Mathematics GU4053 Algebraic Topology
Assignment # 12

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Problem 1.

Let X and Y be connected n-dimensional CW complexes and $f : X \to Y$ a map which induces isomorphisms $\pi_i(X) \to \pi_i(Y)$ for $i \le n$. Let $p_* : \tilde{X} \to X$ and $q_* : \tilde{Y} \to Y$ be covering maps of the universal covers of X and Y. The universal covers can be given an n-dimensional cell complex structure.

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "\tilde{f}"] \arrow[d, "p"] & \tilde{Y} \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Since the map $f \circ p : \tilde{X} \to Y$ is a map from a simply-connected space (which is also locally path-connected since its a CW complex) there is a lift to the covering space of $f \circ p$ to a map $\tilde{f} : \tilde{X} \to \tilde{Y}$. Since p_* and q_* and f_* all induce isomorphisms on π_i for $1 < i \le n$ we know that $\tilde{f}$ also induces isomorphism on π_i for $i \le n$ since $\tilde{f}$ trivially induces isomorphisms on π_0 and π_1 because $\tilde{X}$ and $\tilde{Y}$ are simply-connected. Now, consider the long exact homotopy sequence of the pair $(M_{\tilde{f}}, \tilde{X})$,

$$\pi_i(\tilde{X}) \xrightarrow{\sim} \pi_i(M_{\tilde{f}}) \longrightarrow \pi_i(M_{\tilde{f}}, \tilde{X}) \longrightarrow \pi_{i-1}(\tilde{X}) \xrightarrow{\sim} \pi_{i-1}(M_{\tilde{f}})$$

The maps $\tilde{f}_* : \pi_i(\tilde{X}) \to \pi_i(M_{\tilde{f}}) \cong \pi_i(Y)$ are isomorphisms for each $i \le n$ so $\pi_i(M_{\tilde{f}}, X) = 0$ for each $i \le n$. Therefore, the pair $(M_{\tilde{f}}, X)$ is n-connected so by Hurewicz's theorem we have isomorphisms, $h_i : \pi_i(M_{\tilde{f}}, X) \to H_i(M_{\tilde{f}}, X)$ for $i \le n+1$. In particular, $H_i(M_{\tilde{f}}, X) = 0$ for $i \le n$. Furthermore, the Hurewicz map is natural so,

$$\begin{tikzcd} \pi_{n+1}(M_{\tilde{f}}, \tilde{X}) \arrow[r] \arrow[d, "h_{n+1}"] & \pi_n(\tilde{X}) \arrow[r, "\sim"] \arrow[d, "h_n"] & \pi_n(M_{\tilde{f}}) \arrow[d] \\ H_{n+1}(M_{\tilde{f}}, \tilde{X}) \arrow[r] & H_n(\tilde{X}) \arrow[r] & H_n(M_{\tilde{f}}) \end{tikzcd}$$

and thus the map $\pi_{n+1}(M_{\tilde{f}}, \tilde{X}) \to \pi_n(\tilde{X})$ is the zero map. Because the Hurewicz maps h_{n+1} and h_n are isomorphisms, the map $H_{n+1}(M_{\tilde{f}}, \tilde{X}) \to H_n(\tilde{X})$ is also the zero map. Now, applying the long exact sequence of homology to the pair $(M_{\tilde{f}}, \tilde{X})$,

$$H_{i+1}(M_{\tilde{f}}, \tilde{X}) \xrightarrow{zero} H_i(\tilde{X}) \xrightarrow{\sim} H_i(M_{\tilde{f}}) \xrightarrow{zero} H_i(M_{\tilde{f}}, \tilde{X})$$

we see that the map $\tilde{f}_* : H_i(\tilde{X}) \to H_i(M_{\tilde{f}})$ is an isomorphism for $i \leq n$. However, $\tilde{X}$ and $\tilde{Y}$ are n-dimensional CW complexes so $H_i(\tilde{X}) = H_i(\tilde{Y}) = 0$ for $i > n$. Thus, $\tilde{f}_* : H_i(\tilde{X}) \to H_i(\tilde{Y})$ is an isomorphism for all i. However, $\tilde{X}$ and $\tilde{Y}$ are simply-connected CW complexes so my Whitehead's theorem for homology $\tilde{f}$ is a homotopy equivalence. Consider the induced map $f_* : \pi_i(X) \to \pi_i(Y)$. Since $p_* : \pi_i(\tilde{X}) \to \pi_i(X)$ and $q_* : \pi_i(\tilde{Y}) \to \pi_i(Y)$ are isomorphisms for $i > 1$ and $\tilde{f}_* : \pi_i(\tilde{X}) \to \pi_i(\tilde{Y})$ is an isomorphism because $\tilde{f}$ is a homotopy equivalence by the commutativity of the above diagram, $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i > 1$. However, by assumption, $f_* : \pi_1(X) \to \pi_1(Y)$ is also an isomorphism and both X and Y are connected so $f_* : \pi_0(X) \to \pi_0(Y)$ is trivially an isomorphism. Therefore, the map f induces $f_* : \pi_i(X) \to \pi_i(X)$ isomorphisms for each i and thus by Whitehead's theorem, f is a homotopy equivalence.

