Physics C2801, Fall 2013 Prof. Brian Cole

Issued: **Oct. 10**	Problem Set # 5	Due: **Oct. 17**

Problem 1. Kleppner and Kolenkow problem 4.6

Problem 2. Kleppner and Kolenkow problem 4.8

Problem 3. Kleppner and Kolenkow problem 4.13

Problem 4. Kleppner and Kolenkow problem 4.21

Problem 5. Kleppner and Kolenkow problem 5.2

Problem 6. Kleppner and Kolenkow problem 5.4

Problem 7. Kleppner and Kolenkow problem 5.5

Problem 8. Div, curl, Stoke's theorem.

a. Show by direct calculation in Cartesian coordinates that $\vec{\nabla} \times \left(\vec{\nabla} \phi \right) = 0$ for any scalar function of position, $\phi(\vec{r})$.

b. In class, I stated that if a force satisfies $\vec{\nabla} \times \vec{F} = 0$ then there must be a function U such that $\vec{F} = -\vec{\nabla} U$ (minus sign put in by convention). I was asked about how we could be sure that the function U was unique. There's a standard way to show this uniqueness that's appropriate for you to try. Suppose that U is not unique – which means that there can be another function, call it $V(\vec{r})$, that also satisfies $\vec{F} = -\vec{\nabla} V$ for a given force. Then, we can take the difference between U and V, call it $\Delta(\vec{r})$: $\Delta = V - U$. Then, $V = U + \Delta$. Show that requiring $\vec{F} = \vec{\nabla} \times U = \vec{\nabla} \times V$ constrains Δ to be a constant.

c. Show that any force described by the function $F(\vec{r}) = f(r)\hat{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$, is a conservative force. This result was demonstrated in lecture but it's useful for you to work through it yourself. While you can find expressions for the curl in spherical coordinates, you should first do this problem using the curl expressed in Cartesian coordinates.

d. Show that any force described by the function $F(\vec{r}) = f(r)\hat{\theta}$ where $r = \sqrt{x^2 + y^2}$ is non-conservative. Here, $\hat{\theta}$ is the usual unit vector in the $x - y$ plane. This problem can be viewed as a three-dimensional problem in cylindrical coordinates which consists of our usual two-dimensional polar coordinates augmented by a linear z coordinate and corresponding unit vector $\hat{k}$. While the z coordinate won't enter in your evaluation of the curl, you will need the $\hat{k}$ unit vector to express the curl.

e. Show that Stoke's theorem $\oint \vec{F} \cdot d\vec{r} = \int (\vec{\nabla} \times \vec{F}) \cdot dA$ holds for the form for $\vec{F}(\vec{r})$ given in part d for a circular path of radius r in the $x - y$ plane centered on the origin. To apply Stoke's theorem you will need to integrate the curl over the area of the circle. For those of you that have not done multi-dimensional integration in your calculus class, convince yourself that you can express the (scalar) differential area dA in terms of differential radial and angular intervals dr and $d\theta$ as $dA = rdrd\theta$. To do so, picture arc sections of angular range $d\theta$ of two circles of radial extent r and $r + dr$. The infinitesimal area element dA is the difference between the areas of those two arc sections for infinitesimal dr (use binomial approximation). Then you can write the integral over the circle as a "double integral" over r and θ. The integral over θ is trivial so do it first, then integrate over r.

Problem 9. Energy and Galilean Transformations

This is a topic that I've found to frequently bother students and, yet, is not tackled in any introductory textbook as far as I am aware. The issue is: how does energy conservation hold in two different inertial frames when velocities change by $\vec{v}_G$ between the frames. More explicitly, because K varies as v^2, K will not simply change additively under a Galilean transformation. Which means that if energy conservation is to hold in both frames, the potential energy is somehow going to have to compensate (for conservative force) or, more generally, the work integral is somehow going to have to transform in such way that it accounts for the transformed kinetic energy.

Working this problem out will not only show you how and why the transformation of energy works but will also help us lay the foundation for Lorentz transformations that are an essential part of special relativity. We will start with the transformation of position vectors and velocities. We have seen that if an observer in some inertial frame (unprimed) sees an object with position $\vec{r}$ and velocity $\vec{v}$, the observer in an inertial frame (primed) moving with velocity $\vec{v}_G$ relative to the unprimed frame sees positions and velocities, $\vec{r'} = \vec{r} - \vec{v}_G t$, $\vec{v'} = \vec{v} - \vec{v}_G$.

Now, when evaluating how quantities like K transform between inertial frames, we will re-write the above relationships to express, $\vec{r}$ and $\vec{v}$ in terms of $\vec{r'}$ and $\vec{v'}$,

$$\vec{r} = \vec{r'} + \vec{v}_G t, \;\; \vec{v} = \vec{v'} + \vec{v}_G$$

Then, we can substitute these expressions into any quantity (e.g. K) to express how that quantity in terms of the variables in the primed frame.

a. Let's use this approach to evaluate the transformation of the kinetic energy. Take the expression for K in unprimed coordinates and substitute in the expression for $\vec{v}$ in terms of $\vec{v'}$. Use the fact that the observer in the primed frame would naturally define the kinetic energy in that frame $K' = \dfrac{1}{2} m\vec{v'}^2$ to obtain an expression for K' in terms of K and other quantities. You should end up with an equation describing the relationship between the kinetic energies in the two frames that looks like $K' = K + expr$ where $expr$ represents some expression that you will determine. For a proper statement of transformations, each side of the equation will involve quantities defined in the same frame. So for the transformation equation, $K' = K + expr$, the expression $expr$ should only involve unprimed quantities and/or constants. Use the transformation rules for the velocity to put the equation in this proper form.

