1 Cohomology Review

Definition Let X be a smooth complete variety over $\mathbb{C}$ (a smooth proper scheme over $\mathbb{C}$). There is a corresponding analytic manifold X^{an} whose exact topology depends on the structure map $X \to \operatorname{Spec}(\mathbb{C})$. This gives us access to topological cohomology denoted $H^n_B(X) = H^n(X^{\text{an}}, \mathbb{Q})$.

Definition For each embedding $\sigma : k_0 \to \mathbb{C}$ there is a corresponding $X^\sigma = X \times_\sigma \operatorname{Spec}(\mathbb{C})$ and we write $H^p_\sigma(X) = H^p_B(X^\sigma) = H^p((X^\sigma)^{\text{an}}, \mathbb{Q})$.

Remark. In the case that X is projective, a projective embedding $X \to \mathbb{P}^n$ defines an embedding $X^{\text{an}} \to \mathbb{CP}^n$ which pulls back the canonical Kahler form on $\mathbb{CP}^n$ to give X a Kahler structure. By Hodge theory, this gives a decomposition,

$$H^n_B(X, \mathbb{C}) = H^n_{\text{dR}}(X^{\text{an}}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X)$ can be identified with a complex form of type (p, q) and also with the sheaf cohomology,

$$H^{p,q}(X) = H^p(X, \Omega^q)$$

Definition The algebraic deRham cohomology is given by the hyper cohomology of the deRham complex,

$$H^n_{\text{dR}}(X/k) = \mathbb{H}^n(X, \Omega^\bullet)$$

Theorem 1.1. There is a Hodge-to-deRham spectral sequence,

$$E_1^{p,q} = H^p(X, \Omega^q) \implies \mathbb{H}^{p+q}(X, \Omega^\bullet) = H^{p+q}_{\text{dR}}(X)$$

which gives a filtration on the algebraic deRham cohomology. Furthermore, the continuous map $X \to X^{\text{an}}$ induces an isomorphism,

$$H^n_{\text{dR}}(X) \xrightarrow{\sim} H^n_{\text{dR}}(X^{\text{an}})$$

which sends the filtration of the Hodge-to-deRham spectral sequence to the filtration of $H^n_{\text{dR}}(X^{\text{an}})$ given by Hodge theory.

Remark. In general, let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor and $\mathbf{Ch}\mathbb{A}$ its category of complexes. Then there is a spectral sequence computing the hyperderived functor,

$$E_1^{p,q} = R^q F(C^p) \implies \mathbb{R}^{p+q} F(C^\bullet) = \mathbb{H}^{p+q}(C^\bullet)$$

Proposition 1.2. Consider a resolution (exact sequence) in an abelian category $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$$

and an additive functor $F : \mathcal{A} \to \mathcal{B}$. Then, the derived functors of F on A agree with the hyperderived functors of F on $C^\bullet$,

$$R^p F(A) = \mathbb{R}^p F(C^\bullet)$$

In pariticular, in the category of sheaves on X, given any resolution $\mathscr{F} \to \mathscr{G}^\bullet$ we have,

$$H^p(X, \mathscr{F}) = \mathbb{H}^p(X, \mathscr{G}^\bullet)$$

Proof. We choose a resolution of $C^\bullet$ which is an complex of injectives $I^\bullet$ and a quasi-isomorphism $\alpha : C^\bullet \to I^\bullet$. Consider the diagram,

$$\begin{tikzcd}
 & A \arrow[d, "\varepsilon"] & & & \\
0 \arrow[r] & C^0 \arrow[r] \arrow[d, "\alpha_0"] & C^1 \arrow[r] \arrow[d, "\alpha_1"] & C^2 \arrow[r] \arrow[d, "\alpha_2"] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

Since $A \xrightarrow{\varepsilon} C^\bullet$ is a resolution, the top row is exact except in degree zero where $\ker(C^0 \to C^1) = A$. Since $\alpha : C^\bullet \to I^\bullet$ is a quasi-isomorphism the complex $I^\bullet$ must also be exact in positive degree and at degree zero $\alpha_* : H^0(C^\bullet) \xrightarrow{\sim} H^0(I^\bullet)$ is an isomorphism so $\alpha_0 \circ \varepsilon : A \to \ker(C^0 \to C^1) \to \ker(I^0 \to I^1)$ is an isomorphism. Thus the complex $0 \to A \xrightarrow{\alpha_0 \circ \varepsilon} I^\bullet$ is exact so it is an injective resolution of A. Therefore,

$$R^p F(A) = H^p(F(I^\bullet)) = \mathbb{R}^p F(C^\bullet)$$

□

Remark. When the resolution $A \to C^\bullet$ is acyclic then, applying the spectral sequence,

$$E_1^{p,q} = R^q F(C^p) \implies \mathbb{R}^{p+q} F(C^\bullet)$$

we see that $E_1^{p,0} = F(C^p)$ and all others are zero. Thus, $E_2^{p,0} = H^p(F(C))$ so the spectral sequence converges giving,

$$\mathbb{R}^p F(C^\bullet) = H^p(F(C^\bullet))$$

Together with the previous proposition we conclude,

$$R^p F(A) = H^p(F(C^\bullet))$$

that we can compute derived functors on any acylcic resolution.

Remark. Applying these remarks to the case of a complex manifold X, we consider the resolution of the constant sheaf $\underline{\mathbb{C}}_X$ by the holomorphic differential forms Ω_X^k,

$$0 \longrightarrow \underline{\mathbb{C}}_X \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^2 \longrightarrow \cdots$$

This complex is exact by the Poincare lemma. Thus we have an isomorphism,

$$H^p_{\text{sing.}}(X;\mathbb{C}) = H^p(X, \underline{\mathbb{C}}_X) \xrightarrow{\sim} \mathbb{H}^p(X, \Omega^\bullet_X) = H^p_{\text{dR}}(X)$$

Definition When $k = \bar{k}$ we write the Etale cohomology as,

$$H^n(X, \mathbb{A}_{\mathbb{Q},\text{fin.}}) = \varprojlim H^n_{\text{et}}(X_{\text{et}}, \mathbb{Z}/m\mathbb{Z})$$

Theorem 1.3. For $k = \mathbb{C}$ there is a canonical isomorpihsm,

$$H^n_B(X) \otimes \mathbb{A}_{\mathbb{Q},\text{fin.}} \to H^n_{\text{et}}(X)$$

Therefore $H^n_B(X) \otimes \mathbb{A}_{\mathbb{Q},\text{fin.}}$ is independent of the choice of structure map $X \to \text{Spec}\,(\mathbb{C})$.

Remark. Recall that we have defined an algebraic cycle via the cohomology class of a smooth subvariety $Z \subset X$ of codimension p,

$$\text{cl}\,(Z) \in \text{Hdg}^p(X) = H^{2p}_B(X) \cap H^p(X, \Omega^p)$$

We give an alternative definition in terms of Chern classes.

Definition First, we define a Chern class $c_1 : \text{Pic}(X) \to H^2_{\text{dR}}(X)$ via the following. Consider the map $\mathrm{d}\log : \mathcal{O}_X^\times \to \Omega^1_X$ which takes $f \mapsto f^{-1}\mathrm{d}f$. Then there is a map of complexes,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathcal{O}_X^\times \arrow[r] \arrow[d, "\mathrm{d}\log"] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\mathrm{d}"] & \Omega^1_X \arrow[r, "\mathrm{d}"] & \Omega^2_X \arrow[r] & \cdots
\end{tikzcd}$$

Which gives a map on hypercohomology,

$$H^{n-1}(X, \mathcal{O}_X^\times) = \mathbb{H}^n(X, 0 \to \mathcal{O}_X^\times \to 0 \to \cdots) \to \mathbb{H}^n(X, \Omega^\bullet_X) = H^n_{\text{dR}}(X)$$

Recall that $\text{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ and therefore we have a map,

$$c_1 : \text{Pic}(X) \to H^2_{\text{dR}}(X)$$

Then, note that we may extend this to $c_p : \text{Pic}(X) \to H^{2p}_{\text{dR}}(X)$ via splitting.

Definition For any smooth codimension p subvariety $Z \subset X$ we can define,

$$\text{cl}\,(Z) = \frac{1}{(p-1)!} c_p(\iota_* \mathcal{O}_Z)$$

To make this definition make any sense, we need to note that the Chern class is defined on the Grothendieck group of X which, when X is smooth is equivalent to the Grothendieck group of the category of coherent $\mathcal{O}_X$-modules. This correspondence defines $c_p(\iota_* \mathcal{O}_Z)$ when $\iota_* \mathcal{O}_Z$ is not a vector bundle only a coherent sheaf.

1.1 Basic Properties of Absolutly Hodge Cycles

Remark. We first need to discuss algebraic conntections on bundles. The setup is k_0 is a field of characteristic zero and S is a smooth k_0-scheme.

Definition A k_0-connection on a coherent $\mathcal{O}_S$-nodule $\mathcal{E}$ is a morphism of sheaves of k_0-modules,

$$\nabla : \mathcal{E} \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{E}$$

(not as $\mathcal{O}_S$-modules) which further satisfies the Leibniz rule, for $f \in \mathcal{O}_S(U)$ and $s \in \mathcal{E}(U)$,

$$\nabla(fs) = \mathrm{d}f \otimes e + f\nabla(e)$$

where $\mathrm{d} : \mathcal{O}_S \to \Omega^1_S$ is the canonical map. We define the subsheaf of horizontal sections, $\mathcal{E}^\nabla = \ker \nabla$

Remark. Any connection may be extended to $\mathcal{E}$-valued k-forms,

$$\nabla_k : \Omega^k_S \otimes_{\mathcal{O}_S} \mathcal{E} \to \Omega^{k+1}_S \otimes_{\mathcal{O}_S} \mathcal{E}$$

via,

$$\nabla_k(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^k \omega \wedge \nabla e$$

Definition The connection ∇ defines a corresponding curvature form,

$$\omega_\nabla = \nabla_1 \circ \nabla : \mathcal{E} \to \Omega^2_S \otimes_{\mathcal{O}_S} \mathcal{E}$$

We say that ∇ is flat or integrable if the curvature vanishes $\omega_\nabla = \nabla_1 \circ \nabla = 0$.

Lemma 1.4. The curvature $\omega_\nabla : \mathcal{E} \to \Omega^2_S \otimes_{\mathcal{O}_S} \mathcal{E}$ is a $\mathcal{O}_S$-module map.

Proof. Consider,

$$\begin{aligned}\omega_\nabla(fs) &= \nabla_1(\mathrm{d}f \otimes s + f\nabla s) = \mathrm{dd}f \otimes s - \mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + f\nabla_1 \circ \nabla \\ &= f\nabla_1 \circ \nabla s = f\omega_\nabla(s)\end{aligned}$$

□

Remark. If we write locally,

$$\nabla e = \sum_i f_i \mathrm{d}g_i \otimes s_i$$

then the curvature takes the form,

$$\omega_\nabla(e) = \sum_i (\mathrm{d}f_i \wedge \mathrm{d}g_i \otimes e - f_i \mathrm{d}g_i \otimes \nabla s_i)$$

Proposition 1.5. ∇ is flat iff the $\mathcal{O}_S$-map $Q : \mathrm{Der}(\mathcal{O}_S, \mathcal{O}_S) \to \mathrm{End}(\mathcal{E})$ given by sending D to,

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes \mathrm{id}} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{a \otimes b \mapsto ab} \mathcal{E}$$

is a morphism of Lie algebras.

