Mathematics GU4053 Algebraic Topology
Assignment # 10

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Problem 1.

We have calculated,

$$H_n(S^1 \times S^1) = H_n(T^2) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & n > 2 \end{cases}$$

Similarly, for $n > 0$,

$$H_n(S^1 \vee S^1 \vee S^2) = \tilde{H}_n(S^1 \vee S^1 \vee S^2) = \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2)$$

and since $S^1 \vee S^1 \vee S^2$ is connected, $H_0(S^1 \vee S^1 \vee S^2) \cong \mathbb{Z}$. Therefore, since $H_n(S^n) \cong \mathbb{Z}$ and is zero otherwise (for $n > 0$) we have,

$$H_n(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & n > 2 \end{cases}$$

Therefore, by direct calculation,

$$H_n(S^1 \vee S^1 \vee S^2) \cong H_n(S^1 \times S^1)$$

Furthermore, the universal cover of $S^1 \times S^1$ is $\mathbb{R}^2$ which is contractible. Let $\tilde{X}$ be the universal cover of $S^1 \vee S^1 \vee S^2$. We know that the covering map $p : \tilde{X} \to S^1 \vee S^1 \vee S^2$ induces an isomorphism on higher homotopy groups $p_* : \pi_2(\tilde{X}) \xrightarrow{\sim} \pi_2(S^1 \vee S^1 \vee S^2)$. However, $\pi_2(S^1 \vee S^1 \vee S^2) \neq 0$ so $\tilde{X}$ cannot be contractible because not all its homotopy groups are zero. However, since $S^1 \vee S^1 \vee S^2$ admits a CW complex structure, so does $\tilde{X}$ and thus $\tilde{X}$ is contractible iff $H_n(\tilde{X})) = 0$ for all n which implies that $\tilde{X}$ does not have trivial homology since it is not contractible.

Problem 2.

Take any continuous map $f : S^{2n} \to S^{2n}$. The degree of f cannot equal both 1 and $(-1)^{2n+1} = -1$. Therefore, by Lemma **??** and Lemma **??**, there must either be a point $x \in S^{2n}$ such that $f(x) = x$

or a point such that $f(x) = -x$.

Take any map $f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$. Then, take the covering map $p : S^{2n} \to \mathbb{RP}^{2n}$ which projects $p(x) = p(-x) = [x]$. Therefore, we get a map $f \circ p : S^{2n} \to \mathbb{RP}^{2n}$ but S^{2n} is simply connected (and path-connected and locally path-connected) so by the lifting criterion there exists a map $\tilde{f} : S^{2n} \to S^{2n}$ such that $p \circ \tilde{f} = f \circ p$. By the above result, there exists $x \in S^n$ such that $\tilde{f}(x) = \pm x$. Therefore, $f([x]) = p(\tilde{f}(x)) = p(\pm x) = [x]$ so f has a fixed point. However, if the dimension of projective space is odd then there exist maps with no fixed points. Consider the linear map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by the symplectic matrix,

$$A = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

We see that $A^2 = -I_{2n}$ so $F \circ F(x) = -x$. Suppose that F had an eigenvector $x \in \mathbb{R}^{2n}$ with eigenvalue $\lambda \in \mathbb{R}$. Then we know that $F(x) = \lambda x$ so $F \circ F(x) = \lambda F(x) = \lambda^2 x$ because F is linear. However, $F \circ F = -\mathrm{id}_{\mathbb{R}^{2n}}$ so $\lambda^2 x = -x$ but $x \neq 0$ since x is an eigenvector. Thus, $\lambda^2 = -1$ but $\lambda \in \mathbb{R}$ which is impossible so F has no eigenvectors. First, F is injective (since it is invertible) so F is a map $\mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}^{2n} \setminus \{0\}$. Since F is linear, F descends to the quotient $\mathbb{RP}^{2n-1}$ under $x \sim \lambda x$ as a map $f : \mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ such that $f([x]) = [F(x)]$. If $f([x]) = [F(x)] = [x]$ then we know that $F(x) = \lambda x$ which we have shown to be impossible. Thus, f has no fixed points.

Problem 3.

