1 Week 1

2 Week 2

(a) δ-rings

(b) prisms

(c) prismatic site

(d) Hodge-Tate comparison

(e) *étale* comparison

(f) Nygaard filtration.

2.1 δ-rings

Definition 2.1.1. A δ-ring is a pair $(A, \delta : A \to A)$ where δ is not a ring homomorphism but satisfies the property that,

$$\phi : A \to A \quad \phi(x) = x^p + p\delta(x)$$

is a ring homomorphism.

Remark. We will usually assume that A is a $\mathbb{Z}_{(p)}$-algebra but this is not necessary. On $\mathbb{Z}_{(p)}$ we have $\phi(x) = x$ is the only possible lift so,

$$\delta(x) = \frac{x - x^p}{p}$$

Proposition 2.1.2. $(\mathbb{Z}_{(p)}, \delta)$ is the initial δ-ring.

Remark. Suppose that $p^n = 0$ in A then $\delta'(p^n) = 0$

Proposition 2.1.3. The forgetful functor δ-Rings to Rings has a left and right inverse. The right adjoint is given by Witt vectors the left adjoint is given by free δ-rings

Remark. The free δ-ring structure $A\{x\} = A[x_1, x_2, \dots]$ with $\delta(x_i) = x_{i+1}$

Definition 2.1.4. $\alpha \in A$ is distinguished if $\delta(\alpha)$ is a unit.

Example 2.1.5. The following are distinguished,

(a) $p \in \mathbb{Z}_p$

(b) $x - p \in \mathbb{Z}_p[[x]]$

(c) for $A_{\text{inf}} = W(R^\flat)$ each $\xi \in \ker(A_{\text{inf}} \to R)$ is distinguished.

Definition 2.1.6. A δ-ring is *perfect* if ϕ is an isomorphism.

Proposition 2.1.7. The following categories are equivalent:

(a) perfect p-complete δ-rings

(b) p-complete p-torsion-free rings A with A/p perfect

(c) perfect $\mathbb{F}_p$-algebras.

Proof. $\square$

2.2 Prisms

Definition 2.2.1. A *prism* is a pair (A, I) with $I \subset A$ Cartier divisor such that A is (p, I)-adically complete with $p \in I + \phi(I)A$. The category of prisms is a full subcategory of the category of pairs (A, I) with maps $f : (A, I) \to (B, J)$ meaning $f(I) \subset J$.

Remark. $p \in I + \phi(I)A \iff$ after ind-Zariski localization$I = (\zeta)$ where ζ is distinguished.

Definition 2.2.2. A prism is,

(a) *perfect* if A is perfect

(b) *bounded* if $A[p^\infty] = A[p^n]$.

(c) *orientable* if I is principal

(d) *oriented* if I is given a canonical generator

(e) *crystaline* if $I = (p)$ which implies oriented and bounded.

Proposition 2.2.3. If $f : (A, I) \to (B, J)$ is a morphism of prisms then $f(I)B = J$.

Theorem 2.2.4. There is an equivalence of categories between,

(a) perfect prisms

(b) perfectoid rings.

Remark. Thus perfect prisms are orientable.

Proposition 2.2.5. If (A, I) is perfect then,

$$\mathrm{Hom}\,(A/I, B/J) = \mathrm{Hom}\,((A, I), (B, J))$$

2.3 Tilting Equivalence

For $R \to S$ a map of perfectoid rings, let $R = A/I$ and $S = B/J$ for some perfect prisms. Then $R^\flat = A/p$ and $S^\flat = B/p$. By the lifting we get a unique map $(A, I) \to (B, J)$ and hence a map of δ-rings $A \to B$ which gives a map $A/p \to B/p$ giving the map of tilts $R^\flat \to S^\flat$.

2.4 Prismatic Site

Let X be a p-adic formal scheme. Let (A, I) be a bounded prism with a map $X \to \mathrm{Spf}\,((A/I))$. The site $(X/A)_{\gneqq}$ whose objects are pairs of a prism (B, J) over (A, I) and a map $\mathrm{Spf}\,(B/J) \to X$ over $\mathrm{Spf}\,(A/I)$.

The absolute prismatic site $(X)_{\gneqq}$ are just prisms (B, J) equipped with a map $\mathrm{Spf}\,(B/J) \to X$.

If $X = \mathrm{Spf}\,(A/I)$ then $(X/A)_{\gneqq}$ are just prisms over (A, I).

