1 Week 2: The Swan Conductor and the Grothendieck-Odd-Shafarevich Formula

Review of ramification: Let $\mathcal{O}_K$ be the henselian DVR of characeristic $p > 0$. Let $K = \text{Frac}(\mathcal{O}_K)$ and $\kappa = \mathcal{O}_K/\mathfrak{m}$. We get a tower,

$$K^{\text{sep}} \supset K^{\text{tame}} \supset K^{\text{ur}} \supset K$$

Then the Galois groups are,

$$\text{Gal}(K^{\text{ur}}/K) = \text{Gal}(\kappa^{\text{sep}}/\kappa) \quad \text{Gal}\left(K^{\text{tame}}/K\right) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$$

Then the inertia group is $I = \text{Gal}(K^{\text{sep}}/K^{\text{ur}})$ and the wild inertia group is $P = \text{Gal}(K^{\text{sep}}/K^{\text{tame}})$. From now on, we only care about ramification so assume that $K = K^{\text{ur}}$.

Let L/K be finite Galois with Galois group G.

Definition 1.0.1. The ramification filtration G_i is a decreasing filtration given by,

$$G_i = \{\sigma \in G \mid \sigma(\varpi_L) - \varpi_L \in (\varpi_L)^{i+1}\}$$

Then $G_0 = I$ and G_0/G_1 is the tame inertia. Then G_1 is the wild inertia.

Remark. Let X be a geometrically integral curve over κ and $K = K(X)$. Let $j : U \hookrightarrow X$ be a nonempty open subset. let $\mathbb{F}$ be a finite field of characteristic $\ell \neq p$. Then let $\mathscr{F}$ be a $\mathbb{F}$-local system on U corresponding to a Galois representation $\rho : \text{Gal}(K^{\text{sep}}/K) \to \mathscr{F}_{\bar{\eta}}$. Then $I_x \circlearrowright \mathscr{F}_{\bar{\eta}}$ for each closed point $x \in X$. Then,

(a) if $x \in U$ then $I_x \circlearrowright \mathscr{F}_{\bar{\eta}}$ is trivial

(b) if $x \notin U$ then $I_x \circlearrowright \mathscr{F}_{\bar{\eta}}$ is interesting and we get a Swan conductor $\text{Sw}_x(\mathscr{F})$.

Definition 1.0.2. The Swan conductor $\text{Sw}_x(\mathscr{F})$ is defined as follows. Since $\mathscr{F}$ is a local system over a finite field $V = \mathscr{F}_{\bar{\eta}}$ is finite. Hence the action factors through a finite quotient L/K. Consider the ramification filtration G_i of $G = \text{Gal}(L/K)$. Then,

$$\text{Sw}_x(\mathscr{F}) = \sum_{i \ge 1} \frac{\dim(V/V^{G_i})}{[G_0 : G_i]}$$

which is actually a well-defined integer.

Proposition 1.0.3. The following hold about the Swan conductor,

(a) $\text{Sw}_x(\mathscr{F}) = 0 \iff V$ is tamely ramified at x menaing $P_x \circlearrowright V$ trivially

(b) For $\mathscr{F}$ tamely ramified at x and some other local system $\mathscr{G}$ we have,

$$\text{Sw}_x(\mathscr{F} \otimes \mathscr{G}) = (\text{rank}\,\mathscr{F}) \cdot \text{Sw}_x(\mathscr{G})$$

Proposition 1.0.4. Let $\mathscr{F}$ be a free lisse $\mathcal{O}_E$-local system for some finite $E/\mathbb{Q}_\ell$. Define $\text{Sw}_x(\mathscr{F}) := \text{Sw}_x(\mathscr{F}/\varpi_E\mathscr{F})$ where $\mathscr{F}/\varpi_E\mathscr{F}$ is a κ_E-local system.

Example 1.0.5. (a) Kummer sheaf $\mathcal{L}(\chi)$. For $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a multiplicative character. Consider the Kummer cover $\mathbb{G}_m \to \mathbb{G}_m$ via $x \mapsto x^{q-1}$ with Galois groip $\mathbb{F}_q^\times$. Define $\mathcal{L}(\chi)$ to be the local system on $\mathbb{G}_m$ corresponding to $\pi_1(\mathbb{G}_m) \to \mathbb{F}_q^\times \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$. Then $\mathbb{G}_m \subset \mathbb{P}^1$ has boundary consisting of $\{0, \infty\}$ and the two Swan conductors are,

$$\mathrm{Sw}_0(\mathcal{L}(\chi)) = \mathrm{Sw}_\infty(\mathcal{L}(\chi)) = 0$$

since the group has order coprime to p and thus has no wild ramification.

