1 Feb 11

1.1 Line Bundles

There exists a map,

$$\Gamma(X, \mathcal{L}^{\otimes a}) \otimes \Gamma(X, \mathcal{L}^{\otimes b}) \to \Gamma(X, \mathcal{L}^{\otimes ab})$$

since we have an isomorphism $\mathcal{L}^{\otimes a} \otimes \mathcal{L}^{\otimes b} = \mathcal{L}^{\otimes ab}$. Furthermore, since $\mathcal{L}$ is rank 1 this map is commutative since $s \times s' = s' \otimes s$ since they only differ by a section of $\mathcal{O}_X$. This alows us to define the following graded ring structure.

Definition 1.1. Let $\mathcal{L}$ be an invertable $\mathcal{O}_X$-module, $\mathscr{F}$ any $\mathcal{O}_X$-module and $s \in \mathcal{L}(X)$ a global section. Then we define the following graded ring.

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

and then the following module,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which is a graded $\Gamma_*(X, \mathcal{L})$-module. Furthermore, there is a map,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F})_{(s)} \to \mathscr{F}(X_s) = \Gamma(X_s, \mathscr{F})$$

sending $\frac{t}{s^n} \mapsto t|_{X_s} \otimes (s|_{X_s})^{\otimes -n}$.

Proposition 1.2. Let X be a quasi-compact, quasi-seperated scheme and $\mathscr{F}$ be quasi-coherent. Then the above map is an isomorphism.

Proof. Tag OB5K. (Compare with that Hartshorne Excercise 2.16). $\square$

Example 1.3. Let A be a graded ring such that A is generated by A_1 as a A_0-algebra (e.g. $A = k[X_0, \dots, X_n]$). Let $X = \text{Proj}\,(A)$ and consider the graded module $M = A(n)$ which is the graded module $M_k = A_{k+n}$. Then we can construct the Serre twists,

$$\mathcal{O}_X(n) = \widetilde{M} = \widetilde{A(n)}$$

which is an invertable $\mathcal{O}_X$-module. Furthermore,

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$$

Remark. This will not be invertible and these maps will not be isomorphisms in general when A does not satisfy the required conditions.

Proof. We can decompose,

$$X = \bigcup_{f \in A_1} D_+(f) = \bigcup_{f \in A_1} \text{Spec}\left(A_{(f)}\right)$$

via the given assumptions. We know that,

$$\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{A(n)}|_{D_+(f)} = A\widetilde{[f^{-1}]}_n$$

However it is clear that $A[f^{-1}]_n = A[f^{-1}]_0 \cdot f^n$ so this sheaf is free of rank 1. $\square$

Remark. For $n = 1$ any element $f \in A_1$ gives a global section $f \in \Gamma(X, \mathcal{O}_X(1))$ such that $D_+(f) = X_s$ and hence,

$$\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(1)|_{X_s}$$

Corollary 1.4. In the setting above, further assume that A is generated by finitely many $f \in A_1$ as an A_0-algebra. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ if we set,

$$M = \Gamma_*(X, \mathcal{O}_X(1), \mathscr{F})$$

as a graded A-module via the map,

$$A \to \Gamma_*(X, \mathcal{O}_X(1)) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

Then we get, $\mathscr{F} = \widetilde{M}$.

Proof. Tag □

2 Feb. 13

Definition 2.1. Let X be a scheme and $\mathcal{L}$ an invertivle $\mathcal{O}_X$-module . We say $\mathcal{L}$ is *ample* if X is quasi-compact and $\forall x \in X : \exists n > 0 : s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $x \in X_s$.

Example 2.2. Let $X = \text{Proj}\,(A)$ where A is generated by A_1 as a A_0-algebra and $A_1 = f_1 A_0 + \cdots + f_r A_0$. Then $\mathcal{O}_X(1)$ is invertible and X is covered by $D_+(f_i)$ and is quasi-compact, and $D_+(f_i) = X_{s_i}$ where $s_i \in \Gamma(X, \mathcal{O}_X(1))$ is a section corresponding to f_i.

Proposition 2.3. Let X be quasi-compact and quasi-seperated for $\mathcal{L} \in \text{Pic}\,(X)$ the following are equivalent,

(a). $\mathcal{L}$ is ample

(b). for all $\mathcal{O}_X$-modules $\mathscr{F}$ locally of finite type there exists $n > 0$ s.t. $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections.

Proof. TAG 01Q3. □

Lemma 2.4. $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is ample for any $n > 0$.

Lemma 2.5. If X is affine, and $\mathcal{L}$ is invertible, and $s \in \Gamma(X, \mathcal{L})$ then X_s is affine.

Definition 2.6. A scheme is noetherian if it has a finite open cover by spectra of noetherian rings.

Remark. It is equivalent to require that X is quasi-compact and $\mathcal{O}_X(U)$ is noetherian for each affine open.

Lemma 2.7. A locally noetherian scheme is quasi-seperated.

Proof. If U, V are affines then $U \cap V$ is quasi-compact since every subspace of a noetherian space is quasi-compact. □

Definition 2.8. Let X be a neotherian scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is *coherent* if it is quasi-coherent and locally of finite type.

Remark. It is equivalent to require that locally on affine opens $\mathscr{F}|_U = \widetilde{M}$ for a finitely-generated module M.

Remark. The inclusion functors,

$$\mathfrak{Coh}(\mathcal{O}_X) \subset \mathfrak{QCoh}(\mathcal{O}_X) \subset \mathscr{M}od(\mathcal{O}_X)$$

are exact and preserved under extensions i.e. given a short exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

if $\mathscr{F}_1, \mathscr{F}_2$ are (quasi)-coherent then $\mathscr{F}_2$ is also (quasi)-coherent.

Lemma 2.9. A scheme of finite type over a noetherian scheme is noetherian.

Proof. Since $f : X \to Y$ is finite type f is quasi-compact but Y is quasi-compact open so its preimage X is also quasi-compact. Furthermore, for any affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(B) = V \subset Y$ such that $f(U) \subset V$ we get a ring map $B \to A$ of finite type so $B[x_1, \dots, x_n] \twoheadrightarrow A$ and since B is noetherian we see that A is noetherian so X is quasi-compact and covered by $\mathrm{Spec}(A)$ for noetherian rings A. $\square$

Remark. We want to prove the following theorem. Let R be a noetherian ring, X a projective (or proper) scheme over R (then X is noetherian), and $\mathscr{F}$ a coherent sheaf on X, then,

$$H^i(X, \mathscr{F})$$

is a finite R-module for any i and $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$.

