1 Homework 1

1.1 1

1.1.1 1

Maps $\text{Hom}_k(\mathbb{G}_a, \mathbb{G}_a) = \text{Hom}_k(\text{Spec}(k[t]), \text{Spec}(k[t])) = \text{Hom}_k(k[t], k[t]) = k[t]$. Therefore we need $f \in k[t]$ which are cogroup maps for $k[t] \to k[t] \otimes_k k[t]$ meaning that $f(x+y) = f(x) + f(y)$.

Consider $\text{Hom}_k(\mathbb{G}_m, \mathbb{G}_m) = \text{Hom}_k(\text{Spec}(k[t]), \text{Spec}(k[t])) = \text{Hom}_k(k[t, t^{-1}], k[t, t^{-1}]) = (k[t, t^{-1}])^\times$ such that $f(xy) = f(x)f(y)$.

1.1.2 2

If k is a $\mathbb{Q}$-algebra then any $f \in k[t]$ satisfying $f(x+y) = f(x) + f(y)$ must be linear with zero constant term and thus $f = at$ for $a \in k$ so we get $\text{End}(k)\,\mathbb{G}_a \cong k$.

If k is a field of characteristic $p > 0$ then if $f(x+y) = f(x) + f(y)$ we must have that $f(ax) = af(x)$ for each $a \in \mathbb{F}_p$ which implies that,

$$f(t) = \sum c_j t^{p^j}$$

Suppose that $k = \mathbb{Z}/(p^2)$. (DO THIS CASE!!)

1.1.3 3

If k is a field then $(k[t, t^{-1}])^\times$ consists of elements of the form $f(t) = at^n$ and if $f(xy) = f(x)f(y)$ then $a = 1$ so $\text{End}(\mathbb{G}_m) \cong \mathbb{Z}$.

Now suppose that A is an Artinian local ring and $\kappa = A/\mathfrak{m}$ its residue field. It suffices to prove that every $f \in (A[t, t^{-1}])^\times$ such that $f(xy) = f(x)f(y)$ is of the form $f = t^n$ for some $t \in \mathbb{Z}$. We have shown that the image of f in $(\kappa[t, t^{-1}])^\times$ is of the form t^n for some $n \in \mathbb{Z}$. Then we can consider $g = ft^{-n}$ which is 1 when reduced to the special fiber. To conclude that $g = 1$ we appeal to induction on the length of A. If $\ell_A(A) = 1$ then A must be a field in which case we are done. Since A is Artinian, $\mathfrak{m}^{N+1} = 0$ but $\mathfrak{m}^N \neq 0$ for some N. Then,

$$0 \longrightarrow \mathfrak{m}^N \longrightarrow A \longrightarrow A/\mathfrak{m}^N \longrightarrow 0$$

However, $\mathfrak{m}^N$ is a κ-module. Then $A' = A/\mathfrak{m}^N$ has smaller length so the image of g in $A'[t.t^{-1}]$ equals 1 and thus $g - 1 \in \mathfrak{m}^N[t, t^{-1}]$. However, $g(xy) = g(x)g(y)$ and thus $g(1) = g(1)^2$ but $g(1) \in A^\times$ so $g(1) = 1$. Furthermore,

$$(g(x) - 1)(g(y) - 1) = g(xy) + 1 - g(x) - g(y) = (g(xy) - 1) - (g(x) - 1) - (g(y) - 1)$$

and thus since $(g(x) - 1)(g(y) - 1) = 0$ letting $h = g - 1$,

$$h(xy) = h(x) + h(y)$$

But this is impossible for degree reasons since $h(t^2) = 2h(t)$ so if,

$$h = \sum_{n=-k}^{k} c_n t^n$$

then $c_k = 0$ and $c_{-k} = 0$ since $h(t^2) = 2h(t)$ and thus $h = 0$.