Problem 2.

Let X be an $(n-1)$-connected CW complex with $n > 1$.

(a). Consider the long exact homotopy sequence associated with the pair (X, X^{n+1}),

$$\pi_{n+1}(X^{n+1}) \longrightarrow \pi_{n+1}(X) \longrightarrow \pi_{n+1}(X, X^{n+1}) \longrightarrow \pi_n(X^{n+1}) \longrightarrow \pi_n(X)$$

We know that the pair (X, X^{n+1}) is $(n+1)$-connected. Thus, $\pi_{n+1}(X, X^{n+1}) = 0$. Therefore, the map $\pi_{n+1}(X^{n+1}) \to \pi_{n+1}(X)$ is surjective since the sequence is exact.

Similarly, the long exact sequence of homology assoicated with the pair (X, X^{n+1}) gives,

$$H_{n+1}(X^{n+1}) \longrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, X^{n+1}) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X)$$

but the proof of cellular homology gives us that $H_k(X, X^{n+1}) = 0$ for $k \leq n+1$ and thus $H_{n+1}(X, X^{n+1}) = 0$. By exactness, the map $H_{n+1}(X^{n+1}) \to H_{n+1}(X)$ is surjective.

(b). Because the Hurewicz map is natural, we have a morphism of long exact sequences for the pair (X^{n+1}, X^n),

$$\begin{tikzcd}
\pi_{n+1}(X^n) \arrow[r] \arrow[d, "h_{n+1}"] & \pi_{n+1}(X^{n+1}) \arrow[r] \arrow[d] & \pi_{n+1}(X^{n+1}, X^n) \arrow[r] \arrow[d, "h'_{n+1}"] & \pi_n(X^n) \arrow[d, "h_n"] \\
H_{n+1}(X^n) \arrow[r] & H_{n+1}(X^{n+1}) \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r] & H_n(X^n)
\end{tikzcd}$$

Homology is zero for cell complexes with strictly lower dimensional cells so $H_{n+1}(X^n) = 0$. Thus, h_{n+1} is surjective. Since X is $(n-1)$-connected Hurewicz's theorem gives that h_n is an isomorphism. Furthermore, (X^{n+1}, X^n) is n-connected so Hurewicz's theorem gives that h'_{n+1} is an isomorphism. Therefore, by the 4-lemma, the map $\pi_{n+1}(X^{n+1}) \to H_{n+1}(X^{n+1})$ is a surjection.

(c). Consider the Hurewicz map between the long exact sequences for the pair (X, X^{n+1}),

$$
\begin{tikzcd}
\pi_{n+1}(X^{n+1}) \arrow[r, two heads, "\iota_*"] \arrow[d, two heads, "h'_{n+1}"] & \pi_{n+1}(X) \arrow[r] \arrow[d, "h_{n+1}"] & \pi_{n+1}(X, X^{n+1}) = 0 \arrow[d] \\
H_{n+1}(X^{n+1}) \arrow[r, two heads, "\iota_*"] & H_{n+1}(X) \arrow[r] & H_{n+1}(X, X^{n+1}) = 0
\end{tikzcd}
$$

We have shown that the maps $\iota_* : \pi_{n+1}(X^{n+1}) \to \pi_{n+1}(X)$ and $\iota_* : H_{n+1}(X^{n+1}) \to H_{n+1}(X)$ and the Hurewicz map $h'_{n+1} : \pi_{n+1}(X^{n+1}) \to H_{n+1}(X^{n+1})$ are surjective. However, the diagram commutes so,

$$h_{n+1} \circ \iota_* = \iota_* \circ h'_{n+1}$$

h'_{n+1} and ι_* are surjective so $\iota_* \circ h'_{n+1}$ is surjective. Thus, $h_{n+1} \circ \iota_*$ is also surjective which implies that $h_{n+1} : \pi_{n+1}(X) \to H_{n+1}(X)$ must be surjective as well.

