Math GR6262 Algebraic Geometry
Final Project:
Group Schemes and Vector Bundles

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1 Basic Definitions and Examples

Definition 1.0.1. Let $\mathcal{C}$ be a category with all finite products (including the empty product which is the terminal object 1). Then a group object is a tuple (G, m, e, i) where $G \in \mathcal{C}$ is an object and $m : G \times G \to G$, $e : 1 \to G$, and $i : G \to G$ are morphisms such that the diagrams commute,

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

giving associativity,

$$\begin{tikzcd} G \arrow[r, "\mathrm{id} \times e"] \arrow[d, "e \times \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

giving identity,

$$\begin{tikzcd} G \arrow[r, "(\mathrm{id} \times i) \circ \Delta"] \arrow[d, "(i \times \mathrm{id}) \circ \Delta"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

giving inverses. A morphism of group objects G to G' is a morphism $f : G \to G'$ such that the diagram commutes,

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] \\ G' \times G' \arrow[r, "m'"'] & G' \end{tikzcd}$$

Definition 1.0.2. Let $\mathcal{C}$ be a category with finite products and G a group object in $\mathcal{C}$. Then for $X \in \mathcal{C}$ an action of G on X is a morphism $\rho : G \times X \to X$ such that the following diagrams commute,

$$\begin{tikzcd} G \times G \times X \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\rho"'] & G \times X \arrow[d, "\rho"] \\ G \times X \arrow[r, "\rho"'] & X \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & G \times X \arrow[d, "\rho"] \\ & X \end{tikzcd}$$

In this case we call X a G-object. A morphism of G-objects is a morphism $f : X \to Y$ which is a G-intertwiner i.e. the following diagram commutes,

$$\begin{tikzcd} G \times X \arrow[r, "\rho_X"] \arrow[d, "\mathrm{id}\times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\rho_Y"'] & Y \end{tikzcd}$$

Definition 1.0.3. Let S be a scheme. A group scheme over S is a group object in the category of schemes over S. If a group scheme G acts on a scheme X then we say X is a G-scheme.

Example 1.0.4. The additive group scheme $\mathbb{G}_a$ is the scheme $\mathrm{Spec}\,(\mathbb{Z}[x])$ with operation,

$$\begin{gathered} \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a \\ \mathrm{Spec}\,(\mathbb{Z}[x] \otimes \mathbb{Z}[x]) \to \mathrm{Spec}\,(\mathbb{Z}[x]) \\ \mathbb{Z}[x] \otimes \mathbb{Z}[x] \leftarrow \mathbb{Z}[x] \\ x \otimes 1 + 1 \otimes x \mapsfrom x \end{gathered}$$

We should check that this is actually a group scheme. The identity is the natural map induced by the quotient $\mathbb{Z}[x] \to \mathbb{Z}$ and inverses are given by $\mathbb{Z}[x] \to \mathbb{Z}[x]$ sending $x \mapsto -x$. Then the following diagram commutes,

$$\begin{tikzcd} \mathbb{Z}[x] \otimes \mathbb{Z}[x] \otimes \mathbb{Z}[x] & \mathbb{Z}[x] \otimes \mathbb{Z}[x] \arrow[l, "\mathrm{id}\otimes m"'] \\ \mathbb{Z}[x] \otimes \mathbb{Z}[x] \arrow[u, "m\otimes\mathrm{id}"] & \mathbb{Z}[x] \arrow[l, "m"] \arrow[u, "m"'] \end{tikzcd}$$

because under the two directions,

$$
\begin{aligned}
x &\mapsto (x\otimes 1 + 1\otimes x) \mapsto (x\otimes 1\otimes 1 + 1\otimes(x\otimes 1 + 1\otimes x)) = x\otimes 1\otimes 1 + 1\otimes x\otimes 1 + 1\otimes 1\otimes x \\
x &\mapsto (x\otimes 1 + 1\otimes x) \mapsto ((x\otimes 1 + 1\otimes x)\otimes 1 + 1\otimes 1\otimes x) = x\otimes 1\otimes 1 + 1\otimes x\otimes 1 + 1\otimes 1\otimes x
\end{aligned}
$$

