Mathematics GU4051 Topology
Assignment # 10

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February 17, 2020

Remark 1. For loops $\gamma_1, \gamma_2 : I \to X$ I will use the (old) notation $\gamma_1 * \gamma_2$ to denote the loop,

$$h(t) = \begin{cases} \gamma_2(2t) & t \le \frac{1}{2} \\ \gamma_1(2t-1) & t \ge \frac{1}{2} \end{cases}$$

Problem 1.

Let X be a locally euclidean connected space. If $X = \varnothing$ then X is vacuously path connected because there are no two points to connect. By Lemma 0.1, path-connectedness is an equivalence relation denoted by $\sim$. Suppose that $X \neq \varnothing$ then choose some point $x_0 \in X$ and define the set,

$$U = [x_0] = \{x \in X \mid x_0 \sim x\}$$

For $x \in X$, by the locally Euclidean property, $\exists$ open $x \in V_x \subset X$ such that $f : V_x \to B_1(0) \subset \mathbb{R}^n$ is a homoemorphism. Take any point $y \in V_x$ then $f(x)$ and $f(y)$ are points in $B_1(0)$ which is convex in $\mathbb{R}^n$ and therefore path-connected. Thus, there exists a path $\gamma : I \to B_1(0)$ from $f(x)$ to $f(y)$. Then, take $\delta = f^{-1} \circ \gamma$ which is continuous because f is continuous with continuous inverse. Then, $\delta(0) = f^{-1}(f(x)) = x$ and $\delta(1) = f^{-1}(f(y)) = y$. Thus, there is a path from x to y.

If $x \in U$ then $x_0 \sim x$ and $\forall y \in V_x : x \sim y$ thus $x_0 \sim y$ so $y \in U$. Therefore, $V_x \subset U$ so U is open because every point has an open neighborhood contained in U.

If $x \in X \setminus U$ then $x_0 \not\sim x$ and $\forall y \in V_x : x \sim y$, however, if $x_0 \sim y$ then by transitivity $x_0 \sim x$ which we assumed was false. Thus, $x_0 \not\sim y$ so $y \notin U$ and so $V_x \subset X \setminus U$. Therefore, $X \setminus U$ is open because every point has an open neighborhood contained in U.

Therefore, U is a clopen set but $x_0 \sim x_0$ so $x_0 \in U$ and thus $U \neq \varnothing$. However, X is connected so the only nonempty clopen set is X. Thus $U = X$. Thus, $\forall x, y \in X : x, y \in U$ so $x \sim x_0$ and $y \sim x_0$ so by transitivity, $x \sim y$. Thus, X is path connected. Becausere path-connected always implies connected, the converse holds as well.

Problem 2.

Let X be path connected and take some $x_0 \in X$. First, suppose that all paths with equal endpoints are path-homotopic. In particular, every loop at x_0 is path-homotopic to the trivial loop so the group $\pi_1(X, x_0)$ contains only the identity i.e. $\pi_1(X, x_0) \cong \{e\}$. Conversely, suppose that $\pi_1(X, x_0) \cong \{e\}$. Take $x, y \in X$ and any two paths γ_1, γ_2 between x and y. Because X is path connected, there is an isomorphism given by conjugation with a path from x_0 to x such that $\pi_1(X,x) \cong \pi_1(X,x_0) \cong \{e\}$. Let $\sim$ denote path-homotopy. Now, $(\gamma_2 * \hat{\gamma}_2) * \gamma_1 \sim \gamma_1$ because $\gamma_2 * \hat{\gamma}_2 \sim e_y$ and $e_y * \gamma_1 \sim \gamma_1$ by reparameterization. Again by reparameterization, $(\gamma_2 * \hat{\gamma}_2) * \gamma_1 \sim \gamma_2 * (\hat{\gamma}_2 * \gamma_1)$ but $\hat{\gamma}_2 * \gamma_1$ is a path from x to x and thus a loop. However, the fundamental group at x is trivial so $\hat{\gamma}_2 * \gamma_1 \sim e_x$. Thus, $\gamma_2 * (\hat{\gamma}_2 * \gamma_1) \sim \gamma_2 * e_x \sim \gamma_2$. Therefore,

$$\gamma_1 \sim (\gamma_2 * \hat{\gamma}_2) * \gamma_1 \sim \gamma_2 * (\hat{\gamma}_2 * \gamma_1) \sim \gamma_2$$

Therefore, any two loops with equal endpoints are path-homotopic.

