*Remark.* All rings are commutative and unital.

# 1 Bases and Generating Sets

**Definition 1.0.1.** Let $M$ be an $R$-module. Elements $\alpha_1, \dots, \alpha_n \in M$ define a map $R^n \to M$. We say that,

(a) $\{\alpha_1, \dots, \alpha_n\}$ are $R$-linearly independent or simply *independent* if the map $R^n \to M$ is injective

(b) $\{\alpha_1, \dots, \alpha_n\}$ *span* $M$ if $R^n \to M$ is surjective

(c) $\{\alpha_1, \dots, \alpha_n\}$ is a basis of $M$ if $R^n \to M$ is an isomorphism.

## 1.1 The Case for Vector Spaces

**Lemma 1.1.1.** Let $V$ be a finitely generated $k$-module. Then $V$ has a basis i.e. $V \cong k^n$.

*Proof.* Since $M$ is finitely generated, there is a spaning set defining a surjection $k^n \twoheadrightarrow V$. However, $\square$

# 2 Modules over a PID

**Theorem 2.0.1.** Let $A$ be a PID. Every submodule of a free module of rank $n$ is free of rank at most $n$.

*Proof.* We proceed by induction on $n$. For $n = 1$ we consider submodules $I \subset R$ which are ideals. Since $R$ is a PID then $I = (a)$ for some $a \in A$ and thus $I \cong R$ or $I = (0)$ since $R$ is a domain proving the claim. Now we assume the claim for $n$. Consider a submodule $M \subset R^{n+1}$. Write $R^{n+1} = R \oplus R^n$ and consider the projection $\pi : R^{n+1} \twoheadrightarrow R^n$. Then $N = \pi(M) \subset R^n$ is a free module of rank at most $n$ by the induction hypothesis. Furthermore the map $\pi|_M : M \twoheadrightarrow N$ gives an exact sequence,

$$0 \longrightarrow \ker \pi|_M \longrightarrow M \longrightarrow N \longrightarrow 0$$

but $N$ is free and thus projective so this exact sequence splits givign,

$$M \cong N \oplus \ker \pi|_M$$

Furthermore, $\ker \pi|_M = M \cap (R \oplus 0) \subset R$ so again because $R$ is a PID we find $\ker \pi|_M$ is a free module of rank at most 1. Thus,

$$M = N \oplus \ker \pi|_M$$

is a free module of rank at most $n + 1$. $\square$

**Definition 2.0.2.** Let $R$ be a PID and $M$ a finite free $A$-module, $M' \subset M$ a submodule. A basis $\{v_1, \dots, v_n\}$ of $M$ and a basis $\{a_1 v_1, \dots, a_m v_m\}$ of $M'$ with $a_i \in R \setminus \{0\}$ and $m \leq n$ are called a pair of *aligned* bases. Such a bair of bases gives a map in the category of product modules,

$$\begin{tikzcd} R^m \arrow[r, dashed] \arrow[d, "\sim"] & R^n \arrow[d, "\sim"] \\ M' \arrow[r, hook] & M \end{tikzcd}$$

**Lemma 2.0.3.** Let $R$ be a PID. Any finite free $R$-module with a nonzero submodule $M' \subset M$ of rank $m \le n$ admit a pair of aligned bases. Thus there is a basis $v_1, \dots, v_n \in M$ and nonzero $a_1, \dots, a_m \in A$ such that,

$$M = Av_1 + \cdots + Av_n \quad \text{and} \quad M' = Aa_1 v_1 + \cdots + Aa_m v_m$$

*Proof.* We proceed by induction on the rank $n$ of $M$. For $n = 1$ we have $M' \subset R$ is an ideal and thus $M' = Aa$ since $R$ is a PID giving aligned bases $\{1\}$ and $\{a\}$. Now we assume the claim for $n-1$ and let $M$ be a free $R$-module of rank $n$.

