Math 56: Proofs and Modern Mathematics
Homework 3 Solutions

Naomi Kraushar

October 18, 2021

Problem 1 (Cf. Axler 1.C.10.)**.** Suppose that U_1, U_2 are subspaces of a vector space V. Show that the intersection $U_1 \cap U_2$ is also a subspace of V.

Solution. Since U_1, U_2 are subspaces of V, their intersection is a subset of V. To prove that a subset is a subspace, we need to prove three things: it must contain 0, be closed under addition, and be closed under scalar multiplication.

<u>Contains 0:</u> Since U_1 and U_2 are subspaces of V, both contain the 0 vector in V. Hence their intersection $U_1 \cap U_2$ also contains 0.

<u>Closed under addition:</u> Let u, v be elements of $U_1 \cap U_2$. This means that $u, v \in U_1$ and $u, v \in U_2$. Since U_1 and U_2 are subspaces of V, they are closed under addition, so $u + v \in U_1$ and $u + v \in U_2$. Hence $u + v \in U_1 \cap U_2$, so it is closed under addition.

<u>Closed under scalar multiplication:</u> Let v be an element of $U_1 \cap U_2$, and let λ be a scalar in the ground field. We have $v \in U_1$ and $v \in U_2$, and since U_1 and U_2 are subspaces of V, they are closed under scalar multiplication, so we have $\lambda v \in U_1$ and $\lambda v \in U_2$. Hence $\lambda v \in U_1 \cap U_2$, so it is closed under scalar multiplication. Having proved all three necessary properties, we conclude that $U_1 \cap U_2$ is a subspace of V.

Problem 2 (Cf. Axler 1.C.12.)**.** Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution. Let the two subspaces be U_1 and U_2. Suppose first that one is contained within the other. If $U_1 \subset U_2$, then $U_1 \cup U_2 = U_2$, which is a subspace by assumption. Similarly, if $U_2 \subset U_1$, then $U_1 \cup U_2 = U_1$, which again is a subspace. Hence if one is contained within the other, their union is a subspace.

We now prove the reverse direction. Suppose that neither is contained within the other, so we can find $u_1 \in U_1$, $u_2 \in U_2$ such that $u_1 \notin U_2$, $u_2 \notin U_1$. This means that $u_1, u_2 \in U_1 \cup U_2$; we prove that $u_1 + u_2 \notin U_1 \cap U_2$. Suppose $u_1 + u_2 \in U_1$. Since U_1 is a subspace and $u_1 \in U_1$, we have $-u_1 = (-1)u_1 \in U_1$, so that $(u_1 + u_2) - u_1 = u_2 \in U_1$, which is a contradiction. Similarly, if $u_1 + u_2 \in U_2$, we have $-u_2 \in U_2$, so $(u_1 + u_2) - u_2 = u_1 \in U_2$, which is again a contradiction. Hence $u_1 + u_2$ is in neither U_1 nor U_2, and so is not in $U_1 \cup U_2$, which means that $U_1 \cup U_2$ is not closed under addition and so is not a subspace. Taking the contrapositive, this means that if $U_1 \cup U_2$ is a subspace, one must be contained within the other.

Problem 3 (Cf. Axler 1.C.24.). A function $f : \mathbb{R} \to \mathbb{R}$ is called even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$, and U_o the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Solution. To prove that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$, we need to prove three things: that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$, that $\mathbb{R}^{\mathbb{R}} = U_e + U_o$, and that $U_e \cap U_o = 0$.

$\underline{U_e, U_o \text{ are subspaces}}$: We will start with U_e. First, let $z \in \mathbb{R}^{\mathbb{R}}$ be the function where $z(x) = 0$ for all x. We then have $z(-x) = 0 = z(x)$ for all x, so the zero function is in U_e. Second, let f, g be even functions. Then for all $x \in \mathbb{R}$, we have $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f + g$ is also even, hence U_e is also closed under addition. Finally, let f be an even function, and $\lambda \in \mathbb{R}$ a scalar. Then for all $x \in \mathbb{R}$, we have $(\lambda f)(-x) = \lambda f(-x) = \lambda f(x) = (\lambda f)(x)$, so λf is even, hence U_e is also closed under scalar multiplication, and so is a subspace of $\mathbb{R}^{\mathbb{R}}$.

The proof for U_o is virtually identical. First, let $z \in \mathbb{R}^{\mathbb{R}}$ be the function where $z(x) = 0$ for all x. We then have $z(-x) = 0 = -0 = -z(x)$ for all x, so the zero function is in U_o. Second, let f, g be odd functions. Then for all $x \in \mathbb{R}$, we have $(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x)$, so $f+g$ is also odd, hence U_o is also closed under addition. Finally, let f be an odd function, and $\lambda \in \mathbb{R}$ a scalar. Then for all $x \in \mathbb{R}$, we have $(\lambda f)(-x) = \lambda f(-x) = \lambda(-f(x)) = -\lambda f(x) = (\lambda f)(x)$, so λf is odd, hence U_o is also closed under scalar multiplication, and so is a subspace of $\mathbb{R}^{\mathbb{R}}$.

