1 Pre-Talk

1.1 The Moduli Spaces

Given a finitely presented group π we consider the functor sending a ring A to representations valued in A,

$$\mathrm{Rep}_{\pi,r} : A \mapsto \{\rho : \pi \to \mathrm{GL}_r(A)\}/\text{conjugation.}$$

This is not quite representable. Indeed, it is not even an étale sheaf.

Example 1.1.1. Suppose π is finite and K has characteristic zero. Then $M(\pi, r)$ satisfies the sheaf condition for L/K exactly if all dimension r representations over L with traces in K are defined over K. For example let $\pi = Q_8$ and consider the representation

$$Q_8 \to \mathrm{GL}_2(\mathbb{C})$$

using the standard representation via Pauli matrices. It is a standard that

$$\sigma_i \sigma_j = -\delta_{ij} I + \epsilon_{ijk} \sigma_k$$

So

$$\operatorname{tr} \sigma_i = 0 \qquad \operatorname{tr} \sigma_i \sigma_j = -2\delta_{ij}$$

hence the traces are all real. However, there are not enough independent order four elements in $\mathrm{GL}_2(\mathbb{R})$ for this to descend.

Let's first consider a framed version

$$M^{\square}(\pi, r) : A \mapsto \{\rho : \pi \to \mathrm{GL}_r(A)\}$$

This is clearly represented by an affine scheme (inside $\mathbb{A}^{nr^2}$ where n is the number of generators and impose $\det \neq 0$ and the finite number of relations). Now we can form a stack

$$[M^{\square}(\pi, r)/\mathrm{GL}_r]$$

which represents the groupoid version

$$[M^{\square}(\pi, r)/\mathrm{GL}_r] : A \mapsto [\{\rho : \pi \to \mathrm{GL}_r(A)\}/\text{conjugation.}]$$

by definition. To get the isomorphism classes, we use GIT to form a coarse space

$$M^{\text{all}}(\pi, r) := M^{\square}(\pi, r)//\mathrm{GL}_r$$

From the perspective of GIT stability conditions:

(a) completely reducible (i.e. semisimple) $\iff$ polystable

(b) irreducible $\implies$ stable (and usually the converse)

Recall that $\varphi : X \to X//G$ identifies two points iff they have the same orbit closures and there is a unique polystable point in each fiber. Hence $M(\pi, r)$ "parametrizes semisimple representations". In fact, we can identify

$$M^{\sqcup}(\pi, r) \to M^{\text{all}}(\pi, r)$$

on $\bar{k}$-points with the semisimplification.

1.2 Irreducibility

We can form two subschemes of $M^{\mathrm{all}}(\pi, r)$. The first is functorial

$$M^{\mathrm{irr}}(\pi, r) \subset M^{\mathrm{all}}(\pi, r)$$

which is the open determined by the open of $M^{\sqcup}(\pi, r)$ of *absolutely irreducible* representations meaning $\pi : \pi \to \mathrm{GL}_r(A)$ such that for all geometric points $A \to \bar{k}$ the representation $\rho : \pi \to \mathrm{GL}_r(\bar{k})$ is irreducible.

This will be too limiting for us. Instead, we consider $M^{\mathrm{gen-irr}}(\pi, r)$ to be the closure of $M^{\mathrm{irr}}(\pi, r)(\mathbb{C})$ inside $M^{\mathrm{all}}(\pi, r)$ which we write as $M(\pi, r)$. This is the natural space to work in if we want to consider only representations that deform to a an irreducible representation over characteristic zero.

1.3 Specialization and Tame Fundamental Groups

Theorem 1.3.1. If $\pi = \pi_1(X)$ for X a quasi-projective variety then $\epsilon : M \to \mathrm{Spec}\,(\mathbb{Z})$ is surjective if and only if it is dominant.

Remark. To make this true we need $M = M(\pi, r, \delta)$ to modify slightly our definitions to involve only representations such that $\det \rho^{\delta} = 1$. This technical condition will just come along for the ride at almost every step of the proof.

