1 TODO!!

(a) Finish symplectic geometry course

  (a) figure out if symplectic toric is the same as projective toric variety (projectivity needed to come from a polytope and also to be Kahler)
  (b) review coisotropic reduced and write some notes
  (c) hyperkahler reduction examples
  (d) are there examples of noncompact hyperkahlers?
  (e) work out the kinks in notes on hamiltonian actions

(b) review killing homotopy groups columbia lectures and write some notes

(c) figure out those damn jet bundles and connections on principal bundles

  (a) RMK: $\pi^* E$ is NOT trivial for a vector bundle let alone a fiber bundle. it does get equiped with a canonical section but for a vector bundle this is just the trivial section, only for a principal bundle does giving a section trivialize it.
  (b) role of atiyah sequence vs jet bundle sequence
  (c)

(d) spectral sequences for tor and ext in derived category (FIND MY NOTES ON THIS!)

  (a) application to universal coefficient theorem
  (b) Kunneth spectral sequence
  (c) Kunneth formula for smash product?
  (d) why are derived functors triangulated
  (e) derived functors in terms of Kan extensions (NOTES)

(e) write notes on universal morphisms

(f) G-action of X/Y induces map Descent data X/Y to G-equivariant sheaves

  (a) isomorphism when X/Y is a G-cover i.e. $X \to Y$ is a G-torsor
  (b) write down explicit G-equivariant structure on Ω_X
  (c) Galois descent derive explicit form

(g) Weil restriction

  (a) write down trivialization after going back up
  (b) Galois descent in explicit form

(h) notes on Galois actions on schemes

(i) notes on Frobenii

(j) notes on universal constructions in math with examples

(k) fix notes on Tor in category of sheaves and Tor symmetry (do I need symmetry of flat objects a priori?).

(l) Finish stable homotopy theory course.

(m) Finish vector bundles and connections notes (in AG folder)

  (a) Kahler iff $\nabla I = 0$ where ∇ is the Levi-Civita connection
  (b) Ricci tensor and the trace bullshit
  (c) Riemann-Hilbert and existence of flat frames for integrable connections

2 What I Want to Think About

(a) Flat cohomology equal etale cohomology for smooth (affine groups) apply this to that counting rational points things

(b) work out the details for the group fixing $\mathbb{C}$ inside endomorphism group. What does an integrable structure of this kind look like, how close to a complex manifold can we get? In dimension two this should be exactly a conformal (not necessarily orientable) structure.

(c) FINISH CONFORMAL NOTES!

(d) Hilbert Class Field of curves (ASK BRIAN FOR REFERENCE)

(e) Read about Fredholm index and Riemann-Roch

(f) Cohmology and inclusion-exclusion: cohomology for vectorspaces?

3 Some Questions I Have

(a) Reduction of structure group for a scheme.

  (a) what about the algebraic group $\mathrm{SL}^{\pm} = \det^{-1}(\mu)$ what does reduction of structure group give. For a manifold this is supposed to be a pseudo-volume form but obviously that's not right.
  (b) what about $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \hookrightarrow \mathrm{GL}_2$ from the action $\mathbb{G}_m \circlearrowleft \mathbb{A}^1_{\mathbb{C}}$ restricted giving an action $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \circlearrowleft \mathbb{A}^2_{\mathbb{R}}$. I feel like this should give an almost complex structure. What properties does it have? What about for other fields?
  (c) What is an almost complex structure on a scheme look like?

(b) Is my calculation of an "almost almost complex structure" as reduction of structure group to $\langle \sigma \rangle \ltimes \mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$. For the case $n = 1$ this should be the conformal group justfying that I think this should correspond to the non-oriented case of a complex manifold since Riemann surfaces are exactly oriented conformal manifolds.

4 The Tautological Bundle

Consider the fibre bundle, $\pi : S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$ given by consider ing $S^{2n+1} \subset \mathbb{C}^{n+1}$ and restricting the projection $\mathbb{C}^{n+1} \to \mathbb{P}^n_{\mathbb{C}}$. Then π is a principal S^1-bundle. Consider the tautological representation $\rho : U(1) \to \mathrm{GL}_1(\mathbb{C})$ which is the inclusion $U(1) \hookrightarrow \mathbb{C}^\times$, which gives an associated line bundle $S^{2n+1} \times_\rho \mathbb{C}$. We call this the tautological bundle since its fibre above a point is the line in $\mathbb{C}^{n+1}$ which that point on $\mathbb{P}^n_{\mathbb{C}}$ corresponds to.

To see this explicitly, consider the following bundle,

$$T = \{(L, v) \mid L \in \mathbb{P}^n_{\mathbb{C}} \text{ and } v \in L \subset \mathbb{C}^{n+1}\} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$$

with the projection $\pi : T \to \mathbb{P}^n_{\mathbb{C}}$ via $(L, v) \mapsto L$. I claim that this bundle is isomorphic to the tautological bundle constructed above.

Consider the map $f : S^{2n+1} \times_\rho \mathbb{C} \to T$ via $f : [x, \lambda] \mapsto (\mathrm{Span}(x), \lambda x)$. This is clearly a bundle map since $\pi([x, \lambda]) = \pi(x) = \mathrm{Span}\left(()\, x\right) = \pi(\mathrm{Span}\left(x\right), \lambda x)$. Furthermore it is well-defined because $f([x, \mu\lambda]) = (\mathrm{Span}\left(x\right), \mu\lambda x) = (\mathrm{Span}\left(\mu x\right), \lambda\mu x) = f([\mu x, \lambda])$. We need to check that this map is injective and surjective. First, if $f([x, \lambda]) = f([y, \mu])$ then $\mathrm{Span}\left(x\right) = \mathrm{Span}\left(y\right)$ so $y = \gamma x$ for $\gamma \in \mathbb{C}^\times$ and $\lambda x = \mu y$ so $\lambda = \mu\gamma$ (since these vectors are nonzero) and thus,

$$[x, \lambda] = [x, \gamma\mu] = [\gamma x, \mu] = [y, \mu]$$

For surjectivity note that given (L, v) with $v \in L$ then $L = \mathrm{Span}\left(x\right)$ for $x \in S^{2n+1}$ and $v = \lambda x$ with $\lambda \in \mathbb{C}$ since L is a line. Thus $f([x, \lambda]) = (L, v)$.

The tautological bundle has no nonzero (holomorphic) global sections. However, there are $n + 1$ independent glboal sections of its dual. To see this consder the global $\mathrm{Hom}(T, \mathcal{O}_{\mathbb{P}})$. There exist $n + 1$ idependent functions defined by the $n + 1$ projections $p_k : \mathbb{C}^{n+1} \to \mathbb{C}$ via the construction,

$$T \hookrightarrow \mathcal{O}_{\mathbb{P}}^{n+1} = \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \xrightarrow{p_k} \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C} = \mathcal{O}_{\mathbb{P}}$$

These sections are refered to as X_k ,the k^{th} coordinate function on $\mathbb{P}^n_{\mathbb{C}}$.

Producing the coordinate functions X_k as sections of the dual $X^\vee$ identifies the tautological bundle T with the algebraic twist $\mathcal{O}_{\mathbb{P}}(-1)$ and thus its dual is the Serre twisting sheaf $T^\vee = \mathcal{O}_{\mathbb{P}}(1)$.

5 MATH 275A 2021 Lecture 2

Using the Stern-Gerlach boxes we define spin operators $\hat{S}_i$ on our Hilbert space $H = \mathcal{C}^2$. These have eigenstate $\ker \pm$ along each axis. Furthermore, we have a Hamiltonian $\hat{H}$. For a constant magnetic field, up to a constant,

$$\hat{H} = \hat{S} \cdot \vec{B}$$

For B along the z-direction,

$$\hat{H} = \hat{S}_z B$$

Then the evolution follows the Schrodinger equation,

$$i\partial_t \left|\psi\right\rangle = \hat{H} \left|\psi\right\rangle$$

For any observable (i.e. operator $\hat{A}$) we can define the expected value,

$$\left\langle \hat{A} \right\rangle_\psi = \langle\psi| \hat{A} |\psi\rangle$$

Then,

$$i\partial_t \left\langle \hat{A} \right\rangle_\psi = \left\langle [\hat{A}, \hat{H}] \right\rangle_\psi$$

Now for example, we choose $|\psi(0)\rangle = |+_x\rangle$. Then we expand,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|+_z\rangle + |-_z\rangle)$$

Then applying the evolution operator,

$$|\psi(t)\rangle = e^{-iHt} \frac{1}{\sqrt{2}} (|+_z\rangle + |-_z\rangle) = \frac{1}{\sqrt{2}} \left(e^{-i\frac{B}{2}t} |+_z\rangle + e^{i\frac{B}{2}t} |-_z\rangle \right)$$

Now we consider,

$$i\partial_t \left\langle \hat{S}_x \right\rangle = \langle\psi| \hat{S}_x |\psi\rangle = \left\langle [\hat{S}_x, \hat{H}] \right\rangle = B \left\langle [\hat{S}_x, \hat{S}_z] \right\rangle = -iB\hat{S}_y$$

and therefore,

$$\partial_t \left\langle \hat{S}_x \right\rangle = -B \left\langle \hat{S}_y \right\rangle$$

Likewise,

$$\partial_t \left\langle \hat{S}_y \right\rangle = B \left\langle \hat{S}_x \right\rangle$$

This coupled system has solution,

$$\left\langle \hat{S}_x \right\rangle = \cos(Bt) \quad \text{and} \quad \left\langle \hat{S}_y \right\rangle = \sin(Bt)$$

5.0.1 Operators

Infinite dimensional space $H = L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathcal{C} \mid \int |f|^2 < \infty\}$. We take observables to be "self-adjoint" operators on $H = L^2(\mathbb{R})$. For example, $\hat{x} = x\cdot$ and $\hat{p} = -\partial_x$. However, the eigenfunctions of these operators are not L^2 they are tempered distributions. We say,

$$\left\langle \frac{1}{\sqrt{2\pi}} e^{ipx} \middle| \frac{1}{\sqrt{2\pi}} e^{iqx} \right\rangle = \nabla(p - q)$$

5.0.2 Uncertainty Principle

Define,

$$\nabla\hat{A} = \hat{A} = -\langle\hat{x}\rangle I$$

and likewise for B two self-adjoint operators A, B. Then,

$$\left\langle (\nabla\hat{x})^2 \right\rangle_\psi \left\langle (\nabla\hat{p})^2 \right\rangle_\psi \geq \frac{1}{4} \left| \langle\psi| [\hat{A}, \hat{B}] |\psi\rangle \right|^2$$

For example,

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = iI$$

because,

$$(\hat{x}\hat{p} - \hat{p}\hat{x})\psi = x(-i\partial_x\psi) + i\partial_x(x\psi) = -i\partial_x\psi + i\psi + x\partial_x\psi = i\psi$$

Therefore,

$$\sigma_x^2\sigma_p^2 \geq \frac{1}{4}$$

5.0.3 Angular Momentum

Classical angular momentum $\vec{L} = \vec{x} \times \vec{p}$. We upgrade these to quantum self-adjoint operators. Thus we get, for example,

$$\hat{L}_z = -i(x\partial_y - y\partial_x)$$

Then $L^2 = L_x^2 + L_y^2 + L_z^2$.

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7 Some Connection Musings

Definition 7.0.1. Let $f : E \to X$ be a smooth surjection (in the smooth category, what should it be in the algebraic category?) then an *Erhesmann connection* is a splitting of the sequence of vector bundles,

$$0 \longrightarrow \ker \mathrm{d}f \longrightarrow TE \longrightarrow \pi^* TX \longrightarrow 0$$

where we usually call $V = \ker df$ the vertical bundle. In algebraic language, V is the dual of the relative differentials so the connection corresponds to a splitting of,

$$0 \longrightarrow f^* \Omega_X \longrightarrow \Omega_E \longrightarrow \Omega_{E/X} \longrightarrow 0$$

Remark. Such splittings are supposed to correspond to smooth sections of the map $J^1(E) \to E$. We now explain how this works. Unfortunately, I don't know a good unified language to describe the jet bundles so I will give the algebraic and smooth definitions.

Definition 7.0.2. Given a smooth surjection $f : E \to X$, consider the n-th thickened diagonal, $X \to X_n \to X \times_S X$. Then we consider the functor sending $T \to S$ to pairs of maps $T \to X$ and $T \times_X X_n \to E$ such that the diagram,

$$\begin{tikzcd}
E \arrow[r, "f"] & X \\
T \times_X X_n \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
T \arrow[r] & X
\end{tikzcd}$$

commutes. Then the jet scheme $J_n(E/X)$ with maps $J_n(E/X) \to X$ and $J^n(E/X) \times_X X_n \to E$ represents this functor.

8 Counterexamples In Geometry

Example 8.0.1. The Hopf surface is the compact complex surface $H = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$ where $\mathbb{Z} \circlearrowright \mathbb{C}^2$ via $(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$ for $0 < \lambda < 1$. This surface has $h^{1,0} = 1$ but $h^{0,1} = 0$. Furthermore, H is diffeomorphic to $S^3 \times S^1$. This provides:

(a) a compact complex manifold that is not Kähler

(b) a compact complex manifold without Hodge symmetry

(c) a compact complex manifold that is not symplectic $(H^2(H, \mathbb{Z}) = 0)$

Remark. From the exponential exact sequences,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

we have that,

9 Questions I was asked in interviews

9.1 Oxford

Exercise 9.1.1. Which genus Riemann surfaces have a covering map to a genus 2 Riemann surface.

From Riemann-Hurwitz we have have $2g-2 = 2n$ where $2h-2 = 2$ since $h = 2$. Thus $g = n+1$ so any genus can appear. To show that these are all actually possible, draw a picture with a central donut surrounded by $g-1$ donuts. This maps by cyclic quotienting onto a two holed torus.

9.2 LSGNT

Exercise 9.2.1. Let E be an elliptic cuve over $\mathbb{F}_p$. Given a_p how do you find $\#E(\mathbb{F}_{p^k})$?

The zeta function is,

$$\zeta_E(t) = \frac{t^2 - a_p t + p}{(1-t)(1-pt)}$$

and therefore,

$$\#E(\mathbb{F}_{p^k}) = 1 + p^k - \alpha^k - \beta^k$$

where α and β are the roots of $t^2 - a_p t + p$ which are determined via $\alpha + \beta = a_p$ and $\alpha\beta = p$.

10 Questions

(a) When people write $\mathcal{M}_\ell(\mathbb{C}) = \mathfrak{h}//\mathrm{SL2}\mathbb{Z}$ isn't this wrong because every point of $\mathcal{M}_\ell(\mathbb{C})$ is "stacky" i.e. this groupoid has $\mathbb{Z}/2\mathbb{Z}$ stabilizer at the general point due to the inversion map. However, $\mathfrak{h}//\mathrm{SL2}\mathbb{Z}$ seems to be a setoid at the general point, oh no that's wrong because $-I$ satabilizes each point and stabilizes a lattice but acts on the elliptic curve by inversion. AHH!!! This is why we retain the SL2$\mathbb{Z}$ and don't pass to PSL2$\mathbb{Z}$.

11 The Universal Elliptic Curve and Modular Forms

Let $\mathfrak{h}$ be the upper half plane with coordinate τ which we think of as parametrizing the complex elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. We first make a universal family of elliptic cuves by,

$$\pi : \mathbb{C} \times \mathfrak{h} / \langle (z, \tau) \sim (z+1, \tau) \sim (z+\tau, \tau) \rangle \to \mathfrak{h}$$

This map has fiber over τ equal to $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. However, if $\gamma \in \mathrm{SL2}\mathbb{Z}$ takes $\gamma \cdot \tau = \tau'$ then I claim that the elliptic curves are isomorphic. We want to encode this. The ismorphism comes from the transformation of a positive ordered basis ω_1, ω_2 of a lattice into the form $\tau, 1$ where $\tau = \frac{\omega_1}{\omega_2}$. Then $\gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$ is the image of the basis $\tau, 1$ under γ after dividing by $(c\tau+d)$ so that we are normalzied to $\gamma \cdot \tau, 1$. Therefore, the isomorphism $C_\tau \xrightarrow{\sim} C_{\gamma\tau}$ is given by $z \mapsto (c\tau+d)^{-1} z$. Therefore, we mod out our family to get,

$$\mathcal{C} \to \mathcal{M}$$

where $\mathcal{M} = \mathfrak{h}/\mathrm{SL}2\mathbb{Z}$ and,

$$\begin{aligned}\mathcal{C} &= \mathbb{C} \times \mathfrak{h} / \langle (z,\tau) \sim (z+1,\tau) \sim (z+\tau,\tau) \rangle / \mathrm{SL}2\mathbb{Z} \\ &= \mathbb{C} \times \mathfrak{h} / \left\langle (z,\tau) \sim (z+1,\tau) \sim (z+\tau,\tau) \sim ((c\tau+d)^{-1}z, \gamma \cdot \tau) \right\rangle\end{aligned}$$

Now we consider the vertical cotangent bundle of $\mathcal{C} \to \mathcal{M}$ which is,

$$\Omega_{\mathcal{C}/\mathcal{M}} = \mathbb{C} \times \mathbb{C} \times \mathfrak{h} / \left\langle (\omega, z, \tau) \sim (\omega, z+1, \tau) \sim (\omega, z+\tau, \tau) \sim ((c\tau+d)\omega, (c\tau+d)^{-1}z, \gamma \cdot \tau) \right\rangle$$

Notice that the cotangent fibers change opposite to the coordinate z because the natural forward map is the inverse pullback which scales oppositely. All this is extremely problematic because of the fixed points of $\mathrm{SL}2\mathbb{Z} \circlearrowright \mathfrak{h}$ which give stabilizers and thus too much quotienting we really should be taking groupoid quotients and thus get "stacky" points but the functions in the two cases are basically the same because they reduce to being equivariant maps. Now pulling back this bundle along the zero section of $\mathcal{C} \to \mathcal{M}$ (i.e. the map $\tau \mapsto (0, \tau)$) gives,

$$\omega = e^* \Omega_{\mathcal{C}/\mathcal{M}} = \mathbb{C} \times \mathfrak{h} / (\omega, \tau) \sim ((c\tau+d)\omega, \gamma \cdot \tau) \rangle$$

Therefore, a section of $\omega \to \mathcal{M}$ is an equivariant section of $\mathbb{C} \times \mathfrak{h} \to \mathfrak{h}$ and thus a function $f : \mathfrak{h} \to \mathbb{C}$ such that,

$$f(\gamma \cdot \tau) = (c\tau + d) f(\tau)$$

which is exactly a weight-one modular function!. Therefore, modular functions of weight k correspond to sections of $\omega^{\otimes k}$.

To get modular forms, we need a holomorphy condition at ∞. To get this, we need to extend ω over the boundary to a line bundle on $\overline{\mathcal{M}}$.

12 How many nodes does a curve in $\mathbb{P}^3$ have when projected to $\mathbb{P}^2$

Given a smooth curve $X \subset \mathbb{P}^3$ of genus g we consider a general projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Then X gets a node at some point if the associated line itersects X at multiple points.

For a curve $X \subset \mathbb{P}^2$ consider the projection map $X \to \mathbb{P}^1$. This is ramified exactly at the points where a section $s \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ vanishes as well as its derivative on X. Thus we want to consider $J^1(\mathcal{O}_X(1))$ and the vanishing locus of a general section. But we don't want a general line, we want a general line that vanishes at a fixed point $p \in \mathbb{P}^2$. Call this linear system $V_p \subset \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1))$. Then

13 Viewpoints on Cech Cohomology

13.1 Čech Cohomology as the Cohomology of a Complex

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, $U \in \mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ be a cover of U. We set,

$$U_{i_0, \dots, i_p} = U_{i_0} \times_U \cdots \times_U U_{i_p}$$

Definition 13.1.1. The Čech complex of a presheaf $\mathscr{F}$ on $\mathcal{C}$ is defined by terms,

$$\check{C}^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0, \dots, i_p \in I} \mathscr{F}(U_{i_0, \dots, i_p})$$

giving a complex,

$$0 \longrightarrow \prod_{i_0 \in I} \mathscr{F}(U_i) \xrightarrow{\mathrm{d}} \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \times_U U_{i_1}) \xrightarrow{\mathrm{d}} \cdots$$

where,

$$\mathrm{d}(s)_{i_0, \dots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0, \dots, \hat{i_j}, \dots i_{p+1}}|_{U_{i_0, \dots, i_{p+1}}}$$

This is a complex. Then the Čech cohomology is the cohomology of this complex,

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = H^p(\check{C}^p(\mathfrak{U}, \mathscr{F}))$$

Proposition 13.1.2. Formation of the Čech complex is functorial,

$$\check{C}^\bullet(\mathfrak{U}, -) : \mathbf{PSh}_{\mathcal{O}} \to \mathbf{Ch}(\mathbf{Mod}_{\mathcal{O}(U)})$$

and therefore Čech cohomology is functorial,

$$\check{H}^p(\mathfrak{U}, -) : \mathbf{PSh}_{\mathcal{O}} \to \mathbf{Mod}_{\mathcal{O}(U)}$$

Lemma 13.1.3. If $\mathscr{I}$ is an injective presheaf then $\check{H}^p(\mathfrak{U}, \mathscr{I}) = 0$ for all $p > 0$.

