

*Remark.* Unless otherwise stated, all rings are commutative and unital.

## 1 Definitions

**Definition 1.0.1.** An element  $p \in A$  is prime if  $(p)$  is a prime ideal. Equivalently  $p$  is prime if whenever  $p \mid xy$  either  $p \mid x$  or  $p \mid y$ .

**Definition 1.0.2.** An element  $r \in A$  which is nonzero and not a unit is irreducible if whenever  $r = xy$  either  $x \in A^\times$  or  $y \in A^\times$ .

## 2 Domains

**Definition 2.0.1.** A ring  $A$  is a domain if  $A$  has no zero divisors i.e. if  $ab = 0$  then  $a = 0$  or  $b = 0$ .

**Proposition 2.0.2.** Let  $A$  be a domain then any nonzero prime element is irreducible.

*Proof.* Let  $p \in A$  be a prime. Now suppose that  $p = xy$  for  $x, y \in A$ . Thus,  $p \mid xy$  so (WLOG) we have  $p \mid x$  so  $x = pz$  and thus  $p = pzy$ . However,  $p$  is nonzero and  $A$  is a domain so  $zy = 1$  and thus  $y \in A^\times$  proving that  $p$  is irreducible.  $\square$

## 3 Principal Ideal Domains

**Definition 3.0.1.** A principal ideal domain (PID) is a domain  $A$  such that every ideal is principal.

**Lemma 3.0.2.** If  $A$  is a PID then  $A$  is Noetherian.

*Proof.* Every ideal is principal and thus finitely generated.  $\square$

**Lemma 3.0.3.** Let  $A$  be a PID and  $r \in A$  irreducible then  $(r)$  is maximal and thus  $r$  is prime.

*Proof.* Consider an intermediate ideal  $(r) \subset J \subset A$  then since  $A$  is a PID we have  $J = (a)$  so  $r \in (a)$  and thus  $r = ac$  so either  $a \in A^\times$  in which case  $J = A$  or  $c \in A^\times$  in which case  $J = (r)$  so  $(r)$  is maximal and thus a prime ideal.  $\square$

**Theorem 3.0.4.** Let  $A$  be a PID and not a field then  $\dim A = 1$ .

*Proof.* Any prime ideal  $\mathfrak{p} \subset A$  is principal so  $\mathfrak{p} = (p)$  and  $p$  is prime. Either  $p = 0$  which is prime since  $A$  is a domain or  $p$  is irreducible and so we have shown  $(p)$  is maximal. So every prime ideal is zero or maximal and thus  $\dim A \leq 1$ . If  $\dim A = 0$  then  $(0)$  is maximal so  $A$  is local and any nonzero element is thus invertible so  $A$  is a field.  $\square$

**Theorem 3.0.5** (Kaplansky). Let  $A$  be Noetherian then  $A$  is a principal ideal ring iff every maximal ideal is prime.

**Theorem 3.0.6** (Cohen). A ring  $A$  is Noetherian iff every prime ideal is finitely generated.

**Corollary 3.0.7.** A ring  $A$  is a principal ideal ring iff every prime ideal is principal.

## 4 Unique Factorization Domains

**Definition 4.0.1.** A domain  $A$  is a unique factorization domain (UFD) if every nonzero element has a unique factorization into irreducible elements.

**Definition 4.0.2.** A factorization ring  $A$  is a ring such that every nonzero element has a factorization into irreducible elements.

**Lemma 4.0.3.** If  $A$  is a Noetherian domain then it is a factorization domain.

*Proof.* Take  $a_0 \in A$ . If  $a$  is irreducible, zero, or a unit then we are done. Then we can write,  $a = a_1^{(1)} a_2^{(1)}$  for  $a_1, a_2 \notin A^\times$ . Continuing in this manner we get,

$$(a) \subsetneq (a_1^{(1)}, a_2^{(1)}) \subsetneq (a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, a_4^{(3)}) \subsetneq \cdots$$

(CHECK THIS) This sequence is proper since if  $a = bc$  and  $b \in (a)$  then  $a = arc$  so  $rc = 1$  and thus  $c \in A^\times$  contradicting our construction. However,  $A$  is Noetherian then the sequence must terminate so at some point the factorization must become irreducible.  $\square$

**Theorem 4.0.4.** Let  $A$  be a factorization domain. Then  $A$  is a UFD iff every irreducible is prime.

*Proof.* If  $A$  is a UFD and  $p$  an irreducible. Let  $x, y \in A$  and  $p \mid xy$  then  $p$  is in the factorization of  $xy$  and thus, by uniqueness must be in the factorization of either  $x$  or  $y$  so  $p \mid x$  or  $p \mid y$ .

