Physics GR6047 Quantum Field Theory I
Assignment # 2

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1 Problem 1

Consider the Boltzmann statistics of a gas of N noninteracting highly-relativistic identical particles in a box of volume V. The single particle dispersion relation for these particles is,

$$E = c|p|$$

and therefore the Hamiltonian is,

$$H(q_i, p_i) = \sum_{i=1}^{N} c|p_i|$$

Now the partition function is,

$$\begin{aligned} Z &= \frac{1}{N!}\frac{1}{h^{3N}} \int \mathrm{d}^3 q_1 \cdots \mathrm{d}^3 q_N \, \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_N \, e^{-\beta H} \\ &= \frac{V^N}{N!} Z_1^N \end{aligned}$$

where Z_1 is the single particle momentum partiton function,

$$\begin{aligned} Z_1 &= \frac{1}{h^3} \int \mathrm{d}^3 p \, e^{-\beta c |p|} = \int_0^\infty 4\pi p^2 \mathrm{d}p \, e^{-\beta c p} \\ &= \frac{8\pi}{\beta^3 h^3 c^3} \end{aligned}$$

Therefore,

$$Z = \frac{V^N}{N!} \left(\frac{8\pi}{\beta^3 h^3 c^3} \right)^N$$

First, we compute the free energy,

$$F = -k_B T \log Z = -N k_B T \left[\log \left(\frac{k_B^3 T^3 V}{h^3 c^3} \right) - (\log N - \log (8\pi) - 1) \right]$$

Then we can compute the entropy and pressue,

$$\begin{aligned} P &= -\left(\frac{\partial F}{\partial V} \right)_T = \frac{N k_B T}{V} \\ S &= -\left(\frac{\partial F}{\partial T} \right)_V = N k_B \left[\log \left(\frac{k_B^3 T^3 V}{h^3 c^3} \right) - (\log N - \log (8\pi) - 4) \right] \end{aligned}$$

Next, we compute the energy via,

$$E - \frac{\partial \log Z}{\partial \beta} = 3Nk_B T$$

and therfore the heat capacity is,

$$c_V = \frac{\partial E}{\partial T} = 3Nk_B$$

2 Problem 2

Consider an isotropic quantum rigid rotor with moment of inertia I which has a Hamiltonian,

$$\hat{H} = \frac{L^2}{2I}$$

which has energy levels,

$$E_J = \frac{\hbar^2}{2I} J(J+1)$$

and for each J there is a J-multiplet with $g_J = 2J+1$ states.

(a)

Then we can write down the partition function,

$$Z = \mathrm{Tr}\left(e^{-\beta \hat{H}}\right) = \sum_{J=0}^{\infty} g_J e^{-\beta E_J} = \sum_{J=0}^{\infty} (2J+1) e^{-\frac{\beta\hbar^2}{2I} J(J+1)}$$

For large temperatures we have $u = \frac{\beta\hbar^2}{2I} \ll 1$ and thus the Boltzmann factor is approximatly flat for low J values and thus the degeneracy $2J+1$ is dominant. Thus we can approximate this by an integral since,

$$\sum_{J=0}^{J'} (2J+1) = J'(J'+1) + J' \approx \int_0^{J'} (2J+1) \mathrm{d}J$$

for large J. Therefore,

$$Z \approx \int_0^{\infty} 2(2J+1) e^{-uJ(J+1)} \, \mathrm{d}J$$

Notice that,

$$2J+1 = \frac{\partial}{\partial J}[J(J+1)]$$

and therefore,

$$Z = \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}J} e^{-uJ(J+1)} \mathrm{d}J = \frac{1}{u} = \frac{2I}{\beta\hbar^2}$$

which is unreasonably simple.

(b)

The energy is then,

$$E = -\frac{\partial \log Z}{\partial \beta} = \frac{1}{\beta} = k_B T$$

Therefore, at high temperature,

$$c_V = \frac{\partial E}{\partial T} = k_B$$

(c)

At low temperature, we may apprimate the partition function by taking only the $J = 0$ and $J = 1$ states to be accessable. Then,

$$Z = 1 + 3e^{-2u}$$

Then we have,

$$E = -\frac{\partial \log Z}{\partial \beta} = \frac{\hbar^2}{I} \cdot \frac{3}{e^{\frac{\beta\hbar^2}{I}} + 3} = \frac{\hbar^2}{I} \cdot \frac{3}{e^{\frac{\hbar^2}{Ik_B T}} + 3}$$