Problem 3.

From problem 4 on assignment 3, $\mathbb{C}\backslash\{0\} \cong \mathbb{R}^2\backslash\{(0,0)\} \cong \mathbb{R} \times S$ where

$$S = \{z \in \mathbb{C} \mid |z| = 1\}$$

Let the homeomorphism between these spaces be denoted by $e : \mathbb{C}\backslash\{0\} \to \mathbb{R} \times S$.

Now, $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ given by $z \mapsto z^n$ induces a map $\tilde{f} : \mathbb{R} \times S \to \mathbb{R} \times S$ given by $\tilde{f} = e \circ f \circ e^{-1}$ which takes $(x,z) \mapsto (nx, z^n)$. Therefore, $\tilde{f} = m_n \times p$ where $p = f|_S$ and $m_n : x \mapsto nx$. We proved in lecture that $f|_S$ and m_n are covering maps of S and $\mathbb{R}$ respectively. Therefore, by Lemma 0.2, $\tilde{f} = m_n \times f|_S$ is a covering map of $\mathbb{R} \times S$.

$$\begin{tikzcd}
\mathbb{C} \setminus \{0\} \arrow[r, "e"] \arrow[d, "f"] & \mathbb{R} \times S \arrow[d, "\tilde{f}"] \\
\mathbb{C} \setminus \{0\} \arrow[r, "e"] & \mathbb{R} \times S
\end{tikzcd}$$

Any homeomorphism is a 1-fold covering map so e and e^{-1} are covering maps. Furthermore, m_n is a 1-fold cover and p is an n fold cover. By problem 5, since both e and f are finite-fold convering maps, $f = e^{-1} \circ f \circ e$ is a covering map.

Problem 4.

Let $p : Y \to X$ be a covering map with connected X. Suppose that for some $x_0 \in X$ that $p^{-1}(x_0)$ contains k elements. Define,

$$U = \{x \in X \mid |p^{-1}(x)| = k\}$$

Take $x \in X$. Then x has an evenly covered neighborhood V_x with a homeomorphism $e : p^{-1}(V_x) \to V_x \times \Lambda$ such that the diagram commutes.

$$\begin{tikzcd}
p^{-1}(V_x) \arrow[r, "e"] \arrow[dr, "p"'] & V_x \times \Lambda \arrow[d, "\pi_1"] \\
& V_x
\end{tikzcd}$$

Now, for any $y \in V_x$ we have $e(p^{-1}(y)) = \pi_1^{-1}(y) = \{y\} \times \Lambda$ but e is a bijection so $|\{y\} \times \Lambda| = |p^{-1}(y)|$. Thus $\forall y \in V_x : |p^{-1}(y)| = |\Lambda|$. In particular, every element of V_x has the same covering number. Therefore, if $x \in U$ then $|p^{-1}(x)| = k$ so for any other $y \in V_x$ we have $|p^{-1}(y)| = |p^{-1}(x)| = k$ so $V_x \subset U$. Therefore, U is open because it contains an open neighborhood of each point. Likewise, if $x \notin U$ then $|f^{-1}(x)| \neq k$ so for any $y \in V_x$ we have $|p^{-1}(y)| = |p^{-1}(x)| \neq k$ and thus $y \notin U$. Therefore, $V_x \subset X \setminus U$ and thus $X \setminus U$ is open because it contains an open neighborhood of each point. Therefore, U is clopen but since $x_0 \in U$ we have that $U \neq \varnothing$. However, X is connected so the only nonempty clopen set is X. Therefore $U = X$ and to $\forall x \in X : p^{-1}(x)$ contains k elements.

Problem 5.

Let $f : Z \to Y$ and $g : Y \to X$ be covering maps with finite $g^{-1}(x)$ for every $x \in X$. Now, take $x \in X$ and let U_x be an evenly covered neighborhood of x. Then the preimage $g^{-1}(U_x)$ is a union of disjoint slices, U_i, each homeomorphic to U_x under g. Let y_i be the preimage of x in U_i i.e. $y_i = g|_{U_i}^{-1}(x)$ which exists because $g|_{U_i} : U_i \to U_x$ is a bijection. There are a finite number of slices because $g^{-1}(x)$ is finite but each slice must map to x because they are homeomorphic to U_x under g. Now, each y_i has an evenly covered neighborhood V_i with preimage $f^{-1}(V_i)$ given by the union of disjoint slices, V_i^λ with $\lambda \in \Lambda_i$. Consider the set,