Proof. FIND A GOOD PROOF (e.g. Tag 01EN). □

Proposition 13.1.4. The functors $\check{H}^p(\mathfrak{U}, -) : \mathbf{PSh}_{\mathcal{O}} \to \mathbf{Mod}_{\mathcal{O}(U)}$ form a universal ∇-functor.

Proof. Given an exact sequence of presheaves,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

we get an exact sequence of complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{F}_1) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{F}_2) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{F}_3) \longrightarrow 0$$

because taking sections of presheaves is exact and products are exact in the category of modules. The associated long exact sequnce gives the required connecting maps and exactness showing that $\check{H}^p(\mathfrak{U}, -)$ form a ∇-functor. Futhermore, since $\mathbf{PSh}_{\mathcal{O}}$ has enough injectives and $\check{H}^p(\mathfrak{U}, \mathscr{I}) = 0$ for $p > 0$ we see that $\check{H}^p(\mathfrak{U}, -)$ are effaceable and thus form a universal ∇-functor. □

13.2 Čech Cohomology as a Canonical Resolution

13.3 Čech Cohomology as a Derived Functor on Presheaves

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, $U \in \mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i \in I}$ be a cover of U then define the Cech sections functor,

$$\check{H}^0(\mathfrak{U}, -) : \mathbf{PSh}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}(U)}$$

defined by,

$$\mathscr{F} \mapsto \ker\left(\prod_{i\in I}\mathscr{F}(U_i) \to \prod_{i,j\in I^2}\mathscr{F}(U_i \times_U U_j)\right)$$

Consider the inclusion functor,

$$\iota : \mathbf{Sh}_{\mathcal{O}} \to \mathbf{PSh}_{\mathcal{O}_X}$$

Then there is a commutative diagram of functors,

$$\begin{tikzcd} \mathbf{PSh}_{\mathcal{O}_X} \arrow[rr, "\iota"] \arrow[dr, "\check{H}^0"'] & & \mathbf{Sh}_{\mathcal{O}_X} \arrow[dl, "\Gamma_U"] \\ & \mathbf{Mod}_{\mathcal{O}(U)} & \end{tikzcd}$$

Furthermore, ι is right-adjoint to sheafification which is exact and thus preserves injectives. Therefore, we can apply the Grothendieck spectral sequence to get,

$$E_2^{p,q} = R^p\check{H}^0(\mathfrak{U}, R^q\iota(\mathscr{F})) \implies H^0(U, \mathscr{F})$$

Furthermore, because Γ_V is exact on presheaves, we see that,

$$R^p\Gamma_V = R^p(\Gamma_V \circ \iota) = \Gamma_V \circ R^p\iota$$

and therefore,

$$[(R^p\iota)(\mathscr{F})](V) = \Gamma_V \circ (R^p\iota)(\mathscr{F}) = (R^p\Gamma_V)(\mathscr{F}) = H^p(V, \mathscr{F})$$

Therefore $(R^p\iota)(\mathscr{F})$ is the presheaf $V \mapsto H^p(V, \mathscr{F})$ which we call $\mathcal{H}^p(\mathscr{F})$. Now I claim that the derived functor $R^p\check{H}^0$ agrees with $\check{H}^p$ defined earlier. Since $\check{H}^0$ is left-exact, there are natural isomorphisms,

$$\check{H}^0 \xrightarrow{\sim} R^0\check{H}^0$$

Furthermore, because $\check{H}^p$ and $R^p\check{H}$ are universal ∇-functors, the above map extends to an isomorphism of ∇-functors. Therefore, we derived the Čech to derived spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(\mathscr{F})) \implies H^{p+q}(U, \mathscr{F})$$

Now, we define,

$$\check{H}^0(U, -) = \varinjlim_{\mathfrak{U}} \check{H}^0(\mathfrak{U}, -)$$

Because covers form a filtered poset this filtered colimit is exact. Therefore,

$$\check{H}^p(U, -) := \varinjlim_{\mathfrak{U}} \check{H}^p(\mathfrak{U}, -) = \varinjlim_{\mathfrak{U}} R^p\check{H}^0(\mathfrak{U}, -) = R^p \varinjlim_{\mathfrak{U}} \check{H}^0(\mathfrak{U}, -)$$

Furthermore, since $\check{H}^0(\mathfrak{U}, -) \circ \iota = \Gamma_U$ for any cover we see that $\check{H}^0(U, -) \circ \iota = \Gamma_U$. Therefore, applying the Grothendieck spectral sequence,

$$E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(\mathscr{F})) \implies H^{p+q}(U, \mathscr{F})$$

13.4 The Degree 1 Case

By the locality of cohomology, for each $s \in H^q(V, \mathscr{F})$ there exists a cover $\{V_i\}$ of V such that $s|_{V_i} = 0$ for each i and therefore choosing a small enough refinement $\check{H}^0(U, \mathcal{H}^q(\mathscr{F})) = 0$. Now consider the differentials $\mathrm{d}_r : E_r^{p,q} \to E^{p+r,q-r+1}$ then for $(p,q) = (1,0)$ we see that $\mathrm{d}_r = 0$ for $r > 1$ because $(p+r, q-r+1) = (r+1, 1-r)$ is not first quadrant. Furthermore, if $(p+r, q-r+1) = (1,0)$ then $(p,q) = (1-r, r-1)$ is not first quadrant for $r > 1$ so $\mathrm{d}_r = 0$. Since $E_2^{0,1} = 0$ we see that the $p+q = 1$ terms are converged at the E_2-page. Therefore, from the filtration we see that,

$$\check{H}^1(U, \mathscr{F}) = H^1(U, \mathscr{F})$$

in vast generality.

14 Orientibility versus Two-Sidedness

In multivariable calculus when you learn about the divergence theorem you may be told "this only works for orientable surfaces because you need to choose a normal vector to integrate over in order to compute the flux" this is only true because the ambient $\mathbb{R}^3$ is orientable. Furthermore, you are told that the Möbius strip is non-orientable because it only has one side but this is actually a feature of its embedding in $\mathbb{R}^3$. In general, orientability, which is intrinsic, is not actually the concept being probed but rather two-sidedness, which is relative to the embedding.

Definition 14.0.1. Let M be a manifold. Then an embedded submanifold $X \subset M$ is *two-sided* if the normal bundle $N_M X$ is orientable.

Proposition 14.0.2. Suppose that $X \subset M$ has codimension 1 then X is two sided if and only if it admits a global nonvanishing normal vector field $v \in \Gamma(X, N_M X)$.

Proof. The normal bundle $N_M X$ is a line bundle and thus is orientable if and only if it is trivial if and only if it admits a nonvanishing global section. □

Remark. Thsi motives the terminology because the nonvanishing normal vector field distinguishes between a "positive" side and a "negative" side of the manifold.

Proposition 14.0.3. If M is orientable and $X \subset M$ is an embedded submanifold then X is two-sided if and only if M is orientable.

Proof. From the exact sequence,

$$0 \longrightarrow TX \longrightarrow TM|_X \longrightarrow N_M X \longrightarrow 0$$

we see that,

$$\det TM|_X \cong \det TX \otimes \det N_M X$$

However, TM is orientable so $\det TM$ is trivial and thus,

$$\det TM \cong \det N_M X$$

menaing that one bundle is orientable if and only if the other is orientable. Equivalently, we can use Stiefel-Whitney classes,

$$w_1(TX) + w_1(N_M X) = w_1(TM)|_X = 0$$

and therefore (because these live in $\mathbb{Z}/2\mathbb{Z}$ cohomology),

$$w_1(TX) = w_1(N_M X)$$

Furthermore, the Stiefel-Whitney classes vanish exactly when the bundle is orientable so we see that TX is orientable if and only if $N_M X$ is orientable. $\square$

Remark. In general, we see that,

$$w_1(N_M X) = w_1(TX) + w_1(TM)|_X$$

and therefore we can compute the two-sidedness from the orientability of X together with the pullback of the Stiefel-Whitney class of TM.

Remark. This paper gives lots of examples of non-orientable surfaces such a Möbius strips and Klien bottles with two-sided embeddings into non-orientable 3-manifolds.

15 Stable Parallelizability of Spheres

Proposition 15.0.1. Let X be a n-dimensional oriented surface and $\iota : X \to \mathbb{R}^{n+1}$ an immersion. Then TX is stabily trivial.

Proof. The canonical exact sequence,

$$0 \longrightarrow TX \longrightarrow \iota^* TY \longrightarrow N_Y X \longrightarrow 0$$

splits (every sequence splits) to give $\iota^* TY = TX \oplus N_Y X$ but $TY \cong \varepsilon^{n+1}$ is trivial. Because X and Y are orientable, the embedding $\iota : X \to Y$ is two-sided so $N_Y X$ is orientable. However, $N_Y X$ is a line bundle since $\dim Y = \dim X + 1$ and thus $N_Y X$ is trivial. Therefore,

$$TX \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}$$

and thus TX is stably trivial so X is stably parallelizable. $\square$

Corollary 15.0.2. All spheres have stabily trivial tangent bundles in fact $TS^n \oplus \underline{\mathbb{R}}$ is trivial.

Remark. This shows that $S^n \times \mathbb{R}$ is parallelizable. For the case $n = 2$, this has to be true because it is a theorem of Thurston that every orientable 3-manifold has trivial tangent bundle.

16 Some Questions

16.1 Can I use Miracle Flatness at Only Closed Points

16.2 Do Regular Functions Separate Points

Throughout we assume that X is separated. Otherwise points of X cannot be separated by rational functions let alone regular functions. For simplicitly, we assume that X is noetherian.

I can reduce to the case of an integral scheme as follows. First, if X is nonreduced then $X_{\text{red}} \to X$. Suppose that X_{red} separated points then any function mapping to it on X will work. Now we assume X is reduced. If X is reducible then there are two cases to consider. If $x, y \in X$ lie on different irreducible components then there exist disjoint opens containing the two points so they can clearly be separated by regular functions. If $x, y \in X$ lie in the same irreducible component then we reduce to that irreducible component. If there is some $x, y \in U \subset Z$ open in Z with a regular function separating x, y then (UGH NOT QUITE).

16.2.1 The Integral Case

Let X be an integral separated scheme. Then points of X are separated by rational functions. Indeed, I claim that if $\mathcal{O}_{X,x} = \mathcal{O}_{X,y}$ inside the function field $K(X)$ then $x = y$. In fact, I will show that neither can dominate the other. Suppose that $\mathcal{O}_{X,x} \subset \mathcal{O}_{X,y}$ with $\mathfrak{m}_y \cap \mathcal{O}_{X,x} = \mathfrak{m}_x$ (we say $\mathcal{O}_{X,y}$ dominates $\mathcal{O}_{X,x}$) then there is a valuation ring A dominating $\mathcal{O}_{X,y}$ inside $K(X)$ giving maps,

$$\begin{tikzcd} \mathrm{Spec}\,(K(X)) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,(A) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}\,(\mathbb{Z}) \end{tikzcd}$$

via sending $\mathfrak{m}_A \mapsto x$ and the local ring map $\mathcal{O}_{X,x} \hookrightarrow A$ and by $\mathfrak{m}_A \mapsto y$ and the local ring map $\mathcal{O}_{X,y} \hookrightarrow A$. By the valuative criterion of separatedness, there is at most one such dotted map and thus $x = y$ and $\mathcal{O}_{X,x} = \mathcal{O}_{X,y}$.

Now I claim that if $\mathcal{O}_{X,x} \subset \mathcal{O}_{X,y}$ then x, y are contained in the same affine open and $y \rightsquigarrow x$. Indeed, the prime ideal $\mathfrak{p} = \mathfrak{m}_y \cap \mathcal{O}_{X,x}$ corresponds to some point $y' \in \mathrm{Spec}\,(\mathcal{O}_{X,x})$ which lies in every affine open containing x. Then $\mathcal{O}_{X,x} \subset \mathcal{O}_{X,y'} \subset \mathcal{O}_{X,y}$ because $\mathcal{O}_{X,y'} = (\mathcal{O}_{X,x})_{\mathfrak{p}}$ which are units in $\mathcal{O}_{X,y}$. Then $\mathcal{O}_{X,y'} \hookrightarrow \mathcal{O}_{X,y}$ is local so by our previous result $y = y'$.

I would like to improve this to give a rational function whose value at x is 1 and at y is 0. We need to know that $\mathfrak{m}_x \cap \mathcal{O}_{X,y} \neq \mathfrak{m}_y \cap \mathcal{O}_{X,x}$.

17 Finite Intersection Property

Definition 17.0.1. A collection of sets $\{K_\alpha\}$ has the finite intersection property if every finite intersection is nonempty,

$$K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \neq \varnothing$$

Proposition 17.0.2. A space is compact if and only if every collection $\{K_\alpha\}$ of closed subsets with the finite intersection property has,

$$\bigcap_\alpha K_\alpha \neq \varnothing$$

Proof. We let $U_\alpha = K_\alpha$ where U_α is open iff K_α is closed. Then $\{U_\alpha\}$ has no finite subcover iff each,

$$U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \subsetneq X \iff K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \neq \varnothing$$

iff $\{K_\alpha\}$ has FIP. Furthermore, $\{U_\alpha\}$ is a cover iff,

$$\bigcup_\alpha U_\alpha = X \iff \bigcap_\alpha K_\alpha = \varnothing$$

Therefore every cover has a finite subcover is equivalent to every collection of opens without a finite subcover is not a cover which is equivalent to every collection of closed sets with FIP has nonempty intersection. $\square$

Proposition 17.0.3. Let X be a topological space. Let $\{K_\alpha\}$ be a collection of sets with the FIP such that one of the following holds,

(a) the K_α are closed and some K_{α_0} is compact

(b) X is Hausdorff and K_α are compact.

Then,

$$\bigcap_\alpha K_\alpha \neq \varnothing$$

Proof. Because compact sets are closed in a Hausdorff space condition (b) implies (a). Therefore, it suffices to prove the conclusion assuming (a). By FIT K_{α_0} is nonempty and $\{K_{\alpha_0} \cap K_\alpha\}$ is a collection of closed sets with FIP in the compact set K_{α_0} and therefore,

$$\bigcap_\alpha K_\alpha = \bigcap_\alpha (K_{\alpha_0} \cap K_\alpha) \neq \varnothing$$

by the previous proposition. $\square$

Corollary 17.0.4. Let $I_n \subset \mathbb{R}$ be a sequence of nonempty nested closed intervals. Then,

$$\bigcap_n I_n \neq \varnothing$$

18 When Are Isometries Smooth

Definition 18.0.1. Let $f : X \to Y$ be a map between metric spaces. We say that f is *isometric* if for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

Proposition 18.0.2. Isometric maps are uniformly continuous with uniformly continuous inverse on their image.

Proof. Uniform continuity is immediate because,

$$d_X(x_1, x_2) < \epsilon \iff d_X(f(x_1), f(x_2))$$

Furthermore, f is injective because if $f(x_1) = f(x_2)$ then,

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = 0$$

so $x_1 = x_2$. Finally, it is clear that f^{-1} is also isometric and thus uniformly continuous. $\square$

Proposition 18.0.3. Isometric maps $f : \mathbb{R}^n \to \mathbb{R}^n$ are affine orthogonal transformations.

Proof. First we may translate f such that $f(0) = 0$. For $x, y \in \mathbb{R}^n$ consider,

$$||f(x) - f(y)||^2 = ||f(x)||^2 - 2\langle f(x), f(y)\rangle + ||f(y)||^2 = ||x||^2 - 2\langle f(x), f(y)\rangle + ||y||^2$$

however,

$$||f(x) - f(y)||^2 = ||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2$$

and therefore,

$$\langle f(x), f(y)\rangle = \langle x, y\rangle$$

so f preserves the inner product. Furthermore we can show that f is linear as follows.

$$\begin{aligned} ||f(x+y) - f(x) - f(y)||^2 &= ||f(x+y)||^2 - 2\langle f(x+y), f(x)\rangle - 2\langle f(x+y), f(y)\rangle + ||f(x) + f(y)||^2 \\ &= ||x+y||^2 - 2\langle x+y, x\rangle - 2\langle x+y, y\rangle + ||f(x)||^2 + 2\langle f(x), f(y)\rangle + ||f(y)||^2 \\ &= ||x+y||^2 - 2\langle x+y, x+y\rangle + ||x||^2 + \langle x, y\rangle + ||y||^2 \\ &= ||x = y||^2 - 2||x+y||^2 + ||x+y||^2 = 0 \end{aligned}$$

and therefore $f(x+y) = f(x) + f(y)$. Furthermore,

$$\begin{aligned} ||f(\lambda x) - \lambda f(x)||^2 &= ||f(\lambda x)||^2 - 2\langle f(\lambda x), \lambda f(x)\rangle + \lambda^2 ||f(x)||^2 = ||\lambda x||^2 - 2\lambda \langle \lambda x, x\rangle + \lambda^2 ||x||^2 \\ &= \lambda^2 ||x||^2 - 2\lambda ||x||^2 + ||x||^2 = 0 \end{aligned}$$

and therefore $f(\lambda x) = \lambda f(x)$ so f is linear and $\langle f(x), f(y)\rangle = \langle x, y\rangle$ so f is orthogonal and therefore invertible. □

Theorem 18.0.4 (Myers-Steenrod Theorem)**.** Let M and N be Riemannian manifolds with induced metrics and $\phi : M \to N$ is a surjective distance preserving map then ϕ is a smooth isometry.

19 Tangent Spaces a la EGA

Grothendieck defines the tangent space in a kinda funny way. First define the tangent bundle,

$$T_{X/S} = \mathbb{V}_X(\Omega_{X/S}) = \mathbf{Spec}_X\left(\mathrm{Sym}\left(\Omega_{X/S}\right)\right)$$

Then the tangent space at a point are the $\mathrm{Spec}\left(\kappa(x)\right)$-points over X (which form a vector space). This is,

$$T_{X/S}(x) = \mathrm{Hom}_X\left(\mathrm{Spec}\left(\kappa(x)\right), \mathbb{V}_X(\Omega_{X/S})\right) = \mathrm{Hom}_{\mathcal{O}_X}\left(\Omega_{X/S}, \iota_* \mathcal{O}_{\kappa(x)}\right) = \mathrm{Hom}_{\kappa(x)}\left((\Omega_{X/S})_x \otimes \kappa(x), \kappa(x)\right)$$

is the $\kappa(x)$-dual of $(\Omega_{X/S})_x \otimes_{\mathcal{O}_X x} \kappa(x)$ which does make sense but doesn't agree with the Zariski tangent space in general. Maybe this is better because it has the correct dimension at the generic point unlike the Zariski tangent space.

Grothendieck shows [EGA IV_4 16.5.13.2] that if $\kappa(s) \to \kappa(x)$ is an isomorphism then,

$$T_{X/S}(X) \xrightarrow{\sim} \mathrm{Hom}_{\kappa(x)}\left(\mathfrak{m}'_x/\mathfrak{m}'^2_x, \kappa(x)\right)$$

where $\mathfrak{m}'_x$ is the maximal ideal of the local ring $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$. This means that $T_{X/S}(x) = T_{X_s/s}(x)$ as it should. This is also Pset 9 problem 2 in Johan's first class.

I claim that this result actually holds more generally. Because $T_{X/S}(X) = T_{X_s/s}(x)$ in general I can work with $X_s \to \mathrm{Spec}\left(\kappa(s)\right)$ and assume that X is over a field k. Then I claim that if $\kappa(x)/k$ is a separable algebraic extension then,

$$T_{X/k}(x) \xrightarrow{\sim} \mathrm{Hom}_{\kappa(x)}\left(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x)\right)$$

Indeed this is an immediate consequence of [H, Ex. II 8.1(a)].

Notice that if $\kappa(x)/k$ is not algebraic this immediately fails e.g. the generic $x \in \mathbb{A}^1_k$ point has $\mathfrak{m}_x/\mathfrak{m}_x^2 = (0)$ but $T_{\mathbb{A}^1/k}(x) \cong k(t)$. Furthermore, if $\kappa(x)/k$ is not separable this also fails e.g. consider $X = \mathrm{Spec}\left(\mathbb{F}_p(t^{\frac{1}{p}})\right) \to \mathrm{Spec}\left(\mathbb{F}_p(t)\right)$. Then $\Omega_{X/k} \cong \mathbb{F}_p(t^{\frac{1}{p}})$ but $\mathfrak{m}_x/\mathfrak{m}_x^2 = 0$.

