Mathematics GU4044 Representations of Finite Groups
Assignment # 1

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Problem 1.

Let $v_1 = (2,5)$ and $v_2 = (-1,3)$ then consider the map $G : \mathbb{R}^2 \to \mathbb{R}^2$ such that $G(e_1) = v_1 = 2e_1 + 5e_2$ and $G(e_2) = v_2 = -e_1 + 3e_2$. Using the expansion $G(e_i) = \sum_i C_{ji} e_j$. Therefore, G is represeneted by the matrix,

$$C = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$$

therefore, the inverse transformation cooresponds to the matrix,

$$C^{-1} = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}$$

Now, let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a map such that $F(v_1) = 2v_1$ and $F(v_2) - v_2$. We can rewrite these equations as $F \circ G(e_1) = 2v_1 = 4e_1 + 10e_2$ and $F \circ G(e_2) = -v_2 = e_1 - 3e_2$. Using the above relation, the matrix of $F \circ G$ is,

$$(F \circ G)_m = \begin{pmatrix} 4 & 1 \\ 10 & -3 \end{pmatrix}$$

and therefore, the matrix for F is given by,

$$F_m = (F \circ G)_m C^{-1} = \frac{1}{11} \begin{pmatrix} 7 & 6 \\ 45 & 4 \end{pmatrix}$$

Problem 2.

Let,

$$A = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}$$

(a). $\det A = 5 \cdot (-4) - 7 \cdot (-2) = -6$ and $\mathrm{Tr}\, A = 5 - 4 = 1$

(b).

$$p_A(t) = \det(It - A) = \det \begin{pmatrix} t-5 & +2 \\ -7 & t+4 \end{pmatrix} = (t-5)(t+4) + 14 = t^2 - t - 6$$

Therefore, the roots of p_A are,

$$\lambda_\pm = \frac{1 \pm \sqrt{1 + 4 \cdot 6}}{2} = \frac{1 \pm \sqrt{25}}{2} = 3, -2 \quad \text{so} \quad \lambda_1 + \lambda_2 = 1 = \mathrm{Tr} A \quad \text{and} \quad \lambda_1 \lambda_2 = \frac{1 - 25}{4} = -6 = \det A$$

(c). If λ is an eigenvalue of T then λ^n is an eigenvalue of T^n. This is easily proven by induction. Consider $T^n v = \lambda^n v$ then,

$$T^{n+1}v = T(T^n v) = T(\lambda^n v) = \lambda^n T v = \lambda^{n+1}$$

Therefore, the eigenvalues of A^n are λ_1^n and λ_2^n and then the trace is given by,

$$\text{Tr } A^n = \lambda_1^n + \lambda_2^n = 3^n + (-2)^n$$

Problem 3.

(i). Tr $A = \lambda_1 + \lambda_2$ and Tr $A^2 = \lambda_1^2 + \lambda_2^2$. Therefore, $(\text{Tr } A)^2 - \text{Tr } A^2 = 2\lambda_1\lambda_2 = 2\lambda_1(\text{Tr } A - \lambda_1)$. Thus, $2\lambda_1^2 - 2(\text{Tr } A)\lambda_1 + [(\text{Tr } A)^2 - \text{Tr } A^2]$. Therfore,

$$\lambda_1 = \frac{2\text{Tr } A \pm \sqrt{4(\text{Tr } A)^2 - 4\left[(\text{Tr } A)^2 - \text{Tr } A^2\right]}}{4} = \frac{\text{Tr } A \pm \sqrt{\text{Tr } A^2}}{2}$$

Then the other root,

$$\lambda_2 = \text{Tr } A - \lambda_1 = \frac{\text{Tr } A \mp \sqrt{\text{Tr } A^2}}{2}$$

Therefore, we can determine the eigenvalues up to ordering.

(ii). Let $a = \lambda_1 + \lambda_2$ and $b = \lambda_1\lambda_2$. Then take $f(t) = t^2 - at + b$. First,

$$f(\lambda_1) = \lambda_1^2 - (\lambda_1 + \lambda_2)\lambda_1 + \lambda_1\lambda_2 = \lambda_1^2 - \lambda_1^2 - \lambda_1\lambda_2 + \lambda_1\lambda_2 = 0$$

Similarly,

$$f(\lambda_2) = \lambda_2^2 - (\lambda_1 + \lambda_2)\lambda_2 + \lambda_1\lambda_2 = \lambda_2^2 - \lambda_2^2 - \lambda_1\lambda_2 + \lambda_1\lambda_2 = 0$$

Thus, λ_1 and λ_2 are the roots of f. Futhermore,

$$(t - \lambda_1)(t - \lambda_2) = t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2 = t^2 - at + b = f(t)$$

Problem 4.

Let k be a field and G a group. Take $f \in k[G]$ and $\delta_h \in k[G]$ is the indicator function at h. Then

$$(\delta_h * f)(g) = \sum_{h_1h_2=g} \delta_h(h_1)f(h_2) = \delta_h(h)f(h_2)$$

such that $hh_2 = g$ i.e. $h_2 = h^{-1}g$. Therefore,

$$(\delta_h * f)(g) = f(h^{-1}g)$$

Problem 5.

