

# Mathematics GU4051 Topology

## Assignment # 9

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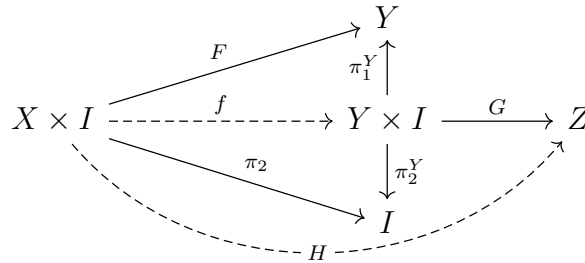
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### Problem 1.

Let  $f_0, f_1 : X \rightarrow Y$  be homotopic and  $g_0, g_1 : Y \rightarrow Z$  be homotopic. Take homotopies for each pair of homotopic functions:  $F : X \times I \rightarrow Y$  which satisfies  $F(x, 0) = f_0(x)$  and  $F(x, 1) = f_1(x)$  and is continuous and  $G : Y \times I \rightarrow Z$  which satisfies  $G(y, 0) = g_0(y)$  and  $G(y, 1) = g_1(y)$  and is continuous. Consider the map  $H : X \times I \rightarrow Z$  given by  $H(x, t) = G(F(x, t), t)$ . Firstly,  $H(x, 0) = G(F(x, 0), 0) = g_0(f_0(x)) = (g_0 \circ f_0)(x)$ . Similarly,  $H(x, 1) = G(F(x, 1), 1) = g_1(f_1(x)) = (g_1 \circ f_1)(x)$ . To show that  $H$  is a homotopy between  $g_0 \circ f_0$  and  $g_1 \circ f_1$  we must show that  $H$  is continuous. The map,

$$f = (F, \pi_2) : X \times I \rightarrow Y \times I$$

is continuous because both  $F : X \times I \rightarrow Y$  and  $\pi_2 : X \times I \rightarrow I$  are continuous so  $G \circ (F, \pi_2)$  is continuous. Also,  $G \circ (F, \pi_2)(x, t) = G(F(x, t), \pi_2(x, t)) = G(F(x, t), t) = H(x, t)$  so  $H$  is continuous. The argument is summarized in the commutative diagram:



Therefore,  $H$  is a homotopy between  $g_0 \circ f_0$  and  $g_1 \circ f_1$ .

### Problem 2.

A note on notation: For  $y_0 \in Y$ , I will use  $\langle y_0 \rangle_X : X \rightarrow Y$  to denote the constant map  $\langle y_0 \rangle_X : x \mapsto y_0$ .

- (a). Let  $L \subset \mathbb{R}$  be a nonempty interval. Take  $x_0 \in L$  and  $\text{id}_L : L \rightarrow L$ . Now, define  $G : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $G(x, y) = x_0 y + x(1 - y)$  which is continuous by analysis. Now, if  $x \in L$  and  $t \in [0, 1]$  then  $x \leq G(x, y) \leq x_0$  or  $x_0 \leq G(x, y) \leq x$  so by the interval property  $G(x, y) \in L$ . Thus,  $F : L \times I \rightarrow L$  given by  $F = G|_{L \times I}$  is a well defined continuous map. Also,  $F(x, 0) = x = \text{id}_L(x)$  and  $F(x, 1) = x_0$  a constant map. Thus,  $\text{id}_L \sim \langle x_0 \rangle$  where  $\langle x_0 \rangle$  represents the constant map  $x \rightarrow x_0$  so  $L$  is contractible.

- (b). Let  $X$  be contractable then there is a homotopy  $F : X \times I \rightarrow X$  such that  $F(x, 0) = x$  and  $F(x, 1) = x_0$  for some  $x_0 \in X$ . Take any  $x_1, x_2 \in X$ . Define  $\gamma : I \rightarrow X$  by,

$$\gamma(t) = \begin{cases} F(x_1, 2t) & t \leq \frac{1}{2} \\ F(x_2, 2 - 2t) & t \geq \frac{1}{2} \end{cases}$$

Because at  $t = \frac{1}{2}$  we have  $F(x_1, 2t) = F(x_1, 1) = x_0$  and  $F(x_2, 2 - 2t) = F(x_2, 1) = x_0$  the path  $\gamma$  is continuous by the glueing lemma. Also,  $\gamma(0) = F(x_1, 0) = x_1$  and  $\gamma(1) = F(x_2, 2 - 2) = F(x_2, 0) = x_2$  so  $\gamma$  is a path from  $x_1$  to  $x_2$  and thus  $X$  is path connected.

