Mathematics GU6308 Algebraic Topology
Assignment # 5

Benjamin Church

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1 The Gysin Sequence

For sphere bundles, there is a particularly nice exact sequence due to Gysin.

Theorem 1.0.1 (Gysin)**.** Let $S^n \hookrightarrow E \xrightarrow{p} B$ be a sphere bundle which is homotopically simple. Then there is an exact sequence,

$$\cdots \xrightarrow{p_*} H_{i+1}(B) \xrightarrow{d} H_{i-n}(B) \xrightarrow{\ell} H_i(E) \xrightarrow{p_*} H_i(B) \xrightarrow{d} H_{i-n-1}(B) \longrightarrow \cdots$$

Furthermore, let $C \in H^{n+1}(B; \pi_n(S^n)) = H^{n+1}(B; \mathbb{Z})$ be the primary obstruction. Then $d(x) = x \frown C$ and ℓ is the map lifting a homology class of B to its preimage in E.

Proof. We consider the homological Serre spectral sequence,

$$E^2_{p,q} = H_p(B, H_q(S^n)) \implies H_{p+q}(E)$$

Note that,

$$E^2_{p,q} = H_p(B, H_q(S^n)) = \begin{cases} H_p(B; \mathbb{Z}) & q = 0, n \\ 0 & q \neq 0, n \end{cases}$$

To choose an isomorphism $\pi_n(S^n) \cong \mathbb{Z}$ we need an orientation of S^n. However, this is not an issue since we have assumed that the fibration is homotopically simple so there is no obstruction to choosing a consistent orientation.

Therefore, the second page of the Serre spectral sequence has two rows. The differential $\mathrm{d}^r_{p,q}$ has bidegree $(-r, r-1)$ so the only relevant differentials occur at page $r = n+1$ giving a differential,

$$\mathrm{d}^{n+1}_{p,0} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$$

Therefore, we can explicitly describe the ∞-page,

$$E^\infty_{p,q} = \begin{cases} H_p(B; \mathbb{Z}) & p < n+1, q = 0 \\ \ker\left(\mathrm{d}^{n+1}_{p,0}\right) & p \geq n+1, q = 0 \\ \operatorname{coker}\left(\mathrm{d}^{n+1}_{p+n+1,0}\right) & q = n \end{cases}$$

Since the spectral sequence converges,

$$E^2_{p,q} \implies H_{p+q}(E)$$

there is a filtration $F_p H_n(E)$ such that,

$$E^{\infty}_{p,q} = F_p H_{p+q}(E)/F_{p-1}H_{p+q}(E)$$

Note that $E^{\infty}_{p,q} \neq 0$ only when $q = 0, n$ so for fixed $i = p + q$ we find,

$$F_i H_i(E)/F_{i-1}H_i(E) = E^{\infty}_{i,0} = \ker\left(\mathrm{d}^{n+1}_{i,0}\right)$$

and

$$F_{i-n}H_i(E)/F_{i-n-1}H_i(E) = E^{\infty}_{i-n,n} = \mathrm{coker}\left(\mathrm{d}^{n+1}_{i+1,0}\right)$$

and all other quotients are zero. Therefore, $F_i H_i(E) = H_i(E)$ and $F_{i-1}H_i(E) = \cdots = F_{i-n}H_i(E)$ and $F_{i-n-1}H_i(E) = 0$. This gives an exact sequence,

$$0 \longrightarrow E^{\infty}_{i-n,n} \longrightarrow H_i(E) \longrightarrow E^{\infty}_{i,0} \longrightarrow 0$$

and therefore,

$$0 \longrightarrow \mathrm{coker}\left(\mathrm{d}^{n+1}_{i+1,0}\right) \longrightarrow H_i(E) \longrightarrow \ker\left(\mathrm{d}^{n+1}_{i,0}\right) \longrightarrow 0$$