(b) Artin-Schrier sheaf $\mathcal{L}(\psi)$ for a nontrivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. We have the Artin-Schreier cover $\mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^q - x$ with Galois group $\mathbb{F}_q$. Define $\mathcal{L}(\psi)$ to be the local system associated to $\pi_1(\mathbb{A}^1) \to \mathbb{F}_q^\times \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times$. Then for $\mathbb{A}^1 \subset \mathbb{P}^1$ and we have,

$$\mathrm{Sw}_\infty(\mathcal{L}(\psi)) = 1$$

To see this, consider the behavior at infinity. We have the equation $y^q - y = x$ let $y = u^{-1}$ and $x = v^{-1}$ so,

$$v = \frac{u^q}{1 - u^{q-1}}$$

and the automorphisms act via $y \mapsto y + a$ so

$$u \mapsto \frac{u}{1 + au} = u - au^2 + a^2u^3 + \cdots$$

which visibly lies in G_1 and not G_2 (for $a \neq 0$) so the entire Galois group is wild inertia of level 1 (besides the trivial element of course). Therefore,

$$\mathrm{Sw}_\infty(\mathcal{L}(\psi)) = \sum_{i \ge 1} \frac{\dim(V/V^{G_i})}{[G : G_i]} = \frac{\dim(V/V^{G_1})}{[G : G_1]} = \dim V = 1$$

1.1 The Trace Formula

Theorem 1.1.1 (Grothendieck-Ogg-Shafarevich)**.** Let $\mathscr{F}$ be a $\overline{\mathbb{Q}}_\ell$-local system on $U \subset X$. Then,

$$\chi_c(U, \mathscr{F}) = \chi_c(U, \overline{\mathbb{Q}}_\ell) \cdot (\operatorname{rank} \mathscr{F}) - \sum_{x \in X \setminus U} \mathrm{Sw}_x(\mathscr{F})$$

Remark. Also we know that $\chi(U, \mathscr{F}) = \chi_c(U, \mathscr{F})$.

1.2 Applications

Definition 1.2.1. Let $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ and $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. Define the *Gauss sum*,

$$G(\chi, \psi) = \sum_{a \in \mathbb{F}_q^\times} \chi(a)\psi(a)$$

Remark. Deligne noticed that,

$$G(\chi, \psi) = \sum_{a \in \mathbb{G}_m(\mathbb{F}_q)} \operatorname{tr}(\mathrm{Frob}_a \mid \mathcal{L}(\chi)_a \otimes \mathcal{L}(\psi)_a)$$

By the Grothendieck-Lefschetz fixed-point formula,

$$G(\chi, \psi) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}\left(\mathrm{Frob} \mid H^i_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi))\right)$$

Notice that,

$$H^0_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = 0 \quad H^2_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = H^0(\mathbb{G}_m, \mathcal{L}(\chi^{-1}) \otimes \mathcal{L}(\psi^{-1}))^\vee = 0$$

since there are no global sections for nontrivial characters. Then we apply the GOS formula,

$$\chi_c(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = -\mathrm{Sw}_0(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) - \mathrm{Sw}_\infty(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi))$$

but both are tamely ramified at 0 and $\mathcal{L}(\chi)$ is tamely ramified at infinity and thus,

$$\chi_c(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = -1$$

and thus,

$$\dim H^1_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = 1$$

Therefore, we see that,

$$G(\chi, \psi) = -\operatorname{tr}\left(\mathrm{Frob} \mid H^1_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi))\right)$$

and is a 1-dimensional space so there is a single eigenvalue. By Weil II we see that this eigenvalue has absolute value $q^{\frac{1}{2}}$ and thus,

$$|G(\chi, \psi)| = q^{\frac{1}{2}}$$

Remark.

$$|G(\chi, \psi)|^2 = \sum_{a,b} \chi(a)\overline{\chi}(b)\psi(a)\overline{\psi}(b) = \sum_{a,b} \chi(a-b)\psi(a)\overline{\psi}(b)$$

