Mathematics GU4051 Topology
Assignment # 7

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Problem 1.

Let X and Y be topological spaces with Y compact and let $\pi : X \times Y \to X$ be given by $\pi : (x, y) \mapsto x$. Let $C \subset X \times Y$ be closed and, assuming that $\pi(C)$ is not closed, consider $z \in \overline{\pi(C)} \backslash \pi(C)$. Now, the preimage satisfies,

$$\pi^{-1}(\{z\}) \subset (X \times Y) \backslash C$$

because if $(x, y) \in \pi^{-1}(\{z\})$ then $x = z$ but $\pi(x, y) \notin \pi(C)$ so $(x, y) \notin C$. For any $y \in Y$ the point $(z, y) \in \pi^{-1}(\{z\})$ because $\pi(z, y) = z$. However, $(X \times Y) \backslash C$ is open so for each $y \in Y$ there exists open sets $U_y \subset X$ and $V_y \subset Y$ such that,

$$(z, y) \subset U_y \times V_y \subset (X \times Y) \backslash C$$

Thus, for any $y \in Y$, $y \in V_y$ so $\mathcal{U} = \{V_y \mid y \in Y\}$ is an open cover of Y. By compactness, there exists a finite subcover, $\mathcal{U}_S$, indexed by a finite set $S \subset Y$. That is,

$$Y = \bigcup_{y \in S} V_y$$

Now, for each $y \in Y$, we have $z \in U_y$ and therefore,

$$z \in \bigcap_{y \in S} U_y = A$$

which is open because S is finite and each U_y is open. However, $z \in \overline{\pi(C)}$ and $z \in A$ which is open in X so $A \cap \pi(C) \neq \varnothing$. Therefore, $\exists t \in A \cap \pi(C)$ so for each $y \in S$ we have $t \in U_y$ and there exists some $y_t \in Y$ such that $(t, y_t) \in C$. However, Y is covered by $\mathcal{U}_S$ so for some $y \in S$ we have $y_t \subset V_{y_S}$ but $t \in U_y$ so $(t, y_t) \in U_y \times V_y$. However, $(t, y_t) \in C$ which contradicts the fact that U_y and V_y were choosen such that $U_y \times V_y \subset (X \times Y) \backslash C$. Thus, $\overline{\pi(C)} \backslash \pi(C)$ is empty but $\pi(C) \subset \overline{\pi(C)}$ so $\overline{\pi(C)} = \pi(C)$ and therefore, $\pi(C)$ is closed.

Problem 2.

Let X be a T_1 space. Suppose that X is countably compact. Because every infinite set contains a countable subset (assuming the axiom of countable choice), it suffices to prove that every infinite countable set has a limit point in X. Let $\Omega \subset X$ be a countable set with an enumeration given by $x_n \in \Omega$ for $n \in \mathbb{N}$. Suppose that Ω has no limit points in X then, each $x_n \notin \overline{\Omega \backslash \{x_n\}}$ so there exists an open $U_n \subset X$ with $x_n \in U_n$ and $U_n \cap (\Omega \backslash \{x_n\}) = \varnothing$ so $U_n \cap \Omega = \{x_n\}$. Futhermore, because Ω has no limit points, Ω is closed so $X \backslash \Omega$ is open. Thus, $V_n = U_n \cup (X \backslash \Omega)$ is open and $X \backslash \Omega \subset V_n$ and $\{x_n\} \in V_n$ however, $\Omega \cap V_n = \Omega \cap U_n = \{x_n\}$ so $V_n = (X \backslash \Omega) \cup \{x_n\}$. Therefore, $\{V_n \mid n \in \mathbb{N}\}$ is an open cover of X because,

$$\bigcup_{n \in \mathbb{N}} V_n = \bigcup_{n \in \mathbb{N}} (X \backslash \Omega) \cup \{x_n\} = (X \backslash \Omega) \cup \bigcup_{n \in \mathbb{N}} \{x_n\} = (X \backslash \Omega) \cup \Omega = X$$

so by countable compactness, there exits a finite subcover indexed by a finite set $S \subset \mathbb{N}$. However,

$$\Omega \cap \bigcup_{n \in S} V_n = \bigcup_{n \in S} \Omega \cap V_n \subset \bigcup_{n \in S} \{x_n\}$$

but $\bigcup\limits_{n \in S} V_n = X$ and $\Omega \subset X$ so $\Omega \subset \{x_n \mid n \in S\}$. However, S is finite and therefore, Ω is finite. Thus, if Ω is infinite then it must have a limit point.