There are important sheaves,

$$\mathcal{O}_{\gneqq} : (B, J) \mapsto B \quad \text{and} \quad \overline{\mathcal{O}_{\gneqq}} : (B, J) \mapsto B/J$$

There is a pushforward map $\nu_* : \mathfrak{Sh}(X/A)_{\gneqq} \to \mathfrak{Sh}(X_{\text{ét}})$. Then the prismatic cohomology is,

$${\gneqq}_{X/A} = R\nu_* \mathcal{O}_{\gneqq} \in D(X_{\text{ét}}, A) \quad \text{and} \quad \overline{\gneqq}_{X/A} = R\nu_* \overline{\mathcal{O}_{\gneqq}} \in D(X_{\text{ét}}, \mathcal{O}_X)$$

Then as A/I-modules,

$$\overline{\gneqq}_{X/A} = {\gneqq}_{X/A} \otimes_A^{\mathbb{L}} A/I$$

2.5 Hodge-Tate Comparison

For any A/I-module M we define,

$$M\{i\} := M \otimes_{A/I} I^i/I^{i+1}$$

Then then there is a distinguished triangle,

$$\overline{\Delta}_{X/A}\{i+1\} \to \Delta_{X/A} \otimes^{\mathbb{L}}_A I^i/I^{i+2} \to \overline{\Delta}_{X/A}\{i\}$$

which gives rise to a differential,

$$\beta_I : H^i(\overline{\Delta}_{X/A}\{i\}) \to H^{i+1}(\overline{\Delta}_{X/A}\{i+1\})$$

and therefore,

$$H^\bullet(\overline{\Delta}_{X/A}\{\bullet\}$$

is a differential-graded ring. Therefore,

$$\eta^0_X : \mathcal{O}_X \to H^0(\overline{\Delta}_{X/A}\{0\})$$

extends to a map,

$$\eta_X : \Omega^\bullet_{X/(A/I)} \to H^\bullet(\overline{\Delta}_{X/A}\{\bullet\})$$

Theorem 2.5.1 (Comparison)**.** If X is smooth the map η_X is an isomorphism (in the derived category).

2.6 Fibers

For $X \to \mathrm{Spf}\,(R)$ we define $X_\eta := X \times_{\mathrm{Spf}(R)} \mathrm{Spa}\left(R[\frac{1}{p}], R\right)$. Then

3 FFFF

4 Sept 29

Definition 4.0.1. Let k be a (noncommutative) ring and A a (noncommutative) k-algebra. If A is a flat k-algebra then the *Hochschild Homology* $HH_\bullet(A/k)$ is defined as the complex associated (via Dold-Kan) to the simplicial ring $SHH_\bullet(A/k)$ with,

$$SHH_n(A/k) = A^{\otimes(n+1)}$$

where $d_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$ for $i < n$ and $d_n(a_0 \otimes \cdots \otimes a_n) = a_n a_0 \otimes \cdots \otimes a_{n-1}$ which is distinct because A is noncommutative.

Remark. Then $HH_0(A/k) = A/[A, A]$ and in the commutative case $HH_1(A/k) = \Omega^1_{A/k}$.

Proposition 4.0.2. There are natural isomorphisms,

$$HH_n(A/k) \xrightarrow{\sim} \mathrm{Tor}_n^{A \otimes_k A^{\mathrm{op}}}(A, A)$$

Definition 4.0.3. We say that an A'-algebra A is étale if A' is projective as an A-module and A' is a projective $A' \otimes_A A'$-module.

Remark. When everything is commutative,

(a) if A' is étale over A then,

$$HH_\bullet(A/k) \otimes_A^{\mathbf{LL}} A' \xrightarrow{\sim} HH_\bullet(A'/k)$$

Theorem 4.0.4. Let A be k-smooth (and everything is commutative). Then,

$$\Omega^\bullet_{A/k} \to \widehat{HH_\bullet}(A/k)$$

is a map of graded rings.

4.1 Extending HH to all k-algebras

Consider the diagram of functors,

$$\begin{tikzcd} & D & \\ P \arrow[ur, "HH"] \arrow[rr] & & k\text{-alg} \arrow[ul, dashed] \end{tikzcd}$$

Where D is the derived category of k-modules. Then $\widetilde{HH_\bullet}(A/k)$ is the left Kan extension. In general, in the commutative case,

$$HH_\bullet(A/k) \xrightarrow{\sim} A \otimes^{\mathbf{LL}}_{A \otimes^{\mathbf{LL}}_k A} A$$

in the derived category.

