Mathematics GU4051 Topology
Assignment # 9

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Problem 1.

Let $f_0, f_1 : X \to Y$ be homotopic and $g_0, g_1 : Y \to Z$ be homotopic. Take homotopies for each pair of homotopic functions: $F : X \times I \to Y$ which satisfies $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ and is continuous and $G : X \times I \to Y$ which satisfies $G(y, 0) = g_0(y)$ and $G(y, 1) = g_1(y)$ and is continuous. Consider the map $H : X \times I \to Z$ given by $H(x, t) = G(F(x, t), t)$. Firstly, $H(x, 0) = G(F(x, 0), 0) = g_0(f_0(x)) = (g_0 \circ f_0)(x)$. Similarly, $H(x, 1) = G(F(x, 1), 1) = g_1(f_1(x)) = (g_1 \circ f_1)(x)$. To show that H is a homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$ we must show that H is continuous. The map,

$$f = (F, \pi_2) : X \times I \to Y \times I$$

is continuous because both $F : X \times I \to Y$ and $\pi_2 : X \times I \to I$ are continuous so $G \circ (F, \pi_2)$ is continuous. Also, $G \circ (F, \pi_2)(x, t) = G(F(x, t), \pi_2(x, t)) = G(F(x, t), t) = H(x, t)$ so H is continuous. The argument is summarized in the commutative diagram:

$$\begin{tikzcd}
& & Y & \\
X \times I \arrow[urr, "F"] \arrow[rr, dashed, "f"] \arrow[drr, "\pi_2"'] \arrow[rrr, dashed, bend right=60, "H"'] & & Y \times I \arrow[u, "\pi_1^Y"'] \arrow[d, "\pi_2^Y"] \arrow[r, "G"] & Z \\
& & I &
\end{tikzcd}$$

Therefore, H is a homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$.

Problem 2.

A note on notation: For $y_0 \in Y$, I will use $\langle y_0 \rangle_X : X \to Y$ to denote the contant map $\langle y_0 \rangle_X : x \mapsto y_0$.

(a). Let $L \subset \mathbb{R}$ be a nonempty interval. Take $x_0 \in L$ and $\mathrm{id}_L : L \to L$. Now, define $G : \mathbb{R}^2 \to \mathbb{R}$ by $G(x, y) = x_0 y + x(1 - y)$ which is continuous by analysis. Now, if $x \in L$ and $t \in [0, 1]$ then $x \le G(x, y) \le x_0$ or $x_0 \le G(x, y) \le x$ so by the interval property $G(x, y) \in L$. Thus, $F : L \times I \to L$ given by $F = G|_{L \times I}$ is a well defined continuous map. Also, $F(x, 0) = x = \mathrm{id}_L(x)$ and $F(x, 1) = x_0$ a constant map. Thus, $\mathrm{id}_L \sim \langle x_0 \rangle$ where $\langle x_0 \rangle$ represents the constant map $x \to x_0$ so L is contractable.

(b). Let X be contractable then there is a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = x_0$ for some $x_0 \in X$. Take any $x_1, x_2 \in X$. Define $\gamma : I \to X$ by,

$$\gamma(t) = \begin{cases} F(x_1, 2t) & t \leq \frac{1}{2} \\ F(x_2, 2 - 2t) & t \geq \frac{1}{2} \end{cases}$$

Because at $t = \frac{1}{2}$ we have $F(x_1, 2t) = F(x_1, 1) = x_0$ and $F(x_2, 2-2t) = F(x_2, 1) = x_0$ the path γ is continuous by the glueing lemma. Also, $\gamma(0) = F(x_1, 0) = x_1$ and $\gamma(1) = F(x_2, 2 - 2) = F(x_2, 0) = x_2$ so γ is a path from x_1 to x_2 and thus X is path connected.