Conversely, let X be limit point compact. Suppose that $\{U_n \mid n \in \mathbb{N}\}$ is a countable open cover of X with no finite subcover. Then, define $x_n \in X \backslash (U_1 \cup \cdots \cup U_n)$ which exists because if $X \backslash (U_1 \cup \cdots \cup U_n) = \varnothing$ then $U_1 \cup \cdots \cup U_n$ is a finite subcover. Let $A = \{x_n \mid n \in \mathbb{N}\}$. Because $\{U_n\}$ is a cover, for any $x \in X$ there exists some N s.t. $x \in U_N$ and then for $i \geq N$ we have $x_i \notin U_N$ because $x_i \notin U_1 \cup \cdots \cup U_N \cup \cdots \cup U_i \supset U_N$. Therefore, $A \cap U_N \subset \{x_n \mid n < N\}$ so $A \cap U_N$ is finite and thus, $C = A \cap U_N \backslash \{x\}$ is also finite. However, X is T_1 so for any $y \in X$, the set $\{y\}$ is closed and thus, by finite unions, C is closed. Therefore, $V = U_N \cap (X \backslash C)$ is open in X but $x \notin C$ and $x \in U_N$ so $x \in V$. $(A \backslash \{x\}) \cap V = (A \backslash \{x\}) \cap U_N \cap (X \backslash C) = (A \backslash \{x\} \cap U_N) \backslash C = \varnothing$. But $x \in V$ so x is not a limit point of A and thus A is an infinite set with no limit points in X contradicting limit point compactness. Therefore, we cannot have any countable cover without a finite subcover i.e. X is countably compact.

Problem 3.

For nonempty $A, B \subset X$ define $d(A, B) = \inf\{d(x, y) \mid x \in A \text{ and } y \in B\}$ and $d(x, A) = d(\{x\}, A)$.

(a). $d(x, A) = 0$ iff $\forall \delta > 0 : \exists y_\delta \in A : d(x, y_\delta) < \delta$ so take any open $U \subset X$ with $x \in U$ then $\exists \delta > 0 : x \in B_\delta(x) \subset U$ so $y_\delta \in B_\delta(x) \subset U$ so $U \cap A \neq \varnothing$. Thus, $x \in \bar{A}$. Conversely, if $x \in \bar{A}$ then $x \in B_\delta(x)$ is open so $B_\delta(x) \cap A \neq \varnothing$ so $\exists y_\delta \in A$ with $d(x, y_\delta) < \delta$ so $d(x, A) = 0$.

(b). Let A be compact then since X is a metric space it is Hausdorff so A is closed. Take $\delta_n = d(x, A) + 1/n$ and $A_n = C_{\delta_n}(x) \cap A$ where $C_\delta(x) = \{y \in X \mid d(x, y) \leq \delta\}$. By Lemma **??** and the intersection of closed sets, A_n is closed and $A_n \subset A$ so A_n is compact. Also, $\delta_n > \delta_{n+1}$ so $C_{\delta_n}(x) \supset C_{\delta_{n+1}}(x)$ and thus $A_n \supset A_{n+1}$. Furthermore, by approximation property, for any $n \in \mathbb{N}$ there exists $y \in A$ s.t. $d(x, y) < d(x, A) + 1/n$ so $y \in C_{\delta_n}(x) \cap A = A_n$. Thus, the sequence is nonempty. Since X is Hausdorff, the intersection of these compact nonempty nested sets is nonempty. Take

$$a \in \bigcap_{n \in \mathbb{N}} A_n$$

Thus, for every n, we have $a \in A_n$ so $a \in C_{\delta_n}(x)$ thus $d(x, a) < d(x, A) + 1/n$ and $a \in A$ so $d(x, A) \leq d(x, a)$. If $d(x, A) < d(x, a)$ then we can choose $n > 1/(d(x, a) - d(x, A))$ so $d(x, a) > d(x, A) + 1/n$ contradicting $a \in A_n$. Therefore, $d(x, a) = d(x, A)$ with $a \in A$.

