   Oct. 25

Let $U_i = \mathrm{Spec}\,(R_i) \subset X$ be an open cover of a scheme. For $x \in U_1 \cap U_2$ wee a basis of open $V \subset U_1 \cap U_2$ and x such that V is base of affine in U_1 and U_2.

Remark. $U_1 \cap U_2$ may fail tor be qc:

$$U = \mathbb{A}_k^\infty \setminus \{0\} = \mathrm{Spec}\,(k[x_1, x_2, \dots]) \setminus \{0\} = \bigcup_{i=1}^\infty D(x_i)$$

Let X be the gluing of $\mathbb{A}_k^\infty$ along $\mathrm{id} : U \to U$ (FINISH)

Proof. Let $U_1 = \mathrm{Spec}\,(R_1)$ and $U_2 = \mathrm{Spec}\,(R_2)$. Choose some principal affine open of X

$$x \in W \subset U_1 \cap U_2$$

then $x \in V_i \subset W$ with $V_1 = \mathrm{Spec}\,((R_1)_{r_1}) \subset W$. Then we can find $x \in \mathrm{Spec}\,((R_2)_{r_2}) = V_2 \subset V_1$. Let's check $V_2 = \{r_2 \neq 0\} \subset U$ is a basic affine open. $\qquad\square$

Theorem 16.0.1. For $f : X \to Y$ the following are equivalent,

(a) there exists an affine open cover $\{\mathrm{Spec}\,(B_i)\}$ of Y such that $f^{-1}(\mathrm{Spec}\,(B_i))$ is covered by affine open $\{\mathrm{Spec}\,(A_{ij})\}$ with A_{ij} a finitely generated B_i-algebra

(b) for every affine open $\mathrm{Spec}\,(B)$ and affine open $\mathrm{Spec}\,(A) \subset f^{-1}(\mathrm{Spec}\,(B))$ then $B \to A$ makes A a finitely generated B-algebra.

Proof. Clearly (b) $\implies$ (a). Now assume (a). Pick affine open $\mathrm{Spec}\,(B) \subset Y$. Then $\mathrm{Spec}\,(B_i) \cap \mathrm{Spec}\,(B)$ is covered by $U_{i\alpha}$ that are basic affine open in $\mathrm{Spec}\,(B_i)$ and $\mathrm{Spec}\,(B)$. Any $\mathrm{Spec}\,((B_i)_{b_i})$ has f-preimage covered by $\mathrm{Spec}\,((A_{ij})_{b_i})$ and $(A_{ij})_{b_i}$ are finite type $(B_i)_{b_i}$-algebras. Thus, each $f^{-1}(U_{i\alpha})$ is covered by, $\mathrm{Spec}\,((A_{ij})_{b_i})$ we know these are also localizations of $B_{b,\alpha}$ where $U_{i\alpha} = \mathrm{Spec}\,(B_{b,\alpha})$. Then $\mathrm{Spec}\,(B)$ is covered by $U_{i\alpha}$ and $B_{b,\alpha}$ are finite type B-algebras. Now we have shown that $f^{-1}(\mathrm{Spec}\,(B))$ is covered by $\mathrm{Spec}\,(R_\gamma)$ with R_γ finitely generated over B.

Now, we can reduce to the case $Y = \mathrm{Spec}\,(B)$. Then consider $\mathrm{Spec}\,(A) \subset X$. Consider $\mathrm{Spec}\,(A) \cap \mathrm{Spec}\,(R_\gamma)$. Any $\mathrm{Spec}\,((R_\gamma)_r)$ also has the same finitness property over B (just include r^{-1} in the generating set). Then by Nike trick, $\mathrm{Spec}\,(A) \cap \mathrm{Spec}\,(R_\gamma)$ is covered by affine opens that are simultaneously basic affine in $\mathrm{Spec}\,(A)$ and $\mathrm{Spec}\,(R_\gamma)$ which are finitely generated B-algebras (since they are principal affine inside $\mathrm{Spec}\,(R_\gamma)$) but also principal affine open in $\mathrm{Spec}\,(A)$. However, $\mathrm{Spec}\,(A)$ is qc so we can take a finite set $\{\mathrm{Spec}\,(A_{f_i})\}$ covering $\mathrm{Spec}\,(A)$ and thus $(f_1, \dots, f_n) = 1$. Since each A_i is finitely generated as a B-algebra.

Thus we reduce to a pure algebra problem. Let $B \to A$ with $f_i \in A$ and $\alpha_1, \dots, \alpha_n \in A$, and $\sum \alpha_i f_i = 1$ with each A_{f_i} is finite type over B. Say A_{f_i} is generated by $\frac{a_{ij}}{f_i^N}$ for common N (by finiteness). Consider $A' = B[f_i, \alpha_i, a_{ij}] \subset A$. By design $f_i \in A'$ generate 1. Furthermore,

$$A'_{f_i} \hookrightarrow A_{f_i}$$

is an equality for each i and $\mathrm{Spec}\,(A')$ is covered by these so each prime $\mathfrak{p}' \subset A'$ does not contain f_i. For any A'-module M, then

$$M_{\mathfrak{p}} = (M_{f_i})_{\mathfrak{p}'_{f_i}}$$

Now $A' \to A$ yields an equality for

$$A'_{\mathfrak{p}'} = (A'_{f_i})_{\mathfrak{p}'_{f_i}} \hookrightarrow (A_{f_i})_{\mathfrak{p}'_{f_i}} = A_{\mathfrak{p}'}$$

so therefore $A' \to A$ is an isomorphism of A'-modules. $\qquad\square$

Definition 16.0.2. We call such maps locally of finite type.

Lemma 16.0.3. For $f : X \to Y$ the following are equivalent:

(a) for all affine open $U \subset Y$ the preimage $f^{-1}(U)$ is pc

(b) For all qc open $U \subset Y$ the preimage $f^{-1}(U)$ is qc

(c) there exists affine open cover $\{U_i\}$ of Y s.t. $f^{-1}(U_i)$ are qc.

Proof. Clearly (b) $\implies$ (a) $\implies$ (c). Therefore we need to show (c) $\implies$ (b). For each $U_i = \mathrm{Spec}\,(B_i)$ also $f^{-1}(\mathrm{Spec}\,((B_i)_{b_i}))$ is qc because $f^{-1}(U_i)$ is covered by finitely many $\mathrm{Spec}\,(A_{ij})$ then $f^{-1}((U_i)_{b_i})$ is covered by $\mathrm{Spec}\,((A_{ij})_{b_i})$ which is quasi-compact. Notice that to prove (b) it suffices to consider affines because any qc open is covered by finitely many affines. Let $U = \mathrm{Spec}\,(B)$ be affine. Then $\mathrm{Spec}\,(B)$ is covered by $\{\mathrm{Spec}\,(B) \cap U_i\}$ is open in U_i hence covered by basis affine opens $U_{i\alpha}$ of U_i. Then finitely many $\{U_{i\alpha}\}$ cover $\mathrm{Spec}\,(B)$ and each $f^{-1}(U_{i\alpha})$ is qc. $\qquad\square$

Definition 16.0.4. We call such maps *quasi-compact*. Say that f is *finite type* if it is qc and locally finite type.

Proposition 16.0.5. For a scheme X the following are equivalent

(a) all $\mathcal{O}_{X,x}$ are reduced

(b) every affine open $\mathrm{Spec}\,(A) \subset X$ has reduced A

(c) there exists an affine open cover $\{\mathrm{Spec}\,(A_i)\}$ of X by reduced A_i

(d) every $\mathcal{O}_X(U)$ is reduced.

Proof. This holds just because $\mathcal{O}_{X,x} = A_{\mathfrak{p}} = (A_i)_{\mathfrak{p}}$ and A is reduced iff $A_{\mathfrak{p}}$ is reduced for all $\mathfrak{p}$. $\qquad\square$

17 Oct. 27

Let k be an algebraically closed field. We want to define $\mathbb{P}^n$ as "classical" projective n-space over k as a gluing of $n+1$ copied of $\mathbb{A}^n$. As a *set*, for V an $(n+1)$-dimensional k-vectorspace,

$$\mathbb{P}(V) := \begin{cases} (V - \{0\})/k^\times = \{\text{lines through the origin in } V\} & \ell = k \cdot (a_0, \ldots, a_n) \\ (V^* - \{0\})/k^\times = \{\text{hyperplanes through the origin in } V\} & \sum_j a_j x_j = 0 \end{cases}$$

We write $[a_0, \ldots, a_n]$ for the equivalence class in $\mathbb{P}^n$ of $(a_0, \ldots, a_n)$. The coordinate "$a_j(p) \in k^n$" is not well-defined because of scaling. However, the condition that "$a_j(p) = 0$" is well-defined. If $a_j(\varphi) \neq 0$ then the ratio $\frac{a_i(p)}{a_j(p)} \in k$ is well-defined.

Definition 17.0.1. Let $U_j = \{a_j \neq 0\} \subset \mathbb{P}^n$. Let $U_j = \mathbb{A}^n$ as sets via,

$$p \mapsto \left(\frac{a_i(p)}{a_j(p)} \right)_{i \neq j}$$

This is called dehomogenization with respect to a_j or x_j. Then $\mathbb{P}^n$ is covered by $U_0, \ldots, U_n$.

Remark. $U_i \cap U_j \subset U_j$ is the condition $\{x_{ij} \neq 0\} \subset \{x_i \neq 0\}$

37

18 Oct. 29

18.1 Proj and More Morphism Properties

Remark. Read handout on irred. components and dimension and codimension for schemes (controlled by primes in $\mathcal{O}_{X,x}$ for $x \in X$).

Example 18.1.1. $\mathbb{N}$-graded rings,

(a) A a commutative ring, $S = A[x_0, \ldots, x_n]$ with $\deg x_i = 1$ so we can write,

$$S = \bigoplus_{d \geq 0} A[x_0, \ldots, x_n]_{(d)}$$

where $A[x_0, \ldots, x_n]_{(d)}$ are the homogeneous polynomials of degree d.

(b) we can also take $S = A[x, y]$ where $\deg x = 2$ and $\deg y = 3$ then the degree decomposition becomes,

$$
\begin{aligned}
S_0 &= A \\
S_1 &= 0 \\
S_2 &= Ax \\
S_3 &= Ay \\
S_4 &= Ax^2 \\
S_5 &= Axy \\
S_6 &= Ax^3 \oplus Ay^2
\end{aligned}
$$

so although the ring is the same, the graded structure is totally different.

Remark. We want to build a scheme $\operatorname{Proj}(S)$. To do so we need three ingredients,

(a) the irrelevant ideal $S_+ = \bigoplus_{m \geq 0} S_m$ (this corresponds to the origin that we are going to throw away).

(b) For any $d > 0$, we can make,

$$S^{(d)} = \bigoplus_{m \geq 0} S_{dm} \subset S$$

where we still give the S_{dm} part degree dm not degree m.

(c) For $f \in S_d$ for $d > 0$ then,

$$S_{(f)} = (S_f)_0$$

where this means the degree zero part of the degree 0 subring of the $\mathbb{Z}$-graded ring S_f (which is graded because f is homogeneous where we say $\deg\left(\frac{a}{f^n}\right) = \deg a - n \deg f$). Notice that $(S^{(d)})_{(f)} = S_{(f)}$ because the only terms have denominator multiples of $\deg f = d$ and thus the numerator in the degree zero part must also.

Example 18.1.2. For $A[x_0, \ldots, x_n]_{(x_i)} = A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ is the coordinate ring of U_i the affine part of projective space.

Remark. An $\mathbb{N}$-graded A-algebra is equipped with $A \to S_0 \subset S$ but this does *not* need to be an equality.

Remark. The most important case is S generated by S_1 over S_0. For example the standard polynomial ring and any of its quotients. Going beyond this is useful but under Noetherian assumptions, often $S^{(d)}$ is generated by S_d over S_0 for $d \gg 0$.

We're going to give a different definition from Hartshorne that turns out to be equivalent.

Definition 18.1.3. The set $\mathrm{Proj}\,(S) = \{\text{homogeneous } \mathfrak{p} \not\supset S_+\}$ so that (0 is not in $\mathbb{P}^n$) and the closed sets are,
$$V(I) = \{\mathfrak{p} \supset I\} \cap \mathrm{Proj}\,(S)$$
where I is a homogeneous ideal. Check that $V(I) \to \mathrm{Proj}\,(S/I)$ via $\mathfrak{p} \mapsto \mathfrak{p} \bmod I$ is an homeomorphism.

Remark. If $\mathfrak{p} \not\supset I$ then there is some $f \in I$ with $f \notin \mathfrak{p}$ and if everything is homogeneous we can choose f to be homogeneous as well. Then the handout shows that by Lemma 1.2 and Thm 1.3 a base of opens is,
$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}\,(S) \mid f \notin \mathfrak{p}\} = \mathrm{Proj}\,(S) \setminus V(f)$$
for homogeneous $f \in S_+$ (the positive degree condition is important).

Proposition 18.1.4. $\mathrm{Proj}\,(S) = \varnothing \iff$ every element of S_+ is nilpotent

Theorem 18.1.5 (2.1 + 2.2)**.** For homogeneous $f \in S_+$ the map,
$$\varphi_f : D_+(f) \to \mathrm{Spec}\,\left(S_{(f)}\right) \quad \text{via} \quad \mathfrak{q} \mapsto \mathfrak{q}_{(f)} = \mathfrak{q}_f \cap S_{(f)}$$
is a homeomorphism. Furthermore, let,
$$T_f = \{\text{homogeneous } f \in S \mid \forall \mathfrak{p} \in D_+(f) : g \notin \mathfrak{p}\}$$
Then,
$$S_{(f)} \xrightarrow{\sim} (T_f^{-1}S)_0$$
so the ring $S_{(f)}$ does not depend on the choice of element representing $D_+(f)$. Moreover, suppose that $D_+(h) \subset D_+(f)$ then $T_f \subset T_h$ so get,
$$S_{(f)} = (T_f^{-1}S)_0 \to (T_h^{-1}S)_0 = S_{(h)}$$
this identifies $S_{(h)}$ with $(S_{(f)})_{\frac{h^{\deg f}}{f^{\deg h}}}$. This enhances $D_+(h) \subset D_+(f)$ to a "basic affine open"
$$\mathrm{Spec}\,\left(S_{(h)}\right) \to \mathrm{Spec}\,\left(S_{(f)}\right)$$

Remark. $D_+(0) = \varnothing$ because $0 \in S_m$ for all m this is okay (we don't say its degree zero).

Proposition 18.1.6. Because $D_+(f)$ and $D_+(g)$ overlap to give $D_+(fg)$ we can define $\mathcal{O}_{\mathrm{Proj}(S)}$ via gluing the structure sheaves on $D_+(f) \xrightarrow{\sim} \mathrm{Spec}\,\left(S_{(f)}\right)$.

Example 18.1.7. $\mathbb{P}^n_A = \mathrm{Proj}\,(A[x_0, \ldots, x_n])$ is covered by the $n + 1$ affine opens $D_+(x_i)$ where $D_+(x_i) \cong \mathbb{A}^n_A$ with coordinates $\frac{x_j}{x_i}$ for $j \neq i$ and $\mathbb{P}^n_A$ is the gluing as in the classical case.

Example 18.1.8. Let $X = \mathrm{Proj}\,(k[X, Y, Z])$ where $\deg X = 2$ and $\deg Y = 3$ and $\deg Z = 4$. This is not smooth.

18.2 Functoriality

Given a graded map $\varphi : S' \to S$ we might hope to make $\operatorname{Proj}(S) \to \operatorname{Proj}(S')$ via $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. However, what if $\varphi^{-1}(\mathfrak{p}) \supset S'_+$. We have to remove some bad locus and get a scheme map,

$$\operatorname{Proj}(S) \setminus V(\varphi(S'_+)S) \to \operatorname{Proj}(S')$$

If $S'_m \to S$ is surjective for all $m \gg 0$ or even $m \gg 0$ and $d \mid m$ for fixed d then $V(\varphi(S'_+)S) = \varnothing$.

Example 18.2.1. We can have,

(a) $\varphi : S' \to S$ is surjective in all degrees $m \gg 0$

(b) $S^{(d)} \hookrightarrow S$

give $\operatorname{Proj}(S) \hookrightarrow \operatorname{Proj}(S')$ a closed immersion in the first case and $\operatorname{Proj}(S) \xrightarrow{\sim} \operatorname{Proj}\left(S^{(d)}\right)$ is an isomorphism in the first case.

Example 18.2.2. $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$ because this is,

$$\bigcap_i A[\underline{X}]_{(X_i)} \subset \mathcal{O}_{\mathbb{P}^n_A}(D_+(X_0 \cdots X_n)) = A[\underline{X}]_{(X_0 \cdots X_n)}$$

Remark.

$$\operatorname{Proj}(S) = \operatorname{Proj}\left(\mathbb{Z} \oplus \left(\bigoplus_{m \geq m_0} S_m \right) \right)$$

this retains its map to $\operatorname{Spec}(S_0)$ because each S_m is an S_0-algebra and the $D_+(f)$ are thus affine schemes over S_0 (the degree zero localizations only involve positive degree stuff).

19 Nov. 1

19.1 Some Examples

Example 19.1.1. Two further examples of Proj "functoriality"

(a) $X = Z(I) \subset \mathbb{P}^N_k$ with ($k = \bar{k}$ and I homogeneous ideal) $F_0, F_1, F_2 \in k[X_0, \ldots, X_N]_d$ for $d > 0$ gives,

$$X \dashrightarrow \mathbb{P}^2_k \text{ via } [F_0(x) : F_1(x) : F_2(x)]$$

which is defined away from the common zeros so it is really defined on $U = \mathbb{P}^2_R \setminus Z(F_0, F_1, F_2)$. This exactly corresponds to $k[Y_0, Y_1, Y_2] \to k[X_0, \ldots, X_n]/I$ via $Y_j \mapsto F_j$ (here we want $\deg Y_j = d$ for it to be a graded map but we can always multiply up the degrees). This corresponds to $\varphi : S' \to S$ has $\varphi(S'_+)S = (F_0, F_1, F_2)$ so this is the locus we need to remove to make the map on Proj.

(b) Let $S = A[X_1, \ldots, X_n]$ and

$$A[T_I : I = (i_1, \ldots, i_N) \text{ s.t } \sum i_j = d] \twoheadrightarrow S^{(d)} \quad \text{via} \quad T_I \mapsto X^I = \prod X_j^{i_j}$$

gives a closed embedding,

$$\mathbb{P}^N_A = \operatorname{Proj}(S) \cong \operatorname{Proj}\left(S^{(d)}\right) \hookrightarrow \mathbb{P}^{\binom{N+d}{N}-1}_A$$

called the *d*-uple or Segre embedding.

(c) An example of the above is $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ given by $[x, y] \mapsto [x^2, xy, y^2]$.

19.2 Further Properties of Schemes and Morphisms

Definition 19.2.1. Let X be a scheme,

(a) X is *integral* if $X \neq \varnothing$ and $\mathcal{O}_X(U)$ is a domain for all open $U \neq \varnothing$

(b) X is *locally noetherian* if there is an open cover by $S_\alpha = \text{Spec}\,(A_\alpha)$ for A_α noetherian

(c) X is *noetherian* if it is locally noetherian and qc.

(d) $f : X \to Y$ is *finite* if there exists an open cover $\{U_\alpha = \text{Spec}\,(B_\alpha)\}$ of Y such that $f^{-1}(U_\alpha) = \text{Spec}\,(A_\alpha)$ is affine with $B_\alpha \to A_\alpha$ a finite ring map.

Remark. The reason we need finite maps to be affine (c.f. finite type which does not) is that when we refine an affine open cover we are localizing which can mess up being finite (although it does not mess up being finite type).

Theorem 19.2.2. X is integral iff X is reduced and $|X|$ is irreducible.

H, II, Prop. 3.1. . In affine case, analogue o what arose when we showed that affine algebraic set $Z \subset \mathbb{A}^n_k$ is irreducible iff $I(Z)$ is prime. $\qquad\square$

Remark. In [H] proof, implicit that if $U \neq \varnothing$ then $\mathcal{O}_X(U) \neq 0$ because U contains a nonempty affine $\text{Spec}\,(A)$ and thus $A \neq 0$ and $\mathcal{O}_X(U) \to A$ is a ring map so it must send $0, 1$ to $0, 1 \in A$ different elements so $\mathcal{O}_X(U) \neq 0$.

Theorem 19.2.3. If X is locally noetherian then all affine open $U = \text{Spec}\,(A) \subset X$ have A noetherian. In particular, $\text{Spec}\,(B)$ is locally noetherian iff B is noetherian.

Proof. X is covered by open $\text{Spec}\,(R_\alpha)$ with R_α noetherian. For each $x \in \text{Spec}\,(A)$ get open neighborhood $\text{Spec}\,(A_f) \subset \text{Spec}\,(A)$ with $A_f \cong (R_\alpha)_r$ for some α and $r \in R_\alpha$ by the Nike trick. Then A_f is noetherian because R_α is noetherian. Furthermore, this is a cover so $(f_1, \ldots, f_n) = A$ (finite because $\text{Spec}\,(A)$ is qc) and each A_{f_j} is Noetherian.

We want to show that A is noetherian. Use acc: for ideal $I_1 \subset I_2 \subset I_3 \subset \cdots$ in A. We have,

$$(I_1)_{f_j} \subset (I_2)_{f_j} \subset \cdots$$

stabilizes in A_{f_j} and since there are finitely many f_j then there is some M such that for all $m > M$,

$$(I_m)_{f_j} = (I_{m+1})_{f_j}$$

for all j. We know that $\mathfrak{p} \in \text{Spec}\,\left(A_{f_{j_0}}\right)$ iff $f_{j_0} \notin \mathfrak{p}$ but these cover so for each $\mathfrak{p} \in \text{Spec}\,(A)$ we have some j_0 with $f_{j_0} \notin \mathfrak{p}$. Thus, for any A-module Q,

$$Q_{\mathfrak{p}} = (Q_{f_{j_0}})_{\mathfrak{p}_{f_{j_0}}}$$

so $(I_m)_{\mathfrak{p}} = (I_{m+1})_{\mathfrak{p}}$ for all $\mathfrak{p}$ for each $m \geq M$ inside $A_{\mathfrak{p}}$ so $I_m = I_{m+1}$ inside A for all $m \geq M$ and thus A is Noetherian. $\qquad\square$

Theorem 19.2.4 (Homework)**.** If $f : X \to Y$ is finite then for all affine open $\text{Spec}\,(B) \subset Y$, then $f^{-1}(\text{Spec}\,(B)) = \text{Spec}\,(A)$ is affine and $B \to A$ is finite.

Proof. Use Nike trick. $\qquad\square$

19.3 Comparison to Classical Varieties

Consider schemes X locally of finite type over K.

Lemma 19.3.1. $x \in X$ is closed iff $\kappa(x)/K$ is algebraic (and thus finite bc X/K is finite type).

Corollary 19.3.2. If $x \in U \subset X$ then $x \in U$ is closed iff $x \in X$ is closed.

Example 19.3.3. For $Y = \operatorname{Spec}(R) = \{\eta = (0), s = (\varpi)\}$ for R a DVR. Let $U = \{\eta\} = \operatorname{Spec}(R_\varphi)$ which is a basic affine open. Use $x = \eta$. Then $x \in U$ is closed by $x \in Y$ is not closed.

Remark. For schemes locally of finite type over $\mathbb{Z}$ a point x is closed iff $\kappa(x)$ is a finite field.

Proof of lemma. For a topological space T, and subset $Z \subset T$ and open cover $\{U_\alpha\}$ of T, then $Z \subset T$ is closed if and only if $Z \cap U_\alpha \subset U_\alpha$ closed. Consider $Z = \{z\}$ so $z \in T$ is closed iff z is closed in each U_α containing z. We apply this to $T = X$ and $\{U_\alpha = \operatorname{Spec}(A_\alpha)\}$ is an affine open cover and $Z = \{x\}$. This reduces to the following: A is a finite type K-algebra $\mathfrak{p} \subset A$ is prime we want to show that $\mathfrak{p}$ is maximal iff $\kappa(\mathfrak{p})/K$ is algebraic.
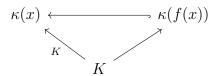
Note that $\kappa(\mathfrak{p}) = \operatorname{Frac}(A/\mathfrak{p})$ but $\operatorname{trdeg}_K(\operatorname{Frac}(A/\mathfrak{p})) = \dim A/\mathfrak{p}$ (see [Mat, 5]) so $\operatorname{trdeg}_K(\operatorname{Frac}(A/\mathfrak{p}))$ iff $\dim A/\mathfrak{p} = 0$ iff $A/\mathfrak{p}$ is a field ($A/\mathfrak{p}$ is a domain). $\qquad\square$

Now we discuss some consequences.

Definition 19.3.4. $X(K) = \operatorname{Hom}(K, \operatorname{Spec}(K)) \, X = \{x \in X \mid K \xrightarrow{\sim} \kappa(x)\}$ giving a section of $X \to \operatorname{Spec}(K)$.

Lemma 19.3.5. For a morphism $f : X \to Y$ of schemes locally of finite type over K the image of a closed point is a closed point. Furthermore, $f : X(K) \to Y(K)$.

Proof. That $f : X(K) \to Y(K)$ is an obvious general fact (for any scheme replacing $\operatorname{Spec}(K)$ even). Furthermore, for $x \in X$ closed,

$$\kappa(x) \longleftarrow \kappa(f(x))$$

But $K \to \kappa(x)$ is finite so $K \to \kappa(f(x))$ is also finite since $\kappa(f(x)) \hookrightarrow \kappa(x)$ and thus $f(x) \in Y$ is closed. $\qquad\square$

19.4 Relation to the Classical Theory

If $T(X) = \{\text{closed points}\}$ with $\mathcal{O}_{T(X)} = \iota^{-1}\mathcal{O}_X$. then $(T(X), \mathcal{O}_{T(X)})$ is quite concrete for $\operatorname{Spec}(A)$ this is just $\operatorname{mSpec}(A)$ with the expected topology and sheaf.

When $K = \overline{K}$ we get an equivalence of categories,

$$\{\text{reduced schemes locally finite type over } K\} \cong \{\text{abstract algebraic sets over } K\}$$

Definition 19.4.1. If L/K is any field extension then the L-valued points are,

$$X(L) = \operatorname{Hom}(K, \operatorname{Spec}(L)) \, X = \{(x, \varphi) \mid x \in X \text{ and } \varphi : \kappa(x) \hookrightarrow L \text{ over } K\}$$

these are something like the "solutions in L to the equations in K".

19.5 Nov. 3

19.5.1 Fiber Products and Universal Structures

Example 19.5.1. Consider $V(x - y^2) \mapsto \mathrm{Spec}\,(k[x])$. Let $B = k[x,y]/(x - y^2) \cong k[y]$ and $A = k[x] \to B$ via $x \mapsto y^2$. Then we compute,

$$B \otimes_A \kappa(P) = k[y] \otimes_{k[x]} k[x]/(x - a) = k[y]/(y^2 - a) = \begin{cases} k \times k & a \neq 0 \\ k[y]/(y^2) & a = 0 \end{cases}$$

At $a = 0$ this is nonreduced and $\dim_k = 2$ showing that it remembers the degree of the point even though there is only one geometric point in the fiber.