Now these are maps,

$$\begin{aligned} \mathrm{d}^{n+1}_{i,0} &: H_i(B;\mathbb{Z}) \to H_{i-n-1}(B;\mathbb{Z}) \\ \mathrm{d}^{n+1}_{i+1,0} &: H_{i+1}(B;\mathbb{Z}) \to H_{i-n}(B;\mathbb{Z}) \end{aligned}$$

Therefore, the following sequence is exact,

$$H_{i+1}(B;\mathbb{Z}) \xrightarrow{\mathrm{d}^{n+1}_{i+1,0}} H_{i-n}(B;\mathbb{Z}) \xrightarrow{\ \ell\ } H_i(E;\mathbb{Z}) \xrightarrow{p_*} H_i(B;\mathbb{Z}) \xrightarrow{\mathrm{d}^{n+1}_{i,0}} H_{i-n-1}(B;\mathbb{Z})$$

These five term sequences glue to form a long exact sequence as follows,

$$\begin{tikzcd} \cdots \arrow[r, "\ell"] & H_{i+1}(E;\mathbb{Z}) \arrow[r, "p_*"] & H_{i+1}(B;\mathbb{Z}) \arrow[r, "\mathrm{d}^{n+1}_{i+1,0}"] \arrow[d, equal] & H_{i-n}(B;\mathbb{Z}) \arrow[d, equal] & & \\ & & H_{i+1}(B;\mathbb{Z}) \arrow[r, "\mathrm{d}^{n+1}_{i+1,0}"] & H_{i-n}(B;\mathbb{Z}) \arrow[r, "\ell"] & H_i(E;\mathbb{Z}) \arrow[r, "p_*"] & \cdots \end{tikzcd}$$

Now that we have demonstrated the existence of the Gysin sequence, we need to identify these maps. First, the morphism of fibrations,

$$\begin{tikzcd} E \arrow[r, "p"'] \arrow[d, "p"] & B \arrow[d, "\mathrm{id}"] \\ B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

induces a morphism on spectral sequences which shows that the map

$$H_i(E;\mathbb{Z}) \to E^2_{p,0} \subset E^2_{p,0} = H_p(B;\mathbb{Z})$$

is induced by $p : E \to B$. Note that this is the map $H_i(E;\mathbb{Z}) \to \ker\left(\mathrm{d}^{n+1}_{i,0}\right) \to H_i(B;\mathbb{Z})$ which appear in our long exact sequence by the properties of a morphism of spectral sequences.

We need to investigate the map $\mathrm{d}^{n+1}_{i,0} : H_i(B;\mathbb{Z}) \to H_{i-n-1}(B;\mathbb{Z})$. To do this, we use the cap product structure on the homological and cohomological spectral sequences $\frown : E^r_{p,q} \times E_r^{p',q'} \to E^r_{p-p',q-q'}$

which reduces to the usual cap product. Furthermore, the cap product structure is compatible with the differential. In particular, for the differential $d^{n+1}_{p,0} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$ this is a map $H_p(B;\mathbb{Z}) \to H_{p-n-1}(B;\mathbb{Z})$ which is cap product with some class $e \in H^{n+1}(B;\mathbb{Z})$ i.e. the map $\frown e : H_p(B;\mathbb{Z}) \to H_{p-n-1}(B;\mathbb{Z})$. Recall the definition of the primary obstruction. There is no obstruction to finding a section $s : B^n \to E$ since $\pi_i(F) = 0$ for $i < n$. Then for each $(n+1)$-cell D^n with an attaching map $f : S^n \to B^n$ we have an inclusion $h : D^{n+1} \to B$. then pulling back the fibration gives $h^*E \to D^{n+1}$. We have a section $s : \partial D^{n+1} \to h^*E$ and since h^*E is locally trivial then a map $s : S^n \to F$. Adding these together gives a cohomology class $H^{n+1}(X;\pi_n(F)) = H^{n+1}(X;\mathbb{Z})$.

