1 Week 1

1.1 Bhatt (Problems 1)

2

Let $\mathcal{A}$ be an abelian category. Any category admits all colimits iff it admits coequalizers and all coproducts (easy exercise). Since $\mathcal{A}$ is abelian it admits cokernels and therefore coequalizers and thus $\mathcal{A}$ admits all colimits iff it admits all direct sums (coproducts).

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(a) the category of finite k-vectorspaces has finite direct sums nut not countable direct sums. Likewise for countably generated vector spaces.

(b) The opposite category of torsion abelian groups.

(c)

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Let $\mathcal{C}$ be the category of torsion abelian groups. It is clear that $\mathcal{C}$ is abelian as kernels and cokernels of torsion groups are torsion since subgroups and quotients are torsion. Furthermore all direct sums exist in $\mathcal{C}$ because elements are zero all but finitely often and thus torsion since the nonzero entries are torsion.

Because $\mathcal{C}$ has cokernels and all coproducts it has all colimits. Furthermore, filtered colimits in **Ab** are exact so they are exact in $\mathcal{C}$ as well. For a generator, consider,

$$X = \bigoplus_{n \in \mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z}$$

Then for each element $a \in A$ for A a torsion abelian group we get a map $X \to A$ whose image contains a sending $\mathbb{Z}/n\mathbb{Z} \to 0$ unless n is the order of a in which case $1 \mapsto a$. Therefore we get a surjection,

$$X^{\oplus A} \twoheadrightarrow A$$

5

Let $\mathcal{A}$ be Grothendieck abelian and I a category. Let $\mathcal{C} = \mathrm{Fun}(I, \mathcal{A})$ be the functor category. Clearly, $\mathcal{C}$ is additive and admits kernels, cokernel, and infinite direct sums (constructed pointwise).

7

Let $\mathcal{A}$ be an abelian category. Now $\mathbf{Ch}(\mathcal{A})$ is the subcategory of functors from $\mathbb{Z}$ as a poset to $\mathcal{A}$ such that the composition of sucessive maps is zero. (CAN WE REDUCE THIS TO PREVIOUS EXERCISE?)

11

Let $\mathcal{A}$ be an abelian category and $\mathcal{A}^{\mathbb{N}} = \mathrm{Hom}\left(\mathbb{N}^{\mathrm{op}}, \mathcal{A}\right)$ the category of projective systems. Assume that $\mathcal{A}$ admits infinite direct sums and products.

(a) Taking limits is right adjoint to the constant diagram functor $\Delta : \mathcal{A} \to \mathcal{A}^{\mathbb{N}}$ defined via $A \mapsto (n \mapsto A)$ with identity transition maps. Therefore $\lim : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}$ preserves limits and thus is, in particular, left exact.

(b) Note that given a projective system $\{X_n\} \in \mathcal{A}^{\mathbb{N}}$,

$$\lim X_n = \ker \left(\prod_{n \in \mathbb{N}} X_n \to \prod_{n \in \mathbb{N}} X_n \right)$$

where on the n^{th} factor the map is the difference of projection $\prod X_{n'} \to X_n$ and $f_n \circ (\prod X_{n'} \to X_{n+1})$ where $f_n : X_{n+1} \to X_n$ is the transition map. (FINISH THIS)

(c)

12

Fairly obvious.

13

Let $\mathcal{A}$ be an abelian category and $f : K^\bullet \to L^\bullet$ be a map in $\mathbf{Ch}\left(\mathcal{A}\right)$. Recall that,

$$C(f) = K[1] \oplus L$$

where the differential is,

$$\mathrm{d}_{C(f)} = \begin{pmatrix} \mathrm{d}_{K[1]} & 0 \\ f[1] & \mathrm{d}_L \end{pmatrix}$$

Specifically, $C(f)^i = K^{i+1} \oplus L^i$ and $\mathrm{d}(x, y) = (-\mathrm{d}x, \mathrm{d}y + f(x))$.