Let $f : S^n \to S^n$ have degree zero. Therefore, $\deg f \neq (-1)^{n+1}$ and $\deg f \neq 1$ so by Lemma **??** there must exist $x \in S^n$ such that $f(x) = x$ and by Lemma **??** there must exist $y \in S^n$ such that $f(y) = -y$.

Let F be a nowhere vanishing continuous vector field on $D^n \subset \mathbb{R}^n$. Consider the continuous map $\tilde{F} : D^n \to S^{n-1}$ given by,

$$\tilde{F}(x) = \frac{F(x)}{|F(x)|}$$

This defines a homotopy between $f = \tilde{F}|_{\partial D^n} : S^{n-1} \to S^{n-1}$ and $\tilde{F}(0)$ a constant map. Therefore, f is nullhomotopic so $\deg f = 0$. By the above result, $\exists x, y \in \partial D^n$ such that $f(x) = x$ and $f(y) = -y$ so,

$$F(x) = x|F(x)| \quad \text{and} \quad F(y) = -y|F(y)|$$

so F points radially outwards at x and radially inwards at y.

Problem 4.

We are given the exact sequence of chain complexes,

$$0 \longrightarrow C(X) \xrightarrow{\ n\ } C(X) \xrightarrow{\ \phi\ } C(X; \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$

This short exact sequence gives rise to a long exact sequence of homology,

$$\cdots \longrightarrow H_k(X) \xrightarrow{\ n\ } H_k(X) \xrightarrow{\ \phi_*\ } H_k(X; \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\ d\ } H_{k-1}(X) \xrightarrow{\ n\ } H_{k-1}(X) \longrightarrow \cdots$$

I can adorn this sequence by using the fact that ϕ_* factors as $\tilde{\phi}_* \circ \pi$ where $\tilde{\phi}_*$ is injective through the quotient by $\ker \phi_* = \mathrm{Im}(n) = nH_k(X)$ by exactness. Furthermore, again by exactness, d factors as a surjective map through $\mathrm{Im}(d) = \ker n = T_n(H_{k-1}(X))$, the n-torsion group. Therefore, the following diagram commutes,

$$\begin{tikzcd}
H_k(X) \arrow[r, "n"] \arrow[dr, "0"] & H_k(X) \arrow[r, "\phi_*"] \arrow[d, "\pi"] & H_k(X;\mathbb{Z}/n\mathbb{Z}) \arrow[r, "d"] \arrow[dr, two heads, "d"] & H_{k-1}(X) \arrow[r, "n"] & H_{k-1}(X) \\
& H_k(X)/nH_k(X) \arrow[ur, hook, "\tilde{\phi}_*"] & & T_n(H_{k-1}(X)) \arrow[u, "\iota"] \arrow[ur, "0"] &
\end{tikzcd}$$

Furthermore, $\phi_* = \tilde{\phi}_* \circ \pi$ so $\mathrm{Im}(\phi_*) = \mathrm{Im}(\tilde{\phi}_*)$ since π is surjective. However, by exactness, $\mathrm{Im}(\phi_*) = \ker d$ and therefore the sequence,

$$0 \longrightarrow H_k(X)/nH_k(X) \xhookrightarrow{\tilde{\phi}_*} H_k(X;\mathbb{Z}/n\mathbb{Z}) \xrightarrow{d} T_n(H_{k-1}(X)) \longrightarrow 0$$

is short exact. Therefore, $H_k(X;\mathbb{Z}/n\mathbb{Z}) = 0$ if and only if $H_k(X)/nH_k(X) = 0$ and $T_n(H_{k-1}(X)) = 0$ if and only if multiplication by n is an automorphism. The last two statements are logically equivalent since multiplication by n is surjective iff $nH_k(X) = H_k$ iff $H_k(X)/nH_k(X) = 0$ and multiplication by n is injective iff $\ker n = T_n(H_{k-1}(X)) = 0$.

By Lemma **??**, we see that the $\mathbb{Z}$-module $H_k(X)$ has the structure of a $\mathbb{Q}$-vectorspace if and only if the multiplication by n is an automorphism for each $n \in \mathbb{Z} \setminus \{0\}$. We have shown that multiplication by n is an automorphism if and only if $H_k(X;\mathbb{Z}/n\mathbb{Z}) = 0$. In fact, we only need to check this for primes p because by integer factorization if multiplication by p is an automorphism for every prime then by composition any nonzero n as a product of primes acts by multiplication as the composition of automorphisms and is thus an automorphism. Therefore $H_k(X)$ extends to a $\mathbb{Q}$-vectorspace if and only if $H_k(X;\mathbb{Z}/p\mathbb{Z}) = 0$ for every prime p.