3 Feb 18

Definition 3.1. An immersion $j : X \to Y$ is a morphism which may be factored as $X \to U \to Y$ where $X \to U$ is a closed immersion and $U \to Y$ is an open immersion.

Definition 3.2. Let R be a ring, and X a scheme over R. We say X is *quasi-projective over* R iff there exists a quasi-compact immersion $j : X \to \mathbb{P}^n_R$ over R.

Remark. If X is proper over R (or just universally closed) then j is automatically a closed immersion since $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is seperated and $X \to \mathrm{Spec}(R)$ is universally closed implies that $j : X \to \mathbb{P}^n_R$ is universally closed and in particular topologically closed and thus closed as an immersion. This gives the following lemma.

Lemma 3.3. X is projective over R iff X is quasi-projective and proper over R.

Theorem 3.4. Let R be a ring and X a scheme over R. The TFAE,

(a). X is quasi-projective over R

(b). X is of finite type over R and X has an ample invertible module $\mathcal{L}$

(c). there exists a quasi-compact open immersion $X \hookrightarrow X'$ with X' projective over R.

Lemma 3.5. Let $j : X \to Y$ be a quasi-compact immersion and $\mathcal{L}$ an ample line bundle on Y. Then $j^*\mathcal{L}$ is an ample line bundle on Y.

Proof. (DO THIS!!) $\square$

Lemma 3.6. Let $j : X \to Y$ be a quasi-compact immersion and X' is scheme-theoretic image. Then $j : X \to X'$ is an open immersion.

Proof. Since j is qc and qs (immersions are seperated) then $j_* \mathcal{O}_X$ is quasi-coherent and thus $\mathscr{I} = \ker(\mathcal{O}_Y \to \mathcal{O}_X)$ is quasi-coherent so we find $X' = V(\mathscr{I})$ (FINSIH THIS) $\square$

Example 3.7. $\mathrm{Spec}\,(k[[x]]) \to \mathrm{Spec}\,(k[x])$ has scheme theoretic image $\mathrm{Spec}\,(k[x])$ since its image contains the generic point. However, its set theoretic image is two points.

Proof. of Theorem (2) $\implies$ (1). Choose $r \geq 0$ and $n \geq 1$ and $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.t.

$$X = \bigcup_{i=0}^{r} X_{s_i}$$

and X_{s_i} affine. Write $X_{s_i} = \mathrm{Spec}\,(A_i)$. Now R is finite type over R so A_i is finite type over R so we may take $a_{i1}, \dots, a_{iN_i} \in A_i$ which generate A_i as an R-algebra. Choose $m \geq 1$ and $s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ such that $a_{ij} = s_{ij} \cdot s_i^{\otimes -m}|_{X_{s_i}}$. Therefore, $s_0^{\otimes m}, \dots, s_r^{\otimes m}, s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ generate $\mathcal{L}^{\otimes mn}$ and therefore define a morphism $\varphi : X \to \mathbb{P}_R^{r + \sum N_i}$. It suffices to check that $X_{s_i} \to D_+(T_i)$ are a closed immersion. This holds because it is given by the ring map,

$$R[\tfrac{T_0}{T_1}, \dots, \tfrac{T_r}{T_i}, \tfrac{T_{ij}}{T_i}] \to A_i = \mathcal{O}_X(X_{s_i})$$

given by $\frac{T_{ij}}{T_i} \to a_{ij}$ which is clearly surjective so $X_{s_i} \to D_+(T_i)$ is a closed immersion. $\square$

Remark. If we had checked that $X_{s_{ij}} \to D(T_{ij})$ we also a closed immersion with $X_{s_{ij}}$ affine then $\varphi : X \to \mathbb{P}_R^N$ would be a *closed* immersion. We checked only that it is locally a closed immersion on X

3.1 Functorial Characterization of $\mathbb{P}_R^n$

Consider the functor, $F : \mathfrak{Sch}_R \to \mathfrak{Set}$ via,

$$T \mapsto \{(\mathcal{L}, s_0, \dots, s_n) \mid \mathcal{L} \in \mathrm{Pic}\,(T)\ \ \mathcal{O}_T^{n+1} \xrightarrow{(s_0, \dots, s_n)} \mathrel{\mkern-14mu}\to \mathcal{L} \ \text{ i.e. } \ s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \text{ generate}\} / \cong$$

where $(\mathcal{L}, s_0, \cdots, s_n) \cong (\mathcal{L}', s_0', \dots, s_n')$ if there is an isomorphism $\alpha : \mathcal{L} \to \mathcal{L}'$ with $\alpha(s_i) = s_i'$.

Theorem 3.8. $\mathbb{P}_R^n$ represents this functor, $\mathrm{Hom}_{\mathfrak{Sch}_R}(T, \mathbb{P}_R^n) = F(T)$.

Proof. Given $\varphi : T \to \mathbb{P}_R^n$ we get $\mathcal{L} = \varphi^* \mathcal{O}_{\mathbb{P}_R^n}(1)$ and $s_i = \varphi^*(T_i)$.

Conversely, given $(\mathcal{L}, s_0, \dots, s_n)$ and $U \subset T$ and $\square$

Theorem 3.9. If R is Noetherian and X is proper over R and $\mathcal{L}$ is ample on X then,

$$X \cong \mathrm{Proj}\,(\Gamma_*(X, \mathcal{L}))$$

and $\Gamma_*(X, \mathcal{L})$ is a finitely-generated graded R-algebra whose degree zero part is a finite R-module.

Remark. We will prove this using cohomology.

4 Cohomology

Theorem 4.1. $\mathbf{Mod}_{\mathcal{O}_X}$ is a Grothendieck abelian category so there are enough injectives.

Definition 4.2. Therefore, we can produce the right-derived functors $H^i(X, -)$ of the global sections functor,

$$\Gamma(X, -) : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\Gamma(X, \mathcal{O}_X)}$$

where $(X, \mathcal{O}_X)$ is a ringed space. Since this is right-exact we find $H^0(X, -) = \Gamma(X, -)$.

Definition 4.3. Furthermore, given a morphism $f : X \to Y$ we can produce $R^i f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$ the right-derived functors of $f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$.

