Mathematics GU4044 Representations of Finite Groups
Assignment # 5

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Problem 1.

(a). The subgroup $\langle(1\,2\,3\,4)\rangle$ on its own acts trivially on $\{1, 2, 3, 4\}$ because given two numbers $x, y \in \{1, 2, 3, 4\}$ then $(1\,2\,3\,4)^k$ takes x to y where $k \equiv y - x \mod 4$. However, D_4 preserves a square so it cannot map adjacent points to non adjacent points. For example, the pair $(1, 2)$ cannot be sent to $(1, 3)$. Thus, D_4 is not doubly transitive.

(b). Consider the standard representation fo D_4 on the vectorspace $\mathbb{C}^4 = W_1 \oplus W_2$ where $W_1 = \{(t_1, t_2, t_3, t_3) \mid \sum_{i=1}^4 t_i = 0\}$. The character is a class function so we need only compute it on a representative from each equivalence class. D_4 has 5 conjugacy classes, using the notation $r = (1\,2\,3\,4)$ and $f = (1\,2)(3\,4)$, namely,

$$[e] = \{e\} \quad [r] = \{r, r^{-1}\} \quad [r^2] = \{r^2\} \quad [f] = \{f, r^2 f\} \quad [rf] = \{rf, r^3 f\}$$

The character of the permutation relation is the number of fixed points, $\chi_{\mathbb{C}[X]}(g) = \#(X^g)$. Thus, $\chi_{\mathbb{C}^4}(e) = 4$ and $\chi_{\mathbb{C}^4}(r) = \chi_{\mathbb{C}^4}(r^2) = 0$ and $\chi_{\mathbb{C}^4}(f) = 0$ and $\chi_{\mathbb{C}^4}(rf) = 2$ because $rf = (1\,2\,3\,4)(1\,2)(3\,4) = (1\,3)$ which fixes 2 and 4. Thus, $\chi_{W_2} = \chi_{\mathbb{C}^4} - \chi_{W_1} = \chi_{\mathbb{C}^4} - 1$.

(c).

$$\langle \chi_{\mathbb{C}^4}, \chi_{\mathbb{C}^4} \rangle = \frac{1}{8}\left(4^2 + 2 \cdot 0 + 0 + 2 \cdot 0 + 2 \cdot 2^2\right) = 3$$

Also,

$$\langle \chi_{W_2}, 1 \rangle = \frac{1}{8}\left((4-1) + 2 \cdot (0-1) + (0-1) + 2 \cdot (0-1) + 2 \cdot (2-1)\right) = 0$$

Likewise,

$$\langle \chi_{W_2}, \chi_{W_2} \rangle = \frac{1}{8}\left((4-1)^2 + 2 \cdot (0-1)^2 + (0-1)^2 + 2 \cdot (0-1)^2 + 2 \cdot (2-1)^2\right) = 2$$

Since this value is not 1, the representation W_2 cannot be irreducible.

(d). Since $\langle \chi_{W_2}, \chi_{W_2} \rangle = 2$ we know that $W_2 = V_1 \oplus V_2$ where V_1 and V_2 must be distinct irreducible representations. Consider the dimension 2 representation V, whose character satisfies $\chi_V(1) = 2$ and $\chi_V(r^2) = -2$ and $\chi_V(g) = 0$ otherwise. Therefore,

$$\langle \chi_V, \chi_{W_2} \rangle = \frac{1}{8}\left(2 \cdot (4-1) + 2 \cdot 0 \cdot (0-1) + (-2) \cdot (0-1) + 2 \cdot 0 \cdot (0-1) + 2 \cdot 0 \cdot (2-1)\right) = 1$$

so there is exactly one copy of V in the decomposition of W_2. Therefore, $W_2 = V \oplus V_2$ where $\dim V_2 = 1$ by dimension counting. Therefore, $V_2 \cong \mathbb{C}(\lambda)$ for some homomorphism $\lambda : D_4 \to \mathbb{C}^\times$. We know that $\lambda(f) \in \mathbb{C}^\times$ has order dividing two so $\lambda(f) = (-1)^s$ and $\lambda(rf) = \lambda(fr^3)$ so $\lambda(r)\lambda(f) = \lambda(r)^3\lambda(f)$ and thus $\lambda(r) = \lambda(r)^3$ so $\lambda(r)$ has order dividing 2 so $\lambda(r) = (-1)^k$. The character of the representation $\mathbb{C}(\lambda)$ is simply λ iself. Calculating the inner product,

$$\langle \chi_\lambda, \chi_{W_2} \rangle = \frac{1}{8}\left(3 + 2 \cdot (-1)^k \cdot (-1) + 1 \cdot (-1) + 2(-1)^s \cdot (-1) + 2 \cdot (-1)^{k+s} \cdot 1\right)$$

