Physics GR8040 General Relativity
Assignment # 2

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1.

Let $\tilde{\epsilon}^{\alpha\beta\gamma\delta}$ be the totally antisymmetric symbol in $d = 1 + 3$ dimensional Minkowski space. Define,

$$\epsilon^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\alpha\beta\gamma\delta}$$

Consider a coordinate transformation $\Lambda^{\mu}_{\mu'}$ which preserves handedness i.e. $\det \Lambda = \det \Lambda^{\mu}_{\mu'} > 0$. Using the well-known cofactor expansion formula,

$$\Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} \Lambda^{\gamma'}_{\gamma} \Lambda^{\delta'}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} \Lambda^{\gamma'}_{\gamma} \Lambda^{\delta'}_{\delta} \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\alpha\beta\gamma\delta} = \frac{\det \Lambda^{-1}}{\sqrt{-g}} \tilde{\epsilon}^{\alpha'\beta'\gamma'\delta'}$$

However,

$$g' = \det g'_{\alpha'\beta'} = \det \left(\Lambda^{\alpha}_{\alpha'} \Lambda^{\beta}_{\beta'} \right) = (\det \Lambda)^2 \det g_{\alpha\beta} = (\det \Lambda)^2 g$$

Therefore, since $\det \Lambda > 0$ we have,

$$\sqrt{-g'} = \sqrt{-g} \det \Lambda$$

and thus,

$$\Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} \Lambda^{\gamma'}_{\gamma} \Lambda^{\delta'}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g'}} \tilde{\epsilon}^{\alpha'\beta'\gamma'\delta'} = \epsilon^{\alpha'\beta'\gamma'\delta'}$$

proving that $\epsilon^{\alpha\beta\gamma\delta}$ transforms as a $(4, 0)$-tensor.

2.

Let a^{ij} be a $(2, 0)$-tensor in Euclidean $\mathbb{R}^3$. Let $\vec{e}_i$ be the coordinate basis vectors for the given (unprimed) coordinates. Then, $a^{ij} \vec{e}_i \otimes \vec{e}_j$ is an invariant object. Therefore,

$$\frac{\partial}{\partial x^k} (a^{ij} \vec{e}_i \otimes \vec{e}_j) = (\partial_k a^{ij}) \vec{e}_i \otimes \vec{e}_j + a^{ij} (\partial_k \vec{e}_i) \otimes \vec{e}_j + a^{ij} \vec{e}_i \otimes (\partial_k \vec{e}_j)$$

must transform tensorially with rank $(0, 1)$ since it is the derivative of an invariant. However, expressing these quantities in terms of Christoffel symbols we find,

$$\begin{aligned}
\frac{\partial}{\partial x^k} (a^{ij} \vec{e}_i \otimes \vec{e}_j) &= (\partial_k a^{ij}) \vec{e}_i \otimes \vec{e}_j + a^{ij} \Gamma^m_{ki} \vec{e}_m \otimes \vec{e}_j + a^{ij} \Gamma^m_{kj} \vec{e}_i \otimes \vec{e}_m \\
&= (\partial_k a^{ij}) \vec{e}_i \otimes \vec{e}_j + a^{lj} \Gamma^i_{kl} \vec{e}_i \otimes \vec{e}_j + a^{il} \Gamma^j_{kl} \vec{e}_i \otimes \vec{e}_j \\
&= \left(\partial_k a^{ij} + a^{lj} \Gamma^i_{kl} + a^{il} \Gamma^j_{kl} \right) \vec{e}_i \otimes \vec{e}_j
\end{aligned}$$

Therefore, since $\vec{e}_i \otimes \vec{e}_j$ transforms tensorially with rank $(0,2)$ and the contraction is an invariant, we must have that

$$\nabla_k a^{ij} = \partial_k a^{ij} + a^{lj}\Gamma^i_{kl} + a^{il}\Gamma^j_{kl}$$

is a tensor of rank $(2,1)$. Furthermore, in Cartesian coordinates, $\Gamma^k_{ij} = 0$ identically so,

$$\nabla_k a^{ij} = \partial_k a^{ij}$$

Any tensor which extends $\partial_k a^{ij}$ must then agree with this expression for $\nabla_k a^{ij}$ in Cartesian coordinates and thus by the tensor property must agree in all coordinate systems.

