Mathematics GU4044 Representations of Finite Groups
Assignment # 6

Benjamin Church

May 9, 2018

Problem 1.

We know that the character of the dual representation satisfies $\chi_{V*} = \overline{\chi_V}$. Therefore,

$$\langle \chi_{V*}, \chi_{V*} \rangle = \langle \overline{\chi_V}, \overline{\chi_V} \rangle = \langle \chi_V, \chi_V \rangle$$

However, a representation is irreducible if and only if $\langle \chi_W, \chi_W \rangle = 1$. Therefore, V^* is irreducible if and only if V is irreducible.

Problem 2.

Suppose G acts on X doubly transitively with $\#(X) = n$. Now consider the action of G on P, the set of ordered distinct pairs of elements in X. By definition, this action must be transitive and thus there is one orbit of size $\#(P) = n^2 - n$. By orbit-stabilizer, $\#(G) = \#(\text{Orb}(x))\#(\text{Stab}(x))$ and thus $\#(P) \mid \#(G)$. Therefore, $n^2 - n \mid \#(G)$. However, the order of G must be positive so $\#(G) \geq n^2 - n$.

Problem 3.

Let G be a nonabelian group of order 6. We know that the number of conjugacy classes is equal to the number of irreducible representation of G. Furthermore, $\sum\limits_{i=1}^{h} d_i^2 = \#(G) = 6$ where d_i is the dimension of the i^{th} irreducible representation. Since G is nonabelian, we cannot have $d_i = 1$ for all i. Therefore, at least one $d_i > 1$. However, $3^2 > 6$ so there must be exactly one 2-dimensional representation. Thus, up to order,$d_1 = 2$ which forces $d_2 = 1$ and $d_3 = 1$ so that $d_1^2 + d_2^2 + d_3^2 = 6$. Thus, there are three irreducible representations and thus three conjugacy classes.

Let G be a nonabelian group of order 8. We know that the number of conjugacy classes is equal to the number of irreducible representation of G. Furthermore, $\sum\limits_{i=1}^{h} d_i^2 = \#(G) = 8$ where d_i is the dimension of the i^{th} irreducible representation. Since G is nonabelian, we cannot have $d_i = 1$ for all i. Therefore, at least one $d_i > 1$. However, $3^2 > 8$ so there must be exactly one 2-dimensional representation. Thus, up to order,$d_1 = 2$. However, the trivial representation is always irredcible so take $d_2 = 1$. Since $8 - d_1^2 - d_2^2 = 3$ the rest of the sum is forced to be $d_3 = 1$, $d_4 = 1$ and $d_5 = 1$ so that $d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8$. Thus, there are five irreducible representations and thus five conjugacy classes.

Problem 4.

Last week we showed that A_4 has exactly 4 irreducible representations. Therefore, A_4 has exactly 4 conjugacy classes. Using the results from last week, we quote the character table,

	e	(1 2 3)	(1 3 2)	(1 2)·(3 4)
χ_0	1	1	1	1
χ_1	1	ζ_3	ζ_3^2	1
χ_2	1	ζ_3^2	ζ_3	1
χ_W	3	0	0	-1

The columns are all clearly orthogonal since $1+\zeta_3+\zeta_3^2=0$ and thus, $1+\zeta_3\overline{(\zeta_3)^2}+\zeta_3^2\overline{\zeta_3}=1+\zeta_3^2+\zeta_3$.

Problem 5.

(a). As given, $\operatorname{Hom}(W_2, W_2) \cong W_2 \otimes W_2$ and thus $\chi_{\operatorname{Hom}(W_2,W_2)} = \chi_{W_2} \cdot \chi_{W_2}$. We can find the character χ_{W_2} by looking at the number of fixed points of each element in S_4. Thus,

$$\chi^2_{W_2}(e) = (4-1)^3 = 9 \quad \chi^2_{W_2}((1\,2)) = 1 \quad \chi_{W_2}((1\,2\,3)) = 0$$
$$\chi_{W_2}((1\,2)(3\,4)) = (-1)^2 \quad \chi^2_{W_2}((1\,2\,3\,4)) = (-1)^2 = 1$$