Remark. Note that $Q(D)$ is in fact a $\mathcal{O}_S$-morphism using the universal property,

$$\mathrm{Der}(\mathcal{O}_S, \mathcal{O}_S) \cong \mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^1_S, \mathcal{O}_S\right)$$

Proof. We need to check that $Q[D_1, D_2] = [Q(D_1), Q(D_2)]$ is equivalent to $\nabla_1 \circ \nabla = 0$. Now,

$$[D_1, D_2] \in \mathrm{Hom}_{\mathcal{O}_S}\left(\Omega^1_S, \mathcal{O}_S\right)$$

is the unique $\mathcal{O}_S$-map such that,

$$[D_1, D_2] \circ \mathrm{d} = D_1 \circ \mathrm{d} \circ D_2 \circ \mathrm{d} - D_2 \circ \mathrm{d} \circ D_1 \circ \mathrm{d}$$

Now consider this action locally,

$$[D_1, D_2] \otimes \mathrm{id} \circ \nabla = \sum_i f_i (D_1 \circ \mathrm{d} \circ D_2 \circ \mathrm{d} - D_2 \circ \mathrm{d} \circ D_1 \circ \mathrm{d})(g_i) \cdot s_i$$

Furthermore,

$$[Q(D_1), Q(D_2)] = (D_1 \otimes \mathrm{id}) \circ \nabla \circ (D_2 \otimes \mathrm{id}) \circ \nabla - (D_2 \otimes \mathrm{id}) \circ \nabla \circ (D_1 \otimes \mathrm{id}) \circ \nabla$$

Again consider its local action,

$$\begin{aligned} Q(D_1) \circ Q(D_2)(e) &= (D_1 \otimes \mathrm{id}) \circ \nabla \left(\sum_i f_i D_2(\mathrm{d}g_i) \cdot s_i \right) \\ &= \sum_i \Big([D_2(\mathrm{d}g_i) D_1(\mathrm{d}f_i) + f_i D_1(\mathrm{d}(D_2(\mathrm{d}g_i)))] \cdot s_i + f_i D_2(\mathrm{d}g_i) D_1(\nabla s_i) \Big) \end{aligned}$$

Now consider,

$$\begin{aligned} &\Big[Q(D_1) \circ Q(D_2) - Q(D_2) \circ Q(D_1)] - Q([D_1, D_2]) \Big](e) \\ &\quad = \sum_i \Big(D_1(\mathrm{d}f_i) D_2(\mathrm{d}g_i) - D_2(\mathrm{d}f_i) D_1(\mathrm{d}g_i) \Big) \cdot s_i \\ &\quad + \sum_i f_i \Big(D_1(\mathrm{d}(D_2(\mathrm{d}g_i))) - D_2(\mathrm{d}(D_1(\mathrm{d}g_i))) \Big) \cdot s_i \\ &\quad + \sum_i \Big(f_i D_2(\mathrm{d}g_i) D_1(\nabla s_i) - g_i D_1(\mathrm{d}g_i) D_2(\nabla s_i) \Big) \\ &\quad - \sum_i f_i (D_1 \circ \mathrm{d} \circ D_2 \circ \mathrm{d} - D_2 \circ \mathrm{d} \circ D_1 \circ \mathrm{d})(g_i) \cdot s_i \\ &\quad = \sum_i \Big(D_1(\mathrm{d}f_i) D_2(\mathrm{d}g_i) - D_2(\mathrm{d}f_i) D_1(\mathrm{d}g_i) \Big) \cdot s_i \\ &\quad + \sum_i \Big(f_i D_2(\mathrm{d}g_i) D_1(\nabla s_i) - g_i D_1(\mathrm{d}g_i) D_2(\nabla s_i) \Big) \\ &\quad = (D_1 \otimes D_2 - D_2 \otimes D_1) \otimes \mathrm{id}_{\mathcal{E}} \circ \omega_\nabla \end{aligned}$$

which is defined on $(\Omega_S^1)^{\otimes 2} \otimes_{\mathcal{O}_S} \mathcal{E}$ but descends to $\Omega_S^2 \otimes_{\mathcal{O}_S} \mathcal{E}$ since it sends the ideal $\omega \otimes \omega \mapsto 0$. Therefore, we see that Q is a Lie algebra map iff

$$\forall D_1, D_2 \in \mathrm{Hom}_{\mathcal{O}_S}\left(\Omega_S^1, \mathcal{O}_S\right) : (D_1 \otimes D_2 - D_2 \otimes D_1) \otimes \mathrm{id}_{\mathcal{E}} \circ \omega_\nabla = 0$$

in particular when $\omega_\nabla = 0$. Furthermore when Q is a Lie algebra map then we must have $\omega_\nabla = 0$ since, for any fixed form, there exists sections of Ω_S^1 which do not kill it. $\square$

Example 1.6. For $\mathcal{E} = \mathcal{O}_S$ we have the universal connection $\mathrm{d} : \mathcal{O}_S \to \Omega_S^1$. Then the statment that d is flat is equivalent to $\mathrm{d}^2 = 0$.

Remark. Recall that given $f : X \to S$ there is an exact sequence of $\mathcal{O}_X$-modules,

$$f^*\Omega_S^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/S}^1 \longrightarrow 0$$

We may define,

$$\Omega_{X/S}^k = \bigwedge^k \Omega_{X/S}^1$$

to give $\Omega_{X/S}^\bullet$, the relative deRham complex of X over S,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/S}^1 \xrightarrow{\mathrm{d}} \Omega_{X/S}^2 \longrightarrow \cdots$$

Definition Now consider a proper smooth morphism $\pi : X \to S$ of smooth varieties. We define its sheaf of relative deRham cohomology by the hyperderived functors applied to the relative de Rham complex,

$$\mathscr{H}_{\mathrm{dR}}^n(X/S) = \mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet)$$

Remark. Note that for the structure map $\pi : X \to \mathrm{Spec}\,(k_0)$ map we have $\pi_*\mathscr{F} = \Gamma(X, \mathscr{F})$ and thus its hyperderived functors are simply hypercohomology of sheaves so,

$$\mathscr{H}_{\mathrm{dR}}^n(X/k_0) = \mathbb{H}^n(\Omega_{S/k_0}^\bullet) = H_{\mathrm{dR}}^n(X/k_0)$$

recovering algebraic de Rham cohomology.

Definition Let S and $\pi : X \to S$ be smooth. Then there is a decreasing filtration,

$$F^p\Omega_X^q = \bigoplus_{p \geq p'} \mathrm{Im}((\pi^*\Omega_S^{p'} \otimes_{\mathcal{O}_X} \Omega_X^{q-p'} \to \Omega_X^q))$$

There is always an exact sequence of sheaves of k_0-modules,

$$0 \longrightarrow F^1/F^2 \longrightarrow F^0/F^2 \longrightarrow F^0/F^1 \longrightarrow 0$$

which, in this case, gives an exact sequence of complexes,

$$0 \longrightarrow \Omega_{X/S}^{\bullet - 1} \otimes_{\mathcal{O}_X} \pi^*\Omega_S^1 \longrightarrow \Omega_X^\bullet / F^2\Omega_X^\bullet \longrightarrow \Omega_{X/S}^\bullet \longrightarrow 0$$

The associated long exact sequence of hypercohomolgy,

$$\begin{array}{ccccccc}
\mathbb{R}^n\pi_*(\Omega^{\bullet-1}_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_S & \longrightarrow & \mathbb{R}^n\pi_*(\Omega^\bullet_X / F^2\Omega^\bullet_X) & \longrightarrow & \mathbb{R}^n\pi_*(\Omega^\bullet_{X/S}) & \xrightarrow{\nabla} & \mathbb{R}^{n+1}\pi_*(\Omega^{\bullet-1}_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_S \\
\| & & & & & & \| \\
\mathbb{R}^{n-1}\pi_*(\Omega^\bullet_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_S & & & & & & \mathbb{R}^n\pi_*(\Omega^\bullet_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_S
\end{array}$$

In partiuclar, the connecting map $\nabla : \mathbb{R}^n\pi_*(\Omega^\bullet_{X/S}) \to \mathbb{R}^n\pi_*(\Omega^\bullet_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_S$ is a flat connection on the relative deRham sheaf, $\mathscr{H}^n_{\mathrm{dR}}(X/S) = \mathbb{R}^n\pi_*(\Omega^\bullet_{X/S})$. We call this connection the Gauss-Manin connection.

Remark. For example, if $f : X \to S$ is etale then we know that $f^*\Omega^1_S \to \Omega^1_X$ is an isomorphism and thus $\Omega^1_{X/S} = 0$. Therefore, the sheaf of relative deRham cohomology is,

$$\mathbb{R}^n\pi_*(\Omega^\bullet_{X/S}) = \mathbb{R}^n\pi_*(0 \to \mathcal{O}_X \to 0 \to \cdots) = R^n\pi_*(\mathcal{O}_X)$$

Then the connecting map $\nabla : R^n\pi_*(\mathcal{O}_X) \to R^n\pi_*(\mathcal{O}_X) \otimes_{\mathcal{O}_S} \Omega^1_S$ is simply induced by the exerior derivative,

$$\nabla = R^n\pi_*(\mathrm{d} : \mathcal{O}_X \to \Omega^1_X)$$

where $\pi_*(\Omega^1_X) = \pi_*\mathcal{O}_X \otimes_{\mathcal{O}_S} \Omega^1_S$.

Remark. If we take $k_0 = \mathbb{C}$ then GAGA implies that,

$$\mathscr{H}^n_{\mathrm{dR}}(X/S)^{\mathrm{an}} \cong \mathscr{H}^n_{\mathrm{dR}}(X^{\mathrm{an}}/S^{\mathrm{an}})$$

and ∇^{an} is a flat connection on $\mathscr{H}^n_{\mathrm{dR}}(X^{\mathrm{an}}/S^{\mathrm{an}})$ so there is a relative deRham complex,

$$0 \longrightarrow \mathcal{O}_X^{\mathrm{an}} \xrightarrow{\mathrm{d}} (\Omega^1_{X/S})^{\mathrm{an}} \xrightarrow{\mathrm{d}} (\Omega^2_{X/S})^{\mathrm{an}} \longrightarrow \cdots$$

However, by Ehresmann's lemma, locally above $s \in S$ we may write $\pi^{-1}(U) = U \times X_s$ and choose U to be contractible. Then, locally, $\Omega^\bullet_{X^{\mathrm{an}}/S^{\mathrm{an}}} = \underline{\mathbb{C}}_X \otimes (\Omega^\bullet_{X_s})^{\mathrm{an}}$ which, using the projection formula, gives,

$$\mathscr{H}^n_{\mathrm{dR}}(X^{\mathrm{an}}/S^{\mathrm{an}}) = \mathbb{R}^n\pi_*(\underline{\mathbb{C}}_X \otimes (\Omega^\bullet_{X_s})^{\mathrm{an}}) = R^n\pi_*(\underline{\mathbb{C}}_X) \otimes \mathcal{O}_S^{\mathrm{an}}$$

In particular, there is a natural connection on this analytic sheaf,

$$\begin{aligned}
\nabla^{\mathrm{an}} &: R^n\pi_*(\underline{\mathbb{C}}_X) \otimes_{\underline{\mathbb{C}}_S} \mathcal{O}_S \to (R^n\pi_*(\underline{\mathbb{C}}_X) \otimes_{\underline{\mathbb{C}}_S} \mathcal{O}_S) \otimes_{\mathcal{O}_S} \Omega^1_S = R^n\pi_*(\underline{\mathbb{C}}_X) \otimes_{\underline{\mathbb{C}}_S} \Omega^1_S \\
\nabla^{\mathrm{an}} &: (\alpha \otimes f) \mapsto \alpha \otimes \mathrm{d}f
\end{aligned}$$

Clearly this connection satisfies $\mathscr{H}^n_{\mathrm{dR}}(X^{\mathrm{an}}/S^{\mathrm{an}})^{\nabla^{\mathrm{an}}} \cong R^n\pi_*(\underline{\mathbb{C}}_X)$. In fact, there is a unique connection satisfing this property which is the GAGA equivalent analytic connection to the algebraic Gauss-Manin connection.

