# 1 Group Actions

**Definition:** Let $G$ be a group acting on a set $X$, call $X$ a $G$-set, then there eixsts a homomorphism $\phi : G \to \mathrm{Sym}(X)$ the group of bijections of $X$ to itself.

For example, $\mathrm{GL}(n,k)$ acts on $k^n$ for a field $k$. However, $\mathrm{GL}(n,k)$ also action on $(k^n)^*$ by the action $A \cdot f = f \circ A^{-1}$. Furthermore, $\mathrm{GL}(n,k)$ acts on $\mathrm{Hom}\,(k^n, k^n)$ by $A \cdot F = A \circ F \circ A^{-1}$.

# 2 Group Representations

**Definition:** A $G$-representation $(V, \rho_V)$ is a group action on a vector space $V$ with a homomorphism $\rho_V : G \to \mathrm{Aut}\,(V)$

**Definition:** A $G$-morphism between $G$-representations $\rho_V$ and $\rho_W$ is a linear map $F : V \to W$ satisfying $F \circ \rho_V(g) = \rho_W(g) \circ F$ for all $g \in G$. The set of all such $G$-morphisms is denoted $\mathrm{Hom}^G\,(V, W)$.

**Definition:** Let $\rho_V : G \to \mathrm{Aut}\,(V)$ be a $G$-representation, then $W \subset V$ is a $G$-invariant subspace if $\rho(g)(W) \subset W$ for all $g \in G$.

**Definition:** A $G$-representation $(V, \rho_V)$ is irreducible if $V \neq \{0\}$ and the only invariant subspaces are $\{0\}$ and $V$.

**Definition:** Given $G$-representations $(V, \rho_V)$ and $(W, \rho_W)$, we can form the following additional $G$-representations,

1. $(V^*, \rho_{V*})$ given by $\rho_{V^*}(g) \cdot \varphi = \varphi \circ \rho_V(g)^{-1}$
2. $(V \oplus W, \rho_V \oplus \rho_W)$ given by,
$$(\rho_V \oplus \rho_W)(g) \cdot (v \oplus w) = (\rho_V(g) \cdot v) \oplus (\rho_W(g) \cdot w)$$
3. $(\mathrm{Hom}\,(V, W)\,, \rho_{\mathrm{Hom}(V,W)})$ given by, $\rho_{\mathrm{Hom}(V,W)} \cdot F = \rho_W(g) \circ F \circ \rho_V(g)^{-1}$. Note, the fixed points, $(\mathrm{Hom}\,(V, W))^G = \mathrm{Hom}^G\,(V, W)$ because $\rho_W(g) \circ F \circ \rho_V(g)^{-1} = F$ for every $g \in G$ if and only if $F$ is a $G$-morphism.
4. $(V \otimes W, \rho_V \otimes \rho_W)$ given by,
$$(\rho_V \otimes \rho_W)(g) \cdot \left( \sum_{i=1}^{n} v_i \otimes w_i \right) = \sum_{i=1}^{n} (\rho_V(g) \cdot v_i) \otimes (\rho_W(g) \cdot w_i)$$

**Lemma 2.1.** If $V$ is a $G$-representation such that $V \neq \{0\}$ then there exists a $G$-invariant subspace $W$ which is an irreducible $G$-representation.

**Lemma 2.2.** Let $F : V \to W$ be a $G$-morphism then $\ker F$ and $\mathrm{Im}(F)$ are invariant subspaces.

*Proof.* Let $V$ and $W$ be $G$-representations and let $F : V \to W$ be a $G$-morphism. Take any $g \in G$. Take, $v \in \ker F$. Then, $F(v) = 0$ and thus, $\rho_W(g)(F(v)) = F(\rho_V(g)(v)) = 0$ so $\rho_V(g)(v) \in \ker F$. Therefore, $\ker F$ is invariant under the action of $\rho_V(g)$ for any $g \in G$. Therefore, $\ker K$ is a $G$-invariant subspace of $V$. Similarly, take $w \in \mathrm{Im}(F)$. Then there exists $v \in V$ such that $F(v) = w$. Therefore, $\rho_W(g)(w) = \rho_W(g)(F(v)) = F(\rho_V(g)(v)) \in \mathrm{Im}(F)$. Therefore, $\rho_V(g)(\mathrm{Im}(K)) \subset \mathrm{Im}(F)$ so $\mathrm{Im}(F)$ is a $G$-invariant subspace of $W$. $\square$

**Lemma 2.3.** Let $F : V \to W$ be a $G$-morphism then,

1. if $V$ is irreducible then $F$ is either 0 or injective.

2. if $W$ if irreducible then $F$ is either 0 or surjective.

3. if $V$ and $W$ are both irreducible then $F$ is either 0 or an isomorphism.

*Proof.* Let $V$ be irreducible. Since $\ker F$ is an invariant subspace, then $\ker F = \{0\}$ or $\ker F = V$ so $F$ is either injective or the zero map. Likewise, let $W$ be irreducible. Since $\mathrm{Im}(F)$ is an invariant subspace, then $\mathrm{Im}(F) = \{0\}$ or $\mathrm{Im}(F) = W$ so $F$ is either the zero map or surjective. $\square$

**Definition:** The notation $(V, \rho_V) \cong (W, \rho_W)$ with shorthand $V \cong W$ mean that there exists a $G$-isomorphism $F : V \to W$ i.e. a bijective $G$-morphism.

**Theorem 2.4** (Schur's Lemma)**.** If $V$ is irreducible then $\mathrm{Hom}^G(V, V) \cong \mathbb{C} \cdot \mathrm{id}$. Also, if $V$ and $W$ are both irreducible then either $V \ncong W$ and $\mathrm{Hom}^G(V, W) = \{0\}$ or $V \cong W$ and $\dim \mathrm{Hom}^G(V, W) = 1$.

*Proof.* Let $F : V \to V$ be a $G$-morphism then $F$ is either zero or an isomorphism because $V$ is irreducible. Then $F$ has an eigenvalue $\lambda$ so consider the $G$-morphism $F - \lambda\mathrm{id}$. However, $\exists v \in V$ such that $F(v) = \lambda v$ so $(F - \lambda\mathrm{id})(v) = 0$ and therefore, $F - \lambda v$ is not injective. However, $V$ is irreducible so $F$ must be the zero map. Thus, $F = \lambda\mathrm{id}$. Furthermore, if every $G$-morphism $F \in \mathrm{Hom}^G(V, W)$ is not an isomorphism then because $V$ and $W$ are irreducible we must have $F = 0$. Thus, if $\mathrm{Hom}^G(V, W) \neq \{0\}$ then there must exist a $G$-isomorphism $F$. In particular, $V \cong W$. Therefore, $\mathrm{Hom}^G(V, W) \cong \mathrm{Hom}^G(V, V) \cong \mathbb{C} \cdot \mathrm{id}$ so $\dim \mathrm{Hom}^G(V, W) = 1$. $\square$

**Corollary 2.5.** $F \in \mathrm{Hom}^G(V, W)$ is either zero or an isomorphism and therefore invertible. Therefore, $\mathrm{Hom}^G(V, W)$ is a division ring.

**Definition:** A $G$-representation $(V, \rho_V)$ is decomposable if $V \cong W_1 \oplus W_2$ where $W_i \neq \{0\}$

**Definition:** A $G$-representation is completely reducible if $V \cong W_1 \oplus \cdots \oplus W_n$ where $W_i$ is irreducible.

