Math GR6262 Algebraic Geometry
Assignment # 5

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1 Problem 1

Let k be a field and X be a scheme over $\text{Spec}\,(k)$. The map $f : X \to \text{Spec}\,(k)$ gives a map on stalks $f^\# : k \to \mathcal{O}_{X,x} \to k(x)$ for each point $x \in X$. By Lemma 3.1, a morphism $\text{Spec}\,(k) \to X$ is determined exactly by specifying a point $x \in X$ and an inclusion $k(x) \to k$. However, a k-rational point is a morphism $\text{Spec}\,(k) \to X$ *as k-schemes* so the diagram,

$$\begin{tikzcd} \text{Spec}\,(k) \arrow[rr, "p"] \arrow[dr, "\text{id}"'] & & X \arrow[dl, "f"] \\ & \text{Spec}\,(k) & \end{tikzcd}$$

is required to commute. This implies that the induced map on stalks is required to commute,

$$\begin{tikzcd} k & & k(x) \arrow[ll, "p^\#"'] \\ & k \arrow[ul, "\text{id}"] \arrow[ur, "f^\#"'] & \end{tikzcd}$$

The commutativity of this diagram shows that $f^\# : k \to k(x)$ must be an isomorphism. Thus given a k-rational point x we have shown that $f^\# : k \to k(x)$ is an isomorphism. Furthermore for any point $x \in X$ if $f^\# : k \to k(x)$ is an isomorphism then its inverse $p^\# : k(x) \to k$ clearly makes the diagram above commute and thus, by the Lemma, induces a morphism of k-schemes $\text{Spec}\,(k) \to X$ so x is k-rational.

To show that all such points are closed, I will prove the stronger fact that if $k(x)$ is a finite extension of k then x is a closed point. Assume $k(x)$ is a finite extension of k. On each affine open $x \in U$, the corresponding prime $\mathfrak{p}$ gives a domain $A/\mathfrak{p}$ and thus inclusions

$$k \hookrightarrow A/\mathfrak{p} \hookrightarrow S_{\mathfrak{p}}^{-1}(A/\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = k(p)$$

showing that $A/\mathfrak{p}$ is a finite-dimensional k-algebra domain and thus a field. Therefore $\mathfrak{p}$ is maximal and thus closed in U. Therefore we have shown that x is closed in every affine open neighborhood. Therefore there exists a closed $C \subset X$ such that $C \cap U = \{x\}$ and thus

$$U^C \cup \{x\} = (U \setminus \{x\})^C = (C^C \cap U)^C = C \cup U^C$$

is closed. Now let $\{U_\alpha\}$ be an affine cover of X. If $x \in U_\alpha$ then we have shown that $U_\alpha^C \cup \{x\}$ is closed otherwise $x \in U_\alpha^C$ so $U_\alpha^C \cup \{x\}$ is closed. Therefore, using the fact that U_α cover X, the set

$$\bigcap_\alpha U_\alpha^C \cup \{x\} = \left(\bigcap_\alpha U_\alpha \right) \cup \{x\} = \varnothing \cup \{x\} = \{x\}$$

is closed.

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2.1

Consider the ring $R = \mathbb{C}$ and the category $\mathbf{Mod}_{\mathbb{C}}$ of $\mathbb{C}$-vectorspaces. Denote by $\mathrm{Hom}_{\overline{\mathbb{C}}}(V, W)$ the $\mathbb{C}$-vectorspace of $\mathbb{C}$-*anti*-linear functions i.e. functions $\varphi : V \to W$ such that $\varphi(\lambda v) = \bar{\lambda}\varphi(v)$ for $\lambda \in \mathbb{C}$. This space is a $\mathbb{C}$-vector space under standard addition and multiplication because $\lambda\varphi$ is still anti-linear.

