1 Locally Free Sheaves

2 Algebraic Vector Bundles

3 Derivations

Definition 3.0.1. Let $\mathscr{A}$ be a sheaf of algebras and $\mathscr{B}$ an $\mathscr{A}$-algebra and $\mathscr{F}$ a $\mathscr{B}$-module. Then an $\mathscr{A}$-derivation $D : \mathscr{B} \to \mathscr{F}$ is a $\mathscr{A}$-module map such that on all local sections,

$$D(fg) = D(f)g + fD(g)$$

Furthermore, we write $\mathscr{D}er_{\mathscr{A}}(\mathscr{B}, \mathscr{F}) \subset \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$ for the $\mathscr{A}$-submodule of derivations.

Definition 3.0.2. If the functor $\mathscr{F} \mapsto \mathscr{D}er_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$ is representable on the category on $\mathscr{B}$-modules then we say the representing pair $(\Omega_{\mathscr{B}/\mathscr{A}}, \mathrm{d})$ is the $\mathscr{B}$-module of $\mathscr{A}$-differentials where,

$$\mathscr{H}om_{\mathscr{A}}\big(\Omega_{\mathscr{B}/\mathscr{A}}, \mathscr{F}\big) = \mathscr{D}er_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$$

and the derivation $\mathrm{d} : \mathscr{B} \to \Omega_{\mathscr{B}/\mathscr{A}}$ is the universal element given by,

$$\mathrm{id} \in \mathscr{H}om_{\mathscr{A}}\big(\Omega_{\mathscr{B}/\mathscr{A}}, \Omega_{\mathscr{B}/\mathscr{A}}\big) = \mathscr{D}er_{\mathscr{A}}\big(\mathscr{B}, \Omega_{\mathscr{B}/\mathscr{A}}\big)$$

Definition 3.0.3. Given morphism of locally ringed spaces $f : X \to S$ we say that $(\Omega_{X/S}, \mathrm{d})$ is the $\mathcal{O}_X$-module of $f^{-1}\mathcal{O}_S$-differentials viewing $\mathcal{O}_X$ as a $f^{-1}\mathcal{O}_S$-algebra via the map $f^{-1}\mathcal{O}_S \to \mathcal{O}_X$.

4 Connections

Remark. Here we have a locally ringed space $X \to S$ over S. We write $\Omega_X = \Omega_{X/S}$ and

Definition 4.0.1. A connection on a vector bundle $\mathcal{E}$ on X in a $\mathcal{O}_S$-linear derivation,

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

Lemma 4.0.2. Suppose that $\nabla_1, \nabla_2 : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ are connections. Then,

$$\nabla_1 - \nabla_2 : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

is a $\mathcal{O}_X$-module map.

Proof. $(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 s - \nabla_2 s) + \mathrm{d}f \otimes s - \mathrm{d}f \otimes s = f(\nabla_1 - \nabla_2)s$. $\square$

Remark. Therefore, the space of connections is a affine subspace of $\mathrm{Hom}\,(\mathcal{E}, \Omega^1_X \mathcal{E})$. Then if $\mathcal{E}$ is finite locally free,

$$\mathrm{Hom}\,\big(\mathcal{E}, \Omega^1_X \mathcal{E}\big) = H^0(X, \Omega^1_X \otimes_{\mathcal{O}_X} \mathscr{E}nd_{\mathcal{O}_S}(\mathcal{E}))$$

Definition 4.0.3. The first Chern class $c_1 : \mathrm{Pic}\,(X) \to H^1(X, \Omega^1) \subset H^2_{\mathrm{dR}}(X)$ is defined by $H^1(X, -)$ applied to the map $\mathrm{dlog} : \mathcal{O}_X^\times \to \Omega^1_X$ defined as $\mathrm{dlog}(f) = f^{-1}\mathrm{d}f$.

Proposition 4.0.4. A line bundle $\mathcal{L}$ admits a connection $\nabla : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$ if and only if $c_1(\mathcal{L}) = 0$.

Proof. A line bundle $\mathcal{L}$ is represented by a Cech cocycle $(U_i, f_{ij}) \in H^1(X, \mathcal{O}_X^\times)$. Then a connection on a line bundle is represented by (U_i, ω_i) with $\omega_i \in \Omega_X^1(U_i)$ where (U_i, s_i) is a trivialization of $\mathcal{L}$ with $\mathcal{O}_{U_i} \xrightarrow{s_i} \mathcal{L}|_{U_i}$ then $s_i|_{U_i \cap U_j} = f_{ij} s_j|_{U_i \cap U_j}$ and $\nabla s_i = \omega_i \otimes s_i$. However, we must have on $U_i \cap U_j$,

$$\nabla s_i = \nabla f_{ij} s_j = f_{ij} \nabla s_j + \mathrm{d} f_{ij} \otimes s_j$$

Therefore,

$$\omega_i \otimes f_{ij} s_j = f_{ij} \omega_j \otimes s_j + \mathrm{d} f_{ij} \otimes s_j$$

and thus,

$$(\omega_i - \omega_j)|_{U_i \cap U_j} = \mathrm{dlog}(f_{ij})$$

Consider the Cech differential $\mathrm{d} : \check{C}^0(\mathfrak{U}, \Omega_X^1) \to \check{C}^1(\mathfrak{U}, \Omega_X^1)$ which takes the sections (ω_i) to the coboundary $(\omega_i - \omega_j)|_{U_{ij}}$. Therefore, such a connection i.e. such a class exists iff the class,

$$c_1(\mathcal{L}) = [\mathrm{dlog}(f_{ij})] \in \check{H}^1(X, \Omega_X^1)$$

is trivial since it is a coboundary. $\square$

5 Differential Operators

Definition 5.0.1. Let $\mathscr{A}$ be a sheaf of algebras and $\mathscr{B}$ an $\mathscr{A}$-algebra and $\mathscr{F}, \mathscr{G}$ be $\mathscr{B}$-modules. Then a differential operator $D : \mathscr{F} \to \mathscr{G}$ of order k is a $\mathscr{A}$-module map such that for all local sections $b \in \Gamma(U, \mathscr{B})$ the map, $D(b \cdot -) - b \cdot D : \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G})$ is a differential operator of order $k-1$. Where a differential operator of order $k = 0$ is a $\mathscr{B}$-linear map $D : \mathscr{F} \to \mathscr{G}$. Furthermore, we write $\mathscr{D}\!\mathit{iff}^k_{\mathscr{B}/\mathscr{A}}(\mathscr{F}, \mathscr{G}) \subset \mathscr{H}\!\mathit{om}_{\mathscr{B}}(\mathscr{F}, \mathscr{G})$ to denote the $\mathscr{B}$-submodule of differential operators of order k.

6 Sheaves of Jets

7 The Atiyah Class

8 Riemann-Hilbert Correspondence