If $\mathbb{C}(\lambda)$ is to be irreducible and in the expansion of W_2 then we need this inner product to be 1. Therefore,

$$(-1)^{k+s} - (-1)^k - (-1)^s = 3$$

which forces k and s to be odd. Therefore, $\lambda(r) = \lambda(f) = -1$ and since $D_4 = \langle r, f \rangle$ the homomorphism is determined on the entire group.

Problem 2.

Let (V, ρ_V) be a G-representation. Consider the dual representation (V^*, ρ_{V^*}) where the action of ρ_{V^*} is defined as, $\rho_{V^*}(g) \cdot \varphi = \varphi \circ \rho_V(g^{-1})$. Thus, $\rho_{V^*}(g) = (\rho_V(g^{-1}))^*$. Thus,

$$\chi_{V_*}(g) = \operatorname{Tr} \rho_{V^*}(g) = \operatorname{Tr} \left(\rho_V(g^{-1})\right)^* = \operatorname{Tr} \rho_V(g^{-1}) = \overline{\operatorname{Tr} \rho_V(g)} = \overline{\chi_V(g)}$$

However, we know that two G-representations V and W are isomorphic if and only if $\chi_V = \chi_W$. Thus,

$$V \cong V^* \iff \chi_V = \chi_{V^*} = \overline{\chi_V}$$

However, $\overline{\chi_V(g)} = \chi_V(g) \iff \chi_V(g) \in \mathbb{R}$. Thus,

$$V \cong V^* \iff \chi_V = \overline{\chi_V} \iff \operatorname{Im}(\chi_V) \subset \mathbb{R}$$

Problem 3.

Let $\lambda : G \to \mathbb{C}^\times$ be a homomorphism. Consider the one-dimensional representation $\mathbb{C}(\lambda)$ inside $\mathbb{C}[G]$. This corresponds to a vector $v \in \mathbb{C}[G]$ such that $\rho_{reg}(g) \cdot v = \lambda(g)v$ for all $g \in G$. We can write any $v \in \mathbb{C}[G]$ as $v = \sum\limits_{h \in G} t_h\, h$ then,

$$\rho_{reg}(g) \cdot v = \sum_{h \in G} t_h\, gh = \sum_{h' \in G} t_{g^{-1}h'}\, h' = \lambda(g)v = \sum_{h \in G} (\lambda(g) \cdot t_h)\, h$$

For these to be equal, the coeficients must be equal since G is a basis of $\mathbb{C}[G]$. Thus, $\lambda(g) \cdot t_h = t_{g^{-1}h}$ for each h and g. Therefore, $\lambda(g^{-1}) \cdot t_e = t_g$ so every constant t_g is determined by the single constant $t_e = c$. Then,

$$v = c \sum_{g \in G} \lambda(g^{-1}) \cdot g$$

In particular, the representation $\mathbb{C}(\lambda)$ inside $\mathbb{C}[G]$ is the span of $\sum\limits_{g \in G} \lambda(g^{-1}) \cdot g$.

Viewing $\mathbb{C}[G]$ as the space of (finitely supported) functions $f : G \to \mathbb{C}$, the function corresponding to v is the map $f(g) = \lambda(g)^{-1}$. Then,

$$(\rho_{reg}(g) \cdot f)(h) = f(g^{-1}h) = \lambda(g^{-1}h)^{-1} = \lambda(g^{-1})^{-1} \cdot \lambda(h)^{-1} = \lambda(g) \cdot \lambda(h)^{-1} = \lambda(g) \cdot f(h)$$

Therefore, on the subspaces spanned by f, the map $\rho_{reg}(g)$ acts as multiplication by $\lambda(g)$.

Problem 4.

Let $\rho_V : G \to \mathrm{Aut}\,(V)$ be a representation and let $\lambda : G \to \mathbb{C}^\times$ be a homomorpism, corresponding to the one-dimensional representation $\mathbb{C}(\lambda)$.