19.0.1 The Submersion Theorem

DO THIS TOMORROW!!

20 Genus and Reduction

Lemma 20.0.1. Let A_1, A_2 be rings. Then there is an equivalence of categories,

$$\mathbf{Mod}_{A_1} \times \mathbf{Mod}_{A_2} \to \mathbf{Mod}_{A_1 \times A_2}$$

given by sending $(M_1, M_2) \mapsto (M_1)_{A_1 \times A_2} \oplus (M_2)_{A_1 \times A_2}$

Proof. Faithfullness is clear. For fullness, notice that if $\phi : M_1 \oplus M_2 \to N_1 \oplus N_2$ is a morphism of $A_1 \times A_2$-modules then letting $e_i \in A_i$ be the identity because $\phi(e_i \cdot m) = e_i \cdot \phi(m)$ we see that ϕ is represented by a diagonal matrix of morphisms proving fullness. Finally, let M be an $A_1 \times A_2$-module. Then $M = e_1 M \oplus e_2 M$ because $e_1 + e_2 = 1$ and $e_1^2 = e_1$ and $e_2^2 = e_2$ and $e_1 e_2 = 0$ and $e_i M$ is naturally an A_i-module proving fullness. □

Lemma 20.0.2. Let A be a Noetherian ring and M a finite A-module such that $\dim \mathrm{Supp}_A(M) = 0$. Then $\mathrm{Supp}_A(M)$ is finite and,

$$M \cong \bigoplus_{\mathfrak{p} \in \mathrm{Supp}_A(M)} M_{\mathfrak{p}}$$

Proof. Because M is finite type we have $\mathrm{Supp}_A(M) = V(\mathrm{Ann}_A(M))$. Let $B = A/\mathrm{Ann}_A(M)$ then B is noetherian and $\dim B = \dim \mathrm{Supp}_A(M) = 0$ so B is Artinian. Therefore $\mathrm{Supp}_A(M) = \mathrm{Spec}(B)$ is finite and consists of the maximal ideals of B. Then by the Chinese remainder theorem,

$$B \cong \prod_{\mathfrak{m} \in \mathrm{Spec}(B)} B_{\mathfrak{m}}$$

and therefore,

$$M \cong \bigoplus_{\mathfrak{m} \in \mathrm{Spec}(B)} M_{\mathfrak{m}}$$

□

Lemma 20.0.3. Let X be a Noetherian scheme and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module with,

$$\dim \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F}) = 0$$

Then, $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F})$ is finite and,

$$\mathscr{F} \cong \bigoplus_{x \in \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F})} (\iota_x)_* \mathscr{F}_x$$

where $\iota_x : \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ is the canonical map and $\mathcal{O}_{X,x}/\mathrm{Ann}_{\mathcal{O}_{X,x}}(\mathscr{F}_x)$ is Artin local.

Remark. Notice that because the $\mathcal{O}_{X,x}$-module $\mathscr{F}_x$ is supported only over the maximal ideal that $(\iota_x)_* \mathscr{F}_x$ is the same (viewing $\mathscr{F}_x$ as an abelian sheaf) as pushing forward along the map $x \hookrightarrow X$.

Proof. Because X is quasi-compact, we can choose a finite affine open cover U_i on which $\mathscr{F}|_{U_i} = \widetilde{M_i}$. Then the result follows immediately from the previous lemma. Notice further that there is a canonical map,

$$\mathscr{F} \to \prod_{x \in \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F})} (\iota_x)_* \mathscr{F}_x$$

from adjunction and the universal property of the product. Finiteness shows that this is a direct sum and we can check locally that it is an isomorphism. (DO THIS BETTER!!) □

Proposition 20.0.4. Suppose that X is a finite type scheme over k and $\mathscr{I} \subset \mathcal{O}_X$ a quasi-coherent ideal sheaf which is supported only at closed points. Let $Z \subset X$ be the closed subscheme determined by Z. Then, $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})$ is a finite set and,

$$\chi(X, \mathcal{O}_X) - \chi(Z, \mathcal{O}_Z) = \sum_{x \in \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})} \dim_k \mathscr{I}_x$$

Proof. Consider the exact sequence of the closed immersion $\iota : Z \hookrightarrow X$,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Therefore,

$$\chi(X, \mathcal{O}_X) - \chi(Z, \mathcal{O}_Z) = \chi(X, \mathscr{I})$$

using that ι is affine so $H^i(X, \iota_* \mathscr{F}) = H^i(Z, \mathscr{F})$. Furthermore, because X is noetherian, $\mathscr{I}$ is coherent and thus $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})$ is closed but only contains closed points. Therefore, writing $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})$ as a union of finitely many irreducible components we see that these must be points (they are irreducible and only contain closed points) and thus $Y = \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})$ consists of a finite number of closed points and is zero dimensional. Therefore,

$$\mathscr{I} \cong \bigoplus_{x \in \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})} (\iota_x)_* \mathscr{I}_x$$

Thus,

$$\chi(X, \mathscr{I}) = \sum_{x \in \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})} \dim_k \mathscr{I}_x$$

where the higher cohomology of $\mathscr{I}$ vanishes because $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})$ is zero dimensional. $\square$

Corollary 20.0.5. Suppose that X is a finite type scheme over k which is reduced at all non-closed points. Then, the ideal sheaf $\mathcal{N}$ of $X_{\mathrm{red}} \hookrightarrow X$ is supported at finitely many points and,

$$\chi(X, \mathcal{O}_X) - \chi(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}) = \sum_{x \in \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I})} \dim_k \mathcal{N}_x$$

Proof. Consider the exact sequence of the closed immersion $X_{\mathrm{red}} \hookrightarrow X$ which is also a homeomorphism,

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_{\mathrm{red}}} \longrightarrow 0$$

Therefore,

$$\chi(X, \mathcal{O}_X) - \chi(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}) = \chi(X, \mathcal{N})$$

Furthermore, because $\mathcal{O}_{X,x}$ is reduced unless $x \in X$ is closed, we see that $\mathrm{Supp}_{\mathcal{O}_X}(\mathcal{N})$ is zero dimensional. Explicitly, there is an exact sequence,

$$0 \longrightarrow \mathcal{N}_x \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X_{\mathrm{red}},x} \longrightarrow 0$$

and $\mathcal{O}_{X_{\mathrm{red}},x} = (\mathcal{O}_{X,x})_{\mathrm{red}}$ so we have $\mathcal{N}_x = \mathrm{nilrad}(\mathcal{O}_{X,x})$ vanishes if and only if $\mathcal{O}_{X,x}$ is reduced. Therefore, $\mathrm{Supp}_{\mathcal{O}_{X,x}}(\mathcal{N})$ is supported only at closed points so we can apply the proposition to conclude. $\square$

Corollary 20.0.6. Let X be a projective scheme with ample line bundle $\mathcal{O}_X(1)$. Let $\mathscr{I}$ be an ideal sheaf on X which is supported only at closed points. Let $Z \subset X$ be the closed subscheme determined by Z. Then $\text{Supp}_{\mathcal{O}_X}(\mathscr{I})$ is a finite set and,

$$P_X(n) - P_Z(n) = \sum_{x \in \text{Supp}_{\mathcal{O}_X}(\mathscr{I})} \dim_k \mathscr{I}_x$$

where P_X and P_Z are the Hilbert polynomials. In particular, if $\dim X > 0$ we see that X and Z have the same dimension and degree.

Proof. This follows from the fact that,

$$0 \longrightarrow \mathscr{I}(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow \iota_* \mathcal{O}_Z(n) \longrightarrow 0$$

is exact and $\mathscr{I}(n) \cong \mathscr{I}$ because $\mathcal{O}_X(n)$ is a line bundle and thus its stalk is free of rank 1 at each $x \in X$. Therefore,

$$P_X(n) - P_Z(n) = \chi(X, \mathcal{O}_X(n)) - \chi(Z, \mathcal{O}_Z(n)) = \chi(X, \mathscr{I})$$

so we conclude by the previous calculation. $\square$

Corollary 20.0.7. Let X be a projective scheme with ample line bundle $\mathcal{O}_X(1)$. Suppose that X is reduced at all non-closed points. Then $\mathcal{N}$ is supported at finitely many points and,

$$P_X(n) - P_{X_{red}}(n) = \sum_{x \in \text{Supp}_{\mathcal{O}_X}(\mathcal{N})} \dim_k \mathcal{N}$$

In particular, if $\dim X > 0$ we see that X and X_{red} have the same dimension and degree.

20.1 The Application to Projective Cones

Let $X \subset \mathbb{P}^n = \text{Proj}\,(S)$ be cut out by the saturated ideal $I \subset S$. Then we consider the projective cone $C(X) = \text{Proj}\,(S[z]/I[z])$. If X is integral then $S/I \hookrightarrow \Gamma_*(\mathcal{O}_X)$ which is a domain and thus S/I is a domain so I is prime. Thus $S[z]/I[z] = (S/I)[x]$ is a domain so $C(X)$ is integral. Likewise if X is reduced then $S/I \hookrightarrow \Gamma_*(\mathcal{O}_X)$ is reduced so I is a radical ideal.

Remark. This only holds for the *saturated* ideal I. For example, consider $I = (xy, x^2) \subset k[x, y]$. Then $I^{\text{sat}} = (x)$ is a prime ideal and $\text{Proj}\,(k[x, y]/I) = \text{Proj}\,(k[x, y]/I^{\text{sat}}) = \text{Spec}\,(k)$ is integral but I is clearly not prime.

If we choose the wrong ideal I then we get the wrong cone $C_I(X) = \text{Proj}\,(S[z]/I[z])$ because $I[z]$ and $I^{\text{sat}}[z]$ need not have the same saturation. However, the problem is completely supported at the origin because I and I^{sat} become equal after localization at any ideal not containing some x_i. Therefore, the closed immersion $C(X) \hookrightarrow C_I(X)$ is cut out by the nilpotent ideal sheaf I^{sat}/I supported at the origin and we see $C_I(X)_{\text{red}} = C(X)_{\text{red}}$. Therefore,

$$P_{C_I(X)}(n) - P_{C(X)}(n) = \dim_k (I^{\text{sat}}/I)_{(\mathfrak{m})}$$

I SHOULD CHECK THIS PART!!!

Example 20.1.1 (H, Example, 9.8.4)**.** we have a twisted cubic degenerating to a nodal cubic with reduced shit. Let X_0 have an affine patch cut out by,

$$I_0 = (z^2, yz, xz, y^2 - x^2(x+1))$$

Then the reduction is cut out by the ideal $N = (z)$ so $\dim_k N/I_0 = 1$. Let $C = (X_0)_{\text{red}}$ be the nodal cubic. Then,

$$P_C(n) = 3n + 0$$

because it is a Cartier divisor in $\mathbb{P}^2$ of degree 3 so,

$$\chi(\mathcal{O}_C) = 1 - g_a = 0$$

Because $X \to \mathbb{A}^1$ is a flat family, P_{X_0} is equal to P_{X_1} where X_1 is the twisted cubic given parametrically by $(t^2 - 1, t^3 - t, t)$ which is isomorphic to the twisted cubic curve (t, t^2, t^3). This has ideal I the relations for the functions s^3, s^2t, st^2, t^3 meaning the kernel of the map,

$$k[x_0, x_1, x_3, x_4] \to k[s, t]$$

sending $x_0 \mapsto s^3, x_1 \mapsto s^2t, x_3 \mapsto st^2, x_4 \mapsto t^3$ and therefore the quotient is isomorphic to its image,

$$S/I \cong k[s^3, s^2t, st^2, t^3]$$

where we give s, t degree $\frac{1}{3}$ (i.e. view it as a subring of $(k[s,t])^{(3)}$) to make this a graded isomorphism. We see that the ideals I_a are homogeneous and prime since they are the kernel of a map of graded domains (the quotient of such a kernel is a subring of a domain and hence a domain) and therefore saturated (if $\mathfrak{p}$ is a prime ideal not containing the irrelevant ideal and $x_i^n f \in \mathfrak{p}$ then either $x_i \in \mathfrak{p}$ for each i or $f \in \mathfrak{p}$). Therefore,

$$\dim_k(S_0/I_0)_d = \dim_k(k[s^4, s^3t, st^3, t^4])_d = \begin{cases} 1 & d = 0 \\ 3d+1 & d > 0 \end{cases}$$

Therefore,

$$P_{X_0} = 3d + 1$$

showing that,

$$P_{X_0}(n) - P_C(n) = 1 = \dim_k(N/I_0)$$

as expected.

21 Local Systems

Remark. We want to prove the following claim: let $\mathcal{L}$ be a local system of A-modules valued in M on a topological space X (or Grothendieck topology) then $\chi(X, \mathcal{L}) = \chi(X, \underline{A})$.

Definition 21.0.1. Suppose that $G_0(A)$ is equipped with a rank function $\text{rank}_A : K_0(A) \to \mathbb{Z}$. Then for a sheaf of A-modules we define,

$$\chi(X, \mathscr{F}) = \sum_{i=0}^{\infty} \text{rank}_A H^i(X, \mathscr{F})$$

when $H^i(X, \mathscr{F})$ are finite A-modules and there is vanishing of $H^i(X, \mathscr{F})$ for sufficiently large i.

Remark. In the case that A is a domain there is always a rank function $M \mapsto \dim_K(M \otimes_A K)$ where $K = \text{Frac}\,(A)$ which descends to $G_0(A)$ because it is additive over short exact sequences.

21.1 The Case of a Topological Space

Proposition 21.1.1 (Mayer-Vietoris)**.**

Proposition 21.1.2. Let $\mathcal{L}$ be a local system valued in an abelian group A on a topological space X then $\chi(X, \mathcal{L}) = \chi(X, \underline{A})$.

Proof. Consider the □

Corollary 21.1.3. If X is a (INSET CORRECT TOP PROPERTY) space and $\mathcal{L}$ is a local system on X valued in an abelian group A then,

$$\chi(X, \mathcal{L}) = \chi(X, \underline{A}) = \sum_{i=0}^{n} \text{rank}$$

21.2 The General Case of a Site

22 Images of Maximal Ideals

Lemma 22.0.1. Let X be a locally finite type scheme over k. Then $x \in X$ is closed if and only if $\kappa(x)/k$ is finite.

Proof. If $x \in X$ is closed then choose an affine open $x \in \text{Spec}\,(A)$ with A a finite type k-algebra. Then $x = \mathfrak{m} \in \text{Spec}\,(A)$ is closed so $\mathfrak{m}$ is maximal so $A/\mathfrak{m}$ is a finitely generated k-algebra and a field so $\kappa(x) = A/\mathfrak{m}$ is a finite k-extension by the Nullstellensatz.

Conversely, if $\kappa(x)/k$ is finite then for every affine open $x \in \text{Spec}\,(A)$ we see that $A/\mathfrak{p} \hookrightarrow (A/\mathfrak{p})_{\mathfrak{p}}$ because $A/\mathfrak{p}$ is a domain but $\kappa(x) = (A/\mathfrak{p})_{\mathfrak{p}}$ is finite dimensional over k and so $A/\mathfrak{p}$ is a finite dimensional k-algebra domain and hence a field so $\mathfrak{p}$ is maximal. Thus x is closed in each $\text{Spec}\,(A)$ and since these cover X we see that $x \in X$ is closed. □

Proposition 22.0.2. A map of schemes locally of finite type over a field k sends closed points to closed points.

Proof. A point $x \in A$ being closed is equivalent to $\kappa(x)/k$ being finite by the Nullstellensatz. Then $\kappa(f(x)) \hookrightarrow \kappa(x)$ so the image of a closed point is closed. □

Remark. In general this is false even for finite type maps. For example, consider $\text{Spec}\,(\mathbb{Q}_p) \to \text{Spec}\,(\mathbb{Z}_p)$ which is finite type since $\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$. However, we give an extension of this result to arbitrary schemes.

Lemma 22.0.3. Let $\varphi : A \to B$ be a finite type ring map and $\mathfrak{m} \subset B$ a maximal ideal and $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$ the image. Then there exists $t \in A \setminus \mathfrak{p}$ such that $\mathfrak{p}A_t$ is maximal.

Proof. Let $K = \text{Frac}\,(A/\mathfrak{p})$. The maximal ideal $\mathfrak{m} \in \text{Spec}\,(B)$ corresponds to a maximal ideal $\bar{\mathfrak{m}} \in \text{Spec}\,(B')$ with $B' = B \otimes_A (A/\mathfrak{p})_{\mathfrak{p}} = B \otimes_A K$ which is the fiber. Then $B'/\mathfrak{m}' = (B/\mathfrak{m})_{\mathfrak{p}} = B/\mathfrak{m}$ because $B/\mathfrak{m}$ is a field. Now B' is a finte type K-algebra because $A \to B$ is finite type and thus $K \to B \otimes_A K$ is finite type. Therefore, by the Nullstellensatz, $B/\mathfrak{m}$ is a finite extension of K. Hence through $A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = K \to B/\mathfrak{m}$ we see that $B/\mathfrak{m}$ is a finite $A_{\mathfrak{p}}$-module.

Choose generators $x_1, \dots, x_n \in B/\mathfrak{m}$ as an A-algebra. Write $B/\mathfrak{m} = A[x_1, \dots, x_n]/I$. Since $B/\mathfrak{m}$ is finite over $A_{\mathfrak{p}}$ each x_i is integral over $A_{\mathfrak{p}}$ so it satisfies some monic $p_i \in A_{\mathfrak{p}}[x]$. Let $t \in A \setminus \mathfrak{p}$ be the product of the denominators of the coefficients of all p_i. Then $p_i \in A_t[x]$ and thus $x_i \in B/\mathfrak{m}$ is integral over A_i and hence $A/\mathfrak{m}$ is a finite A_t-module since $B/\mathfrak{m}$ is generated by finitely many integral elements as an A-algebra and hence as an A_t-algebra.

Consider $A_t/\mathfrak{p}A_t = (A/\mathfrak{p})_t$. Since $\mathfrak{p}B \subset \mathfrak{m}$ we see that $B/\mathfrak{m}$ is a finite $A_t/\mathfrak{p}A_t$-module. Then $(A/\mathfrak{p})_t \subset K \subset B/\mathfrak{m}$ and thus K is finite over $(A/\mathfrak{p})_t$. Therefore, $(A/\mathfrak{p})_t$ is a field because $(A/\mathfrak{p})_t \subset K$ is an integral extension of domains with K a field and hence $(A/\mathfrak{p})_t = K$ since $K = \mathrm{Frac}\,(A/\mathfrak{p})$. Therefore $\mathfrak{p}A_t$ is a maximal ideal. ☐

Proposition 22.0.4. Let $f : X \to Y$ be a locally finite type map of schemes. The image of a locally closed point is locally closed.

Proof. Let $x \in X$ be locally closed. Choose some affine open $U = \mathrm{Spec}\,(B)$ with $x \in U$ closed and $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ for $\mathrm{Spec}\,(A) \subset Y$ affine open. It corresponds to some maximal ideal $\mathfrak{m} \in \mathrm{Spec}\,(A)$ and therefore under the finite type ring map $\varphi : A \to B$ there is $t \in A \setminus \mathfrak{p}$ with $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$ such that $f(x) = \mathfrak{p}$ is closed in $\mathrm{Spec}\,(A_t)$ by the lemma. ☐

Remark. This is a corollary of Chevallay's theoren.

23 Degree of a Pullback of Curves

Proposition 23.0.1. Let $f : X \to Y$ be a finite locally free morphism of proper schemes over k. Let $\mathcal{E}$ be a vector bundle on Y then,

$$\chi(X, f^*\mathcal{E}) =$$

(HMMMMMM)

24 Unimodular Lattices

Definition 24.0.1. Let $(V, \langle -, - \rangle)$ be a real inner-product space with $n = \dim V$ finite. Then a *lattice* is a subgroup $\Lambda \subset V$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$.

Definition 24.0.2. Let Λ be a lattice. Then we define the *dual Lattice*,

$$\Lambda^* \subset V^* \quad \text{where } \Lambda^* = \{\varphi \in V^* \mid \forall \gamma \in \Lambda : \varphi(\gamma) \in \mathbb{Z}\}$$

However, V is equipped with an inner product and under the natural isomorphism $V \xrightarrow{\sim} V^*$ defined by $v \mapsto \langle v, - \rangle$ we can identify,

$$\Lambda^* \subset V \quad \text{via} \quad \Lambda^* = \{v \in V \mid \forall \gamma \in \Lambda \mid \langle v, \gamma \rangle \in \mathbb{Z}\}$$

Thus we can write,

$$\begin{tikzcd} \Lambda^* \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}\,(\Lambda, \mathbb{Z}) \arrow[d] \\ V^* \arrow[r] & \mathrm{Hom}\,(\Lambda, \mathbb{R}) \end{tikzcd}$$

Definition 24.0.3. The *covolume* or DEFINE

Proposition 24.0.4. $|\Lambda| \cdot |\Lambda^*| = 1$

Proof. DO THIS!! $\square$

Definition 24.0.5. A lattice Λ is,

(a) *integral* if $\langle \gamma, \gamma' \rangle \in \mathbb{Z}$ for all $\gamma, \gamma' \in \Lambda$

(b) *unimodular* if $|\Lambda| = 1$

(c) *even* if $||\gamma||^2 \in 2\mathbb{Z}$ for all $\gamma \in \Lambda$

(d) *self-dual* if $\Lambda^* = \Lambda$ inside V.