Remark. Note the reason we passed to M is so that each component hits $\mathrm{Spec}\,(\mathbb{Q})$ by definition. Therefore the above statemnt is equivalent to saying

$$M(\mathbb{C}) \neq \varnothing \iff \forall \ell : M(\overline{\mathbb{Z}}_\ell) \neq \varnothing$$

In fact, Helene proves more: that these $\overline{\mathbb{Z}}_\ell$-points can be chosen to pass through $M^{\mathrm{irr}}_{\mathbb{Q}}$.

Remark. In fact, this theorem is an obstruction to groups arising from geometry since not every character variety satisfies this property. For example, the groups

$$\Gamma_\ell = \langle a, b \mid a^{\ell(\ell-1)} b a^{-\ell} b^{-2} \rangle$$

has structure map $\epsilon : M \to \mathrm{Spec}\,(\mathbb{Z})$ with image $\mathrm{Spec}\,(\mathbb{Z}) \backslash \{\mathrm{Spec}\,(\mathbb{F}_\ell)\}$.

How are we going to prove this? We are going to think about representations that factor as

$$\begin{tikzcd} \pi \arrow[r, "\rho"] \arrow[d] & \mathrm{GL}_r(\mathbb{C}) \arrow[d, "\tau"] \\ \hat{\pi} \arrow[r, "\text{cont.}"] & \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell) \end{tikzcd}$$

If this exists then by continuity and compactness of $\hat{\pi}$, up to conjugation, $\hat{\pi} \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ lands in $\mathrm{GL}_r(\overline{\mathbb{Z}}_\ell)$ so we are done. However, the representation we started with probably does not fit into such a diagram. The game will be to "approximate" ρ by – for each ℓ – a representation of the above form.

For $\ell \gg 0$ it turns out this is easy just by generic smoothness of $\epsilon : M \to \mathrm{Spec}\,(\mathbb{Z})$. To get the other primes, we need some technology: companions for arithmetic representations. This technology is for representations of the fundamental group of a variety over $\mathbb{F}_p$. Since $\hat{\pi}_1 = \pi_1^{\text{ét}}(X_{\mathbb{C}})$ we can spread out X over characteristic p and use Grothendieck specialization maps to obtain a representation of a variety over $\mathbb{F}_p$. However, the specialization map only exists when X is proper. To handle the quasi-projective case, we need the tame fundamental group.

1.4 Tame Fundamental Groups

1.5 Arithmetic Representations

1.6 Companions

1.7 Local Structure: de Jong's Theorem

Drinfelds solution to de Jong's conjecture allows us to approximate by arithmetic representations of $\pi_1^t(X_{\overline{\mathbb{F}}_p})$ for $p \gg 0$.

2 Helene

2.1 Preliminaries

Let B be an effective divisor on a projective smooth variety Y over $\mathbb{C}$. We set

$$B = \sum v_j E_j$$

where E_j are the irreducible components of B. If we suppose that the divisors is assoicated to a positive power $\mathcal{L}^{\otimes d}$ of a line bundle $\mathcal{L}$ meaning $\mathcal{L}^d = \mathcal{O}_Y(\sum v_j E_j)$ we write for $i \geq 0$

$$\mathcal{L}^{(i)} := \mathcal{L}^i \otimes \mathcal{O}_Y(-\sum \left\lfloor v_j \cdot i \cdot d^{-1} \right\rfloor \cdot E_j)$$

If $0 \leq i < d$ the definition of $\mathcal{L}^{(i)}$ involves those E_j for whcich $v_j \geq 2$. When we want to highlight the role of the reduced diviso D of B we write

$$B = D + \sum_{v_j \geq 2} v_j \cdot E_j \text{ or } D = \sum_{v_j = 1} v_j \cdot E_j$$