Notice that if the fibration $p : E \to B$ has a section $s : B \to E$ then by $p \circ s = \mathrm{id}_B$ we immediately see that $p_* \circ s_* = \mathrm{id}$ on homology so the map $p_* H_i(E) \to H_i(B)$ is surjective. Therefore, $d = 0$ and the Gysin sequence splits into short exact sequences,

$$0 \longrightarrow H_{i-n}(B) \longrightarrow H_i(E) \xrightarrow{p_*} H_i(B) \longrightarrow 0$$

In particular, this shows that whenever $p : E \to B$ has a section, the class $e = 0$. Focus on degree $i = n+1$ in which we have shown there is a sequence,

$$H_1(B) \longrightarrow H_{n+1}(E) \xrightarrow{p_*} H_{n+1}(B) \xrightarrow{\frown e} H_0(B)$$

Choose a section $s : B^n \to E$ on the n-skeleton of B. Then the map $\smile e$ takes an $n+1$-cell D^{n+1} to the degree of the map $\partial D^{n+1} \to F$ (as a map $S^n \to S^n$). Thus e is the primary obstruction. $\square$

2 Postnikov and Whitehead Towers

Definition 2.0.1. Let X be a path-connected space. Then a *Postnikov* tower $\mathcal{X}$ is a diagram of spaces,

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d, "q_2"] \\
 & X_2 \arrow[d, "q_1"] \\
X \arrow[ruu, "s_3"] \arrow[ru, "s_2"'] \arrow[r, "s_1"'] & X_1
\end{tikzcd}$$

Such that,

(a) $s_n : X \to X_n$ induces an isomorphism $(s_n)_* : \pi_i(X) \to \pi_i(X_n)$ for $i \leq n$

(b) $\pi_i(X_n) = 0$ for $i > n$.

A morphism of Postnikov towers is a morphism of diagrams.

Remark. Note that any morphism of Postnikov towers $f : \mathcal{X} \to \mathcal{X}'$ induces a weak homotopy equivalence $f_n : X_n \to X'_n$ because the diagram,

$$\begin{tikzcd} \pi_i(X_n) \arrow[rr, "(f_n)_*"] & & \pi_i(X'_n) \\ & \pi_i(X) \arrow[ul, "(s_n)_*"] \arrow[ur, "(s'_n)_*"'] & \end{tikzcd}$$

commutes and either $\pi_i(X_n) = \pi_i(X'_n) = 0$ for $i > n$ or the upward maps are isomorphism for $i \le n$ so $(f_n)_* : \pi_i(X_n) \xrightarrow{\sim} \pi_n(X'_n)$ is an isomorphism.

Proposition 2.0.2. Let X be a path-connected CW complex. Then X admits a Postnikov tower $\mathcal{X}$ of CW complexes which is unique up to homotopy equivalence.

Proof. Starting with the constant map $q_0 : X \to X_0 = *$ we construct a tower,

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_3 \arrow[d, "q_2"] \\ & X_2 \arrow[d, "q_1"] \\ X \arrow[uur, "s_3"] \arrow[ur, "s_2"] \arrow[r, "s_1"'] \arrow[dr, "s_0"'] & X_1 \arrow[d, "q_0"] \\ & X_0 \end{tikzcd}$$

Given a map $s_n : X \to X_n$ such that $(s_n)_* : \pi_i(X) \to \pi_i(X_n)$ with $i \le n$ and $\pi_i(X_n) = 0$ for $i > n$. For each $S^{n+1} \to X$ generating $\pi_{n+1}(X)$ define X_{n+1} by attaching $(n+2)$-cells via the attaching maps $S^{n+1} \to X$ to kill $\pi_{n+1}(X)$ and for each $S^{n+i} \to X$ generating $\pi_{n+i}(X)$ attach an $(n+i+1)$-cell via the attaching maps $S^{n+i+1} \to X$ to kill $\pi_{n+i}(X)$. Then we have a map $s_{n+1} : X \hookrightarrow X_{n+1}$ satisfying the required properties. Furthermore, since $\pi_i(X_n) = 0$ for $i > n$, the map $X \to X_n$ lifts to $X \to X_{n+1} \to X_n$ using Lemma 4.7 in Hatcher and noting that $X_{n+1} \setminus X$ is built from cells of dimension at least $n+2$.

