1 Introduction

1.1 Lue Pan

(a) Shimura varieties and Hodge theory (Complex geometry, Lie groups, automorphic forms)

(b) p-adic geometry

  (a) starting with Tate's p-divisible groups paper

(c) p-adic functional analysis

(d) representation theory of eveloping algebras.

1.2 Schedule

(a) introduction of methods of differential geometry in the study of perfectoid modular curves

Theorem 1.2.1 (Pan, 2022)**.** Let E be a finite extension of $\mathbb{Q}_p$ and $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ a continuous representation (using the p-adic topology). Suppose,

(a) $\mathrm{Hom}_{E[G_{\mathbb{Q}}]}\left(\rho, \hat{H}^1(K^p, E)\right) \neq 0$ where $K = K^p K_p \subset \mathrm{GL}_2(\mathbb{A}_f)$ with $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ is compact open

(b) and $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$ is a decomposition group and $\rho|_{G_{\mathbb{Q}_p}}$ is de Rham of Hodge-Tate weights $0, k$ for $k \in \mathbb{Z}^+$.

Then ρ arises from a (classical) cusp $(k+1)$-form and an eigenvector for the Hecke operators.

Remark. Arises from a classical cusp form means arises from the following theorem.

Theorem 1.2.2 (Eichler-Shimura, Deligne)**.** Let f a cusp form of weight $k+1$ and level N, eigenvector for the Hecke operators. Write,

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$$

with $a_1 = 1$. Then there is a $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ with for $q \not| Np$ we have $\mathrm{Tr}\left(\rho_f(\mathrm{Frob}_q)\right) = a_q$, and $\rho_f|_{G_{\mathbb{Q}_p}}$ is de Rham of Hodge-Tate weights $0, k$.

1.3 Alterior Motives

1.3.1 Simpson Correspondence

The Simpson correspondence. For a complex variety with a local system $\alpha : \pi_1(Z) \to \mathrm{GL}_n(\mathbb{C})$ then Simpson produced a Higgs field (nonabelian Hodge theory for $n > 1$). There is a p-adic simpson correspondence (Faltings, Abber-Gros-Traji) which studies,

$$\alpha : \pi_1(Z) \to \mathrm{GL}_n(\bar{\mathbb{Q}_l})$$

and produces p-adic Higgs field. However, in the Simpson correspondence we need by holomorphic and antiholomorphic differentiation. Lue Pan develops a p-adic version of this.

1.3.2 Perfectoid Geometry

In a perfectoid ring, the map $(-)^p : R \to R$ is surjective (by construction) and therefore, $\mathrm{d}f = \mathrm{d}g^{p^n} = p^n g^{p^n-1}\mathrm{d}g = 0$ so we need to clarify what it means to do differential geometry in a perfectoid setting.

Also want to clarify relation between complex and p-adic differential operators for p-adic L-functions.

1.3.3 p-adic representation theory of p-adic groups

complex nonabelian Hodge theory is related to the theory of complex Lie groups. Similarly, we might hope to make progress on p-adic groups beyond $\mathrm{GL}_2(\mathbb{Q}_p)$ using nonabelian Hodge theory.

1.4 Fontain-Mazur Conjecture

Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be a continuous representation. Suppose,

(a) ρ is unramified at all but finitely many primes (this is implied by (1) in Pan 2022.

(b) $\rho|_{G_{\mathbb{Q}_p}}$ is de Rham of Hodge-Tate weights $0, k$.

Then $\rho = \rho_f$ for some cuspidal Hecke eigenform of weight $k+1$.

Remark. Emerton proved this special case of Fontain-Mazur (only thinking about dimension 2 here) with some condition on the irreducibility of the residual representation.

Suppose that ρ is the representation on the Tate module of an elliptic curve A,

$$\rho : G_{\mathbb{Q}} \to \mathrm{Aut}\left(\varprojlim_n A[p^n]\right)$$

Then for $k = 1$ this satisfies the conditions of the theorem and thus we get that ρ is modular which implies Tanayama-Shimura and hence FLT.

2 Review of Modular Forms

Going back 200 years. We consider,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

and $\mathfrak{h} = \{z = x + iy \mid y > 0\} \subset \mathbb{C}$ is the upper half plane. Then we have an action,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Then for $N \geq 3$ the open modular curve $Y(N) = \mathfrak{h}/\Gamma(N)$ and $X(N) = Y(N) \cup \{\text{cusps / points at infinity}\}$ which has a unique algebraic structure. Then $X(N)$ is defined over $\mathbb{Q}(\zeta_N)$ is a moduli space for elliptic curves with torsion structure. Consider $\Omega = \Omega^1_{X(N)}$ is the cotangent bundle. Consider $\Omega(S)$ are the differential 1-forms with simple poles at S (the set of cusps). There is a natural line bundle ω on $X(N)$ such that $\omega^{\otimes 2} \cong \Omega(S)$.

For $k \in \mathbb{Z}$ the space of modular forms $\mathcal{M}_k(\Gamma(N)) = \Gamma(X(N), \omega^{\otimes k}) = H^0(X(N), \omega^{\otimes k})$. For $k > 0$ there are lots of modular forms. For $k < 0$ there are none. For $k = 0$ there are only constants. Then the cusp forms $\mathcal{S}_k(\Gamma(N)) \hookrightarrow \mathcal{M}_k(\Gamma(N))$ are the modular forms vanishing at the cusps. There is a Hecke algebra acting on the cusp forms. An eigenvector is called an eigenform. For $k \geq 2$, Deligne produces from an eigenform a "modular" representation ρ_f. and for $k = 1$ this is via Deligne-Serre.

Consider $\mathfrak{h} \subset \mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ by adding ∞ via $z \mapsto [z : 1]$. From the action of $G = \mathrm{PGL}_2(\mathbb{C})$ we get,

$$\gamma \cdot [z : 1] = [az + b : cz + d] = \left[\frac{az + b}{cz + d} : 1\right]$$

Notice that the inclusion is invariant under $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{PGL}_2(\mathbb{C})$. Then $\mathbb{P}^1$ has a collection of G-equivariant line bundles called $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ for $n \in \mathbb{Z}$.

$$\iota^*(\mathcal{O}(n))/\Gamma(N) = \omega^{-n}|_{Y(N)}$$

for $n = \deg \mathcal{O}(n)$.

Let $\tau : G \to \mathrm{GL}(V)$ be any algebraic representation. Let G act on $V \times \mathbb{P}^1$ via $g \cdot (v, x) = (\tau(g)v, g \cdot x)$ then the projectionn to $\mathbb{P}^1$ is equivariant. Therefore we can pull back and descent to $Y(N)$ to get $\widetilde{V}$. Then,

$$\nabla : V \otimes \mathcal{O}_{\mathbb{P}^1} \to V \otimes \Omega^1_{\mathbb{P}^1}$$

given by $v \otimes \varphi \mapsto v \otimes \mathrm{d}\varphi$. This is somehow equivariant so we get an integrable connection,

$$\widetilde{\nabla} : \widetilde{V} \to \widetilde{V} \otimes \Omega^1_{Y(N)}$$

By Deligne, we can extend this to $X(N)$ so that ∇ acquires logarithmic poles on the cusps.