1.1.4 4

Let A be a complete Noetherian local ring and $f \in (A[t, t^{-1}])^\times$. Then under $A \to A/\mathfrak{m}^k$ we see that $f \mapsto t^n$ for a fixed n by using (iii) (the fixedness of n comes from the composition $A \to A/\mathfrak{m}^k \to \kappa$ and $f = t^n$ in $\kappa[t, t^{-1}]$). However, the maps $A \to A/\mathfrak{m}^n$ are mutually injective because A is complete so $f = t^n$.

Now let A be any Noetherian local ring and $f \in (A[t, t^{-1}])^\times$. Consider the injection $A \to \hat{A}$ (this is injective because if $x \mapsto 0$ under each $A \to A/\mathfrak{m}^k$ then $x = \mathfrak{m}^k$ for all k so $x = 0$ by the Krull intersection theorem) then we see that $f = t^n$ since it is in $\hat{A}[t, t^{-1}]$.

Now let A be any local ring and $f \in (A[t, t^{-1}])^\times$ and consider A' to be the ring generated by the coefficients of f and f^{-1} over $\mathbb{Z}$. Then localizing at $\mathfrak{m} \cap A'$ we get a Noetherian local subring $A'' \subset A$ such that $f \in (A''[t, t^{-1}])^\times$ and thus $f = t^n$.

Now consider any ring A and $f \in (A[t, t^{-1}])^\times$. Then for any ideal $\mathfrak{p} \in \mathrm{Spec}\,(A)$ we have $f_\mathfrak{p} \in (A_\mathfrak{p}[t, t^{-1}])^\times$ so $f_\mathfrak{p} = t^{n_\mathfrak{p}}$ giving a function $n : \mathrm{Spec}\,(A) \to \mathbb{Z}$ taking $\mathfrak{p} \mapsto n_\mathfrak{p}$. However, if $f_\mathfrak{p} = t^{n_\mathfrak{p}}$ then there is some $u \in A \setminus \mathfrak{p}$ such that $u(f_\mathfrak{p} - t^{n_\mathfrak{p}}) = 0$ which implies that $f = t^n$ in $A_u[t, t^{-1}]$ and thus n is constant on $D(u)$ with $\mathfrak{p} \in D(u)$.

1.2 2

Let V be a finite-dimensional vector space over a field k.

1.2.1 1

Should this be $\mathrm{Sym}\,(V^*)$ or $\mathrm{Sym}\,(V)^*$? It is clear that $\mathrm{Sym}\,(V^*)$ is the set of funtorial functions on V that are sums of products of linear maps while $\mathrm{Sym}\,(V)^*$ contains divided power structures.

1.2.2 2

Consider the functor,

$$\mathrm{End}\,(V)\,(R) = \mathrm{End}\,(V_R)$$

However, $\mathrm{Sym}\,(-)$ is the left adjoint to the forgetful functor from k-algebras to k-modules. Then,

$$\mathrm{Hom}_{k\text{-alg}}\left(\mathrm{Sym}\,(\mathrm{End}\,(V)^*)\,, R\right) = \mathrm{Hom}_k\left(\mathrm{End}\,(V)^*\,, R\right) = \mathrm{End}\,(V) \otimes_k R = \mathrm{End}\,(V_R)$$

functorially and thus $\mathrm{Sym}\,(\mathrm{End}\,(V)^*)$ represents the functor $\mathrm{End}\,()\,V$.

1.2.3 3

Define $\det \in \mathrm{Sym}\,(\mathrm{End}\,(V)^*)$ as follows. Let $n = \dim V$ then define the ring map $\mathrm{End}\,(V) \to \mathrm{End}\,(\bigwedge^n V)$ via $\varphi \mapsto \wedge^n \varphi$. This defines a natural map of algebras,

$$\mathrm{Sym}\left(\mathrm{End}\left(\bigwedge^n V\right)^*\right) \to \mathrm{Sym}\,(\mathrm{End}\,(V)^*)$$

However, $\bigwedge^n V$ is one-dimensional and thus,

$$\mathrm{End}\left(\bigwedge^n V\right) \cong k$$

canonically via the canonical basis element $\mathrm{id} \in \mathrm{End}\left(\bigwedge^n V\right)$. Then the map,

$$\mathrm{Sym}\left(\mathrm{End}\left(\bigwedge^n V\right)^*\right) \to \mathrm{Sym}\left(\mathrm{End}\left(V\right)^*\right)$$

sends $\mathrm{id} \mapsto \det$.