Furthermore, the diagram commutes,

$$
\begin{tikzcd}
\mathbb{Z}[x] & \mathbb{Z}[x]\otimes\mathbb{Z}[x] \arrow[l, "\Delta\circ(\mathrm{id}\otimes e)"'] \\
\mathbb{Z}[x]\otimes\mathbb{Z}[x] \arrow[u, "\Delta\circ(e\otimes\mathrm{id})"] & \mathbb{Z}[x] \arrow[l, "m"] \arrow[u, "m"'] \arrow[ul, "\mathrm{id}"]
\end{tikzcd}
$$

because under the two directions,

$$
\begin{aligned}
x &\mapsto (x\otimes 1 + 1\otimes x) \mapsto \Delta(x\otimes 1) = x \\
x &\mapsto (x\otimes 1 + 1\otimes x) \mapsto \Delta(1\otimes x) = x
\end{aligned}
$$

Finally, the diagram commutes,

$$
\begin{tikzcd}
\mathbb{Z}[x] & \mathbb{Z}[x]\otimes\mathbb{Z}[x] \arrow[l, "\Delta\circ(\mathrm{id}\otimes i)"'] \\
\mathbb{Z}[x]\otimes\mathbb{Z}[x] \arrow[u, "\Delta\circ(i\otimes\mathrm{id})"] & \mathbb{Z}[x] \arrow[l, "m"] \arrow[u, "m"'] \arrow[ul, "e"]
\end{tikzcd}
$$

because under the two directions,

$$
\begin{aligned}
x &\mapsto (x\otimes 1 + 1\otimes x) \mapsto \Delta(x\otimes 1 - 1\otimes x) = 0 \\
x &\mapsto (x\otimes 1 + 1\otimes x) \mapsto \Delta(-x\otimes 1 + 1\otimes x) = 0
\end{aligned}
$$

Example 1.0.5. The multiplicative group scheme $\mathbb{G}_m$ is the scheme $\operatorname{Spec}(\mathbb{Z}[x,x^{-1}])$ with multiplication

$$
\begin{gathered}
\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \\
\operatorname{Spec}\left(\mathbb{Z}[x,x^{-1}]\otimes\mathbb{Z}[x,x^{-1}]\right) \to \operatorname{Spec}\left(\mathbb{Z}[x,x^{-1}]\right) \\
\mathbb{Z}[x,x^{-1}]\otimes\mathbb{Z}[x,x^{-1}] \leftarrow \mathbb{Z}[x,x^{-1}] \\
x\otimes x \leftarrow\!\shortmid x
\end{gathered}
$$

and inverse induced by the map $\mathbb{Z}[x,x^{-1}] \to \mathbb{Z}[x,x^{-1}]$ sending $x \mapsto x^{-1}$.

Example 1.0.6. There is an action $\mathbb{G}_m^k$ on $\mathbb{A}_k^n$ via the ring map,

$$
\begin{gathered}
\mathbb{G}_m^k \times \mathbb{A}_k^n \to \mathbb{A}_k^n \\
k[z,z^{-1}]\otimes k[x_1,\dots,x_n] \leftarrow k[x_1,\dots,x_n] \\
z\otimes x \leftarrow\!\shortmid x
\end{gathered}
$$

This is the scaling action $\lambda\cdot(a_1,\dots,a_n) = (\lambda a_1,\dots,\lambda a_n)$.

Lemma 1.0.7. The base change of a group scheme is a group scheme.

Proof. Base change is a limit which commutes with limits (in particular finite products). It is clear that any functor preserving products preserves group objects. ☐

Lemma 1.0.8. If G is a group scheme over S and X is a scheme over S then the X-points of G i.e. the set $G(X) = \mathrm{Hom}_S(X, G)$ is naturally a group.

Proof. The functor $\mathrm{Hom}_S(X, -) : \mathbf{Sch}_S \to \mathbf{Set}$ is continuous, thus preserves products, and thus preserves group objects. Therefore, $\mathrm{Hom}_S(X, G)$ is a group object in **Set** which is a group. ☐

Definition 1.0.9. The additive and multiplicative group schemes in the category of schemes over S are $\mathbb{G}_a^S = \mathbb{G}_a \times S$ and $\mathbb{G}_m^S = \mathbb{G}_m \times S$ respectively.