3.

Consider Euclidean space E^3 with shperical coordinates (r, θ, ϕ). We can parametrize Cartesian coordinates via,

$$(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta)$$

Therefore, we can find the unit vectors via,

$$\vec{e}_i = \frac{\partial \vec{R}}{\partial q^i} = \frac{\partial}{\partial q^i}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right)$$

In terms of spherical coordinates these give,

$$\begin{aligned}
\vec{e}_r &= \frac{\partial}{\partial r}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right) = \hat{\boldsymbol{\imath}}\sin\theta\cos\phi + \hat{\boldsymbol{\jmath}}\sin\theta\sin\phi + \hat{\boldsymbol{k}}\cos\theta \\
\vec{e}_\theta &= \frac{\partial}{\partial \theta}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right) = \hat{\boldsymbol{\imath}}r\cos\theta\cos\phi + \hat{\boldsymbol{\jmath}}r\cos\theta\sin\phi - \hat{\boldsymbol{k}}r\sin\theta \\
\vec{e}_\phi &= \frac{\partial}{\partial \phi}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right) = -\hat{\boldsymbol{\imath}}r\sin\theta\sin\phi + \hat{\boldsymbol{\jmath}}r\sin\theta\cos\phi
\end{aligned}$$

From these we may compute the metric,

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad \text{and} \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Then the Christoffel symbols may be computed from the formula,

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{km}\left(\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}\right)$$

Plugging in, I find,

$$\Gamma^r_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{pmatrix} \qquad \Gamma^\theta_{ij} = \begin{pmatrix} 0 & r^{-1} & 0 \\ r^{-1} & 0 & 0 \\ 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad \Gamma^\phi_{ij} = \begin{pmatrix} 0 & 0 & r^{-1} \\ 0 & 0 & \cot\theta \\ r^{-1} & \cot\theta & 0 \end{pmatrix}$$

4.

Consider the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which satisfies,

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu \quad \text{and} \quad \partial_\mu * F^{\mu\nu} = 0$$

The electromagnetic tensor takes on the explict form,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Consider the transformation associated to a rotation about the y-axis,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Then the 2-form F as a matrix transforms as,

$$F' = \Lambda F \Lambda^\top = \begin{pmatrix} 0 & -E_x\cos\theta - E_z\sin\theta & -E_y & -E_z\cos\theta + E_x\sin\theta \\ E_x\cos\theta + E_z\sin\theta & 0 & B_z\cos\theta - B_x\sin\theta & -B_y \\ E_y & -B_z & 0 & B_x\cos\theta + B_z\sin\theta \\ E_z\cos\theta - E_x\sin\theta & B_y & -B_x\cos\theta - B_z\sin\theta & 0 \end{pmatrix}$$

Therefore the fields $\mathbf{E}$ and $\mathbf{B}$ transform as vectors under rotation,

$$\begin{aligned} &E_x \mapsto E_x\cos\theta + E_z\sin\theta \quad E_y \mapsto E_y \quad E_z \mapsto E_z\cos\theta - E_x\sin\theta \\ &B_x \mapsto B_x\cos\theta + B_z\sin\theta \quad B_y \mapsto B_y \quad B_z \mapsto B_z\cos\theta - B_x\sin\theta \end{aligned}$$

Now consider the transformation associated to a boost along the z-axis,

$$\Lambda = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$$

Then the 2-form F as a matrix transforms as,

$$\begin{aligned} F' &= \Lambda F \Lambda^\top \\ &= \begin{pmatrix} 0 & -E_x\cosh\eta + B_y\sinh\eta & -E_y\cosh\eta - B_x\sinh\eta & -E_z \\ E_x\cos\eta - B_y\sinh\eta & 0 & B_z & -B_y\cosh\eta + E_x\sinh\eta \\ E_y\cosh\eta + B_x\sinh\eta & -B_z & 0 & B_x\cosh\eta + E_y\sinh\eta \\ E_z & B_y\cosh\eta - E_x\sinh\eta & -B_x\cosh\eta - E_y\sinh\eta & 0 \end{pmatrix} \end{aligned}$$