**Lemma 2.6.** Let $G$ be a finite group and $V$ a $G$-representation, the map $p : V \to V$ given by,

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v)$$

is a $G$-invariant projection $p : V \to V^G$.

*Proof.* If $v \in V^G$ then,

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) = \frac{1}{|G|} \sum_{g \in G} v = v$$

Furthermore, for any $v \in V$ consider,

$$\rho_V(h) \circ p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(h) \circ \rho_V(g)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(hg)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) = p(v)$$

so $p(v) \in V^G$. Therefore, $\mathrm{Im}(p) = V^G$. Furthermore,

$$p \circ \rho_V(g)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ \rho_V(h)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(gh)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) = p(v)$$

Thus, $p \circ \rho_V(g) = \rho_V(g) \circ p$ for all $g \in G$. ☐

**Theorem 2.7** (Maschke)**.** If $G$ is a finite group and $W \subset V$ are $G$-representations then there exists a $G$-invariant complement $W' \subset V$ of $W$ and thus $V = W \oplus W'$.

*Proof.* Let $p_0 : V \to V$ be a projection onto $W$. Then, $p_0 \in \mathrm{Hom}\,(V, V)$ so by the above lemma applied to the $G$-representation $(\mathrm{Hom}\,(V, V), \rho_{\mathrm{Hom}(V,V)})$, the map,

$$p_0 \mapsto p = \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V,V)}(g) \cdot p_0 = \frac{1}{|G|} \sum_{g \in G} \rho_V \circ p \circ \rho_V^{-1}$$

is a projection map $\mathrm{Hom}\,(V, V) \to (\mathrm{Hom}\,(V, V))^G = \mathrm{Hom}^G\,(V, V)$. Thus, $p$ is a $G$-invariant projection from $V$ to $W$ since $p(w) = w$. Therefore, $V \cong W \oplus \ker p$. ☐

**Corollary 2.8.** If $G$ is a finite group then every nonzero $G$-representation is completely reducible.

**Corollary 2.9.** If $G$ is a finite abelian group then any $G$-representation is a sum of 1-dimensional representations.

*Proof.* It suffices to prove that every irreducible $G$-representation is 1-dimensional. Let $W$ be an irreducible $G$-representation. However, since $G$ is abelian, $\rho_W(g)$ is a $G$-morphism in $\mathrm{Hom}^G\,(V, V) \cong \mathbb{C}$ so $\rho_W(g) = \lambda(g) \in \mathbb{C}$. Then, $\rho_W(g)(w) = \lambda(g)w$ so $\mathrm{span}\{w\}$ is a nonempty $G$-invariant subspace. However $W$ is irreducible so $W = \mathrm{span}\{w\}$ which has dimension 1. ☐

**Corollary 2.10.** Let $A \in \mathrm{GL}(n, \mathbb{C})$ and suppose that $A$ has finite order then $A$ is diagonalizable.

*Proof.* $A$ defines a representation of $\mathbb{Z}/N\mathbb{Z}$ where $N$ is the order of $A$. Therefore, $\mathbb{C}^n$ is the sum of 1-dimensional $G$-invariant subspaces which are eigenspaces. Therefore, the eigenvectors of $A$ span $\mathbb{C}^n$. $\square$

**Corollary 2.11.** Let $\rho_V$ be a $G$-representation of a finite group $G$ then $\forall g \in G$ we can diagonalize $\rho_V(g)$ and its eigenvalues are roots of unity of order dividing $|G|$.

*Proof.* Because $G$ is finite, and $g \in G$ has finite order and $\mathrm{ord}(g) \mid |G| =$ so $\rho_V(g)$ has order dividing $n$ and is thus diagonalizable. Furthermore if $v$ is an eigenvector, $\rho_V(g) \cdot v = \lambda v$ then $\rho_V(g)^n \cdot v = \lambda^n v$ but $\rho_V(g^n) = \rho_V(e) = \mathrm{id}$ so $\lambda^n v = v$ and thus $\lambda^n = 1$ since $v \neq 0$ so $\lambda$ is a root of unity. $\square$

# 3 Group Characters

**Definition:** If $(V, \rho_V)$ is a $G$-representation, the character is the map $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\rho_V(g))$.

**Lemma 3.1.** Let $(V, \rho_V)$ be a $G$-representation with character $\chi$ then,

1. $\chi(e) = \mathrm{Tr}(\mathrm{id}) = \dim V$
2. $\chi(hgh^{-1}) = \mathrm{Tr}(\rho_V(h)\rho_V(g)\rho_V(h)^{-1}) = \mathrm{Tr}(\rho_V(h)) = \chi(g)$. Thus, $\chi$ is a function on conjugacy classes.
3. $\chi(g^{-1}) = \overline{\chi(g)}$ because $\rho(g)$ is diagonalizable with norm-1 eigenvalues.

**Lemma 3.2.** Let $(V, \rho_V)$ and $(W, \rho_W)$ be $G$-representations with character $\chi_V$ and $\chi_W$ then,

1. $\chi_{V \oplus W} = \chi_V + \chi_W$
2. $\chi_{V^*} = \overline{\chi_V}$
3. $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
4. $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W$

**Lemma 3.3.**

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

*Proof.* The map,

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v)$$

is a $G$-invariant projection $p : V \to V^G$ so $\mathrm{Tr}(p) = \dim V^G$. However,

$$\mathrm{Tr}(p) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho_V(g)) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

$\square$

**Corollary 3.4.** Applying this fact to $\mathrm{Hom}\,(V, W)$, then,

$$\dim\left(\mathrm{Hom}\,(V,W)^G\right) = \dim \mathrm{Hom}^G\,(V,W) = \frac{1}{|G|}\sum_{g\in G} \chi_{\mathrm{Hom}(V,W)}(g) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi}_V(g)\chi_W(g)$$

**Corollary 3.5.** By Schur's lemma,

$$\dim \mathrm{Hom}^G\,(V,W) = \begin{cases} 1 & V \cong W \\ 0 & \text{else} \end{cases}$$

Therefore,

$$\frac{1}{|G|}\sum_{g\in G} \overline{\chi}_V(g)\chi_W(g) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \begin{cases} 1 & V \cong W \\ 0 & \text{else} \end{cases}$$

where I have used the fact that the sum is real because it is equal to an integer.

