Mathematics GU4044 Representations of Finite Groups
Assignment # 10

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Problem 1.

Let G be a nonabelian group of order pq with $p < q$ and $q \equiv 1 \mod p$. Let $x \in G$ have order q and $y \in G$ such that $yxy^{-1} = x^t$ for $t \in (\mathbb{Z}/q\mathbb{Z})^\times$ with order p. Finally, $H = \langle x \rangle$ is the Sylow q-subgroup.

(a). Let $\lambda : H \to \mathbb{C}^\times$ be a homomomorphism. Given a generator $x \in H$ write $\lambda(x) = e^{2\pi i a/q}$ for some $0 \le a \le q-1$. Consider,

$$\lambda \circ i_y(x) = \lambda(yxy^{-1}) = \lambda(x^t) = e^{2\pi i a t/q} = \lambda_{ta}(x)$$

Furthermore,

$$\lambda \circ i_{y^k}(x) = \lambda \circ (i_y)^{\circ k}(x) = \lambda(x^{t^k}) = e^{2\pi i a t^k/q} = \lambda_{at^k}(x)$$

Now, suppose that $a \neq 0$. Take any $z \in G$. Suppose that $\lambda \circ i_z = \lambda$ let $zxz^{-1} \in x^r$. Then, $\lambda \circ i_z(x) = e^{2\pi i a r/q} = e^{2\pi i a/q}$ and thus $ar \equiv a \mod q \implies r \equiv 1 \mod q \implies x^r = 1$. Therefore $zxz^{-1} = e$ which implies that $z \in H$ since $|C_x|$ must divide pq and cannot be pq since G is nonabelian and $H \subset C_x$. Conversely, if $z \in H$ then since H is abelian, $i_z = \mathrm{id}_H$ and thus $\lambda \circ i_z = \lambda$.

(b). Consider V an irreducible G-representation of dimension p. We know that $\mathrm{Res}^G_H V$ is the sum of p irreducible 1-dimensional representations W_i since H is abelian. Givn λ define $\lambda_k = \lambda \circ i_{y^k}$ with character χ_k. However, using Frobenius reciprocity,

$$\left\langle \chi_k, \chi_{\mathrm{Res}^G_H V} \right\rangle = \left\langle \mathrm{Ind}^G_H \chi_k, \chi_V \right\rangle = \left\langle \mathrm{Ind}^G_H \chi_\lambda, \chi_V \right\rangle$$

because $\mathrm{Ind}^G_H \chi_\lambda$ is a class function on G and i_{y^k} does not change G conjugacy classes. In the above formula $\mathrm{Ind}^G_H \chi_\lambda$ is the character of the induced representation $\mathrm{Ind}^G_H \mathbb{C}(\lambda)$. Therefore, if λ is a homomorphism corresponding to a 1-dimensional summand of $\mathrm{Res}^G_H V$ then so are all $\mathbb{C}(\lambda \circ i_z)$. However, λ_{y^k} and $\lambda_{y^{k'}}$ will be distinct if $k \not\equiv k' \mod p$ since y has multiplicative order p. Therefore, $k = 0, 1, \dots, p-1$ gives all p one dimensional summands of $\mathrm{Res}^G_H V$. Furthermore, $V \cong \mathrm{Ind}^G_H W_i$ so there is a correspondence between p-dimensional irreducible G-representations and homomorphisms $\lambda : H \to \mathbb{C}^\times$ modulo $\lambda \sim \lambda \circ i_{y^k}$. We know that the set of homomorphisms $\lambda : H \to \mathbb{C}^\times$ is,

$$\hat{H} \cong (\mathbb{Z}/q\mathbb{Z})^\times$$

and we are taking the quotient by the subgroup generated by $1 \circ i_y$ since $i_{y^k} = (i_y)^{\circ k}$. However, y has multiplicative order p and this so does i_y. Therefore, the p-dimensional irreducible G-representations are in one-to-one correspondence with the set,

$$(\mathbb{Z}/q\mathbb{Z})^\times / \langle 1 \circ i_y \rangle$$

In particular, there are $(q-1)/p$ of them.

Problem 2.

