

1 Week 1: Intro part of Chapter 2 in Bosch

\mathbb{Q} has no analysis (not complete and totally disconnected so there are no non-constant paths). We can complete to get \mathbb{Q}_p but still totally disconnected so we still don't get nice analysis. However, we can do rigid analytic geometry.

1.1 Absolute Values

Definition 1.1.1. Let K be a field. A *non-Archimedean absolute value* $|\bullet|$ is a function $|\bullet| : K \rightarrow \mathbb{R}_{\geq 0}$ such that,

- (a) $|x| = 0 \iff x = 0$
- (b) $|xy| = |x| \cdot |y|$
- (c) $|x + y| \leq \max\{|x|, |y|\}$.

Remark. This is equivalent to a valuation ν on K by $\nu(x) = -\log |x|$.

Definition 1.1.2. An absolute value is,

- (a) *trivial* if its image is $\{0, 1\}$
- (b) *discrete* if $|K^\times|$ is discrete.

Definition 1.1.3. Let K be a field with a NAAV, this gives rise to a distance function $d(x, y) = |x - y|$ which defines a topology on K . We say K is *complete* if every Cauchy sequence converges.

Proposition 1.1.4. If $|a| \neq |b|$ then $|a + b| = \max\{|a|, |b|\}$.

Proof. Assume $|b| < |a|$. If $|a + b| < |a|$ then $|a| = |a + b - b| \leq \max\{|a + b|, |b|\} < |a|$ which is a contradiction so we see $|a + b| \geq |a|$ and $|a + b| \leq \max\{|a|, |b|\} = |a|$ so we conclude. \square

Proposition 1.1.5. Consider a sequence of partial sums,

$$b_n = \sum_{i=0}^n a_i$$

Then b_n converges iff $\lim_{n \rightarrow \infty} a_n = 0$.

Proposition 1.1.6. In K ,

- (a) all triangles are isosceles
- (b) all points inside the disk are a center
- (c) K is totally disconnected.

Proof. Let,

$$D_r^-(x) = \{z \in K \mid d(x, z) < r\} \quad \text{and} \quad D_r^+(x) = \{z \in K \mid d(x, z) \leq r\}$$

Both will turn out to be clopen. Let $U = D_r^+(x)$ then for any $y, z \in D_r^+(x)$ we have $d(y, x) \leq \max\{d(x, y), d(x, z)\} < r$ and thus $D_r^+(y) \subset D_r^+(x)$ and conversely so they are equal. Furthermore

U is clopen because for any $y \in U^C$ we see that $D_r^\pm(y)$ is disjoint from U since otherwise they would be the same disk. Therefore, U^C is open.

Even worse, $\partial D^+(x)$ is open since it is the difference of two clopen sets. Indeed, if $z \in D_r^-(x)$ then $d(y, z) = \max\{d(x, y), d(x, z)\} = r$ since $d(x, y) = r$ and $d(x, z) < r$ so they are not equal. Thus $z \in \partial D^+(x)$ so we see that it is open.

Let S be a subsets and $x, y \in S$ with $x \neq y$. Let $\delta = \frac{1}{2}d(x, y)$ and $S_1 = S \cap D_r^-(x)$ and $S_2 = S \setminus D_r^-(x)$ then $S = S_1 \sqcup S_2$ but these are both clopen in S so S is disconnected. \square

Definition 1.1.7. Call a function $f : U \rightarrow K$ for open $U \subset K$ *locally analytic* if for each $x \in U$ there is a ball $D_r^-(x)$ on which f is a convergent power series.

Example 1.1.8. Let $U = D_1^\pm(0)$ then as have seen,

$$U = \cup_{x \in U} D^-(x, 1)$$

Therefore U is a disjoint union of disks D_i . Given a convergent power series f on D_i consider $f : U \rightarrow K$. Then we can set $f|_{D_i} = f_i$ then f is locally analytic for any choices of f_i . This is bad.

Proposition 1.1.9. If V is a f.d. K -vectorspace we can define a norm $|\bullet|_{\max}$ on V . Fix a basis v_1, \dots, v_n then $|v|_{\max} = \max\{|\alpha_i|\}$ for $v = \alpha_1 v_1 + \dots + \alpha_n v_n$. All bases give equivalent norms. Furthermore, all norms on V are equivalent.

Corollary 1.1.10. If L/K is algebraic and K is complete then all norms extending $|\bullet|_K$ are equal.

