1 Lang-Nishimura

Theorem 1.0.1 (Lang-Nishimura)**.** Let $f : X \dashrightarrow Y$ be a rational map of k-varieties with Y proper. If X has a smooth k-point then Y has a point.

Proof. First we prove the case that X is a curve. Shrink to the smooth locus $U \subset X$ which intersects some generic point since X has a smooth point $x \in X$ and U is open. Hence we get a rational map $U \dashrightarrow Y$ which extends to $U \to Y$ since U is a regular curve and Y is proper.

Now we reduce to the curve case. We may shrink X such that it affine and integral with $x \in X(k)$ a smooth k-point. The goal is to show that there exists a (nonproper) curve $C \to X$ mapping to X whose image intersects the locus of definition of $f : X \dashrightarrow Y$ and contains a lift $x' \in C(k)$ as a smooth k-point of C. There is an étale neighborhood $U \to X$ of x with a lift $x' \in U(k)$ with an étale map $U \to \mathbb{A}^n_k$. Let $V \subset X$ be the domain of f then pushing and pulling gives a dense open of $\mathbb{A}^n_k$. Therefore, choose a line $L \subset \mathbb{A}^n_k$ through the origin intersecting this locus. Then the preimage $L' \subset U$ is a smooth curve passing through x' and hence $L' \to X$ satisfies the hypotheses. $\square$

Example 1.0.2. The condition that $x \in X(k)$ is a *smooth point* is necessary. For example, consider,

$$X = \mathrm{Proj}\left(\mathbb{R}[X, Y, Z]/(X^2 + Y^2)\right)$$

and let $Y = \mathbb{P}^1_{\mathbb{C}}$ be its normalization and consider the inverse of the normalization $X \dashrightarrow Y$. Now X contains a nonsmooth $\mathbb{R}$-point $[0 : 0 : 1] \in X(\mathbb{R})$ but Y does not have an $\mathbb{R}$-point.

2 $\mathbb{E}_8$ lattice

Let $X = \mathrm{Bl}_{P_1,\dots,P_9}(\mathbb{P}^2)$ be the blowup at 9 points sufficiently general so there is a unique cubic C through these points and it is smooth. Then there there is a genus 1 curve $\widetilde{C} \subset X$ which is the strict transform of the unique conic through the points $P_1, \dots, P_9$. Let $E_1, \dots, E_9$ be the exceptional divisors. Then,

$$\widetilde{C} = 3H - (E_1 + \cdots + E_9)$$

so indeed we see that $\widetilde{C}^2 = 0$. Now the claim is that the lattice,

$$\mathrm{Pic}\,(X) = \mathrm{NS}\,(X)$$

contains the $\mathbb{E}_8$ lattice as a subquotient. Indeed,

$$\left\langle \widetilde{C} \right\rangle^{\perp} / \left\langle \widetilde{C} \right\rangle \cong \mathbb{E}_8$$

3 Root Stacks