Suppose that τ is integral, meaning $V_{\mathbb{Z}} \subset V$ is a lattice stabilized by the action of $\mathrm{SL}(2, \mathbb{Z}) \subset \mathrm{GL}(2, \mathbb{Z})$. Then,

$$\widetilde{V}_{\mathbb{Z}} = (V_{\mathbb{Z}} \times \mathfrak{h})/\Gamma(N)$$

Then $\widetilde{V}_n = \widetilde{V}_{\mathbb{Z}}/p^n$ for $n \geq 1$. Define,

$$H^1_!(Y(N), \widetilde{V}_n) = \mathrm{im}\,(H^1_c(Y(N), \widetilde{V}_n) \to H^1(Y(N), \widetilde{V}_n))$$

Then, Eichler-Shimura relations say,

$$H^1_!(Y(N), \widetilde{V}) \otimes \mathbb{C} \xrightarrow{\sim} \mathcal{S}_{k+2}(\Gamma(N)) \oplus \overline{\mathcal{S}_{k+2}(\Gamma(N))}$$

this is a sort of Hodge decomposition which is equivariant for the Hecke operators. Let,

$$H^1_!(Y(N), \widetilde{V} \otimes \mathbb{Q}_p) = \varprojlim_n H^1_!(Y(N), \widetilde{V}_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

This is étale cohomology and therefore is equipped with an action of $G_{\mathbb{Q}}$. Consider, $Y(Np^m)$ for $m \to \infty$. If $N \mid N'$ then get a Galois cover $Y(N') \to Y(N)$. Now we consider the inverse system of $Y(Np^m)$ whose limit has covering group $\mathrm{SL}(2, \mathbb{Z}_p)$. Then the completed cohomology is,

$$\hat{H}^1_!(\Gamma(N), E) = \varinjlim_n \varprojlim_m H^1_!(Y(Np^m), \widetilde{V}_N) \otimes_{\mathbb{Z}_p} E$$

First taking the limit in the level and then taking the inverse limit. The order of limits is important. This is p-adically complete (a p-adic Banach space over E) but the other order of limits does not give a p-adically complete space. This is equipped with a continuous $\mathrm{Gal}\left(()\,\overline{\mathbb{Q}}/\overline{\mathbb{Q}}(\zeta_{Np^\infty}\right) \times \mathrm{SL}(2, \mathbb{Z}_p)$ action. This is H^1 of a perfectoid space $\mathfrak{X}_N$ and there is a Hodge-Tate map,

$$\pi_{HT} : \mathfrak{X}_N \to \mathbb{P}^1_{\hat{\overline{\mathbb{Q}}}}$$

of adic spaces which is "anti-holomorphic" relative to $\iota : \mathfrak{h} \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$. The story is, differential operators on $Y(Np^m)$ pull back to operators on the completed cohomology and same with along $\iota : \mathfrak{h} \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$. However, these operators only act on the locally analytic elements.

3 Sep. 8

3.1 Benefits of registering for this course

(a) NO DRAWBACKS (homework, exams, etc)

(b) Sends a message to the administration

(c) You get messages from me (mixed perhaps): will be away Sep. 29th

3.2 Shimura Varieties as Moduli Spaces for Hodge Structures (Complex Theory)

Modular curve - parameter space for elliptic curves,

$$E(\mathbb{C}) = \mathbb{C}/\Lambda \quad \Lambda \subset \mathbb{C} \text{ is a lattice}$$

Any smooth projective curve of genus 1 over $\mathbb{C}$ is isomorphic to $\mathbb{C}/\Lambda$ for some Λ. Furthermore, if $\alpha \in \mathbb{C}^\times$ then $\alpha : \mathbb{C} \to \mathbb{C}$ takes $\Lambda \mapsto \alpha\Lambda$ so $\mathbb{C}/\Lambda \cong \mathbb{C}/\alpha\Lambda$ but these are the only relations.

Orient $\mathbb{C}$ such that $\{1, i\}$ is a positive $\mathbb{R}$-basis. Let,

$$\Omega = \{(\omega, \omega') \in \mathbb{C}^2 \mid \{\omega, \omega'\} \text{ is an oriented } \mathbb{R}\text{-basis}\}$$

Then $\mathrm{SL}_2(\mathbb{Z}) \circlearrowright \Omega$ via,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\omega, \omega') = (a\omega + b\omega', c\omega + d\omega')$$

fixing the lattice $\mathbb{Z}\omega \oplus \mathbb{Z}\omega'$. Thus $\Omega/\mathrm{SL}_2(\mathbb{Z})$ is the set of lattices in $\mathbb{C}$. Therefore, the set of complex elliptic curves is,

$$\mathrm{SL}_2(\mathbb{Z})\backslash\Omega/\mathbb{C}^\times$$

We are assuming $\tau = \frac{\omega'}{\omega} \in \mathfrak{h}$ and therefore, we can write $(\omega, \omega') \sim (1, \tau)$. Furthermore, we can write $\omega = ai + b$ and $\omega' = ci + d$ and,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})_+$$

so we see that $\Omega \cong \mathrm{GL}_2(\mathbb{R})_+$. Then $\mathrm{GL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}$ and the stabilizer of $\tau = i$ is $\mathbb{C}^\times$. Therefore, we get,

$$\mathrm{SL}_2(\mathbb{Z})\backslash\Omega/\mathbb{C}^\times \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$$

An alternative way to think about this is,

$$\mathbb{Z}^2 \hookrightarrow \mathbb{Q}^2 \hookrightarrow \mathbb{R}^2$$

with a varying complex structure on $\mathbb{R}^2$.

Definition 3.2.1. A complex structure on $\mathbb{R}^2$ is a homomorphism,

$$h : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$$

such that the eigenvalues of $h(z)$ are z and $\bar{z}$. Such an h extends to a homomorphism of $\mathbb{R}$-algebras,

$$\mathbb{C} \to \mathrm{End}\left(\mathbb{R}^2\right)$$

Then h defnes an isomorphism $\iota_h : \mathbb{R}^2 \to \mathbb{C}$.

Remark. Then $\mathbb{C}/\iota_h(\mathbb{Z}^2)$ is an elliptic curve and this gives everything from varying the complex structure rather than the lattice.

For all $z \in \mathbb{C}\backslash\mathbb{R}$ the map $h(z)$ has two distinct eigenvalues $z, \bar{z}$. Let $V_h^{-1,0}$ and $V_h^{0,-1}$ in $V \otimes_{\mathbb{R}} \mathbb{C}$ be the eigenspaces. Then,

$$V \otimes_{\mathbb{Q}} \mathbb{C} \cong V^{-1,0} \oplus V^{0,-1}$$

and $\overline{V^{-1,0}} = V^{0,-1}$.

Remark. The standard complex structure is given by,

$$h_0 : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R}) \quad h_0(x+iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

In this case, $V^{-1,0} = \mathbb{C} \cdot (1, i)$ and $V^{0,-1} = \mathbb{C}(1, -i)$.