(a) Let $A^\bullet \in \mathbf{Ch}\left(\mathcal{A}\right)$ be a complex. Consider,

$$g \in \mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}\left(C(f), A^\bullet\right)$$

Then $g^i = (k^i, h^i)$ where $k^i : K^{i+1} \to A^i$ and $h^i : L^i \to A^i$ which satisfy,

$$g^{i+1} \circ \mathrm{d}_C^i = \mathrm{d}_A^{i+1} \circ g^i$$

Explicitly,

$$-k^{i+1} \circ \mathrm{d}_K^{i+1}(x) + h^{i+1} \circ \mathrm{d}_L^i(y) + h^{i+1} \circ f^{i+1}(x) = \mathrm{d}_A^{i+1} \circ (k^i(x) + h^i(y))$$

Setting $x = 0$ we find that,

$$h^{i+1} \circ \mathrm{d}_L^i(y) = \mathrm{d}_A^{i+1} \circ h^i(y)$$

and therefore $h \in \mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}\left(L^\bullet, A^\bullet\right)$. Setting $y = 0$ we find that,

$$h^{i+1} \circ f^{i+1}(x) = \mathrm{d}_A^{i+1} \circ k^i(x) + k^{i+1} \circ \mathrm{d}_K^{i+1}(x)$$

therefore k is a nullhomotopy of $h \circ f$ so we see that,

$$\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}\left(C(f), -\right) = \{k : L^\bullet \to A^\bullet \text{ and } h : K^{\bullet+1} \to A^\bullet \mid h \text{ is a nullhomotopy of } k \circ f\}$$

(b) If L is acyclic then from the long exact sequence for the exact triangle,

$$K \xrightarrow{f} L \to C(f) \to K[1]$$

shows that $H^i(L) \to H^i(C(f))$ is an isomorphism.

(c)

1.2 Bhatt Lectures

2.4

Let $\mathcal{C}$ be a category such that $\mathcal{C}$ is enriched over **Ab** with finite coproducts. Given $f, g : A \to B$ there exists a map $f + g : A \to B$. To show that being abelian is a property, we must describe $f + g$ in terms of internal properties of the category. That is, there is a unique additive structure on any additive category.

Consider the map $A \to A \oplus A \to B$ defined by,

$$(f, g) \circ (\iota_1 + \iota_2) = (f, g) \circ \iota_1 + (f, g) \circ \iota_2 = f + g$$

Therefore, it suffices to show that $h = \iota_1 + \iota_2$ is internal to the category. There are zero maps $A \to 0$ (where 0 is the initial object) b/c $\mathrm{Hom}_{\mathcal{C}}(A, 0)$ has an identity. Then $(\mathrm{id}, 0) \circ h = \mathrm{id} + 0 = \mathrm{id}$ and $(0, \mathrm{id}) \circ h = \mathrm{id}$. Call $\pi_1 = (\mathrm{id}, 0)$ and $\pi_2 = (0, \mathrm{id})$ then these make $A \oplus A$ a product and h the diagonal so h is unique.

To prove this consider $a : C \to A$ and $b : C \to B$ then $q = \iota_1 \circ a + \iota_2 \circ b$ satisfies $\pi_1 \circ q = a$ and $\pi_2 \circ q = b$. Furthermore, let $q' : C \to A \oplus B$ be any map with this property. Then $q' = (\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \circ q' = \iota_1 \circ a + \iota_2 \circ b$ becuase,

$$(\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \circ \iota_i = \iota_i + 0 = \iota_i$$

and thus $(\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) = \mathrm{id}$ because $A \oplus B$ is a coproduct.