$$S = \bigcap_{i=1}^{n} g(V_i \cap U_i) \subset U_x$$

Now, $V_i \cap U_i \subset U_i$ and g is a homeomorphism between U_i and U_x so $g(V_i \cap U_i)$ is open in U_i and therefore open in X because U_x is open in X. Thus, S is open because it is the finite intersection of open sets. Also, $y_i \in V_i$ and $y_i \in U_i$ so $x \in g(V_i \cap U_i)$ because $g(y_i) = x$. Thus, $x \in S$. Consider $g^{-1}(S) \subset g^{-1}(U_x)$. Let $S_i = g^{-1}(S) \cap U_i = g|_{U_i}^{-1}(S)$ i.e. the component of the preimage in each slice. Now, $S \subset g(V_i \cap U_i)$ so $S_i \subset g|_{U_i}^{-1}(g(V_i \cap U_i)) = V_i \cap U_i$ because $g|_{U_i}$ is a bijection on its image. Therefore, $f^{-1}(S_i) \subset f^{-1}(V_i \cap U_i) \subset f^{-1}(V_i)$. We can decompose this preimage into each slice as $S_i^\lambda = f^{-1}(S_i) \cap V_i^\lambda = f|_{V_i^\lambda}^{-1}(S_i)$. Now, we need to show that $(g \circ f)^{-1}(S)$ is a disjoint union over these sets S_i^λ and that when restricted to S_i^λ that $g \circ f$ is a homeomorphism to S. First,

$$\bigcup_{\lambda \in \Lambda_i} S_i^\lambda = \bigcup_{\lambda \in \Lambda_i} f^{-1}(S_i) \cap V_i^\lambda = f^{-1}(S_i) \cap f^{-1}(V_i) = f^{-1}(S_i \cap V_i)$$

because the preimage of V_i under f is split into slices V_i^λ. Further,

$$\begin{aligned}\bigcup_{i=1}^{n} f^{-1}\left(S_i \cap V_i\right) &= f^{-1}\left(\bigcup_{i=1}^{n} S_i \cap V_i\right) = f^{-1}\left(\bigcup_{i=1}^{n} g^{-1}\left(S\right) \cap U_i \cap V_i\right) \\ &= f^{-1}\left(g^{-1}\left(S\right) \cap \bigcup_{i=1}^{n} U_i \cap V_i\right) = f^{-1}(g^{-1}\left(S\right)) = \left(g \circ f\right)^{-1}\left(S\right)\end{aligned}$$

where I have used the fact that if $x \in g^{-1}(S)$ then $g(x) \in g(U_i \cap V_i)$ for each i but $x \in g^{-1}(U_x)$ so x is in some slice U_j and thus $x \in U_j \cap V_j$ because on the slice g is injective and therefore $x \in \bigcap_{i=1}^{n}(U_i \cap V_i)$ so $g^{-1}(S) \subset \bigcap_{i=1}^{n}(U_i \cap V_i)$. The sets S_i^λ are clearly disjoint because if $\lambda \neq \lambda'$ then $S_i^\lambda \subset V_i^\lambda$ and $S_i^{\lambda'} \subset V_i^{\lambda'}$ which are disjoint slices. Also, if $i \neq j$ then S_i and S_j are disjoint because they are contained in U_i and U_j respectively which are disjoint slices. Therefore, S_i^λ and S_j^λ are disjoint because they are contained in the preimages of disjoint sets which are disjoint (since $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \varnothing$). Finally, we need to show that $g \circ f$ is a homeomorphism restricted to S_i^λ. The map $\tilde{f} = f|_{V_i^\lambda} : V_i^\lambda \to V_i$ is a homeomorphism and thus, its restriction to $S_i^\lambda = \tilde{f}^{-1}(S_i) \subset V_i^\lambda$ is a homeomorphism to the image S_i. Also, $S_i = S^{-1}(g|_{U_i}) \subset U_i$ and $\tilde{g} = g|_{U_i} : U_i \to U_x$ is a homeomorphism so its restriction to S_i is also a homeomorphism to the image S. Thus, $(g \circ f)|_{S_i^\lambda} = g|_{S_i} \circ f|_{S_i^\lambda}$ is a homeomorphism to S. Finally, we have that every point x has an open neighborhood S such that $(g \circ f)^{-1}(S)$ is the union of disjoint slices S_i^λ on which $g \circ f$ is a homeomorphism S and thus, $g \circ f$ is a covering map.