Now, take any loop  $\gamma$  at  $x_0$ . Now, define,

$$G(x, t) = \begin{cases} F(x_0, 2xt) & x \leq \frac{1}{2} \\ F(\gamma(4x - 2), t) & \frac{1}{2} \leq x \leq \frac{3}{4} \\ F(x_0, (4 - 4x)t) & x \geq \frac{3}{4} \end{cases}$$

At  $x = \frac{1}{2}$ ,  $F(x_0, 2xt) = F(x_0, t)$  and  $F(\gamma(4x - 2), t) = F(\gamma(0), t) = F(x_0, t)$ . Similarly, at  $x = \frac{3}{4}$ ,  $F(\gamma(4x - 2), t) = F(\gamma(1), t) = F(x_0, t)$  and  $F(x_0, (4 - 4x)t) = F(x_0, t)$ . Therefore, by the glueing lemma,  $G$  is continuous. Also, let  $\delta(x) = F(x_0, x)$  which is a loop at  $x_0$  because  $\delta(0) = F(x_0, 0) = x_0$  and  $F(x_0, 1) = x_0$ .

$$\begin{aligned} G(x, 0) &= \begin{cases} F(x_0, 0) = x_0 & x \leq \frac{1}{2} \\ F(\gamma(4x - 2), 0) = \gamma(4x - 2) & \frac{1}{2} \leq x \leq \frac{3}{4} \\ F(x_0, 0) = x_0 & x \geq \frac{3}{4} \end{cases} \\ &= (e_{x_0} * (\gamma * e_{x_0}))(x) \\ G(x, 1) &= \begin{cases} F(x_0, 2x) = \delta(2x) & x \leq \frac{1}{2} \\ F(\gamma(4x - 2), 1) = x_0 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ F(x_0, (4 - 4x)) = \delta(4 - 4x) & x \geq \frac{3}{4} \end{cases} \\ &= (\delta * (e_{x_0} * \delta^{-1}))(x) \\ G(0, t) &= \begin{cases} F(x_0, 0) & 0 \leq \frac{1}{2} \\ F(\gamma(-2), t) & \frac{1}{2} \leq 0 \leq \frac{3}{4} \\ F(x_0, (-4x)t) & 0 \geq \frac{3}{4} \end{cases} \\ &= F(x_0, 0) = x_0 \\ G(1, t) &= \begin{cases} F(x_0, 2t) & 1 \leq \frac{1}{2} \\ F(\gamma(2), t) & \frac{1}{2} \leq 1 \leq \frac{3}{4} \\ F(x_0, 0) & 1 \geq \frac{3}{4} \end{cases} \\ &= F(x_0, 0) = x_0 \end{aligned}$$

Therefore,  $e_{x_0} * (\gamma * e_{x_0})$  and  $\delta * (e_{x_0} * \delta^{-1})$  are path-homotopic so  $[e_{x_0} * (\gamma * e_{x_0})] = [\delta * (e_{x_0} * \delta^{-1})]$ . However,  $[e_{x_0} * (\gamma * e_{x_0})] = [e_{x_0}] * [\gamma] * [e_{x_0}] = [\gamma]$  because  $[e_{x_0}]$  is the identity of  $\pi_1(X, x_0)$ . Furthermore,  $[\delta * (e_{x_0} * \delta^{-1})] = [\delta] * [e_{x_0}] * [\delta]^{-1} = [\delta] * [\delta]^{-1} = [e_{x_0}]$  because the reversed path generates the inverse homotopy class. Thus,  $[\gamma] = [e_{x_0}]$  but  $\gamma$  was arbitrary so every

element of  $\pi_1(X, x_0)$  is the identity. Now, for any other base point  $x \in X$  we know that  $\pi_1(X, x) \cong \pi_1(X, x_0)$  with isomorphism induced by conjugation with a path from  $x_0$  to  $x$ . Therefore,  $\pi_1(X, x) \cong \pi_1(X, x_0) \cong \{e\}$ .