(i). Let V_1 be a vector space with basis $v_1, \ldots, v_n$ and V_2 a vector space with basis $w_1, \ldots, w_m$. Then, consider the set of vectors, $(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)$. Take any vector $(v, w) \in V_1 \oplus V_2$ then $v \in V_1$ and $w \in V_2$ so these vectors can be expressed in terms of the respective bases,

$$v = c_1 v_1 + \cdots + c_n v_n \quad \text{and} \quad w = d_1 w_1 + \ldots d_m w_m$$

for constants in the common field, $c_i, d_j \in k$. Therefore,

$$(v, w) = (c_1 v_1 + \cdots + c_n v_n, d_1 w_1 + \ldots d_m w_m) = c_1(v_1, 0) + \cdots + c_n(v_n, 0) + d_1(0, w_1) + \cdots + d_m(0, w_m)$$

so these vectors span $V_1 \oplus V_2$. Futhermore, if there exist constants $c_i, d_j \in k$ such that,

$$c_1(v_1, 0) + \cdots + c_n(v_n, 0) + d_1(0, w_1) + \cdots + d_m(0, w_m) = (v, w) = 0_{V_1 \oplus V_2} = (0, 0)$$

then we know that,

$$v = c_1 v_1 + \cdots + c_n v_n = 0 \quad \text{and} \quad w = d_1 w_1 + \ldots d_m w_m = 0$$

therefore, by the linear independence of the bases $\{v_1\}$ and $\{w_i\}$ we know that all $c_i = d_j = 0$. Therefore, all the coeficients are forced to be zero so this set of vectors is independent. Therefore, $(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)$ form a basis of $V_1 \oplus V_2$.

(ii). Let V be a vector space and $W \subset V$ a subspace with basis $w_1, \ldots, w_\alpha$. This is extended to a basis $w_1, \ldots w_\alpha, w_{\alpha+1}, \ldots, w_n$ of V. An arbitrary element of V/W can be written as $v + W$ for some $v \in V$. Therefore, there exist coeficients $c_i \in k$ such that $v = c_1 w_1 + \cdots + c_n w_n$. Thus,

$$v + W = c_1 w_1 + \cdots + c_\alpha w_\alpha + c_{\alpha+1} w_{\alpha+1} + \cdots + c_n w_n + W = c_{\alpha+1}(w_{\alpha+1} + W) + \cdots + c_n(w_n + W)$$

because $w_i \in W$ for $1 \leq i \leq \alpha$ so $w_i + W = W$. Thus, $w_{\alpha+1} + W, \ldots, w_n + W$ spans V/W. Futhermore, suppose that there exist coeficients $c_i \in k$ such that,

$$c_{\alpha+1}(w_{\alpha+1} + W) + \cdots + c_n(w_n + W) = c_{\alpha+1} w_{\alpha+1} + \cdots + c_n w_n + W = W$$

then the vector $c_{\alpha+1} w_{\alpha+1} + \cdots + c_n w_n \in W$ so it can be expressed in terms of the basis $w_1, \ldots, w_\alpha$. Therefore, there exist coeficients such that,

$$c_{\alpha+1} w_{\alpha+1} + \cdots + c_n w_n = c_1 w_1 + \cdots + c_\alpha w_\alpha \quad \text{so} \quad c_1 w_1 + \cdots + c_\alpha w_\alpha - [c_{\alpha+1} w_{\alpha+1} + \cdots + c_n w_n] = 0$$

However, $w_1, \ldots w_\alpha, w_{\alpha+1}, \ldots, w_n$ is a basis of V so by linear independence, all $c_i = 0$. Therefore, $w_{\alpha+1} + W, \ldots, w_n + W$ is independent and thus a basis of V/W.

Problem 6.

Let V be a vectorspace, $W \subset V$ a subspace, and $p : V \to W$ a projection such $\text{Im}(p) = W$ and $\forall w \in W : p(w) = w$. Let $p' : V \to V$ be the map $p'(v) = v - p(v)$.

(a). For $v \in V$, we have $p'(v) = v \iff v - p(v) = v \iff p(v) = 0 \iff v \in \ker p$

(b). If $v \in \text{Im}(p')$ then $\exists u \in V$ such that $p'(u) = u - p(u) = v$. Then, $p(v) = p(u) - p(p(u)) = p(u) - p(u) = 0$ so $v \in \ker p$. Therefore, $\text{Im}(p') \subset \ker p$. Futhermore, by (a), if $v \in kerp$ then $p'(v) = v$ so $v \in \text{Im}(p)$ so $\text{Im}(p') = \ker p$.

(c). Take $v \in \ker p'$ then $p'(v) = v - p(v) = 0$ so $p(v) = v$ but $\text{Im}(p) \subset W$ so $v \in W$. Thus, $\ker p' \subset W$. Futhermore, if $v \in W$ then $p(v) = v$ so $p'(v) = v - p(v) = 0$ so $v \in \ker p'$. Thus, $W \subset \ker p'$ so $\ker p' = W$.

Problem 7.

Let V be a vector space and let $p, p' : V \to V$ be linear maps such that $p + p' = \mathrm{id}_V$ and $p \circ p' = p' \circ p = 0$. Let $W = \mathrm{Im}(p)$. Take $v \in \ker p'$ then $p'(v) = 0$ but $p(v) + p'(v) = v$ so $p(v) = v$ and thus $v \in \mathrm{Im}(p) = W$. Similarly, take $v \in W$ then $\exists u \in V$ such that $p(u) = v$ then $p' \circ p(u) = 0$ so $p'(v) = 0$ and thus $v \in \ker p'$. Therefore $W = \ker p'$. By an exactly analogous argument with p and p' swapped, we have that $W' = \mathrm{Im}(p') = \ker p$.

Finally, if $v \in W \cap W'$ then $v \in \ker p$ and $v \in \ker p'$ so $p(v) + p'(v) = v$ but $p(v) = p'(v) = 0$ so $v = 0$. Thus, $W \cap W' = \{0\}$. Also, because $p + p' = \mathrm{id}_V$, the function $p + p'$ is surjective which implies that $\mathrm{Im}(p) + \mathrm{Im}(p') = V$ and thus $W + W' = V$. Therefore, $V = W \oplus W'$.