2 Local Systems

Proposition 2.1. Let $\mathcal{E}$ be a vector bundle on X with a flat connection

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

Then $\mathcal{E}^\nabla = \ker \nabla$ is a local system.

Proof. Since $\mathcal{E}$ is locally free, we can find a cover of trivializing neighbrohoods U such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$. Then $\nabla : \mathcal{O}_U^{\oplus n} \to (\Omega^1_U)^{\oplus n}$ is a connection. Define,

$$\nabla e_j = \sum_{i=1}^n \omega_{ij} \otimes e_i$$

where $\omega_{ij} \in \Omega^1_X(U)$ is a form. This uniquely defines the connection since,

$$\begin{aligned}
\nabla(f_1, \ldots, f_n) = \nabla \left(\sum_{i=1}^n f_i e_i \right) &= \sum_{i=1}^n (f_i \nabla e_i + \mathrm{d} f_i \otimes e_i) \\
&= \sum_{i,j=1}^n \omega_{ij} \otimes f_j e_i + (\mathrm{d} f_1, \ldots, \mathrm{d} f_n)
\end{aligned}$$

Therefore, $\mathcal{E}^\nabla$ is given locally by $(f_1, \ldots, f_n)$ solving the linear system of differential equations,

$$\mathrm{d} f_i + \sum_{j=1}^n \omega_{ij} f_j = 0$$

The condition of flatness is that,

$$\nabla_1 \circ \nabla = 0$$

which locally is,

$$\begin{aligned}
\nabla_1 \circ \nabla(f_1, \ldots, f_n) &= \nabla_1 \left(\sum_{i,j=1}^n \omega_{ij} \otimes f_j e_i + \sum_{j=1}^n \mathrm{d} f_j \otimes e_j \right) \\
&= \sum_{i,j=1}^n [\mathrm{d}\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \nabla(f_j e_i)] + \sum_{i=1}^n [\mathrm{d}\mathrm{d} f_i \otimes e_i - \mathrm{d} f_j \wedge \nabla e_j] \\
&= \sum_{i,j=1}^n \left[\mathrm{d}\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \left(\mathrm{d} f_j \otimes e_i + f_j \sum_{k=1}^n \omega_{ki} \otimes e_k \right) \right] - \sum_{i,j=1}^n [\mathrm{d} f_j \wedge \omega_{ij} \otimes e_i] \\
&= \sum_{i,j=1}^n \left[\mathrm{d}\omega_{ij} \otimes e_i - \sum_{k=1}^n \omega_{ij} \wedge \omega_{ki} \otimes e_k \right] f_j \\
&= \sum_{i,j=1}^n \left[\mathrm{d}\omega_{ij} + \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} \right] \otimes f_j e_i
\end{aligned}$$

So the curvature ω_∇ is given by coefficients,

$$\Theta_{ij} = \mathrm{d}\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj}$$

This vanishing is exactly the criterion in Frobenius' theorem for integrability. $\square$

3 Principle B

Proposition 3.1. Let $k_0 \subset \mathbb{C}$ have finite transcendence degree over $\mathbb{Q}$ and X be a complete smooth variety over a field k that is finitely generated over k_0. Let ∇ be the Gauss-Manin connection on $\mathscr{H}^n_{\mathrm{dR}}(X)$ relative to $X \to \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k_0)$.

If $t \in H^n_{\mathrm{dR}}(X)$ is rational relative to all embeddings $k \hookrightarrow \mathbb{C}$ then $\nabla t = 0$.

Proof. Let A be a finite-type k_0-algebra and $\pi : X_A \to \mathrm{Spec}\,(A)$ a smooth proper map with generic fibre $X_{(0)} = X \to \mathrm{Spec}\,(k)$ and such that t extends to $\Gamma(\mathrm{Spec}\,(A)\,, \mathscr{H}^n_{\mathrm{dR}}(X/\mathrm{Spec}\,(A))$. After bse change via $k_0 \hookrightarrow \mathbb{C}$ to $S = \mathrm{Spec}\,(A_{\mathbb{C}})$ there are maps,

$$X_S \to S \to \mathrm{Spec}\,(\mathbb{C})$$

and a global section $t' = t \otimes 1$ of $\mathscr{H}^n_{\mathrm{dR}}(X_S^{\mathrm{an}}/S^{\mathrm{an}})$. We need to show that $(\nabla \otimes 1)t' = 0$. However, if we recall that,

$$\mathscr{H}^n_{\mathrm{dR}}(X_S^{\mathrm{an}}/S^{\mathrm{an}}) = \mathbb{R}^n \pi_*^{\mathrm{an}}(\Omega^\bullet_{X_S^{\mathrm{an}}/S^{\mathrm{an}}}) = (R^n \pi_* \mathbb{C}) \otimes_{\underline{\mathbb{C}}} \mathcal{O}_{S^{\mathrm{an}}} = H^n(X_S^{\mathrm{an}}, \underline{\mathbb{C}}) \otimes_{\underline{\mathbb{C}}} \mathcal{O}_{S^{\mathrm{an}}}$$

and that the Gauss-Manin connection kills exactly those sections purely in,

$$\mathscr{H}^n(X_S^{\mathrm{an}}, \underline{\mathbb{C}}_X) = R^n \pi_*(\underline{C}_X)$$

An embedding $\sigma : k \hookrightarrow \mathbb{C}$ gives a point $\mathrm{Spec}\,(\mathbb{C}) \to \mathrm{Spec}\,(A)$ of s. Since t is rational,

$$t(s) \in H^n(X_s^{\mathrm{an}}, \mathbb{Q}) \subset H^n_{\mathrm{dR}}(X_s^{\mathrm{an}})$$

Then locally on S we have $\mathscr{H}^n_{\mathrm{dR}}(X^{\mathrm{an}}/S^{\mathrm{an}}) = R^n \pi_*(\underline{\mathbb{C}}_X) \otimes \mathcal{O}_S^{\mathrm{an}}$ which is locally free and $\mathscr{H}^n(X^{\mathrm{an}}, \underline{\mathbb{C}}_X)$ gives its sheaf of locally constant sections. However, t takes rational vatonal values on the closed points which are dense so it must be locally constant and thus $t \in \mathscr{H}^n(X^{\mathrm{an}}, \underline{\mathbb{C}}_X)$ so $\nabla t = 0$. $\square$

Definition Let $\pi : X \to S$ e a proper smooth map of smooth varieties / $\mathbb{C}$ with S connected. Then,

$$\mathscr{H}^n_{\mathrm{et}}(X/S)(m) = \varprojlim_r (R^n \pi^*_{\mathrm{et}} \mu_r^{\otimes m}) \otimes_Z \mathbb{Q}$$

and

$$\mathscr{H}^n_{\mathbb{A}}(X/S)(m) = \mathscr{H}^n_{\mathrm{dR}}(X/S)(m) \times \mathscr{H}^n_{\mathrm{et}}(X/S)(m)$$

and

$$\mathscr{H}^{2p}_B(X/S)(p) = R^{2p} \pi_*^{\mathrm{an}} \mathbb{Q}(p)$$

Remark. By Ehresmann's lemma we can locally write $\pi^{-1}(U) = U \times X_s$ with U contractible. Therefore, by Kunneth,

$$H_B^{2p}(X/S)(p)(U) = H^{2p}(\pi^{-1}(U), \mathbb{Q}(p)|_U) = H_B^{2p}(X_s, \mathbb{Q}(p)) \otimes_{\mathbb{Q}} H^0(U, \mathbb{Q}(p) = H_B^{2p}(X_s, \mathbb{Q}(p))$$

since U is contractible. This is a constant sheaf so $H_B^{2p}(X/S)(p)$ is a local system. A similar arument holds for the other sheaves.

Theorem 3.2 (Principle B)**.** Let $t \in \Gamma(\mathscr{H}_{\mathbb{A}}^{2p}(X/S)(p))$ such that $\nabla t_{\mathrm{dR}} = 0$. If $(t_{\mathrm{dR}})_s \in F^0 H_{\mathrm{dR}}^{2p}(X_s)(p)$ for each $s \in S$ and t_s is an absolute Hodge cyclce in $H_{\mathbb{A}}^{2p}(X_s)(p)$ for some s then it is an absolute Hodge cycle for every s.

Proof. We suppose that t_s is an absolute Hodge cycle for some some $s \in S$. For any $s' \in S$ we need to show that $t_{s'}$ is absolutly Hodge meaning that it is rational relative to every isomorphism $\sigma : \mathbb{C} \to \mathbb{C}$. However, such an isomorphism gives a morphism $\sigma\pi : \sigma X \to \sigma S$ and a section $\sigma(t)$ of $\mathscr{H}_{\mathbb{A}}^{n}(\sigma X / \sigma S)(p)$. We know that $\sigma(t)_{\sigma s}$ is rationl and we must show that $\sigma(t)_{\sigma s'}$ is rational. It suffices to prove this for $\sigma = \mathrm{id}$ given that there is some σ for which this global rationality holds.

First, consider the component t_{dR} of t (relative the the construction of $\mathscr{H}_{\mathbb{A}}^{n}(\sigma X/\sigma S)(p)$ as a product. By assumption $\nabla t_{\mathrm{dR}} = 0$ so t_{dR} is a global section of $\mathscr{H}^{2p}(X^{\mathrm{an}}, \underline{\mathbb{C}}_X)$ which we have shown is the vanishing of the analytic Gauss-Manin connection. Since t_{dR} is rational at one point, it must be rational at every point since $\mathscr{H}^{2p}(X^{\mathrm{an}}, \underline{\mathbb{C}}_X)$ is locally constant and X^{an} is connected.