Therefore the fields $\mathbf{E}$ and $\mathbf{B}$ transform under z-boosts by,

$$\begin{aligned} &E_x \mapsto E_x\cosh\eta - B_y\sinh\eta \quad E_y \mapsto E_y\cosh\eta + B_x\sinh\eta \quad E_z \mapsto E_z \\ &B_x \mapsto B_x\cosh\eta + E_y\sinh\eta \quad B_y \mapsto B_y\cosh\eta - E_x\sinh\eta \quad B_z \mapsto B_z \end{aligned}$$

5.

The electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies,

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu \quad \text{and} \quad \partial_\mu * F^{\mu\nu} = 0$$

In components,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \qquad * F^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

Furthermore, the charge current four vector has components,

$$j^\nu = (c\rho, \mathbf{j})$$

Therefore, the first equation becomes,

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad -\frac{\partial}{\partial t}\mathbf{E} + \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

And the second becomes,

$$\nabla \cdot \mathbf{B} = 0 \qquad \frac{\partial}{\partial t}\mathbf{B} + \nabla \times \mathbf{E} = 0$$

6.

(a)

The fact that any exact form is closed is equivalent to the condition $\mathrm{dd}\omega = 0$ for any p-form ω. Consider,

$$(\mathrm{dd}\omega)_{\mu_1 \dots \mu_{p+2}} = (p+2)(p+1)\partial_{[\mu_1}\partial_{[\mu_2}\omega_{\mu_3 \dots \mu_{p+2}]]}$$

Now, each term $\partial_{[\mu_i}\partial_{\mu_j}\omega_{\sigma_1 \dots \sigma_p]}$ is symmetric in i and j since partial derivatives commute. However, the entire form is antisymmetric in all variables so each of these terms vanishes. Thus, $\mathrm{dd}\omega = 0$.

(b)

Let ω be a p-form. Consider,

$$\nabla_{[\mu_1}\omega_{\mu_2 \dots \mu_{p+1}]} = \partial_{[\mu_1}\omega_{\mu_2 \dots \mu_{p+1}]} + \Gamma^\nu_{[\mu_1 \mu_2}\omega_{\nu \dots \mu_{p+1}]} + \cdots + \Gamma^\nu_{[\mu_1 \mu_{p+1}}\omega_{\mu_2 \dots \nu]}$$

However $\Gamma^\alpha_{\mu\nu}$ is symmetric in $\mu \iff \nu$ and therefore each additional term must give zero when anti-symmetrized over indices including both lower indices of Γ. Therefore,

$$\nabla_{[\mu_1}\omega_{\mu_2 \dots \mu_{p+1}]} = \partial_{[\mu_1}\omega_{\mu_2 \dots \mu_{p+1}]} = (\mathrm{d}\omega)_{\mu_1 \dots \mu_{p+1}}$$

7.

Let ω be a p-form and η be a q-form. We need to show that $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta$. I will prove this claim by induction on q. First, suppose that $q = 0$ then $\eta = f$ some smooth function and thus,

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}(f\omega))_{\mu_1 \dots \mu_{p+1}]} = (p+1)\partial_{[\mu_1}(f\eta_{\mu_2 \dots \mu_{p+1}]}) = (p+1)(\partial_{[\mu_1}\eta_{\mu_2 \dots \mu_{p+1}]})f + (p+1)(\partial_{[\mu_1} f)\eta_{\mu_2 \dots \mu_{p+1}]}$$

