1 Flag Varieties

Definition 1.1. Let V be a vectorspace then the Grassmaniann $G_d(V)$ is the space of d-dimensional subspaces of V. For any subspace $S \subset V$ of complementary dimension $n-d$ we define,

$$G_d(V)_S = \{W \in G_d(V) \mid W \cap S = \{0\}\}$$

Thus, writing $V = W_0 \oplus S$ for some fixed W_0 then projection $W \to W_0$ defines an isomorphism,

$$G_d(V)_S \xrightarrow{\sim} \mathrm{Hom}\,(W_0, S)$$

These form an affine open cover of $G_d(V)$ with its variety structure with,

$$G_d(V)_S \cong \mathbb{A}(\mathrm{Hom}\,(W_0, S))$$

Furthermore this shows that $G_d(V)_S$ is smooth with tangent space,

$$T_{W_0} G_d(V) = \mathrm{Hom}\,(W_0, S) = \mathrm{Hom}\,(W_0, V/W_0)$$

Definition 1.2. We extend the above discussion to chain s of subspaces. Let $\mathbf{d} = (d_1, \dots, d_r)$ be a sequence of integers with $n > d_1 > d_2 > \cdots > d_r > 0$ and let $G_{\mathbf{d}}(V)$ be the space of flags,

$$F : \; V \supset V^1 \supset \cdots \supset V^r \supset 0$$

with $\dim V^i = d_i$. The map,

$$G_{\mathbf{d}}(V) \xrightarrow{F \mapsto (V^i)} \prod_i G_{d_i}(V) \subset \prod_i \mathbb{P}\left(\bigwedge^{d_i} V \right)$$

gives an embedding of $G_{\mathbf{d}}(V)$ inside $\prod_i G_{d_i}(V)$ showing that $G_{\mathbf{d}}(V)$ is a projective variety.

2 The Hodge Filtration

Given any Hodge structure of weight n there is a filtration,

$$F^\bullet : \; F^{-n} \supset F^1 \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \qquad\qquad F^p = \bigoplus_{r \ge p} V^{r,s} \subset V_{\mathcal{C}}$$

Note that when $p + q = n$ we have,

$$\overline{F^q} = \bigoplus_{s \ge q} \overline{V^{s,r}} = \bigoplus_{S \ge q} V^{r,s} = \bigoplus_{r \le p} V^{r,s}$$

Therefore,

$$V^{p,q} = F^p \cap \overline{F^q}$$

Recall that Hodge structures correspond to representations of the Deligne torus, $h : \mathbb{S} \to \mathbb{G}_m^{\mathbb{R}}$ which is a map of $\mathbb{R}$-algebraic groups. Recall that h acts on $V^{p,q}$ on real points $z \in \mathbb{S}(\mathbb{R}) = \mathcal{C}^\times$ via $h(z) \cdot v = z^{-p} \bar{z}^{-q} \cdot v$ for $v \in V^{p,q}$.

3 Variations of Hodge Structures

4 Shimura Data

Proposition 4.1. Let $\varphi : G \to H$ be a surjective map of algebraic groups over $\mathbb{R}$. Then $\varphi(\mathbb{R}) : G(\mathbb{R})^+ \to H(\mathbb{R})^+$ is surjective.

Proof. Viewing this as a map of Lie groups, it suffices to show that $\mathrm{d}\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is surjective. In that case $\mathrm{Im}(\varphi)$ is open (since it is a local diffeomorphism) and closed since it is an open subgroup. Therefore, $\mathrm{Im}(\varphi) \supset H(\mathbb{R})^+$ the connected component of the identity.

(Q? Why is it surjective on tangent spaces?) □

Remark. For a reductive group G with center $Z = Z(G)$ and a torus T there is an exact diagram,

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & G^{\mathrm{der}} \arrow[d] \arrow[dr] & & \\
1 \arrow[r] & Z \arrow[r] \arrow[dr] & G \arrow[r, "\mathrm{ad}"] \arrow[d, "\nu"] & G^{\mathrm{ad}} \arrow[r] & 1 \\
 & & T \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}$$

Then $Z \cap G^{\mathrm{der}}$ is the center of G^{der} and $G^{\mathrm{der}} = \ker(G \to T)$ where T is the maximal abelian quotient. (Q? why don't we call $G^{\mathrm{der}} = G^{\mathrm{ab}}$ in this case?)