1.3 Kloosterman Sums

Fix $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. For $n \geq 1$ and $a \in \mathbb{F}_q$ define the Kloosterman sum,

$$K_{n,a} = \sum_{x_1 \cdots x_n = a} \psi(x_1 + \cdots + x_n)$$

Trivial bound,

$$|K_{n,a}| \leq q^{n-1}$$

Deligne gives,

$$|K_{n,a}| \leq n q^{\frac{n-1}{2}}$$

Write the Kloosterman sums as sums of traces of Frobenius. Let,

$$V_a^{n-1} = \{x_1 \cdots x_n = a\} \subset \mathbb{A}^n$$

which is smooth for $a \neq 0$. Consider the maps $\sigma : \mathbb{A}^n \to \mathbb{A}$ and $\pi : \mathbb{A}^n \to \mathbb{A}$ taking the sum and product respectively. We use the sheaves $\mathscr{F} = \iota^* \sigma^* \mathcal{L}(\psi)$ where $\iota : V_a^{n-1} \subset \mathbb{A}^n$ is the inclusion. Then $\mathrm{Frob}_x \circlearrowright \mathscr{F}_{\bar{x}}$ via $\psi(x_1 + \cdots + x_n)$. Therefore, by the Grothendieck trace formula,

$$K_{n,a} = \sum_{i=0}^{2n-2} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_x \mid H^i_c(V_a^{n-1}, \mathscr{F})\right)$$

Then Deligne showed the following.

Theorem 1.3.1 (Deligne). (a) $H_c^i(V_a^{n-1}, \mathscr{F}) = 0$ for $i \neq n-1$

(b) $\dim H_c^i(V_a^{n-1}, \mathscr{F}) = n$.

Corollary 1.3.2. Then by Weil II we see that $|K_{n,a}| \leq nq^{\frac{n-1}{2}}$.

Theorem 1.3.3 (Deligne). (a) the Kloosterman sheaf $\mathrm{Kl}_n := R^{n-1}\pi_! \mathscr{F})$ satisfies $\mathrm{Kl}_n|_{\mathbb{G}_m}$ is lisse of rank n

(b) direct image of Kl_n on $\mathbb{P}^1$ has stalk 0 at ∞

(c) $\dim (\mathrm{Kl}_n)_0 = 1$

(d) $\mathrm{Sw}_0(\mathrm{Kl}_n|_{\mathbb{G}_m}) = 0$ has unipotent monodrom with a single Jordan block

(e) $\mathrm{Sw}_0(\mathrm{Kl}_n|_{\mathbb{G}_m}) = 1$.

2 Connected Affine Varieties over $\mathbb{F}_p$ are $K(\pi, 1)$

Let X be a nice topological space with $x \in X$. Get a category of pointed covering spaces:

$$(X', x') \to (X, x)$$

with $(\widetilde{X}, \widetilde{x})$ universal cover. Get a map,

$$\rho^* : \pi_1(X, x) - \text{Sets} \to \mathfrak{Sh}(X)$$

This induces a bunch of maps,

$$\rho^q : H^q(\pi_1(X, x), M) \to H^1(X, \rho^* M)$$

for any $\pi_1(X, x)$-module M.

Proposition 2.0.1. Let X be connected. The following are equivalent (and give the definitino of X being a $K(\pi, 1)$ space),

(a) $\pi_i(X) = 0$ for all $i > 1$

(b) $\widetilde{X}$ is weakly contractible

(c) all the maps ρ^q are isomorphisms

(d) for all locally constant sheaves F and $\omega \in H^q(X, F)$ with $q > 0$ ther eis a covering space $f : X' \to X$ such that $f^*\omega = 0$.

Here let X be qcqs and has finitely many connected components.

Definition 2.0.2. X is a $K(\pi, 1)$ if the map,

$$H^q(\pi_1(X, \bar{x}), F_{\bar{x}}) \xrightarrow{\sim} H^q(X, F)$$

is an isomorphism for all F lcc abelian sheaves.

Remark. Historically, Artin proved the comparison theorem for étale cohomology and singular cohomology over $\mathbb{C}$ using this stuff because Artin neighborhoods are $K(\pi, 1)$.

Theorem 2.0.3. Every affine connected variety over $\mathbb{F}_p$ is a $K(\pi, 1)$.