Example 19.5.2. An elliptic curve family. Consider k of characteristic not 2 and 3. Then $a(t), b(t) \in k[t]$ with $\Delta(t) = 4a^3 - 27b^2 \neq 0$. Consider,

$$E_{t_0} = \{y^2 z = x^3 + a(t_0)xz^2 + b(t_0)z^3\} \subset \mathbb{P}_k^2$$

This is smooth when $\Delta(t_0) \neq 0$ for $t_0 \in k$. Let $A = k[t][\Delta^{-1}]$. Then, let
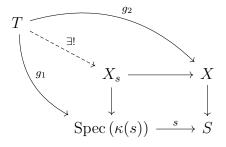
$$\mathcal{E} = \{y^2 z = x^3 + a(t)z^2 + b(t)z^3\}$$

$$
\begin{array}{ccccc}
\mathcal{E}_{t_0} & \longrightarrow & \mathcal{E} & \hookrightarrow & \mathbb{P}_A^3 \\
\downarrow & & \downarrow & \swarrow & \\
\mathrm{Spec}\,(k) & \xrightarrow{\,t_0\,} & \mathrm{Spec}\,(A) & &
\end{array}
$$

we want a way to put a natural scheme structure on the topological fiber $\mathcal{E}_{t_0}$ so it agrees with E_{t_0}.

How do we find the correct scheme structure on $f^{-1}(s)$. We define it using the universal property,

$$
\begin{array}{ccc}
T & & \\
& \overset{g_2}{\searrow} & \\
g_1 \downarrow \quad \overset{\exists!}{\dashrightarrow} & X_s \longrightarrow & X \\
& \downarrow & \downarrow \\
& \mathrm{Spec}\,(\kappa(s)) \xrightarrow{\,s\,} & S
\end{array}
$$

given T and maps $g_1 : T \to \mathrm{Spec}\,(\kappa(s))$ and $g_2 : T \to X$ that make the diagram commute then there exists a unique $T \to X_s$ making the diagram commute.
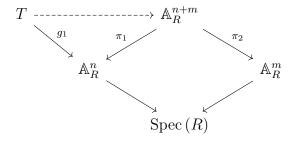
I can get more, the infinitesimal fibers, say $S = \mathrm{Spec}\,(R)$, for R is local and $s = \{\mathfrak{m}\}$ closed point $k = R/\mathfrak{m}$ then,

$$
\begin{array}{ccccc}
X_s & \longrightarrow & X_{s,R} & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{Spec}\,(k) & \longrightarrow & \mathrm{Spec}\,(R) & \xrightarrow{\,s\,} & S
\end{array}
$$

Example 19.5.3. Direct product over a base. For R a ring. Consider,

$$
\begin{array}{ccc}
T & \dashrightarrow & \mathbb{A}_R^{n+m} \\
\end{array}
$$

Given maps $T \to \mathbb{A}_R^n$ and $R \to \mathbb{A}_R^m$ that agree over $\mathrm{Spec}\,(R)$ I claim there is a unique map $g : T \to \mathbb{A}_R^{n+m}$. The map g_1 is equivalent to $R[x_1, \ldots, x_n] \to \mathcal{O}_T(T)$ as R-algebras and thus equivalent to an n-tuple $f_1, \ldots, f_n \in \mathcal{O}_T(T)$. Thus, $(g_1, g_2) \iff T \to \mathbb{A}_R^{n+m}$ over R naturally.

19.6   The Construction of Fiber Products

For an S-scheme X, get a contravariant functor $h_{X/S} : \mathbf{Sch}_S \to \mathbf{Set}$ via $T \mapsto h_X(T) = \mathrm{Hom}\,(S, T)\,X$. We think of this as T-valued solutions to X over S when $S = \mathrm{Spec}\,(R)$ is a Noetherian ring and $X = \mathrm{Spec}\,(A)$ is finitely presented over S then it is literally given by polynomial equations with coefficients in R and thus for $T = \mathrm{Spec}\,(B)$ maps $T \to X$ over S are exactly B-valued solutions to the equations defining X where the structure map $R \to B$ determines how the coefficients are interpreted in B.
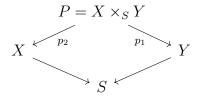
Brian's oddball nonsequitor humor coupled with random profanity makes his class a very disorienting experience.

Example 19.6.1. For S-scheme T, we see $\mathbb{A}_S^{n+m}(T) = \mathbb{A}_S^n(T) \times \mathbb{A}_S^m(T)$ (these are just sets). This tells us that $h_{\mathbb{A}_S^{n+m}}$ is naturally isomorphic to $h_{\mathbb{A}_S^n} \times h_{\mathbb{A}_S^m}$ so we do see a hint of a product structure. However, usually (including $k = \bar{k}$) there is no homeomorphism $|\mathbb{A}^{n+m}| \to |\mathbb{A}^n| \times |\mathbb{A}^m|$ it's not even bijective!

Let's look at the $n = m = 1$ case. For any irreducible curve in $\mathbb{A}^2$ that is not parallel to one of the axes defines a generic point in $\mathbb{A}^2$ which is not in $|\mathbb{A}^1| \times |\mathbb{A}^1|$.

The goal: given S-schemes X, Y we see a "relative direct product" $X \times_S Y \to S$ with s-fiber equal to $X_s \times_{\kappa(s)} Y_s$ whatever that's supposed to be.

Definition 19.6.2. Given, two S-schemes $X \times_S Y$ we construct,

$$
\begin{array}{ccc}
 & P = X \times_S Y & \\
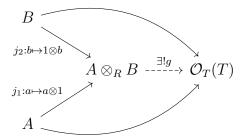X & & Y \\
 & S &
\end{array}
$$

with the universal property that given $g_1 : T \to X$ and $g_2 : T \to Y$ over S there is a unique map $g : T \to P$ such that $g_i = p_i \circ g$. We can say that $h_P(T) \xrightarrow{\sim} h_X(T) \times h_Y(T)$ naturally in T via the maps p_1, p_2. The triple (P, p_1, p_2) is called the *fiber product* of X and Y if it exists.
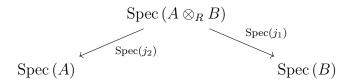
Example 19.6.3. Suppose that $S = \mathrm{Spec}\,(R)$ and $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$. Then,

$$
h_X(T) = \left\{ \begin{array}{c} T \longrightarrow \mathrm{Spec}\,(A) \\ \downarrow \\ \mathrm{Spec}\,(R) \end{array} \right\} = \mathrm{Hom}\,(R, A)\,\mathcal{O}_T(T)
$$

Likewise $h_Y(T) = \mathrm{Hom}\,(R, B)\,\mathcal{O}_T(T)$. However, notice that tensor product satisfies the opposite universal property,

$$
\begin{array}{c}
B \\
\quad\searrow{\scriptstyle j_2: b \mapsto 1 \otimes b} \\
\qquad A \otimes_R B \dashrightarrow{\scriptstyle \exists! g} \mathcal{O}_T(T) \\
\quad\nearrow{\scriptstyle j_1: a \mapsto a \otimes 1} \\
A
\end{array}
$$

over R. Therefore, in the affine case we get the fiber product,

$$
\begin{array}{ccc}
& \mathrm{Spec}\,(A \otimes_R B) & \\
\swarrow{\scriptstyle \mathrm{Spec}(j_2)} & & \searrow{\scriptstyle \mathrm{Spec}(j_1)} \\
\mathrm{Spec}\,(A) & & \mathrm{Spec}\,(B)
\end{array}
$$

over $\mathrm{Spec}\,(R)$.

Next time we use the affine case and suitable affine covers to build $X \times_S Y$ in general. Furthermore, we Weil see that,

$$
X_S \times \mathrm{Spec}\,(\kappa(s)) \xrightarrow{\sim} s\text{-fiber}
$$

is a homeomorphism.

19.7 Viewpoints on the Fiber Product

Given X, Y over S. Then $F : \mathbf{Sch}_S \to \mathbf{Set}$ is contravariant $T \mapsto X(T) \times Y(T)$ so $F = h_X \times h_Y$. Then the fiber product $(P, p_1, p_2) \in F(P) = X(P) \times Y(P)$ is a representing object for F since,

$$
\theta_P : h_P \to F
$$

given by sending $\mathrm{Hom}\,(S, T)\,P \to F(T) = X(T) \times Y(T)$ sending $g \mapsto (p_1 \circ g, p_2 \circ g)$. The universal property is exactly that θ is a natural isomorphism of functors.

Lemma 19.7.1. Let $\mathcal{C}$ be a category $F : \mathcal{C} \to \mathbf{Set}$ contravariant. Given a representing object $\xi \in \mathcal{C}$ and $\theta : F \xrightarrow{\sim} \mathrm{Hom}\,(\mathcal{C}, -)\,\xi$ then,

$$
\theta_\xi : \mathrm{Hom}\,(\mathcal{C}, \xi)\,\xi \xrightarrow{\sim} F(\xi)
$$

sending $\mathrm{id}_\xi \mapsto \alpha_\xi \in F(\xi)$. Then (α_ξ, θ) is the universal structure in the following sense meaning for any $T \in \mathcal{C}$ and any $t \in F(T)$ there is a unique map $T \to \xi$ such that $F(\xi) \to F(T)$ sending $\alpha_\xi \mapsto t$.

Proof. For any $f : T \to \xi$ because θ is a natural isomorphism, there is a diagram,

$$
\begin{array}{ccc}
F(T) & \xleftarrow{\;\theta_T\;} & \mathrm{Hom}\,(\mathcal{C}, T)\,\xi \\
{\scriptstyle F(f)}\big\uparrow & & \big\uparrow{\scriptstyle -\circ f} \\
F(\xi) & \xleftarrow{\;\theta_\xi\;} & \mathrm{Hom}\,(\mathcal{C}, \xi)\,\xi
\end{array}
$$

then $\theta_\xi(\mathrm{id}) = \alpha_\xi$ by definition. Then by commutativity, $F(f)(\alpha_\xi) = \theta_T(f)$. $\qquad\square$

19.8   Yoneda's Lemma and Fiber Products

Example 19.8.1. Suppose that $S = \mathbb{A}^2$ and $X = \{x = 0\}$ and $Y = \{x = 1\}$ two parallel lines. Then $X \times_S Y = \varnothing$. Indeed,

$$k[x,y]/(x) \otimes_{k[x,y]} k[x,y]/(x-1) = k[x,y]/(x, x-1) = 0$$

In general,

$$\mathrm{Spec}\,(A/I) \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(A/J) = \mathrm{Spec}\,((A/I \otimes_A A/J)) = \mathrm{Spec}\,(A/(I+J))$$

Lemma 19.8.2. Let $F : \mathcal{C} \to \mathbf{Set}$ be a contravariant functor and a representing object: $\xi \in \mathcal{C}$ with $\theta : F \xrightarrow{\sim} h_\xi = \mathrm{Hom}\,(\mathcal{C}, -)\,\xi$ then $\theta : F(\xi) \xrightarrow{\sim} \mathrm{Hom}\,(\mathcal{C}, \xi)\,\xi$ so there is $\alpha_\xi \in F(\xi)$ corresponding to id_ξ. Then (α_ξ, θ_ξ) is a universal structure: for any $t \in F(T)$ there is a unique map $f_t : T \to \xi$ (the classifying map) so that $f_t^* \alpha_\xi = t$.

Example 19.8.3. For $\mathcal{C} = \mathbf{Sch}_R$ for $F(T) = \mathcal{O}(T)^n \cong \mathrm{Hom}\,(\mathcal{C}, -)\,\mathbb{A}_R^n$ then the representing object is $\alpha_\xi = (x_1, \ldots, x_n) \in F(\mathbb{A}_R^n)$.
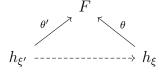
Example 19.8.4. $\mathbb{P}_k^n$ "classifies" lines in k^{n+1} or L^{n+1} for all L/k but this is not enough to actually define a moduli problem because we need to consider its B-points for all k-algebras B or its T-points for all k-schemes T. Then $\mathbb{P}_k^n$ gets a "universal line" which will correspond to a line bundle with canonical generating sections.

Remark. We say that such F are representable and it represented by ξ via α_ξ. Furthermore, we can reconstruct θ from α_ξ via,

$$\theta_T : F(T) \xrightarrow{\sim} \mathrm{Hom}\,(\mathcal{C}, T)\,\xi$$

sending $t \mapsto f_t$.

Corollary 19.8.5. If (ξ, α_ξ) and (ξ', α_ξ') are two structures representing F then,

arises from a unique isomorphism $\xi' \xrightarrow{\sim} \xi$ sending $\alpha_\xi \mapsto \alpha_{\xi'}$.

Proof. We get $h_{\xi'}(\xi') \xrightarrow{\sim} F(\xi') \xrightarrow{\sim} h_\xi(\xi') = \mathrm{Hom}\,(\mathcal{C}, \xi')\,\xi$ then the image of id gives a map $f : \xi' \to \xi$ such that $\mathrm{id}_\xi' \mapsto \alpha_\xi' \mapsto f$ so $F(f) : \alpha_\xi \mapsto \alpha_{\xi'}$. Likewise we get $f' : \xi \to \xi'$ then we check via uniqueness that $f \circ f' = \mathrm{id}_\xi$ and $f' \circ f = \mathrm{id}_{\xi'}$ because they preserve α_ξ and α_ξ'. $\square$

Lemma 19.8.6 (Yoneda)**.** Let $X, Y \in \mathcal{C}$ then any natural transformation $\theta : h_X \to h_Y$ arises as $f \circ -$ for a unique map $f : X \to Y$.

Proof. Consider,

$$h_X(X) \xrightarrow{\theta_X} h_Y(X)$$

then $\mathrm{id}_X \mapsto f \in \mathrm{Hom}\,(\mathcal{C}, X)\,Y$. This is the unique thing that can work because if $\theta(-) = \tilde{f} \circ -$ so $\theta_X(\mathrm{id}_X) = \tilde{f} \circ \mathrm{id}_X = \tilde{f}$ so $\tilde{f} = f$. Now we need to show that $\theta = f \circ -$. Let $T \in \mathcal{C}$ and $x \in h_X(T)$ meaning $x : T \to X$. Then consider,

$$h_X(T) \xrightarrow{\theta_T} h_Y(T)$$

$$\scriptstyle{-\circ x}\Big\uparrow \qquad\qquad \Big\uparrow\scriptstyle{-\circ x}$$

$$h_X(X) \xrightarrow{\theta_X} h_Y(X)$$

Therefore, $\mathrm{id} \mapsto x \mapsto \theta_T(x)$ and also $\mathrm{id} \mapsto \theta_X(\mathrm{id}) = f \mapsto f \circ x$ and therefore $\theta_T(x) = f \circ x$ so $\theta = f \circ -$ proving what we wanted. $\qquad\square$

Remark. How can this linguistic triviality be useful! The utility is we can often find a rich subcategory $\mathcal{C}' \subset \mathcal{C}$ (e.g. affine schemes inside schemes) to be like a "base for the topology" to build natural transformations $F_1 \to F_2$ between functors on $\mathcal{C}$ even if $\xi \notin \mathcal{C}'$.

19.9 Construction of Fiber Products

We've done the affine case (universal on all schemes over S. Now we consider the general setting, X, Y are S-schemes. We want to glue together the fiber products for compatible affine opens.

First suppose that the fiber product $X \times_S Y$ exists. Then for any $U \subset X$ and $V \subset Y$ and $W \subset S$ compatibly meaning $U \to W$ and $V \to W$. Then I claim that $p_1^{-1}(U) \cap p_2^{-1}(V) \subset X \times_S Y$ is $U \times_W V$.

20 Nov. 8

20.1 Applications of Fiber Products

Some loose ends:

(a) $(X \times_S S') \times_{S'} S'' \cong X \times_S S''$ over S''. To see this, evaluate the functor of points on S''-schemes.

(b) $|X \times_Z Y| \twoheadrightarrow |X| \times_{|Z|} |Y|$ is always surjective although not usually injective. Given $x, y \mapsto z$ then we can form $\kappa(x) \otimes_{\kappa(z)} \kappa(y) \neq 0$ because these are nonzero vectorspaces over $\kappa(z)$. Thus, $\mathrm{Spec}\left(\kappa(x) \otimes_{\kappa(z)} \kappa(y)\right)$ is nonempty and thus there is a point mapping to (x, y).

(c) Think of $f : X \to Y$ as "family of schemes" $\{X_y\}_{y \in Y}$ as $\kappa(y)$-schemes.

(d) $X \otimes_A B$ or X_B denotes $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$.

20.2 Applications of Fiber Products

20.2.1 Remarks

(a) $\mathbb{A}_R^n = \mathbb{A}_{\mathbb{Z}}^n \otimes_{\mathbb{Z}} R$ as R-schemes

(b) $\mathbb{P}_R^n = \mathbb{P}_{\mathbb{Z}}^n \otimes_{\mathbb{Z}} R$ as R-schemes

(c) For $A \to S_0$ (so the base change does not mess up the graded pieces) $\mathrm{Proj}(S) \otimes_A A' = \mathrm{Proj}(S \otimes_A A')$.

20.2.2 Scheme Theoretic Intersection

For $Z, Z' \hookrightarrow X$ closed subschemes then $Z \times_X Z' \to X$ is a closed immersion with $|Z \times_X Z'| = |Z| \cap |Z'|$. Indeed, WLOG $X = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(A/I)$ and $Z' = \operatorname{Spec}(A/I')$ then,

$$Z \cap Z' := Z \times_X Z' = \operatorname{Spec}(A/I \otimes_A A/I') = \operatorname{Spec}(A/(I + I'))$$

This also has a good mapping property: $T \to X$ factors through Z and Z' iff it factors through $Z \times_X Z'$. However, if T is nonreduced then it may not factor through the reduced structure on $(Z \cap Z')$ so this scheme structure is the "right one".

Remark. If X, Z, Z' are smooth over $k = \bar{k}$ and $x \in Z \cap Z'$ with $T_x(Z)$ and $T_x(Z') \subset T_x(X)$ are transverse then $Z \cap Z'$ is smooth at x.

20.2.3 Cartesian

We say that a commutative square is Cartesian if,

$$
\begin{array}{ccc}
P & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & S
\end{array}
$$

the natural map $P \to X \times_S Y$ is an isomorphism. Fiber products interact well with base change in the sense that,

$$
\begin{array}{ccc}
X' \times_{S'} Y' & \longrightarrow & X \times_S Y \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
$$

is Cartesian where $X', Y' \to S'$ are the base changes of $X, Y \to S$. To see this, evaluate on an S'-scheme T'. Then,

$$((X \times_S Y) \times_S S')_{S'}(T') = (X \times_S Y)_S(T') = X_S(T') \times Y_S(T') = X'(T') \times Y'(T') = (X' \times_{S'} Y')(T')$$

where the natural map $X'(T') \to X_S(T')$ is given by "forgetting" the S'-structure or composing with $X' \to X$ are bijective by the universal property.

Key example, let $S' = \operatorname{Spec}(\kappa(s)) \to S$ then $(X \times_S Y)_s = X_s \otimes_{\kappa(s)} Y_s$.

20.2.4 Flatness

Let $f : X \to Y$ be a flat morphism of locally noetherian schemes. Take $x \in X$ and $y = f(x)$. Then,

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X_y,x}$$

For a proof [Mat, Thm. 15.1] use that if $\varphi : A \to B$ is a flat map of noetherian rings and $\mathfrak{p} \subset B$ is a prime and $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$ then,

$$\dim B_{\mathfrak{p}} = \dim A_{\mathfrak{q}} + \dim B \otimes_A \kappa(\mathfrak{q})$$

and $B \otimes_A \kappa(\mathfrak{q}) = (B/\mathfrak{q}B)_{\mathfrak{q}}$.

In general, for non-flat maps there is an inequality,

$$\dim \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X_y,x}$$

20.2.5 The Diagonal

Definition 20.2.1. For $f : X \to S$ the *relative diagonal* is $\Delta_f : X \to X \times_S X$ given by the identity maps $X \to X$.

Remark. In the affine case, $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ then $\Delta : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(B \otimes_A B)$ is induced by the map $\delta : B \leftarrow B \otimes_A B$ given by sending $b_1 \otimes b_2 \mapsto b_1 b_2$. To see this, the projections $\mathrm{Spec}\,(B \otimes_A B) \to \mathrm{Spec}\,(A)$ take $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$ and therefore since Δ is a ring map such that it sends $b_1 \otimes 1 \mapsto b_1$ and $1 \otimes b_2 \mapsto b_2$ because it is the identity when composed with the projections. Therefore, $b_1 \otimes b_2 \mapsto b_1 b_2$. Therefore, $\delta : B \otimes_A B \to B$ is surjective so $\Delta_{B/A}$ is a closed immersion.

Remark. In general, $(\Delta_{X/S}) \times_S S' = \Delta_{X'/S'}$ which is easily checked on S'-pints. For $\{S_i\}$ an open cover of S then $\Delta_{X/S} : X \to X \times_S X$ restricts under base change to the Δ_{X_i/S_i} for $X_i = f^{-1}(S_i)$ with $f : X \to S$.

Thus, we fix an open affine cover $\{X_{ij}\}_{j \in J_i}$ of $X_i = f^{-1}(S_i)$. Then,

$$
\begin{array}{ccc}
X & \xrightarrow{\ \Delta\ } & X \times_S X \\
\uparrow & & \uparrow \\
\Delta^{-1}(V_{ij}) \longrightarrow X_{ij} & \xrightarrow{\ \Delta\ } & X_{ij} \times_{S_i} X_{ij} = V_{ij}
\end{array}
$$

Then this is actually Cartesian. Furthermore, let,

$$
V = \bigcup_{i,j} V_{ij} \subset X \otimes_S X
$$

is an open subscheme. Then $\Delta(X) \subset V$ and because X_{ij} is affine $\Delta : X_{ij} \to X_{ij} \times_{S_i} X_{ij}$ is a closed immersion. Since being a closed immersion is a local property on the target we get,

$$
X \underset{\text{closed im}}{\hookrightarrow} V \underset{\text{open im}}{\hookrightarrow} X \times_S X
$$
with Δ the composite.

Definition 20.2.2. Say that $g : Y \to Z$ is an *immersion* if

(a) g is a homeomorphism onto $g(Y) \subset Z$ which is locally closed

(b) for some (equivalently all) open $V \subset Z$ with $g(Y) \subset V$ closed and $Y \to V$ is a closed immersion.

Lemma 20.2.3. An immersion is a closed immersion if and only if it has closed image.

Definition 20.2.4. Say X is S-separated when $\Delta_{X/S}(X) \subset X \times_S X$ is closed or equivalently $\Delta_{X/S}$ is a closed immersion.

20.3 Nov. 10

20.4 Separatedness

For $f : X \to S$ we factored the diagonal,

$$X \xrightarrow[\text{closed im}]{} V \xrightarrow[\text{open im}]{} X \times_S X$$
$$\overset{\Delta}{\overgroup{}}$$

and therefore say that $\Delta_{X/S}$ is an immersion. Immersions have reasonable properties: stable under base change and composition. Furthermore a closed immersion is exactly an immersion with closed image. However, notice that an immersion with open image is usually <u>not</u> an open immersion e.g.

$$Y_{\text{red}} \hookrightarrow Y$$

If $\{S_i\}$ is an open cover of S, and $X_i = f^{-1}(S_i) \subset X$, then pulling back $\Delta_f; X \to X \times_S X$ over S_i gives exactly,

$$X_i \xrightarrow{\Delta_{X_i/S_i}} X_i \times_{S_i} X_i = (X \times_S X)|_{S_i}$$
$$S_i$$

Since closedness can be checked locally *on the target* then Δ_f is a closed immersion if and only if each Δ_{X_i/S_i} is a closed immersion iff each $X_i \to S_i$ is separated. Therefore, separatedness is "local on the base".

Proposition 20.4.1. Affine maps are separated.

Proof. Work locally on the base then we reduce to the case of a map of affine schemes which we know is separated. $\qquad\square$

Example 20.4.2. Both $\mathbb{P}^1_k$ and the line with a double origin are the gluing of $\mathbb{A}^1_k$ and $\mathbb{A}^1_k$ along $U_1 = \{x \neq 0\} = U_2$ via either the map $j : U_i \to U_i$ by $x \mapsto x$ or $x \mapsto x^{-1}$. Consider the graph,
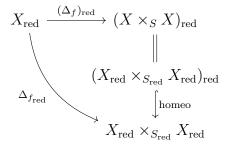
$$\Gamma_j \subset \mathbb{A}^1_k \times \mathbb{A}^1_k = \mathbb{A}^2_k$$

In the first case, $\Gamma_j = \{x = y, x \neq 0\}$ which is not closed but in the second case $\Gamma_j = \{xy = 1\}$ which is closed in $\mathbb{A}^2$.

Example 20.4.3. We will see that $\text{Proj}\,(S)$ is always separated over $\text{Spec}\,(S_0)$.

Proposition 20.4.4. A morphism of schemes $f : X \to S$ is separated iff $f_{\text{red}} : X_{\text{red}} \to S_{\text{red}}$ is separated.
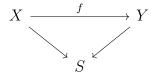
Proof. We just need to show that Δ_f has closed image. However, the following map is the same topologically,

$$X_{\text{red}} \xrightarrow{(\Delta_f)_{\text{red}}} (X \times_S X)_{\text{red}}$$
$$\|$$
$$(X_{\text{red}} \times_{S_{\text{red}}} X_{\text{red}})_{\text{red}}$$
$$\downarrow \text{homeo}$$
$$X_{\text{red}} \times_{S_{\text{red}}} X_{\text{red}}$$

with $\Delta_{f_{\text{red}}}$ the diagonal map from X_{red}.

Therefore Δ_f has closed image if and only if $(\Delta_f)_{\text{red}}$ has closed image if and only if $\Delta_{f_{\text{red}}}$ has closed image. $\qquad\square$

20.5   Graph Construction

Given an S-morphism,

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
& \searrow \qquad \swarrow & \\
& S &
\end{array}
$$

yields a morphism $\Gamma_f : X \to X \times_S Y$ given by the pair (id, f). Thus, on T points it is given by $x \mapsto (x, f(x))$ where $f(x) = f \circ x$. Then the following diagram,

$$
\begin{array}{ccc}
X & \xrightarrow{\Gamma_f} & X \times_S Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times \mathrm{id}_Y} \\
Y & \xrightarrow{\Delta_{X/Y}} & Y \times_S Y
\end{array}
$$

is Cartesian. By Yoneda, we can check the Cartesian property at the level of sets of T-points (basically this is just the definition of this diagram being a limit). On T-points, given (x, y) and y' such that $(f(x), y) = (y', y')$ then $y = y' = f(x)$ so (x, y) and y' come from the unique point $x \mapsto (x, f(x))$ and $x \mapsto f(x)$.

If Y is separated over S then $\Delta_{Y/S}$ is a closed immersion so by base change Γ_f is also a closed immersion. In general, $\Delta_{Y/S}$ is always an immersion so Γ_f is always an immersion.

Therefore, for $Y \to S$ separated,

$$
\mathrm{Hom}\,(S, X)\,Y \xrightarrow{\sim} \{Z \hookrightarrow X \times_S Y \mid \pi_1 : Z \xrightarrow{\sim} X\} \quad \mathrm{via}\, f \mapsto \Gamma_f
$$

Conversely, given $Z \hookrightarrow X \times_S Y$ then consider,

$$
\begin{array}{ccc}
Z & \lhook\joinrel\longrightarrow & X \times_S Y \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \pi_2} \\
X & \dashrightarrow & Y
\end{array}
$$

Gives a morphism $X \to Y$ by inverting $Z \to X$. Furthermore, $Z = \Gamma_f$ because they have the same T-points.

A special case is $X = S$ then we are considering sections $\sigma : S \to Y$ of $\pi : Y \to S$ meaning $\pi \circ \sigma = \mathrm{id}$. Then we see that,

$$
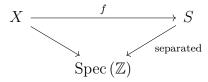Y(S) = \{Z \hookrightarrow Y \mid Z \xrightarrow{\sim} S\}
$$

Theorem 20.5.1. We have the following,

(a) any monomorphism $f : X \to S$ is separated

(b) $X \xrightarrow{f} Y \xrightarrow{g} Z$ if f, g are separated then $g \circ f$ is separated

(c) if $f : X \to Y$ is separated then any base change $f' : X' \to Y'$ is separated

(d) if $f : X \to Y$ and $f' : X' \to Y'$ are separated S-maps then $f \times f' : X \times X' \to Y \times Y'$ is also separated. (e.g. if $Y = S = Y'$ then $X \times_S X'$ is S-separated if X and X' are)

(e) $X \xrightarrow{f} Y \xrightarrow{g} Z$ if $g \circ f$ is separated then f is separated.

