1 Quasi-Coherent Sheaves

Recall that for a DM-stack we defined the small étale site:

Definition 1.0.1. Let $\mathscr{X}$ be a DM-stack. Then the *small étale site* $\mathscr{X}_{\text{ét}}$ of $\mathscr{X}$ is the category of schemes equipped with an étale map $U \to \mathscr{X}$. A covering is $\{U_i \to U\}$ over $\mathscr{X}$ such that $\sqcup_i U_i \to U$ is surjective.

Then for a sheaf $\mathscr{F}$ on $\mathscr{X}_{\text{ét}}$ we defined its global sections,

$$\Gamma(\mathscr{X}, \mathscr{F}) := \mathrm{Hom}_{\mathfrak{Sh}(\mathscr{X}_{\text{ét}})}(1, \mathscr{F})$$

where 1 is the terminal sheaf (the sheafification of $U \mapsto *$).

Remark. This definition works nicely for $\mathscr{X}$ DM and naturally generalizes the étale site $X_{\text{ét}}$ of a scheme. However, there is a glaring flaw if we attempt to extend this definition to Artin stacks there is a catastrophic failure: $\mathscr{X}_{\text{ét}}$ could be empty! For example, $(B\mathbb{G}_m)_{\text{ét}}$ is empty. Indeed, DM-stacks are exactly those stacks with schemes as étale neighborhoods. To remedy this we could take the smooth site of $\mathscr{X}$. To stay in the world of étale cohomology we consider a hybrid site where the schemes are smooth over $\mathscr{X}$ but the covers are all étale.

Definition 1.0.2. Let $\mathscr{X}$ be an algebraic stack. Then the *lisse-étale site* $\mathscr{X}_{\ell-\text{ét}}$ is the category of schemes smooth over $\mathscr{X}$ with *arbitrary* maps of schemes over $\mathscr{X}$. A covering $\{U_i \to U\}$ is a collection of morphisms such that $\sqcup_i U_i \to U$ is surjective or étale.

Definition 1.0.3. Let $\mathscr{F}$ be a sheaf on $\mathscr{X}_{\ell-\text{ét}}$ then,

$$\Gamma(\mathcal{U}, \mathscr{F}) = \mathrm{Hom}_{\mathfrak{Sh}(\mathcal{U}_{\ell-\text{ét}})}\left(1_{\mathfrak{U}}, \mathscr{F}|_{\mathcal{U}_{\ell-\text{ét}}}\right)$$

where $1_{\mathcal{U}}$ is the *indicator sheaf* of the smooth $\mathscr{X}$-stack $\mathcal{U} \to \mathscr{X}$ the sheafification of the constant sheaf $\underline{*}$. This is the terminal object of $\mathcal{U}_{\ell-\text{ét}}$. This can be computed by choosing a smooth presentation,

$$R \rightrightarrows U \to \mathcal{U}$$

and setting,

$$\Gamma(\mathfrak{U}, \mathscr{F}) = \mathrm{eq}\left(\mathscr{F}(U) \rightrightarrows \mathscr{F}(R)\right)$$

Definition 1.0.4. The structure sheaf $\mathcal{O}_{\mathscr{X}}$ is defined via,

$$\mathcal{O}_{\mathscr{X}}(U) = \Gamma(U, \mathcal{O}_U)$$

is a ring object in the abelian category $\mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}})$. We therefore define the abelian category $\mathbf{Mod}_{\mathcal{O}_{\mathscr{X}}}$. Given a morphism $f : \mathscr{X} \to \mathcal{Y}$ of algebraic stacks there are morphisms of topoi,

$$\begin{tikzcd} \mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}}) \arrow[r, bend left, "f_*"] & \mathbf{Ab}(\mathcal{Y}_{\ell-\text{ét}}) \arrow[l, bend left, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mod}_{\mathcal{O}_{\mathscr{X}}} \arrow[r, bend left, "f_*"] & \mathbf{Mod}_{\mathcal{O}_{\mathcal{Y}}} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Given two $\mathcal{O}_{\mathscr{X}}$-modules $\mathscr{F}$ and $\mathscr{G}$ we define the tensor product $\mathscr{F} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathscr{G}$ as the sheafification of,

$$U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_{\mathscr{X}}(U)} \mathscr{G}(U)$$

and the Hom sheaf $\mathscr{H}\!om_{\mathcal{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{G})$ as the sheaf,

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_U}\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$$

where $\mathscr{F}|_U$ means the restriction to the site $U_{\ell-\text{ét}}$ (note this is much more data than the restriction to U_{Zar}).

1.1 Quasi-Coherent Sheaves

As above we denote by $\mathscr{F}|_U$ the restriction of $\mathscr{F}$ to $U_{\ell-\text{ét}}$ and $\mathscr{F}|_{U_{\text{Zar}}}$ the restriction to U_{Zar}. Then we define,

Definition 1.1.1. Let $\mathscr{X}$ be an algebraic stack. A $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{F}$ is *quasi-coherent* if:

(a) for every smooth $U \to \mathscr{X}$ the restriction $\mathscr{F}|_{U_{\text{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\text{Zar}}}$-module

(b) for every morphism $f : V \to U$ of smooth $\mathscr{X}$-schemes, the induced morphism,

$$f^*(\mathscr{F}|_{U_{\text{Zar}}}) \to \mathscr{F}_{V_{\text{Zar}}}$$

is an isomorphism.

Remark. The above definition can be made in any site which refines the Zariski topology on each of its opens. However, in this generality such an object is usually called a *crystal in quasi-coherent sheaves* and the term *quasi-coherent* in an arbitrary site is reserved for the notion developed below. However, in most sites the two notions agree.

Definition 1.1.2. In an arbitrary ringed site $(\mathcal{C}, \mathcal{O})$ (or even an arbitrary ringed topos) a $\mathcal{O}$-module $\mathscr{F}$ is *quasi-coherent* if for each object $U \in \mathcal{C}$ there exists a cover $\{U_i \to U\}$ such that $\mathscr{F}|_{\mathcal{C}/U_i}$ is a *presentable* $\mathcal{O}$-module meaning there exists a presentation,

$$\bigoplus_J \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus_I \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \mathscr{F}|_{\mathcal{C}/U_i} \longrightarrow 0$$

We call the abelian subcategory of such sheaves $\text{QCoh}(\mathcal{C}) \subset \mathbf{Mod}_{\mathcal{O}_{\mathcal{C}}}$.

Definition 1.1.3. Let S be a scheme and $\mathcal{C} \subset \mathbf{Sch}_S$ a subcategory. Consider the presheaf of rings,

$$\mathcal{O} : \mathcal{C}^{\text{op}} \to \text{Ring}$$
$$(T \to S) \mapsto \Gamma(T, \mathcal{O}_T)$$

This is a sheaf for the fpqc topology. Furthermore, for any sheaf $\mathscr{F}$ on S_{Zar} there is a presheaf,

$$\mathcal{O} : \mathcal{C}^{\text{op}} \to \text{Ab}$$
$$(f : T \to S) \mapsto \Gamma(T, f^*\mathscr{F})$$

which is a $\mathcal{O}$-module. Furthermore, if $\mathscr{F}$ is quasi-coherent then $\mathscr{F}^a$ is a fpqc sheaf by descent.

Theorem 1.1.4 (Tag 03OJ)**.** Let S be a scheme. Let $\mathcal{C}$ be a site such that,

(a) $\mathcal{C}$ is a full subcategory of $\mathbf{Sch}_S$

(b) any Zariski covering of $T \in \mathcal{C}$ can be refined by a covering of $\mathcal{C}$

(c) $\text{id} : S \to S$ is an object of $\mathcal{C}$ (so it particular $\mathcal{C}$ has a terminal object)

(d) every covering of $\mathcal{C}$ is an fpqc covering of schemes

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and there is an equivalence of categories,

$$\text{QCoh}(S) \xrightarrow{\sim} \text{QCoh}(\mathcal{C})$$
$$\mathscr{F} \mapsto \mathscr{F}^a$$

Proof. This is basically a rephrasing of fpqc descent. $\square$

Proposition 1.1.5. Let $\mathscr{F}$ be a $\mathcal{O}_{\mathscr{X}_{\ell-\text{ét}}}$-module. Then the following are equivalent,

(a) $\mathscr{F}$ is quasi-coherent in the general sense

(b) $\mathscr{F}$ is quasi-coherent in the crystal sense.