Now define,

$$T_Q = \frac{\hbar^2}{Ik_B}$$

and thus,

$$E = k_B T_Q \cdot \frac{3}{e^{\frac{T_Q}{T}} + 3}$$

Thus,

$$c_V = k_B \left(\frac{T_Q}{T}\right)^2 \cdot \frac{3e^{\frac{\hbar^2}{Ik_B T}}}{(e^{\frac{T_Q}{T}} + 3)^2}$$

Then, at low temperatures, we have,

$$c_V \approx 3k_B \left(\frac{T_Q}{T}\right)^2 e^{-\frac{T_Q}{T}}$$

which dies exponentially because the system is gapped. This approximaton is valid for,

$$T \ll T_Q$$

3 Problem 3

Consider an ideal gas of classical identical particles with mass m and no internal degrees of freedom enclosed in a cylinder of radius b and length L. The cylinder is rotating with angular velocity ω about is symmetry axis. The ideal gas is in thermal equilibrium at temperature T in the rotating coordinate system.

(a)

We first need to compute the Hamiltonian in a rotating coordinate system. First, we write down the Lagrangian for a single particle,

$$\mathcal{L} = \tfrac{1}{2}m\dot{\vec{r}}^2$$

in the lab frame coordinates. Then, in terms of the rotating coordinate system,

$$\dot{\vec{r}} = R(t)(\dot{\vec{q}} + \vec{\omega} \times \vec{q})$$

where $R(t)$ is the rotation matrix sending the rotating frame coordinates to the lab frame coodinates. Therefore,

$$\mathcal{L} = \tfrac{1}{2}m(\dot{\vec{q}} + \vec{\omega} \times \vec{q})^2$$

Then the canonical momentum associated to q is,

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m(\dot{\vec{q}} + \vec{\omega} \times \vec{q})_j$$

Incidentally, we could write down the Euler Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_j = \frac{\partial \mathcal{L}}{\partial q_j}$$

which gives,

$$m\ddot{\vec{q}} + m(\vec{\omega} \times \dot{\vec{q}}) + m(\dot{\vec{\omega}} \times \vec{q}) = m(\dot{\vec{q}} + \vec{\omega} \times \vec{q}) \times \vec{\omega}$$

which allows us to identify the pseudoforces,

$$F_{\text{pseudo}} = -m(\dot{\vec{\omega}} \times \vec{q}) - 2m(\vec{\omega} \times \dot{\vec{q}}) - m\vec{\omega} \times (\vec{\omega} \times \vec{q})$$

Back to our problem of computing the Hamiltonian. We have,

$$H = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L} = \tfrac{1}{m}\vec{p} \cdot (\vec{p} - \vec{\omega} \times \vec{q}) - \tfrac{1}{2m}\vec{p}^2 = \frac{(\vec{p} - \vec{\omega} \times \vec{q})^2}{2m} - \frac{(\vec{\omega} \times \vec{q})^2}{2m}$$

Thus, the Hamiltonian for our gas is,

$$H(\vec{q}_i, \vec{p}_i) = \sum_{i=1}^{N} \frac{1}{2m} \left[(\vec{p}_i - \vec{\omega} \times \vec{q}_i)^2 - (\vec{\omega} \times \vec{q}_i)^2 \right]$$

(b)

Now we can compute the partition function,

$$Z = \frac{1}{N!h^{3N}} \int \mathrm{d}^3 q_1 \cdots \mathrm{d}^3 q_N \, \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_N \, \exp\left[-\frac{\beta}{2m} \sum_{i=1}^{\infty} \left((\vec{p}_i - \vec{\omega} \times \vec{q}_i)^2 - (\vec{\omega} \times \vec{q}_i)^2 \right) \right]$$

Notice that these integrals factor since the Hamiltonian for each particle decouples (no interactions). Furthermore, I can seperate the integrals into position and momentum parts as follows,

$$Z = \frac{1}{N!h^{3N}} \left(\int \mathrm{d}^3 q \, \exp\left[\frac{\beta}{2m} (\vec{\omega} \times \vec{q})^2 \right] \int \mathrm{d}^3 p \, \exp\left[-\frac{\beta}{2m} (\vec{p} - \vec{\omega} \times \vec{q})^2 \right] \right)^N$$

The integral over p appears to involve q as well but since we integrate over p first we may consider q fixed and perform a change of variables,

$$\vec{p}' = \vec{p} - \vec{\omega} \times \vec{q}$$

and the q-dependence vanishes,

$$Z = \frac{1}{N!h^{3N}} \left(\int \mathrm{d}^3 q \, \exp\left[\frac{\beta}{2m}(\vec{\omega} \times \vec{q})^2\right] \int \mathrm{d}^3 p \, \exp\left[-\frac{\beta}{2m}\vec{p}^2\right] \right)^N$$