Now I can swap the order of the indices in the last term by introducing p swaps and thus p sign flips,

$$\mathrm{d}(\omega \wedge \eta) = (p+1)(\partial_{[\mu_1}\eta_{\mu_2 \dots \mu_{p+1}]})f + (p+1)(-1)^p \eta_{[\mu_2 \dots \mu_{p+1}}(\partial_{\mu_1]} f) = \mathrm{d}\omega \wedge f + (-1)^p \omega \wedge \mathrm{d}f$$

Where the factor $(p+1) = \frac{(p+1)!}{p!1!}$ is absorbed into the definition of $\omega \wedge \mathrm{d}f$. Now we assume the induction hypothesis that $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta$ for forms of any p and fixed q which we are inducting on. Now consider the $p+1$-form $\eta \wedge \beta$ where β is a 1-form. Clearly this is not a general $p+1$-form however we have shown that the above formula respects linear combination and scaling by smooth functions. Therefore, it suffices to prove the induction step for $p+1$-forms which can be writen as $\eta \wedge \beta$ since any $p+1$-form can then be built from linear combinations and scaling by smooth functions. Therefore, I must show,

$$\mathrm{d}(\omega \wedge \eta \wedge \beta) = \mathrm{d}(\omega \wedge \eta) \wedge \beta + (-1)^{p+q}(\omega \wedge \eta) \wedge \mathrm{d}\beta$$

for 1-forms. Given this,

$$\begin{aligned}
\mathrm{d}(\omega \wedge \eta \wedge \beta) &= \mathrm{d}\omega \wedge \eta \wedge \beta + (-1)^p \omega \wedge \mathrm{d}\eta \wedge \beta + (-1)^{p+q} \omega \wedge \eta \wedge \beta \\
&= \mathrm{d}\omega \wedge (\eta \wedge \beta) + (-1)^p \omega \wedge [\mathrm{d}\eta \wedge \beta + (-1)^q \eta \wedge \mathrm{d}\beta] \\
&= \mathrm{d}\omega \wedge (\eta \wedge \beta) + (-1)^p \omega \wedge \mathrm{d}(\eta \wedge \beta)
\end{aligned}$$

Proving the induction step. Therefore it suffices to prove the claim for $q = 1$ i.e. that for any p-from ω and any 1-form η that,

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta$$

To show this,consider

$$(\mathrm{d}(\omega \wedge \eta))_{\mu_1 \dots \mu_{p+2}} = (p+1)^2 \partial_{[\mu_1}(\omega_{[\mu_2 \dots \mu_{p+1}}\eta_{\mu_{p+2}]]})$$

Because ω is totally antisymmetric,

$$(p+1)\omega_{[\mu_2 \dots \mu_{p+1}}\eta_{\mu_{p+2}]} = \sum_i (-1)^{p+2-i} \omega_{\mu_2 \dots \mu_{p+2}} \eta_{\mu_i}$$

and therefore,

$$\begin{aligned}
(\mathrm{d}(\omega \wedge \eta))_{\mu_1 \dots \mu_{p+2}} &= (p+1)\sum_i (-1)^{p+2-i} \partial_{[\mu_1}\omega_{\mu_2 \dots \mu_{p+2}}\eta_{\mu_i]} = (p+1)\sum_i (-1)^{p+2-i} \partial_{[\mu_1}\omega_{\mu_2 \dots \mu_{p+2}}\eta_{\mu_i]} \\
&= \sum_{i,j} (-1)^{p+2-i}(-1)^{j-1} \partial_{\mu_j}(\omega_{\mu_2 \cdots \mu_{p+2}}\eta_{\mu_i}) \\
&= \sum_{i,j} (-1)^{p+2-i}(-1)^{j-1} (\partial_{\mu_j}\omega_{\mu_2 \cdots \mu_{p+2}}\eta_{\mu_i} + \omega_{\mu_2 \cdots \mu_{p+2}}\partial_{\mu_j}\eta_{\mu_i}) \\
&= \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta
\end{aligned}$$

where the factor of $(-1)^p$ comes from properly reordering the indices by swapping the index of the derivative operator through the p indices of ω.