Remark. The proof in Hartshorne is crazy and uses valuative criterion.

Remark. Given a diagram,

$$X \xrightarrow{\quad f \quad} S$$

with arrows to $\mathrm{Spec}\,(\mathbb{Z})$, labeled "separated"

therefore we see from the theorem that $X \to S$ is separated iff $X \to \mathrm{Spec}\,(\mathbb{Z})$ is separated.

Proof. (a) f is monic iff Δ_f is an isomorphism in which case Δ_f is a closed immersion. For (b) consider,

$$X \xrightarrow{\Delta_{g \circ f}} X \times_Z X$$

For (c) if you make a base change $\Delta_f \times_Y Y' = \Delta_{f'}$ and the base change of a closed immersion is a closed immersion.

For (d) then $X \times_S X' \xrightarrow{\mathrm{id}_X \times f'} X \times_S Y' \xrightarrow{f \times \mathrm{id}_{Y'}} Y \times_S Y'$ but the first is a base change of f' and the second of f so we conclude by the previous. $\qquad \square$

Corollary 20.5.2. Then the following also follow,

(a) if $f : X \to Y$ is separated and Y is separated and $U \subset X$ affine open and $V \subset Y$ affine open then $U \cap f^{-1}(V) \subset X$ is affine.

(b) If X is separated and $U, V \subset X$ affine open then $U \cap V$ is affine.

Proof. It is clear that (a) $\implies$ (b) by taking $X = Y$. Then consider,

$$
\begin{array}{ccc}
U \cap f^{-1}(V) & \lhook\joinrel\longrightarrow & U \times_{\mathbb{Z}} V \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;\Gamma_f\;} & X \times_{\mathbb{Z}} Y
\end{array}
$$

Since Γ_f is a closed immersion we see that $U \cap f^{-1}(V) \hookrightarrow U \times_{\mathbb{Z}} V$ is a closed immersion. However, $U \times_{\mathbb{Z}} V$ is affine and thus any closed subscheme is affine. $\qquad \square$

20.6 Separatedness criterion

Lemma 20.6.1. Let $\{U_i\}$ be an affine open cover of R-scheme X. Then $X \to \mathrm{Spec}\,(R)$ is separated if and only if,

(a) $U_i \cap U_j$ is affine

(b) $\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \twoheadrightarrow \mathcal{O}_X(U_i \cap U_j)$ is surjective

Proof. Consider the Cartesian diagram,

$$
\begin{array}{ccc}
U_i \cap U_j & \xrightarrow{\;\varphi_{ij}\;} & U_i \times_R U_j \\
\downarrow & & \uparrow \\
X & \xrightarrow{\;\Delta\;} & X \times_R X
\end{array}
$$

Therefore, Δ is a closed immersion if and only if each φ_{ij} is a closed immersion. $\square$

Proposition 20.6.2. $\operatorname{Proj}(S) \to \operatorname{Spec}(S_0)$ is separated.

Proof. Consider the open affine cover $\{U_i = D_+(f_i)\}$ for $f_i \in S_+$. Then $D_+(f) \cap D_+(g) = D_+(fg)$ which is affine. Thus we just need to consider,

$$S_{(f)} \otimes_{S_0} S_{(g)} \to S_{(fg)}$$

However, $S_{(fg)} = (S_{(g)})_{\frac{f^n}{g^m}}$ so the map is given by,

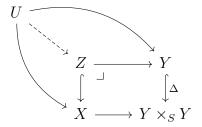$$\left(\frac{g^m}{f^n}\right)^r \otimes \alpha \mapsto \frac{\alpha}{\left(\frac{f^n}{g^m}\right)^r}$$

which is surjective. $\square$

21 Nov. 12

21.1 Applications of Separatedness

Proposition 21.1.1. Let $f, g : X \to Y$ be morphisms with Y separated. Let $j : U \hookrightarrow X$ be an open subscheme that is schematically dense meaning $\mathcal{O}_X \hookrightarrow j_*\mathcal{O}_U$ is injective. Then if $f|_U = g|_U$ then $f = g$.

Proof. Consider,

because $\Delta_{Y/S} : Y \to Y \times_S Y$ is a closed immersion so $Z \to X$ is a closed immersion and therefore we get $U \hookrightarrow Z \hookrightarrow X$. We want to deduce that $Z = X$. This is equivalent to the condition that $\mathscr{I}_Z = (0)$. However, $\mathcal{O}_X \to \iota_*\mathcal{O}_Z \to j_*\mathcal{O}_U$ is injective and thus $\mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is injective meaning that $\mathscr{I}_Z = \ker(\mathcal{O}_X \to \iota_*\mathcal{O}_Z) = (0)$. Alternatively, $\mathscr{I}_Z \subset \mathcal{O}_X \to j_*\mathcal{O}_U$ is zero because $Z \cap U = U$ and this is the ideal sheaf of the closed $Z \cap U$ in U. $\square$

21.2 Triangular Implication of Properties

Say that $\mathcal{P}$ is some property of scheme maps. Consider conditions:

(a) All closed immersions (e.g. all isomorphisms) are $\mathcal{P}$

(b) $\mathcal{P}$ is stable under compositions

(c) If $f : X \to Y$ and $f' : X' \to Y'$ over S are $\mathcal{P}$, then so it $f \times f' : X \times_S X' \to Y \times_S Y'$

(d) $\mathcal{P}$ is stable under base change

(e) for $f : X \to Y$ and $g : Y \to Z$ if $g \circ f$ is $\mathcal{P}$ and g is separated then f is $\mathcal{P}$.

(f) If f is $\mathcal{P}$ then f_{red} is $\mathcal{P}$.

Example 21.2.1. finite, affine, finite type, closed immersion all satisfy (a) (b) (d) but open immersion violates (a).

Theorem 21.2.2. Assume that $\mathcal{P}$ satisfies (a) and (b) then (c) $\iff$ (d) and when they hold then (d) and (e) hold.

Proof. For (d) $\implies$ (c) we just factor $X \times_S X' \xrightarrow{f \times \mathrm{id}_{Y'}} Y \times_S X' \xrightarrow{\mathrm{id}_Y \times f'} Y \times_S Y'$ which is $f \times_S f'$ so $f \times_S f'$ satisfies (d) by (b) and (d). Now suppose (c) then consider $X \to S$ then $X \times_S S' \to S \times_S S' \xrightarrow{\sim} S$ is $\mathcal{P}$ by (c) and (a) for id and isomorphism and (b).

Now we grant (c) and (d). Consider the diagram,

$$
\begin{array}{ccc}
 & & \overset{f}{\frown} \\
X & \xrightarrow{\;\Gamma_f\;} & X \times_Z Y \xrightarrow{\;\mathrm{pr}_2\;} Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times \mathrm{id}_Y} \\
Y & \xrightarrow{\;\Delta_g\;} & Y \times_Z Y
\end{array}
$$

since $f = \mathrm{pr}_2 \circ \Gamma_f$ and Γ_f is the base change of Δ_g which is a closed immersion (because g is separated) and thus is $\mathcal{P}$ by (a) and (c) and π_2 is the base change of $f \circ g : X \to Z$ via $g : Y \to Z$ so by (c) it has $\mathcal{P}$. Thus by (b) we see that $f = \pi_2 \circ \Gamma_f$ is $\mathcal{P}$.

Finally, consider,

$$
\begin{array}{ccc}
X_{\mathrm{red}} & \hookrightarrow & X \\
\downarrow{\scriptstyle f_{\mathrm{red}}} & & \downarrow{\scriptstyle f} \\
Y_{\mathrm{red}} & \xrightarrow{\;g\;} & Y
\end{array}
$$

Since $g \circ f_{\mathrm{red}}$ is the composition of f and a closed immersion it is $\mathcal{P}$ and g is a closed immersion and this separated so by (v) we have f_{red} satisfies $\mathcal{P}$. $\qquad\square$

21.3 Quasi-Separatedness

In EGA "scheme" means "separated scheme" and "pre-scheme" means "scheme". A useful weakening of separatedness, discovered in EGA IV is the following.

Definition 21.3.1. A morphism $f : X \to S$ is *quasi-separated* if $\Delta_f : X \to X \times_S X$ is quasi-compact.

Remark. For $X \to \mathrm{Spec}\,(A)$ and U, V are affine open then consider,

$$
\begin{array}{ccc}
U \cap V & \longrightarrow & U \times_A V \\
\downarrow & & \uparrow \\
X & \xrightarrow{\;\Delta_f\;} & X \times_A X
\end{array}
$$

and thus by base change $U \cap V \to U \times_A V$ is quasi-compact but $U \times_A V$ is affine so $U \cap V$ is quasi-compact.

Remark. IF $\{U_i\}$ is an *affine* open cover of X and $\{U_i \times_A U_j\}$ is a cover for $X \times_A X$ then f is quasi-separated iff all $U_i \cap U_j$ are quasi-compact.

If $\{S_i\}$ is an open cover of S and $X_i = f^{-1}(S_i)$ then Δ_f is quasi-compact iff all $X_i \to X_i \times_{S_i} X_i = (X \times_S X)|_{S_i}$ are quasi-compact for $f_i : X_i \to S_i$. Therefore quasi-separatedness is local on the base/target.

Lemma 21.3.2. quasi-separatedness is preserved by base change and composition.

Proof. Let $f : X \to Y$ and $g : Y \to Z$. Then,

$$
\begin{array}{ccc}
X & \xrightarrow{\Delta_{g \circ f}} & X \times_Z X \\
& \Delta_f \searrow & \uparrow j \\
& & X \times_Y X
\end{array}
$$

we say that j is a base change of Δ_g which is quasi-compact and thus j is quasi-compact and f is quasi-compact so we see that $\Delta_{g \circ f}$ is quasi-compact so $g \circ f$ is quasi-separated. Likewise,

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S
\end{array}
$$

Then $\Delta_{f'}$ is the base change of Δ_f which is quasi-compact and thus Δ_f is quasi-compact so f' is quasi-separated. $\square$

Proposition 21.3.3. If $f : X \to S$ for locally noetherian X is quasi-separated.

Proof. WLGO let $S = \mathrm{Spec}\,(A)$ so we want $U \cap V$ to be quasi-compact for affine opens $U, V \subset X$. But U, V are noetherian so $U \cap V$ is open in $U \cap V$ and thus quasi-compact. $\square$

Corollary 21.3.4. Consider the diagram,

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & \mathrm{Spec}\,(R') \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & \mathrm{Spec}\,(R)
\end{array}
$$

If f is locally of finite type and R is noetherian then f' is quasi-separated.

Proof. Since f is locally finite type then X is locally noetherian so by above f is quasi-separated to by base change f' is quasi-separated. $\square$

Remark. However, R' could be some enormous non-noetherian ring.

Remark. Consider,

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & \mathrm{Spec}\,(A) \\
& \searrow \qquad \swarrow & \\
& \mathrm{Spec}\,(\mathbb{Z}) &
\end{array}
$$

Then f is quasi-separated iff $X \to \mathrm{Spec}\,(\mathbb{Z})$ is quasi-separated so we just say X is quasi-separated in the case that it is quasi-separated over an affine.

We want a notion of "relative compactness" not quasi-compactness because this doesn't give the correct properties for non-Hausdorff topologies. For locally compact Hausdorff spaces a continuous map $f : X \to Y$ is *proper* if $f^{-1}(C)$ is compact for each compact $C \subset Y$. We want an analogue for schemes.

22   Nov. 15 Projectivity and Properness

22.1   Projectivity

Definition 22.1.1. Let S be a scheme. Let $\mathbb{A}^n_S = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$ and $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$ (this is a gluing of $n+1$-copies of $\mathbb{A}^n_S$ in "usual" way). Over open affine $\mathrm{Spec}\,(R) \subset S$, this recovers $\mathbb{A}^n_R$ and $\mathbb{P}^n_R$.

Definition 22.1.2. Say $f : X \to S$ is *projective* if there exists.

$$X \overset{j}{\hookrightarrow} \mathbb{P}^n_S$$
$$\searrow^{f} \quad \downarrow$$
$$S$$

where j is a closed immersion. If such diagram exists with j a quasi-compact immersion then $X \to S$ is quasi-projective.
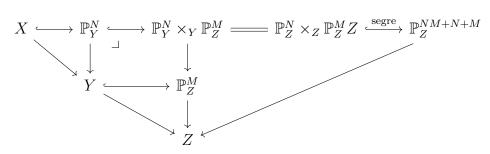
Remark. If j is quasi-compact notice that quasi-protectiveness is equivalent to the existence of j that is an immersion.

Remark. In EGA, there is a more general version of projectivity but also fails to be local on the base.

Proposition 22.1.3. The following hold:

(a) quasi-projective $\implies$ separated and finite type

(b) projective and quasi-projective are preserved under base change

(c) closed immersions are projective

(d) projective and quasi-projective are stable under composition.

Proof. (a) through (c) are evident. Let's look at (d). Consider $f : X \to Y$ and $g : Y \to Z$. Then consider,

$$X \hookrightarrow \mathbb{P}^N_Y \hookrightarrow \mathbb{P}^N_Y \times_Y \mathbb{P}^M_Z = \mathbb{P}^N_Z \times_Z \mathbb{P}^M_Z Z \overset{\text{segre}}{\hookrightarrow} \mathbb{P}^{NM+N+M}_Z$$
$$Y \hookrightarrow \mathbb{P}^M_Z$$
$$Z$$

where heuristically the Segre embedding is given by $([x_i], [y_i]) \mapsto [x_i y_j]$. $\qquad \square$

Remark. Intersection theory gives a lot of geometric information on projective schemes $X \hookrightarrow \mathbb{P}^n$ using "hyperplane slices".

22.2 Elimination Theory

Remark. Elimination theory is the study of taking a system of multivariable polynomial equations and producing polynomial conditions that decide if they have a common solution.

Theorem 22.2.1 (Fundamental Thm of Elimination Theory)**.** Given homogeneous $f_1, \ldots, f_m \in k[\underline{a}][X_0, \ldots, X_n]$ there exists $g_1, \ldots, g_r \in k[\underline{a}]$ such that for $\underline{a}_0 \in k^N$, there exists nonzero $\bar{k}$-solution to $\{f_i(\underline{a}_0, \underline{X})\}$ iff all $g_j(\underline{a}_0) = 0$.

Remark. We reinterpret this fact as,

$$\mathrm{Proj}\,(k[\underline{a}][\underline{X}]/(f_1, \ldots, f_n)) \to \mathrm{Spec}\,(k[\underline{a}])$$

has Zariski closed image. Equivalently, because the left is a closed subscheme of $\mathbb{P}^n_{k[\underline{a}]}$ we require that the map,

$$\mathbb{P}^n_{k[\underline{a}]} \to \mathrm{Spec}\,(k[\underline{a}])$$

is closed. This means that $\mathbb{P}^n_k \to \mathrm{Spec}\,(k)$ is closed after any base-change $S \to \mathrm{Spec}\,(k)$ for affine S of finite type (because every finite type k-algebra is a quotient of a polynomial ring).

Definition 22.2.2. A morphism $f : X \to Y$ is *universally closed* if for any $Z \to Y$, the base change $f' : X_Z \to Z$ is closed (topologically).

Definition 22.2.3. A morphism $f : X \to Y$ is *proper* if it is separated, finite type, and universally closed.

Remark. We will see that $\mathbb{P}^n_\mathbb{Z} \to \mathrm{Spec}\,(\mathbb{Z})$ is proper. This is much stronger than standard elimination theory. In particular we recover that $\mathbb{P}^n_R \to \mathrm{Spec}\,(R)$ is closed giving "elimination theory over any ring".
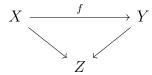
22.2.1 Merits of Properness over Projectivity

(a) properness is insensitive to nilpotents, but not projectivity (properness is more topological and is simpler for deformation theory).

(b) properness is local on the base (and indeed stability under base change)

(c) properness has a functorial characterization

(d) there are deep characterizations of *other* properties of maps via properness [EGA, IV$_4$, 18.12].

  (a) finite $\iff$ proper and quasi-finite

  (b) integral $\iff$ affine and universally closed (Exercise 35, Ch.5 in [AM])

  (c) closed immersion $\iff$ proper monomorphism

Remark. From the definitions we see that,

(a) closed immersions are proper

(b) composition of proper is proper

(c) proper is stable under base change

therefore we also see that (d) - (f) from last time also hold. In particular, given,

$$X \xrightarrow{\quad f \quad} Y$$
$$\searrow \qquad \swarrow$$
$$Z$$

with $Y \to Z$ separated and finite type. Then $X \to Y$ is proper when $X \to Z$ is proper (like image of compact is compact). For example, if f is quasi-projective,

$$X \xhookrightarrow{\ j\ } \mathbb{P}^n_S$$

then f is proper iff f proper iff j is a closed immersion iff f is projective.

23 Valuative Criterion

(Inspired by the notion of "completeness" in Weil's foundations for abstract varieties).

Theorem 23.0.1 (Valuative Criterion). Let $f : X \to Y$ be a quasi-separated finite type morphism. Consider valuation rings V with $K = \mathrm{Frac}\,(V)$ and a diagram,

$$\mathrm{Spec}\,(K) \xrightarrow{\ x_K\ } X$$
$$\downarrow \qquad \dashrightarrow \quad \downarrow f$$
$$\mathrm{Spec}\,(V) \xrightarrow{\ y\ } Y$$

Then,

(a) f is separated iff all such diagrams can be filled in at most one way (like uniqueness of limits in Hausdorff spaces)

(b) f is proper iff all such diagrams can be filled in a unique way (like existence of limits in a compact space)

Remark. Consider,

(a) $V = k[[t]]$ then $\mathrm{Spec}\,(V)$ is the AG version of the unit disc then $K = V[t^{-1}]$ and $\mathrm{Spec}\,(K)$ is the AG version of the punctured unit disk. Thus the valuative criterion asks about extending sections over 0.

(b) For noetherian Y it suffices to use DVRs.

(c) f is separated $\iff \Delta_X$ is proper so it allows (a) to reduced to (b).

23.1 Killer App

Show that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ is proper (we know already that it is separated and finite type).

For any DVR V with $K = \operatorname{Frac}(V)$ we want $\mathbb{P}^n_{\mathbb{Z}}(V) \to \mathbb{P}^n_{\mathbb{Z}}(K)$ is bijective. For a local ring A, maps $\operatorname{Spec}(A) \to \mathbb{P}^n_{\mathbb{Z}}$ must land in one $\mathbb{A}^n_{\mathbb{Z}}$ because the only open containing the closed point is all of $\operatorname{Spec}(A)$ so if $\mathfrak{m}$ lands in $\mathbb{A}^n_{\mathbb{Z}}$ then all of $\operatorname{Spec}(A)$ does (the preimage of $\mathbb{A}^n_{\mathbb{Z}}$ is an open containing $\mathfrak{m}$). This yields,

$$\mathbb{P}^n_{\mathbb{Z}}(A) = \frac{A^{n+1} \setminus \mathfrak{m}^{n+1}}{A^\times}$$

we can interpret this as saying,

$$
\begin{array}{ccc}
 & & \mathbb{A}^{n+1}_{\mathbb{Z}} \setminus \{0\} \\
 & \overset{\exists}{\nearrow} & \downarrow \\
\operatorname{Spec}(A) & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}}
\end{array}
$$

But by clearing denominators,

$$\frac{V^{n+1} \setminus (\varpi)^{n+1}}{V^\times} \xrightarrow{\sim} \frac{K^{n+1} \setminus \{0\}}{K^\times}$$

24 Nov. 17 Hilbert Functors and Quasi-Coherent Sheaves

To get a handle on properness i n general settings, we use Chow's lemma.

Lemma 24.0.1. For $f : X \to S$ with S noetherian and f separated of finite type. Then there exists $\pi : X' \twoheadrightarrow X$ projective surjective birational (there are dense opens $U \subset X$ and $U' \subset X'$ such that $\pi : U' \xrightarrow{\sim} U$ is an isomorphism) such that $X' \to S$ is quasi-projective.

Remark. We see that $X \to S$ is proper iff $X' \to S$ is surjective.

Remark. For $S = \operatorname{Spec}(\mathbb{C})$. We can use Chow's lemma to show that a separated finite type $X \to \operatorname{Spec}(\mathbb{C})$ is proper iff $X(\mathcal{C})$ is compact for the complex topology. We use Chow's lemma to reduce to showing that an immersion of finite type $\mathbb{C}$-schemes is closed iff its image on $\mathbb{C}$-points is closed for complex topology.

24.1 Hilbert Functors

Remark. Grothendieck named these for Hilbert because the moduli space is stratified by Hilbert polynomial. Chow defined the "Chow variety" which encoded "Chow forms" in projective space that, up to some numerical invariants, classified certain subvarieties. However this was not a functorial construction and only worked for certain subvarieties not all subschemes. Grothendieck's Hilbert scheme actually represents the correct functor.

Definition 24.1.1. Let $f : X \to S$ be proper with S noetherian. Let $\underline{\operatorname{Hilb}}_{X/S} : \mathbf{Sch}_S \to \mathbf{Set}$ be the functor,

$$\underline{\operatorname{Hilb}}_{X/S}(T) = \{Z \hookrightarrow X_T \mid \mathscr{I}_Z \subset \mathcal{O}_{X_I} \text{ is locally finitely generated and } Z \to T \text{ is flat}\}$$

and for $T' \to T$ we define $\underline{\operatorname{Hilb}}_{X/S}(T) \to \underline{\operatorname{Hilb}}_{X/S}(T')$ via $(Z \hookrightarrow X_T) \mapsto (Z \times_T T' \hookrightarrow X_{T'})$.

Remark. The locally finite generated assumption is redundant for T locally noetherian and the flatness is redundant for $T = \mathrm{Spec}\,(k)$.

Example 24.1.2. For $T = S$ then,

$$\underline{\mathrm{Hilb}}_{X/S}(S) = \{Z \hookrightarrow X \mid \mathscr{I}_Z \subset \mathcal{O}_X \text{ locally finitely generated and } Z \to S \text{ flat }\}$$

so for $S = \mathrm{Spec}\,(k)$ we see that,

$$\underline{\mathrm{Hilb}}_{X/S}(S) = \{Z \hookrightarrow X\}$$

Definition 24.1.3. If $\underline{\mathrm{Hilb}}_{X/S}$ is representable then there is a representing scheme $\mathrm{Hilb}_{X/S}$ called the *Hilbert scheme of* $f : X \to S$ equipped with the "universal closed subscheme" of X,

$$
\begin{array}{ccc}
\mathcal{Z} & \longrightarrow & \mathrm{Hilb}_{X/S} \times_S X \\
& \searrow & \downarrow \\
& & \mathrm{Hilb}_{X/S}
\end{array}
$$

meaning for any $Z \hookrightarrow X_T$ there is a unique map $T \to \mathrm{Hilb}_{X/S}$ with $\mathcal{Z} \times_{\mathrm{Hilb}_{X/S}} T = Z$ inside X_T.

Remark. For $s : \mathrm{Spec}\,(k) \to S$ the fibers of $\mathcal{Z}$ over $\mathrm{Hilb}_{X/S}(k)$ *are* the closed subschemes of X_s.

Theorem 24.1.4 (G)**.** If f is projective then $\mathrm{Hilb}_{X/S}$ exists and is a countable disjoint union of quasi-projective S-schemes.

Proof. For a DVR V and $K = \mathrm{Frac}\,(V)$. Consider $\varphi : \mathrm{Spec}\,(V) \to S$ we want

$$\mathrm{Hilb}_{X/S}(V) \to \mathrm{Hilb}_{X/S}(K)$$

bijective. This means for,

$$X' = X \times_\varphi \mathrm{Spec}\,(V)$$

we want,

$$\{V\text{-flat closed } Z \hookrightarrow X'\} \to \{\text{closed } \widetilde{Z} \hookrightarrow X'_X\}$$

sending $Z \mapsto Z_K$ to be bijective. $\qquad \square$

Lemma 24.1.5. For any finite-type V-scheme X',

$$\{V\text{-flat closed } Z \hookrightarrow X'\} \to \{\text{closed } \widetilde{Z} \hookrightarrow X'_X\}$$

where $X = X'_K$.

Proof. By uniqueness aspect it suffices to prove this statement locally because then we get unique gluing. Thus let $X' = \mathrm{Spec}\,(A)$ be affine. Then we consider,

$$\{I \subset A \text{ with } A/I \text{ torsion-free }\} \to \{\widetilde{I} \subset A_K\}$$

We define the inverse via $\widetilde{I} \mapsto I = A \cap \widetilde{I}$ and $A/I \hookrightarrow A_K/\widetilde{I}$ so A/I is torsion-free. Then we check that these are inverse constructions. $\qquad \square$

24.2 Quasi-Coherent Sheaves

For a scheme X, seek special class of $\mathcal{O}_X$-modules $\mathscr{F}$ so that for affine opens $\operatorname{Spec}(A) \subset X$ then sheaf $\mathscr{F}|_{\operatorname{Spec}(A)}$ "is an A-module".

Remark. When building the structure sheaf on $\operatorname{Spec}(A)$ only the A-module structure was used in the construction. Thus given an A-module M and a basic affine $U = D(f)$ then we can define $M_U = S_U^{-1} M$ where,

$$S_U = \{a \in A \mid \forall u \in U : a(u) \neq 0\}$$

and $U \mapsto M_U$ is a B-sheaf.

Definition 24.2.1. Then $\widetilde{M}$ is the $\mathcal{O}_A$-module arising from M so,

$$\Gamma(D(f), \widetilde{M}) = M_f$$

as A_f-modules.

Proposition 24.2.2. Let $X = \operatorname{Spec}(A)$. For any $\mathcal{O}_A$-module $\mathscr{G}$ and A-module M there is a natural isomorphism,

$$\operatorname{Hom}\left(\mathcal{O}_A, \widetilde{M}\right) \mathscr{G} = \operatorname{Hom}(A, M) \mathscr{G}(X)$$

Proof. Consider,

$$
\begin{array}{ccc}
M & \longrightarrow & \Gamma(X, \mathscr{G}) \\
\downarrow & & \downarrow \\
M_f & \dashrightarrow & \Gamma(D(f), \mathscr{G})
\end{array}
$$

and $\Gamma(D(f), \mathscr{G})$ is an A_f-module so $M_f \to \Gamma(D(f), \mathscr{G})$ is uniquely determined by $M \to \Gamma(X, \mathscr{G})$. $\quad\square$

Definition 24.2.3. Let $f : X \to Y$ be a map of ringed spaces. We have $f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$ given by $\mathscr{F} \mapsto f_*\mathscr{F}$ and $f_*\mathscr{F}$ is a $f_*\mathcal{O}_X$-module so given a $\mathcal{O}_Y$-module structure via $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Then we define $f^* : \mathbf{Mod}_{\mathcal{O}_Y} \to \mathbf{Mod}_{\mathcal{O}_X}$ via,

$$\mathscr{G} \mapsto f^*\mathscr{G} = \mathcal{O}_X f^{-1} \otimes_{f^{-1}\mathcal{O}_Y} \mathscr{G}$$

using the adjoint map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ which is a sheaf of rings. Then tensor product gives $f^*\mathscr{G}$ a $\mathcal{O}_X$-module structure.

