1 Introduction

The result is interesting for two reasons: first because we can do it for all rational an elliptic surfaces rather than those having $-K_X$ nef and seconly because X is covered by unirational surfaces. This is another indication of the strong distinction between rational and unirational surfaces in positive characteristic.

2 Clemens Conjecture

Lemma 2.0.1. Let $f : S \to X$ be a morphism from a smooth birationally ruled surface to a smooth 3-fold. Suppose $\varphi : \mathcal{L} \hookrightarrow \wedge^2 \Omega_X$ is a line bundle embedded in $\wedge^2 \Omega_X$ and $\mathcal{L}$ has a nonzero section s. Let $\overline{S} = \operatorname{im} f$ then one of the following must hold:

(a) $\overline{S} \subset V(s)$

(b) $f^*(\mathcal{L} \otimes \mathcal{O}_X(\overline{S}))$ intersects non-positively with the general fiber of $S \to C$

(c) $\overline{S} \subset V(\varphi)$

Proof. Suppose (a) does not hold. Because $H^0(S, \omega_S) = 0$ since S is ruled and $f^*\mathcal{L}$ has a nonzero section because we are not in case (a), the composition is zero

$$f^*\mathcal{L} \to f^* \wedge^2 \Omega_X \to \omega_S$$

since ω_S has no sections and $f^*\mathcal{L}$ is big.

Now consider the sequence

$$0 \to \mathcal{C} \to f^*\Omega_X \to \Omega_S$$

Let $\overline{S}$ be the image of S. Then we have a sequence,

$$0 \to \mathcal{C} \to \Omega_X|_{\overline{S}} \to \Omega_{\overline{S}} \to 0$$

and the sequence is left exact because $\overline{S}$ is a prime divisor and hence is Cartier and so $\mathcal{C}$ is a line bundle. Consider the exact sequence

$$0 \to f^*\mathcal{C} \to f^*\Omega_X \twoheadrightarrow \mathscr{F} \subset \Omega_S$$

where $\Omega_S/\mathscr{F}$ has support over the exceptional locus of $S \to \overline{S}$. Then I claim there is a sequence

$$0 \to \mathscr{F} \otimes \mathcal{C} \to \wedge^2 f^*\Omega_X \to \omega_S$$

Indeed, consider the map $f^*\Omega_X \otimes \mathcal{C} \to \wedge^2 f^*\Omega_X$. I claim this surjects onto the kernel. Indeed, if $\alpha \wedge \beta \mapsto 0$ then $\alpha - \lambda\beta$ is in the kernel. Therefore, $\alpha \wedge \beta = (\alpha - \lambda\beta) \wedge \beta$ thus is in the image of the claimed map. Moreover, since $\mathcal{C} \otimes \mathcal{C}$ maps to zero we get a map $\mathscr{F} \otimes \mathcal{C} \to \wedge^2 f^*\Omega_X$. This is injective because $\mathcal{C}$ is a line bundle and $\mathscr{F}$ is torsion-free and rank 1 so we can check injectivity at the generic point.

Therefore, since $f^*\mathcal{L} \to \wedge^2 f^*\Omega_X \to \omega_S$ is zero we get that the map factors through $f^*\mathcal{L} \to \mathscr{F} \otimes \mathcal{C}$. Hence, if the map $f^*\mathcal{L} \to \wedge^2 f^*\Omega_X$ is nonzero then we get an embedding

$$f^*\mathcal{L} \hookrightarrow \Omega_S \otimes f^*\mathcal{C}$$

We need that $f^*(\mathcal{L} \otimes \mathcal{C}^\vee)$ is big since Ω_S cannot contain a big line bundle. Indeed, there is map $S \to C$ whose general fiber is $\mathbb{P}^1$. Then we know $\Omega_S|_F \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ but a big line bundle must restrict positively to the generic fiber. $\square$

If X is singular this might be an issue unless the singularities are not so bad that forms do not extend to the resolution

Note if $S \to X$ hits a singular point of X that needs to be resolved then the modification to the normal bundle is only over exceptional loci of S I think and therefore do not interact with the general fiber of S maybe?? Unless the map contracts something to the singularity which seems very possible.

3 Chang and Ran

let $X \subset \mathbb{P}^4$ be a general quintic hypersurface. Let it be a general hyperplane section of $Y \subset \mathbb{P}^5$ another fixed quintic. Let $S \to X$ be a smooth surface of negative kodiara dimension mapping birationally onto its image in X. There are two cases:

(a) either S fills Y as we move H

(b) S extends to a divisor of Y such that S is a section.

I THINK they show (b) does not occur and when $-K_S$ is nef (a) does not occur either.