Notice that this construction only relied on the choice of a zero map $A \to 0$. However, the identity of $0 : \mathrm{Hom}_{\mathcal{C}}(0, 0)$ must be $\mathrm{id}_0 : 0 \to 0$ because 0 is initial so this set has a unique element. Therefore, for any $f : A \to 0$ we have $f = \mathrm{id}_0 \circ f = 0 \in \mathrm{Hom}_{\mathcal{C}}(A, 0)$ because id_0 is the identity of the group and $- \circ f : \mathrm{Hom}_{\mathcal{C}}(0, 0) \to \mathrm{Hom}_{\mathcal{C}}(A, 0)$ is a group map. Therefore, $\mathrm{Hom}_{\mathcal{C}}(A, 0)$ has a single element so there is no choice of zero map $A \to 0$.

Since there is a unique map $A \to 0$ we see that 0 is initial and final.

2.11

Solved in Bhatt problems 2,3.

2.20

(a) Bhatt problems 7

(b) Let $\mathcal{A}$ be a Grothendieck abelian category. We construct the injective resolution inductively. First, $X \to (X \hookrightarrow I(X))$ is functorial. Assume there is a functorial assignment,

$$X \mapsto (X \hookrightarrow I^0(X) \to I^1(X) \to \cdots \to I^n(X))$$

Then consider

$$\operatorname{coker}(I^{n-1}(X) \to I^n(X)) \hookrightarrow I(\operatorname{coker}(I^{n-1}(X) \to I^n(X))) = I^{n+1}(X)$$

which is funtorial in X because cokernels and $C \mapsto I(C)$ is thus giving,

$$X \mapsto (X \hookrightarrow I^0(X) \to I^1(X) \to \cdots \to I^{n+1}(X))$$

2 Week 2

2.1 Bhatt (Problems 1)

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(a) a

2.2 Bhatt (Problems 2)

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2.3 Bhatt (Lectures)

2.25

6.12

6.13

2.4 Tsai (Problems)

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3 Week 4

3.1 Bhatt (Problems 3)

1

Let $\mathcal{D}$ be a triangulated category equiped with a t-structure. Let $X, Y \in \mathcal{D}^{\heartsuit}$. Recall that,

$$\mathrm{Ext}^{-n}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, Y[-n])$$

Suppose that $n > 0$, since $Y \in \mathcal{D}^{\ge 0}$ we see that $Y[-n] \in \mathcal{D}^{\ge n} \subset \mathcal{D}^{\ge 1}$ and furthermore $X \in \mathcal{D}^{\le 0}$ and therefore,

$$\mathrm{Ext}^{-n}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, Y[-n]) = 0$$

when $n > 0$.

2

Let X be a topological space and $K \in D(X)$,

(a) Consider $X = \mathbb{P}^1$ and $K = \mathcal{O}_X \oplus \mathcal{O}_X(-2)[1]$ Then we consider,

$$\begin{aligned}\mathrm{Hom}_{D(X)}(K|_U, K|_U) &= \mathrm{Hom}_{D(X)}(\mathcal{O}_U, \mathcal{O}_U) \oplus \mathrm{Hom}_{D(X)}(\mathcal{O}_U, \mathcal{O}_U(-2)[1]) \\ &\oplus \mathrm{Hom}_{D(X)}(\mathcal{O}_U(-2)[1], \mathcal{O}_U) \oplus \mathrm{Hom}_{D(X)}(\mathcal{O}_U(-2)[1], \mathcal{O}_U(-2)[1]) \\ &= \Gamma(U, \mathcal{O}_U) \oplus H^1(U, \mathcal{O}_U(-2)) \oplus \Gamma(U, \mathcal{O}_U)\end{aligned}$$

which is not a sheaf because of the $H^1(U, \mathcal{O}_U(-2))$ term. We use,

$$\mathrm{Hom}_{D(X)}(\mathcal{O}_U, \mathcal{O}_U(-2)[1]) = \mathrm{Ext}^1_{D(X)}(\mathcal{O}_U, \mathcal{O}_U(-2)) = H^1(U, \mathcal{O}_U(-2))$$

(b) Suppose that $\text{Ext}^i_{K|_U}(K|_U, =) 0$ for all $i < 0$ and open $U \subset X$.