- (c). Let  $f_0, f_1 : X \rightarrow Y$  be continuous and let  $Y$  contractable. Then there exists a homotopy  $G : Y \times I \rightarrow Y$  such that  $G(y, 0) = y$  and  $G(y, 1) = y_0$  for some  $y_0 \in Y$ . Define,  $F : X \times I \rightarrow Y$  by,

$$F(x, t) = \begin{cases} G(f_0(x), 2t) & t \leq \frac{1}{2} \\ G(f_1(x), 2 - 2t) & t \geq \frac{1}{2} \end{cases}$$

First,  $G(f_0(x), 2t)$  and  $G(f_1(x), 2 - 2t)$  are continuous by composition of continuous functions. Now, because on the closed set  $X \times \{\frac{1}{2}\}$ , we have  $G(f_0(x), 2t) = G(f_0(x), 1) = y_0$  and  $G(f_1(x), 2 - 2t) = G(f_1(x), 1) = y_0$  then  $F$  is continuous by the glueing lemma. Also,  $F(x, 0) = G(f_0(x), 0) = f_0(x)$  and  $F(x, 1) = G(f_1(x), 0) = f_1(x)$  so  $F$  is a homotopy from  $f_0$  to  $f_1$  so  $f_0 \sim f_1$ .

An alternative proof goes as follows. Take continuous  $f_0, f_1 : X \rightarrow Y$ . Because  $Y$  is contractable,  $\text{id}_Y \sim \langle y_0 \rangle_Y$  where  $y_0$  is some fixed point  $y_0 \in Y$ . Now,  $f_0 \sim f_0$  so by the problem 1, we have  $f_0 = \text{id}_Y \circ f_0 \sim \langle y_0 \rangle_Y \circ f_0 = \langle y_0 \rangle_X$  by Lemma ???. Similarly,  $f_1 \sim f_1$  so  $f_1 = \text{id}_Y \circ f_1 \sim \langle y_0 \rangle_Y \circ f_1 = \langle y_0 \rangle_X$ . Thus,  $f_0 \sim \langle y_0 \rangle_X$  and  $f_1 \sim \langle y_0 \rangle_X$  so  $f_0 \sim f_1$  by transitivity.

- (d). Let  $g_0, g_1 : X \rightarrow Y$  be continuous, let  $X$  be contractable, and let  $Y$  be path-connected. Because  $X$  is contractable, there exists a point  $x_0 \in X$  such that  $\text{id}_X \sim \langle x_0 \rangle_X$ . Then, because  $g_0 \sim g_0$  we know that  $g_0 = g_0 \circ \text{id}_X \sim g_0 \circ \langle x_0 \rangle_X = \langle g_0(x_0) \rangle_X$  by Lemma ???. Similarly,  $g_1 = g_1 \circ \text{id}_X \sim g_1 \circ \langle x_0 \rangle_X = \langle g_1(x_0) \rangle_X$ . However, because  $Y$  is path-connected, by Lemma ??, all constant functions are homotopic so  $\langle g_0(x_0) \rangle_X \sim \langle g_1(x_0) \rangle_X$ . Thus, by transitivity,  $g_0 \sim \langle g_0(x_0) \rangle_X \sim \langle g_1(x_0) \rangle_X \sim g_1$  so  $g_0$  and  $g_1$  are homotopic.

### Problem 3.

- (a). Take  $S \subset \mathbb{R}^2$  to be the axes,  $S = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ . This set is star-convex because any point  $P \in S$  lies either on the  $x$ -axis or the  $y$ -axis. Either way, the segment  $\overline{PO} \subset S$ , where  $O = (0, 0)$  is the origin, because it is a subset of the corresponding axis. However, take  $P = (1, 0)$  and  $Q = (0, 1)$ . Both  $P, Q \in S$  but  $\overline{PQ} \not\subset S$  because  $(\frac{1}{2}, \frac{1}{2}) \notin S$  so  $S$  is nonconvex.
- (b). Let  $T \subset \mathbb{R}^2$  be the graph of a parabola,  $T = \{(t, t^2) \mid t \in \mathbb{R}\}$ . Then  $T$  is not star-convex because it contains no nontrivial line segments. However,  $T$  is contractable. Consider the map  $G : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by  $G(x, y, z) = (x(1 - z), y(1 - z)^2)$  which is continuous by analysis. Also if  $x^2 = y$  then  $(x(1 - z))^2 = x^2(1 - z)^2 = y(1 - z)^2$  so  $\text{Im}(G|_{T \times I}) \subset T$ . Therefore, the map  $F : T \times I \rightarrow T$  given by  $F((x, x^2), t) = G(x, x^2, t)$  is continuous. Also,

$$F((x, x^2), 0) = G(x, x^2, 0) = (x, x^2) \quad F((x, x^2), 1) = (x(1 - 1), x^2(1 - 1)) = (0, 0)$$

so  $F$  is a homotopy from  $\text{id}_T$  to the constant map from  $T$  to  $(0, 0)$ .

