Mathematics GU6308 Algebraic Topology
Assignment # 4

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1 Maps of Hopf Invariant One

Recall that the Hopf invariant is a integer $h(f) \in \mathbb{Z}$ defined for maps $f : S^{2n-1} \to S^n$ as follows.

Definition 1.0.1. Let $f : S^{2n-1} \to S^n$ be a continuous map. Then consider $C_f = D^{2n} \cup_f S^n$. Choosing generators we have $H^n(C_f; \mathbb{Z}) = \alpha\mathbb{Z}$ and $H^{2n}(C_f; \mathbb{Z}) = \beta\mathbb{Z}$. Then,

$$\alpha^2 \in H^{2n}(C_f; \mathbb{Z}) \implies \alpha^2 = h(f)\beta$$

Remark. Notice that when n is odd $\alpha^2 = \alpha \smile \alpha = 0$ since α has odd degree. Therefore, we may restrict ourself to considering maps $f : S^{4n-1} \to S^{2n}$.

Proposition 1.0.2. The Hopf invariant gives a homomorphism $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$ with the following properties,

(a) if n is odd then $h = 0$ (since $\alpha \smile \alpha = 0$ in odd n).

(b) for the Hopf fibration $H : S^3 \to S^2$ then $C_f = S^2 \cup_H D^4 = \mathbb{CP}^2$ and $H^*(\mathbb{CP}^2; \mathbb{Z}) = \mathbb{Z}[x]/(x^3)$ so the generator of $H^2(\mathbb{CP}^2; \mathbb{Z})$ squares to the generator of $H^4(\mathbb{CP}^2; \mathbb{Z})$ which implies that $h(H) = 1$. In particular, $h : \pi_3(S^2) \xrightarrow{\sim} \mathbb{Z}$ sending $H \mapsto 1$.

Our main result is the following.

Theorem 1.0.3. For all n, there exists a map $f : S^{4n-1} \to S^{2n}$ with Hopf invariant: $h(f) = 2$.

To prove this theorem, we consider the following spaces.

1.1 The James Restricted Product

Definition 1.1.1. Let (X, e) be a based topological space. Define the *James restriced product* as the following quotient space,

$$J_k(X) = X^k / \sim$$

where we identify $(x_1, \dots, x_i, e, \dots, x_k) \sim (x_1, \dots, e, x_i, \dots, x_k)$. Furthermore, we can define the total James space, $J(X) = \varinjlim J_m(X)$.

Example 1.1.2. We have $J_1(X) = X$ and $J_2(X) = X \times X/(x, e) \sim (e, x)$.

When X is a CW complex, $J_m(X)$ inherents a CW complex structure from the product CW structure on X. Explicitly, we glue together the subcomplexes with one coordinate fixed at e. These James restricted products are especially interesting for us in the case of spheres in which case the cohomology is particularly easy to understand.

Theorem 1.1.3. Fix even $n > 0$. Then $H^p(J(S^n); \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ whenever $n \mid p$. Let $\alpha_k \in H^{nk}(J(S^n); \mathbb{Z})$ be a generator. Then for each $k \geq 1$ we have $\alpha_1^k = k! \cdot \alpha_k$.

Proof. (GIVE PROOF) □

1.2 The Proof

We consider, explicitly, the space $J_2(S^n) = S^n \times S^n/(x, e) \sim (e, x)$. Consider the cell structure,

$$S^n = \{e\} \cup D^n$$

Then we get a cell decomposition,

$$J_2(S^n) = \{e\} \cup D^n \cup D^{2n} = S^n \cup D^{2n}$$

since the product cells $\{e\} \times D^n$ and $D^n \times \{e\}$ are glued together.

2 K-Theory of Projective Space

2.1 K-Theory

Proposition 2.1.1. $K^*(X) \cong K(X \times S^1)$

2.2 G-Spaces

Definition 2.2.1. Let G be a topological group. A *G-space* is a topological space along with a continous action $\rho : G \times X \to X$. A *morphism* of G-spaces is a continous map $f : X \to Y$ which commutes with the G-action. We say a vector bundle $\pi : E \to X$ is a *G-bundle* if E is a G-space with a linear action and $\pi : E \to X$ is a morphism of G-spaces.

Proposition 2.2.2. Suppose that $G \circlearrowright X$ freely. Then there is an equivalence of categories between the category of G-vector bundles on X and the category of vector bundles on X/G.

Proof. □

Definition 2.2.3. Let G be a finite discrete group and X a G-space. Let $\mathrm{Vect}_G(X)$ denote the category of G-vector bundles on X. The set of isomorphism classes is a commutative monoid under $\oplus$. Then let $K_G(X)$ be the group completion which is a ring under $\otimes$.

Example 2.2.4. If $G = 1$ then $K_G(X) = K(X)$.

Example 2.2.5. If $X = *$ then $\mathrm{Vect}_G(X)$ is the category of finite dimensional G-representations. Then $K_G(X) = R(X)$ which is the Grothendieck group of G-representations.

2.3 Thom Isomorphism

Definition 2.3.1. Let $E \to X$ be a vector bundle. Then we define the unit sphere bundle $S(E)$ and the unit ball bundle $B(E)$. Then the *Thom space* is $X^E = B(E)/S(E)$. Note that,

$$K(B(E), S(E)) = \tilde{K}(X^E)$$

Furthermore, the exterior bundle $\Lambda^*(E)$ defines a vector bundle $\lambda_E \in \tilde{K}(X^E)$.

Proposition 2.3.2. Let E be a decomposable vector bundle over X. Then $\tilde{K}^*(X^E)$ is a free $K^*(X)$-module with λ_E as generator.