Theorem 1.1.11. If L/K is algebraic and K is complete then there is a unique extension of $|\bullet|_K$ to L .

Proof. For $\alpha \in L$ let $F = K(\alpha)$ then define,

$$|\alpha|_L = |N_{K(\alpha)/K}(\alpha)|^{\frac{1}{\deg \alpha}}$$

\square

Proposition 1.1.12. For any field K with a NAAV can form \hat{K} a complete NAAV containing $K \hookrightarrow \hat{K}$ as a dense subspace. Then we extend the absolute value from \hat{K} to $\overline{\hat{K}}$.

Remark. If L/K is algebraically closed then we get an embedding $L \hookrightarrow \overline{\hat{K}}$ but this is not unique it depends on the embedding. Thus without completeness we cannot get the unique extension property.

Remark. $\overline{\hat{K}}$ may fail to be complete but $\widehat{\overline{\hat{K}}}$ is both complete and algebraically complete.

Proposition 1.1.13. Let K be a field with a nontrivial complete NAAV. Then $|\bullet|_K$ extends uniquely to \overline{K} and all finite subextensions are complete.

Definition 1.1.14. Let $B^n(\overline{k}) = \{(x_1, \dots, x_n) \in \overline{K} \mid |x_i| \leq 1\}$.

Lemma 1.1.15. A formal power series,

$$f(\zeta) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \zeta^\alpha \in K[[\zeta_1, \dots, \zeta_n]]$$

converges on $B^n(\overline{K})$ iff

$$\lim_{|\alpha| \rightarrow \infty} |c_\alpha| = 0$$

Definition 1.1.16. The Tate algebra,

$$T_n = K \langle \zeta_1, \dots, \zeta_n \rangle$$

consists of all formal power series which are convergent on $B^n(\overline{K})$.

Definition 1.1.17. The *Gauss norm* on T_n is defined setting for,

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \zeta^\alpha$$

the norm,

$$|f| = \max\{|c_\alpha|\}$$

Lemma 1.1.18. $|fg| = |f| \cdot |g|$.

Proof. It is clear from the non-Arch property that $|fg| \leq |f| \cdot |g|$. Let $r = \{x \in K \mid |x| \leq 1\}$ which is a local ring with maximal ideal $\mathfrak{m} = \{x \in K \mid |x| < 1\}$. Let $\kappa = R/\mathfrak{m}$. Let $S = R \langle \zeta_1, \dots, \zeta_n \rangle \subset T_n$ be power series with coefficients in R . Equivalently, $f \in T_n$ such that $|f| \leq 1$. The map $R \rightarrow \kappa$ extends to $\pi : R \langle \zeta_1, \dots, \zeta_n \rangle \rightarrow \kappa[\zeta_1, \dots, \zeta_n]$ which lands in polynomials because $c_\alpha \rightarrow 0$ so eventually $c_\alpha \in \mathfrak{m}$ for each $f \in R \langle \zeta_1, \dots, \zeta_n \rangle$. Let $\pi(f) = \bar{f}$. Consider $f, g \in T_n$ with $|f| = |g| = 1$ then $f, g, fg \in S$. Then $\pi(fg) = \bar{f}\bar{g} \neq 0$ since norm 1 means at least one coefficient is not in \mathfrak{m} so $\pi(f) \neq 0$ and $\pi(g) \neq 0$ and since $\kappa[\zeta_1, \dots, \zeta_n]$ is a domain either. Thus $|fg| \geq 1$ and hence $|fg| = 1$. Otherwise, we can just factor out the largest coefficient to reduce to the norm 1 case. \square

Proposition 1.1.19. T_n is complete.

1.2 Schedule

- (a) Week 2 : rest of chapter 2
- (b) Week 3: 3.1 - 3.2 Ben
- (c) Week 4: 3.3

2 Affinoid algebras and Spaces

Let K be a complete nonarch field. Let,

$$T_n = K \langle \zeta_1, \dots, \zeta_n \rangle$$

Then $\text{mSpec}(T_n) = B^n(\overline{K})/\sim$.

Definition 2.0.1. An *affinoid* K -algebra is T_n/I for some ideal $I \subset T_n$.

Proposition 2.0.2. Affinoid algebras are,

- (a) Noetherian
- (b) Jacobson
- (c) satisfies Noether normalization meaning there is a finite injection $T_d \hookrightarrow A$.