**Definition:** For $f_1, f_2 \in \mathbb{C}[G]$ define the Hermitian inner product,

$$\langle f_1, f_2 \rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

**Proposition 3.6.** Therefore, for irreducible representations $(V, \rho_V)$ and $(W, \rho_W)$ with characters $\chi_V$ and $\chi_W$ then,

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & V \cong W \\ 0 & \text{else} \end{cases}$$

**Corollary 3.7.** Let $V$ be a completely reducible representation, $V = \bigoplus_{i=1}^{n} V_i^{m_i}$ with $V_i \cong V_j$ only if $i = j$ then,

$$\langle \chi_V, \chi_V \rangle = \sum_{i,j} m_i m_j \left\langle \chi_{V_i}, \chi_{V_j} \right\rangle = \sum_{i=1}^{n} m_i^2$$

**Corollary 3.8.** Let $V$ be a completely reducible $G$-representation, $V = \bigoplus_{i=1}^{n} V_i^{m_i}$ with $V_i \cong V_j$ only if $i = j$ and $W$ an irreducible $G$-representation then,

$$\langle \chi_W, \chi_V \rangle = \begin{cases} m_i & W \cong V_i \\ 0 & \text{else} \end{cases}$$

*Proof.* We have, $\chi_V = \sum_{i=1}^{n} m_i \chi_{V_i}$. Thus,

$$\langle \chi_W, \chi_V \rangle = \sum_{i=1}^{n} m_i \langle \chi_W, \chi_{V_i} \rangle = \begin{cases} m_i & W \cong V_i \\ 0 & \text{else} \end{cases}$$

since by hypothesis $i \neq j \implies V_i \ncong V_j$. $\square$

**Corollary 3.9.** *$V$ is irrediucible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

**Theorem 3.10.** *Let $G$ be finite, then a $G$-representation $V$ is determined up to isomorphism by $\chi_V$. That is, $V \cong W \iff \chi_V = \chi_W$.*

*Proof.* If $V \cong W$ then there exists an isomorphism $F : V \to W$ such that $F \circ \rho_V(g) = \rho_W(g) \circ F$ and thus $\rho_V(g) = F^{-1} \circ \rho \circ F$. Thus,

$$\chi_V = \mathrm{Tr}(\rho_V(g)) = \mathrm{Tr}\big(F^{-1} \circ \rho \circ F\big) = \mathrm{Tr}(\rho_W(g)) = \chi_W(g)$$

Conversely, suppose that $\chi_V = \chi_W$. Then, because $G$ is finite, we can write any $G$-representations as,

$$V = \bigoplus_{i=1}^{n} V_i^{m_i} \qquad W = \bigoplus_{i=1}^{n} W_i^{k_i}$$

Therefore, $\chi_V = \sum\limits_{i=1}^{n} m_i\, \chi_{V_i}$. Consider

$$\langle \chi_{V_i}, \chi_W \rangle = \langle \chi_{V_i}, \chi_V \rangle = \langle \chi_{V_i}, \chi_V \rangle = m_i$$

but $V_i$ is irreducible so $\langle \chi_{V_i}, \chi_W \rangle = m_i$ implies that some factor $W_j^{k_j}$ is isomorphic to $V_i$ and $m_i = k_j$. Therefore, up to order, the expansions of $V$ and $W$ are equal. Thus, $V \cong W$. $\square$

**Definition:** The regular representation is $\rho_{reg} : G \to \mathbb{C}[G]$ given by $\rho(g)v = g \cdot v$. Call the character of this representation $\chi_{reg} = \chi_{\mathbb{C}[G]}$.

**Lemma 3.11.** *Let $G$ act on $X$ and let $(\mathbb{C}[X], \rho)$ be the permutation $G$-representation. Then,*

$$\chi_{\mathbb{C}[X]}(g) = \#(X^g)$$

*Proof.* We know that $\rho(g) \cdot x = g \cdot x$ so

$$\mathrm{Tr}(\rho(\sigma)) = \sum_{i=1}^{|X|} \mathbf{1}(g \cdot x = x) = \#(X^g)$$

$\square$

**Corollary 3.12.**

$$\chi_{reg}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

*Proof.* A group acts freely on itself ($gh = h \implies g = e$) so there cannot be any fixed points of $G$ for any map except $\rho(e)$ which fixes every element. $\square$

**Lemma 3.13.** $\langle \chi_V, \chi_{reg} \rangle = \dim V$

*Proof.*

$$\langle \chi_V, \chi_{reg} \rangle = \frac{\chi_V(e)|G|}{|G|} = \chi_V(e) = \dim V$$

$\square$

**Theorem 3.14.** Write,

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{n} V_i^{d_i}$$

If $W$ is an irreducible $G$-representation then $W \cong V_i$ for some $i$. Furthermore, $\dim V_i = d_i$.

*Proof.* Let $W$ be irreducible, then $\langle \chi_W, \chi_{reg} \rangle = \dim W > 0$ and therefore by corollary **??**, $W \cong V_i$ for a unique $i$. However, $\dim V_i = \langle \chi_{V_i}, \chi_{reg} \rangle = d_i$. $\square$

**Corollary 3.15.**

$$\dim \mathbb{C}[G] = |G| = \sum_{i=1}^{n} (d_i)^2$$

**Corollary 3.16.** For any $g \in G$,

$$\sum_{i=1}^{n} d_i \cdot \chi_{V_i}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

*Proof.* Because,

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{n} V_i^{d_i}$$

the character factors as,

$$\chi_{reg}(g) = \sum_{i=1}^{n} d_i \cdot \chi_{V_i}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

$\square$

**Theorem 3.17.** If $G$ is a finite group, then there are finitely many irreducible $G$-representations.

*Proof.* Every irreducible $G$-representation must be isomorphic so a factor of the regular representation. Equivalently, the sum of the squares of the dimensions of all irreducible $G$-representations is $|G|$ which is, in particular, finite. $\square$

**Proposition 3.18.** Let $G$ be abelian, then every representation is one-dimensional so $d_i = 1$. Thus, $\sum_{i=1}^{n} d_i^2 = n = |G|$. So there are exactly $|G|$ irreducible $G$-representations.

# 4 The Permutation Representation

# 5 Class Functions

**Definition:** $f : G \to \mathbb{C}$ is a class function if $f$ is constant on conjugay classes or equivalently, $\forall g, h \in G : f(hgh^{-1}) = f(g)$.

**Definition:** $Z \subset \mathbb{C}[G]$ is the vectorspace of class functions.

**Proposition 5.1.** $f_{Cl(x)}$ is the characteristic function of $[x]$ which is,

$$f_{Cl(x)}(g) = \begin{cases} 1 & g \in Cl(x) \\ 0 & g \notin Cl(x) \end{cases}$$

form a basis of $Z$.

**Proposition 5.2.**

$$\left\langle f_{Cl(x)}, f_{Cl(y)} \right\rangle = \begin{cases} \frac{|Cl(x)|}{|G|} & Cl(x) = Cl(y) \\ 0 & \text{else} \end{cases}$$

**Definition:** For $f \in \mathbb{C}[G]$ the map, $F_{V,f} : V \to V$ is defined by,

$$F_{V,f} = \sum_{g \in G} f(g) \rho_V(g)$$

**Lemma 5.3.** If $f$ is a class function, $F_{V,f}$ is a $G$-morphism. If in addition, $V$ is irreducible, then $F_{V,f} = t \cdot \text{id}$ where,

$$t = \frac{|G| \cdot \langle f, \overline{\chi}_V \rangle}{\dim V}$$

*Proof.* $F_{V,f}$ if a $G$-morphism if and only if $\forall h \in G$ we have $\rho_V(h) \circ F_{V,f} \circ \rho_V(h)^{-1} = F_{v,f}$. Expanding,

$$\rho_V(h) \circ F_{V,f} \circ \rho_V(h)^{-1} = \sum_{g \in G} f(g) \rho_V(h) \circ \rho_g \circ \rho_V(h)^{-1} = \sum_{g \in G} f(g) \rho_V(hgh^{-1}) = \sum_{g \in G} f(h^{-1}gh) \rho_V(g) = F_{V,f}$$

because $f$ is a class function.