Lemma 24.0.6. A lattice Λ is self-dual if and only if Λ is integral and unimodular.

Proof. If Λ is integral, $\Lambda \subset \Lambda^*$ and if Λ is unimodular then $|\Lambda^*| = |\Lambda| = 1$ proving that $\Lambda = \Lambda^*$. Conversely, if $\Lambda = \Lambda^*$ then $|\Lambda| = |\Lambda^*| = 1$ and $\Lambda \subset \Lambda^*$ proving that Λ is unimodular and integral. $\square$

Definition 24.0.7. Let Λ be a lattice. We define the theta function,

$$\Theta_\Lambda : \mathfrak{h} \to \mathbb{C}$$

via the infinite summation,

$$\Theta_\Lambda(\tau) = \sum_{\gamma \in \Lambda} e^{i\pi\tau||\gamma||^2}$$

Proposition 24.0.8. The summation form of Θ_Λ is everywhere absolutely convergent on $\mathfrak{h}$.

Proof. Notice that,

$$|e^{i\pi\tau||\gamma||^2}| = e^{-\pi||\gamma||^2 \operatorname{Im}(\tau)}$$

Since $\operatorname{Im}(\tau) > 0$ we see that $0 < e^{-\pi \operatorname{Im}(\tau)} < 1$ and therefore because the number of lattice points of bounded norm grows polynomially the sum is convergent. $\square$

Theorem 24.0.9 (Poisson Summation)**.** Let $f : V \to \mathbb{C}$ be a Schwartz function with Fourier transform $\hat{f} : V^* \to \mathbb{C}$. Then,

$$\sum_{\gamma \in \Lambda} f(\gamma) = \frac{1}{|\Lambda|} \sum_{\varphi \in \Lambda^*} \hat{f}(\varphi)$$

Proposition 24.0.10. Let Λ be a lattice. For any $\tau \in \mathfrak{h}$,

$$\Theta_{\Lambda^*}(-1/\tau) = \left(\frac{\tau}{i}\right)^{\frac{n}{2}} |\Lambda| \cdot \Theta_\Lambda(\tau)$$

Proof. This is a direct application of Poisson summation for $f(v) = e^{i\pi\tau||v||^2}$. A direct calculation shows that,

$$\hat{f}(v) = \left(\frac{i}{\tau}\right)^{\frac{n}{2}} e^{-i\pi||v||^2/\tau}$$

Then,

$$\Theta_\Lambda(\tau) = \sum_{\gamma \in \Lambda} f(\gamma) = \frac{1}{|\Lambda|} \sum_{\varphi \in \Lambda^*} \hat{f}(\varphi) = \frac{1}{|\Lambda|} \left(\frac{i}{\tau}\right)^{\frac{n}{2}} \Theta_{\Lambda^*}(-1/\tau)$$

$\square$

Corollary 24.0.11. If Λ is self-dual then,

$$\Theta_\Lambda(-1/\tau) = \left(\frac{\tau}{i}\right)^{\frac{n}{2}} \Theta_\Lambda(\tau)$$

Proposition 24.0.12. If Λ is integral then $\Theta_\Lambda(\tau+2) = \Theta_\Lambda(\tau)$. If Λ is even then $\Theta_\Lambda(\tau+1) = \Theta_\Lambda(\tau)$.

Proof. If $||\gamma||^2 \in \mathbb{Z}$ then,

$$e^{i\pi(\tau+2)||\gamma||^2} = e^{2\pi i||\gamma||^2} e^{i\pi\tau||\gamma||^2} = e^{i\pi\tau||\gamma||^2}$$

Likewise, if $||\gamma||^2 \in 2\mathbb{Z}$ then,

$$e^{i\pi(\tau+1)||\gamma||^2} = e^{\pi i||\gamma||^2} e^{i\pi\tau||\gamma||^2} = e^{i\pi\tau||\gamma||^2}$$

□

Corollary 24.0.13. If Λ is self-dual and even then Θ_Λ is modular.

Theorem 24.0.14. Let Λ be an even integral unimodular lattice. Then $8 \mid \dim \Lambda$.

Proof. Since integral unimodular lattices are self-dual we see that Θ_Λ is modular. □

Proof. Let $S, T \in \mathrm{SL}2\mathbb{Z}$ describe $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$. The relation $(ST)^3 = \mathrm{id}$ describes the trajectory,

$$\tau \mapsto -\frac{1}{\tau} \mapsto \frac{\tau - 1}{\tau} \mapsto \frac{\tau}{1-\tau} \mapsto \frac{1}{1-\tau} \mapsto \tau - 1 \mapsto \tau$$

Using the modularity properties,

$$\Theta_\Lambda(\tau) = \left(\frac{i}{\tau - 1}\right)^{\frac{n}{2}} \left(\frac{\tau-1}{i\tau}\right)^{\frac{n}{2}} \left(\frac{\tau}{i}\right)^{\frac{n}{2}} \Theta_\Lambda(\tau) = \left(\frac{1}{i}\right)^{\frac{n}{2}} \Theta_\Lambda(\tau)$$

Since $\Theta_\Lambda(\tau) \neq 0$ because,

$$\Theta_\Lambda(i) = \sum_\Lambda e^{-\pi||\gamma||^2} > 0$$

and therefore, we must have $i^{\frac{n}{2}} = 1$ and hence n is divisible by 8. □

Remark. All these numbers lie in a wedge on the complex plane (indeed $\mathrm{Re}(z) > 0$) and thus the function $(-)^{\frac{n}{2}}$ is well-defined and is multiplicative.

25 Gradient Descent

25.1 Convex Functions

Definition 25.1.1. Let $\Omega \subset V$ be a convex subset of a real vectorspace. Then $f : \Omega \to \mathbb{R}$ is *convex* if,

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y)$$

for all $t \in [0, 1]$ and *strictly convex* if for $x \neq y$,

$$f((1-t)x + ty) < (1-t)f(x) + tf(y)$$

for $t \in (0, 1)$.

Proposition 25.1.2. Suppose that $f : \Omega \to \mathbb{R}$ is convex and M the set of local minima. Then $f(M)$ is at most one point. If f is strictly convex then M is at most one point.

Remark. $f = 0$ is convex but not strictly covex shoing that M can be large. Furthermore, M can be empty even for strictly convex functions. For example for $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x$.

Proof. Suppose that $x, y \in M$ with $x \neq y$ and suppose that $f(x) \neq f(y)$. WLOG let $\nabla = f(x) - f(y) > 0$. Then,

$$g(t) = f((1-t)x + ty) \leq (1-t)f(x) + tf(y) = f(x) - t\nabla$$

By assumption, there exists some $\epsilon > 0$ such that x is the minimum of f on $B_\epsilon(x)$. Choosing,

$$t = \frac{\epsilon}{2||x-y||}$$

We see that $(1-t)x + ty = x + t(y-x) \in B_\epsilon(x)$ and therefore,

$$f(x) \leq f((1-t)x + ty) \leq f(x) - t\nabla$$

which is a contradiction to $t > 0$ and $\nabla > 0$. Thus $f(M)$ is empty or a single point. Furthermore, if f is strictly convex then let $x, y \in M$ and assume that $x \neq y$. We showed that $f(x) = f(y)$ so we see that for,

$$t \leq \frac{\epsilon}{||x-y||}$$

we see that,

$$f(x) \leq f((1-t)x + ty) < (1-t)f(x) + tf(y) = f(x)$$

which is a contradiction. Thus $x = y$. $\square$

Proposition 25.1.3. If f is convex and $(\nabla f)_x = 0$ then x is a global minimum.

Proof. For any $y \in \Omega$ let $\nabla = f(y) - f(x)$. It suffices to prove that $\nabla \geq 0$. Consider,

$$g(t) = f((1-t)x + ty) \leq (1-t)f(x) + tf(y) = f(0) + \nabla t$$

Notice that $g'(0) = 0$ and,

$$\frac{g(t) - g(0)}{t} \leq \nabla$$

and thus taking the limit,

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} \leq \nabla$$

and thus $\nabla \geq g'(0) = 0$. $\square$

Corollary 25.1.4. Let f be convex and differentiable at $x \in \Omega$. Then for any $y \in \Omega$,

$$f(y) \geq f(x) + (y - x) \cdot (\nabla f)_x$$

Proof. Define,

$$g(y) = f(y) - f(x) - (y - x) \cdot (\nabla f)_x$$

Then we see that g is convex and $(\nabla g)_x = 0$ so x is a global minimum of g and $g(x) = 0$ so $g \geq 0$ proving the claim. $\square$

25.2 Lipschitz Condition

Definition 25.2.1. We say that $f : \Omega \to \mathbb{R}$ is Lipschitz-differentiable if f is differentiable and there is a constant $L > 0$ such that for all $x, y \in \Omega$,

$$||(\nabla f)_x - (\nabla f)_y|| \leq L||x - y||$$

Definition 25.2.2. Let $f : \Omega \to \mathbb{R}$ be a differentiable function. Then for some fixed $\eta > 0$ we define the *gradient descent iteration* of x as,

$$x^+ = x - \eta(\nabla f)_x$$

Lemma 25.2.3. If f is Lipschitz-differentiable and $\eta \leq L^{-1}$ then,

$$-\tfrac{3}{2}\eta||(\nabla f)_x||^2 \leq f(x^+) - f(x) \leq -\tfrac{1}{2}\eta||(\nabla f)_x||^2$$

Proof. Define,

$$g(t) = f((1-t)x + tx^+) = f(x - t\eta(\nabla f)_x)$$

Because g is continuously differentiable since f is continuously differentiable,

$$f(x^+) - f(x) = g(1) - g(0) = \int_0^1 g'(t)\mathrm{d}t$$

However,

$$g'(t) = -\eta(\nabla f)_x \cdot (\nabla f)_{x(t)}$$

Therefore,

$$\begin{aligned}
|f(x^+) - f(x) + \eta||(\nabla f)_x||^2| &= \eta\left|(\nabla f)_x \cdot \int_0^1 [(\nabla f)_x - (\nabla f)_{x(t)}]\mathrm{d}t\right] \\
&\leq \eta||(\nabla f)_x|| \int_0^1 ||(\nabla f)_x - (\nabla f)_{x(t)}||\mathrm{d}t \\
&\leq \eta L||(\nabla f)_x|| \int_0^1 ||x - x(t)||\mathrm{d}t \\
&= \eta^2 L||(\nabla f)_x||^2 \int_0^1 t\mathrm{d}t
\end{aligned}$$

$$\& = \tfrac{1}{2}\eta^2 L||(\nabla f)_x||^2$$

If we take $\eta < L^{-1}$ then completing the proof we find,

$$|f(x^+) - f(x) + \eta||(\nabla f)_x||^2| \leq \tfrac{1}{2}\eta||(\nabla f)_x||^2$$

$\square$

Remark. Therefore, defining a sequence $x_{k+1} = x_k^+$ then either x_k hits a point with $(\nabla f)_{x_k} = 0$ at which point the sequence stabilizes or $f(x_k)$ forms a strictly decreasing sequence.

Remark. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$. This has a constant derivative and hence we can take L to be any positive constant. However, we get the sequence $x_k = x_0 - k\eta$ does not converge and gives an unbounded decreasing sequence $f(x_k)$.

Corollary 25.2.4. Suppose that in addition, f is bounded below. Then the decreasing sequence $f(x_k)$ must converge to a limit. However, I claim that the sequence x_k need not converge to a limit. Indeed, consider the following example.

Example 25.2.5. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} 0 & x \le 0 \\ \varphi(x) & 0 < x < 1 \\ \frac{1}{x} & x \ge 1 \end{cases}$$

where φ is a smooth function which interpolates to make f smooth. Then f has second derivative bounded so it is Lipschitz-differentiable and notice that f is even strictly convex on $[1, \infty)$. However, let $x_0 = 1$. Then we get the recurrence,

$$x_{k+1} = x_k + \frac{\eta}{x_k^2}$$

This is increasing so we just need to show that it is unbounded. Suppose that $x_k < n$ for all n. Then $x_{k+1} - x_k > \frac{\eta}{n^2}$ so for $N > \eta^{-1} n^2 (n - x_k)$ we would have $x_{k+N} \ge n$ giving a contradiction proving that this sequence is indeed unbounded. However, f on $\mathbb{R}$ we see that f achieves it minimum but is not convex on this region. We will see that if we have both convexity and achieving a minimum then this cannot happen.

Remark. Now we suppose that f achieves it minimum at some point x^*. Then we want to bound how quickly the distance between x and x^* decreases under $x \mapsto x^+$.

Proposition 25.2.6. Suppose that $f : \Omega \to \mathbb{R}$ is convex and Lipschitz-differentiable with $\eta < L^{-1}$. Fix $x^* \in \Omega$ then,

$$||x - x^*||^2 - ||x^+ - x^*||^2 \ge 2\eta(f(x^+) - f(x^*)) \ge 0$$

Proof. We expand,

$$||x^+ - x^*||^2 = ||(x^+ - x) + (x - x^*)||^2 = ||x^+ - x||^2 + 2(x^+ - x) \cdot (x - x^*) + ||x - x^*||^2$$

Therefore,

$$||x - x^*||^2 - ||x^+ - x^*||^2 = -2(x^+ - x^*) \cdot (x - x^*) - ||x^+ - x||^2 = 2\eta(\nabla f)_x \cdot (x - x^*) - \eta^2 ||(\nabla f)_x||^2$$

However, by previous lemmas,

$$f(x^*) - f(x) \ge (\nabla f)_x \cdot (x^* - x)$$

and also,

$$f(x^+) - f(x) \le -\tfrac{1}{2}\eta ||(\nabla f)_x||^2$$

Therefore,

$$||x - x^*||^2 - ||x^+ - x^*||^2 \ge 2\eta(f(x) - f(x^*)) + 2\eta(f(x^+) - f(x)) = 2\eta(f(x^+) - f(x^*))$$

proving the claim. ☐

Remark. If x^* is a global (local suffices since f is convex) minimum of x^* (and hence f is bounded below) The above proposition implies that the seqeuence $||x - x^*||$ is decreasing.

Proposition 25.2.7. Suppose that $f : \Omega \to \mathbb{R}$ is convex and has a global minimum (and hence f is bounded below) and f is Lipschitz-differentiable with $\eta < L^{-1}$. Furthermore assume that the unit ball in V is precompact (e.g. if V is finite dimensional) then $x_k \to x^*$ converges where x^* is a global minimum and,

$$f(x_k) - f(x^*) \le \frac{||x_0 - x^*||^2}{2\eta k}$$

Proof. Let x' achieve the global minimum. Then the previous lemma shows that $x_k \in B_r(x')$ where $r = 2\eta(f(x_0) - f(x'))$. Since $\overline{B_r(x')}$ is compact we see that x_k has a convergent subsequence $x_{k_i} \to x^*$. I claim that x^* is a global minimum of f. First, since $f^* = f(x')$ is a global minimum, f is bounded below and thus the decreasing sequence $f(x_k)$ converges and hence is Cauchy. Therefore,

$$\lim_{k \to \infty} ||(\nabla f)_{x_k}||^2 \le 2\eta^{-1} \lim_{k \to \infty} |f(x_{k+1}) - f(x_k)| = 0$$

Therefore,

$$||(\nabla f)_{x^*}|| \le ||(\nabla f)_{x^*} - (\nabla f)_{x_k}|| + ||(\nabla f)_{x_k}|| \le L||x^* - x_k|| + ||(\nabla f)_{x_k}||$$

The first term goes to zero on the subsequence x_{k_i} and the second term goes to zero and therefore $(\nabla f)_{x^*} = 0$ so x^* is a global minimum of f. Therefore we can apply the previous lemma to conclude,

$$||x_k - x^*||^2 - ||x_{k+1} - x^*||^2 \ge 2\eta(f(x_{k+1}) - f(x^*)) \ge 0$$

meaning that,

$$||x_{k+1} - x^*|| \le ||x_k - x^*||$$

Therefore, since $x_{k_i} \to x^*$ we see that $x_k \to x^*$ converges. Finally, because $f(x_k)$ is a decreasing sequence,

$$\begin{aligned} f(x_k) - f(x^*) &\le \frac{1}{k} \sum_{j=1}^{k} [f(x_i) - f(x^*)] \le \frac{1}{2\eta k} \sum_{j=1}^{k} \left(||x_{j-1} - x^*||^2 - ||x_j - x^*||^2 \right) \\ &= \frac{1}{2\eta k} \left(||x_0 - x^*||^2 - ||x_k - x^*||^2 \right) \le \frac{||x_0 - x^*||^2}{2\eta k} \end{aligned}$$

□

26 Relationships between Geometric Categories

Remark. We consider the properties of the following morphisms of categories,

$$\mathbf{AffSch} \hookrightarrow \mathbf{Sch} \hookrightarrow \mathbf{LRS} \hookrightarrow \mathbf{RingSp} \to \mathbf{Top} \to \mathbf{Set}$$

27 G-Structure on the Cotangent Bundle

Consider an action $G \circlearrowright X$ over a base S.

The map $a_G = (\mathrm{id}, a) : G \times X \to G \times X$ defines a morphism $\psi : a_G^* \Omega_{G \times X/S} \xrightarrow{\sim} \Omega_{G \times X/S}$. Now we have,

$$\Omega_{G \times X/S} \cong \pi_1^* \Omega_{G/S} \oplus \pi_2^* \Omega_{X/S}$$

Furthermore,

$$a_G^* \Omega_{G \times X/S} \cong \pi_1^* \Omega_{G/S} \oplus a^* \Omega_{X/S}$$

Therefore ψ is a matrix,

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{21} \\ \psi_{12} & \psi_{22} \end{pmatrix}$$

Furthermore, $\pi_1 \circ a_G = \pi_1$ so $\psi \circ a_G^* \mathrm{d}\pi_1 = \mathrm{d}\pi_1$ which is the inclusion $\pi_1^* \Omega_{G/S} \to \Omega_{G \times X/S}$ and therefore ψ on the $\pi_1^* \Omega_{G/S}$ factor is trivial meaning the matrix is of the form,

$$\psi = \begin{pmatrix} \mathrm{id} & \psi_{21} \\ 0 & \psi_{22} \end{pmatrix}$$

therefore,

$$\psi_{22} : a^* \Omega_{X/S} \to \pi_2^* \Omega_{X/S}$$

is an isomorphism since the matrix is an isomorphism giving the required structure.

28 Vector Bundles on Vector Bundles

Given a vector bundle $\pi : E \to X$ and a vector bundle $\pi' : E' \to E$ is it true that $\pi' : E' \to X$ in, in some sense, a vector bundle.

If E' is pulled back from a vector bundle $V \to X$ then I think E' as an X-vector bundle is $V \oplus E$. Therefore if $s : X \to E$ is the zero section, we should ask if $E' = \pi^* s^* E$.

In the topological category this is true because $\pi : E \to X$ is a homotopy equivalence and indeed $s \circ \pi$ is homotopic to the identity and therefore $E' \cong \mathrm{id}^* E' \cong \pi^* s^* E'$.

However, this is false in the holomorphic/algebraic category. Indeed consider $X = \mathbb{P}^1$ and the trivial bundle so $E = \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1$. Then consider the vector bundle,

$$0 \longrightarrow \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{E} \longrightarrow \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0$$

corresponding to the extension class,

$$\begin{aligned} \xi \in \mathrm{Ext}^1_E \left(\pi_1^* \mathcal{O}_{\mathbb{P}^1}(-1), \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \right) = H^1(E, \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-2)) &= H^1(\mathbb{P}^1, (\pi_1)_* \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-2)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \otimes_k k[t] \cong k[t] \end{aligned}$$

which is given by $\xi = t$. Therefore, at $t = 0$ we have the trivial extension so $\mathcal{E}_t \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and for $t \neq 1$ we have the nontrivial extension $\mathcal{E}_t \cong \mathcal{O}^{\oplus 2}$.