Example 1.0.10. Let k be an algebraically closed field and consider the group schemes $\mathbb{G}_a = \mathrm{Spec}\,(k[x])$ and $\mathbb{G}_m = \mathrm{Spec}\,(k[x, x^{-1}])$ over $\mathrm{Spec}\,(k)$. Then, as abelian groups, there are bijections,

$$\begin{aligned} \mathbb{G}_a &\to k \\ (x - \mu) &\mapsto \mu \\ \mathbb{G}_m &\to k^\times \\ (x - \mu) &\mapsto \mu \end{aligned}$$

(since $(x) \notin \mathrm{Spec}\,(k[x, x^{-1}]) = D(x) \subset \mathrm{Spec}\,(k[x])$). I claim these maps are isomorphisms.

Definition 1.0.11.

$$\mathbb{GL}_n = \mathrm{Spec}\left(\mathbb{Z}[\{x_{ij} \mid 1 \le i, j \le n\}]_{(\det(x_{ij}))}\right)$$

with multiplication defined via,

$$\begin{aligned} \mathbb{GL}_n \times \mathbb{GL}_n &\to \mathbb{GL}_n \\ \mathrm{Spec}\left(\mathbb{Z}[x_{ij}]_{(\det(x_{ij}))} \otimes \mathbb{Z}[x_{ij}]_{(\det(x_{ij}))}\right) &\to \mathrm{Spec}\left(\mathbb{Z}[x_{ij}]_{(\det(x_{ij}))}\right) \\ \mathbb{Z}[x_{ij}]_{(\det(x_{ij}))} \otimes \mathbb{Z}[x_{ij}]_{(\det(x_{ij}))} &\leftarrow \mathbb{Z}[x_{ij}]_{(\det(x_{ij}))} \\ \sum_k x_{ik} \otimes x_{kj} &\leftarrow\!\shortmid x_{ij} \end{aligned}$$

Remark. In the case $n = 1$ we have $\mathrm{GL}_n(\mathbb{Z}) = \mathrm{Spec}\left(\mathbb{Z}[x]_{(x)}\right) = \mathrm{Spec}\,(\mathbb{Z}[x, x^{-1}]) = \mathbb{G}_m$.

Example 1.0.12. There is a defining action of $\mathbb{GL}_n$ on $\mathbb{A}^n$ defined by,

$$\begin{aligned} \mathbb{GL}_n \times \mathbb{A}^n &\to \mathbb{A}^n \\ \mathrm{Spec}\left(\mathbb{Z}[x_{ij}]_{(\det(x_{ij}))} \otimes \mathbb{Z}[y_1, \dots, y_n]\right) &\to \mathrm{Spec}\,(\mathbb{Z}[y_1, \dots, y_n]) \\ \mathbb{Z}[x_{ij}]_{(\det(x_{ij}))} \otimes \mathbb{Z}[y_1, \dots, y_n] &\leftarrow \mathbb{Z}[y_1, \dots, y_n] \\ \sum_k x_{ik} \otimes y_k &\leftarrow\!\shortmid y_i \end{aligned}$$

Lemma 1.0.13. Let X be an S scheme. Then the group schemes $\mathbb{G}_m$ and $\mathbb{G}_a$ have X-points,

$$\begin{aligned} \mathrm{Hom}_S\left(X, \mathbb{G}_a^S\right) &= \Gamma(X, \mathcal{O}_X) \\ \mathrm{Hom}_S\left(X, \mathbb{G}_m^S\right) &= \Gamma(X, \mathcal{O}_X^\times) \\ \mathrm{Hom}_S\left(X, \mathbb{GL}_n^S\right) &= \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X)) \end{aligned}$$

Proof.

$$\begin{aligned}\mathrm{Hom}_S\left(X, \mathbb{G}_a^S\right) &= \mathrm{Hom}_S\left(X, S\right) \times \mathrm{Hom}\left(X, \mathbb{G}_a\right) = \mathrm{Hom}\left(X, \mathbb{G}_a\right) \\ &= \mathrm{Hom}\left(\mathbb{Z}[x], \Gamma(X, \mathcal{O}_X)\right) = \Gamma(X, \mathcal{O}_X)\end{aligned}$$

since any ring map $\mathbb{Z}[x] \to R$ is determined uniquely by the image of x. Similarly,

$$\begin{aligned}\mathrm{Hom}_S\left(X, \mathbb{G}_m^S\right) &= \mathrm{Hom}\left(X, \mathbb{G}_m\right) \\ &= \mathrm{Hom}\left(\mathbb{Z}[x, x^{-1}], \Gamma(X, \mathcal{O}_X)\right) = \Gamma(X, \mathcal{O}_X^\times)\end{aligned}$$

since any ring map $\mathbb{Z}[x, x^{-1}] \to R$ is determined uniquely by the image of $x \in R^\times$.