This gives an exact sequence,

$$1 \longrightarrow Z \cap G^{\mathrm{der}} \longrightarrow Z \times G^{\mathrm{der}} \longrightarrow G \longrightarrow 1$$

Definition 4.2. We define the cohomology $H^1(\mathbb{Q}, G)$ as crossed modulo principal homomorphisms $\mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right) \to G(\mathbb{Q}^{\mathrm{ab}})$. (Q? Is $\mathbb{Q}^{\mathfrak{a}} = \mathbb{Q}^{\mathrm{ab}}$? How to make sense of this? Is this like taking an abelian version of the etale site and viewing G as a sheaf on this site and taking cohomology?)

Proposition 4.3. For a reductive group G over $\mathbb{R}$ the Lie group $G(\mathbb{R})$ has only finitely many connected components.

Proof. Using the above map, an exact sequence of real algebraic groups,

$$1 \longrightarrow N \longrightarrow G' \longrightarrow G \longrightarrow 1$$

with $N \subset Z(G')$ gives rise to an exact sequence,

$$\pi_0(G'(\mathbb{R})) \longrightarrow \pi_0(G(\mathbb{R})) \longrightarrow H^1(\mathbb{R}, N)$$

(Q? I don't understand this?? If it were ordinary cohomology which could be related to π_1 then I understand but how group cohomology? Are you using the fibration seqeunce?) □

Theorem 4.4 (real approximation)**.** Let G be a connected algebraic group over $\mathbb{Q}$. Then $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$.

4.1 The Data

Definition 4.5. A Shimura datum is a pair (G, X) with G a reductive group over $\mathbb{Q}$ and X a set of $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ as real algebraic groups. The pair must satisfy the foloowing conditions,

(a). for all $h \in X$ the Hodge structure defined on $\mathrm{Lie}\,(G_{\mathbb{R}})$ via,

$$\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{Aut}\,(\mathrm{Lie}\,(G_{\mathbb{R}}))$$

has type $\{(-1, 1), (0, 0), (1, -1)\}$.

(b). for all $h \in X$ the composition $\mathrm{ad} \circ h$ takes $\mathrm{ad}(h(i))$ to the Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$ (i.e. if $-B(X, \mathrm{ad}(h(i)) \cdot Y)$ is a positive-definite bilinear form where B is the killing form)

(c). G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Remark. The first condition says that $\mathrm{Ad} \circ h : \mathbb{S} \to \mathrm{Aut}\,(\mathrm{Lie}\,(G_{\mathbb{R}}))$ acts only via the characters $z/\bar{z}, 1, \bar{z}/z$.

Remark. Note that in contrast to the connected case, G is reductive (rather than semi-simple), h has target $G_{\mathbb{R}}$ (rather than $G^{\mathrm{ad}}_{\mathbb{R}}$), and X is a full $G(\mathbb{R})$-conjugacy class (not a connected component).

Proposition 4.6. Let (G, X) be a Shimura datum and X^+ be a connected component of $X' = \{\mathrm{ad} \circ h : \mathbb{S} \to \mathscr{G}^{\mathrm{ad}}_{\mathbb{R}} \mid h \in X\}$ (regarded as a $G(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$). Then (G^{der}, X^+) is a connected Shimura datum. In particular, X is a finite disjoint union of hermitian symmetric domains.

4.2 Shimura Varieties

Lemma 4.7. Let (G, X) be a Shimura datum, for every connected component X^+ of X there is a natural bijection,

$$G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}_f) \xrightarrow{\sim} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)$$

Proof. The map is surjective because $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$. ☐

Lemma 4.8. For every open subgroup $K \subset G(\mathbb{A}_f)$ the set $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$, the set $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ is finite.

Definition 4.9. Gor a compact open subgroup $K \subset G(\mathbb{A}_f)$, consider the double coset space,

$$\mathrm{Sh}_K\,(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$$

in which $G(\mathbb{Q})$ acts on X and $G(\mathbb{A}_f)$ on the left, and K acts on $G(\mathbb{A}_f)$ on the right:

$$q(x, a)k = (qx, qak) \quad q \in G(\mathfrak{q}) \quad x \in X \quad a \in G(\mathbb{A}_f) \quad k \in K$$

Lemma 4.10. Let C be a set of representatives for the double coset space,

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$$

and let X^+ be a commnected component of X. Then,

$$G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \cong \bigsqcup_{g \in C} \Gamma_g \backslash X^+$$

where Γ_g is the subgroup $gKg^{-1} \cap G(\mathbb{Q})_+$ of $G(\mathbb{Q})_+$. When we endow X with its usual topology and $G(\mathbb{A}_f)$ with its adelic topology this is a homeomorphism.