Therefore, $\mathcal{E}$ is not the pullback of any vector bundle on $\mathbb{P}^1$ otherwise its fibers would be constant. However, it is not clear if $\mathcal{E}$ can be thought of as a vector bundle over $\mathbb{P}^1$.(WAIT)

29 Why the Stabilizers in the Moduli Space of Elliptic Curves are important

"The moduli space of elliptic curves is just $\mathbb{A}^1$ because an elliptic curve is determined uniquely by its j-invariant". This is wrong even over $\mathbb{C}$ but it becomes clear if we try to work in a field that doesn't contain all square roots (or even $\mathbb{Z}$). If k does not have all square roots then it is false that E is determined by $j(E)$. Indeed,

$$y^2 = x^3 + ax + b$$

and

$$y^2 \cdot d = x^3 + ax + b$$

where $d \in k$ is a nonsquare are nonisomorphic curves which become isomorphic over $k(\sqrt{d})$. Indeed, we see that away from the special values $j(E) \neq 0, 1728$ the elliptic curves with fixed j-invariant are classified by $k^\times/(k^\times)^2$ and for $j = 0, 1728$ they are classified by $k^\times/(k^\times)^6$ and $k^\times/(k^\times)^4$.

Indeed, since these curves become isomorphic over $\bar{k}$ they are all forms of some E_0. Then $\text{Aut}\,(E_0) \cong \mu_2, \mu_6, \mu_4$ (note I think it really should be $\mathbb{Z}/2\mathbb{Z}$ not μ_2 but these are the same in characteristic not 2 and indeed I am not sure what happens for $p \le 5$ anyway) as above and therefore isomorphism classes of curves over k with fixed $j(E)$ are classfieid by,

$$H^1(k, \mu_n) \cong k^\times/(k^\times)^n$$

by Kummer theory.

What does this mean at the level of stacks. A morphism $\text{Spec}\,(k) \to \mathcal{M}_{1,1}$ landing in the topological point determined by $j(E)$ factors through the residual Gerbe which is usually $B(\mathbb{Z}/2\mathbb{Z})$ (at the special points it is $B\mu_6$ or $B\mu_4$) and therefore there are actually many such maps classified by $(\mathbb{Z}/2\mathbb{Z})$-bundles on k i.e. Galois 2-covers which are, via Kummer theory, classified by $H^1(k, \mathbb{Z}/2\mathbb{Z}) = k^\times/(k^\times)^2$ and ditto for the special points. Therefore, we indeed see that the stabilizers of this stack carry useful arithmetic data.

30 E-Fibrations

For simplicitly assume that k has characteristic zero. I think everything here works in every characteristic except for possibly 2 and 3.

Definition 30.0.1. Let E be an elliptic curve over a field k. An E-fibration is a smooth proper morphism $\mathcal{E} \to S$ with S over k such that each geometric fiber $\mathcal{E}_K$ is isomorphic to E_K as a curve (not as a group scheme). We say that an E-fibration is trivial if it is isomorphic to $E \times S$ as an S-scheme.

Lemma 30.0.2. If $\mathcal{E} \to S$ has a section then it is trivial after a finite étale cover.

Proof. If $\mathcal{E} \to S$ has a section then it is a family of elliptic curves over a base and therefore defines a morphism $S \to \mathcal{M}_{1,1}$ whose image is topologically a single point. Since $\mathcal{M}_{1,1}$ is a DM-stack this implies that there is a finite étale cover $S' \to S$ such that $S' \to \mathcal{M}_{1,1}$ is constant (the residual gerbe is finite étale) and thus $\mathcal{E}' \to S'$ is trivial. □

Proposition 30.0.3. An E-fibration is locally trivial in the étale topology.

Proof. Because $\mathcal{E} \to S$ is smooth, étale locally on S it has sections and hence after a further finite étale extension it is trivial. □

Remark. Here is a weird argument which works in some more general cases but only when S is regular. We can similarly define an F-fibration for any scheme F over a field k. Suppose that F is smooth and has smooth aut scheme $\text{Aut}\,(F)$. Then if S is regular I claim that every F-fibration over S is locally trivial in the étale topology.

Let $X \to S$ be an F-fibration with regular S. Then consider $I = \text{Isom}_S(X, S \times_k F)$ which is automatically an $\text{Aut}\,(F)$-pseudo-torsor. I want to show that I is actually an étale torsor. The strategy is to use the fact that (representable) pseudotorsors for a smooth group scheme which are fppf are localy trivial in the étale topology. This is because using fppf descent and the pseudotorsor condition the morphisms is smooth and hence admits sections locally in the étale topology locally trivializing it as a torsor. Thus it suffices to show that $I \to S$ is fppf. Becuase the geometric fibers of $X \to S$ are all isomorphic to F_K the fibers of $I \to S$ are isomorphic to $\mathrm{Aut}\,(F_K)$ and hence have constant dimension. In particular, $I \to S$ is surjective. (I DONT THINK THE NEXT STEP WORKS) I want to use miracle flatness but I could be highly singular so ...

Proposition 30.0.4. Let S be a curve. Then every E-fibration over S is trivial after a finite (not necessarily étale) extension.

Proof. $\mathcal{E} \times_S \mathcal{E} \to \mathcal{E}$ has a section so if we can produce a subscheme $S' \subset \mathcal{E}$ which is finite over S then $\mathcal{E}_{S'} \to S'$ admits a section and hence is trivial after an additional finite étale extension. There is an étale open $U \to S$ with a section and hence a map $U \to \mathcal{E}$ over S. Taking the scheme theoretic image gives a closed subscheme $S' \subset \mathcal{E}$ such that $S' \to S$ is nonconstant and hence finite since it is a map of curves. □

Example 30.0.5. Consider $x \in E$ a nontorsion point. Then take $E \times \mathbb{P}^1 \to \mathbb{P}^1$ and identify the fibers over ± 1 via translation by x to give $X \to S$ where S is a nodal curve. This exists as a scheme by (REF) but is not projective (example on p.198 of Raynaud's thesis). In fact it is easy to see that X cannot be projective over S. Otherwise X would be projective and thus would admit a line bunle $\mathcal{L}$. Pulling back gives a line bundle $\mathcal{L}'$ on $E \times \mathbb{P}^1$ which takes the form $\mathcal{L}_1 \otimes \mathcal{O}(n)$ for some n and $\mathcal{L}_1 \in \mathrm{Pic}\,(E)$. However, the line bundle $\mathcal{L}_1$ must be isomorphic to its pullback by $\varphi : E \to E$ given by translation by x. Now,

$$\varphi \mathcal{O}_E([p_1] + \cdots + [p_n]) = \mathcal{O}_E([p_1 + x] + \cdots + [p_n + x]) = \mathcal{O}_E([p_1] + \cdots + [p_n] + n[x] - n[e])$$

and therefore we need that $n[x] = n[e]$ meaning that $nx = e$ so x must be a torsion point giving a contraction unless $\mathcal{L}_1 = \mathcal{O}_E$. Therefore $\mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1}(n)$ which is not ample. Since $E \times \mathbb{P}^1 \to X$ is finite and $\mathcal{L}$ is ample, its pullback $\mathcal{L}'$ is ample giving a contradiction.

Remark. Since an E-fibration is étale locally trivial they are forms of E and therefore classified by $H^1_{\text{ét}}(S, \mathrm{Aut}\,(E))$.

Proposition 30.0.6. An E-fibration is equivalent to the data of an E-torsor after a finite étale extension of the base.

Proof. Consider the sequence,

$$1 \longrightarrow E \longrightarrow \mathrm{Aut}\,(E) \longrightarrow G \longrightarrow 1$$

I claim that G is a finite étale group scheme over k. (IS THIS TRUE!!) Therefore, there is an exact sequence,

$$H^1(S, E) \longrightarrow H^1(S, \mathrm{Aut}\,(E)) \longrightarrow H^1(S, G)$$

Given a class $[\mathcal{E} \to S] \in H^1(S, \mathrm{Aut}\,(E))$ its image in $H^1(S, G)$ is killed by a finite étale extension (since G is finite étale every G-torsor is finite étale and thus kills itself) hence we can assume that $[\mathcal{E} \to S]$ is in the image of $H^1(S, E) \to H^1(S, \mathrm{Aut}\,(E))$ after a finite étale extension of the base. □

Lemma 30.0.7. If $[\mathcal{E} \to S] \in H^1(S, E)$ is torsion then it is killed by a finite étale extension.

Proof. Consider the sequence,

$$0 \longrightarrow E[n] \longrightarrow E \xrightarrow{n} E \longrightarrow 0$$

where $E[n]$ is a finite étale group scheme over k (IS THIS TRUE). Therefore we get a sequence,

$$H^1(S, E[n]) \longrightarrow H^1(S, E) \xrightarrow{n} H^1(S, E)$$

therefore if $[\mathcal{E} \to S] \in H^1(S, E)$ is torsion it lies in some kernel and hence in the image of some $[c'] \in H^1(S, E[n])$ which is killed by a finite étale cover because $E[n]$ is a finite étale group scheme. □

Proposition 30.0.8. If S is regular then every E-fibration over S is projective and trivial in the finite étale topology.

Proof. By [Raynaud, Cor. XIII 2.4] every E-torsor is projective and torsion in $H^1(S, E)$ so this follows from the previous results. □

(WANT AN EXAMPLE OF NORMAL SURFACE WITH LOC-TORSION TORSION WHICH IS NOT TORSION)

31 Review of Basics

Remark. The Zariski topology on $\text{Spec}\,(A)$ makes the relations between I and Z definitional.

Proposition 31.0.1. For any $Q \subset \text{Spec}\,(A)$,

$$\overline{Q} = V(\bigcap Q) = V(I(Q)) \quad \text{where} \quad I(Q) = \{f \in A \mid \forall \mathfrak{p} \in S : f \in \mathfrak{p}\}$$

Proof. By definition, if $\mathfrak{p} \in Q$ then $I(Q) \subset \mathfrak{p}$ so $\mathfrak{p} \in V(I(Q))$ meaning $S \subset V(I(Q))$. Furthermore, if $Q \subset V(J)$ for some ideal J then $J \subset \mathfrak{p}$ for each $\mathfrak{p} \in Q$ meaning,

$$J \subset \bigcap_{\mathfrak{p} \in S} \mathfrak{p} = I(Q)$$

and thus $V(I(Q)) \subset V(J)$ proving the claim. □

Proposition 31.0.2. for any $S \subset A$ we have $I(V(S)) = \sqrt{S}$ where,

$$\sqrt{S} = \bigcap_{\mathfrak{p} \supset S} \mathfrak{p}$$

Proof. This is by definition since $V(S) = \{\mathfrak{p} \mid \mathfrak{p} \supset S\}$ and

$$I(V(S)) = \{f \in A \mid \forall \mathfrak{p} \in V(S) : f \in \mathfrak{p}\}$$

□

Proposition 31.0.3. Let $\varphi^* : \text{Spec}\,(B) \to \text{Spec}\,(A)$ be a map of affine schemes. Then,

(a) $(\varphi^*)^{-1}(D(f)) = D(\varphi(f))$

(b) $(\varphi^*)^{-1}(V(I)) = V(\varphi(I))$

(c) $\overline{\varphi^*(V(I))} = V(\varphi^{-1}(I))$

Proof. First,

$$\varphi^*(\mathfrak{p}) \in D(f) \iff f \notin \varphi^*(\mathfrak{p}) \iff \varphi(f) \in \mathfrak{p} \iff \mathfrak{p} \in D(\varphi(f))$$

Second,

$$\varphi^*(\mathfrak{p}) \in V(I) \iff \varphi^*(\mathfrak{p}) \supset I \iff \mathfrak{p} \supset \varphi(I) \iff \mathfrak{p} \in V(\varphi(I))$$

For (c),

$$\overline{\varphi^*(V(I))} = V(\bigcap \varphi^*(V(I))) = V(\bigcap_{\mathfrak{p} \supset I} \varphi^{-1}(\mathfrak{p})) = V(\varphi^{-1}(\bigcap_{\mathfrak{p} \supset I})) = V(\varphi^{-1}(\sqrt{I})) = V(\sqrt{\varphi^{-1}(I)}) = V(\varphi^{-1}(I))$$

where,

$$\varphi^{-1}(\sqrt{I}) = \sqrt{\varphi^{-1}(I)}$$

because,

$$x \in \varphi^{-1}(\sqrt{I}) \iff \varphi(x) \in \sqrt{I} \iff \varphi(x)^n \in I \iff \varphi(x^n) \in I \iff x^n \in \varphi^{-1}(I) \iff x \in \sqrt{\varphi^{-1}(I)}$$

□

Remark. Dually, one might expect $\varphi^*(D(f))^\circ = D(\varphi^{-1}(f))$ but this is false. Indeed, consider $\varphi : \mathbb{Z} \to \mathbb{Q}$ then $\varphi^*(D_{\mathbb{Q}}(2)) = \{(0)\}$ has empty interior but $D_{\mathbb{Z}}(2)$ is nonempty.

Remark. Some of the sets on the RHS are not ideals but it is clear that for any subset $S \subset A$ we have $D(S) = D(SA)$ with SA the ideal generated by S.

Remark. For any multiplicative subset $S \subset A$ we have $\mathrm{Spec}\,(S^{-1}A) \to \mathrm{Spec}\,(A)$ identifies the subset where $S \cap \mathfrak{p} = \varnothing$ in general this cannot be expressed as the condition $I \not\subset \mathfrak{p}$ for some ideal $I \subset A$ so this is not open. For example, let $A = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$ then $S \cap \mathfrak{p} = \varnothing$ if and only if $\mathfrak{p} = (0)$ but we need some $I \neq (0)$ which is contained in every prime, impossible since $\mathbb{Z}$ is reduced.

Corollary 31.0.4. If $\varphi : A \to B$ satisfies $\ker \varphi \subset \mathrm{nilrad}\,(A)$ if and only if $\varphi^* : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is dominant.

Proof. Since $\varphi^{-1}(0) = \ker \varphi$ we have,

$$\overline{\varphi^*(\mathrm{Spec}\,(B))} = V(\ker \varphi)$$

and the result follows. Let's do this directly. First, let $\ker \varphi \subset \mathrm{nilrad}\,(A)$ we need to show any nonempty open $U \supset \mathrm{Spec}\,(A)$ conatins some element of the image. Choose $D(f) \subset U$ with $f \notin \mathrm{nilrad}\,(A)$ so $D(f)$ is nonempty then $\varphi(f) \notin \mathrm{nilrad}\,(B)$ (see the remark) so $D(\varphi(f))$ is nonempty and $D(\varphi(f)) \to D(f)$ proving the claim. Conversely, if φ^* is dominant and $f \in \ker \varphi$ then if $D(f)$ is nonempty there is some $\mathfrak{p} \in \mathrm{Spec}\,(B)$ mapping into $D(f)$ meaning $\varphi(f) \notin \mathfrak{p}$ but $\varphi(f) = 0$ giving a contradiction so $D(f)$ is empty menaing $f \in \mathrm{nilrad}\,(A)$. □

Remark. Notice that,

$$\varphi^{-1}(\mathrm{nilrad}\,(B)) \subset \mathrm{nilrad}\,(A) \iff \ker \varphi \subset \mathrm{nilrad}\,(A)$$

The forward implication is obvious. Converely, if $\ker \varphi \subset \mathrm{nilrad}\,(A)$ then if $\varphi(x) \in \mathrm{nilrad}\,(A)$ then $\varphi(x)^n = 0$ so $\varphi(x^n) = 0$ and hence $x^n \in \ker \varphi \subset \mathrm{nilrad}\,(A)$ so $x \in \mathrm{nilrad}\,(A)$ proving the claim.

32 Factoring Through a Point

Proposition 32.0.1. Let $f : X \to Y$ be a morphism of schemes with $f(X) = \{y\}$ one point. Then there is a factorization,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X_{\text{red}} \arrow[u, hook] \arrow[r] & \text{Spec}\,(\kappa(y)) \arrow[u, hook] \end{tikzcd}$$

Proof. It suffices to show that $X_{\text{red}} \to Y$ factors through $\text{Spec}\,(\kappa(y))$. Since $f(X) = \{y\}$ reduce to affine opens $U \to V$ is $\varphi^* : \text{Spec}\,(B) \to \text{Spec}\,(A)$ image is $\mathfrak{p} \subset A$. Thus $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ for all primes $\mathfrak{q} \subset B$. Thus,

$$\mathfrak{p} = \varphi^{-1}(\text{nilrad}\,(B))$$

and also if $f \notin \mathfrak{p}$ then $\varphi(f) \notin \mathfrak{q}$ so $\varphi(f)$ is a unit. Therefore we get a factorization,

$$A \to (A/\mathfrak{p})_{\mathfrak{p}} \to B/\text{nilrad}\,(B) = B_{\text{red}}$$

□

33 The Stack of All Curves is Not Separated

Remark. What do we mean by separated for a map of stacks? First of all, for a map of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ the diagonal Δ_f is always representable by algebraic spaces (Tag 04XS) so we can talk about its properties as long as we know about the property for algebraic spaces. It might seem natural to impose that the diagonal is a closed embedding but it turns out this is too restrictive, in particular this will imply that a separated DM-stack is an algebraic space (I THINK) for the following reason.

Proposition 33.0.1. The following is a 2-pullback diagram,

$$\begin{tikzcd} \text{Isom}_{\mathscr{X}}\,(a, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "(a\,b)"] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times \mathscr{X} \end{tikzcd}$$

Proof. □

Remark. In particular, if Δ is a closed immersion then we require that $\text{Isom}_{\mathscr{X}}\,(a, b) \to T$ is a closed immersion. Consider the case that $T = \text{Spec}\,(k)$ then this says that all objects in $\mathscr{X}$ have only trivial automorphisms. In particular, if $\mathscr{X}$ is an algebraic-stack then it is an algebraic space (Tag 04SZ). This is clearly not satisfactory.

Remark. One way to fix this problem is to notice the following: if $f : X \to Y$ is a map of schemes then because $\Delta_{X/Y} : X \to X \times_Y X$ is an immersion always the following are equivalent,

(a) f is separated

(b) $\Delta_{X/Y}$ is a closed immersion

(c) $\Delta_{X/Y}$ is finite

(d) $\Delta_{X/Y}$ is proper

(e) $\Delta_{X/Y}$ is universally closed

(f) $\Delta_{X/Y}$ is closed.

Therefore, we could take any of these as a definition for stacks and retain the same notion for schemes (and algebraic spaces by similar reasoning). Because topological conditions behave somewhat badly for stacks, finite and proper are the convenient notions. Following the stacks project we take the latter although some authors prefer finiteness of the diagonal.

Definition 33.0.2. A map of algebraic stacks $f : \mathscr{X} \to \mathscr{Y}$ is,

(a) *separated* if $\Delta_f : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is proper

(b) *quasi-separated* if $\Delta_f : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is quasi-compact and quasi-separated.

where properties of Δ_f are in the sense of representable maps.

Remark. There are two ways to know we have the "correct" definition of separatedness. The first is that, when the map f is representable by algebraic spaces, this definition agrees with the separated in the sense of representable maps.

Lemma 33.0.3. Let $f : \mathscr{X} \to \mathscr{Y}$ be a representable (by algebraic spaces) morphism of algebraic stacks. Then the following are equivalent,

(a) f is separated in the sense of representable maps

(b) Δ_f is a clsoed immersion

(c) Δ_f is proper

(d) Δ_f is universally closed.

Proof. This is basically because these properties are the same for maps of algebraic spaces and are preserved under base change. See Tag 04YS for details. □

Remark. Part of the reason we require Δ_f to be proper and not just universally closed is that although Δ_f is automatically finite type it can be nonseparated. For example, let G be a nonseparated group algebraic space over $S = \text{Spec}(k)$ (see Tag 06E9) and consider $\mathscr{X} = BG = [S/G] \to S$. Then consider,

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d] \\ BG \arrow[r, "\Delta"] & BG \times_S BG \end{tikzcd}$$

shows that Δ is not separated.

Remark. The best reason we know that we have the right notion of separatedness is that the following valuative criterion holds.

Proposition 33.0.4. Let $f : \mathscr{X} \to \mathscr{Y}$ be a quasi-separated morphism of algebraic stacks. If f satisfies the uniqueness part of the valuative criterion then f is separated where this criterion is for every 2-commutative diagram,

$$
\begin{tikzcd}
\mathrm{Spec}\,(K) \arrow[r] \arrow[d, "j"] & \mathscr{X} \arrow[d, "f"] \\
\mathrm{Spec}\,(A) \arrow[r] \arrow[ur, dashed] & \mathscr{Y}
\end{tikzcd}
$$

with A a valuation ring and $K = \mathrm{Frac}\,(A)$ the category of maps $\mathrm{Spec}\,(A) \to \mathscr{X}$ equipped with natural transformations making the diagrams 2-commute is equivalent to a (-1)-cateogry (either empty or one element).

