# 1 Definitions

**Definition 1.0.1.** A *pre-log structure* on $X$ is a sheaf of commutative monoids $\mathcal{M}$ on $X_{\text{ét}}$ along with a morphism of sheaves of monoids,

$$\exp : \mathcal{M} \to (\mathcal{O}_X, \times)$$

A morphism of pre-log structues $\alpha : (\mathcal{M}_1, \exp_1) \to (\mathcal{M}_2, \exp_2)$ is a morphism of sheaves of monoids $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ such that the diagram,

$$\begin{tikzcd} \mathcal{M}_1 \arrow[rr, "\alpha"] \arrow[dr, "\exp_1"'] & & \mathcal{M}_2 \arrow[dl, "\exp_2"] \\ & \mathcal{O}_X & \end{tikzcd}$$

commutes.

*Remark.* From now on, when we write $\mathcal{O}_X$ in the category of monoids we mean $(\mathcal{O}_X, \times)$.

**Example 1.0.2.** Some examples,

(a) $\mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$

(b) $\mathcal{O}_X \to \mathcal{O}_X$

**Example 1.0.3.** Suppose that $P$ is a commutative monoid and $\exp : P \to \Gamma(W, \mathcal{O}_W)$ is a map then we get a pre-log structure $\exp : \underline{P} \to \mathcal{O}_W$ by adjunction.

For a monoid $P$ let $X_P = \text{Spec}\,(k[P])$ then we see,

$$\text{Hom}\,(W, X_P) = \{\text{pre-log structures } \underline{P} \to \mathcal{O}_W\}$$

**Definition 1.0.4.** A pre-log structure $\exp : \mathcal{M} \to \mathcal{O}_X$ is called a *log structure* if $\exp^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism.

*Remark.* In particular, there is a unique morphism $\alpha : \mathcal{O}_X^\times \to \mathcal{M}$ making the diagram,

$$\begin{tikzcd} \mathcal{O}_X^\times \arrow[r, "\alpha"] \arrow[dr, hook] & \mathcal{M} \arrow[d, "\exp"] \\ & \mathcal{O}_X \end{tikzcd}$$

commute so all log-structures "lie between" $\mathcal{O}_X^\times$ and $\mathcal{O}_X$ in the sense that $\mathcal{O}_X^\times$ is the initial object and $\mathcal{O}_X$ the terminal object of the categories of log structures.

**Definition 1.0.5.** A *log scheme* $(X, \mathcal{M})$ is a scheme $X$ equipped with a log structure $\mathcal{M}$.

**Proposition 1.0.6.** Let $Z \subset X$ be a closed subset. Let $\mathcal{M}_Z \subset \mathcal{O}_X$ be the subsheaf of functions invertible on $U = X \setminus Z$. Then $\mathcal{M}_Z$ is a log structure.

*Proof.* This just says that if $f \in \mathcal{O}_X(V)$ then it is invertible on $U$ which is obvious by restriction. $\square$

## 1.1 Logification

**Proposition 1.1.1.** There is a left-adjoint $\mathcal{M} \mapsto \mathcal{M}^{\log}$ to the forgetful functor $\{\log_X\} \to \{\text{pre-}\log_X\}$ given by,

$$\begin{tikzcd}
\exp^{-1}(\mathcal{O}_X^\times) \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \arrow[ddr, bend left, "\exp"] & \\
\mathcal{O}_X^\times \arrow[r] \arrow[drr, bend right] & \mathcal{M}^{\log} \arrow[dr, dashed, "\exp'"] \arrow[ul, phantom, "\ulcorner"] & \\
& & \mathcal{O}_X
\end{tikzcd}$$

*Proof.* (DO THIS!!!) $\square$

**Definition 1.1.2.** Given a morphism $\pi : X \to Y$ and a pre-log structure $\exp : \mathcal{M} \to \mathcal{O}_Y$ on $Y$. Then the pullback is $\pi^{-1}\mathcal{M} \to \pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. If $\mathcal{M}$ is a log structure on $Y$ then the pullback is,

$$\pi^\flat \mathcal{M} = (\pi^{-1}\mathcal{M})^{\log}$$

By the universal property there is a map,

$$\begin{tikzcd}
\pi^{-1}\mathcal{M} \arrow[rr] \arrow[dr] & & \mathcal{O}_X \\
& \pi^\flat \mathcal{M} \arrow[ur] &
\end{tikzcd}$$

*Remark.* There is a tautological log structure $\underline{P}^{\log} \to \mathcal{M}_{X_P}$. In the toric case $P = \sigma^\vee \cap M$ we can define $Z = X_\sigma \setminus T$ where $T = \text{Spec}\,(k[M])$ is the torus then I claim that $\underline{P}^{\log} = \mathcal{M}_Z$. (PROVE THIS!!)

*Remark.* Consider a morphism $W \to X_P$ then the log structure $\underline{P}^{\log} \to \mathcal{O}_W$ is $\pi^\flat \mathcal{M}_P$. (PROVE TIS!!)

**Proposition 1.1.3.** Let $\pi : X \to Y$ be a morphism. Then the following diagram commutes,

$$\begin{tikzcd}
\{\text{pre-}\log_Y\} \arrow[r, "\pi^{-1}"] \arrow[d, "\log"] & \{\text{pre-}\log_X\} \arrow[d, "\log"] \\
\{\log_Y\} \arrow[r, "\pi^\flat"] & \{\log_X\}
\end{tikzcd}$$

*Proof.* IS THIS ACTUALLY TRUE?? $\square$

**Definition 1.1.4.** A morphism $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ of *log schemes* is a morphism of schemes $f : X \to Y$ and a morphism of sheaves of monoids $\alpha$ such that,

$$\begin{tikzcd}
\mathcal{N} \arrow[r, "\alpha"] \arrow[d] & \pi_* \mathcal{M} \arrow[d] \\
\mathcal{O}_Y \arrow[r] & \pi_* \mathcal{O}_X
\end{tikzcd}$$

commutes. Equivalently this is a morphism of log structures $\alpha : \pi^\flat \mathcal{N} \to \mathcal{M}$ because then automatically the diagram,

$$\begin{tikzcd}
\pi^{-1}\mathcal{N} \arrow[r] \arrow[d] & \pi^{\flat}\mathcal{N} \arrow[r] & \mathcal{M} \arrow[d] \\
\pi^{-1}\mathcal{O}_Y \arrow[rr] & & \mathcal{O}_X
\end{tikzcd}$$

commutes.