It is better to consider $\mathfrak{h}^\pm = \mathfrak{h} \cup \bar{\mathfrak{h}} = \mathbb{C}\backslash\mathbb{R}$ and $\mathrm{GL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}^\pm$ by,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Choose γ such that, $\gamma^{-1} h(i) \gamma = h_0(i)$ ger a map,

$$\pi : \{\text{complex structures}\} \to \mathfrak{h}^\pm \quad \pi(h) = \tau_h = \gamma(i)$$

Check that if γ' is another choice that $k = (\gamma')^{-1}\gamma$ centralizes h_0 and thus belongs to $h_0(\mathbb{C})^\times$. Then τ_h is independent of the choice of γ such that $\gamma^{-1} h(i) \gamma = h_0(i)$.

The upshot,

$$\{\text{complex structures}\} \cong \mathrm{GL}_2(\mathbb{R})/K_\infty \cong \mathfrak{h}^\pm$$

where

$$K_\infty = k_0(\mathbb{C}^\times) \subset \mathrm{GL}_2(\mathbb{R})$$

Therefore, these sets acquire the structure of a complex manifold. Also $V_h^{0,-1} \subset \mathbb{C}^2$ is a varible line defining a point $p_h \in \mathbb{P}^1(\mathbb{C}) = \mathbb{P}(V_{\mathbb{C}})$. If $[\alpha : \beta]$ is a homogeneous coordinate on $\mathbb{P}^1$ and $t = \alpha/\beta$ the inhomogeneous coordinate,

$$\mathrm{GL}_2(\mathbb{R})/K_\infty \hookrightarrow \mathbb{P}(V_{\mathbb{C}})$$

Furthermore, $\iota_h : \mathbb{R}^2 \to \mathbb{C}$ extends by linearity to $V_{\mathbb{C}} = \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$. Then we get an isomorphism $V_{\mathbb{C}}/V_h^{0,-1} \cong \mathbb{C}$. Thus the elliptic curve parametrized by h becomes,

$$E_h(\mathbb{C}) = \iota_h(\mathbb{Z}^2)\backslash\mathbb{C} \cong \mathbb{Z}^2\backslash\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{C}/V_h^{0,-1}$$

However,

$$(\mathbb{Z}^2 \times \mathrm{SL}_2(\mathbb{Z}))\backslash\mathfrak{h}^{\pm} \times V_{\mathbb{C}}/V^{0,-1}$$

is not a family of elliptic curves because of the fixed points. The fibers are $E_h/\operatorname{Stab} h$. Therefore, we replace $\mathrm{SL}_2(\mathbb{Z})$ by $\Gamma(N)$ for $N \geq 3$ to get trivial stabilizers.

Then $Y_N = \Gamma(N)\backslash\mathfrak{h}^+$ and $G = \mathrm{GL}(2)$ let $\widetilde{V}_{\mathbb{Z}} = \Gamma(N)\backslash\mathfrak{h}^+ \times \mathbb{Z}^2$ where $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z}) \circlearrowleft \mathbb{Z}^2$ by the standard action. Then $\Gamma(N)$ acts trivially on $\mathbb{Z}^2/N\mathbb{Z}^2$ and let $\widetilde{V}[N] = \widetilde{V}_{\mathbb{Z}}/N\widetilde{V}_{\mathbb{Z}}$ is a trivial $\mathbb{Z}/N\mathbb{Z}$-local system of rank 2. We see that,

$$\widetilde{V}_{\mathbb{Z}}|_{E_h} \cong H_1(E_h, \mathbb{Z})$$

then we see,

$$\widetilde{V}[N]|_{E_h} \cong E_h[N]$$

is trivialized over Y_N so $Y_N(\mathbb{C}) = \{(E, \alpha) \mid \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$ plus condition of the Weil pairing$\}$ level N structure. For $E/\mathbb{C}$ and elliptic cuve. There exists Y_N^* defined over $\mathbb{Q}$ representing the Moduli problem,

$$Y_N^*(F) = \{(E, \alpha) \mid \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]\}$$

for $F \supset \mathbb{Q}$. Furthermore, $Y_N^*(\mathbb{C})$ is $(\mathbb{Z}/N\mathbb{Z})^\times$ copies of $Y_N(\mathbb{C})$ permuted by the Galois action since the Weil-paring requires the choice of a root of unity ζ_N. Therefore, Y_N is algebraic and defined over $\mathbb{Q}(\zeta_N)$ but not over $\mathbb{Q}$.

3.3 Some Algebraic Geometry

Let $f : Z \to X$ be a proper smooth morphis of $\mathbb{Q}$-varieties of relative dimension d. Then $\widetilde{H}^d = R^d f_* \mathbb{Z}$ is a local system with $(R^d f_* \mathbb{Z})_* \cong H^d(Z_x, \mathbb{Z})$. In a neighborhood of x, this is trivial. If we suppose this is free of rank m then $\widetilde{H}^d/N\widetilde{H}^d$ is a free $\mathbb{Z}/N\mathbb{Z}$-modules of rank N. This fives a representation of $\pi_1(X(\mathbb{C}), x_0)$ on $(\mathbb{Z}/N\mathbb{Z})^m$. If $K \subset \pi_1(X, x_0)$ is the kernel, then there is a covering space $X_N/X(\mathbb{C})$ with group $\pi_1(X, x_0)/K$. Then by Riemann existence X_N is a variety.

Let $N_0 \geq 3$. Then for any N we have, $Y_{N_0 N} \to Y_{N_0}$ trivializes $\widetilde{V}[NN_0]$.

Define,

$$\hat{H}^i(N_0, \mathbb{Z}_p) = \varprojlim_m \varinjlim_n H^i(X_{N_0 p^n}, \mathbb{Z}/p^m\mathbb{Z})$$

completed cohomology.

3.4 Hodge Structres

Definition 3.4.1. Let $V/\mathbb{Q}$ be a finite dimensional vector space. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$. A *Hodge structure* V pure of weight w is a decomposition,

$$V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

Remark. We have seen that elliptic curves over $\mathbb{C}$ are in correspondence with weight -1-Hodge structures.

Definition 3.4.2. A morphism of Hodge structures $g : V \to V'$ is a $\mathbb{Q}$-linear map such that $g_{\mathbb{C}}(V^{p,q}) \subset V'^{p,q}$. Then we get a category of Hodge structures. If we allow for mixed weights we also get direct sums.

Proposition 3.4.3. There is an equivalence of categories,

$$\{(V, h : \mathbb{C}^{\times} \to \mathrm{Aut}\,(V_{\mathbb{R}})) \cong \{\text{Hodge structures}\}$$

where h is a homomorphism of $\mathbb{R}$-linear algebraic groups (in particular all the eigenvalues of $h(z)$ are of the form $z^p \bar{z}^q$).

4 Hodge Stuctures

If G is an algebraic group then $\mathrm{Rep}_F(G)$ of finite dimensional G-representations forms a catgory where F is a field. In fact, this is a symmetric monoidal category:

(a) additive: direct sums exist

(b) F-linear it is enriched over F-vectorspaces

(c) there is a neutral object $\rho : G \to \mathrm{Aut}\,(V)$ trivial

(d) there is a monoidal functor $\otimes$ such that the neutral element is an identity

(e) symmetric $V \otimes W \cong W \otimes V$.