Proof. C.f. 06WK. Let $\mathcal{C} = \mathscr{X}_{\ell-\text{ét}}$. Suppose that $\mathscr{F}$ satisfies (a). Then the restriction of $\mathscr{F}$ is quasi-coherent on $\mathcal{C}_{/U}$ and thus by the previous lemma $\mathscr{F}|_{\mathcal{C}} = (\mathscr{F}|_{U_{\text{Zar}}})^a$ and therefore satisfies (b). Given (b) take any $U \to \mathscr{X}$ smooth. Then we know $\mathscr{F}|_{U_{\text{Zar}}}$ is quasi-coherent so there is an affine Zariski open cover $\{U_i \to U\}$ such that $\mathscr{F}|_{(U_i)_{\text{Zar}}}$ is presented. Then the claim is that $\mathscr{F}|_{\mathcal{C}/U_i}$ is also presented. Indeed, the comparison map induced by $f : V \to U$ is an isomorphism the presentation pulls back to give a presentation of $\mathscr{F}|_{\mathcal{C}/U_i}$. $\square$

1.2 Descent Data

Definition 1.2.1. Let (U, R, s, t, c, e) be a groupoid scheme over S where $s, t : R \rightrightarrows U$ are the source and target maps and $c : R \times_{s,U,t} R \to R$ is the composition and $e : U \to R$ is the identity. Then the category of *descent data* consists of the category of pairs $(\mathscr{F}, \varphi)$ where $\mathscr{F}$ is a sheaf on U and φ is an isomorphism,

$$\varphi : t^*\mathscr{F} \xrightarrow{\sim} s^*\mathscr{F}$$

such that $e^*\varphi = \text{id}$ and satisfying the cocycle condition,

$$c^*\varphi = \pi_2^*\varphi \circ \pi_1^*\varphi$$

as morphisms of sheaves on $R \times_{s,U,t} R$.

Example 1.2.2. For any cover $U \to X$ we can form the "Cech groupoid" $U \times_X U \rightrightarrows U$ whose composition is given by projection,

$$(U \times_X U) \times_{\pi_1, U, \pi_2} (U \times_X U) = U \times_X U \times_X U \to U \times_X U \qquad ((a, b), (c, a)) \mapsto (c, a, b) \mapsto (c, b)$$

For this we recover the ordinary notion of a descent datum.

Example 1.2.3. Let $G \circlearrowright X$ be an action of an algebraic group on a scheme. Then there a groupoid $G \times X \rightrightarrows X$ whose composition $G \times G \times X \to G \times X$ is given by multiplication in the group.

For this we will recover the notion of G-equivariance.

Proposition 1.2.4. Let $R \rightrightarrows U$ be a smooth presentation of an algebraic stack $\mathscr{X}$ by schemes. There is an equivalence of categories,

$$\text{QCoh}(\mathscr{X}) \to \text{DD}_{\text{QCoh}}(U/R) \quad \mathscr{F} \mapsto (\mathscr{F}|_{U_{\text{Zar}}}, \varphi)$$

where $\text{DD}_{\text{QCoh}}(U/R)$ is the category of descent data for quasi-coherent sheaves along the groupoid $R \rightrightarrows U$.

Proof. For any smooth map $V \to \mathscr{X}$ there is a further smooth refinement $V' \to V$ such that $V' \to \mathscr{X}$ factors through $U \to \mathscr{X}$. Hence, applying descent to $V' \to V$, any quasi-coherent sheaf $\mathscr{F}$ on $\mathscr{X}_{\ell-\text{ét}}$ is determined by its descent data over $R \rightrightarrows U$. $\square$

Definition 1.2.5. Let $G \circlearrowright X$ be an action of a group scheme on a scheme (or algebraic space). The category of G-equivariant sheaves is defined as the category of descent data for the groupoid $G \times X \rightrightarrows X$.

Remark. Some standard diagram chasing shows that this is formally the same as the usual definition of a G-equivariant sheaf in [Stacks]. In the case that G is a finite constant group it is easy to check that this agrees with the naive notion in terms of compatible isomorphisms between the pullbacks along the action by elements $g \in G$.

Proposition 1.2.6. There is an equivalence of categories,

$$\mathrm{QCoh}([X/G]) \to \mathrm{QCoh}_G(X)$$

Proof. This is a special case of the previous proposition. □

1.3 Examples

Example 1.3.1. Let $\mathscr{X} \to S$ be a DM-stack. Then the sheaf,

$$\Omega_{\mathscr{X}/S} : (U \to \mathscr{X}) \mapsto \Gamma(U, \Omega_{U/S})$$

is quasi-coherent since any morphism $f : V \to U$ in $\mathscr{X}_{\text{ét}}$ is étale so the map,

$$f^*\Omega_{U/S} \xrightarrow{\sim} \Omega_{V/S}$$

is an isomorphism. However, if $\mathscr{X} \to S$ is not DM we don't have access to $\mathscr{X}_{\text{ét}}$ nor can we define $(\Omega_{X/S})^a$ on X_{fppf} as we can for a scheme since there is no Zariski or étale site to define this sheaf over for a bootstrap. There is still a sheaf of $\mathcal{O}_{\mathscr{X}_{\ell-\text{ét}}}$-modules,

$$\Omega_{\mathscr{X}/S} : (U \to \mathscr{X}) \mapsto \Gamma(U, \Omega_{U/S})$$

but it is not quasi-coherent. This is the sort of sheaf the stacks project calls *locally quasi-coherent* meaning that it is quasi-coherent when restricted to $U_{\text{ét}}$ for any $U \to \mathscr{X}$.

Remark. Indeed, it is not clear that an Artin stack $\mathscr{X} \to S$ should have any good notion of a cotangent bundle $\Omega_{\mathscr{X}/S}$. For example, consider $\mathscr{X} = \mathbf{B}\mathbb{G}_m$ which is smooth of relative dimension -1 so what should $\Omega_{\mathscr{X}/S}$ even be? It can't be a vector bundle of rank -1 can it! To fix this conundrum, we either work with $\Omega_{\mathscr{X}/S}$ as defined above which is not quasi-coherent and hence does not have a well-defined rank or we define the cotangent complex $\mathbb{L}_{\mathscr{X}/S} \in D^{\le 1}_{\mathrm{QCoh}}(\mathscr{X})$ (technically it's an ind-object in this generality) [Champs Algebriques, Chapter 17] which encodes the deformation theory of $\mathscr{X}$. Note that unlike for a scheme, $\mathbb{L}_{\mathscr{X}/S}$ can be supported in degree 1. In fact, the following are equivalent,

(a) $\mathscr{X} \to S$ is DM

(b) $\mathcal{H}^1(\mathbb{L}_{\mathscr{X}/S}) = 0$

Proof: [Champs Algebriques, Cor. 17.9.2].

1.4 Picard Groups

Let $\mathscr{X}$ be an algebraic stack. Then $\mathrm{Pic}\,(\mathscr{X})$ denotes the set of isomorphism classes of line bundles on $\mathscr{X}$ which becomes an abelian group under $\otimes$.

Example 1.4.1. If G is an affine algebraic k-group then $\mathrm{Pic}\,(\mathbf{B}G) = \mathrm{Hom}_{\mathrm{gp}}\,(G, \mathbb{G}_m)$ is the group of characters. For example,

(a) $\mathrm{Pic}\,(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$

(b) $\mathrm{Pic}\,(\mathbf{B}\mathrm{GL}_n) = \mathbb{Z}$

(c) $\mathrm{Pic}\,(\mathbf{B}\mathrm{PGL}_n) = \{0\}$.

This is because line bundles on $\mathbf{B}G$ are the same as line bundles on $\mathrm{Spec}\,(k)$ along with descent data i.e. a G-action. This is the same as a 1-dimensional G-representation.

Example 1.4.2. Consider the action, $\mathbb{G}_m \circlearrowright \mathbb{A}^n$ with weights $d_1, \dots, d_n$. Let the *weighted projective stack* be the DM-stack (at least if $p \nmid d_i$),

$$\mathcal{P}(d_1, \dots, d_n) = [(\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m]$$

Here let k be a field of characteristic not dividing any d_i or 2 or 3.