3.1 (a)

Consider the sequences,

$$0 \to T_S \to f^*T_X \to N_f \to 0$$

and

$$0 \to N_f \to N_{\tilde{f}} \to L \to 0$$

where $\tilde{f}$ is the composite

$$S \xrightarrow{f} X \hookrightarrow Y$$

and $L = f^*\mathcal{O}(1)$.

Note that the second sequence splits in any neighbrohood of a fiber of f. Let $\tau = (N_f)_{\text{tors}}$ which is supported purely in codimension 1 (because T_S has corank 1 in f^*T_X). Since S fills Y we see that $N_{\tilde{f}}$ is generated generically by global sections. Thus

$$c_1(N_{\tilde{f}}/\tau) = c_1(N_{\tilde{f}}) - c_1(\tau)$$

is nef WHY? maybe I don't know what generically globally generated means in this context?

3.2 (b)

4 Wang 2000

Let X be a non-singular complete intersection of type $(m_1, \ldots, m_k)$ in a Grassmanian $G(r, n+1)$ such that $\dim X \geq 3$ and $m = m_1 + \cdots + m_k \geq n+1$, and supposet $\overline{D} \subset X$ is an irreducible and reduced divisor. Let $f : D \to \overline{D} \subset X$ be a desingularization, ℓ denote the dimension of D and $L = f^*\mathcal{O}_G(1)$. Obviously, L is big and nef. Let K_D be the canonical bundle of D. Let S and Q be the universal subbundle and universal quotient bundle on G.

Proposition 4.0.1. X does not cotain any reduced irreducible divisor which admits a designularization having

$$H^0(K_D \otimes f^*Q^\vee) = 0 \quad \text{and} \quad H^1(K_D - L^{\otimes m_i}) = 0$$

for any all $i = 1, \ldots, k$.

4.1 Reflexive Sheaves

Let $\mathscr{F}^{\vee\vee}$ be the double dual of $\mathscr{F}$. A coherent sheaf $\mathscr{F}$ is reflexive if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism. Define the singularity set of $\mathscr{F}$ to be the locus where $\mathscr{F}$ is not free over the local ring.

It is well-known that the sigularity set of a torsion-free sheaf on D is in codimension ≥ 2. Moreover, the singularity set of a reflexitve sheaf on D is in codimension ≥ 3. It is also well-known that, in general, any reflexive rank 1 sheaf on an integral locally factoral scheme is a line bundle.

4.2 The Proof

Assume such $\overline{D}$ exists. Consider the sequence

$$0 \to Q^\vee \to \mathcal{O}_G^{n+1} \to S^\vee \to 0$$

Pull this back and tensor with f^*Q to get

$$0 \to f^*Q \otimes f^*Q^\vee \to (f^*Q)^{n+1} \to f^*T_G \to 0$$

The top cohomology

$$h^\ell(f^*Q) = h^0(K_D \otimes f^*Q^\vee) = 0$$

vanishes by assumption and hence $H^\ell(f^*T_G) = 0$. Now we pull back the normal bundle sequence of X

$$0 \to f^*T_X \to f^*T_G \to \bigoplus L^{\otimes m_i} \to 0$$

Note that we need the smoothness of X to get the above sequence. Then we have,

$$h^{\ell-1}(L^{\otimes m_i}) = h^1(K_D - L^{\otimes m_i}) = 0$$

also by assumption and hence using this and the above calculation

$$H^\ell(f^*T_X) = 0$$

Next, consider the defining sequence of the normal sheaf

$$0 \to T_D \to f^*T_X \to N_f \to 0$$

with the above three sequences we obtain

$$H^\ell(N_f) = 0$$

and

$$c_1(N_f) = K_D + (n + 1 - m)L$$

where

$$m = m_1 + \cdots + m_k$$

Let $N_f^{\vee\vee}$ be the double dual of N_f which is a line bundle. The image of $N_f \to N_f^{\vee\vee}$ is torsion-free. The singularity set of the image is in codimension ≥ 2 so there is an exact sequence

$$0 \to \tau \to N_f \to N_f^{\vee\vee} \to \phi \to 0$$

with $\dim \mathrm{Supp}\,(\phi) \leq 0$. Devide these into sequences

$$0 \to \tau \to N_f \to \psi \to 0$$

and

$$0 \to \psi \to N_f^{\vee\vee} \to \phi \to 0$$

Then $H^\ell(N_f) = 0$ implies that likewise

$$H^\ell(N_f^{\vee\vee}) = 0$$

because $H^\ell(\phi) = 0$ by dimension reasons. On the other hand, we have

$$c_1(N_f^{\vee\vee}) = K_D + (n+1-m)L - c_1(\tau)$$

Note that $c_1(\tau)$ is always effective. Therefore,

$$h^\ell(N_f^{\vee\vee}) = h^0(K_D - N_f^{\vee\vee}) = h^0((m-n-1)L + c_1(\tau)) > 0$$

which is a contradiction.