Hodge structures correspond to $\mathrm{Rep}_{\mathbb{R}}(\mathcal{S})$ along with $\mathbb{Q}$-structure.

We saw that $\mathfrak{h}^{\pm}$ is a parameter space for Hodge structure with $V_{\mathbb{C}} \cong V^{-1,0} \oplus V^{0,1}$ or equivalently for Hodge structures on GL_2-representations.

Theorem 4.0.1. Let $\rho_0 : G \to \mathrm{GL}(V)$ is a faithful representation of a reductive group G (over characteristic zero) then any representation $\rho : G \to \mathrm{GL}(W)$ is a direct summand of $V^{\otimes a} \otimes (V^*)^{\otimes b}$ for some a and b.

Remark. Applying this to $V = \mathbb{Q}^2$ with natural GL_2-structure. Then $W \subset V^{\otimes a} \otimes (V^*)^{\otimes b}$. Then a Hodge structure on V determines on V^* and thus on $V^{\otimes a} \otimes (V^*)^{\otimes b}$. Then clearly, for any $h : \mathcal{S} \to \mathrm{GL}_2$ the direct summand as GL_2-representations is also stable under the $\mathcal{S}$-action and hence defines a Hodge structure (really we also need to worry about the $\mathbb{Q}$-structure).

Definition 4.0.2. A Hodge structure on a G-representation $\rho : G \to \mathrm{GL}(V)$ is a map $h : \mathcal{S} \to G$ which then endows V with a Hodge structure.

4.1 Real Algebraic Groups

Definition 4.1.1. Let $\mathfrak{g}$ be a Lie algebra. Then for $\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$ we have,

$$\text{ad}([X, Y]) = \text{ad}(X) \cdot \text{ad}(Y) - \text{ad}(Y) \circ \text{ad}(X)$$

Then the Killing form is,

$$B(X, Y) = \text{Tr}\left((\text{ad}(X) \cdot \text{ad}(Y))\right)$$

which is a symmetric bilinear form invariant under $\text{Aut}(\mathfrak{g})$.

Definition 4.1.2. A complex Lie algebra $\mathfrak{g}$ is *semisimple* if the Killing form is nondegenerate.

Definition 4.1.3. A *real form* of a complex Lie algebra $\mathfrak{g}$ is a Lie algebra $\mathfrak{g}_0$ over $\mathbb{R}$ such that $\mathfrak{g} \cong \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

Remark. A real form is equivalent to a $\mathbb{C}$-antilinear involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ in which case,

$$\mathfrak{g}_0 = \mathfrak{g}^\sigma = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$$

Theorem 4.1.4. Lie subgroups of G correspond to Lie subalgebras of $\mathfrak{g} = \text{Lie}(G)$.

Definition 4.1.5. Say the real form $\mathfrak{g}_0$ is *compact* if the assocated subgroup of G is compact.

Remark. We can recover an algebraic group G with $\text{Lie}(G) = \mathfrak{g}$ as the subgroup $G = \text{Aut}(\mathfrak{g})^\circ/\text{scalars} \subset \text{GL}(\mathfrak{g})$ then $\text{Lie}(G) = \mathfrak{g}$. (WHY?)

Lemma 4.1.6. $\mathfrak{g}_0$ is compact iff the Killing form $B_\mathfrak{g}$ is negative-definite on $\mathfrak{g}$.

Proof. If $B_\mathfrak{g}$ is negative-definite, then G_0 the adjoint group of $\mathfrak{g}_0$ maps faithfully to $\text{Aut}(()\,\mathfrak{g}_0)$ and preserves $B_\mathfrak{g}$ hence is compact (since its image lies inside $O(\mathfrak{g}, B_\mathfrak{q})$ which is compact. Conversely, if G_0 is compact then $\mathfrak{g}_0$ admits a G_0-invariant (positive-definite) inner produce $\langle -, - \rangle$ via integrating over an arbitrary non-degenerate inner product on $\mathfrak{g}$. Now $\langle -, - \rangle$ is G_0-invariant which implies that for infinitessimal transformations,

$$\forall X \in \mathfrak{g}_0 : \langle \text{ad}(X)Y, Z \rangle = -\langle Y, \text{ad}(X)Z \rangle$$

To see this, consider,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \exp(\text{ad}(X)(Y)), \exp(\text{ad}(X)(Z)) \rangle = 0$$

The nonzero eigenvalues of a real skew-symmetric matrix are pure imaginary. Then $\text{ad}(X)^2$ is smmetric with eigenvalues ≤ 0 so $B_\mathfrak{q}(X, X) = \text{Tr}\left(()\,\text{ad}(X)^2\right) \leq 0$ and is zero iff all eigenvalues are zero and thus $\text{ad}(X) = 0$ so it is negative-definite. ☐

Theorem 4.1.7. Let $\mathfrak{g}$ be a complex semisimple $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then there is a compact real form $\mathfrak{g}_u$ such that $\mathfrak{h}$ is stable under the involution. Then there is a real vector space $\mathfrak{h}_\mathbb{R}$ which is a real form of $\mathfrak{h}$ (just a real basis since $\mathfrak{h}$ is abelian) and $i\mathfrak{h}_\mathbb{R}$ is a maximal abelian subalgebra of $\mathfrak{g}_u$. In fact, $\mathfrak{h}_\mathbb{R}$ is spanned by the coroot vectors H_α for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ with $\alpha(H) = B_\mathfrak{g}(H_\alpha, H)$.

Theorem 4.1.8. $\mathfrak{g}, \mathfrak{g}_u$ as above and τ is the conjugation associated to $(\mathfrak{g}, \mathfrak{g}_u)$. Let σ be a second conjugation of $\mathfrak{g}$. There is a 1-parameter group $t \mapsto A_t \subset \text{Aut}(\mathfrak{g})$ such that $A(0) = \text{id}$ and $A(1)\tau A(1)^{-1}$ commutes with σ.

Corollary 4.1.9. Any two compact real forms of $\mathfrak{g}$ are conjugate by an element of $\mathrm{Aut}\,(()\,\mathfrak{g})^\circ$.

Remark. Let $\mathfrak{h} = \mathrm{Lie}(H)$ then,

$$\Delta \subset X^*(H) = \mathrm{Hom}\,(H, \mathbb{G}_m) \to \mathrm{Hom}\,(\mathfrak{h}, \mathbb{C}) \cong X^*(H) \otimes_{\mathbb{Z}} \mathbb{C}$$

Definition 4.1.10. Let $\mathfrak{g}$ be a real semisimple Lie algebra. A *Cartan involution* is an involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{k} = \mathfrak{g}^{\sigma = 1}$ and $\mathfrak{p} = \mathfrak{g}^{\sigma = -1}$ has $B_{\mathfrak{g}}$ negative-definite and positive-definite respectively and gives a Cartan decomposition,

$$\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{p}$$

Theorem 4.1.11. Let $\mathfrak{g}$ be real semisimple. Then $\mathfrak{g}$ has a Cartan involution and any two are conjugate by $\mathrm{Aut}\,(\mathfrak{g})^\circ$.

