1 Introduction

Definition 1.0.1. A *valuation ring* is a domain A contained in a field K such that for all nonzero $x \in K$ either $x \in A$ or $x^{-1} \in A$.

Remark. Clearly in this case $K = \text{Frac}(A)$.

Proposition 1.0.2. Valuation rings are local.

Proof. Let $A \subset K = \text{Frac}(A)$ be a valuation ring and define,

$$\mathfrak{m} = \{x \in A \mid x^{-1} \notin A \text{ or } x = 0\}$$

I claim that $\mathfrak{m} \subset A$ is an ideal. If $x \in \mathfrak{m}$ and $y \in A$ then $yx \in \mathfrak{m}$ since if $(yx)^{-1} \in A$ then $(yx)^{-1}y = x^{-1} \in A$ which is not the case. Furthermore, if $x, y \in \mathfrak{m}$ then $x + y \in A$ and suppose $(x+y)^{-1} \in A$ (if $x + y \neq 0$ in which case we are done). Now either $x/y \in A$ or $y/x \in A$, without loss of generality, take $x/y \in A$. Then $x + y = y(1 + x/y)$ and thus,

$$y^{-1} = \frac{1 + x/y}{x + y} \in A$$

contradicting the fact that $y \in \mathfrak{m}$. Therefore, $x + y \in \mathfrak{m}$ so $\mathfrak{m}$ is an ideal. Furthermore, it is clear that if $x \in A \setminus \mathfrak{m}$ then $x \in A^\times$ so A is local. □

Definition 1.0.3. Let $A, B \subset K$ be two local domains contained in a field K. Then we say B *dominates* A if $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ i.e. $A \hookrightarrow B$ is a *local* extension of domains.

Proposition 1.0.4. Let A be a local domain. Then A is a valuation rings iff A is maximal with respect to domination of local subrings of $K = \text{Frac}(A)$.

Proof. Let $B \subset K$ be a local domain dominating A i.e. $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. Then take $x \in B \setminus A$. Since $x \in B \subset K$ we must have $x^{-1}A$ so $x^{-1} \in \mathfrak{m}_A$ however, $\mathfrak{m}_A = \mathfrak{m}_B \cap A$ so $x^{-1} \in \mathfrak{m}_B$ but $x \in B$ contradicting the fact that $\mathfrak{m}_B$ is a maximal ideal contradicting the existence of $x \in B \setminus A$ so $B = A$.

conversely, assume that $A \subset K = \text{Frac}(A)$ is a local domain maximal with respect to domination. Take $x \in K$ and assume $x \notin A$ then take $A' = A[x]$ so $A' \supsetneq A$. Furthermore, for any prime $\mathfrak{p} \subset A'$ we can take $A \subset A' \subset A'_{\mathfrak{p}} \subset K$ and if $\mathfrak{p} \cap A = \mathfrak{m}$ this contradicts the maximality of A so there are no primes of A' lying above $\mathfrak{m}$. Thus $V(\mathfrak{m}A') = \varnothing$ so $\mathfrak{m}A' = A'$. Therefore,

$$1 = \sum_{i=0}^{d} t_i x^i$$

for $t_i \in \mathfrak{m}$ and thus,

$$(1 - t_0)(x^{-1})^d - \sum_{i=1}^{d} t_i (x^{-1})^{d-i} = 0$$

so x^{-1} is integral over A. Thus $A'' = A[x^{-1}] \subset K$ is finite over A and thus $\text{Spec}(A') \to \text{Spec}(A)$ is surjective so there is an ideal $\mathfrak{p} \subset A''$ lying over $\mathfrak{m}$. And thus $A \subset A''_{\mathfrak{p}}$ is a local extension but A is a valuation ring so $A = A''_{\mathfrak{p}}$ and thus $x^{-1} \in A$. □

Proposition 1.0.5. Let $A \subset K$ be a local domain inside a field K. Then there exists a valuation ring B dominating A with fraction field K.

Proof. Let I be a totally ordered set and A_i a totally ordered, via domination, chain of local domains indexed by I. Then consider,

$$B = \bigcup_i A_i$$

I claim that B is a local ring. If $x, y \in B$ then $x, y \in A_i$ for some $i \in I$ and thus $x + y, xy \in A_i \subset B$. Furthermore, let,

$$\mathfrak{m} = \bigcup_i \mathfrak{m}_i$$

then for the same reason $\mathfrak{m}$ is an ideal. Furthermore, if $x \in B \setminus \mathfrak{m}$ then $x \in A_i \setminus \mathfrak{m}_i$ for some $i \in I$ so x is a unit in A_i and thus x has an inverse in B so B is a local ring with maximal ideal $\mathfrak{m}$. By Zorn's lemma, there exists a maximal local subring of K with respect ot domination containing A. Thus, it suffices to show that if $K \supsetneq \mathrm{Frac}\,(A)$ then A is not maximal since then the maximal element will satisy $K = \mathrm{Frac}\,(A)$ and thus will be a valuation ring.