(a) The map $\mathrm{Pic}\,(\mathbf{B}\mathbb{G}_m) \to \mathrm{Pic}\,(\mathcal{P}(d_1, \dots, d_n))$ induced by the canonical $\mathbb{G}_m$-bundle is an isomorphism. Indeed, this reduces to classifying $\mathbb{G}_m$-actions on $\mathcal{O}_{\mathbb{A}^n \setminus \{0\}}$. By Hartogs' these correspond to $\mathbb{G}_m$-actions on $\mathcal{O}_{\mathbb{A}^n}$ and thus to different grading of $A = k[x_1, \dots, x_n]$ as an A-module with x_i given weight d_i. These are just overall shifts $A(d)$ i.e. putting 1 in degree d. This is the same as the pullback of the bundle over $\mathbf{B}\mathbb{G}_m$ corresponding to $\mathbb{G}_m \xrightarrow{z^d} \mathbb{G}_m$.

(b) Using Weierstrass models we get an isomorphism,

$$\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4, 6)$$

Therefore, $\mathrm{Pic}\left(\overline{\mathcal{M}}_{1,1}\right) = \mathbb{Z}\omega_{\mathcal{C}/\mathcal{M}}$

(c) Then it turns out that,

$$\mathrm{Pic}\,(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z}\omega_{\mathcal{C}/\mathcal{M}}$$

This is because the discriminant Δ is a section of $\mathcal{O}(12)$ which is nowhere vanishing for smooth families.

1.5 Global Quotients and the Resolution Property

Definition 1.5.1. An algebraic stack $\mathscr{X}$ is a *global quotient stack* if there is an isomorphism $\mathscr{X} \cong [U/\mathrm{GL}_n]$ where U is an algebraic space. This is equivalent to asking for the existence of a GL_n-bundle $U \to \mathscr{X}$ where U is an algebraic space. By definition this is the same as a representable morphism $\mathscr{X} \to \mathbf{B}\mathrm{GL}_n$.

Proposition 1.5.2. Let $\mathscr{X} \to \mathcal{Y}$ be a surjective, flat, and projective morphism of noetherian algebraic stacks. If $\mathscr{X}$ is a quotient stack then $\mathcal{Y}$ is a quotient stack.

Definition 1.5.3. A noetherian algebraic stack has the *resolution property* if every coherent sheaf if a quotient of a vector bundle.

A smooth or quasi-projective scheme over a field has the resolution property. Any noetherian normal $\mathbb{Q}$-factorial scheme with affine diagonal also has the resolution property.

Proposition 1.5.4. Let G be an affine algebraic k-group with an action $G \circlearrowright U$ on a quasi-projective k-scheme U. Assume that there is an ample line bundle $\mathcal{L}$ with a G-action. Then $[\mathrm{Spec}\,(A)\,/G]$ has the resolution property.

Remark. While not every line bundle $\mathcal{L}$ on a normal k-scheme admits a G-action, it turns out there is always some positive power such that $\mathcal{L}^{\otimes n}$ has a G-action.

Proof. The G-line bundle $\mathcal{L}$ corresponds to a line bundle on $[U/G]$ which is ample which respect to the morphism $p : [U/G] \to \mathbf{B}G$ since relative ampleness can be checked after smooth covers (it can be reduced to a fiberwise condition). For a coherent sheaf $\mathscr{F}$ on $[U/G]$ the natural map,

$$\mathcal{L}^{-\otimes N} \otimes p^* p_*(\mathcal{L}^{\otimes N} \otimes \mathscr{F}) \twoheadrightarrow \mathscr{F}$$

is surjective for $N \gg 0$ since relative ampleness implies global generation of $\mathcal{L}^{\otimes N} \otimes \mathscr{F}$. The pushforward $p_*(\mathcal{L}^{\otimes N} \otimes \mathscr{F})$ is quasi-coherent on $\mathbf{B}G$ hence a G-representation. We can hence write it as an increasing union of finite-dimensional G-representations V_i and obtain,

$$\operatorname*{colim}_i (\mathcal{L}^{-\otimes N} \otimes p^* V_i) \twoheadrightarrow \mathscr{F}$$

since $\mathscr{F}$ is coherent, this stabilizes at some stage meaning,

$$\mathcal{L}^{-\otimes N} \otimes p^* V_i \twoheadrightarrow \mathscr{F}$$

at some finite stage i. □

Theorem 1.5.5 (Totaro-Gross)**.** Let $\mathscr{X}$ be a quasi-separated normal algebraic stack of finite type over k. Assume that the stabilizer group at every closed point is smooth and affine. Then the following are equivalent:

(a) $\mathscr{X}$ has the resolution property

(b) $\mathscr{X} \cong [U/\mathrm{GL}_n]$ with U quasi-affine

(c) $\mathscr{X} \cong [\mathrm{Spec}\,(A)\,/G]$ with G an affine algebraic group.

In particular, $\mathscr{X}$ has affine diagonal.

Remark. The normality hypothesis on $\mathscr{X}$ and smoothness hypothesis on the stabilizers are unnecessary. However, the affineness hypothesis on the stabilizers is necessary. For example, $\mathbf{B}E$ the classifying stack of an elliptic curve has the resolution property.

1.6 Sheaf Cohomology

Lemma 1.6.1. If $\mathscr{X}$ is an algebraic stack, the categories $\mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}})$ and $\mathbf{Mod}_{\mathscr{X}}$ have enough injective. If $\mathscr{X}$ is quasi-separated then $\mathrm{QCoh}(\mathscr{X})$ has enough injectives.

Definition 1.6.2. Let $\mathscr{X}$ be an algebraic stack and $\mathscr{F}$ a sheaf on $\mathscr{X}_{\ell-\text{ét}}$. The *cohomology groups* $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ are the derived functors of,

$$\Gamma(\mathscr{X}, -) : \mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}}) \to \mathbf{Ab}$$

applied to $\mathscr{F}$,

$$H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F}) = R^i\Gamma(\mathscr{X}, \mathscr{F})$$

Definition 1.6.3. Given a smooth covering $\mathfrak{U} = \{\mathcal{U}_i \to \mathscr{X}\}_{i\in I}$ of algebraic stacks and an abelian presheaf $\mathscr{F}$ on $\mathscr{X}_{\ell-\text{ét}}$ the *Cech complex* $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ of $\mathfrak{U}$ with respect to $\mathfrak{U}$ is,

$$\check{C}^n(\mathfrak{U}, \mathscr{F}) = \prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathscr{F}(\mathcal{U}_{i_0} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathcal{U}_{i_n})$$

with differential,

$$\mathrm{d}^n : \check{C}^n(\mathfrak{U}, \mathscr{F}) \to \check{C}^{n+1}(\mathfrak{U}, \mathscr{F}) \quad (s_{i_0,\dots,i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_{\hat{k}}^* s_{i_0,\dots,\hat{i}_k,\dots,i_n} \right)_{i_0,\dots,i_{n+1}}$$

where the projection $p_{\hat{k}}$ forgets the t^{th} coordinate. The *Čech cohomology* of $\mathscr{F}$ with respect to $\mathfrak{U}$ is,

$$\check{H}^i(\mathfrak{U}, \mathscr{F}) := H^i(\check{C}^\bullet(\mathfrak{U}, \mathscr{F}))$$

Theorem 1.6.4. Let $\mathscr{X}$ be an algebraic stack and $\mathscr{F}$ a quasi-coherent sheaf on $\mathscr{X}_{\ell-\text{ét}}$. Then for any cover $\mathfrak{U} = \{\mathcal{U}_i \to \mathscr{X}\}_{i\in I}$ there exists a spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, H^q(-, \mathscr{F})) \implies H^{p+q}(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$$

where $H^q(-, \mathscr{F})$ is the presheaf $U \mapsto H^q(U_{\ell-\text{ét}}, \mathscr{F})$.