We now consider these two integrals in detail. First,

$$Z_p \int \mathrm{d}^3 p \, \exp\left[-\frac{\beta}{2m}\vec{p}^2\right] = \left(\int_{-\infty}^{\infty} \mathrm{d}p \, e^{-\frac{\beta}{2m}p^2} \right)^3 = \left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}}$$

Next we need to integrate over the cylinder,

$$\begin{aligned} Z_q = \int \mathrm{d}^3 q \, \exp\left[\frac{\beta}{2m}(\vec{\omega} \times \vec{q})^2\right] &= \int_0^b \mathrm{d}r \int_0^L \mathrm{d}z \int_0^{2\pi} r \mathrm{d}\phi \, e^{-\frac{\beta}{2m}r^2\omega^2} \\ &= 2\pi L \int_0^b r \mathrm{d}r \, e^{-\frac{\beta}{2m}r^2\omega^2} = 2\pi L \int_0^b \frac{2m}{\beta\omega^2} \frac{\mathrm{d}}{\mathrm{d}r} e^{-\frac{\beta}{2m}r^2\omega^2} \mathrm{d}r \\ &= \frac{4m\pi L}{\beta\omega^2}\left(1 - e^{-\frac{\beta b^2\omega^2}{2m}}\right) \end{aligned}$$

Putting everything together, we find,

$$Z = \frac{1}{N!}\left(\frac{4m\pi L}{\beta h^3 \omega^2}\right)^N \left(1 - e^{-\frac{\beta b^2\omega^2}{2m}}\right)^N \left(\frac{2\pi m}{\beta}\right)^{\frac{3N}{2}}$$

(c)

The density of particles at a given radius is given by the operator,

$$\rho_r(\vec{q}_1, \ldots, \vec{q}_N) = \sum_{i=1}^{N} \delta(r_i - r)$$

The average particle density is thus given by the expectation of this operator,

$$\rho(r) = \langle \rho_r \rangle = \frac{1}{N!h^{3N}} \int \mathrm{d}^3 q_1 \, \cdots \mathrm{d}^3 q_N \, \mathrm{d}^3 p_1 \, \cdots \mathrm{d}^3 p_N \, \rho_r(\vec{q}_1, \ldots, \vec{q}_N) P(\vec{q}_1, \ldots, \vec{q}_N, \vec{p}_1, \ldots, \vec{p}_N)$$

where,

$$P(\vec{q}_1, \ldots, \vec{q}_N, \vec{p}_1, \ldots, \vec{p}_N)) = \frac{1}{Z} e^{-\beta H(\vec{q}_1, \ldots, \vec{q}_N, \vec{p}_1, \ldots, \vec{p}_N)}$$

The above integral reduces to,

$$\begin{aligned} \rho(r) &= \frac{1}{Z_q} \sum_{i=1}^{N} \int \mathrm{d}^3 q_i \, \left[\delta(r_i - r) e^{\frac{\beta}{2m}(\vec{\omega} \times \vec{q}_i)^2}\right] \\ &= \frac{N}{Z_q} \int_0^b \mathrm{d}r' \int_0^L \mathrm{d}z \int_0^{2\pi} r' \mathrm{d}\phi \, \delta(r' - r) e^{-\frac{\beta}{2m}r'^2\omega^2} \\ &= \frac{N}{Z_q}(2\pi L) \int_0^b r' \mathrm{d}r' \, \delta(r' - r) e^{-\frac{\beta}{2m}r'^2\omega^2} \\ &= \frac{N}{Z_p}(2\pi L) \, r e^{-\frac{\beta}{2m}r^2\omega^2} \end{aligned}$$

Therefore, we have shown that,

$$\rho(r) = \frac{N\omega^2}{2mk_B^2T^2}\left(1 - e^{-\frac{b^2\omega^2}{2mkT}}\right)^{-1} re^{-\frac{r^2\omega^2}{2mkT}}$$

4 Problem 4

Consider a material of n independent particles inside a weak magnetic field H. Each particle has spin J and a magnetic moment μm_J for $m_J \in \{-J, -J+1, \ldots, J\}$. The system is in thermal equilibrium at constant temperature T.