This exhausts all cycle types and therefore all the conjugacy classes. Therefore,

$$\langle \chi_{W^2}, 1\rangle = \frac{1}{24}\left[1\cdot 9 + 6\cdot 1 + 8\cdot 0 + 3\cdot 1 + 6\cdot 1\right] = 1$$

Similarly, we know that $\langle \chi_V, 1\rangle = \dim V^G$ and thus,

$$\left\langle \chi_{\operatorname{Hom}(W_2,W_2)}, 1\right\rangle = \dim\left(\operatorname{Hom}(W_2,W_2)\right)^G = \dim \operatorname{Hom}^G(W_2,W_2)$$

Since W_2 is irreducible, by Schur's Lemma, $\dim \operatorname{Hom}^G(W_2, W_2) = 1$. Therefore,

$$\langle \chi_{W_2}, \chi_{W_2}\rangle = \left\langle \chi_{\operatorname{Hom}(W_2,W_2)}, 1\right\rangle = 1$$

(b). Now,

$$\langle \chi_{W^2}, \chi_{W^2}\rangle = \frac{1}{24}\left[1\cdot 9^2 + 6\cdot 1^2 + 8\cdot 0^2 + 3\cdot 1^2 + 6\cdot 1^2\right] = 4$$

There are five cycle types of S_4 and therefore five irreducible representations of S_4. From part (a), we know that $\left\langle \chi^2_{W_2}, 1\right\rangle = 1$ which implies that the trivial representation is a summand of $\operatorname{Hom}(W_2, W_2)$ with multiplicity 1. Furthermore, $\left\langle \chi^2_{W_2}, \chi^2_{W_2}\right\rangle = 4 = \sum_{i=1}^r m_i^2$. But we know that $m_1 = 1$ for the trivial representation so $m_1 = m_2 = m_3 = m_4 = 1$. Therefore, $\operatorname{Hom}(W_2, W_2)$ is the sum of exactly four distinct irreducible representations each with multiplicity 1. We know that for the trivial representation $d_1 = 1$. There cannot be any other one-dimensional representation in the sum. If there were another one-dimemsional representation V in the decomposition, then,

$$\left\langle \chi^2_{W_2}, \chi_V\right\rangle \geq 1$$

However, $\chi_{W_2}^2$ is positive and the real part of χ_V must be less than 1 for some values of g for χ_V to not be the trivial homomorphism. Therefore,

$$\langle \chi_{W_2}^2, \chi_V \rangle < \langle \chi_{W_2}^2, 1 \rangle = 1$$

which is a contradiction. Therefore, the is a unique one-dimensional representation. By dimension counting, the remaining three representations mut give $3^2 - 1 = 8$ dimensions. The only way this is possible using irreducible representations of S_4 which are at most three dimensional is to have one two-dimensional representation and two three-dimensional representations in the decomposition of $\mathrm{Hom}\,(W_2, W_2)$.

(c). First compute the inner product,

$$\langle \chi_{W_2}^2, \chi_{W_2} \rangle = \frac{1}{24}\left[3^3 + 6 \cdot 1^3 + 8 \cdot 0^3 + 3 \cdot (-1) + 6 \cdot (-1)^3\right] = 1$$

Therefore, W_2 appears with multiplicity 1 in the decomposition of $\mathrm{Hom}\,(W_2, W_2)$. Similarly, consider the character of $\epsilon \otimes W_2$,

$$\langle \chi_{W_2}^2, \epsilon \otimes \chi_{W_2} \rangle = \frac{1}{24}\left[3^3 - 6 \cdot 1^3 + 8 \cdot 0^3 + 3 \cdot (-1) - 6 \cdot (-1)^3\right] = 1$$

Therefore, $\epsilon \otimes W_2$ appears with multiplicity 1 in the decomposition of $\mathrm{Hom}\,(W_2, W_2)$. Because we know there are four irreducible representations each with multiplicity one in the decomposition of $\mathrm{Hom}\,(W_2, W_2)$, we may write,

$$\mathrm{Hom}\,(W_2, W_2) = \mathbb{C}(1) \oplus W \oplus W_2 \oplus (\epsilon \otimes W_2)$$

where W is a yet to be determined S_4-representation. By dimension counting,

$$\dim W = \dim \mathrm{Hom}\,(W_2, W_2) - \dim W_2 - \dim \epsilon \otimes W_2 - \dim \mathbb{C}(1) = 3^2 - 3 - 3 - 1 = 2$$

From general theory, we know there is a unique S_4-representation with dimension 2 which therefore must be W.