(c). Define $B_\delta(A) = \{x \in X \mid d(x, A) < \delta\}$. Let $x \in B_\delta(A)$ then $d(x, A) < \delta$ so $\epsilon = \delta - d(x, A) > 0$. Thus, by the approximtion property, there exist $a \in A$ such that $d(x, a) < d(x, A) + \epsilon = \delta$ so $x \in B_\delta(a) \subset \bigcup_{a \in A} B_\delta(a)$. Thus, $B_\delta(A) \subset \bigcup_{a \in A} B_\delta(a)$.

Conversely, if $x \in \bigcup_{a \in A} B_\delta(a)$ then for some $a \in A$ we have $x \in B_\delta(a)$ so $d(x, a) < \delta$ but $d(x, A)$ is the infimum of all such numbers so $d(x, A) \le d(x, a) < \delta$ so $x \in B_\delta(A)$. Therefore, $B_\delta(A) = \bigcup_{a \in A} B_\delta(a)$.

(d). Let A be compact and $U \subset X$ be open such that $A \subset U$. Because U is open, for each $a \in A$, $\exists \delta_a > 0 : B_{\delta_a}(a) \subset U$. Now, $\forall a \in A : a \in B_{\frac{1}{2}\delta_a}(a)$ so $\{B_{\frac{1}{2}\delta_a}(a) \mid a \in A\}$ is a open cover of A. By compactness, there exists a finite subcover indexed by $S \subset A$. Then, $\delta = \frac{1}{2} \min_{a \in S} \delta_a$ exists and is positive. Let $x \in B_\delta(A) = \bigcup_{a \in A} B_\delta(a)$ then, there exists $a_0 \in A$ such that $x \in B_\delta(a_0)$ but $a_0 \in A$ so there must exist $a' \in S$ so that $a_0 \in B_{\frac{1}{2}\delta_{a'}}(a')$. Thus,

$$d(x, a') < d(x, a_0) + d(a_0, a') < \delta + \frac{1}{2}\delta_{a'} \le \delta_{a'}$$

because $a' \in S$ so $\delta \le \frac{1}{2}\delta_{a'}$. Therefore, $x \in B_{\delta_{a'}}(a') \subset U$ by the definition of $\delta_{a'}$ so $B_\delta(A) \subset U$.

(e). Take $\mathbb{Z}^+ \subset \mathbb{R}$ and $U = \bigcup_{n \in \mathbb{Z}^+} B_{\frac{1}{n}}(n)$. $\mathbb{Z}^+$ is closed in $\mathbb{R}$ because $\mathbb{R}/\mathbb{Z}^+ = (-\infty, 1) \cup \bigcup_{n \in \mathbb{Z}^+} (n, n+1)$ which is a union of open sets and thus open. Also, each $\forall n \in \mathbb{Z}^+ : n \in B_{\frac{1}{n}}(n)$ thus $\mathbb{Z}^+ \subset U$. However, suppose that $B_\delta(\mathbb{Z}^+) = \bigcup_{n \in \mathbb{Z}^+} B_\delta(n) \subset U$ then choose $n > \frac{1}{\delta}$. Now,

$$B_\delta\big(\mathbb{Z}^+\big) \cap B_\delta(n) \subset U \cap B_\delta(n)$$

but, $B_\delta(\mathbb{Z}^+) \cap B_\delta(n) = B_\delta(n)$ because $B_\delta(n) \subset B_\delta(\mathbb{Z}^+) = \bigcup_{a \in \mathbb{Z}^+} B_\delta(n)$ and $U \cap B_\delta(n) = B_{\frac{1}{n}}(n)$ because $\frac{1}{n} < \delta$. Therefore, $B_\delta(n) \subset B_{\frac{1}{n}}(n)$ which contradicts $\frac{1}{n} < \delta$. Thus, for every $\delta > 0$, $B_\delta(\mathbb{Z}^+) \not\subset U$.

Problem 4.

Let $f : X \to X$ be an isometry of a compact metric space X. We know that f is injective and continuous.

