1 Remedial Curve Theory

1.1 Geometric Irreducibility of Generic Fibers

Lemma 1.1.1 (Tag 0553)**.** Let $f : X \to Y$ be a morphism of schemes. Assume,

(a) Y is irreducible with generic point η,

(b) X_η is geometrically irreducible

(c) f is of finite type

then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibers.

Lemma 1.1.2. Let $f : X \to Y$ be a morphism of schemes. Suppose that,

(a) X and Y are integral

(b) X is normal

(c) the fibers of f are geometrically connected (e.g. $f_* \mathcal{O}_X = \mathcal{O}_Y$)

then the generic fiber $X_\eta \to \text{Spec}\,(\kappa(\eta))$ is geometrically irreducible.

Proof. $X_\eta / \kappa(\eta)$ is geometrically irreducible iff $\kappa(\eta)$ is separable closed in $\kappa(\xi)$. This follows from Tag 054Q and Tag 0G33. Let $\alpha \in \kappa(\xi)$ be separably algebraic over $\kappa(\eta)$ i.e. a root of a separable polynomial $p \in \kappa(\eta)[x]$. There is a coordinate ring A of Y where all the denominators of p are invertible. We claim that $A[\alpha] \subset B$ where B is any coordinate ring of X containing A. Indeed, α is integral over A and hence over B so by normality $\alpha \in B$ so we get morphisms,

$$X_A \to \text{Spec}\,(A[\alpha]) \to \text{Spec}\,(A)$$

but the fibers of $X_A \to \text{Spec}\,(A)$ are geometrically connected so we must have $\alpha \in A$ since otherwise the fibers of $\text{Spec}\,(A[\alpha]) \to \text{Spec}\,(A)$ and hence $X_A \to \text{Spec}\,(A)$ are not geometrically irreducible. $\square$

Example 1.1.3. We cannot ensure geometric reducedness of the fiber via Stein factorization however. Indeed, consider,

$$X = \text{Proj}\,(\mathbb{F}_p[s,t][X,Y,Z]/(X^p + sY^p + tZ^p)) \to \text{Spec}\,(\mathbb{F}_p[s,t]) = Y$$

satisfies $f_* \mathcal{O}_X = \mathcal{O}_Y$ and X is normal but the generic fiber,

$$X = \text{Proj}\,(\mathbb{F}_p(s,t)[X,Y,Z]/(X^p + sY^p + tZ^p)) \to \text{Spec}\,(\mathbb{F}_p(s,t))$$

is not geometrically reduced. Indeed, alhough $\mathbb{F}_p(s,t)$ is algebraically closed in,

$$\text{Frac}\,(\mathbb{F}_p(s,t)[x,y]/(x^p + sy^p + t))$$

it is not separable since separability implies reducedness fo the base change by the field extension $\mathbb{F}_p(s^{\frac{1}{p}}, t^{\frac{1}{p}})$.

1.2 Genera of Curves

Definition 1.2.1. A *curve* C over k is a separated finite type scheme over k of pure dimension 1.

Definition 1.2.2. Let X be a proper curve over k. The *arithmetic genus* of X is,

$$p_a(X/k) := \dim_k H^1(X, \mathcal{O}_X)$$

If $H^0(X, \mathcal{O}_X) = K$ is a field then we write,

$$p_a(X) := \dim_K H^1(X, \mathcal{O}_X)$$

Remark. The arithmetic genus is stable under field extension by flat base change. However, if X admits $X \to \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ then the arithmetic genus of X viewed over k is $[k' : k]$ times the arithmetic genus of X viewed over k'. The point of the second definition is that when it it applies the base field is unambigious.

Definition 1.2.3. Let X be a curve which is a disjoint union of finitely many smooth proper curves over an algebraically closed field k. Then the *geometric genus* (or just *genus*) of X is,

$$g(X) := p_a(X/k) = \sum_{i=1}^{n} p_a(C_i/k)$$

Definition 1.2.4. Let X be a proper curve over a field k. Consider $\widetilde{X}$ which is the normalization of $(X_{\bar{k}})_{\mathrm{red}}$. This is a disjoint union of finitely many smooth proepr curves C_i over $\bar{k}$. Thus we can define,

$$g(X/k) := g(\widetilde{X})$$

If $H^0(X, \mathcal{O}_X) = K$ is a field then we set,

$$g(X) := g(X/k)$$

Remark. The geometric genus is stable under field extension by definition. However, notice that $g(X/k)$ does depend on the base field. If X admits $X \to \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ then the geometric genus of X viewed over k is $[k' : k]$ times the geometric genus of X viewed over k'. The point of the second definition is that when it it applies the base field is unambigious.