4.3 Main Theorem

For $r = 1$ we identify $G(1, n+1) = \mathbb{P}^n$.

Proposition 4.3.1. A nonsingular complete intersection X of type $(m_1, \ldots, m_k)$ in $\mathbb{P}^n$ for $n \geq 4$ such that

$$m = m_1 + \cdots + m_k \geq n+1$$

does not contain a reduced irreducible divisor which admits a desingularization having $H^0(K_D - L) = 0$ and $H^1(K_D - m_i L) = 0$ for all $i = 1, \ldots, k$.

We get thiis immediately if we identify $\mathbb{P}^n$ with $G(n, n+1)$.

Theorem 4.3.2. A non-singular complete intersection X of type $(m_1, \ldots, m_k)$ in $\mathbb{P}^n$ such that $\dim X \geq 3$ and $m = m_1 + \cdots + m_k \geq n+1$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

Proof. If $-K_D$ is nef, $-K_D + L$ and $-K_D + m_i L$ are nef and big. Therefore by Kawamata-Viehweg vanishing we obtain

$$H^0(K_D - L) = 0 \quad H^1(K_D - m_i L) = 0$$

for all i. Note that $\dim D = \dim X - 1 \geq 2$ so we may apply the vanishing results. □

5 Wang's Thesis Filling Result

6 Mori's Construction

Let S be a scheme, $t \in \mathcal{O}_S$. Let $f, g \in \mathcal{O}_S[x_0, \dots, x_n]$ be homogeneous polynomials of degrees cd and d respectively such that $g^c - f$ is not identically zero in $\kappa(s)$ for any $s \in S$. The scheme

$$Z = V(y^c - f, ty - g) \subset \mathbb{P}_S(x_0, \dots, x_n, y) = \mathbb{P}_S(1, \dots, 1, d)$$

defines a family of weighted complete intersections over S. If $s \in S$ and $t(s) \neq 0$ then the fiber Z_s is isomophic to the hypersurface

$$V(g(s)^c - t(s)^c f(s)) \subset \mathbb{P}_{\kappa(s)}(x_0, \dots, x_n)$$

If $t(s) = 0$ then the fiber Z_s is isomorphic to a μ_c-cover of the hypersurface $V(g(s))$ branched over $V(f(s))$.

6.1 μ_c-cyclic covers

For polynomials $f, g \in k[x_0, \dots, x_n]$ as before we consider

$$Z = V(y^c - f, g) \subset \mathbb{P}_k(x_0, \dots, x_n, y) = \mathbb{P}_k(1, \dots, 1, d)$$

which is a μ_c-cover of the hypersurface $V(g)$ branched along $V(f)$.

The map $Z \to V(f) \to \mathbb{P}^n$ is given by the projection $\mathbb{P}(1, \dots, 1, d) \to \mathbb{P}^n$ away from $[0 : \dots : 0 : 1]$ and therefore $\mathcal{O}_Z(1)$ is unambiguous and is an ample line bundle (it is a line bundle since it is pulled back from $\mathcal{O}_{\mathbb{P}^n}(1)$ and ample since it is pulled back from a closed embedding into $\mathbb{P}(1, \dots, 1, d)$).

We are going to be interested in the case $g = x_0$ and f has degree c. In this way we can just drop g and reduce the number of variables and work in an unweighted projective space since $\deg y = d = 1$.

Note that we can always increase the number of variables $k[x_0, \dots, x_n, x_{n+1}]$ and extend to $\tilde{f}, \tilde{g} \in k[x_0, \dots, x_n, x_{n+1}]$ that restrict to f, g when we set $x_{n+1} = 0$. Hence we get extensions of Z in a larger weighted projective space.

6.2 Non-filling case

Let f be a generic polynomial of degree c in $k[x_0, \dots, x_n]$ and suppose that $D \subset Z$ is a uniruled divisor. Then if we extend Z^m for $m \geq n - 1$ to a generic higher degree cyclic cover then there is not for all m an extension of a divisor $D^m \subset Z^m$ such that $D \subset Z$ is a hyperplane slice.

Proof. We need D^m is uniruled. We can get this in two ways:

(a) Mori-Miyaoka (works in all characteristics)

(b) don't prove it in general just take the image of the Hilbert scheme component for slices of Z^m in Z^m this is either a divisor or everything, if its everything then we get sections of $N_{\tilde{f}}$ and everything.