Proof. Consider the commutative diagram of functors,

$$\begin{tikzcd} \mathrm{Sh}(\mathscr{X}_{\ell-\text{ét}}) \arrow[r, hook, "a"] \arrow[dr, "\Gamma"'] & \mathrm{PSh}(\mathscr{X}_{\ell-\text{ét}}) \arrow[d, "\check{H}^0"] \\ & \mathbf{Ab} \end{tikzcd}$$

Notice that $\check{C}^\bullet(\mathfrak{U}, -)$ is exact in the category of presheaves which shows that $\check{H}^\bullet(\mathfrak{U}, -)$ forms a δ-functor. In fact, since $\check{H}^i(\mathfrak{U}, \mathscr{I}) = 0$ for $i > 0$ and any injective sheaf (this is a very general fact, see Tag 03OR) it is a universal δ-functor. Now the inclusion a takes injectives to injectives because sheaves form a reflexive subcategory (maps to a sheaf factors through the sheafification). Therefore, we apply the Grothendieck spectral sequence so it suffices to compute $R^q a(\mathscr{F})$ of a sheaf $\mathscr{F}$. Since the functor $(-) \mapsto \Gamma(U, -)$ is exact on presheaves we see that,

$$R^q a(\mathscr{F})(U) = R^q\Gamma(U, \mathscr{F}) = H^q(U, \mathscr{F})$$

so we conclude. $\square$

Theorem 1.6.5. If X is an affine scheme and $\mathscr{F}$ is a quasi-coherent $\mathcal{O}_{\mathscr{X}_{\ell-\text{ét}}}$-module then,

$$H^i(X_{\ell-\text{ét}}, \mathscr{F}) = \begin{cases} \Gamma(X, \mathscr{F}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. We refine to affine coverings $\{\text{Spec}\,(B) \to \text{Spec}\,(A)\}$ then $\mathscr{F}$ is quasi-coherent (in all equivalent notions) and hence arises from some A-module M. To show that $\check{H}^{>0} = 0$ for this covering we show that the Amistur complex,

$$0 \longrightarrow M \longrightarrow M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \otimes_A B \longrightarrow \cdots$$

is exact. Indeed, after applying $B \otimes_A -$ which is faithfully flat this complex obtains a nullhomotopy. Now to conclude, we can either apply Cartan's criterion (Tag 03F9) or use hypercoverings and the fact that hypercover Cech cohomology computes derived functor cohomology. □

Proposition 1.6.6. Let $\mathscr{X}$ be an algebraic stack with affine diagonal and $\mathscr{F}$ be a quasi-coherent sheaf. If $\mathfrak{U} = \{U_i \to \mathscr{X}\}$ is an étale covering with each U_i affine, then $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F}) = \check{H}^i(\mathfrak{U}, \mathscr{F})$.

Proof. Follows immediately from the Cech-to-derived spectral sequence and the above. □

Remark. To remove the "affine diagonal" condition we need to use hypercovers. Indeed, if $U_\bullet \to \mathscr{X}$ is a simplicial hypercover in $\mathscr{X}_{\ell-\text{ét}}$ where each $U_\bullet$ is an affine scheme and $\mathscr{F}$ is quasi-coherent then,

$$H^i(\mathscr{X}, \mathscr{F}) = \check{H}^i(U_\bullet, \mathscr{F})$$

Proposition 1.6.7. Let X be a scheme (or a DM-stack with a sheaf on $\mathscr{X}_{\text{ét}}$) with affine diagonal[1] and $\mathscr{F}$ a quasi-coherent sheaf. Then,

$$H^i(X, \mathscr{F}) = H^i(X_{\ell-\text{ét}}, \mathscr{F}_{\ell-\text{ét}})$$

for all i where $\mathscr{F}_{\ell-\text{ét}}$ is the $\mathcal{O}_{X_{\ell-\text{ét}}}$-module defined by,

$$\mathscr{F}_{\ell-\text{ét}}(U) = \Gamma(U, f^* \mathscr{F})$$

for a smooth map $f : U \to X$. (In the stack case it is pullback under $f : \mathscr{X}_{\ell-\text{ét}} \to \mathscr{X}_{\text{ét}}$).

Proof. Choose an affine Zariski cover U of X (affine étale cover U of $\mathscr{X}$) by the assumption on the diagonal we see that,

$$H^i(X_{\ell-\text{ét}}, \mathscr{F}) = \check{H}^i(\mathrm{U}, \mathscr{F}) = H^i(X, \mathscr{F})$$

(and similarly for $\mathscr{X}$). The affine diagonal condition is to ensure that projects in the Cech complex are affine and hence have vanishing $H^{>0}$. However, this condition is not necessary. We can always choose a Zariski hypercover $U_\bullet \to X$ by affines and similar arguments show that,

$$H^i(X_{\ell-\text{ét}}, \mathscr{F}) = \check{H}^i(U_\bullet, \mathscr{F}) = H^i(X, \mathscr{F})$$

□

Proposition 1.6.8. Let $\mathscr{X}$ be an algebraic stack.

(a) $\mathscr{F}$ is an $\mathcal{O}_{\mathscr{X}}$-module then $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ agrees with $R^i\Gamma : \mathbf{Mod}_{\mathcal{O}_{\mathscr{X}}} \to \mathbf{Ab}$ computed in the category of $\mathcal{O}_{\mathscr{X}}$-modules.

[1] If we use hypercovers (see the discussion in the proof then we can remove this condition.

(b) If $\mathscr{X}$ has affine diagonal and $\mathscr{F}$ is a quasi-coherent sheaf on $\mathscr{X}$, then the cohomology $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ agrees with $R^i\Gamma : \mathrm{QCoh}(\mathscr{X}) \to \mathbf{Ab}$ computed in the category of quasi-coherent modules.

For a morphism $f : \mathscr{X} \to \mathcal{Y}$ of algebraic stacks (resp. quasi-compact morphism of algebraic stacks with affine diagonals) then (a) (resp. (b)) holds also for $R^i f_* \mathscr{F}$ of an $\mathcal{O}_{\mathscr{X}}$-module (resp. quasi-coherent sheaf).

Remark. If $\mathscr{X}$ does not have affine diagonal, then the sheaf cohomology $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ of a quasi-coherent sheaf may differ from the derived functor $R^i\Gamma(\mathscr{X}, -) : \mathrm{QCoh}(\mathscr{X}) \to \mathbf{Ab}$.

Proposition 1.6.9. If $\mathscr{X}$ is an algebraic stack and $\mathscr{F}_i$ is a directed system of abelian sheaves in $\mathscr{X}_{\ell-\text{ét}}$ then $\operatorname{colim}_i H^i(\mathscr{X}, \mathscr{F}_i) \to H^i(\mathscr{X}, \operatorname{colim}_i \mathscr{F}_i)$ is an isomorphism.

2 July 8 Affine GIT and Good moduli spaces

2.1 Good Moduli Spaces

Definition 2.1.1. A quasi-compact quasi-separated morphism $\pi : \mathscr{X} \to X$ from an algebraic stack to an algebraic space is a *good moduli space* if,

(a) $\mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathscr{X}}$

(b) $\pi_* : \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(X)$ is exact.

Example 2.1.2. Let G be linearly k-reductive (meaning taking invairants of representations is exact) then $G \circlearrowright \mathrm{Spec}\,(A)$ gives a diagram,

$$\begin{tikzcd} \mathrm{Spec}\,(A) \arrow[d] \arrow[dr] & \\ {[\mathrm{Spec}\,(A)\,/G]} \arrow[r, "\pi"] & \mathrm{Spec}\left(A^G\right) \end{tikzcd}$$

Then π satisfies the properties of a good moduli space morhism. Indeed,

(a) $\Gamma([\mathrm{Spec}\,(A)\,/G], \mathcal{O}_{[\mathrm{Spec}(A)/G]}) = A^G$

(b) and $\pi_* : \mathrm{QCoh}([\mathrm{Spec}\,(A)\,/G]) \to \mathrm{QCoh}(\mathrm{Spec}\left(A^G\right))$ is exact since taking invariants of a G-representation is exact.

Example 2.1.3. $[\mathbb{A}^1/\mathbb{G}_m] \to \mathrm{Spec}\,(k)$ is a good moduli space

Example 2.1.4. $[\mathbb{P}^1/\mathbb{G}_m] \to \mathrm{Spec}\,(k)$ is NOT a good moduli space. Indeed (2) fails because $\pi_* = H^0$ is not exact for $\mathbb{G}_m$-equivariant sheaves on $\mathbb{P}^1$. Here two closed points specializes to 1 closed point downstairs. We will see this is a problem.

Theorem 2.1.5. Let $\pi : \mathscr{X} \to X$ be a good moduli space and $\mathscr{X}$ be q-sep over a scheme S.