Remark. $\mathbf{Ab}(X) = \mathbf{Mod}_{\underline{\mathbb{Z}}}$ so we may apply the theory of cohomology of $\mathcal{O}_X$-modules to the ringed space $(X, \underline{\mathbb{Z}})$ to get a cohomology theory for abelian sheaves.

Lemma 4.4 (locality of cohomology)**.** Given $\xi \in H^p(X, \mathscr{F})$ with $p > 0$ there exists an open covering,

$$X = \bigcup_{i \in I} U_i$$

s.t. $\xi|_{U_i} = 0$ for each $i \in I$.

Proof. $\square$

Remark. The pullback is defined as follows,

5 Feb 20

5.1 Cech Cohomology

For any open covering $\mathfrak{U}$ of a space X and a sheaf $\mathscr{F}$ there is a simplicial abelian group,

$$\prod_{i_0 \in I} \mathscr{F}(U_{i_0})$$

Then $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is the complex associated to the cosimplicial object.

Example 5.1. Given an exact sequence of sheaves,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

An obstruction to lifting a section $s \in \Gamma(X, \mathscr{H})$ is a cocycle in $\check{C}^1(\mathfrak{U}, \mathscr{F})$.

Lemma 5.2. Cech cohomology vanishes on injective objects in the category of presheaves.

Corollary 5.3. As a functor ON THE CATEGORY OF PRESHEAVES $\check{H}^i(\mathfrak{U}, -)$ are the right-derived functors of $\check{H}^0(\mathfrak{U}, -)$.

Lemma 5.4. Given a ringed space, $(X, \mathcal{O}_X)$ and B is a basis of top and Cov a set of coverings s.t.

(a). $\mathfrak{U}$ in cov implies that its union and all finite intersections are in B

(b). for U basis the coverings of U in Cov are cofinal

If $\mathscr{F} \in \mathscr{Mod}(\mathcal{O}_X)$ and

$$(*) \forall \mathfrak{U} \in Cov : \ \check{H}^p(\mathfrak{U}, \mathscr{F}) = 0$$

Then $H^p(\mathfrak{U}, \mathscr{F}) = 0$ for any U in the basis.

6 Feb 25

Lemma 6.1. Let $\mathfrak{U}$ be an open covering of X and $\mathscr{F} \in \mathscr{Mod}(\mathcal{O}_X)$ s.t. $H^p(U_{i_1} \cap \cdots U_{i_n}, \mathscr{F}) = 0$ for all finite intersections. Then $H^p(X, \mathscr{F}) = \check{H}^p(X, \mathscr{F})$ for all $p \geq 0$.

Proof. See proof in Hartshorne Ex. It goes as follows,

(a). Use an exact sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{G} \longrightarrow 0$$

with $\mathscr{I}$ injective.

(b). Show for any sheaf $\check{H}^0(X, \mathscr{F}) = H^0(X, \mathscr{F})$ just by the sheaf property.

(c). By the assumptions, there is an exact sequence on check complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{I}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{G}) \longrightarrow 0$$

(d). this gives a long exact sequence of Cech cohomology

(e). use this exact sequence plus $\check{H}^p(\mathfrak{U}, \mathscr{I}) = 0$ for $p > 0$ (since flasque) to show that $\check{H}^p(\mathfrak{U}, \mathscr{G}) = \check{H}^{p+1}(\mathfrak{U}, \mathscr{F})$ and $\check{H}^1(\mathfrak{U}, \mathscr{F}) = \operatorname{coker} \check{H}^0(\mathfrak{U}, \mathscr{I}) \to \check{H}^0(\mathfrak{U}, \mathscr{G})$

(f). use long exact sequence of $H^p(U_{i_0, \dots, i_n}, -)$ to show that $\mathscr{G}$ also satisfies the hypotheses.

(g). use long exact sequence of $H^p(X, -)$ to show that the above hold for usual cohomology.

(h). then by induction we get $\check{H}^{p+1}(\mathfrak{U}, \mathscr{F}) = \check{H}^p(\mathfrak{U}, \mathscr{G}) = H^p(X, \mathscr{G}) = H^{p+1}(X, \mathscr{F})$ and the base case holds since they are both kernels.

□

Corollary 6.2. Let X be a scheme whose diagonal is affine (for example a separated scheme). Let $\mathfrak{U}$ be a covering of affine opens and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then,

$$H^p(X, \mathscr{F}) = \check{H}^p(X, \mathscr{F})$$

Remark. There is a Cech to cohomology spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \underline{H}^q(\mathscr{F})) \implies H^{p+q}(X, \mathscr{F})$$

Corollary 6.3. Let $f : X \to Y$ be a squasi-compact quasi-separated morphism of schemes. Then $R^i f_*$ sends quasi-coherent mofules to quasi-coherent modules.

Lemma 6.4. Let $f : X \to Y$, $F \in \mathscr{Mod}(\mathcal{O}_X)$ then $R^p f_* \mathscr{F}$ is the sheaf associated to the presheaf,

$$V \mapsto H^i(f^{-1}(V), \mathscr{F})$$

Proposition 6.5. We define the following modifications to the Cech complex,

$\check{C}^\bullet_{\text{alt}}$ is elements of the form $(s_{i_0 \dots i_p})$ which are antisymmetric and vanish if any two indicies agree and the ordered check complex for a total order $<$ on I,

$$\check{C}^p_{\text{ord}} = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0 \cdots i_p})$$

There are the following relations between Cech complexes,

$$\begin{tikzcd} \check{C}^\bullet_{\text{alt}}(\mathfrak{U}, \mathscr{F}) \arrow[r, "\text{include}"] \arrow[rr, bend left] & \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \arrow[r, "\text{project}"] & \check{C}^\bullet_{\text{ord}}(\mathfrak{U}, \mathscr{F}) \end{tikzcd}$$

the curves arrow is an isomorphism of complexes and the horizontal arrows are homotopy equivalences.