Remark. Because Γ_g is a congrence subgroup of $G(\mathbb{Q})$, its image in $G^{\mathrm{ad}}(\mathbb{Q})$ is arithmetic (WHAT?) and so its image in $\mathrm{Aut}\,(X^+)$ is arithmetic. Moreove, when K is sufficently small, Γ_g will be neat for all $g \in C$ and so its image in $\mathrm{Aut}\,(X^+)^+$ will also be near and hence torsion-free. Then $\Gamma_G \backslash X^+$ is an arithmetic locally symmetric variety, and $\mathrm{Sh}_K\,(G,X)$ is a finite disjoint union of such varieties. Moreover, for an inclusion $K' \subset K$ of sufficiently small compact open subgroups of $G(\mathbb{A}_f)$, the natural map $\mathrm{Sh}_{K'}\,(G,X) \to \mathrm{Sh}_K\,(G,X)$ is regular. Thus, when we vary K, we get an inverse system of algebraic vareties $(\mathrm{Sh}_K\,(G,X))_K$. There is a natural action of $G(\mathbb{A}_f)$ on the system as follows: for $g \in G(\mathbb{A}_f)$ we send $K \mapsto g^{-1}Kg$ which sends compact open subgroups to compact open subgroups then,

$$T(g) : \mathrm{Sh}_K\,(G,X) \to \mathrm{Sh}_{g^{-1}Kh}\,(G,X)$$

Definition 4.11. Let (G,X) be a Shimura datum. A Shimura variety relative to (G,X) is a variety of the form $\mathrm{Sh}_K\,(G,X)$ for some compact open subgroup $K \subset G(\mathbb{A}_f)$.

The *Shimura variety* $\mathrm{Sh}\,(G,X)$ attached to the Shimura datum (G,X) is the inverse system of varieties $(\mathrm{Sh}_K\,(G,X))_K$ endowed with the $G(\mathbb{A}_f)$ action.

4.3 Morphisms of Shimura Varieties

Definition 4.12. Let (G,X) and (G',X') be Shimura data,

(a). a morphism of Shimura data $(G,X) \to (G',X)$ is a homomorphism $G \to G'$ of algebraic groups sending $X \twoheadrightarrow X'$

(b). A morphism of Shimura varieties $\mathrm{Sh}\,(G,X) \to \mathrm{Sh}\,(G',X')$ is an inverse system of regular maps of algebraic varieties equivariant with respect to the $G(\mathbb{A}_f)$-action.

Theorem 4.1. A morphism of Shimura data $(G,X) \to (G',X')$ defines a morphism $\mathrm{Sh}\,(G,X) \to \mathrm{Sh}\,(G',X')$ of Shimura varieties which is a closed immersion if $G \to G'$ is injection.

Remark. What does it mean here to be a closed immersion? Does this mean that each map in the sequence is a closed immersion? Okay, it means that for sufficiently small K it becomes a closed immersion.

4.4 The Structure of a Shimura Variety

5 Symplectic Space

Definition 5.1. Let k be a field ($\mathrm{char}(k) \neq 2$) then a *symplectic space* (V,ψ) is a $2n$-dimensional k-vectorspace V and a nondegenerate alternative 2-form ψ.

We say that a subspace $W \subset V$ is totally isotropic if $\psi(W,W) = 0$.

A symplectic basis of V is a basis $\{e_{\pm i}\}$ such that,

$$\begin{aligned} \psi(e_{+i}, e_{-i}) &= 1 \quad \text{for} 1 \le i \le n \\ \psi(e_i, e_j) &= 0 \quad \text{for} j \neq \pm i \end{aligned}$$

Definition 5.2. For a symplectic space (V,ψ) we define $\mathrm{GSp}\,(\psi)$ to be endomorphisms of V which preserve ψ up to scaling,

$$\mathrm{GSp}\,(\psi)\,(k) = \{g \in \mathrm{GL}_n(V) \mid \psi(gu,gv) = v(g) \cdot \phi(u,v) \quad v(g) \in k^\times\}$$

Remark. Here $\mathrm{GSp}(\psi)$ is a sort of conformal version of the symplectic group.