Lemma 3.1.1. If the cohomology sheaves $H^i(K) = 0$ for all $i < d$ then $U \mapsto \mathbb{H}^d(U, K)$ is a sheaf.

Proof. $K \cong \tau^{\geq d} K$ is an equivalent so we may assume K is zero in deg $< d$. Then choose a quis $K \xrightarrow{\sim} I$ for an injective resolution. Then,

$$\mathbb{H}^d(U, K) = \ker\left(I^d(U) \to I^{d+1}(U)\right)$$

and therefore $H^d(-, K) = \ker\left(I^d \to I^{d+1}\right)$ is a sheaf. □

Let L, K be complexes. Assume that $\text{Ext}^i_{D(X)}(L|_U, K|_U) = 0$ for $i < 0$ and $U \subset X$ open. Now $H^i(\text{RHom}(L, K))$ is the sheafification of,

$$U \mapsto \text{Ext}^i_{D(U)}(L|_U, K|_U)$$

(c)

3

Let $\mathcal{A}$ be an abelian category with enough projectives. Assume that $\text{Ext}^2_{\mathcal{A}}(X, Y) = 0$ for all $X, Y \in \mathcal{A}$.

(a) Let $K \in D^b(\mathcal{A})$. Choose a projective resolution $P \to K$

(b)

4

Let $D^b_f(k)$ be the derived category of bounded complexes of k-vectorspaces with finitely generated cohomology.

Lemma 3.1.2. A t-structure is determined by $\mathcal{D}^{\leq 0}$.

Proof. We can recover

$$\mathcal{D}^{\geq 1} = \{K \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}\left(\mathcal{D}^{\leq 0}, X\right)\}$$

□

5

Let $\mathcal{D}$ be a triangulared category with a t-structure. Let $K \in \mathcal{D}$ be a direct summand of $L \in \mathcal{D}^{\leq 0}$. Consider,

$$L = K \oplus F$$

We know $\tau^{\geq 1}(K \oplus F) = 0$ but $\tau^{\geq 1}$ is a left adjoint and thus preserves colimits so $\tau^{\geq 1}(K) \oplus \tau^{\geq 1}(F) = 0$ therefore $\tau^{\geq 1}(K) = 0$ so $K \in \mathcal{D}^{\leq 0}$.

6

4 Week 4

(DOOOOO THHHHIIIIIS!!!!!)

5 Week 7

5.1 Tsai (Problems 1)

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5.2 Tsai (Problems 2)

1

Let $E \subset \mathbb{P}^2$ be an elliptic curve. Let $V \subset \mathbb{A}^3$ be the corresponding affine cover over E. Let o be the origin and $U := V \setminus \{o\}$ be the smooth locus of V. Wrtie $\iota : \{o\} \hookrightarrow V$ and $j : U \hookrightarrow V$ the embeddings. We want to compute $\iota^* Rj_* \underline{\mathbb{Q}}_U$.

(DO THIISSS!!)

2

As in the last problem we want to show that $D_V \underline{\mathbb{Q}}_V[2] \ncong \underline{\mathbb{Q}}_V[2]$.

(DO THIS!!!!)

3

As in the last problem, we want to show that,

$$R\Gamma(V, \tau_{\leq -1} Rj_* \underline{\mathbb{Q}}_U[2]))$$

is dual to

$$R\Gamma_c(V, \tau_{\leq -1}(Rj_* \underline{\mathbb{Q}}_U[2]))$$

4

Let X be a smooth complete variety over $\mathcal{C}$ and let $x \in X$ be a fixed point. Let $\iota_x : \{x\} \hookrightarrow X$ be the inclusion. We want to compute $\mathscr{F} = \mathrm{RHom} \iota_x \underline{\mathbb{Q}\mathbb{Q}}_X$.