Now consider the algebra,

$$A = \mathrm{Sym}\left(\mathrm{End}\left(V\right)^*\right)[\overset{-1}{\det}]$$

Then we see,

$$\mathrm{Hom}_{k\text{-alg}}\left(A, R\right) = \{\varphi : \mathrm{End}\left(V\right)^* \to R \mid \varphi(\det) \in R^\times\} = \{\varphi \in \mathrm{End}\left(V\right) \otimes_k R \mid \det \varphi \in R^\times\}$$

which is exactly $\mathrm{Aut}\left(V\right)\left(R\right)$.

1.2.4 4

Let $B : V \times V \to k$ be a bilinear form. Consider the subfunctor $\mathrm{Aut}\left(V, B\right) \subset \mathrm{Aut}\left(V\right)$ of points preserving B. It is clear that $B(g \cdot v, g \cdot u) = B(v, u)$ is a closed condition.

Let B be nondegenerate. We say that $T : V_R \to V_R$ is a B-similitude if $B_R(Tv, Tu) = \mu(T)B_R(v, u)$ for all $v, u \in V_R$ and some $\mu(T) \in R^\times$. Since B is nondegenerate, for each $v \in V$ there is $v' \in V$ such that $B(v, v') = 1$ and thus $B_R(v \otimes 1, v' \otimes 1) = 1$. Then, $B_R(Tv \otimes 1, Tv' \otimes 1) = \mu(T) \cdot B_R(v \otimes 1, Tv' \otimes 1) = \mu(T)$ so $\mu(T)$ is uniquely determined by T and B. Consider the functor sending $R \mapsto \{B\text{-similitudes}\}$. Consider the closed subscheme,

$$H \subset V \times \mathbb{G}_m$$

defined in coodinates,

$$V \times \mathbb{G}_m = k[x_{ij}][\overset{-1}{\det}][t, t^{-1}]$$

as the vanishing of (let B be represented by S_{ij}) the equations,

$$x_{\ell i} S_{ij} x_{jk} = t S_{\ell k}$$

for each pair (ℓ, k).

1.3 3 dO THIS

1.3.1 1

Let X be a connected scheme of finite type over a field k and $x : \mathrm{Spec}\left(k\right) \to X$ is a rational point. Let k'/k be a finite extension. Since $\mathrm{Spec}\left(k'\right) \to \mathrm{Spec}\left(k\right)$ is flat and finite we see that $X_{k'} \to X$ is flat and finite and thus open and closed. Consider the fiber over x,

$$\begin{tikzcd}
\mathrm{Spec}\left(k'\right) \arrow[r] \arrow[d] & \mathrm{Spec}\left(k\right) \arrow[d] \\
X_{k'} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}\left(k'\right) \arrow[r] & \mathrm{Spec}\left(k\right)
\end{tikzcd}$$

since the bottom square is cartesian and the outer square is trivially cartesian we see that the top square is cartesian. Therefore, the fiber over x consists of a single point with residue field k'. Now if $X_{k'} = C_1 \cup C_2$ is a disjoint union of clopen sets. Then under $f : X_{k'} \to X$ we see that $f(C_1) \cup f(C_2) = X$ and $f(C_i)$ are clopen so we need to show that $f(C_1) \cap f(C_2) = \varnothing$. Since the fibers over $X(k)$ are single poitns we see that each fiber is contained in exactly one of C_1 or C_2 so $f(C_1) \cap f(C_2) \cap X(k) = \varnothing$. Without loss of generality, the fiber over x is contained in C_1 and thus $x \notin f(C_2)$ but X is connected and $f(C_2)$ is clopen so $f(C_2) = \varnothing$ and thus $C_2 = \varnothing$ meaning that $X_{k'}$ is connected.