Define the contravariant functor $F : \mathbf{Mod}_{\mathbb{C}} \to \mathbf{Mod}_{\mathbb{C}}$ given by $F(V) = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ what I might call the *anti*-dual space. For maps $f : V \to W$ and $f \in \mathrm{Hom}_{\overline{\mathbb{C}}}(W, \mathbb{C})$ take $F(f) : \varphi \mapsto \varphi \circ f$. Then, since f is $\mathbb{C}$-linear and φ is $\mathbb{C}$-anti-linear,

$$\varphi \circ f(\lambda v) = \varphi(\lambda f(v)) = \bar{\lambda}\varphi \circ f(v)$$

so $\varphi \circ f \in \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$. Clearly, F is additive since function composition commutes with addition. Finally, consider, $F(\lambda f)(\varphi) = \varphi \circ (\lambda f) = \bar{\lambda}(\varphi \circ f)$. However, the map $(\lambda \cdot F(f))(\varphi) = \lambda(\varphi \circ f)$ is not equal, so F is *not* $\mathbb{C}$-linear but rather $\mathbb{C}$-*anti*-linear.

2.2

Let R be a commutative ring and N and R-module. Consider the functor $F : \mathbf{Mod}_R \to \mathbf{Mod}_R$ given by $F(M) = M \otimes_R N$. We know that F is left-adjoint to the internal hom functor $\mathrm{Hom}_R(N, -)$ i.e. there is a natural isomorphism,

$$\mathrm{Hom}_R(M \otimes_R N, K) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, K))$$

Therefore, by general abstract nonsense (see Lemma 3.2) F preserves all colimits. In particular F preserves cokernels and therefore is right-exact and F preserved all coproducts and thus all direct sums in the category $\mathbf{Mod}_R$. Finally, take a map $f : M \to M'$ and consider $F(rf) = (rf) \otimes \mathrm{id}_N : M \otimes_R N \to M' \otimes_R N$. However,

$$((rf) \otimes \mathrm{id}_N)(m \otimes n) = (rf(m)) \otimes n = r(f(m) \otimes n) = r(f \otimes \mathrm{id}_N)(m \otimes n)$$

and therefore $F(rf) = rF(f)$ so F is R-linear.

2.3

Let $F : \mathbf{Mod}_R \to \mathbf{Mod}_R$ be a R-linear, right-exact functor preserving all direct sums. First we will consider the action of F on free modules. Let I be some index set and take,

$$P = \bigoplus_{i \in I} R$$

Because F preserves arbitrary direct sums (coproducts) we have,

$$F(P) = F\left(\bigoplus_{i \in I} R\right) = \bigoplus_{i \in I} F(R) = \bigoplus_{i \in I} (R \otimes_R F(R)) = \left(\bigoplus_{i \in I_1} R\right) \otimes_R F(R) = P \otimes_R F(R)$$

where we have used the fact that tensor product commutes with arbitrary direct sums. We now need to show that these functors are *naturally* equivalent on free objects. Let $\eta_P : F(P) \to P \otimes_R F(P)$ be the isomorphism constructed above. Let,

$$P_1 = \bigoplus_{i \in I_1} R \qquad P_2 = \bigoplus_{i \in I_2} R$$

then consider a map $f : P_1 \to P_2$. Since P_1 is free this map is equivalent to a sequence of maps $f_i : R \to P_2$ for $i \in I_1$. Using the explicit construction of the coproduct in the category $\mathbf{Mod}_R$ we have maps,

$$R \xhookrightarrow{\iota_i} \bigoplus_{i \in I_1} R \xrightarrow{f} \bigoplus_{r \in I_2} R \hookrightarrow \prod_{i \in I_2} R \xrightarrow{\pi_j} R$$

notate the composition by $f_{ij} : R \to R$ and $f_i = f \circ \iota_i : R \to P_2$. Since f_{ij} is an R-module map it is uniquely determined by $f_{ij}(1) = r_{ij} \in R$. We may intrinsically define the projection maps $\pi_j : R^{I_2} \to R$ via the universal property applied to the maps $\mathrm{id}_R : R_i \to R_i$ on factor i and the zero map on all other factors. Therefore, because F preserves the universal property of the coproduct it preserves these projection and inclusion maps. Now consider the diagram,

