Math GR6262 Algebraic Geometry
Assignment # 2

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1 02CJ

Let k be a field. We need to show that the following k-algebras,

(a). The k-algebras $k[x_1, \dots, x_n]$ and $k[x_1, \dots, x_{n+1}]$ have dimension n and $n+1$ respectively. Therefore, these k-algebras can not be isomorphic.

(b). Consider the variety $V = V(ab + cd + ef) \subset \mathbb{A}^6_k$. Then V is singular at the origin (since the partial derivatives vanish). Therefore, $V \not\cong \mathbb{A}^5_k$ since $\mathbb{A}^5_k$ is smooth. Thus their coordinate rings, $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \dots, x_5]$ cannot be isomorphic.

(c). Consider the k-algebras $A = k[a, b, c, d, e, f]$ and $I = (ab + cd + ef)$ and $B = A/I = k[a, b, c, d, e, f]/(ab + cd + ef)$. Then, $\dim A = 6$. We use the fact that

$$\dim A \geq \dim A/I + \mathbf{ht}\,(I)$$

A is a domain so (0) is prime. Since $I \supsetneq (0)$ any prime above I has height at least 1 so $\mathbf{ht}\,(I) > 0$. Therefore,

$$\dim B = \dim A/I \leq \dim A - \mathbf{ht}\,(I) < \dim A$$

Thus $A \not\cong B$.

2 0E9D

Consider he affine curve X given by $t^2 = s^5 + 8$ in $\mathbb{C}^2$ with coordinates s, t. Let $x \in X$ be the point $(1, 3)$ and let $U = X \setminus \{x\}$. Consider the function on U,

$$f(s,t) = \begin{cases} \dfrac{t+3}{s-1} & (s,t) \in D(s-1) \\ \dfrac{s^4 + s^3 + s^2 + s + 1}{t-3} & (s,t) \in D(t-3) \end{cases}$$

Because the sets $D(s-1)$ and $D(t-3)$ intersected with U are standard opens, f is a regular function as long as its case agree on the overlap. This is the case because on the variety X,

$$t^2 = s^5 + 8 \implies t^2 - 9 = s^5 - 1 \implies (t-3)(t+3) = (s-1)(s^4 + s^3 + s^2 + s + 1)$$

Therefore, on $D(s-1) \cap D(t-3)$ we have $s \neq 1$ and $t \neq 3$ so we can write,

$$\frac{t+3}{s-1} = \frac{(t+3)(t-3)}{(s-1)(t-3)} = \frac{(s-1)(s^4+s^3+s^2+s+1}{(s-1)(t-3)} = \frac{s^4+s^3+s^2+s+1}{t-3}$$

Therefore, f is regular on $U \subset X$. However, f cannot be extened to a regular function on $\mathbb{A}^2_{\mathbb{C}}$ since it diverges approaching the point $(s,t) = (1,3)$. All global regular functions on $\mathbb{A}^1_{\mathbb{C}}$ are polynomials but f cannot be written without a denominator.

3 0E9E

Let $E \subset \mathbb{C}^n$ be a finite subset. Then since points are closed (in the Zariski topology) the set $U = \mathbb{C}^n \setminus E$ is open. Consider a regular function f on U. Then there exists an open cover $\mathcal{V} = \{V_\alpha\}$ of U such that on V_α we can write $f = g_\alpha / h_\alpha$ with $V_\alpha \subset D(h_\alpha) = \{p \in \mathbb{C}^n \mid h_\alpha(p) \neq 0\}$. Since all open sets intersect (because $\mathbb{C}^n$ is irreducible), U must be connected. Let $U_\alpha = \{p \in D(h_\alpha) \mid f(p) = g_\alpha(p)/h_\alpha(p)\}$. Since $U_\alpha \supset V_\alpha$ we know that the sets U_α cover U. Furthermore if $W = U_\alpha \cap U_\beta$ is nonempty, then $f = g_\alpha/h_\alpha = g_\beta/h_\beta$ on the open set W which implies that $W \subset V(g_\alpha b_\beta - g_\beta h_\alpha)$. However, since $\mathbb{C}^n$ is irreducible no nonempty open can be contained in a proper closed. Since the vanishing locus of a polynomial is, by definition, closed, either $V(g_\alpha h_\beta - g_\beta h_\alpha) = \mathbb{C}^n$, implying that $g_\alpha h_\beta = g_\beta h_\alpha$, or $W = \varnothing$. In the first case, $U_\alpha = U_\beta$ and in the second case U_α and U_β are disjoint. Therefore, the sets $\{U_\alpha\}$ form a partition of U. Since U is connected there must be a single $U = U_\alpha$ meaning that $f = g/h$ with $h \neq 0$ on all of U.