Proposition 1.2.5. The geometric genus is a birational invarian of proper curves over k.

Proof. This is almost by definition. Let $f : X \dashrightarrow Y$ be a birational map of curves meaning there is a dense open on which it becomes an isomorphism. Then by functoriality this gives a birational map $f : \widetilde{X} \dashrightarrow \widetilde{Y}$ which is an isomorphism since both sides are collections of regular curves over $\bar{k}$. Hence $g(X) = g(Y)$. □

Lemma 1.2.6. Let $f : X \to Y$ be a nonconstant map of proper regular curves over an algebrcially closed field k. Then $g(X) \geq g(Y)$.

Proof. Riemann-Hurwitz and Frobenius tricks [Hartshorne, Chapter IV] □

Proposition 1.2.7. Let $f : X \to Y$ be a dominant map of proper curves over a field k. Then $g(X/k) \geq g(Y/k)$.

Proof. By definition, we set $\widetilde{X}$ to be the normalization of $(X_{\bar{k}})_{\text{red}}$ and then $g(X/k) = g(\widetilde{X})$. Then the induced map $f : \widetilde{X} \to \widetilde{Y}$ is also surjective since it is dominant (because this is preserved by base change and reduction and normalization) and proper. Therefore, each component of $\widetilde{Y}$ is hit by some component of $\widetilde{X}$ so we reduce to the previous lemma and conclude,

$$g(X/k) \geq g(Y/k)$$

□

Example 1.2.8. Say $E = \text{Proj}\left(\mathbb{R}[X,Y,Z]/(Y^2Z - X^3 - xZ^2)\right)$ is an elliptic curve over $\mathbb{R}$. It is important that we consider the genus of $E_{\mathbb{C}}$ *as a curve over* $\mathbb{R}$ as 2 and not 1 because,

$$X = \text{Proj}\left(\mathbb{R}[X,Y,Z]/((Y^2Z - X^3)^2 + (XZ^2)^2)\right)$$

has normalization $E_{\mathbb{C}}$. However, X has genus 2 since $H^0(X, \mathcal{O}_X) = \mathbb{R}$ so we must view it over $\mathbb{R}$ and to compute its genus we base change to $X_{\mathbb{C}}$ then our definition will give genus 2. If we want the map $E_{\mathbb{C}} \to X$ to satisfy the above lemma we must have $g(E_{\mathbb{C}}/\mathbb{R}) = 2$.

Corollary 1.2.9. Let $f : X \to Y$ be a dominant map of proper curves over k with,

$$k \to H^0(Y, \mathcal{O}_Y) \to H(X, \mathcal{O}_X)$$

isomorphisms. Then $g(X) \geq g(Y)$.

Remark. The above example shows that the assumption on the fields is necessary.

1.3 Birational Maps of Curves and the Relationship Between Genera

Lemma 1.3.1. Let C, S be proper integral curves over k which are birational over k. Suppose that S is regular. Let $k_C = H^0(C, \mathcal{O}_C)$ and $k_S = H^0(S, \mathcal{O}_S)$. Then the genera satisfy,

(a) $g(C) = g(S)$

(b) $p_a(C) \geq p_a(S)$

and if one of the following holds,

(a) $p_a(C) = p_a(S)$ with $p_a(S) > 0$

(b) $p_a(C) = p_a(S) = 0$ and $k_C = k_S$

then $C \cong S$ so C is regular.

Proof. We have already seen that g is a birational invariant for all curves. Now focus on p_a. Given a birational map $S \overset{\sim}{\dashrightarrow} C$ we can extend it to a birational morphism $S \to C$ since S is regular. The morphism $f : S \to C$ is automatically finite since it is a non-constant map of proper curves. In particular, f is affine so for each $y \in C$ we may choose an affine open $y \in V \subset C$ whose preimage $U = f^{-1}(V)$ is also affine. On sheaves, this gives a map of domains $\mathcal{O}_C(V) \to \mathcal{O}_S(U)$ which localizes to an isomorphism on the fraction fields. However, the localization map of a domain is injective so $\mathcal{O}_C(V) \hookrightarrow \mathcal{O}_S(U)$ is an injection. This shows that $\mathcal{O}_C \to f_*\mathcal{O}_S$ is an injection of sheaves which is generically an isomorphism. Extending to an exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_* \mathcal{O}_S \longrightarrow \mathscr{C} \longrightarrow 0$$

where $\dim \mathrm{Supp}\,(\mathscr{C}) = 0$ and hence $H^1(C, \mathscr{C}) = 0$. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, \mathscr{C}) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow 0$$