Proof. the steps are:

(a) Establish "Bertini for lcc sheaves"

(b) Show $\mathbb{A}_k^n$ is $K(\pi, 1)$

(c) Etale things over $\mathbb{A}_k^n$ are $K(\pi, 1)$

(d) Henselian pairs and $K(\pi, 1)$

(e) General case.

$\square$

Corollary 2.0.4. $\pi_1(\mathbb{A}_k^n) \ncong \pi_1(\mathbb{A}_k^m)$.

Proof. Both are $K(\pi, 1)$ and thus the cohomological dimension of $\pi_1(\mathbb{A}_k^n)$ is the max q s.t. $H^q(\mathbb{A}_k^n, F) \neq 0$. By artin vanishing this is at most n. However, $H^1(\mathbb{A}_k^1, \mathbb{F}_p) \neq 0$ so by Kunneth get $H^n(\mathbb{A}_k^n, F) \neq 0$ and thus the cohomological dimension is n so for $n \neq m$ the groups have nonequal cohomological dimensions. $\square$

Proposition 2.0.5. Let X be a normal k-scheme. The following are equivalent,

(a) X is a $K(\pi, 1)$

(b) $\pi_i^{\text{ét}}(X) = 0$ for $i > 0$

(c) $\widetilde{X}$ is weakly contractible whatever this means

(d) for every lcc sheaf F and $\omega \in H^q(X, F)$ there is a finite étale cover $f : X' \to X$ with $f^*\omega = 0$.

Proposition 2.0.6. Let $f : Y \to X$ be a finite étale cover. Then Y is $K(\pi, 1)$ iff X is $K(\pi, 1)$.

2.1 The Proof

2.2 Bertini Theorem

Proposition 2.2.1. Let K be an infinite extension of $\mathbb{F}_p$, and F an lcc $\mathbb{F}_\ell$-sheaf on $\mathbb{A}_k^{n+1}$. Let $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^n$ be the projection onto the first n coordinates. Then there exists an automorphism ρ of $\mathbb{A}^{n+1}$ st ρ^*F is well-aligned.

Definition 2.2.2. Let $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^n$ then F is *well-aligned* if $R^i\pi_*\mathscr{F}$ are locally constant and formation commutes with base change.

2.3 The Case of Affine Space

Assume that k is infinite (why is this allowed?). Let F be lcc abelain sheaf on $\mathbb{A}_k^{n+1}$. Then WTS for $\zeta \in H^q(\mathbb{A}_k^{n+1}, F)$ there exsits a finite étale surjection such that $f^*\zeta = 0$. For $q = 1$ this is always true. Assume $q > 1$. Assume F is a $\mathbb{F}_\ell$-vs sheaf. Let $\ell = 0$ then consider,

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a \longrightarrow 0$$

and $H^q_{\text{ét}}(X, \mathbb{G}_a) = H^q(X, \mathcal{O}_X) = 0$ by Serre vanishing for $q > 0$. Therefore $H^1(X, \mathbb{F}_p) = 0$ for $q > 1$. For $q = 1$ kill any torsor by going to some finite étale cover.

For $\ell \neq p$ we use Bertini and induction on n. Consider $\pi : \mathbb{A}_k^{n+1} \to \mathbb{A}_k^n$ such that $R^i\pi_* F$ is locally constant and formation commutes with base change. Artin vanishing $R^{>1}\pi_* = 0$ and therefore using Leray,

$$H^i(\mathbb{A}^n, R^j\pi_* F) \implies H^{i+j}(\mathbb{A}^{n+1}, F)$$

this gives an exact sequence,

$$\cdots \longrightarrow H^q(\mathbb{A}^n, \pi_* F) \longrightarrow H^q(\mathbb{A}^{n+1}, F) \longrightarrow H^{q-1}(\mathbb{A}^n, R^1\pi_* F) \longrightarrow \cdots$$ The image $\zeta_0 \in H^{q-1}(\mathbb{A}^n, R^1\pi_* F)$ is killed by some cover $f^* : Y \to \mathbb{A}^n$. Therefore, replacing $\mathbb{A}^n$ by Y we can asssume that $f^*\zeta$ lies in the kernel $H^q(Y, R^j\pi_* F)$ then $Y \to \mathbb{A}^n$ is finite étale cover so Y is $K(\pi, 1)$ and hence after a futher cover we can kill ζ.

2.4 Step 3

Given an étale map $U \to \mathbb{A}^n$ then there exists finite étale $U \to \mathbb{A}^n$, use Noetherian normalization but add p-powers.