(a). Define the map $\lambda \otimes \rho_V : G \to \mathrm{Aut}\,(V)$ by,

$$(\lambda \otimes \rho_V)(g) \cdot v = \lambda(g)\rho_V(g) \cdot v$$

Since $\lambda(g) \neq 0$ the linear map $\lambda(g)\rho_V(g)$ is invertible. Likewise, view $\lambda(g)$ as a linear map on V given by $\lambda(g)(v) = \lambda(g) \cdot v$. Thus, $(\lambda \otimes \rho_V)(g) = \lambda(g) \circ \rho_V(g)$ so, since it is a composition of linear maps, $(\lambda \otimes \rho_V)(g)$ is linear. Finally, take any $g, h \in G$ and consider,

$$\begin{aligned}(\lambda \otimes \rho_V)(gh)v &= \lambda(gh)\rho_V(gh)v = \lambda(g)\lambda(h) \cdot \rho_V(g) \circ \rho_V(h)v = \lambda(g)\rho_V(g)(\lambda(h)\rho_V(h)v) \\ &= (\lambda \otimes \rho_V)(g) \circ (\lambda \otimes \rho_V)(h)v\end{aligned}$$

where I have used the fact that $\rho_V(g)$ is linear. Thus, $\lambda \otimes \rho_V$ is a homomorphism to $\mathrm{Aut}\,(V)$ and thus a G-representation.

(b). Suppose that $W \subset V$ is a G-invariant subspace under ρ_V. Then for $w \in W$ consider $(\lambda \otimes \rho_V)(g)w = \lambda(g) \cdot \rho_V(g)w$ but $\rho_V(g)w = w' \in W$ and $\lambda(g)w' \in W$ so $(\lambda \otimes \rho_V)(g)w \in W$. Therefore, W is a G-invariant subspace under $\lambda \otimes \rho_V$. Likewise let W be a G-invariant subspace under $\lambda \otimes W$ and take $w \in W$. Then, consider $(\lambda \otimes \rho_V)(g)w = \lambda(g) \cdot \rho_V(g)w \in W$. Since W is invariant under scaling and $\lambda(g) \neq 0$, take $\lambda(g)^{-1} \cdot (\lambda(g) \cdot \rho_V(g)w) = \rho_V(g)w \in W$. Therefore, W is a G-invariant subspace under ρ_V.

We have shown that the G-invariant subspaces under these two representations correspond. Therefore, (V, ρ_V) is irreducible $\iff$ $(V, \lambda \otimes \rho_V)$ is irreducible.

(c). $\chi_{\lambda\otimes\rho} = \mathrm{Tr}\,\lambda(g)\rho_V(g) = \lambda(g)\mathrm{Tr}\,\rho_V(g) = \lambda(g)\chi_V$ since the trace scales linearly under scalar multiplication. Therefore, $\chi_{\lambda\otimes\rho} = \chi_V \iff \lambda(g)\chi_V(g) = \chi_V(g)$ iff either $\lambda(g) = 1$ or $\chi_V(g) = 0$. However, two G-representations are isomorphic if and only if their characters agree. Thus, ρ_V and $\lambda \otimes \rho_V$ are isomorphic iff for every g either $g \in \ker \lambda$ or $\chi_V(g) = 0$.

(d). The trace of the S_n-representation (W_2, ρ_{W_2}) where $W_2 = \{(t_1, \dots, t_n) \mid \sum_{i=1}^n t_i = 0\}$ is given by $\chi_{W_2}(g) = \chi_{\mathbb{C}^n}(g) - 1 = \#(\{1, \cdots, n\}^g) - 1$. We explicitly calculated that standard character of S^3 on the previous homework. $\chi_{st}(e) = 3$ and $\chi_{st}(\sigma) = 0$ and $\chi_{st}(\tau) = 1$. Therefore, $\chi_{W_2}(e) = 2$ and $\chi_{W_2}(\sigma) = -1$ and $\chi_{W_2}(\tau) = 0$. Because the character is a class function, these values determine the character everywhere. In particular, χ_{W_2} is zero on all the odd permutations. By the above criterion, $\epsilon \otimes \rho$ is therefore isomorphic to ρ. However, for $n \geq 4$, since $\chi_{W_2}(g) = \#(\{1, \cdots, n\}^g) - 1$ character of $(1\,2)$ is $n - 3 \geq 1$ for $n \geq 4$. However, $(1\,2)$ is an odd permutation ($(1\,2) \notin \ker \epsilon$) and $\chi((1\,2)) \neq 0$ so $\epsilon \otimes \rho$ and ρ cannot be isomorphic by the above criterion.