Now, take any loop γ at x_0. Now, define,

$$G(x, t) = \begin{cases} F(x_0, 2xt) & x \leq \frac{1}{2} \\ F(\gamma(4x - 2), t) & \frac{1}{2} \leq x \leq \frac{3}{4} \\ F(x_0, (4 - 4x)t) & x \geq \frac{3}{4} \end{cases}$$

At $x = \frac{1}{2}$, $F(x_0, 2xt) = F(x_0, t)$ and $F(\gamma(4x - 2), t) = F(\gamma(0), t) = F(x_0, t)$. Similarly, at $x = \frac{3}{4}$, $F(\gamma(4x - 2), t) = F(\gamma(1), t) = F(x_0, t)$ and $F(x_0, (4 - 4x)t) = F(x_0, t)$. Therefore, by the glueing lemma, G is continuous. Also, let $\delta(x) = F(x_0, x)$ which is a loop at x_0 because $\delta(0) = F(x_0, 0) = x_0$ and $F(x_0, 1) = x_0$.

$$\begin{aligned} G(x, 0) &= \begin{cases} F(x_0, 0) = x_0 & x \leq \frac{1}{2} \\ F(\gamma(4x - 2), 0) = \gamma(4x - 2) & \frac{1}{2} \leq x \leq \frac{3}{4} \\ F(x_0, 0) = x_0 & x \geq \frac{3}{4} \end{cases} \\ &= (e_{x_0} * (\gamma * e_{x_0}))(x) \\ G(x, 1) &= \begin{cases} F(x_0, 2x) = \delta(2x) & x \leq \frac{1}{2} \\ F(\gamma(4x - 2), 1) = x_0 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ F(x_0, (4 - 4x)) = \delta(4 - 4x) & x \geq \frac{3}{4} \end{cases} \\ &= (\delta * (e_{x_0} * \delta^{-1}))(x) \\ G(0, t) &= \begin{cases} F(x_0, 0) & 0 \leq \frac{1}{2} \\ F(\gamma(-2), t) & \frac{1}{2} \leq 0 \leq \frac{3}{4} \\ F(x_0, (-4x)t) & 0 \geq \frac{3}{4} \end{cases} \\ &= F(x_0, 0) = x_0 \\ G(1, t) &= \begin{cases} F(x_0, 2t) & 1 \leq \frac{1}{2} \\ F(\gamma(2), t) & \frac{1}{2} \leq 1 \leq \frac{3}{4} \\ F(x_0, 0) & 1 \geq \frac{3}{4} \end{cases} \\ &= F(x_0, 0) = x_0 \end{aligned}$$

Therefore, $e_{x_0} * (\gamma * e_{x_0})$ and $\delta * (e_{x_0} * \delta^{-1})$ are path-homotopic so $[e_{x_0} * (\gamma * e_{x_0})] = [\delta * (e_{x_0} * \delta^{-1})]$ However, $[e_{x_0} * (\gamma * e_{x_0})] = [e_{x_0}] * [\gamma] * [e_{x_0}] = [\gamma]$ because $[e_{x_0}]$ is the identity of $\pi_1(X, x_0)$. Furthermore, $[\delta * (e_{x_0} * \delta^{-1})] = [\delta] * [e_{x_0}] * [\delta]^{-1} = [\delta] * [\delta]^{-1} = [e_{x_0}]$ because the reversed path generates the inverse homotopy class. Thus, $[\gamma] = [e_{x_0}]$ but γ was arbitrary so every element of $\pi_1(X, x_0)$ is the identity. Now, for any other base point $x \in X$ we know that $\pi_1(X, x) \cong \pi_1(X, x_0)$ with isomorphism induced by conjugation with a path from x_0 to x. Therefore, $\pi_1(X, x) \cong \pi_1(X, x_0) \cong \{e\}$.