Proposition 2.4.1. If $R = k[x_1, \dots, x_n, x_{n+1}, \dots, x_r]/I$ if $x_1, \dots, x_n \in R$ are algebraically independent then there are $y_1, \dots, y_n \in R$ such that R is finite over $k[x_1 + y_1^p, \dots, x_n + y_n^p]$.

Therefore, we can transform any étale cover of $\mathbb{A}^n$ into a finite étale cover and hence is a $K(\pi, 1)$.

2.5 Step 4

Definition 2.5.1. (A, I) is a *Henselian pair* if every étale A-algebra A' there is an isomorphism,

$$\mathrm{Hom}_A(A', A) \xrightarrow{\sim} \mathrm{Hom}_A(A', A/I)$$

Proposition 2.5.2. Given any pair (A, I) can construct an initial henselian pair (A^h, I^h) with $I^h = I \cdot A^h$ via,

$$A^h = \varinjlim_{(B,J)} B$$

where the limit is taken over $A \to B$ étale with $\sigma : B \to A/I$ such that $\ker \sigma = J$.

Theorem 2.5.3 (Gabber)**.** Let (X, I) be a henselian pair. Let $X = \mathrm{Spec}\,(A)$ and $X_0 = \mathrm{Spec}\,(A/I)$ and $\iota : X_0 \hookrightarrow X$. Then the following are equivalences,

$$\iota^* : \mathrm{FEt}(X) \xrightarrow{\sim} \mathrm{FEt}(X_0)$$
$$\iota^* : \mathrm{LCC}(X) \xrightarrow{\sim} \mathrm{LCC}(X_0)$$
$$\iota^* : H^q(X, F) \xrightarrow{\sim} H^q(X_0, \iota^* F)$$

Corollary 2.5.4. X is a $K(\pi, 1)$ iff X_0 is a $K(\pi, 1)$.

Proof. Let $\zeta \in H^q(X, F)$, want $f : X' \to X$ finite étale st $f^*\zeta = 0$ but this follows from the existence for ζ_0 by the equivalences. $\square$

2.6 Step 5: the General Case

Let $X = \mathrm{Spec}\,(A)$ be a connected affine scheme over $\mathbb{F}_p$. Consider all finite subsets $S \subset A$ and notice that,

$$A = \bigcup_{S \subset A} \mathbb{F}_p[S]$$

It suffices to show that $\mathrm{Spec}\,(\mathbb{F}_p[S])$ is a $K(\pi, 1)$ because of spreading out things we know any cohomology class will be defined over a finite level so we get a finite étale cover which kills it which we can pull back to $\mathrm{Spec}\,(A)$.

Assume A is finitely presented take $X \hookrightarrow \mathbb{A}^n_{\mathbb{F}_p} = \mathrm{Spec}\,(P)$ cut out by I. Consider the Hensilization (P^h, I^h). Then P^h is a direct limit of rings B with $P \to B$ étale so by step 3 $\mathrm{Spec}\,(B)$ is $K(\pi, 1)$ and thus $\mathrm{Spec}\left(P^h\right)$ is a $K(\pi, 1)$. By step 4, since P^h is a $K(\pi, 1)$ we see that $X = P/I = P^h/I^h$ is a $K(\pi, 1)$.

2.7 Proof of Bertini

Theorem 2.7.1. Let $X \to S$ be projective with geometrically connected fibers smooth of relative dimension 1 and $\iota : S \to X$ a section, F lcc $\mathbb{F}_\ell$-sheaf on $U = X \setminus \iota(S)$. Supose that $\mathrm{Sw}_{\iota(s)}(F|_{U_{\bar{s}}})$ is independent of s. Then $R^q f_* F$ and $R^q f_! F$ are locally constant and formation commutes with base change.

Proposition 2.7.2. Let X be a smooth k-scheme and $D \subset X$ divisor and let $U = X \setminus D$ and F is lcc abelian sheaf on U. Then there exists a dense open $T^\circ \subset D \times_X \mathbb{P}(T_X)$ such that if $(x, \ell) \in T^\circ$ then for all smooth curves $C, C' \subset X$ st $C \cap D = C' \cap D = \{x\}$ and $T_x C = \ell = T_X C'$ then $\mathrm{Sw}_x(F|_{C \setminus \{x\}}) = \mathrm{Sw}_x(F|_{C' \setminus \{x\}})$.