Conversely, if  $A$  is a factorization domain and every irreducible is prime then given two factorizations of  $x$  each irreducible must, by primality, divide an irreducible in the other factorization so they are equal. (DO THIS BETTER)  $\square$

**Corollary 4.0.5.** If  $A$  is a PID then  $A$  is a UFD.

*Proof.* If  $A$  is a PID then it is Noetherian and thus a factorization domain. Furthermore, every irreducible is prime so  $A$  is a UFD.  $\square$

### 4.1 Height One Prime Ideals

**Proposition 4.1.1.** Let  $A$  be Noetherian. Then any principal prime ideal has height at most one.

*Proof.* Let  $\mathfrak{p} = (p) \subset A$  be a principal prime ideal. Then consider the localization which is  $A_{(p)}$  Noetherian and the unique maximal ideal  $pA_{(p)}$  is principal. Take  $N = \text{nilrad}(A_{(p)})$  then,

$$\dim A_{(p)}/N = \dim A_{(p)} = \text{ht}(\mathfrak{p})$$

but  $A_{(p)}/N$  is a Noetherian domain and the unique maximal ideal  $pA_{(p)}$  is principal so  $A_{(p)}/N$  is a PID and thus  $\dim A_{(p)}/N \leq 1$ .  $\square$

**Proposition 4.1.2.** If  $A$  is a UFD then every prime ideal of height one is principal.

*Proof.* Let  $\mathfrak{p} \subset A$  be a prime ideal with  $\text{ht}(\mathfrak{p}) = 1$ . Take any nonzero element  $x \in \mathfrak{p}$  and consider its factorization into irreducibles. Since  $\mathfrak{p}$  is prime some irreducible factor  $p \mid x$  must be in  $\mathfrak{p}$  so  $(p) \subset \mathfrak{p}$ . Since  $A$  is a UFD all irreducibles are prime so  $(p) \subset \mathfrak{p}$  is prime. However  $\text{ht}(\mathfrak{p}) = 1$  and  $(p) \neq (0)$  so  $(p) = \mathfrak{p}$  and thus  $\mathfrak{p}$  is principal.  $\square$

**Theorem 4.1.3.** Let  $A$  be a Noetherian domain. Then  $A$  is a UFD iff every height one prime ideal is principal.

*Proof.* We showed one direction above. Conversely, suppose every height one prime ideal is principal. Since  $A$  is a Noetherian domain, it suffices to show that each irreducible is prime. Let  $r$  be irreducible and consider a minimal prime  $\mathfrak{p} \supset (r)$ . Then by Krull's Hauptidealsatz,  $\mathfrak{p}$  has height one so by our assumption  $\mathfrak{p} = (p)$  is principal. However,  $(r) \subset (p)$  so  $p \mid r$  but  $r$  is irreducible so we must have  $(r) = (p) = \mathfrak{p}$  and thus  $r$  is prime.  $\square$

**Theorem 4.1.4** (Krull's Hauptidealsatz). Let  $I \subset A$  be an ideal in a Noetherian ring  $A$  with  $n$  generators then any minimal prime ideal  $\mathfrak{p} \supset I$  has height at most  $n$ .

## 5 Simple Modules

**Definition 5.0.1.** A nonzero  $R$ -module is *simple* if it has no nontrivial submodules.

**Proposition 5.0.2.** Let  $R$  be a ring and  $M$  an  $R$ -module. Then the following are equivalent,

- (a)  $M$  is simple
- (b)  $\ell_R(M) = 1$
- (c)  $M = R/\mathfrak{m}$  for some maximal ideal  $\mathfrak{m} \subset R$ .