$$\begin{aligned}\mathrm{Hom}_S\left(X, \mathbb{GL}_n^S\right) &= \mathrm{Hom}\left(X, \mathbb{GL}_n\right) \\ &= \mathrm{Hom}\left(\mathbb{Z}[x_{ij}]_{(\det(x_{ij}))}, \Gamma(X, \mathcal{O}_X)\right) = \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))\end{aligned}$$

since a ring map $\mathbb{Z}[x_{ij}]_{(\det(x_{ij}))} \to R$ is exactly determined by a matrix of elements a_{ij} which are the images of x_{ij} such that the determinant polynomial $\det(x_{ij})$ is mapped to a unit: $\det(a_{ij}) \in R^\times$. ☐

Remark. In particular, let $S = \mathrm{Spec}\,(k)$ then by the lemma, the geometric points of these group schemes are,

$$\begin{aligned}\mathrm{Hom}_S\left(S, \mathbb{G}_a^S\right) &= \Gamma(S, \mathcal{O}_S) = k \\ \mathrm{Hom}_S\left(S, \mathbb{G}_m^S\right) &= \Gamma(S, \mathcal{O}_S^\times) = k^\times\end{aligned}$$

which, in the case $k = \bar{k}$ correspond to the closed points as we computed before.

2 Vector Bundles on Schemes

Remark. Given a scheme S and a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ Recall the relative spectrum, $\mathbf{Spec}_S\left(\mathcal{A}\right)$. The relative spectrum over S may be characterized as representing the functor,

$$F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$$

defined by sending a scheme T to the set of pairs (f, g) of morphisms $f : T \to S$ and $\mathcal{O}_T$-algebra morphisms $g : f^*\mathcal{A} \to \mathcal{O}_T$. The universal element $\xi \in F(\mathbf{Spec}_S\left(\mathcal{A}\right))$ is thus a pair of canonical maps,

$$\pi : \mathbf{Spec}_S\left(\mathcal{A}\right) \to S \text{ and (by adjunction) } g : \mathcal{A} \to \pi_*\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}$$

It turns out that when $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra then $g : \mathcal{A} \to \pi_*\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}$ is an isomorphism of $\mathcal{O}_S$-algebras (Tag 01LX). The explicit isomorphism,

$$\eta_X : \mathrm{Hom}_X\left(\mathbf{Spec}_S\left(\mathcal{A}\right), \to\right) F(X)$$

is given by sending $s : X \to \mathbf{Spec}_S\left(\mathcal{A}\right)$ to $F(s)(\xi) = (\pi \circ s, g \circ \pi_* s^{\#})$.

Definition 2.0.1. Let X be a scheme. A *vector bundle* over X is an affine morphism $\pi : V \to X$ such that $\pi_*\mathcal{O}_V$ is a graded $\mathcal{O}_X$-algebra,

$$\pi_*\mathcal{O}_V = \bigoplus_{n \geq 0} \mathcal{E}_n$$

such that $\mathcal{E}_0 = \mathcal{O}_X$ and the natural maps,

$$\mathrm{Sym}^n_{\mathcal{O}_X}(\mathcal{E}_1) \longrightarrow \mathcal{E}_n$$

are isomorphisms for all $n \neq 0$.

Given a morphism of schemes $g : X \to Y$ a *bundle map* $f : V_X \to V_Y$ of vector bundles V_X over X and V_Y over Y is a commutative diagram of schemes,

$$\begin{tikzcd} V_X \arrow[r, "f"] \arrow[d, "\pi_X"'] & V_Y \arrow[d, "\pi_Y"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

such that the induced sheaf map $(\pi_Y)_*\mathcal{O}_{V_Y} \to g_*(\pi_X)_*\mathcal{O}_{V_X}$ is a map of *graded* sheaves. In particular, if we take the map $\mathrm{id}_X : X \to X$ then a morphism of vector bundles over X is a morphism $f : V_1 \to V_2$ such that $\pi_2 \circ f = \pi_1$ and $(\pi_2)_*\mathcal{O}_{V_2} \to (\pi_1)_*\mathcal{O}_{V_1}$ is a morphism of graded sheaves.

