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1.1 Line Bundles

There exists a map,

$$\Gamma(X, \mathcal{L}^{\otimes a}) \otimes \Gamma(X, \mathcal{L}^{\otimes b}) \to \Gamma(X, \mathcal{L}^{\otimes ab})$$

since we have an isomorphism $\mathcal{L}^{\otimes a} \otimes \mathcal{L}^{\otimes b} = \mathcal{L}^{\otimes ab}$. Furthermore, since $\mathcal{L}$ is rank 1 this map is commutative since $s \times s' = s' \otimes s$ since they only differ by a section of $\mathcal{O}_X$. This alows us to define the following graded ring structure.

Definition Let $\mathcal{L}$ be an invertable $\mathcal{O}_X$-module, $\mathscr{F}$ any $\mathcal{O}_X$-module and $s \in \mathcal{L}(X)$ a global section. Then we define the following graded ring.

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

and then the following module,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which is a graded $\Gamma_*(X.\mathcal{L})$-module. Furthermore, there is a map,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F})_{(s)} \to \mathscr{F}(X_s) = \Gamma(X_s, \mathscr{F})$$

sending $\frac{t}{s^n} \mapsto t|_{X_s} \otimes (s|_{X_s})^{\otimes -n}$.

Proposition 1.1. Let X be a quasi-compact, quasi-seperated scheme and $\mathscr{F}$ be quasi-coherent. Then the above map is an isomorphism.

Proof. Tag OB5K. (Compare with that Hartshorne Excercise). $\square$

Example 1.2. Let A be a graded ring such that A is generated by A_1 as a A_0-algebra (e.g. $A = k[X_0, \dots, X_n]$). Let $X = \text{Proj}\,(A)$ and consider the graded module $M = A(n)$ which is the graded module $M_k = A_{k+n}$. Then we can construct the Serre twists,

$$\mathcal{O}_X(n) = \widetilde{M} = \widetilde{A(n)}$$

which is an invertable $\mathcal{O}_X$-module. Furthermore,

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$$

Remark. This will not be invertible and these maps will not be isomorphisms in general when A does not satisfy the required conditions.

Proof. We can decompose,

$$X = \bigcup_{f \in A_1} D_+(f) = \bigcup_{f \in A_1} \text{Spec}\left(A_{(f)}\right)$$

via the given assumptions. We know that,

$$\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{A(n)}|_{D_+(f)} = \widetilde{A[f^{-1}]_n}$$

However it is clear that $A[f^{-1}]_n = A[f^{-1}]_0 \cdot f^n$ so this sheaf is free of rank 1. $\square$

Remark. For $n = 1$ any element $f \in A_1$ gives a global section $f \in \Gamma(X, \mathcal{O}_X(1))$ such that $D_+(f) = X_s$ and hence,

$$\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(1)|_{X_s}$$

Corollary 1.3. In the setting above, further assume that A is generated by finitely many $f \in A_1$ as an A_0-algebra. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ if we set,

$$M = \Gamma_*(X, \mathcal{O}_X(1), \mathscr{F})$$

as a graded A-module via the map,

$$A \to \Gamma_*(X, \mathcal{O}_X(1)) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

Then we get, $\mathscr{F} = \widetilde{M}$.

Proof. Tag □

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Definition Let X be a scheme and $\mathcal{L}$ an invertivle $\mathcal{O}_X$-module . We say $\mathcal{L}$ is *ample* if X is quasi-compact and $\forall x \in X \exists n > 0, s\ in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $x \in X_s$.

Example 2.1. Let $X = \mathrm{Proj}\,(A)$ where A is generated by A_1 as a A_0-algebra and $A_1 = f_1 A_0 + \cdots + f_r A_0$. Then $\mathcal{O}_X(1)$ is invertible and X is covered by $D_+(f_i)$ and is quasi-compact, and $D_+(f_i) = X_{s_i}$ where $s_i \in \Gamma(X, \mathcal{O}_X(1))$ is a section corresponding to f_i.

Proposition 2.2. Let X be quasi-compact and quasi-seperated for $\mathcal{L} \in \mathrm{Pic}\,(X)$ the following are equivalent,

(a). $\mathcal{L}$ is ample

(b). for all $\mathcal{O}_X$-modules $\mathscr{F}$ locally of finite type there exists $n > 0$ s.t. $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections.

Proof. TAG 01Q3. □

Lemma 2.3. $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is ample for any $n > 0$.

Lemma 2.4. If X is affine, and L is invertible, and $s \in \Gamma(X, \mathcal{L})$ then X_s is affine.

Definition A scheme is noetherian if it has a finite open cover by spectra of noetherian rings.

Remark. It is equivalent to require that X is quasi-compact and $\mathcal{O}_X(U)$ is noetherian.

Lemma 2.5. A locally noetherian scheme is quasi-seperated.

Proof. If U, V are affines then $U \cap V$ is quasi-compact since every subspace of a noetherian space is quasi-compact. □

Definition Let X be a neotherian scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is *coherent* if it is quasi-coherent and locally of finite type.

Remark. It is equivalent to require that locally on affine opens $\mathscr{F}|_U = \widetilde{M}$ for a finitely-generated module M.

Remark. The inclusion functors,

$$\mathfrak{Coh}\,(\mathcal{O}_X) \subset \mathfrak{QCoh}\,(\mathcal{O}_X) \subset \mathfrak{Mod}\,(\mathcal{O}_X)$$

are exact and preserved under extensions i.e. given a short exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

if $\mathscr{F}_1, \mathscr{F}_2$ are (quasi)-coherent then $\mathscr{F}_2$ is also (quasi)-coherent.

Lemma 2.6. A scheme of finite type over a noetherian scheme is noetherian.

Proof. Since $f : X \to Y$ is finite type f is quasi-compact but Y is quasi-compact open so its preimage X is also quasi-compact. Furthermore, for any affine opens $\mathrm{Spec}\,(A) = U \subset X$ and $\mathrm{Spec}\,(B) = V \subset Y$ such that $f(U) \subset V$ we get a ring map $B \to A$ of finite type so $B[x_1, \dots, x_n] \twoheadrightarrow A$ and since B is noetherian we see that A is noetherian so X is quasi-compact and covered by $\mathrm{Spec}\,(A)$ for noetherian rings A. $\square$

Remark. We want to prove the following theorem. Let R be a noetherian ring, X a projective (or proper) scheme over R (then X is noetherian), and $\mathscr{F}$ a coherent sheaf on X, then,

$$H^i(X, \mathscr{F})$$

is a finite R-module for any i and $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$.