*Proof.* The first two are equivalent by definition. Clearly if  $\mathfrak{m} \subset R$  is maximal then  $R/\mathfrak{m}$  is simple. Now suppose that  $M$  is simple and take a nonzero  $x \in M$ . Then  $(x) = M$  by simplicity so consider  $I = \ker(R \xrightarrow{x} M) = \text{Ann}_A(x) = \{r \in R \mid rx = 0\}$ . Since  $M = Rx$  we know that  $M \cong R/I$ . However, by the lattice isomorphism theorem, submodules of  $R/I$  correspond to ideals above  $I$  so since  $M$  is simple we must have  $I$  maximal.  $\square$

## 6 Artinian Modules

**Definition 6.0.1.** An  $R$ -module  $M$  is *noetherian/artinian* if it satisfies the ascending/descending chain condition on submodules.

**Theorem 6.0.2.** An  $R$ -module  $M$  has finite length iff it is both noetherian and artinian.

*Proof.* If  $M$  has finite length then clearly it is noetherian and artinian since chains of submodules are bounded in length. Alternatively, simple modules are noetherian and artinian so given a composition series we see that  $M$  is noetherian and artinian by repeated extension. Now, conversely, assume that  $M$  is noetherian and artinian. By the artinian property we can take a minimal nonzero submodule  $M_1 \subset M$ . Then  $M_1$  is simple. Either  $M/M_1$  is simple or we may repeat to get  $M_2 \supset M_1$  and  $M_2/M_1$  is simple. Thus we get an ascending chain  $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$  with  $M_{i+1}/M_i$  simple. Since  $M$  is Noetherian, this must terminate at  $M_n = M$  so we get a finite length composition series showing that  $M$  has finite length.  $\square$

## 7 Artinian Rings

**Definition 7.0.1.** A ring  $A$  is *artinian* if it satisfies the descending chain condition on ideals: given a chain of ideals,

$$I_0 \supset I_1 \supset I_2 \supset \cdots$$

the chain stabilizes  $I_{n+i} = I_n$ .

*Remark.*  $A$  is artinian iff it is artinian as a module over itself.

**Proposition 7.0.2.** An artinian ring has finitely many maximal ideals.

*Proof.* Let  $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \dots$  be a list of maximal ideals. Then consider the chain,

$$\mathfrak{m}_1 \supset \mathfrak{m}_1 \mathfrak{m}_2 \supset \mathfrak{m}_1 \mathfrak{m}_2 \mathfrak{m}_3 \supset \cdots$$

By the artinian condition, we must have  $\mathfrak{m}_1 \cdots \mathfrak{m}_n = \mathfrak{m}_1 \cdots \mathfrak{m}_n \mathfrak{m}_{n+1}$  for some  $n$ . But then by prime avoidance  $\mathfrak{m}_{n+1}$  must be one of  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$  since  $\mathfrak{m}_{n+1} \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n$  so  $\mathfrak{m}_{n+1} \supset \mathfrak{m}_i$  and  $\mathfrak{m}_i$  is maximal.  $\square$

**Proposition 7.0.3.** Let  $A$  be an artinian ring. Then every prime ideal is maximal so  $\dim A = 0$ .

*Proof.* Let  $\mathfrak{p}$  be prime and  $x \notin \mathfrak{p}$ . Consider the chain,

$$(x) \supset (x^2) \supset (x^3) \supset \cdots$$

By the artinian condition  $(x^n) = (x^{n+1})$  for some  $n$  so  $x^n = rx^{n+1}$  for some  $r \in A$ . Thus  $x^n(rx - 1) = 0$ . However,  $x^n \notin \mathfrak{p}$  so  $rx - 1 \in \mathfrak{p}$  and thus  $x \in A/\mathfrak{p}$  is invertible so  $A/\mathfrak{p}$  is a field and thus  $\mathfrak{p}$  is maximal.  $\square$

**Proposition 7.0.4.** Let  $A$  be artinian. Then  $\text{nilrad}(A)$  is a nilpotent ideal.

*Proof.* Let  $I = \text{nilrad}(A)$ . Consider the chain of ideals,

$$I \supset I^2 \supset I^3 \supset \cdots$$

By the artinian condition,  $I^{n+1} = I^n$  for some  $n$ .