Remark. There is a homomorphism $v : \mathrm{GSp}(\psi) \to \mathbb{G}_m$ sending $g \mapsto v(g)$. Then clearly $\ker \nu = \mathrm{Sp}(\psi)$. Furthermore, $\mathrm{Sp}(\psi) \hookrightarrow \mathrm{GSp}(\psi)$ is the derived subgroup giving a diagram,

$$\begin{tikzcd} & \mathrm{Sp}(\psi) \arrow[d] \arrow[dr] & \\ \mathbb{G}_m \arrow[r] \arrow[dr] & \mathrm{GSp}(\psi) \arrow[r, "\mathrm{ad}"] \arrow[d, "\nu"] & \mathrm{GSp}(\psi)^{\mathrm{ad}} \\ & \mathbb{G}_m & \end{tikzcd}$$

5.1 Shimura Datum Attached to a Symplectic Space

Let (V, ψ) be a symplectic space over $\mathbb{Q}$ and let $G(\psi) = \mathrm{GSp}(\psi)$ and $S(\psi) = \mathrm{Sp}(\psi) = G^{\mathrm{der}}$.

Now let J be a complex structure on $V(\mathbb{R})$ such that $\psi(Ju, Jv) = \psi(u, v)$. Then clearly, $J \in S(\psi)(\mathbb{R})$. For each $z \in \mathbb{C}^\times$ we get $h_J(z) \in G(\psi)(\mathbb{R})$ and it lies in $S(\psi)(\mathbb{R})$ if $|z| = 1$ (UNDERSTAND THIS?). We say that J is *positive* if $\psi_J(u, v) := \psi(u, Jv)$ is positive-definite.

Let $X^\pm$ denote the set of positive and respectively negatively complex constructures on $V(\mathbb{R})$ such that $\psi(Ju, Jv) = \psi(u, v)$ for all $u, v \in V$ and let $X(\psi) = X^+ \sqcup X^-$. Then $G(\psi)(\mathbb{R})$ acts on X according to,

$$(g, J) \mapsto gJg^{-1}$$

and the stabilizer in $G(\psi)(\mathbb{R})$ of X^+ is,

$$G(\psi)(\mathbb{R})^+ = \{g \in G(\psi)(\mathbb{R}) \mid v(g) > 0\}$$

Proposition 5.3. Then the pair $(G(\psi), X(\psi))$ satisfies the axioms of the Shimura datum and furthermore satisfies the futher additional axioms,

SV2' for all $h \in X$ then $\mathrm{ad}(h(i))$ is a Cartan involution on $G_\mathbb{R}/w_X(\mathbb{G}_m)$ (not just $G_\mathbb{R}^{\mathrm{ad}}$)

(a). the weight map $w_X : \mathbb{G}_m \to G_\mathbb{R}$ is defined over $\mathbb{Q}$

(b). the group $Z(\mathbb{Q}) \hookrightarrow Z(\mathbb{A}_{\mathrm{fin}})$ is discrete

(c). the torus Z° splits over a CM-field

where the weight homomorphism $w_X : \mathbb{G}_m\mathbb{R} \to Z(G)_\mathbb{R}^\circ \subset G_\mathbb{R}$ is a map over $\mathbb{R}$ of $\mathbb{Q}$-tori.

Remark. We call $(G(\psi), X(\psi))$ the Siegel Shimura datum associated to a symmetric space (V, ψ).

5.2 The Siegel Modular Variety

Let $(G, X) = (G(\psi), X(\psi))$ be the Shimura datum defined by a symplectic space (V, ψ) over $\mathbb{Q}$. Then the Siegal modular variety of (V, ψ) is simply $\mathrm{Sh}(G, X)$.

Now let $V(\mathbb{A}_{\mathrm{fin}}) = V \otimes_\mathbb{Q} \mathbb{A}_{\mathrm{fin}}$ thne $G(\mathbb{A}_{\mathrm{fin}})$ is the group of $\mathbb{A}_{\mathrm{fin}}$-linear automorphisms of $V(\mathbb{A}_{\mathrm{fin}})$ perserving ψ up to multiplication by an element of $\mathbb{A}_{\mathrm{fin}}^\times$.

Let $K \subset G(\mathbb{A}_{\mathrm{fin}})$ be a compact open subgroup of $G(\mathbb{A}_{\mathrm{fin}})$ and let $\mathscr{H}_K$ denote the set of triples $((W, h), s, \eta K)$ where, (a). (W, h) is a rational Hodge structure of type $(-1, 0)$ and $(0, -1)$

(b). one of s or $-s$ is a polarization for (W, h)

(c). ηK is a K-orbit of $\mathbb{A}_{\text{fin}}$-linear isomorphism $V(\mathbb{A}_{\text{fin}}) \to W(\mathbb{A}_{\text{fin}})$ under which ψ corresponds to s up to $\mathbb{A}_{\text{fin}}^\times$-scaling.

