1 Motivation

Definition 1.0.1. A *quiver* is a finite oriented graph.

Definition 1.0.2. A *quiver representation* M is a vectorspace M_i for each $i \in Q_0$ and a map $M_a : M_{s(a)} \to M_{t(a)}$ for $a \in Q_1$. A morphism $\varphi : M \to N$ of quiver representations is a set of morphisms $\varphi_i : M_i \to N_i$ commuting with the M_a and N_a.

Definition 1.0.3. A quiver Q is acyclic if there do not exist cycles.

Definition 1.0.4. Let $\mathcal{M}_{d,S}^{\theta\text{-ss}}$ be the moduli stack of θ-ss representations of Q of dimension $d \in \mathbb{N}^{Q_0}$ (meaning $\dim M_i = d_i$).

In what follows, let S be a noetherian scheme.

Theorem 1.0.5. If Q is acyclic there exists a projective adequate moduli space $\mathcal{M}_{d,S}^{\theta\text{-ss}} \to M_{d,S}^{\theta\text{-ss}}$.

Theorem 1.0.6. Let Q be any quiver and $\theta = -\langle -, \beta \rangle$ then there exists a separated adequate moduli space and a morphism $M_{d,S}^{\theta\text{-ss}} \to M_{d,S}$ which is proper.

Theorem 1.0.7. Let k be a field, Q any quiver. For a natural construction $\mathcal{L}_\theta \in \operatorname{Pic}\left(\mathcal{M}_{d,S}^{\theta\text{-ss}}\right)$. There exists an explicit bound m_0 such that $\mathcal{L}_\theta^{\otimes m}$ is globally generated for $m \geq m_0$.

2 Moduli Stacks

A Stack $\mathscr{X}$ is a pseudofunctor $\mathscr{X} : \mathbf{Sch}_S^{\text{ét}} \to \text{Gpd}$ with a descent condition.

Definition 2.0.1. An stack is *algebraic* if,

(a) $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable by algebraic spaces

(b) there exists a smooth surjective morphism from a scheme $U \to \mathscr{X}$.

Definition 2.0.2. We define,

$$\mathcal{M}_{d,S}(T) = \{\mathscr{F}_i \text{ loc free on } T \text{ of rank } d_i \text{ with } \varphi_a : \mathscr{F}_{s(a)} \to \mathscr{F}_{t(a)}\}$$

Proposition 2.0.3. $\mathcal{M}_{d,S} \cong [R_d / G_d]$ where,

$$R_d = \prod_{a \in Q_1} \mathbb{A}_S^{d_{t(a)} d_{s(a)}} \quad G_d = \prod_{i \in Q_0} \mathrm{GL}_{d_i}$$

Therefore we have a presentation $\mathbb{A}_S^N \twoheadrightarrow \mathcal{M}_{d,S}$.

Corollary 2.0.4. $\mathcal{M}_{d,S}$ is an algebraic stack smooth and finite type over S'.

Proposition 2.0.5. $\mathcal{M}_{d,S}$ has affine diagonal.

Definition 2.0.6. A *stability function* $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is a group homomorphism.

Remark. Given a stability function we will write $\theta(M) := \theta(\dim M)$ where $\dim M \in \mathbb{Z}^{Q_0}$ is the vector of dimensions.

Definition 2.0.7. $M \in \mathcal{M}_{d,S}$ is *θ-semistable* if $\theta(M) = 0$ and $\forall M' \subset M$ we have $\theta(M') \leq 0$.

Proposition 2.0.8. $\mathcal{M}_{d,S}^{\theta\text{-ss}} \hookrightarrow \mathcal{M}_{d,S}$ is an open subfunctor.

Example 2.0.9. If $\theta = 0$ then $\mathcal{M}_{d,S}^{\theta\text{-ss}} = \mathcal{M}_{d,S}$.

3 Characterizing Semistability

Over $S = \mathrm{Spec}\,(k)$ and $k = \bar{k}$.

Definition 3.0.1. $\mathcal{M}_{d,k}$ has the universal representation $\mathcal{E} = (\mathcal{E}_i)$. The *determinantal line bundle* is,

$$\mathcal{L}_\theta = \bigotimes_{i \in Q_0} (\det \mathcal{E}_i)^{\otimes -\theta_i}$$

where $\theta_i = \theta(e_i)$.

Let Q be a quiver and M, N be two representations. Every M has a 2-step projective resolution and thus,

$$\langle M, N \rangle := \chi(QMN)$$

is the Euler pairing. This only depends on the dimension vectors $\dim M$ and $\dim N$.

Assume that $\theta = -\langle -, \beta \rangle$ for $\beta \in \mathbb{N}^{Q_0}$ and $\theta(d) = 0$ so $d\beta = 0$.

For $V \in \mathcal{M}_\beta$ then,

$$Q\mathcal{E}V \otimes \mathcal{O}_{\mathcal{M}_d} = [K^0 \xrightarrow{\mathrm{d}} K^1]$$

We can take $\det d : \det K^0 \to \det K^1$ whcih corresponds to a section $\sigma_V \in H^0(\det K^{0^\vee} \otimes \det K^1) = H^0(\det \mathcal{E}V \otimes \mathcal{O}) = \mathcal{L}_\theta$

Proposition 3.0.2. $M \in \mathcal{M}_d$ is θ-ss for $\theta = --\beta$ iff there is $m > 0$ and $V \in \mathcal{M}_{m\beta}$ s,t, $QMV = 0$.

Proof. $QMV = 0$ iff d is an isomorphism at the point M iff $\det \mathrm{d}$ is nonzero. This this is equivalent to σ_V being nonzero. Thus we see that $\mathcal{L}_t heta^{\otimes m}$ is globally generated on $\mathcal{M}_d^{\theta\text{-ss}}$. ☐

Remark. $\mathcal{L}_{m\theta} = \mathcal{L}_\theta^{\otimes m}$.

4 Good and adequate Moduli Space

For $\theta = 0$ we have $\mathcal{M}_d^{\theta\text{-ss}} = \mathcal{M}_d$.

Proposition 4.0.1. Q is acyclic implies that adequate moduli of $\mathcal{M}_{d,S} \to S$ is isomorphic to S.

Definition 4.0.2. Let $\mathscr{X}$ be quasi-separated. A *good moduli space* of $\mathscr{X}$ is $f : \mathscr{X} \to X$ (qcqs) to be an algebraic space X s.t.

(a) $f_* \mathcal{O}_{\mathscr{X}} = \mathcal{O}_X$

(b) f is "cohomologically affine" meaning f_* is exact on $\mathfrak{QCoh}\,(\mathscr{X})$.

Example 4.0.3. If $G \circlearrowright X$, say $X \to Y$ is a good quotient. Then $[X/G] \to Y$ is a good moduli space.

Definition 4.0.4. $f : \mathscr{X} \to X$ is an adequate moduli space if,

(a) $f_* \mathcal{O}_{\mathscr{X}} = \mathcal{O}_X$

(b) f is "adequately affine" meaning for any $\mathcal{A} \twoheadrightarrow \mathcal{B}$ of quasi-coherent algebras, étale-locally $\sqrt{f_* \mathcal{A}} \twoheadrightarrow f_*$. (??)

Theorem 4.0.5 (Alper)**.** Let $f : \mathscr{X} \to X$ be an adequate moduli space

(a) f is surjective, universall closed

(b) over $k = \bar{k}$ the map f induces a bijection on closed points

(c) if X is a scheme implies that it is initial in $\mathbf{Sch}_S$ under $\mathscr{X}$

(d) base change of an adequate moduli space is homeom. to the adequate moduli space of the base change.