Math 56: Proofs and Modern Mathematics
Homework 8 Solutions

Naomi Kraushar

December 3, 2021

Problem 1 (Abbott, Exercise 2.2.1)**.** What happens if we reverse the order of quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges to* x *if there exists an* $\varepsilon > 0$ *such that for all* $N \in \mathbb{N}$ *it is true that* $n \geq N$ *implies* $|x_n - x| < \varepsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described by this strange definition?

Solution. An example of a vercongent sequence might be $a_n = 1$, which verconges to 1 (in fact any $\varepsilon > 0$ will work).

An example of a vercongent sequence might be $a_n = (-1)^n$, which runs $-1, 1, -1, 1, \ldots$. This verconges to 1: if we choose $\varepsilon = 3$, then every term satisfies $|a_n - 1| < 3$.

A sequence can verconge to two different values. The sequence $a_n = (-1)^n$ verconges to both 1 and -1: if we choose $\varepsilon = 3$, every term satisfies $|a_n - 1| < 3$ and $|a_n + 1| < 3$. (In fact, if a sequence verconges to some value, it verconges to every value.)

The property of "vercongence" is the same as being bounded. We can see this as follows: if (x_n) is bounded, then $|x_n| < M$ for all n, so x_n verconges to 0. Conversely, if x_n verconges to x, there exists some $\varepsilon > 0$ such that $|x_n - x| < \varepsilon$ for all $n \geq 1$, i.e., for all n. This means that $-\varepsilon < x_n - x < \varepsilon$ for all n, so $x - \varepsilon < x_n < x + \varepsilon$ for all n. Hence $|x_n| < \max\{|x - \varepsilon|, |x + \varepsilon|\}$ for all n, so (x_n) is bounded.

Problem 2 (Abbott, Exercise 2.2.2)**.** Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit. (a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$, (b) $\lim \frac{2n^2}{n^3+3} = 0$, (c) $\lim \frac{\sin(n^2)}{n^{1/3}} = 0$

Solution. (a) We have

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{1}{10+25n}\right| = \frac{1}{10+25n}.$$

Fix arbitrary $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $\frac{1}{10+25N} < \varepsilon$. Then for $n \geq N$, we have $10 + 25n \geq 10 + 25N$, so that $\frac{1}{10+25n} \leq \frac{1}{10+25N}$. Hence for all $n \geq N$ we have

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \frac{1}{10+25n} \leq \frac{1}{10+25N} < \varepsilon.$$

Hence the sequence converges to the proposed limit.

(b) We have

$$\left|\frac{2n^2}{n^3+3}\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n},$$

using the fact that $n^3 + 3 > n^3 > 0$, so $0 < \frac{1}{n^3+3} < \frac{1}{n^3}$. Fix arbitrary $\varepsilon > 0$. Let N be such that $\frac{2}{N} < \varepsilon$. Then for $n \geq N$, we have $\frac{2}{n} \leq \frac{2}{N}$, so for all $n \geq N$, we have

$$\left|\frac{2n^2}{n^3+3}\right| < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

Hence the sequence converges to the proposed limit.

(c) We have

$$\left|\frac{\sin(n^2)}{n^{1/3}}\right| \leq \frac{1}{n^{1/3}},$$

using the fact that $|\sin x| \leq 1$ for all real x. Fix arbitrary $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N^{1/3}} < \varepsilon$. For $n \geq N \geq 1$, we have $n^{1/3} \geq N^{1/3}$, so $\frac{1}{n^{1/3}} \leq \frac{1}{N^{1/3}}$. Hence for all $n \geq N$ we have

$$\left|\frac{\sin(n^2)}{n^{1/3}}\right| \leq \frac{1}{n^{1/3}} \leq \frac{1}{N^{1/3}} < \varepsilon.$$

Hence the sequence converges to the proposed limit.

Problem 3 (Abbott, Exercise 2.2.6)**.** Prove Theorem 2.2.7, uniqueness of limits. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Solution. Fix arbitrary $\varepsilon > 0$. Since $a_n \to a$, there exists some N_1 such that $n \geq N_1$ implies $|a_n - a| < \varepsilon/2$; similarly, since $a_n \to b$, there exists N_2 such that $n \geq N_2$ implies $|a_n - b| \leq \varepsilon/2$. Then, applying these facts and the triangle inequality, we have

$$|a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \varepsilon/2 + \varepsilon/2 = \epsilon.$$

We can do this for all $\varepsilon > 0$, so we have $|a - b| < \varepsilon$ for all $\varepsilon > 0$. Since $|a - b| \geq 0$, this means that we must have $|a - b| = 0$, so $a = b$.

Problem 4 (Abbott, Exercise 2.3.1)**.** Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Solution. (a) If $x_n \to 0$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq \mathbb{N}$ implies $|x_n| < \varepsilon^2$. This means that $|\sqrt{x_n}| < \varepsilon$, so indeed $\sqrt{x_n} \to 0$.

