0.1 Rational Connectedness

Definition 0.1.1. A variety X is *rationally connected* if there is a rational curve connecting a general pair of points on X.

Definition 0.1.2. A variety X is *rationally chain connected* if there is a chain of $\mathbb{P}^1$ though every general pair of rational points.

There are two issues in formalizing,

(a) If k is finite then there can be rational curves though all rational points but not closed points

(b) If characteristic of k is positive, rational connectedness can come from some inseparable family.

Definition 0.1.3. X is *rationally connected* if there is a family U of rational curves on X meaning a map $u : U \times \mathbb{P}^1 \to X$ such that $u^{(2)} : U \times \mathbb{P}^1 \times \mathbb{P}^1 \to X \times X$ is dominant. We say that X is *separably rationally connected* if $u : U \times \mathbb{P}^1 \to X$ can be chosen with $u^{(2)}$ is smooth.

Remark. If $\mathrm{char}(k) = 0$ and X is smooth then the following are equivalent,

(a) X is rationally chain connected

(b) X is rationally connected

(c) X is separably rationally connected.

Remark. However, in positive characteristic, SRC is the "nicest" notion.

Theorem 0.1.4 (de Jong - Starr)**.** If $\pi : X \to B$ is proper, flat with normal SRC fibers and B is a curve then π admits a section.

Theorem 0.1.5 (Tian-Zong)**.** Chow groups of 1-cycles on X SRC are generated by rationa lcurves.

Theorem 0.1.6. If X is smooth and SRC then $H^1(X, \mathcal{O}_X) = 0$.

0.2 Fano Varieties

Definition 0.2.1. A normal $\mathbb{Q}$-Gorenstein variety X is *Fano* if $-K_X$ is ample.

Example 0.2.2. Some Fano varieties,

(a) $\mathbb{P}^n$

(b) $\mathrm{Gr}(k, n)$

(c) $X = G/P$.

(d) low degree complete intersection in the above.

Fano varieties have a lot of rational curves. This uses Mori bend and break.

Proposition 0.2.3. If X is a smooth Fano variety, then X is rationally chain-connected.

Remark. Question: given X smooth Fano, is X SRC in positive characteristic.

Remark. The smoothness condition is essential here. Kollar gives an example of a singular Fano variety that is not SRC.

Example 0.2.4. Some more examples,

(a) $\mathbb{P}^n$ is SRC by lines

(b) $X = G/P$ homogeneous with P reduced is also SRC

(c) If P is not reduces then G/P is more mysterious but not provably not SRC (yet)

Theorem 0.2.5 (Chen-Zhu, Tian)**.** A general Fano complete intersection in $\mathbb{P}^n$ is SRC.

Proof. Tian notes that these varieties are separably uniruled and shows that if they aren't SRC, their tangent bundle is unstable but in general these tangent bundle is stable.

Chen-Zhu treat this as a problem in log geometry. □

Remark. Don't say much about how the rational curves involved.

0.3 Free Curves and Very Free Curves

Definition 0.3.1. Let $\iota : \mathbb{P}^1 \to X$ be an immersion into the smooth locus on X. Let $C = \iota(\mathbb{P}^1)$. We say that,

(a)

(b) C is *free* if $\iota^* \mathcal{T}_X$ is globally generated

(c) C is *very free* if $\iota^* \mathcal{T}_X$ is ample.

Remark. By the structure of bundles on $\mathbb{P}^1$,

$$\iota^* \mathcal{T}_X = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(e_i)$$

Then C is free iff every $e_i \geq 0$ and C is very free iff every $e_i \geq 1$.

Theorem 0.3.2. If X is smooth then,

(a) X is SRC iff it has a very free curve

(b) X is separably uniruled iff it has a free curve.

Theorem 0.3.3 (Costum - S)**.** Let $X = \mathrm{Gr}(k, n)$ and let Y be a general Fano complete intersection in X of hypersurfaces Y_i, each of class $d_i H$ where H is the ample class with $d_i \geq 3$. Then Y is SRC, and has a very free curve of the "lowest possible degree"