Using Schur's Lemma, if $V$ is irreducible then because $F_{V,f}$ is a $G$-morphism we know that $F_{V,f} = t \cdot \text{id}$. Thus, $\text{Tr}(F_{V,f}) = \text{Tr}(t \cdot \text{id}) = t \dim V$. However,

$$\text{Tr}(F_{V,f}) = \sum_{g \in G} f(g) \text{Tr}(\rho_V(g)) = \sum_{g \in G} f(g) \chi_V(g) = |G| \langle f, \overline{\chi_V} \rangle$$

Therefore, $t \dim V = |G| \langle f, \overline{\chi_V} \rangle$. ☐

**Proposition 5.4.** If $f$ is a class function then $\langle f, \chi_V \rangle = 0$ for all irreducible $V$ implies that $f = 0$. Furthermore, if $V_1, \cdots, V_n$ are the irreducible $G$-representations up to isomorphism then $\chi_{V_1}, \cdots, \chi_{V_n}$ are a basis for $Z$. Finally, $n$ is the number of conugacy classes of $G$.

*Proof.* If $V$ is irreducible then $V^*$ is irreducible so $\langle f, \chi_{\overline{V}} \rangle = 0$ and thus $F_{V,f} = 0 \cdot \mathrm{id} = 0$ for all irreducible $V$. However, $F_{V_1 \oplus V_2, f} = F_{V_1,f} + F_{V_2,f} = 0$ so by induction $F_{W,f} = 0$ for all $G$-representations. In particular, $F_{\mathbb{C}[G],f} = 0$ that is,

$$F_{\mathbb{C}[G],f} = \sum_{g \in G} f(g) \rho_{reg}(g) = 0$$

so applied to 1,

$$F_{\mathbb{C}[G],f} = \sum_{g \in G} f(g) \rho_{reg}(g)(1) = \sum_{g \in G} f(g) \cdot g = 0$$

and therefore $f = 0$ because $\mathbb{C}[G]$ is a free vectorspace over $G$.

By orthogonality conditions, $\left\langle \chi_{V_i}, \chi_{V_j} \right\rangle = \delta_{ij}$ and thus these characters are linearly independent. Consider the subspace of $Z$ orthogonal to all $\chi_{V_i}$. However, we have shown that if $\langle f, \chi_{V_i} \rangle = 0$ for all irreducible representations $V_i$ then $f = 0$. Thus, the orthogonal complement is empty so the set $\{\chi_{V_1}, \dots, \chi_{V_n}\}$ spans $Z$ and thus $\dim V = n$.

However, the functions $f_{Cl(x)}$ form a basis of $Z$. Therefore, $\dim Z = n$ is the number of conjugacy classes of $G$. ☐

**Proposition 5.5.** $G$ is abelian if and only if every irreducible $G$-representation is one-dimensional.

*Proof.* If $d_i = 1$ then $\sum_{i=1}^n d_i^2 = n = |G|$ so there are $|G|$ conjugacy classes and thus $G$ is abelian. We have already proved the converse. ☐

**Proposition 5.6.** We having the following orthogonality relationship on $G$ over the set of irreducible characters,

- $$\forall x \in G : \quad \sum_{i=1}^{h} |\chi_{V_i}(x)|^2 = \frac{|G|}{|Cl(x)|}$$

- $$\forall x, y \in G : y \notin Cl(x) : \quad \sum_{i=1}^{h} \chi_{V_i}(x) \overline{\chi_{V_j}}(y) = 0$$

# 6 Fourier Inversion on Groups

## 6.1 The Structure of $\mathbb{C}[G]$

**Definition:** A $K$-algebra is a $K$-vectorspace $A$ together with a $K$-bilinear map donoted by $B : A \times A \to A$ where $B(a,b) \mapsto ab$.

**Proposition 6.1.** If $A$ is an *associative unital* $K$-algebra, then $A$ has a ring structure.

*Proof.* $(a_1+a_2)b = B(a_1+a_2,b) = B(a_1,b)+B(a_2,b) = a_1b+a_2b$. The other properties are similar. $\square$

**Definition:** A homomorphism of $K$-algebras is a $K$-linear map $F : A \to A'$ such that $F(B(a,b)) = B'(F(a),F(b))$. In particular, if $A$ is an associative unital algebra then $F$ is a linear ring homomorphism.

**Proposition 6.2.** A $G$-representation $(V,\rho_V)$ induces a homomorphism of $\mathbb{C}$-algebras $\rho_V : \mathbb{C}[G] \to \mathrm{End}\,(V) = \mathrm{Hom}\,(V,V)$ given by,

$$\rho_V\left(\sum_{g\in G} t_g \cdot g\right) = \sum_{g\in G} t_g \cdot \rho_V(g)$$

or alternatively given a map $f : G \to \mathbb{C}$ define,

$$\rho_V(f) = \sum_{g\in G} f(g)\rho_V(g)$$

**Proposition 6.3.** Let $V = \mathbb{C}[G]$ then the regular representation induces a homomorphism $\rho_{\mathbb{C}[G]} : \mathbb{C}[G] \to \mathrm{End}\,(\mathbb{C}[G])$. This map is given by $\rho_{\mathbb{C}[G]}(\alpha)(\beta) = \alpha\beta$.

**Theorem 6.4** (Weddenburn)**.** Define $\rho : \mathbb{C}[G] \to \mathrm{End}\,(V_1) \times \cdots \times \mathrm{End}\,(V_h)$ where $V_1, \cdots, V_h$ enumerates all the irreducible $G$-representations by the map,

$$\rho(\alpha) = (\rho_{V_1}(\alpha), \cdots, \rho_{V_h}(\alpha))$$

where $\rho_{V_i}(\alpha) = \sum\limits_{g\in G} \alpha(g)\rho_V(g)$ for $\alpha \in \mathbb{C}[G]$. Then, $\rho$ is an isomorphism of $\mathbb{C}$-algebras.

*Proof.* $\dim \mathbb{C}[G] = |G|$ and $\dim\left(\mathrm{End}\,(V_1) \times \cdots \times \mathrm{End}\,(V_h)\right) = \dim \mathrm{End}\,(V_1) + \cdots + \dim \mathrm{End}\,(V_h) = (\dim V_1)^2 + \cdots + (\dim V_h)^2 = d_1 + \cdots + d_h^2 = |G|$. Therefore, to prove that $\rho$ is an isomorphism of $\mathbb{C}$-algebras it suffices to prove that $\rho$ is an injective $\mathbb{C}$-algebra homomorphism. Suppose that $\rho(\alpha) = 0$ then $\rho_{V_i}(\alpha) = 0$ for all $i$. Therefore, $\rho_V(\alpha) = 0$ for every representation because we have shown this for every irreducible component. In particular, $\rho_{\mathbb{C}[G]}(\alpha) = 0$ and in particular $\rho_{\mathbb{C}[G]}(\alpha)(1) = \alpha = 0$ so $\alpha = 0$. Therefore $\rho$ is injective and thus an isomorphism. $\square$

**Theorem 6.5** (Hard)**.** Suppose $K$ is a field of characteristic zero then,

$$K[G] \cong \mathrm{End}\,(D_1) \times \cdots \times \mathrm{End}\,(D_h)$$

where $D_i$ is not necessarily a field but a division ring.

