Mathematics GU4053 Algebraic Topology
Assignment # 4

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Note. My order of path concatenation follows Hatcher,

$$\gamma * \delta(x) = \begin{cases} \gamma(2x) & x \le \frac{1}{2} \\ \delta(2x-1) & x \ge \frac{1}{2} \end{cases}$$

Problem 1.

Let $f : X \to Y$ and $g : Y \to Z$ be fibrations. Given any space A, a map $p : A \to X$ and a homotopy $h : A \times I \to Z$ consider the diagram,

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
A \arrow[u, "p"] \arrow[ur, "f \circ p"] \arrow[rr, hook] & & A \times I \arrow[u, "h"'] \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}$$

Because g is a fibration, there exists a lift $\tilde{h} : A \times I \to Y$ of h matching $f \circ p$ such that the diagram commutes. Now, rewrite the commutative diagram as,

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
A \arrow[u, "p"] \arrow[r, hook] & A \times I \arrow[ul, dashed, "\tilde{\tilde{h}}"] \arrow[u, "\tilde{h}"] \arrow[ur, "h"]
\end{tikzcd}$$

which gives a lift $\tilde{\tilde{h}}$ of $\tilde{h}$ at p because f is a fibration. Therefore, there is a map $\tilde{\tilde{h}} : A \times I \to X$ which makes the diagram commute. That is, $\tilde{\tilde{h}}(a, 0) = p(a)$ and $g \circ f \circ \tilde{\tilde{h}} = g \circ \tilde{h} = h$. Thus, $g \circ f$ is a fibration.

Likewise, let $f : X \to Y$ and $g : Y \to Z$ be cofibrations. Given any space A, a map $p : Z \to A$ and a homotopy $h : X \times I \to A$ consider the diagram,

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, hook] & & Y \arrow[r, "g"] \arrow[dl, "p \circ g"] \arrow[dd, hook] & Z \arrow[dd, hook] \arrow[dll, bend right, "p"] \\
& A & & \\
X \times I \arrow[rr, "f \times \mathrm{id}_I"] \arrow[ur, "h"] & & Y \times I \arrow[r, "g \times \mathrm{id}_I"] \arrow[ul, dashed, "\tilde{h}"] & Z \times I
\end{tikzcd}
$$

Because f is a cofibration, there exists an extension $\tilde{h} : Y \times I \to A$ of h lifting to $p \circ g$ such that the diagram commutes. Now, rewrite the commutative diagram as,

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd, hook] & Y \arrow[rr, "g"] \arrow[dd, hook] & & Z \arrow[dl, "p"] \arrow[dd, hook] \\
& & A & \\
X \times I \arrow[r, "f \times \mathrm{id}_I"] & Y \times I \arrow[rr, "g \times \mathrm{id}_I"] \arrow[ur, "\tilde{h}"] & & Z \times I \arrow[ul, dashed, "\tilde{\tilde{h}}"]
\end{tikzcd}
$$

which gives an extension $\tilde{\tilde{h}}$ of $\tilde{h}$ lifting p because f is a cofibration. Therefore, there is a map $\tilde{\tilde{h}} : Z \times I \to A$ which makes the diagram commute. Thus, $g \circ f$ is a cofibration.

Problem 2.

Let $f : X \to Y = \{*\}$ be a continuous (constant map). Given a map $g : A \to X$ and a homotopy $h : A \times I \to Y$. However, h must be a constant map so we should take $\tilde{h} : A \times I \to X$ given by $\tilde{h}(x, t) = g(x)$. Then, $f \circ \tilde{h} = h$ because both are constant maps. Also, $\tilde{h}(x, 0) = g(x)$ by construction. Thus, the following diagram commutes,

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
A \arrow[u, "p"] \arrow[r, hook] & A \times I \arrow[u, "h"'] \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}
$$

which means that f is a fibration.

Problem 3.

Problem 4.

Problem 5.

Problem 6.

Let X be a CW complex equal to an increasing union of subcomplexes,

$$X_1 \subset X_2 \subset X_3 \subset X_4 \subset \cdots \qquad \text{and} \qquad X = \bigcup_{i=1}^{\infty} X_i$$

such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. I claim that the complex $X = \varinjlim X_i$ is the direct limit (colimit) of the system,

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow X_4 \hookrightarrow \cdots$$

(PROVE THIS)

The projection functor $\pi : \text{Top} \to \text{hTop}$ which maps topological spaces to themselves and maps maps to homotopy classes is cocontinuous. (PROVE THIS) Therefore,

$$\pi(X) = \pi(\varinjlim X_i) = \varinjlim \pi(X_i)$$

However, each inclusion map ι is nullhomotopic so the map $j_i : X_i \to X$ is also nullhomotopic because $j_i = \iota_i \circ j_{i+1}$ but ι_i is nullhomotopic so j_i is also nullhomotopic. Because X is the limit of this diagram, there must be a unique map $\text{id}_X : X \to X$ which commutes with every cone.

Problem 7.

Consider the pointed space (X, x_0) and the suspension $\Sigma X \cong X \times I / (X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)$. Consider the homotopy, $X \times I \to \Sigma X$ defined by $h(x,t) = (x, t/2)$. Then, $h(x,0) = (x,0)$ but $(x_1, 0) \sim (x_2, 0)$ so $h(x,0)$ is constant. Also, $h(x,1) = (x, 1/2) = \iota(x)$. Therefore, $\iota : X \hookrightarrow \Sigma X$ is nullhomotopic.

Problem 8.

Let X be a pointed CW complex. The infinite suspension of X is given by,

$$\Sigma^\infty(X) = \bigcup_{i=1}^{\infty} \Sigma^i(X)$$

By problem 7, the inclusion maps in the following chain are nullhomotopic,

$$\Sigma X \hookrightarrow \Sigma^2 X \hookrightarrow \Sigma^3 X \hookrightarrow \Sigma^4 X \hookrightarrow \Sigma^5 X \hookrightarrow \Sigma^6 X \hookrightarrow \cdots$$

which, by problem 7, implies that the total complex $\Sigma^\infty(X)$ is contractable.