(c). Let $f_0, f_1 : X \to Y$ be continuous and let Y contractable. Then there exists a homotopy $G : Y \times I \to I$ such that $G(y, 0) = y$ and $G(y, 1) = y_0$ for some $y_0 \in Y$. Define, $F : X \times I \to Y$ by,

$$F(x,t) = \begin{cases} G(f_0(x), 2t) & t \leq \frac{1}{2} \\ G(f_1(x), 2-2t) & t \geq \frac{1}{2} \end{cases}$$

First, $G(f_0(x), 2t)$ and $G(f_1(x), 2 - 2t)$ are continuous by composition of continuous functions. Now, because on the closed set $X \times \{\frac{1}{2}\}$, we have $G(f_0(x), 2t) = G(f_0(x), 1) = y_0$ and $G(f_1(x), 2 - 2t) = G(f_1(x), 1) = y_0$ then F is continuous by the glueing lemma. Also, $F(x, 0) = G(f_0(x), 0) = f_0(x)$ and $F(x, 1) = G(f_1(x), 0) = f_1(x)$ so F is a homotopy from f_0 to f_1 so $f_0 \sim f_1$.

An alternative proof goes as follows. Take continuous $f_0, f_1 : X \to Y$. Because Y is contractable, $\mathrm{id}_Y \sim \langle y_0 \rangle_Y$ where y_0 is some fixed point $y_0 \in Y$. Now, $f_0 \sim f_0$ so by the problem 1, we have $f_0 = \mathrm{id}_Y \circ f_0 \sim \langle y_0 \rangle_Y \circ f_0 = \langle y_0 \rangle_X$ by Lemma 0.1. Similarly, $f_1 \sim f_1$ so $f_1 = \mathrm{id}_Y \circ f_1 \sim \langle y_0 \rangle_Y \circ f_1 = \langle y_0 \rangle_X$. Thus, $f_0 \sim \langle y_0 \rangle_X$ and $f_1 \sim \langle y_0 \rangle_X$ so $f_0 \sim f_1$ by transitivity.

(d). Let $g_0, g_1 : X \to Y$ be continuous, let X be contractable, and let Y be path-connected. Because X is contractable, there exists a point $x_0 \in X$ such that $\mathrm{id}_X \sim \langle x_0 \rangle_X$. Then, because $g_0 \sim g_0$ we know that $g_0 = g_0 \circ \mathrm{id}_X \sim g_0 \circ \langle x_0 \rangle_X = \langle g_0(x_0) \rangle_X$ by Lemma 0.1. Similarly, $g_1 = g_1 \circ \mathrm{id}_X \sim g_1 \circ \langle x_0 \rangle_X = \langle g_1(x_0) \rangle_X$. However, because Y is path-connected, by Lemma 0.2, all constant functions are homotopic so $\langle g_0(x_0) \rangle_X \sim \langle g_1(x_0) \rangle_X$. Thus, by transitivity, $g_0 \sim \langle g_0(x_0) \rangle_X \sim \langle g_1(x_0) \rangle_X \sim g_1$ so g_0 and g_1 are homotopic.

Problem 3.

(a). Take $S \subset \mathbb{R}^2$ to be the axes, $S = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$. This set is star-convex because any point $P \in S$ lies either on the x-axis or the y-axis. Either way, the segment $\overline{PO} \subset S$, where $O = (0, 0)$ is the origin, because it is a subset of the corresponding axis. However, take $P = (1, 0)$ and $Q = (0, 1)$. Both $P, Q \in S$ but $\overline{PQ} \notin S$ because $(\frac{1}{2}, \frac{1}{2}) \notin S$ so S is nonconvex.

