Lemma 0.1. *Let $H_1, H_2 \lhd G$ be normal subgroups with quotient maps $\pi_i : G \to G/H_i$ and consider the maps,*

$$\varphi_{i,j} : H_i \hookrightarrow G \overset{\pi_j}{\twoheadrightarrow} G/H_j$$

Then $\varphi_{1,2}$ is surjective iff $\varphi_{2,1}$ is surjective.

Proof. Consider the commutative diagram with exact rows and columns,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H_1 \cap H_2 \arrow[r, hook] \arrow[d, hook] & H_1 \arrow[r, two heads] \arrow[d, hook] \arrow[dr, "\varphi_{1,2}"] & K_1 \arrow[r] \arrow[d, hook, "\bar{\varphi}_{1,2}"] & 0 \\
0 \arrow[r] & H_2 \arrow[r, hook] \arrow[d, two heads] \arrow[dr, "\varphi_{2,1}"] & G \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"] & G/H_2 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & K_2 \arrow[r, hook, "\bar{\varphi}_{2,1}"] \arrow[d] & G/H_1 \arrow[r, two heads] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $K_i = H_i/(H_1 \cap H_2)$ and the maps $\bar{\varphi}_{i,j} : K_i \to G/H_j$ are induced by the maps $\varphi_{i,j}$ and are injective by the first isomorphism theorem. Exactness and commutativity are obvious except at C which I have yet to define! By commutativity and surjectivity, $\operatorname{Im} \bar{\varphi}_{i,j} = \pi_j(H) \lhd \operatorname{Im} \pi_j = G/H_j$ so $\operatorname{Im} \bar{\varphi}_{i,j}$ is a normal subgroup and thus $\operatorname{coker} \bar{\varphi}_{i,j} = (G/H_j)/\operatorname{Im} \bar{\varphi}_{i,j}$ exists. Take $C = \operatorname{coker} \bar{\varphi}_{1,2}$. Furthermore, the exactness of columns gives a surjective map $G/H_1 \to C$ which makes the bottom right square commute (see Lemma **??**). By the nine lemma, the bottom row is exact proving that $C = \operatorname{coker} \bar{\varphi}_{2,1}$. Finally, by exactness,

$$\bar{\varphi}_{1,2} \text{ is an isomorphism} \iff C = 0 \iff \bar{\varphi}_{2,1} \text{ is an isomorphism}$$

But $\varphi_{i,j}$ is a surjection iff $\bar{\varphi}_{i,j}$ is an isomorphism so $\varphi_{1,2}$ is surjective iff $\varphi_{2,1}$ is surjective. □

Lemma 0.2. *Consider the commutative diagram with exact rows,*

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"] & B \arrow[r, "s"] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & 0 \\
A' \arrow[r, "r'"] & B' \arrow[r, "s'"] & C' \arrow[r] & 0
\end{tikzcd}$$

If f and g are defined then there exists a map h making the diagram commute. Furthermore, if g is surjective then h is surjective.

Proof. Take $c \in C$ then lift to $b \in B$ then define $h(c) = s'(g(b))$. Since $c = s(b)$ clearly this makes the diagram commute. This map is defined up to an element of $\ker s = \operatorname{Im} r$ since if $b' \in B$ maps $s(b') = c = s(b)$ then $b^{-1}b' \in \ker s = \operatorname{Im} r$ so $b^{-1}b' = r(a)$ for some $a \in A$ and thus, by commutativity, $s'(g(b^{-1}b')) = s'(g(r(a))) = s'(r'(f(a))) = 0$ since $\ker s' = \operatorname{Im} r'$. Thus, $s'(g(b)) = s'(g(b'))$ so h is well-defined. If g is surjective then $s' \circ g = h \circ s$ is surjective (since s' is) so h is surjective. □

This can probably be proven this with some variant of the four or five lemma but I needed diagram chasing practice anyway. For fun, I will consider the other case I did not end up needing.

Lemma 0.3. *Consider the commutative diagram with exact rows,*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "r"] \arrow[d, dashed, "f"] & B \arrow[r, "s"] \arrow[d, "g"] & C \arrow[d, "h"] \\
0 \arrow[r] & A' \arrow[r, "r'"] & B' \arrow[r, "s'"] & C'
\end{tikzcd}$$

If g and h are defined then there exists a map f making the diagram commute. Furthermore, if g is injective then h is injective.

Proof. Take some $a \in A$ then by commutativity, $s'(g(r(a))) = h(s(r(a))) = 0$ since $\ker s = \operatorname{Im} r$. Thus, $g(r(a)) \in \ker s' = \operatorname{Im} r'$ so $g(r(a)) = r'(a')$ for some a'. Define $f(a) = a'$ which is well-defined because $r'(a') = g(r(a))$ and r' is injective. Furthermore, if g is injective then $g \circ r = r' \circ f$ is injective (since r is) so f is injective. $\square$

Now for the main result.

Proposition 0.4. *Let $p : G \to G'$ be surjective and $H \triangleleft G$ a normal subgroup. Then there exist coset representatives for G/H with fixed image in G' if and only if $p(H) = G'$. Furthermore, we if this holds, we may take the coset representatives to be trivial in G'.*

Proof. A set $S \subset G$ contains a full set of coset represenatives for G/H if $\pi(S) = G/H$. Therefore, we require that $\pi(p^{-1}(x)) = G/H$ for some $x \in G'$. Since we must hit the identity, $H \cap p^{-1}(x) \neq \varnothing$ so there exits $h \in H$ such that $p(h) = x$. Thus, $p^{-1}(x) = h \ker p$ so $\pi(p^{-1}(h)) = \pi(h)\pi(\ker p) = \pi(\ker p)$ so we may take $h = e$. The conclusion holds if and only if $\pi(\ker p) = G/H$.

Take $H_1 = H$ and $H_2 = \ker p$ in Lemma 0.1 and thus,

$$\operatorname{Im} \varphi_{2,1} = \pi(\ker p) = G/H \iff \operatorname{Im} \varphi_{1,2} = \pi_2(H) = G/\ker p$$

but the map p naturally factors through $G/\ker p$ as,

$$\begin{tikzcd}
H \arrow[r, hook] & G \arrow[rr, two heads, "p"] \arrow[dr, two heads, "\pi_2"'] & & G' \\
& & G/\ker p \arrow[ur, "\sim"'] &
\end{tikzcd}$$

so $p(H) = G' \iff \pi_2(H) = G/\ker p$. $\square$