1 The Tautological Bundle

Consider the fibre bundle, $\pi : S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$ given by consider ing $S^{2n+1} \subset \mathbb{C}^{n+1}$ and restricting the projection $\mathbb{C}^{n+1} \to \mathbb{P}^n_{\mathbb{C}}$. Then π is a principal S^1-bundle. Consider the tautological representation $\rho : U(1) \to \mathrm{GL}_1(\mathbb{C})$ which is the inclusion $U(1) \hookrightarrow \mathbb{C}^\times$, which gives an associated line bundle $S^{2n+1} \times_\rho \mathbb{C}$. We call this the tautological bundle since its fibre above a point is the line in $\mathbb{C}^{n+1}$ which that point on $\mathbb{P}^n_{\mathbb{C}}$ corresponds to.

To see this explicitly, consider the following bundle,

$$T = \{(L, v) \mid L \in \mathbb{P}^n_{\mathbb{C}} \text{ and } v \in L \subset \mathbb{C}^{n+1}\} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$$

with the projection $\pi : T \to \mathbb{P}^n_{\mathbb{C}}$ via $(L, v) \mapsto L$. I claim that this bundle is isomorphic to the tautological bundle constructed above.

Consider the map $f : S^{2n+1} \times_\rho \mathbb{C} \to T$ via $f : [x, \lambda] \mapsto (\mathrm{Span}(x), \lambda x)$. This is clearly a bundle map since $\pi([x, \lambda]) = \pi(x) = \mathrm{Span}\left(()\, x\right) = \pi(\mathrm{Span}\left(x\right), \lambda x)$. Furthermore it is well-defined because $f([x, \mu\lambda]) = (\mathrm{Span}\left(x\right), \mu\lambda x) = (\mathrm{Span}\left(\mu x\right), \lambda\mu x) = f([\mu x, \lambda])$. We need to check that this map is injective and surjective. First, if $f([x, \lambda]) = f([y, \mu])$ then $\mathrm{Span}\left(x\right) = \mathrm{Span}\left(y\right)$ so $y = \gamma x$ for $\gamma \in \mathbb{C}^\times$ and $\lambda x = \mu y$ so $\lambda = \mu\gamma$ (since these vectors are nonzero) and thus,

$$[x, \lambda] = [x, \gamma\mu] = [\gamma x, \mu] = [y, \mu]$$

For surjectivity note that given (L, v) with $v \in L$ then $L = \mathrm{Span}\left(x\right)$ for $x \in S^{2n+1}$ and $v = \lambda x$ with $\lambda \in \mathbb{C}$ since L is a line. Thus $f([x, \lambda]) = (L, v)$.

The tautological bundle has no nonzero (holomorphic) global sections. However, there are $n + 1$ independent glboal sections of its dual. To see this consder the global $\mathrm{Hom}(T, \mathcal{O}_{\mathbb{P}})$. There exist $n+1$ idependent functions defined by the $n+1$ projections $p_k : \mathbb{C}^{n+1} \to \mathbb{C}$ via the construction,

$$T \hookrightarrow \mathcal{O}_{\mathbb{P}}^{n+1} = \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \xrightarrow{p_k} \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C} = \mathcal{O}_{\mathbb{P}}$$

These sections are refered to as X_k ,the k^{th} coordinate function on $\mathbb{P}^n_{\mathbb{C}}$.

Producing the coordinate functions X_k as sections of the dual $X^\vee$ identifies the tautological bundle T with the algebraic twist $\mathcal{O}_{\mathbb{P}}(-1)$ and thus its dual is the Serre twisting sheaf $T^\vee = \mathcal{O}_{\mathbb{P}}(1)$.

2 Hilbert Spaces

A *norm* on a -vector space V is a function $||\bullet|| : V \to$ such that,

1. $||v|| \geq 0$
2. $||v|| = 0 \iff v = 0$

3. $||v + u|| \leq ||v|| + ||u||$
4. $||\alpha v|| = \alpha ||v||$ for any $\alpha \in$.

A Banach space is a normed vector space $|| \bullet || : V \rightarrow^{+}$