Remark. We show how to explicitly construct this induced morphism. The map of schemes gives $f^\# : \mathcal{O}_{V_Y} \to f_*\mathcal{O}_{V_Y}$. Then apply the functor $(\pi_Y)_*$ which gives a morphism, $(\pi_Y)_*f^\# : (\pi_Y)_*\mathcal{O}_{V_Y} \to (\pi_Y)_*f_*\mathcal{O}_{V_Y}$ however, $\pi_Y \circ f = g \circ \pi_X$ giving the desired morphism,

$$(\pi_Y)_*f^\# : (\pi_Y)_*\mathcal{O}_{V_Y} \to g_*(\pi_X)_*\mathcal{O}_{V_Y}$$

Remark. Vector bundles are important because we can associate them to (quasi)coherent sheaves which will give our most important examples.

Definition 2.0.2. Let X be be a scheme and $\mathscr{F}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Then the associated vector bundle $\mathbf{V}(\mathscr{F})$ over X is the scheme over X with structure morphism,

$$\pi : \mathbf{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X}(\mathscr{F})\right) \to X$$

Then by definition,

$$\pi_*\mathcal{O}_{V(\mathscr{F})} = \mathrm{Sym}_{\mathcal{O}_X}(\mathscr{F}) = \bigoplus_{n\geq 0} \mathrm{Sym}^n_{\mathcal{O}_X}(\mathscr{F})$$

which makes $\pi_*\mathcal{O}_{V(\mathscr{F})}$ a graded $\mathcal{O}_X$-algebra where we may recover $\mathscr{F}$ in degree 1.

Theorem 2.0.3. There is an anti-equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of vector bundles over X.

Proof. (Sketch) We have shown that given a quasi-coherent sheaf $\mathscr{F}$ we can construct a vector bundle $V(\mathscr{F})$ and that $(\pi_*V(\mathscr{F}))_1 = \mathscr{F}$ so the functor $V \to (\pi_*\mathcal{O}_V)_1$ recovers the original sheaf. I claim that the functors $\mathbf{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X}(-)\right)$ and $V \to (\pi_*\mathcal{O}_V)_1$ give this anti-equivalence. We should check that the above construction can reproduce any vector bundle over X. Given such a vector bundle $\pi : V \to X$, we know that $\pi_*\mathcal{O}_V$ is a graded $\mathcal{O}_X$-algebra such that we have graded isomorphisms,

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}_1) \to \pi_*\mathcal{O}_V = \bigoplus_{n\geq 0} \mathcal{E}_n$$

By Tag 01LY in the stacks project, since $\pi : V \to X$ is an affine morphism and thus quasi-compact and separated there is a canonical morphism,

$$V \longrightarrow \mathbf{Spec}_X\left(\pi_*\mathcal{O}_V\right) = \mathbf{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X}\left(\mathcal{E}_1\right)\right) = \mathbf{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X}\left((\pi_*\mathcal{O}_V)_1\right)\right)$$

Lastly, this first map is an isomorphism because $\pi : V \to X$ is affine (Tag 01S8). To see this take any affine open $U \subset X$ then we know the canonical map $V \to \mathbf{Spec}_X\left(\pi_*\mathcal{O}_V\right)$ restricts to,

$$\pi^{-1}(U) \to \mathrm{Spec}\left(\Gamma(\pi^{-1}(U), \mathcal{O}_V)\right)$$

However, π is affine so $\pi^{-1}(U) \subset V$ is affine open meaning that,

$$\pi^{-1}(U) = \mathrm{Spec}\left(\Gamma(\pi^{-1}(U), \mathcal{O}_V)\right)$$

and the canonical map is the identity because it is, by definition, induced by the identity ring map on $\Gamma(\pi^{-1}(U), \mathcal{O}_V)$. Thus we have found,

$$V \cong \mathbf{Spec}_X\left(\pi_*\mathcal{O}_V\right) = \mathbf{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X}\left((\pi_*\mathcal{O}_V)_1\right)\right)$$

We should also show that these functors are fully faithful but I will leave the proof here. $\square$