Lemmas

Lemma 0.1. Two points being path-connected is an equivalence relation.

Proof. Take any $c \in X$ then the path $\gamma : I \to X$ given by $\gamma(t) = c$ is continuous because it is constant (Problem 1, Assignment 2) and therefore a path from c to c. Thus, $c \sim c$. Next, suppose that $x \sim y$ then there exists a continuous map $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Consider $\delta(t) = \gamma(1-t)$ which is continuous because $r : t \mapsto 1 - t$ is continuous and $\delta = \gamma \circ r$. Also, $\delta(0) = \gamma(1) = y$ and $\delta(1) = \gamma(0) = x$. Thus, $y \sim x$. Finally, let $x \sim y$ and $y \sim z$ then there exist paths $\gamma_1, \gamma_2 : I \to X$ with $\gamma_1(0) = x$ and $\gamma_1(1) = y$ and $\gamma_2(0) = y$ and $\gamma_2(1) = z$. Consider the function, $\delta : I \to X$,

$$\delta(t) = \begin{cases} \gamma_1(2t) & t \leq \frac{1}{2} \\ \gamma_2(2t-1) & t \geq \frac{1}{2} \end{cases}$$

At $t = \frac{1}{2}$, $\gamma_1(2t) = \gamma_1(1) = y$ and $\gamma_2(2t-1) = \gamma_2(0) = y$ so by the glueing lemma, δ is continuous. Furthermore, $\delta(0) = \gamma_1(0) = x$ and $\delta(1) = \gamma_2(1) = z$. Therefore, δ is a path from x to z so $x \sim z$. $\square$

Lemma 0.2. Let $p_1 : Y_1 \to X_1$ and $p_2 : Y_2 \to X_2$ be covering maps. Then, $p_1 \times p_2 : Y_1 \times Y_2 \to X_1 \times X_1$ is a covering map.

Proof. Let $p_1 : Y_1 \to X_1$ and $p_2 : Y_2 \to X_2$ be covering maps. Take a point $(x_1, x_2) \in X_1 \times X_2$. Then there are evenly covered open neighborhoods U_1 and U_2 of x_1 and x_2 under the maps p_1 and p_2 respectively. Thus, there exist homeomorphisms and discrete topological spaces such that the following top and bottom triangles in the following diagram commute,

$$\begin{tikzcd}
 & & & U_1 & \\
 & p_1^{-1}(U_1) \arrow[rr, "e_1"] \arrow[urr, "p_1"] & & U_1 \times \Lambda_1 \arrow[u, "\pi_1^1"'] & \\
p_1^{-1}(U_1) \times p_2^{-1}(U_2) \arrow[ur, "\pi_1^p"] \arrow[dr, "\pi_2^p"'] \arrow[rrr, dashed, "e_1 \times e_2"] \arrow[rrrr, dashed, bend right=60, "p_1 \times p_2"'] & & & U_1 \times \Lambda_1 \times U_2 \times \Lambda_2 \arrow[u, "\pi_1^\Lambda"'] \arrow[d, "\pi_2^\Lambda"] \arrow[r, dashed, "\pi_1^1 \times \pi_1^2"] & U_1 \times U_2 \arrow[uul, "\pi_1^U"'] \arrow[ddl, "\pi_2^U"] \\
 & p_2^{-1}(U_2) \arrow[rr, "e_2"] \arrow[drr, "p_2"'] & & U_2 \times \Lambda_2 \arrow[d, "\pi_1^2"] & \\
 & & & U_2 &
\end{tikzcd}$$

The map $e_1 \times e_2$ is the unique map induced by $e_1 \circ \pi_1^p$ and $e_2 \circ \pi_2^p$. The map $p_1 \times p_2$ is induced by $e_1 \circ \pi_1^p$ and $e_2 \circ \pi_2^p$. The map $p_1 \times p_2$ is induced by $p_1 \circ \pi_1^p$ and $p_2 \circ \pi_2^p$. Finally, the map $\pi_1^1 \times \pi_1^2$ is induced by $\pi_1^1 \circ \pi_1^\Lambda$ and $\pi_1^2 \circ \pi_2^\Lambda$. However,

$$\pi_1^U \circ (\pi_1^1 \times \pi_1^2) \circ (e_1 \times e_2) = \pi_1^1 \circ \pi_1^\Lambda \circ (e_1 \times e_2) = \pi_1^1 \circ e_1 \circ \pi_1^p = p_1 \circ \pi_1^p$$