Remark. See Tag 0CL9 we define what this uniqueness criterion means. The category of dotted arrows has objects (a, α, β) where $a : \mathrm{Spec}\,(A) \to \mathscr{X}$ is a morphism and $\alpha : a \circ j \to x$ and $\beta : y \to f \circ a$ are two morphisms witnessing the commutativity of the triangles. These must be compatible in the sense that,

$$
\begin{tikzcd}
 & f \circ a \circ j \arrow[dr, "\mathrm{id}_f * a"] & \\
y \circ j \arrow[ur, "\beta * \mathrm{id}_j"] \arrow[rr, "\gamma"] & & f \circ x
\end{tikzcd}
$$

commutes. A morphism $(a, \alpha, \beta) \to (a', \alpha', \beta')$ is a 2-arrow $\theta : a \to a'$ compatible with commutativity meaning $\alpha = \alpha' \circ (\theta * \mathrm{id}_j)$ and $\beta = (\mathrm{id}_f * \theta) \circ \beta$.

33.1 The Stack of All Curves

Remark. Work in the category of k-schemes so all our stacks are equipped with a forgetful map to $\mathrm{Spec}\,(k)$.

Definition 33.1.1. The stack of all curves $\mathcal{M}^{\mathrm{all}}$ is the fibered category of flat proper finitely presented morphism of algebraic spaces $\pi : \mathcal{C} \to S$ whose geometric fibers are 1-dimensional.

Example 33.1.2. Consider $A = k[[t]]$ and $K = \mathrm{Frac}\,(A) = k((t))$. Then consider,

$$X = \mathrm{Proj}\left(A[x, y, z]/(zy^2 - x^3 - t^4 xz^2)\right) \to \mathrm{Spec}\,(A)$$

which is flat because it is an integral scheme over a DVR and hence torsion-free as an A-module. The special fiber,

$$X_k \cong \mathrm{Proj}\left(k[x, y, z]/(zy^2 - x^3)\right)$$

is a nodal curve while the generic fiber,

$$X_k \cong \mathrm{Proj}\left(K[x, y, z]/(zy^2 - x^3 - t^4 xz^2)\right) \cong \mathrm{Proj}\left(K[x, y, z]/(zy^2 - x^3 - xz^2\right) \cong E_K$$

where the isomorphism takes,

$$x \mapsto t^2 x \quad y \mapsto t^3 y \quad z \mapsto z$$

is an elliptic curve base changed from the elliptic curve over k,

$$E = \mathrm{Proj}\left(k[x, y, z]/(zy^2 - x^3 - xz^2)\right)$$

Therefore, this shows that $\mathcal{M}^{\mathrm{all}}$ is not separated because it violates the valuative criterion,

$$\begin{tikzcd}
\mathrm{Spec}\,(K) \arrow[r, "x"] \arrow[d, "j"'] & \mathcal{M}^{\mathrm{all}} \arrow[d, "f"] \\
\mathrm{Spec}\,(A) \arrow[r, "y"] \arrow[ur, dashed] & \mathrm{Spec}\,(k)
\end{tikzcd}$$

Such a diagram corresponds to a curve $C \to \mathrm{Spec}\,(K)$ and because $\mathrm{Spec}\,(k)$ is a scheme there are no nontrivial 2-morphisms so the square is commutative with a unique 2-morphism $\gamma = \mathrm{id}$. Lifts are tripples (a, α, β) where a corresponds to a curve $\mathcal{C} \to \mathrm{Spec}\,(A)$ and $\alpha : a \circ j \to x$ is an isomorphism $\alpha : \mathcal{C}_K \to C$ over K and $\beta : y \to f \circ x$ must be the identity because $\mathrm{Spec}\,(k)$ is representable. Finally these much satisfy,

$$\gamma = (\mathrm{id}_f * \alpha) \circ (\beta * \mathrm{id}_j)$$

which is automatic because these are 2-morphisms on $\mathrm{Spec}\,(k)$ (also $\gamma = \mathrm{id}$ and $\beta = \mathrm{id}$ and $\mathrm{id}_f * \alpha = \mathrm{id}$ since $\mathcal{M}^{\mathrm{all}} \to \mathrm{Spec}\,(k)$ collapses all isomorphisms to id). Therefore, the category of arrows is equivalent to the category of curves $\mathcal{C} \to \mathrm{Spec}\,(A)$ equipped with an isomorphism $\alpha : \mathcal{C}_K \to C$ with morphismsas isomorphisms of families respecting the identification of the special fiber. Therefore, this is exactly the category of models. However, above we have X and E_A which are nonisomorphic models (have nonisomorphic generic fibers) so $\mathcal{M}^{\mathrm{all}}$ is not separated.

Here is another way to understand the failure of separatedness in this example. The models X, E_A determine morphisms $a, b : \mathrm{Spec}\,(A) \to \mathcal{M}^{\mathrm{all}}$ so consider,

$$\begin{tikzcd}
\mathrm{Isom}_A\,(X, E_A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\,(A) \arrow[d] \\
\mathcal{M}^{\mathrm{all}} \arrow[r] & \mathcal{M}^{\mathrm{all}} \times_k \mathcal{M}^{\mathrm{all}}
\end{tikzcd}$$

and we know $\mathrm{Isom}_A\,(X, E_A)$ is nonempty over the generic fiber but empty on the special fiber so its image in $\mathrm{Spec}\,(A)$ is not closed and hence $\mathrm{Isom}_A\,(X, E_A) \to \mathrm{Spec}\,(A)$ is not proper proving that Δ_f is not proper.

Remark. The semi-stable reduction theorem tells us that a semi-stable reduction when it exists is always unique and also after a finite extension does exist. This proves the uniqueness and existence parts of the valuative criteria for properness for the stack of semi-stable curves.

34 Moduli of Smooth Fanos

Definition 34.0.1. A family of smooth fanos is a smooth proper finitely presented morphism $\pi : X \to S$ such that each fiber X_s is a smooth Fano meaning a smooth projective variety with ample $\omega_{X_s}^{\vee}$.

35 Isotrivial Families

Definition 35.0.1. A *polarized family* if a proper, flat, finitely presented morphism $\pi : X \to S$ equipped with a relatively ample invertible sheaf $\mathcal{L}$ on X. A morphism of polarized families is a cartesian diagram,

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

and an isomorphism $\varphi : f^*\mathcal{L} \to \mathcal{L}'$.

Remark. Since $\pi : X \to T$ is finte type and $\mathcal{L}$ is π-relatively ample there is a (Zariski) open cover $T_i \to T$ such that $\mathcal{L}$ is ample for $X_i \to T_i$ and thus we get a closed embedding $X_i \hookrightarrow \mathbb{P}^N_{T_i}$ over T_i defined by $\mathcal{L}^{\otimes n_i}$ for some $n_i > 0$. Therefore, π is locally projective (in the sense of Hartshorne).

Definition 35.0.2. The stack of polarized proper schemes $\mathcal{M}_{\text{pol}}$ is the stack of polarized families. Explicitly, it is the category fibered over $(\mathbf{Sch}_{\mathbb{Z}})_{\text{fppf}}$ whose objects are pairs $(X \to S, \mathcal{L})$ with,

(a) $X \to S$ a proper, flat, finitely presented morphism

(b) $\mathcal{L}$ an invertible $\mathcal{O}_X$-module relatively ample for $X \to S$

and morphisms $(X' \to S', \mathcal{L}') \to (X \to S, \mathcal{L})$ are given by (f, g, φ) with,

(a) $f : X' \to X$ and $g : S' \to S$ morphisms of schemes such that,

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

is a commutative cartesian diagram,

(b) $\varphi : f^*\mathcal{L} \to \mathcal{L}'$ is an isomorphism.

Theorem 35.0.3. The fibered category $\mathcal{M}_{\text{pol}}$ is a locally noetherian algebraic stack and the canonical morphism $\mathcal{M}_{\text{pol}} \to \text{Spec}(\mathbb{Z})$ is quasi-separated and locally of finite presentation.

Proof. See Tag 0D4X and Tag 0DPU. Then $\mathcal{M}_{\text{pol}}$ is locally noetherian because $\text{Spec}(\mathbb{Z})$ is and we apply Tag 06R6.

DO THE IDEA!! $\square$

Lemma 35.0.4. The morphism $\mathscr{I}_{\mathcal{M}_{\text{pol}}} \to \mathcal{M}_{\text{pol}}$ is quasi-compact.

Proof. For each morphsim $T \to \mathcal{M}_{\text{pol}}$ from a scheme defining a polarized family $(X \to T, \mathcal{L})$ we get the 2-fiber square,

$$\begin{tikzcd} \text{Aut}_{\mathcal{M}_{\text{pol}}}(X) \arrow[r] \arrow[d] & T \arrow[d] \\ \mathscr{I}_{\mathcal{M}_{\text{pol}}} \arrow[r] & \mathcal{M}_{\text{pol}} \end{tikzcd}$$

therefore it suffices to show that $\text{Aut}_{\mathcal{M}_{\text{pol}}}(X) \to T$ is quasi-compact. Since quasi-compactness is local on the base, we may assume that $\pi : X \to T$ is projective with $X \hookrightarrow \mathbb{P}^n_T$ via $\mathcal{L}$. Since polarized automorphisms of X fix $\mathcal{L}$ we see that any automorphism of X extens to an automorphism of $\mathbb{P}^n_T$ giving a map of sheaves,

$$\text{Aut}_{\mathcal{M}_{\text{pol}}}(X) \to \text{PGL}_{n+1}$$

whose kernel is given by automorphisms of X which fix $\mathbb{P}^n_T$ and hence of the form $(\text{id}, \text{id}, \varphi)$ where $\varphi : \mathcal{L} \to \mathcal{L}$ is an automorphism. Therefore, we get a sequence,

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Aut}_{\mathcal{M}_{\mathrm{pol}}}(X) \longrightarrow \mathrm{PGL}_{n+1}$$

(DO TRANSPORTERS EXIST IN GENRAL?) □

Remark. This is quite special to automorphisms of polarized varieties. For example, $A = E \times E$ where E is an ordinary elliptic curve has $\mathrm{Aut}(A) = \mathrm{GL}_2(\mathbb{Z})$ which is not quasi-compact but A has finitely many polarized automorphisms.

Lemma 35.0.5. Let $\mathscr{X}$ be a locally noetherian algebraic stack then for each $x \in |X|$ the residual gerbe $\mathscr{Z}_x$ of $\mathscr{X}$ at x exists and $\mathscr{Z}_x \to \mathscr{X}$ is a closed embedding.

Proof. □

Corollary 35.0.6. For each $x \in |\mathcal{M}_{\mathrm{pol}}|$ the residual gerbe $\mathscr{Z}_x$ of $\mathcal{M}_{\mathrm{pol}}$ at x exists and $\mathscr{Z}_x \to \mathcal{M}_{\mathrm{pol}}$ is a closed embedding.

Proof. Using lemma Tag 06UH.
DO THE IDEA □

35.1 Using the Gerbes

Lemma 35.1.1. Let $\mathscr{X}$ be a reduced algebraic stack and $\mathscr{Y}$ be a locally noetherian algebraic stack with $y \in |\mathscr{Y}|$ is a closed point. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks such that $f(|\mathscr{X}|) = \{y\}$ then there is a factorization,

$$\begin{tikzcd} \mathscr{X} \arrow[rr, "f"] \arrow[dr] & & \mathscr{Y} \\ & \mathbb{Z}_y \arrow[ur] & \end{tikzcd}$$

Proof. Because $\mathbb{Z}_y \hookrightarrow \mathscr{Y}$ is a closed substack with $|\mathbb{Z}_y| = \{y\}$, we can apply Tag 050B. □

Proposition 35.1.2. Let $f : T \to \mathscr{X}$ be a morphism from a reduced noetherian Jacobson scheme to a locally noetherian algebraic stack

35.2 WORK TO DO

Here is a problem, because of the extra $\mathbb{G}_m$ factor the DM-locus is empty. We would like to consider the DM-locus inside here and see the stack of curves living inside or something. Can we rigidify this stack somehow. What sort of level structure will kill these automorphisms of the line bundle.

36 Is Syntomic the topology that was promised?

I want the smallest topology containing $\mathrm{Spec}(A[x]/(x^p - a)) \to \mathrm{Spec}(A)$ in characteristic p and Zariski covers. I would also be interested in the "finite" version of this topology.

Perhaps there is a good notion of a "finite flat topology".

37 Connections on Principle Bundles

Proposition 37.0.1. Let $\pi : G \to S$ be a group scheme. Then,

$$\Omega_{G/S} = \pi^* e^* \Omega_{G/S}$$

so if we set,

$$\omega_{G/S} = e^* \Omega_{G/S}$$

then,

$$\Omega_{G/S} = \pi^* \omega_{G/S}$$

Furthermore, by the projection form,

$$\pi_* \Omega_{G/S} = \omega_{G/S} \otimes_{\mathcal{O}_S} \mathcal{O}_G$$

and thus if π is $\mathcal{O}$-connected (e.g. for G/S an abelian scheme) then,

$$\pi_* \Omega_{G/S} = \omega_{G/S}$$

Proof. Consider the Cartesian diagram,

$$\begin{tikzcd}
G \times_S G \arrow[rrd, bend left, "m"] \arrow[rd, dashed] \arrow[rdd, bend right, "\pi_1"'] & & \\
& G \times_S G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & G \arrow[d] \\
& G \arrow[r] & S
\end{tikzcd}$$

because the dashed arrow is an isomorphism, the outside square is Cartesian so $m^* \Omega_{G/S} = \pi_1^* \Omega_{G/S}$. Then,

$$\pi^* e^* \Omega_{G/S} = (e \circ \pi, \mathrm{id})^* \pi_1^* \Omega_{G/S} = (e \circ \pi, \mathrm{id})^* m^* \Omega_{G/S} = \mathrm{id}^* \Omega_{G/S} = \Omega_{G/S}$$

□

Remark. If $S = \mathrm{Spec}\,(k)$ then $\omega_{G/S} = \mathfrak{g}$ is the Lie algebra.

Definition 37.0.2. Let P be an object on an S-scheme $\pi : X \to S$. Then an S-connection on P is an isomorphism $\varphi : \pi_1^* P \to \pi_2^* P$ of objects over $X^{(1)}$ such that $\Delta^* \varphi = \mathrm{id}$ where,

$$X^{(1)} \hookrightarrow X \times_S X$$

is the first infinitesimal diagonal. Consider,

$$X \hookrightarrow X \times_S X \times_S X$$

and the first-order neighborhood $X_3^{(1)}$ equipped with three projections $\pi_{ij} : X_3^{(1)} \to X^{(1)}$. We say that φ is *integrable* if it satisfies the cocycle condition,

$$\pi_{23}^* \varphi \circ \pi_{12}^* \varphi = \pi_{13}^* \varphi$$

Proposition 37.0.3. Let $G \to S$ be a smooth group scheme and $\pi : P \to X$ be a G-bundle. Consider the sequence,

$$0 \longrightarrow \pi^* \Omega_{X/S} \longrightarrow \Omega_{P/S} \longrightarrow \Omega_{P/X} \longrightarrow 0$$

This is an exact sequence of descent data and therefore arises as π^* of an exact sequence,

$$0 \longrightarrow \Omega_X \longrightarrow Q \longrightarrow \mathrm{ad}(P) \longrightarrow 0$$

where $\mathrm{ad}(P)$ is the adjoint bundle,

$$\mathrm{ad}(P) = P \times_G \omega_{G/S}$$

defined via the adjoint action of G on $\omega_{G/S}$.

38 When are Flag Varities Toric?

Let G be a simple reductive group and $P \subset G$ a parabolic subgroup. When is the flag variety G/P toric? We know that any algebraic group is unirational (at least if G is reductive or k is perfect see Lemma 7.2.3 in Brian's second course) and thus G/P is unirational so it might seem reasonable that it could be toric.

Proposition 38.0.1. If G is simple with trivial center and $P \subset G$ is a parabolic subgroup then $\mathrm{Aut}\,(G/P) = G \rtimes A$ where A is a finite group determined by the automorphisms of the Dynkin diagram of G except for the following exceptional cases,

(a) $X = G_2/U_2$ where $\mathrm{Aut}^0(\mathrm{X}) = \mathrm{SO}_7$

(b) $X = \mathrm{Sp}_r/\mathrm{Sp}_{r-1}U_1$ where $\mathrm{Aut}^0(\mathrm{X}) = \mathrm{PSL}_{2\mathrm{r}}$

(c) $X = \mathrm{SO}_{2n+1}/U_n$ where $\mathrm{Aut}^0(\mathrm{X}) = \mathrm{PSO}_{2\mathrm{n}+2}$.

Proof. There is a reference here. □

Therefore, except for these cases the maximal torus of $\mathrm{Aut}\,(X)$ is the maximal torus $T \subset G$. Hence if $\dim T < \dim G/P$ then it is impossible for G/P to be toric. For example, let $G = \mathrm{PGL}_n$ and $B \subset G$ the standard Borel so that G/B is the complete flag variety. Then $\dim T = n - 1$ and $\dim G/B = \frac{1}{2}n(n-1)$ so we see that,

$$\dim T < \dim G/B \iff n > 2$$

and indeed for $n = 2$ we get $G/B = \mathbb{P}^1$ which is toric.

This proves that projective bundles over toric varieties need not be toric. Indeed, $F_n = \mathrm{PGL}_n/B$ is an iterated projective bundle over $\mathrm{Spec}\,(k)$ and hence at some point a projective bundle must take a toric variety to a non-toric variety since F_n is not toric for $n > 2$. For $n = 3$ we get a counter-example for the projectivization of a rank 2 vector bundle on $\mathbb{P}^2$ in fact $F_3 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{Q})$ where $\mathcal{Q} = \Omega_{\mathbb{P}^2}(1)$ is the canonical subbundle and $\mathbb{P}^2$ is toric but F_3 is not toric by the above discussion.

39 May 27 Orders of Points on genus 1 curves

Definition 39.0.1. Let $X \to \mathrm{Spec}\,(k)$ be a k-scheme. For a point $x \in X$, let $\deg(x) = [\kappa(x) : k]$ and,

(a) $\mathrm{radix}\, X = \min\{\deg x \mid x \in X\}$

(b) $\mathrm{ind}\, X = \gcd\{\deg x \mid x \in X\}$.

Proposition 39.0.2. Let G be a k-group and T a G-torsor. Then $\mathrm{ind}\, T$ is the gcd of the degrees of all extensions such that T becomes trivial.

Proof. $T_{k'}$ is trivial iff $T(k') \neq \varnothing$ proving the claim. ☐

Proposition 39.0.3. If $T \in H^1(k, G)$ is torson, denote its order $\mathrm{per}\, T$, then $\mathrm{per}\, T \mid \mathrm{ind}\, T$.

Proof. It suffices to show that if $T_{k'}$ is trivial then $\mathrm{per}\, T \mid [k' : k]$. Indeed, using that,

$$H^1(k, G) \xrightarrow{\text{res}} H^1(k', G) \xrightarrow{\text{cor}} H^1(k, G)$$

is multiplication by $n = [k' : k]$ we see that $T \in H^1(k, G)[n]$ and hence its order $\mathrm{per}\, T \mid n$. ☐

Proposition 39.0.4. Let C be a genus 1 curve over a field k. Then $\forall x \in C : \mathrm{ind}(C) \mid \deg x$ so in particular $\mathrm{radix}\, C = \mathrm{ind}\, C$.

Proof. Let $x \in C$ be a point achiving the minimum and $y \in C$ another point. Assume the theorem is false then $\gcd(\deg x, \deg y) < \deg x$. Therefore, there is a divisor,

$$D = p[x] + q[y]$$

for $p, q \in \mathbb{Z}$ such that $0 < \deg D < \deg x$. Then by Riemann-Roch,

$$h^0(C, \mathcal{O}_C(D)) = \deg D > 0$$

meaning that D is equivalent to an effective divisor proving there is some point with smaller degree than x. ☐

Proposition 39.0.5. Let k be a field and E an elliptic curve over k such that for every finite extension k'/k and all sufficiently large primes $p \gg 0$ the abelian group $k'^{\times}/(k'^{\times})^n$ is infinite and $E(k)/pE(k)$ is finite (e.g. k is a number field and E is any elliptic curve). Then for any $n > 0$ there is an E-torsor C with $\mathrm{ind}\, C \geq n$.

