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1.1 Mapping Class Groups

Definition 1.1.1. Let S be a surface. Then,

$$\mathrm{Map}\,(S) = \mathrm{Homeo}^{+}(S, \partial S)/\mathrm{Isotopy}(S, \partial S)$$

is orientation-preserving homeomorphisms fixing the boundary up to boundary-preserving isotopies.

Example 1.1.2. $\mathrm{Map}\,(D^2) = 0$. Given any map $f : D^2 \to D^2$ we define,

$$F(x,t) = \begin{cases} tf(x/t) & |x| \le t \\ x & \text{else} \end{cases}$$

is an isotopy to the constant map.

Remark. This works for D^2 with one puncture. This also works for $(D^n, \partial D^n)$. However, it is not known if this works in the differentiable category for $n \ge 4$.

1.2 Alexander Method

Let S be a surface. Find some curves $\{\gamma_i\}$ on S such that $S \setminus \bigcup r_i$ is a union of disks or once-punctured disks. If $f \in \mathrm{Map}\,(S)$ takes $f(\gamma_i) \sim \gamma_i$ then there exists an isotopy I of S such that $I(\bigcup f(\gamma_i)) = \bigcup \gamma_i$. If the action of $I(f)$ on the graph $\bigcup \gamma_i$ fixes each vertex edge and preserves the orientation of the edges then $f \sim \mathrm{id}$.

Example 1.2.1. Define $\Phi : \mathrm{Map}\,(T^2) \to \mathrm{SL}(2, \mathbb{Z})$ via its action of $H_1(T^2; \mathbb{Z}) = \mathbb{Z}^2$ preserving orientation. Then we can use the Alexander method to show that this is an isomorphism.

Definition 1.2.2. A curve is an essential, simple, closed curve (sometimes arcs) meaning it is not homotopic to a point or puncture and has no self-intersections.

Definition 1.2.3. The geometric intersection number,

$$i([\alpha], [\beta]) = \min\{|a \cap b| \mid a \in [\alpha] \text{ and } b \in [\beta] \text{ intersect transversally}\}$$

where $[\alpha]$ and $[\beta]$ are homotopy classes. Then the algebraic intersection number,

$$\hat{i}([\alpha], [\beta])$$

is the signed intersection which is a homotopy invariant so we don't need to minimize. Therefore,

$$i([\alpha], [\beta]) \equiv \hat{i}([\alpha], [\beta]) \mod 2$$

Lemma 1.2.4. a, b transverse simple closed curves a, b are in minimal position iff there is no bigon (embedded disk bounded by arcs contained in the two curves) between a and b.

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The mapping class group of the torus is $\mathrm{SL}2, \mathbb{Z}$.

Theorem 2.0.1 (Degn-Licorice)**.** The mapping class group is generated by Dehn twists.

2.1 The Word Problem

Given $(G, g_1, \ldots, g_n)$ is there an algorithm to check if any given $w = \prod g_i$ is trivial?

Remark. The answer only depends on G not on the generating set for G.

Remark. There exists a group G for which the word problem is unsolvable.

Theorem 2.1.1. The word problems for mapping class groups of surfaces are solvable.

Proof. Let $G = \mathrm{MCG}(\Sigma)$ be the mapping class group and $T_{a_1}, \ldots, T_{an}$ Dehn twists for curves $a_1, \ldots, a_n$ generating G. Define Γ to be the graph of $\cup a_i$. Given $\varphi \in \mathrm{MCG}(\Sigma)$ then $\varphi(\Gamma) \subset \Sigma$.

Then φ is the identity if and only if $\varphi(\Gamma) \sim \Gamma$ are isotopic by the Alexander method. This works if the a_i are pairwise minmal (no bigons) and the a_i fill Σ

How to prove finite generation? Quasi-stabilizer generation theorem. Suppose $G \circlearrowright X$ and $D \subset X$ which satisfies $GD = X$. Then the "quasi-stabilizer" of D generates G. We will use this by the action of G on a curve complex X. $\square$

2.2 The Quasi-Stabilizer Theorem

Remark. Recall that,

$$\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}$$

Then for a set we could define,

$$\{\mathrm{Stab}(S) = \{g \mid gS \subset S\}$$

Now we define the quasi-stabilizer,

$$\mathrm{QStab}(S) = \{g \mid gS \cap S \neq \varnothing\}$$

Definition 2.2.1. Let Σ be a surface, and $c(\Sigma)$ is the flag complex defined by the following 1-skeleton (graph):

(a) vertices are isomorphism class of essential simple closed curves

(b) edges from a to b iff $i(a, b) = 0$

then the flag complex adds all possible maximal simplices with edges defined by this graph.

Theorem 2.2.2. Let G act on a connected topological space X. Suppose that $D \subset X$ an open tralation domain (meaning $GD = X$ e.g. a fundamental domain) then $\langle \mathrm{QStab}(D) \rangle = G$.

Proof. Let $g \in G$. assume that $gD \cap \langle \mathrm{Stab}(D) \rangle D \neq \varnothing$. There exists $s \in \langle \mathrm{QStab}(D) \rangle$ such that $gD \cap sD \neq \varnothing \iff s^{-1}gD \cap D \neq \varnothing$. Therefore $s^{-1}g \in \mathrm{QStab}(D)$ so $g \in s\mathrm{QStab}(D) \subset \langle \mathrm{QStab}(D) \rangle$. Therefore $(G \setminus \langle \mathrm{QStab}(D) \rangle)D$ is disjoint from $\langle \mathrm{QStab}(D) \rangle D$ which is nonempty and therefore the first must be empty by connectedness proving the claim. $\square$

Theorem 2.2.3. Let X be a connected simplicial complex, $G \circlearrowright X$. If $D \subset X$ is a subcomplex translational domain, then $\mathrm{QStab}(D)$ generates G.

Proof. Use induction. $\square$

2.3 Finite Generation of Mapping Class Groups

Theorem 2.3.1. For all $g \geq 0$ the mapping class group $\operatorname{MCG}(\Sigma_g)$ is finitely generated by the Dehn twisits by non-separating essential simple closed curves.

Proof. For $g \leq 1$ we knkow $\operatorname{MCG}(\Sigma_g)$ explicitly. Assume $g \geq 2$. Then $\operatorname{MCG}(\Sigma_g) \circlearrowright \hat{N}(\Sigma_g)$ the connected curve complex (I forgot which one). Suppose $\operatorname{MCG}(\Sigma_{g-1})$ is finitely generated etc. Then let a be a curve in Σ_g and b a curve with $i(a, b) = 1$. Then $T_a T_b(a) = b$ and therefore if $g \cdot a = b$ then g is in the quasi-stabilizer for the segment a to b. $\square$