Definition 0.0.1. Take $\alpha \in \mathbb{C}$ to be an algebraic integer with minmal polynomial,

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_n$$

Then,

$$\mathrm{tr}\,(\alpha) = \sum_{\text{conjugates}} \sigma(\alpha) = -a_{n-1}$$

Then the absolute trace is,

$$\mathrm{atr}\alpha = \frac{1}{\deg \alpha} \sum_{\text{conjugates}} \sigma(\alpha) = -\frac{a_{n-1}}{n}$$

Call α totally positive (TPAI = totally positive algebraic integer) if all roots of p are positive reals (if $\mathbb{Q}(\alpha)$ is totally real and α always embeds as a positive number).

There are TPAIs with atr $= 1, \frac{3}{2}, \frac{5}{3}, \ldots$ approaching 2. If $\zeta^p = 1$ and $\zeta \neq 1$ and p is an odd prime then $\zeta + \zeta^{-1} + 2$ as atr < 2.

0.1 Schur-Siegel-Smyth Trace Problem

Choose an enummeration $\alpha_1, \alpha_2, \ldots$ of TPAI define,

$$\lambda_{SSS} = \liminf_{k \to \infty} \mathrm{atr}(\alpha_k) \leq k$$

We want to show that $\lambda_{SSS} = 2$.

0.2 Serre

There are lots of results like $\lambda_{SSS} > 1.79$. Serre showed that their method cannot prove $\lambda_{SSS} > \lambda_{\text{Serre}}$ where $\lambda_{\text{Serre}} \simeq 1.899$.

0.3 What Alex Showed

Proved that $1.809 > \lambda_{SSS} > 1.802$ so the conjecture is false.

0.4 Methods

We two methods.

0.4.1 Integrality of Resultants

Suppose α is an AI with conjugates $\alpha_1, \ldots, \alpha_n$. Take $\mathbb{Q}$ be an integer polynomial and suppose that $Q(\alpha) \neq 0$ then,

$$\prod_{i \leq n} |Q(\alpha_i)| \geq 1$$

0.4.2 Integrality of Discriminants

$$\prod_{i<j} |\alpha_i - \alpha_j|^2 \geq 1$$

0.4.3 Analysis

Definition 0.4.1. Given $\alpha \in \mathbb{C}$ take δ_α to be the Borel measure on $\mathbb{C}$ defined by,

$$\delta_\alpha(Y) = \begin{cases} 1 & \alpha \in Y \\ 0 & \alpha \notin Y \end{cases}$$

Given $P(z) = a_n(z-\alpha_1)\cdots(z-\alpha_n)$ then we define,

$$\mu_P = \tfrac{1}{n}(\delta_{\alpha_1} + \cdots + \delta_{\alpha_n})$$

Proposition 0.4.2. If α is an AI with min poly P and given an int poly Q with $P + Q$ then,

$$\int \log|Q(z)| \mathrm{d}\mu_O \geq 0$$

0.4.4 Smyth Approach

Find integer polynomials $Q_1, \ldots, A_N$ and positive numbers $a_1, \ldots, a_N, \lambda$ so that ,

$$t \geq \lambda + \sum_{i \leq N} a_i \log\left(Q_i(t)\right.$$

for $t \geq 0$. Suppose α is TPAI with min poly P and $p \not| Q_1, \ldots, Q_N$ then,

$$\operatorname{atr}(\alpha) = \int t \mathrm{d}\mu_P(t) \geq \int \lambda \mathrm{d}\mu_P + \sum_{i \leq N} a_i \int \log|Q_i| \mathrm{d}\mu_P \geq \lambda$$

The exception list is exactly the roots of the $Q_1, \ldots, Q_N$.

Proposition 0.4.3. Suppose every probability measure μ on $\mathbb{R}^{\geq 0}$ satisfying,

(a) $\int \log|Q_i| \mathrm{d}\mu \geq 0$

(b) $I(\mu) = \int \int \log|z - w| \mathrm{d}\mu(z) \mathrm{d}\mu(w) \geq 0$

also satisfies,

$$\int t \mathrm{d}\mu \geq \lambda$$

Then $\lambda_{SSS} \geq \lambda$.

0.4.5 Schur

Uses no polynomials but incorporates the discriminant.

0.4.6 Siegel

Uses only one polynomial $Q_1(x) = x$.

0.4.7 Alexander Smith

Uses about 10 polynomials. Key ingredients in the proof, given a sequence of prob. measures $\mu_1, \mu_2, \ldots$ on $\mathbb{P}^1_{\mathbb{C}}$ there is a measure μ and subsequence, $n_1, n_2, \ldots$, so $\mu_{n_k} \to \mu$ converge in the weak-$*$ topology (this is because $\mu_i(\mathbb{P}^1_{\mathbb{C}}) = 1$ for all i so this is in the unit ball). Also for every $f : \mathbb{C}^\infty \to \mathbb{R}$ continuous then,

$$\lim_{k \to \infty} \int f \mathrm{d}\mu_{n_k} = \int f \mathrm{d}\mu$$

Remark. This is just because $C(X)^*$ is the space of finite sined measures on X and the probability measures are the unit sphere which is inside the unit ball which is weak-$*$ compact. For what it's worth $(L^1)^* = L^\infty$ and $(L^\infty)^*$ are finite sined measures absolutely continuous with respect to the measure defining L^∞.

Theorem 0.4.4.

Let Σ be a finite union of