Proof. Let $\mathfrak{g}_u$ be a compact real form of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}^{\sigma}$ and $\mathfrak{g}_u = \mathfrak{g}_{\mathbb{C}}^{\tau}$ where σ and τ are anti-linear involutions. By the theorem, we may assume that σ and τ commute. Then τ defines an involution of $\mathfrak{g}$ since it commutes with σ. Then the fixed points are $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{g}_u$ which is maximal compact and τ is a Cartan involution on $\mathfrak{g}$. ☐

4.2 Deligne's Axioms for Shimura Data

Definition 4.2.1. A Shimura datum is a pair (G, X) where G is a reductive group over $\mathbb{Q}$ and X is a $G_{\mathbb{R}}$-conjugacy class of homomorphism $h : \mathcal{S} \to G_{\mathbb{R}}$ satisfying,

(a) $\forall h \in X$ we have $\mathrm{im}\, h|_{\mathbb{G}_{m,\mathbb{R}}} \subset Z(G)$

(b) $\forall h \in X : \mathrm{Ad}(h(i))$ is a Cartan involuton of $\mathfrak{g}^{\mathrm{ad}} = \mathrm{Lie}(G^{\mathrm{ad}}(\mathbb{R}))$ where $G^{\mathrm{ad}} = G/Z(G)$.

(c) For any $h \in X : \mathrm{Ad} \circ h : \mathcal{S} \to \mathrm{GL}(\mathfrak{g}_{\mathbb{R}})$ is a Hodge structure such that $\mathfrak{g}^{p,q} = 0$ unless $(p, q) \in \{(0, 0), (-1, 1), (1, -1)\}$.

Remark. For $h \in X$ let $K_h = Z_{G_{\mathbb{R}}}(h(\mathcal{S}))$ then $\mathfrak{k}_h = \mathrm{Lie}(K_h) = \mathfrak{g}^{0,0}$ is is the part where $\mathcal{S}$ acts trivially through h. Furthermore, $\mathrm{ad}(h(i))$ actis as -1 pn $\mathfrak{g}^{-1,1}$ or $\mathfrak{g}^{1,-1}$.

(a) $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_h \oplus \mathfrak{p}_h$ is a Cartan decomposition

(b) K_h/Z_G is maximal compact in $G^{\mathrm{ad}}(\mathbb{R})$

(c) $\mathfrak{p}_h \cong \mathfrak{p}_h^+ \oplus \mathfrak{p}_h^-$ with $\mathfrak{p}_h^- = \mathfrak{g}^{1,-1}$ and $\mathfrak{p}_h^+ = \mathfrak{g}^{-1,1}$.

Then X has a $G(\mathbb{R})$-invariant complex structure. And G/K is a symmetric space with an action of G. This has a complex structure invariant under the G-action iff X is constructed from a Shimura datum.

5 September 15

To produce $\mathbb{C}$-local systems on a connected X the best way is to take a smooth proper morphism $p : Y \to X$ then $\mathscr{F} = R^i p_* \underline{\mathbb{C}}$ then $\mathscr{F}_x = \cong H^i(Y_x, \mathbb{C})$.

Let $X = \widetilde{X}/\Gamma$ where $\widetilde{X}$ is simply connected. Then $\Gamma \cong \pi_1(X, x)$ for any $x \in X$. If $\rho : \Gamma \to \mathrm{GL}(W)$ for W finite dimensional over $\mathbb{C}$. Then we can take $\widetilde{W} = (\widetilde{X} \times W)/\Gamma$ which is a local system on X whose associated representation is ρ.

For $\widetilde{X}$ a hermitian symmetric space for $G(\mathbb{R})^\circ$. Then for $\Gamma \subset G(\mathbb{Q})$ discrete such that,

(a) $\Gamma \circlearrowright \widetilde{X}$ has no fixed points

(b) $\widetilde{X}/\Gamma$ is compact

(c) $\widetilde{X}/\Gamma$ has finite invariant volume.

Then we get not only a complex manifold but actually a variety.

Let G be a reductive algebraic group over $\mathbb{Q}$. Let X be a $G(\mathbb{R})$-conjugacy class of maps $h : \mathcal{S} \to G_{\mathbb{R}}$ satisfying,

(a) $\mathrm{ad}(h(i))$ is a Cartan involution of $\mathfrak{g}$

(b) $h(\mathbb{G}_{m,\mathbb{R}}) \subset Z_G$

(c) $\forall z \in \mathcal{S}(\mathbb{R})$ then $h(z)$ has eigenvalues $z/\bar{z}, 1, \bar{z}/z$.

Remark. Recall that $\mathcal{S} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}})$ and thus contains $\mathbb{G}_{m,\mathbb{C}} \subset \mathcal{S}$. This corresponds to the lattice $\mathbb{Z}^2$ with complex conjugation $\sigma(a,b) = (b,a)$ with the invariant map $\mathbb{Z}^2 \to \mathbb{Z}$ sending $(a,b) \mapsto a+b$.

Example 5.0.1. Let $G = \mathbb{G}_{m,\mathbb{R}}$ and $h(z) = z\bar{z} = N(z)$.

Definition 5.0.2. $\mathrm{GU}(p,q) = \{g \in \mathrm{GL}_{p+q}(\mathbb{C}) \mid g^\dagger I_{p,q} g = \nu(g) I_{p,q}\}$ where $I_{p,q}$ is the matrix for the standard quadratic form with signature (p,q). Then there is an exact sequence,

$$1 \longrightarrow \mathrm{SO}(p,q) \longrightarrow \mathrm{GU}(p,q) \longrightarrow \mathbb{G}_m \longrightarrow 1$$

Then we let,

$$h_0(z) = \begin{pmatrix} zI_p & 0 \\ 0 & \bar{z} I_q \end{pmatrix}$$

such that $h_0(i) = iI_{p,q}$. Then $U(p,q) = \ker \nu$.

6 Sept 20

Let G be a group over $\mathcal{O}$ and X a $G(\mathbb{R})$-conjugacy class of $h : \mathcal{S} \to \mathscr{G}_{\mathbb{R}}$ with the required properties. Consider $K \subset G(\mathbb{A}_{\text{fin}})$ then,

$$\mathfrak{Sh}_K(G,X) = G(\mathbb{Q})\backslash(X \times \mathscr{G}(\mathbb{A}_{\text{fin}})/K$$

For $K' \subset K$ get $\mathfrak{Sh}_{K'}(G,X) \to \mathfrak{Sh}_K(G,X)$. If K' is normal in K then,

$$\mathfrak{Sh}_{K'}(G,X) \to \mathfrak{Sh}_K(G,X)$$

is a $/K'$-covering. Write $K = K^{[} \cdot K_p$ with,

$$K^p \subset \prod_{q \neq p}' G(\mathcal{O}_q)$$

Fix K_p let $K_{p,n} \to \{1\}$ then consider,

$$S(K^p) = \varprojlim_n \mathfrak{Sh}_{K^p K_{p,n}}(G,X)$$

Note that for affine schemes,

$$\varprojlim_N \operatorname{Spec}(R_n) = \operatorname{Spec}\left(\varinjlim_n R_n\right)$$

Then $S(K^p) \to \mathfrak{Sh}_{K^p K_{p,n}}(G,X)$ is a pro-étale cover with group $K_{p,n}$. We cam also from the adic space $\hat{S}(K^p)$.