Proposition 24.2.4. We have adjointness,

$$\operatorname{Hom}(\mathcal{O}_X, f^*\mathscr{G}) \mathscr{F} = \operatorname{Hom}(\mathcal{O}_Y, \mathscr{G}) f_*\mathscr{F}$$

Proof. This is just,

$$\operatorname{Hom}(\mathcal{O}_X, f^*\mathscr{G}) \mathscr{F} = \operatorname{Hom}\left(f^{-1}\mathcal{O}_Y, f^{-1}\mathscr{G}\right) \mathscr{F} = \operatorname{Hom}(\mathcal{O}_Y, \mathscr{G}) f_*\mathscr{F}$$

where we used the adjointness $\operatorname{Hom}(, f^{-1}\mathscr{G}) \mathscr{F} = \operatorname{Hom}(, \mathscr{G}) f_*\mathscr{F}$ and the fact that it preserves the module structure. $\quad\square$

Theorem 24.2.5. For $f : X \to Y$ with $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$.

(a) $M \mapsto \widetilde{M}$ for A-modules is fully faithfully and exact

(b) $\widetilde{M_1} \otimes_{\mathcal{O}_Y} \widetilde{M_2} \xrightarrow{\sim} (\widetilde{M_1 \otimes_A M_2})$

(c) $\varinjlim \widetilde{M_i} = \varinjlim \widetilde{M_i}$ (e.g. for direct sums)

(d) $f_*(\widetilde{N}) = \widetilde{N_A}$ with N_A being N viewed as an A-module

(e) $f^*(\widetilde{M}) = \widetilde{B \otimes_A M}$

(f) $\widetilde{\text{Hom}(A, M')}\, M = \mathscr{H}\!om_{\mathcal{O}_Y}\!\left(\widetilde{M'}, \widetilde{M}\right)$ for finitely presented M'.

Definition 24.2.6. We say that $\mathscr{F}$ is quasi-coherent on a scheme X if $\mathscr{F}|_U = \widetilde{M}$ on affine opens.

25   Nov. 19

Remark.
$$\text{Hilb}^n_{X/k}(T) = \{Z \hookrightarrow X_T \mid \pi : Z \to T \text{ finite flat of degree } n\}$$

with T locally noetherian is "compactified configuration space". For X a smooth projective curve then $\text{Hilb}^n_{X/k} = X^n/\Sigma_n$.

25.1   Localization Criterion

Proposition 25.1.1. Let $X = \text{Spec}(A)$ and $\mathscr{F}$ is an $\mathcal{O}_X$-module. For any $a \in A$ the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(D(a), \mathscr{F})$ is linear over $A \to A_f$ and thus factors,

$$\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})_a \xrightarrow{\theta_a} \Gamma(D(a), \mathscr{F})$$

Then $\mathscr{F}$ is quasi-coherent iff each θ_a is an isomorphism.

Proof. Let $M = \Gamma(X, \mathscr{F})$ so the map $\text{id}_M : M \to \Gamma(X, \mathscr{F})$ induces $\varphi : \widetilde{M} \to \mathscr{F}$. If $\mathscr{F}$ is quasi-coherent then φ is an isomorphism. If φ is an isomorphism then by definition $\mathscr{F}$ is quasi-coherent. By φ is an isomorphism iff it is locally an isomorphism on $D(a)$ but this is the map $M_a \to \Gamma(X, \mathscr{F})_a \xrightarrow{\theta_a} \Gamma(D(a), \mathscr{F})$ and $M = \Gamma(X, \mathscr{F})$ so the first map is an isomorphism and thus φ is an isomorphism over $D(a)$ iff θ_a is an isomorphism. $\square$

Definition 25.1.2. A $\mathcal{O}_X$-module $\mathscr{F}$ is *finite type* if there exists an open cover $\{U_i\}$ of X and $\mathcal{O}_{U_i}^{\oplus n_i} \twoheadrightarrow \mathscr{F}|_{U_i}$ for all i.

Remark. For open $U \hookrightarrow X$ there is a crazy ideal sheaf $j_!(\mathcal{O}_U) \subset \mathcal{O}_X$ which is almost never quasi-coherent. But $\mathcal{O}_X/j_!(\mathcal{O}_U)$ is finite type and not (usually) quasi-coherent.

Proposition 25.1.3. For $X = \text{Spec}(A)$ and $\mathscr{F} = \widetilde{M}$ quasi-coherent then $\mathscr{F}$ is finite type iff M is finitely generated.

Proof. If M is finitely generated then $A^{\oplus n} \twoheadrightarrow M$ gives $\mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathscr{F}$ and thus it is globally finite type and thus finite type.

Conversely, if $\mathscr{F}$ is finite then we can refine the open cover by basic affines which has a finite subcover since X is quasi-compact. Thus we get $D(a_i)$ with,

$$\mathcal{O}_{A_{a_i}}^{\oplus n_i} \twoheadrightarrow \mathscr{F}|_{D(a_i)} = \widetilde{M_{a_i}}$$

Therefore, we get a map $A_a^{\oplus n} \twoheadrightarrow M_{a_i}$ so M_{a_i} is locally finitely generated and thus finitely generated. $\square$

Definition 25.1.4. For X locally noetherian, an $\mathcal{O}_X$-module $\mathscr{F}$ is *coherent* if $\mathscr{F}$ is quasi-coherent and finite type.

Remark. In the non-noetherian case, this is the wrong condition. Warning [H,p.111] does this without requiring locally noetherian: BAD.

Theorem 25.1.5.

quasi-coherence is stable under $\otimes, \varinjlim$ and affine pushforward

if $\varphi : \mathscr{F} \to \mathscr{G}$ is a $\mathcal{O}_X$-linear map of quasi-coherent sheaves then $\ker \varphi$ and $\operatorname{coker} \varphi$ and $\operatorname{im} \varphi$ are quasi-coherent.

If X is locally noetherian then (b) also holds for coherent sheaves as does (a) for $\otimes$ and *finite* pushforward. If $\mathscr{F}$ and $\mathscr{G}$ are coherent then $\mathscr{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is coherent.

Proof. These are all of local nature (we assumed the maps are affine). Then we apply properties of $\widetilde{(-)}$. $\square$

Example 25.1.6. For closed immersion $\iota : Y \hookrightarrow X$ with X a scheme then $\mathscr{I}_Y = \ker(\mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Y)$ is quasi-coherent ideal sheaf. In reverse, if $\mathscr{I} \subset \mathcal{O}_X$ is quasi-coherent, then $\mathscr{I} = \mathscr{I}_Y$ for a unique closed subscheme $Y \hookrightarrow X$.

Remark. It is very fruitful to associate an ideal sheaf $\mathscr{I}_{\{x\}} \subset \mathcal{O}_X$ to a closed point $x \in X$ on a smooth curve.

Proposition 25.1.7. Let X be locally noetherian $\mathscr{F}, \mathscr{G}$ coherent on X then,

(a) Given $\varphi : \mathscr{F} \to \mathscr{G}$ with $\varphi_x : \mathscr{F}_x \xrightarrow{\sim} \mathscr{G}_x$ then $\varphi|_U : \mathscr{F}|_U \to \mathscr{G}|_U$ is an isomorphism on some open neighborhood

(b) Any $\mathcal{O}_{X,x}$-linear map $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ comes from $\varphi_U : \mathscr{F}|_U \to \mathscr{G}|_U$ for *some* open U and has a uniquely determined germ.

These together mean that the natural map,

$$\mathscr{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_x \to \mathscr{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

is an isomorphism.

Proof. We see that $(\ker \varphi)_x = \ker \varphi_x = 0$ and $(\operatorname{coker} \varphi)_x = \operatorname{coker} \varphi_x = 0$ and $\operatorname{coker} \varphi$ and $\ker \varphi$ are coherent. Now, if $\mathscr{K}$ is finite type and $\mathscr{K}_x = 0$ then $\mathscr{K}|_U = 0$ for some neighborhood U (because we just need to kill all the generators. Thus $\ker \varphi|_U = \operatorname{coker} \varphi|_U = 0$ so φ_U is an isomorphism.

We use a local finite presentation,

63

$$\mathcal{O}_U^{\oplus s} \longrightarrow \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

near x to reduce to the case $\mathcal{F} = \mathcal{O}_U^{\oplus m}$. $\qquad\square$

Theorem 25.1.8. For $f : X \to Y$ a morphism of schemes,

    (a) f^* of a quasi-coherent is quasi-coherent and of a coherent is coherent (in loc. noetherian case)

    (b) if f is qcqs then f_* of a quasi-coherent is quasi-coherent

Proof. For (a) WLOG X, Y are affine. For (b) use an exact sequence trick to reduce to affine case. $\qquad\square$

Proposition 25.1.9. Consider a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

if $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent then $\mathcal{F}$ is coherent.

26   Nov. 29

26.1   Quasi-Coherent Sheaves on Proj

Just like how Proj is not a functor for general maps of graded rings we will see that we lose information when we pass from graded modules to quasi-coherent sheaves.

Fix some notation. Let,

$$S = \bigoplus_{n \geq 0} S_n$$

an $\mathbb{N}$-graded ring, $\mathrm{Proj}\,(S)$ is covered by affine opens,

$$D_+(f) = \mathrm{Spec}\left(S_{(f)}\right)$$

for homogeneous elements $f \in S_+$.

Remark. If $\{f_i\} \subset S_+$ generate,

$$\bigoplus_{n \geq n_0} S_n$$

then $\{D_+(f_i)\}$ cover $\mathrm{Proj}\,(S)$ since we can rewrite any fraction with a large enough denominator.

    Let M be a $\mathbb{Z}$-graded S-module,

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

meaning $S_a \times M_b \to M_{a+b}$.

Example 26.1.1. Consider the following examples,

    (a) let $M = I \subset S$ a homogeneous ideal

    (b) $M = S(r)$ is the S-module with $M_d = S_{r+d}$ so $S(r)_0 = S_r$ for any $r \in \mathbb{Z}$

Example 26.1.2. Let $S = A[x_0, \ldots, x_n]$ then,

$$0 \longrightarrow S(-1) \xrightarrow{x_0} S \longrightarrow S/(x_0) \longrightarrow 0$$

is an exact sequence of graded rings. Notice we needed to introduce the twist by -1 to make the map $S(-1) \to S$ a graded map since $1 \mapsto x_0$ so 1 needs to have degree 1 in $S(-1)$.

Definition 26.1.3. Define $\widetilde{M}$ on $\mathrm{Proj}\,(S)$ by $\widetilde{M}|_{D_+(f)} = \widetilde{M_{(f)}}$ on $\mathrm{Spec}\,\left(S_{(f)}\right) = D_+(f)$ where $M_{(f)}$ is the degree zero submodule of the $\mathbb{Z}$-graded module M_f. The gluing are given by,

$$\left(M_{(f)}\right)_{\frac{g^{\deg f}}{f^{\deg f}}} = M_{(fg)} = \left(M_{(g)}\right)_{\frac{f^{\deg g}}{g^{\deg f}}}$$

Remark. This process loses low degree information from M. Indeed, we can write,

$$\frac{m}{f^r} = \frac{mf^n}{f^{r+n}}$$

then mf^n can be made to have positive degrees. Therefore, if we get,

$$M' = \bigoplus_{n \geq n_0} M_n$$

then $M' \hookrightarrow M$ yields $\widetilde{M'} \xrightarrow{\sim} \widetilde{M}$. Therefore, if $M_n = 0$ for $n \gg 0$ then $\widetilde{M} = 0$.

Remark. We define, $M \otimes_S N = (M \otimes_{\mathbb{Z}} N)/(sm \otimes n - m \otimes sn)$ for m, n, s all homogeneous. This solves the problem of $M \otimes_S N$ not having an obvious grading because the graded pieces of M, N are not S-modules but rather S_n-modules which is not what the tensor-product is over.

Theorem 26.1.4. The following hold in the setting of associated sheaves on $\mathrm{Proj}\,(S)$,

(a) the functor $M \mapsto \widetilde{M}$ is exact and preserves $\varinjlim$

(b) for $\mathfrak{p} \in \mathrm{Proj}\,(S)$ the stalk is $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$ the degree zero part of the localization of M at the homogeneous elements of $S_+ \setminus \mathfrak{p}$.

(c) $\widetilde{M}$ is quasi-coherent

(d) the canonical map $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}$ built from $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ (which is an isomorphism if $f \in S_1$ but otherwise not necessarily surjective because it is not clear how to split up fractions) is an isomorphism when S is generated by S_1 over S_0

Proof. Compute over $D_+(f)$. $\square$

Corollary 26.1.5. If S is generated by S_1 over S_0 then for homogeneous ideal $I \subset S$,

$$\widetilde{IM} = \widetilde{I} \cdot \widetilde{M}$$

considering the ideal sheaf $\widetilde{I} \subset \widetilde{S} = \mathcal{O}_X$.

Proof. $I \otimes_S M \twoheadrightarrow IM$ and under the hypothesis we see that $\widetilde{IM}$ is the image of $\widetilde{I} \otimes_{\mathcal{O}_X} \widetilde{M} \to \widetilde{M}$. $\square$

Remark. For the interaction of $\widetilde{}$ with Hom f_* and f^* see section 1-2 of "further properties of $\widetilde{M}$".

Example 26.1.6. Consider the exact sequence,

$$0 \longrightarrow S(-1) \xrightarrow{x_0} S \longrightarrow S/(x_0) \longrightarrow 0$$

for $S = A[x_0, \dots, x_n]$. This yields the sequence,

$$0 \longrightarrow \widetilde{S(-1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} j_*(\mathcal{O}_{\mathbb{P}^{n-1}}) \longrightarrow 0$$

identifies $\widetilde{S(-1)}$ with $\mathscr{I}_H$ as $\mathcal{O}_{\mathbb{P}^n_A}$-modules where $H = V(x_0)$ is the hyperplane cut out by x_0. Now applying global sections,

$$0 \longrightarrow \Gamma(\widetilde{S(-1)}) \longrightarrow \Gamma(\mathcal{O}_{\mathbb{P}^n_A}) \longrightarrow \Gamma(\mathcal{O}_{\mathbb{P}^{n-1}})$$

but $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$ and thus the second map is $\mathrm{id} : A \to A$ so $\Gamma(\widetilde{S(-1)}) = \ker \mathrm{id} = 0$. Thus $\widetilde{S(-1)}$ has no global sections. But,

$$AX_0 \oplus \cdots \oplus AX_n = S(1)_0 \xrightarrow{\sim} \Gamma(\widetilde{S(1)})$$

Remark. Usually, $M_0 \to \Gamma(\widetilde{M})$ is not an isomorphism.

Example 26.1.7. When S is not generated by S_1 over S_0, weird things can happen: $S = k[X_1, X_2, X_3]$ where $\deg x_0 = i$. Let $I = (X_1)$. Then we get an ideal sheaf $\widetilde{I}$ on $\mathrm{Proj}\,(S) = X$. However, $\widetilde{I}$ is not locally principal. Look at $D_+(f)$ for $f = X_2$. Then,

$$D_+(f) = \mathrm{Spec}\,\left(S_{(X_2)}\right) \xrightarrow{\sim} k[u, v, w]/(uw - v^2)$$

where,

$$u \mapsto \frac{X_1^2}{X_2}$$
$$v \mapsto \frac{X_1 X_3}{X_2^2}$$
$$w \mapsto \frac{X_3^2}{X_2^3}$$

Then we see that $\Gamma(D_+(f), \widetilde{I}) = I_{(f)} = (u, v)$.

Remark. Some questions,

(a) If S is noetherian, is $\mathrm{Proj}\,(S)$ noetherian?

(b) If S is noetherian and M is finite, is $\widetilde{M}$ coherent?

Notice that S_+ has a *finite* generating set $\{f_1, \dots, f_r\}$ of homogeneous elements (S_+ is finitely generated by noetherian property and then each generator has a finite set of generators). For homogeneous $f \in S_+$ and $\deg f = d > 0$, is $S_{(f)}$ noetherian? Notice this is not immediate because noetherianness is not preserved under taking subrings (every domain lies in a field). However, let $S^{(d)} = \bigoplus_{d \mid n} S_n \subset S$ this is a S_0-subalgebra. Then,

$$S_{(f)} \xrightarrow{\sim} S^{(d)}/(f - 1)$$

via $\frac{h}{f^m} \mapsto h$ in S_{dm}. Therefore, it is enough to show that $S^{(d)}$ is noetherian for all $d \geq 0$. Notice that,

$$S_0 \xrightarrow{\sim} S/S_+$$

so S_0 is noetherian. If $d_i = \deg f_i > 0$ then $S_0[X_1, \ldots, X_r] \twoheadrightarrow S$ for $\deg X_i = d_i$. Therefore, S is finitely generated as a S_0-algebra. Can leverage properties of $S_0[X_1, \ldots, X_r]$ to deduce affirmative answers to all these questions. The key part is that for $d \gg 0$ we have $S^{(d)}$ generated by S_d over S_0 and also $M^{(d)}$ is $S^{(d)}$-finite if M is S-finite and $M^{(d)} = \widetilde{M}$. Then we see that,

$$\operatorname{Proj}(S) \xrightarrow{\sim} \operatorname{Proj}\left(S^{(d)}\right) \hookrightarrow \mathbb{P}_{S_0}^{n-1}$$

Proposition 26.1.8. If S is generated by S_1 over S_0 and M is S-finite,

$$\widetilde{M} = 0 \iff M_n = 0 \text{ for all } n \gg 0$$

Remark. Is S_1-generating hypothesis necessary. It turns out NO [EGA II, 2.7.3(ii)] removes it when S_+ is finitely generated over S_0. EXCEPT let,

$$S = k[X^2]$$

and $M = S(1)$ then it is zero in even degrees so $\widetilde{M} = 0$ because S is supported in even degrees. There is an error in EGA where they cite a result with these hypotheses but forget them.

27 Dec. 1 The Serre Twist, I

Example 27.0.1. $S = A[X_0, \ldots, X_n]$ then,

$$S(1)_0 = AX_0 \oplus \cdots \oplus AX_n \xrightarrow{\sim} \Gamma(\mathbb{P}_A^n, \widetilde{S(1)})$$

Each,

$$\Gamma(\mathbb{P}_A^n, \widetilde{S}(1)) \to \Gamma(D_+(X_i), \widetilde{S(1)}) \to \Gamma(D_+(X_0 \cdots X_n), \widetilde{S(1)}) \subset A[X_0, \ldots, X_n]_{(X_0 \cdots X_n)}$$

This equals,

$$S(1)_{(x_i)} = S_{(x_i)} X_i = A[\tfrac{X_0}{X_i}, \ldots, \tfrac{X_n}{X_i}] X_i$$

Then,

$$\Gamma(\mathbb{P}_A^n, \widetilde{S(1)}) = \bigcap_i A[\tfrac{X_\bullet}{X_i}] X_i = AX_0 \oplus \cdots \oplus AX_n$$

For S an $\mathbb{N}$-graded ring $X = \operatorname{Proj}(S)$ for Z-graded modules M, N have,

$$M_{(f)} \otimes_{S_{(f)}} N_{(f)} = (M \otimes_S N)_{(f)}$$

yields $\widetilde{M} \otimes \widetilde{N} \to \widetilde{M \otimes S}$.

For T a $\mathbb{Z}$-graded S-algebra, $\widetilde{T}$ is a $\mathcal{O}_X$-module with maps $T \otimes T \to T$ giving $\widetilde{T} \otimes \widetilde{T} \to \widetilde{T \otimes T} \to \widetilde{T}$.

Definition 27.0.2. For $n \in \mathbb{Z}$ then $\mathcal{O}_X(n) = \widetilde{S(n)}$ (depending on S not just X).

Proof. For $\mathcal{O}_X$-module $\mathscr{F}$, define the Serre twists $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. $\qquad \square$

Remark. Consider $\mathscr{F} \mapsto \mathscr{F}(n)$ as sheaf version of $M \mapsto M(n)$ and $\Gamma(X, \mathscr{F}(n))$ is the sheaf version of $M \mapsto M_n$.

Theorem 27.0.3. Assume S is generated by S_1 over S_0. Let M be a $\mathbb{Z}$-graded module. Then,

(a) $\widetilde{M}(n) \xrightarrow{\sim} \widetilde{M(n)}$

(b) $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \xrightarrow{\sim} \mathcal{O}_X(n+m)$ making all the right diagrams commute

(c) $\mathcal{O}_X(n)$ are invertible (locally free of rank 1) where we have seen,

$$\mathcal{O}_X(1)^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X(n)$$

and,

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-n) \xrightarrow{\sim} \mathcal{O}_X(0) = \mathcal{O}_X$$

meaning that $\mathcal{O}_X(-n) \cong \mathcal{O}_X(n)^\vee$.

Proof.

$$\widetilde{M}(n) = \widetilde{M} \otimes \mathcal{O}_X(n) = \widetilde{M} \otimes \widetilde{S(n)} \xrightarrow{\sim} \widetilde{M \otimes_S S(n)} = \widetilde{M(n)}$$

where the tensor commuting with tilde map is an isomorphism since S_1 generates. Furthermore,

$$\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) = \widetilde{S(n)} \otimes \widetilde{S(m)} \xrightarrow{\sim} \widetilde{S(n)(m)} = \widetilde{S(n+m)} = \mathcal{O}_X(n+m)$$

$\square$

Example 27.0.4. When $X = \mathrm{Proj}\,(A[T_0, \dots, T_n])$ then $\mathcal{O}_X(1) = \widetilde{S(1)}$ has global sections,

$$\Gamma(X, \mathcal{O}_X(1)) = AT_0 \oplus \cdots \oplus AT_n$$

This is the universal data on $\mathbb{P}^n_A$ in sense of Exercise B(y) HW 9. Furthermore,

$$\mathcal{O}_X(1)|_{D_+(T_i)} = \mathcal{O}_{D_+(T_i)} \cdot T_i$$

and likewise,

$$\mathcal{O}_X(n)|_{D_+(T_i)} = \mathcal{O}_{D_+(T_i)} \cdot T_i^n$$

Example 27.0.5. Ex. 2.2 in "Further Properties of $\widetilde{M}$"). The d-uple embedding $\iota : \mathbb{P}^n_A \hookrightarrow \mathbb{P}^N_A$ defined by $[t_0 : \cdots : t_n] \mapsto [t^I]_{\sum t_i = d}$. This pulls back hyperplanes to degree d hypersurfaces. Therefore we get,

$$\iota^* \mathcal{O}_{\mathbb{P}^N_A}(1) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n_A}(d)$$

Example 27.0.6. Let $k = \bar{k}$. We've seen that $\mathcal{O}_{\mathbb{P}^n_k}(-1)$ is (isomorphic to) the ideal sheaf of every hyperplane $H \subset \mathbb{P}^n_A$ as $\mathcal{O}_{\mathbb{P}^n}$-modules (since $\mathrm{GL}_{n+1}(k) \circlearrowright k[x_0, \dots, x_n]$ as a graded ring). Consider a smooth closed curve $j : X \hookrightarrow \mathbb{P}^2_k$ which is not a line. Let $L \subset \mathbb{P}^2$ be a line. Then $\mathscr{I}_L \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^2}(-1)$. Then applying ι^* we get,

$$j^* \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow j^* \mathcal{O}_{\mathbb{P}^2}(-1) = \mathcal{O}_X$$

has image exactly $\mathscr{I}_{X \cap L} \subset \mathcal{O}_X$ using a local calculation. But $\mathcal{O}_X$ is a sheaf of Dedekind domains and $\mathscr{I}_{X \cap L}$ is a nonzero ideal and thus invertible. Therefore,

$$j^* \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\sim} \mathscr{I}_{X \cap L}$$

is an isomorphism.

Definition 27.0.7. Let S be an $\mathbb{N}$-graded ring and $X = \mathrm{Proj}\,(S)$ and $\mathscr{F}$ a $\mathcal{O}_X$-module. Define,

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{F}(n))$$

is a graded module over,

$$\Gamma_*(\mathcal{O}_X, \mathcal{O}_X(n))$$

which is an $\mathbb{N}$-graded ring but may not be generated by degree 1.

Remark. We have,

$$S = \bigoplus_{n \geq 0} S_n = \bigoplus_{n \geq 0} (S(n))_0 \to \Gamma_*(\mathcal{O}_X)$$

as $\mathbb{N}$-graded rings. Therefore $\Gamma_*(\mathscr{F})$ is a graded S-module.

Remark. We saw that the above map is an isomorphism if $S = A[T_0, \ldots, T_n]$.

Now we assume that S is generated by S_1 over S_0. For an $\mathbb{N}$-graded modules M then consider,

$$\alpha_M : M = \bigoplus_{n \geq 0} M_n = \bigoplus_{n \geq 0} M(n)_0 \to \bigoplus_{n \geq 0} \Gamma(X, \widetilde{M(n)}) = \Gamma_*(\widetilde{M})$$

which is linear over,

$$\alpha_S : S \to \Gamma_*(\mathcal{O}_X)$$

For a $\mathcal{O}_X$-module $\mathscr{F}$ we get,

$$\beta_{\mathscr{F}} : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

using,

$$\Gamma_*(\mathscr{F})_{(f)} \to \Gamma(D_+(f), \mathscr{F})$$

for $f \in S_d$ over $S_{(f)}$. This map gives,

$$\Gamma_*(\mathscr{F})_{(f)} \to \Gamma(D_+(f), \mathscr{F}(dr)) \otimes_{S_{(f)}} S(-rd)_{(f)}$$

given by sending,

$$\frac{m}{f^r} \mapsto m \otimes \frac{1}{f^r}$$

and the right-hand side is equal to,

$$\Gamma(D_+(f), \mathscr{F}(dr)) \otimes_{S_{(f)}} S(-rd)_{(f)} = \Gamma(D_+(f), \mathscr{F}(dr)) \otimes \Gamma(D_+(f), \mathcal{O}_X(-rd)) \to \Gamma(D_+(f), \mathscr{F}(dr)(-dr))$$
$$= \Gamma(D_+(f), \mathscr{F})$$

28   Dec. 3

The ring $\Gamma_*(\mathscr{F}, \mathcal{L})$ is a graded module over $\Gamma_*(\mathcal{O}_X, \mathcal{L}) = \mathfrak{S}$. Then we construct a canonical map,

$$\beta_f : \Gamma_*(\mathscr{F}, \mathcal{L})_{(f)} \to \Gamma(X_f, \mathscr{F})$$

defined by sending,

$$\frac{s_n}{f^n} \mapsto a_n$$

for $s_n \in \Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n})$ where $s_n|_{X_f} = f^{\otimes n} a_n$ because,

$$s_n|_{X_f} \in \Gamma(X_f, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) \quad \text{and} \quad \mathcal{L}^{\otimes n}|_{X_f} \cong f^{\otimes n} \cdot \mathcal{O}_{X_f}$$

In the key example, $\Gamma_*(\mathscr{F})$ is an S-module via,

$$S \to \Gamma_*(\mathcal{O}_X)$$

Lemma 28.0.1. if X is qcqs then β_f is an isomorphism.

Proof. Focus on qc and separated. Let $\{U_i\}$ be a finite affine open cover of X trivializing $\mathcal{L}$ and such that $U_i \cap U_j$ affine because we assumed that X is separated. We reduce the problem to the affine case. The β_f construction is <u>functorial</u> with respect to restriction along open immersions $U \hookrightarrow X$ since $U_{(f|_U)} = U \cap X_f$. We have a diagram,

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma_*(\mathcal{F}, \mathcal{L})_{(f)} & \longrightarrow & \prod_i \Gamma(\mathcal{F}_i, \mathcal{L}_i) & \longrightarrow & \prod_{i,j} \Gamma_*(\mathcal{F}_{ij}, \mathcal{F}_{ij}) \\
& & \downarrow{\scriptstyle \beta_f} & & \downarrow{\scriptstyle \prod \beta_{f_i}} & & \downarrow{\scriptstyle \prod_{ij} \beta_{f_{ij}}} \\
0 & \longrightarrow & \Gamma(X_f, \mathcal{F}) & \longrightarrow & \prod_i \Gamma((U_i)_{f_i}, \mathcal{F}_i) & \longrightarrow & \Gamma((U_{ij})_{f_{ij}}, \mathcal{F}_{ij})
\end{array}
$$

commutes and therefore we reduce to showing that the β map is an isomorphism on affines because the diagram proves that β_f is an isomorphism on X.