We need to write S_3 and S_4 as unions of conjugacy classes.

$$S_3 = [e] \cup [(1\,2)] \cup [(1\,2\,3)] \quad \text{and} \quad S_4 = [e] \cup [(1\,2)] \cup [(1\,2\,3)] \cup [(1\,2\,3\,4)] \cup [(1\,2)(3\,4)]$$

with sizes,

$$|S_3| = 6 = 1 + 3 + 2 \quad \text{and} \quad |S_4| = 24 = 1 + 6 + 8 + 6 + 3$$

Note that $\chi_{W_2} = \chi_{\mathbb{C}^3} - 1$ and $\chi_{V_3} = \chi_{\mathbb{C}^4} - 1$.

(a). The S_4-representations $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$ are defined by the acton of the homomorphisms $1, \varepsilon : S_4 \to \mathbb{C}^\times$ which clearly restrict to $1, \varepsilon : S_3 \to \mathbb{C}^\times$ and thus $\mathrm{Res}^{S_4}_{S_3} \mathbb{C} = \mathbb{C}$ and $\mathrm{Res}^{S_4}_{S_3} \mathbb{C}(\varepsilon) = \mathbb{C}(\varepsilon)$.

(b). We will compute the characters,

$$\left\langle \chi_{\mathrm{Res}^{S_4}_{S_3} V_3}, \chi_{W_2 \oplus \mathbb{C}} \right\rangle_{S_3} = \frac{1}{6}\left[3 \cdot 3 + 3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0\right] = 2$$

However, the sum of the squared multiplicities of $W_2 \oplus \mathbb{C}$ is 2 and likewise,

$$\left\langle \chi_{\mathrm{Res}^{S_4}_{S_3} V_3}, \chi_{\mathrm{Res}^{S_4}_{S_3} V_3} \right\rangle_{S_3} = \frac{1}{6}\left[3 \cdot 3 + 3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0\right] = 2$$

so $\mathrm{Res}^{S_4}_{S_3} V_3$ must also have 2 irreducible components with multiplicity 1. Thus, each component must be isomorphic so,

$$\mathrm{Res}^{S_4}_{S_3} V_3 \cong W_2 \oplus \mathbb{C}$$

Furthermore, consider,

$$\left\langle \chi_{\mathrm{Res}^{S_4}_{S_3} V_3 \otimes \varepsilon}, \chi_{W_2 \oplus \mathbb{C}(\epsilon)} \right\rangle_{S_3} = \frac{1}{6}\left[3 \cdot 3 - 3 \cdot (1 - 2) \cdot 1 + 2 \cdot 0 \cdot 0\right] = 2$$

However, the sum of the multiplicities of $W_2 \oplus \mathbb{C}(\varepsilon)$ is 2 and likewise,

$$\left\langle \chi_{\mathrm{Res}^{S_4}_{S_3} V_3 \otimes \varepsilon}, \chi_{\mathrm{Res}^{S_4}_{S_3} V_3 \otimes \varepsilon} \right\rangle_{S_3} = \frac{1}{6}\left[3 \cdot 3 + 3 \cdot (-1) \cdot (-1) + 2 \cdot 0 \cdot 0\right] = 2$$

so $\mathrm{Res}^{S_4}_{S_3} V_3 \otimes \varepsilon$ must be the sum of two disjoint irreducible representations. Therefore,

$$\mathrm{Res}^{S_4}_{S_3} V_3 \otimes \varepsilon \cong W_2 \oplus \mathbb{C}(\varepsilon) \cong (W_2 \oplus \mathbb{C}) \otimes \varepsilon$$

because $W_2 \otimes \varepsilon \cong W_2$ since S_3 has a unique 2-dimensional irreducible representation.

(c). Consider the S_3 representation $\operatorname{Ind}_{S_3}^{S_4}\mathbb{C}$. Using Frobenius reciprocity,

$$\left\langle \chi_{\operatorname{Ind}_{S_3}^{S_4}\mathbb{C}}, \chi_{V_3 \oplus \mathbb{C}} \right\rangle_{S_4} = \left\langle \chi_{\mathbb{C}}, \chi_{\operatorname{Res}_{S_3}^{S_4} V_3 \oplus \mathbb{C}} \right\rangle_{S_3}$$

However, we can calculate,

$$\left\langle \chi_{\mathbb{C}}, \chi_{\operatorname{Res}_{S_3}^{S_4} V_3 \oplus \mathbb{C}} \right\rangle_{S_3} = \frac{1}{6}\left[1 \cdot 4 + 3 \cdot (1 \cdot 2) + 2 \cdot (1 \cdot 1)\right] = 2$$