Therefore,

$$p_a(C/k) \geq p_a(S/k)$$

and the fields $k_C = H^0(C, \mathcal{O}_C)$ and $k_S = H^0(S, \mathcal{O}_S)$ satisfy $k_C \hookrightarrow k_S$. Therefore, dividing by the respective degrees of the extensions gives,

$$p_a(C/k_C) \geq p_a(S/k_S)$$

since $[k_C : k] \leq [k_S : k]$. Now if $p_a(C) = p_a(S)$ and are nonzero then the two constituent inequalities are equalities meaning $p_a(C/k) = p_a(S/k)$ and $k_C = k_S$. Indeed whenever this holds, the extact sequence shows that $H^0(C, \mathscr{C}) = 0$ so $\mathscr{C} = 0$ since it is supported on an affine scheme. Therefore $f : S \to C$ is an isomorphism since it is affine and $\mathcal{O}_C \xrightarrow{\sim} f_* \mathcal{O}_S$ is an isomorphism. $\square$

Example 1.3.2. The normalization map,

$$\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Proj}\left(\mathbb{R}[X, Y, Z]/(X^2 + Y^2)\right)$$

gives an example where $p_a(C) = p_a(S) = 0$ but the map is not an isomorphism since $k_C \hookrightarrow k_S$ is not an isomorphism.

Proposition 1.3.3. Let C be a proper integral curve over k. Then $g(C) \leq p_a(C)$. If C is smooth, this is an equality. If equality holds and C is geometrically reduced then C is smooth.

Proof. Change the field such that $k = H^0(C, \mathcal{O}_C)$. Then $p_a(C) = p_a(C_{\bar{k}})$ by flat base change. Then $\widetilde{C} \to (C_{\bar{k}})_{\mathrm{red}}$ satisfies the above hypotheses so,

$$g(C) = p_a(\widetilde{C}/\bar{k}) \leq p_a((C_{\bar{k}})_{\mathrm{red}}/\bar{k}) \leq p_a(C_{\bar{k}}/\bar{k}) = p_a(C)$$

Now if C is smooth then so is $C_{\bar{k}}$ so the above are equalities. Now if $g(C) = p_a(C)$ the above are equalities. This implies that $\widetilde{C} \xrightarrow{\sim} (C_{\bar{k}})_{\mathrm{red}}$ is an isomorphism so if C is geometrically reduced then $C_{\bar{k}} \cong \widetilde{C}$ and hence C is smooth. $\square$

1.4 Degenerations of Curves

Notation: let $(R, \mathfrak{m}, \kappa)$ be a DVR with fraction field $K = \mathrm{Frac}\,(R)$. Let $S = \mathrm{Spec}\,(R)$. For $X \to S$ let $X_\eta = X_K$ be the generic fiber and let $X_s = X_\kappa$ the special fiber.

Definition 1.4.1. A *degeneration of curves* is a proper flat family $X \to S = \mathrm{Spec}\,(R)$ over a DVR R where X_η is an integral normal projective curve over $K = \mathrm{Frac}\,(R)$. If X is normal we say that X is a *model* of X_η over R.

Lemma 1.4.2. The total space X of a degeneration of curves is integral.

Proof. We need to show that every affine open $\mathrm{Spec}\,(A) = U \subset X$ has A a domain. Indeed, $R \to A$ is flat so $A \hookrightarrow A_K$ is injective but A_K is an affine open of X_K which in integral so A_K and hence A is a domain. $\square$

Lemma 1.4.3. Let $f : X \to Y$ be a proper flat map of integral schemes with Y normal. Then the following are equivalent,

(a) $f_* \mathcal{O}_Y = \mathcal{O}_Y$

(b) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = \kappa(\eta)$

Proof. Indeed, $f_* \mathcal{O}_X$ is a finite $\mathcal{O}_Y$-algebra and since X is integral it is a sheaf of domains. We need to show that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism which is a local question so we reduce to $\mathrm{Spec}\,(A) \subset Y$ and $\mathrm{Spec}\,(B) \subset X$ such that $A \to B$. Then we have maps $A \to (f_* \mathcal{O}_X)(A) \to B$ and $A \to B$ is flat hence injective since they are domains. Hence $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective. Furthermore, by flat base change,

$$H^0(X_\eta, \mathcal{O}_{X_\eta}) = (f_* \mathcal{O}_X)_\eta$$

so if (b) holds then $(f_* \mathcal{O}_X)_\eta = \kappa(\eta)$. Since $\mathcal{O}_Y$ is normal and $f_* \mathcal{O}_X$ is integral over $\mathcal{O}_Y$ we see that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism since it is contained in the fraction field. □