Problem 5.

Consider the transfer sequence associated to the covering map $p : S^\infty \to \mathbb{RP}^\infty$,

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\tau_*"] & H_n(S^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "p_*"] & H_n(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[dll] \\
& H_{n-1}(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\tau_*"] & H_{n-1}(S^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[r] & H_{n-1}(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

However, S^∞ is contractible $\tilde{H}_n(S^\infty;\mathbb{Z}/2\mathbb{Z}) = 0$ for each n. Therefore, we have isomorphisms,

$$H_n(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H_{n-1}(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z})$$

for each n. However, since $\mathbb{RP}^\infty$ is path-connected we know that $H_0(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore, $H_n(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for each n.

Problem 6.

Let $f : C \to D$ be a morphism of chain complexes. Consider the complex $(C_f)_n = C_{n-1} \oplus D_n$ with the boundary map $\partial_n(x, y) = (-\partial_{C,n-1}(x), f_{n-1}(x) + \partial_{D,n}(y))$. Consider,

$$\begin{aligned}
\partial_n \circ \partial_{n+1}(x, y) &= \partial_n(-\partial_{C,n}(x), f_n(x) + \partial_{D,n+1}(y)) \\
&= (\partial_{C,n-1} \circ \partial_{C,n}(x), -f_{n-1}(\partial_{C,n}(x)) + \partial_{C,n} \circ f_n(x) + \partial_{D,n} \circ \partial_{D,n+1}(y)) \\
&= (0, -f_{n-1}(\partial_{C,n}(x)) + \partial_{C,n} \circ f_n(x)) = 0
\end{aligned}$$

because f is a morphism of chain complexes. Therefore, $\mathrm{Im}(\partial_{n+1}) \subset \ker \partial_n$ so C_f is a complex.

Problem 7.

Define the morphism of chain complexes $j : D \to C_f$ by $j_n(y) = (0, y)$ and a morphism of chain complexes, $d : C_f \to C[-1]$ by $d_n(x, y) = (-1)^n x$ where $(C[-1])_n = C_{n-1}$ and $\partial_{C[-1]} = -\partial_C$. These are maps of complexes because,

$$\partial_n \circ j_n(y) = \partial_n(0, y) = (0, \partial_{D,n}(y)) = j_{n-1} \circ \partial_{D,n}(y)$$

and likewise,

$$d_{n-1} \circ \partial_n(x, y) = d_{n-1}(-\partial_{C,n-1}(x), f_{n-1}(x) + \partial_{D,n}(y)) = (-1)^n \partial_{C,n-1}(x) = \partial_{C[-1],n} \circ d_n(x, y)$$

Clearly, j is injective and d is surjective. Furthermore, $\ker d_n = \{(0, y) \mid y \in D_n\} = \mathrm{Im}(j_n)$. Therefore,

$$0 \longrightarrow D \xrightarrow{\ j\ } C_f \xrightarrow{\ d\ } C[-1] \longrightarrow 0$$

is an exact sequence of complexes.

Problem 8.

Let $f : C \to D$ be a morphism of complexes. Consider the long exact sequence of homology corresponding to the above exact sequence of chain complexes,

$$\cdots \longrightarrow H_n(D) \xrightarrow{j_*} H_n(C_f) \xrightarrow{d_*} H_{n-1}(C) \xrightarrow{f_*} H_{n-1}(D) \xrightarrow{j_*} H_{n-1}(C_f) \longrightarrow \cdots$$

I claim that the linking map $H_n(C[-1]) = H_{n-1}(C) \to H_{n-1}(D)$ is equal to f_*. To see this, take $c \in C_{n-1}$ then $[c] \mapsto [a]$ where $a \in D_{n-1}$ is such that $c = d_n(x, y)$ for $(x, y) \in (C_f)_n$ and $\partial_{C_f,n}(x, y) = j_{n-1}(a)$. However,

$$\partial_{C_f,n}(x, y) = (-\partial_{C,n-1}(x), f_{n-1}(x) + \partial_{D,n}(y)) = j_{n-1}(a) = (0, a)$$