Now suppose that $\mathcal{X}$ is a CW Postnikov tower for X and $\mathcal{X}'$ be the tower constructed above via attaching cells to X. It suffices to show there is a morphism $\mathcal{X} \to \mathcal{X}'$ of Postnikov towers since such a morphism is a weak homotopy equivalence on X_n and are CW complexes so any weak homotopy equivalence is automatically a homotopy equivalence. Such a morphism is given by Hatcher Proposition 4.18. $\square$

Remark. For each $q_n : X_{n+1} \to X_n$ we can expand $X_n \hookrightarrow X'_n$ to give a fibration $q'_n : X'_{n+1} \to X'_n$ fitting into the diagram,

$$\begin{tikzcd} X_{n+1} \arrow[r, hook] \arrow[d, "q_n"] & X'_{n+1} \arrow[d, "q'_n"] \\ X_n \arrow[r, hook] & X'_n \end{tikzcd}$$

Therefore, we may assume that $q_n : X_{n+1} \to X_n$ is a fibration in the definition of a Postnikov tower. This allows us to investigate the fiber $F_{n+1} \hookrightarrow X_{n+1} \xrightarrow{q_n} X_n$ via the long exact sequence,

$$\pi_{i+1}(X_{n+1}) \xrightarrow{q_*} \pi_{i+1}(X_n) \longrightarrow \pi_i(F_{n+1}) \longrightarrow \pi_i(X_{n+1}) \xrightarrow{q_*} \pi_i(X_n)$$

For $i > n-1$ we have $\pi_{i+1}(X_n) = 0$ and for $i > n+1$ we have $\pi_i(X_{n+1}) = 0$. Therefore, for $i > n+1$ we have $\pi_i(F_{n+1}) = 0$. Furthermore, the diagram,

$$\begin{tikzcd}
\pi_i(X_{n+1}) \arrow[rr, "(q_{n+1})_*"] & & \pi_i(X_n) \\
& \pi_i(X) \arrow[ul, "(s_{n+1})_*"] \arrow[ur, "(s'_n)_*"'] &
\end{tikzcd}$$

Therefore, for $i \le n$ both upward maps are isomorphisms so $(q_{n+1})_* : \pi_i(X_{n+1}) \to \pi_i(X_n)$ is an isomorphism. Therefore, $\pi_i(F_{n+1}) = 0$ from the long exact sequence when $i < n$. Thus, we only need to consider the cases $i = n, n+1$. For $i = n$ we get,

$$0 \longrightarrow \pi_n(F_{n+1}) \longrightarrow \pi_n(X_{n+1}) \xrightarrow{\sim} \pi_n(X_n)$$

and thus $\pi_n(F_{n+1}) = 0$. For $i = n+1$ we get,

$$0 \longrightarrow \pi_{n+1}(F_{n+1}) \longrightarrow \pi_{n+1}(X_{n+1}) \xrightarrow{q_*} 0$$

and thus $\pi_{n+1}(F_{n+1}) = \pi_{n+1}(X_{n+1}) = \pi_n(X)$. Thus, we find that $F_{n+1} = K(\pi_n(X), n)$.