Furthermore, an isomorphism $b : ((W, h), s, \eta K) \to ((W', h'), s', \eta' K)$ of triples is an isomorphism $b : (W, h) \to (W', h')$ of rational Hodge structures sending s to cs' for $c \in \mathbb{Q}^\times$ and $b \circ \eta = \eta'$ modulo K.

6 Shimura Varieties of Hodge Type

Remark. For a symplectic space (V, ψ) denote the associated Shimura data by $(G(\psi), X(\psi))$.

Definition 6.1. A Shimura datum (G, X) is of *Hodge type* if there is a symplective space (V, ϕ) over $\mathbb{Q}$ and an injection $\rho : G \hookrightarrow G(\psi)$ sending X into $X(\psi)$. The Shimura variety $\mathrm{Sh}\,(G, K)$ is then of *Hodge type*.

Remark. The embedding $\rho : G \hookrightarrow G(\psi)$ composes with $\nu : G(\psi) \to \mathbb{G}_m$ to give a character $\nu : G \to \mathbb{G}_m$ of G.

Proposition 6.2. Let $\mathbb{Q}(r)$ be the Hodge struture on $\mathbb{Q}$ with G acting by $g \cdot x = \nu(g)^r \cdot g$. Then for each $h \in X$ we see that $(\mathbb{Q}(r), \nu \circ h)$ is a rational Hodge structure of type $(-r, -r)$ so this agrees with previous notation.

Proposition 6.3. There exist multilinear maps $t_1, \dots, t_n : V \times \cdots \times V \to \mathbb{Q}(r_i)$ such that G is the subgroup of $G(\psi)$ fixing these maps.

Proof. This is an application of Chevalley's theorem since this is equivalent to finding a set of tensors which give G as the group fixing these tesnors. $\square$

Remark. Let (G, X) be of Hodge type an schoose an embedding $(G, X) \hookrightarrow (G(\psi), X(\psi))$ for some symplectic space (V, ψ) and multilinear maps $t_1, \dots, t_n : V \times \cdots \times V \to \mathbb{Q}(r_i)$ with G their stabilizing subgroup.

We now define a set $\mathscr{H}_K$ of triples, $((W, h), (s_i), \eta K)$ as before satisfying,

(a). (W, h) is a rational Hodge structure of type $(-1, 0), (0, -1)$

(b). s_0 or $-s_0$ is a polarization for (W, h)

(c). $s_1, \dots, s_n$ are multilinear maps $s_i : W \times \cdots \times W \to \mathbb{Q}(r_i)$

(d). ηK is a K-orbit of isomorphisms $V(\mathbb{A}_{\text{fin}}) \to W(\mathbb{A}_{\text{fin}})$ under which ψ is s_0 up to $\mathbb{A}_{\text{fin}}^\times$-scaling and t_i corresponds to s_i.

and further require that, there is an isomorphism $a : W \to V$ sending s_0 to ψ up to $\mathbb{Q}^\times$-scaling and s_i corresponds to t_i and h to an element of X.

Proposition 6.4. The complex points $\mathrm{Sh}_K\,(\mathbb{C})$ classify the elements of $\mathscr{H}_K$ up to isomorphism.

Definition 6.5. Now let A be an abelian variety over $\mathbb{C}$ and $W = H_1(A, \mathbb{Q})$. Then we have seen that,

$$H^m(A, \mathbb{Q}) = \mathrm{Hom}\left(\bigwedge^m W, \mathbb{Q}\right)$$

We say that $t \in H^{2r}(A, \mathbb{Q})$ is a *Hodge tensor for A* if the corresponding map,

$$W^{\otimes 2r} \to \bigwedge^{2r} W \to \mathbb{Q}(r)$$

is a morphism of Hodge structures.

Remark. Now we define a similar set of tuples whose moduli we can describe.

Definition 6.6. Let $(G, X) \hookrightarrow (G(\psi), X(\psi))$ and $t_1, \ldots, t_n$ as above. Then let M_K be the following set of triples $(A, (s_i), \eta K)$ where,

(a). A is an abelian variety over $\mathbb{C}$

(b). s_0 or $-s_0$ is a polarization for the rational Hodge structure $H_1(A, \mathbb{Q})$

(c). $s_1, \ldots, s_n$ are Hodge tensors for A (or its powers WHAT DOES THAT MEAN?)