Definition 6.0.1. A map of Shimura data $(H,Y) \to (G,X)$ is a homomorphism $\phi : H \to G$ and a map of complex analytic spaces $Y \to X$ which is equivariant for the $H_{\mathbb{R}}$-action.

Remark. This gives rise to a map of Shimura varities,

$$\mathfrak{Sh}_{K \cap H(\mathbb{A}_{\mathrm{fin}})}(H,Y) \to \mathfrak{Sh}_K(G,X)$$

of quasi-projective algebraic varities over $E(H,Y) \supset E(G,X)$.

The complex structure is given by $X \hookrightarrow \hat{X} = G(\mathbb{C})/P(\mathbb{C})$ the Borel embedding where for $h \in X$ we have $P = P_h$ is the subgroup with $\mathrm{Lie}(P_h) = \mathfrak{k}_{h,\mathbb{C}} \oplus \mathfrak{p}_h^-$ with $\mathfrak{p}_h^- = \mathfrak{g}^{1,-1}$.

Remark. Recall any algebraic map $h : \mathcal{S} \to \mathrm{GL}(W)_{\mathbb{R}}$ with W defined over $\mathbb{Q}$ gives a Hodge structure on W.

Let $\hat{X}$ be a G-homogeneous space (over a number field E). Consider G-equivariant vector bundles on $\hat{G}$ meaning $V \to \hat{X}$ which is locally isomorphic to $U \times \mathbb{G}_a^n$ for an open $U \subset \hat{X}$ and the transition maps are algebraic. If the transition functions lie in $\mathrm{GL}(n,E)$ then it is a local system. Furthermore, we want it to be equivariant meaning it is equipped with a fiberwise linear action $G \circlearrowright V$ such that $\pi : V \to \hat{X}$ is equivariant.

$$\begin{tikzcd}
G \times V \arrow[r] \arrow[d] & V \arrow[d] \\
G \times \hat{X} \arrow[r] & \hat{X}
\end{tikzcd}$$

Suppose that $\hat{X} = G/P$ (in full generality $\hat{X}$ might not have E-points and hence the parabolic might lie over some extension). Let x be a fixed point of $P \circlearrowright G$ meaning $x \in \hat{X}$. Then we get $V \mapsto V_x$ with $P \circlearrowright V_x$. This defines an equivalence of $\otimes$-categories,

$$\{\text{equivariant vector bundles on } \hat{X}\} \xrightarrow{\sim} \mathrm{Rep}_{E(x)}(P)$$

To go the other way, given a representation $\rho : P \to \mathrm{GL}(W)$ then we let,

$$V = (G \times V)/P \to G/P = \hat{X} \quad \text{ with } (g,v) \cdot p = (gp, \rho(p)^{-1}v)$$

Consider the restriction map, $\mathrm{Rep}(G) \to \mathrm{Rep}(P)$ then given $\rho : G \to \mathrm{GL}(W)$ we get,

$$(G \times W)/P \cong (G/P) \times W$$

If we have $\rho : K = P/R_n(P) \to \mathrm{GL}(W)$ then ρ is completely reducible and thus,

$$V(\rho) = (G \times W)/P$$

is completely reducible.

Let V be a homogeneous vector bundle on $\hat{X}$. For $\beta : X \hookrightarrow \hat{X}$ consider,

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & G(\mathbb{Q})\backslash\beta^*(V) \times G(\mathbb{A}_{\text{fin}})/K \arrow[d] \\ \mathfrak{Sh}_K(G,X) \arrow[r, equal] & G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_{\text{fin}})/K \end{tikzcd}$$

Therefore we get a functor $V \mapsto [V]_K$ from homogeneous vector bundles on $\hat{X}$ to "automorphic" vector bundle son $\mathfrak{Sh}_K(G,X)$.

Remark. There is often an extra axiom: $\text{rank}_{\mathbb{R}} Z_G = \text{rank}\, Z_G$ to exclude the following example: $G = \text{Res}_{\mathbb{Q}}^F(\mathbb{G}_m)$ for F totally real. Since G is commutative then $X = \{h\}$ there is trivial conjugation. Then,

$$F^\times\backslash(h \times V \times \mathbb{A}_{\text{fin},F}^\times/K$$

will have some fixed points. Equivalently we can consider vectorbundles over $\hat{X}$ with trivial $Z_G^0 = \ker$ action which ensures that the above quotient is actually a vector bundle.

Definition 6.0.2. $\mathfrak{Sh}(G,X) = \varprojlim_{K \subset G(\mathbb{A}_{\text{fin}})} \mathfrak{Sh}_K(G,X)$ on which $G(\mathbb{A}_{\text{fin}})$ acts. Then we get $\langle [V]_K \rangle = \varprojlim_K [V_K$ is equivariant under $G(\mathbb{A}_{\text{fin}})$.

For any h, then $r \circ h : \mathcal{S} \to \text{GL}(W)$ defines a Hodge structure on W which we write,

$$W_{\mathbb{C}} = \bigoplus_{p,q} W_h^{p,q}$$

If $g \in G(\mathbb{R})$ then $r \circ g(h)$ gives the Hodge structure,

$$W = \bigoplus_{p,q} W_{g(h)}^{p,q} \quad \text{with} \quad W_{g(h)}^{p,q} = r(g) \cdot W_h^{p,q}$$

Recall,

$$W^{p,q} = F^p W \cap \overline{F^q W}$$

But W also has a filtration by P-invariant subspaces. Writing,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$$

Then,

$$\mathfrak{g}^{a,b} \otimes W^{p,q} \subset W^{p+a,q+b}$$

WHAT!!

Let $r : G \to \text{GL}(W)$ and,

$$\widetilde{W}_K = G(\mathbb{Q})\backslash(X \times W_F \times G(\mathbb{A}_{\text{fin}}))/K$$

is a local system in F-vectorspaces where as $[W]$ is an algebraic vector bundle attached to $\hat{X} \times W$ as homogeneous vector bundles. Then $F^p[W] = [F^p W]$ the Hodge filtration on $[W]$ over the Shimura variety.

Example 6.0.3. Let $(G,X) = (\text{GL}(2), \mathfrak{h}^\pm)$ and $\hat{X} = \mathbb{P}^1$ and $W = \text{Sym}_{(k)V}$ where $V = \mathbb{Q}^2$ with standard representation ρ_{st}. Then,

$$\mathfrak{Sh}_{K(N)}(\text{GL}(2), \mathfrak{h}^\pm) = Y(N)$$

For $N \geq 3$ there is a universal curve $p : \mathcal{E} \to Y(N)$ then $[V] = (R^1 p_* \underline{\mathbb{Q}})^\vee$ or really the flat sections of the relative de Rham vector bundle under the Gauss-Manin connection so $[V]_x = H^1_{\mathrm{dR}}(\mathcal{E}_x)^\vee$.