$$\begin{tikzcd}
F(P_1) \arrow[r, "F(f)"] \arrow[d, "\sim"] & F(P_2) \arrow[d, "\sim"] \\
\bigoplus_{i \in I_1} F(R) \arrow[r, "\oplus F(f_i)"] \arrow[d, "\eta_1"] & \bigoplus_{i \in I_2} F(R) \arrow[d, "\eta_2"] \\
P_1 \otimes_R F(R) \arrow[r, "f \otimes \mathrm{id}_{F(R)}"] & P_2 \otimes_R F(R)
\end{tikzcd}$$

The upper square commutes giving a natural isomorphism because F preserves arbitrary direct sums. We must show that the lower square commutes. The maps η_P take a sequence

$$(a_i) \in \bigoplus_{i \in I} F(R)$$

under the isomorphisms,

$$\bigoplus_{i \in I} F(R) \longrightarrow \bigoplus_{i \in I} (R \otimes_R F(R)) \longrightarrow \left(\bigoplus_{r \in I} R\right) \otimes_R F(R)$$

to

$$(a_i) \mapsto (1 \otimes a_i) \mapsto \sum_{i \in I} \delta_i \otimes a_i$$

where I have defined the sequence $\delta_i = \iota_i(1) \in P = R^I$. Then,

$$(f \otimes \mathrm{id}_{F(R)}) \circ \eta_1((a_i)) = (f \otimes \mathrm{id}_{F(R)}) \left(\sum_{i \in I_1} \delta_i \otimes a_i \right) = \sum_{i \in I_1} f(\delta_i) \otimes a_i = \sum_{i \in I_1} f_i(1) \otimes a_i$$

Because, $f(\delta_i) = f \circ \iota_i(1) = f_i(1)$.

Next, consider,

$$\eta_2 \circ \oplus F(f)((a_i)) = \eta_2 \left(\sum_{i \in I_1} F(f_i)(a_i) \right) = \sum_{i \in I_1} \eta_2(F(f_i)(a_i)) = \sum_{i \in I_1} \sum_{j \in I_2} \delta_j \otimes F(\pi_j) \circ F(f_i)(a_i)$$

because projecting a sequence in $F(R)^{I_2}$ to its components uses the map $F(\pi_2)$ which lifts $\mathrm{id}_{F(R)} : F(R) \to F(R)$ exactly on factor i and zero elsewhere. Therefore,

$$F(\pi_j) \circ F(f_i) = F(\pi_j \circ f_i) = F(f_{ij})$$

However, $f_{ij} = r_{ij}\mathrm{id}_R$ so, using the fact that F is an R-linear functor, we find that,

$$F(f_{ij}) = F(r_{ij}\mathrm{id}_R) = r_{ij}F(\mathrm{id}_R) = r_{ij}\mathrm{id}_{F(R)}$$

Therefore,

$$\eta_2 \circ \oplus F(f)((a_i)) = \sum_{i \in I_1} \sum_{j \in I_2} \delta_j \otimes r_{ij} a_i = \sum_{i \in I_1} \sum_{j \in I_2} r_{ij} \delta_j \otimes a_i$$

Furthermore, summing over the support of $f_i(1)$ we find,

$$f_i(1) = \sum_{j \in I_2} \iota_j \circ \pi_j \circ f_i(1) = \sum_{j \in I_2} \iota_j \circ f_{ij}(1) = \sum_{j \in I_2} \iota_j(r_{ij}) = \sum_{j \in I_2} r_{ij} \iota_j(1) = \sum_{j \in I_2} r_{ij} \delta_j$$

Finally,

$$\eta_2 \circ \oplus F(f)((a_i)) = \sum_{i \in I_1} \sum_{j \in I_2} r_{ij} \delta_j \otimes a_i = \sum_{i \in I_1} f_i(1) \otimes a_i = (f \otimes \mathrm{id}_{F(R)}) \circ \eta_1((a_i))$$

which proves that these isomorphisms are natural.