- (c). Let  $S \subset \mathbb{R}^n$  be star-convex. Therefore,  $\exists \mathbf{x} \in S$  such that  $\forall \mathbf{y} \in S$  the segment  $\overline{\mathbf{x}\mathbf{y}} \subset S$ . Consider the function  $G : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  given by  $G(\mathbf{y}, t) = (1 - t)\mathbf{y} + t\mathbf{x}$ . By analysis,  $G$  is

continuous. Also, for  $t \in [0, 1]$  we have  $G(\mathbf{y}, t) \in \overline{\mathbf{x}\mathbf{y}}$  so  $G(\mathbf{y}, t) \in S$ . Thus,  $\text{Im}(G|_{S \times I}) \subset S$  so the function  $F : S \times I \rightarrow S$  given by  $F(\mathbf{y}, t) = G(\mathbf{y}, t)$  is continuous and well defined. Also,  $F(\mathbf{y}, 0) = \mathbf{y} = \text{id}_S(\mathbf{y})$  and  $F(\mathbf{y}, 1) = \mathbf{x}$  which is a constant function. Thus,  $\text{id}_S$  is homotopic to the constant function mapping to  $\mathbf{x}$ . Therefore,  $S$  is contractible.

## Problem 4.

For  $S \subset X$  let  $f : X \rightarrow S$  be a retraction. Take  $x_0 \in S$  and any loop  $\gamma : I \rightarrow S$  at  $x_0$ . Now we can lift the loop  $\gamma$  into the ambient space  $X$  simply by defining  $\tilde{\gamma} : I \rightarrow X$  by  $\tilde{\gamma}(t) = \gamma(t)$ . Consider the homomorphism induced by the retraction,  $f_* : \pi_1(X, x_0) \rightarrow \pi_1(S, f(x_0))$ . However,  $x_0 \in S$  and  $f|_S = \text{id}_S$  so  $f(x_0) = x_0$ . Thus,  $f_* : \pi_1(X, x_0) \rightarrow \pi_1(S, x_0)$ . Now, consider  $f_*([\tilde{\gamma}]) = [f \circ \tilde{\gamma}]$  then we have,  $f \circ \tilde{\gamma} : I \rightarrow S$  and  $f(\tilde{\gamma}(t)) = f(\gamma(t)) = \gamma(t)$  because  $\gamma(t) \in S$  and  $f|_S = \text{id}_S$ . Thus,  $f_*([\tilde{\gamma}]) = [\gamma]$ . However,  $\gamma$  was an arbitrary loop at  $x_0$  so the function  $f_*$  is surjective because the equivalence class of any loop is in the image.

## Problem 5.

The projections  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are continuous and thus induce homomorphisms  $f_1 : \pi_1(X \times Y, x \times y) \rightarrow \pi_1(X, x)$  and  $f_2 : \pi_1(X \times Y, x \times y) \rightarrow \pi_1(Y, y)$  because  $\pi_1(x \times y) = x$  and  $\pi_2(x \times y) = y$ . Using Lemma ??, define the homomorphism,

$$F : \pi_1(X \times Y, x \times y) \rightarrow \pi_1(X, x) \times \pi_2(Y, y)$$

by  $F = (f_1, f_2)$ . It remains to show that  $F$  is a bijection. Let

$$G : \pi_1(X, x) \times \pi_2(Y, y) \rightarrow \pi_1(X \times Y, x \times y)$$

be given by  $G([\gamma], [\delta]) = [\Gamma]$  where  $\Gamma = (\gamma, \delta) : I \rightarrow X \times Y$ . Lemma ?? shows that this function maps loops to loops with the correct base points and is well defined on path-homotopy equivalence classes. Now,

$$G \circ F([\Gamma]) = G([\pi_1 \circ \Gamma], [\pi_2 \circ \Gamma]) = [(\pi_1 \circ \Gamma, \pi_2 \circ \Gamma)] = [\Gamma]$$