(d). We need to show that the Hurewicz map on a path-connected (0-connected) CW complex does not necessarily induce as surjection $h_2 : \pi_2(X) \to H_2(X)$. Consider the torus, $T^2 = S^1 \times S^1$. We know that $\pi_2(T^2) \cong \pi_2(S^1) \times \pi_2(S^1) = 0$. However, we have calculated in class that $H_2(T^2) \cong \mathbb{Z}$. Therefore, the map $h_2 : \pi_2(T^2) \to H_2(T^2)$ cannot be surjective.

Problem 3.

Let C and D be chain complexes of abelian groups. Define the chain complex $C \otimes D$ with chains $(C \otimes D)_n = C_n \otimes D_n$ and a boundary map,

$$\partial_n(x \otimes y) = (\partial_C x) \otimes y + (-1)^n x \otimes (\partial_D y)$$

Consider the composition of boundary maps,

$$
\begin{aligned}
\partial_n \circ \partial_{n+1}(x \otimes y) &= \partial_n \left((\partial_C x) \otimes y + (-1)^{n+1} x \otimes (\partial_D y) \right) = \partial_n((\partial_C x) \otimes y) + (-1)^{n+1} \partial_n(x \otimes (\partial_D y)) \\
&= (\partial_C^2 x) \otimes y + (-1)^n (\partial_C x) \otimes (\partial_D y) + (-1)^{n+1} (\partial_C x) \otimes (\partial_D y) + (-1)^{n+n+1} x \otimes (\partial_D^2 y) \\
&= (-1)^n \left[(\partial_C x) \otimes (\partial_D y) - (\partial_C x) \otimes (\partial_D y) \right] \\
&= 0
\end{aligned}
$$

where I used the fact that the boundard maps on C and D statisfy $\partial_C^2 x = \partial_D^2 y = 0$. Since the boundary map ∂_n is a homomorphism and it is zero on each $x \otimes y$ we have shown that $\partial_{n+1} \circ \partial_n = 0$ on $(C \otimes D)_{n-1}$.

Problem 4.

Let F be a field and X a space such that $H_i(X; F)$ has finite dimension for all i. Define the Poincare series,

$$p_X(t) = \sum_i \left(\dim_F H_i(X; F) \right) t^i$$

If X and Y are spaces with Poincare series p_X and p_Y. Consider the homology,

$$H_i(X \coprod Y; F) = H_i(X; F) \oplus H_i(Y; F)$$

Therefore,

$$\dim_F H_i(X \coprod Y; F) = \dim_F (H_i(X; F) \oplus H_i(Y; F)) = \dim_F H_i(X; F) + \dim_F H_i(Y; F)$$

Thus, the Poincare series become,

$$\begin{aligned}
p_{X \coprod Y}(t) &= \sum_i \left(\dim_F H_i(X \coprod Y; F)\right) t^i \\
&= \sum_i (\dim_F H_i(X; F))\, t^i + \sum_i (\dim_F H_i(Y; F))\, t^i = p_X(t) + p_Y(t)
\end{aligned}$$

Similarly, reduced homology commutes with wedge product,

$$\tilde{H}_i(X \vee Y; F) = \tilde{H}_i(X; F) \oplus \tilde{H}_i(Y; F)$$

Therefore,

$$\dim_F \tilde{H}_i(X \vee Y; F) = \dim_F \left(\tilde{H}_i(X; F) \oplus H_i(Y; F)\right) = \dim_F \tilde{H}_i(X; F) + \dim_F \tilde{H}_i(Y; F)$$

However,

$$H_i(X; F) \cong \begin{cases} \tilde{H}_i(X; F) & i > 0 \\ \tilde{H}_i(X; F) \oplus \mathbb{Z} & i = 0 \end{cases}$$

which implies that,

$$\dim_F H_i(X; F) \cong \begin{cases} \dim_F \tilde{H}_i(X; F) & i > 0 \\ \dim_F \tilde{H}_i(X; F) + 1 & i = 0 \end{cases}$$

Putting these facts together,

$$\begin{aligned}
\dim_F H_i(X \vee Y; F) &= \begin{cases} \dim_F \tilde{H}_i(X; F) + \dim_F \tilde{H}_i(Y; F) & i > 0 \\ \dim_F \tilde{H}_i(X; F) + \dim_F \tilde{H}_i(Y; F) + 1 & i = 0 \end{cases} \\
&= \begin{cases} \dim_F \tilde{H}_i(X; F) + \dim_F \tilde{H}_i(Y; F) & i > 0 \\ \dim_F \tilde{H}_i(X; F) + \dim_F \tilde{H}_i(Y; F) - 1 & i = 0 \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
p_{X \vee Y}(t) &= \sum_i (\dim_F H_i(X \vee Y; F))\, t^i \\
&= \sum_{i>0} (\dim_F H_i(X; F))\, t^i + \sum_{i>0} (\dim_F H_i(Y; F))\, t^i + (H_0(X; F) + H_0(Y; F) - 1)\, t^0 \\
&= \sum_i (\dim_F H_i(X; F))\, t^i + \sum_i (\dim_F H_i(Y; F))\, t^i - 1(t^0) \\
&= p_X(t) + p_Y(t) - 1
\end{aligned}$$