To prove the proposition, take $M \in \mathbf{Mod}_R$ and take the first two terms of any free resolution of M,

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

Now applying both the functor F and the functor $(-) \otimes_R F(R)$ to this sequence and using the natural isomorphism defined above gives a commutative diagram,

$$\begin{tikzcd}
F(P_1) \arrow[r, "F(f)"] \arrow[d, "\eta_{P_1}"] & F(P_0) \arrow[r] \arrow[d, "\eta_{P_2}"] & F(M) \arrow[r] \arrow[d, dashed] & 0 \\
P_1 \otimes_R F(R) \arrow[r, "f \otimes \mathrm{id}_{F(R)}"] & P_0 \otimes_R F(R) \arrow[r] & M \otimes_R F(R) \arrow[r] & 0
\end{tikzcd}$$

with exact rows because by F and $(-) \otimes_R F(R)$ are right-exact i.e. preserve cokernels. Therefore, because the downward maps are isomorphisms then the induced map $F(M) \to M \otimes_R F(R)$ is an isomorphism. Because a morphism $M \to N$ lifts to a morphism of free resolutions over M and N this constructed isomorphism is natural. Therefore $F \cong (-) \otimes_R F(R)$.

2.4

Let I be some infinite index set and consider the functor $F : \mathbf{Mod}_R \to \mathbf{Mod}_R$ given by $F(X) = \mathrm{Hom}_R\left(R^I, X\right)$ where

$$R^I = \bigoplus_{i \in I} R$$

I claim that F is R-linear, right-exact but does not preserve arbitrary direct sums and thus cannot be tensor product by any fixed module (since that functor does preserve direct sums). First, take $r \in R$ and $f : A \to B$ then for $\varphi : R^I \to A$ we have $F(rf) : F(A) \to F(B)$ takes $F(rf) : \varphi \mapsto rf \circ \varphi = r(f \circ \varphi) = rF(f)$. Furthermore, from the definition of an abelian category, the hom functor is additive. Next, R^I is a free R-module and therefore a projective which is equivalent to the functor $F(-) = \mathrm{Hom}_R\left(R^I, -\right)$ being exact (and in particular right-exact). Finally, consider,

$$F\left(\bigoplus_{i \in I} R\right) = F(R^I) = \mathrm{Hom}_R\left(R^I, R^I\right)$$

We have the map $\mathrm{id}_{R^I} \in \mathrm{Hom}_R\left(R^I, R^I\right)$. However, I claim that,

$$\mathrm{id}_{R^I} \notin \bigoplus_{i \in I} \mathrm{Hom}_R\left(R^I, R\right)$$

because id_{R^I} is nonzero projected onto each factor $p_i : R^I \to R$ and since I is infinite this cannot be an element of the direct sum which only contains sequences with finite support. Therefore,

$$F\left(\bigoplus_{i \in I} R\right) = \mathrm{Hom}_R\left(R^I, R^I\right) \neq \bigoplus_{i \in I} \mathrm{Hom}_R\left(R^I, R\right) = \bigoplus_{i \in I} F(R)$$

so F does not preserve arbitrary coproducts and thus cannot be the tensor product functor with any fixed module.

3 Lemmas

Lemma 3.1. Let X be a scheme and K a field. A morphism $\mathrm{Spec}\,(K) \to X$ is the same as specifying a point $p \in X$ and an inclusion $\iota : k(p) \to K$ where $k(p) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field at x.