**Lemma 6.6.** The center $Z(\mathbb{C}[G]) \cong Z$ the set of class functions.

*Proof.* Suppose $g \in Z(\mathbb{C}[G])$ if and only if $\forall g \in \mathbb{C}[G]$ we have $f * g = g * f$. Thus,

$$f \in Z(\mathbb{C}[G]) \iff \delta_x * f = f * \delta_x \iff f(x^{-1}y) = f(yx^{-1}) \iff f(h) = f(xhx^{-1}) \iff f \in Z$$

$\square$

**Remark 1.** We will sometimes refer to $\rho : \mathbb{C}[G] \to \operatorname{End}(V_1) \times \cdots \times \operatorname{End}(V_h)$ as the Fourier transform.

**Proposition 6.7.** For $(A_1, \cdots, A_n) \in \operatorname{End}(V_1) \times \cdots \times \operatorname{End}(V_h)$ we have,

$$\rho^{-1}(A_1, \cdots, A_n) = \sum_{g \in g} t_g \cdot g$$

where

$$t_g = \frac{1}{|G|} \sum_{i=1}^{h} d_i \operatorname{Tr}\big(\rho_{V_i}(g^{-1}) \cdot A_i\big)$$

*Proof.* We know that $\rho$ is an isomorphism so $\rho$ takes any basis of $\mathbb{C}[G]$ to an basis of $End V_1 \times \cdots \times \operatorname{End}(V_h)$. $\square$

## Classical Finite Fourier Analysis

Let $G$ be an abelian group.

**Definition:** The dual group is $\hat{G} = \{\lambda : G \to \mathbb{C}^\times \mid \lambda \text{ is a homo.}\}$ with pointwise multiplication.

**Proposition 6.8.** $|\hat{G}| = |G|$

*Proof.* Suppose the group $G$ is cyclic, all its irreducible representations are finite. Therefore, there is a one-to-one correspondence between irreducible representations and homomorphisms $\lambda : G \to \mathbb{C}^\times$. However, there are exactly $|G|$ irreducible representations because in an abelian group every element defines a distinct conjugacy class. $\square$

**Proposition 6.9.** For a finite group $G \cong \hat{G}$ (but not naturally) and $G \cong \hat{\hat{G}}$ naturally.

**Definition:** The Fourier transform is a map $\mathbb{C}[G] \to \mathbb{C}[\hat{G}]$ given by $f \mapsto \hat{f}$ where,

$$\hat{f}(\lambda) = |G| \langle f, \lambda \rangle = \sum_{g \in G} f(g) \lambda(g)$$

**Proposition 6.10.** The Fourier transform satisfies,

- $\widehat{f_1 * f_2} = \hat{f}_1 \cdot \hat{f}_2$

- Inversion: $f = \frac{1}{|G|} \sum\limits_{\lambda \in G} \hat{f}(\lambda) \cdot \lambda$ such that $f = \hat{\hat{f}}$ up to normalization.

- $\langle f_1, f_2 \rangle = \frac{1}{|G|} \left\langle \hat{f}_1, \hat{f}_2 \right\rangle$

*Proof.* Because $\lambda$ forms a unitary basis,

$$f = \sum_{\lambda \in \hat{G}} \langle f, \lambda \rangle \cdot \lambda = \frac{1}{|G|} \sum_{\lambda} \hat{f}(\lambda) \cdot \lambda$$

Furthermore,

$$\langle f_1, f_2 \rangle = \sum_{\lambda \in \hat{G}} \langle f_1, \lambda \rangle \langle \lambda, f_2 \rangle = \sum_{\lambda \in \hat{G}} \langle f_1, \lambda \rangle \langle \lambda, f_2 \rangle = \frac{1}{|G|^2} \sum_{\lambda \in \hat{G}} \hat{f}_1(\lambda) \overline{\hat{f}_2}(\lambda) = \frac{1}{|G|} \left\langle \hat{f}_1, \hat{f}_2 \right\rangle$$

$\square$

**Theorem 6.11.** Let $G$ be a finite abelian group then the map,

$$ev : G \to \hat{\hat{G}}$$

is an isomorphism and $ev : f \mapsto \hat{\hat{f}} = |G| f(g^{-1})$.

# 7 One-Dimensional Representations

**Theorem 7.1.** Let $G$ be finite. The number of one-dimensional representations of $G$ is the order of $G^{ab}$.

*Proof.* Any one-dimensional representation is given by a homomorphism $\lambda : G \to \mathbb{C}^\times$. However, $\mathbb{C}^\times$ is abelian so such homomorphisms are in one-to-one correspondence with homomorphisms $G^{ab} \to \mathbb{C}^\times$ i.e. to the group $\widehat{G^{ab}}$. Therefore, the number of one-dimensional representations is $|G^{ab}|$ and thus this number divides $|G|$. $\square$

**Lemma 7.2.** A subgroup $N \triangleleft G$ such that $N \subset G'$ and $G/N$ is abelian then $N = G'$

*Proof.* We know that $G/N$ is abelian and $\pi : G \to G/N$ is a homomorphism so $G' \subset \ker \pi = N$. Thus, $N = G'$. $\square$

# 8 Product Groups

**Theorem 8.1.** Let $\rho_{V_1}$ be an irreducible $G_1$-representation and $\rho_{V_2}$ be an irreducible $G_2$-representation then $\rho_{V_1 \otimes V_2} : G_1 \times G_2 \to \mathrm{Aut}\,(V_1 \otimes V_2)$ given by,

$$\rho_{V_1 \otimes V_2}(g_1, g_2) = \rho_{V_1}(g_1) \otimes \rho_{V_2}(g_2)$$

is an irreducible $G_1 \times G_2$ representation and every irreducible $G_1 \times G_2$ representation is of this form.

*Proof.* The chracter is given by,

$$\chi_{V_1\otimes V_2}(g_1,g_2) = \mathrm{Tr}(\rho_{V_1\otimes V_2}(g_1,g_2))) = \mathrm{Tr}(\rho_{V_1}(g_1))\cdot\mathrm{Tr}(\rho_{V_2}(g_2)) = \chi_{V_1}(g_1)\cdot\chi_{V_2}(g_2)$$

Therefore,

$$\begin{aligned}\langle\chi_{V_1\otimes V_2},\chi_{V_1\otimes V_2}\rangle &= \frac{1}{|G_1\times G_2|}\sum_{(g_1,g_2)\in G_1\times G_2}|\chi_{V_1\otimes V_2}(g_1,g_2)|^2\\ &= \frac{1}{|G_1||G_2|}\sum_{g_1\in G}|\chi_{V_1}(g_1)|^2\sum_{g_2\in G}|\chi_{V_2}(g_2)|^2 = \langle\chi_{V_1},\chi_{V_1}\rangle\cdot\langle\chi_{V_2},\chi_{V_2}\rangle = 1\end{aligned}$$

and therefore $\rho_{V_1\otimes V_2}$ is irreducible.

Furthermore, (WIP) □

# 9 Burnside's Theorem

**Definition:** $c(x) = |Cl(x)|$ is the size of the conjugacy class of $x$.

**Lemma 9.1.** If $G$ is finite and $\rho_V$ is a $G$-representation, then $\chi_V(g)$ is an algebraic integer.

*Proof.* We know that $\rho_V(g)$ is diagonalizable and each eigenvalue is a root of unity because $\rho_V(g)^n = \rho_V(g^n) = \rho_V(e) = \mathrm{id}$. Therefore, $\chi_V(g) = \mathrm{Tr}(\rho_V(g))$ is the sum of roots of unity which is an algebraic integer. □

**Theorem 9.2.** Let $V$ be an irreducible $G$-representation with $\dim V = d_V$ then for all $g\in G$ the number $\frac{c(g)}{d_V}\chi_V(g)$ is an algebraic integer.