Definition 2.0.3. Given $\alpha : T_n \twoheadrightarrow A$ define the *residue norm*

$$|f|_\alpha = \inf_{f \in \alpha^{-1}(u)} |f|$$

This is the quotient norm of the map of Banach spaces $\alpha : T_n \rightarrow A$.

Proposition 2.0.4. (a) $|\bullet|_\alpha$ is a complete K -algebra norm, α is continuous and open.

(b) for each $u \in A$ there exists $f \in \alpha^{-1}(u)$ such that $|f| = |u|_\alpha$.

Definition 2.0.5. The *supremum seminorm* is,

$$|f|_{\text{sup}} = \sup_{x \in \text{mSpec}(A)} |f(x)|$$

Remark. f is nilpotent iff $|f|_{\text{sup}} = 0$.

Proposition 2.0.6. Let $\varphi : V \rightarrow A$ be a map of affinoid K -algebras then $|\varphi(b)|_{\text{sup}} \leq |b|_{\text{sup}}$ for all b .

Proof. We have $\text{mSpec}(A) \rightarrow \text{mSpec}(B)$ by Nllstellensatz. View $\varphi(b)$ as the pullback of b to $\text{mSpec}(A)$. \square

Proposition 2.0.7. For T_n the sup norm is the Gauss norm.

Proof. Use maximum principle: for all $f \in T_n$ there is $x \in B^n(\overline{K})$ s.t. $|f(x)| = |f|$. \square

Proposition 2.0.8. Let $\alpha : T_n \twoheadrightarrow A$ be a presentation. Then $|f|_{\text{sup}} \leq |f|_\alpha$ for all $f \in A$. Thus the sup norm is finite.

Proof. Pick $g \in \alpha^{-1}(f) \subset T_n$ such that $|g| = |f|_\alpha$. Then for each $x \in \text{mSpec}(A)$,

$$|f(x)| = |g(x)| \leq |g|_{\text{sup}} = |g| = |f|_\alpha$$

\square

Lemma 2.0.9. For a polynomial,

$$P(\zeta) = \zeta^r + c_1 \zeta^{r-1} + \cdots + c_r = \prod_{j=1}^r (\zeta - \alpha_j)$$

with $c_i \in K$ and $\alpha_i \in \overline{K}$ then $\max_j |\alpha_j| = \max_i |c_i|^{\frac{1}{i}}$.

Proof. Use elementary symmetric polynomials. \square

2.1 Some Lemmas

Lemma 2.1.1 (13). Let $T_d \hookrightarrow A$ be a finite injection such that A is torsion-free as a T_d -module. Fix $f \in A$ then,

- (a) there is a unique monic polynomial $p_f(\zeta) = \zeta^r + a_1 \zeta^{r-1} + \cdots + a_r \in T_d[\zeta]$ of minimal degree with $p_f(f) = 0$

- (b) Fix $x \in \text{mSpec}(T_d)$ and let $y_1, \dots, y_s \in \text{mSpec}(A)$ be the preimages. Then $\max_j |f(y_j)| = \max_j |a_j(x)|^{\frac{1}{j}}$
- (c) $|f|_{\text{sup}} = \max_j |a_j|^{\frac{1}{j}}$.

Lemma 2.1.2 (14). Let $\varphi : B \rightarrow A$ be finite. Then for $f \in A$ there exists a polynomial,

$$f^r + b_1 f^{r-1} + \dots + b_r = 0$$

with $b_j \in B$ such that,

$$|f|_{\text{sup}} = \max_j |b_j|^{\frac{1}{j}}$$

Theorem 2.1.3 (Maximum Principle). For $f \in A$ there is $x \in \text{mSpec}(A)$ such that $|f(x)| = |f|_{\text{sup}}$.

Proof. □

Theorem 2.1.4 (Power Boundedness). Fix $\alpha : T_n \rightarrow A$ and $f \in A$. The following are equivalent,

- (a) $|f|_{\text{sup}} \leq 1$
- (b) there is an equation $f^r + a_1 f^{r-1} + \dots + a_r = 0$ with $|a_i|_{\alpha} \leq 1$
- (c) the sequence $|f^n|_{\alpha}$ is bounded.

Corollary 2.1.5. The following are equivalent,

- (a) $|f|_{\text{sup}} < 1$
- (b) $|f^n|_{\alpha} \rightarrow 0$ (meaning f is topologically nilpotent).

