1 Calculus

1.1

Compute the following derivatives:

(a.) $\frac{\mathrm{d}}{\mathrm{d}x}x^2$

(b.) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + ax + b)$

(c.) $\frac{\mathrm{d}}{\mathrm{d}x}\tan x$

1.2

Compute the following definite and indefinite integrals:

(a.)

$$\int (x^3 + x + 1)\,\mathrm{d}x$$

(b.)

$$\int_0^{\pi/2} \sin x\,\mathrm{d}x$$

(c.)

$$\int_2^{\infty} \frac{3x^2 - 1}{x(x^2 - 1)}\,\mathrm{d}x$$

(d.)

$$\int_1^{\infty} \frac{3x^2 - 1}{\sqrt{x(x^2 - 1)}}\,\mathrm{d}x$$

(e.)

$$\int \frac{1}{x^2 - 1}\,\mathrm{d}x$$

1.3

Demonstrate uning limits why

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$

use your favorite definition of e^x and justify all limits used.

1.4

Do the following series converge or diverge?

(a.)

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

(b.)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

(c.)

$$\sum_{n=1,m=1}^{\infty} \frac{1}{n^2+m^2}$$

(d.)

$$\prod_{p \text{ primes}} \frac{1}{1-p^{-2}}$$

2 Polynomials

2.1

Find all complex roots of the following polynomials

(a.) $x^3 - 6x^2 + 11x - 6$

(b.) $x^6 - 1$

(c.) $x^4 - 2x^2 + 6$

2.2

Find as many integer solutions to

$$y^2 = x^3 + x + 1$$

as you can.

2.3

How many integer solutions does $x^2 - 2y^2 = 1$ have?

3 Complex Numbers

3.1

How many real roots does $x^3 + x + 1$ have? How many complex roots does $x^3 + x + 1$ have? What is the sum of the roots? What is the product?

3.2

Find and justify reasonable values for the following expressions (here $i = \sqrt{-1}$)

(a.) i^i

(b.) $\log(-1)$

(c.) $\log i$

4 Algebra

Definition: A group G is a set with a binary operation $\circ$ which satisfies,

1. associativity, $x \circ (y \circ z) = (x \circ y) \circ z$
2. there exists an identity $e \in G$ such that $e \circ g = g \circ e = g$ for any $g \in G$
3. for each $g \in G$ there exists an inverse $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$

4.1

Which of the following pairs of a set and a binary operation are groups? If not, why?

(a.) the natural numbers $\{0, 1, 2, \dots\}$ with addition $+$

(b.) the integers $\mathbb{Z}$ with addition $+$

(c.) the integers $\mathbb{Z}$ with multiplication $\cdot$

(d.) the rational numbers $\mathbb{Q}$ with addition $+$

(e.) the rational numbers $\mathbb{Q}$ with multiplication

(f.) the nonzero rational numbers $\mathbb{Q}^{\times}$ with addition $+$

(g.) the nonzero rational numbers $\mathbb{Q}^{\times}$ with multiplication $\cdot$

(h.) the permutations of the numbers $\{1, \dots, n\}$ with composition of functions

(i.) pairs of integers (a, b) with $a, b \in \mathbb{Z}$ and addition $(a, b) + (c, d) = (a + c, b + d)$.

Definition: A group G is *finitely generated* if there exists a finite set $S \subset G$ such that every element in $g \in G$ can be expressed as a finite combination of elements of S (and the inverses of elements in S) i.e. $g = s_1 \circ \cdots \circ s_n$ for $s_1, \dots, s_n \in S \cup S^{-1}$ where $S^{-1} = \{s^{-1} \mid s \in S\}$.

4.2

Of the groups you identified above, which are finitely generated? If yes, provide a finite generating set. If no, justifty why no finite generating set exists.