Theorem 2.3.3. Let X be a G-space such that $K^1_G(X) = 0$ and E be a decomposable G-vector bundle. Let $S(E)$ be the associated sphere bundle then there is an exact sequence,

$$0 \longrightarrow K^1_G(S(E)) \longrightarrow K^0_G(X) \xrightarrow{\varphi} K^0_G(X) \longrightarrow K^0_G(S(E)) \longrightarrow 0$$

where φ is multiplication by,

$$\lambda_{-1}[E] = \sum (-1)^i \lambda^i [E]$$

Proof. Consider the pair $(B(E), S(E))$ where $B(E)$ is the unit ball bundle. Then there is a long exact sequence in K-theory,

$$K^{-1}_G(B(E)) \longrightarrow K^{-1}_G(S(E)) \longrightarrow K^0_G(B(E), S(E)) \longrightarrow K^0_G(B(E)) \longrightarrow K^0_G(S(E)) \longrightarrow K^{-1}_G(B($$

but $B(E)$ is homotopy equivalent to X. Therefore, we get $K^1_G(B(E)) = K^1_G(X) = 0$ and $K^0_G(B(E)) = K^0_G(X)$ so we see,

$$0 \longrightarrow K^1_G(S(E)) \longrightarrow K^0_G(x) \longrightarrow K^0_G(X) \longrightarrow K^0_G(S(E)) \longrightarrow K^{-1}_G(B(E), S(E)) \longrightarrow 0$$

why $K^0_G(B(E), S(E)) = K^0_G(X)$ (USE PREVIOUS PROP) □

Proposition 2.3.4. Let $X = *$ then $K^1_G(X) = 0$.

Proof. (SHOW THIS!) □

Corollary 2.3.5. Let G be a cyclic group and E a G-module with $S(E)$ having a free G-action. Then there is an exact sequence,

$$0 \longrightarrow K^1(S(E)/G) \longrightarrow R(G) \longrightarrow R(G) \longrightarrow K^0(S(E)/G) \longrightarrow 0$$

2.4 Application to the Case of Projective Space

Remark. For $E = \mathbb{C}^n$ we have $S(E) = S^{2n-1}$. Let $G = \mathbb{Z}/2\mathbb{Z}$ which acts freely on E via $x \mapsto -x$. Then G acts on $S(E)$ freely via $x \mapsto -x$, the antipodal action. Therefore, $S(E)/G = \mathbb{RP}^{2n-1}$. This will alow us to apply the above sequence. First we need to understand the representation theory of G. First, recall that by Maschke's theorem, G-representations are semi-simple so need only understand irreducible representations.

Theorem 2.4.1. Let G be a finite abelian group. Then all irreducible G-representations are one-dimensional i.e. are characters.

Proof. Let $\rho : G \to \mathrm{Aut}\,(V)$ be an irreducible G-representation. Then for any $g, h \in G$ we have,

$$\rho(g) \circ \rho(h) = \rho(gh) = \rho(hg) = \rho(h) \circ \rho(g)$$

Therefore, $\rho(g) : V \to V$ is a G-morphism. Since V is irreducible, by Shur's Lemma, $\rho(g) = \lambda_g \mathrm{id}$ and thus $\rho : G \to \mathbb{C}^\times$ is a character. □

Example 2.4.2. Representations of $G = \mathbb{Z}/2\mathbb{Z}$ are thus direct sums of characters. The characters $\rho : G \to \mathbb{C}^\times$ are determined by the image of 1. We must have $\rho(1) = \pm 1$. These options are 1 the trivial character and ρ the nontrivial character. Furthermore, $\rho \otimes \rho : G \to \mathbb{C}^\times$ is trivial since $(-1)^2 = 1$. Therefore, representations are sums,

$$n + m\rho := 1 \oplus \cdots 1 \oplus \rho \oplus \cdots \oplus \rho$$

for $n, m \geq 0$ with the relation $\rho^{\otimes 2} = 1$. Thus, taking the group completion we find,

$$R(G) = \mathbb{Z}[\rho]/(\rho^2 - 1)$$

Furthermore, the map $R(G) \to R(G)$ is given by,

$$\lambda_{-1}[E] = \sum (-1)^i \rho^i = (1 - \rho)^n$$

Proposition 2.4.3. We have $\tilde{K}^0(\mathbb{RP}^{2n-1}) = \mathbb{Z}/2^{n-1}\mathbb{Z}$ and $K^1(\mathbb{RP}^{2n-1}) = \mathbb{Z}$.

Proof. Applying the exact sequence,

$$0 \longrightarrow K^1(\mathbb{RP}^{2n-1}) \longrightarrow \mathbb{Z}[\rho]/(\rho^2 - 1) \longrightarrow \mathbb{Z}[\rho]/(\rho^2 - 1) \longrightarrow K^0(\mathbb{RP}^{2n-1}) \longrightarrow 0$$

We change variables $\rho = \rho - 1$ then $\sigma^2 = -2\sigma$ and the map sends $1 \mapsto \sigma^n = (-2)^{n-1}\sigma$. Then the kernel is,

$$K^1(\mathbb{RP}^{2n-1}) \cong \mathbb{Z}$$

Finally, the cokernel is,

$$K^0(\mathbb{RP}^{2n-1}) = \mathbb{Z}[\sigma]/(\sigma^2 + 2\sigma, (-2)^{n-1}\sigma) = \mathbb{Z} \oplus \mathbb{Z}/2^{n-1}\mathbb{Z}$$

□