1 Godeaux-Serre Varities

The main theorem:

Theorem 1.0.1. k field and G is a finite commutative k-group scheme then there exists a smooth projective geometrically connected 3-dimensional k-scheme X such that,

$$\mathrm{Pic}^{\tau}_{X/k} \cong G^{\vee}$$

1.1 Other examples

(a) For any finite abstract group G there exists X with $\pi_1^{\text{ét}}(X_{\bar{k}}) \cong G$

(b) failure of Hodge symmetry in characteristic p

(c) failure of lifting of surfaces in char p

(d) if done in families then jumping of Hodge and de Rham numbers in mixed or equal char p

1.2 Finite Commutative Group Schemes

If H is a finite abstract group then then there is a finite constant k-group H.

If H is a finte abstract group with a $G_k = \mathrm{Gal}\left(k^{\text{sep}}/k\right)$-action α then we get a finite étale group scheme H_α representing the functor,

$$H_\alpha(k') = H^{\mathrm{Gal}(k^{\text{sep}}/k')}$$

for k' finite seperable over k.

Proposition 1.2.1. This is an equivalence of categories between finite étale k-groups and finite groups with a Galois action.

Remark. In characteristic 0, all finite group schemes are étale and thus all arise from the above constructions.

Example 1.2.2. μ_n over $\mathbb{Q}$ corresponds to $\mu_n(\overline{\mathbb{Q}})$ as a $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$-module.

Example 1.2.3. μ_p and α_p in characteristic p are (nontrivial) connected finite group schemes

Remark. A sequence of k-groups,

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{\pi} G'' \longrightarrow 1$$

is exact if it is an exact sequence of fppf sheaves. This is equivalent to $\pi : G \to G''$ a faithfully flat surjection and G' is its kernel.

Theorem 1.2.4 (Connected-étale sequence)**.** There is a canonical short exact sequence,

$$1 \longrightarrow G^0 \longrightarrow G \xrightarrow{\pi} G/G^0 \longrightarrow 1$$

where G^0 is connected and G/G^0 is étale. Furthermore, if k is perfect then this sequence splits.

Remark. What is G/G^0? At first it is just the fppf quotient sheaf. However, we want this to be representable.

Theorem 1.2.5 (SGA3, Exp V, Thm. 4.1)**.** Let X be a quasi-projective scheme over k and G is a finite k-group scheme acting on X. Then the ringed space quoteint X/G is a quasi-projective scheme and the map $\pi : X \to X/G$ is finite. If G acts freely then π is a G-torsor (meaning $\pi : G \to X/G$ if fppf and $G \times X \to X \times_{X/G} X$ is an isomorphism).

Remark. In general, if G acts freely, then "all good properties" of X descend to X/G e.g.

(a) smoothness

(b) normality

(c) cohen-macaullayness

by faithfully flat descent.

Remark. Even if $G \circlearrowright X$ is not free X/G exists (in the generality of the theorem) and X/G is the coarse space of the stack $[X/G]$.

1.3 Cartier Duality

For a finite commutative k-group scheme G we can construct the dual,

$$G^\vee = \mathscr{H}om(G, \mathbb{G}_m)$$

where this is the sheaf,

$$G^\vee(S) = \mathrm{Hom}_{\mathrm{gp}}\left(G_S, \mathbb{G}_m\right)$$

It turns out that $G^\vee$ is representable by $\mathrm{Spec}\left(k[G]^*\right)$ (see Mumford's book on abelian varities).

Theorem 1.3.1. The functor $G \mapsto G^\vee$ is an anti-equivalence sending short exact sequences to short exact sequences and $G \to (G^\vee)^\vee$ is an isomorphism.

Example 1.3.2. $(\mathbb{Z}/n\mathbb{Z})^\vee = \mathscr{H}om(\mathbb{Z}/n\mathbb{Z}, \mu_n) = \mu_n$. Furthermore, $\alpha_p^\vee = \alpha_p$. By double duality $\mu_n^\vee = \mathbb{Z}/n\mathbb{Z}$.

Remark. Therefore we can classify finite group schemes into four types: étale- étale, étale- infinitessimal, infinitessimal - étale, infinitessimal - infinitessimal. Examples of these three types are $\mathbb{Z}/n\mathbb{Z}$ for $p \not| n$ and $\mathbb{Z}/p\mathbb{Z}$ and μ_p and α_p.

