Mathematics 257B Symplectic Geometry
Assignment # 1

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April 29, 2022

1 Problem 1

(a) Given a 1-from α define its graph,

$$\Gamma_\alpha = \{(x, \alpha_x) \mid x \in X\}$$

We claim that Γ_α is Lagrangian if and only if $\mathrm{d}\alpha = 0$. Because Γ_α is half-dimensional, it suffices to show that $\omega_{\text{can}}|_{\Gamma_\alpha} = 0$ if and only if $\mathrm{d}\alpha = 0$. Denote the section determined by α as $s_\alpha : X \to T^*X$. Indeed, the tautological form satisfies the tautological property (λ_{can} is the universal 1-form) that $\alpha = s_\alpha^* \lambda_{\text{can}}$ and therefore $\mathrm{d}\alpha = s_\alpha^* \mathrm{d}\lambda_{\text{can}} = -s_\alpha^* \omega_{\text{can}}$. Since s_α is an isomorphism onto Γ_α we see that

$$\omega_{\text{can}}|_{\Gamma_\alpha} = 0 \iff s_\alpha^* \omega_{\text{can}} = 0 \iff \mathrm{d}\alpha = 0$$

Now we check that λ_{can} is universal. Indeed,

$$(s_\alpha^* \lambda_{\text{can}})_x = (\lambda_{\text{can}} \circ \mathrm{d}s_\alpha)_x = (\lambda_{\text{can}})_{(x,\alpha_x)} \circ (\mathrm{d}s_\alpha)_x = \alpha_x \circ (\mathrm{d}\pi)_{(x,\alpha_x)} \circ (\mathrm{d}s_\alpha)_x = \alpha_x \circ \mathrm{id} = \alpha_x$$

so we win.

(b) Let $Y \subset X$ be a submanifold and consider the conormal bundle,

$$L_Y = \{\alpha \in T^*X|_Y \mid \alpha|_{TY} = 0\}$$

We need to consider $TL_Y \subset T(T^*X)$ and show that $\omega_{\text{can}}|_{L_Y} = 0$ since,

$$\dim L_Y = \dim Y + \operatorname{rank} T^*M - \operatorname{rank} TY = \dim M = \tfrac{1}{2} \dim T^*M$$

Even better, I claim that $\lambda_{\text{can}}|_{L_Y} = 0$. Indeed, at $(x, \alpha) \in L_Y$ we know,

$$(\lambda_{\text{can}})_{(x,\alpha)} = \mathrm{d}\pi^* \alpha = \alpha \circ \mathrm{d}\pi_{(x,\alpha)}$$

and therefore because $\alpha_{TY} = 0$ using the inclusion $\iota_{L_Y} : L_Y \hookrightarrow T^*X$,

$$(\lambda_{\text{can}}|_{L_Y})_{(x,\alpha)} = \alpha \circ \mathrm{d}\pi \circ \mathrm{d}\iota_{L_Y} = 0$$

since $\pi \circ \iota_{L_Y}$ is the projection $L_Y \to Y \subset X$ and hence $\mathrm{d}(\pi \circ \iota_{L_Y}) : TL_Y \to TY \subset TX$.

(c) Let $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ be a diffeomorphism and consider the graph morphism,

$$g_\varphi : M_1 \to M_1 \times M_2$$

which is a closed embedding. Its image be the embedded submanifold Γ_φ. Since φ is a diffeomorphism $\dim M_1 = \dim M_2$ so Γ_φ is half-dimensional. Thus, $\Gamma_\varphi \subset M_1 \times M_2$ is Lagrangian for $(M_1, \times M_2, \omega)$ with $\omega = \omega_1 \oplus (-\omega_2)$ if and only if $\omega|_{\Gamma_\varphi} = 0$. Since $g_\varphi : M_1 \to \Gamma_\varphi$ is an isomorphism we see that,

$$\Gamma_\varphi \text{ is Lagrangian} \iff \omega|_{\Gamma_\varphi} = 0 \iff g_\varphi^* \omega = 0 \iff g_\varphi^*(\omega_1 \oplus (-\omega_2)) = \omega_1 - \varphi^* \omega_2 = 0$$
$$\iff \omega_1 = \varphi^* \omega_2 \iff \varphi \text{ is a symplectomorphism}$$

2 Problem 2

(a) Let $W \subset V$ be a linear subspace of a symplectic space (V, ω). Let,

$$K = \ker \omega|_W = W \cap W^\omega$$

By definition ω is a well-defined 2-form on W/K. It suffices to show that ω on W/K is nondegenerate. Suppose that $\omega([w], -) = 0$ then $w \in K$ so $[w] = 0$ by definition proving the claim.