We suppose that the divisor B is strict normal crossings (SNC). The section s of $\mathcal{L}^{\otimes d}$ with support B then defines a sheaf of $\mathcal{O}_Y$-modules

$$\mathcal{A} = \bigoplus_{i=0}^{d-1} \mathcal{L}^{-\otimes i}$$

which the strucrure of an algebra. The multiplication is defined by

$$\mathcal{L}^{-i} \oplus \mathcal{L}^{-j} \to \mathcal{L}^{-i} \otimes \mathcal{L}^{-j} \to \mathcal{L}^{-i-j}$$

and we identify $\mathcal{L}^{-d} \hookrightarrow \mathcal{O}_Y$ by the dual of s. Let W denote the normalization of $\mathbf{Spec}_Y(\mathcal{A})$ and $V \to W$ a resolution of singularities so we get a diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[dr, "f"'] & W \arrow[d, "\tau"] \\ & Y \end{tikzcd}$$

The variety W is called the d-th root of the divisor B.

Lemma 2.1.1. W has rational singularities. In particular W is Cohen-Macaulay and τ is flat. Furthermore

$$\tau_* \mathcal{O}_W = \bigoplus_{i=0}^{d-1} (\mathcal{L}^{(i)})^{-1} \quad \mathrm{R}f_* \mathcal{O}_V = \bigoplus_{i=0}^{d-1} (\mathcal{L}^{(i)})^{-1}[0]$$

Theorem 2.1.2. Let $Y, B, \mathcal{L}$ be as above and let ω_Y be the canonical bundle. Suppose that the Kodiara dimension of $\mathcal{L}$ satisfies $\kappa(\mathcal{L}) = \dim Y = n$ and $\mathcal{L}$ is is generated by global sections. Then

$$H^q(Y, \omega_Y \otimes \mathcal{L}^{(i)} \otimes \mathcal{L}^k) = 0$$

for all $k > 0, q > 0$ and $i \geq 0$.

Proof. Serre duality and Kawamata-Vieweg vanishing for V. □

Theorem 2.1.3. Let $Y, B, \mathcal{L}$ as above. Suppose that $\kappa(\mathcal{O}_Y(D)) = \dim Y$. Then

$$H^q(Y, \omega_Y \otimes \mathcal{L}^{(i)}) = 0$$

for all $q > 0$ and $d > i > 0$.

Proof. The proof is based on three facts:

(a) the caluation of $\mathcal{L}^{(i)}$ in (2.2)

(b) the symmetry of Hodge numbers on V

(c) the formation of differential forms with logarithmc poles on a divisor with normal crossings.

□

2.2 •

Let X^0 be quasi-projective smooth subvariety of dimension $m \geq 1$ in $\mathbb{P}^n$. Let Z be the closure and $\pi : X \to Z$ a birational map such that X is smooth projective. In the proof of Theorem I, we construct sections of certain line bundles on X which we want to identify with the restriction to X^0 of polynomial functions on $\mathbb{P}^n$. We do this using the "section hunting" proposition as follows. Let X' be the normalization of Z and write

$$\begin{array}{ccccc} X & & & & \\ \downarrow & \searrow^{\pi} & & & \\ X' & \xrightarrow{\pi'} & Z & \hookrightarrow & \mathbb{P}^n \end{array}$$

for the corresponding maps. Let U be the smooth locus of X'. For any variety U' with a mrophism $\varphi : U' \to \mathbb{P}^n$ we set $\mathcal{O}_{U'}(1) = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$. For any $\ell<$ call θ_ℓ the composition of the canonical maps

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \to H^0(Z, \mathcal{O}_Z(\ell)) \to H^0(X', \mathcal{O}_{X'}(\ell)) \to H^0(U, \mathcal{O}_U(\ell))$$

Proposition 2.2.1. There is an injection

$$j : \omega_U \to \mathcal{O}_U(\deg X^0 - m - 2)$$

such that for all k, the image under j of $H^0(U, \omega_U \otimes \mathcal{O}_U(k))$ inside $H^0(U, \mathcal{O}_U(\deg X^0 - m - 2 + k))$ is contaied in the image of $\theta_{\deg X^0 - m - 2 + k'}$.