Remark. Cartier duality works for finite commutative group schemes over any base.

1.4 Picard Schemes

For X a k-scheme, define the functor,

$$\mathrm{Pic}^{\mathrm{pre}}_{X/k}(S) = \mathrm{Pic}\left(X \times_k S\right)$$

and define $\mathrm{Pic}_{X/k}$ to be the fppf sheafification.

Theorem 1.4.1. If X is projective and geometrically intergral then $\mathrm{Pic}_{X/k}$ is represented by a locally finite type k-scheme and $\mathrm{Pic}_{X/k}(\bar{k})/\mathrm{Pic}^0_{X/k}(\bar{k})$ is a finitely generated abelian group.

Remark. We denote $\mathrm{Pic}^0_{X/k}$ the identity component. And $\mathrm{Pic}^\tau_{X/k}$ the union of components which are torsion in $\mathrm{Pic}_{X/k}/\mathrm{Pic}^0_{X/k}$ which is finite type.

Example 1.4.2. If X is a curve then $\mathrm{Pic}_{X/k}$ is smooth. To see this we can verify the infinitessimal criterion. If A is Artin local and $I \subset A$ is square zero then there is an exact sequence of sheaves,

$$1 \longrightarrow \mathcal{O}_X \otimes I \longrightarrow \mathcal{O}_{X_A}^\times \longrightarrow \mathcal{O}_{X_{A/I}}^\times \longrightarrow 0$$

and from the long exact sequence of cohomology,

$$H^1(X_A, \mathcal{O}_{X_A}^\times) \longrightarrow H^1(X_{A/I}, \mathcal{O}_{X_{A/I}}^\times) \longrightarrow H^2(X, \mathcal{O}_X \otimes I)$$

but $\dim X = 1$ so $H^2(X, \mathcal{O}_X \otimes I) = 0$ and therefore the map $\mathrm{Pic}\,(X_A) \to \mathrm{Pic}\left(X_{A/I}\right)$ is surjective.

Example 1.4.3. If X is smooth, then $\mathrm{Pic}^\tau_{X/k}$ is proper. Verify this using the valuative criterion.

Example 1.4.4. If X is a smooth curve of genus g then $\mathrm{Pic}^0_{X/k}$ is a smooth proper commutative group scheme of dimension g. We have,

$$T_0 \mathrm{Pic}_{X/k} \cong H^1(X, \mathcal{O}_X)$$

Use the previous construction with $A = k[\epsilon]$ and,

$$T_0 \mathrm{Pic}_{X/k} = \ker\left(\mathrm{Pic}_{X/k}(k[\epsilon]) \to \mathrm{Pic}_{X/k}(k)\right)$$

Remark. If $X(k) \neq \varnothing$ then $\mathrm{Pic}_{X/k}(S) = \mathrm{Pic}(X_S)/\mathrm{Pic}\,(S)$.

Remark. When X is not proper, why is $\mathrm{Pic}_{X/k}$ not nec. representable. The functorial criterion for loc. fin. pres: if Y is a scheme then Y is lpf iff for any direct limit $A = \varinjlim A_i$ we have $Y(A) = \varinjlim Y(A_i)$.

Using spreading out, $\mathrm{Pic}_{X/k}$ always satisfies this but $T_0 \mathrm{Pic}_{X/k} = H^1(X, \mathcal{O}_X)$ is infinite dimensional so it can't be the tangent space of a finite type scheme.

Example 1.4.5. For X a nodal curve over k and Y a cuspidal curve over k we have $\mathrm{Pic}^0_{X/k} = \mathbb{G}_m$ and $\mathrm{Pic}^0_{Y/k} = \mathbb{G}_a$. "Pinching two points adds a $\mathbb{G}_m$" and "collapsing a tangent vector introduces a $\mathbb{G}_a$".

Lemma 1.4.6. If $X \subset \mathbb{P}^n$ is a complete intersection of dimension ≥ 3, then $\mathrm{Pic}_{X/k} \cong \mathbb{Z}$.