For $V/\mathbb{P}^1 = \mathrm{GL}_2/B$ is isomorphic to a sum of $\mathcal{O}(k)$ for some k and $\Omega = \mathcal{O}(-2)$ so it pulls back to Ω on the modular curve which is $\omega^{\otimes 2}$. Therefore, $[\mathcal{O}(k)] = \omega^{\otimes(-k)}$.

For $r : G \to \mathrm{GL}(W)$ and $\hat{X} \times W = W \otimes \mathcal{O}_{\hat{X}}$ as algebraic vector bundles. There is a trivial connection on here. Therefore, we get $\nabla : [W] \to [W] \otimes \Omega^1_{\mathfrak{Sh}(G,X)}$ which is an integrable connection on the Shimura variety and,

$$\nabla(F^p[W]) \subset F^{p-1}[W] \otimes \Omega^1_{\mathfrak{Sh}(G,X)}$$

which we check using Lie algebras, another manifestation of Griffiths transversality. Using the de Rham complex associated to $[W]$ we get an equivalence in the derived category,

$$\widetilde{W}(\mathbb{C}) \sim [0 \to [W] \to [W] \otimes \Omega^1 \to \cdots \to [W] \otimes \Omega^d \to 0]$$

for holomorphic vectorspaces. Therefore,

$$H^i(\mathfrak{Sh}, \widetilde{W}) = \mathscr{H}^i_{\mathrm{dR}}(\mathfrak{Sh}, [W])$$

7 Sept 27

7.1 Perfectoid Spaces

Let $C_p = \hat{\overline{\mathbb{Q}}}_p$ for any characteristic zero field complete with repsecte to the p-adic topology and algebraically closed. Let $[K : \mathbb{Q}_p] < \infty$ and $G = \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/K\right)$. Let $\rho : G \to \mathrm{GL} V$ and $\rho_{C_p} : G \to \mathrm{GL} V \otimes_{\mathbb{Q}_p} C_p$ the action is diagonal.

Let K_{00}/K be with Galois group $\mathbb{Z}_p$, totally ramified (e.g. $K = \mathbb{Q}_p(\zeta_{p^n})^{\mathbb{F}_p^\times}$.

Definition 7.1.1. A *pseudo-uniformizer* ϖ is an extension of $\mathbb{Q}_p$ is a topologically nilpotent unit meaning,

$$\lim_{n \to \infty} \varpi^n = 0$$

Remark. This is called topologically nilpotent because it says that for any open neighborhood U of 0 there is $\varpi^n \in U$ for sufficiently large n.

Proposition 7.1.2. For any L/K_∞ with integer ring $\mathcal{O}_L$ we have,

$$\mathrm{Tr}\,()\, L/K_\infty \mathcal{O}_L \supset \mathfrak{m}_\infty$$

which is called being almost unramified or almost etale.

Proof. Let $L = L_0 K_\infty$ with L_0/K and $L_h = L_0 K_h$ gives $({}_{L_n/K_N}) \to 9$ as $n \to \infty$. Therefore, it $\square$

8 Oct. 4 Sen's Theory

Recall the situation: $\mathbb{Q}_p \subset K \subset K_\infty \subset C$ with $\Gamma = \mathrm{Gal}\,(K_\infty/K)$.

Corollary 8.0.1. The inflation map,

$$H^1$$

⊠$(\Gamma, \mathrm{GL}(n, \hat{K}_\infty) \to H^1$⊠(G, GL(n, C))

is an isomorphism.

Remark. We are using the fact that $\hat{K}_\infty = C^H$ where $H = \mathrm{Gal}\,(()\,C/K_\infty)$.

Remark. H^1⊠(H, GL(n, C)) $= 1$

Now we take W a finite dimensional C-vectorspace with $n = \dim_C W$. We equip it with the structure of a semi-linear G-representaion $g \cdot (\lambda w) = g(\lambda)g(w)$ for $w \in W$ and $g \in G$ and $\lambda \in C$.

Proposition 8.0.2. Let $\hat{W}_\infty = W^H$ is an n-dimensional $\hat{K}_\infty$ subspace. Then the inclusion $\hat{W}_{\infty \otimes_{\hat{K}_\infty} \to W}$ is an isomorphism.

Proof. This is Galois descent ☐

Proposition 8.0.3 (Decompletion)**.** The inclusion $H^1(\mathbb{G}_a, \mathrm{GL}(n, K_\infty)) \to H^1(\Gamma, \mathrm{GL}(n, \hat{K}_\infty))$.

Proof. The map $g \mapsto U_g$ descends to a cocycle $g \in \Gamma$ maps to $U_g \in \mathrm{GL}(n, \hat{K}_\infty)$. But it γ is a topological generator then,

$$U_\gamma \in \mathrm{GL}(n, K_\infty) = \bigcup_r \mathrm{GL}(n, K_r)$$

Therefore, because this is a topological generator we can choose a uniform r such that,

$$\forall g \in \Gamma : U_g \in \mathrm{GL}(n, K_r)$$

☐

Proposition 8.0.4. There is a K_r-representation W_r of Γ of dimension n such that $W_r \otimes_{K_r} \hat{K}_\infty \cong \hat{W}_\infty$.

Corollary 8.0.5. Let $W_\infty \subset \hat{W}_\infty$ be the sset of all vectors whose Γ-orbit is contained in a K-vector space of finite dimension. Then $W_r \otimes_{K_r} K_\infty = W_\infty$ and hence $W_\infty \otimes_{K_\infty} \hat{K}_\infty = \hat{W}_\infty$.

Proof. Clearly $\dim_{K_\infty} W_r \otimes_{K_r} K_\infty = \dim_{K_r} W_r = n$. Also, $W_\infty \supset W_r \otimes_{K_r} K_\infty$ so $\dim W_\infty \geq n$. On the other hand, Sen proves that any element of $\hat{K}_\infty$ whose Γ-orbit is finite is contained in K_∞. One writes $D_{\mathrm{Sen}}(W) = W_\infty$ is an n-dim v.s. over K_∞ attached to the G-action on W. ... ☐

Remark. Write $\log \chi$ for the map $G \to \Gamma_r \xrightarrow{\sim} \mathbb{Z}_p$ for $\Gamma_r = \mathrm{Gal}\,(K_\infty/K_r)$. Then γ_r acts linearly on W_r.

Definition 8.0.6. The Sen operator $\phi = \phi_W$ is the K_r-linear endomorphism fo W_r whose matrix in the basis $\{e_1, \dots, e_r\}$ is given by,

$$\Phi = \frac{\log U_{\gamma_r}}{\log \chi(\gamma_r)}$$

is independent of the choice of γ_r.