Similarly,

$$\pi_2^U \circ (\pi_1^1 \times \pi_1^2) \circ (e_1 \times e_2) = \pi_1^2 \circ \pi_2^\Lambda \circ (e_1 \times e_2) = \pi_1^2 \circ e_2 \circ \pi_2^p = p_2 \circ \pi_2^p$$

Therefore, $(\pi_1^1 \times \pi_1^2) \circ (e_1 \times e_2)$ satisfies the properties of unique product map defined by $p_1 \circ \pi_1^p$ and $p_2 \circ \pi_2^p$. Thus, $(\pi_1^1 \times \pi_1^2) \circ (e_1 \times e_2) = p_1 \times p_2$ so the entire diagram commutes. Finally, $p_1^{-1}(U_1) \times p_2^{-1}(U_2) = (p_1 \times p_2)^{-1}(U_1 \times U_2)$ because

$$(p_1 \times p_2)(x_1, x_2) \in U_1 \times U_2 \iff (p_1(x_1), p_2(x_2)) \in U_1 \times U_2 \iff x_1 \in p_1^{-1}(U_1) \text{ and } x_2 \in p_2^{-1}(U_2)$$

and there is a natrual isomorphism between $U_1 \times \Lambda_1 \times U_2 \times \Lambda_2$ and $(U_1 \times U_2) \times (\Lambda_1 \times \Lambda_2)$ and $\Lambda_1 \times \Lambda_2$ is a discrete space because each $\{\lambda_1\} \times \{\lambda_2\} = (\lambda_1, \lambda_2)$ is open. Therefore, the product of the neighborhoods about any point is evenly covered by the product map. $\square$

Addendum to Problem 3.

For completeness, I will also exhibit the covering explicitly. Take $U = \mathbb{C} \setminus R(e^{i\theta_0})$ where, $R(z) = \{zt \mid t \in \mathbb{R}^+\}$. Now,

$$f^{-1}(U) = \mathbb{C} \setminus \left(\bigcup_{k=1}^{n} R(e^{\frac{i}{n}(2\pi k + \theta_0)}) \right) = \bigcup_{k=1}^{n} \left\{ te^{i\theta} \mid t \in \mathbb{R}^+ \text{ and } \theta \in \left(2\pi \tfrac{k}{n} + \tfrac{\theta_0}{n}, 2\pi \tfrac{k+1}{n} + \tfrac{\theta_0}{n}\right) \right\} = \bigcup_{k=1}^{n} U_k$$

Because any point $re^{\frac{i}{n}(2\pi k + \theta_0)}$ maps to $t^n e^{2\pi i + i\theta_0} = t^n e^{i\theta_0} \in R(e^{i\theta_0})$. These sets are disjoint by construction. For any $z = te^{i\theta} \in U$ write $z = te^{i(\theta - \theta_0)} e^{i\theta}$ where $\theta - \theta_0 \in (0, 2\pi)$. Now let $g_k(z) = t^{1/n} e^{\frac{i}{n}(2\pi k + \theta)}$ where the form of the domain ensures that $g_k(z) \in U_k$. By analysis f and g are continuous. Now, $f \circ g_k(z) = te^{2\pi i k + \theta} = te^{\theta} = z$ and if $z \in U_k$ then $z = te^{\frac{i}{n}(2\pi k + \theta)}$ with $\theta > \theta_0$ so

$g_k \circ f(z) = g_k(t^n e^{i\theta}) = te^{\frac{i}{n}(2\pi k+\theta)}$ because $\theta - \theta_0 > 0$ so the angle is ihe proper form. Therefore, $f|_{U_k}$ and g_k are inverse functions and therefore $f|_{U_k}$ is a bijection to U. Thus, f restricted to U_k is a continuous bijection onto U with continuous inverse so we have shown that $f^{-1}(U)$ is partitioned into slices which are homeomorphic under f to U. Since every $z \in \mathbb{C}\backslash\{0\}$ is in $\mathbb{C}\backslash R(iz)$ because z and iz are not positive real multiples of each other, we have that every point has an evenly covered neighborhood so f is a covering map since U is an open set.