Proposition 1.4.4. Let $X \to S$ be a degeneration of curves. Consider the following properties,

(a) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically integral

(b) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically irreducible

(c) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically connected

(d) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = \kappa(\eta)$

(e) $f_* \mathcal{O}_X = \mathcal{O}_S$

then the following implications hold,

$$\begin{tikzcd}
(a) \arrow[r] \arrow[d] & (d) \arrow[d, leftrightarrow] \\
(b) \arrow[u, bend left, "X_\eta \text{ geom.red.}"] \arrow[d] & (e) \arrow[dl] \\
(c) \arrow[u, bend left, "X \text{ normal}"] &
\end{tikzcd}$$

In particular, if X is normal and X_η is geometrically reduced all the properties are equivalent.

Proof. The only nontrivial implications are:

- $(a) \implies (d)$ is Tag 0BUG (8)
- $(d) \implies (e)$ is exactly Lemma 1.4.3
- $(c) \implies (b)$ is Lemma 1.1.2 and the fact that geometric connectedness of fibers can be checked generically in universally open (e.g. flat finitely presented) families [EGA IV, Cor. 15.5.4].

□

Remark. Even if $f_* \mathcal{O}_X = \mathcal{O}_S$ we don't necessarily have that X_η is geometrically reduced e.g. Example **??**.

1.5 Controlling the Arithmetic Genus in Families

1.5.1 Setup

Let $X \to S$ be a normal degeneration of curves. Then consider the following data. Let $\Gamma_i \subset X_s$ be the (reduced) irreducible componetns of the special fiber and the following κ-algebras,

(a) $A = H^0(X_s, \mathcal{O}_{X_s})$

(b) $\kappa' = H^0((X_s)_{\text{red}}, \mathcal{O}_{(X_s)_{\text{red}}})$

(c) $\kappa_i = H^0(\Gamma_i, \mathcal{O}_{\Gamma_i})$

where A is an Artin local κ-algebra and κ' and κ_i are finite field extensions of κ by Tag 0BUG (1) since these schemes are connected and the second two are reduced.

1.5.2 Results in Any Characteristic

For simplicitly, we should either restrict in this section to the case where the curve $C = X_\eta$ over K is smooth and $H^0(C, \mathcal{O}_C) = K$ or the base DVR R is excellent. I think the results should be true without this assumption but I want to quote the stacks project which makes this assumption without caution and use strong desingularization which either relies on smoothness of C or excellence in an important way (see [Liu, Cor. 8.3.51] and [Liu, 8.3.44]).

Proposition 1.5.1. Let $X \to S$ be a regular degeneration of curves. For any irreducible component,

$$p_a(\Gamma_i/\kappa_i) \le p_a(X_K/K)$$

Proof. By Tag 0C68 and Tag 0C69 the effective Cartier divisor,

$$C = \sum_{i=1}^{r} (m_i/d) C_i$$

where m_i is the multiplicity of C_i and $d = \gcd(m_i)$ satisfies,

(a) $H^0(D, \mathcal{O}_D) = \kappa_D$ is a field and

(b) $\chi(X_s, \mathcal{O}_{X_s}) = d\chi(D, \mathcal{O}_D)$

and hence,

$$g - 1 = d[\kappa_D : \kappa](g_D - 1)$$

where $g = p_a(X_K/K)$ and $g_D = p_a(D/\kappa_D)$. Therefore,

$$g_D = \frac{g-1}{d[\kappa_D : \kappa]} + 1 \le g$$

since either $g = 0$ in which case $g_D = 0$ and $d = [\kappa_D : \kappa] = 1$ or $g > 0$ in which case we see that $g_D \le g$. Furthermore, since $(X_s)_{\text{red}} = D_{\text{red}}$ we see that $(X_s)_{\text{red}}$ is a κ_D-scheme and,

$$H^1(X_s, \mathcal{O}_D) \twoheadrightarrow H^1(X_s, \mathcal{O}_{(X_s)_{\text{red}}})$$

so we conclude that,

$$p_a((X_s)_{\text{red}}/\kappa') \le p_a((X_s)_{\text{red}}/\kappa_D) \le p_a(D/\kappa_D) = g_D \le g = p_a(X_K/K)$$