Let $X = \operatorname{Spec}(A)$ and $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{F} = \widetilde{A}$. Then,

$$
\Gamma_*(\mathcal{F}, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X^{\otimes n}) = \bigoplus_{n \geq 0} M = M[T]
$$

Then $f \in A$ is some element and,

$$
\mathfrak{S} = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X^{\otimes n}) = A[T]
$$

Then we have the map,

$$
\beta_f : M[T]_{(fT)} \to \Gamma(X_f, \widetilde{M}) = M_f
$$

but the degree 0 fractions in $M[T]_{fT}$ have to be M_f so this map is an isomorphism. $\qquad\square$

Corollary 28.0.2. For $Y \hookrightarrow \mathbb{P}_A^n = X$ a closed subscheme. Then,

$$
I = \Gamma_*(\mathscr{I}_Y) \hookrightarrow \Gamma_*(\mathcal{O}_X) = A[T_0, \ldots, T_n]
$$

is an ideal and $Y = \operatorname{Proj}(S/I) \hookrightarrow \operatorname{Proj}(S)$ as schemes.

Corollary 28.0.3. If S_1 is S_0-finite and generates S over S_0 and $\mathcal{F}$ is quasi-coherent, finite type on $X = \operatorname{Proj}(S)$ then $\mathcal{F} = \widetilde{m}$ for S-finite M.

Proof. Choose M such that $\widetilde{M} = \mathcal{F}$. Consider $\{M_i\}$ directed system of finitely generated graded S-submodule of M. Then,

$$
\varinjlim_i M_i = M
$$

Because tilde interacts well with limits and submodules,

$$
\widetilde{M_i} \subset \widetilde{M} = \mathcal{F}
$$

and,

$$
\varinjlim \widetilde{M_i} = \mathcal{F}
$$

Enough to show that $\{\widetilde{M_i}\}$ on X stabilizes. Since X is qc it is enough to check on finitely many affine opens $\operatorname{Spec}(A)$ covering X. But $\mathcal{F}|_{\operatorname{Spec}(A)} = \widetilde{N'}$ for finite N' because $\mathcal{F}$ is finite type. Therefore,

$$
\widetilde{M_i}|_{\operatorname{Spec}(A)} \subset \widetilde{N'}
$$

and N' is finitely generated so eventually this hits each generator of N' so we only need finitely many. $\qquad\square$

Remark. For noetherian S, it is true that $\Gamma_*(\mathcal{F})$ is S-finite but the proof uses sheaf cohomology.

29 Jan. 7

For work with $\mathrm{Cl}\,(X)$ we assume that X is smooth in codimension ≤ 1 meaning the singular locus has codimension ≥ 2. To connected $\mathrm{Cl}\,(X)$ to $\mathrm{Pic}\,(X)$ we need the following property.

Definition 29.0.1. Say X is *locally factorial* if X is noetherian and all $\mathcal{O}_{X,x}$ are UFDs.

Example 29.0.2. If $X = \mathrm{Spec}\,(A)$ for A a UFD then all $S^{-1}A$ are UFDs and hence all local rings are UFDs.

Example 29.0.3. $X = \mathbb{P}^n_R$ is locally factorial for R locally factorial.

Definition 29.0.4. A locally noetherian scheme X is *regular* if all $\mathcal{O}_{X,x}$ are regular.

Definition 29.0.5. Say that a noetherian ring A is *regular* if all $A_\mathfrak{m}$ are regular for $\mathfrak{m} \subset A$ maximal.

Example 29.0.6. If $k = \bar{k}$, and A is a *smooth* k-algebra (in the Jacobian sense) then A is regular.

Theorem 29.0.7 (Serre)**.** Let A be a regular noetherian local ring and $\mathfrak{p} \subset A$ a prime. Then $A_\mathfrak{p}$ is regular.

Corollary 29.0.8. If A is a regular noetherian ring then all $A_\mathfrak{p}$ are regular.

Proof. Choose a maximal ideal $\mathfrak{m} \supset \mathfrak{p}$ then $A_\mathfrak{p} = (A_\mathfrak{m})_{\mathfrak{p}_\mathfrak{m}}$ and $A_\mathfrak{m}$ is regular by hypothesis so $A_\mathfrak{p}$ is regular. $\qquad\square$

Example 29.0.9. If $k = \bar{k}$ and X is a finite type k-scheme then k is smooth if and only if X is regular.

Theorem 29.0.10. Regular local rings are UFDs. Therefore regular schemes are locally factorial.

29.1 Equivalence of Picard Group and Class Group

For the rest of today, X is noetherian, integrally, and locally factorial.

Remark (H)*.* uses an intermediate notion of a Cartier divisor which we avoid. Furthermore, [H] assumes that X is separated which is not needed [EGA IV$_4$, 21.6.10(ii)].

Remark. The role of separateness: if $Y \neq Y'$ are prime divisors then $\mathrm{Spec}\,(\mathcal{O}_{X,\eta_Y})\,, \mathrm{Spec}\,(\mathcal{O}_{X,\eta'_Y}) \to X$ have $\eta_Y \neq \eta_{Y'}$ and therefore the valuations of Y and Y' on $K(X)$ (using the valuative criterion of separatedness).

Proposition 29.1.1. For X as above, if $Y \subset X$ is a prime divisor then $\mathscr{I}_Y \subset \mathcal{O}_X$ is invertible.

Proof. Since $\mathscr{I}_Y \subset \mathcal{O}_X$ is coherent it suffices to show that $(\mathscr{I}_Y)_x \subset \mathcal{O}_{X,x}$ is a principal ideal. For $x \notin Y$ we know that $(\mathscr{I}_Y)_x = \mathcal{O}_{X,x}$ so we are done. Otherwise, $(\mathscr{I}_Y)_x = \mathcal{O}_{X,x}$ is a height 1 prime ideal and $\mathcal{O}_{X,x}$ is a UFD so $(\mathscr{I}_Y)_x$ is principal and therefore we are done. $\qquad\square$

Remark. Now we build a map $\mathrm{Div}\,(X) \to \mathrm{Pic}\,(X)$ via $[Y] \mapsto (\mathscr{I}_Y)^{\otimes -1}$ and extending linearly with the tensor product. We need to show this map is surjective and has kernel exactly equal to the principal divisors.

Remark. Consider the map,

$$\bigotimes_j \mathscr{I}_{Y_j}^{\otimes m_j} \twoheadrightarrow \prod \mathscr{I}_{Y_j}^{m_j}$$

for $m_g \geq 0$ where the right hand side is a product of ideals. However, this is a surjection of invertible sheaves and therefore an isomorphism.

Remark. For an open $U \subset X$ we can write,

$$\mathscr{I}_Y(U) = \begin{cases} \mathcal{O}_X(U) & U \cap Y = \varnothing \\ \{f \in \mathcal{O}_X(U) \mid \mathrm{ord}_Y(f) \geq 1\} & U \cap Y \neq \varnothing \end{cases}$$

We need to make the dual of the ideal more concrete. Consider the generic point $\eta : \mathrm{Spec}\,(K(X)) \to X$. This is an affine map so $\eta_*(K(X)) = \underline{K(X)}$ (this is a fancy way of saying that $\mathrm{Frac}\,(A) = K(X)$ for all affine opens $\mathrm{Spec}\,(A) \subset X$). Then we have,

$$\mathscr{I}_Y \hookrightarrow \mathcal{O}_X \to \eta_*\eta^*\mathcal{O}_X = \eta_*(K(X)) = \underline{K(X)}$$

gives $(\mathscr{I}_Y)_\eta = K(X)$. Then,

$$\mathscr{I}_Y^{\otimes -1} \to \eta_*(\eta^*\mathscr{I}_Y^{\otimes -1}) = \eta_*((\mathscr{I}_Y)_\eta^{\otimes -1}) = \eta_*(K(X)^\vee) \cong \eta_*(K(X)) = \underline{K(X)}$$

Proposition 29.1.2. For $U \neq \varnothing$,

$$\mathscr{I}_Y^{\otimes -1}(U) = \begin{cases} \mathcal{O}_X(U) & U \cap Y = \varnothing \\ \{0\} \cup f \in K(X)^\times \mathrm{div} f|_U \geq -[Y \cap U]\} & Y \cap Y \neq \varnothing \end{cases}$$

inside $K(X)$ where this condition on f is equivalent by Hartongs to $f \in \mathcal{O}_X(U \setminus Y)$ and $\mathrm{ord}_Y(f) \geq -1$.

Definition 29.1.3. For $D \in \mathrm{Div}\,(X)$ define,

$$\mathcal{O}_X(D) \subset \underline{K(X)}$$

to be the sheaf,

$$\Gamma(U, \mathcal{O}_X(D)) = \begin{cases} \mathcal{O}_X(U) & U \cap \mathrm{Supp}\,(D) = \varnothing \\ \{0\} \cup \{f \in K(X)^\times \mid \mathrm{div} f|_U + D|_U \geq 0\} \end{cases}$$

Example 29.1.4. $\mathcal{O}_X(Y) = \mathscr{I}_Y^{-1}$ for prime Y.

Proposition 29.1.5. Let,

$$D = \sum n_j Y_j$$

then,

$$\mathcal{O}_X(D) \cong \bigotimes_j \mathcal{O}_X(Y_j)^{\otimes n_j}$$

Remark. In [H] they write $\mathcal{L}(D)$ instead.

Lemma 29.1.6. For $f \in K(X)^\times$ then the map $\underline{K(X)} \xrightarrow{f} \underline{K(X)}$ carries,

$$\mathcal{O}_X(\mathrm{div} f) \xrightarrow{\sim} \mathcal{O}_X$$

Proof. Check on stalks and use that $\mathcal{O}_{X,x}$ is a UFD to write $f \in K(X)^\times$ as a reduced fraction. $\square$

Corollary 29.1.7. We get a map $\mathrm{Cl}\,(X) \to \mathrm{Pic}\,(X)$ via $D \mapsto \mathcal{O}_X(D)$.

Proposition 29.1.8. The above map is an isomorphism.

Proof. For $\mathcal{L} \in \mathrm{Pic}\,(X)$ consider,

$$\mathcal{L} \hookrightarrow \eta_*(\eta^*\mathcal{L}) = \eta_*(\mathcal{L}_\eta) \cong \eta_*(K(X)) = \underline{K(X)}$$

$\square$

30 Jan. 10

30.1 Automorphisms of $\mathbb{P}^n$

There is an action of $\mathrm{GL}_{n+1}(k)$ on $k[t_0, \ldots, t_n]$ respecting the grading which gives an induced right action on $\mathbb{P}^n_k$. Then on $\mathbb{P}^n(\bar{k}) = (\bar{k}^{n+1} - \{0\})/\bar{k}^\times$ this is effect of transpose on $M \in \mathrm{PGL}_{n+1}(\bar{k})$. Therefore we see that,

$$\mathrm{PGL}_{n+1}(k) \subset \mathrm{Aut}\,(\mathbb{P}^n_k)$$

is an inclusion.

Proposition 30.1.1. This is an equality for automorphisms over k.

Proof. Say $\varphi : \mathbb{P}^n_k \xrightarrow{\sim} \mathbb{P}^n_k$ is an automorphism. Consider $\varphi^*\mathcal{O}_{\mathbb{P}^n_k}(1) \in \mathrm{Pic}\,(\mathbb{P}^n_k)$. Furthermore, because φ is an isomorphism φ^* is an isomorphism on $\mathrm{Pic}\,(\mathbb{P}^n_k)$ and thus $\varphi^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ is a generator. However, $\varphi^*\mathcal{O}_{\mathbb{P}^n_k}(1) \not\cong \mathcal{O}_{\mathbb{P}^n_k}(-1)$ because the left is generated by global sections (this property is preserved by pullback in general) but the right has no nonzero global sections. Therefore, there is an isomorphism $\alpha : \varphi^*\mathcal{O}_{\mathbb{P}^n_k}(1) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n_k}(1)$. Furthermore, we pull back the canonical global sections,

$$\varphi^*(t_j) \mapsto \sum_i M_{ij} t_i$$

where M_{ij} is the matrix we want because the map φ is determined by the generating global sections $\alpha(\varphi(t_j))$. Since φ is an isomorphism M must be invertible since $\{\alpha(\varphi(t_j))\}$ must be a k-basis of $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$. $\square$

Remark. Applications of $\mathrm{Cl}\,(X) \cong \mathrm{Pic}\,(X)$ in Ex. 6.10.2 and Remark. 6.10.3 in [H] (see FGA Explained for Picard schemes).

30.2 Ample Line Bundle

Definition 30.2.1. For $f : X \to Y$ separated, finite type. We say that a line bundle $\mathcal{L}$ is *very ample* over Y if $\mathcal{L} \cong j^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ for some Y-immersion $j : X \hookrightarrow \mathbb{P}^n_Y$.

Definition 30.2.2. Let X be a noetherian scheme. Say $\mathcal{L}$ on X is *ample* if for all coherent $\mathcal{F}$ on X there is an integer $n_{\mathcal{F}}$ such that for all $n \geq n_{\mathcal{F}}$,

$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

is a surjection meaning $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

Remark. We can check this by global sections spanning $(\mathscr{F} \otimes \mathcal{L}^{\otimes n})_x$ over $\mathcal{O}_{X,x}$ for each $x \in X$ or equivalently by Nakayama that it spans $(\mathscr{F} \otimes \mathcal{L}^{\otimes n})_x \otimes \kappa(x)$ over $\kappa(x)$ for all $x \in X$. By coherence, this shows that for sufficiently small opens $\Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n})$ generates $(\mathscr{F} \otimes \mathcal{L}^{\otimes n})|_U$ over $\mathcal{O}_U$.

Example 30.2.3. Consider the following line bundles,

(a) if X is affine then $\mathcal{O}_X$ is ample

(b) if $X = \mathbb{P}^n_k$ then $\mathcal{O}_X(-1)$ is not ample, because use $\mathscr{F} = \mathcal{O}_X$ then $\mathcal{O}_X(-1)^{\otimes n} = \mathcal{O}_X(-n)$ has no global sections

(c) for immersion $j : X \hookrightarrow \mathbb{P}^n_A$ then $\mathcal{L} = j^*\mathcal{O}(1)$ is ample (very ample over a ring implies ample).

The last one needs a proof.

Proposition 30.2.4. Let $j : X \hookrightarrow \mathbb{P}^n_A$ be an immersion then $\mathcal{L} = j^*\mathcal{O}(1)$ is ample (very ample over a ring implies ample).

Proof. Let $\overline{X} \subset \mathbb{P}^N_A$ be the schematic closure. Then $X \hookrightarrow \overline{X}$ is an open subscheme. By [Ex. 5.15(b)] given a coherent sheaf $\mathscr{F}$ on X there is a coherent sheaf $\mathscr{G}$ on $\overline{X}$ with $\mathscr{G}|_X = \mathscr{F}$. Then we apply Serre's theorem to $\mathscr{G}$. Then for all $n \gg 0$ we have $\mathscr{G}(n)$ is generated by global sections. Therefore $\mathscr{F}(n) = \mathscr{G}(n)|_X$ is also generated by global sections proving that $\mathcal{O}_X(1)$ is ample. $\qquad\square$

Lemma 30.2.5. For X noetherian, $\mathcal{L}$ invertible on X then if $\mathcal{L}^{\otimes m_0}$ is ample for some $m_0 > 0$ then $\mathcal{L}^{\otimes m}$ is ample for all $m > 0$.

Proof. Fix m. Let $\mathscr{F}$ be coherent. Then,

$$\mathscr{F} \otimes (\mathcal{L}^{\otimes m})^{\otimes n} = (\mathscr{F} \otimes \mathcal{L}^{\otimes mr}) \otimes \mathcal{L}^{\otimes m_0 qm}$$

for $n = m_0 q + r$ with $0 \leq r < m_0$. There are finitely many such coherent $(\mathscr{F} \otimes \mathcal{L}^{\otimes mr})$ and $\mathcal{L}^{\otimes m_0}$ is ample so for sufficiently large n we have sufficiently large q such that all these sheaves are generated by global sections. $\qquad\square$

Remark. Using the Veronese embedding if $\mathcal{L}$ is very ample then $\mathcal{L}^{\otimes m}$ is very ample for all $m > 0$.

Remark. For smooth connected curve X over $k = \bar{k}$ write $\mathcal{L} = \mathcal{O}_X(D)$ then Riemann-Roch shows that $\mathcal{L}$ is ample if and only if $\deg D > 0$.

Theorem 30.2.6. For X of finite type over A a noetherian ring. Then a line bundle $\mathcal{L}$ on X is ample if and only if $\mathcal{L}^{\otimes m}$ is very ample over A for some $m > 0$.

Proof. One direction is clear. If $\mathcal{L}^{\otimes m}$ is very ample then $\mathcal{L}^{\otimes m}$ is ample so $\mathcal{L}$ is ample. The other direction is in the notes. $\qquad\square$

31   Cech Cohomology

Let X be a topological space, $\mathfrak{U}$ an open cover and $\mathscr{F} \in \mathbf{Ab}(X)$. There are three notions of "Čech complex" for $\mathscr{F}$ with respect to $\mathfrak{U}$. However, they all have the same cohomology.

Definition 31.0.1. For $n \geq 0$ define *the unrestricted Cech complex,*

$$\check{C}^n(\mathfrak{U}, \mathscr{F}) = \prod_{\underline{i} \in I^{n+1}} \mathscr{F}(U_{\underline{i}})$$

where $U_{(i_0,\dots,i_n)} = U_{i_0} \cap \cdots \cap U_{i_n}$ with differential,

$$\check{\mathrm{d}}^n : \check{C}^n(\mathfrak{U}, \mathscr{F}) \to \check{C}^{n+1}(\mathfrak{U}, \mathscr{F})$$

is defined,

$$\check{\mathrm{d}}(\xi)_{(i_0,\dots,i_{n+1})} = \sum_{\ell=0}^{n+1} (-1)^\ell \mathrm{res}_\ell(\xi_{i_0,\dots,\hat{i}_\ell,\dots,i_{n+1}})$$

where $\mathrm{res}_\ell : \mathscr{F}(U_{(i_0,\dots,\hat{i}_\ell,\dots,i_{n+1})}) \to \mathscr{F}(U_{\underline{i}})$ is the restriction map.

Definition 31.0.2. The *alternating Cech complex,*

$$C_{\mathrm{alt}}^n(\mathfrak{U}, \mathscr{F}) = \{(\xi_{\underline{i}}) \in \check{C}^n(\mathfrak{U}, \mathscr{F}) \mid \xi_{\underline{i}} \text{ is alternating}\}$$

meaning $\xi_{\underline{i}} = 0$ if there are repeated indices and $\xi_{\sigma(\underline{i})} = \mathrm{sing}(\sigma)\xi_{\underline{i}}$.

Definition 31.0.3. Fix a well-ordering on I. Then there is a quotient complex,

$$C^n(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_n} \mathscr{F}(U_{\underline{i}})$$

called the *ordered Cech complex* which depends on I and the ordering.

Proposition 31.0.4. The map $C_{\mathrm{alt}}^\bullet \to \check{C}^\bullet \to C^\bullet$ is an isomorphism of complexes.

Proof. Alternating chains have the same information. $\square$

Example 31.0.5. Let $\mathfrak{V} = \{V_0, V_1\}$. The ordered Cech complex is,

$$\mathscr{F}(V_0) \times \mathscr{F}(V_1) \to \mathscr{F}(V_{01}) \to 0$$

where the differential is,

$$(s_0, s_1) \mapsto s_1 - s_0$$

Example 31.0.6. Let $\mathfrak{U} = \{U_0, U_1, U_2\}$ then $C^\bullet(\mathfrak{U}, \mathscr{F})$ is,

$$\mathscr{F}(U_0) \times \mathscr{F}(U_1) \times \mathscr{F}(U_2) \to \mathscr{F}(U_{12}) \times \mathscr{F}(U_{02}) \times \mathscr{F}(U_{01}) \to \mathscr{F}(U_{012}) \to 0$$

The differentials take,

$$(s_0, s_1, s_2) \mapsto (s_2 - s_1, s_2 - s_0, s_1 - s_0) \quad (t_{12}, t_{02}, t_{01}) \mapsto (t_{12} - t_{02} + t_{01})$$

Definition 31.0.7. We define the cohomology of this complex as,

(a) $\check{H}^n(\mathfrak{U}, \mathscr{F}) = H^n(\check{C}^\bullet(\mathfrak{U}, \mathscr{F}))$

(b) $H_{\mathrm{alt}}^n(\mathfrak{U}, \mathscr{F}) = H^n(C_{\mathrm{alt}}^\bullet(\mathfrak{U}, \mathscr{F}))$

(c) $H^n(\mathfrak{U}, \mathscr{F}) = H^n(C^\bullet(\mathfrak{U}, \mathscr{F}))$.

Remark (H). denotes $H^n(\mathfrak{U}, \mathscr{F})$ as $\check{H}^n(\mathfrak{U}, \mathscr{F})$ unfortunately.

Remark. If $\#I = N$ then $C^m(\mathfrak{U}, \mathscr{F}) = 0$ for $n > N$ because there are no strictly ordered lists of size greater than N.

Example 31.0.8. Let $X = \mathbb{P}^1_A$ and $\mathscr{F} = \mathcal{O}_X$. Then $\mathfrak{U} = \{U_0 = D_+(t_0), U_1 = D_+(t_1)\}$. Therefore, the Cech complex is,

$$A[x] \times A[x^{-1}] \xrightarrow{\mathrm{d}} A[x, x^{-1}]$$
$$(f(x), g(x^{-1})) \mapsto g(x^{-1}) - f(x)$$

Then,

$$H^0 = \ker \mathrm{d} = \{(f, g) \mid f(x) = g(x^{-1})\} = \{(a, a) \mid a \in A\}$$

and likewise,

$$H^1 = \operatorname{coker} \mathrm{d} = 0$$

since d is extremely surjective because every element of $A[x, x^{-1}]$ can be split into positive and negative degrees.

Theorem 31.0.9. The maps $C^\bullet_{\mathrm{alt}} \hookrightarrow \check{C}^\bullet \to C^\bullet$ are each chain homotopy equivalences.

Corollary 31.0.10. Hence, there are natural isomorphisms,

$$H^n_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} \check{H}^n(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^n(\mathfrak{U}, \mathscr{F})$$

31.1 Refinement

Definition 31.1.1. Let $\mathfrak{U}$ be an open cover. We say that $\mathfrak{V}$ is a refinement if there exists a map of index sets $\tau : J \to I$ such that $V_j \subset U_{\tau i}$ (note: there are usually many τ and the τ need not be injective).

Remark. We want to relate $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ to $\check{C}^\bullet(\mathfrak{V}, \mathscr{F})$. This will not work for the ordered Cech complex.

Definition 31.1.2. Define $\tau : J^{n+1} \to I^{n+1}$ by $(j_0, \ldots, j_n) \mapsto (\tau(j_0), \ldots, \tau(j_n))$ and $\tau^\bullet : \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \to \check{C}^\bullet(\mathfrak{V}, \mathscr{F})$ by,

$$\tau^n(\xi)_{\underline{j}} = \operatorname{res}^{U_{\tau(\underline{j})}}_{V_{\underline{j}}} (\xi_{\tau(\underline{j})}) \in \mathscr{F}(V_{\underline{j}})$$

Theorem 31.1.3. The maps $\tau^\bullet : \check{H}^n(\mathfrak{U}, \mathscr{F}) \to \check{H}^n(\mathfrak{V}, \mathscr{F})$ is independent of τ.

Definition 31.1.4. The *Čech cohomology* of $\mathscr{F}$ on X is,

$$\check{H}^n(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} \check{H}^n(\mathfrak{U}, \mathscr{F})$$

32 Feb. 9

Cech resolution of sheaves.

Proposition 32.0.1. The complex,

$$0 \longrightarrow \mathscr{F} \longrightarrow \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$$

is exact so $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ forms a resolution of $\mathscr{F}$.

Proof. Exactness at $\mathscr{F}$, and $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ has been seen so we just need to show exactness of $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ and above. This is local on X so we may restrict to $V \subset X$ and $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})|_V = \check{\mathscr{C}}^\bullet(V \cap \mathfrak{U}, \mathscr{F}|_V)$ so we take $V \in \mathfrak{U}$ which gives a cover. Apply this to $V = U_\alpha$ for each $\alpha \in I$ and rename $U_\alpha \cap U_i$ as U_i to reduce to case $U_\alpha = X$ for some $\alpha \in I$.

We build a homotopy from id to 0 on $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$. Meaning maps $k^n : \check{\mathscr{C}}^n(\mathfrak{U}, \mathscr{F}) \to \check{\mathscr{C}}^{n-1}(\mathfrak{U}, \mathscr{F})$ such that $\mathrm{d}^{n-1} \circ k^n + k^{n+1} \circ \mathrm{d}^n = \mathrm{id}$ on $\check{\mathscr{C}}^n(\mathfrak{U}, \mathscr{F})$ for $n \geq 1$. We need,

$$k^n : \prod_{\underline{i} \in I^{n+1}} \Gamma(U_{\underline{i}}, \mathscr{F}) \to \prod_{\underline{i} \in I^n} \Gamma(U_{\underline{i}}, \mathscr{F})$$

defined by sending,

$$\xi \mapsto (\xi_{(\alpha, i_0, \dots, i_n)})_{(i_0, \dots, i_n)}$$

Then one can check this is a nulhomotopy. $\qquad\square$

Remark. We really proved: $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is exact whenever some $U_\alpha = X$.

Definition 32.0.2. Use $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ to connected $\check{H}^\bullet(\mathfrak{U}, \mathscr{F})$ to $H^\bullet(X, \mathscr{F})$. Choose an injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}$ in abelian sheaves to get a diagram,

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}) \\
 & & \| & & \vdots \\
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{I}^\bullet
\end{array}
$$

Therefore there is a morphism of complexes $\varphi : \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ unique up to homotopy. This gives a map $\varphi : \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \to \Gamma(X, \mathscr{I}^\bullet)$ and thus a *canonical* map on cohomology,

$$\theta_{\mathfrak{U}}^n : \check{H}^n(\mathfrak{U}, \mathscr{F}) \to H^n(X, \mathscr{F})$$

For $n = 0$ this is the identity map on $\Gamma(X, \mathscr{F})$. For $n > 0$ next time we will discuss when this is an isomorphism.

Remark. What happens when we refine the cover? Suppose that $\mathfrak{V}$ refines $\mathfrak{U}$. Then we check that $\tau_W^\bullet : \check{C}^\bullet(W \cap \mathfrak{U}, \mathscr{F}) \to \check{C}^n(W \cap \mathfrak{V}, \mathscr{F})$ respect open inclusions $W' \hookrightarrow W$ so we get,

$$\tau^\bullet : \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}) \to \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$$

respecting $\mathrm{id}_{\mathscr{F}}$ on degree 0 so giving a map of resolutions. This depends on τ but not on homology. Given $\mathscr{I}^\bullet$, if we choose $\psi : \check{\mathscr{C}}^\bullet(\mathfrak{V}, \mathscr{F}) \to \mathscr{I}^\bullet$ as resolutions then we can *pick* φ to be,

$$\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}) \xrightarrow{\tau^\bullet} \check{\mathscr{C}}^\bullet(\mathfrak{V}, \mathscr{F}) \xrightarrow{\psi} \mathscr{I}^\bullet$$

and therefore get a commutative diagram,

$$
\begin{array}{ccc}
\check{H}^n(\mathfrak{U}, \mathscr{F}) & \xrightarrow{\quad\text{can}\quad} & \check{H}^n(\mathfrak{V}, \mathscr{F}) \\
 & \searrow{\scriptstyle \theta_{\mathfrak{U}}^n} \qquad {\scriptstyle \theta_{\mathfrak{V}}^n}\swarrow & \\
 & H^n(X, \mathscr{F}) &
\end{array}
$$

These allow us to apply $\varinjlim_{\mathfrak{U}}$ to give a map,

$$\theta^n \check{H}^n(X, \mathscr{F}) \to H^n(X, \mathscr{F})$$

Next time we will discuss when this map is an isomorphism.