(a) π is surjective and universally closed

(b) if $\mathbb{Z}_1, \mathbb{Z}_2 \subset \mathscr{X}$ are closed substacks then $\pi(\mathbb{Z}_1 \cap \mathbb{Z}_2) = \pi(\mathbb{Z}_1) \cap \pi(\mathbb{Z}_2)$ where this denotes scheme-theoretic (or rather stack-theoretic) images and intersections. In particular, for geometric points $x_1, x_2 \in \mathscr{X}(k)$ then $\pi(x_1) = \pi(x_2)$ iff $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \varnothing$.

(c) If $\mathscr{X}$ is noetherian then X is also noetherian.

(d) If $\mathscr{X}$ is of finite type over S and S is noetherian then X is finite type over S and π_* preserves coherent sheaves.

(e) If $\mathscr{X}$ is noetherian then π is initial for maps to algebraic spaces.

Corollary 2.1.6. Let G be linearly reductive group over $k = \bar{k}$ and $\tilde{\pi} : U = \mathrm{Spec}\,(A) \to U//G = \mathrm{Spec}\left(A^G\right)$ then,

(a) $\tilde{\pi}$ is surjective and for any G-invariant closed $Z \subset U$ (meaning corresponding to a closed substack of $[U/G]$) then $\pi(Z) \subset U//G$ is closed. This property remains true after base change

(b) given G-invariant closed $Z_1, Z_2 \subset U$ then $\pi(Z_1 \cap Z_2) = \pi(Z_1) \cap \pi(Z_2)$ so if $x_1, x_2 \in U(k)$ then $\tilde{\pi}(x_1) = \tilde{\pi}(x_2)$ iff $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \varnothing$.

(c) If A is Noetherian so is A^G. If A is f.g. over k then AG^G is f.g. over k and for any f.g. A-module M with a G-action, M^G is a f.g. A^G-module

(d) if A is noetherian then $\tilde{\pi}$ is initial for G-invariant maps to algebraic spaces.

2.2 Cohomologically Affine Morphisms

Definition 2.2.1. A quasi-compact quasi-separated morphism $f : \mathscr{X} \to \mathcal{Y}$ is *cohomologically affine* if $f_* : \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact. We say $\mathscr{X}$ is *cohomologically affine* if $\mathscr{X} \to \mathrm{Spec}\,(\mathbb{Z})$ is.

Example 2.2.2. (a) an affine morphism is cohomologically affine

(b) an affine algebraic group G/k is linearly reductive iff $\mathbf{B}G \to \mathrm{Spec}\,(k)$ is cohomologically affine

Lemma 2.2.3. Consider a diagram,

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{f} & \mathscr{X} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

(a) if f is faithfully flat and π' is a GMS then π is a GMS

(b) if $\mathcal{Y}$ has quasi-affine diagonal and π is a GMS then π' is a GMS

Proof. (a) by flat base change $\pi'_* f^* \cong g^* \pi_*$ and f^* is exact and g^* is faithfully exact so we conclude. For (b), first consider the case that g is quasi-affine. Then factor as,

$$\mathcal{Y}' \hookrightarrow \mathbf{Spec}_{\mathcal{Y}}\,(f_* \mathcal{O}_{\mathcal{Y}'}) \to \mathcal{Y}$$

where the second map is affine by definition. If g is affine then g_* is faithfully exact. If g is an open immersion then for $F' \twoheadrightarrow G'$ in $\mathrm{QCoh}(\mathscr{X}')$ we conclude that $G := \mathrm{im}\,(f_* F' \to f_* G')$ since $f^* f_* = \mathrm{id}$ we get $f^* G = G'$ and π_* is exact so $\pi_* f_* F' \to \pi_* F$ so pullback via g then,

$$g^* \pi_* f_* F' = \pi'_* f^* f_* F' = \pi'_* F' \twoheadrightarrow g^* \pi_* G = \pi'_* f^* G = \pi'_* G$$

the first by flat basechange. Thus we conclude. To do the general case, assume $\mathcal{Y}$ and $\mathcal{Y}'$ are quasi-compact and choose a smooth presentation $Y = \mathrm{Spec}\,(A) \to \mathcal{Y}$ which is quasi-affine since $\mathcal{Y}$ is. Then $\mathcal{Y}'_Y \to \mathcal{Y}$ is faithfully flat so by (a) suffices to show that $\mathscr{X}'_Y \to \mathcal{Y}'_Y$ is cohomologically affine. By (a) again we can pass to a smooth cover $Y' \to \mathcal{Y}'_Y$ and reduce to a morphism of affine schemes which is hence affine so we win. $\square$

Corollary 2.2.4. If $f : \mathscr{X} \to \mathcal{Y}$ is representable and $\mathcal{Y}$ has quasi-affine diagonal. If f is cohomologically affine then f is affine.

Proof. Suffices to prove this for a map $f : X \to Y$ of schemes with Y affine. However, this is just Serre's criterion for affineness. $\square$

2.3 First Properties of GMS

Lemma 2.3.1. Consider a diagram,

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "f"] \arrow[d, "\pi'"] & \mathscr{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with X', X quasi-separated algebraic spaces and X has quasi-affine diagonal.

(a) if g is faithfully flat and π' is a GMS then π is a GMS

(b) if π is a GMS then π' is a GMS.

Now assume that π is a GMS then,

(a) there is a projection formula for $F \in \mathrm{QCoh}(\mathscr{X})$ and $G \in \mathrm{QCoh}(X)$ then,

$$\pi_* \mathscr{F} \otimes G \xrightarrow{\sim} \pi_*(F \otimes \pi^* G)$$

In particular,

$$G \xrightarrow{\sim} \pi_* \pi^* G$$

(b) $F \in \mathrm{QCoh}(\mathscr{X})$ then,

$$g^* \pi_* F \xrightarrow{\sim} \pi'_* f^* F$$

(c) for any $\mathscr{I} \subset \mathcal{O}_X$ a quasi-coherent ideal sheaf then,

$$\mathscr{I} \xrightarrow{\sim} \pi_*(\pi^{-1} \mathscr{I} \cdot \mathcal{O}_{\mathscr{X}})$$

Proof. For (a) and g flat then $g^*(\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathscr{X}})$ is just $\mathcal{O}_{X'} \to \pi'_* \mathcal{O}_{\mathscr{X}'}$ by flat base change. Thus if g is faithfully flat then we get,

$$\mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathscr{X}} \iff \mathcal{O}_{X'} \xrightarrow{\sim} \pi'_* \mathcal{O}_{\mathscr{X}'}$$

Furthermore, the lemma says that cohomological flatness descends under g proving (a). We also showed that (b) holds for flat base change. To prove it for arbitrary base change we first prove the projection formula. Indeed, let $U \to X$ be an étale presentation with U disjoint union of affine schemes. Then $\pi_U : \mathscr{X}_U \to U$ is a GMS by flat case. Then the pullback of $\mathrm{id} \to \pi_* \pi^*$ is $\mathrm{id} \to \pi_{U*} \pi_U^*$. We can assume that $X = \mathrm{Spec}\,(A)$ and consider,

$$G_2 \to G_1 \to G \to 0$$

is a free presentation. Then consider,

$$\begin{tikzcd}
\pi_* F \otimes G_2 \arrow[r] \arrow[d, equal] & \pi_* F \otimes G \arrow[r] \arrow[d, equal] & \pi_F \otimes G \arrow[r] \arrow[d] & 0 \\
\pi_*(F \otimes \pi^* G_2) \arrow[r] & \pi_*(F \otimes \pi^* G_1) \arrow[r] & \pi_*(F \otimes \pi^* G) \arrow[r] & 0
\end{tikzcd}$$

using that π_* is exact and the locally free form of the projection formula. Then we conclude the projection formula by the 5 lemma. (DO THE REST) ☐

Lemma 2.3.2. Let $\pi : \mathscr{X} \to X$ be a GMS and X is quasi-separated. Then,

(a) for $\mathcal{A} \in \mathrm{QCoh}(\mathscr{X})$ then $\mathbf{Spec}_{\mathscr{X}}\,(\mathcal{A}) \to \mathbf{Spec}_X\,(\pi_* \mathcal{A})$ is a GMS

(b) $\mathbb{Z} \subset \mathscr{X}$ is a closed substack defined by $\mathscr{I}$ then $Z = \pi(\mathbb{Z}) \subset X$ satisfies that $\mathbb{Z} \to Z$ is a GMS.