Thus, it suffices to prove the rationality of the other factor t_{et}. Since the relative cohomology sheaves defined above are local systems, for any point s we have a monodromy action of $\pi_1(S, s)$ on their stalks at s whose fixed points are those germs which extend globally. In particular, this induces isomorphism,

$$\Gamma(S, \mathscr{H}_B^{2p}(X/S)(p)) \cong H_B^{2p}(X_s)^{\pi_1(S,s)}$$
$$\Gamma(S, \mathscr{H}_{\mathrm{et}}^{2p}(X/S)(p)) \cong H_{\mathrm{et}}^{2p}(X_s)^{\pi_1(S,s)}$$

Then consider the diagram,

$$\begin{tikzcd}
\Gamma(S, \mathscr{H}_B^{2p}(X/S)(p)) \arrow[r, hook] \arrow[d, "\sim"] & \Gamma(S, \mathscr{H}_B^{2p}(S/X)(p)) \otimes \mathbb{A}_{\mathrm{fin}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \Gamma(S, \mathscr{H}_{\mathrm{et}}^{2p}(X/S)(p)) \arrow[d, "\sim"] \\
H_B^{2p}(X_s)(p)^{\pi_1(S,s)} \arrow[r, hook] \arrow[d, hook] & H_B^{2p}(X_s)(p)^{\pi_1(S,s)} \otimes \mathbb{A}_{\mathrm{fin}} \arrow[r, "\sim"] \arrow[d, hook] & H_{\mathrm{et}}^{2p}(X_s)(p)^{\pi_1(S,s)} \arrow[d, hook] \\
H_B^{2p}(X_s) \arrow[r, hook] & H_B^{2p}(X_s) \otimes A_{\mathrm{fin}} \arrow[r, "\sim"] & H_{\mathrm{et}}^{2p}(X_s)(p)
\end{tikzcd}$$

We have $t_{\mathrm{et}} \in \Gamma(S, \mathscr{H}_{\mathrm{et}}^{2p}(X/S)(p))$ which is rational at s so its image in $H_{\mathrm{et}}^{2p}(X_s)(p)$ lies in $H_B^{2p}(X_s)(p)$. Now we need the following lemma which alows us to conclude that $t_{\mathrm{et}} \in \Gamma(S, \mathscr{H}_B^{2p}(X/S)(p))$ and thus $(t_{\mathrm{et}})_{s'} \in H_B^{2p}(X_s)(p) \subset H_{\mathrm{et}}^{2p}(X_s)(p)$ for all s' completing the theorem. $\square$

Lemma 3.3. Let $W \hookrightarrow V$ be an inclusion of vectorspaces. Let Z be a third vectorspace and take nonzero $z \in Z$. Wmbed V in $V \otimes Z$ via $v \mapsto v \otimes z$. Then, in $V \otimes Z$,

$$(W \otimes V) \cap (V \otimes z) = W \otimes z$$

Proof. This is clear if we choose a basis e_i for W which extends to a basis of V. Then any $x \in V \otimes Z$ has a unique expansion,

$$x = \sum e_i \otimes z_i$$

If $x \in W \otimes Z$ then $z_i = 0$ for each e_i not in W and if $x \in V$ then $z_i = z$ for each nonzero z_i. $\square$

Remark. The proof of principle B concludes taking $Z = \mathbb{A}_{\text{fin}}$ and $z = 1$ over the inclusion $H^{2p}_B(X_s)^{\pi_1(S,x)(p)} \to H^{2p}_B(X_s)(p)$. The lemma then implies that, in $H^{2p}_{\text{et}}(X_s)(p)$,

$$\begin{aligned}\Gamma(S, \mathscr{H}^{2p}_B(X/S)(p)) \cap H^{2p}_B(X_s)(p) &= [H^{2p}_B(X_s)(p)^{\pi_1(S,s)} \otimes \mathbb{A}_{\text{fin}}] \cap H^{2p}_B(X_s)(p) \\ &= H^{2p}_B(X_s)(p)^{\pi_1(S,s)} = \Gamma(S, \mathscr{H}^{2p}_B(X/S)(p))\end{aligned}$$

so we get a global rational section.

4 The Main Theorem

Theorem 4.1 (Deligne)**.** Let X be an abelian variety over an algebraically closed field k and $t \in H^{2p}_{\mathbb{A}}(X)(p)$. If t is a Hodge cycle relative to some embedding $\sigma : k \hookrightarrow \mathbb{C}$ then it is a Hodge cycle with repsect to every embedding. That is, every Hodge cycle is absolutly Hodge.

5 Hodge Structures and Mumford-Tate Groups

5.1 The Deligne Torus

Remark. Let $T \to S$ be a morphism of schemes. Given an S-scheme X and a T-scheme Y,

$$\text{Hom}_T(Y, X \times_S T) = \text{Hom}_S(Y, X)$$

where,

$$\begin{tikzcd}
Y \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, dashed, bend right] & & \\
& X \times_S T \arrow[r] \arrow[d] & T \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Definition Let $T \to S$ be a morphism of schemes. Given an T-scheme X we define the restriction of scalars functor $\mathcal{R}_{T/S}(X) : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ via,

$$Y \mapsto X(Y \times_S T) = \mathrm{Hom}_T\,(Y \times_S T, X)$$

When the functor $\mathcal{R}_{T/S}(X)$ is representable in $\mathbf{Sch}_S$ then we call the (unique up to unique isomorphism) S-scheme representing it $X' = \mathrm{Res}_{T/S}(X_T)$ such that,

$$\mathcal{R}_{T/S}(X) = \mathrm{Hom}_S\left(-, \mathrm{Res}_{T/S}(X)\right)$$

In this case, we have an isomorphism of functors,

$$\mathrm{Hom}_T\,(- \times_S T, X) = \mathrm{Hom}_S\left(-, \mathrm{Res}_{T/S}(X)\right)$$

which makes $\mathrm{Res}_{T/S}(X)$ be right-adjoint to extension of scalars functor,

$$Y_S \mapsto Y_S \times_S T$$

Remark. Starting with $\mathbb{G}_m^A = \mathrm{Spec}\,(A[z, z^{-1}])$ we define some algebraic groups as follows.

Definition The Deligne torus $\mathbb{S}$ is an algebraic group over $\mathbb{R}$ defined as,

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m^{\mathbb{C}}$$

where $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ is restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$.

Remark. We may characterize $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ as the right-adjoint to base change so the S-points are,

$$\begin{aligned}\mathbb{S}(S) &= \mathrm{Hom}_{\mathbb{R}}\left(S, \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m^{\mathbb{C}}\right) = \mathbb{G}_m^{\mathbb{C}}(S \times_{\mathbb{R}} \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}\left(S \times_{\mathbb{R}} \mathbb{C}, \mathbb{G}_m^{\mathbb{C}}\right) \\ &= \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}[z, z^{-1}], \Gamma(S \times_{\mathbb{R}} \mathbb{C})\right) = \Gamma(S \times_{\mathbb{R}} \mathbb{C})^{\times}\end{aligned}$$

In particular, the $\mathbb{R}$-points of $\mathbb{S}$ are,

$$\mathbb{S}(\mathbb{R}) = \mathbb{G}_m^{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^{\times}$$

Furthermore, the $\mathbb{C}$-points of $\mathbb{S}$ are,

$$\mathbb{S}(\mathbb{C}) = \mathbb{G}_m^{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}[z, z^{-1}], \mathbb{C} \oplus i\mathbb{C}\right) = \mathbb{C}^{\times} \times i\mathbb{C}^{\times}$$

Definition We define a set of characters and cocharacters of $\mathbb{S}$. First we define the character,

$$\mathrm{Nm} : \mathbb{S} \to \mathbb{G}_m^{\mathbb{R}}$$

on $\mathbb{R}$-points $\mathbb{S}(\mathbb{R}) \to \mathbb{G}_m^{\mathbb{R}}(\mathbb{R})$ as $\mathbb{C}^{\times} \to \mathbb{R}^{\times}$ via $z \mapsto z\bar{z}$.

Furthermore, we define the cocharacter,

$$w : \mathbb{G}_m^{\mathbb{R}} \to \mathbb{S}$$

on $\mathbb{R}$-points $\mathbb{G}_m^{\mathbb{R}}(\mathbb{R}) \to \mathbb{S}(\mathbb{R})$ by the natural inclusion $\mathbb{R}^{\times} \hookrightarrow \mathbb{C}^{\times}$.

Lastly, we define a $\mathbb{C}$-cocharacter,

$$\mu : \mathbb{G}_m^{\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$$

on $\mathbb{C}$-points via $\mathbb{G}_m^{\mathbb{C}}(\mathbb{C}) \to \mathbb{S}_{\mathbb{C}}(\mathbb{C})$ as $\mu(z) = (z, i)$ where,

$$\mathbb{S}_{\mathbb{C}}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}\,(\mathbb{C}, \mathbb{S} \times_{\mathbb{R}} \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}\,(\mathbb{C}, \mathbb{S}) = \mathbb{S}(\mathbb{C}) = \mathbb{C} \oplus i\mathbb{C}$$

5.2 Hodge Structures

Definition Let V be a finite-dimensional $\mathbb{Q}$-vectorspace. A $\mathbb{Q}$-*rational Hodge structure* of weight n on V is a decomposition,

$$V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $V^{q,p} = \overline{V^{p,q}}$.

Definition A Hodge structure defines a cocharacter,

$$\mu : \mathbb{G}_m^{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$$

via $\mu(z)v^{p,q} = z^{-p}v^{p,q}$ for $v^{p,q} \in V^{p,q}$.

Furthermore, $\overline{\mu(z)} \cdot v^{p,q} = \bar{z}^{-q}v^{p,q}$ commutes with the action of $\mu(z)$. Therefore, we may take their product to give a map of real algebraic groups,

$$h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

via $h(z)v^{p,q} = z^{-q}\bar{z}^{-q}v^{p,q}$. where $\mathbb{C}^{\times}$ is the algebraic group,

$$\mathrm{Spec}\left(\mathbb{C}[x, x^{-1}]\right) \to \mathrm{Spec}\left(\mathbb{R}\right)$$

Remark. Conversely, any homomorphism of $\mathbb{R}$-algebraic groups $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ which, on $\mathbb{R}$, restricts to $r \mapsto r^{-n}\mathrm{id}_V$ defines a Hodge structure of weight n on V by taking $V^{p,q}$ to be the eigenspace of eigenvalue $z^{-p}\bar{z}^{-q}$ for $h(z)$ i.e.,

$$V^{p,q} = \{v \in V_{\mathbb{C}} \mid \forall z \in \mathbb{S}(\mathbb{R}) : h(z) \cdot v = z^{-p}\bar{z}^{-q}v\}$$

Definition The Weil operator $C \in \mathrm{GL}(V_{\mathbb{R}})$ of a Hodge structure (V, h) is $C = h(i)$.

Proposition 5.1. Given a Hodge structure on V there is a decreasing filtration of $V_{\mathbb{C}}$ via,

$$F^pV = \bigoplus_{p' \geq p} V^{p',n-p'}$$

(ASK RAYMOND ABOUT TATE TWISTS AND THIS HODE STRUCTURE)

Example 5.2. For any m we define a Hodge structure of weight $-2m$ denoted $\mathbb{Q}(m)$ via taking $\mathbb{Q}(m)_{\mathbb{C}} = \mathbb{Q}(m)^{-m,-m}$

5.3 Mumford-Tate Groups

Definition The Mumford-Take group $M(V)$ associatd to Hodge structure (V, h) is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ such that,

$$\mathrm{Im}(h)(\mathbb{R}) \subset M(V)(\mathbb{R})$$

Example 5.3. For $\mathbb{Q}(m)$ as a Hodge structure the map $h : \mathbb{C}^\times \to \mathrm{GL}_1(\mathbb{R})$ is given by $h(z) = |z|^{-m}$ which is surjective for $m \neq 0$. Thus, for $n \neq 0$ we have,

$$M_h = \mathbb{G}_m^{\mathbb{Q}}$$

and for $n = 0$ it is $\mathrm{Spec}\,(\mathbb{Q})$ the trivial $\mathbb{Q}$-group scheme.

(BADDD)

Proposition 5.4. Let V be a $\mathbb{Q}$-vectorspace with Hodge structure h of weight n. The tensor space,

$$T = V^{\otimes m_1} \otimes V^{\vee \otimes m_2} \otimes \mathbb{Q}(1)^{\otimes m_3}$$

has a Hodge structure of weight $(m_1 - m_2)n - 2m_3$. Then the Mumford-Tate group G of (V, h) is the subgroup of $\mathrm{GL}_n(V) \times \mathbb{G}_m$ fixing all rational tensors of type $(0, 0)$ in T.

Proof. For any $t \in T$ the element t is of type $(0,)$ iff it is fixed by $\mu(\mathbb{G}_m)$ so $M_h = H'$. We will now prove that characters of H lift and thus $H = H'$. □

5.4 DO IT RIGHT

Remark. Let (V, h) be a Hodge structure of weight d. Then the tensor space,

$$T^{m,n}(V) = \bigoplus_{j=1}^{n} V^{\otimes m_j} \otimes (V^\vee)^{\otimes n_j}$$

is a Hodge structure of weight,

$$N = \sum_{j=1}^{n} (m_j - n_j) d$$

Furthermore, let $M(V)$ be the Mumford-Tate group of (V, h) i.e. the intersection of all $\mathbb{Q}$-algebraic subgroups of $\mathrm{GL}(V)$ whose $\mathbb{R}$-points contain $\mathrm{Im}(h)$.

