1 Affine Curves with No Immersions

Exercise 1.1. Find a smooth affine curve C with no immersion $C \hookrightarrow \mathbb{P}^2_k$. Unlike in plane case, $K_X \neq 0$ is not an obstruction ($g = 3$ example).

Exercise 1.2. An easier version, find a curve with no immersion $C \hookrightarrow \mathbb{A}^2_k$. My guess is that here $K_X \neq 0$ is an obstruction but I can't prove it.

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2 Curves with Intersection at One Point in Ambiant Space

Exercise 2.1. First question: given a curve C and a closed point $P \in C$ when is there a function $f \in \Gamma(C, \mathcal{O}_C)$ such that $V(f) = \{P\}$ (topologically it may be nonreduced that is fine).

For affine curves: the previous question is asking: given a one-dimensional Noetherian domain A and a maximal ideal $\mathfrak{m}_0 \in \operatorname{Spec}(A)$ when is there $f \in A$ such that $\sqrt{(f)} = \mathfrak{m}_0$.

Suppose that A is a Dedekind domain. Then by unique factorization, $\sqrt{(f)} = \mathfrak{m}_0$ iff $(f) = \mathfrak{m}_0^n$ meaning that this is possible iff $\mathfrak{m}_0$ is torsion in the ideal class group. Chooing any nontorision point of an elliptic curve is then a counter example.

Is the class group not being torsion the only obstruction?

Exercise 2.2. Given a closed immersion $\iota : C \hookrightarrow S$ with S a surface, given a point $P \in C$ when does there exist a closed curve $Z \subset S$ such that $Z \cap C = \{P\}$?

First, an example. Consider the surface $S = C \times_k \mathbb{P}^1_k$ with $C \hookrightarrow S$ via the closed point $0 \in \mathbb{P}^1_k$. Then for any closed point $P \in C$ we can consider the curve $\{P\} \times_k \mathbb{P}^1_k \subset C \times_k \mathbb{P}^1_k$ and $\{P\} \times_k \mathbb{P}^1_k \cap C = \{P\}$. Therefore, this is certainally possible for an arbitrary curve. Perhaps we need to fix

The affine curve $Z \subset X$ is defined by some sheaf of ideals $\mathscr{I} \subset \mathcal{O}_S$. Then $\iota^{-1}\mathscr{I} \cdot \mathcal{O}_C$ gives the sheaf of ideals for $\iota^{-1}(Z)$.

First, consider $S = \mathbb{A}^2_k$ then we have $C = V(f)$ for some $f \in k[x, y]$ and $Z = V(g)$ for some $g \in k[x, y]$ we can assume these are irreducible so C and Z are integral curves. Then, under the map $k[x, y] \to k[x, y]/(f)$ we get $\iota^{-1}\mathscr{I} \cdot \mathcal{O}_C = g \cdot k[x, y]/(f) = (g)$ and thus we reduce exactly to the previous problem for the affine curve $C = \operatorname{Spec}(k[x, y]/(f))$.

Likewise, consider $S = \mathbb{P}^2_k$ then we have $Z = V(I)$ for a height one homogeneous prime ideal I (CHECK THIS). I am guessing that we can take $I = (f)$ since $k[X_0, X_1, X_2]$ is a UFD so height one primes are principal (MAKE SURE f is HOMOGENEOUS). Then $Z = \operatorname{Proj}(k[X_0, X_1, X_2]/(g))$ and I think $\iota^{-1}(Z) = \operatorname{Proj}(k[X_0, X_1, X_2]/(f, g))$ or equivalently $V(g)$ in $C = \operatorname{Proj}(k[X_0, X_1, X_2]/(f))$. So now reduce to when $V(g)$ on a projective curve have support at only one point.

3 Weakly but not Strongly Toric Curve

Find a curve which is toric (i.e. weakly Δ-nondegenerate for some Δ) but is never nondegnerate i.e. (never Δ-nondegenerate for any Δ).

4 Affines in the Plane

Exercise 4.1. Given an example of an immersed curve $C \hookrightarrow \mathbb{A}^2_k$ such that C is not (closed) embedded in any affine open of $\mathbb{A}^2_k$. Likewise, give an example of an immersed curve $C \hookrightarrow \mathbb{P}^2_k$ such that C is not (closed) embedded in any affine open of $\mathbb{P}^2_k$.

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5 Why is this sor hard

Proposition 5.1. There exists a smooth affine curve C over k with no immersion $C \hookrightarrow \mathbb{A}^2_k$ and, in particular, no immersion $C \hookrightarrow \mathbb{G}_m k^2$. Thus, there are smooth affine curves which are not affine plane curves.

Proof. First, we show that if $j : C \hookrightarrow \mathbb{A}^2_k$ is an immersion then $\Omega_{C/k} \cong \mathcal{O}_C$ is trivial. We can factor j as $C \hookrightarrow \overline{C} \hookrightarrow \mathbb{A}^2_k$ into an open immersion followed by a closed immersion [Stacks, Tag 03DQ]. Then $\overline{C}$ □

Take an algebraically closed field k. Show the following,

(a). if $C \hookrightarrow \mathbb{P}^2_K$ is an immersion then there is a plane curve $\bar{C} \subset \mathbb{A}^2_k$ (closed immersion) an an open immersion $C \hookrightarrow \bar{C}$.

(b). for any closed curve $\bar{C} \subset \mathbb{A}^2_k$ we have $\Omega_{\bar{C}/k} = \mathcal{O}_C$

(c). thus, since $C \hookrightarrow \bar{C}$ is étale we have $\Omega_{C/k} = \mathcal{O}_C$ so it suffices to construct a smooth affine curve with nontrivial canonical bundle $\Omega_{C/k}$.

(d). Choose a curve C with genus $g(C) \geq 2$ then $\deg \Omega_{C/k} \geq 2$ and choose a point $P \in C$ such that $K_X \not\sim (2g-2)[P]$ for any $k \in \mathbb{Z}$.

(e). Show that $U = C \setminus \{P\}$ is affine,

(f). Then $U \hookrightarrow C$ is étale so $\Omega_{U/k} = f^* \Omega_{C/k}$ so $K_C \sim [P]$.

(g). Show that this is nontrivial using the exact sequence,

$$\mathbb{Z} \longrightarrow \mathrm{Cl}\,(C) \longrightarrow \mathrm{Cl}\,(U) \longrightarrow 0$$

the first map sending $1 \mapsto [P]$ so we need to show that $K_X \not\sim (2g-2)[P]$ for any k.

6 Supersingular Stuff Here

7 TODO

Milne's Notes on Etale Cohomology
Write notes on GAGA and analytification of a Scheme
Finish K-book
Finish Milne's Notes on Shimura Varieties