Now for every extension of fields k'/k we know that,

$$\varinjlim_{k' \supset k'' \supset k} k'' = k'$$

over the finite extensions k''/k. However, $\mathrm{Spec}\,(-)$ is a right adjoint $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ and thus preserves limits (this is a limit in $\mathbf{Ring}^{\mathrm{op}}$) and thus we see that.

$$\mathrm{Spec}\,(k') = \varprojlim_{k' \supset k'' \supset k} \mathrm{Spec}\,(k'')$$

and products commute with limits so we see that,

$$X_{k'} = \varprojlim_{k' \supset k'' \supset k} X_{k''}$$

Since for each $k_1 \supset k_2$ the map $X_{k_1} \to X_{k_2}$ is surjective and furthermore the map $X_{k'} \to X_{k_1}$ is surjective

1.3.2 2

Let X and Y be geometrically connected of finite type over k. Then it suffices to show that $(X \times_k Y) \times_k \bar{k}$ is connected. However,

$$(X \times_k Y) \times_k \bar{k} = (X \times_k \bar{k}) \times_{\bar{k}} (Y \times_k \bar{k})$$

and then I claim that if X and Y are connected and finite type over an algebraically closed field k then $X \times_k Y$ is connected. Since X and Y are connected the standard affine cover $U_i \times V_j$ overlap eachother and thus it suffices to show the claim for affine X and Y. Thus we need to show that if A and B are finitely generated k-algebras with prime nilradical then $A \otimes_k B$ has prime nilradical.

Suppose that X and Y are connected but not necessarily geometrically connected over $k = \mathbb{Q}$. We can take $X = \mathrm{Spec}\,(\mathbb{Q}(i))$ and $Y = \mathrm{Spec}\,(\mathbb{Q}(i))$ and then

$$X \times_k Y = \mathrm{Spec}\,(\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i)) = \mathrm{Spec}\,(\mathbb{Q}(i)) \sqcup \mathrm{Spec}\,(\mathbb{Q}(i))$$

is not connected.

1.4 4 DO THIS

Let G be a group scheme of finite type over k.

1.4.1 1

Let $(G_{\overline{k}})_{\text{red}}$ be the closed subscheme of $G_{\overline{k}}$. To show that various maps factor through $(G_{\overline{k}})_{\text{red}} \hookrightarrow G_{\overline{k}}$ it suffices to show that $(G_{\overline{k}})_{\text{red}} \times (G_{\overline{k}})_{\text{red}}$ is reduced (since obviously $(G_{\overline{k}})_{\text{red}}$ and $\text{Spec}\left(\overline{k}\right)$ are reduced). Since reducedness and smothness are local properties we reduce to the affine case that A is a finite type k-algebra then $B = (A_{\overline{k}})_{\text{red}}$ then then tensor product of reduced $\bar{k}$-algebras is reduced. To see this, consider,

$$B \to \prod_{\mathfrak{p} \text{ minimal}} B_{\mathfrak{p}}$$

which is injective because for a reduced ring the associated primes are exactly the mininal primes. Then $B_{\mathfrak{p}}$ is a field because $\mathfrak{p}$ is a minimal prime. Since B is flat over k we can suppose that B is a finite product of fields (B is finite type and thus noetherian) so it suffices to reduce to the case of a domain and we know that the tensor product of domains over an algebraically closed field is a domain.

Now we need to show that $H = (G_{\bar{k}})_{\text{red}}$ is smooth. However, since $\bar{k}$ is algebraically closed and H is reduced, by generic smoothness there is a smooth point and then by translation every point is smooth.