(d). ηK is a K-orbit of $\mathbb{A}_{\mathrm{fin}}$-linear isomorphisms $V(\mathbb{A}_{\mathrm{fin}}) \to V_f(A)$ (WHAT IS THIS) sending ψ onto s_0 up to $\mathbb{A}_{\mathrm{fin}}^{\times}$-scaling and each t_i to s_i

which further satisfies the condition:

there exists an isomorphisms $a : H_1(A, \mathbb{Q}) \to V$ sending s_0 to ψ up to $\mathbb{Q}^{\times}$-scaling and s_i to t_i and h to an element of X.

Theorem 6.7. The complex points $\mathrm{Sh}_K(\mathbb{C})$ classify the elements of M_K up to isomorphism.

Remark. Let $A(\mathbb{C}) = \mathbb{C}^g/\Lambda$ then,

$$H^m(A, \mathbb{Q}) = \mathrm{Hom}\left(\bigwedge^m \Lambda, \mathbb{Q}\right)$$

and for $T = T_0 A$ we have $\Lambda \otimes \mathbb{C} = T \oplus \bar{T}$. Therefore,

$$H^m(A, \mathbb{C}) = \mathrm{Hom}\left(\bigwedge^m (\Lambda \otimes \mathbb{C}), \mathbb{C}\right) = \mathrm{Hom}\left(\bigoplus_{p+q=m} \bigwedge^p T \otimes \bigwedge^q \bar{T}, \mathbb{C}\right) = \bigoplus_{p+q=m} H^{p,q}$$

where we identify,

$$H^{p,q} = \mathrm{Hom}\left(\bigwedge^p T \otimes \bigwedge^q \bar{T}, \mathbb{C}\right)$$

This construction of the Hodge structure on H^m does agree with the usual abstract construction from Hodge theory.

Proposition 6.8. A Hodge tensor on A is an element of,

$$H^{2r}(A, \mathbb{Q}) \cap H^{r,r} \subset H^{2r}(A, \mathbb{C})$$

Recall that the Hodge conjecture predicts the Hodge tensors are spaned over $\mathbb{Q}$ by the algebraic classes. We can show this here for $r = 1$. Consider the exponential sequence,

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_A \xrightarrow{\exp} \mathcal{O}_A^{\times} \longrightarrow 0$$

which on cohomology gives,

$$\begin{tikzcd}
H^1(A, \mathcal{O}_A^{\times}) \arrow[r] \arrow[d, equal] & H^2(A, \mathbb{Z}) \arrow[r] & H^2(A, \mathcal{O}_A) \\
\mathrm{Pic}\,(A) \arrow[ur, dashed, "c_1"] & &
\end{tikzcd}$$

Note that $\alpha \in H^2(A, \mathbb{Z})$ maps to zero in $H^2(A, \mathcal{O}_A) = H^{0,2}$ iff it maps to zero in $H^{2,0}$. Therefore,

$$\mathrm{Im}\,(c_1) = H^2(A, \mathbb{Z}) \cap H^{1,1}$$

thus we see that the space $H^{1,1}$ in *integral* (even better than rational) cohomology is exactly the subgroup of classes of algbraic line bundles or equivalently of divisors i.e. algebraic cylces.

Remark. Since degree zero bundles map to zero, the first map $\mathrm{Pic}\,(A) \to H^2(A, \mathbb{Z})$ factos through $\mathrm{Pic}\,(A)\,/\mathrm{Pic}^0(A) = NS(A)$ the Neron-Severi group of connected components of $\mathrm{Pic}\,(A)$. Furthermore, from before we have,

$$H^2(A, \mathbb{Z}) = \mathrm{Hom}\left(\bigwedge^2 H_1(A, \mathbb{Z}), \mathbb{Z}\right)$$

Then we get an injection,

$$NS(A) \hookrightarrow H^2(A, \mathbb{Z}) = \mathrm{Hom}\left(\bigwedge^2 H_1(A, \mathbb{Z}), \mathbb{Z}\right)$$

Note that a *polarization* of A is an element of $NS(A)$ mapping to a polarization of $H_1(A, \mathbb{Z})$. (LOOK AT PREVIOUS DEF OF POLARIZATION AND COMPARE!!)