1 A

1.1 $\mathbb{Z}$-HS

Pure of weight n is a lattice $V_{\mathbb{Z}}$ with a decreasing filtration,

$$V_{\mathcal{C}} = F_0 \supset F^1 \supset \cdots \supset F^n = \{0\}$$

where $V_{\mathcal{C}} = F^p \oplus \overline{Fn - p + 1}$. Comparison between the lattice and the filtration gives the period matrix.

Polarization: $Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ nondegenerate $(-1)^n$-symmetric such that

(a) $Q(F^p, F^{n-p+1}) = 0$ or equivalently the decomposition $V = \bigoplus V^{p,q}$ is orthogonal

(b) $i^{2p-n}Q(\xi, \xi) > 0$ for nonzero $\xi \in V^{p,n-p} := F^p \cap \overline{F^{n-p}}$.

1.2 Variations of PHS

Of weight n over a complex manifold S. A VHS $\mathcal{V}$ consists of

(a) a $\mathbb{Z}$-local system $V_{\mathbb{Z}}$ over S

(b) $Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ a pairing

(c) a filtration $F^\bullet \subset V := V_{\mathbb{Z}} \otimes \mathcal{O}_S$ by holomorphic vector bundles

(d) a connection $\nabla : V \to V \otimes \nabla$ with $V_{\mathbb{Z}} = \ker \nabla$

such that

(a) on each fiber the filtration and Q gives a PHS

(b) $\nabla F^p \subset F^{p-1} \otimes \Omega^1_S$ for all p

The first two give the data of a representation,

$$\rho : \pi_1(S, s_0) \to \text{Aut}\left(() \left(V_{\mathbb{Z}}\right)_{s_0}, Q_{s_0}\right)$$

and we define the monodromy group

$$M = (\overline{\rho(\pi_1)}^{\text{Zar}})^\circ$$

Remark. If $\pi : X \to S$ is smooth projective then it defines a VHS on cohomology with $V_{\mathbb{Z}} = (R^n\pi_*\underline{\mathbb{Z}})/\text{tors}$ and $\mathscr{F}^p =^n \pi_*\Omega^{\bullet \geq p}_{X/S}$. The connection is induced by

$$\pi^*\Omega^1_S \otimes \Omega^{\bullet \geq p-1}_X[1] \to \Omega^{\bullet \geq p}_X \to \Omega^{\bullet \geq p}_{X/S}$$

Remark. Let $S = \mathbb{P}^1 \backslash \Sigma$ where Σ is a finite set of points and $V_{\mathcal{C}}$ is irreducible and $h^{n,0} \neq 0$ then there is a section $\mu \in \Gamma(\mathbb{P}^1, \mathscr{F}^n_e)$ (where $\mathscr{F}^n_e$ is the extension to $\mathbb{P}^1$) such that $(V, \nabla) \cong \mathcal{D}/\mathcal{D}L$ for some $L \in \mathbb{C}[D, t]$ is a $\mathcal{D}$-module. This L is called the Picard-Fuchs operator. The periods:

$$\mu\gamma = \pi_\gamma(t)$$

for $\gamma \in \Gamma(S^{\text{an}}, V^\vee_{\mathbb{Z}})$satisfy $L\pi_\gamma = 0$.

Remark. Let $S = \Delta^*$ and let T be the monodromy operator around the look. Then T is quasi-unipotent meaning $T = T_{ss}Y_u$ such that $T_{ss}^m = I$ and $(T_u - I)^k = 0$ and $[T_{ss}, T_u] = 0$. Write $N = \log T_u$.

($\mathbb{Q}-$)LMHS ψ_t basechange such that T is unipotent then $\mathscr{F}^\bullet \subset V$ extends to

$$\mathscr{F}_e^\bullet \subset V_e := e^{-\frac{\log(t)}{2\pi i}N} V_{\mathbb{Q}} \otimes \mathcal{O}_S$$

which is well-defined over Δ. Then N is part of an ${}_2$-tripple (N, Y, N^+). Then $V = \mathcal{V}_e|_{t=0}$ has two filtrations

(a) $F_e^\bullet|_{t=0}$

(b) $W_\bullet = W(N)[-n]$

this defines a mixed hodge structure.

1.3 Example

Conifold point: A_1 singularity on a CY 3-fold.

1.4 hypergeometric variations

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ be numbers in $(0,1] \cap \mathbb{Q}$ and $\alpha_i \neq \beta_j$ for all i, j. We suppose that the $\underline{\alpha}$ and $\underline{\beta}$ satisfy,

$$q_\infty(\lambda) := \prod_j (\lambda - e^{2\pi i \alpha_j}) \in \mathbb{Q}[\lambda]$$

and

$$q_0(\lambda) := \prod_j (\lambda - e^{2\pi i \beta_j}) \in \mathbb{Q}[\lambda]$$

Theorem 1.4.1. There exists a geometrric $\mathbb{Q}$-PVHS $\mathcal{V}_{\alpha,\beta}$ over $\mathbb{P}^1_z \backslash \{0, 1, \infty\}$ with $L = \prod(D + \beta_j - 1) - z\prod(D_{\alpha j})$ such that q_0, q_∞ are the char polys o T_0 and T_∞ and period

$$\prod = \sum_{k \geq 0} \frac{\prod[\alpha_j]_k}{\prod[\beta_j]_k} z^k$$

where $[\alpha]_k$ is the rising factorial $\alpha(\alpha+1)\cdots(\alpha+k)$. There is an interesting formula for the hodge numbers in terms of a zig-zag diagram. Furthermore, the monodromy group is

(a) $\{1\}$ weight zero (i.e. if α, β are intertwined: they alternate in order)

(b) Sp_r for odd weight

(c) $\mathrm{SO}(h_{\text{even}}, h_{\text{odd}})$ for even weight,