4.1 (a)

First we need to compute the partition function. The Hamiltnian is,

$$\hat{H} = \sum_{i=1}^{n} \mu \hat{S}_i \cdot H$$

and thus,

$$Z = \sum_{m\in[-J,J]^n} e^{-\beta\mu H \sum_{i=1}^n m_i} = \left(\sum_{m\in[-J,J]} e^{-\beta\mu m}\right)^n$$

However, we can sum the geometric,

$$\sum_{m\in[-J,J]} e^{-\beta\mu H m} = \frac{e^{\beta\mu HJ} - e^{-\beta\mu H(J+1)}}{1 - e^{-\beta\mu H}} = \frac{e^{\beta\mu H(J+\frac{1}{2})} - e^{-\beta\mu H(J+\frac{1}{2}}}{e^{\frac{1}{2}\beta\mu H} - e^{-\frac{1}{2}\beta\mu H}} = \frac{\sinh\left(\beta\mu H(J+\frac{1}{2})\right)}{\sinh\left(\frac{1}{2}\beta\mu H\right)}$$

We can compute the average of the particle magnetic moment over the statistical ensemble,

$$\langle M\rangle = \left\langle \sum_{i=1}^{n} \mu m_i \right\rangle = \frac{1}{Z}\sum_{m\in[-J,J]^n}\sum_{i=1}^{n} \mu m_i\, e^{-\beta\mu H \sum\limits_{i=1}^{n} m_i} = -\frac{1}{ZH}\frac{\mathrm{d}Z}{\mathrm{d}\beta} = -\frac{1}{H}\frac{\mathrm{d}\log Z}{\mathrm{d}\beta} = \frac{E}{H}$$

So in this case we see that the magnetization is also equal to the proportionality between energy and applied field. Computing,

$$-E = \frac{\mathrm{d}\log Z}{\mathrm{d}\beta} = \tfrac{1}{2}\mu H\left[\coth\left(\tfrac{1}{2}\beta\mu H\right) - (2J+1)\coth\left(\beta\mu H(J+\tfrac{1}{2})\right)\right]$$

Therefore,

$$M = \tfrac{1}{2}\mu\left[(2J+1)\coth\left(\beta\mu H(J+\tfrac{1}{2})\right) - \coth\left(\tfrac{1}{2}\beta\mu H\right)\right]$$

We can also compute the magnetic susceptibility which is the thermodynamic conjugate variable to the magnetizing field H,

$$\langle M\rangle = \left.\frac{\partial E}{\partial H}\right|_S = \left.\frac{\partial F}{\partial H}\right|_T$$

4.2 (b)

Now we apply the series,

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + O(x^5)$$

Then we find, at high T,

$$\begin{aligned} M &= \tfrac{1}{2}\mu\left[\frac{2}{\beta\mu H} + \frac{\beta\mu H}{6}(2J+1)^2 - \frac{(\beta\mu H)^3}{360}(2J+1)^4 + O(\beta^5) - \frac{2}{\beta\mu H} - \frac{\beta\mu H}{6} + \frac{(\beta\mu H)^3}{360} + O(\beta^5)\right] \\ &= \mu\left[\frac{(2J+1)^2-1}{12}\cdot\frac{\mu H}{k_B T} - \frac{(2J+1)^4-1}{720}\cdot\left(\frac{\mu H}{k_B T}\right)^3 + O(T^{-5})\right] \end{aligned}$$

5 Problem 7

Consider a classical system in phase space with $3N$ dimensions and a Hamiltonian,

$$H(\vec{r}_r, \vec{p}_i)$$

Conisder the application of an external magnetic field which has the effect of transforming the momenta,

$$\vec{p}_i \mapsto \vec{p}_i - \tfrac{e}{c}\vec{A}(\vec{r}_i)$$

Then we consider the partition function,

$$Z = \frac{1}{N!h^{3N}}\int \mathrm{d}^3r_1 \cdots \mathrm{d}^3r_N\, \mathrm{d}^3p_1 \cdots \mathrm{d}^3p_N \,\exp\left[-\beta H(r_i, p_i - \tfrac{e}{c}\vec{A}_i(r_i))\right]$$

We perform the integral over momenta first,

$$Z = \frac{1}{N!h^{3N}}\int \mathrm{d}^3r_1 \cdots \mathrm{d}^3r_N \int \mathrm{d}^3p_1 \cdots \mathrm{d}^3p_N \,\exp\left[-\beta H(\vec{r}_i, \vec{p}_i - \tfrac{e}{c}\vec{A}(\vec{r}_i))\right]$$

Notice that we may treat the r_i as fixed in the momentum integral and thus we can view $\vec{A}(\vec{r}_i)$ as a constant in this integral. Therefore, we are free to perform a substitution setting $p_i' = p_i - \frac{e}{c}\vec{A}$ and then the partition function loses its dependence on A,

$$Z = \frac{1}{N!h^{3N}}\int \mathrm{d}^3r_1 \cdots \mathrm{d}^3r_N \int \mathrm{d}^3p_1' \cdots \mathrm{d}^3p_N' \,\exp\left[-\beta H(\vec{r}_i, \vec{p'}_i)\right]$$

therefore we have proven that all thermodynamic potentials derivable from Z must be independent of the applied magnetic field.