Theorem 8.0.7. Sen's operator is the unique K_∞-linear endomorphism fo $W = D_{\text{Sen}}(W)$ wuch that for all $w \in W_\infty$ there exists $\Gamma_W \subset \Gamma$ open such that $\forall \sigma \in \Gamma_W$,

$$\sigma(e) = [\exp(\phi(\log \chi(r)))](w)$$

Theorem 8.0.8 (Sen)**.** Let V be a $\mathbb{Q}_p$-vectorspace and $\rho : \text{Gal}\left(()\,\overline{\mathbb{Q}}_p/K\right) \to \text{GL}(V)$ a linear represetnation. Let $W = V \otimes C$ and $\phi = \phi_W$ its SEn operator. Suppose that residue field of K is algebraically closed. Then $\text{Lie}(\rho(\text{Gal}\left(()\,\overline{\mathbb{Q}}_p/K\right)))$ is the smallest $\mathbb{Q}_p$-rational subspace $S \subset \text{End}_{\mathbb{Q}_p}(V)$ such that $\phi \in S \otimes C$.

Remark. We want a geometric version where the residue field may not be perfect.

9 Oct 6

Definition 9.0.1. A *strict p-ring* is a p-adically complete ring S, flat over $\mathbb{Z}_p$ such that S/pS is perfect.

Proposition 9.0.2. Suppose that S is a strict p-ring and $A = S/pS$. Then there is a multiplicative Teichmuller lift $A \to S^\times\{0$ given by,

$$a \mapsto [a] = \varprojlim_n \tilde{a}^{p^n}$$

Theorem 9.0.3 (Witt)**.** The functor $S \mapsto S/pS$ from,

$$\{\text{strict p-rings}\} \to \{\text{perfect } \mathbb{F}_p\text{-algebas}\}$$

is an equivalence of catefories. The inverse map $A \mapsto W(A)$ is called the ring of Witt vectors. Any $w \in W(A)$ has a unique expansion,

$$w = \sum_{n \ge 0} [a_n(w)] p^n$$

with $a_n(w) \in A$.

10 Oct. 11

Remark. No class next tuesday.

Consider the perfectoid modular curve $X_\infty(N)$ over $X(N)$ (with $p \not| N$) then we get a vector bundle $\widetilde{M} = (X_\infty(N) \times M)/G$ where $G = \text{GL}(2, \mathbb{Z}_p)$ is the Galois group of the perfectoid cover. To $\widetilde{M}$ there is an associated Sen operator.

Working over C a complete algebraically closed field of characteristic zero and p-adic normalized such that $|\bullet|_C$ restricts to the standard norm on $\mathbb{Q}_p$. Consider $\mathcal{O}_C \subset C$ then $(C, \mathcal{O}_C)$ and $X = (A, A^+)$ is a 1-dimensional smooth affinoid adic space over $(C, \mathcal{O}_C)$.

Definition 10.0.1. Profinite inverse limit: let $\mathbb{X}$ be a perfectoid space and $\mathbb{X}_i$ for $i \in I$ a filtered system of noetherian adic spaces over C with maps,

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\varphi_i"] \arrow[dr, "\varphi_j"] & \mathbb{X}_i \arrow[d] \\ & \mathbb{X}_j \end{tikzcd}$$

We say that $\mathbb{X} \sim \varprojlim \mathbb{X}_i$ if,

(a) $|\mathbb{X}| \to \varprojlim |\mathbb{X}_i|$ is a homeomorphism

(b) for any $x \in \mathbb{X}$ and $\varphi_i(x) = x_i$ the map on residue fields,

$$\varinjlim_i k(x_i) \to k(x)$$

has dense image.

Remark. If the limit exists as a perfectoid space then it is unique. However, it does not always exist as a perfectoid space.

Example 10.0.2. $\mathbb{A}^1(C) \supset D_\alpha$ closed disk of radius α. Then consider $A = C\langle X \rangle$ is the ring of functions on D_1. Furthermore, $\sup_{D_\alpha} f$ is a norm which corresponds to a point. If we write,

$$f = \sum a_i x^i$$

then $\sup |a_i|$ is another norm.

11 Mysterious Functor

We want a functor relating,

$$H^n_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_p) \otimes K$$

and the deRham cohomology,

$$H^n_{\text{dR}}(X/K)$$

Theorem 11.0.1 (Fantaine-Messing, and others)**.** There is a canonical isomorphism,

$$H^n_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H^n_{\text{dR}}(X/k)_k B_{\text{dR}}$$

of Galois modules.

12 Oct 27

Definition 12.0.1. A *strict p-ring* is a p-adically complete ring S flat over $\mathbb{Z}_p$ such that S/pS is perfect.

Remark. Can replace $\mathbb{Z}_p$ by $\mathbb{Z}_{(p)}$.

If S is a strict p-ring and $A = S/pS$ then,

$$A \to S^\times \cup \{0\} \quad a \mapsto [a] = \lim_n \tilde{a}^{p^n}$$

is well-defined and multiplicative where $\tilde{a}$ is an arbitrarily chosen lift.

Theorem 12.0.2 (Witt)**.** The map $S \mapsto S/pS$ is an equivalence of categories,

$$\{\text{strict}p\text{-rings}\} \to \{\text{perfect}\mathbb{F}_p\text{-algebras}\}$$

The inverse map $A \mapsto W(A)$ is the ring of Witt vectors. Any $w \in W(A)$ can be written uniquely as,

$$w = \sum_{n \geq 0} [a_n] p^n$$

for $a_n \in A$.

Example 12.0.3. For $A = \mathbb{F}_{p^r}$ with $r \geq 1$ thne $W(A)$ is the ring of integers of $\mathbb{Q}(\zeta_{p^r-1})$ in that case $[a_n] \in \mu_{p^r-1} \cup \{0\}$ for all $a_n \in \mathbb{F}_{p^r}$.

Example 12.0.4. If $A = \mathbb{F}_q[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ then $\widetilde{A} = \mathbb{F}_q[x_i^{\pm \frac{1}{p^r}}]_{i,r}$ is a perfect ring.

Remark. Can prove Witt's theorem by showing that deformations of a perfect $\mathbb{F}_p$-algebra to characteristic zero are unique. Then the unique deformation will be $W(A)$.

12.1 Addition and Multiplication

Consider an infintie sequence of variables α_i, β_j corresponding to α_i^p and α_j^p. Let

12.2 Universal Property

Let S be p-adically complete, p-torsion-free. Consider a map of multiplicative monoids $\varphi : R \to S$ such that the composition $R \to S \to S/p$ is a ring homomorphism. Then $\varphi : R \to S$ factors uniquely through $[] : R \to W(R)$.

12.3 •

Let $C = \mathbb{C}_p$ be the complete normed field and $\mathcal{O}_C$ the elements of norm ≤ 1. Define $\mathcal{O}_C^\flat$ the *tilt* of $\mathcal{O}_C$ via,

$$\mathcal{O}_C^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_C$$

These maps are surjective. In fact, the reduction map,

$$\mathcal{O}_C^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_C \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} C/p$$

where the inverse map takes,

$$(a_n \mod p) \mapsto b_n = \lim_m a_{n+m}^{p^m}$$