Let $A \subset K$ be a local domain with $\mathrm{Frac}\,(A) \subsetneq K$. Choose $t \in K \backslash \mathrm{Frac}\,(A)$. If t is transcendental over $\mathrm{Frac}\,(A)$ then $A[t]_{(t,\mathfrak{m})}$ dominates A otherwise $A[t]$ is finite over A and thus $\mathrm{Spec}\,(A[t]) \to \mathrm{Spec}\,(A)$ is surjective so there is a prime $\mathfrak{p} \subset A[t]$ above $\mathfrak{m}$ and thus $A[t]_{\mathfrak{p}}$ dominates A. $\square$

2 Valuative Criteria

Theorem 2.0.1. A morphism of schemes $f : X \to S$

3 Zariski-Riemann Spaces

Definition 3.0.1. Let k be a field and K a finitely generated extension of k and $A \subset K$ a sub k-algebra. Then consider the category $\mathcal{C}(K, A)$ of proper models of K over $\mathrm{Spec}\,(A)$ with birational morphisms. Explicitly, $\mathcal{C}(K, A)$ is the category whose objects are proper k-varieties over $\mathrm{Spec}\,(A)$ together with a map $\mathrm{Spec}\,(K) \to X$ at the generic point inducing an isomorphism $K(X) \xrightarrow{\sim} K$. Furthermore, $\mathrm{Spec}\,(K) \to X \to \mathrm{Spec}\,(A)$ corresponds to the fixed inclusion $A \hookrightarrow K$. Morphisms $f : X \to Y$ are diagrams,

$$\begin{tikzcd} & \mathrm{Spec}\,(K) \arrow[ld] \arrow[rd] & \\ X \arrow[rr, "f"] \arrow[rd] & & Y \arrow[ld] \\ & \mathrm{Spec}\,(A) & \end{tikzcd}$$

meaning that $f(\xi_X) = \xi_Y$ and f induces an A-morphism $K(Y) \to K(X)$ compatible with the isomorphism $K(Y) \xrightarrow{\sim} K$ and $K(X) \xrightarrow{\sim} K$,

$$\begin{tikzcd}
 & K & \\
K(Y) \arrow[ur, "\sim"] \arrow[rr, "f^*"] & & K(X) \arrow[ul, "\sim"'] \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and thus $f^* : K(Y) \to K(X)$ is an isomorphism showing that f is a birational morphism.

Now we define the Zariski Riemann space,

$$\mathrm{ZR}(K, A) = \varprojlim_{X \in \mathcal{C}(K,A)} X$$

where the limit is taken in the category of locally ringed spaces.

Example 3.0.2. Let $\mathrm{trdeg}_k(K) = 1$ then $\mathrm{ZR}(K, k) = C$ where C is the unique complete regular k-curve with function field K since $\mathcal{C}(K, k)$ contains the single object XC and no nontrivial automorphisms since any map $f : \mathcal{C} \to \mathcal{C}$ in $\mathcal{C}(K, k)$ must fix the function field and thus is the identity.

Theorem 3.0.3. There is a natural identification of the points of $\mathrm{ZR}(K, A)$ with the valuation rings of K containing A.

Proof. Consider a point $x \in \mathrm{ZR}(K, A)$ which corresponds to a point $x_i \in X_i$ in each $X_i \in \mathcal{C}(K, A)$ compatible with the maps. Since X_i is integral, $\mathcal{O}_{X_i,x_i}$ is a local ring contained in K under the identification $K(X_i) \xrightarrow{\sim} K$. For a morphism $f : X_i \to X_j$ we get a diagram,

$$\begin{tikzcd}
 & A \arrow[dl, hook'] \arrow[dr, hook] & \\
\mathcal{O}_{X_j,x_j} \arrow[rr] \arrow[d, hook] & & \mathcal{O}_{X_i,x_i} \arrow[d, hook] \\
K(X_j) \arrow[rr, hook] \arrow[dr, "\sim"'] & & K(X_i) \arrow[dl, "\sim"] \\
 & K &
\end{tikzcd}$$

therefore $f^* : \mathcal{O}_{X_j,x_j} \to \mathcal{O}_{X_i,x_i}$ is a local extension of domains containg A and contained in K i.e. $\mathcal{O}_{X_i,x_i}$ dominates $\mathcal{O}_{X_j,x_j}$. Then we may define,

$$B = \varinjlim_{i \in I} \mathcal{O}_{X_i,x_i}$$

$\square$