*Proof.* Define the map $\rho_V : \mathbb{C}[G]\to\mathrm{End}\,(V)$ by,

$$\rho_V(f) = \sum_{g\in G} f(g)\rho_V(g)$$

We know that since $V$ is irreducible if $f$ is a class function then,

$$\rho_V(g) = \frac{|G|\,\langle f,\overline{\chi_V}\rangle}{\dim V}\cdot\mathrm{id}$$

Since $\delta_{Cl(x)}$ is a class function,

$$\rho_V(\delta_{Cl(x)}) = \frac{|G|\,\left\langle\delta_{Cl(x)},\overline{\chi_V}\right\rangle}{d_V}\cdot\mathrm{id}$$

but we know that,

$$\left\langle\delta_{Cl(x)},\overline{\chi_V}\right\rangle = \frac{1}{|G|}\sum_{g\in G}\delta_{Cl(x)}(g)\chi_V(g) = \frac{1}{|G|}\sum_{g\in Cl(x)}\chi_V(g) = \frac{c(x)}{|G|}\chi_V(x)$$

since $\chi_V$ is a class function. Therefore,

$$\rho_V(\delta_{Cl(x)}) = \frac{c(x)}{d_V}\chi_V(x) \cdot \mathrm{id}$$

Therefore,

$$\frac{c(x)}{d_V}\chi_V(x)$$

is the eigenvalue of the map $\rho_V(\delta_{Cl(x)}$ which must be an algebraic integer. $\square$

**Theorem 9.3** (Frobenius)**.** If $V$ is irreducible then $d_V \mid |G|$.

*Proof.* $\langle \chi_V, \chi_V \rangle = 1$ so $|G| = \sum_{g\in G} \chi_V(g)\overline{\chi_V(g)}$. We wrtite $G$ as the disjoint union over conjugacy classes. Thus,

$$|G| = \sum_{i=1}^{n} \sum_{g\in Cl(x_i)} \chi_V(g)\overline{\chi_V(g)} = \sum_{i=1}^{h} c(x_i)\chi_V(x_i)\overline{\chi_V(x_i)}$$

Therefore,

$$\frac{|G|}{d_V} = \sum_{i=1}^{h} \left( \frac{c(x_i)\chi_V(x_i)}{d_V} \right) \overline{\chi_V(x_i)}$$

is the sum of products of algebraic integers and thus an algebraic integer. Therefore, $|G|/d_V$ is an algebraic integer but also rational. therefore $|G|/d_V \in \mathbb{Z}$ so $d_V \mid |G|$. $\square$

**Lemma 9.4.** Let $\lambda_1, \cdots, \lambda_d$ be roots of unity. Then,

1. $|\lambda_1 + \cdots + \lambda_d| \leq d$ with equality iff $\lambda_1 = \cdots = \lambda_d$.
2. $\alpha = \frac{1}{d}(\lambda_1 + \cdots + \lambda_d)$ is an algebraic integer if and only if $\alpha = 0$ or $\lambda_1 = \cdots = \lambda_d$.

*Proof.* $\square$

**Lemma 9.5.** Let $G$ be finite and $V$ any $G$-representation of dimension $d = d_V$ then,

1. $\forall g \in G : |\chi_V(g)| \leq d_V$ with equality iff $\rho_V(g) = \frac{\chi_V(g)}{d_V}\mathrm{id}$
2. $\forall g \in G : \chi_V(g) = d_V \iff \rho_V(g) = \mathrm{id} \iff g \in \ker \rho_V$.

*Proof.* We know that $\rho_V(g)$ is diagonalizable with eigenvalues which are roots of unity. Therefore $\chi_V(g) = \lambda_1 + \cdots + \lambda_d$. Thus, $|\chi_V(g)| \leq d_V$ with equality iff $\lambda_1 = \cdots = \lambda_d = \frac{\chi_V(g)}{d_V}$ so $\rho_V(g) = \frac{\chi_V(g)}{d_V}\mathrm{id}$. Furthermore,

$$\chi_V(g) = d_V \implies |\chi_V(g)| = d_V \implies \rho_V(g) = \frac{\chi_V(g)}{d_v}\mathrm{id} = \mathrm{id}$$

And clearly if $\rho_V(g) = \mathrm{id}$ then $\chi_V(g) = \mathrm{Tr}(\mathrm{id}) = d_V$. $\square$

**Corollary 9.6.** A finite group $G$ is not simple iff there exists a nontrivial irreducible $G$-representation $V$ such that $\exists g \in G\backslash\{e\} : \chi_V(g) = \chi_V(e) = d_V$.

*Proof.* $G$ is not simple if there exists $N \triangleleft G$ such that $N$ is nontrivial and proper. Therefore, $G/N$ is not isomorphic to $G$ or $\{e\}$. Therefore, there must exist a nontrivial representation $\rho_V : G/N \to \mathrm{Aut}\,(V)$ of $G/N$ which lifts under $\pi : G \to G/N$ to a representation $\pi^*\rho_V = \rho_V \circ \pi : G \to \mathrm{Aut}\,(V)$.

Converseley, choose $\rho_V$ which is a nontrivial irreducible $G$-representation such that $\exists g \in G\backslash\{e\} : \chi_V(g) = \chi_V(e) = d_V$. Then, $\ker \rho_V \triangleleft G$ but $\ker \rho_V \neq G$ since $\rho_V$ is nontrivial. However, there exists $g \in G\backslash\{e\}$ such that $\chi_V(g) = d_V$ which implies that $g \in \ker \rho_V$ so $\ker \rho_V$ is nontrivial. Thus, $G$ is not simple becuase $\ker \rho_V$ is a nontrivial proper subgroup. □

**Proposition 9.7.** Let $G$ be a finite group, let $V$ be an irreducible $G$-representation suppose that $\gcd\,(c(g), d_V) = 1$ then $\chi_V(g) = 0$ or $\rho_V(g) = \lambda \cdot \mathrm{id}$.

*Proof.* Since $gcd(c(x), d_V) = 1$ we know that $\exists a, b \in \mathbb{Z}$ such that $ac(x) + bd_V = 1$ but,

$$\frac{\chi_V(g)}{d_V} = (ac(x) + bd_V)\frac{\chi_V(g)}{d_V} = a\left(\frac{c(x)\chi_V(g)}{d_V}\right) + b\chi_V(g)$$

which is the sum of algebraic integers. Thus, $\frac{\chi_V(g)}{d_V}$ is an algebraic integer. However, $\chi_V(g) = \lambda_1 + \cdots + \lambda_d$ is a sum of roots of unity. Therefore, since $\frac{1}{d}(\lambda_1 + \cdots + \lambda_d)$ is an algebraic integer, we know that $\lambda_1 + \cdots + \lambda_d = 0$ so $\chi_V(g) = 0$ or $\lambda_1 = \cdots = \lambda_d$ so $\chi_V(g) = \lambda \cdot \mathrm{id}$. □

**Corollary 9.8.** Let $G$ be a finite simple nonabelian group and $V$ a nontrivial irreducible $G$-representation then $\gcd\,(c(g), d_V) = 1 \implies \chi_V(g) = 0$.