7 Feb. 27

Proposition 7.1. Let R be a Noetherian ring and $\mathscr{F}$ a coherent sheaf on $\mathbb{P}^n_R$. Then,

(a). $\exists r \geq 0 : \exists m \in \mathbb{Z}$ and a surjection $\mathcal{O}_X(m)^{\oplus r} \twoheadrightarrow \mathscr{F}$

(b). $H^i(\mathbb{P}^n_R, \mathscr{F}) = 0$ for $i \notin [0, n]$

(c). $H^i(\mathbb{P}^n_R, \mathscr{F})$ is a finite R-module

(d). for $i > 0$, $H^i(\mathbb{P}^n_R, \mathscr{F}(d)) = 0$ for any $d \geq d_0(\mathscr{F})$

(e). $\bigoplus_{d \geq 0} H^0(\mathbb{P}^n_R, \mathscr{F}(d))$ is a finite $P = R[T_0, \dots, T_n]$-module.

Proof. Recall that $\mathcal{O}_X(1)$ is ample so $\mathscr{F} \otimes \mathcal{O}_X(d)$ is generated by global sections for sufficiently large d and thus we get $\mathcal{O}_X^{\oplus r} \twoheadrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ and thus $\mathcal{O}_X(-d)^{\oplus r} \twoheadrightarrow \mathscr{F}$.

Note that $\mathbb{P}^n_R = \bigcup_i D_+(T_i)$ which is an open cover of $n+1$ affines so by Cech cohomology, cohomology vanishes above n.

Now we apply descending induction since $H^{n+1}(\mathbb{P}^n_R, \mathscr{F}) = 0$. Now we assume (3) and (4) for degree $k+1$. For a coherent sheaf $\mathscr{F}$ consider the exact sequence,

$$0 \longrightarrow \mathscr{G}(d) \longrightarrow \mathcal{O}_X(m+d)^{\oplus n} \longrightarrow \mathscr{F}(d) \longrightarrow 0$$

then, from the LES we get,

$$H^k(\mathbb{P}^n_R, \mathcal{O}_X(m+d)^{\oplus r}) \longrightarrow H^k(\mathbb{P}^n_R, \mathscr{F}(d)) \longrightarrow H^{k+1}(\mathbb{P}^n_R, \mathscr{G}(d))$$

For the case $d = 0$ we assume that $H^{k+1}(\mathbb{P}^n_R, \mathscr{G})$ is a finite R-module and, by computation, so is $H^k(\mathbb{P}^n_R, \mathcal{O}_X(m)^{\oplus r})$ and thus $H^k(\mathbb{P}^n_R, \mathscr{F})$ is a finite R-module. For $d \gg 0$ then we assume that $H^{k+1}(\mathbb{P}^n_R, \mathscr{G}(d)) = 0$ for sufficiently large d. Futhermore, for $k > 0$ we computed that $H^k(\mathbb{P}^n_R, \mathcal{O}_X(m)^{\oplus r}) = 0$ for $d \geq m$ and thus we see that $H^k(\mathbb{P}^n_R, \mathscr{F}(d)) = 0$ for sufficiently large d proving (3) and (4).

Finally, we also use descending induction and conisder the exact sequence,

$$\bigoplus_{d\geq 0} H^0(\mathbb{P}^n_R, \mathcal{O}_X(m+d)^{\oplus r}) \longrightarrow \bigoplus_{d\geq 0} H^0(\mathbb{P}^n_R, \mathscr{F}(d)) \longrightarrow \bigoplus_{d\geq 0} H^0(\mathbb{P}^n_R, \mathscr{G}(d))$$

By computation, the first term is a submodule of a finite P-module and the last term is zero is sufficiently large degrees. Thus the middle term M has a f.g. P-submodule M' such that M/M' is finite as an R-module so M is a f.g. P-module. □

Lemma 7.2. Let $f : X \to Y$ be an affine morphism of schemes. Then $H^p(X, \mathscr{F}) = H^p(Y, f_*\mathscr{F})$ for $\mathscr{F}$ quasi-coherent.

Proof. We use the Grothendieck spectral sequence and not that for $f : X \to Y$ affine and $\mathscr{F}$ quasi-coherent we have $R^p f_*\mathscr{F} = 0$ for $p > 0$ since quasi-coherent higher cohomology vanishes on affine schemes. □

Example 7.3. If X is a projective scheme over a Noetherian ring R. For closed immersion $X \hookrightarrow \mathbb{P}^n_R$,

$$H^i(X, \mathscr{F}) = H^i(\mathbb{P}^n_R, j_*\mathscr{F})$$

for quasi-coherent $\mathcal{O}_X$-modules.

Lemma 7.4. If $\mathscr{F} : X \to Y$ is finite and X and Y are Noetherian then f_* preserves coherent sheaves.

Proof. Since f is affine it preserves quasi-coherent modules. Since the morphism is additionally finite on rings so it changes finite modules to finite modules on the affine open level. □

Corollary 7.5. For any coherent $\mathscr{F}$ on a scheme X projective over Noetherian R then the above proposition holds with $\mathscr{F}(d) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ where $\mathcal{L}$ is an ample line bundle.

Remark. Let X be Noetherian over Noetherian R then let $n = \max\{\dim X_s \mid s \in \operatorname{Spec}(R)\}$ then $H^i(X, \mathscr{F}) = 0$ for $i > n$. Warning, this is not true for quasi-projective X over a Noetherian ring. For example, consider $X = \mathbb{A}^2_{\mathbb{Q}} \setminus \{0\} \to \mathbb{A}^2_{\mathbb{Q}}$ is quasi-projective over $R = \mathbb{Q}[x, y]$ but X does not have finitely generated cohomology.

Lemma 7.6. Let X be projective over a field k then X has an open cover by $\dim X + 1$ affines.

Proof. Choose $X \hookrightarrow \mathbb{P}^n_k$ show that we can find $F \in k[T_0, \dots, T_n]_d$ s.t. $\dim(X \cap V(F)) < \dim X$. Namely, choose F not vanishing at the generic points of X by graded prime avoidance. Then we can repeat to get,

$$X \cap V(F_1) \cap \cdots \cap V(F_{\dim X+1}) = \varnothing$$

and thus,

$$X = (X \cap D_+(F)) \cup \cdots \cup (X \cap D_+(F_{\dim X+1})$$

where these factors are affine. □

Corollary 7.7. $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$ for $\mathscr{F}$ quasi-coherent on X projective over a field.

Theorem 7.8 (Grothendieck)**.** If $(X, \mathcal{O}_X)$ is a Noetherian ringed space then $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$ and any $\mathcal{O}_X$-module $\mathscr{F}$.