Lemma 5.5. There are morphism of $\mathbb{R}$-algebraic subgroups,

$$\mathbb{S} \xrightarrow{\psi} M(V)_{\mathbb{R}} \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$$

Conversely, given any $\mathbb{Q}$-vectorspace H with an algebraic representation,

$$\rho : M(V) \to \mathrm{GL}(H)$$

gives H a Hodge structure via,

$$\mathbb{S} \xrightarrow{\psi} M(V)_{\mathbb{R}} \xrightarrow{\rho} \mathrm{GL}(H_{\mathbb{R}})$$

Proposition 5.6. Let $H \subset T^{m,n}(V)$ be any rational subspace. Then H is a Hodge substructure iff H is stable under $M(V)$. Furthermore, a rational vector $t \in T^{m,n}(V)$ is of type $(0,0)$ iff it is fixed by $M(V)$.

Proof. If H is stable under the action of the Mumford-Tate group then it becomes a representation $\rho : M(V) \to \mathrm{GL}(H)$ since it is rational this gives a Hogde structure on H.

Conversely, suppose that $V \subset T^{m,n}(V)$ is a substructure then consider its stabilizer $G_H \subset \mathrm{GL}(V)$ which is a $\mathbb{Q}$-algebraic subgroup since H is rational. Moreover, $(G_H)_{\mathbb{R}}$ contains $\mathrm{Im}(h)$ because as a Hodge structure it splits into eigenspaces of h so is preserved by its image. Thus $M(V) \subset G_H$ by definition so $M(V)$ preserves V.

Likewise, it is clear that t is fixed by the action of $\mathbb{S}(\mathbb{R})$ iff t is of Hodge type $(0,0)$. Thus it suffices to prove that t is fixed by $\mathbb{S}(\mathbb{R})$ iff it is fixed by $M(V)$. A similar argument will show this.

First, if t is fixed by $M(V)$ then it is fixed by $M(V)(\mathbb{R})$ which contains $\mathrm{Im}(h)$ and thus t is fixed by $\mathbb{S}(\mathbb{R})$.

Conversely, if t is fixed by $\mathbb{S}(\mathbb{R})$ then its stabilizer $G_t \subset \mathrm{GL}(V)$ is a $\mathbb{Q}$-algebraic subgroup since t is rational. Furthermore, by assumption, $\mathrm{Im}(h) \subset (G_t)(\mathbb{R})$ and thus $M(V) \subset G_t$ by definition showing that $M(V)$ fixes t. ☐

Corollary 5.7. The space $\mathrm{End}\,(V)$ is an algebraic $M(V)$-rep and therfore a Hodge structure. Furthermore, the type-$(0,0)$ Hodge classes are exactly morphisms of Hodge structures since they must commute with the action of $\mathbb{S}$. Therefore,

$$\mathrm{Hom}_{\mathrm{HS}}\,(V,V) = \mathrm{End}\,(V)^{M(V)}$$

5.5 Polarization

Definition A polarization ψ of (V,h) is a morphism of Hodge structures,

$$\psi : V \times V \to \mathbb{Q}(-n)$$

such that $\psi(x, Cy)$ on $V_{\mathbb{R}}$ is an inner product where $C = h(i)$ is the Weil operator.

Remark. Under the canonical isomorphism,

$$\mathrm{Hom}\,(V \otimes V, \mathbb{Q}(-n)) \cong V^{\vee} \otimes V^{\vee}(-n)$$

a polarization is a tensor of bidegree $(0,0)$ because it is a morphism of Hodge structures and thus is fixed by the Mumford-Tate group G,

$$\forall v, v' \in V : \forall (g_1, g_2) \in G(\mathbb{Q}) : \psi(g_1 v, g_1 v') = g_2^n \psi(v, v')$$

Remark. Let $C = h(i)$ be a Weil operator. For $v^{p,q} \in V^{p,q}$ we have $Cv^{p,q} = i^{-p+q}v^{p,q}$ and thus C^2 acts as $(-1)^n$ on all of V where $n = p + q$ is the weight of V.

Definition Let H be a real algebraic group with an involution σ of $H_{\mathbb{C}}$. Then a real-form of H is a real algebraic group H_σ and an isomorphism $H_{\mathbb{C}} \to (H_\sigma)_{\mathbb{C}}$ sending complex conjugation to the action of σ on complex conjugation on $H(\mathbb{C})$.

Theorem 5.8. The Mumford-Tate group $M(V)$ is connected and if (V, h) is polarizable then $M(V)$ is reductive.

Proof. $M(V)$ is clearly connected else its connected component of the identity would be a smaller $\mathbb{Q}$-algebraic subgroup also satisfying the property that its $\mathbb{R}$-points contain $\mathrm{Im}(h)$ (because $\mathbb{S}$ is connected the image must lie in this connected component). Now, we use the fact that a connected algebraic group is reductive if it has a faithful semisimple representation. We will show that the tautological representation $M(V) \hookrightarrow \mathrm{GL}(V)$ which is clearly faithful is also semisimple when V is polarizable. $\square$

Proposition 5.9. If V is polarizable then $M(V) \subset \mathrm{GL}(V)$ is semisimple.

Proof. We will prove that a real algebraic group H is semisimple if it has a *compact* real-form. It suffices to show that H_σ is semisimple. By the unitarian trick, any finite-dimensional H-rep has an H_σ-invariant positive definite symmetric form via,

$$\langle u, v \rangle_0 = \int_{H_\sigma} \langle h \cdot u, h \cdot v \rangle$$

to conclude that every finite-dimensional H_σ-rep is semisimple. This implies that H_σ is reductive.

Thus, it suffices to prove that the Mumford-Tate group has a *compact* real-form (the compactness here is the magic ingredient). Consider the special Mumford-Take group of (V, h),

$$G^0 = \ker (G \to \mathbb{G}_m)$$

and G^1 be the smallest $\mathbb{Q}$-reational subgroup of $\mathrm{GL}(V) \times \mathbb{G}_m$ (WHY THIS GROUP) such that $G^1_{\mathbb{R}}$ constains $h(U^1)$ where U^1 is the $\mathbb{R}$-algebraic groups whose $\mathbb{R}$-points are $S^1 \subset \mathbb{C}^\times$. Then, $G^1 \subset G^0 \subset G$ since,

$$G^1_{\mathbb{R}} \cdot h(C^\times) = G_{\mathbb{R}} \text{ and } h(U^1) = \ker (h(C^\times) \to \mathbb{G}_m$$

so $G^0 = G^1$ and thus G^0 is connected since G^1 is.

Since $C = h(i)$ acts trivially on $\mathbb{Q}(1)$ we know $C \in G^0(\mathbb{R})$. Furthermore C^2 acts as $(-1)^n$ on V and thus is in the center of $G^0(\mathbb{R})$. The inner automorphism $a_C : g \mapsto C^{-1}gC$ of $G_{\mathbb{R}}$ is therefore an involution since its square satisfies,

$$a_C^2(g) = C^{-2}gC^2 = g$$

because C^2 is in the center.

Now let ψ be a polarization of V. For $u, v \in V_{\mathbb{C}}$ and $g \in G^0(\mathbb{C})$ we have,

$$\psi(u, C\bar{v}) = \psi(gu, gC\bar{v}) = \psi(g, CC^{-1}gC\bar{v}) = \psi(gu, C\overline{a_C(\bar{g})v})$$

Thus, the positive-definition bilinear form $\phi(u, v) = \psi(u, C\bar{v})$ on $V_{\mathbb{R}}$ is invariant under the G^0-real-form $G^0_{a_C}$ since the action of $\bar{g}$ is sent to $a_C(\bar{g})$ under the the isomorphism $G^0_{\mathbb{C}} \to (G^0_{a_C})_{\mathbb{C}}$. Since $G^0_{a_C}$ has an invariant inner-product on V it must be compact. (ASK HARRIS ABOUT THAT) $\square$

5.6 Characterizing Subgroups

Here let G be a reductive algebraic group over a field k of characteristic zero and let V_α be a faithful faimly of finite-dimensional representations of G over k such that $G \to \prod \mathrm{GL}(V_\alpha)$ is injecitve. We may define a tensor algebra,

$$T^{m,n} = \bigotimes_\alpha V_\alpha^{\otimes m(\alpha)} \otimes \bigotimes_\alpha (V_\alpha^\vee)^{\otimes n(\alpha)}$$

which is also a finite G-rep.

Definition Then for any algebraic subgroup $H \subset G$ we write H' for the subgroup fixing all tensors appearing in some T fixed by H. That is, H' is the largest subgroup $H \subset H'$ which fixes every tensor fixed by H.

Definition Given an algebraic group G over k we define its character group,

$$X_k(G) = \mathrm{Hom}_k\left(G, \mathbb{G}_m^k\right)$$

Theorem 5.10. We have the following,

(a). Every finite G-rep is contained in a sum of $T^{m,n}$

(b). Every subgroup $H \subset G$ is the stabilizer of a line D is some finite G-rep.

(c). If $H \subset G$ is reductive or $X_k(G) \to X_k(H)$ is surjective then $H = H'$.

Proof. Let W be a finite G-rep and W_0 be the trivial rep on the underlying space of W. There is a morphism of G-reps, $W \to W_0 \otimes_k k[G] \cong k[G]^{\dim W}$ so it suffices to prove that the regular representation can be expressed in terms of tensors.

There must be a finite sum $V = \bigoplus_\alpha V_\alpha$ such that the action $G \to \mathrm{GL}(V)$ is faithful then embed,

$$\mathrm{GL}(V) \to \mathrm{End}\,(V) \times \mathrm{End}\,(V^\vee)$$

identifying $\mathrm{GL}(V)$ with a closed subvariety of $\mathrm{End}\,(V) \times \mathrm{End}\,(V^\vee)$ (FIX)

Let $I \subset \Gamma(G, \mathcal{O}_G)$ be the ideal of global functions on G whose value is zero on H. Consider the regular G-representation $k[G]$ (FIX)

The subgroup H is the stabilizer of a line D in some G-representation V which, by (a), we may take to be a direct sum of tensor representations $T^{m,n}$. Now suppose that H is reductive then V must be a semisimple H-representation so we can write $V = W \oplus D$ for some H-representation W. Furthermore, dualizing $V^\vee = W^\vee \oplus D^\vee$. Since H is the stabilizer of D $\square$

(WHAT IS THE POINT)

Lemma 5.11. Every $\mathbb{Q}$-character of H (above) extends to $\mathrm{GL}(V) \times \mathbb{G}_m$

Proof. Any $\mathbb{Q}$-character restricted to $\mathbb{G}_m$ is $\mathbb{Q}(n)$ for some n. After tensoring with $\mathbb{Q}(-n)$ we find that the character is trivial on $\mu(\mathbb{G}_m)$. But H as the minimal subgroup must act trivially then we use the fact that trivial characters extend. $\square$

(OF THIS)

Theorem 5.12. Let $G \subset \mathrm{GL}(V)$ be the subgroup of all elements which fix every $(0,0)$-hodge class in every tensor space $T^{m,n}(V)$. Then $M(V) = G$.

Proof. We have shown that $M(V) \subset G$. Furthermore, $M(V)' = G$ since $(0,0)$-tensors are exactly the tensors fixed by the Mumford-Tate group and thus G is the group of all elements fixing all tensors fixed by $M(V)$. Now we use the general fact about reductive groups that if G is reductive and $H \subset G$ is a reductive subgroup then $H' = H$. $\square$

5.7 Back to Principle B

Remark. We need a slightly stronger version of Principle B proved as a corellary.