Problem 5.

Let $N \lhd G$ and (V, ρ) be a G/N-representation of G and let $\pi : G \to G/N$ be the quotient map. Suppose that $W \subset V$ is G/N-invariant then $\rho(gN) \cdot W \subset W$ for any $gN \in G/N$. Then, for any $g \in G$ the map $\rho \circ \pi(g) \cdot W = \rho(gN) \cdot W \subset W$. Thus, W is G-invariant. Conversely, if W is G-invariant, then $\rho \circ \pi(g) \cdot W \subset W$ but $\rho \cdot \pi(g) = \rho(gN)$ so W is also G/N-invariant. Therefore, the set of G/N-invariant subspaces of V w.r.t the map ρ is equal to the set of G-invariant subspaces of V w.r.t the map $\rho \circ \pi$. Therefore,

$$(V, \rho) \text{ is an irreducible } G/N\text{-representation} \iff (v, \rho \circ \pi) \text{ is an irreducible } G\text{-representation}$$

Problem 6.

Let $H = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \lhd A_4$

(a). Since $|A_4/H| = |A_4|/|H| = 12/4 = 3$, the quotient A_4/H must be cyclic of order 3. Furthermore, $\mathbb{Z}/3\mathbb{Z}$ is generated by any nonzero element and since $(1\;2\;3) \notin H$ we have that $(1\;2\;3)$ generates A_4/H.

(b). Since $|A_4/H| = 3 = \sum_{i=1}^{g} d_i^2$ we must have three one-dimensional irreducible representations of A_4/H. Each of these representations lift to one-dimensional irreducible representations of A_4. These representations of $\mathbb{Z}/3\mathbb{Z}$ are generated by homomorphisms $\lambda : \mathbb{Z}/3\mathbb{Z} \to \mathbb{C}^\times$ which are uniquely characterized by an integer k such that $\lambda(1) = \zeta_3^k$ since $\lambda(1)$ has order dividing 3. For these one-dimensional representations $\chi_\rho = \lambda$. However, the characters of these lifts are simply given by, $\chi_{\rho\circ\pi} = \chi_\rho \circ \pi$. Thus, the characters of these one-dimensional A_4-representations are, $\chi_k(g) = \chi_\rho(gH) = (\zeta_3^k)^m$ where $gH = (1\;2\;3)^m H$.

(c). $(1\;3\;2) = (1\;2\;3)^2$ and thus $\chi_1((1\;3\;2)) = \zeta_3^2 \neq \zeta_3 = \chi_1((1\;2\;3))$. However, any character is a class function so $(1\;2\;3)$ and $(1\;3\;2)$ cannot lie in the same conjugacy class.

(d). The character χ_{W_2} on A_4 is the same as the character for S_4 under the standard representation restricted to elements of A_4. Thus, $\chi_{A_4} = \chi_{st} - 1$. However, χ_{st} simply counts the fixed points of the action of a permutation. The only elements with fixed points are three cycles and two cycles, the latter of which are not contained in A_4. Therefore, $\chi_{W_2}(e) = 3$ and $\chi_{W_2}((a\;b\;c)) = 1 - 1 = 0$ and $\chi_{W_2}(g) = 0 - 1 = -1$ otherwise. Since there are eight three-cycles,

$$\langle \chi_{W_2}, \chi_{W_2} \rangle = \frac{1}{12}\left(3^2 + 8 \cdot 0^2 + 3 \cdot (-1)^2\right) = \frac{12}{12} = 1$$

Therefore, W_2 is an irreducible A_4-representation.

(e). We know that $|A_4| = \sum_{i=1}^{g} d_i^2$. However, in part (b) we found three one-dimensional irreducible representations and in part (e) we found a three-dimensional irreducible representation. Furthermore, $1 + 1 + 1 + 3^2 = 12 = |A_4|$ so we have found every irreducible A_4-representation up to isomorphism.