1 Manifolds with any Finite Fundamental Group

Remark. For loops $\gamma_1, \gamma_2 : I \to X$ we will use the notation $\gamma_1 * \gamma_2$ to denote the loop,

$$h(t) = \begin{cases} \gamma_1(2t) & t \leq \frac{1}{2} \\ \gamma_2(2t-1) & t \geq \frac{1}{2} \end{cases}$$

Definition An action of of a group G on a topological space X is a homomorphism $A : G \to \text{Homeo}(X)$. Equivalently, one may define a map $\varphi : G \times X \to X$ and let $\varphi_g(x) = \varphi(g, x)$ such that $\varphi_e = \text{id}_X$ and $\varphi_{gh} = \varphi_g \circ \varphi_h$ and φ_g is a continuous map. Because $\varphi_{g^{-1}}$ is also continuous and $\varphi_g \circ \varphi_{g^{-1}} = \varphi_{g^{-1}} \circ \varphi_g = \varphi_e = \text{id}_e$ then each map is a homeomorphism of X to itself so $g \mapsto \varphi_g$ is a homomorphism from G to Homeo(X).

Definition Let G be a group acting on a topological space X then X/G is the quotient space under the equivalence relation $x \sim y \iff \exists g \in G : g \cdot x = y$.

Remark. For $x \in X$, let $[x]_G$ denote the equivalence class under a group action and for $\gamma : I \to X$ let $[\gamma]$ denote the equivalence class under path-homotopy.

Definition A group G acts freely on a set X if every stabilizer is trivial. Equivalently, if for some $x \in G$ we have $g \cdot x = h \cdot x$ then $(h^{-1}g) \cdot x = x$ so $g = h$.

Definition A group action on X is properly discontinuous if for any $x \in X$ there exists an open neighborhood $x \in U$ such that $(g \cdot U) \cap U = \varnothing$ for each $g \neq e$.

Lemma 1.1. Let the action of G on X be properly discontinuous, then X is a covering space of X/G with the covering map $\pi : X \to X/G$.

Proof. Take an open set $U \subset X$ and consider $\pi(U)$. Then, $\pi(x) \in \pi(U)$ if and only if $\exists y \in U$ such that $x \sim y \iff \exists g \in G : x = g \cdot y \iff x \in g \cdot U$. Therefore,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

which is open because each g acts as an open map (in fact a homeomorphism). By the definition of X/G then $\pi(U)$ is open so π is an open map. Take a point $x_0 \in X$ and because the action is properly discontinuous, there exists an open $x_0 \in U$ such that $(g \cdot U) \cap U = \varnothing$ for each $g \neq e$. Consider $V = \pi(U) \subset X/G$ which is open. Since for $g \neq h$, we have $(h^{-1}g) \cdot U \cap U = \varnothing$ then $(g \cdot U) \cap (h \cdot U) = \varnothing$ so the slices are disjoint. Finally, take $x, y \in g \cdot U$ then if $\pi(x) = \pi(y)$ we have $[x] = [y]$ so $x = h \cdot y$ for some $g \in G$. But since $y \in g \cdot U$ then $x \in hg \cdot U$ and $x \in g \cdot U$ so $h = e$ and thus $x = y$ because for $g \neq e$ the sets $hg \cdot U$ and $g \cdot U$ are disjoint since $hg \neq g$. Therefore, $\pi|_{g \cdot U}$ is injective but it is trivialy surjective onto $V = \pi(U) = \pi(g \cdot U)$. Furthermore, π is an open continuous map and thus a homeomorphism when restricted to U. Therefore, V is an openly covered neighborhood of $[x]_G$ so π is a covering map of X/G. $\square$

Theorem 1.2. Let X be a simply connected and a let the action of G on X be free and properly discontinuous. Then $\pi_1(X/G, [x_0]_G) \cong G$.

Proof. Fix $x_0 \in X$, then take $g \in G$ and let $\gamma_g : I \to X$ be a path from x_0 to $g \cdot x_0$. Such a path exists because X is path-connected. Take the projection map $\pi : X \to X/G$ given by $\pi(x) = [x]_G$. These paths project to loops in the quotient space, $\eta_g = \pi \circ \gamma_g$ which is a loop because $\eta_g(0) = \pi(x_0) = [x_0]$ and $\eta_g(1) = \pi(g \cdot x_0) = [g \cdot x_0] = [x_0]$ and action by g is a continuous map.

Define the map $\phi : G \to \pi_1(X/G, [x_0]_G)$ given by $\phi : g \mapsto [\pi \circ \gamma_g]$. Take $g, h \in G$ and consider the path $\delta = \gamma_g * (g \cdot \gamma_h)$ where $(h \cdot \gamma_g)(t) = h \cdot \gamma_g(t)$ with endpoints:

$$\gamma_g * (g \cdot \gamma_h)(0) = \gamma_g(0) = x_0 \text{ and } \gamma_g * (g \cdot \gamma_h)(1) = (g \cdot \gamma_h)(1) = g \cdot (h \cdot x_0) = (gh) \cdot x_0$$

Therefore, because X is simply connected, $\delta \sim \gamma_{gh}$ and thus,

$$\pi \circ \delta = (\pi \circ \gamma_g) * (\pi \circ (g \cdot \gamma_h)) \sim \pi \circ \gamma_{gh} = \eta_{gh}$$

However, $\pi \circ \gamma_g = \eta_g$ and $\pi \circ (g \cdot \gamma_h)(t) = \pi(g \cdot \gamma_h(t)) = [g \cdot \gamma_h(t)]_G = [\gamma_h(t)]_G = \eta_h(t)$ because the orbits are equivalence classses. Thus, $\pi \circ (g \cdot \gamma_h) = \eta_h$ so $\eta_g * \eta_h \sim \eta_{gh}$. Therefore, $\phi(gh) = [\eta_{gh}] = [\eta_g * \eta_h] = [\eta_g][\eta_h] = \phi(g)\phi(h)$ so ϕ is a homomorphism. It remains to show that ϕ is a bijection.