Proof. It suffices to prove that for any sufficiently large prime ℓ there is a nontrivial $C \in H^1(k, E)[\ell]$ since then $\mathrm{per}\, C = \ell$ and thus $\mathrm{ind}\, C \geq \ell$. Choose ℓ large enough to be invertible in k (i.e. not the characteristic) then consider the sequence,

$$0 \longrightarrow E[n] \longrightarrow E \xrightarrow{\ n\ } E \longrightarrow 0$$

exact in the étale topology. Thus we find an exact sequence,

$$0 \longrightarrow E(k)/\ell E(k) \longrightarrow H^1(k, E[\ell]) \longrightarrow H^1(k, E)[\ell] \longrightarrow 0$$

Since the first term is finite, to show that $H^1(k, E)[\ell] \neq 0$ it suffices to show that $H^1(k, E[\ell])$ is infinite. Let k'/k be a field extension such that $E[\ell]$ is split and k' has all ℓ^{th}-roots of unity. Then from the inflation-restriction sequence,

$$0 \longrightarrow H^1(\mathrm{Gal}\,(k'/k)\,, E[\ell]^{G_{k'}}) \longrightarrow H^1(k, E[\ell]) \longrightarrow H^1(k', E[\ell]) \longrightarrow H^2(\mathrm{Gal}\,(k'/k)\,, E[\ell]^{G_{k'}})$$

However, $\mathrm{Gal}\,(k/'k)$ is finite and $E[\ell]^{G_{k'}}$ is a finite abelian group (equipped with the trivial action) so $H^1(\mathrm{Gal}\,(k'/k)\,, E[\ell]^{G_{k'}})$ and $H^2(\mathrm{Gal}\,(k'/k)\,, E[n]^{G_{k'}})$ are finite. Therefore, $H^1(k, E[\ell])$ is infinite iff $H^1(k', E[\ell]) = \mathrm{Hom}\,(G_{k'}, E[\ell])$ is infinite. However, $E[\ell]_{k'} \cong (\mathbb{Z}/\ell\mathbb{Z})^2 \cong \mu_\ell^2$ and therefore from the Kummer sequence,

$$H^1(k', \mu_\ell) = (k'^\times)/(k'^\times)^\ell$$

is infinite proving that $H^1(k', E[\ell])$ is infinite. □

Remark. Can we strengthen the statement to $\mathrm{ind}\, C = n$?

40 Alternative Proofs for the HW

Lemma 40.0.1. Let A be a Noetherian domain with $\dim A = 1$ and $\mathfrak{p} \subset A$ a nonzero prime. Then there exists $f \in \mathrm{Frac}\,(A)$ with $f \notin A_\mathfrak{p}$ but $f \in A_\mathfrak{q}$ for all primes $\mathfrak{q} \neq \mathfrak{p}$.

Proof. Let $K = \mathrm{Frac}\,(A)$ and take any $f \in \mathrm{Frac}\,(K)$ such that $f \notin A_\mathfrak{p}$ (e.g. p^{-1} for any $p \in \mathfrak{p}$). Consider the ideal

$$I = (A : f) = \{x \in A \mid xf \in A\}$$

Then if $x \in I$ we have $xf \in A$ so if $x \in A \setminus \mathfrak{p}$ then $f = \frac{xf}{x} \in A_\mathfrak{p}$. Since $f \notin A_\mathfrak{p}$ we must have $I \subset \mathfrak{p}$. Since A is Noetherian and I is proper it has a primary decomposition,

$$I = \mathfrak{q}_0 \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

such that $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Therefore,

$$\sqrt{I} = \sqrt{\mathfrak{q}_0} \cap \cdots \cap \sqrt{\mathfrak{q}_n} = \mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_n$$

which implies that $\mathfrak{p}_0 \ldots, \mathfrak{p}_n \in V(I)$. Furthermore, $\dim A = 1$ so each prime $\mathfrak{p}_i$ is maximal and thus $V(I) = \{\mathfrak{p}_0, \ldots, \mathfrak{p}_n\}$ since if some prime $\mathfrak{q} \supset I$ then $\mathfrak{q} \supset \sqrt{I}$ and thus $\mathfrak{q} \supset \mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_n$ but $\mathfrak{q}$ is prime so $\mathfrak{q} \supset \mathfrak{p}_i$ for some but $\mathfrak{p}_i$ is maximal so $\mathfrak{q} = \mathfrak{p}_i$. In particular there are a finite number of primes above I and since $\mathfrak{p} \in V(I)$ we can take $\mathfrak{p}_0 = \mathfrak{p}$ WLOG.

By prime avoidance $\mathfrak{p}_i \not\subset \bigcup_{j \neq i} \mathfrak{p}_j$ and thus there exist elements, $a_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$. Then let $\tilde{a} = \prod_{i=1}^n a_i$ and thus $a_0 \tilde{a} \in \mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_n = \sqrt{I}$ so $(a_0 \tilde{a})^N \in I$ for some positive integer N. Consider $I' = (A : \tilde{a}^N f) \supset I$. Since $a_0^N \tilde{a}^N \in I$ we know that $a_0^N (\tilde{a}^N f) \in A$ so $a_0^N \in I'$. However, $a_0 \notin \mathfrak{p}_i$ for $i > 0$ and thus neither is a_0^N so $I' \not\subset \mathfrak{p}_i$ for $i > 0$. But since $I' \supset I$ we have $V(I') \subset V(I)$ so $V(I') = \{\mathfrak{p}\}$. Furthermore,

$$g \in A_\mathfrak{q} \iff \exists s \in A \setminus \mathfrak{q} : sg \in A \iff (A : g) \not\subset \mathfrak{q}$$

Therefore $a_0^N f \notin A_\mathfrak{p}$ but $a_0^N f \in A_\mathfrak{q}$ for each prime $\mathfrak{q} \neq \mathfrak{p}$. □

Proposition 40.0.2. If A is a Noetherian domain with $\dim A = 1$ and $\mathfrak{p} \subset A$ nonzero prime. Then $\mathfrak{p}^{-1} \neq A$.

Proof. Choose f as above such that $f \notin A_\mathfrak{p}$ but $f \in A_\mathfrak{q}$ for all $\mathfrak{q} \neq \mathfrak{p}$. Since $\dim A_\mathfrak{p} = 1$ and $A_\mathfrak{p}$ is local we know that for any nonzer $a \in A_\mathfrak{p}$ we have $\sqrt{(a)} = \mathfrak{p}^n$. Therefore, there is some n such that $f\mathfrak{p}^n \subset A$. Choose n minimal. Then $f\mathfrak{p}^{n-1} \not\subset A$ so choose some $x \in f\mathfrak{p}^{n-1} \setminus A$ but $x\mathfrak{p} \subset A$ because $f\mathfrak{p}^n \subset A$ so $x \in \mathfrak{p}^{-1} \setminus A$ proving the claim. □

Remark. If $A = k[t^2, t^3] \subset k[t]$ and $\mathfrak{p} = (t^2, t^3)$ then $\mathfrak{p}^{-1} = \widetilde{A}$ so $t \in \mathfrak{p}^{-1}$ meaning $\mathfrak{p}^{-1} \neq A$ but $\mathfrak{p}^{-1}$ is contained in the normalization $\widetilde{A}$. However, notice that $\mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}$. Indeed, this is the generic behavior in the following sense.

Proposition 40.0.3. Let A be a Noetherian domain with $\dim A = 1$ and $\mathfrak{p} \subset A$ a nonzero prime. Suppose $\mathfrak{p}\mathfrak{p}^{-1} \subsetneq A$ then $\mathfrak{p}^{-1} \subset \widetilde{A}$. In particular $\mathfrak{p}^{-1} \backslash A$ which is nonempty, consists of integral elements which do not lie in A.

Proof. Suppose $\mathfrak{p}\mathfrak{p}^{-1} \subsetneq A$. Since $\mathfrak{p}\mathfrak{p}^{-1}$ is an ideal containing $\mathfrak{p}$ and $\mathfrak{p}$ is maximal then $\mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}$ so $\mathfrak{p}^{-1} = B$ is a ring since if $x, y \in B$ then $xy\mathfrak{p} \subset x\mathfrak{p} \subset \mathfrak{p}$ so $xy \in B$. However, $\mathfrak{p}^{-1}$ is a fractional ideal and hence a finite A-module so B is integral over A and $B \subset K$ so $B \subset \widetilde{A}$. □

41 Gimbal Lock

Lemma 41.0.1. A submersion $f : M \to N$ between compact connected manifolds is a fiber bundle.

Proof. To apply Ehresmann's theorem we need to show that f is proper and surjective. Indeed, since M is compact f is automatically proper (preimage of a compact set is closed and hence compact). Furthermore, $f(M)$ is closed because M is compact but f is a submersion and hence open (since it is an $\mathbb{R}^n$-bundle locally on the source) so $f(M)$ is clopen and N is connected so $f(M) = N$. □

We are wondering if there are surjective submersions $M = (S^1)^n \to \mathbb{RP}^3$. It is automatically a fiber bundle. For $n = 3$ this is a covering map so injective on π_1 which is impossible. For $n = 4$ it is a fiber bundle and the fiber F is a compact 1-manifold and hence $F = S^1$. Consider the fibration sequence,

$$\pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \pi_1(\mathbb{RP}^3) \longrightarrow 1$$

but this gives an exact sequence,

$$\mathbb{Z} \to \mathbb{Z}^4 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

which is impossible. For $n = 5$ the fiber F is a compact surface. Then consdier the exact sequence,

$$\pi_3(\mathbb{RP}^3) \longrightarrow \pi_2(F) \longrightarrow \pi_2(M) \longrightarrow \pi_2(\mathbb{RP}^3) \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \pi_1(\mathbb{RP}^3) \longrightarrow 1$$

which gives,

$$\mathbb{Z} \longrightarrow \pi_2(F) \longrightarrow \pi_2(M) \longrightarrow 0 \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

but the map $\pi_1(F) \to \pi_1(M)$ needs to be injective and index 2. This is impossible since $\pi_1(M) = \mathbb{Z}^5$ but $\pi_1(F)$ is nonabelian if it has rank bigger than 2. We could keep going but this is clearly unsustainable.

Next thing to try is go to universal covers,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, dashed] \arrow[d] & S^3 \arrow[d] \\ M \arrow[r] & \mathbb{RP}^3 \end{tikzcd}$$

Since $M \to \mathbb{RP}^3$ is a surjective submersion so is $f : \mathbb{R}^n \to S^3$. So we ask the following question,

Remark. For which m, n do there exist surjective submersions $\mathbb{R}^m \to S^n$?

The problem is these exist for all $m \geq n$ and are clearly impossible for $m < n$. Indeed, since S^n is geodesically complete, the map $\exp : \mathbb{R}^n \to S^n$ is a surjective submersion and we can simply compose with the projection $\mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$.

Ah but notice since $M \to \mathbb{RP}^3$ is a fibratio then $\mathbb{R}^n \to \mathbb{RP}^3$ is also a fibration. So the lift to S^3 should also be a fibration (check this). Then we see that the homotopy fiber is ΩS^3 which is not homotopy equivalent to a manifold (infinite cohomology amplitude).

Another strategy is to take the fiber product,

$$\begin{tikzcd} \widetilde{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S^3 \arrow[d] \\ M \arrow[r] & \mathbb{RP}^3 \end{tikzcd}$$

which is a manifold since the maps are surjective submersions. Then by coversing space theory $\widetilde{M} \cong (S^1)^n$ also since the finite index subgroups of $\mathbb{Z}^n$ are all isomorphic to $\mathbb{Z}^n$. Therefore, we reduce to a surjective submersion $M \to S^3$. This is better because S^3 is simply connected so we don't need to mess around with local coefficients when we apply the Serre spectral sequence. Since $M \to S^3$ is a fiber bundle by Ehresman's theorem we consider the following situation, what are the possible fiber bundles,

$$F \to (S^1)^n \to S^m$$

we have the Serre spectral sequence,

$$E_2^{p,q} = H^p(S^m, H^q(F)) \implies H^{p+q}(M) = [H^\bullet(S^1)^{\otimes n}]^{p+q}$$

By the universal coeffiicent theorem, there is an exact sequence,

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\left(H_{p-1}(S^m), H^q(F)\right) \longrightarrow H^p(S^m, H^q(F)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(H_p(S^m), H^q(F)\right) \longrightarrow 0$$

but since $H_i(S^m)$ is a free $\mathbb{Z}$-module, the first term dies and we see that,

$$H^p(S^m, H^q(F)) = \begin{cases} H^q(F) & p = 0, m \\ 0 & \text{else} \end{cases}$$

Therefore, the spectral sequence has two columns. This gives the Wang sequence,

$$\cdots \longrightarrow H^{i-1}(F) \xrightarrow{\partial} H^{i-m}(F) \xrightarrow{\alpha} H^i(M) \xrightarrow{\iota^*} H^i(F) \xrightarrow{\partial} H^{i+1-m}(F) \longrightarrow \cdots$$

In a bounded spectral sequence we also have the following "Euler characteristic" formula, given,

$$E_2^{p,q} \implies H^{p+q}$$

we have,

$$\sum_{n \geq 0} (-1)^n \dim H^{p+q} = \sum_{p,q \geq 0} (-1)^{p+q} \dim E_r^{p,q}$$

Therefore, in the cases when,

$$H^p(B, H^q(F)) = H^p(B) \otimes H^q(F)$$

as does in the case that $X = S^m$ then we see that,

$$\chi(E) = \chi(B) \cdot \chi(F)$$

In particular, we have,

$$\chi(S^m) \cdot \chi(F) = \chi(M) = \chi((S^1)^n) = \chi(S^1)^n = 0$$

Therefore, if m is even we must have $\chi(F) = 0$. Okay this is not so helpful.

Let's actually go back to the fibration homotopy exact sequence. Notice that the map $\pi_k(M) \to \pi_k(S^m)$ is always zero for $m > 1$. This is somehow supposed to imply that the homotopy fiber F is (rationally) homotopy equivalent to $M \times \Omega S^m$ (I'm not sure why) but the homology of this is not finite support and thus F cannot be homotopic to a manifold meaning $M \to S^m$ cannot be a submersion.

Maybe we can see this from the Wang sequence. For large i we have $H^i(M) = 0$ so we get $H^i(F) \xrightarrow{\sim} H^{i+1-m}(F)$ which shifts up dimension infinitely if we can get it nonzero in the range $i > \dim M$. Therefore, we should consider $i = \dim M = n$. Then $H^n(M) \neq 0$ but if we assume $H^n(F) = 0$ (it is a lower dimensional submanifold) so indeed we have $H^{n-m}(F) \twoheadrightarrow H^n(M)$ is an isomorphism (fundamental classes) so HMMM seems hard !!!

42 Orientation-Reversing Maps

The rational Pontryagin classes are topological invariants in the following sense. If $f : X \to Y$ is a diffeomorphism then $f^* p_i(Y) = p_i(X)$ using the map $f^* : H^{4i}(Y, \mathbb{Q}) \to H^{4i}(X, \mathbb{Q})$. However, notice that this does not exactly mean that Pontryagin numbers are preserved because of orientation issues. If $f : X \to X$ is orientation reversing then it reverses the Pontryagin numbers.

Indeed, $p_1 = c_1(X)^2 - 2c_2(X)$. Say X is a 2-dimensional complex manifold. Then $p_1 \in H^4(X, \mathbb{Q})$ so there is a canonical $\deg p_1 \in \mathbb{Z}$ which is given the orientation on X. Hovever, there exist orientation-reversing homeomorphisms $f : X \to X$ which means $f^* p_1 = -p_1$. This means that if we transport the complex structure along f we get a complex manifold X' such that $p_1(X') = -p_1(X)$. Since $c_2(X) = \chi(X)$ is a topological invariant we see that,

(a) $c_2(X') = c_2(X)$

(b) $c_1^2(X') = 4c_2(X) - c_1^2(X)$

such that $p_1(X') = -p_1(X)$. Indeed, this is theorem 9 of "TOPOLOGICALLY INVARIANT CHERN NUMBERS OF PROJECTIVE VARIETIES".

43 fpqc sheaf that does not sheafify

Remark. A reference for this is: BASICALLY BOUNDED FUNCTORS AND FLAT SHEAVES by WILLIAM C. WATERHOUSE.

The fpqc site has the problem that given a scheme X, there is no cofinal *set* of fpqc covers of X.

Indeed, consider $X = \mathrm{Spec}\,(k)$. Then let $X_I = \mathrm{Spec}\,(k(t_i)_{i \in I})$ for any index set. Then consider,

$$\{X_I \to X\}_{I \in \mathrm{Set}}$$

raning over all sets which is an fpqc cover. Any refinement of this cover $\{V_j \to X\}_{j \in J}$ by a set J must have for each I there is some j such that $V_j \to X_I \to X$ factors $V_j \to X$ but then $\mathrm{Spec}\,(k(t_i)_{i \in I}) \to V_j$ menaing that V_j has a point with residue field $\#\kappa(x) \geq \#I$ and thus the residue fields of the schemes V_j must have unbounded cardinality. This is impossible if J is a set since each scheme has a set of residue fields.

Definition 43.0.1. Consider the presheaf $F : \mathrm{Sch}_{\mathrm{fpqc}} \to \mathrm{Set}$ defined as follows,

$$F(X) \to \{\text{locally constant } f : X \to \mathrm{Card} \mid f(x) \leq \#\kappa(x)\}$$

where Card is the proper class of all cardinals. If $f : X \to Y$ is a map then $\kappa(f(x)) \hookrightarrow \kappa(x)$ so the pullback $F(Y) \to F(X)$ is well-defined.

Remark. It is not immediately apparent that F is set (rather than proper class) valued. Indeed, since the scheme X has a set of points and hence $\#\kappa(x)$ for $x \in X$ is bounded by some cardinal λ and then,

$$F(X) = \{\text{locally constant } f : X \to \lambda \mid f(x) \leq \#\kappa(x)\}$$

which is a set.

Proposition 43.0.2. F has no fpqc-sheafification.

Proof. Suppose there was a sheaf G and a universal morphism $\varphi : F \to G$. Furthermore, for any cardinal λ consider the presheaf,

$$L_\lambda(X) = \{\text{locally constant } f : X \to \lambda\}$$

Sending cardinals less than λ to themselves and others to zero we get a map,

$$F \to L_\lambda$$

For each X we have argued that $F(X) \to L_\lambda(X)$ is injective. Now L_λ is a sheaf. Indeed, it is represented by λ-copies of $\mathrm{Spec}\,(\mathbb{Z})$. By the universal property we get,

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr] & & F_\lambda \\
& G \arrow[ur, dashed] &
\end{tikzcd}$$

and $F(X) \hookrightarrow F_\lambda(X)$ is injective for some λ and thus $F \hookrightarrow G$ is an injection.

Let $f : U \to X$ be an fpqc cover. Then by the sheaf condition,

$$\begin{tikzcd}
G(X) \arrow[r] & G(U) \arrow[r] & G(U \times_X U) \\
F(X) \arrow[r] \arrow[u, hook] & F(U) \arrow[r] \arrow[u, hook] & F(U \times_X U) \arrow[u, hook]
\end{tikzcd}$$

where the top row is an equalizer. Now suppose that X, U are fields. Then $\text{Spec}\,(U)$ is a single point so the sets maps $U \times_X U \to U$ are set-theoretically the same. Thus $F(U)$ is the equalizer of the two maps $F(U) \to F(U \times_X U)$ and therefore $F(U) \to G(U)$ factors through $G(X) \to G(U)$ hence $F(U) \hookrightarrow G(X)$. However, for a field $U = \text{Spec}\,(K)$ we have $\#F(U) \geq \#K$ so for a fixed field X we can take arbitrarily large cardinalities for K and $F(U) \hookrightarrow G(X)$ which contradicts that $G(X)$ is a set. $\square$

Remark. How does this work if we had universes? The sheafification is basically a universe-sized colimit and so we would get that G is universe-sized so if we changed the universe G would increase unboundedly. Indeed, for a fixed cardinality λ which is a universe, we just get that $G = F_\lambda$ since all schemes have residue fields bounded by λ and alternatively we can always cover a scheme such that its residue fields are arbitrarily large.

44 h-Cobordism

Definition 44.0.1. An h-cobordism between n-manifolds M, N is a cobordism W such that both inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

Remark. In particular, M and N are homotopy equivalent.

Remark. Unlike other types of cobordisms, h-cobordism classes don't usually form a group. For oriented manifolds, there is a coboridsm $M \sqcup \overline{M}$ to $\varnothing$ by using a tube $M \times [0, 1]$ with both boundary components thought of as the source of the cobordism. However, this is not an h-coborhism unless M is contractible.

Theorem 44.0.2 (Smale)**.** Let $n \geq 5$ and W a compact $(n+1)$-dimensional h-cobordism between M and N in $C = \mathbf{Diff}, \mathbf{PL}, \mathbf{Top}$ such that W, M, N are simply-connected, then W is C-trivial meaning isomorphic in C to $M \times [0, 1]$ rel boundary. In particular, M and N are isomorphic in C.

45 Spectral Norm

Let A be an $\mathbb{R}$-algebra. Then we define the spectrum of $f \in A$ via,

$$\sigma(f) = \{\lambda \in \mathbb{R} \mid f - \lambda \text{ is not invertible}\}$$

Example 45.0.1. If $A = C^\infty(M)$ then $\sigma(f) = \text{im}\, f$. Indeed, $f - \lambda$ is invertible iff it is nonzero iff $\lambda \notin \text{im}\, f$.

Example 45.0.2. If A is the ring of linear endomorphisms of a vectorspace V. Then $\sigma(\varphi)$ for $\varphi \in A$ is the set of eigenvectors. Indeed, $\varphi - \lambda$ is not invertible iff $\det(\varphi - \lambda) = 0$.