Consider the poset of ideas $S = \{\varphi(M') \mid \varphi \in \mathrm{Hom}_R(M, R)\}$ ordered with respect to inclusion. Because $R$ is Noetherian $S$ contains a maximal element $I = \varphi_0(M')$ for some $\varphi_0 \in \mathrm{Hom}_R(M, R)$. Furthermore, since $R$ is a PID, $I = (a)$ for some $a \in R$. Thus we must have $a = \varphi_0(v')$ for some $v' \in M'$. For any $\varphi \in \mathrm{Hom}_R(M, R)$ consider $a_\varphi = \varphi(v')$. Since $R$ is a PID, $(a_\varphi) + (a) = (d)$ so we can write $xa_\varphi + ya = d$ and thus $(x\varphi + y\varphi_0)(v') = d$ meaning that $(a) \subset (d) \subset (x\varphi + y\varphi_0)(M')$ but $(a) = I$ is maximal in $S$ so we must have $(a) = (d)$ and thus $a_\varphi \in (a)$. Therefore we have shown,

$$\{\varphi(v') \mid \varphi \in \mathrm{Hom}_R(M, R)\} \subset (a) = I$$

Choose a basis $e_1, \dots, e_n \in M$ and write,

$$v' = c_1 e_1 + \cdots + c_n e_n \quad \text{for} \quad c_1, \dots, c_n \in R$$

Then consider the dual basis $\{e_i^* \in \mathrm{Hom}_R(M, R)\}$ such that $e_i^*(e_j) = \delta_{ij}$. Then $e_i^*(v') = c_i \in (a)$ so we can write $c_i = ab_i$ for $b_i \in R$. Let,

$$v = b_1 e_1 + \cdots + b_n e_n$$

and thus $v' = av$. Then $\varphi(v') = a$ so $a\varphi_0(v) = a$ so $\varphi_0(v) = 1$ since $R$ is a domain. Therefore, $\varphi_0 : M \twoheadrightarrow R$ is surjective. Since $R$ is a free and thus projective $R$-module, the sequence,

$$0 \longrightarrow \ker \varphi_0 \longrightarrow M \xrightarrow{\varphi_0} R \longrightarrow 0$$

splits with $R \to M$ via $1 \mapsto v$ so $M = Rv \oplus \ker \varphi_0$ as an internal direct sum. Simultaneously, $\varphi_0(M') = Ra$ so we get an exact sequence,

$$\begin{tikzcd}
0 \arrow[r] & \ker \varphi_0 \arrow[r] & M \arrow[r, "\varphi_0"] & R \arrow[r] \arrow[l, bend left] & 0 \\
0 \arrow[r] & \ker \varphi_0|_{M'} \arrow[r] \arrow[u, hook] & M' \arrow[r, "\varphi_0"] \arrow[u, hook] & Ra \arrow[r] \arrow[l, bend left] \arrow[u, hook] & 0
\end{tikzcd}$$

splits with $Ra \to M'$ via $a \mapsto av = v'$. Therefore, we get compatible decompositions i.e. an inclusion,

$$\begin{tikzcd}
M = \ker \varphi_0 \oplus R \\
M' = \ker \varphi_0|_{M'} \oplus Ra \arrow[u, hook]
\end{tikzcd}$$

in the category of products defined by the inclusions $\ker \varphi_0|_{M'} \subset \ker \varphi_0$ and $Ra \subset R$. Then $\ker \varphi_0|_{M'} \subset \ker \varphi_0$ are free modules of rank $n-1$ and $m-1$ respectively so by the induction hypothesis, $\ker \varphi_0$ and $\ker \varphi_0|_{M'}$ have aligned bases $\{v_2, \dots, v_n\}$ and $\{a_2 v_2, \dots, a_m v_m\}$ for $a_2, \dots, a_m \in R$. Then, $\{v, v_2, \dots, v_n\}$ and $\{av, a_2 v_2, \dots, a_m v_m\}$ give aligned bases for $M' \subset M$. $\square$

**Theorem 2.0.4** (Structure Theorem for Modules over a PID)**.** Let $R$ be a PID and $M$ a finitely generated $R$-module. Then,

$$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_n)$$

*Proof.* Since $M$ is finitely generated, there is a map $\varphi : R^n \twoheadrightarrow M$. Then $\ker \varphi \subset R^n$ is a free module of rank $m \leq n$. Therefore, we get an exact sequence,

$$0 \longrightarrow R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

We can choose aligned bases for $R^m \cong \ker \varphi \subset R^n$ so that the map $R^m \to R^n$ is in the category of products i.e. represented by a diagonal matrix such that $e_i \mapsto a_i e_i$ for $i = 1, \dots, m$. Therefore,

$$M \cong \frac{R \oplus \cdots \oplus R}{Ra_1 \oplus \cdots \oplus Ra_m} = R/(a_1) \oplus \cdots \oplus R/(a_m) \oplus R^{n-m}$$

$\square$