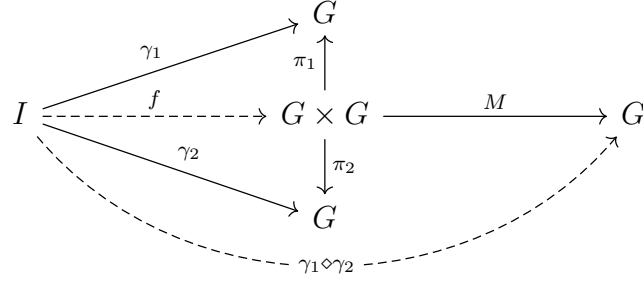
In the last line, I used the fact that, for any function  $\gamma : I \rightarrow X \times Y$ , the map  $(\pi_1 \circ \gamma, \pi_2 \circ \gamma) = \gamma$ . Also,

$$F \circ G([\gamma], [\delta]) = F([\Gamma]) = ([\pi_1 \circ \Gamma], [\pi_2 \circ \Gamma]) = ([\gamma], [\delta])$$

where I have used the fact that  $\Gamma = (\gamma, \delta)$  so  $\pi_1 \circ \Gamma = \gamma$  and  $\pi_2 \circ \Gamma = \delta$ . Therefore,  $G$  is the inverse function of  $F$  so  $F$  must be a bijection. Therefore,  $F$  is an isomorphism.

## Problem 6.

- (a). Let  $G$  be a topological group with a multiplication function  $M : G \times G \rightarrow G$  which takes  $M(x, y) = x \cdot y$ . Let  $\gamma_1, \gamma_2 : I \rightarrow G$  be continuous loops based at  $e$ . Then, let  $\gamma_1 \diamond \gamma_2 : I \rightarrow G$  be given by  $(\gamma_1 \diamond \gamma_2)(t) = \gamma_1(t) \cdot \gamma_2(t)$ . This is a loop at  $e$  because  $(\gamma_1 \diamond \gamma_2)(0) = \gamma_1(0) \cdot \gamma_2(0) = e \cdot e = e$  and  $(\gamma_1 \diamond \gamma_2)(1) = \gamma_1(1) \cdot \gamma_2(1) = e \cdot e = e$ . This function is also continuous because,  $f = (\gamma_1, \gamma_2)$  is continuous thus  $\gamma_1 \diamond \gamma_2 = f \circ M$  is continuous by composition of continuous functions.



Now,

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_2(2t) & t \leq \frac{1}{2} \\ \gamma_1(2t - 1) & t \geq \frac{1}{2} \end{cases}$$

Let  $f : [0, \frac{1}{2}] \times I \rightarrow I^2$  given by,

$$f(x, t) = (tx, (2 - t)x)$$

which is continuous and let  $g : [\frac{1}{2}, 1] \times I \rightarrow I^2$  be given by

$$g(x, t) = ((2 - t)x + t - 1, 1 + t(x - 1))$$

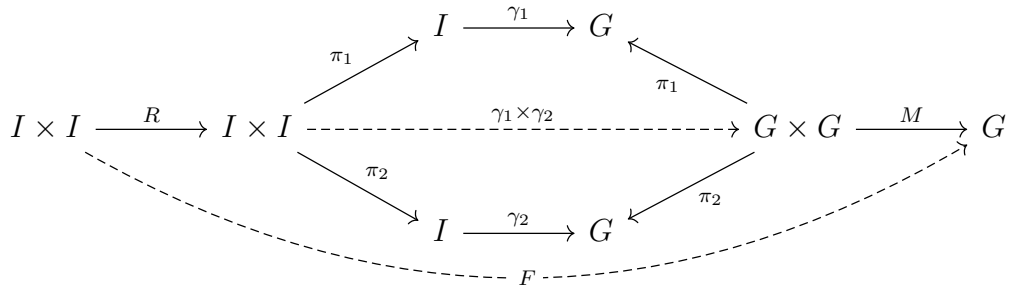
which is also continuous. Also, at  $x = \frac{1}{2}$ ,

$$\begin{aligned}
f(x, t) &= (\tfrac{1}{2}t, \tfrac{1}{2}(2 - t)) \\
g(x, t) &= ((2 - t)\tfrac{1}{2} + t - 1, 1 + t(\tfrac{1}{2} - 1)) = (\tfrac{1}{2}t, \tfrac{1}{2}(2 - t))
\end{aligned}$$

so by the glueing lemma, the function  $R : I^2 \rightarrow I^2$  given by,

$$R(x, t) = \begin{cases} f(x, t) & x \leq \frac{1}{2} \\ g(x, t) & x \geq \frac{1}{2} \end{cases}$$

is continuous. Define,  $F = M \circ (\gamma_1 \times \gamma_2) \circ R$  which is a well defined continuous map detailed in the commutative diagram below.