4.2 Cyclic Homology

Work with A flat over k. Then $HH_n(A/k)$ has a circle action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on,

$$HH_n(A/k) = A^{\otimes(n+1)}$$

given by the brading of tensor product (exchanging terms),

$$t_n : a_0 \otimes \cdots \otimes a_n = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

Then we define,

$$N = \sum_{i=0}^{n} (-1)^i t_n^i : A^{\otimes(n+1)} \to A^{\otimes(n+1)}$$

and likewise,

$$B = (1 - (-1)^n t_n) s_0 N : A^{\otimes n} \to A^{\otimes n} \to A^{\otimes(n+1)} \to A^{\otimes(n+1)}$$

where s_0 is a degeneracy map. Write $\pm t$ for $(-1)^n t_n$.

Lemma 4.2.1. (a) $(1 - \pm t)b' = n(1 - \pm t)$

(b) $b'N = Nb$

(c) $s_0 b' + b's = 1$

(d) $B^2 = 0$

(e) $Bb = -bB$.

Therefore, B gives a map on the homology of $HH_\bullet(A/k)$ such that,

$$\begin{tikzcd} HH_n(A/k) \arrow[r, "B"] & HH_{n+1}(A/k) \\ \Omega^n(A/k) \arrow[r, "\mathrm{d}"] \arrow[u] & \Omega^{n+1}(A/k) \arrow[u] \end{tikzcd}$$

commutes.

5 Intro to Σ

(a) W_S-modules and definition

(b) points and divisors

(c) contracting prop. of F

(d) global sections

(e) line bundles.

Prismatisation functor $X \mapsto X^{\gneqq}$. Then,

$$D^\bullet_{\gneqq}(X) \xrightarrow{\sim} D^\bullet(X^{\gneqq})$$

This is like how sheaves on the de Rham site correspond to sheaves on the de Rham stack.

Let $X = \mathrm{Spf}\,(\mathbb{Z}_p)$ then,

$$\gneqq_X = \mathbb{Z}_p[[x]][0]$$

Maps $X \to \mathrm{Spf}\,(\mathbb{Z}_p)$ should correspond to maps $X^{\gneqq} \to \Sigma$.

5.1 Witt Vectors

Let $W = \mathrm{Spec}\,(\mathbb{Z}[x_0, x_1, \dots]) \to \mathrm{Spf}\,(\mathbb{Z}_p)$ Then,

$$Z_p[x_0, x_1, \dots] = \mathrm{Spec}\,(\mathbb{Z}_p\{x\})$$

is the free δ-ring. Let $Z \subset W$ be cut out by $p = x_0 = 0$ and $x_1 \neq 0$. Then $W_{\mathrm{prim}} = \widehat{W_Z}$ is the completion. Then,

$$W_{\mathrm{prim}} = \mathrm{Spf}\left(\mathbb{Z}_p[x_0, x_1, \dots][x_1^{-1}]^\vee, (p, x_0)\right)$$

Then W_{prim} is Frobenius stable so we get a diagram,

$$\begin{tikzcd} W_{\mathrm{prim}} \arrow[r, "F"] \arrow[d] & W_{\mathrm{prim}} \arrow[d] \\ W \arrow[r, "F"] & W \end{tikzcd}$$

Also W is a ring scheme. Consider,

$$W^\times \times W \to W \quad \text{via } (\lambda, x) \mapsto \lambda^{-1}x$$

Then get $W^\times \to W_{\text{prim}} \to W_{\text{prim}}$. Therefore we can define the following stack.

Definition 5.1.1. Let $\Sigma = [W_{\text{prim}}/W^\times]$ meaning it is the stackification of the functor,

$$R \mapsto W_{\text{prim}}(R)/W^\times(R)$$

Definition 5.1.2. Let $W_S = W \times_{\text{Spf}(\mathbb{Z}_p)} S$. Then a W_S-module is a commutative affine group scheme with an action of W_S.

Definition 5.1.3. Then M is invertible if fpqc locally isomorphic to W_S. This is equivalent to a $W_S^\times$-torsor.