Now we need to consider the homology of the spaces S^n, $\mathbb{RP}^n$, $\mathbb{RP}^\infty$, $\mathbb{CP}^n$, $\mathbb{CP}^\infty$, and M_g, the orientable surface of genus g. From the universal coeficient theorem there is a short exact sequence,

$$0 \longrightarrow H_n(X) \otimes_{\mathbb{Z}} F \longrightarrow H_n(X;F) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}\left(H_{n-1}(X), F\right) \longrightarrow 0$$

However, Tor above 0 of a field vanishes so we get an isomorphism,

$$H_n(X;F) \cong H_n(X) \otimes_{\mathbb{Z}} F$$

However, we have calculated in class the homology with coeficients in $\mathbb{Z}$ for each of these spaces,

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = n, 0 \\ 0 & i \neq n, 0 \end{cases}$$

Therefore,

$$H_i(S^n; F) = \begin{cases} F & i = n, 0 \\ 0 & i \neq n, 0 \end{cases}$$

and thus,

$$p_{S^n}(t) = 1 + t^n$$

Furthermore,

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & i \text{ odd } 0 < i < n \\ 0 & \text{else} \end{cases}$$

However, $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} F = 0$ so,

$$H_i(\mathbb{RP}^n; F) = \begin{cases} F & i = 0 \\ F & i = n \text{ and } n \text{ is odd} \\ 0 & i \text{else} \end{cases}$$

and thus,

$$p_{\mathbb{RP}^n} = \begin{cases} 1 + t^n & n \text{ is odd} \\ 1 & \end{cases}$$

Furthermore, in the case of $n = \infty$ the homology is the same except with no upper bound,

$$H_i(\mathbb{RP}^\infty; F) = \begin{cases} F & i = 0 \\ 0 & i > 0 \end{cases}$$

and thus $p_{\mathbb{RP}^\infty} = 1$. Next, in class we used cellular homology to calculate,

$$H_i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & i \text{ even } 0 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

Since $\mathbb{Z} \otimes_{\mathbb{Z}} F \cong F$ we get,

$$p_{\mathbb{CP}^n}(t) = 1 + t^2 + t^4 + \cdots + t^{2n}$$

As before, the homology of the infinite-dimensional complex projective plane is the same except without an upper bound,

$$H_i(\mathbb{CP}^\infty; F) = \begin{cases} F & i \text{ even} \\ 0 & \text{else} \end{cases}$$

and thus,

$$p_{\mathbb{CP}^\infty}(t) = 1 + t^2 + t^4 + \cdots = \frac{1}{1-t^2}$$

Finally, the orientable surface of genus g denoted by M_g has homology with coeficients in $\mathbb{Z}$ given by,

$$H_i(M_g) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & \text{else} \end{cases}$$

Since $\mathbb{Z}^{2g} \otimes_{\mathbb{Z}} F = F^{2g}$ we have,

$$p_{M_g}(t) = 1 + 2gt + t^2$$

Problem 5.

Let F be a field. Then, $\mathrm{Tor}_1^F = 0$ so the Künneth formula givse a natural isomorphism,

$$H_n(X \times Y; F) \cong \bigoplus_{p+q=n} H_p(X; F) \otimes_F H_q(Y; F)$$

We know that the dimension of the tensor product of vectorspaces is the product of dimensions. Thus,

$$\dim_F \left(H_p(X; F) \otimes_F H_q(Y; F)\right) = \dim_F \left(H_p(X; F)\right) \cdot \dim_F \left(H_q(Y; F)\right)$$

Using the definition of the Poincere sequence,

$$\begin{aligned} p_{X \times Y}(t) &= \sum_i \left(\dim_F H_i(X \times Y; F)\right) t^i \\ &= \sum_i \sum_{p+q=i} \left(\dim_F \left(H_p(X; F) \otimes_F H_q(Y; F)\right)\right) t^i \\ &= \sum_i \sum_{p+q=i} \left[\dim_F \left(H_p(X; F)\right) \cdot \dim_F \left(H_q(Y; F)\right)\right] t^i \\ &= \left(\sum_p \dim_F \left(H_p(X; F)\right) t^p\right) \left(\sum_q \dim_F \left(H_q(Y; F)\right) t^q\right) \\ &= p_X(t) \cdot p_Y(t) \end{aligned}$$