2.3 Proof of Theorem I

We consider the situation of Part 1. Let X^0 be smooth and quasi-projective of dimension m and Z the closure. Let $\{X^0_j\}$ be (integral) subvarities of X^0 of dimensions n_j and Z_j the closures inside Z.

We write X' for the normalization of Z (corrected from X). Then choose a resolution $\pi : X \to X'$.

We construct a desingularization of the divisor $V(s)$ associated to the section s in Theorem I.

3 Cubic Fourfolds

$X \subset \mathbb{P}^5$ a smooth cubic four-fold. First we consider the Hodge diamond. By Lefschetz we just need to understand the middle row.

(a) $H^4(X, \mathcal{O}_X) = H^4(X, \omega_X(3)) = 0$ by Kodaira vanishing

(b) for $H^3(X, \Omega^1_X)$ we use

$$0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^5}|_X \to \Omega^1_X \to 0$$

so by Kodaira vanishing we get

$$H^3(X, \Omega^1_X) \xrightarrow{\sim} H^4(X, \mathcal{O}_X(-3)) = H^4(X, \omega_X) = \mathbb{C}$$

(c) $\chi_{\text{top}}(X) = \deg c_4(\mathcal{T}_X)$. Thus we get

$$h^{22} + 6 = \deg c_4$$

and we use the SES

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^5}|_X \to \mathcal{O}(3) \to 0$$

and hence

$$c(\mathcal{T}_X) = c(\mathcal{T}_{\mathbb{P}^5})/(1+3H) = \frac{(1+H)^6}{1+3H} = 1 + 3H + 6H^2 + 2H^3 + 9H^4$$

and $\deg H^4 = 3$ so $\deg c_4 = 27$ and thus $h^{22} = 21$.

The question is: when is X rational? For $X_d \subset \mathbb{P}^{n+1}$ surface if $d = 1, 2$ then it is rational. Therefore, $d = 3$ is the first interesting case.

(a) if X_3 is a curve it has genus 1 so is not rational

(b) if X_3 is a surface then it is rational

(c) if X_3 is a 3-fold it is not rational (Clemens-Griffiths)

(d) if X_3 is a 4-fold ... well this is interesting

(e) if X_3 has $\dim X_3 > 5$ or something it is rational

Example 3.0.1. Fix two planes:

$$P_1 = \{u = v = w = 0\} \quad P_2 = \{x = y = z = 0\}$$

in $\mathbb{P}^5$ and let X be a cubic 4-fold containing P_1, P_2. Consider

$$\varphi : P_1 \times \mathbb{P}^2 \dashrightarrow X$$

given by

$$(p, q) \mapsto (\ell_{p,q} \cap X) \setminus \{p, q\}$$

there is a unique extra intersection point since the line intersects in three points. More precisely, φ is defined outside the locus at which $\ell_{p,q} \subset X$ which is a surface. We can always write $X = V(F_1 + F_2)$ where F_1 has bidegree $(2, 1)$ and F_2 has bidegree $(1, 2)$ (wrt the variables x, y, z and u, v, w) usually there is a bidegree $(0, 3)$ and $(3, 0)$ part but these are zero if it contains the planes. Then the non-defined locus S is a K3 surfaces $V(F_1, F_2) \subset P_1 \times P_2$.

Example 3.0.2. Suppose X contains a plane then there is a map

$$q : \mathrm{Bl}_P(X) \to \mathbb{P}^2$$

projecting away from the plane. The fibers are quadric surfaces (these are the residuals of the intersection of a 3-space containing P with X). Then X is rational if q admits a rational section since then it is birational to $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Consider $F_1(q)$ be the relative Fano scheme of lines for the map q. This is a fibration over $\mathbb{P}^2$. The map

$$F_1(q) \to \mathbb{P}^2$$

has general fiber a disjoint union of two lines. The stein factorization

$$F_1(q) \to S \to \mathbb{P}^2$$

gives a degree 2 cover $S \to \mathbb{P}^2$ branched over a sextic and S is a K3 surfaces. And $F_1(q) \to S$ is a smooth conic bundle.