Remark. Since we can always choose $\mathcal{O}_X = \underline{\mathbb{Z}}$ in the above theorem applies to all abelian sheaves.

Lemma 7.9. If X is qc and qs then for $\mathscr{F}_i$ quasi-coherent and I an *arbitrary* index set,

$$H^p(X, \bigoplus_{i \in I} \mathscr{F}_i) = \bigoplus_{i \in I} H^p(X, \mathscr{F})$$

Remark. The above is always true in general for *finite* I since biproducts preserve exact sequences and injectives.

Proof. It is enough to show the above for Cech cohomology for finite affine open covers. Thus, iti s enough to show that,

$$\left(\bigoplus_{i \in I} \mathscr{F}_i \right)(U) = \bigoplus_{i \in I} \mathscr{F}_i(U)$$

If X is affine open in X (WAIT WHAT??) □

7.1 Duality

Lemma 7.10. Let R be a ring, M an R-module, and X qc + sep over R. And some $n \geq 0$ such that $H^{n+1}(X, \mathscr{F})$ for all $\mathscr{F}$ quasi-coherent. Then, the functor $F : \mathfrak{QCoh}(\mathcal{O}_X) \to \mathbf{Mod}_R$ via $\mathscr{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathscr{F}), N)$ is representable by some $\omega_{X/R,M} \in \mathfrak{QCoh}(\mathcal{O}_X)$. That is,

$$F(-) = \mathrm{Hom}_{\mathcal{O}_X}\left(-, \omega_{X/R,M}\right)$$

Example 7.11. For $X = \mathrm{Spec}\,(A)$ then we have $\widetilde{N} \mapsto \mathrm{Hom}_R(N_R, M)$. Then,

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_A(N, \mathrm{Hom}_R(A, M))$$

so we would have $\omega_{A/R,M} = \widetilde{\mathrm{Hom}_R(A, M)}$.

Proof. First note that F acts on direct sums as,

$$F\left(\bigoplus_{i \in I} \mathscr{F}_i\right) = \mathrm{Hom}_R\left(H^n(X, \bigoplus_{i \in I} \mathscr{F}_i), M\right) = \mathrm{Hom}_R\left(\bigoplus_{i \in I} H^n(X, \mathscr{F}_i), M\right) = \prod_{i \in I} \mathrm{Hom}_R(H^n(X, \mathscr{F}_i), M)$$

Furthermore, F takes epis to monos since given an exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \twoheadrightarrow \mathscr{F}_3 \longrightarrow 0$$

then we get,

$$H^n(X, \mathscr{F}_2) \twoheadrightarrow H^n(X, \mathcal{D}_3) \longrightarrow H^{n+1}(X, \mathscr{F}_1) = 0$$

These together shows that F takes all small colimits to products. Then if F satisfies some mild set-theoretic condition then the adjoint functor theorem gives $\omega_{X/R,M}$ as a funtor on M. The ideal goes as follows. We take,

$$\omega_{X/R,M} = \mathrm{colim}_{\mathcal{C}}\, \mathscr{F}$$

where $\mathcal{C}$ is a category of pairs $(\mathscr{F}, \alpha)$ where $\mathscr{F}$ is a quasi-coherent sheaf and $\alpha \in F(\mathscr{F})$ and $\mathrm{Hom}_{\mathcal{C}}((\mathscr{F}, \alpha), \mathscr{G}, \beta)) = \varphi : \mathscr{F} \to \mathscr{G}$ and $\varphi^* \beta = \alpha$. However, this category is big so we cannot take a total colimit over it. We must resolve this set-theoretic issue.

In the case R is Noetherian and X is finite type over R then any quasi-coherent $\mathscr{F}$ can be writen as a filtered colimit,

$$\mathscr{F} = \mathrm{colim}_{i \in I}\, \mathscr{F}_i$$

with $\mathscr{F}_i$ coherent. This means that in the colimit defining $\omega_{X/R,M}$ we can restrict to only coherent $\mathscr{F}$ and there is a set of isomorphism classes of coherent sheaves. □

8 Mar 3

Remark. Here X will be a Noetherian scheme.

Lemma 8.1. Let X be a Noetherian scheme. Any presheaf on $\mathfrak{QCoh}(\mathcal{O}_X)$ which transforms colimits into limits is representable.

Lemma 8.2. Any quasi-coherent module $\mathscr{F}$ on X is a filtered colimit of coherent $\mathcal{O}_X$-modules. (In fact $\mathscr{F}$ is the rising union of its coherent submodules).

Corollary 8.3. For any $\mathscr{F} \in \mathfrak{QCoh}(\mathcal{O}_X)$ there exists an exact sequence,

$$\bigoplus_{j\in J} \mathscr{G}_j \longrightarrow \bigoplus_{i\in I} \mathscr{F}_i \longrightarrow 0$$

where $\mathscr{F}_i$ and $\mathscr{G}_j$ are coherent.

Lemma 8.4. There is a set of isomorphism classes of coherent $\mathcal{O}_X$-modules.

Proposition 8.5. Let X be finite type over R Noetherian. Let n be an integer s.t. $H^{n+1}(X, \mathscr{F}) = 0$ for any $\mathscr{F} \in \mathfrak{QCoh}(\mathcal{O}_X)$. Then, for any R-module M, the functor,

$$\mathscr{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathscr{F}), M)$$

is representable by $\omega_{X/R,M,n} \in \mathfrak{QCoh}(\mathcal{O}_X)$ i.e.

$$\mathrm{Hom}_R(H^n(X, \mathscr{F}), M) = \mathrm{Hom}_{\mathcal{O}_X}\left(\mathscr{F}, \omega_{X/R,M,n}\right)$$

functorially in $\mathscr{F} \in \mathfrak{QCoh}(\mathcal{O}_X)$.

Remark. For any integer p and $\mathscr{F} = \mathrm{colim}\, \mathscr{F}_i$ is a filted colimit of $\mathcal{O}_X$-modules on a Noetherian scheme (or qcqs scheme) we have,

$$H^p(X, \mathscr{F}) = \mathrm{colim}\, H^p(X, \mathscr{F}_i)$$

Theorem 8.1. If k is a field and $n \geq 0$. Then $\omega_{\mathbb{P}^n_k/k,k,n} = \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$. In particular,

$$H^n(\mathbb{P}^n_k, \mathscr{F})^\vee = \mathrm{Hom}_{\mathcal{O}_X}\left(\mathscr{F}, \mathcal{O}_{\mathbb{P}^n_k}(-n-1)\right)$$

functorially in $\mathscr{F} \in \mathfrak{QCoh}\left(\mathcal{O}_{\mathbb{P}^n_k}\right)$.