(b). Let $T \subset \mathbb{R}^2$ be the graph of a parabola, $T = \{(t, t^2) \mid t \in \mathbb{R}\}$. Then T is not star-convex because it contains no nontrivial line segements. However, T is contractable. Consider the map $G : \mathbb{R}^3 \to \mathbb{R}^2$ given by $G(x, y, z) = (x(1 - z), y(1 - z)^2)$ which is continuous by analysis. Also if $x^2 = y$ then $(x(1 - z))^2 = x^2(1 - z)^2 = y(1 - z)^2$ so $\operatorname{Im} G|_{T \times I} \subset T$. Therefore, the map $F : T \times I \to T$ given by $F((x, x^2), t) = G(x, x^2, t)$ is continuous. Also,

$$F((x,x^2),0) = G(x,x^2,0) = (x,x^2) \qquad F((x,x^2),1) = (x(1-1), x^2(1-1)) = (0,0)$$

so F is a homotopoy from id_T to the constant map from T to $(0, 0)$.

(c). Let $S \subset \mathbb{R}^n$ be star-convex. Therefore, $\exists \mathbf{x} \in S$ such that $\forall \mathbf{y} \in S$ the segment $\overline{\mathbf{xy}} \subset S$. Consider the function $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by $G(\mathbf{y}, t) = (1 - t)\mathbf{y} + t\mathbf{x}$. By analysis, G is continuous. Also, for $t \in [0,1]$ we have $G(\mathbf{y},t) \in \overline{\mathbf{xy}}$ so $G(\mathbf{y},t) \in S$. Thus, $\operatorname{Im} G|_{S\times I} \subset S$ so the funtion $F : S \times I \to S$ given by $F(\mathbf{y},t) = G(\mathbf{y},t)$ is continuous and well defined. Also, $F(\mathbf{y},0) = \mathbf{y} = \mathrm{id}_S(\mathbf{y})$ and $F(\mathbf{y},1) = \mathbf{x}$ which is a contant function. Thus, id_S is homotopic to the constant function mapping to $\mathbf{x}$. Therefore, S is contractable.

Problem 4.

For $S \subset X$ let $f : X \to S$ be a retraction. Take $x_0 \in S$ and any loop $\gamma : I \to S$ at x_0. Now we can lift the loop γ into the ambient space X simply by defining $\tilde{\gamma} : I \to X$ by $\tilde{\gamma}(t) = \gamma(t)$. Consider the homomorphism induced by the retraction, $f_* : \pi_1(X, x_0) \to \pi_1(S, f(x_0))$. However, $x_0 \in S$ and $f|_S = \mathrm{id}_S$ so $f(x_0) = x_0$. Thus, $f_* : \pi_1(X, x_0) \to \pi_1(S, x_0)$. Now, consider $f_*([\tilde{\gamma}]) = [f \circ \tilde{\gamma}]$ then we have, $f \circ \tilde{\gamma} : I \to S$ and $f(\tilde{\gamma}(t)) = f(\gamma(t)) = \gamma(t)$ because $\gamma(t) \in S$ and $f|_S = \mathrm{id}_S$. Thus, $f_*([\tilde{\gamma}]) = [\gamma]$. However, γ was an aribitrary loop at x_0 so the function f_* is surjective because the equivalence class of any loop is in the image.

Problem 5.

The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous and thus induce homomorphisms $f_1 : \pi_1(X \times Y, x \times y) \to \pi_1(X, x)$ and $f_2 : \pi_1(X \times Y, x \times y) \to \pi_1(Y, y)$ because $\pi_1(x \times y) = x$ and $\pi_2(x \times y) = y$. Using Lemma 0.3, define the homomorphism,

$$F : \pi_1(X \times Y, x \times y) \to \pi_1(X, x) \times \pi_2(Y, y)$$

by $F = (f_1, f_2)$. It remains to show that F is a bijection. Let

$$G : \pi_1(X, x) \times \pi_2(Y, y) \to \pi_1(X \times Y, x \times y)$$

be given by $G([\gamma], [\delta]) = [\Gamma]$ where $\Gamma = (\gamma, \delta) : I \to X \times Y$. Lemma 0.4 shows that this function maps loops to loops with the correct base points and is well defined on path-homotopy equivalence classes. Now,

$$G \circ F([\Gamma]) = G([\pi_1 \circ \Gamma], [\pi_2 \circ \Gamma]) = [(\pi_1 \circ \Gamma, \pi_2 \circ \Gamma)] = [\Gamma]$$