Proof. Indeed (b) is the special case of (a) where $\mathcal{A} = \mathcal{O}_{\mathscr{X}} / \mathscr{I}$. For (a) we see that,

$$\mathbf{Spec}_{\mathscr{X}}\,(\mathcal{A}) \to \mathscr{X} \times_X \mathbf{Spec}_X\,(\pi_* \mathcal{A}) \to \mathbf{Spec}_X\,(\pi_* \mathcal{A})$$

the first is affine and the second is cohomologically affine by base change. ☐

I SHOULD HAVE BEEN CAREFUL ABOUT π vs im since I MEAN SCHEME THEORETIC IMAGE

Proof of Theorem. (a) if $\mathscr{X}$ is quasi-sep then so is X. Then for all $x \in X(k)$ use Lemma 6.3.20 (b) $\mathscr{X} \times_X \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k)$ is a GMS. Then $\Gamma(\mathscr{X}_x, \mathcal{O}_{\mathscr{X}_x}) = k$ so $\mathscr{X}_x \neq \varnothing$ and hence π is surjective. To prove universal closedness consider $\mathbb{Z} \subset \mathscr{X}$ a closed substack use Lemma 6.3.22 (b) then $\mathbb{Z} \to \pi(\mathbb{Z})$ is a GMS and hence surjective hence $\pi(\mathbb{Z})$ the scheme theoretic image is just equal to the image and hence the image is closed. Then use preservation under base change to get universally closed. (b) if $\mathbb{Z}_i \subset \mathscr{X}$ are defined by $\mathscr{I}_i$ then apply π_* to the sequence,

$$0 \longrightarrow \mathscr{I}_1 \longrightarrow \mathscr{I}_1 + \mathscr{I}_2 \longrightarrow \mathscr{I}_2 / (\mathscr{I}_1 \cap \mathscr{I}_2) \longrightarrow 0$$

so we get,

$$\pi_* \mathscr{I}_1 + \pi_* \mathscr{I}_2 \cong \pi^*(\mathscr{I}_1 + \mathscr{I}_2)$$

and hence,

$$\mathrm{im}(\mathbb{Z}_1) \cap \mathrm{im}(\mathbb{Z}_2) = \mathrm{im}(\mathbb{Z}_1 \cap \mathbb{Z}_2)$$

☐

2.4 Finite Typeness of GMS

Definition 2.4.1. A morphism $f : X \to Y$ of schemes is *universally submersive* if it is surjective and Y has the quotient topology ($U \subset Y$ open iff $f^{-1}(U)$ is open) and this is true after any base change.

Lemma 2.4.2. Valuative criterion:

$$\begin{tikzcd}
\mathrm{Spec}\,(R') \arrow[r, dashed] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}\,(R) \arrow[r] & Y
\end{tikzcd}$$

if $X \to Y$ is universally submersive then there exists R'/R extensions of DVRs lifitng.

Example 2.4.3. The following maps are universally submersive,

(a) fppf covering

(b) good moduli space morphism

Proof. □

3 Determinantal Line Bundles

Recall that k is an algebraically closed field of characteristic zero.

Let X be a smooth, projective and connected curve over k. Let $\mathcal{S}$ be an algebraic stack over k. Then consider the diagram,

$$\begin{tikzcd}
& X \times \mathcal{S} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X & & \mathcal{S}
\end{tikzcd}$$

Since $X \times \mathcal{S} \to \mathcal{S}$ is representable by schemes we can use cohomology and base change theorem. If $\mathcal{E}$ is a vector bundle on $X \times \mathcal{S}$ then $\mathbf{R}p_*\mathcal{E}$ is a refect complex on $\mathcal{S}$ with amplitude in $[0,1]$ since this is true for the projection map $X \times T \to T$ from a test scheme. what does amplitude mean

Proposition 3.0.1. Let k be a field, S a k-scheme of finite type and $f : X \to S$ a smooth projective morphism of relative dimension n. If F is a flat coherent sheaf on X then there is a locally free resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to F \to 0$$

such that $R^n f_* F_\nu$ is locally free for $\nu = 0, \dots, n$ and $R^i f_* F_\nu = 0$ for $i \neq n$ and $\nu = 0, \dots, n$. Moreover, there is a natural quasi-isomorphism

$$[R^n f_* F_n \to R^n f_* F_{n-1} \to \cdots \to R^n f_* F_0] \to (\mathbf{R} f_* F)[n]$$

Proof. Let $\mathcal{O}_X(1)$ be an f-very ample line bundle on X. Since f is a Cohen-Macaulay morphism (it is even smooth) we have a Serre duality pairing,

$$R^i f_* \mathscr{F}(-m) \times R^{n-i} f_*(\mathscr{F}^\vee(m) \otimes \omega_{X/Y}) \to R^n f_* \omega_{X/Y} = \mathcal{O}_X$$

which is perfect. Using the mumford complex, there is some m_0 such that for $m \geq m_0$ we have $R^{n-i} f_*(\mathscr{F}^\vee(m) \otimes \omega_{X/Y}) = 0$ for all $i < n$ and hence $R^i f_* \mathscr{F}(-m) = 0$. Define S-flat sheaves K_ν and G_ν inductively as follows. Let $K_0 := F$. Assume that K_ν has been constructed. Since this is a bounded family, for sufficiently large $m \gg m_0$ all fibers $(K_\nu)_s$ are m-regular. Hence, by cohomology and base change $f_* K_\nu(m)$ is locally free (since its higher cohomology vanishes) and there is a natural surjection $G_\nu := f^*(f_* K_\nu(m))(-m) \to K_\nu$ (surjective on fibers by m-regularity). Then G_ν is locally free and

$$R^i f_* G_\nu = f_* K_\nu(m) \otimes R^i f_* \mathcal{O}_X(-m)$$

by the projection formula. In particular, $R^n f_* G_\nu$ is locally free and the other direct image sheaves vanish. Finally, define

$$K_{\nu+1} := \ker(G_\nu \to K_\nu)$$

Therefore we get an infinite locally free resolution $G_\bullet \to F$. Since for each $s \in S$

$$(K_n)_s = \ker((G_{n-1})_s \to (G_{n-2})_s)$$

and the fiber X_s is regular of dimension n it has global dimension n and therefore $(K_n)_s$ is locally free since it is the n^{th} syzygy module in the locally free resolution of the locally free sheaf F_s. Hence K_n is itself locally free by Nakayama[2]. Therefore, we can truncate to get the locally free resolution $F_\bullet$. The last statement follows from the first. Indeed, the quasi-isomorphism,

$$[F_n \to \cdots \to F_0] \to F[0]$$

gives the desired result after applying $\mathbf{R} f_*$ and showing that the natural map WHERE DOES IT COME FROM?

$$\mathbf{R} f_*[F_n \to \cdots \to F_0] \to [R^n f_* F_n \to \cdots \to R^n f_* F_0][-n]$$

is a quasi-isomorphism. Indeed, the derived functor spectral sequence shows that,

$$E_2^{p,q} = \mathcal{H}^p(R^q f_* F_\bullet) \implies \mathcal{H}^{p+q}(\mathbf{R} f_* F_\bullet)$$

but the only nonzero part of the E_2 page is the column (p, n) and thus

$$E^{p,n} = \mathcal{H}^p(R^n f_* F_\bullet) = \mathcal{H}^{p+n}(\mathbf{R} f_* F_\bullet)$$

showing that the natural map is a quasi-isomorphism. □

WHY CAN WE APPLY THIS TO A REPRESENTABLE MAP OF STACKS

Let $\mathcal{E}$ be a vector bundle on $X \times \mathcal{S}$ then there exists a short exact sequence,

[2] Suppose that $f : X \to S$ is a morphism and F is a coherent $\mathcal{O}_X$-module. If F_x is flat over $\mathcal{O}_{S,s}$ then F is locally free at x if and only if F_s is locally free at $x \in X_s$. Recall that F is locally free at x if and only if F_x is free over $\mathcal{O}_{X,x}$ since F is coherent. Then if $(F_s)_x = F_x / \mathfrak{m}_s F_x$ is free we lift the basis to

$$0 \to K \to \mathcal{O}_X x^{\oplus r} \to F_x \to 0$$

but F_x is flat over $\mathcal{O}_{S,s}$ so this stays exact when applying $- \otimes \mathcal{O}_{S,s}/\mathfrak{m}_s$ so we see that $K/\mathfrak{m}_s K = 0$ and hence $K/\mathfrak{m}_x K = 0$ so by Nakayama's lemma $K = 0$.