Proof. It suffices to show for $\mathscr{F}$ coherent. Pick a resolution,

$$\bigoplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^k_k}(e_j) \longrightarrow \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^n_k}(d_i) \longrightarrow \mathscr{F} \longrightarrow 0$$

Since $H^n(\mathbb{P}^n_k, -)$ is right exact (by dimension vanishing) we get,

$$\bigoplus_{j=1}^{s} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^k_k}(e_j)) \longrightarrow \bigoplus_{j=1}^{r} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_i)) \longrightarrow H^n(\mathbb{P}^n_k, \mathscr{F}) \longrightarrow 0$$

Then taking k-linear duals,

$$\begin{tikzcd}
\bigoplus_{j=1}^{s} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^k_k}(e_j))^\vee \arrow[d, equal] & \bigoplus_{j=1}^{r} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_i))^\vee \arrow[l] \arrow[d, equal] & H^n(\mathbb{P}^n_k, \mathscr{F}) \arrow[l] \arrow[d, equal, dashed] & 0 \arrow[l] \\
\bigoplus_{j=1}^{s} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^k_k}(-e_j - n - 1)) & \bigoplus_{j=1}^{r} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-d_i - n - 1)) \arrow[l, "t"] & \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}\big(\mathscr{F}, \mathcal{O}_{\mathbb{P}^n_k}(-n-1)\big) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Note that,

$$\mathcal{O}_{\mathbb{P}^n_k}(-d-n-1) = \mathscr{H}\!om_{\mathcal{O}_{\mathbb{P}^n_k}}\big(\mathcal{O}_{\mathbb{P}^n_k}(d), \mathcal{O}_{\mathbb{P}^n_k}(-n-1)\big)$$

gives the above "transpose" map t above by functoriality in the first argument along with the fact,

$$H^0(\mathbb{P}^n_k, \mathscr{H}\!om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathscr{F}, \mathscr{G})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$$

□

9 March 5

9.1 Serre Duality for $\mathbb{P}^n_k$ Continued.

Write $\omega = \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$ and $t : H^n(\mathbb{P}^n_k, \omega) \to k$ via the Check class,

$$\frac{1}{T_0 \cdots T_n} \mapsto 1$$

Then we know that ω represents the functor,

$$\mathscr{F} \mapsto H^n(\mathbb{P}^n_k, \mathscr{F})^\vee$$

on $\mathfrak{QCoh}\,(\mathcal{O}_X)$ with universal object t.

Theorem 9.1. For coherent modules $\mathscr{F}$, there is an isomorphism,

$$H^{n-i}(\mathbb{P}^n_k, \mathscr{F})^\vee = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \omega)$$

Proof. Both sides are contravariant δ-functors in $\mathscr{F}$ so it suffices to show that both are universal for which it suffices to show that both are coeffecable. For any coherent sheaf $\mathscr{F}$ we can find,

$$\mathcal{O}_{\mathbb{P}^n_k}(-q)^{\bigoplus r} \twoheadrightarrow \mathscr{F}$$

and then for $i > 0$ we know,

$$H^{n-i}(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-q)) = 0 \quad \text{and} \quad \mathrm{Ext}^i_{\mathcal{O}_X}\big(\mathcal{O}_{\mathbb{P}^n_k}(-q), \omega\big) = H^i(\mathbb{P}^n_k, \omega(q)) = H^i(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-n-1+q)) = 0$$

for sufficiently large $q \gg 0$ using our Cech calculations. □

Lemma 9.1. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module. Then.

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E}, \mathscr{G}) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{G})$$

where $\mathcal{E}^\vee = \mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

Proof. Choose an injective resolution $\mathscr{G} \to \mathscr{I}^\bullet$ then,

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathscr{I}^\bullet) = \Gamma(X, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathscr{I}^\bullet)) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{I}^\bullet)$$

Now I claim that $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{I}^\bullet$ is an injective resolution over $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{G}$. To see this, we use,

$$\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{I}^\bullet) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} -, \mathscr{I}^\bullet)$$

but $I^\bullet$ is injective and $\mathcal{E}$ is flat so this is an exact functor. Taking cohomology of the first equality proves the lemma. ☐

Remark. We could also just say, $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, -) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$ so taking their derived functors gives the same thing. However, $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} -$ is exact so taking derived functors of $\Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$.

Remark. The perfect pairings,

$$\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathscr{F}, \omega) \times H^{n-i}(\mathbb{P}^n_k, \mathscr{F}) \to H^n(\mathbb{P}^n_k, \omega) \xrightarrow{t} k$$

factors through $H^n(\mathbb{P}^n_k, \omega)$. The first map can be realized as composition of ext classes or a cup product.

Remark. If $\mathscr{F}$ is locally free then we have a diagram,

$$\begin{tikzcd}
H^i(\mathbb{P}^n_k, \mathscr{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \omega) \times H^{n-i}(\mathbb{P}^n_i, \mathscr{F}) \arrow[r] \arrow[d] & k \\
H^n(\mathbb{P}^n_k, \mathscr{F} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathscr{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \omega) \arrow[r] & H^n(\mathbb{P}^n_k, \omega) \arrow[u, "t"]
\end{tikzcd}$$

which gives the same pairing using the unique evaluation pairing,

$$\mathscr{F} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathscr{F}^\vee = \mathscr{F} \otimes_{\mathbb{P}^n_k} \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}\big(\mathscr{F}, \mathcal{O}_{\mathbb{P}^n_k}\big) \to \mathcal{O}_{\mathbb{P}^n_k}$$

9.2 Dualizing Sheaves in General

Definition 9.2. Let X be proper over k and $\dim X = n$. A *dualizing sheaf* (ω_X, t) is a pair consisting of a coherent $\mathcal{O}_X$-module ω_X and a map $t : H^n(X, \omega_X) \to k$ which represents the functor,

$$\mathscr{F} \mapsto H^n(X, \mathscr{F})^\vee$$

Remark. We have proven, by abstract nonsense, that such a *quasi-coherent* dualizing sheaf exists but now we want to know when such a module is actually *coherent*.