In the last line, I used the fact that, for any function $\gamma : I \to X \times Y$, the map $(\pi_1 \circ \gamma, \pi_2 \circ \gamma) = \gamma$. Also,

$$F \circ G([\gamma], [\delta]) = F([\Gamma]) = ([\pi_1 \circ \Gamma], [\pi_2 \circ \Gamma]) = ([\gamma], [\delta])$$

where I have used the fact that $\Gamma = (\gamma, \delta)$ so $\pi_1 \circ \Gamma = \gamma$ and $\pi_2 \circ \Gamma = \delta$. Therefore, G is the inverse function of F so F must be a bijection. Therefore, F is an isomorphism.

Problem 6.

(a). Let G be a topological group with a multiplcation function $M : G \times G \to G$ which takes $M(x, y) = x \cdot y$. Let $\gamma_1, \gamma_2 : I \to G$ be continuous loops based at e. Then, let $\gamma_1 \diamond \gamma_2 : I \to G$ be given by $(\gamma_1 \diamond \gamma_2)(t) = \gamma_1(t) \cdot \gamma_2(t)$. This is a loop at e because $(\gamma_1 \diamond \gamma_2)(0) = \gamma_1(0) \cdot \gamma_2(0) = e \cdot e = e$ and $(\gamma_1 \diamond \gamma_2)(1) = \gamma_1(1) \cdot \gamma_2(1) = e \cdot e = e$. This function is also continuous because, $f = (\gamma_1, \gamma_2)$ is continuous thus $\gamma_1 \diamond \gamma_2 = f \circ M$ is continuous by composition of continuous functions.

$$\begin{tikzcd}
 & G & \\
I \arrow[ur, "\gamma_1"] \arrow[r, dashed, "f"] \arrow[dr, "\gamma_2"'] \arrow[rr, dashed, bend right=60, "\gamma_1\diamond\gamma_2"'] & G \times G \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[r, "M"] & G \\
 & G &
\end{tikzcd}$$

Now,

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_2(2t) & t \leq \frac{1}{2} \\ \gamma_1(2t-1) & t \geq \frac{1}{2} \end{cases}$$

Let $f : [0, \frac{1}{2}] \times I \to I^2$ given by,

$$f(x,t) = (tx, (2-t)x)$$

which is continuous and let $g : [\frac{1}{2}, 1] \times I \to I^2$ be given by

$$g(x,y) = \big((2-t)x + t - 1, 1 + t(x-1)\big)$$

which is also continuous. Also, at $x = \frac{1}{2}$,

$$\begin{aligned} f(x,t) &= \left(\tfrac{1}{2}t, \tfrac{1}{2}(2-t)\right) \\ g(x,t) &= \left((2-t)\tfrac{1}{2} + t - 1, 1 + t(\tfrac{1}{2} - 1)\right) = \left(\tfrac{1}{2}t, \tfrac{1}{2}(2-t)\right) \end{aligned}$$

so by the glueing lemma, the function $R : I^2 \to I^2$ given by,

$$R(x,t) = \begin{cases} f(x,t) & x \leq \frac{1}{2} \\ g(x,t) & x \leq \frac{1}{2} \end{cases}$$

is continuous. Define, $F = M \circ (\gamma_1 \times \gamma_2) \circ R$ which is a well defined continuous map detailed in the commutative diagram below.

$$\begin{tikzcd}
 & & I \arrow[r, "\gamma_1"] & G & & \\
I \times I \arrow[r, "R"] \arrow[rrrrr, dashed, bend right=40, "F"'] & I \times I \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rrr, dashed, "\gamma_1 \times \gamma_2"] & & & G \times G \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[r, "M"] & G \\
 & & I \arrow[r, "\gamma_2"] & G & &
\end{tikzcd}$$