Example 45.0.3. If A is the ring of bounded linear endomorphism of a Banach space V then $\sigma(T)$ is the spectrum which is a closed set. For a compact operator T this is the closure of the set of eigenvalues.

Definition 45.0.4. The *spectral norm* $||f||$ of an element $f \in A$ is defined as,

$$||f|| = \sup_{\lambda \in \sigma(f)} |\lambda|$$

46 Closed Points of Stacks

Consider the stack $[\mathbb{A}^1/\mathbb{G}_m]$. This has two points, an open point and a closed point with stabilizer $\mathbb{G}_m$. This means that any $\mathrm{Spec}\,(\mathbb{C}) \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ which does not map through zero hits the open point. Weird!

What happens to the generic point. Well, it also maps to the open point. Indeed, we can see that $\mathrm{Spec}\,(K) \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ determines a map $(\mathbb{G}_m)_K \to \mathbb{A}^1$ given by the base change of the action to the generic point. This is isomorphic as a K-torsor with a map to $\mathbb{A}^1$ to the base change of the standard inclusion $\mathbb{G}_m \to \mathbb{A}^1$. Indeed, these two maps correspond to $k[t] \to K[t, t^{-1}]$ via either $t \mapsto 1 \otimes t$ or $t \mapsto t \otimes t$. However, these two maps are isomorphic over K by just sending $1 \otimes t$ to $t \otimes t$, this is fine its just multiplying my an element of the field.

This sort of thing shows that in the stack of polarized varities there are many nonclosed $\mathbb{C}$-points. For example, a conic degenerates to two lines but all conics are equal. This means the pair $(\mathbb{P}^1, \mathcal{O}(2))$ is not closed. However, I think $(\mathbb{P}^1, \mathcal{O}(1))$ is closed since it forms a closed orbit of the Hilbert scheme. I think the closed orbits of the PGL-action on the Hilbert scheme will correspond to the closed points of the stack. Certainly, the open orbits correspond to open points since the map from the Hilbert scheme is a PGL-torsor hence smooth so it is open.

However, this does not endanger our argument that if we have an integral ft k-scheme S with a map $f : S \to \mathscr{X}$ to a stack such that all closed points map to a single point $x \in |\mathscr{X}|$ then $f(|S|) = \{x\}$. Indeed, it follows from the following lemma.

Lemma 46.0.1. Let $f : X \to Y$ a continuous map and $x_0 \rightsquigarrow x_1$ then $f(x_0) \rightsquigarrow f(x_1)$.

Proof. $x_1 \in \overline{\{x_0\}}$ and thus $f(x_1) \in f(\overline{\{x_0\}}) \subset \overline{\{f(x_0)\}}$. □

Therefore, for any $s \in S$ we see that $f(x) \rightsquigarrow x$. However, also in S the closed points are dense so $f(S) \subset \overline{f(S_{\text{closed}})} = \overline{\{x\}}$. However, if $\mathscr{X}$ is quasi-separated then $|\mathscr{X}|$ is sober (Tag 0DQQ) meaning that x is the unique generic point of $\overline{\{x\}}$ and thus $f(s) = x$. This seems to not need Chevalley's theorem at all.

Indeed, the stack of polarized schemes is quasi-separated by Tag 0DPU. However, this does allow all but one closed points to map to x and then the remaining closed point to map to x' s.t. $x \rightsquigarrow x'$. In this case the generic points will still map to x. This is what happens when an isotrivial family degenerates for example.

The philosophy is that the open points have larger dimension since the points with large stabilizers have very negative dimension. In this sense, the closed special objects should be the ones with exceptionally large automorphism groups. For example, in polarized schemes, the locus with Hilbert polynomial $2t + 1$ in $\mathbb{P}^2$ corresponds to $[\mathbb{P}^5/\mathrm{PGL}_3]$. Now the orbits of $\mathrm{PGL}_3 \circlearrowright \mathbb{P}^5$ are,

(a) a 5-dimensional open corresponding to the smooth conics: $B\mathrm{PGL}_2$ with residual stabilizer PGL_2

(b) a 4-dimensional locally closed corresponding to the union of two lines: $B(B_2 \times B_2)$ where $B_2 \subset \mathrm{PGL}_2$ is a borel

(c) a 2-dimensional closed orbit corresponding to double lines: BG where G is the 6-dimensional stabilizer group of a line in $\mathbb{P}^2$ under the PGL_3-action which is an extension of PGL_2 by an extension of $\mathbb{G}_m$ by $\mathbb{A}^2$ corresponding to the extra automorphisms arising automorphisms of the nilpotents.

Remark. Let $X \subset \mathbb{P}^2$ be a double line. Then $\mathscr{I}_X = \mathcal{O}(-2)$ and $\mathscr{I}_{X_{\text{red}}} = \mathcal{O}(-1)$ so the nilpotent sequence is,

$$0 \longrightarrow \mathcal{O}_{X_{\text{red}}}(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_{\text{red}}} \longrightarrow 0$$

giving $H^0(X, \mathcal{O}_X) = k$ as expected from the ideal sheaf sequence on $\mathbb{P}^2$. However, there are more automorphisms that just PGL_2 since the isomorphism $f^*\mathcal{O}_X \to \mathcal{O}_X$ is just a morphism of sheaves of $\mathbb{C}$-algebras not of $\mathcal{O}_X$-modules. Since $f^*\mathcal{O}_X \cong X$ abstractly and nilpotents must map to nilpotents we need to consider morphisms of diagrams,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{X_{\text{red}}}(-1) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_{\text{red}}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{X_{\text{red}}}(-1) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_{\text{red}}} & \longrightarrow & 0
\end{array}$$

so the inner morphism is an isomorphism but it is not determined by the outer morphisms. We see that,

$$\mathrm{Hom}_{\mathrm{Rings}}(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_{X_{\text{red}}}(-1))$$

and the conormal sequence gives,

$$0 \longrightarrow \mathcal{O}_{X_{\text{red}}}(-1) \longrightarrow \Omega_X \longrightarrow \Omega_{X_{\text{red}}} \longrightarrow 0$$

therefore we get a sequence,

$$0 \longrightarrow H^1(\mathcal{O}_{X_{\text{red}}}(1)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_{X_{\text{red}}}(-1)) \longrightarrow H^1(\mathcal{O}_{X_{\text{red}}}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_{X_{\text{red}}}, \mathcal{O}_{X_{\text{red}}}(-1))$$

and there is a base-change spectral sequence,

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_{X_{\text{red}}}}\left(\Omega_{X_{\text{red}}}, \mathscr{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_{X_{\text{red}}}, \mathscr{F})\right) \implies \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\Omega_{X_{\text{red}}}, \mathscr{F})$$

and we see that,

$$\mathrm{Ext}^1_{\mathcal{O}_{X_{\text{red}}}}(\Omega_{X_{\text{red}}}, \mathcal{O}_{X_{\text{red}}}(-1)) = H^1(X_{\text{red}}(1)) = 0$$

and likewise,

$$\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{X_{\text{red}}}, \mathcal{O}_{X_{\text{red}}}(-1)) = 0$$

because the extension class of $\mathcal{O}_X$ is trivial so from the exact sequence,

$$\mathscr{E}xt^0_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_{X_{\text{red}}}) \xrightarrow{\sim} \mathscr{E}xt^0_{\mathcal{O}_X}(\mathcal{O}_{X_{\text{red}}}(-1), \mathcal{O}_{X_{\text{red}}}(-1)) \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{X_{\text{red}}}, \mathcal{O}_{X_{\text{red}}}(-1)) \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_{X_{\text{r}}}$$

and the last term is zero since $\mathcal{O}_X$ is locally free. Therefore,

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_{X_{\text{red}}}, \mathcal{O}_{X_{\text{red}}}(-1)) = 0$$

which implies that,

$$\dim \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_{X_{\text{red}}}(-1)) = \dim H^1(\mathcal{O}_{X_{\text{red}}}(1)) + \dim H^1(\mathcal{O}_{X_{\text{red}}}) = 3$$

as expected. Therefore, there are $3 + 3 = 6$ dimensions of automorphism of X.

47 Some Topos Theory

47.1 Preliminaries

47.2 Cohomology

Definition 47.2.1. Let $\mathcal{E}$ be a topos. Then there is a unique geometric morphism $\gamma : \mathcal{E} \to \mathcal{S}$ such that,

$$\gamma^*(S) = \sum_{s \in S} 1 \quad \text{and} \quad \gamma^*(E) = \mathrm{Hom}_{\mathcal{E}}(1, E)$$

We usually write Δ for γ^* and Γ for γ_*.

Remark. In a sheaf topos, this recovers the notions of constant sheaf and global sections.

Definition 47.2.2. Let $\mathcal{E}$ be a topos, write $\mathbf{Ab}(\mathcal{E})$ for the abelian category of abelian group objects in $\mathcal{E}$. Then the global sections functor induces a left-exact (since it admits a left-adjoint Δ) functor,

$$\Gamma : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$$

Define the cohomology groups as the right-derived functors,

$$H^n(\mathcal{E}, A) = R^n\Gamma(A)$$

For any object $B \in \mathcal{E}$, we can define cohomology in the slice topos,

$$H^n_{\mathcal{E}}(B, A) := H^n(\mathcal{E}_{/B}, B^*(A))$$

where $B^* : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}(\mathcal{E}_{/B})$ is the base change functor $E \mapsto (E \times B \to B)$.

Proposition 47.2.3. We have,

$$\mathrm{Ext}^n_{\mathcal{E}}(B, -) := R^n\mathrm{Hom}_{\mathcal{E}}(B, -) = R^n\mathrm{Hom}_{\mathbf{Ab}(\mathcal{E})}(\mathbb{Z}[B], -) = \mathrm{Ext}^n_{\mathbf{Ab}(\mathcal{E})}(\mathbb{Z}[B], -)$$

Then,

$$\mathrm{Ext}^n_{\mathcal{E}}(B, -) = H^n(B, -)$$

Proof. This is because,

$$\Gamma(\mathcal{E}_{/B}, B^*(-)) = \mathrm{Hom}_{\mathcal{E}_{/B}}(B, B^*(-)) = \mathrm{Hom}_{\mathcal{E}}(B, -)$$

since B is the terminal object of $\mathcal{E}_{/B}$. Where,

$$\mathbb{Z}[B] = \bigoplus_{n \in \mathbb{Z}} B$$

in $\mathbf{Ab}(\mathcal{E})$ is left-adjoint to $\mathbf{Ab}(\mathcal{E}) \hookrightarrow \mathcal{E}$. $\square$

Example 47.2.4. In a sheaf topos $\mathcal{E}$ on a site $\mathcal{C}$, the cohomology of a representable object h^X is,

$$H^n(h^X, A) = R^n\mathrm{Hom}_{\mathcal{E}}\left(h^X, A\right) = R^n\Gamma(X, A) = H^n(X, A)$$

is cohomology of X in the usual sense of derived functors of $\Gamma(X, -)$.

Remark. However, there are sites $\mathcal{C}$ without a terminal object. In these sites, there isn't a representable "global sections" however, the topos $\mathcal{E} = \mathfrak{Sh}(\mathcal{C})$ has a global sections functor,

$$\Gamma(\mathcal{C}, -) := \mathrm{Hom}_{\mathcal{E}}(1, -)$$

Example 47.2.5. The étale or Zariski site of a nontrivial stack usually does not have a terminal object. For example, let G be finite étale group scheme over a base S. Then consider the stack BG of G-tosors. The étale site $(BG)_{\text{ét}}$ of BG is the site of étale maps $U \to BG$ from k-schemes equiped with étale covers. Concretely, this is the category of pairs (U, P) where P is a G-torsor over U and U is an étale k-scheme (the map $U \to BG$ being étale is equivalent to $P \to S$ being étale or equivalently $U \to S$ being étale). The morphisms $(U, P) \to (U', P')$ are equivariant diagrams,

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & P' \arrow[d] \\
U \arrow[r] & U'
\end{tikzcd}$$

These automatically make $P \xrightarrow{\sim} U \times_{U'} P'$. There is no terminal object since there is no scheme with a universal G-torsor. This is why we needed to define the stack! Therefore, we need to define global sections as,

$$\Gamma((BG)_{\text{ét}}, -) = \mathrm{Hom}_{\mathfrak{Sh}(BG)_{\text{ét}}}(1, -)$$

meaning compatible choices of a section on every object of $(BG)_{\text{ét}}$. Let's compute,

$$H^i((BG)_{\text{ét}}, \mathbb{Z})$$

(DO EXAMPLE!!!)

47.3 Covers

Definition 47.3.1. Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism in a topos $\mathcal{E}$. Then f is a *cover* if f is an epimorphism. We say that a family $\{\mathscr{F}_\alpha \to \mathscr{G}\}$ is a cover if $\sqcup \mathscr{F}_\alpha \to \mathscr{G}$ is.

Definition 47.3.2. A set $\{X_\alpha\}$ of objects $X_\alpha \in \mathcal{E}$ is a *cover* of $\mathcal{E}$ if the canonical maps $\{X_\alpha \to 1\}$ is a covering family.

Proposition 47.3.3. The collection of covers forms a Grothendieck topology on any topos $\mathcal{E}$.

Proof. We need to check the following axioms:

(a) if $f : \mathscr{G}' \to \mathscr{G}$ is any morphism and $\{\mathscr{F}_\alpha \to \mathscr{G}\}$ is a covering family then,

$$\begin{tikzcd}
\mathscr{F}'_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{F}_\alpha \arrow[d] \\
\mathscr{G}' \arrow[r] & \mathscr{G}
\end{tikzcd}$$

is a covering family. Indeed, pullbacks commute with coproducts (pullback is a left-adjoint by the existence of Power objects and hence preserves colimits) and preserve epimorphisms [SGL, Prop. 7.3]

(b) if $\{\mathscr{F}_\alpha \to \mathscr{G}\}$ is a covering family and $\{\mathscr{F}_{\alpha\beta} \to \mathscr{F}_\alpha\}$ are covering families then $\{\mathscr{F}_{\alpha\beta} \to \mathscr{G}\}$ is a covering family. This is just compatibility reindexing coproducts and that compositions of epis are epis

(c) If $f : \mathscr{F} \to \mathscr{G}$ is an isomrophism then f is an epi and hence a cover.

□

Proposition 47.3.4. Let $\mathcal{E} = \mathfrak{Sh}(\mathcal{C})$ where $\mathcal{C}$ is a site. Then if $f : Y \to X$ is a morphism in $\mathcal{C}$ then,

$$f : h^Y \to h^X \text{ is a cover} \iff f : Y \to X \text{ is refined by a cover}$$

Proof. Suppose $f : h^Y \to h^X$ is an epimorphism. Then, for $\mathrm{id} \in h^X(X)$ there is a covering family $\{U_\alpha \to X\}$ such that $\mathrm{id}|_{U_\alpha}$ which is $(U_\alpha \to X) \in h^X(U_\alpha)$ is in the image of $h^Y(U_\alpha) \to h^X(U_\alpha)$. Explicitly, this means there are maps,

$$\begin{tikzcd} & Y \arrow[d] \\ U_\alpha \arrow[ur] \arrow[r] & X \end{tikzcd}$$

so $X' \to X$ is refined by a cover.

Conversely, if $f : X' \to X$ is refined by a cover then for any $T \to X$ we can base change to $T_\alpha \to U_\alpha$ so we get a cover $\{T_\alpha \to T\}$ such that $T_\alpha \to T \to X$ factors through $T_\alpha \to U_\alpha \to Y$ meaning $h^Y \to h^X$ is an epimorphism. □

Remark. It may seem like this is enlarging the covering families too much. However, this is not true because the following lemma says we do not restrict our category of sheaves.

Lemma 47.3.5. Let $f : Y \to X$ be refined by a cover. Then any sheaf $\mathscr{F}$ on $\mathcal{C}$ satisfies the sheaf condition for f.

Proof. This is standard, see Stacks (2022) notes. □

47.4 Hypercovers

Now we want to define hypercovers internal to the topos. (DO THIS DEFINITION!!) Verdier!

Proof. □

47.5 Quasi-Coherent Sheaves

Definition 47.5.1. Let $(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$ be a ringed topos. Then we say a $\mathcal{O}_{\mathcal{E}}$-module $\mathscr{F} \in \mathcal{E}$ is *quasi-coherent* if there exists a cover $\{U_i\}$ of $\mathcal{E}$.

48 Small Contractions

We want to study the structure of birational maps $f : X \to Y$. From experience with smooth varities we expect the exceptional locus to be a divisor. First, we define the exceptional locus.

Definition 48.0.1. Let $f : X \to Y$ be a birational map of varieties. Then there exists a largest open $U \subset Y$ such that $f : f^{-1}(U) \to U$ is an isomorphism. Then the *exceptional locus* if the closed subscheme,

$$\mathrm{Ex}\,(f) = X \setminus f^{-1}(U)$$

Proposition 48.0.2 (Kollar-Mori, Cor. 2.63)**.** If $f : X \to Y$ is birational where X is projective and Y is $\mathbb{Q}$-factorial then $\mathrm{Ex}\,(f)$ is pure codimension 1.

Proof. Heuristically, f is projective hence is a blowup at some ideal $\mathscr{I} \subset \mathcal{O}_Y$. Using the methods [Hartshorne, Ex. 7.11] (which only requires that Y has $\mathbb{Q}$-factorial singularities) we modify $\mathscr{I}$ such that it has support equal to $Y \setminus U$ where U is the largest open over which f is an isomorphism. Therefore, $\mathrm{Ex}\,(f)$ is the total transform of $V(\mathscr{I})$ which is a Cartier divisor by the definition of blowing up. □

Example 48.0.3. Let $Y = \mathrm{Spec}\,(k[x, y, z, w]/(xy - zw))$ be the affine cone over a the quadric surface $Q = \mathrm{Proj}\,(k[x, y, z, w]/(xy - zw))$. Thus Y which has an isolated singularity at the origin which is not $\mathbb{Q}$-factorial. Indeed, consider the prime divisor,

$$D = V(x, z)$$

Then I claim that nD is never Cartier for $n \neq 0$. Indeed, the vanishing of each coordinate function x, y, z, w contains components.

Set $\widetilde{X} = \mathrm{Bl}_0 Y$ which has exceptional fiber Q. We can blow down along the two rulings to get two smooth 3-folds,

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & X^+ \arrow[dl] \\ & Y & \end{tikzcd}$$

These can be described as the blowups along $I = (x, z)$ and $I^+ = (x, w)$. Since the exceptional of $\widetilde{X} \to Q$ is codimension 1 then by contracting Q to a curve on X and X^+ we see that these blowups $X \to Y$ and $X^+ \to Y$ has codimension 2 exceptional divisors.

Let's compute this in coordinates. By symmetry, it suffices to consider $X \to Y$ which is the blowup of $I = (x, z)$. Then,

$$\mathrm{Bl}_I(A) = A[u, v]/(uz - vx, uy - vw)$$

Then we get two charts for X,

$$U_0 = \mathrm{Spec}\left(A[\tfrac{u}{v}]/(\tfrac{u}{v}z - x, \tfrac{u}{v}y - w)\right) = \mathrm{Spec}\left(k[y, z, \tfrac{u}{v}]\right)$$
$$U_1 = \mathrm{Spec}\left(A[\tfrac{v}{u}]/(z - \tfrac{v}{u}x, y - \tfrac{v}{u}w)\right) = \mathrm{Spec}\left(k[x, w, \tfrac{v}{u}]\right)$$

so we indeed see that X is smooth (in fact it is locally affine space). The fiber over I is,

$$f^{-1}(V(I)) = \text{Proj}\left(k[y,w][u,v]/(uy - vw)\right)$$

However, this is *not* the exceptional locus since I is invertible on $Y \setminus \{0\}$. Indeed, the exceptional locus is exactly over the origin since these are blowdowns of $\widetilde{X}$. Then the exceptional locus is,

$$E = \text{Proj}\left(\text{Bl}_I(A)/\mathfrak{m}\text{Bl}_I(A)\right) = \text{Proj}\left(k[u,v]\right)$$

which is a copy of $\mathbb{P}^1$.

Definition 48.0.4. A *small contraction* is a birational map $f : X \to Y$ with $\text{codim}\left(\text{Ex}\left(f\right), X\right) \geq 2$.

Remark. We have seen if Y is not $\mathbb{Q}$-Cartier there are often small contractions. However, this does not always happen. For example, if Y is the projective cone over a degree d plane curve, this is normal and projective but not $\mathbb{Q}$-factorial. However, there is no small contraction over Y. Indeed, since $\dim Y = 2$, such a small contraction would have zero dimensional exceptional locus. However, Y is normal so any birational map has connected fibers. Thus, we see that small contractions are a dimension ≥ 3 phenomenon.