*Proof.* $G$ is simple so $\rho_V$ is injective since $\ker \rho_V$ is normal and $\rho_V$ is nontrivial. Therefore, take $g$ as in the condition, if $\chi_V(g) \neq 0$ then $\rho_V(g) = \lambda \cdot \mathrm{id}$. Therefore, $\rho_V(g) \in Z(\mathrm{Aut}\,(V))$ so $\mathrm{Im}(\rho_V)$ is abelian so $G' \subset \ker \rho_V = \{e\}$. Therefore $G' = \{e\}$ which implies that $G/G' \cong G$ is abelian which contradicts the assumption that $G$ is nonabelian. Thus, $\chi_V(g) = 0$. □

**Theorem 9.9.** Let $G$ be a nonabelian finite simple group let $g \in G\backslash\{e\}$ then $c(g)$ is not a prime power.

*Proof.* Suppose that $|Cl(g)| = p^a$ for some prime $p$. If $a = 0$ then $a \in Z(G)$ but $Z(G) \neq G$ because $G$ is nonabelian so $Z(G)$ is a nontrivial proper normal subgroup contradicting simplicity. Let $V$ be an irreducible $G$-representation. If $\gcd\,(c(x), d_V) = 1$ then $\chi_V(g) = 0$. Therefore, if $p \not| \, d_V$ then $\chi_V(g) = 0$ so either $p \mid d_V$ or $\chi_V(g) = 0$. Consider,

$$\chi_{\mathrm{reg}}(g) = 0 = \sum_{i=1}^{h} d_i \chi_{V_i}(g) = 1 + \sum_{i=2}^{h} d_i \chi_{V_i}(g)$$

However, $\chi_V(g) = 0$ or $p \mid d_i$ so $\frac{d_i \chi_{V_i}(g)}{p}$ is an algebraic integer. Therefore,

$$\frac{1}{p} \sum_{i=2}^{h} d_i \chi_{V_i}(g) = -\frac{1}{p}$$

is an algebraic integer but $\frac{-1}{p}$ is rational so it would need to be in $\mathbb{Z}$ which is clearly false. Thus, $|Cl(g)| = p^a$ is false. $\square$

**Theorem 9.10** (Burnside)**.** If $|G| = p^a q^b$ for primes $p, q$ and $a, b \geq 1$ then $G$ is not simple.

*Proof.* Assume that $G$ is simple. We know that $G$ cannot be abelian because $G$ does not have prime order. However, for all $g \in G$ we know that $c(g)$ is not a prime power. However,

$$|G| = p^a q^b = \sum_{i=1}^{h} |Cl(x_i)| = 1 + \sum_{i \geq 2}^{h} |Cl(x_i)|$$

However, the nontrivial conjugacy classes divide $p^a q^b$ and cannot be prime powers so they each must be divisible by $pq$. Thus,

$$p^a q^b = 1 + \sum_{i \geq 2}^{h} |Cl(x_i)| \equiv 1 \bmod p \quad \text{and} \quad p^a q^b = 1 + \sum_{i \geq 2}^{h} |Cl(x_i)| \equiv 1 \bmod q$$

which are clearly contradictions. $\square$

# 10 Induced Representations

**Definition:** Let $G$ be a finite group and $H \subset G$ a subgroup then the induced representation,

$$\mathrm{Ind}_H^G(W) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

as a left $\mathbb{C}[G]$ module thus a $G$-representation. Alternatively,

$$\mathrm{Ind}_H^G(W) = \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W) = \{f : G \to W \mid f(hg) = \rho_W(h) f(g)\}$$

**Proposition 10.1.** Properties of the induced representation.

1. $$\mathrm{Ind}_H^G(\mathbb{C}) \cong \mathbb{C}[G/H]$$

2. $$\mathrm{Ind}_G^G(V) \cong V$$

**Remark 2** (Notation)**.** Let $x_1, \cdots, x_n$ be representatives for $G/H$. Then, $gx_i \in gx_iH = x_{j(i,g)}H$ so $gx_i = x_{j(i,g)} h_i(g)$

We want to determine the structure $\mathrm{Ind}_H^G(W)$.

**Definition:** For $w \in W$, let $F_{i,w} : G \to W$ be given by,

$$F_{i,w}(g) = \rho_W(h)^{-1}(w)$$

where $g = x_i h \in x_i H$ and zero otherwise.

**Proposition 10.2.** Properties of $F_{i,w}$,

1. $F_{i,w} \in \mathrm{Ind}_H^G(W)$
2. $F_{i,w_1+w_2} = F_{i,w_1} + F_{i,w_2}$
3. $F_{i,t\cdot w} = t \cdot F_{i,w}$
4. $W^{(i)} = \{F_{i,w} \mid w \in W\}$ is a vector subspace of $\mathrm{Ind}_H^G(W)$ and,
$$W^{(i)} = \{F \in \mathrm{Ind}_H^G(W) \mid F(g) = 0 \text{ if } g \notin x_i H\}$$
5. $\forall F \in \mathrm{Ind}_H^G(W)$ we have $F = \sum_{i=1}^{k} F_{i,w_i}$ where $w_i = F(x_i)$.
6. We have the isomorphism of vectorspaces,
$$\mathrm{Ind}_H^G(W) \cong \bigoplus_{i=1}^{k} W^{(i)}$$
Therefore,
$$\dim \mathrm{Ind}_H^G(W) = k \dim W = [G : H] \dim W$$

**Proposition 10.3.**
$$\rho_{\mathrm{Ind}_H^G(W)}(g) \cdot F_{i,w} = F_{j(i,g),\rho_W(h_i(g))\cdot w}$$

*Proof.* Consider, $\rho(g)\cdot F_{i,w}(x_\ell) = F_{i,w}(g^{-1}x_\ell)$. Now, $g^{-1}x_\ell \in x_i H$ so $x_\ell \in g x_i H = x_j H$ therefore zero unless $\ell = j$. Assume that $\ell = j$ then $\rho(g) \cdot F_{i,w}(x) = F_{i,w}(g^{-1}x$ but $x \in x_j H$ so $x = x_j h$ □

**Theorem 10.4** (Frobenius Reciprocity)**.**

$$\mathrm{Hom}^H\left(W, \mathrm{Res}_H^G(U)\right) \cong \mathrm{Hom}^G\left(\mathrm{Ind}_H^G(W), U\right)$$

**Theorem 10.5.** For any class functions $f_1 : H \to \mathbb{C}$ and $f_2 : G \to \mathbb{C}$ we have,

$$\left\langle f_1, \mathrm{Res}_H^G(f_2)\right\rangle_H = \left\langle \mathrm{Ind}_H^G(f_1), f_2\right\rangle_G$$

*Proof.* and the right hand side is,

$$\left\langle \mathrm{Ind}_H^G(f_1), f_2\right\rangle_G = \frac{1}{|G|}\sum_{g\in G} \mathrm{Ind}_H^G(f_1)(g)\overline{f_2(g)} = \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{x\in G}\tilde{f}_1(x^{-1}gx)\overline{f_2(g)}$$