Choose a compatible complex structure J on V and thus we get a metric,

$$g(v, w) = \omega(v, Jw)$$

It is clear that,

$$J(W^\omega) = W^\perp$$

Furthermore, g restricts to a metric on K and therefore defines an isomorphism $q^{-1} : K \to K^*$ via $v \mapsto g(v, -)$. Notice that, $JK \subset (W + W^\omega)^\perp \subset K^\perp$ because if $v \in W$ and $u \in W^\omega$ and $k \in K$ then,

$$g(Jk, v + u) = \omega(k, v) + \omega(k, u) = 0$$

because $k \in W \cap W^\omega$. Now consider the map,

$$\Phi : (W/K) \oplus (W^\omega/K) \oplus (K \oplus K^*) \to V$$

defined by,

$$([w], [u], v, \varphi) \mapsto w + u + v + Jq(\varphi)$$

where w and u are the unique representative in $W \cap K^\perp$ and $W^\omega \cap K^\perp$ so the map is well-defined. I claim this map is injective. Suppose that,

$$w + u + v + Jq(\varphi) = 0$$

Since $w + u + Jq(\varphi) \in K^\perp$ and $v \in K$ we see that $v = 0$ so,

$$w + u + Jq(\varphi) = 0$$

Since $Jq(\varphi) \in (W + W^\omega)^\perp$ and $w + u \in W + W^\omega$ we see that $\varphi = 0$ and $w + u = 0$ so $w, u \in W \cap W^\perp = K$ but also both lie in $K^\perp$ so $u = w = 0$ so the map is indeed injective.

Since the two sides have the same dimension, Φ is an isomorphism. Now we need to check that Φ is a symplectomorphism,

$$\Phi : (W/K, \omega) \oplus (W^\omega/K, \omega) \oplus (K \oplus K^*, \omega_{\text{can}}) \to (V, \omega)$$

Indeed, consider,

$$\begin{aligned}
\omega(w+u+v+Jq(\varphi), w'+u'+v'+Jq(\varphi')) &= \omega(w,w') + \omega(u,w') + \omega(v,w') - g(q(\varphi),w') \\
&+ \omega(w,u') + \omega(u,u') + \omega(v,u') - g(q(\varphi),u') \\
&+ \omega(w,v') + \omega(u,v') + \omega(v,v') - g(q(\varphi),v') \\
&+ g(w,q(\varphi')) + g(u,q(\varphi')) + g(v,q(\varphi')) + \omega(q(\varphi),q(\varphi')) \\
&= \omega(w,w') + \omega(u,u') - \varphi(v') + \varphi'(v)
\end{aligned}$$

Because,

$$\omega(u,w') = \omega(v,w') = \omega(w.u') = \omega(v,u') = \omega(w,v') = \omega(u.v') = \omega(v,v') = 0$$

by paring W with W^ω. Furthermore, $q(\varphi), q(\varphi') \in K$ so because $w, u, w', u' \in K^\perp$ we have,

$$g(q(\varphi),w') = g(q(\varphi),u') = g(q(\varphi),v') = g(w,q(\varphi')) = g(u,q(\varphi')) = 0$$

Likewise $\omega(q(\varphi),q(\varphi')) = 0$ since K is isotropic. Therefore,

$$\Phi^*\omega = \omega \oplus \omega \oplus \omega_{\text{can}}$$

proving the claim.

(b) Let W be a submanifold of a symplectic manifold (M, ω) such that

$$K = (TW) \cap (TW)^\omega = \ker \omega|_{TW}$$

is constant rank. Choose an almost complex structure J on TM compatible with ω. Since the previous map Φ_J is canonical, the same formula defines a map of symplectic bundles,

$$\Phi_J : (TW/K, \omega) \oplus (TW^\omega/K, \omega) \oplus (K \oplus K^*, \omega_{\text{can}}) \xrightarrow{\sim} TM|_W$$

which is fiberwise an isomorphism and therefore is an isomorphism of vector bundles.