1.4.2 2

Let k be an imperfect field k and G an algebraic group scheme over k. Then G_{red} need not be a closed algebraic subgroup of G. This happens when $G_{\text{red}} \times_k G_{\text{red}}$ is not reduced and thus it does not map into G_{red}. (FIND EXAMPLE)

1.4.3 3

Let k be imperfect and characteristic $p > 0$. Choose $\alpha \in k \setminus k^p$ then let,

$$f = x_0^0 + a x_1^p + \cdots + a^{p-1} x_{p-1}^p - 1$$

and consider,

$$G = \text{Spec}\left(k[x_0, \ldots, x_{p-1}]/(f)\right)$$

with the group operation,

$$(x \cdot y)_n = \sum_{p+q=n} x_p y_q$$

This works because,

$$G = \ker\left(\text{Nm} : \text{Res}_k^{k(a^{\frac{1}{p}})}(\mathbb{G}_m) \to \mathbb{G}_m\right)$$

I claim that f is not a power over k. Indeed, because the degree of f is prime it would have to be $f = g^p$ but this implies that $a = \alpha^p$ which is not true by hypothesis. Therefore, G is reduced.

Now after base changing to $k(a^{\frac{1}{p}})$ we see that,

$$f = (x_0 + a^{\frac{1}{p}} x_1 + \cdots + a^{\frac{p-1}{p}} x_{p-1} - 1)^p$$

and thus $G \times_k k(a^{\frac{1}{p}}) = \text{Spec}\left(k[x_0, \ldots, x_{p-1}]/(f)\right)$ is not reduced and thus not smooth.

1.4.4 4

Consider the subscheme,

$$\mu_n = \ker\left(\mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m\right)$$

which is a subgroup because kernels always are. Clearly, the kernel is closed. Let $K = \ker(G \to H)$. Then $K \times K \subset G \times G \to G$ maps to K because,

$$K \times K \to G \to H$$

is the map $K \times K \to \mathrm{Spec}\,(k) \to H$ and thus factors through $K \to G$ by the universal property of the kernel which is a fiber product.

Consider the map $\deg : \mathrm{GL}_N \to \mathbb{G}_m$ and the preimage $G = \det^{-1} \mu_n \subset \mathrm{GL}_N$ is always a k-subgroup by the universal property (its the exact same argument as for the kernel). Expliclty, let $K \subset H$ be a closed subgroup and $f : G \to H$ a morphism of algebraic groups. Let $\tilde{K} = f^{-1}(K)$ the pullback which is a closed subscheme of G then consider the diagram,

$$\begin{tikzcd}
\tilde{K} \times \tilde{K} \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "m"] & \tilde{K} \arrow[r] \arrow[d] & G \arrow[d, "f"] \\
K \times K \arrow[r, "m"] & K \arrow[r] & H
\end{tikzcd}$$

therefore multiplication factors through $\tilde{K} \times \tilde{K} \to \tilde{K}$. The same trick works for inversion.

It is clear that $\mathrm{SL}_N \subset G$ and thus assuming that SL_N is connected we see that $\mathrm{SL}_N \subset G^0$. Furthermore, as long as $p \not| n$ we see that μ_n is disconnected with a reduced point at each root of unity contained in k and $\mu_n^0 = \mathrm{Spec}\,(k)$ the trivial group scheme at the origin. Thus, $G^0 = \ker \det = \mathrm{SL}_N$.

For $k = \mathbb{Q}$ and $n = 5$ the group scheme $G \setminus G^0$ is the fiber over the one nonidentity point of $\mu_5 = \mathrm{Spec}\,(k[x]/(x^5 - 1))$ which is the point $\eta = \mathrm{Spec}\,(k[x]/(x^4 + x^3 + x^2 + x + 1))$. Therefore, the preimage is isomorphic as a scheme over k to $\mathrm{SL}_N \times_k \eta$ which is connected because it is just SL_N over η viewed as a k-scheme. However, over $\overline{\mathbb{Q}}$ we see that η splits into four points and thus $G \setminus G^0 = \det^{-1}(\eta)$ must have at least four components after base change to $\overline{\mathbb{Q}}$.