Proof. Let $(f, f^\#) : \mathrm{Spec}\,(K) \to X$ be a morphism. Then take the image $\{p\} = f((0))$. Furthermore, we have a sheaf map,

$$f^\# : \mathcal{O}_X(U) \to \mathcal{O}_{\mathrm{Spec}(K)}(f^{-1}(U)) = \begin{cases} K & p \in U \\ 0 & p \notin U \end{cases}$$

Consider the commutative diagram,

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{f^\#} & \mathcal{O}_{\mathrm{Spec}(K)}(f^{-1}(U)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{X,x} & \xrightarrow{f^\#_x} & \mathcal{O}_{\mathrm{Spec}(K),(0)} \end{array}$$

On opens U with $p \notin U$ clearly the map $f^\# : \mathcal{O}_X(U) \to \mathcal{O}_{\mathrm{Spec}(K)}(f^{-1}(U))$ is the zero map. Otherwise, the map $\mathcal{O}_{\mathrm{Spec}(K)}(f^{-1}(U)) \to \mathcal{O}_{\mathrm{Spec}(K),(0)}$ is the identity. Therefore, the above diagram determines $f^\# = f^\#_x \circ \mathrm{res}_{U,x}$ uniquely from the stalk map

$$f^\#_x : \mathcal{O}_{X,x} \to \mathcal{O}_{\mathrm{Spec}(K),(0)} = K$$

Furthermore, $f^\#_x$ must be a local so $f^\#_x(\mathfrak{m}_x) = (0)$ since (0) is maximal in K. Therefore, this map factors through $k(p) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Therefore, $f^\#$ is determined from the map $k(p) \to K$ (which is an inclusion) via the canonical composition,

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x \xrightarrow{f^\#_x} K$$

□

Lemma 3.2. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors and left F be left adjoint to G that is $F \dashv G$. Then F preserves all colimits and G preserves all limits.

Proof. Let $\mathscr{I}$ be a fixed category and $J : \mathcal{I} \to \mathcal{C}$ some diagram. Let $\Delta : \mathcal{C} \to \mathcal{C}^{\mathscr{I}}$ be the constant functor (taking $A \in \mathcal{C}$ to the constant functor with image A). Then I claim that $\mathrm{colim} : \mathcal{C}^{\mathscr{I}} \to \mathcal{C}$ is left adjoint to $\Delta : \mathcal{C} \to \mathcal{C}^{\mathscr{I}}$. Therefore, for any $X \in \mathcal{D}$, consider,

$$\mathrm{Hom}_{\mathcal{D}}\left(F(\mathrm{colim}\, J), X\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(\mathrm{colim}\, J, G(X)\right) \cong \mathrm{Hom}_{\mathcal{C}^{\mathscr{I}}}\left(J, \Delta \circ G(X)\right)$$

Any natural transformation $\eta : J \to \Delta \circ G(X)$ is a set of maps $\eta_A : J(A) \to G(X)$ for each $A \in J$ such that,

$$\begin{tikzcd} J(A) \arrow[r, "J(f)"] \arrow[d, "\eta_A"'] & J(B) \arrow[d, "\eta_B"] \\ G(X) \arrow[r, "\mathrm{id}_{G(X)}"'] & G(X) \end{tikzcd}$$

However, the natural equivalence,

$$\mathrm{Hom}_{\mathcal{D}}\left(F(X), Y\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(X, G(Y)\right)$$

gives an equivalent natural transformation $\eta' : F \circ J \to \Delta(X)$. Therefore we have shown that,

$$\mathrm{Hom}_{\mathcal{C}^{\mathscr{I}}}\left(J, \Delta \circ G(X)\right) \cong \mathrm{Hom}_{\mathcal{D}^{\mathscr{I}}}\left(F \circ J, \Delta(X)\right)$$

Therefore, we have,

$$\mathrm{Hom}_{\mathcal{D}}\left(F(\mathrm{colim}\, J), X\right) \cong \mathrm{Hom}_{\mathcal{D}^{\mathscr{I}}}\left(F \circ J, \Delta(X)\right) \cong \mathrm{Hom}_{\mathcal{D}}\left(\mathrm{colim}(F \circ J), X\right)$$

Furthermore, by the injectivity of the Yoneda embedding there is a natural equivalence $F(\mathrm{colim}\, J) \cong \mathrm{colim}\,(F \circ J)$ so F is cocontinuous. The case for right adjoints is exactly dual. □