If W is Lagrangian then $K = TW$ so we recover an isomorphism,

$$(TM|_W, \omega) \cong (TW \oplus T^*W, \omega_{\text{can}})$$

If W is symplectic then $K = 0$ so we recover,

$$(TM|_W, \omega) \cong (TW, \omega) \oplus ((TW)^\omega, \omega)$$

(c) Let W be a symplectic submanifold of (M, ω). Since W is symplectic,

$$(TM|_W, \omega) \cong (TW, \omega) \oplus ((TW)^\omega, \omega)$$

Consider two symplectic forms ω_1, ω_2 on M which agree as symplectic structures on TW and $(TW)^\omega$ meaning, from the above decomposition (which holds for any symplectic form) they must agree as symplectic structures on $TM|_W$. Therefore, we may apply the relative Moser theorem (since M is compact) to conclude that there exist tubular neighborhoods U_0 and U_1 of $W \subset M$ and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi|_W = \text{id}$ and $\varphi^*\omega_1 = \omega_2$. This proves that, up to diffeomorphism, the symplectic structure on a sub tubular neighborhood of U is determined by the data of the symplectic structure on TW and $(TW)^\omega$.

3 Problem 3

(a) Let $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ with the family of symplectic forms,

$$\omega_\lambda = \lambda\omega_0 \oplus \lambda^{-1}\omega_0$$

for $\lambda > 0$. These all have the same volume form because,

$$\omega_\lambda^{\wedge 2} = (\lambda\omega_0 \oplus \lambda^{-1}\omega_0) \wedge (\lambda\omega_0 \oplus \lambda^{-1}\omega_0) = \omega_0 \otimes \omega_0 \in \Omega^4(\mathbb{CP}^1 \times \mathbb{CP}^1) = \Omega^2(\mathbb{CP}^1) \otimes \Omega^2(\mathbb{CP}^1)$$

This is much clearer if we write,

$$\omega_\lambda = \lambda \pi_1^* \omega_0 + \lambda^{-1} \pi_2^* \omega_0$$

for $\pi_i : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ and notice that,

$$(\pi_i^* \omega_0)^{\wedge 2} = \pi_i^* \omega_0^{\wedge 2} = 0$$

because $\dim \mathbb{CP}^1 = 2$. Therefore,

$$\omega_\lambda^{\wedge 2} = \pi_1^* \omega_0 \wedge \pi_2^* \omega_0$$

is constant. We need to show that (X, ω_λ) are not symplectomorphic for all λ. Consider a diffeomorphism $f : X \to X$ which induces an automorphism,

$$f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

and since $H^2(X, \mathbb{Z}) = \mathbb{Z}^{\oplus 2}$ by Kunneth we see that f is given by a $\mathrm{GL}(2, \mathbb{Z})$ matrix. Then,

$$f^* : H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R})$$

is induced by the same integer matrix (the map tensored with $\mathbb{R}$). Now I claim that $[\omega_\lambda] \in H^2_{\mathrm{dR}}(X)$ is not in the $\mathrm{GL}(2, \mathbb{Z})$-orbit of $[\omega_1] \in H^2_{\mathrm{dR}}(X)$. Because $e = [\omega_0] \in H^2_{\mathrm{dR}}(\mathbb{CP}^1)$ is a generator we see that $[\omega_\lambda] = \lambda e_1 + \lambda^{-1} e_2$ which is not in the $\mathbb{Z}$-lattice for $\lambda > 1$ proving that these cannot be symplectomorphic.

(b) Let $X = \overline{\mathbb{CP}^2}$ with the opposite orientation meaning we choose the distinguished element $-[\mathbb{CP}^2] \in H^4(X, \mathbb{Z})$ as an orientation. Suppose that ω is a symplectic form inducing the correct orientation. Then $[\omega] \in H^2(X, \mathbb{R})$ and $\omega^{\wedge 2}$ is a volume form inducing the correct orientation meaning that $[\omega]^2 < 0$ but this is not possible because under $H^2(X, \mathbb{R}) \cong \mathbb{R}$ the cup product is multiplication and squares of real numbers are always positive.