Mathematics GU4044 Representations of Finite Groups Assignment # 12

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Problem 1.

Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. Consider the partition $\lambda^\top = (\mu_1, \dots, \mu_m)$. We know that μ_i is the length downward in the Young diagram t of λ. However, the number of rows in which a square appears in column i is exactly the number of λ_j which are greater than or equal to i. However, the rows of $t^\top$ are the columns of t so μ_i is the number of λ_j greater than or equal to i.

Problem 2.

(a). Consider the five partitions of 4,

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1)$$

We know that $S^(4)$ is the trivial representation, $S^{(3,1)}$ is the standard irreducible representation and $S^{(1,1,1,1)}$ to the representation $\mathbb{C}(\epsilon)$.

(b). For $\lambda = (2,2)$ the Young subgroup S_λ is the subgroup permuting inside the partition into two equal groups. Thus, $S_\lambda \cong S_2 \times S_2$ since S_λ acts transitivly inside the partition. Thus, $\#(S_4/S_\lambda) = 4$. Consider the tabloids of type λ,

$$\begin{array}{|c|c|} \hline i & j \\ \hline \bullet & \bullet \\ \hline \end{array}$$

Given fixed i and j the bottom row is the missing two elements in $\{1,2,3,4\}$ and is thus determined up to order. Therefore, chooing a canonical order $i < j$ will uniquely determine the tabloid since the rows are determined up to permutation. Call this tabloid $[ij]$.

(c). Consider the tableau,

$$t = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Which has corresponding tabloid $[t] = [12]$. The column stabilizer of this tableau is the subgroup of S_4 which preserves $\{1,3\}$ and $\{2,4\}$ and is therefore generated by the 2-cycles $(1\,3)$ and $(2\,4)$. The corresponding polytabloid is given by,

$$E_{12} = A_t([t]) = \sum_{\sigma \in C_t} \epsilon(\sigma)[\sigma \cdot t] = [12] - [23] - [14] + [34]$$

Likewise, consider the tableau and associated polytabloids,

$$t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \implies E_{13} = A_t([t]) = [13] - [23] - [14] + [24]$$

$$t = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} \implies E_{14} = A_t([t]) = [14] - [24] - [13] + [23]$$

$$t = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} \implies E_{23} = A_t([t]) = [23] - [13] - [24] + [14]$$

$$t = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \implies E_{24} = A_t([t]) = [24] - [14] - [23] + [13]$$

$$t = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \implies E_{34} = A_t([t]) = [34] - [14] - [23] + [12]$$

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \implies E'_{12} = A_t([t]) = [12] - [13] - [24] + [34]$$

Therefore, $E'_{12} = E_{12} - E_{13}$.

(d). First, let t be a tableau and C_t be the column stabilizer. Then we know that $C_{\tau\cdot t} = \tau C_t \tau^{-1}$. Therefore, since $\sigma \in \tau C_t \tau^{-1} \iff \tau^{-1}\sigma\tau \in C_t$ we have,

$$e_{\tau\cdot t} = A_{\tau\cdot t}([\tau \cdot t]) = \sum_{\sigma \in \tau C_t \tau^{-1}} \epsilon(\sigma)[(\sigma\tau)\cdot t] = \sum_{\sigma \in C_t} \epsilon(\tau^{-1}\sigma\tau)[(\tau\sigma)\cdot t] = \tau \cdot A_t([t]) = \tau \cdot e_t$$

Consider the tableaux,

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \ell \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline j & i \\ \hline \ell & k \\ \hline \end{array}$$

This pair corresponds to swapping the entire columns. Therefore, the two are related by the element $\tau = (i\,j)(k\,\ell)$. By the above argument, their associated polytabloids are related by,

$$e_{\tau\cdot t} = \tau \cdot E_{ij} = \tau \cdot ([ij] - [kj] - [i\ell] + [k\ell]) = [ji] - [\ell i] - [jk] + [\ell k] = [ij] - [kj] - [i\ell] + [k\ell]$$

Therefore, these two tableaux have the same associated polytabloids. Futhermore, consider the tableaux,

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \ell \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline i & j \\ \hline \ell & k \\ \hline \end{array}$$

The two are related by the element $\tau = (k\,\ell)$. Thus, their polytabloids are related by,

$$e_{\tau\cdot t} = \tau \cdot E_{ij} = \tau \cdot ([ij] - [kj] - [i\ell] + [k\ell]) = [ij] - [\ell j] - [ik] + [k\ell] = E_{ij} - E_{ik}$$

where E_{ik} is given by acting with $\tau = (j\,k)$,

$$E_{ik} = e_{\tau\cdot t} = \tau \cdot E_{ij} = \tau \cdot ([ij] - [kj] - [i\ell] + [k\ell]) = [ik] - [jk] - [i\ell] + [j\ell]$$