Remark. What about the δ-functor structure.

Example 32.0.3. Suppose that X is a paracompact Hausdorff space so locally finite $\mathfrak{U}$ are cofinal. In [Warner, 5.33] it is proved that $\check{H}^\bullet(X, \mathscr{F})$ is equipped with the structure of an erasable δ-functor making θ into a morphism of δ-functors and hence an isomorphism.

Example 32.0.4. Let X be a separated scheme and $\mathfrak{U} = \{U_i\}$ is an affine cover (such $\mathfrak{U}$ are cofinal). Then by separatedness every $U_{\underline{i}}$ is affine. Given a short exact sequence,

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

Then since $U_{\underline{i}}$ are affine we get an exact sequence,

$$0 \longrightarrow \Gamma(U_{\underline{i}}, \mathscr{F}') \longrightarrow \Gamma(U_{\underline{i}}, \mathscr{F}) \longrightarrow \Gamma(U_{\underline{i}}, \mathscr{F}'') \longrightarrow 0$$

Therefore, taking the product we see that,

$$0 \longrightarrow \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}') \longrightarrow \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}'') \longrightarrow 0$$

is an exact sequence of resolutions. Using Cartan-Eilenberg resolutions, can show that a short exact sequence of resolutions maps to a short exact sequence of injective resolutions. Therefore, we get a diagram,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}') & \longrightarrow & \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}) & \longrightarrow & \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}'') & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{I}'^\bullet & \longrightarrow & \mathscr{I}^\bullet & \longrightarrow & \mathscr{I}''^\bullet & \longrightarrow & 0
\end{array}
$$

Applying $\Gamma(X, -)$ this stays exact because we showed the top sequence is exact and the second sequence is made of injectives. Taking the associated long exact sequence of these short exact sequences of complexes, we get a δ-functor structure (for quasi-coherent and affine $\mathfrak{U}$) compatible with $\theta_{\mathfrak{U}}^\bullet$.

33 Feb. 11

Consider the functors,

$$\mathrm{Ab}(X) \xrightarrow{F} \mathrm{Ch}_{\geq 0}(\mathrm{Ab}(X)) \xrightarrow{H^0} \mathrm{Ab}$$

where $F : \mathscr{F} \mapsto \check{C}^\bullet(\mathfrak{U}, \mathscr{F})$.

We have a δ-functor $\{H^n\}_{n \geq 0}$ on $\mathrm{Ch}_{\geq 0}(\mathrm{Ab}) \to \mathrm{Ab}$. from the snake lemma and this is erased by injectives so it is the derived functors of H^0. What about $R^q F$. Consider $T^q : \mathrm{Ab}(X) \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab})$ sending,

$$T^q(\mathscr{F}) = C^\bullet(\mathfrak{U}, \underline{H}^q(\mathscr{F}))$$

where $\underline{H}^q(\mathscr{F}) : V \mapsto H^q(V, \mathscr{F})$. To make $\{T^q\}_{q \geq 0}$ into a δ-functor, given a short exact sequence,

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

Then we get long exact sequences of cohomology on each $U_{\underline{i}}$ and taking products, this gives a long exact sequence of the $C^n(\mathfrak{U}, \underline{H}^\bullet(\mathscr{F}))$. This is compatible with $\check{\mathrm{d}}^n$ so we get a long exact sequence in $\mathrm{Ch}_{\geq 0}(\mathrm{Ab})$.

Lemma 33.0.1. We $\{T^\bullet\} \cong \{R^\bullet F\}$ uniquely as δ-functors extending $T^0 = F$.

Proof. We just need to show that $\{T^\bullet\}$ is an erasable δ-functor. In particular, it suffices to show that $T^q(\mathscr{I}) = 0$ for all $q \geq 1$ and $\mathscr{I} \in \mathrm{Ab}(X)$ an injective. However, this is clear because $H^q(U, \mathscr{I}) = 0$ for all $q \geq 1$ and any open U (because $\mathscr{I}|_U$ is injective as well) and therefore

$$C^\bullet(\mathfrak{U}, \underline{H}^q(\mathscr{I})) = 0$$

$\square$

Lemma 33.0.2. $F(\mathscr{I})$ is H^0-acyclic for $\mathscr{I} \in \mathrm{Ab}(X)$ is injective. In particular, $\check{C}^\bullet(\mathfrak{U}, \mathscr{I})$ is exact in degrees $n \geq 1$. This holds if $\mathscr{I}$ is just flasque.

Proof. I claim that $\check{\mathscr{C}}^n(\mathfrak{U}, \mathscr{I})$ are all flasque and thus it computes $H^\bullet(X, \mathscr{I})$ meaning,

$$H^n(\check{C}^\bullet(\mathfrak{U}, \mathscr{I})) = H^n(\Gamma(X, \check{\mathscr{C}}^n(\mathfrak{U}, \mathscr{I}))) = H^n(X, \mathscr{I})$$

and $H^n(X, \mathscr{I}) = 0$ for $n \geq 0$. Indeed,

$$\check{\mathscr{C}}^n(\mathfrak{U}, \mathscr{I}) = \prod_{\underline{i} \in I^{n+1}} (j_{\underline{i}})_* (\mathscr{I}|_{U_{\underline{i}}})$$

is flasque because restrictions of flasque are flat, pushforward of flasque are flasque, and products of flasque sheaves are flasque. $\square$

34 Feb. 14

34.1 Affine Vanishing and Its Applications

Theorem 34.1.1. For $X = \mathrm{Spec}\,(A)$ and $\mathscr{F}$ quasi-coherent on X, then $H^n(X, \mathscr{F}) = 0$ for all $n > 0$.

Proof. First, I claim that $\check{H}^n(\mathfrak{U}, \mathscr{F}) = 0$ for $n > 0$ and $\mathfrak{U}$ a finite affine cover. We know that,

$$0 \longrightarrow \mathscr{F} \longrightarrow \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$$

is a resolution so $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ is exact for $n > 0$. However,

$$\check{\mathscr{C}}^n(\mathfrak{U}, \mathscr{F}) = \prod_{\underline{i} \in I^{n+1}} (j_{\underline{i}})_* (\mathscr{F}|_U)$$

is quasi-coherent because the cover is finite. Therefore,

$$\check{C}^\bullet(\mathfrak{U}, \mathscr{F}) = \Gamma(X, \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F}))$$

is also exact for $n > 0$ since X is affine proving the vanishing.

For $n = 1$ we know that,

$$H^1(X, \mathscr{F}) = \check{H}^1(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) = 0$$

because finite affine covers are cofinal under refinement since X is affine (and hence quasi-compact). Now we proceed by induction (over all pairs of an affine scheme and a quasi-coherent sheaf) in n. Assume $n \geq 2$ and the result is established for all $1 \leq q \leq n - 1$. In particular,

$$\underline{H}^q(\mathscr{F})(U_{\underline{i}}) = H^q(U_{\underline{i}}, \mathscr{F}) = 0$$

for $1 \leq q \leq n - 1$. Now we apply the Cech-to-derived spectral sequence for $\mathfrak{U}$,

$$E_2^{p,q}(\mathfrak{U}) = \check{H}^p(\mathfrak{U}, \underline{H}^q(\mathscr{F})) \implies H^{p+q}(X, \mathscr{F})$$

We have $E_2^{p,q}(\mathfrak{U}) = 0$ for $1 \leq q \leq n-1$ by the previous calculation. Furthermore because $\check{H}^p(\mathfrak{U}, \mathscr{F}) = 0$ for all $p > 0$ we see that $E_2^{p,0} = 0$ for all $q > 0$. Therefore we see that $H^n(X, \mathscr{F}) = E_\infty^{0,n}$ because all others with $p + q = 0$ are zero. We don't know,

$$E_2^{0,n}(\mathfrak{U}) = \check{H}^0(\mathfrak{U}, \underline{H}^n(\mathscr{F}))$$

However, if we pass to the limit $\varinjlim_{\mathfrak{U}}$ to get,

$$E_2^{p,q} = \check{H}^p(X, \underline{H}^q(\mathscr{F})) \implies H^{p+q}(X, \mathscr{F})$$

and thus $E_2^{0,n} = 0$ proving our claim. $\square$

35 Serre's Converse

Theorem 35.0.1. Let X be quasi-compact. If $H^1(X, \mathscr{I}) = 0$ for all quasi-coherent $\mathscr{I} \subset \mathcal{O}_X$ then X is affine.

Proof. WLOG $X \neq \varnothing$. Let's show that X has a closed point. Apply Zorn's lemma to the poset of nonempty closed subsets $Z \subset X$ ordered by reverse inclusion to get a *minimal* such Z (using the finite intersection property for qc to conclude that the intersection of a chain is nonempty). Such Z is *indiscreet* (otherwise it would have a proper closed subset). I claim Z is a point. Using the reduced structure Z is a scheme. Choose a nonempty affine open $U \subset Z$ then $U = Z$ then a maximal ideal of $\mathcal{O}(U)$ is a closed point so Z is a single point.

We will now use the affineness criterion [H, ChII, Ex. 2.17(b)]. Seek to cover X by finitely many open $X_{f_j} = \{x \in X \mid f_j(x) \neq 0\}$ for $f_1, \ldots, f_j \in \Gamma(X, \mathcal{O}_X)$ such that X_{f_j} are affine. By quasi-compactness, we just need to show that for each closed point $P \in X$ there is exists an affine open X_f containing P. Indeed, let

$$\Omega = \bigcup_{P \in X^{\mathrm{closed}}} X_{f_P}$$

This is open and it contains every closed point so $X \setminus \Omega$ is a closed subset (and hence quasi-compact) with no closed points (closed point of closed subset is closed) so $X \setminus \Omega$ is empty. Then by quasi-compactness Ω is a finite union of such X_{f_P} proving the claim.

Let $P \in X$ be closed and U an affine open neighborhood and let $Y = (X \setminus U)_{\mathrm{red}}$. It suffices to find $f \in \mathscr{I}_Y(X)$ with $f(P) \neq 0$ since then $X_f \subset U$ and thus $X_f = D(f)$ in the affine scheme U and hence is affine and also $P \in X_f$. Consider the sequence,

$$0 \longrightarrow \mathscr{I}_{Y \cup \{P\}} \longrightarrow \mathscr{I}_Y \longrightarrow \kappa(P) \longrightarrow 0$$

which is exact by checking on $X \setminus \{P\}$ and on U. Applying global sections,

$$0 \longrightarrow \mathscr{I}_{Y \cup \{P\}}(X) \longrightarrow \mathscr{I}_Y(X) \longrightarrow \kappa(P) \longrightarrow H^1(X, \mathscr{I}_{Y \cup \{P\}})$$

but $H^1(X, \mathscr{I}_{Y \cup \{P\}}) = 0$ and thus $1 \in \kappa(P)$ is in the image of $\mathscr{I}_Y(X) \to \kappa(P)$ so such a function $f \in \mathscr{I}_Y(X)$ exists proving our claim. $\qquad \square$

Lemma 35.0.2. If X is separated and qc with a cover by $r + 1$ affine opens, then $H^i(X, \mathscr{F}) = 0$ for $i > r$ and all quasi-coherent $\mathscr{F}$.

Proof. We know $H^i(X, \mathscr{F}) = \check{H}^i(\mathfrak{U}, \mathscr{F})$ for the cover $\mathfrak{U} = \{U_0, \ldots, U_r\}$. Using the ordered Cech complex, $C^i(\mathfrak{U}, -) = 0$ for $i > r$ and therefore $H^i(X, \mathscr{F}) = 0$. $\qquad \square$

Proposition 35.0.3. Let X be projective over a noetherian ring A. Let $\mathscr{F} \in \mathfrak{Coh}(X)$. Then,

(a) all $H^i(X, \mathscr{F})$ are A-finite

(b) Fix $j : X \hookrightarrow \mathbb{P}_A^r$ and $\mathcal{O}_X(1) = j^*(\mathcal{O}_{\mathbb{P}_A^r}(1))$ then $H^i(X, \mathscr{F}(n)) = 0$ for all $n > n_{\mathscr{F}}$ and $i > 0$.

Proof. Use descending induction on i. Fix $j : X \hookrightarrow \mathbb{P}_A^r$. Reduce to $X = \mathbb{P}_A^r$ because $H^i(X, \mathscr{F}(n)) = H^i(\mathbb{P}_A^r, j_*\mathscr{F}(n))$ and $j_*\mathscr{F}(n) = (j_*\mathscr{F})(n)$ and is coherent.

For $i > r$ everything is zero so we are done. Now suppose that (a) holds for all coherent sheaves in $i + 1$. By Serre's theorem,

$$0 \longrightarrow \mathscr{G} \longrightarrow \bigoplus_{k=1}^N \mathcal{O}(-m_k) \longrightarrow \mathscr{F} \longrightarrow 0$$

Note $\mathscr{G}$ is coherent! Taking the long exact sequence,

$$\bigoplus_{k=1}^N H^i(\mathbb{P}_A^r, \mathcal{O}(-m_k)) \longrightarrow H^i(\mathbb{P}_A^r, \mathscr{F}) \longrightarrow H^{i+1}(\mathbb{P}_A^r, \mathscr{G})$$

The first term is A-finite by an explicit calculation and the last term by the induction hypothesis. Therefore $H^i(\mathbb{P}_A^r, \mathscr{F})$ is A-finite.

For (b) suppose we know for degree $i + 1$ for all coherent sheaves. Only H^j for $j \le r$ mattes so we can use descending induction. We apply the previous sequence twisted by $- \otimes \mathcal{O}(n)$ and taking the long exact sequence gives,

$$\bigoplus_{k=1}^N H^i(\mathbb{P}_A^r, \mathcal{O}(n - m_k)) \longrightarrow H^i(\mathbb{P}_A^r, \mathscr{F}(n)) \longrightarrow H^{i+1}(\mathbb{P}_A^r, \mathscr{G}(n))$$

We can choose n such that $\mathscr{G}(n)$ has vanishing higher cohomology and for $n > m_k$ for each k then both terms are zero so $H^i(\mathbb{P}_A^r, \mathscr{F}(n)) = 0$ for $n \gg 0$ depending only on $\mathscr{F}$. $\qquad \square$

36 Ampleness and Higher direct Images

Theorem 36.0.1 (Serre)**.** Let X be proper over a Noetherian ring A, $\mathcal{L}$ is invertible on X. Then $\mathcal{L}$ is ample if and only if $\mathscr{F} \in \mathfrak{Coh}(X)$ then $H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ and $n \gg 0$.

Proof. We proved $\implies$ last time. Now we prove the other direction (which will not actually involve properness). Build $n_1 \geq 1$ so that $\mathcal{L}^{\otimes n_1}$ is generated by global sections and thus the same holds for $\mathcal{L}^{\otimes n_1 q}$ for $q \geq 1$. For a closed point $x \in X$. Consider the exact sequence,

$$0 \longrightarrow \mathscr{I}_x \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n}(x) \longrightarrow 0$$

where the last term is a skyscraper. Since $\mathcal{L}$ is flat we have $\mathscr{I}_x \mathcal{L}^{\otimes n} \cong \mathscr{I}_x \otimes \mathcal{L}^{\otimes n}$. The cohomology sequence gives,

$$H^0(X, \mathcal{L}^{\otimes n}) \longrightarrow \mathcal{L}^{\otimes n}(x) \longrightarrow H^1(X, \mathscr{I}_x \otimes \mathcal{L}^{\otimes n})$$

By the vanishing, there is $N_x \geq 1$ so that if $n \geq N_x$ then $H^0(X, \mathcal{L}^{\otimes n}) \twoheadrightarrow \mathcal{L}^{\otimes n}(x)$ and hence by Nakayama, $\mathcal{L}_x^{\otimes n}$ is generated by global sections. By coherence, there is an open W_x neighborhood of x such that $\mathcal{L}^{\otimes N_x}|_{W_x}$ is generated by global section $\Gamma(X, \mathcal{L}^{\otimes N_x})$. Also the same happens on W_x for all $\mathcal{L}^{\otimes N_x q}$ for $q \geq 1$. Then,

$$\bigcup_{x \in X} W_x \subset X$$

is open and contains all closed points so it is all of X because X is qc and thus there is a finite collection $W_1, \ldots, W_m$ that cover m. Then,

$$n_1 = \prod_{j=1}^{m} N_{x_j}$$

gives $\mathcal{L}^{\otimes n_1}$ generated by global sections.

Step 2: choose $\mathscr{F} \in \mathfrak{Coh}(X)$ and $x \in X$ a closed point. Seek an open $x \in U_x$ and $n_x \geq 1$ such that $\Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes(n_x + r)})$ generates $\mathscr{F} \otimes \mathcal{L}^{\otimes(n_x + r)}|_{U_x}$ for all $0 \leq r \leq n_1 - 1$. We know that for all $n \geq n_{x, \mathscr{F}}$ depending on x and $\mathscr{F}$ we know that,

$$H^1(X, \mathscr{I}_x \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

and therefore we get a short exact sequence,

$$0 \longrightarrow \mathscr{I}_x \mathscr{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathscr{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow (\mathscr{F} \otimes \mathcal{L}^{\otimes n})(x) \longrightarrow 0$$

to get long exact sequence of cohomology giving,

$$\Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) \longrightarrow (\mathscr{F} \otimes \mathcal{L}^{\otimes n})(x) \longrightarrow H^1(X, \mathscr{I}_x \mathscr{F} \otimes \mathcal{L}^{\otimes n})$$

so by the vanishing an Nakayama we see that $\Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n})$ generates $(\mathscr{F} \otimes \mathcal{L}^{\otimes n})_x$ over $\mathcal{O}_{X,x}$. The spreading out to the finite number of coherent sheaves $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$ for $n = n_{x, \mathscr{F}} + r$ for $0 \leq r \leq n_1 - 1$ then we can spread out to some $U_{x, \mathscr{F}}$ which satisfies the hypotheses.

Step 3: finale. Pick a coherent sheaf $\mathscr{F} \in \mathfrak{Coh}(X)$ we want $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$ to be generated by global sections for all $n \gg 0$. For closed point $x \in X$ and $n \geq n_{x, \mathscr{F}}$ we write,

$$n = n_x + (n_1 q + r)$$

for $q \geq 0$ and $0 \leq r \leq n_1 - 1$ so we have,

$$\mathscr{F} \otimes \mathcal{L}^{\otimes n} = (\mathscr{F} \otimes \mathcal{L}^{\otimes(n_x+r)}) \otimes (\mathcal{L}^{\otimes n_1})^{\otimes q}$$

The second term is generated by global sections by step 1 and the second is generated by global sections over $U_{x,\mathscr{F}}$. Thus, if $n \geq n_x$ then $(\mathscr{F} \otimes \mathcal{L}^{\otimes n})|_{U_x}$ is generated by $\Gamma(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n})$. Because the U_x cover X and X is qc finitely many of then $U_{x_1}, \ldots, U_{x_b}$ covers X. Therefore, we can take $N = \max n_{x_j}$ and we win. $\square$

Definition 36.0.2. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ map of ringed spaces, the *higher direct images* $R^i f_* : \mathbf{Mod}_X \to \mathbf{Mod}_Y$ are the derived functors of f_*.

Remark. Let Y be a point, then $R^i f_* = H^i(X, -)$.

Lemma 36.0.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ map of ringed spaces,

  (a) $R^i f_*$ is unaffected by passing to $\mathbf{Ab}(X) \to \mathbf{Ab}(Y)$

  (b) $R^i f_* \mathscr{F}$ is the sheafification of $V \mapsto H^i(f^{-1}(V), \mathscr{F})$ for opens $V \subset Y$.

Proof. Notice that (b) $\implies$ (a) because cohomology works in the category of abelian sheaves. For (b) notice that this sheafification gives a δ-functor and agreement for $i = 0$ is clear. This is erasable because $H^i(f^{-1}(V), \mathscr{I}) = 0$ for any injective sheaf $\mathscr{I}$. $\square$

Lemma 36.0.4. For $X \to \mathrm{Spec}(A)$ quasi-compact separated, $\mathscr{F} \in \mathfrak{QCoh}(X)$ then for $a \in A$ the map $H^i(X, \mathscr{F}) \to H^i(X_a, \mathscr{F}_a)$ is linear for $A \to A_a$ and thus gives,

$$H^i(X, \mathscr{F})_a \to H^i(X_a, \mathscr{F}_a)$$

this is an isomorphism. Therefore, $R^i f_* \mathscr{F} = \widetilde{H^i(X, \mathscr{F})}$.

Remark. Therefore $R^i g_*$ preserves quasi-coherence for any qc separated $g : X \to Y$ because $R^i g_*(\mathscr{F})|_V \cong R^i(g_V)_*(\mathscr{F}|_{g^{-1}(V)})$ for open sets $V \subset Y$ and shrink to affine.

Proof. Choose a finite affine open cover $\mathfrak{U} = \{U_0, \ldots, U_r\}$ of X so all $U_{\underline{i}}$ are also affine. Then we have $C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{U}_a, \mathscr{F}_a)$ by restriction inducing the map $\check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{U}_a, \mathscr{F}_a)$ which agrees with $\alpha : H^i(X, \mathscr{F}) \to H^i(X_a, \mathscr{F}_a)$. However, $C^\bullet(\mathfrak{U}, \mathscr{F})_a \xrightarrow{\sim} C^\bullet(\mathfrak{U}_a, \mathscr{F}_a)$ is an isomorphism because these are finite products and $\mathscr{F}$ is quasi-coherent so $\Gamma(U, \mathscr{F})_a \xrightarrow{\sim} \Gamma(U_a, \mathscr{F}_a)$ for any affine a. Finally, localization is exact so we get an isomorphism on cohomology. $\square$

Corollary 36.0.5. Let $f : X \to Y$ be projective with Y locally noetherian then $R^i f_*$ preserves coherence.

Proof. $R^i f_*$ preserves quasi-coherence. This is a local question so we may assume that $Y = \mathrm{Spec}(A)$ then $(R^i f_*)\mathscr{F} = \widetilde{H^i(X, \mathscr{F})}$ but $H^i(X, \mathscr{F})$ are A-finite. $\square$

Theorem 36.0.6 (Coherence of Higher Direct Images)**.** For a proper map $f : X \to Y$ with Y locally noetherian, $R^i f_*$ preserves coherence.

Proof. Because this is local on Y, this is equivalent to when $Y = \mathrm{Spec}(A)$ for a noetherian ring A we have that $H^i(X, \mathscr{F})$ is A-finite since $R^i f_* \mathscr{F} = \widetilde{H^i(X, \mathscr{F})}$ for any $\mathscr{F} \in \mathfrak{Coh}(X)$. $\square$

37 Feb. 23

37.1 Duality Formalism

37.1.1 Coherence of Higher Direct Images

Theorem 37.1.1. For X proper over noetherian A, then all $H^i(X, \mathscr{F})$ are A-finite for $\mathscr{F} \in \mathfrak{Coh}\,(X)$.

Proof. We may assume $X \neq \varnothing$. We proceed by Noetherian induction. Suppose that there is a counterexample $\mathscr{G}$ then $\mathrm{Supp}\,(\mathscr{G}) \subset X$ is some closed subset so by Noetherian induction, we can pick a minimal such example and pass to the closed subscheme X' defined by $\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{G})$. Then $\mathscr{G} = j_*\mathscr{G}'$ for $\mathscr{G}' \in \mathfrak{Coh}\,(X')$ and $H^\bullet(X, \mathscr{G}) = H^\bullet(X', \mathscr{G}')$ where $j : X' \hookrightarrow X$ is the closed immersion. Now $X' \to \mathrm{Spec}\,(A)$ is proper because $X' \hookrightarrow X$ is closed. Therefore, we may assume that for all proper closed subsets the theorem holds. Replace X by X'.

Now we apply Chow's lemma to get $\pi : X' \to X$ projective and birational and $f : X' \to \mathrm{Spec}\,(A)$ is proper. Using properness of π there is an open $U \subset X$ such that $\pi^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism. Choose $\mathscr{F} \in \mathfrak{Coh}\,(X)$ consider,
$$\alpha : \mathscr{F} \to \pi_*\pi^*\mathscr{F}$$
and set $\mathscr{F}' = \pi^*\mathscr{F} \in \mathfrak{Coh}\,(X')$. First, $\alpha|_U$ is an isomorphism so $\mathscr{K} = \ker \alpha$ and $\mathscr{C} = \mathrm{coker}\,\alpha$ are supported on $Z = X \setminus U$. Since π is projective, $\pi_*\mathscr{F}'$ is coherent so $\mathscr{K}$ and $\mathscr{C}$ are also coherent. Therefore we get short exact sequences of coherent sheaves,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{Q} \longrightarrow \pi_*\mathscr{F}' \longrightarrow \mathscr{C} \longrightarrow 0$$

For $\mathscr{G} \in \mathfrak{Coh}\,(X)$ let $\mathcal{P}(\mathscr{G})$ be the property that all $H^i(X, \mathscr{G})$ are A-finite. Notice that $\mathcal{P}$ satisfies the two-out-of-three property for short exact sequences using the long exact sequence of cohomology. By the induction, hypothesis, $\mathscr{K}$ and $\mathscr{C}$ satisfy $\mathcal{P}$. Thus it suffices to show that $\pi_*\mathscr{F}'$ satisfies $\mathcal{P}$. However, by the Leray spectral sequence,

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathscr{F}') \implies H^{p+q}(X', \mathscr{F}')$$

By the projectivity of X' we know that $H^{p+q}(X', \mathscr{F}')$ is A-finite. Furthermore, the $R^q\pi_*\mathscr{F}'$ are coherent sheaves (by projectivity of π) and ford $q > 0$ they are supported on $X \setminus U$ and hence $E_2^{p,q}$ are A-finite for $q > 0$ by the induction assumption. Therefore $E_\infty^{p,q}$ is finite for $q > 0$ automatically since A is noetherian. Furthermore, $E_\infty^{p,0}$ can only be modified by taking quotients from A-submodules a finite number of times before the differential leaves the positive quadrant and hence since $E_\infty^{p,0}$ is A-finite we see that $E_2^{p,0}$ is A-finite proving the claim. $\qquad\square$

37.2 Coherent Duality

Now we want to motivate formulations of Serre duality for the cohomology of coherent sheaves on smooth projective schemes X of pure $\dim = d$ over a field k (we allow for nonconnected schemes to allow for non-geometrically connected varieties).

Definition 37.2.1. A scheme X over a field k is *smooth* if X is locally finite type over k and $X_{\bar{k}}$ is regular.

Remark. In HW9, develop the notion of smooth morphism $f : X \to S$ to any locally noetherian S in analogy with submersions in differential geometry.