Remark. An S-point of $[W_{\text{prim}}/W^\times]$ is by definition an $W^\times$-torsor over S equipped with a map to $(W_{\text{prim}})_S$. This is the same data as a W_S-module M with a map $\xi : M \to W_S$ factoring through $(W_{\text{prim}})_S$ which is the same as having fibers in kernel of ξ_1 but in kernel of ξ_2.

Definition 5.1.4. A better definition of Σ is therefore,

$$\Sigma : S \mapsto \{(M, \xi) \mid M \text{ invertible } W_S\text{-module and } \xi : M \to W_S \text{ is primitive}\}$$

which is a category fibered in groupoids.

6 Oct 13

Definition 6.0.1. A *generalized Cartier divisor* $(\mathscr{I}, \alpha)$ is a pair of an invertible $\mathcal{O}_X$-module $\mathscr{I}$ equipped with a map (not necessarily injective),

$$\alpha : \mathscr{I} \to \mathcal{O}_X$$

Let $\text{Cart}(X)$ be the groupoid of generalized Cartier divisors.

Remark. A generalized Cartier divisor $(\mathscr{I}, \alpha)$ is a Cartier divisor exactly when α is injective.

Proposition 6.0.2. Given $f : X \to Y$ there is a map $f^* : \text{Cart}(Y) \to \text{Cart}(X)$ taking $\alpha : \mathscr{I} \to \mathcal{O}_Y$ to $f^*\alpha : f^*\mathscr{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$ making Cart a pre-stack.

Remark. If f is non-flat it need not pullback Cartier divisors to Cartier divisors. This is why we introduce generalized Cartier divisors.

Proposition 6.0.3. Cart is an algebraic stack in the fpqc topology. Indeed $\text{Cart} = [\mathbb{A}^1/\mathbb{G}_m]$.

Remark. We will write $\text{Cart}(R) := \text{Cart}(\text{Spec}\,(R))$.

Now let R be p-nilpotent and consider $W(R) \to R$ giving a map $\text{Cart}(W(R)) \to \text{Cart}(R)$ which is functorial in R. Then $\widetilde{WCart}$ is the stack sending $R \mapsto \text{Cart}(W(R))$. Thus there is a map $\widetilde{\text{WCart}} \to \text{Cart}$.

7 Oct 27

Definition 7.0.1. A morphism $f : X \to S$ is *syntomic* if,

(a) f is locally of finite presentation

(b) f is flat

(c) f is a locally a complete intersection morphism meaning for each $x \in X$ there is an affine open $x \in U \to V$ with ring map $A \to B$ is a local complete intersection.

Proposition 7.0.2. Let $f : X \to S$ be flat and finitely presented. Then the following are equivalent,

(a) f is syntomic

(b) $H_i(\mathbb{L}_{X/S}) = 0$ for $i < 0$ and $H_0(\mathbb{L}_{X/S})$ is locally free.

Remark. Note that if we requred $\mathbb{L}_{X/S} \xrightarrow{\sim} \Omega_{X/S}[0]$ in the derived category for a locally-free $\Omega_{X/S}$ then f would be smooth. We are requring less.

Lemma 7.0.3. (a) smooth maps are syntomic

(b) syntomic morphisms are stable under composition and base change

(c) Consider a diagram,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

with f flat and X, Y smooth over S then f is syntomic

(d) if A is noetherian, and $X \to \text{Spec}\,(A)$ is syntomic then X locally is of the form,

$$\text{Spec}\,(A[T_1, \dots, T_n]/(f_1, \dots, f_r))$$

with $f_1, \dots, f_r$ Koszul-regular.

Corollary 7.0.4. If p is zero on A then the map,

$$\phi A[T_1, \dots, T_d] \to A[T_1, \dots, T_d]$$

given by $T_i \mapsto T_i^p$ is syntomic and faithfully flat.

Proof. Flat because ϕ is finite locally free (in fact globally free) and then faithfully flat by surjectivity. Then we conclude by (c) of the previous lemma. □

Corollary 7.0.5. The sequence,

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0$$

is exact in the syntomic topology.

Proof. The only nontrivially part is the surjectivity of $p^n : \mathbb{G}_m \to \mathbb{G}_m$ as sheaves on the syntomic topology. This holds because $p^n : \mathbb{G}_m \to \mathbb{G}_m$ is syntomic and faithfully flat. Thus for any map $X \to \mathbb{G}_m$ we pullback to get a surjective syntomic cover of X lifting the map. □