Proof. If $|f|_{\text{sup}} = 0$ then f is nilpotent so f is topologically nilpotent. If $0 < |f|_{\text{sup}} < 1$ then by Lemma 14 there is some $r > 0$ and $c \in K^\times$ s.t. $|f^r|_{\text{sup}} = |c|$ and thus $c^{-1}f^r$ has sup norm 1 so by the theorem it is power bounded. Hence $|f^{nr}| \leq M c^n \rightarrow 0$ since $|c| < 1$. □

3 Affinoid Subdomains

Remark. Let A be an affinoid K -algebra and $X = \text{Sp}A$ the associated affinoid K -space. The Zariski topology is the topology whose closed sets are generated by the zero locus of the functions $f \in A$ meaning the open sets are of the form,

$$D(f) = \{x \in X \mid f(x) \neq 0\}$$

However, this topology is extremely coarse. For example, because our functions are analytic, in a one-dimensional domain, rigidity means that we can only remove finitely many points to get an open set. To go beyond algebraic geometry and get an analytic theory we include inequalities in the definition of our basic open sets.

Definition 3.0.1. Let $X = \text{Sp}A$ be an affinoid K -space. Let,

$$X(f; \epsilon) = \{x \in X \mid |f(x)| \leq \epsilon\}$$

For any $f \in A$ and $\epsilon \in \mathbb{R}_{>0}$. The topology generated by these sets is called the *canonical topology*.

Remark. This is the coarsest topology for which the functions $f : A \rightarrow \mathbb{A}_K^1$ and the absolute values $|\bullet| : \mathbb{A}_K^1 \rightarrow \mathbb{R}_{>0}$ are continuous (DO THIS)

Proposition 3.0.2. The canonical topology is generated by sets of the form $X(f; 1)$.

Proof. It is clear we only need to produce sets of the form $X(f; \epsilon_n)$ where $\epsilon_n \rightarrow 0$ is some sequence of real numbers. Choose a nonzero topologically nilpotent element c and set $\epsilon_n = |c^n|$. Then $X(f, \epsilon_n) = X(fc^{-n}; 1)$. \square

Lemma 3.0.3. For an affinoid K -space $X = \text{Sp}A$ consider an element $f \in A$ and a point $x \in \text{Sp}A$ where $f(x) \neq 0$. Set $\epsilon = |f(x)| > 0$. Then there exists an element $g \in A$ such that $g(x) = 0$ and $X(g) \subset \{y \in X \mid |f(y)| = \epsilon\}$.

Proof. Let $\mathfrak{m}_x \subset A$ be the corresponding maximal ideal. Let $P \in K[X]$ be the minimal polynomial of $f \in A/\mathfrak{m}_x$ over K . Write,

$$P(\zeta) = \prod_{i=1}^n (\zeta - \alpha_i)$$

Since we can embed $A/\mathfrak{m}_x \hookrightarrow \overline{K}$ sending any of the roots α_i of \bar{f} to any other and the uniqueness of the valuation we see that,

$$|\alpha_i| = |\bar{f}| = |f(x)| = \epsilon$$

Consider $g = P(f) \in A$. Then $g(x) = P(f(x)) = 0$ and if $|g(y)| < \epsilon^n$ suppose that $|f(y)| \neq \epsilon$. Then,

$$|f(y) - \alpha_i| = \max\{|f(y)|, |\alpha_i|\} \geq |\alpha_i| = \epsilon$$

since these have distinct norm by assumption. Thus,

$$|g(y)| = |P(f(x))| = \prod_{i=1}^n |f(y) - \alpha_i| \geq \epsilon^n$$

contradicting the choice of y . Therefore, choosing $c \in K^\times$ with $|c| < \epsilon^n$ we have $X(gc^{-1}) \subset \{y \in X \mid |f(y)| < \epsilon\}$. \square

Corollary 3.0.4. Let $\text{Sp}A$ be an affinoid K -space. Then for $f \in A$ and $\epsilon > 0$ the following sets are open in the canonical topology,

$$\begin{aligned} &\{x \in \text{Sp}A \mid f(x) \neq 0\} \\ &\{x \in \text{Sp}A \mid |f(x)| \leq \epsilon\} \\ &\{x \in \text{Sp}A \mid |f(x)| = \epsilon\} \\ &\{x \in \text{Sp}A \mid |f(x)| \geq \epsilon\} \end{aligned}$$

Proposition 3.0.5. Let $\varphi : \text{Sp}B \rightarrow \text{Sp}A$ be a morphism of affinoid K -spaces with $\varphi^* : A \rightarrow B$ the associated morphism of K -algebras. Then for $f_1, \dots, f_r \in A$,

$$\varphi^{-1}(X(f_1, \dots, f_r)) = X(\varphi^*(f_1), \dots, \varphi^*(f_r))$$

And thus φ is continuous in the canonical topology.