Which implies that $V_3 \oplus \mathbb{C}$ appears without multiplicity in the expansion of $\operatorname{Ind}_{S_3}^{S_4}\mathbb{C}$ as a sum of irreducible representations. However, since $\dim \operatorname{Ind}_{S_3}^{S_4}\mathbb{C} = [S_4 : S_3] \dim \mathbb{C} = 4$, by dimension counting,

$$\operatorname{Ind}_{S_3}^{S_4}\mathbb{C} \cong V_3 \oplus \mathbb{C}$$

An identical argument shows that,

$$\left\langle \chi_{\operatorname{Ind}_{S_3}^{S_4}\mathbb{C}(\varepsilon)}, \chi_{(V_3 \oplus \mathbb{C}) \otimes \varepsilon} \right\rangle_{S_4} = \left\langle \chi_{\mathbb{C}(\varepsilon)}, \chi_{\operatorname{Res}_{S_3}^{S_4} (V_3 \oplus \mathbb{C}) \otimes \varepsilon} \right\rangle_{S_3} = 2$$

since each term in the inner product is modified only by a multiplication by -1 for each ε factor and thus an overall factor of 1. Therefore, by dimension counting,

$$\operatorname{Ind}_{S_3}^{S_4}\mathbb{C}(\varepsilon) \cong (V_3 \oplus \mathbb{C}) \otimes \varepsilon \cong (V_3 \otimes \varepsilon) \oplus \mathbb{C}(\varepsilon)$$

(d). Using Frobenius reciprocity,

$$\left\langle \chi_{\operatorname{Ind}_{S_3}^{S_4} W_2}, V_3 \right\rangle_{S_4} = \left\langle \chi_{W_2}, \chi_{\operatorname{Res}_{S_3}^{S_4} V_3} \right\rangle_{S_3} = 1$$

since we know that,

$$\operatorname{Res}_{S_3}^{S_4} V_3 \cong W_2 \oplus \mathbb{C}$$

and thus W_2 is an irreducible factor of $\operatorname{Res}_{S_3}^{S_4} V_3$ without multiplicity. Therefore, V_3 is an irreducible factor of $\operatorname{Ind}_{S_3}^{S_4} W_2$ without multiplicity. Furthermore,

$$\left\langle \chi_{\operatorname{Ind}_{S_3}^{S_4} W_2}, V_3 \otimes \varepsilon \right\rangle_{S_4} = \left\langle \chi_{W_2}, \chi_{\operatorname{Res}_{S_3}^{S_4} (V_3 \otimes \varepsilon)} \right\rangle_{S_3} = 1$$

since

$$\operatorname{Res}_{S_3}^{S_4} (V_3 \otimes \varepsilon) \cong W_2 \otimes \mathbb{C}(\varepsilon)$$

and thus $V_3 \otimes \varepsilon$ is also an irreducible factor of $\operatorname{Ind}_{S_3}^{S_4} W_2$ without multiplicity.

Using more Frobenius reciprocity,

$$\left\langle \chi_{\operatorname{Ind}_{S_3}^{S_4} W_2}, \mathbb{C} \right\rangle_{S_4} = \left\langle \chi_{W_2}, \chi_{\operatorname{Res}_{S_3}^{S_4} \mathbb{C}} \right\rangle_{S_3} = \langle \chi_{W_2}, \chi_{\mathbb{C}} \rangle_{S_3} = 0$$

and likewise,

$$\left\langle \chi_{\operatorname{Ind}_{S_3}^{S_4} W_2}, \mathbb{C}(\varepsilon) \right\rangle_{S_4} = \left\langle \chi_{W_2}, \chi_{\operatorname{Res}_{S_3}^{S_4} \mathbb{C}(\varepsilon)} \right\rangle_{S_3} = \left\langle \chi_{W_2}, \chi_{\mathbb{C}(\varepsilon)} \right\rangle_{S_3} = 0$$

by (a) and the fact that $W_2 \not\cong \mathbb{C}$ and $W_2 \not\cong \mathbb{C}(\varepsilon)$. Therefore, $\mathbb{C}$ and $\mathbb{C}(\varepsilon)$ are not summands of $\operatorname{Ind}_{S_3}^{S_4} W_2$. However, by $\dim \operatorname{Ind}_{S_3}^{S_4} W_2 = [S_4 : S_3] \dim W_2 = 8$ so by dimension counting, the fact that no 1-dimensional irreducible S_4-representation is a summand, and the fact that V_3 and $V_3 \otimes \varepsilon$ have multiplicity 1,