$\underline{\mathbb{R}^{\mathbb{R}} = U_e + U_o}$: What this means is that every element in $\mathbb{R}^{\mathbb{R}}$ can be written as a sum of an element in U_e and an element in U_o. Let f be an arbitrary function in $\mathbb{R}^{\mathbb{R}}$. Define the new functions f_e, f_o by

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \qquad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

We have

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{2f(x)}{2} = f(x),$$

so $f = f_e + f_o$. We now show that f_e is even and f_o is odd: we have

$$f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x),$$

so f_e is indeed even, and

$$f_o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x),$$

so f_o is indeed odd. Hence every function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an even function and an odd function; in other words, $\mathbb{R}^{\mathbb{R}} = U_e + U_o$.

$\underline{U_e \cap U_o = \{0\}}$: Finally, we need to prove that the intersection of these two subspaces contains only the 0 element (recall that this is equivalent to the sum we found above being *unique*). Let f be a function in $U_e \cap U_o$, so that f is both even and odd. This means that $f(-x) = f(x)$ and $f(-x) = -f(x)$ for all x, so we have $f(-x) = -f(-x)$ for all x.

Rearranging this equation gives $2f(-x) = 0$ for all x, so $f(-x) = 0$ for all x, so f is indeed the 0 function, and $U_e \cap U_o = \{0\}$.

Having proven all the necessary conditions, we conclude that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Problem 4. Let V be the real vector space of continuous functions $f : [0,1] \to \mathbb{R}$. Show that $U = \{f \in V : \int_0^1 f(x)dx = 0\}$ is a subspace of V.

Solution. To prove that a subset is a subspace, we need to prove three things: it must contain 0, be closed under addition, and be closed under scalar multiplication. Contains 0: The 0 element in the vector space V is the function $z : [0,1] \to \mathbb{R}$ where $z(x) = 0$ for all x. We have

$$\int_0^1 z(x)dx = \int_0^1 0dx = 0,$$

so this is indeed in U.

Closed under addition: Let f, g be two functions in U, so $\int_0^1 f(x)dx = 0$ and $\int_0^1 g(x)dx = 0$. We then have

$$\int_0^1 (f+g)(x)dx = \int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = 0 + 0 = 0,$$

so $f + g \in U$. Hence U is closed under addition.

Closed under scalar multiplication: Let f be a function in U, so $\int_0^1 f(x)dx = 0$, and let λ a scalar in the ground field $\mathbb{R}$. We then have

$$\int_0^1 (\lambda f)(x)dx = \int_0^1 \lambda f(x)dx = \lambda \int_0^1 f(x)dx = \lambda \cdot 0 = 0,$$

so $\lambda f \in U$. Hence U is closed under scalar multiplication.

Having proved all three necessary properties, we conclude that U is a subspace of V.

Solution. This is TRUE. One way we can find such a basis is using the previous problem. The standard basis for $\mathcal{P}_4(\mathbb{F})$ is $1, x, x^2, x^3$. Let $v_1 = x^2, v_2 = x^3, v_3 = 1, v_4 = x$. By the previous problem, $x^2 + x^3, x^3 + 1, 1 + x, x$ is also a basis for $\mathcal{P}_3(\mathbb{F})$, and none of these has degree 2.

Note: this is not the only possible basis where none of the polynomials has degree 2, e.g. another possibility is $1, x, x^2 + x^3, x^3$.

Problem 5 (Axler 2.A.11)**.** Suppose that $v_1, \ldots, v_m$ are linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ are linearly independent if and only if $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

Solution. Suppose first that $w \notin \operatorname{span}(v_1, \ldots, v_m)$; we want to show that $v_1, \ldots, v_m, w$ are linearly independent. Consider the equation

$$a_1 v_1 + \cdots + a_m v_m + bw = 0.$$

If $b \neq 0$, we can divide by b and rearrange to get

$$w = \frac{a_1}{b} v_1 + \cdots + \frac{a_m}{b} v_m.$$

Hence $w \in \operatorname{span}(v_1, \ldots, v_m)$, which contradicts our initial assumption, so b must be 0, and we are left with the equation $a_1 v_1 + \cdots + a_m v_m = 0$. Since $v_1, \ldots, v_m$ are linearly independent, we have $a_1 = 0, \ldots, a_m = 0$. Hence all our coefficients must be 0 and $v_1, \ldots, v_m, w$ are linearly independent.

Conversely, suppose that $w \in \operatorname{span}(v_1, \ldots, v_m)$, so $w = a_1 v_1 + \cdots + a_m v_m$ for some scalars $a_1, \ldots, a_m$. This means that we have

$$a_1 v_1 + \cdots + a_m v_m - w = 0,$$

so that $v_1, \ldots, v_m, w$ are not linearly independent, since the coefficient of w here is $1 \neq 0$.