It remains to show that h is a constant. We know that $D(h) \supset U$ and thus $V(h) \subset E$ which is a finite set of points. However, by Lemma 6.1, any nonconstant polynomial over $\mathbb{C}^n$ for $n \geq 2$ has infinitely many zeros. Thus h is constant so $f = g/h$ is a polynomial.

4 0E9F

Let $X \subset \mathbb{C}^n$ be an affine cone. Consider $I(X) \subset \mathbb{C}[x_1, \dots, x_n]$ the ideal of functions vanishing on X. For each $p \in I(X)$ we can write it as a sum of homogeneous terms. Define the ideal $[p] = (t_1) + \cdots + (t_\ell)$ where $t_1, \dots, t_\ell$ are the homogeneous terms of p. Then $[p] \supset (p)$ and $[p]$ is a homogeneous ideal. Consider the ideal,

$$I = \sum_{p \in I(X)} [p]$$

Clearly, $I \supset I(X)$ and I is homogeneous. It remains to show that $I \subset I(X)$ in order to prove that $I(X)$ is a homogeneous ideal. Take $p \in I(X)$ then $p = t_1 + \cdots + t_\ell$. Also take some $a \in X$ then for all $\lambda \in \mathbb{C}$ we must have $\lambda a \in X$ and since $p \in I(X)$ we have,

$$p(\lambda a) = \sum_{i=1}^{\ell} t_i(\lambda a) = \sum_{i=1}^{\ell} t_1(a) \lambda^{d_i} = 0$$

Where d_i is the degree of the homogeneous polynomial t_i. Thus, for fixed a, the function $p(\lambda a)$ is a polynomial in λ so $p(\lambda a) = 0$ for all $\lambda \in \mathbb{C}$ implies that it is the zero polynomial. Therfore $t_i(a) = 0$ for each i. Since $a \in X$ was arbitrary, we have shown that $t_i \in I(X)$ for each i and thus $[p] \subset I(X)$ which implies that $I = I(X)$ so $I(X)$ is homogeneous.

5 Additional Problem

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{1}{1+x^2}$. Then f is a quotient of polynomials but not a polynomial (since its tylor series does not terminate at $x = 0$).

6 Lemmas

Lemma 6.1. Suppose that $f : \mathbb{C}^n \to \mathbb{C}$ is a nonconstant polynomial. If $n \geq 2$ then f has uncountably infinitely many roots.

Proof. Since f is nonconstant it must have positive degree in some variable, WLOG let that variable be x_1. Then write,

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{d} q_i(x_2, \ldots, x_n) x_1^i$$

If $q_1, \ldots, q_n$ have a common zero at $(\tilde{x}_2, \ldots, \tilde{x}_n)$ then $f(x_1, \tilde{x}_2, \ldots, \tilde{x}_n) = 0$ for all uncountably infinitely many $x_1 \in \mathbb{C}$. Otherwise, for each $x_2, \ldots, x_n$ the polynomial

$$g_{(x_2, \ldots, x_n)}(x_1) = f(x_1, x_2, \ldots, x_n)$$

has nonzero degree and thus, since $\mathbb{C}$ is algebraically closed, has a root at some $x_1 \in \mathbb{C}$. Thus, there exists a root of $g_{(x_2, \ldots, x_n)}$ for each choice of $x_2, \ldots, x_n \in \mathbb{C}^{n-1}$ and therefore $\forall x_2, \ldots, x_n \in \mathbb{C}^{n-1} : \exists x_1 \in \mathbb{C} : f(x_1, x_2, \ldots, x_n) = 0$ so the set of roots is at least the size of $\mathbb{C}^{n-1}$ which is uncountably infinite. $\square$