X is the universal cover of X/G so any path $\delta : I \to X/G$ can be lifted to a a unique path $\gamma : I \to X$ up to a choice of initial point. Thus, if δ is a loop at $[x_0]_G$ then there exists a unique path $\gamma : I \to X$ such that $\pi \circ \gamma = \delta$ and $\gamma(0) = x_0$. However, $\pi \circ \gamma(1) = \delta(1) = [x_0]_G$ so $[\gamma(1)]_G = [x_0]_G$ thus $\exists g \in G : \gamma(1) = g \cdot x_0$. Because X is simply connected, $\gamma \sim \gamma_g$ since they share endpoints. Finlly, $\phi(g) = [\pi \circ \gamma_g] = [\pi \circ \gamma] = [\delta]$ so the map ϕ is surjective. Finally, take $g, h \in G$ and suppose that $\phi(g) = \phi(h)$ then $\pi \circ \gamma_g \sim \pi \circ \gamma_h$. By the homotopy lifting lemma, these loops lift to unique path-homotopic paths in X with initial point x_0. However, γ_g and γ_h already satisfy the projection property and therefore must be the unique lifts so $\gamma_g \sim \gamma_h$. In particular, $\gamma_g(1) = \gamma_h(1)$ because they are path homotopic so $g \cdot x_0 = h \cdot x_0$ but because G acts freely on X this implies that $g = h$. Therefore, ϕ is a bijection. ☐

Lemma 1.3. A free action of a finite group on a Hausdorff space is properly discontinuous.

Proof. Take $x \in X$ and, because the action is free, for each $g \neq e$ we have $g \cdot x \neq x$ so because X is Hausdorff, there exist open sets U_g and V_g such that $x \in U_g$ and $g \cdot x \in U_g$ and $U_g \cap V_g$. Now, let,

$$U = \bigcap_{g \in G \setminus \{e\}} (U_g \cap g^{-1} \cdot V_g)$$

which is open because the intersection is finite. Also, for each g, $x \in U_g$ and $g \cdot x \in V_g$ so $x \in g^{-1} \cdot V_g$. Thus, $x \in U$. Now, take any $g \neq e$. We have $U \subset U_g$ and $U \subset g^{-1} \cdot V_g$ so $g \cdot U \subset V_g$. However, U_g and V_g are disjoint so U and $g \cdot U$ are disjoint. ☐

Lemma 1.4. Any quotient of a compact connected space is compact and connected.

Proof. Let X be compact and connected. Then, $\pi : X \to X/\sim$ is continuous and surjective. Therefore, $X/\sim$ is the image of a compact and conected set and is thus compact and connected. ☐

Theorem 1.5. For any finite group G, there exists a compact connected manifold with fundamental group G.

Proof. By considering $n \times n$ permutation matrices in $\mathrm{SU}(n)$ we get an embedding of S_n inside $\mathrm{SU}(n)$. However, by Cayley's theorem, any group with order n can be embedded as a subgroup of S_n. Therefore, we have the embeddings,

$$G \hookrightarrow S_n \hookrightarrow \mathrm{SU}(n)$$

Let $\Gamma \subset \mathrm{SU}(n)$ be the embedded copy of G. Γ acts on $\mathrm{SU}(n)$ by left multiplication which is a topological action because $\mathrm{SU}(n)$ is a topological group and thus has continuous multiplcation. Futhermore, if $g \cdot h = h$ then $gh = h$ so $g = e$. Thus, Γ acts freely on $\mathrm{SU}(n)$. However, $\mathrm{SU}(n)$ is a simply connected compact manifold. In particular, $\mathrm{SU}(n)$ is a Hausdorff space and Γ is finite so the action is properly discontinuous. Therefore, since $\mathrm{SU}(n)$ is simply connected, $\pi_1(\mathrm{SU}(n)/\Gamma, [x_0]) \cong \Gamma \cong G$. Furthermore $\mathrm{SU}(n)/\Gamma$ is a compact connected space because it is a quotient of $\mathrm{SU}(n)$ which is compact and connected. Finally, we must show that $\mathrm{SU}(n)/\Gamma$ is a manifold. (WIP) □

Theorem 1.6. For any finite cyclic group G, there exists a compact connected 3-manifold with fundamental group G.

Proof. Consider the matrix,

$$M = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} \in \mathrm{SU}(2)$$

Let $\Gamma = \langle M \rangle \subset \mathrm{SU}(2)$. Since M has order n, $\Gamma \cong C_n$. By an identical argument to above, $\pi_1(\mathrm{SU}(2)/\Gamma, [x_0]) \cong \Gamma \cong C_n$ and $\mathrm{SU}(2)/\Gamma$ is compact and connected. It remains to show that $\mathrm{SU}(2)/\Gamma$ is a 3-manifold. (WIP) □