(e). Consider three standard tableaux,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

and call their associated polytabloids e_1 and e_2 respectively. We know that permuting the elements in the columns corresponds to acting wit an element of the column stabilizer. However, if $\tau \in C_t$ then,

$$\tau \cdot A_t([t]) = \sum_{\sigma \in C_t} \epsilon(\sigma)[(\tau\sigma) \cdot t] = \epsilon(\tau) \sum_{\sigma' \in C_t} \epsilon(\sigma')[\sigma' \cdot t] = \epsilon(\tau) A_t([t])$$

Therefore, permuting the elemnts of the columns only changes the sign of a polytabloid. Furthermore, by the previous problem, swapping the columns leaves the polytabloid invariant. Finally, permuting just the bottom row gives a polytabloid wich is the sum of the origional and the polytabloid with one diagonal reversed (up to sign). However, permuting elements in the columns and swapping columns allows for any tableau to be reduced to one of the three standard tableaux. Furthermore, from above, $e_3 = e_1 - e_2$. Thus, e_1 and e_2 span S^λ. However, we have shown above that e_1 and e_2 are not multiples of eachother. Thus, $\dim S^{(2,2)} = 2$.

Problem 3.

Let $\lambda = (n-2, 1, 1) \vdash n$. Any tableau of type λ is of the form,

$$\begin{array}{|c|c|c|c|c|c|}\hline i & \bullet & \bullet & \bullet & \cdots & \bullet \\ \hline j \\ \cline{1-1} k \\ \cline{1-1}\end{array}$$

(a). The column stabilizer C_t is the subgroup S_3 acting on $(i\ j\ k)$. Consider the polytabloid,

$$A([t]) = E_{ijk} = \sum_{\sigma \in C_t} \epsilon(\sigma)\sigma \cdot [t] = [jk] - [kj] + [ki] + [ij] - [ji]$$

where the positive terms are ones requiring an even number of swaps.

(b). If $\tau \in C_t$ then,

$$\cdot A_t([\tau \cdot t]) = \sum_{\sigma \in C_t} \epsilon(\sigma)[\sigma \cdot (\tau \cdot t)] = \epsilon(\tau) \sum_{\sigma \in C_t} \epsilon(\sigma\tau)[(\sigma\tau) \cdot t] = \epsilon(\tau) A_t([t])$$

Furthermore ,cyclicly permuting the elements $(i\,j\,k)$ is equivalent to actig acting on the tabloid by a 3-cycle which is even. Thus,

$$E_{ijk} = E_{jki} = E_{kij}$$

Furthermore, swapping any two of $(i\ j\ k)$ is equivalent to acting on the tabloid by a 2-cycle which is odd. Thus,

$$E_{jik} = -E_{ijk}$$

This means that the Sprecht representation is spanned by the polytabloids E_{ikj} where we can restic to the case $1 \leq i < j < k \leq n$.

(c). Furthermore,

$$
\begin{aligned}
E_{ijk} - E_{ij\ell} + E_{ik\ell} &= [jk] - [kj] + [ki] - [ik] + [ij] - [ji] \\
&\quad - ([j\ell] - [\ell j] + [\ell i] - [i\ell] + [ij] - [ji]) \\
&\quad + [k\ell] - [\ell k] + [\ell i] - [i\ell] + [ik] - [ki] \\
&= [k\ell] - [\ell k] + [jk] - [kj] + [\ell j] - [j\ell] = E_{jk\ell}
\end{aligned}
$$

Therefore, we can generate the span of all E_{ijk} with a fixed i. Thus, we only need the span of E_{1jk} where $1 < j < k \leq n$. Thus, $S^{(n-2,1,1)}$ is spanned by $n-1$ choose 2 elements taking any two j, k of the numbers $2, \cdots, n$. An arbitrary element can be written as,

$$E_{1jk} = [jk][kj] + [k1] - [1k] + [1j] - [j1]$$

However, the element $[1k]$ only appears in the exansion of E_{1ik} and E_{1ki}. Summs of these terms over i contain the elements $[1i]$ respectivly. Thus, the only one which can appear in any expansion of E_{1jk} is E_{1jk} because the only elements of the form $[1i]$ which appear are $[1j]$ and $[1k]$ and elements of this form cannot cancel in the sum because only E_{1ik} contains the element $[1i]$ in the expansion. Therefore, E_{1jk} cannot be written in terms of the other $E_{1j'k'}$ so this spanning set is a basis. Thus,

$$\dim S^{(n-2,1,1)} = \binom{n-1}{2}$$