1 April 25

Remark. Here $k = \bar{k}$ but the theorem is still true if k is an arbitrary ring.

1.1 Elementary Symmetric Polynomials

Consider the "root map" $\Phi : \mathbb{A}^n \to \mathbb{A}^n$ which takes a tuple of roots $(\alpha_1, \dots, \alpha_n)$ to the coefficients (up to sign) of the monic polynomial,

$$(x - \alpha_1) \cdots (x - \alpha_n)$$

which are elementary symmetric polynomials $s_i(\alpha_1, \dots, \alpha_n)$. We want to show it is the quotient map of the permutation action $S_n \circlearrowright \mathbb{A}^n$. Notice that the action of $S_n \circlearrowright \mathbb{A}^n$ is not free at points with repeated coefficients which lie above polynomial having repeated roots. Nevertheless, its quotient is actually nonsingular and isomorphic to $\mathbb{A}^n$. This is due to the "fundamental theorem of elementary symmetric polynomials".

Theorem 1.1.1. Let $S_n \circlearrowright k[x_1, \dots, x_n]$ by permutation. Then,

$$(k[x_1, \dots, x_n])^{S_n} = k[s_1, \dots, s_n]$$

where the algebra on the right is freely generated by the elementary symmetric polynomials.

Proof. The map $\Phi : \mathbb{A}^n \to \mathbb{A}^n$ determined by $A = k[s_1, \dots, s_n] \to k[x_1, \dots, x_n] = B$ which takes a tuple of roots $(\alpha_1, \dots, \alpha_n)$ to the coefficients (up to sign) of the monic polynomial,

$$(x - \alpha_1) \cdots (x - \alpha_n)$$

is dominant because every polynomial has a root in $k = \bar{k}$ and thus $k[s_1, \dots, s_n] \to k[x_1, \dots, x_n]$ is injective. Now we consider the inclusion of fraction fields,

$$\mathrm{Frac}\,(A) = k(s_1, \dots, s_n) \subset k(x_1, \dots, x_n) = \mathrm{Frac}\,(B)$$

which is the splitting field of the "universal polynomial",

$$X^n - s_1 X^{n-1} + \cdots + (-1)^n s_n = \prod_{i=1}^{n} (X - x_i)$$

and hence is Galois with $G \hookrightarrow S_n$ but clearly the permutation action $S_n \circlearrowright \mathrm{Frac}\,(B)$ is via field automorphisms fixing $\mathrm{Frac}\,(A)$ so we see that $G = S_n$ so by Galois theory,

$$\mathrm{Frac}\,(A) = \mathrm{Frac}\,(B)^G$$

Then because $A = k[s_1, \dots, s_n]$ is integrally closed,

$$A^{S_n} = \mathrm{Frac}\,(B)^{S_n} \cap B = \mathrm{Frac}\,(A) \cap B = A$$

□

Remark. Geometrically this proof is doing the following. Because the map $\Phi : \mathbb{A}^n \to \mathbb{A}^n$ is S_n-invariant it factors as,

$$\mathbb{A}^n \to \mathbb{A}^n/S_n \to \mathbb{A}^n$$

Using Galois theory we show that $\mathbb{A}^n/S_n \to \mathbb{A}^n$ is generically (meaning on the function fields) an isomorphism (you could also see this geometrically because S_n acts transitively on the fibers of Φ and thus $\mathbb{A}^n/S_n \to \mathbb{A}^n$ is bijective so it must have "generic degree 1" meaning it is an isomorphism on the function fields) but it is also finite so $\mathbb{A}^n/S_n \to \mathbb{A}^n$ is an isomorphism because $\mathbb{A}^n$ is normal.

Remark. Notice that if $f : X \to Y$ is a bijective map of varieties which is generically an isomorphism (meaning is an isomorphism on function fields) it does not follow that f is an isomorphism. For example,

$$\mathrm{Spc}(k[t]) \to \mathrm{Spc}(k[x,y]/(y^2 - x^3)) \quad \text{where} \quad x \mapsto t^3 \text{ and } y \mapsto t^2$$

is an isomorphism on function fields because $t = \frac{x}{y}$ but is not an isomorphism of rings. Therefore, we used that $\mathbb{A}^n$ is normal (meaning the ring $k[x_1, \dots, x_n]$ is integrally closed) crucially in the proof.