Remark. Consider the case that X is the disjoint union of a curve and a surface. Then $H^2(X, -)$ ignores cohomology on the curve since it vanishes above $H^1(X, -)$. Thus the dualizing sheaf will be zero on the curve. To fix this one looks for a dualizing complex,

$$\omega_X^\bullet \in D^b(\mathfrak{QCoh}(\mathcal{O}_X))$$

such that $H^i(X, \mathscr{F})$ is dual to $\operatorname{Ext}^{-i}_{\mathcal{O}_X}(\mathscr{F}, \omega_X^\bullet)$.

Theorem 9.3. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}^n_k$,

$$\iota_* \omega_X \cong \mathscr{E}xt^c_{\mathcal{O}_X}\big(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k}\big)$$

where $c = n - \dim X$ is the codimension.

Lemma 9.4. Let $\iota : X \to Y$ be a clsoed immersion of schemes then $\iota_* : \mathfrak{QCoh}\left(\mathcal{O}_X\right) \to \mathfrak{QCoh}\left(\mathcal{O}_X\right)$ defines an equivalence of categories onto its image which is the full subcategory of quasi-coherent $\mathcal{O}_Y$-modules $\mathscr{F}$ such that $\mathscr{I} \cdot \mathscr{F} = 0$ for $\mathscr{I} = \ker\left(\mathcal{O}_Y \to \iota_* \mathcal{O}_X\right)$.

Remark. If X and Y are Noetherian schemes, then the above holds also for coherent modules.

Remark. If $f : X \to Y$ is an affine morphism, $\mathfrak{QCoh}\left(\mathcal{O}_X\right)$ is the category of pairs $(\mathscr{F}, \gamma)$ with $\mathscr{F} \in \mathfrak{QCoh}\left(\mathcal{O}_Y\right)$ and $\gamma : f_* \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathscr{F} \to \mathscr{F}$ gives $\mathscr{F}$ a $f_* \mathcal{O}_X$-module structure

10 Mar. 12

Lemma 10.1. Let A be Noetherian and M, N be finite-presentation A-modules and $X = \mathrm{Spec}\,(A)$. Then,

$$\mathscr{H}om_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right) = \widetilde{\mathrm{Hom}_A(M, N)}$$

Proof. The isomorphism,

$$\mathrm{Hom}_A(M, N)_f = \mathrm{Hom}_{A_f}(M_f, N_f)$$

for finitely-presented modules patch together on the open sets $D(f)$ to give an isomorphism,

$$\widetilde{\mathrm{Hom}_A(M, N)} = \mathscr{H}om_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$$

□

Lemma 10.2. Let A be Noetherian and M, N be finite A-modules and $X = \mathrm{Spec}\,(A)$. Then,

$$\mathscr{E}xt^i_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right) = \widetilde{\mathrm{Ext^i_A\,(M, N)}}$$

Proof. This holds for $i = 0$ by the above. Then we apply dimension-shifting to prove this in general. Given a □

Lemma 10.3. For $p < \dim P - \dim X$ we have,

$$\mathscr{E}xt^p_{\mathcal{O}_X}(\iota_* \mathcal{O}_X, \omega_P) = 0$$

Proof. This reduced to the algebra question, given $B = k[x_1, \ldots, x_n] \twoheadrightarrow A$ then,

$$\mathrm{Ext}^p_B\,(A, B) = 0$$

for $p < \dim B - \dim A$. To see this, recall we have $\iota : X \hookrightarrow P = \mathbb{P}^n_k$ then $X \cap D_+(T_i) \subset X$ and $D_+(T_i) = \mathrm{Spec}\,(B)$. Then, $\omega_P|_{D_+(T_i)} = \mathcal{O}_X|_{D_+(T_i)} = \widetilde{B}$. Furthermore, $\iota : X \hookrightarrow P$ is affine (closed immersion) so $X \cap D_+(T_i) = \mathrm{Spec}\,(A)$ for $A = B/I$.

Since B is Cohen-Macaullay we have vanishing for,

$$\mathrm{depth}_I(A) \geq \dim B - \dim A$$

□

Proof. $\square$

Theorem 10.4. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}^n_k$,

$$\iota_* \omega_X \cong \mathscr{E}xt^c_{\mathcal{O}_X}\big(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k}\big)$$

where $c = n - \dim X$ is the codimension.

Proposition 10.5. If $\iota : X \to Y$ is a closed immersion then $\iota^* \iota_* \mathscr{F} = \mathscr{F}$ for any $\mathcal{O}_X$-module $\mathscr{F}$ and if $\mathscr{I} \cdot \mathscr{G} = 0$ for some $\mathcal{O}_Y$-module $\mathscr{F}$ then $\mathscr{G} = \iota_* \iota^* \mathscr{G}$ where $\mathscr{I} = \ker(\mathcal{O}_Y \to \iota_* \mathcal{O}_X)$.

11 Local Property

Definition 11.1. A property P of ring maps is *local* if,

(a). $P(R \to A) \implies P(R_f \to A_f)$ for all $f \in R$

(b). $P(R_f \to A)$ for some $f \in R$ then $P(R \to A_a)$ for any $a \in A$

(c). if $P(R \to A_{a_i})$ for $(a_1, \dots, a_r) = A$ then $P(R \to A)$.

Definition 11.2. We say a morphism of schemes $f : X \to Y$ is locally P for some local property P if for each $x \in X$ there is an affine open $U = \mathrm{Spec}\,(A)$ with $x \in U \subset X$ and $V = \mathrm{Spec}\,(R)$ with $V \subset Y$ with $f(U) \subset V$ such that $P(R \to A)$.

Lemma 11.3. If $f : X \to Y$ is locally P then for any affine opens $U \subset X$ and $V \subset Y$ with $f(U) \to V$ then $P(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$.

Remark.

12 Smooth Maps

Definition 12.1. A ring map $R \to A$ is *smooth* if it is of finite presentation,

$$A \cong R[x_1, \dots, x_n]/I$$

where I is finitely generated. Then consider,

$$I/I^2 \xrightarrow{\mathrm{d}} \bigoplus_{i=1}^n A\mathrm{d}x_i$$

given by,

$$f \mapsto \mathrm{d}f = \sum \frac{\partial f}{\partial x_i} \mathrm{d}x_i$$

Then d is injective and its cokernel is a projective A-module. Since $K = \mathrm{coker}\,\mathrm{d}$ is projective an finitely generated then it is locally free so it has a rank function. We say that $R \to A$ is *smooth of relative dimension* n if K is of constant rank n.