Thus, the function  $F : I^2 \rightarrow G$  is given by,

$$F(x, t) = \begin{cases} \gamma_1(tx) \cdot \gamma_2((2 - t)x) & x \leq \frac{1}{2} \\ \gamma_1((2 - t)x + t - 1) \cdot \gamma_2(1 + t(x - 1)) & x \geq \frac{1}{2} \end{cases}$$

Finally, using the fact that  $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1) = e$  so products with these elements do nothing.

$$\begin{aligned}
F(x, 0) &= \begin{cases} \gamma_1(0) \cdot \gamma_2(2x) & x \leq \frac{1}{2} \\ \gamma_1(2x-1) \cdot \gamma_2(1) & x \geq \frac{1}{2} \end{cases} = \begin{cases} \gamma_2(2x) & x \leq \frac{1}{2} \\ \gamma_1(2x-1) & x \geq \frac{1}{2} \end{cases} = (\gamma_1 * \gamma_2)(x) \\
F(x, 1) &= \begin{cases} \gamma_1(x) \cdot \gamma_2(x) & x \leq \frac{1}{2} \\ \gamma_1(x) \cdot \gamma_2(x) & x \geq \frac{1}{2} \end{cases} = (\gamma_1 \diamond \gamma_2)(x) \\
F(0, t) &= \begin{cases} \gamma_1(0) \cdot \gamma_2(0) & 0 \leq \frac{1}{2} \\ \gamma_1(t-1) \cdot \gamma_2(1-t) & 0 \geq \frac{1}{2} \end{cases} = \gamma_1(0) \cdot \gamma_2(0) = e \\
F(1, t) &= \begin{cases} \gamma_1(t) \cdot \gamma_2(2-t) & 1 \leq \frac{1}{2} \\ \gamma_1(1) \cdot \gamma_2(1) & 1 \geq \frac{1}{2} \end{cases} = \gamma_1(1) \cdot \gamma_2(1) = e
\end{aligned}$$

Therefore,  $F$  is a path-homotopy from  $\gamma_1 * \gamma_2$  to  $\gamma_1 \diamond \gamma_2$ . Therefore,

$$[\gamma_1] \diamond [\gamma_2] = [\gamma_1 \diamond \gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_1] * [\gamma_2]$$

Because  $*$  is well-defined on equivalence classes we have that  $\diamond$  is also a well-defined operation on equivalence classes and gives the same group structure.

- (b). Let  $\gamma_1, \gamma_2 : I \rightarrow G$  be loops at  $e$ . In an analogous fashion to part (a) but with the components of the output flipped, define  $f : [0, \frac{1}{2}] \times I \rightarrow I^2$  given by,

$$f(x, t) = ((2-t)x, tx)$$

which is continuous and let  $g : [\frac{1}{2}, 1] \times I \rightarrow I^2$  be given by

$$g(x, y) = (1+t(x-1), (2-t)x+t-1)$$

which is also continuous. Also, at  $x = \frac{1}{2}$ ,

$$\begin{aligned}
f(x, t) &= (\tfrac{1}{2}(2-t), \tfrac{1}{2}t) \\
g(x, t) &= (1+t(\tfrac{1}{2}-1), (2-t)\tfrac{1}{2}+t-1) = (\tfrac{1}{2}(2-t), \tfrac{1}{2}t)
\end{aligned}$$

so by the glueing lemma, the function  $R : I^2 \rightarrow I^2$  given by,

$$R(x, t) = \begin{cases} f(x, t) & x \leq \frac{1}{2} \\ g(x, t) & x \geq \frac{1}{2} \end{cases}$$

is continuous. Define,  $F = M \circ (\gamma_1 \times \gamma_2) \circ G$  which is a well defined continuous map detailed in the commutative diagram below.