Thus, the function $F : I^2 \to G$ is given by,

$$F(x,t) = \begin{cases} \gamma_1(tx) \cdot \gamma_2((2-t)x) & x \leq \frac{1}{2} \\ \gamma_1((2-t)x + t - 1) \cdot \gamma_2(1 + t(x-1)) & x \geq \frac{1}{2} \end{cases}$$

Finally, using the fact that $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1) = e$ so products with these elements do nothing.

$$F(x,0) = \begin{cases} \gamma_1(0)\cdot\gamma_2(2x) & x \le \frac{1}{2} \\ \gamma_1(2x-1)\cdot\gamma_2(1) & x \ge \frac{1}{2} \end{cases} = \begin{cases} \gamma_2(2x) & x \le \frac{1}{2} \\ \gamma_1(2x-1) & x \ge \frac{1}{2} \end{cases} = (\gamma_1 * \gamma_2)(x)$$

$$F(x,1) = \begin{cases} \gamma_1(x)\cdot\gamma_2(x) & x \le \frac{1}{2} \\ \gamma_1(x)\cdot\gamma_2(x) & x \ge \frac{1}{2} \end{cases} = (\gamma_1 \diamond \gamma_2)(x)$$

$$F(0,t) = \begin{cases} \gamma_1(0)\cdot\gamma_2(0) & 0 \le \frac{1}{2} \\ \gamma_1(t-1)\cdot\gamma_2(1-t) & 0 \ge \frac{1}{2} \end{cases} = \gamma_1(0)\cdot\gamma_2(0) = e$$

$$F(1,t) = \begin{cases} \gamma_1(t)\cdot\gamma_2(2-t) & 1 \le \frac{1}{2} \\ \gamma_1(1)\cdot\gamma_2(1) & 1 \ge \frac{1}{2} \end{cases} = \gamma_1(1)\cdot\gamma_2(1) = e$$

Therefore, F is a path-homotopy from $\gamma_1 * \gamma_2$ to $\gamma_1 \diamond \gamma_2$. Therefore,

$$[\gamma_1] \diamond [\gamma_2] = [\gamma_1 \diamond \gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_1] * [\gamma_2]$$

Because $*$ is well-defined on equivalence classes we have that $\diamond$ is also a well-defined operation on equivalence classes and gives the same group structure.

(b). Let $\gamma_1, \gamma_2 : I \to G$ be loops at e. In an analogous fashion to part (a) but with the components of the output flipped, define $f : [0, \frac{1}{2}] \times I \to I^2$ given by,

$$f(x,t) = ((2-t)x, tx)$$

which is continuous and let $g : [\frac{1}{2}, 1] \times I \to I^2$ be given by

$$g(x,y) = \big(1 + t(x-1), (2-t)x + t - 1\big)$$

which is also continuous. Also, at $x = \frac{1}{2}$,

$$\begin{aligned} f(x,t) &= \big(\tfrac{1}{2}(2-t), \tfrac{1}{2}t\big) \\ g(x,t) &= \big(1 + t(\tfrac{1}{2}-1), (2-t)\tfrac{1}{2} + t - 1\big) = \big(\tfrac{1}{2}(2-t), \tfrac{1}{2}t\big) \end{aligned}$$

so by the glueing lemma, the function $R : I^2 \to I^2$ given by,

$$R(x,t) = \begin{cases} f(x,t) & x \le \frac{1}{2} \\ g(x,t) & x \le \frac{1}{2} \end{cases}$$

is continuous. Define, $F = M \circ (\gamma_1 \times \gamma_2) \circ G$ which is a well defined continuous map detailed in the commutative diagram below.