Proof. This is immediate from the fact that $\varphi^*(f)(x) = f(\varphi(x))$ under the canonical inclusion $A/\mathfrak{m}_{\varphi(x)} \hookrightarrow B/\mathfrak{m}_x$ from the diagram,

$$\begin{array}{ccc}
A & \xrightarrow{\varphi^*} & B \\
\downarrow & & \downarrow \\
A/\mathfrak{m}_{\varphi(x)} & \longrightarrow & B/\mathfrak{m}_x
\end{array}$$

□

Definition 3.0.6. Let $X = \text{Sp}A$ then,

- (a) $X(f_1, \dots, f_r) = \{x \in X \mid |f_i(x)| \leq 1\}$ is called a Weierstrass domain
- (b) $X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{x \in X \mid |f_i(x)| \leq 1, |g_j(x)| \geq 1\}$ is called a Laurent domain
- (c) $X(f_1/f_0, \dots, f_r/f_0) = \{x \in X \mid |f_j(x)| \leq |f_0(x)|\}$ for $f_0, \dots, f_r \in A$ without common zeros is called a rational domain.

Lemma 3.0.7. The above domains are all open in $X = \text{Sp}A$ for the canonical topology.

Example 3.0.8. It is necessary to assume that $f_0, \dots, f_r \in A$ have no common zero. Indeed, for $X = \text{Sp}K \langle \zeta_1 \rangle$ and choose a constant $c \in K$ with $0 < |c| < 1$ then,

$$\{x \in X \mid |\zeta_1(x)| \leq |c\zeta_1(x)|\}$$

consists of the single point where $\zeta_1(x) = 0$ which is not open.

Definition 3.0.9. An *affinoid subdomain* is a subset $U \subset X$ such that,

- (a) there is a morphism of affinoid K -spaces $\iota : X' \hookrightarrow X$ such that $\iota(X') \subset U$
- (b) U is universal in the sense that for any $\iota' : Y \rightarrow X$ with $\iota'(Y) \subset U$ there is a unique factorization $f : Y \rightarrow X'$ such that $\iota' = \iota \circ f$.

Lemma 3.0.10. For the embedding $\iota : X' \rightarrow X$ with $X = \text{Sp}A$ and $X' = \text{Sp}A'$ then,

- (a) ι is injective and $\iota(X') = U$ so it gives a bijection $\iota : X' \xrightarrow{\sim} U$
- (b) for any $x \in X'$ the map ι^* induces an isomorphism of affinoid K -algebras,

$$A/\mathfrak{m}_{\iota(x)}^n \xrightarrow{\sim} A'/\mathfrak{m}_x^n$$

- (c) for $x \in X'$ we have $\mathfrak{m}_x = \mathfrak{m}_{\iota(x)}A'$.

Proof. The universal property applied to one-point spaces gives injectivity and surjectivity of $X' \xrightarrow{\sim} U$. □

Proposition 3.0.11. The Weierstrass, Laurent, and rational domains are examples of open affinoid subdomains. We call these *special* affinoid subdomains.

Proof. For $f_1, \dots, f_r \in A$ consider,

$$A \langle f_1, \dots, f_r \rangle = A \langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_i - f_i)$$

is an affinoid K -algebra where we have “forced” power series in $\{f_i\}$ with coefficients tending to zero to converge. This is equivalent to bounding $|f_i(x)| \leq 1$ and thus the natural map $A \rightarrow A \langle f_1, \dots, f_r \rangle$ gives $\text{Sp}A \langle f_1, \dots, f_r \rangle = X(f_1, \dots, f_r) \rightarrow X$. Indeed,

$$|\varphi^*(f_i)(y)| = |f_i(\varphi(y))|$$

We claim this is a universal presentation. □

Proposition 3.0.12. Let $\varphi : Y \rightarrow X$ be a morphism of affinoid K -spaces and $X' \hookrightarrow X$ an affinoid subdomain. Then $Y' = \varphi^{-1}(X')$ is an affinoid subdomain of Y and the morphism restricts to,

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\varphi} & X \end{array}$$

if X' is Weierstrass, Laurent or rational in X , so is Y' in Y .