$$\begin{array}{ccccccc}
& & I & \xrightarrow{\gamma_1} & G & & \\
& \nearrow \pi_1 & & & & \nwarrow \pi_1 & \\
I \times I & \xrightarrow{R} & I \times I & \xrightarrow{\gamma_1 \times \gamma_2} & G \times G & \xrightarrow{M} & G \\
& \searrow \pi_2 & & & & \swarrow \pi_2 & \\
& & I & \xrightarrow{\gamma_2} & G & & \\
& & \searrow F & & \swarrow & & 
\end{array}$$

Thus, the function  $F : I^2 \rightarrow G$  is given by,

$$F(x, t) = \begin{cases} \gamma_1((2-t)x) \cdot \gamma_2(tx) & x \leq \frac{1}{2} \\ \gamma_1(1+t(x-1)) \cdot \gamma_2((2-t)x+t-1) & x \geq \frac{1}{2} \end{cases}$$

Finally, using the fact that  $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1) = e$  so products with these elements do nothing.

$$\begin{aligned} F(x, 0) &= \begin{cases} \gamma_1(2x) \cdot \gamma_2(0) & x \leq \frac{1}{2} \\ \gamma_1(1) \cdot \gamma_2(2x-1) & x \geq \frac{1}{2} \end{cases} = \begin{cases} \gamma_1(2x) & x \leq \frac{1}{2} \\ \gamma_2(2x-1) & x \geq \frac{1}{2} \end{cases} = (\gamma_2 * \gamma_1)(x) \\ F(x, 1) &= \begin{cases} \gamma_1(x) \cdot \gamma_2(x) & x \leq \frac{1}{2} \\ \gamma_1(x) \cdot \gamma_2(x) & x \geq \frac{1}{2} \end{cases} = (\gamma_1 \diamond \gamma_2)(x) \\ F(0, t) &= \begin{cases} \gamma_1(0) \cdot \gamma_2(0) & 0 \leq \frac{1}{2} \\ \gamma_1(1-t) \cdot \gamma_2(t-1) & 0 \geq \frac{1}{2} \end{cases} = \gamma_1(0) \cdot \gamma_2(0) = e \\ F(1, t) &= \begin{cases} \gamma_1(2-t) \cdot \gamma_2(t) & 1 \leq \frac{1}{2} \\ \gamma_1(1) \cdot \gamma_2(1) & 1 \geq \frac{1}{2} \end{cases} = \gamma_1(1) \cdot \gamma_2(1) = e \end{aligned}$$

Therefore,  $F$  is a path-homotopy from  $\gamma_2 * \gamma_1$  to  $\gamma_1 \diamond \gamma_2$ .

- (c). From the previous parts,  $\gamma_1 \diamond \gamma_2 \sim \gamma_2 * \gamma_1$  and also,  $\gamma_1 \diamond \gamma_2 \sim \gamma_1 * \gamma_2$  therefore,  $\gamma_1 * \gamma_2 \sim \gamma_2 * \gamma_1$  by transitivity. Therefore,  $[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_2 * \gamma_1] = [\gamma_2] * [\gamma_1]$  so the fundamental group  $\pi_1(G, e)$  is abelian.

## Lemmas

**Lemma 0.1.** Let  $g : X \rightarrow Y$  be any function, with  $x_0 \in X$  and  $y_0 \in Y$  then  $\langle y_0 \rangle_Y \circ g = \langle y_0 \rangle_X$  and  $g \circ \langle x_0 \rangle_X = \langle g(x_0) \rangle_Y$

*Proof.* For all  $x \in X$  we have  $(\langle y_0 \rangle_Y \circ g)(x) = \langle y_0 \rangle_Y(g(x)) = y_0$  thus  $\langle y_0 \rangle_Y \circ g = \langle y_0 \rangle_Y$ . Also, for any  $x \in X$  we have  $(g \circ \langle x_0 \rangle_X)(x) = g(x_0)$  thus  $g \circ \langle x_0 \rangle_X = \langle g(x_0) \rangle_Y$ .  $\square$