Remark. Smoothness satisfies the following,

(a). local

(b). preserved under composition

(c). preserved uncer base change

Example 12.2. Take $R \to R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ such that,

$$\det \left(\frac{\partial f_j}{\partial x_i} \right)_{\substack{i=1,\dots,c \\ j=1,\dots,c}}$$

maps to an invertible element of A. Then,

$$\frac{(f_1, \dots, f_c)}{(f_1, \dots, f_c)^2} \to \bigoplus_{i=1}^{n} A\mathrm{d}x_i \to \text{coker}$$

In differential geometry, we have,

$$\begin{tikzcd}
f^{-1}(\{0\}) \arrow[r, hook] & \mathcal{C}^n \arrow[d, "f"] \\
\{0\} \arrow[r, hook] & \mathcal{C}^n
\end{tikzcd}$$

Then $f^{-1}(\{0\})$ is a smooth manifold by the implicit function theorem. We call this situation standard smooth.

Lemma 12.3. A map $R \to A$ is smooth if and only if there exist a_i s.t. $(a_1, \dots, a_r) = A$ and $R \to A_{a_i}$ is standard smooth.

Definition 12.4. For a standard smooth ring map, $R \to A$ we can factor,

$$\begin{tikzcd}
R \arrow[r] \arrow[dr] & R[x_{n+1}, \dots, x_n] \arrow[d] \\
& R[x_1, \dots, x_n]/(f_1, \dots, f_c)
\end{tikzcd}$$

Then the downward map is étale.

Definition 12.5. A smooth morphism of schemes is a morphism which is locally smooth.

Remark. Using the previous lemma, for any smooth morphism of schemes $X \to Y$ it is locally standard smooth so we can factor,

$$\begin{tikzcd}
X \arrow[d, "f"] & U \arrow[l, hook'] \arrow[d] \arrow[dr] & \\
& & \mathbb{A}^d \arrow[dl] \\
Y & V \arrow[l, hook'] &
\end{tikzcd}$$

Definition 12.6. A variety X over k is smooth iff $X \to \mathrm{Spec}\,(k)$ is smooth.

Definition 12.7. A locally noetherian scheme X is *regular* or *nonsingular* iff $\mathcal{O}_{X,x}$ is regular at each $x \in X$.

Remark. For locally Noetherian schemes it suffices to check regularity on the closed points.

Theorem 12.8. If $X \to \mathrm{Spec}\,(k)$ is smooth then X is regular.

Theorem 12.9. If k is perfect then a variety is smooth iff it is regular.

Example 12.10. Let $k = \mathbb{F}_p(t)$ then take $\mathrm{Spec}\,(k[x]/(y^2 - (x^p - t)))$ which is regular but not smooth. Consider,

$$\mathrm{d}(y^2 - x^p + t) = 2y\mathrm{d}y + 0$$

and thus we have,

$$(f)/(f^2) \mapsto A\mathrm{d}x \oplus A\mathrm{d}y$$

which is injective but the cokernel is $A \oplus A/yA$ but A/yA has torsion so cannot be projective and thus not smooth.

However, we just need to check regularity at $(y, x^p - t) = (y) \subset A$ which is a height one ideal and generated by one element so $A_{(y)}$ is regular.

13 Differentials

Definition 13.1. For a ring map $\varphi : R \to A$ the A-module of differentials $\Omega_{A/R}$ is generated by the symbols $\mathrm{d}a$ for $a \in A$ such that,

(a). $\mathrm{d}(a_1 + a_2) = \mathrm{d}a_1 + \mathrm{d}a_2$

(b). $\mathrm{d}a_1 a_2 = \mathrm{d}a_1 \cdot a_2 + a_1 \cdot \mathrm{d}a_2$

(c). $\mathrm{d}r = 0$ for $r \in R$

Then $\mathrm{d}_{R/A} : R \to \Omega_{R/A}$ is the universal derivation meaning that $\Omega_{R/A}$ represents the functor $\mathrm{Der}_R(A, -)$ i.e.

$$\mathrm{Hom}_A\big(\Omega_{A/R}, M\big) \cong \mathrm{Der}_R(A, M)$$

via $(f : \Omega_{A/R} \to M) \mapsto f \circ \mathrm{d}_{A/R}$

(O8RL, O8RT)

Definition 13.2. Given a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ then there is a universal derivation $\mathrm{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$ s.t.

$$\mathrm{Hom}_{\mathcal{O}_X}\big(\Omega_{X/Y}, \mathscr{F}\big) \cong \mathrm{Der}_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathscr{F})$$

Where a derivation $\varphi : \mathcal{O}_X \to \mathscr{F}$ is an abelian map such that $\varphi(fs) = f\varphi(s) + \varphi(f)s$ and under $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ we send $s \in \mathcal{O}_Y(U)$ to $\varphi(s) = 0$.

Lemma 13.3. For a morphism of schemes $f : X \to Y$ we have,

$$\begin{tikzcd} X \arrow[d, "f"] & U = \mathrm{Spec}\,(A) \arrow[l] \arrow[d] \\ Y & V = \mathrm{Spec}\,(R) \arrow[l] \end{tikzcd}$$

Then we have $\Omega_{X/Y}|_U = \widetilde{\Omega_{A/R}}$.

13.1 The Diagonal

(Tag O1R1) Consider $R \to A$ then consider the map,

$$\Omega_{A/R} \xrightarrow{\sim} J/J^2$$

via $\mathrm{d}a \mapsto a \otimes 1 - 1 \otimes a$ where $J = \ker(A \otimes_R A \to A)$ via $a \otimes b \mapsto ab$. This situation generalizes to Schemes in which,

$$\Omega_{X/Y} = \Delta^*_{X/Y}(\mathscr{J})$$

where $\mathscr{J}$ is the sheaf of ideals of $\Delta : X \to X \times_Y X$ i.e. $\mathscr{J} = \ker(\mathcal{O}_{X \times_Y X} \to \Delta^*_{X/Y}\mathcal{O}_X)$. This is the conormal sheaf of $\Delta_{X/Y} : X \to X \times_Y X$.