Therefore, $\partial_{C,n-1}(x) = 0$ and $f_{n-1}(x) + \partial_{D,n}(y) = a$ but $(-1)^n x = c$ so $a = (-1)^n f_{n-1}(c) + \partial_{D,n}(y)$ so $[a] = (-1)^n [f_{n-1}(c)] = (-1)^n f_*([c])$. Therefore, up to sign, the linking map is f_*. From the above exact sequence, $f_* : H_n(C) \to H_n(D)$ is an isomorphism for all n if and only if $H_{n+1}(C_f) = 0$ for all n. Therefore, f is a quasi-isomorphism if and only if the complex C_f is a cyclic.

Lemmas

Lemma 0.1. Let $f : S^n \to S^n$ have no point at which $f(x) = -x$ then $\deg f = 1$.

Proof. Suppose $\forall x : f(x) \neq -x$. Thus, the line $f(x)$ to x does not pass though the origin. Therefore, the map $H(x,t) = [(1-t)f(x) + tx]/|(1-t)f(x) + tx|$ is a homotopy between f and the identity. Thus, $\deg f = \deg \mathbf{1} = 1$. $\square$

Lemma 0.2. Let $f : S^n \to S^n$ have no fixed points then $\deg f = (-1)^{n+1}$.

Proof. If the map $f : S^n \to S^n$ has no fixed points then the map $-f : S^n \to S^n$ satisfies Lemma **??** and thus $\deg(-f) = 1$. However, $\deg(-f) = \deg(-1) \cdot \deg f = (-1)^{n+1} \deg f$. Therefore, $\deg f = (-1)^{n+1}$. $\square$

Lemma 0.3. Let R be an integral domain and K its field of fractions. Then an R-module extends to a K-vectorspace if and only if for each $r \in R \backslash \{0\}$ the multiplication by r map is an automorphsm.

Proof. Suppose M is an R-module. If M is the restriction of a K-vectorspace then for any $r \in R \setminus \{0\} \subset K^\times$ we have $r^{-1} \in K$. Therefore, for $x \in M$ we have $r^{-1} \cdot (r \cdot x) = (r^{-1} r) \cdot x = x$ and $r \cdot (r^{-1} \cdot x) = (r r^{-1}) \cdot x = x$ so multiplication by r is a bijection and thus an automorphism of the abelian group M by scalar distributivity.

Convsersely, suppose that the map $f_r : M \to M$ given by $f_r(x) = r \cdot x$ is an automorphism for each $r \in R \setminus \{0\}$ then we can define a K-vectorspace M by the action of $\frac{a}{b} \in K$ via,

$$\frac{a}{b} \cdot x = (f_b^{-1} \circ f_a)(x) = f_b^{-1}(a \cdot x)$$

Clearly, if $r \in R \subset K$ then $r \cdot_K M = f_r(x) = r \cdot M$ so this K-vectorspace restricts to the given R-module. We need to check that this is a honest-to-god vectorspace. First, if $\frac{a}{b} = \frac{a'}{b'}$ then $ab' = a'b$ so,

$$b \cdot \frac{a'}{b'} \cdot x = b \cdot f_{b'}^{-1}(a' \cdot x) = f_{b'}^{-1}(ba' \cdot x) = f_{b'}^{-1}(b'a \cdot x) = (f_{b'}^{-1} \circ f_{b'})(a \cdot x) = a \cdot x \implies \frac{a'}{b'} \cdot x = f_b^{-1}(a \cdot x) = \frac{a}{b} \cdot x$$

so the action is well-defined. Commutativity of f_a and f_b gives the remaining properties. Take any $p, q \in K$ then,

$$p \cdot (q \cdot x) = (pq) \cdot x \quad (p + q) \cdot x = p \cdot x + q \cdot x$$

and we know that $f_b^{-1} \circ f_a$ is the composition of homomorphisms and thus $p \cdot (x + y) = p \cdot x + p \cdot y$. Therefore, M is a K-vectorspace. $\square$