$$0 \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E} \longrightarrow 0$$

such that $R^0 p_* \mathcal{E}^{j-1} = 0$ and $K^j = \mathbf{R}^1 \mathcal{E}^{j-1}$ is locally free for $j = 0, 1$ (Note: I think there is a typo in the indices in the notes) therefore there is a quasi-isomorphism,

$$[K^0 \to K^1] \to \mathbf{R} p_* \mathcal{E}$$

Definition 3.0.2. If $\mathcal{E}$ is a vector bundle on $X \times \mathcal{S}$ and $[K^0 \to K^1]$ is a two-term complex of locally free sheaves quasi-isomorphic to $\mathbf{R} p_* \mathcal{E}$ then we define the line bundle,

$$\det \mathbf{R} p_* \mathcal{E} := \det K^0 \otimes (\det K^1)^\vee$$

We could also make this definition for any perfect complex on $\mathcal{S}$. The rank of a perfect complex is defined as the alternating sum of the ranks of a representative. This is independent of the choice of representative because localizing at the generic point reduces to the case of finite dimensional vectorspaces for whcih Euler characteristic of a bounded complex coincides with the alternating sum of dimensions. If $\operatorname{rank}(\mathbf{R} p_* \mathcal{E}) = 0$ then by definition $\operatorname{rank} K^0 = \operatorname{rank} K^1$ so the dual $(\det \mathbf{R} p_* \mathcal{E})^\vee$ is then equipped with a section given by the determinant of the map $K^0 \to K^1$.

Lemma 3.0.3. The definition of $\det \mathbf{R} p_* \mathcal{E}$ is independent of the choice of representative perfect complex.

Lemma 3.0.4. Given an exact sequence,

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

then,

$$\det \mathbf{R} p_* \mathcal{E} = (\det \mathbf{R} p_* \mathcal{E}') \otimes (\det \mathbf{R} p_* \mathcal{E}'')$$

Proof. We find a complex $\mathcal{E}^\bullet = [\mathcal{E}^{-1} \to \mathcal{E}^0]$ and $\mathcal{E}^\bullet \to \mathcal{E}$ a quasi-isomorphism as above. Choose $\mathcal{E}'^\bullet$ and $\mathcal{E}''^\bullet$ similarly. In fat, it follows from the construction that these resolutions may be choosen comptabily so that there is a short exact sequence of complexes,

$$0 \longrightarrow \mathcal{E}'^\bullet \longrightarrow \mathcal{E}^\bullet \longrightarrow \mathcal{E}''^\bullet \longrightarrow 0$$

compatible with the quasi-isomorphisms to the previous sequence. Taking cohomology, we find a short exact sequence

$$0 \longrightarrow K'^\bullet \longrightarrow K^\bullet \longrightarrow K''^\bullet \longrightarrow 0$$

of complex of locally free sheaves on $\mathcal{S}$. The result follows from the multiplicativity of determinants in short exact sequences of locally free sheaves,

$$\det K^0 = \det K'^0 \otimes \det K''^0 \qquad \det K^1 = \det K'^1 \otimes \det K''^1$$

and therefore,

$$\det K^\bullet := \det K^0 \to (\det K^1)^\vee = \det K'^0 \to \det K''^0 \otimes (\det K'^1 \otimes \det K''^1)^\vee = \det K'^\bullet \otimes (\det K''^\bullet)^\vee$$

□

We will apply this construction to the case $\mathcal{S} = \mathcal{M}_X(r, d)$ and $\mathcal{E} = \mathcal{E}_{\text{univ}} \otimes q^* V$ where V is a vector bundle on X.

Definition 3.0.5. For a vector bundle V on X we define the *determinantal line bundle*

$$\mathcal{L}_V := (\det \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V)^\vee$$

on $\mathcal{M}_X(r,d)$ associated to V. If $\chi(X, E \otimes V) = 0$ for all $[E] \in \mathcal{M}_X(R,d)$ then the rank of $\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V)$ is zero and we define the section

$$s_V \in \Gamma(\mathcal{M}_X(r,d), \mathcal{L}_V)$$

Remark. Since $\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V)$ is perfect, its construction commutes with base change. In particular, its restrciction to a k-point $[E] \in \mathcal{M}_X(r,d)$ is identified with the two-term complex $\mathbf{R}\Gamma(X, \mathcal{E} \otimes V)$. If moreover

$$\chi(X, E \otimes V) = h^0(X, E \otimes V) - h^1(X, E \otimes V) = 0$$

we see that indeed $\operatorname{rank} \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V) = 0$.

Remark. Note that,

$$\deg E \otimes V = (\deg E)(\operatorname{rank} V) + (\operatorname{rank} E)(\deg V)$$

and therefore by Riemann-Roch $\chi(X, E \otimes V) = 0$ if and only if,

$$d \operatorname{rank} V + r \deg V + (1-g) r \operatorname{rank} V = 0$$

or equivalently,

$$(*) \quad \mu(E) \operatorname{rank} V + \deg V + (1-g) \operatorname{rank} V = 0$$

Notice that this is actually just a numerical condition on V that only depends on the slope of E. Therefore, if $(*)$ holds then $\chi(X, E \otimes V) = 0$ for all vector bundles E with fixed slope $\mu(E)$.

The construnction of the determinantal line bundle defines a morphism,

$$\det : \mathcal{M}_X(r,d) \to \mathscr{Pic}^d_X$$

such that,

$$(\mathrm{id}_X \times \det)^* \mathcal{P} = p^* \det \mathcal{E}_{\text{univ}}$$

where $\mathcal{P}$ is the Poincare bundle on $\mathscr{Pic}^d_X \times X$.

Proposition 3.0.6. The following hold:

(a) the assigment $V \mapsto \mathcal{L}_V$ induces a group homomorphism,

$$K_0(X) \to \operatorname{Pic}(\mathcal{M}_X(r,d))$$

meaning the isomorphism class of $\mathcal{L}_V$ depends only on $\operatorname{rank} V$ and $\det V$

(b) if V and W are vector bundles of the same rank and degree then there exists a line bundle $\mathcal{N}$ on $\mathscr{Pic}^d_X$ such that,

$$\mathcal{L}_W \cong \mathcal{L}_V \otimes \det^* \mathcal{N}$$

where $\det^*$ is the pullback along the map $\det : \mathcal{M}_X(r,d) \to \mathscr{Pic}^d_X$.

Proof. For an exact sequence of vector bundles on X,

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

we get an exact sequence using flatness of q and the fact that $\mathcal{E}_{\text{univ}}$ is a vector bundle,

$$0 \longrightarrow q^*V_1 \otimes \mathcal{E}_{\text{univ}} \longrightarrow q^*V_2 \otimes \mathcal{E}_{\text{univ}} \longrightarrow q^*V_3 \otimes \mathcal{E}_{\text{univ}} \longrightarrow 0$$

which thus induces an isomorphism,

$$\mathcal{L}_{V_2} \cong \mathcal{L}_{V_1} \otimes \mathcal{L}_{V_3}$$

proving the factorization of the map of monoids through $K_0(X)$,

$$\begin{tikzcd} \mathrm{Vect}_X \arrow[rr] \arrow[dr] & & \mathrm{Pic}\,(\mathcal{M}_X(r,d)) \\ & K_0(X) \arrow[ur, dashed] & \end{tikzcd}$$

Then the consequence follows from the isomorphism $K_0(X) = \mathbb{Z} \oplus \mathrm{Pic}\,(X)$ given by $V \mapsto (\mathrm{rank}\, V, \det V)$.