Let $Y = (X_s)_{\text{red}}$ and consider the finite map $\pi : \bigsqcup_i \Gamma_i \to Y$ splitting the irreducible components. This gives a sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \prod \mathcal{O}_{\Gamma_i} \longrightarrow \mathscr{C} \longrightarrow 0$$

and therefore since $\dim \mathrm{Supp}\,(\mathscr{C}) = 0$ we have,

$$H^1(Y, \mathcal{O}_Y) \twoheadrightarrow \bigoplus_i H^1(\Gamma_i, \mathcal{O}_{\Gamma_i})$$

and hence,

$$p_a(\Gamma_i/\kappa_i) \le p_a(\Gamma_i/\kappa') \le p_a(Y/\kappa') \le p_a(X_K/K)$$

□

Proposition 1.5.2. Let $X \to S$ be a normal degeneration of curves and $\Gamma_i \subset X_s$ a (reduced) irreducible component which is *normal* then,

$$p_a(\Gamma_i/\kappa_i) \le p_a(X_K/K)$$

Proof. Let X be a normal degeneration of curves. Then there exists a strong desingularization [Liu, Cor. 8.3.51] $\pi : \widetilde{X} \to X$ meaning it is an isomorphism on the regular locus of X. Then $\widetilde{X} \to S$ is a regular model of $\widetilde{X}_\eta = X_\eta$ and hence verifies the inequality for *every* irreducible component via Proposition 1.5.1. However, since $\widetilde{X} \to X$ is an isomorphism away from a finite set of points, for each $\Gamma_i \subset X_s$ there is an irreducible component $\widetilde{\Gamma}_i \subset \widetilde{X}_s$ mapping birationally onto Γ_i. However, Γ_i is a normal curve so the birational map $\widetilde{\Gamma}_i \to \Gamma_i$ is an isomorphism and hence we conclude. □

2 Application to Specializing Genus of Fibrations

As before let $(R, \mathfrak{m}, \kappa)$ be a DVR with fraction field $K = \mathrm{Frac}\,(R)$ and spectrum $S = \mathrm{Spec}\,(R)$.

Proposition 2.0.1. Let $f : X \to Y$ be a morphism of flat proper S-schemes. Suppose that,

(a) X and Y are integral and normal

(b) X_s decomposes into (reduced) irreducible components $X_1, \dots, X_r$ with multiplicites m_i

(c) Y_s is integral

(d) the map $f_s : X_s \to Y_s$ is dominant

(e) f has relative dimension 1

Let $\xi \in Y$ and $\eta \in Y_s$ be the generic points. Then, the fiber X_η of f decomposes into (reduced) irreducble components $\Gamma_1, \dots, \Gamma_r$ with multiplicities m_i such that Γ_i is the generic fiber of $X_i \to Y_s$. Thus, for each normal X_j we have,

$$p_a(\Gamma_j/\kappa_j) \le p_a(X_\xi/\kappa(\xi))$$

where $\kappa_j = H^0(\Gamma_j, \mathcal{O}_{\Gamma_j})$. and thus the Stein factorization,

$$X_j \to B \to Y_s$$

has $X_j \to B$ a fibration by curves with generic fiber of genus $\le p_a(X_\xi/\kappa(\xi))$.

Proof. Since X is integral, the fiber dimension can only jump on a codim ≥ 2 locus. In fact, the following lemma gives a strengthening of generic flatness. Now $(X_s)_\eta = X_\eta$ and the generic points of Γ_i and of X_i are the same in X and hence they come with the same multiplicity. Indeed, this is a purely local situation. Consider the diagram of ring maps,

$$\begin{tikzcd} A & & B \arrow[ll] \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

with $\mathfrak{m} \subset R$ the maximal ideal. Then $\mathfrak{m}B$ is prime by assumption. Let $S = B \backslash (\mathfrak{m}B)$. Then,

$$(R/\mathfrak{m}) \otimes_R A \otimes_B (S^{-1}B) = [(B/\mathfrak{m}) \backslash \{0\}]^{-1}(A/\mathfrak{m}A)$$

has as its local rings those points of the special fiber of X mapping to η. By flatness (see below) each generic point of X_s maps to η so we get the assumed decomposition. Now consider $X_D \to \operatorname{Spec}(D)$ where $D = \mathcal{O}_{Y,\eta}$ is a DVR since Y is regular in codimension 1. Since X is irreducible and normal and $X_D \to X$ is a localization then X_D is also irreducible and normal. Finally, $X_R \to \operatorname{Spec}(D)$ is proper by basechange and flat because it is dominant by assumption. Therefore, we conclude by an application of Propositione 1.5.2. $\square$