The way to interpret this expression is that if an observer in the unprimed frame determines that an object has a kinetic energy with value K, the observer in the primed frame will find that the object has a different kinetic energy, K', with the difference between K and k' given by the expression $expr$.

b. Now, let's suppose that both observers watch an object moving under the influence of some applied forces and they study the motion of the object between times t_1 and t_2. The observer in the unprimed frame will observe a change in kinetic energy due to the work on the object, $\Delta K \equiv K_2 - K_1$. Find an expression for $\Delta K'$ seen in the unprimed frame in terms of ΔK, other primed quantities and any relevant constants.

c. Now let's consider what happens to the potential energy function. The transformation of explicit functions of position can be confusing at first because two things happen due to the transformation: 1) the value of the function at the same point in space may be different in the two frames, 2) the dependence of the function on coordinate can change between the two frames.

We will show here that for the force to be the same in different inertial frames, the potential energy function must have the same value *at the same point in space* in the two frames (except for an additive constant which has no physical significance). It is this last qualification that makes relating the potential energy function between the two frames conceptually difficult. Lets refer to some point in space symbolically as p. The position vector for p in the unprimed frame is $\vec{r^p}$ and the position vector in the primed frame is $\vec{r'^p}$. You know how these two position vectors are related. Let U be the potential energy energy as seen in the unprimed frame and U' be the potential energy seen in the primed frame. We will show that U and U' are equal to each other at the point p. Now, we can express both functions in terms of either set of coordinates because $\vec{r^p}$ and $\vec{r'^p}$ have a well-defined relationship. So, for example, to express the equality of the potential energies in the two frames we can write (dropping the "p"s for simplicity)

$$\begin{aligned} U(\vec{r}) &= U'(\vec{r}) & (1) \\ U(\vec{r'}) &= U'(\vec{r'}) & (2) \\ U(\vec{r} - \vec{v}_G t) &= U'(\vec{r'}) & (3) \end{aligned}$$

The first two equalities expresses that U and U' are equal at the same point in space expressed in either set of coordinates. If the $U(\vec{r'})$ or $U'(\vec{r})$ trouble you, remember that both $\vec{r}$ and $\vec{r'}$ refer to the same point in space as long as $\vec{r'} = \vec{r} - \vec{v}_G t$. So, we can use either set of coordinates though $U(\vec{r}$ and $U'(\vec{r'})$ are the most natural forms – and the ones we need to use in evaluating the mechanics in a given frame. The third equality above shows **one** example of substituting the relationship between the coordinates into one of the potential energy functions to determine the position dependence in one frame in terms of the position dependence in the other frame.

Now, how can we be certain, or how can we demonstrate that the first two equalities above hold – namely that the potential energy is the same as seen in the two frames at the same point in space. To show this, we will assume a possible difference and then see that this difference would violate Galilean relativity. So, suppose $U' = U + g$ where g is a function of position. I have left off the functional dependence to avoid confusion over coordinates. Now, let's use $\vec{r'}$ in all three functions, but then write $\vec{r'}$ in terms of $\vec{r}$. Now evaluate $-\vec{\nabla}'U'$ – i.e. the gradient of U' with respect to the primed coordinates and show that g would make the forces in the two frames different unless g is constant. You will need to use the chain rule when evaluating (e.g.) $-\vec{\nabla}'U$.

d. Now, consider $E = K + U$. For problems where all forces are conservative, we should have $dE/dt = 0$. We want to show that in the primed frame we obtain $dE'/dt = 0$. First, let's take the one-dimensional reduction of the results you obtained above (i.e. $\vec{r} \to x$, $\vec{r'} \to x'$, etc.). Show that if you evaluate dE'/dt you get zero when $F = m\dot{v}$ and $F = -dU/dx$ in the unprimed frame. *Hint: it may be helpful to take the time derivative of one of the equations relating U and U'.* You should find that dU/dt and dU'/dt differ and that difference cancels the extra factor appearing in the kinetic energy.

e. Now extend your analysis in part d to three dimensions using the gradient, e.g. $\vec{F} = -\vec{\nabla} U$. Note that the chain rule works the same for partial derivatives as it does for total derivatives. Namely, if we think of U' as a function of some dummy variable w which for our problem is $w = f(\vec{r'})$, then $\partial U'(w)/\partial x = dU'/dw \, \partial w/\partial x$. You will also need to use the result that I showed you in lecture for a total time derivative of an arbitrary function of position $h(x, y, z, t)$

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} + \frac{\partial h}{\partial z}\frac{dz}{dt} + \frac{\partial h}{\partial t}. \tag{4}$$

Show using vector quantities that $dE'/dt = 0$, i.e. that energy is conserved in the primed frame if it is conserved in the unprimed frame.