Mathematics GU4044 Representations of Finite Groups
Assignment # 11

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April 23, 2018

Problem 1.

(i) Let V be the 3-dimensional A_4-representation. A_4 has four conjugacy classes,

$$[e] \quad [(1\,2\,3)] \quad [(1\,3\,2)] \quad [(1\,2)(3\,4)]$$

Therefore, using the fact that $\chi_V = \chi_{st.} - 1$.

$$\langle \psi_2(\chi_V), 1 \rangle = \frac{1}{\#(A_4)} \sum_{g \in A_4} \chi_V(g^2) = \frac{1}{12} \left[1 \cdot 3 + 4 \cdot 0 + 4 \cdot 0 + 3 \cdot 3 \right] = 1$$

(ii) Let $G = D_n = \langle r, s \mid r^n = s^2 = e, rs = sr^{-1} \rangle$. Any element in D_n can be written in the form $r^i s^j$ for $0 \le i < n$ and $0 \le j \le 1$. Any element of the form $r^i s$ satisfies $(r^i s)^2 = r^i s r^i s = r^i r^{-i} s^2 = e$. Furthermore, $(r^i)^2 = r^{2i}$. Let V be a 2-dimensional irreducible D_n-representation. Then, $\chi_V(r^i s) = \chi_V(e) = 2$. However,

$$|\chi_V(g^2)| \le \dim V = 2$$

Therefore,

$$\langle \psi_2(\chi_V), 1 \rangle = \frac{1}{|D_n|} \sum_{g \in D_n} \chi_V(g^2) = \frac{1}{2n} \left[2 \cdot (n+1) + \sum_{i=0}^{n-1} \chi_V(r^{2i}) \right] > 0$$

because,

$$\left| \sum_{i=0}^{n-1} \chi_V(r^{2i}) \right| < 2n$$

This implies that V is defined over $\mathbb{R}$.

Problem 2.

(i) Let V be a complex vector space and let $\psi : V \to V^*$ be a conjugate linear map which is an isomorphism as a real map. Define the map,

$$\gamma : V \oplus V^* \to V \oplus V^*$$

via $\gamma(v, \ell) = (\psi^{-1}(\ell), \psi(v))$. I claim that γ is a conjugate-linear involution. First,

$$\begin{aligned}\gamma(\alpha v_1 + \beta v_2, \alpha \ell_1 + \beta \ell_2) &= (\psi^{-1}(\alpha \ell_1 + \beta \ell_2), \psi(\alpha v_1 + \beta v_2)) \\ &= (\bar{\alpha}\psi^{-1}(\ell_1) + \bar{\beta}\psi^{-1}(\ell_2), \bar{\alpha}\psi(v_1) + \bar{\beta}\psi(v_2)) \\ &= \bar{\alpha}(\psi^{-1}(\ell_1), \psi(v_1)) + \bar{\beta}(\psi^{-1}(\ell_2), \psi(v_2))\end{aligned}$$

Furthermore,

$$\gamma^2(v, \ell) = \gamma(\psi^{-1}(\ell), \psi(v)) = (\psi^{-1}(\psi(v)), \psi(\psi^{-1}(\ell))) = (v, \ell)$$

since ψ is a bijection. Thus, γ is a conjugate-linear involution which we have shown defines a complex structure on $V \oplus V^*$.

(ii) Let V be a G-representation and H a G-invariant positive definite Hermitian form. Define the map $\psi : V \to V^*$ by $\psi(v)(w) = H(w, v)$. Then, $\psi(\alpha v + \beta u)(w) = H(w, \alpha v + \beta u) = \bar{\alpha} H(w, v) + \bar{\beta} H(w, u)$. Furthermore, V and V^* have the same dimension since V is finite dimensional. Therefore, it suffices to show that ψ is an injection. Suppose that $\psi(v) = 0$ then $H(w, v) = 0$ for all w. However, $H(v, v) = 0 \iff v = 0$ so $v = 0$. Thus, ϕ is a bijection and thus an isomorphism as a real-linear map. Therefore, by (i) the map ψ gives rise to a conjugate-linear γ involution γ which puts a real structure on $V \oplus V^*$. However, H is G-invariant so $\psi(g \cdot v)(g \cdot w) = H(g \cdot w)(g \cdot v) = H(w)(v) = \psi(v)(w)$. Thus, $\rho_{V^*}(g)^{-1} \circ \psi \circ \rho_V(g) = \psi$ so ψ is a G-morphism between V and V^*. Therefore,

$$\begin{aligned}\rho_{(V \oplus V^*)}(g)^{-1} \circ \gamma \circ \rho_{(V \oplus V^*)}(g)(v, \ell) &= (\rho_V(g)^{-1} \circ \psi^{-1}(\rho_{V^*}(g) \cdot \ell), \rho_{V^*}(g)^{-1} \circ \psi(\rho_V(g) \cdot v)) \\ &= (\psi^{-1}(\ell), \psi(v))\end{aligned}$$

Therefore, γ is a G-morphism i.e. γ commutes with the G-representation on $V \oplus V^*$. Thus, $V \oplus V^*$ is defined over $\mathbb{R}$ as a G-representation.

Problem 3.

Let p and q be odd primes with $p < q$ and $q \equiv 1 \bmod p$. Let G be a nonabelian group of order pq. From HW. 8 problem 2 we know that G has $h = \frac{1}{p}(q-1) + p$ conjugacy classes. Write $q = pr + 1$ for some integer r. Thus, $h = r + p$. Now,

$$pq - p = p(q-1) = p^2 r$$

however, since p is odd, $p^2 \equiv 1 \bmod 8$. However, $r = \frac{1}{p}(q-1)$ is even because p is odd and $q - 1$ is even. Therefore, $r = 2s$. But $2p^2 \equiv 2 \bmod 16$ since $8 \mid p^2 - 1$ and thus $16 \mid 2p^2 - 2$. Thus, $p^2(2s) \equiv 2s \bmod 16$. Therefore, $p^2 r \equiv r \bmod 16$ so $pq - p \equiv r \bmod 16$ and thus,

$$pq \equiv r + p = h \bmod 16$$