Proposition 3.0.3. q admits a rational section iff $r : F_1(q) \to S$ admits a rational section (i.e. it is a trivial Brauer class on S).

Definition 3.0.4. A polarized K3 surface (X, L) is associated with X if there exists a surface T on X non-homologous to a complete intersection such that $\langle h^2, T\rangle^\perp \subset H^4(X, \mathbb{Z})$ is isomorphic to $\langle L\rangle^\perp \subset H^2(S, \mathbb{Z})(-1)$.

Conjecture 3.0.5. Let X be a cubic 4-fold. Then X is rational iff it admits an associated K3 surface.

It is known that admitting an associated K3 surface is equivalent to $F_1(X)$ being birational to a Moduli space of stable sheaves on a K3.

4 Twisted Intermediate Jacobian Fibrations

Setup: $X \subset \mathbb{P}^5$ smooth cubic 4-fold. Let $B := \{[H] \mid H \subset \mathbb{P}^5\} \cong (\mathbb{P}^5)^\vee$. Then we get a fibration

$$p : \mathcal{Y} \to B$$

whose fibers are $X \cap H_b$ for $b \in B$ called the universal hyperplane section. Recall: cohomology of the generic fiber which is a cubic 3-fold

$$H^i(Y, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0, 2, 6 \\ \mathbb{Q}^{\oplus 10} & i = 4 \\ 0 & i = \text{odd} \end{cases}$$

The intermediate Jacobian associated a smooth cubic 3-fold Y is given by the Hodge filtration

$$H^3(Y, \mathbb{C}) \supset F^1 \supset F^2 \supset 0$$

then we define

$$J(Y) = \frac{(F^2 H^3)^\vee}{H^3(Y, \mathbb{Z})}$$

is a ppav of dimension 5. Goal to do this for the family $p : \mathcal{Y} \to B$. What about the singular fibers?

Proposed candiate: $(R^2 p_* \Omega^1_{\mathcal{Y}})/R^3 p_* \mathbb{Z}_{\mathcal{Y}}$

Remark. Note that $R^2 p_* \Omega^1_{\mathcal{Y}/B} = R^2 p_* \Omega^1_{\mathcal{Y}}$ because $R^1 p_* \mathcal{O}_{\mathcal{Y}} = 0$ for all $i > 0$.

4.1 •

Over the smooth locus $U \subset B$ of p we have a VHS

$$(\Lambda_U := R^3 p_* \mathbb{Z}_{\mathcal{Y}_U}, \Lambda_{\mathbb{C}}, F^\bullet \Lambda_{\mathbb{C}})$$

and so we can associate an intermediate Jacobian

$$J(\Lambda_U) := \frac{(F^2 \Lambda_{\mathbb{C}})^\vee}{R^3 p_* \mathbb{Z}_{\mathcal{Y}}} \cong \frac{R^2 p_* \Omega_{\mathcal{Y}}}{R^3 p_* \mathbb{Z}_{\mathcal{Y}}}$$

Proposition 4.1.1. The injection

$$\Lambda \to \Lambda_{\mathbb{C}} \to (F^2 \Lambda)^\vee$$

extends to an injection

$$\Lambda := R^3 p_* \mathbb{Z}_{\mathcal{Y}} \to R^2 p_* \Omega^1_{\mathcal{Y}}$$

Remark. $R^2 p_* \Omega^1_{\mathcal{Y}}$ is a locally free sheaf isomorphic to Ω^1_B.