(b) Suppose that $x_n \to x \neq 0$, since we already dealt with the zero case. Note that since $x_n \geq 0$, we must have $x > 0$, since otherwise we would have a gap of at least $|x|$ between every x_n and x. Since $x_n \to x$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq \mathbb{N}$ implies $|x_n - x| < \varepsilon\sqrt{x}$. This gives us

$$\begin{aligned}
|\sqrt{x_n} - \sqrt{x}| &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} && \text{(using } (\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x}) = x_n - x\text{)} \\
&\leq \frac{|x_n - x|}{\sqrt{x}} \\
&\text{(since } \sqrt{x_n} \geq 0\text{, so } \sqrt{x_n} + \sqrt{x} \geq \sqrt{x} \text{ and therefore } \tfrac{1}{\sqrt{x_n}+\sqrt{x}} \leq \tfrac{1}{\sqrt{x}}\text{)} \\
&< \frac{\varepsilon\sqrt{x}}{\sqrt{x}} && \text{(using the limit definition as above)} \\
&= \varepsilon.
\end{aligned}$$

Hence for $x_n \to x \neq 0$, we have $\sqrt{x_n} \to \sqrt{x}$ as required.

Problem 5 (Abbott, Exercise 2.3.3)**.** (Squeeze theorem) Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution. Method 1. Fix arbitrary $\varepsilon > 0$. Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|x_n - l| < \varepsilon$; similarly, since $z_n \to z$, there exists $N_2 \in N$ such that $n \geq N_2$ implies $|z_n - l| < \varepsilon$. Let $N = \max\{N_1, N_2\}$, so that if $n \geq N$, we have $|x_n - l| < \varepsilon$ and $|z_n - l| < \varepsilon$. We can rearrange these inequalities as we have done before: $|x_n - l| < \varepsilon$ is the same as saying $l - \varepsilon < x_n < l + \varepsilon$; similarly, $|z_n - l| < \varepsilon$ is equivalent to $l - \varepsilon < z_n < l + \varepsilon$. Using the "squeezing" inequality, we have, for all $n \geq N$,

$$l - \varepsilon < x_n \leq y_n \leq z_n < l + \varepsilon,$$

so $|y_n - l| < \varepsilon$. Hence $y_n \to l$ as required.

Method 2. I saw this when grading and liked it, so I'm adding it here. Fix arbitrary $\varepsilon > 0$. Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|x_n - l| < \varepsilon/3$; similarly, since $z_n \to z$, there exists $N_2 \in N$ such that $n \geq N_2$ implies $|z_n - l| < \varepsilon/3$. Let $N = \max\{N_1, N_2\}$, so that if $n \geq N$, we have $|x_n - l| < \varepsilon/3$ and

$|z_n - l| < \varepsilon/3$. Then for $n \geq N$, we have

$$\begin{aligned}
|y_n - l| &= |y_n - x_n + x_n - l| \\
&\leq |y_n - x_n| + |x_n - l| && \text{(by the triangle inequality)} \\
&\leq |z_n - x_n| + |x_n - l| && \text{(since } z_n \geq y_n \geq x_n) \\
&= |z_n - l + l - x_n| + |x_n - l| \\
&\leq |z_n - l| + |x_n - l| + |x_n - l| && \text{(by the triangle inequality again)} \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

Hence $y_n \to l$ as well.

Problem 6 (Abbott, Exercise 2.3.8)**.** Let $(x_n) \to x$ and let p be a polynomial.

(a) Show that $p(x_n) \to p(x)$.

(b) Find an example of a function f and a convergent sequence $(x_n) \to x$ such that $f(x_n)$ converges but not to $f(x)$.

Solution. (a) This follows directly from Theorem 2.3.3 (The algebra of limits). Explicitly, write p out as $p(t) = \sum_{i=0}^{m} a_i t^i$. We then have

$$\begin{aligned}
\lim p(x_n) &= \lim \left(\sum_{i=0}^{m} a_i x_n^i \right) && \text{(using our definition of } p) \\
&= \sum_{i=0}^{m} \lim(a_i x_n^i) && \text{(Theorem 2.3.3(a), limit of sum is sum of limits)} \\
&= \sum_{i=0}^{m} a_i \lim(x_n^i) \\
& && \text{(Theorem 2.3.3(b), limit of scalar multiple is scalar multiple of limit)} \\
&= \sum_{i=0}^{m} a_i \lim(x_n)^i && \text{(Theorem 2.3.3(c), limit of product is product of limits)} \\
&= p(x) && \text{(definition of } p \text{ and } \lim x_n = x.)
\end{aligned}$$

Hence $p(x_n) \to p(x)$ as required.

(b) Define the following function:

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Now let x_n be a sequence of rational numbers converging to $\sqrt{2}$ (an example is given in Problem 5). We have $x_n \to \sqrt{2}$, but $f(x_n) = 1$ for all n, so $f(x_n) \to 1 \neq 2$.

(There is a property of functions such that if f has this property and $x_n \to x$, then $f(x_n) \to f(x)$. This property is called *continuity*.)