Consider  $J = \{x \in A \mid xI^n = 0\}$ . If  $J \neq R$  we can choose  $J' \supsetneq J$  minimal (using the artinian property). Then take  $y \in J'$  so by minimality  $J' = J + (y)$ . Suppose  $J + I(y) = J'$  then, since  $J \subset \text{Jac}(A)$  and  $(y)$  is finitely generated, by Nakayama,  $J' = J + I(y) = J$  which is false so  $J \subset J + I(y) \subsetneq J'$  and thus  $J = J + I(y)$  by minimality so  $I(y) \in J$ . Therefore,  $y \cdot I^{n+1} = 0$  but  $I^{n+1} = I^n$  so  $y \cdot I^n = 0$  and thus  $y \in J$  contradicting our situation so  $J = R$  and thus  $I^n = 0$ .  $\square$

**Proposition 7.0.5.** Every artinian ring is a product of local artinian rings:  $A_{\mathfrak{m}_i} = A/\mathfrak{m}_i^n$ .

*Proof.* Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_r$  be the maximal ideals. Then we know that  $\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r} = 0$  for some integers  $n_1, \dots, n_r \in \mathbb{Z}$ . Therefore, by the Chinese remainder theorem,

$$A = A/(\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r}) = \prod_{i=1}^r A/\mathfrak{m}_i^{n_i}$$

Furthermore,  $A/\mathfrak{m}_i^{n_i}$  is local because  $\mathfrak{m}_i$  is the only maximal ideal above  $\mathfrak{m}_i^{n_i}$ . Furthermore,

$$A_{\mathfrak{m}_i} = (A/\mathfrak{m}_i^{n_i})_{\mathfrak{m}_i} = A/\mathfrak{m}_i^{n_i}$$

since  $A \setminus \mathfrak{m}_i$  is not contained in any maximal ideal of  $A/\mathfrak{m}_i^{n_i}$  and thus is invertible.  $\square$

**Proposition 7.0.6.** A ring  $A$  is artinian iff it has finite length as a module over itself.

*Proof.* If  $A$  has finite length as an  $A$ -module then it satisfies both the ascending and descending chain conditions on  $A$ -submodules i.e. ideals thus  $A$  is both noetherian and artinian. Conversely, let  $A$  be artinian. Since  $A$  is a finite product of local artinian rings we may reduce to the case that  $A$  is local artinian with maximal ideal  $\mathfrak{m}$ . Since  $\text{nilrad}(A) = \mathfrak{m}$  then  $\mathfrak{m}^n = 0$  for some  $n$  so we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

Then  $\mathfrak{m}^i/\mathfrak{m}^{i+1}$  is a  $A/\mathfrak{m}$ -module and,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

which must be finite since  $\mathfrak{m}^i/\mathfrak{m}^{i+1}$  is an artinian module and thus must have finite dimension else there would be a nonterminating descending chains. Thus from the series  $A$  has finite length.  $\square$

**Theorem 7.0.7.** A ring  $A$  is artinian iff  $A$  is noetherian and  $\dim A = 0$ .

*Proof.* If  $A$  is artinian then it has finite length over itself and thus is noetherian. Also every prime is maximal so  $\dim A = 0$ . Conversely, suppose that  $A$  is noetherian and  $\dim A = 0$ . Then  $\text{Spec}(A)$  is a noetherian topological space which has finitely many irreducible components so  $A$  has finitely many minimal primes which are also maximal since  $\dim A = 0$ . Thus  $A$  has finitely many primes all of which are maximal. Since  $\dim A = 0$  we have  $I = \text{Jac}(A) = \text{nilrad}(A)$  so any  $f \in I$  is nilpotent so  $I$  is nilpotent because  $A$  is noetherian so  $I$  is finitely generated. Thus by the Chinese remainder theorem  $A$  is a finite product of local rings so we reduce to the case that  $A$  is local with maximal ideal  $\mathfrak{m}$ . Then we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

but  $\mathfrak{m}^i/\mathfrak{m}^{i+1}$  is a finite  $A/\mathfrak{m}$ -module since  $A$  is noetherian so,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is finite and thus  $\ell_A(A)$  is finite from the series showing that  $A$  is artinian.  $\square$