Proof. Consider the exponential sequence

$$0 \to \mathbb{Z}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}^\times_{\mathcal{Y}} \to 0$$

and we get

$$R^3 p_* \mathbb{Z}_{\mathcal{Y}} \cong R^2 p_* \mathcal{O}^\times_{\mathcal{Y}} \xrightarrow{\text{dlog}} R^2 p_* \Omega^1_{\mathcal{Y}}$$

Step 2: need to show $R^3 p_* \mathbb{Z}_{\mathcal{Y}}$ is an irreducible sheaf. □

Therefore we can define

$$J := \frac{R^2 p_* \Omega^1_{\mathcal{Y}}}{R^3 p_* \mathbb{Z}_{\mathcal{Y}}}$$

is an abelian sheaf on B.

4.2 Hodge Modules

Schnell: complex analytic Neron model. Recall, given a VHS of weight $2k+1$ and level 1 (meaning there is only two steps in the Hodge filtration $\Lambda \supsetneq F^k \supsetneq F^{k+1} \supsetneq 0$). Call it $(\Lambda, \Lambda_{\mathbb{C}}, F^\bullet \Lambda_{\mathbb{C}})$. On $U \subset B$ we have $(\Lambda_U, F^\bullet \Lambda)$ a VHS.

$$J(\Lambda_U) = \frac{(F^{k+1}\Lambda_{\mathbb{C}})^\vee}{\Lambda_U}$$

On B, let $\mathcal{M}$ be the minimal extension of $\Lambda_{\mathbb{C}}$ as a Hodge module

$$J(\mathcal{M}) := \frac{(F_{-k-1}\mathcal{M})^\vee}{j_*\Lambda_U}$$

Schnell shows:

(a) total space is Hausdorff

(b) its formation commutes with smooth base change $B' \to B$

(c) Extends admissible normal functions without singularities w/o singularities (??)

Proposition 4.2.1. Back to our VHS $(\Lambda_U, F^\bullet \Lambda_{\mathbb{C}}$

(a) $(F_{-k-1}\mathcal{M})^\vee \cong R^2 p_* \Omega^1_{\mathcal{Y}}$

(b) $j_*\Lambda_U \cong \Lambda \cong R^3 p_* \mathbb{Z}_{\mathcal{Y}}$

Proof. Main input: decomposition theorem

$$\mathrm{R}p_*\mathbb{Q}_{\mathcal{Y}}[8] = \mathbb{Q}_B[5][3] \oplus \mathbb{Q}_B[5][1] \oplus \mathrm{R}^3 p_* \mathbb{Z}_{\mathcal{Y}}[5] \oplus \mathbb{Q}_B[5][-1] \oplus \mathbb{Q}_B[5][-3] \oplus K$$

we show that $K = 0$. Upshot $IC(\Lambda_U) = \Lambda[5]$. Moreover we get the Hodge-Module theoretic decomposition theorem

$$p_+\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_B[3] \oplus \mathcal{O}_B(-1)[1] \oplus \mathcal{M} \oplus \mathcal{O}_B(-2)[-1] \oplus \mathcal{O}_B(-3)[-2]$$

Saito:

$$\mathrm{gr}^F_{-k}\mathrm{DR}(p_+\mathcal{O}_{\mathcal{Y}}) \cong \mathrm{R}p_*\mathrm{gr}^F_{-k}\mathrm{DR}(\mathcal{O}_{\mathcal{Y}})$$

For $k=1$ we get

$$\mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{O}_{\mathcal{Y}}) = \Omega^1_{\mathcal{Y}}[7]$$

and therefore by Saito

$$\mathrm{R}p_*\Omega^1_{\mathcal{Y}}[7] \cong \Omega^1_B[7] \oplus \mathcal{O}_B[6] \oplus \mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{M})$$

therefore

$$\mathrm{gr}^F_{-1}\mathrm{DR}(\mathcal{M}) \cong R^2 p_* \Omega^1_{\mathcal{Y}}$$

□