(a). Suppose that there exists $a \in X$ such that $a \notin f(X)$. Then because X is compact and f is continuous, $f(X)$ is compact. However, X is a metric space so it is Hausdorff. The set $\{a\}$ is compact because it is finite. Since X is Hausdorff, there exists open sets seperating $\{a\}$ and $f(X)$. Specifically, $\exists U, V \in \mathcal{T}_X$ s.t. $\{a\} \subset U$, $f(X) \subset V$ and $U \cap V = \varnothing$. Since $a \in U$ is open, $\exists \epsilon > 0 : a \in B_\epsilon(x) \subset U$. Thus, $B_\epsilon(x) \cap V = \varnothing$ so $B_\epsilon(x) \cap f(X) = \varnothing$. Because $d(f(x), f(y)) = d(x, y)$, by induction, $d(f^n(x), f^n(y)) = d(x, y)$. Now, consider $x_0 = a$ and $x_{n+1} = f(x_n)$. By induction, $x_n = f^n(a)$. For natural numbers $n > m$ consider,

$$d(x_n, x_m) = d(f^n(a), f^m(a)) = d(f^{n-m}(a), a) \ge \epsilon$$

The last inequalty holds because $n - m \ge 0$ so we have $f^{n-m}(a) \in f(X)$ but $B_\epsilon(a) \subset X \backslash f(X)$ so $f^{n-m}(a)) \notin B_\epsilon(a)$. Thus, no subsequence of $\{x_n\}$ can have a limit in X because any ball of radius $\epsilon/2$ can contain at most one x_i. This contradicts the sequential compactness of X which follows from the fact that X is a compact metric space. Thus, $f(X) = X$.

(b). We have that $f : X \to X$ is a continuous bijection but X is compact and X is Hausdorff because it is a metric space. Thus, f is a homeomorphism.

Problem 5.

Let X be a compact metric space and let $f : X \to X$ be a contraction i.e. for $c \in [0, 1)$,

$$\forall x, y \in X : d(f(x), f(y)) \le c \cdot d(x, y)$$

(a). Let $U \subset X$ be open. Consider $x \in f^{-1}(U)$ and, equivalently, $f(x) \in U$. Because U is open, $\exists \delta > 0 : f(x) \in B_\delta(f(x)) \subset U$. Suppose that $y \in B_\delta(x)$ then $d(x, y) < \delta$ so,

$$d(f(x), f(y)) \le c \cdot d(x, y) < c\delta < \delta$$

therefore $f(y) \in B_\delta(f(x)) \subset U$. Thus, $y \in f^{-1}(U)$. Therefore, $x \in B_\delta(x) \subset f^{-1}(U)$ so $f^{-1}(U)$ is open. Thus, f is continuous.

(b). Since f is continuous and X is compact, $f(X) \subset X$ is compact. Now, suppose that $f^n(X) \subset X$ is compact, then $f(f^n(X)) = f^{n+1}(X)$ is compact by continuity. Thus, by induction, $f^n(X)$ is compact for all $n \in \mathbb{N}$. Consider,

$$C = \bigcap_{n \in \mathbb{N}} f^n(X)$$

Now if $x \in f^{n+1}(X)$ then $x \in f^n(f(X))$ but $y = f(X) \in X$ so $x = f^n(y)$ thus $x \in f^n(X)$. Therefore, $f^{n+1}(X) \subset f^n(X)$. Futhermore, assuming X is nonempty each $f^n(X)$ is nonempty. Thus, this sequence of nested nonempty compact sets in a metric space which is thus Hausdorff has a nonempty intersection. Thus, $C \neq \varnothing$. Take $x \in C$ thus for every $n \in \mathbb{N}$, $x \in f^n(X)$ so there is a sequence y_n s.t. $x = f^n(y_n)$.

I claim that for any $a, b \in X$ we have $d(f^n(a), f^n(b)) \le c^n \cdot d(a, b)$. We proceed by induction: for $n = 1$ this is the definition of a contration. Suppose it holds for n then,

$$d(f^{n+1}(a), f^{n+1}(b)) = d(f(f^n(a)), f(f^n(b))) \le c \cdot d(f^n(a), f^n(b)) \le c^{n+1} \cdot d(a, b)$$

so the claim holds by induction. Now, consider,

$$d(x, f(x)) = d(f^n(y_n), f^n(f(y_n))) \le c^n \cdot d(y_n, f(y_n))$$

However, X is a compact metric space so by Lemma **??** it is bounded. Thus, there exists some $B \in \mathbb{R}^+$ s.t. $\forall a, b \in X : d(a, b) < B$. Therefore,

