Mathematics 257B Symplectic Geometry
Assignment # 2

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1 Problem 1

1.1 Chern Classes are Determined by Connected Component of the Almost Complex Structure

Let M be a smooth manifold of even dimension which admits an almost complex structure (for example if M is symplectic). I claim that for any smooth path J_t of almost complex structures, the Chern classes,

$$c_k(TM) \in H^{2k}(M, \mathbb{Z})$$

are constant. Indeed, the complex vector bundles (TM, J_0) and (TM, J_1) are always isomorphic. To see this, notice that such a path J_t defines a complex structure on the vector bundle $\pi_1^* TM$ on $M \times \mathbb{R}$ whose action on $T_p M$ at (p, t) is,

$$(a + ib) \cdot v = av + bJ_t v$$

By Proposition 1.7 of Hatcher's Vector Bundles this implies that on the sections $M \times \{0\}$ and $M \times \{1\}$ the complex vector bundles are isomorphic proving the claim.

We can rephrase this in terms of classifying spaces. The path of almost complex structures (TM, J_t) defines a homotopy of classifying maps,

$$f_t : M \to \mathrm{BGL}_n(\mathbb{C})$$

between the classifying maps f_0 and f_1 of the complex vector bundles (TM, J_0) and (TM, J_1) and hence these define isomorphic vector bundles. This proof is equivalent because f_t is just the classifying map of $(\pi_1^* TM, J)$,

$$f : M \times \mathbb{R} \to \mathrm{BGL}_n(\mathbb{C})$$

1.2 Invariance Under Choice of Tamed Structure

Now we show that if (M, ω) is symplectic then $c_k(TM)$ is independent of the choice of tamed almost complex structure J. Indeed, the space of tamed structures is contractible and hence path connected so this follows immediately from our previous result.

1.3 Invariance Under Symplectic Deformation

Let ω_t be a symplectic deformation on M. By the previous discussion, to conclude that $c_k(TM)$ are independent of t it suffices to show there exists a path J_t of almost complex structures which are tamed for ω_t. This follows from continuity in the polar decomposition.

2 Problem 2

(a) Let $f_n(z) = [z^2, z, \frac{1}{n}]$. The limit $n \to \infty$ is not-well-defined at $z = 0$ and thus we need to catch a bubble. Rescale to let $w = nz$ then,

$$f_n(w) = [\tfrac{1}{n^2}w^2, \tfrac{1}{n}w, \tfrac{1}{n}] = [\tfrac{1}{n}w^2, w, 1]$$

which is not well-defined at $w = \infty$ in the limit. Therefore, we get a limit consisting of two degree one maps $f_\infty(z) = [z, 1, 0]$ and $f_\infty(w) = [0, w, 1]$ which glue at $z = 0$ and $w = \infty$.

(b) Let,

$$f_n(z) = [z(z - \tfrac{1}{n}), z, \tfrac{1}{n}]$$

The limit $n \to \infty$ is not-well-defined at $z = 0$ and thus we need to catch a bubble. Rescale to let $w = nz$ then,

$$f_n(w) = [\tfrac{1}{n^2}w(w-1), \tfrac{1}{n}w, \tfrac{1}{n}] = [\tfrac{1}{n}z(z-1), z, 1]$$

which is not well-defined at $z = \infty$ in the limit. Therefore, we get a limit consisting of two degree one maps $f_\infty(z) = [z, 1, 0]$ and $f_\infty(w) = [0, w, 1]$ which glue at $z = 0$ and $w = \infty$.

