Mathematics GU4051 Topology
Assignment # 6

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Problem 1.

(a). Let $\mathcal{S} \subset \mathcal{T}$ be topologies on a set X. Suppose that X is compact under $\mathcal{T}$. Now let

$$\{U_\lambda \in \mathcal{S} \mid \lambda \in \Lambda\}$$

be an open cover of X consisting of sets of $\mathcal{S}$. Then, each $U_\lambda \in \mathcal{S} \subset \mathcal{T}$ so this is an open cover of X under the topology $\mathcal{T}$. Therefore, by compactness, there exists an open subcover consisting of $\{U_\lambda \mid \lambda \in \Lambda\}$ which are open sets in $\mathcal{S}$ by definition. Thus, X is compact under $\mathcal{S}$. So compactness under $\mathcal{T}$ implies compactness under $\mathcal{S}$. However, the converse is false because the indiscrete topology is a subset of any topology and under the indiscrete topology any set is compact. However, we can choose X to be noncompact under the origional topology.

(b). I believe that I misinterpreted this question. I intially thought that $\mathcal{S}$ and $\mathcal{T}$ were *any* two topologies not the two related ones above. However, after working on that problem for many hours I realized that, in this case, the statment is false. To not feel that I wasted by life, I will both solve the indended problem and give a counterexample I constructed to the more general case.

First, the real question. Let $(X, \mathcal{S})$ and $(X, \mathcal{T})$ be compact Hausdorff spaces with $\mathcal{S} \subset \mathcal{T}$. Consider the identity map $\iota : (X, \mathcal{T}) \to (X, \mathcal{S})$ which takes $\iota : x \mapsto x$. This function is continuous because if $U \in \mathcal{S}$ then $i^{-1}(U) = U \in \mathcal{S} \subset \mathcal{T}$. Thus, ι is a continuous map from a compact space to a Hausdorff space and therefore, for closed $C \subset X$ under $\mathcal{T}$, $\iota(C) = C$ is closed under $\mathcal{S}$. Thus, if $U \in \mathcal{T}$ then $X \setminus U$ is closed in $\mathcal{T}$ and then by above also in $\mathcal{S}$ so $X \setminus (X \setminus U) = U$ is open in $\mathcal{S}$. Therefore, $\mathcal{T} \subset \mathcal{S}$ so $\mathcal{S} = \mathcal{T}$. This can be stated more elegantly as: continuous bijections from compact to Hausdorff are homeomorphisms and thus ι is a homeomorphism so $\mathcal{S} = \mathcal{T}$.

Now for a counterexample. To get inequivalent Hausdorff spaces even without considering compactness, we must consider an infinite set because all Hausdorff spaces are T_1 and thus discrete on a finite set. Let $X = [0, 1]$ and $\mathcal{S}$ be the standard topology on X which we know is compact and metrizable so Hausdorff. Now let,

$$\mathcal{T} = \{U \subset X \mid 1 \notin U \text{ or } (1 \in U \text{ and } X \setminus U \text{ is finite})\}$$

This can be summed up as the discrete topology on $[0, 1)$ plus the indiscrete topology restricted to each open set containing 1. We have to show three things: $\mathcal{T}$ is a topology, $(X, \mathcal{T})$ is

Hausdorff, and $(X, \mathcal{T})$ is compact. First, $\varnothing, X \in \mathcal{T}$ because $1 \notin \varnothing$ and $1 \in X$ and $X \backslash X$ is finite. Now, take a collection of open sets $\{U_\lambda \in \mathcal{T} \mid \lambda \in \Lambda\}$. If 1 is not an element of any U_λ then their union still does not contain 1 so it is open. If for some λ_0, $1 \in U_{\lambda_0}$ then the union contains U_{λ_0} so it contains 1 and has a complement smaller than U_{λ_0} which is already finite so the union is also open. Suppose the collection is finite. If some U_λ does not contain 1 then the intersection also does not and thus is open. Else, 1 is in every U_λ so each $X \backslash U_\lambda$ is finite. However,

$$X \setminus \bigcap_{\lambda \in \Lambda} U_\lambda = \bigcup_{\lambda \in \Lambda} X \backslash U_\lambda$$

which is a finite union of finite sets and therefore, finite. Thus, the intersection contains 1 and has finite complement and so is open. Next, we must show that $(X, \mathcal{T})$ is Hausdorff. Let $x \neq y$, so WLOG $x \neq 1$. Then, take $U = \{x\}$ and $V = X \backslash \{x\}$. U an V are disjoint and $x \in U$ and $y \in V$ because $x \neq y$. However, $1 \notin U$ so U is open and $X \backslash V = \{x\}$ is finite with $1 \in V$ so V is open. Therefore, $(X, \mathcal{T})$ is Hausdorff. Finally, take any open cover of X, $\{U_\lambda \in \mathcal{T} \mid \lambda \in \Lambda\}$. Now, $1 \in X$ so, because this collection is a cover, there is some $\lambda_0 \in \Lambda$ s.t. $1 \in U_{\lambda_0}$ and thus, $X \backslash U_{\lambda_0}$ is finite because U_{λ_0} is open. For each $x \in X \backslash U_{\lambda_0}$, $\exists \lambda_x \in \Lambda$ s.t. $x \in U_{\lambda_x}$. Thus,

$$X = U_{\lambda_0} \cup \bigcup_{x \in X \backslash U_{\lambda_0}} U_{\lambda_x}$$

which is a finite union so we have found a finite subcover. Therefore, $(X, \mathcal{T})$ is compact and Hausdoff but $\{0\}$ is open with respect to $\mathcal{T}$ so this is not the same as the standard topology. Note, this consitruction is equivalent to the one-point compactification of $[0, 1)$ with the discrete topology.