$$\begin{tikzcd}
 & & I \arrow[r, "\gamma_1"] & G & & \\
I \times I \arrow[r, "R"] \arrow[rrrrr, dashed, bend right, "F"] & I \times I \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rrr, dashed, "\gamma_1 \times \gamma_2"] & & & G \times G \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[r, "M"] & G \\
 & & I \arrow[r, "\gamma_2"] & G & &
\end{tikzcd}$$

Thus, the function $F : I^2 \to G$ is given by,

$$F(x,t) = \begin{cases} \gamma_1((2-t)x) \cdot \gamma_2(tx) & x \leq \frac{1}{2} \\ \gamma_1(1 + t(x-1)) \cdot \gamma_2((2-t)x + t - 1) & x \geq \frac{1}{2} \end{cases}$$

Finally, using the fact that $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1) = e$ so products with these elements do nothing.

$$F(x,0) = \begin{cases} \gamma_1(2x) \cdot \gamma_2(0) & x \leq \frac{1}{2} \\ \gamma_1(1) \cdot \gamma_2(2x-1) & x \geq \frac{1}{2} \end{cases} = \begin{cases} \gamma_1(2x) & x \leq \frac{1}{2} \\ \gamma_2(2x-1) & x \geq \frac{1}{2} \end{cases} = (\gamma_2 * \gamma_1)(x)$$

$$F(x,1) = \begin{cases} \gamma_1(x) \cdot \gamma_2(x) & x \leq \frac{1}{2} \\ \gamma_1(x) \cdot \gamma_2(x) & x \geq \frac{1}{2} \end{cases} = (\gamma_1 \diamond \gamma_2)(x)$$

$$F(0,t) = \begin{cases} \gamma_1(0) \cdot \gamma_2(0) & 0 \leq \frac{1}{2} \\ \gamma_1(1-t) \cdot \gamma_2(t-1) & 0 \geq \frac{1}{2} \end{cases} = \gamma_1(0) \cdot \gamma_2(0) = e$$

$$F(1,t) = \begin{cases} \gamma_1(2-t) \cdot \gamma_2(t) & 1 \leq \frac{1}{2} \\ \gamma_1(1) \cdot \gamma_2(1) & 1 \geq \frac{1}{2} \end{cases} = \gamma_1(1) \cdot \gamma_2(1) = e$$

Therefore, F is a path-homotopy from $\gamma_2 * \gamma_1$ to $\gamma_1 \diamond \gamma_2$.

(c). From the previous parts, $\gamma_1 \diamond \gamma_2 \sim \gamma_2 * \gamma_1$ and also, $\gamma_1 \diamond \gamma_2 \sim \gamma_1 * \gamma_2$ therefore, $\gamma_1 * \gamma_2 \sim \gamma_2 * \gamma_1$ by transitivity. Therefore, $[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_2 * \gamma_1] = [\gamma_2] * [\gamma_1]$ so the fundamental group $\pi_1(G, e)$ is abelian.

Lemmas

Lemma 0.1. *Let $g : X \to Y$ be any function, with $x_0 \in X$ and $y_0 \in Y$ then $\langle y_0 \rangle_Y \circ g = \langle y_0 \rangle_X$ and $g \circ \langle x_0 \rangle_X = \langle g(x_0) \rangle_X$*

Proof. For all $x \in X$ we have $(\langle y_0 \rangle_Y \circ g)(x) = \langle y_0 \rangle_Y (g(x)) = y_0$ thus $\langle y_0 \rangle_Y \circ g = \langle y_0 \rangle_Y$. Also, for any $x \in X$ we have $(g \circ \langle x_0 \rangle_X)(x) = g(x_0)$ thus $g \circ \langle x_0 \rangle_X = \langle g(x_0) \rangle_X$. □

Lemma 0.2. *If Y is path connected then any two constant functions from X to Y are homotopic.*

