Mathematics GU4051 Topology
Assignment # 1

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Problem 1.

(a). $(-\infty, a) \cup (b, \infty)$ is open in $\mathbb{R}$:
Let $x \in (-\infty, a)$ then $x < a$ so take $\delta = a - x$ so that whenever $|y - x| < \delta$, $y < \delta + x = a$ then $y \in (-\infty, a)$. Therefore, $B_\delta(x) \subset (-\infty, a)$ so $(-\infty, a)$ is open.

Similarly, let $x \in (b, \infty)$ then $b < x$ so take $\delta = x - b$ so that whenever $|y - x| < \delta$ then $y > x - \delta = b$ so $y \in (b, \infty)$. Therefore, $B_\delta(x) \subset (b, \infty)$ so (b, ∞) is open. So as a union of open sets, $(-\infty, a) \cup (b, \infty)$ is open.

For $a < b$, $S = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty))$ is not open in $\mathbb{R}$:
Take $a \in S$ (since $a \not< a$ and $a < b$) then suppose that $\exists \delta \in \mathbb{R}^+ : B_\delta(a) \subset S$ then let $x = a - \frac{1}{2}\delta < a$ thus $x \in (-\infty, a)$ so $x \notin S$ a contradiction because $|x - a| < \delta$ so $x \in B_\delta(a) \subset S$.

(b). $\mathbb{Z}$ is not open in $\mathbb{R}$:
Take $0 \in \mathbb{Z}$ then suppose that $\exists \delta \in \mathbb{R}^+ : B_\delta(0) \subset \mathbb{Z}$ but since $B_\delta(0)$ is an interval, $\exists x \in B_\delta(0) \setminus \mathbb{Q}$ thus $x \notin \mathbb{Q} \supset \mathbb{Z}$ a contradiction because $x \in B_\delta(0) \subset \mathbb{Z}$.

$\mathbb{R} \setminus \mathbb{Z}$ is open in $\mathbb{R}$:
Since for any $x \in \mathbb{R}$, $\exists n \in \mathbb{Z} : n \le x < n + 1$ we have $\mathbb{R} \setminus \mathbb{Z} = \bigcup\limits_{n \in \mathbb{Z}} (n, n+1)$ but each $(n, n+1)$ is open so the union is open.

(c). $\mathbb{Q}$ is not open in $\mathbb{R}$:
Take $q \in \mathbb{Q}$ and suppose $\exists \delta \in \mathbb{R}^+ : B_\delta(q) \subset \mathbb{Q}$ then since $B_\delta(q)$ is an interval, $\exists x \in B_\delta(q) \setminus \mathbb{Q}$ so $x \notin \mathbb{Q}$ which is a contradiction because $x \in B_\delta(q) \subset \mathbb{Q}$.

$\mathbb{R} \setminus \mathbb{Q}$ is not open in $\mathbb{R}$:
Take $r \in \mathbb{R} \setminus \mathbb{Q}$ and suppose $\exists \delta \in \mathbb{R}^+ : B_\delta(q) \subset \mathbb{R} \setminus \mathbb{Q}$ then since $B_\delta(q)$ is an interval, $\exists x \in B_\delta(q) \cap \mathbb{Q}$ so $x \in \mathbb{Q}$ which is a contradiction because $x \in B_\delta(q) \subset \mathbb{R} \setminus \mathbb{Q}$ so $x \in \mathbb{Q}$.

(d). $S = \{1/n \mid n \in \mathbb{Z}^+\}$ is not open in $\mathbb{R}$:
Take $x = 1 \in S$ and suppose that $\exists \delta \in \mathbb{R}^+ : B_\delta(1) \subset S$ then take $y = 1 + \frac{1}{2}\delta$ then $y > \sup(S) = 1$ so $y \notin S$ but $|y - x| < \delta$ so $y \in B_\delta(1) \subset S$ which is a contradiction.

$\mathbb{R} \setminus S$ is not open in $\mathbb{R}$:
For all $n \in \mathbb{Z}^+$, $1/n \neq 0$ so $0 \in \mathbb{R} \setminus \S$ so suppose $\exists \delta \in \mathbb{R}^+ : B_\delta(0) \subset \mathbb{R} \setminus S$. But by the unboundedness of $\mathbb{Z}$ there exists $k \in \mathbb{Z}^+$ s.t. $0 < 1/k < \delta$ and $1/k \in S$ but then $1/k \in B_\delta(0) \subset \mathbb{R} \setminus S$ which is a contradiction.

Problem 2.

(a). $f(x) = |x|$ is continuous:
given $\epsilon > 0$ take $\delta = \epsilon$. Whenever $|x - y| < \delta$ then $|f(x) - f(y)| = ||x| - |y|| \leq |x - y| < \delta = \epsilon$ thus $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

(b). $g(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$ is not continuous:

$U = (-\frac{1}{2}, \frac{1}{2}) \subset \mathbb{R}$ is open in $\mathbb{R}$ but $g^{-1}(U) = \mathbb{Q}$ since $g(\mathbb{Q}) = \{0\} \subset U$ and if $x \notin \mathbb{Q}$ then $g(x) = 1 \notin U$. But $\mathbb{Q}$ is not open in $\mathbb{R}$ so g cannot be continuous.

Problem 3.

$f : \mathbb{R} \to \mathbb{R}$ is continuous iff $f^{-1}(V)$ is closed for any closed $V \subset \mathbb{R}$

Proof. By Lemma 0.1, $f^{-1}(\mathbb{R} \setminus V) = \mathbb{R} \setminus f^{-1}(V)$.

Now suppose that f is continuous. Then let $V \subset \mathbb{R}$ be closed so $\mathbb{R} \setminus V$ is open. By continuity, $f^{-1}(\mathbb{R} \setminus V) = \mathbb{R} \setminus f^{-1}(V)$ is open and therefore, $f^{-1}(V)$ is closed.