Theorem 5.13. Let $\pi : X \to S$ be a smooth proper map of smooth varieties over $\mathbb{C}$ with S connected and let V be a local subsyttem of $R^{2p}\pi_*\mathbb{Q}(p)$ such that V_s consists purely of $(0,0)$-cycles for all s and consistens of absolute Hodge cycles at at least one $s \in S$. Then V_s consists of absolute Hodge cycles for all $s \in S$.

Proof. If V is constant i.e. if the map $\Gamma(S, V) \to V_s$ is bijective then this follows immedietly from the above argument. However, we may reduce the general case to this as follows.

By Hodge theory on S^{an}, at each point $s \in S$ the stalk $(R^{2p}\pi_*\underline{\mathbb{Q}}(p))_s$ has a Hodge structure and a polarization which, since $R^{2p}\pi_*\mathbb{Q}(p)$ is a local system, glue to give a form,

$$\psi : R^{2p}\pi_*\underline{\mathbb{Q}}(p) \times R^{2p}\pi_*\underline{\mathbb{Q}}(p) \to \underline{\mathbb{Q}}(-p)$$

which at each point is a polarization on the Hodge structure $(R^{2p}\pi_*\mathbb{Q}(p))_s$. On the rational $(0,0)$-subspace,

$$(R^{2p}\pi_*\underline{\mathbb{Q}}(p))_S \cap (R^{2p}\pi_*\underline{\mathbb{C}}(p))_s^{0,0}$$

the form is symmetric, bilinear, rational and positive definite. Since V_s everywhere consists of $(0,0)$-cycles this is a form defined on V_s. Since monodromy preserves the form, the image of $\pi_1(S, s_0)$ in $\mathrm{Aut}(V_{s_0})$ is finite because it is discrete and lies inside the compact group preserving the form. Therefore, after passing to a finite covering we can ensure that $\pi_1(S, s_0)$ acts trivially on V_{s_0} implying that V is globally constant. □

6 Principle A

Definition Let X_α be a family of complete smooth varieties over k. We define tensor spaces,

$$\begin{aligned} T_{\mathrm{dR}} &= \left(\bigotimes_\alpha H^{m(\alpha)}_{\mathrm{dR}}(X_\alpha) \right) \otimes \left(\bigotimes_\alpha H^{n(\alpha)}_{\mathrm{dR}}(X_\alpha)^\vee \right)(m) \\ T_{\mathrm{dR}} &= \left(\bigotimes_\alpha H^{m(\alpha)}_{\mathrm{et}}(X_\alpha) \right) \otimes \left(\bigotimes_\alpha H^{n(\alpha)}_{\mathrm{et}}(X_\alpha)^\vee \right)(m) \\ T_{\mathbb{A}} &= T_{\mathrm{dR}} \times T_{\mathrm{et}} \end{aligned}$$

Finally, given an inclusion $k \hookrightarrow \mathbb{C}$ we get a Betti tensor space,

$$T_\sigma = \left(\bigotimes_\alpha H^{m(\alpha)}_{\sigma}(X_\alpha) \right) \otimes \left(\bigotimes_\alpha H^{n(\alpha)}_{\sigma}(X_\alpha)^\vee \right)(m)$$

We say that an element $t \in T_{\mathbb{A}}$ is,

(a). rational relative to σ if its image in $T_{\mathbb{A}} \otimes_{k \times \mathbb{A}_{\mathrm{fin}}} (\mathbb{C} \times \mathbb{A}_{\mathrm{fin}})$ lies in the subspace T_σ

(b). is a Hodge cycle relative to σ if it is rational relative to σ and its first component lies in F^0 meaning it lies in the subspace generated by,

$$F^0 H^{2p}_{\mathrm{dR}}(X)(p) = H^{p,p}_{\mathrm{dR}}(X) \subset H^{2p}_{\mathrm{dR}}(X)(p) \times H^{2p}_{\mathrm{et}}(X)(m)$$

(c). is absolutly Hodge if it is a Hodge cycle relative to each $\sigma : k \hookrightarrow \mathbb{C}$.

Theorem 6.1 (Principle A)**.** Let X_α be a family of varieties over $\mathbb{C}$ and,

$$T = \bigotimes_\alpha H^{n_\alpha}_B(X_\alpha) \otimes H^{n_\alpha}_B(X_\alpha)^\vee \otimes \mathbb{Q}(1)$$

Let $t_i \in T_i$ be absolute Hodge cycles and let G be the subgroup of,

$$\prod_{\alpha, n_\alpha} \mathrm{GL}(H^{n_\alpha}_B(X_\alpha)) \times \mathbb{G}_m$$

fixing all t_i. If $t \in T$ and is fixed by G then it is an aboslute Hodge cycle.

Remark. We first need a lemma.

(FIX THIS SECTION ON TORSORS)

Lemma 6.2. Let G be an algebraic group over $\mathbb{Q}$ and P be a G-torsor of isomorphism $H_\sigma^\alpha \to H_\tau^\alpha$ where these are families of $\mathbb{Q}$-rational G-reps. Let T_σ and T_τ be tensor spaces of H_σ and H_τ. Then P defines a map $T_\sigma^G \to T_\tau$.

Proof. Locally, for the etale topology on Spec$(\mathbb{Q})$, (MEANING WE CAN CHOSE AN ETALE COVERING SUCH THAT THIS IS THE CASE?) points of P give isomorphisms $T_\sigma \to T_\tau$. Furthermore, the restriction to T_σ^G is idependent of the point since P is a G-torsor. Therefore, this map descends to $T_\sigma^G \to T_\tau$. □

Proof. We define our groups over k with an isomorphism $\sigma : k \hookrightarrow \mathbb{C}$. Let $\tau : k \hookrightarrow \mathbb{C}$ be any other isomorphism. We may assume that t and t_i belong to the same tensor space T then because the t_i are absolute Hodge cylces, they lie in T_σ for each σ. Then there are inclusions of cohomology,

$$\begin{tikzcd} H_\sigma(X_\alpha) \arrow[dr, hook] & & H_\tau(X_\alpha) \arrow[dl, hook'] \\ & H_\sigma(X_\alpha) \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}}) & \end{tikzcd}$$

defined by these isomorphisms. These inclusions follow from the identification of $H_\sigma(X_\alpha) \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}})$ with the etale cohomology which is independent of the choice of embedding $k \hookrightarrow \mathbb{C}$. These induce maps on the tensors,

$$\begin{tikzcd} T_\sigma \arrow[dr, hook] & & T_\tau \arrow[dl, hook'] \\ & T \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}}) & \end{tikzcd}$$

Now, define a functor,

$$P(R) = \{p : H_\sigma \times R \xrightarrow{\sim} H_\tau \otimes R \mid p : \text{p preserves each absolute Hodge cylces}\}$$

Recall that, by definition, an absolute Hodge cycle corresponds to another absolute Hodge cycle for each embedding $k \hookrightarrow \mathbb{C}$ so the condition above make sense, p should itentify $t_i \in T_\sigma$ with its corresponding absolute Hodge cylce in T_τ.

The inclusions demonstrate that $P(\mathbb{C} \times \mathbb{A}_{\mathbb{Q},\text{fin.}})$ is nonempty and since $H_\sigma \otimes R$ and $H_\tau \otimes \mathbb{R}$ are G-representations we get a G-action on $P(R)$. Since G is the group fixing exactly the absolute Hodge cycles, we can see that P is a G-torsor.

If we apply the previous lemma we obtain a map $T_\sigma^G \to T_\tau$ making the following diagram commute,

$$\begin{tikzcd} T_\sigma^G \arrow[r] \arrow[d, hook] & T_\tau \arrow[d, hook] \\ T_\sigma \arrow[r, hook] & T \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}}) \end{tikzcd}$$

Therefore, the map $T_\sigma^G \to T_\tau$ is injective we must have $t \in T_\tau$ since it lies in T_σ^G by hypothesis. Thus t is rational relative to all σ.

It remains to show that the component t_{dR} of $T \otimes \mathbb{C} = T_{\mathrm{dR}}$ lies in the filtration $F^0 T_{\mathrm{dR}}$. For a rational $s \in T_{\mathrm{dR}}$,

$$s \in F^0 T_{\mathrm{dR}} \iff s \text{ is fixed by } \mu(\mathbb{C}^\times)$$

where $\mu(\mathbb{C}^\times)$ corresponds to the real action defining the Mumford-Tate group. Since, by hypothesis, $(t_i)_{\mathrm{dR}} \in F^0$ we know that $G \supset \mu(\mathbb{C}^\times)$ since $\mu(\mathbb{C}^\times)$ must fix all of them. Clearly then if t is fixed by G we must have t fixed by $\mu(\mathbb{C}^\times)$ and thus $t_{\mathrm{dR}} \in F^0 T_{\mathrm{dR}}$. $\square$

7 Construction of Some Absolute Hodge Cycles

7.1 Hermitian Forms

Remark. Recall that a number field E is a CM-filed if for each embedding $E \hookrightarrow \mathbb{C}$ complex conjugation induces a nontrivial automorphism on E independently on the embedding. The fixed field is then a totally real field F and E/F has degree 2.

Definition If E is a CM-field and V is a K-vectorspace then a sesquilinear form $\phi : V \times V \to \mathcal{E}$ is Hermitian if $\phi(v, w) = \overline{\phi(w, v)}$.

Remark. For any embedding $\tau : F \hookrightarrow \mathbb{R}$ we obtain a Hermitian form ϕ_τ on $V_\tau = V \otimes_\tau \mathbb{R}$. Let a_τ and b_τ be the dimensios of the maximal subspaces of V_τ on which ϕ_τ is positive definite and negatice dfinite respectively.

Furthermore, ϕ defines a Hermitan form on the top forms $\Lambda^{\dim V} V \cong E$ which must be an E-Hermitian form on E and thus is given by an element $f \in F$ defined up to $\mathrm{Nm}_{E/F} E^\times$. We call this the discriminant.

Remark. Let $(v_1, \dots, v_d)$ be an orthogonal basis for ϕ and $\phi(v_i, v_i) = c_i$. Then a_τ is the number of i s.t. $\tau c_i > 0$ and b_τ is the number of i s.t. $\tau c_i < 0$ and $f = c_1 \cdots c_n$. If ϕ is nondegenrate, then $f \in F^\times / \mathrm{Nm}_{E/F} E^\times$ and,

$$a_\tau + b_\tau = \dim V \qquad \mathrm{sign}(\tau f) = (-1)^{b_\tau}$$

Proposition 7.1. Given, for each embedding $\tau : F \hookrightarrow \mathbb{C}$, a tripple (a_τ, b_τ) and $f \in F^\times / \mathrm{Nm}_{E/F} E^\times)$ satisfying the above. Then there exists a unique pair (V, ϕ) a non-degenerate Hermitian form ϕ on an E-vectorspace V with invariants (a_τ, b_τ) with respect to $\tau : F \hookrightarrow \mathbb{R}$ and f.

Definition A Hermitian space (V, ϕ) of dimension d is *split* if it satisfies the equivalent conditions,

(a). $a_\tau = b_\tau$ for all τ and $f = (-1)^{d/2}$

(b). there is a totally isotropic subspace of V of dimnsion $d/2$ (for each $v \in W$: $\phi(v, v) = 0$).