Problem 2.

(a). Let $\mathcal{T}$ be the cofinite topology on X. If $X = \varnothing$ then it is trivially compact (there is only one open set). Take any open cover of X, $\{U_\lambda \in \mathcal{T} \mid \lambda \in \Lambda\}$. Now, $\exists x_0 \in X$ so, because this collection is a cover, there is some $\lambda_0 \in \Lambda$ s.t. $x_0 \in U_{\lambda_0}$ and thus, $X \backslash U_{\lambda_0}$ is finite because U_{λ_0} is open and non-empty. For each $x \in X \backslash U_{\lambda_0}$, since the collection is a cover, $\exists \lambda_x \in \Lambda : x \in U_{\lambda_x}$ and thus,

$$X = U_{\lambda_0} \cup \bigcup_{x \in X \backslash U_{\lambda_0}} U_{\lambda_x}$$

which is a finite union so we have found a finite subcover. Therefore, $(X, \mathcal{T})$ is compact.

(b). Let $\mathbb{R}$ have the cocountable topology $\mathcal{T}$. Now consider the collection open sets

$$U_z = \mathbb{R} \setminus (\mathbb{Z} \setminus \{z\})$$

which are indexed by $\mathbb{Z}$. Each of these sets is indeed open because $\mathbb{Z} \setminus \{z\}$ is countable so $\mathbb{R} \setminus (\mathbb{Z} \setminus \{z\})$ is cocountably open. Now, $U_z = (\mathbb{R} \setminus \mathbb{Z}) \cup \{z\}$ so

$$\bigcup_{z \in \mathbb{Z}} U_z = (\mathbb{R} \setminus \mathbb{Z}) \cup \bigcup_{z \in \mathbb{Z}} \{z\} = (\mathbb{R} \setminus \mathbb{Z}) \cup \mathbb{Z} = \mathbb{R}$$

and thus $\{U_z \mid z \in \mathbb{Z}\}$ is a open cover of $\mathbb{R}$. However, take any finite subcover indexed by a finite set $S \subset \mathbb{Z}$. Then let $m = \max S + 1$ which exists because S is finite. Thus, $m \notin S$ so

$U_m \notin \{U_z \mid z \in S\}$ but m is an integer so $m \in U_z$ only if $z = m$ so $m \notin \bigcup_{z \in S} U_z$. Therefore, no finite subset of $\{U_z \mid z \in \mathbb{Z}\}$ can cover $\mathbb{R}$ so $\mathbb{R}$ is noncompact.

Problem 3.

Let $\{A_n \mid n \in \mathbb{N}\}$ be a collection of compact, connected subsets of X which is a Hausdorff space such that $A_n \supset A_{n+1}$. Now, consider the set

$$A = \bigcap_{n \in \mathbb{N}} A_n$$

(a). If some A_k were empty then $A \subset A_k = \varnothing$ so A is empty. Suppose that A is empty. Define $K_n = X \setminus A_n$. Since X is Hausdorff and A_n is compact, A_n must be closed so K_n is open. However,

$$\bigcup_{n \in \mathbb{N}} K_n = X \setminus \bigcap_{n \in \mathbb{N}} A_n = X \setminus A = X \supset A_0$$

because $A = \varnothing$ by hypothesis. Therefore, $\{K_n \mid n \in \mathbb{N}\}$ is an open cover of A_0 but A_0 is compact so there is a finite subcover indexed by $S \subset \mathbb{N}$ such that

$$\bigcup_{n \in S} K_n = X \setminus \bigcap_{n \in S} A_n \supset A_0$$

therefore, $A_0 \cap \bigcap_{n \in S} A_n = \varnothing$ because $x \in A_0 \implies x \in X \setminus \bigcap_{n \in S} A_n \implies x \notin \bigcap_{n \in S} A_n$. However, S is finite so $m = \max S$ exists. For any $n \le m$ we have $A_n \supset A_m$ which follows by induction from the property $A_n \supset A_{n+1}$. Thus, $A_m \subset A_0 \cap \bigcap_{n \in S} A_n = \varnothing$ because A_m is contained in every set in the intersection. Thus, $A_m = \varnothing$ and therefore, A is empty iff some A_m is empty.

(b). Each A_n is closed because it is a compact subset of a Hausdorff space and therefore, A is closed because it is the arbitrary intersection of closed sets. However, $A \subset A_0$ and A_0 is compact so A being closed must also be compact.

(c). Suppose that A is disconnected, then $A = C \cup D$ with C and D nonempty disjoint clopen (in A) sets. Since A is compact and $C, D \subset A$ are closed, then C and D are compact. However, X is a Hausdorff space so there are disjoint open sets U, V such that $C \subset U$ and $D \subset V$. Now if $A_n \subset U \cup V$ then because U and V are open and disjoint, A_n would be disconnected as long as $A_n \cap U \neq \varnothing$ which holds because $C \subset A \subset A_n$ is not empty and $C \subset U$. Likewise, $A_n \cap D \neq \varnothing$. Thus, $A_n \setminus (U \cup V) \neq \varnothing$ and the set is compact because $A_n \setminus (U \cup V)$ is closed in A_n (since $U \cup V$ is open) which is compact. Furthermore, $A_n \supset A_{n+1} \implies A_n \setminus (U \cup V) \supset A_{n+1} \setminus (U \cup V)$ so by part (a),

$$\bigcap_{n \in \mathbb{N}} (A_n \setminus (U \cup V)) = \left(\bigcap_{n \in \mathbb{N}} A_n \right) \setminus (U \cup V) = A \setminus (U \cup V)$$

is non empty. However, $A = C \cup D \subset U \cup V$ which is a contradiction. Thus, A is connected.