Suppose that $f^{-1}(V)$ is closed for any closed $V \subset \mathbb{R}$
Let $V \subset \mathbb{R}$ be open. Then $\mathbb{R} \setminus V$ is closed, since $V = \mathbb{R} \setminus (\mathbb{R} \setminus V)$ is open, so $f^{-1}(\mathbb{R} \setminus V) = \mathbb{R} \setminus f^{-1}(V)$ is closed. Therefore, $\mathbb{R} \setminus (\mathbb{R} \setminus f^{-1}(V)) = f^{-1}(V)$ is open. Thus, $V \subset \mathbb{R}$ is open $\implies f^{-1}(V)$ is open so f is continuous. $\square$

Problem 4.

False. Let $f(x) = 0$ then $f^{-1}(V) = \begin{cases} \varnothing & 0 \notin V \\ \mathbb{R} & 0 \in V \end{cases}$ which is always open in $\mathbb{R}$ so f is continuous. However, $\mathbb{R}$ is open but $f(\mathbb{R}) = \{0\}$ is not open.

Problem 5.

(a). Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open. Take $\mathbf{x} \in U \times V$ then $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$ where $\mathbf{x_1} \in U$ and $\mathbf{x_2} \in V$.

Now since U and V are open, $\exists \delta_1, \delta_2 \in \mathbb{R}^+ : B_{\delta_1}(\mathbf{x_1}) \subset U$ and $B_{\delta_2}(\mathbf{x_2}) \subset V$.

Take $\delta = \min\{\delta_1, \delta_2\}$ so that for $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^{m+n}$ if $|\mathbf{y} - \mathbf{x}| < \delta$ then $|\mathbf{y_1} - \mathbf{x_1}|^2 + |\mathbf{y_2} - \mathbf{x_2}|^2 \leq \delta^2$ therefore, $|\mathbf{y_1} - \mathbf{x_1}| < \delta \leq \delta_1$ and $|\mathbf{y_2} - \mathbf{x_2}| < \delta < \delta_2$ so $\mathbf{y_1} \in B_{\delta_1}(\mathbf{x_1}) \subset U$ and $\mathbf{y_2} \in B_{\delta_2}(\mathbf{x_2}) \subset V$ so $\mathbf{y} \in U \times V$.
Therefore, $B_\delta(\mathbf{y}) \subset U \times V$ so $U \times V$ is open.

(b). No. Take $m = n = 1$ and $S = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x \neq y\} \subset \mathbb{R}^2$.

Now take $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x - y$ so f is linear so, by Lemma 0.2, f is continuous. Since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is open, $f^{-1}(\mathbb{R} \setminus \{0\}) = S$ is open because

$$f^{-1}(\{0\}) = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x = y\}$$

However, suppose $S = U \times V$ with $U, V \subset \mathbb{R}$ then since $(1, 0), (0, 1) \in S$ we have $0 \in U$ and $0 \in V$ so $(0, 0) \in U \times V = S$ which is a contradiction.

Problem 6.

Let $L \subset \mathbb{R}^2$ be a line given by $L = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } ax + by = c\}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = ax + by$ is linear and thus continuous (by Lemma 0.2). Since $\mathbb{R} \setminus \{c\} = (-\infty, c) \cup (c, \infty)$ is open, $f^{-1}(\mathbb{R} \setminus \{c\}) = \mathbb{R} \setminus L$ is open because $f^{-1}(\{c\}) = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } ax + by = c\}$.
Now let $\{L_1, \ldots, L_n\}$ be a finite collection of lines and $S = \bigcup_{i=1}^{n} L_i$. Then by DeMorgan,

$$\mathbb{R} \setminus S = \bigcap_{i=1}^{n} \mathbb{R} \setminus L_i$$

but each $\mathbb{R} \setminus L_i$ is open so $\mathbb{R} \setminus S$ is open as a finite intersection of open sets.

Lemmas

Lemma 0.1. For $f : X \to Y$ and $V \subset Y$, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$

Proof. Let $x \in f^{-1}(Y \setminus V)$ then $f(x) \in Y \setminus V$ so $f(x) \notin V$ thus $x \notin f^{-1}(V)$ so $x \in X \setminus f^{-1}(V)$ since $f^{-1}(Y \setminus V) \subset X$.

Also if $x \in X \setminus f^{-1}(V)$ then $f(x) \notin V$ but $f(x) \in Y$ (because $\operatorname{Im}(f) \subset Y$) so $f(x) \in Y \setminus V$ so $f(x) \in Y \setminus V$ therefore, $x \in f^{-1}(Y \setminus V)$.

Thus, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. □

Lemma 0.2. if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear then f is uniformly continuous

Proof. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $g(\mathbf{x}) = \begin{cases} |f(\mathbf{x})|/|\mathbf{x}| & \mathbf{x} \neq \vec{0} \\ 0 & \mathbf{x} = \vec{0} \end{cases}$ is bounded (proven in Honors Math). Thus $\exists M \in \mathbb{R}^+ : \forall \mathbf{v} \in \mathbb{R}^n : |f(\mathbf{v})| < M|\mathbf{v}|$ so f is Lipschitz.

Given $\epsilon > 0$ take $\delta = \frac{1}{M}\epsilon$.
If $|\mathbf{x} - \mathbf{y}| < \delta$ then $|f(\mathbf{x}) - f(\mathbf{y})| = |f(\mathbf{x} - \mathbf{y})| < M|\mathbf{x} - \mathbf{y}| < M\delta = \epsilon$

Therefore, $|\mathbf{x} - \mathbf{y}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$ $\square$