Rewriting,

$$\left\langle \mathrm{Ind}_H^G\left(f_1\right), f_2\right\rangle_G = \frac{1}{|G|\cdot|H|}\sum_{g,x\in G}\tilde{f}_1(x^{-1}gx)\overline{f}_2(g)$$
$$= \frac{1}{|G|\cdot|H|}\sum_{g,x\in G}\tilde{f}_1(g)\overline{f}_2(xgx^{-1}) = \frac{1}{|H|}\sum_{g\in G}\tilde{f}_1(g)\overline{f}_2(g)$$

where I have used the fact that $f_2$ is a $G$-class function. However, $\tilde{f}(g) = 0$ unless $g \in h$ so the left hand side becomes,

$$\left\langle \mathrm{Ind}_H^G\left(f_1\right), f_2\right\rangle_G = \frac{1}{|H|}\sum_{h\in H} f_1(h)\overline{f_2(h)} = \left\langle f_1, \mathrm{Res}_H^G\left(f_2\right)\right\rangle_H$$

$\square$

**Corollary 10.6.**

$$\left\langle \chi_W, \chi_{\mathrm{Res}_H^G(U)}\right\rangle_H = \left\langle \chi_{\mathrm{Ind}_H^G(W)}, \chi_U\right\rangle_G$$

**Theorem 10.7** (Projection Formula)**.**

$$\mathrm{Ind}_H^G\left(W\otimes \mathrm{Res}_H^G\left(U\right)\right) = \left(\mathrm{Ind}_H^G\left(W\right)\right)\otimes U$$

**Corollary 10.8.**

$$\mathrm{Ind}_H^G\left(\mathrm{Res}_H^G\left(V\right)\right) = \mathrm{Ind}_H^G\left(\mathrm{Res}_H^G\left(\mathbb{C}\otimes V\right)\right) = \mathbb{C}[G/H]\otimes V$$

**Definition:**

**Theorem 10.9.** Suppose that $W$ is irreducible then $\mathrm{Ind}_H^G\left(W\right)$ is irreducible if and only if $\forall x \in G\backslash H$ the representations $W$ and $W_x$ are not isomorphic $G$-representations.

*Proof.*

$$\left\langle \chi_{\mathrm{Ind}_H^G(W)}, \chi_{\mathrm{Ind}_H^G(W)}\right\rangle_G = \left\langle \chi_W, \chi_{\mathrm{Res}_H^G\left(\mathrm{Ind}_H^G(W)\right)}\right\rangle_H$$

$\square$

**Definition:** Let $H \subset G$ and $[G : H] = 2$ then define the homomorphism $\epsilon : G \to \{\pm 1\} \subset \mathbb{C}^\times$ by,

$$\epsilon(g) = \begin{cases} 1 & g \in H \\ 0 & g \notin H \end{cases}$$

**Theorem 10.10.** Let $V$ be an irreducible $G$-representation, $W = \mathrm{Res}_H^G\left(V\right)$ and let $V\otimes\epsilon$ correspond to $\epsilon\rho_V$. Then, exactly one of the following holds,

1. $V \cong V\otimes\epsilon$ and $W \cong W' \oplus W'_x$ where $W'$ is irreducible and $W' \ncong W'_x$ and $V \cong \mathrm{Ind}_H^G\left(W'\right) \cong \mathrm{Ind}_H^G\left(W'_x\right)$.
2. $V \ncong V\otimes\epsilon$ and $W \cong W_x$ is irreducible and $\mathrm{Ind}_H^G\left(W\right) \cong V\otimes\left(V\otimes\epsilon\right)$.

## 11 Real Representations

**Definition:** A $G$-representation $\rho_V : G \to \mathrm{Aut}(V)$ is real if $V$ is an $\mathbb{R}$-vectorspace.

**Proposition 11.1.** If $\rho_V$ is a real representation then $V \cong V^*$ as a $G$-reprsentation.

*Proof.* If $\rho_V$ is real then $\chi_V$ is real so $\chi_V = \overline{\chi_V}$ and thus $V \cong V^*$. $\square$

**Remark 3.** The condition $V \cong V^*$ is not sufficient to show that $\rho_V$ is the complexification of a real representation.

**Theorem 11.2.** Let $V$ be an irreducible $G$-representation then,

1. $V \ncong V^*$ and $V$ cannot be defined over $\mathbb{R}$ if and only if $(\mathrm{Bil}\, V)^G = 0$.
2. $V \cong V^*$ and $V$ cannot be defined over $\mathbb{R}$ if and only if $\dim \left(\bigwedge^2 V^*\right)^G = 1$.
3. $V \cong V^*$ and $V$ can be defined over $\mathbb{R}$ if and only if $\dim (\mathrm{Sym}\, V)^G = 1$.

*Proof.* We know that $\mathrm{Bil}\, V \cong \mathrm{Hom}(V, V^*)$ so $(\mathrm{Bil}\, V)^G = \mathrm{Hom}^G(V, V^*) = 0$ if and only if $V \ncong V^*$.

Furthermore, $\square$

## 12 Representations of the Symmetric Group

**Remark 4.** For any $n$ we always have the 1-dimensional (irreducible) representations $\mathbb{C}$ and $\mathbb{C}(\epsilon)$ and the $n$-dimensional permutation representation $\mathbb{C}^n \cong \mathbb{C} \oplus V$ where $V$ is an $(n-1)$-dimensional irreducible $S_n$-representation.

**Lemma 12.1.** Any $\sigma \in S_n$ can be written as a unique product of disjoint nontrivial cycles $\gamma_1 \cdots \gamma_k$ ordered by length. The cycle type of $\sigma$ is $(n_1, \cdots, n_k)$ where $n_i$ is the length of $\gamma_i$. Futhermore, there is a one-to-one correspondence between cycle types and conjugacy classes.

**Definition:** $\lambda$ is a partition of $n$ written as $\lambda \vdash n$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ such that,

$$\sum_{i=1}^{\ell} \lambda_i = n$$

**Proposition 12.2.** Every $\sigma \in S_n$ determines a partition of $n$. Furthermore, the action of $\langle \sigma \rangle$ on $S_n$ by partition $S_n$ into orbits of sizes $\lambda_i, \ldots, \lambda_\ell$.

**Proposition 12.3.** Conjugacy classes of $S_n$ are indexed by partitions $\lambda \vdash n$.

**Definition:** The Young Subgroup of a partition $\lambda \vdash n$ is the group $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ where $\sigma \in S_\lambda$ means that $\sigma$ preserves the partition $\lambda$ of the set $\{1, \cdots, n\}$.

**Definition:** For each $\lambda \vdash n$ we get an $S_n$-representation,

$$M^\lambda = \mathbb{C}[S_n/S_\lambda] = \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C})$$

For example, for the extreme partitions $\lambda = (n)$ we have $S_\lambda = S_n$ so $M^{(n)} = \mathbb{C}[S_n/S_n] = \mathbb{C}$. Furthermore, if $\lambda = (1, \cdots, 1)$ then $S_\lambda = \{e\}$ so $M^{(1,\cdots,1)} = \mathbb{C}[S_n]$ the regular representation.

**Definition:** Given two partitions $\lambda, \mu \vdash n$ then $\lambda$ dominates $\mu$ writen as $\lambda \trianglerighteq \mu$ if,

$$\forall i : \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

**Proposition 12.4.** Domination is a partial order on the set of paritions of $n$ and for any $\lambda \vdash n$ we have $(n) \trianglerighteq \lambda \trianglerighteq (1, \cdots, 1)$