Remark. For M a compact $\mathbb{C}$-manifold of pure dimension d there is a trace map,

$$t : H^d(M, \Omega_M^d) \xrightarrow{\sim} H^{2d}(M, \mathbb{C}) \xrightarrow{\sim} H^{2d}_{\mathrm{dR}}(M, \mathbb{C}) \to \mathbb{C}$$

where the last map,

$$\alpha \mapsto \frac{1}{(2\pi i)^d} \int_M \alpha$$

And for a holomorphic vector bundle $\mathcal{E}$ on M there is a pairing,

$$H^j(X, \mathcal{E}) \times H^{d-j}(M, \mathcal{E}^\vee \otimes \Omega_M^d) \xrightarrow{\smile} H^d(M, \mathcal{E} \otimes E^\vee \otimes \Omega_M^d) \xrightarrow{t} H^d(M, \Omega_M^d)$$

Remark. Serre duality will be a version of this in algebraic geometry over any field k with two caveats,

(a) Allow any coherent $\mathscr{F}$, not just locally free, so we need to rewrite,

$$H^i(X, \mathcal{E}^\vee \otimes \Omega_X^d) = \mathrm{Ext}_X^i\left(\mathcal{E}, \Omega_X^d\right)$$

(b) Give up pinning down explicit trace map,

$$t : H^d(X, \Omega_X^d) \to k$$

Definition 37.2.2. Let $(X, \mathcal{O}_X)$ be a ringed space over a ring A. Then define,

$$\mathrm{Ext}_X^\bullet(\mathscr{F}, -) = R^\bullet \mathrm{Hom}(X, \mathscr{F}) -$$

as functors $\mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_A$. Therefore, for an injective resolutions $\mathscr{G} \to \mathscr{I}^\bullet$ then,

$$\mathrm{Ext}_X^n(\mathscr{F}, \mathscr{G}) = H^n(\mathrm{Hom}(\mathcal{O}_X, \mathscr{F})\, \mathscr{I}^\bullet)$$

Remark. We don't have enough projectives but for $\varphi : \mathscr{F} \to \mathscr{F}'$, get $\mathrm{Hom}(X, \mathscr{F}') - \to \mathrm{Hom}(X, \mathscr{F}) -$ and therefore,

$$\mathrm{Ext}_X^\bullet(\mathscr{F}', -) \to \mathrm{Ext}_X^\bullet(\mathscr{F}, -)$$

We can upgrade this to a morphism of δ-functors.

Remark. Some facts,

(a) For $X = \mathrm{Spec}(A)$ noetherian,

$$\mathrm{Ext}_X^i\left(\widetilde{M}, \widetilde{N}\right) = \mathrm{Ext}_A^i(M, N)$$

when M is A-finite.

(b) $\mathrm{Ext}_X^\bullet(\mathcal{O}_X, -) = H^\bullet(X, -)$ because it is the derived functor of $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) - = \Gamma(X, -)$

(c) $\mathrm{Ext}_X^1(\mathscr{F}'', \mathscr{F}')$ is in bijection with isomorphism classes of extensions,

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

38 Feb. 25

Let X be proper of pure dim d over k. Say a coherent sheaf on X is *dualizing* if it represents the contravariant functor $\mathscr{F} \mapsto H^d(X, \mathscr{F})^\vee$ on $\mathfrak{Coh}\,(X)$ and denote it as $\omega_{X/k}$ and all have "universal" $t : H^d(X, \mathscr{F})^\vee \xrightarrow{\sim} k$ called the trace map.

Proposition 38.0.1. Let $P = \mathbb{P}_k^d$ and $\omega_P = \Omega_{P/k}^d \cong \mathcal{O}_P(-d-1)$. Then there exists $t : H^d(P, \omega_P) \xrightarrow{\sim} k$. Then,

(a) (ω_P, t) is dualizing meaning $\mathrm{Hom}\,(P, \mathscr{F})\,\omega_P \to H^d(P, \mathscr{F})^\vee$ is an isomorphism.

(b) there is a unique δ-functor isomorphism $\mathrm{Ext}_P^j(\mathscr{F}, \omega_P) \xrightarrow{\sim} H^{d-j}(P, \mathscr{F})^\vee$ for $\mathscr{F} \in \mathfrak{Coh}\,(P)$ recovering the above for $j = 0$.

Proof. For (i), both sides turn right exact sequences in $\mathscr{F}$ into left-exact sequences of k-vectorspaces (because $H^{d+1}(X, -) = 0$). By Serre's theorem, we have,

$$\mathcal{E} = \mathcal{O}(-q)^{\oplus n} \twoheadrightarrow \mathscr{F}$$

Therefore, applying this to the kernel we get an exact sequence,

$$\mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathscr{F} \longrightarrow 0$$

where the first two are vector bundles. Therefore, there is a diagram of exact sequences,

$$
\begin{array}{ccccc}
0 \longrightarrow \mathrm{Hom}\,(P, \mathscr{F})\,\omega_P & \longrightarrow & \mathrm{Hom}\,(P, \mathcal{E})\,\omega_P & \longrightarrow & \mathrm{Hom}\,(P, \mathcal{E}')\,\omega_P \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow H^d(P, \mathscr{F})^\vee & \longrightarrow & H^d(P, \mathcal{E})^\vee & \longrightarrow & H^d(P, \mathcal{E}')^\vee
\end{array}
$$

therefore, it suffices to show that the maps are isomorphisms for $\mathscr{F} = \mathcal{O}_P(-q)$ for all $q \gg 0$ since it commutes with direct sums. In this case,

$$
\begin{array}{c}
\mathrm{Hom}\,(P, \mathcal{O}(-q))\,\mathcal{O}(-d-1) \longrightarrow H^d(P, \mathcal{O}(-q))^\vee \\
\| \\
H^0(P, \mathcal{O}(q-d-1))
\end{array}
$$

and this is the map arising from the perfect pairing,

$$H^0(P, \mathcal{O}(q-d-1)) \times H^d(P, \mathcal{O}(-q)) \to H^d(P, \mathcal{O}(-d-1)) \xrightarrow{\sim} k$$

To get a unique δ-functor map,

$$\mathrm{Ext}_{\mathscr{F}}^j(\omega_P, \to) H^{d-j}(P, \mathscr{F})^\vee$$

agreeing with $j = 0$ and that it is an isomorphism it suffices to show that both sides are coerasable on $\mathfrak{Coh}\,(P)$ meaning each $\mathscr{F}$ admits $\mathcal{E} \twoheadrightarrow \mathscr{F}$ for $\mathcal{E} \in \mathfrak{Coh}\,(P)$ which kills the functors for $j > 0$. We take $\mathcal{E} = \mathcal{O}(-q)^{\oplus n}$ which surject. We need to show these kills for $q > 0$ and $n \geq 1$. However, by compatibility with direct sums it suffices to show that,

$$\mathrm{Ext}_P^j(\mathcal{O}(-q), \omega_P) = H^j(P, \mathcal{O}(q-d-1)) = 0$$

for $q > 0$ and all $j > 0$ (even $j = d$) and likewise,

$$H^{d-j}(P, \mathcal{O}(-q))^\vee = 0$$

for $j > 0$ because for $0 < j < d$ these always vanish and for $j = d$ then this is zero because $q > 0$. $\square$

38.1 Ext Sheaves

Definition 38.1.1. For a ringed space X, and $\mathscr{F} \in \mathbf{Mod}_X$ define $\mathscr{E}xt_X^\bullet(\mathscr{F}, -) : \mathbf{Mod}_X \to \mathbf{Mod}_X$ to be the derived functor of $\mathscr{H}om_X(\mathscr{F}, -) : \mathbf{Mod}_X \to \mathbf{Mod}_X$. For $\varphi : \mathscr{F} \to \mathscr{F}'$ we get $\mathscr{H}om_X(\mathscr{F}', -) \to \mathscr{H}om_X(\mathscr{F}, -)$ via $\alpha \mapsto \alpha \circ \varphi$ so we get $\mathscr{E}xt_X^\bullet(\mathscr{F}', -) \to \mathscr{E}xt_X^\bullet(\mathscr{F}, -)$. We will also upgrade this to a δ-functor.

Proposition 38.1.2.   (a) for open $U \subset X$ we have,

$$\mathscr{E}xt_X^\bullet(\mathscr{F}, \mathscr{G}) \,|_U \cong \mathscr{E}xt_U^\bullet(\mathscr{F}|_U, \mathscr{G}|_U)$$

bi-δ-functorially

(b) for $\mathcal{E}$ locally free of finite rank $\mathscr{E}xt_X^\bullet(\mathscr{F}, \mathcal{E} \otimes \mathscr{G}) = \mathcal{E} \otimes \mathscr{E}xt_X^\bullet(\mathscr{F}, \mathscr{G}) = \mathscr{E}xt_X^\bullet(\mathcal{E}^\vee \otimes \mathscr{F}, \mathscr{G})$

(c) there is a spectral sequence,

$$E_2^{p,q} = H^p(X, \mathscr{E}xt_X^q(\mathscr{F}, \mathscr{G})) \implies \mathrm{Ext}_X^{p+q}(\mathscr{F}, \mathscr{G})$$

(d) if X is a locally noetherian scheme and $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent then all $\mathscr{E}xt_X^q(\mathscr{F}, \mathscr{G})$ are quasi-coherent and,

$$\mathscr{E}xt_X^q(\mathscr{F}, \mathscr{G})_x = \mathrm{Ext}_{\mathcal{O}_{X,x}}^q(\mathscr{F}_x, \mathscr{G}_x)$$

and if X is additionally affine $X = \mathrm{Spec}\,(A)$ and $\mathscr{F} = \widetilde{M}$ and $\mathscr{G} = \widetilde{N}$ then,

$$\mathscr{E}xt_X^q(\mathscr{F}, \mathscr{G}) = \widetilde{\mathrm{Ext}_A^q(M, N)}$$

Proposition 38.1.3. Let $j : X \hookrightarrow P = \mathbb{P}_k^N$ be a closed immersion with pure dimension d so pure codimension $c = N - d$. Then,

(a) $\mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P) = 0$ for $i < c$

(b) $\omega_{X/k} = \mathscr{E}xt_P^c(j_*\mathcal{O}_X, \omega_P)$ is dualizing on X.

39   Theorem on Formal Functions

Remark. In topology, a continuous map $f : X \to Y$ is universally closed (called "proper" in Bourbaki) if for all continuous map $Z \to Y$ the map $X \times_Y Z \to Z$ is closed. By [Lang, Ch.II, Ex. 22-26] univ. closed iff closed + qu-compact fibers iff closed + preimage of quasi-compact is quasi-compact.

Remark. For locally compact Hausdorff X, Y, closedness is automatic so proper if and only if the preimage of quasi-compact is quasi-compact.

Definition 39.0.1. Say a continuous map $f : X \to Y$ is *(topologically) proper* if it is universally closed and separated.

Remark. By [SGA1, Exp XII, Prop. 3.2(v)] for $f : X \to Y$ over $\mathbb{C}$ between finite type $\mathbb{C}$-schemes then f is proper if and only if $X(\mathcal{C}) \to Y(\mathcal{C})$ is proper with the analytic topology.

Remark. For continuous map $f : X \to Y$ of topological spaces, $\mathscr{F} \in \mathbf{Ab}X$ if $y \in Y$ then get,

$$R^i f_*(\mathscr{F})_y = \varinjlim_{U \ni y} H^i(f^{-1}(U), \mathscr{F}) \to H^i(X_y, \mathscr{F}_y)$$

where the first equality holds because sheafification preserves stalks. Note that $\mathscr{F}_y = \mathscr{F}|_{X_y}$.

Theorem 39.0.2 (Topological Proper Base Change)**.** If f is (topologically) proper then all,

$$\theta_y^i : R^i f_*(\mathscr{F})_y \to H^i(X_y, \mathscr{F}|_{X_y})$$

are isomorphisms.

SGA4I, ExpVbis, Cor. 4.12. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. We want a variant of this theorem for schemes. First we give a variant construction of the base change map for locally ringed spaces.

Definition 39.0.3. For $f : X \to Y$ a map of locally ringed spaces, make X_y a locally ringed space using $\mathcal{O}_{X_y} = \mathcal{O}_X|_{X_y} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$ (probably this is the fiber product in LRS). For $\mathscr{F} \in \mathbf{Mod}_X$ define $\mathscr{F}_y \in \mathbf{Mod}_{X_y}$ by

$$\mathscr{F}_y = (\mathscr{F}|_{X_y}) \otimes_{\mathcal{O}_{Y,y}} \kappa(y) = j_y^*(\mathscr{F})$$

Then we define,

$$R^i f_*(\mathscr{F})_y \to H^i(X_y, \mathscr{F}|_{X_y}) \to H^i(X_y, \mathscr{F}_y)$$

which is linear over $\mathcal{O}_{Y,y} \to \kappa(y)$ giving an induced map,

$$\theta_y^i : R^i f_*(\mathscr{F})_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y) \to H^i(X_y, \mathscr{F}_y)$$

Example 39.0.4. If $f : X \to Y$ is quasi-compact separated morphism of schemes and $\mathscr{F} \in \mathfrak{QCoh}(X)$. Assume $Y = \mathrm{Spec}(A)$ (the construction is local over the base anyway) then we get,

$$H^i(X, \mathscr{F}) \otimes_A \kappa(y) \to H^i(X_y, \mathscr{F}_y)$$

Remark. One more variant (the epsilon variant): infinitesimal fibers! For $n \geq 0$.,

$$X_{y,n} = (X_y, \mathcal{O}_X \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1}) \quad \text{and} \quad \mathscr{F}_{y,n} = \mathscr{F}|_{X_y} \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1})$$

We likewise get a base change map,

$$\theta_{y,n}^i : R^i f_*(\mathscr{F})_y/\mathfrak{m}_y^{n+1} \to H^i(X_{y,n}, \mathscr{F}_{y,n})$$

Remark. Focus on a proper map of local. noetherian schemes $f : X \to Y$ and $\mathscr{F} \in \mathfrak{Coh}(X)$ so $R^i f_*(\mathscr{F}) \in \mathfrak{Coh}(Y)$ and thus $R^i f_*(\mathscr{F})_y$ is $\mathcal{O}_{Y,y}$-finite.

39.1 Completions

Remark. We are about to take completions so lets review some properties.

Proposition 39.1.1. For M a finite A-module with A noetherian and an ideal $I \subset A$, then the completion,

$$\hat{M} = \varprojlim M/I^{n+1}M$$

is a module over $\hat{A}$ which is noetherian and,

(a) $A \to \hat{A}$ is flat (faithfully flat if A is local with maximal ideal I)

(b) $M \otimes_A \hat{A} \to \hat{M}$ is an isomorphism.

Mat, Thms 8.7, 8.8. □

Remark. The map $\mathbb{Z} \to \mathbb{Z}_p$ is not faithfully flat so we need A to be local.

Definition 39.1.2. Pass to the inverse limit of $\theta^i_{y,n}$ then,

$$\hat{\theta}^i_y R^i f_*(\mathscr{F})_y \otimes_{\mathcal{O}_{Y,y}} \widehat{\mathcal{O}_{Y,y}} \cong \widehat{R^i f_*(\mathscr{F})}_y \to \varprojlim_n H^i(X_{y,n}, \mathscr{F}_{y,n})$$

Remark. In an ideal world, all $\theta^i_{y,n}$ would be isomorphisms. Grothendieck gave two ways to get satisfactory results:

(a) base change theorems for cohomology with flatness results

(b) Direct isomorphism for $\hat{\theta}^i_y$ with no further assumptions (even get more than with $\mathcal{O}_{Y,y}$!)

Theorem 39.1.3 (Formal Functions). For $f : X \to \operatorname{Spec}(A)$ proper with A noetherian and $\mathscr{F} \in \mathfrak{Coh}(X)$ and $I \subset A$ na ideal let,

$$X_n = X \otimes_A A/I^{n+1} \quad \mathscr{F}_n = \mathscr{F} \otimes_A A/I^{n+1} \in \mathfrak{Coh}(X_n)$$

Then the maps,

$$\hat{\theta}^i_y : H^i(X, \mathscr{F}) \otimes_A \hat{A} \to \varprojlim_n H^i(X_n, \mathscr{F}_n)$$

is an isomorphism.

Remark. (a) For $A = \mathcal{O}_{Y,y}$ and $I = \mathfrak{m}_y$ and $X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y})$ then get $\hat{\theta}^i_y$ isomorphism in preceding discussion since coherent cohomology commutes with flat base change $\operatorname{Spec}(A_y) \to \operatorname{Spec}(A)$)

(b) Since $A \to \hat{A}$ is flat, $H^i(X, \mathscr{F}) \otimes_A \hat{A} \cong H^i(X_{\hat{A}}, \mathscr{F}_{\hat{A}})$ and therefore this is a statement this is a statement over a complete base. This is basically formal GAGA.

(c) $A = \mathbb{Z}[[q]]$ and $I = (q)$ is widely used in the arithmetic of modular forms...

Theorem 39.1.4. If $f : X \to Y$ with Y local noetherian is proper + quasi-finite then f is finite.

Proof. By properness, $f_* \mathcal{O}_X$ is $\mathcal{O}_Y$-finite (even coherent). If f were affine, then $X = \mathbf{Spec}_Y(f_* \mathcal{O}_X)$ is finite over Y so it suffices to show that f is affine. We work locally on the base so assume that $Y = \operatorname{Spec}(A)$. Thus it suffices to show that X is affine. Since f is quasi-compact and separated, X is quasi-compact and separated. We will check the cohomology criterion $H^i(X, \mathscr{F}) = 0$ for all $i > 0$ and $\mathscr{F} \in \mathfrak{Coh}(X)$. Enough to show that $H^i(X, \mathscr{F})_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \operatorname{mSpec}(A)$. By finiteness (proper + coherent) and NAK it suffices to show that $H^i(X, \mathscr{F}) \otimes_A A/\mathfrak{m} = 0$ and since,

$$\widehat{H^i(X, \mathscr{F})} \otimes_{\hat{A}} \hat{A}/\mathfrak{m} = H^i(X, \mathscr{F}) \otimes_A A/\mathfrak{m}$$
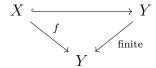
it suffices to show that the completion is zero. By the theorem on formal functions,

$$\widehat{H^i(X, \mathscr{F})} = \varprojlim H^i(X_n, \mathscr{F}_n)$$

but the fiber spaces X_n are finite discrete spaces and thus have vanishing higher cohomology completing the theorem. □

40 Mar. 4

Theorem 40.0.1 (Zariski's Main Theorem)**.** For noetherian Y and $f : X \to Y$ quasi-finite and separated then there exists a diagram,

$$
\begin{array}{ccc}
X & \hookrightarrow & Y \\
& \searrow^{f} \quad \swarrow_{\text{finite}} & \\
& Y &
\end{array}
$$

where $Y \hookrightarrow Y$ is an open immersion.

EGA III, Thm 4.4.3. □

Remark. This is useful. Used to show that étale maps $f : X \to Y$ Zariski locally looks like $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ where $B = (A[T]/(f))_{f'h}$ for monic $f \in A[T]$.

Remark. Let X be the line with two origins then the map $f : X \to \mathbb{A}^1$ is quasi-finite but not separated and indeed cannot be an open in a finite cover of $\mathbb{A}^1$ because such covers are affine over $\mathbb{A}^1$ and hence affine and hence separated but of course X is not separated.

Theorem 40.0.2 (Stein)**.** Let $f : X \to Y$ be proper then let $Y' = \mathbf{Spec}_Y\,(f_*\mathcal{O}_X)$. Because f is proper $f_*\mathcal{O}_X$ is coherent and thus $\pi : Y' \to Y$ is finite. Furthermore, $f' : X \to Y'$ has $f'_*\mathcal{O}_X = \mathcal{O}_Y$ by construction. Then f' has connected fibers.

Proof. The only thing to show is the following. □

Theorem 40.0.3 (Zariski Connectedness)**.** If $f : X \to Y$ is proper and $f_*\mathcal{O}_X = \mathcal{O}_Y$ then f has connected fibers.

Proof. Assume $f : X \to Y$ is proper with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Choose $y \in Y$. Want X_y is connected. Pass to $X \times_Y \mathrm{Spec}\,(\mathcal{O}_{Y,y}) \to \mathrm{Spec}\,(\mathcal{O}_{Y,y})$ this is a flat base change along $\mathrm{Spec}\,(\mathcal{O}_{Y,y}) \to Y$ and this commutes with cohomology so we can reduce to the case $Y = \mathrm{Spec}\,(R)$ with $f_*\mathcal{O}_X = \mathcal{O}_Y$ and R is local noetherian where $y = \mathfrak{m}$ is the closed point. We want to deduce that X_0 is connected. Consider the completion $R \to \hat{R}$ and apply flat base change and the special fiber does not change but preserves the cohomology condition so we can assume that R is complete. The theorem of formal functions says that,

$$R = H^0(X, \mathcal{O}_X) = \varprojlim H^0(X_n, \mathcal{O}_{X_n})$$

Assume that X_0 is disconnected write $X_0 = U_0 \sqcup V_0$ for nonempty open disjoint U_0 and V_0. Then we get nontrivial idempotents $e_0|_{U_0} = 1$ and $e_0|_{V_0} = 0$. All the X_n have compatible separations so get compatible sequence of e_n and get $e = \varprojlim e_n \in R$ is an idempotent but it is not 0 or 1 at any finite level and therefore cannot be 0 or 1 but R is local and thus $\mathrm{Spec}\,(R)$ is connected giving a contradiction. □

Remark. Consider,

(a) In HW10, show the fibers are actually geometrically connected.

(b) Consider $f : X \to Y$ surjective proper with X integral and Y normal noetherian and connected. Assume X_η satisfies $H^0(X_\eta, \mathcal{O}_{X_\eta}) = \kappa(\eta) = K(Y)$ (e.g. if X_η is geometrically connected and geometrically reduced over η) then $f_*\mathcal{O}_X = \mathcal{O}_Y$ because $f_*\mathcal{O}_X$ is finite over $\mathcal{O}_Y$ inside the function field and hence integral but Y is normal so we get an equality.

Example 40.0.4. $Y = \mathbb{A}^1_k = \mathrm{Spec}\,(k[t])$ and $X = \mathrm{Proj}\,(k[t][x,y,z]/(xy - t^2 z^2))$ so the fibers over $t \neq 0$ are irreducible hyperbolas and for $t = 0$ the fiber is a reducible but connected union of two lines. Indeed $f_* \mathcal{O}_X = \mathcal{O}_Y$ so this connectedness follows from Zariski's theorem.

Remark. Now we turn to the proof of the theorem on formal functions.

Theorem 40.0.5. Let $f : X \to \mathrm{Spec}\,(A)$ be proper with A noetherian. Let $I \subset A$ be an ideal and $\hat{A}$ the I-adic completion of A. For $\mathscr{F} \in \mathfrak{Coh}\,(X)$ we have,

$$H^i(X, \mathscr{F}) \otimes_A \hat{A} = \widehat{H^i(X, \mathscr{F})} \xrightarrow{\sim} \varprojlim_n H^i(X_n, \mathscr{F}_n)$$

is a topological isomorphism for all $i \geq 0$.

Proof. First notice that,

$$H^i(X_n, \mathscr{F}_n) = H^i(X, \mathscr{F}/I^{n+1}\mathscr{F})$$

Consider the long exact sequence associated to,

$$0 \longrightarrow I^n \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_{n-1} \longrightarrow 0$$
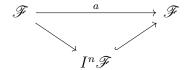
which gives,

$$\cdots \longrightarrow H^i(X, I^n \mathscr{F}) \xrightarrow{u_n} H^i(X, \mathscr{F}) \longrightarrow H^i(X, \mathscr{F}_{n-1}) \xrightarrow{\delta} H^{i+1}(X, I^n \mathscr{F}) \xrightarrow{v_n} H^{i+1}(X, \mathscr{F}) \longrightarrow \cdots$$

this produces compatible short exact sequences,

$$(*)_n \qquad 0 \longrightarrow H^i(X, \mathscr{F})/\mathrm{im}\,u_n \longrightarrow H^i(X, \mathscr{F}_{n-1}) \xrightarrow{\delta} \ker v_n \longrightarrow 0$$

We have $I^n H^i(X, \mathscr{F}) \subset \mathrm{im}\,u_n$ because for $a \in I^n$ we have,

The transition maps,

$$H^i(X, \mathscr{F})/\mathrm{im}\,u_{n+1} \to H^i(X, \mathscr{F})/\mathrm{im}\,u_n$$

are surjective since $\mathrm{im}\,u_{n+1} \subset \mathrm{im}\,u_n$ (so it satisfies M-L condition) and therefore applying $\varprojlim_n$ is exact giving a short exact sequence,

$$0 \longrightarrow \varprojlim_n H^i(X, \mathscr{F})/\mathrm{im}\,u_n \longrightarrow \varprojlim_n H^i(X, \mathscr{F}_{n-1}) \longrightarrow \varprojlim_n \ker v_n \longrightarrow 0$$

We want $\{\mathrm{im}\,u_n\}$ to define the I-adic topology on $H^i(X, \mathscr{F})$ and for big $d > 0$ we want

$$\ker v_{n+d} \to \ker v_h$$

vanish for $n \geq d$ and thus $\varprojlim_n \ker v_n = 0$ (however the first statement is stronger).

We introduce a graded module:

$$S = \bigoplus_{n \geq 0} I^n$$

which is noetherian $\mathbb{N}$-graded and generated by S_1 over S_0. Let,

$$M = \bigoplus_{n \geq 0} H^i(X, I^n \mathscr{F})$$

is an $\mathbb{N}$-graded S-module. Consider the map,

$$S_m \times H^i(X, I^n \mathscr{F}) \xrightarrow{*} H^i(X, I^{n+m} \mathscr{F}) \quad \text{via} \quad (a, \xi) \mapsto H^i(\mu_a)(\xi)$$

where $\mu_a : I^n \mathscr{F} \to I^{n+m} \mathscr{F}$ given by multiplication by $a \in S_n = I^n$. Then,

$$M = H^i(X, \bigoplus i^n \mathscr{F}) = H^i(X, g_* \mathscr{F}_S)$$

where

$$
\begin{array}{ccc}
X_S & \xrightarrow{\ \ g\ \ } & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}\,(S) & \longrightarrow & \mathrm{Spec}\,(A)
\end{array}
$$

because g is an affine map so we get,

$$H^i(X, g_* \mathscr{F}_S) = H^i(X_S, \mathscr{F}_S)$$

which is S-finite because $X_S \to \mathrm{Spec}\,(S)$ is proper and $\mathscr{F}_S \in \mathfrak{Coh}\,(X_S)$. But $M \twoheadrightarrow \bigoplus \mathrm{im}\, u_n = N$ is S-linear quotient so N is S-finite graded. For big n we have $N_{n+1} = S_1 \cdot N_n$ so we get $\mathrm{im}\, u_{n+1} = I \,\mathrm{im}\, u_n$ inside $H^i(X, \mathscr{F})$ which means that this sequence is I-adic. $\qquad\square$

41 Mar 7

41.1 Base Change for Cohomology

Zariski connectedness theorem also allows us to conclude (even in characteristic p) that the moduli space of curves $\mathcal{M}_g$ is connected. Deligne and Mumford compactified the moduli space and used Zariski connectedness to show that the new moduli space is irreducible.

Last time we saw that in the formal limit cohomology and base change commute for proper maps. This time we talk about base change theorems without the formal limit but with flatness assumptions.

Consider a proper map $f : X \to S$ with S locally noetherian, $\mathscr{F} \in \mathfrak{Coh}\,(X)$ which we will usually take to be flat over S.

Definition 41.1.1. For a map of schemes $f : X \to S$ we say that a $\mathcal{O}_X$-module $\mathscr{F}$ is flat over S if $\mathscr{F}_x$ is flat over $\mathcal{O}_{X, f(x)}$ for each $x \in X$. If $\mathscr{F}$ is quasi-coherent this is equivalent to $\mathscr{F}(U)$ being a flat $\mathcal{O}_Y(V)$-module for affines $U \to V$).

Example 41.1.2. If $f : X \to S$ is flat and $\mathscr{F}$ is locally free then f is flat over S.

Consider the morphisms,

$$\theta_s^i : (R^i f_* \mathscr{F})(s) \to H^i(X_s, \mathscr{F}_s)$$

We want to know when these are isomorphisms and relate it to local freeness of the higher pushforwards. Hopefully we will also so some natural examples where θ^i fails to be an isomorphism.

Remark. Often these questions arise in many algebraic situations for example:

(a) S is a "moduli scheme" and $X \to S$ is a "universal family" (e.g. of curves, abelian varieties, etc) and $\mathscr{F} = \mathcal{O}_X$ (or for f smooth also $(\Omega^1_{X/S})^{\otimes q}$ called the Hodge bundle). Then $(R^i f_*)(\mathscr{F})$ arise for intersection theory on S and, in the case of elliptic curves, for the arithmetic of modular forms.