$$\operatorname{Ind}_{S_3}^{S_4} W_2 \cong V_3 \oplus (V_3 \otimes \varepsilon) \oplus V$$

where $\dim V = 2$ since $\dim V_3 = \dim (V_3 \otimes \varepsilon) = 3$. However, S_4 has a unique 2-dimensional representation up to isomorphism so,

$$\operatorname{Ind}_{S_3}^{S_4} W_2 \cong V_3 \oplus (V_3 \otimes \varepsilon) \oplus V_2$$

Problem 3.

We already know that $V = \operatorname{Ind}_H^G W$ is irreducible and $\operatorname{Res}_H^G V = W \oplus W_x$ if and only if $W \not\cong W_x$. It remains to prove that in the case that $W \cong W_x$ that the entire statement (ii) follows.

Suppose that $W \cong W_x$ then by above $V = \operatorname{Res}_H^G W$ is irreducible. Since $[G : H] = 2$ take coset representatives H and xH. Because H has index 2 the subgroup H is normal in G so $z^{-1}gz \in H \iff g \in H$. Therefore, $i_z(g) \in H \iff g \in H$. Now we apply the formula for the character of the induced representation, if $g \notin H$

$$\chi_V(g) = \frac{1}{H} \sum_{z^{-1}gz \in H} \chi_W(z^{-1}gz \in H) = 0$$

Otherwise,

$$\chi_V(h) = \chi_{\operatorname{Res}_H^G V}(h) = \chi_W(h) + \chi_{W_x}(h)$$

This implies that,

$$\langle \chi_V, \chi_V \rangle_G = \frac{\#(H)}{\#(G)} \left\langle \chi_{\operatorname{Res}_H^G V}, \chi_{\operatorname{Res}_H^G V} \right\rangle_H = \frac{1}{2} \langle \chi_W + \chi_{W_x}, \chi_W + \chi_{W_x} \rangle = \frac{1}{2} \cdot 4 = 2$$

where I have used the fact that $W \cong W_x$ are both are irreducible so each of the four inner products of characters is 1. This implies that $V \cong V_1 \oplus V_2$ where V_1 and V_2 are irreducible. Because $V = \operatorname{Ind}_H^G W$ we can apply Frobenius reciprocity to conclude that,

$$\langle V, \chi_{V_i} \rangle = \langle \operatorname{Ind}_H^G W, \chi_{V_i} \rangle = \left\langle \chi_W, \chi_{\operatorname{Res}_H^G V_i} \right\rangle = 1$$

which shows that,

$$\operatorname{Res}_H^G V_i \cong W \implies \operatorname{Ind}_H^G \operatorname{Res}_H^G V_i \cong \operatorname{Ind}_H^G W$$

Therefore, we have that,

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G V_1 \cong \operatorname{Ind}_H^G W \cong \operatorname{Ind}_H^G \operatorname{Res}_H^G V_2$$

From class we have the formula,

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G V_1 \cong V_1 \otimes \mathbb{C}[G/H] = V_1 \otimes (\mathbb{C} \oplus \mathbb{C}(\varepsilon)) = V_1 \oplus (V_1 \otimes \varepsilon)$$

because $G/H \cong \mathbb{Z}/2\mathbb{Z}$ since it has index 2. The same argument shows that,

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G V_1 \cong V_2 \oplus (V_2 \otimes \varepsilon)$$

However, we know that these are isomorphic,

$$V_1 \otimes (V_1 \otimes \varepsilon) \cong V_2 \otimes (V_2 \otimes \varepsilon)$$

But we have shown that V has no multiplicity so $V_1 \ncong V_2$ which implies that $V_2 \cong V_1 \otimes \varepsilon$ since it is the only other irreducible factor. This proves that $V \cong V_1 \oplus (V_1 \otimes \varepsilon)$ where V_1 is an irreducible G-representation and $V_1 \ncong V_1 \otimes \varepsilon$ and that $W \cong \operatorname{Res}_H^G V_1 \cong \operatorname{Res}_H^G V_1 \otimes \varepsilon \cong W$. If this is true then V is not irreducible so $W \cong W_x$. Therefore, we have (ii) exactly when $W \cong W_x$ and (i) exactly when $W \ncong W_x$.