1 Kodaira Dimension

Definition 1.0.1. Let X be a smooth projective variety over k with canonical bundle ω_X. Then we define the *plurigenera* of X to be,

$$p_n(X) = \dim_k H^0(X, \omega_X^{\otimes n})$$

Furthermore, we define the *Kodaira* dimension $\kappa(X)$ as the minimal integer d such that the plurigenera satisfy $p_n(X) \in O(n^d)$ and $\kappa(X) = -\infty$ if $p_n(X) = 0$ for all $n > 0$.

Definition 1.0.2. We say that a variety is *of general type* if $\kappa(X) = \dim X$.

Proposition 1.0.3. For smooth projective curves X over k of genus g we have,

$$p_\ell(X) = \begin{cases} 0 & g = 0 \\ 1 & g = 1 \\ g & \ell = 1 \\ (2\ell - 1)(g-1) & g \ge 2, \ell > 1 \end{cases}$$

and therefore,

$$\kappa(X) = \begin{cases} -\infty & g = 0 \\ 0 & g = 1 \\ 1 & g \ge 2 \end{cases}$$

Proof. For $g = 0$ we know $\deg \omega_X^{\otimes \ell} = -\ell < 0$ so $H^0(X, \omega_X^{\otimes \ell}) = 0$ and thus $\kappa(X) = -\infty$. For $g = 1$ we know $\omega_X = \mathcal{O}_X$ and thus $p_\ell(X) = 1$ for all ℓ so $\kappa(X) = 0$.

Now consider $g \ge 2$. For any $\mathcal{L} \in \mathrm{Pic}\,(X)$ we know that $H^0(X, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$ so if $\deg \mathcal{L} > 2g - 2$ then $H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee) = 0$ and thus by Riemann-Roch,

$$\dim_k H^0(X, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

In particular, for $\mathcal{L} = \omega_X^{\otimes \ell}$ we have $\deg \omega_X^{\otimes \ell} = (2g-2)\ell$ so for $g \ge 2$ and $\ell > 1$ we have,

$$p_\ell(X) = \dim_k H^0(X, \omega_X^{\otimes \ell}) = (2g-2)\ell + (1-g) = (2\ell - 1)(g-1)$$

Also, for $\ell = 1$ we get $H^0(X, \omega_X) = g$. Therefore, $p_\ell(X) \sim \ell$ so $\kappa(X) = 1$. □

Remark. In particular, a curve is general type iff $g \ge 2$.

Proposition 1.0.4. Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface of degree d. Then,

$$\kappa(X) = \begin{cases} -\infty & d < n+1 \\ 0 & d = n+1 \\ n-1 & d > n+1 \end{cases}$$

Therefore, X is of general type iff $d > n+1$.

Proof. Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface of degree d. Then $\omega_X = \mathcal{O}_X(d-n-1)$. Thus,

$$\omega_X^{\otimes \ell} = \mathcal{O}_X((d-n-1)\ell)$$

There is an exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

then twisting we get,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(d(\ell-1)-\ell(n+1)) \longrightarrow \mathcal{O}_{\mathbb{P}}((d-n-1)\ell)) \longrightarrow \omega_X^{\otimes \ell} \longrightarrow 0$$

Then we can compute,

$$p_\ell(X) = H^0(X, \omega_X^{\otimes \ell})$$

from the long exact sequence,

$$H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}}(a)) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}}(b)) \longrightarrow H^0(X, \omega_X^{\otimes \ell}) \longrightarrow H^1(X, \mathcal{O}_{\mathbb{P}}(a))$$

where $a = d(\ell-1)-(n+1)\ell$ and $b = (d-n-1)\ell$. For the case $n > 2$ we get vanishing of H^1 for line bundles in general. In the case $n = 2$ we can apply our results for curves to conclude. Now,

$$p_\ell(X) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(b)) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(a)) = \binom{(d-n-1)\ell+n}{n} - \binom{(d-n-1)\ell+n-d}{n}$$

Therefore, we have three cases depending on the sign of $d-n-1$. If $d = n+1$ then $\omega_X \cong \mathcal{O}_X$ is the trivial bundle so $p_\ell(X) = 1$ and thus $\kappa(X) = 0$. If $d < n+1$ then $p_\ell(X) = 0$ for $\ell > 0$ and thus $\kappa(X) = -\infty$. If $d > n+1$ then $p_\ell(X)$ is a degree $n-1$ polynomial in ℓ so $\kappa(X) = \dim X = n-1$. $\square$

2 Weighted Projective Spaces

Definition 2.0.1. For an $(r+2)$-tuple $(q_0, \ldots, q_{r+1})$ we define the *weighted projective space* over k,

$$\mathbb{P}_k(q_0, \ldots, q_{r+1}) = \operatorname{Proj}\left(k[x_0, \ldots, x_{r+1}]\right)$$

where the ring $R = k[x_0, \ldots, x_{r+1}]$ is graded with $\deg x_i = q_i$. Clearly, $\mathbb{P}_k(dq_0, \ldots, q_{r+1}) \cong \mathbb{P}_k(q_0, \ldots, q_{r+1})$ by scaling degrees. Thus we may assume that the ideal $(q_0, \ldots, q_{r+1}) = \mathbb{Z}$.

2.1 Line Bundles and Divisors

2.2 Toric Construction

Proposition 2.2.1. The weighted projective space $\mathbb{P}_k(q_0, \ldots, q_{r+1})$ is a toric variety with torus,

$$D_+(x_0 \cdots x_{r+1}) = \operatorname{Spec}\left(k[y_1^{\pm 1}, \ldots, y_{r+1}^{\pm 1}]\right)$$

where y_i is a monomial in $x_j^{\pm 1}$.

Proof. We know that,

$$D_+(x_0 \cdots x_{r+1}) = \operatorname{Spec}\left(k[x_0, \ldots, x_{r+1}]\left[\frac{1}{x_0 \cdots x_{r+1}}\right]_0\right)$$

Then monomials in,

$$\tilde{R} = k[x_0, \ldots, x_{r+1}]\left[\frac{1}{x_0 \cdots x_{r+1}}\right]_0$$

are of the form,

$$y = \prod_{i=0}^{r+1} x_i^{a_i}$$

for some $(r+2)$-tuple $a_i \in \mathbb{Z}$ such that,

$$\deg y = \sum_{i=0}^{r+1} q_i a_i = 0$$

Thus, the allowed monomials are given by $(r+2)$-tuples in $K \subset \mathbb{Z}^{r+2}$ the kernel of $\mathbb{Z}^{r+2} \to \mathbb{Z}$. There is an exact sequence,

$$0 \longrightarrow K \hookrightarrow \mathbb{Z}^{r+2} \twoheadrightarrow \mathbb{Z} \longrightarrow 0$$

where the map $\mathbb{Z}^{r+2} \twoheadrightarrow \mathbb{Z}$ via $(q_i) \mapsto \sum q_i a_i$ is surjective since the ideal $(q_0, \dots, q_{r+1}) = \mathbb{Z}$. Since $\mathbb{Z}$ is projective, this sequence splits so,

$$\mathbb{Z}^{r+2} = K \oplus \mathbb{Z}$$

and thus K is free of rank $r+1$ which implies that there are generators y_i for $1 \le i \le r+1$ such that $\tilde{R} = k[y_1^{\pm 1}, \dots, y_{r+1}^{\pm 1}]$. $\square$

Now we consider how weighted projective space may be constructed from combinatorial toric fan or polytope data.

2.3 Weighted Hypersurfaces