(b) Given a variety (lets say a curve for concreteness) defined over $\mathbb{Q}$ then we want to spread out to an "arithmetic family" over $S = \mathrm{Spec}\,(R)$ for R a DVR with fraction field K (in our case some number field). Then for $X_K \hookrightarrow \mathbb{P}^n_K$ we can take the schematic closure of $X_K \hookrightarrow \mathbb{P}^n_K \hookrightarrow \mathbb{P}^n_R$ giving closed $X \hookrightarrow \mathbb{P}^n_R$. Now the map $\mathcal{O}_X \to \mathcal{O}_{X_K}$ is injective so $\mathcal{O}_X$ is R-torsion free and hence $X \to \mathrm{Spec}\,(R)$ is proper and flat. If the generic fiber is geometrically connected and reduced, then by the Zariski connectedness theorem we know the special fiber is also geometrically connected which is our first step towards making it "nice" by blowing up. Supposing,
$$H^1(X, \mathcal{O}_X) \otimes_R k \to H^1(X_0, \mathcal{O}_{X_0})$$
is an isomorphism then we can get a handle on "improving" the special fiber.

Concrete question:

Example 41.1.3. For a smooth proper map $f : X \to S$ with all $X_{\bar{s}}$ connected curves, is the function,
$$s \mapsto h^0(X_s, \Omega^1_{X_s/\kappa(s)}) = h^1(X_s, \mathcal{O}_{X_s})$$
sending s to the genus of the fiber locally constant on S?

Example 41.1.4. A variant of this: for a closed subscheme $X \hookrightarrow \mathbb{P}^N_S$ such that $X \to S$ is flat, is the Hilbert polynomial,
$$s \mapsto \chi_s(n) = \sum_j (-1)^j h^j(X_s, \mathcal{O}_{X_s}(n)) \in \mathbb{Q}[n]$$

locally constant in s as a rational polynomial.

The answer to both questions is yes (and the second will prove the first). However, $h^j(X_s, \mathscr{F}_S)$ can jump even though the Euler characteristics do not.

Example 41.1.5. Let C be a smooth proper geometrically connected curve, with genus $g > 0$, over a field k. Let $p \in C(k)$ be a rational point. Consider $X = C \times C$ and $S = C$ and let $f : X \to S$ be the second projection. This map is smooth because it is the base change of $C \to \mathrm{Spec}\,(k)$ along $C \to \mathrm{Spec}\,(k)$. On X there is are divisors: the diagonal $\Delta \subset C \times C$ and the constant section $\{p\} \times C \subset X$. Consider the invertible sheaf
$$\mathcal{L} = \mathcal{O}_X(\Delta - \{p\} \times C) \cong \mathcal{O}_X(\Delta) \otimes \mathcal{O}_X(\{p\} \times C)^{\otimes -1} = \mathscr{I}_\Delta^{\otimes -1} \otimes \mathscr{I}_\sigma$$

where $\sigma : S \to X$ is the section sending $x \mapsto (p, x)$. Now $X_s = C \otimes_k \kappa(s)$ and we would like that $\mathcal{L}_s \cong \mathcal{O}_C(s - p_{\kappa(s)})$. This works because the formation of $\mathscr{I}_\Delta$ and $\mathscr{I}_\sigma$ are ideal sheaves of sections $S \hookrightarrow X$ and therefore are flat over the base so the formation of their ideal sheaves commutes with base change. Indeed, given the situation,

$$
\begin{array}{ccc}
Z & \xrightarrow{\ j\ } & Y \\
 & \searrow_{\text{flat}} & \downarrow \\
 & & T
\end{array}
$$

then the sequence,

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathcal{O}_Y \longrightarrow j_*\mathcal{O}_Z \longrightarrow 0$$

is exact after any base change $T' \to T$ because $j_*\mathcal{O}_Z$ is flat over T and so the sequence remains exact after any tensor product because Tor with coefficients in $j_*\mathcal{O}_Z$ vanishes.

Returning to our situation, what can we say about $h^i(\mathcal{L}_s)$ for $i = 0, 1$.

i = 0

$$H^0(X_s, \mathcal{L}_s) = \Gamma(C_s, \mathcal{O}(s-p)) = \begin{cases} 0 & s \neq p \text{ and } g > 0 \\ k & s = p \end{cases}$$

i = 1

$$h^1(X_s, \mathcal{L}_s) = -\chi(X_s, \mathcal{L}_s) + h^0(X_s, \mathcal{L}_s) = \deg_{\kappa(s)}(s-p) + 1 - g + h^0(X_s, \mathcal{L}_s) = h^0(X_s, \mathcal{L}_s) + 1 - g$$

and therefore is just $h^0(X_s, \mathcal{L}_s)$ but with a constant shift and therefore has the same jumping behavior. This is a constant because,

$$g_s = \text{genus}(C_s) = \dim_{\kappa(s)} h^0(X_s, \Omega^1_{C_s/\kappa(s)}) = \dim_k h^0(C, \Omega^1_{C/k}) = g$$

Now let's look at the higher direct images $f_*\mathcal{L}$ and $R^1f_*(\mathscr{F})$ on $S = C$. Since X is integral and $X \to S$ is dominant so $f_*\mathscr{F}$ is $\mathcal{O}_C$-torsion-free because its sections are sections on X. Then $f_*\mathcal{L} = 0$ because of the torsion-freeness it suffices to check that the generic fiber is zero but by flat base change,

$$(f_*\mathcal{L})_\eta = H^0(C_\eta, \mathcal{L}_\eta) = \Gamma(C_\eta, \mathcal{O}(\eta - p_\eta)) = 0$$

While we cannot use cohomology and base change to a random point, we can apply it to the generic point because $\eta \to C$ is flat. Using an upcoming result (next time), vanishing of $h^1(X_s, \mathcal{L}_s)$ for $s \in C \setminus \{p\}$ does imply that $R^1f_*(\mathscr{F})|_{C \setminus \{p\}} = 0$ so we show that $R^1f_*(\mathscr{F}) \in \mathfrak{Coh}(C)$ is some skyscraper sheave supported on $\{p\}$. I claim however that $R^1f_*(\mathcal{L}) \neq 0$. We will show that,

$$\theta^1_p : (R^1f_*\mathcal{L})(p) \to H^1(C, \mathcal{L}_p) \cong H^1(C, \mathcal{O}) \neq 0$$

is surjective. Localize to $\text{Spec}(\mathcal{O}_{C,p}) \to C$ so consider $f : Y \to \text{Spec}(R)$ for a DVR R whose fibers are 1-dimensional and f is proper and flat. Then there is an exact sequence,

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow k \longrightarrow 0$$

which by flatness of $\mathcal{L}$ over R gives an exact sequence of sheaves on Y,

$$0 \longrightarrow \mathfrak{m} \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_0 \longrightarrow 0$$

and then the long exact sequence of cohomology gives,

$$H^1(X, \mathcal{L}) \xrightarrow{\theta} H^1(Y_0, \mathcal{L}_0) \xrightarrow{\delta} H^2(Y, \mathfrak{m} \otimes_R \mathcal{L})$$

By the theorem on formal functions,

$$H^2(Y, \mathscr{F}) \otimes_R \hat{R} = \varprojlim H^2(Y_n, \mathscr{F}_n) = 0$$

because $\dim Y_n = 1$ for all n but $R \to \hat{R}$ is faithfully flat and therefore $H^2(Y, \mathscr{F}) = 0$.

41.2 General Base Change Map

Consider the diagram of ringed spaces,

$$
\begin{array}{ccc}
X' & \xrightarrow{\ g'\ } & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
S' & \xrightarrow{\ g\ } & S
\end{array}
$$

And let $\mathscr{F}$ be a $\mathcal{O}_X$-module and $\mathscr{F}' = g'^*\mathscr{F}$. Then there is a map,

$$\theta^i : g^* R^i f_*(\mathscr{F}) \to R^i f'_* \mathscr{F}'$$

This arises from the presheaves,

$$\underline{H}^i(\mathscr{F}) : U \to H^i(X_U, \mathscr{F})$$

$$\underline{H}^i(\mathscr{F}') : U' \to H^i(X'_{U'}, \mathscr{F}')$$

and there is a natural map,

$$\underline{H}^i(\mathscr{F})(U) \to \underline{H}^i(\mathscr{F}')(g^{-1}(U)) = g_*(\underline{H}^i(\mathscr{F}'))(U)$$

so we get $\underline{H}^i(\mathscr{F}) \to g_*(\underline{H}^i(\mathscr{F}'))$ and therefore applying sheafification,

$$
\begin{array}{ccc}
\underline{H}^i(\mathscr{F}) & \longrightarrow & g_*(\underline{H}^i(\mathscr{F}')) & \longrightarrow & g_*(R^i f'_*(\mathscr{F}')) \\
\downarrow & & & \nearrow & \\
R^i f_*(\mathscr{F}) & & & &
\end{array}
$$

yields θ^i. In scheme setting for quasi-coherent $\mathscr{F}$, over affine opens $\mathrm{Spec}\,(A) \subset S$ and $\mathrm{Spec}\,(A') \subset S'$ mapping to $\mathrm{Spec}\,(A)$ we get,

$$H^i(X_A, \mathscr{F}) \to H^i(X'_{A'}, \mathscr{F}')$$

which is linear over $A \to A^1$, so this gives,

$$A' \otimes_A H^i(X_A, \mathscr{F}) \to H^i(X'_{A'}, \mathscr{F}')$$

Remark. Such θ^i being an isomorphism for all $S' \to S$ is closely linked to local freeness of $R^i f_*(\mathscr{F})$ and the surjectivity of θ^i_s for all s. Next time we will see this and how the jumping phenomena are generic along closed subschemes.

42 Cohomology and Base Change II

The theorem on formal functions only assumed properness not flatness but only gives an isomorphism on *formal* neighborhoods not Zariski neighborhoods. Today we prove cohomology and base change for flat sheaves.

Consider a proper map $f : X \to S$ for locally noetherian S, and $\mathscr{F} \in \mathfrak{Coh}\,(X)$ that is flat over S. We have the base change maps,

$$\theta^i_s : (R^i f_* \mathscr{F})(s) \to H^i(X_s, \mathscr{F}_s)$$

for $s \in S$. The main goal is the following theorem:

Theorem 42.0.1 (Base Change). Suppose that $\theta^i_{s_0}$ is surjective for some $s_0 \in S$ (with fixed i). Then,

(a) The map $\theta^i_{s_0}$ is an isomorphism and there is an open $U \subset S$ neighborhood of s_0 such that θ^i_s is an isomorphism for all $s \in U$.

(b) $\theta^{i-1}_{s_0}$ is surjective if and only if $R^i f_*(\mathscr{F})_{s_0}$ is a free $\mathcal{O}_{s_0}$-free module (iff $R^i f_*(\mathscr{F})$ is locally free at s_0 by coherence).

When both θ^i_s and θ^{i-1}_s are surjective for all $s \in U$ then $R^i f_*(\mathscr{F})|_U$ commutes with arbitrary base change (for any open $\mathrm{Spec}\,(A) \subset U$ we have $H^i(X_A, \mathscr{F}_A) \otimes_A A' \xrightarrow{\sim} H^i(X_{A'}, \mathscr{F}_{A'})$ for any A-algebra A'.

Corollary 42.0.2. Extremely useful example [EGAIII2, 7] Assume $H^1(X_{s_0}, \mathscr{F}_{s_0}) = 0$. By (1) then $R^1 f_*(\mathscr{F})_{s_0} = 0$ by Nakayama so $R^1 f_*(\mathscr{F})|_U = 0$ on some open neighborhood U of s_0. Thus this is locally free so by (2) we know that θ^0_s is surjective for all $s \in U$ and hence an isomorphism. By by definition, θ^{-1}_s is always surjective and thus we conclude that $f_*(\mathscr{F})|_U$ is locally free and commutes with all base change.

42.1 Setup for the Proof of Cohomology and Base Change

For the proof of the base change theorem, we may replace $S = \mathrm{Spec}\,(A)$ for a noetherian ring A since everything is local on the base. Now we have $R^i f_*(\mathscr{F}) = \widetilde{H^i(X, \mathscr{F})}$ which is coherent. To understand these groups we choose a finite affine open cover $\mathfrak{U} = \{U_i\}$ of X such that,

$$H^i(X, \mathscr{F}) = H^i(C^\bullet) \quad \text{where} \quad C^\bullet = C^\bullet(\mathfrak{U}, \mathscr{F})$$

for the ordered Cech cover. Notice that $C^\bullet$ is a bounded complex and each term is comprised of a finite product of groups of the form $\Gamma(U_{\underline{i}}, \mathscr{F}))$ which are flat A-modules by the assumption that $\mathscr{F}$ is flat. Hence each C^i is A-flat because it is a finite product of flat A-modules. Furthermore, by properness, all $H^i(C^\bullet)$ are finite A-modules (although the terms $C^\bullet$ are very much not finite).

Our goal is to understand when $H^i(C^\bullet)$ commutes with $- \otimes_A A'$ (e.g. $A' = k(s) = A/\mathfrak{m}$). Explicitly, what we mean is, when is the natural map,

$$A' \otimes_A H^i(C^\bullet) \to H^i(A' \otimes_A C^\bullet)$$

an isomorphism. We will turn this into a concrete problem of free A-modules and matrices using the following lemma.

Lemma 42.1.1. Consider a bounded complex $C^\bullet$ of flat A-modules (say with amplitude $[0, N]$) with all $H^p(C^\bullet)$ finite. Then there exists a complex $K^\bullet$ of A-modules with the same amplitude as $C^\bullet$ along with a map $g : K^\bullet \to C^\bullet$ such that,

(a) all K^p are finite free for $1 \le p \le N$ and K^0 is finite flat over A (or equivalently finite locally free as an A-module)

(b) for any A-module M the natural map $H^p(K^\bullet \otimes_A M) \xrightarrow{\sim} H^p(C^\bullet \otimes M)$ is an isomorphism for all p.

96

Proof. Idea: descending inductive construction. Choose generators of the cohomology and map a free module onto lifts of these generators in $C^\bullet$. [H, Ch. III, Lemma 12.9] except $K^\bullet$ is not bounded below. For the truncation in degree zero, see [Mum, Ch II, Section 5, 1st Lemma 1]. $\qquad\square$

Corollary 42.1.2. The map $s \mapsto \chi(X_s, \mathscr{F}_s)$ is locally constant.

Proof. WLOG take $S = \operatorname{Spec}(A)$ and get a complex $K^\bullet$ as above. Shrinking the base we may assume that all $K^\bullet$ are free. Then,

$$H^p(X_s, \mathscr{F}_s) = H^p(K^\bullet(s))$$

for $M(s) = M \otimes_A k(s)$. However, the complex,

$$0 \longrightarrow K^0(s) \longrightarrow \cdots \longrightarrow K^N(s) \longrightarrow 0$$

is a bounded complex of finite $k(s)$-vector spaces and therefore,

$$\sum_p (-1)^p h^p(K^\bullet(s)) = \sum_p (-1)^p \dim K^p(S) = \sum_p (-1) \operatorname{rank}_A(K^p)$$

is constant in s. $\qquad\square$

Definition 42.1.3. Say a function $\varphi : |S| \to \mathbb{Z}$ is *upper semicontinuous* if $\{s \in S \mid \varphi(s) \geq n\}$ is closed.

Remark. Informally, this means that φ "jumps up along closed sets".

Corollary 42.1.4 (Semicontinuity)**.** The map $s \mapsto h^p(X_s, \mathscr{F}_s)$ is upper semicontinuous.

Proof. Passing to a small enough affine $S = \operatorname{Spec}(A)$ we can assume that $K^\bullet$ is a complex of free A-modules. We have a complex,

$$0 \longrightarrow K^0 \xrightarrow{\delta^0} K^1 \xrightarrow{\delta^1} \cdots \longrightarrow K^N \longrightarrow 0$$

where each δ^p is presented as a matrix. By design,

$$h^p(X_s, \mathscr{F}_s) = \dim_{k(s)} H^p(K^\bullet(s)) = \dim \ker \delta^p(s) - \dim \operatorname{im} \delta^{p-1}(s)$$

By the rank-nullity theorem for $\delta^p(s)$ this is,

$$\dim K^p(s) - \dim \operatorname{im} \delta^p(s) - \dim \operatorname{im} \delta^{p-1}(s) = \operatorname{rank}_A K^p - (\operatorname{rank} \delta^p(s) + \operatorname{rank} \delta^{p-1}(s))$$

Sufficient to show that $s \mapsto \operatorname{rank} \delta^i(s)$ is lower semicontinuous (meaning $\{s \in S \mid \operatorname{rank} \delta^i(s) \leq m\}$ is closed. This says that all $(m+1) \times (m+1)$ minors vanish which is a closed condition. $\qquad\square$

Remark. The preceding proof shows that if $s \mapsto h^p(X_s, \mathscr{F}_s)$ is *constant* then the ranks $\operatorname{rank} \delta^p(s)$ and $\operatorname{rank} \delta^{p-1}(s)$ never jump (for connected S) because *a priori* they are upper semicontinuous but their sum is constant and therefore these are constant.

Lemma 42.1.5 (Rank Lemma)**.** Let Y be a reduced noetherian scheme and $\varphi : \mathcal{E}' \to \mathcal{E}$ be a map of vector bundles on Y (e.g. $\widetilde{\delta^p}; \widetilde{K^p} \to \widetilde{K^{p+1}}$). Then the following are equivalent,

  (a) $\operatorname{rank} \varphi(y) : \xi'(y) \to \xi(y)$ is constant r for $y \in Y$

(b) φ factors as $\xi' \twoheadrightarrow \nu \hookrightarrow \xi$ onto some subbundle ν (meaning $\nu \hookrightarrow \xi$ is a local direct summand) with rank $\nu = r$.

Proof. (2) $\implies$ (1) is easy. And (1) $\implies$ (2) by [Mum, Ch II, Section 5, 2$^{\text{nd}}$ Lemma 2]. $\qquad \square$

Remark. Reducedness is essential. Consider $A = k[\epsilon]$ and,

$$\varphi = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}$$

Then $\varphi(y) = 0$ with rank 0 but $\varphi : A^{\oplus 2} \to A^{\oplus 2}$ does not have locally free image.

Remark. φ strongly rank r implies that $\ker \varphi \subset \xi'$ is also locally free and a subbundle and all $\ker \varphi$ $\operatorname{im} \varphi = \nu$ and $\operatorname{coker} \varphi$ all commute with arbitrary base change.

Theorem 42.1.6 (Grauert). If $s \mapsto h^p(X_s, \mathscr{F}_s)$ is constant and S is reduced then $R^p f_*(\mathscr{F})$ is locally free and commutes with every base change.

Proof. WLOG we assume $S = \operatorname{Spec}(A)$ and get $K^\bullet$ with all K^p finite free. In the proof of semicontinuity, we saw that δ^p and δ^{p-1} have constant fiber rank. Since S is reduced, by the rank lemma, we get that δ^p and δ^{p-1} are strongly of constant rank. Therefore,

$$\operatorname{im} \delta^{p-1} \subset \ker \delta^p \subset K^p$$

are each subbundles of K^- so $\operatorname{im} \delta^{p-1} \hookrightarrow \ker \delta^p$ is a subbundle inclusion so $H^p(K^\bullet)$ is locally free and commutes with any base change. $\qquad \square$

43 Proof and Applications of Base Change Theorem

Proof. For proper $f : X \to S$ with S locally noetherian and $\mathscr{F} \in \mathfrak{Coh}(X)$ flat over S assume that $\theta_{s_0}^i : (R^i f_*(\mathscr{F}))(s_0) \to H^i(X_{s_0}, \mathscr{F}_{s_0})$ is surjecitve. We want to show that,

(a) $\theta_{s_0}^i$ is an isomorphism

(b) θ_s^i is surjective and hence an isomorphism for all s near s_0.

By shrinking our base we can assume $S = \operatorname{Spec}(A)$ and have $K^\bullet$ complex with all terms free universally computing cohomology. Therefore, the map in question is,

$$\theta_{s_0}^i : H^i(K^\bullet)(s_0) \to H^i(K^\bullet(s_0))$$

the natrual map for tensor product and cohomology. Observe that localization is exact so $H^i(K^\bullet)(s_0) = H^i(K_{s_0}^\bullet)/\mathfrak{m}_{s_0}$. Thus $\theta_{s_0}^i$ arises from the surjective map of complexes,

$$K_{s_0}^\bullet \twoheadrightarrow K^\bullet(s_0)$$

The map on the H^i fits into the sequence,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im} \delta_{s_0}^{i+1} & \longrightarrow & \ker \delta_{s_0}^i & \longrightarrow & H^i(K_{s_0}^\bullet) & \longrightarrow & 0 \\
 & & \downarrow{\gamma} & & \downarrow{\alpha} & & \downarrow{\beta} & & \\
0 & \longrightarrow & \operatorname{im} \delta^{i-1}(s_0) & \longrightarrow & \ker \delta^i(s_0) & \longrightarrow & H^i(K^\bullet(s_0)) & \longrightarrow & 0
\end{array}
$$

then γ is surjective because the map of complexes is surjective. Therefore, by the snake lemma β is surjective if and only if α is surjective. However, β is surjective if and only if $\theta^i_{s_0}$ is surjective because it is a factorization. Nor a miracle: surjectivity of α if and only if $\widetilde{\delta^i} : \widetilde{K}^i \to \widetilde{K}^{i+1}$ being strongly of constant rank at s_0. $\qquad\square$

Lemma 43.0.1 (Strong Rank Lemma)**.** For Y a locally ringed space, $\varphi : \mathcal{E}' \to \mathcal{E}$ a map of vector bundles on Y, then for $y_0 \in Y$ the following are equivalent,

(a) φ is strongly of constant rank near y_0

(b) $\ker \varphi_{y_0} \twoheadrightarrow \ker \varphi(y_0)$

(c) $\mathrm{coker}\, \varphi$ is local free near y_0.

Proof. The equivalence of (a) and (c) is an easy exercise. The rest is in the handout. $\qquad\square$

Completion of the proof. Therefore $\theta^i_{s_0}$ is surjective if and only if δ^i being strongly of constant rank over some open neighborhood $s_0 \in U$. Therefore, in this case θ^i_s is surjective for all $s \in U$ and $\delta^i|_U : K^i_U \to K^{i+1}_U$ has $\ker \delta^i|_U \subset K^i_U$ is locally free and a subbundle. Formation comuting with any base change on U. However,

$$H^i(K^\bullet_U) = \mathrm{coker}\,(K^{i-1}_U \to \ker \delta^i|_U)$$

is $H^i(K^\bullet_U)$ commmutes with any base change on U, so θ^i_s is an isomorphism for all $s \in U$. Now we apply the strong rank lemma to $\delta^{i-1}|_U : K^{i-1}_U \to \ker \delta^i|_U$ which is a map of vector bundles. The cokernel $H^i(K^\bullet|_U)$ is locally free near s_0 if and only if δ^{i-1} is strongly of constant rank at s_0 which by the previous argument is equivalent to $\theta^{i-1}_{s_0}$ being surjective. $\qquad\square$

44 Applications

[H] gives many applications of RR to intrinsic and extrinsic geometry of smooth proper geometrically connected curves X over k but only when $k = \bar{k}$. Using fpqc descent, we can often reduce suitable statments over k to the setting $\bar{k}$.

Theorem 44.0.1. For general k, Weil divisor D on X,

$$\mathcal{O}_X(D) \text{ is ample} \iff \deg_k D > 0$$

Furthermore, if $\deg_k D \geq 2g + 1$ then $\mathcal{O}_X(D)$ is very ample.

Proof. In handout on RR calculations, reduce to $k = \bar{k}$ so that closed points are rational. The handout describes $g = 0$ and $g = 1$ (with a k-point) over any k. $\qquad\square$

Definition 44.0.2. For a locally noetherian S, an *elliptic curve* over S is a pointed relative genus 1 curve.

Proposition 44.0.3. Then $s \mapsto \mathrm{genus}(X_s)$ is locally constant on S using cohomology and base change for $\mathcal{O}_X$.

Remark. It is true that $f_*(\Omega^1_{X/S})$ is a rank r vector bundle commuting with base change but the proof is actually hard when S is nonreduced (can't use Grauert).

Definition 44.0.4. An elliptic curve over locally noetherian S is a genus-1 curve $E \to S$ equipped with $e \in E(S)$.

We want to build a commutative S-group law on (E, e)-relativizing the moedthod via divisors over fields. We'll use the invertible sheaves on E_T for $T \to S$ and base change theorems. By Yoneda, to make $m : E \times E \to E$ an S-group with identity e is suffices to build a commutative group law on $(E(T), e_T)$ for locally noetherian $T \to S$ that is functionial in T.

Remark. By Exercise B(iii) HW10, uniqueness and functoriallity of the group law on (E, e) reduecs to the geometric fibers.

For $P \in E(T)$ notice that $E(T) = E_T(T)$ so we consider $P : T \to E_T$ as a section of $E_T \to T$ and this is proper so P is a closed immersion and thus P has an ideal sheaf $\mathscr{I}_P$ that is invertible and commutes with base change (HW10, Exercise B(i)). For $\mathcal{L} = \mathscr{I}_P^{\otimes -1}$ on E_T then $\mathcal{L}_t = \mathcal{O}_{E_t}(P(t))$ has degree 1.

Theorem 44.0.5. Any invertible $\mathcal{L}$ on E with $\deg_{\kappa(s)}(\mathcal{L}_s) = 1$ for all $s \in S$ is of the form,

$$\mathscr{I}_P^{\otimes -1} \otimes f^* \mathcal{M}$$

for a unique section $P \in E(S)$ and line bundle on the base $\mathcal{M} \in \operatorname{Pic}(S)$.

Remark. Over a field, the above is proved using RR: $\deg_k D = 1$ and $g = 1$ implies that $D \sim P$ for a unqiue $P \in E(k)$.

Corollary 44.0.6. Let $\mathcal{L}_1 \approx \mathcal{L}_2$ if $\mathcal{L}_1 \cong \_2 \otimes f^* \mathcal{M}$ for some $\mathcal{M} \in \operatorname{Pic}(S)$. Cosndier $P, Q \in E(T) = E_T(T)$ then get $\mathscr{I}_P$ and $\mathscr{I}_Q$ on E_T therefore,

$$\mathcal{L} = \mathscr{I}_P^{\otimes -1} \otimes \mathscr{I}_Q^{\otimes -1} \otimes \mathscr{I}_{e_T}$$

has fiberwise degree 1 so $\mathcal{L} \approx \mathscr{I}_{P \oplus Q}^{-1}$ for a unqiue $P \oplus Q \in E(T)$. Clearly this is commutative and $P \oplus e_T = P$. Furthermore we get,

$$E(T) \hookrightarrow \operatorname{Pic}(E_T) / \operatorname{Pic}(T)$$

sending $P \mapsto \mathscr{I}_P^{\otimes -1} \otimes \mathscr{I}_{e_T}$ which sends the $\oplus$ operation on $E(T)$ to the $\otimes$ operation on the target. Therefore $\oplus$ is associative. Finally, we need to check that inverses exist. Consider $\mathcal{L} \cong \mathscr{I}_P \otimes \mathscr{I}_{e_T}^{\otimes -2}$ which has fiber degree 1 and thus $\mathcal{L} \approx \mathscr{I}_{P'}^{\otimes -1}$ for a unique $P' \in E(R)$. Therefore,

$$\mathscr{I}_P \otimes \mathscr{I}_{e_T}^{\otimes -1} \approx \mathscr{I}_{P'}^{\otimes -1} \otimes \mathscr{I}_{e_T} \implies \mathscr{I}_P^{\otimes -1} \otimes \mathscr{I}_{P'}^{\otimes} \otimes \mathscr{I}_{e_T} \approx \mathscr{I}_{e_T}^{\otimes -1}$$

and therefore $P \oplus P' = e_T$.