$$d(x, f(x)) < c^n \cdot B$$

for every $n \in \mathbb{N}$. However, $c < 1$ so if $d(x, f(x)) > 0$ then there exists some $n \in \mathbb{N}$ s.t. $c^n \cdot B < d(x, f(x))$ which contradicts the above formula. Thus, $d(x, f(x)) = 0$ so $f(x) = x$. This point is unique because if both x and y are fixed by f i.e. $f(x) = x$ and $f(y) = y$, we would have $d(f(x), f(y)) = d(x, y)$ but $d(f(x), f(y)) \le cd(x, y)$ so $d(x, y) \le cd(x, y)$ thus either $1 \le c$ or $d(x, y) = 0$. Since we know $c < 1$ we must have $d(x, y) = 0$ and thus $x = y$. Therefore, x is the unique point such that $f(x) = x$.

Problem 6.

Take $X = \mathbb{R}^+$ with the subspace topology in $\mathbb{R}$ with the standard topology. Define the harmonic series

$$x_n = \sum_{k=1}^{n} \frac{1}{k}$$

with $x_0 = 0$ and let $V_n = (x_n, x_{n+2})$ and $U_n = \bigcup_{i=0}^{n} V_n$ with $U_0 = (0, x_2)$. Suppose that $U_n = (0, x_{n+2})$ then $U_{n+1} = (0, x_{n+2}) \cup (x_{n+1}, x_{n+3}) = (0, x_{n+3})$ because x_n is an increasing sequence. Thus, by induction, $U_n = (0, x_{n+2})$. Since the harmonic series diverges to infinity, for any $r \in \mathbb{R}^+$ there exists $n \in \mathbb{N}$ s.t. $r < x_n < x_{n+2}$. Therefore, $r \in U_n$ so $r \in V_k$ for some $k \leq n$. Therefore, $\{V_n \mid n \in \mathbb{N}\}$ is a open cover of $\mathbb{R}^+$. However, suppose there existed a Lebesgue number δ. Then take $n + 1 > \frac{1}{\delta}$ and, by the definition of a Lebesgue number, we must have $B_\delta(x_{n+1}) \subset V_n$ because $x_{n+1} \in (x_n, x_{n+2})$ and no other U_k. However, $|x_{n+1} - x_n| = \frac{1}{n+1} < \delta$ so $x_n \in B_\delta(x_{n+1}) \subset V_n$ but $x_n \notin (x_n, x_{n+2}) = V_n$ which is a contradiction. Thus, there cannot exist a Lebesgue number for this cover.

Lemmas

Lemma 0.1. In a metric space X the set $C_\delta(x) = \{y \in X \mid d(x, y) \leq \delta\}$ is closed.

Proof. Take $U = X \setminus C_\delta(x)$ then $y \in U$ iff $d(x, y) > \delta$. For any $y \in U$ we have $d(x, y) > \delta$ so take $\epsilon = d(x, y) - \delta$ and then for any $z \in B_\epsilon(y)$ we have $d(z, y) < \epsilon$ but $d(x, y) < d(x, z) + d(z, y) < d(x, z) + \epsilon$ so $d(x, z) > d(x, y) - \epsilon = \delta$. Thus, $z \notin C_\delta(x)$ so $z \in U$. Therefore, $y \in B_\epsilon(y) \subset U$ so U is open and thus $C_\delta(x)$ is closed. □

Lemma 0.2. Let X be a compact metric space then $\exists B \in \mathbb{R}^+ : \forall x, y \in X : d(x, y) < B$.

Proof. If $X = \varnothing$ we are done. Else, take $x_0 \in X$, consider the open cover $\{B_\delta(x_0) \mid \delta \in \mathbb{R}^+\}$ by compactness, there is a finite subcover indexed by a finite set $S \subset \mathbb{R}^+$ s.t.

$$\bigcup_{\delta \in S} B_\delta(x_0) = X$$

But S is finite so $\Delta = \max_{\delta \in S} \delta$ exists. Then,

$$x \in X \implies \exists \delta \in S : x \in B_\delta(x_0) \implies d(x, x_0) < \delta \leq \Delta$$

So define $B = 2\Delta$ then for any $x, y \in X$ we have $d(x, y) < d(x, x_0) + d(x_0, y) < \Delta + \Delta = B$. □