**Proposition 7.0.8.** Let  $A$  be an artinian ring. Then,

$$\ell_A(A) = \sum_{i=1}^r \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i})$$

*Proof.* We can write,  $A = A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r}$  and thus the formula immediately follows.  $\square$

**Proposition 7.0.9.** Any finite dimensional  $k$ -algebra is artinian.

*Proof.* By dimensionality arguments every descending chain stabilizes.  $\square$

**Proposition 7.0.10.** Let  $A \rightarrow B$  be a local map and  $M$  an  $B$ -module of finite length. Then,

$$\ell_A(M) = \ell_B(M) \cdot [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

and in particular  $\ell_A(M)$  is finite if  $\kappa(\mathfrak{m}_B)$  is a finite extension of  $\kappa(\mathfrak{m}_A)$ .

*Proof.* Consider a composition series,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

Then  $M_i/M_{i-1}$  is a simple  $A$ -module so  $M_i/M_{i-1} \cong B/\mathfrak{m}_B = \kappa(\mathfrak{m}_B)$  since  $B$  is local. Therefore,

$$\ell_A(M) = \sum_{i=1}^n \ell_A(M_i/M_{i-1}) = \sum_{i=1}^n \ell_A(\kappa(\mathfrak{m}_B)) = n \cdot [\kappa(B_{\mathfrak{m}}) : \kappa(A_{\mathfrak{m}})]$$

where  $\ell_A(\kappa(\mathfrak{m}_B)) = \ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$  because  $A \rightarrow B$  is local and,

$$\ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = \dim_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

□

**Corollary 7.0.11.** If  $A$  is a local artinian finite type  $k$ -algebra. Then,

$$\dim_k A = \ell_A(A) \cdot \dim_k (A/\mathfrak{m})$$

in particular  $A$  is a finite  $k$ -module.

*Proof.* Viewing  $A$  as a module over itself we know it has finite length since  $A$  is artinian. Furthermore,  $A/\mathfrak{m}$  is a field finitely generated over  $k$  and thus a finite extension of  $k$  by the Nullstellensatz. Then applying the previous result we conclude. □

**Corollary 7.0.12.** Let  $A$  be an artinian finite type  $k$ -algebra. Then,

$$\dim_k A = \sum_{i=1}^r \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}) \cdot \dim_k (A/\mathfrak{m}_i)$$

*Proof.* Since  $A$  is artinian we can write,

$$A = \prod_{i=1}^r A_{\mathfrak{m}_i}$$

where  $A_{\mathfrak{m}_i}$  are the local artinian factors associated to the finitely many prime ideals  $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ . The result follows from above by additivity of the dimensions. □

## 8 Cohen-Macaulay Rings

### 8.1 Dimension

**Proposition 8.1.1.** Let  $(A, \mathfrak{m})$  be a Noetherian local ring and  $f \in \mathfrak{m}$ . Then,

$$\dim A/(f) \geq \dim A - 1$$

with equality iff  $f$  is a nonzero divisor.

*Proof.* <https://math.stackexchange.com/questions/2085779/the-dimension-modulo-a-principal-ideal-in-a-noetherian-local-ring> □

### 8.2 Depth

### 8.3 Properties

**Proposition 8.3.1.** Let  $(A, \mathfrak{m})$  be a Noetherian local ring and  $f \in \mathfrak{m}$  a nonzero divisor. Then  $A$  is Cohen-Macaulay iff  $A/(f)$  is Cohen-Macaulay.

*Proof.* We have  $\text{depth}(A/(f)) = \text{depth}(A) - 1$  and  $\dim A/(f) = \dim A - 1$ . □