Proof. Ihe claim that $\mathrm{Pic}\,(X) \cong \mathbb{Z}$ is a *Leftschetz-Theorem* (SGA2, Exp. XII, Cor. 3.7). The only other point is to show that it is étale which follows from $H^1(X, \mathcal{O}_X) = 0$. ☐

Proposition 1.4.7. For X a complete intersection $\dim X = d \geq 1$ and $N \geq 0$ then,

$$H^i(X, \mathcal{O}_X(-N)) = 0$$

for $1 \leq i \leq d-1$ and,

$$H^0(X, \mathcal{O}_X(-N)) = \begin{cases} k & N = 0 \\ 0 & \end{cases}$$

Proof. Induction. ☐

Lemma 1.4.8. If $\pi : Y \to X$ is a G-torsor, then $\ker\left(\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{Y/k}\right) \cong G^\vee$.

Proof. By fppf descent for line bundles: a line bundle on X is the same as a G-linearized line bundle on Y and the map forgets the linearization. ☐

1.5 The Main Construction

We want to construct X smooth projective $\dim X = 3$ such that $\mathrm{Pic}^\tau_{X/k} \cong G$. In light of the previous lemmas, it would be enough to find a complete intersection $X \subset \mathbb{P}^n$ with a free action of G such that X/G is smooth because then because this is a finite group scheme,

$$\mathrm{Pic}^\tau_{X/G} = \ker\left(\mathrm{Pic}^\tau_{X/G} \to \mathrm{Pic}^\tau_X\right) \cong G^\vee$$

because $\mathrm{Pic}^\tau_X = 0$.

(a) Find a projective space $P = \mathbb{P}^n$ with an action of G which is free away from codim ≥ 3.

(b) Let $Z = P/G$ be a projective k-schemem and a finite map $\pi : P \to P/G$. If $U \subset P$ is the free locus for the action of G, then $U/G \hookrightarrow Z$ is smooth and open.

(c) Bertini's theorem (Poonen's if k is finite) shows that after slicing by finitely many hypersurfaces $H_1, \dots, H_m$ so that if $Y = Z \cap H_1 \cap \cdots \cap H_m$ then Y is smooth and geometrically integral of dimension 3. Then we get a Cartesian diagram,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Given an ample line bundle $\mathcal{L}$ on Z then $\pi^*\mathcal{L}$ is ample on P (because π is finite) so $\pi^{-1}(H_i)$ is also a hypersurface in P so X is a complete intersection and $X \subset U$ so $X \to Y$ is a G-torsor.

To do (1) let G act on $\mathbb{P}((k[G]^{\otimes n})^*)$ for some integer n ($n = 3$ will work). Then the free locus of this action is complementary codimension ≥ 3 if $n \geq 3$. We will prove this when G has no nontrivial proper subgroups (if $k = \bar{k}$ this is equivalent to $G = \mu_\ell$ or $\mathbb{Z}/p\mathbb{Z}$ or α_p) but it's more difficult in general. If K is a field over k then any point $x \in P(K)$ can be lifted to $\varphi : G^n \to \mathbb{A}^1$ which is nonzero. This is because,

$$\mathbb{P}(K) = (K[G^n] \setminus \{0\})/K^*$$

and an element of $K[G^n]$ defines a map $G^n \to \mathbb{A}^1$ which has to be nonzero. The assumption shows that if $G_x \neq 0$ then $G_x = G$, so the G-invariance means,

$$\varphi(g + g_1, g + g_2, g + g_3) = \eta(g)\varphi(g_1, g_2, g_3)$$

for some $\eta : G \to \mathbb{G}_m$ because we lifted. Therefore we can define,

$$\psi(g_1, g_2, g_3) = \eta(-g_1)\varphi(g_1, g_2, g_3)$$

This is G-invariant if and only if ψ factors through $G^3 \to G^2$ but now $k[G^2]$ is codim $(\dim G)^3 - (\dim G)^2 \geq 4$ so the condition that $G_x \neq 0$ gives large enough codimension.

Remark. There is an issue with extending this proof that is for a finite commutative group scheme there might be ∞-many distinct subgroups. E.g. α_p^2 contas $\ker F_L$ for any line L passing though $0 \in \mathbb{A}^2$.