Proposition 3.0.13. Let X be an affinoid K -space and $U, V \subset X$ be affinoid subdomains. Then $U \cap V$ is an affinoid subdomain. If U, V are Weierstrass, Laurent, or rational then the same is true of $U \cap V$.

Corollary 3.0.14. Weierstrass domains are Laurent and Laurent domains are rational.

Proof. The first is clear. Laurent domains are intersections of rational domains $X(f/1)$ and $X(1/g)$ and hence rational. \square

Proposition 3.0.15. If $U \subset X$ is a rational subdomain. Then there is a Laurent domain $U' \subset X$ such that $U \subset U'$ as a Weierstrass domain.

Proof. Let $U = X(f_1/f_0, \dots, f_r/f_0)$ with no common zero $f_0, \dots, f_r \in A$. Since f_0 is nonzero on U , by the maximum principle we can choose $c \in K^\times$ such that $|cf_0(x)| \geq 1$ for all $x \in U$ so $U' = X((cf_0)^{-1})$. \square

Proposition 3.0.16. Let X be an affinoid K -space. If $V \subset X$ is a Weierstrass (resp. rational) domain and $U \subset V$ a Weierstrass (resp. rational) domain. Then $U \subset X$ is a Weierstrass (resp. rational) domain in X .

Remark. Transitivity does not hold for Laurent domains. Otherwise every rational domain would be Laurent by the previous.

Lemma 3.0.17. Let $\varphi : Y \rightarrow X$ be a morphism of affinoid K -spaces and $x \in X$ a point with associated maximal ideal $\mathfrak{m} \subset A$.

- (a) assume that $\varphi^* : A/\mathfrak{m} \rightarrow B/\mathfrak{m}B$ is surjective then there is a special affinoid subdomain $X' \hookrightarrow X$ containing x such that $\varphi' : Y' \rightarrow X'$ with $Y' = \varphi^{-1}(X')$ is a closed immersion (meaning the ring map is surjective).
- (b) Assume that φ^* induces isomorphisms $A/\mathfrak{m}^n \xrightarrow{\sim} B/\mathfrak{m}^n B$ for all n . Then there is a special affinoid subdomain $X' \hookrightarrow X$ containing x such that the resulting morphism $\varphi' : Y' \rightarrow X'$ is an isomorphism.

Corollary 3.0.18. Let $U \rightarrow X$ be a morphism of K -spaces making U an affinoid subdomain of X . Then U is open in X and the canonical topology of X restricts to the canonical topology of U .

Of Lemma. \square

Proof. We have showed that the morphism $U \rightarrow X$ satisfies the conditions of the above lemma. Then we use the fact that special affinoid subdomains generate the canonical topology. \square

Theorem 3.0.19 (Gerritzen-Grauert). Let X be an affinoid K -space and $U \subset X$ an affinoid subdomain. Then U is a finite union of rational subdomains.

Remark. Why don't we consider non-maximal ideals. One reason is $K_{\mathfrak{p}} = \text{Frac}(A/\mathfrak{p})$ is not an affinoid K -algebra (else it would be finite by Noetherian normalization) and it is not complete so we can't take affinoid $K_{\mathfrak{p}}$ -algebras.

Even worse, non-maximal ideals do not behave well under restricting to affinoid subdomains.

Example 3.0.20. Let $X = \text{Sp}T_1$ and consider the Weierstrass subdomain,

$$U = \{x \in X \mid |\zeta(x)(\zeta(x) - 1)| \leq \epsilon\} \subset X$$

For some $\epsilon = |c| < 1$ with $c \in K$. Then U is the disjoint union of the Weierstrass subdomains,

$$U_1 = \{x \in X \mid |\zeta(x)| \leq \epsilon\} \quad \text{and} \quad U_2 = \{x \in X \mid |\zeta(x) - 1| \leq \epsilon\}$$

Then,

$$T_1 \langle c^{-1}\zeta(\zeta - 1) \rangle \cong T_1 \langle c^{-1}\zeta \rangle \times T_1 \langle c^{-1}(\zeta - 1) \rangle$$

Therefore, we see that the zero ideal of T_1 (the “generic” point of X) gives rise to two different points of U namely the “generic” points of U_1 and U_2 . The issue is basically that,

$$\text{Spec} \left(T_1 \langle c^{-1}\zeta(\zeta - 1) \rangle \right) \rightarrow \text{Spec} (T_1)$$

is not an open immersion of schemes.