Lemma 7.2. Let k be a field, k' an etale k-algebra (a finite product of finite separable extensions of k) and V a f.g. free k'-module. Then,

(a). The map,
$$f \mapsto \mathrm{Tr}_{k'/k} \circ f : \mathrm{Hom}_{k'}(V, k') \to \mathrm{Hom}_k(V, k)$$
is an isomorphism of k-vectorspaces.

(b). $\bigwedge_{k'}^n V$ is a direct summand of $\bigwedge_k^n V$ naturally.

Proof. The trace map $\mathrm{Tr}_{k'/k} : k' \times k' \to k$ is nondegenerate (HOW IS THIS A PAIRING). The map $f \mapsto \mathrm{Tr}_{k'/k} \circ f$ is injective and then onto because the spaces are of the same dimension.

There are obvious maps,
$$\bigwedge_k^n V \to \bigwedge_{k'}^n V$$
$$\bigwedge_k^n V^\vee \to \bigwedge_{k'}^n V^\vee$$
where here we deine the dual of k'-modules as,
$$V^\vee = \mathrm{Hom}_{k'}(V, k') = \mathrm{Hom}_k(V, k)$$
(WHAT?) □

7.2 Conditions to Consist of Absolute Hodge Cycles

Remark. In this section we will be in the following situation.

Definition Let A be an abelian variety over $\mathbb{C}$ and E a CM field with a homomorphism $\nu : E \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $= \dim_E H_1(A, \mathbb{Q})$ which has an E-vectorspace structure via ν. Thus, $2 \dim A = d[E : \mathbb{Q}]$.

Proposition 7.3. The analytic space A^{an} is a compact complex Lie group which is a complex torus. Let $\mathfrak{g}$ be the lie Algebra then there is an $\mathbb{R}$-linear map $\mathfrak{g} \to H_1(A^{\mathrm{an}}, \mathbb{R})$ sending a tangent vector to the homology class defined by its geodesic (ASK HARRIS ABOUT THIS). Now $\mathfrak{g}$ is a complex vectorspace so $H_1(A^{\mathrm{an}}, \mathbb{R})$ inherents a complex structure given by an $\mathbb{R}$-linear endomorphism $J : H_1(A^{\mathrm{an}}, \mathbb{R}) \to H_1(A^{\mathrm{an}}, \mathbb{R})$.

Proposition 7.4. Hoge theory gives a hodge structure on $H^1(A^{\mathrm{an}}, \mathbb{R})$ which is determined by a map $h : \mathbb{S} \to \mathrm{GL}(H^1(A, \mathbb{R}))$.

Now, on a complex torus of $\dim_{\mathbb{R}}(A^{\mathrm{an}}) = 2g$ there are isomorphisms,
$$H^1(A^{\mathrm{an}}, \mathbb{R})^\vee \xrightarrow{\sim} \bigwedge^{2g-1} H^1(A^{\mathrm{an}}, \mathbb{R}) \xrightarrow{\sim} H^{2g-1}(A^{\mathrm{an}}, \mathbb{R}) \xrightarrow{\sim} H_1(X, \mathbb{R})$$

This identification gives an isomorphism,

$$\mathrm{GL}(H^1(A^{\mathrm{an}}, \mathbb{R})) \cong \mathrm{GL}(H_1(A, \mathbb{R}))$$

under which $h(i) \mapsto J$.

Proposition 7.5. Consider the decomposition,

$$E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \mathbb{C}$$

$$e \otimes z \mapsto (\sigma \mapsto \sigma(e) \cdot z)$$

Tensoring by $H^1_B(A) = H^1(A^{\mathrm{an}}, \mathbb{Q})$ we find,

$$H^1_B(A) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} H^1_B(A) \otimes_\sigma \mathbb{C}$$

where,

$$H^1_B(A) = H^1(A^{\mathrm{an}}, \mathbb{Q})$$

is an E-vectorspace and $e \in E$ acts on $H^1_B(A) \otimes_\sigma \mathbb{C}$ via $\sigma(e)$. Since E repsects the Hodge structure on $H^1_B(A)$ each $H^1_{E,\sigma}(A) = H^1(A^{\mathrm{an}}, \mathbb{Q}) \otimes_\sigma \mathbb{C}$ acquires a Hodge structure,

$$H^1_{E,\sigma}(A) = H^{1,0}_{E,\sigma}(A) \oplus H^{0,1}_{E,\sigma}(A)$$

Define,

$$a_\sigma = \dim_{\mathbb{C}} H^{1,0}_{E,\sigma}(A) \quad \text{and} \quad b_\sigma = \dim_{\mathbb{C}} H^{1,0}_{E,\sigma}(A) \quad \text{thus} \quad a_\sigma + b_\sigma = d$$

Proposition 7.6. The subspace,

$$\bigwedge\nolimits_E^d H^1_B(A) \subset H^d(A^{\mathrm{an}}, \mathbb{Q})$$

has pure bidegree $(\frac{d}{2}, \frac{d}{2})$ iff $a_\sigma = b_\sigma$ for each $\sigma \in \mathrm{Hom}\,(E, \mathbb{C})$.

Proof. For a complex torus, we have,

$$H^d(A^{\mathrm{an}}, \mathbb{Q}) \cong \bigwedge\nolimits_{\mathbb{Q}}^d H^1(A^{\mathrm{an}}, \mathbb{Q})$$

so a previous lemma identifies,

$$\bigwedge\nolimits_E^d H^1(A^{\mathrm{an}}, \mathbb{Q}) \subset \bigwedge\nolimits_{\mathbb{Q}}^d H^1(A^{\mathrm{an}}, \mathbb{Q})$$

as a direct summand. Then consider,

$$\begin{aligned}
\left(\bigwedge\nolimits_E^d H^1_B(A)\right) \otimes_{\mathbb{Q}} \mathbb{C} &\cong \bigwedge\nolimits_{E \otimes_{\mathbb{Q}} \mathbb{C}}^d \left(H^1_B(A) \otimes_{\mathbb{Q}} \mathbb{C}\right) \\
&\cong \bigoplus_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \bigwedge\nolimits_{\mathbb{C}}^d (H^1(A^{\mathrm{an}}, \mathbb{Q}) \otimes_\sigma \mathbb{C})) \\
&\cong \bigoplus_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \bigwedge\nolimits_{\mathbb{C}}^d (H^{1,0}_{E,\sigma}(A) \oplus H^{0,1}_{E,\sigma}(A)) \\
&\cong \bigoplus_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \bigwedge\nolimits_{\mathbb{C}}^{a_\sigma} H^{1,0}_{E,\sigma}(A) \oplus \bigwedge\nolimits_{\mathbb{C}}^{b_\sigma} H^{0,1}_{E,\sigma}(A)
\end{aligned}$$

Thus, we have decomposed this subspace into a sum of pure bidegree $(a_\sigma, 0)$ and $(0, b_\sigma)$ proving the proposition. □

Remark. In the case $a_\sigma = b_\sigma$ then,

$$\left(\bigwedge_E^d H_B^1(A)\right)\left(\tfrac{d}{2}\right)$$

(ASK HARRIS WHY TATE TWIST HERE?) consists of Hodge cycles. We want to know when this consists of absolute Hodge cycles.

Lemma 7.7. If $A = A_0 \otimes_{\mathbb{Q}} E$ for some abelian variety A_0 of dimension $\frac{d}{2}$ then,

$$\left(\bigwedge_E^d H_B^1(A)\right)\left(\tfrac{d}{2}\right) \subset H^d(A^{\mathrm{an}}, \mathbb{Q})\left(\tfrac{d}{2}\right)$$

consists of absolute Hodge cycles.

Proof. Consier the diagram,

$$\begin{array}{ccccc}
H_B^d(A_0)(\frac{d}{2}) \otimes_{\mathbb{Q}} E & \longrightarrow & H_B^d(A_0)(\frac{d}{2}) \otimes_{\mathbb{Q}} E & & \\
\downarrow\wr & & \downarrow\wr & & \\
\left(\bigwedge_E^d H_B^1(A_0 \otimes_{\mathbb{Q}} E)\right)(\frac{d}{2}) & \longrightarrow & \left(\bigwedge_{E\otimes\mathbb{A}}^d H_{\mathbb{A}}^1(a_0 \otimes_{\mathbb{Q}} E)\right)(\frac{d}{2}) & \hookrightarrow & H_{\mathbb{A}}^d(A_0 \otimes E)(\frac{d}{2})
\end{array}$$

The vertical maps are induced by the isomorphism $H_B^1(A_0) \otimes_{\mathbb{Q}} E \xrightarrow{\sim} H_B^1(A_0 \otimes_{\mathbb{Q}} E)$. There is a similar diagram for each embedding $\sigma : E \hookrightarrow \mathbb{C}$ and thus the image of the bottom map must be stable with respect to a choice of $\sigma : E \hookrightarrow \mathbb{C}$. Therefore, the Hodge cycles,

$$\left(\bigwedge_E^d H_B^1(A_0 \otimes_{\mathbb{Q}} E)\right)\left(\tfrac{d}{2}\right) \subset H_B^d(A_q \otimes_Q E)\left(\tfrac{d}{2}\right)$$

are indeed absolutly Hodge. (ASK HARRIS ABOUT THIS PROOF)? I don't understand it. □

7.3 Riemann Forms

Definition A Hermitian form H on a complex vectorspace V is a complex bilinear form $H : \overline{V} \times V \to \mathbb{C}$ (sesquilinear on H) which satisfies,

$$H(u, v) = \overline{H(v, u)}$$

Lemma 7.8. Let V be a complex vectorspace. There is a one-to-one correspondence between Hermitian forms H on V and real-valued skew-symmetric forms E on V.

Proof. The correspondence is given by,

$$\begin{aligned}
H \mapsto E_H \qquad & E_H(u, v) = \mathrm{Im}(H(u, v)) \\
E \mapsto H_E \qquad & H_E(u, v) = E(iu, v) + iE(u, v)
\end{aligned}$$

□

Definition A Riemann form $E : V \times V \to \mathbb{R}$ on a complex vectorspace V is an antisymmetric $\mathbb{R}$-bilinear form such that,

(a). $E(iu, iv) = E(u, v)$

(b). the corresponding Hermitan form H_E is positive definite.

Definition A complex torus $X = V/\Lambda$ is *polarizable* if there exists an antisymmetric form $E : \Lambda \times \Lambda \to \mathbb{Z}$ such that $E_{\mathbb{R}} : V \times V \to \mathbb{R}$ (using that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$) is a Riemann form.

(IS THIS EQUIVALENT TO THE POLARIZATION OF THE HODGE STRUCTURE $H_1(X, \mathbb{Q})$)

Theorem 7.9. A complex torus $X = V/\Lambda$ is of the form A^{an} for some abelian variety A iff X is polarizable.

(DOES THIS IMPLY THAT ALL ABELIAN VARIETIES ARE POLARIZABLE IN THE FOLLOWING SENSE)

Definition A polarization of an abelian variety A is an isogeny $\lambda : A \to A^\vee$ such that

Remark. We can identify, $A^\vee = \text{Pic}^0(A)$.

Proposition 7.10. For each line bundle $\mathcal{L}$ on A/k there is an associated morphism $\phi_{\mathcal{L}} : A \to A^\vee$ which is an isogeny if $\mathcal{L}$ is ample.