Problem 6.

(a). Let S_2 act on $V^{\otimes 2}$ by premuting tensor products. The character of any permutation action is given by the number of fixed points. e fixes everything so $\chi_{V^{\otimes 2}}(e) = \dim V^{\otimes 2} = d^2$. However, the flip, $(1\,2)$ only fixes elements of the form $v \otimes v$. This subspace is canonically isomorphic to V so $\chi_{V^{\otimes 2}}((1\,2)) = \dim V = d$.

If we write $\chi_{V^{\otimes 2}} = A \cdot 1 + B \cdot \varepsilon$ then $\chi_{V^{\otimes 2}}(e) = A + B = d^2$ and $\chi_{V^{\otimes 2}}((1\,2)) = A - B = d$. Therefore, $A = \frac{1}{2}(d^2 + d)$ and $B = \frac{1}{2}(d^2 - d)$. These are the multiplicities of the one-dimensional representations, $\mathbb{C}(1)$ and $\mathbb{C}(\epsilon)$ when we write,

$$V^{\otimes 2} \cong \mathbb{C}(1)^A \oplus \mathbb{C}(\lambda)^B$$

(b). Now let S_3 act on $V^{\otimes 3}$ by premuting tensor products. The character of any permutation action is given by the number of fixed points. e fixes everything so $\chi_{V^{\otimes 3}}(e) = \dim V^{\otimes 3} = d^3$. However, the flip, $(1\,2)$ only fixes elements of the form $v \otimes v \otimes w$. This subspace is canonically isomorphic to $V \otimes W$ so $\chi_{V^{\otimes 3}}((1\,2)) = \dim V \otimes W = d^2$. Furthermore, the thee-cycle $(1\,2\,3)$ only fixes elements of the form $v \otimes v \otimes v$ which is a subspace canonically isomorphic to V. Therefore, $\chi_{V^{\otimes 2}}((1\,2\,3)) = \dim V = d$. This fixes the character on all conjugacy classes.

Write $\chi_{V^{\otimes 3}} = A \cdot 1 + B \cdot \varepsilon + C \cdot \chi_{W_2}$ where W_2 is the irreducible two-dimensional permutation representation of S_3. Then, we know,

$$\begin{aligned}\chi_{V^{\otimes 3}}(e) &= d^3 = A + B + 2C \\ \chi_{V^{\otimes 3}}((1\,2)) &= d^2 = A - B \\ \chi_{V^{\otimes 3}}((1\,2\,3)) &= d = A + B - C\end{aligned}$$

Therefore,

$$A = \tfrac{1}{6}(d^3 + 3d^2 + 2d) \quad B = \tfrac{1}{6}(d^3 - 3d^2 + 2d) \quad C = \tfrac{1}{3}(d^3 - d)$$

Therefore, if we write,

$$V^{\otimes 3} = \mathbb{C}(1)^A \oplus \mathbb{C}(\epsilon)^B \oplus W_2^C$$

Then we have found the multiplicities,

$$A = \tfrac{1}{6}(d^3 + 3d^2 + 2d) \quad B = \tfrac{1}{6}(d^3 - 3d^2 + 2d) \quad C = \tfrac{1}{3}(d^3 - d)$$

(c). Now, let S_n act on $V^{\otimes n}$. We have seen that the character of an $\sigma \in S_n$ element of S_n is d^f where f is the dimension of the subspace fixed by g. Let $g = \gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_t$ be the product of disjoint cycles. The number of fixed points of a cycle of length is ℓ_i is $n - \ell_i$ and thus the dimension of the fixed subspace is $n + 1 - \ell_i$. To be a fixed point of g, a vector must be fixed by every cycle γ_i. Each cycle subtracts a factor of $\ell_i - 1$. Therefore, the character of σ is,

$$\chi_{V^{\otimes n}}(\sigma) = d^{n+t-\sum_{i=1}^{t} g_i}$$