Let $h : \mathbb{P}^1 \to D$ be a rational curve through a general point of D and consider

$$h_m : \iota_m \circ h \quad \text{with } \iota : D \hookrightarrow D^m$$

Consider the sequence

$$0 \to N_{h_{n+1}} \to N_{h_m} \to h^* \mathcal{O}(1)^{\oplus(m-n)} \to 0$$

where the last term is the normal bundle of $D^{n+1} \hookrightarrow D^m$ since D^{n+1} is a linear slice.

Now $N_{h_{n+1}}$ is semipositive HERE WE NEED TO USE THE ARGUMENT OF CLEMENS MAYBE D^{n+1} IS GENERIC (every quotient has nonnegative degree) and hence nonspecial. Ths means a general deformation $\hat{h_m}$ of h_m for $m \gg 0$ will be linearly normal (meaning that $H^0(\mathbb{P}^N, \mathcal{O}(1)) \to H^0(\mathbb{P}^1, \hat{h}_m^* \mathcal{O}(1))$ is surjective) and hence projectively normal because $\hat{h}_m^* \mathcal{O}(1)$ is 0-regular with respect to itself. This is just because for degree less than N a generic rational curve in $\mathbb{P}^N$ is projectively normal (and degenerate) BUT IT IS NOT GENERIC SINCE IT IS IN D^m WHAT IS UP?

Let $L = g_m^* \mathcal{O}(1)$ which is independent of m.

The previous sequence implies that

$$c_1(N_{h_{n+1}} \otimes L^\vee) = c_1(N_{h_m} \otimes L^\vee)$$

is also independent of m. Likewise,

$$c_1(\hat{h}_m^* N_{g_m}(-1)) = c_1(h_m^* N_{g_m}(-1)) = c_1(h_{n+1}^* N_{g_{n+1}}(-1))$$

where $g_m : D^m \hookrightarrow X^m$ is the inclusion, becuase $h_m^* N_{g_m} = h_{m+1}^* N_{g_{m+1}}$ and then

$$c_1(h_{n+1}^* N_{g_{n+1}}(-1)) = \deg h_{n+1}^* K_{D^{n+1}}) \leq -2$$

WHERE DOES THE -2 COME FROM I think the point is that because D^{n+1} is uniruled there should be a nonzerp map

$$h_{n+1}^* K_{D^{n+1}} \to K_{\mathbb{P}^1}$$

NEED SEPARABLY UNIRULED FOR THIS, IS IT TRUE? because

$$0 \to T_{D^{n+1}} \to T_{X^{n+1}}|_{D^{n+1}} \to N_{g_{n+1}} \to 0$$

and $\det T_{X^{n+1}} = \mathcal{O}(1)$ so $\det N_{g_{n+1}}(-1) = K_{D^{n+1}}$.

Now consider the sequence

$$0 \to N_{\hat{h}_m} \to N_{g_m \circ \hat{h}_m} \to \hat{h}_m^* N_{g_r} \to 0$$

From above

$$H^1(\hat{h}_m^* N_{g_m}(-1)) \neq 0$$

and therefore

$$H^1(N_{g_m \circ h_m}(-1)) \neq 0$$

this gives a contradiction from the following lemma.

Lemma 6.2.1. Let $r : \mathbb{P}^1 \to X^n \subset \mathbb{P}^{n+1}$ be a projectively normal rational curve on a smooth hypersuface. Then there exists an extension $X^m \supset X^n$ in $\mathbb{P}^{m+1}$ such that the map $r_m : \mathbb{P}^1 \to X^m$ has $H^1(N_{r_m}(-1)) = 0$.

Proof. Consider a potential extension $X^m \supset X^n$ defined by $y^c - F$ and $j : X^m \to \mathbb{P}^{m+1}$ the inclusion. Then we have an exact sequence

$$0 \to N_{r_m}(-1) \to N_{j \circ r_m}(-1) \to r_m^* \mathcal{O}(c-1) \to 0$$

and the natural map

$$\delta : H^0(\mathcal{T}_{\mathbb{P}^{m+1}}(-1)) \to H^0(N_{j \circ r_m}(-1)) \to H^0(r_m^* \mathcal{O}(c-1))$$

is given by

$$\frac{\partial}{\partial x_i} \mapsto \frac{\partial F}{\partial x_i} \quad \frac{\partial}{\partial y} \mapsto 0$$

Since $N_{j \circ r_m}(-1)$ is semipositive (it is a quotient of $\mathcal{T}_{\mathbb{P}^{m+1}}(-1)$) it has vanishng H^1 and hence $H^1(N_{r_m}(-1)) = 0$ if and only if δ is surjective. SHIT MEGA SHIT ☐

CAN WE DO IT WITH GENERIC CYCLIC COVERS RATHER THAN GENERIC HYPER-SURFACES?

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