To prove the second statment, using the first we may assume that $V = \mathcal{O}_X^{\oplus r_V - 1} \oplus \mathcal{O}_X(D)$ for some divisor D on X and similarly for W. Moreover, writing $D = D_1 - D_2$ as a difference of effective divisors we see that,

$$[\mathcal{O}_X(D)] = [\mathcal{O}_X] + [\mathcal{O}_{D_1}] - [\mathcal{O}_{D_2}]$$

in $K_0(X)$. Therefore, the classes of V and W in $K_0(X)$ differ only by the class of a divisor of degree 0 DOESNT THE PROOF WORK FOR DIVISORS OF ANY DEGREE?. Thus by the additivity of the derminantal construction, it suffices to prove that,

$$\det \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*\mathcal{O}_x) \cong \det{}^* \mathcal{N}'$$

for some line bundle $\mathcal{N}'$ on $\mathscr{Pic}^d_X$ where $x \in X$ is a closed point. Viewing $\mathcal{E}_x$ and $\mathcal{P}_x$ as sheaves on $\mathcal{M}_X(r,d)$ and $\mathscr{Pic}^d_X$ respectively as pullback along the sections $\mathcal{S} \to X \times \mathcal{S}$ defined by $x \in X$ then,

$$\det \mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*\mathcal{O}_x) = \det \mathcal{E}_x = \det{}^* \mathcal{P}_x$$

which proves the claim. □

Definition 3.0.7. Let E be a vector bundle on X. We say that E is *cohomology-free* if $h^0(X,E) = h^1(X,E) = 0$.

Proposition 3.0.8. If $(*)$ holds for V then the following are equivalent,

(a) the section $s_V \in \Gamma(\mathcal{M}_X(r,d), \mathcal{L}_V)$ is nonzero at $[E]$

(b) $E \otimes V$ is cohomology-free

Proof. The morphism $\det j : \det K_0 \to \det K_1$ of line bundles is nonzero at the point $[E] \in \mathcal{M}_X(r,d)$ if and only if the morphism of vector bundles $j : K - 0 \to K_1$ is an isomorphism at $[E]$. Since $\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V)$ is quasi-isomorphic to $[K_0 \to K_1]$ we see that j is an isomorphism if and only if $(\mathbf{R}p_*(\mathcal{E}_{\text{univ}} \otimes q^*V))_{[E]} = 0$ if and only if $h^0(X, E \otimes V) = h^1(X, E \otimes V) = 0$ by cohomology and base change. □

Remark. While $\mathcal{L}_V$ only depends on $\mathrm{rank}\, V$ and $\det V$ the section s_V *does* depend on V itself since it can detect cohomology-freeness as above. We will leverage this fact to produce enough sections of $\mathcal{L}_V$ to establish ampleness. Notice also that, under the assumption that $\chi(X, E \otimes V) = 0$ the vanishing of $H^0(X, E \otimes V)$ is equivalent to the vanishing of $H^1(X, E \otimes V)$.

Now we specialize this discussion to the open substack $\mathcal{M}_X^{ss}(r,d) \subset \mathcal{M}_X(r,d)$. The goal is to prove that the good mouli space $M_X^{ss}(r,d)$ is projective. Our candidate ample line bundle is the descent of some $\mathcal{L}_V$.

Proposition 3.0.9. The determinantal line bundle $\mathcal{L}_V$ associated to a vector bundle V satisfying $(*)$ descends to $M_X^{ss}(r,d)$ uniquely meaning there exists a unique line bundle $L_V \in \mathrm{Pic}\,(M_X^{ss}(r,d))$ such that $\mathcal{L}_V \cong \phi^* L_V$.

Proof. By Thorem 3.5(iv), we must show that stabilizers of $\mathcal{M}_X^{ss}(r,d)$ act trivially on the fibers of $\mathcal{L}_V$. By Theorem 3.12(ii), the closed points of $\mathcal{M}_X^{ss}(r,d)$ correspond to polystable bundles,

$$E = \bigoplus_{j=1}^{n} E_j^{\oplus m_j}$$

where the E_i are pairwise nonisomorphic stable bundles. Since E is also semistable the slopes of the E_i must all be equal so $\mu(E_i) = \mu(E) = \frac{d}{r}$. Since these are nonisomorphic stable bundles with the same slope there are no nonzero morphisms between them (we have $\mathrm{Hom}\,(E_i, E_j) = k \cdot \delta_{ij}$). Since $\mathrm{End}\,(E_j) = k$ we see that,

$$\mathrm{Aut}\,(E) = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_n}$$

The fiber of $\mathcal{L}_V|_{[E]}$ is identified with

$$\det \mathbf{R}\Gamma(X, E \otimes V) = \prod_{i=1}^{n} (\det H^i(X, E \otimes V))^{\otimes(-1)^i}$$

An element $(g_1, \ldots, g_n) \in \mathrm{Aut}\,(E)$ act on,

$$\det H^i(X, E \otimes V) \cong \bigotimes_{j=1}^{n} (\det H^i(X, E_j \otimes V))^{\otimes m_j}$$

by multiplication with,

$$\prod_{j=1}^{n} \det\,(g_i)^{\dim H^i(X, E_j \otimes V)}$$

and thus on $\mathcal{L}_V|_{[E]}$ by multiplication with,

$$\prod_{j=1}^{n} \det\,(g_j)^{\chi(X, E_j \otimes V)}$$

but each E_j has slope $\mu(E_j) = \frac{d}{r}$ so by $(*)$ we have $\chi(X, E_j \otimes V) = 0$ since it implies vanishing for all E with $\mu(E) = \frac{d}{r}$ and hence the action is trivial. $\square$

3.0.1 Moduli of Vector Bundles with Fixed Determinant

Let L be a line bundle on X of degree d corresponding to a closed point of $\mathscr{Pic}_X^d$ represented by a morphism $[L] : \mathrm{Spec}\,(k) \to \mathscr{Pic}_X$ IS EVERY k-point closed? in Pic? I Think so since X/k is SMOOTH so we should have separatedness. Then consider the diagram,

$$\begin{tikzcd}
\mathcal{M}_X(r,L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_X(r,d) \arrow[d, "\det^*"] \\
\mathrm{Spec}\,(k) \arrow[r, "L"] & \mathscr{Pic}_X^d
\end{tikzcd}$$

Explicitly, $\mathcal{M}_X(r, L)$ is the stack of pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a vector bundle on $X \times S$ of rank r and degree d fiberwise equipped with an isomorphism $\varphi : \det \mathcal{E} \xrightarrow{\sim} L|_{X \times S}$. The condition of constant degree d on fibers is implied by determinant isomorphism which is why d is left out of the notation.

Corollary 3.0.10. For a line bundle L of degree d on X, the restriction of the determinantal line bundle L_V to $M_X^{ss}(r, L)$ only depends on the rank and degree of V.

Proof. We have commuting diagrams,

$$\begin{tikzcd} \mathcal{M}_X^{ss}(r, L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_X^{ss}(r, d) \arrow[d] \\ \mathrm{Spec}\,(k) \arrow[r, "L"] & \mathscr{P}\!ic_X^d \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{M}_X^{ss}(r, L) \arrow[r, hook] \arrow[d] & \mathcal{M}_X^{ss}(r, d) \arrow[d] \\ M_X^{ss}(r, L) \arrow[r, hook] & M_X^{ss}(r, d) \end{tikzcd}$$

where in the second square, both vertical arrows are hood moduli space morphisms. If V and W are vector bundles of equal rank and degree on X, both satisfying condition (10) then there exists a line bundle $\mathcal{N}$ on $\mathscr{P}\!ic_X^d$ such that $\mathcal{L}_W \cong \mathcal{L}_V \otimes \det^* \mathcal{N}$. The left diagram shows that, restriction to $\mathcal{M}_X^{ss}(r, L)$, this isomrophism becomes $\mathcal{L}_W \cong \mathcal{L}_V$. The right diagram now shows that the restriction of L_V and L_W to $M_X^{ss}(r, L)$ become isomrophic after pulling back to $\mathcal{M}_X^{ss}(r, L)$, so the restrictions must be isomorphic by the uniqueness of the descent along the good moduli space morphism $\mathcal{M}_X^{ss}(r, L) \to M_X^{ss}(r, L)$. $\square$

Theorem 3.0.11 (Drézet-Narasimhan)**.** There exist isomorphisms,

$$\mathrm{Pic}\,(M_X^{ss}(r, d)) \cong \mathrm{Pic}\left(\mathrm{Pic}_X^d\right) \oplus \mathbb{Z}$$
$$\mathrm{Pic}\,(M_X^{ss}(r, L)) \cong \mathbb{Z}$$

In the first line, $\mathbb{Z}$ is WHAT IS IT GENERATED BY? In the second line, $\mathbb{Z}$ is generated by the determinantal line bunle L_V where V is chosen to be of minimal rank,