Proof. We define a map $\phi_{\mathcal{L}} : A(\overline{k}) \to \text{Pic}(A)$ via $\phi_{\mathcal{L}}(x) = t_x^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}$. First, via the Theorem of the Square, for $x, y \in A(\overline{k})$,

$$t_x^* \mathcal{L} \otimes_{\mathcal{O}_A} t_y^* \mathcal{L} = t_{x+y}^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}$$

Therefore,

$$(t_x^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}) \otimes_{\mathcal{O}_A} (t_y^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}) = t_{x+y}^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}$$

so ϕ is a group homomorphism. Furthermore, $\deg t_x^* \mathcal{L} = \deg \mathcal{L}$ since the map $t_x : A \to A$ is an isomorphism. (IS THIS TRUE?) Therefore, $\deg \phi_{\mathcal{L}}(x) = 0$ so the image is contained in $\text{Pic}^0(A) = A^\vee(\overline{k})$. ☐

Definition A polarization of A is an isogeny $\phi : A \to A^\vee$ such that $\phi_{\overline{k}} : A_{\overline{k}} \to A_{\overline{k}}^\vee$ is of the form $\phi_{\mathcal{L}}$ for some ample line bundle $\mathcal{L}$ on $A_{\overline{k}}$. Deriving from a line bundle gives symmetry $\phi = \phi^\vee$ and ampleness is a positivity condition.

Definition Let A be an abelian variety with a polarization $\phi : A \to A^\vee$. Since ϕ is an isogeny, it has an "inverse element" in the algebra $\phi^{-1} \in \text{Hom}(A^\vee, A) \otimes \mathbb{Q}$. (This follows from inverting the multiplication by n maps). Then we define the Rosati involution of the endomorphism algebra $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ via,

$$\alpha^\dagger = \phi^{-1} \circ \alpha^\vee \circ \phi \quad \text{for} \quad \alpha \in \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Remark. The Rosati involution depends on the choice of polarization.

Theorem 7.11. A polarization θ on A is determined by a Riemann form ϕ on $H_1(A^{\mathrm{an}}, \mathbb{Q})$. Two forms ϕ, ϕ' determine the same polarization iff $\exists a \in \mathbb{Q}^\times : \phi' = a\phi$. In this case, the Rosati involution is determined by,

$$\forall u, v \in H_1(A^{\mathrm{an}}, \mathbb{Q}) : \phi(\alpha(u), v) = \phi(u, \alpha^\dagger(v)) \qquad \alpha \in \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Proof. (HOW DOES ONE PROVE THIS?) □

Theorem 7.12. Let A be an abelian variety over $\mathbb{C}$ and $\nu : E \to \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ the inclusion of a CM-field with $d = \dim_E H^1(A^{\mathrm{an}}, \mathbb{Q})$. Suppose there exists a polarization θ for A such that,

(a). the Rosati involution of θ induces complex conjugation on $E \subset \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$

(b). ther exists a split E-Hermitian form ϕ on $H_1(A^{\mathrm{an}}, \mathbb{Q})$ and $f \in E^\times$ with $\overline{f} = -f$ such that $\phi(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(f\phi(x, y))$ is a Riemann form for θ.

Then the subspace,

$$\left(\bigwedge\nolimits_E^d H^1_B(A)\right)(\tfrac{d}{2}) \subset H^d(A^{\mathrm{an}}, \mathbb{Q})(\tfrac{d}{2})$$

consists of absolute Hodge cycles.

7.4 Shimura Varieties

8 The Proof for Abelian Varieties of CM Type

Definition The Mumford-Tate group $M(A)$ of an abelian variety A is the Mumford tate group of the rational Hodge structure $H_1(A, \mathbb{Q})$.

Definition An abelian variety is of CM-type if $M(A)$ is abelian.

Remark. Any abelian variety A is isogenous to a product of simple abelian varieties A_α and A is CM-type iff each A_σ is CM-type since the Mumford-Tate group of the product $M(A)$ is contained in the product of $M(A_\alpha)$ and projects fully onto each. Therefore, it will suffice to study simple abelian varieties of CM-type.

Lemma 8.1. Let A be an abelian variety. Then $\mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the subalgebra of elements in $\mathrm{End}\,(H_1(A^{\mathrm{an}}, \mathbb{Q}))$ preserving the Hodge structure. Furthermore, preserving the Hodge structure is equivalent to commuting with the image of $\mu : \mathbb{G}_m \to \mathrm{GL}(H_1(A^{\mathrm{an}}, \mathbb{C}))$.

Proof. (PROVE THIS) □

Proposition 8.2. A simple abelian varity over $\mathbb{C}$ is of CM-type iff $E = \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a commutative field over which $H_1(A, \mathbb{Q})$ has dimension 1. In this case, E is a CM-field and the Rosati involution on E for any polarization of A is complex conjugation on $E \subset \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. Let A be an abelian variety with $E \subset \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that,

$$\dim_E H_1(A, \mathbb{Q}) = 1$$

Then $\mu(\mathbb{G}_m)$ commutes with $E \otimes \mathbb{R}$ in $\mathrm{End}\,(H_1(A^{\mathrm{an}}, \mathbb{R}))$ because the Hodge structure is compatible with the E-vectorspace structure. (WHY THOUGH) The subspace $(E \otimes_{\mathbb{Q}} \mathbb{R}) \subset \mathrm{GL}(H_1(A^{\mathrm{an}}, \mathbb{R}))$ is all diagonal matricies (since $H_1(A^{\mathrm{an}}, \mathbb{R})$ is dimension one over E) and since anything that commutes with all diagonal matrices must itself be diagonal, we have $h(\mathbb{S}) \subset (E \otimes_{\mathbb{Q}} \mathbb{R})^\times$ which implies that $M(A) \subset \mathbb{G}_{E^\times}$ where $\mathbb{G}_{E^\times} \subset \mathrm{GL}(H_1(A^{\mathrm{an}}, \mathbb{Q}))$ is the commutative $\mathbb{Q}$-algebraic subgroup defined by $\mathbb{G}_{E^\times}(F) = (E \otimes_{\mathbb{Q}} F)^\times$ and thus whose $\mathbb{R}$-points are $(E \otimes_{\mathbb{Q}} \mathbb{R})$ containing $h(\mathbb{S})$. Therefore $M(A) \subset \mathbb{G}_{E^\times}$ is abelian since $\mathbb{G}_{E^\times}$ is a commutative group scheme. (I believe that $\mathbb{G}_{E^\times} = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m^E)$ IS THIS CORRECT?)

Conversely, let A be simple and of CM-type an $\mu : \mathbb{G}_m \to \mathrm{GL}(H_1(A^{\mathrm{an}}, \mathbb{C}))$ define the Hodge structure on $H_1(A^{\mathrm{an}}, \mathbb{C})$. Since A is simple, $E = \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division ring of degree $\le \dim_{\mathbb{Q}} H_1(A^{\mathrm{an}}, \mathbb{Q})$ over $\mathbb{Q}$. (COMPLETE THIS PROOF!?) $\square$

8.1 The Proof For CM Case

Let A_α be a finite family of abelian varieties of CM-type. We need to show that every Hodge cycle in,

$$T_{\mathbb{A}} = \left(\bigotimes_\alpha H^1_{\mathbb{A}}(X_\alpha)^{\otimes m_\alpha} \right) \otimes \left(\bigotimes_\alpha H^1_{\mathbb{A}}(X_\alpha)^{\vee \otimes n_\alpha} \right)(m)$$

is an absolute Hodge cycle. According to Principal A the group G^{AH} fixing all absolute Hodge cycles fixes exactly the absolute Hodge cycles. Thus it suffices to prove that the subgroup $G^H \subset G^{AH}$ fixing all Hodge cycles is equal to G^{AH}.

9 Proof of the Main Theorem

Let A be an abelian variety over $\mathbb{C}$ and t_α for $\alpha \in I$ be Hodge cycles on A. We need to show that these are absolute Hodge cycles. Since we know the result in the case that A is CM-type it suffices to prove the following.

Proposition 9.1. There exists a connected smooth algebraic variety $S/\mathbb{C}$ and an abelian scheme $\pi : Y \to S$ such that,

(a). for some $s_0 \in S$ the fibre $Y_{s_0} = A$

(b). for some $s_1 \in S$ the fibre Y_{s_1} is of CM-type

(c). the cycles t_α extend to rational cycles of bidegree $(0,0)$ on Y. Explicitly, suppose that,

$$t_\alpha \in H^1_B(A)^{\otimes m(\alpha)} \otimes H^1_B(A)^{\vee \otimes n(\alpha)}$$

then there is a section t of,

$$(R^1\pi_*\underline{\mathbb{Q}})^{\otimes m(\alpha)} \otimes (R^1\pi_*\underline{\mathbb{Q}})^{\otimes n(\alpha)}$$

over a finite cover $\tilde{S} \to S$ such that for some $\bar{s}_0$ over s_0 we have $t_{\bar{s}_0} = t_\alpha$ and for all $\tilde{s} \in \tilde{S}$ we have,

$$t_{\tilde{s}} \in H^1_B(Y_{\tilde{s}})^{\otimes m(\alpha)} \otimes H^1_B(Y_{\tilde{s}})^{\vee \otimes n(\alpha)}$$

is a Hodge cycle.

Proof. S will be a Shimura Variety. Extend the set of AH cylces such that some t_α is a polarization of A and let $H = H_1(A, \mathbb{Q})$. Now we consider $G \subset \mathrm{GL}_H(\times)\mathbb{G}_m$ fixing t_α. Since the hodge character must act trivially on t_α then it defines a character $h_0 : \mathbb{C}^\times \to G(\mathbb{R})$.

Define,

$$X = \{h : \mathbb{C}^\times \to G(\mathbb{R}) \mid h \text{ is conjugate to } h_0 \in G(\mathbb{R})\}$$

For each $h \in X$ we get a new Hodge structue of H relative to which t_α has bidegree $(0,0)$ since h fixes it. Let $F^0(h) = H^{0,-1} \subset H \otimes \mathbb{C}$ in this new Hodge structue. Sending $h \mapsto F^0(h)$ is a map $X \to \mathrm{Gr}_k(H \otimes \mathbb{C})$ as real manifolds. The map is injective becaues the filtration completely determines a hodge structure. Consider the centralizer K_∞ of h_0. Then,

$$\begin{tikzcd}
T_{h_0}(X) \arrow[r, equal] & \mathrm{Lie}(G_{\mathbb{R}})/\mathrm{Lie}(K_\infty) \arrow[r, hook] \arrow[d, equal] & \mathrm{End}\,(H \otimes \mathbb{C})\,/F^0\mathrm{End}\,(H \otimes \mathbb{C}) \arrow[r, equal] & T_{\phi(h_0)}\mathrm{Gr}_k(H \otimes \mathbb{C}) \\
& \mathrm{Lie}(G_{\mathbb{C}})/F^0\mathrm{Lie}(G_{\mathbb{C}}) \arrow[ur, hook] & &
\end{tikzcd}$$

where the Filtration on $\mathrm{End}\,(H \otimes \mathbb{C})$ is given by the Hodge structure h_0 on H. Then, X is a complex manifold.

To each $h \in X$ we attach a complex torus given by the double cosets $F^0(h) \setminus H \otimes \mathbb{C}/H(\mathbb{Z})$ where $H(\mathbb{Z})$ is a fixed lattice inside H. In particular, at h_0 we get,

$$F^0(h_0) \setminus H \otimes \mathbb{C}/H(\mathbb{Z}) = T_0(A)/H(\mathbb{Z})$$

These tori form a family $B \to X$. Then define the group,

$$\Gamma_n = \{g \in G(\mathbb{Q}) \mid (g - q)H(\mathbb{Z}) \subset nH(\mathbb{Z})\}$$

for some. For sufficiently large n Baily and Borel show that $S = X/\Gamma$ is an algebraic variety, in particular a Shimura variety. $\square$

10 Ideal for Next Semester

That paper on Slopes of powers of Frobenius on crystalline cohomology.

Course on crystalline cohomology.

Course on Shimura varieties.

Study supersingular curves or K3 surfaces.