Physics GR6037 Quantum Mechanics I
Assignment # 1

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Problem 1.

$$\psi(x,y) = A \sum_{j=-N/2}^{N/2} e^{i\sqrt{x^2+(y-jd)^2}\frac{p}{\hbar}}$$

(a). $\sqrt{x^2+(y-jd)^2} = \sqrt{x^2+y^2-2yjd+j^2d^2} = r\sqrt{1-(2yjd-j^2d^2)/r^2} = r - jd\sin\theta + O(\frac{1}{r}) \approx r - jd\sin\theta$. Thus,

$$\psi(x,y) = A \sum_{j=-N/2}^{N/2} e^{i(r-jd\sin\theta)\frac{p}{\hbar}}$$

(b).

$$\psi(x,y) = Ae^{ir\frac{p}{\hbar}} \sum_{j=-N/2}^{N/2} e^{-ijd\sin\theta\frac{p}{\hbar}} = Ae^{i(r+\frac{N}{2}d\sin\theta)\frac{p}{\hbar}} \frac{1-e^{-i(N+1)d\sin\theta\frac{p}{\hbar}}}{1-e^{-id\sin\theta\frac{p}{\hbar}}}$$

Therefore, taking the norm square, $|\psi(x,y)|^2$

$$= |A|^2 \left| \frac{1-e^{-i(N+1)d\sin\theta\frac{p}{\hbar}}}{1-e^{-id\sin\theta\frac{p}{\hbar}}} \right|^2 = |A|^2 \left| \frac{e^{i\frac{N+1}{2}d\sin\theta\frac{p}{\hbar}} - e^{-i\frac{N+1}{2}d\sin\theta\frac{p}{\hbar}}}{e^{i\frac{1}{2}d\sin\theta\frac{p}{\hbar}} - e^{-i\frac{1}{2}d\sin\theta\frac{p}{\hbar}}} \right|^2 = |A|^2 \frac{\sin^2\left(\frac{N+1}{2}d\sin\theta\frac{p}{\hbar}\right)}{\sin^2\left(\frac{1}{2}d\sin\theta\frac{p}{\hbar}\right)}$$

Take $D \gg d$ then $r \approx D$ and $\sin\theta = \frac{y}{D}$ so

$$|\psi(x,y)|^2 = |A|^2 \frac{\sin^2\left(\frac{y}{D}\frac{(N+1)pd}{2\hbar}\right)}{\sin^2\left(\frac{y}{D}\frac{pd}{2\hbar}\right)}$$

(c). Let $f(y)$ be the probability amplitude of the incident wave at the screen. Also, $f(y) = 0$ for $|y| > d$ then,

$$|\psi(x,y)|^2 = A\int_{-d}^{d} e^{i\sqrt{x^2+(y-y')^2}\frac{p}{\hbar}} f(y')\,\mathrm{d}y' = A\int_{-\infty}^{\infty} e^{i\sqrt{x^2+(y-y')^2}\frac{p}{\hbar}} f(y')\,\mathrm{d}y'$$

because $f(y) = 0$ on the added domain. As before, we approximate: $\sqrt{x^2+(y-y')^2} \approx r - y'\sin\theta$ so $|\psi(x,y)|^2 =$

$$\left|A\int_{-\infty}^{\infty} e^{i\frac{p}{\hbar}(r-y'\sin\theta)}f(y')\,\mathrm{d}y'\right|^2 = \left|Ae^{i\frac{p}{\hbar}r}\right|\left|\int_{-\infty}^{\infty} e^{-i\frac{p}{\hbar}y'\sin\theta}f(y')\,\mathrm{d}y'\right|^2 = |A|^2\left|\tilde{f}\left(\frac{p\sin\theta}{\hbar}\right)\right|^2$$

Take $x = D \gg d$ then $r = \sqrt{D^2+y^2}$ and $\sin\theta = \frac{y}{r}$ so $|\psi(x,y)|^2 = |A|^2\left|\tilde{f}\left(\frac{p_y}{\hbar}\right)\right|^2$ where $p_y = p\sin\theta = p\frac{y}{\sqrt{D^2+y^2}} \approx p\frac{y}{D}$.

Problem 2.

(a). Let $\mathcal{H}$ be finite dimensional and $O\colon \mathcal{H}\to\mathcal{H}$ be linear.

Claim: $\ker O^\dagger = (\mathrm{Im}\, O)^\perp$:

Proof: if $v \in \ker O^\dagger$ then $O^\dagger v = 0$ so $\left\langle u, O^\dagger v\right\rangle = 0$ for any $u \in \mathcal{H}$. Thus, $\langle Ou, v\rangle = 0$ so $v \in (\mathrm{Im}\, O)^\perp$. Likewise, if $v \in (\mathrm{Im}\, O)^\perp$ then for any $u \in \mathcal{H}$, $\langle Ou, v\rangle = 0$ so $\left\langle u, O^\dagger v\right\rangle = 0$. Now take $u = O^\dagger v$ then $\left\langle O^\dagger v, O^\dagger v\right\rangle = 0$ so $O^\dagger v = 0$ thus $v \in \ker O^\dagger$.