**Lemma 0.2.** If  $Y$  is path connected then any two constant functions from  $X$  to  $Y$  are homotopic.

*Proof.* If  $X$  is empty then all functions from  $X$  are homotopic. Let  $X$  be nonempty, let  $g_0, g_1 : X \rightarrow Y$  be constant then  $g_0(X) = \{y_0\}$  and  $g_1(X) = \{y_1\}$ . Since  $Y$  is path connected, there exists a path  $\gamma : I \rightarrow Y$  from  $g_0(x_0)$  to  $g_1(x_0)$ . Define the function  $G : X \times I \rightarrow Y$  by,  $G = \gamma \circ \pi_2$  which is continuous as a composition of continuous maps. Then,  $G(x, 0) = \gamma(0) = y_0 = g_0(x)$  and  $G(x, 1) = \gamma(1) = y_1 = g_1(x)$ . Thus,  $G$  is a homotopy from  $g_0$  to  $g_1$ .  $\square$

**Lemma 0.3.** Let  $G$ ,  $H_1$ , and  $H_2$  be groups with homomorphisms  $f_1 : G \rightarrow H_1$  and  $f_2 : G \rightarrow H_2$  then there is a unique homomorphism  $F : G \rightarrow H_1 \times H_2$  given by  $F = (f_1, f_2)$  such that  $\pi_1 \circ F = f_1$  and  $\pi_2 \circ F = f_2$ . In other words, the product  $H_1 \times H_2$  satisfies the following universal property:

$$\begin{array}{ccc} & & H_1 \\ & \nearrow f_1 & \uparrow \pi_1 \\ G & \xrightarrow{F} & H_1 \times H_2 \\ & \searrow f_2 & \downarrow \pi_2 \\ & & H_2 \end{array}$$

*Proof.* For  $g, h \in G$ , we have,

$$F(gh) = (f_1(gh), f_2(gh)) = (f_1(g)f_1(h), f_2(g)f_2(h)) = (f_1(g), f_2(g)) * (f_1(h), f_2(h)) = F(g) * F(h)$$

Thus,  $F$  is a homomorphism. Let  $K : G \rightarrow H_1 \times H_2$  be any homomorphism satisfying  $\pi_1 \circ K = f_1$  and  $\pi_2 \circ K = f_2$  then for any  $g \in G$  we have  $K(g) \in H_1 \times H_2$  so  $K(g) = (h_1, h_2)$  for  $h_1 \in H_1$  and  $h_2 \in H_2$  and  $\pi_1 \circ K(g) = h_1 = f_1(g)$  and  $\pi_2 \circ K(g) = h_2 = f_2(g)$  so  $K(g) = (f_1(g), f_2(g)) = F(g)$ .  $\square$

**Lemma 0.4.** Let  $\gamma_0, \gamma_1 : I \rightarrow X$  be path-homotopic loops at  $x_0$  and let  $\delta_0, \delta_1 : I \rightarrow Y$  be path-homotopic loops at  $y_0$  then  $\Gamma_0 = (\gamma_0, \delta_0) : I \rightarrow X \times Y$  and  $\Gamma_1 = (\gamma_1, \delta_1) : I \rightarrow X \times Y$  are path-homotopic loops at  $(x_0, y_0)$ .

*Proof.* Because  $\gamma_0$  and  $\delta_0$  are continuous,  $\Gamma_0$  is also continuous.  $\Gamma_0$  is a loop at  $(x_0, y_0)$  because  $\Gamma_0(0) = (\gamma_0(0), \delta_0(0)) = (x_0, y_0)$  and  $\Gamma_0(1) = (\gamma_0(1), \delta_0(1)) = (x_0, y_0)$ . An identical argument shows that  $\Gamma_1$  is a loop at  $(x_0, y_0)$ . Take path-homotopies  $F : I^2 \rightarrow X$  and  $G : I^2 \rightarrow Y$  for  $\gamma_0 \sim \gamma_1$  and  $\delta_0 \sim \delta_1$  respectively. Now, consider  $H = (F, G) : I^2 \rightarrow X \times Y$  which is continuous because  $F$  and  $G$  are continuous. Also,

$$\begin{aligned} H(0, t) &= (F(0, t), G(0, t)) = (x_0, y_0) \\ H(1, t) &= (F(1, t), G(1, t)) = (x_0, y_0) \\ H(x, 0) &= (F(x, 0), G(x, 0)) = (\gamma_0(x), \delta_0(x)) = \Gamma_0(x) \\ H(x, 1) &= (F(x, 1), G(x, 1)) = (\gamma_1(x), \delta_1(x)) = \Gamma_1(x) \end{aligned}$$

Therefore,  $H$  is a path-homotopy between  $\Gamma_0$  and  $\Gamma_1$ .  $\square$