(c) Let,

$$f_n(z) = [z^2 - \tfrac{1}{n^2}, z - \tfrac{1}{n^2}, \tfrac{1}{n}]$$

The limit $n \to \infty$ is not-well-defined at $z = 0$ and thus we need to catch a bubble. Rescale to let $w = nz$ then,

$$f_n(w) = [\tfrac{1}{n^2}(z^2 - 1), \tfrac{1}{n}z - \tfrac{1}{n^2}, \tfrac{1}{n}] = [\tfrac{1}{n}(z^2 - 1), z - \tfrac{1}{n}, 1]$$

which is not well-defined at $z = \infty$ in the limit. Therefore, we get a limit consisting of two degree one maps $f_\infty(z) = [z, 1, 0]$ and $f_\infty(w) = [0, w, 1]$ which glue at $z = 0$ and $w = \infty$.

3 Problem 3

Let $\mathcal{M}_{g,n}$ be the Deligne-Mumford moduli space of stable genus g curves with n marked points.

(a) Let $x_0, x_1 \in \Sigma$ be two points. I claim there exists a disk $D \subset \Sigma$ containing $x_0, x_1 \in D^\circ$ in the interior. Given this it is always possible to find a homeomorphism (even a diffeomorphism!) $D \to D$ which fixes the boundary sending $x_0 \mapsto x_1$ by using bump functions. This gives a homeomorphism $\Sigma \to \Sigma$ sending $x_0 \mapsto x_1$. If $g = 0$ then $\Sigma = S^2$ so removing a point not equal to x_0 or x_1 gives the required disk. Otherwise, choose a basis of homology cycles on Σ not intersecting x_0 and x_1 and cutting along these Σ is homoeomorphic to a $4g$-sided polygon which is convex and hence x_0 and x_1 are contained in some common disk.

However, if Σ has a node then no homeomorphism can take a node to a non-node since these have topologically distinct neighborhoods (a node is not locally euclidean).

(b) We need to show that any pair of genus g surfaces Σ with n marked points $(\Sigma, x_1, \dots, x_n)$ are homeomorphic. The same argument as previously reduces to the case of n distinct points $x_1, \dots, x_n \in D^\circ$ in the interior of a disk. These points may be moved arbitrarily while fixing the boundary. I will draw the types on another page.

(c) Consider the graph G whose vertices are the irreducible components and whose edges correspond to nodes. This graph has nodes labeled by their genus g. The number of cycles is,

$$\#\text{cycles} = \#E - \#V + 1$$

and $E = N$ is the set of nodes and $V = C$ is the set of components so the genus becomes,

$$g(G) = \sum_{c \in C} g_c + \#N - \#C + 1$$

Now let $C = C_0 \sqcup C_1 \sqcup C_{\ge 2}$ be the components of genus $g = 0$ and $g = 1$ and $g \ge 2$ respectively. Thus,

$$g(G) = \sum_{c \in C_{\ge 2}} (g_c - 1) - \#C_0 + \#N + 1$$

Furthermore, the stability condition says that each genus 0 component has at least three marked points or nodes and each genus 1 component at least 1 meaning,

$$3\#C_0 + \#C_1 \le 2\#N + n$$

because each node may count on two components or twice if it is a self-intersection but each marked point lies on exactly one irreducible component (since it is required to be a nonsingular point). Therefore,

$$\sum_{c \in C_{\ge 2}} 3(g_c - 1) - 3(g - 1) + 3\#N = 3\#C_0 \le 2\#N + n - \#C_1$$

which implies that,

$$\#N + \#C_1 + \sum_{c \in C_{\ge 2}} 3(g_c - 1) \le 3g - 3 + n = \dim \mathcal{M}_{g,n}$$

In particular, since all numbers on the right hand side are non-negative,

$$\#N \le \dim \mathcal{M}_{g,n}$$

and I claim that equality is possible. For the cases in question, I gave explicit topological types with $\dim \mathcal{M}_{g,n}$ nodes. Furthermore,

$$3\#C_0 + \#C_1 \le 2\#N + n = 2g - 2 + n - \sum_{c \in C_{\ge 2}} 2(g_c - 2) + 2\#C_0$$

and therefore,

$$\#C_0 + \#C_1 + \sum_{c \in C_{\ge 2}} 2(g_c - 1) \le 2g - 2 + n$$