Proof. If X is empty than all functions from X are homotopic. Let X be nonempty, let $g_0, g_1 : X \to Y$ be constant then $g_0(X) = \{y_0\}$ and $g_1(X) = \{y_1\}$. Since Y is path connected, there exists a path $\gamma : I \to Y$ from $g_0(x_0)$ to $g_1(x_0)$. Define the function $G : X \times I \to Y$ by, $G = \gamma \circ \pi_2$ which is continuous as a composition of contiuous maps. Then, $G(x,0) = \gamma(0) = y_0 = g_0(x)$ and $G(x,1) = \gamma(0) = y_1 = g_1(x)$. Thus, G is a homotopy from g_0 to g_1. □

Lemma 0.3. *Let G, H_1, and H_2 be groups with homomorphisms $f_1 : G \to H_1$ and $f_2 : G \to H_2$ then there is a unique homomorphism $F : G \to H_1 \times H_2$ given by $F = (f_1, f_2)$ such that $\pi_1 \circ F = f_1$ and $\pi_2 \circ F = f_2$. In other words, the product $H_1 \times H_2$ satisfies the following universal property:*

$$\begin{tikzcd}
& & H_1 \\
G \arrow[urr, "f_1"] \arrow[rr, dashed, "F"] \arrow[drr, "f_2"'] & & H_1 \times H_2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
& & H_2
\end{tikzcd}$$

Proof. For $g, h \in G$, we have,

$$F(gh) = (f_1(gh), f_2(gh)) = (f_1(g)f_1(h), f_2(g)f_2(h)) = (f_1(g), f_2(g)) * (f_1(h), f_2(h)) = F(g) * F(h)$$

Thus, F is a homomorphism. Let $K : G \to H_1 \times H_2$ be any homomorphism satisfying $\pi_1 \circ K = f_1$ and $\pi_2 \circ K = f_2$ then for any $g \in G$ we have $K(g) \in H_1 \times H_2$ so $G(g) = (h_1, h_2)$ for $h_1 \in H_1$ and $h_2 \in H_2$ and $\pi_1 \circ K(g) = h_1 = f_1(g)$ and $\pi_2 \circ K(g) = h_2 = f_2(g)$ so $K(g) = (f_1(g), f_2(g)) = F(g)$. □

Lemma 0.4. *Let $\gamma_0, \gamma_1 : I \to X$ be path-homotopic loops at x_0 and let $\delta_0, \delta_1 : I \to Y$ be path-homotopic loops at y_0 then $\Gamma_0 = (\gamma_0, \delta_0) : I \to X \times Y$ and $\Gamma_1 = (\gamma_1, \delta_1) : I \to X \times Y$ are path-homotopic loops at (x_0, y_0).*

Proof. Becuase γ_0 and δ_0 are continuous, Γ_0 is also continuous. Γ_0 is a loop at (x_0, y_0) because $\Gamma_0(0) = (\gamma_0(0), \delta_0(0)) = (x_0, y_0)$ and $\Gamma_0(1) = (\gamma_0(1), \delta_0(1)) = (x_0, y_0)$. An identical argument shows that Γ_1 is a loop at (x_0, y_0). Take path-homotopies $F : I^2 \to X$ and $G : I^2 \to Y$ for $\gamma_0 \sim \gamma_1$ and $\delta_0 \sim \delta_1$ respectively. Now, consider $H = (F, G) : I^2 \to X \times Y$ which is continuous because F and G are continuous. Also,

$$\begin{aligned}
H(0,t) &= (F(0,t), G(0,t)) = (x_0, y_0) \\
H(1,t) &= (F(1,t), G(1,t)) = (x_0, y_0) \\
H(x,0) &= (F(x,0), G(x,0)) = (\gamma_0(x), \delta_0(x)) = \Gamma_0(x) \\
H(x,1) &= (F(x,1), G(x,1)) = (\gamma_1(x), \delta_1(x)) = \Gamma_1(x)
\end{aligned}$$

Therefore, H is a path-homotopy between Γ_0 and Γ_1. □