Definition 2.0.3. Let X be a connected CW complex and $\mathcal{X}$ its Postnikov tower. Then we define a completion,

$$\hat{X} = \varprojlim_n X_n$$

2.1 Whitehead Towers

Definition 2.1.1. Let X be a connected space. Then a *Whitehead tower* $\mathcal{X}$ is a sequence of spaces,

$$\cdots \longrightarrow X^3 \xrightarrow{q^3} X^2 \xrightarrow{q^2} X^1 \xrightarrow{q^1} X^0 =\!=\!= X$$

where X^n is n-connected and the morphism $q^n : X^n \to X^{n-1}$ induces an isomorphism,

$$(q^n)_* : \pi_i(X^n) \xrightarrow{\sim} \pi_i(X^{n-1})$$

for all $i \geq n+1$. A morphism $f : \mathcal{X} \to \mathcal{X}'$ of Whitehead towers is a morphism $f^n : X^n \to X'^n$ of sequences such that $f^0 : X^0 \to X^0$ is the identity.

Remark. Given any morphism $f : \mathcal{X} \to \mathcal{X}'$ of Whitehead towers, we have a commuting square,

$$\begin{tikzcd}
\pi_i(X^{n+1}) \arrow[r, "q_*^{n+1}"] \arrow[d, "f_*^{n+1}"'] & \pi_i(X^n) \arrow[d, "f_*^n"] \\
\pi_i(X'^{n+1}) \arrow[r, "q'^{n+1}_*"'] & \pi_i(X'^n)
\end{tikzcd}$$

For $i \geq n+2$ the maps q_*^{n+1} are isomorphism. For $i \leq n+1$ we have $\pi_i(X^{n+1}) = \pi_i(X'^{n+1}) = 0$ and thus if f_*^n is an isomorphism for all i then f_*^{n+1} is also an isomorphism for each i. Furthermore, since $f^0 = \text{id}$, by induction we see that $f_*^n : \pi_i(X^n) \to \pi_i(X^n)$ is an isomorphism for each i. Thus any morphism $f : \mathcal{X} \to \mathcal{X}'$ on Whitehead towers gives a weak homotopy equivalence $f^n : X^n \to X'^n$ on each X^n.

Theorem 2.1.2. Let X be a connected CW complex. Then X admits a Whitehead tower which is unique up to homotopy equivalence.

Proof. We may take a Postnikov tower of fibrations for X,

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d, "q_2"] \\
 & X_2 \arrow[d, "q_1"] \\
X \arrow[r, "s_1"'] \arrow[ur, "s_2"'] \arrow[uur, "s_3"] & X_1
\end{tikzcd}$$

Consider the map $s_n : X \to X_n$. Then we define the Whitehead tower X^n to be the homotopy fiber of $s_n : X \to X_n$. Therefore, we get a fibration $X^n \hookrightarrow N_{s_n} \to X_n$ where N_{s_n} is homotopy equivalent to X. Furthermore, $\pi_i(N_{s_n}) \xrightarrow{\sim} \pi_i(X_n)$ is an isomorphism for $i \leq n$ and $\pi_{n+1}(X_n) = 0$ so by the long exact sequence of a fibration, we find that $\pi_i(X^n) = 0$ for $i \leq n$. Furthermore, for $i > n$ we know $\pi_i(X_n) = 0$ and thus $\pi_i(X^n) \to \pi_i(X)$ is an isomorphism for $i > n$. Using the functoriality of homotopy fibers, we get a diagram,

$$\begin{tikzcd}
\pi_i(X^{n+1}) \arrow[rr] \arrow[dd] \arrow[dr] & & \pi_i(X^n) \arrow[dd] \arrow[dl] \\
 & \pi_i(X) \arrow[dl] \arrow[dr] & \\
\pi_i(X_{n+1}) \arrow[rr, "(q_{n+1})_*"] & & \pi_i(X_n)
\end{tikzcd}$$

For $i > n + 1$ the upper triangle are isomorphism proving that the spaces X^n form a Whitehead tower. Using a similar argument we see that Whitehead towers and Postnikov towers are dual and existence and uniqueness of Whitehead towers thus carries over from our discussion of Postnikov towers. $\square$