Because $\mathcal{H}$ is finite dimensional,

$$\dim\mathcal{H} = \dim\mathrm{Im}\, O + \dim\,(\mathrm{Im}\, O)^\perp = \dim\mathrm{Im}\, O + \dim\ker O^\dagger$$

By rank-nulty,

$$\dim\mathcal{H} = \dim\mathrm{Im}\, O + \dim\ker O$$

thus,

$$\dim\mathrm{Im}\, O + \dim\ker O = \dim\mathrm{Im}\, O + \dim\ker O^\dagger$$

Therefore,

$$\dim\ker O = \dim\ker O^\dagger$$

(b). Let H be the hilbert subspace of $L^2(\mathbb{R})$ spaned by the solutions to a one-dimensional quantum harmonic oscillator. Then any state is a sum, $\langle\psi| = \sum_{n=0}^{\infty} c_n\langle n|$. Now $\hat{a}\langle n| = \sqrt{n}\langle n-1| \neq 0$ for $n > 0$ and $\hat{a}\langle 0| = 0$ so $\dim\ker\hat{a} = 1$.

However, $\hat{a}^\dagger\langle n| = \sqrt{n+1}\langle n+1|$ which is always non-zero. Thus, $\dim\ker\hat{a}^\dagger = 0$.

(c). Claim: For any operator $\hat{U}$, $\ker U^\dagger U = \ker U$.

Proof: Trivially, $\ker U \subset \ker U^\dagger U$. Thus, consider $v \in \ker U^\dagger U$ the $U^\dagger U v = 0$ so $\left\langle v, U^\dagger U v\right\rangle = 0$. However, $\left\langle v, U^\dagger U v\right\rangle = \langle Uv, Uv\rangle = 0$. Thus, $Uv = 0$ so $v \in \ker u$. Thus, $\ker U^\dagger U \subset \ker U$.

In particular, $\ker O_\alpha^\dagger O_\alpha = \ker O_\alpha$ and $\ker O_\alpha O_\alpha^\dagger = \ker O^\dagger$

Claim: the λ eigenspaces V_λ of $O_\alpha O_\alpha^\dagger$ and $O_\alpha^\dagger O_\alpha$ are isomorphic for $\lambda \neq 0$.

Proof: I claim that $\frac{1}{\sqrt{\lambda}}O_\alpha : V_\lambda^{O_\alpha^\dagger O_\alpha} \to V_\lambda^{O_\alpha O_\alpha^\dagger}$ is an isomorphism. First, if $v \in V_\lambda^{O_\alpha^\dagger O_\alpha}$ then

$O_\alpha^\dagger O_\alpha v = \lambda v$ so $(O_\alpha O_\alpha^\dagger) O_\alpha v = \lambda(O_\alpha v)$ thus $O_\alpha v \in V_\lambda^{O_\alpha O_\alpha^\dagger}$ so the transformation is well defined. Second, if $v \in V_\lambda^{O_\alpha^\dagger O_\alpha}$ then $O_\alpha^\dagger O_\alpha v = \lambda v$ therefore, $(\frac{1}{\sqrt{\lambda}} O_\alpha^\dagger)(\frac{1}{\sqrt{\lambda}} O_\alpha) v = v$. Similarly, $v \in V_\lambda^{O_\alpha O_\alpha^\dagger}$ then $O_\alpha O_\alpha^\dagger v = \lambda v$ therefore, $(\frac{1}{\sqrt{\lambda}} O_\alpha)(\frac{1}{\sqrt{\lambda}} O_\alpha^\dagger) v = v$ so we have constructed an inverse operator of $\frac{1}{\sqrt{\lambda}} O_\alpha$, namely, $\frac{1}{\sqrt{\lambda}} O_\alpha^\dagger$ thus the transformation is a bijection. Since O_α is linear, the transformation is an isomorphism.

Because the operators are continuous functions of α, the eigenvalues $\lambda(\alpha)$ are also continuous functions. Suppose that at two values of α one of the product operators have different kernels. Without loss of generality, take $\ker O_\alpha O_\alpha^\dagger \neq \ker O_{\alpha'} O_{\alpha'}^\dagger$. Then take some $\lambda(\alpha)$ to have a root at α'. For $\lambda(\alpha) \neq 0$ there is a one-to-one correspondance between the $\lambda(\alpha)$ eigenspaces of $O_\alpha O_\alpha^\dagger$ and $O_\alpha^\dagger O_\alpha$. In particular, $\lambda(\alpha)^{O_\alpha O_\alpha^\dagger} = \lambda(\alpha)^{O_\alpha^\dagger O_\alpha}$ when the eigenvalues are non-zero. However, these functions are continuous and since they are equal arbitrarily close to α' they must both be zero for identical values of α. Therefore, corresponding eigenvalues and eigenspaces are isomorphic for all α even when $\lambda(\alpha) = 0$. Whenever $\lambda(\alpha) = 0$, an eigenvector of $O_\alpha O_\alpha^\dagger$ adds one to the dimension of $\ker O_\alpha O_\alpha^\dagger$ because $O_\alpha O_\alpha^\dagger$ is self-adjoint and therefore the distinct eigenvalues correspond to orthogonal and thus linearly independent spans. Orthognality of eigenvectors with distinct eigenvalues is preserved even when both eigenvalues go to zero because the vectors are continuous and orthogonal so they cannot jump from being orthogonal to dependent. $\dim\ker O_\alpha^\dagger O_\alpha - \dim\ker O_\alpha O_\alpha^\dagger$ is constant. Then $\dim\ker O_\alpha^\dagger O_\alpha = \dim\ker O_\alpha$ and $\dim\ker O_\alpha O_\alpha^\dagger = \dim\ker O_\alpha^\dagger$ therefore $\mathrm{ind}(O_\alpha) = \dim\ker O_\alpha - \dim\ker O_\alpha^\dagger$ is contant.