Example 2.0.4. Let $X = \mathbb{A}^n_R$ over some ring R. Then,

$$\mathrm{Sym}_{\mathcal{O}_X}\left(\mathcal{O}_X\right) = \mathrm{Sym}_{R[x_1,\dots,x_n]}\left(R[x_1,\dots,x_n]\right)^{\sim} = R[x_1,\dots,x_n,x_{n+1}]^{\sim}$$

$$\mathbf{V}(\mathcal{O}_X) = \mathbf{Spec}_X\left(R[x_1,\cdots,x_n,x_{n+1}]^{\sim}\right) = \mathrm{Spec}\left(R[x_1,\cdots,x_n,x_{n+1}]\right) = \mathbb{A}^{n+1}_R$$

with the projection $\pi : \mathbb{A}^{n+1}_R \to \mathbb{A}^n_R$ induced by the embedding $R[x_1,\dots,x_n] \to R[x_1,\dots,x_n,x_{n+1}]$. This recovers nicely the picture of $\mathbb{A}^{n+1}$ as a line bundle over $\mathbb{A}^n$ whose sections are exactly regular functions on $\mathbb{A}^n$.

Lemma 2.0.5. Let X be a scheme and $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Take $\pi : \mathbf{V}(\mathscr{F}) \to X$ its associated vector bundle. Then there is a canonical correspondence between sections $s : X \to \mathbf{V}(\mathscr{F})$ (such that $\pi \circ s = \mathrm{id}_X$) and global sections of the dual sheaf $\mathscr{F}^\vee$. That is,

$$\mathrm{Hom}_X\left(X, \mathbf{V}(\mathscr{F})\right) = \Gamma(X, \mathscr{F}^\vee) = \mathrm{Hom}_{\mathcal{O}_X}\left(\mathscr{F}, \mathcal{O}_X\right)$$

Proof. The associated vector bundle is constructed as,

$$\mathbf{V}(\mathscr{F}) = \mathbf{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X}\left(\mathscr{F}\right)\right)$$

and recall that the relative spectrum represents the functor F defined at the beginning of the section. Denote $\mathcal{A} = \mathrm{Sym}_{\mathcal{O}_X}\left(\mathscr{F}\right)$. Sections $s : X \to \mathbf{V}(\mathscr{F})$ correspond to pairs $(f : X \to X, g : \mathcal{A} \to f_*\mathcal{O}_X)$ where we require $f = \mathrm{id}_X$ since $f = \pi \circ s = \mathrm{id}_X$ because the corresponding map is a section. Therefore, sections $s : X \to \mathbf{V}(\mathscr{F})$ correspond conically to $\mathcal{O}_X$-algebra maps $g : \mathrm{Sym}_{\mathcal{O}_X}\left(\mathscr{F}\right) \to \mathcal{O}_X$. However, such a map of algebras is uniquely determined by its action in degree 1 i.e. by a morphism $\mathscr{F} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules which is exactly a global section of the dual sheaf $\mathscr{F}^\vee = \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. $\square$

Definition 2.0.6. Let $\pi : V \to Y$ be a vector bundle and $f : X \to Y$ a morphism of schemes. The *pullback bundle* along f, denoted f^*V, is the bundle over X given by base change $\pi_X : V \times_Y X \to X$ which is the pullback in the diagram,

$$\begin{tikzcd} V \times_Y X \arrow[r] \arrow[d, "\pi_X"] & V \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Lemma 2.0.7. The pullback bundle is a vector bundle and the map $f^*V \to V$ is a bundle map.

Proof. We will explicitly demonstrate this for the case of interest by the following. $\square$

Lemma 2.0.8. Let Y be a scheme and $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_Y$-module. Given a morphism of schemes $f : X \to Y$, the relative spectrum base changes as,

$$X \times_Y \mathbf{Spec}_Y(\mathcal{A}) = \mathbf{Spec}_X(f^*\mathcal{A})$$

Proof. A pair $(a : T \to X, g : a^*f^*\mathcal{A} \to \mathcal{O}_T)$ is canonically the same as a pair $(f \circ a : T \to Y, g : (f \circ a)^*\mathcal{A} \to \mathcal{O}_T)$ i.e. a pair $(a' : T \to Y : (a')^* : \mathcal{A} \to \mathcal{O}_T)$ such that a' factors through $f : X \to Y$ as $a' = f \circ a$. By the representation, such a pair can be identified with a map $\tilde{a} : T \to \mathbf{Spec}_Y(\mathcal{A})$ such that the map $a' = \pi \circ \tilde{a}$ factors through $f : X \to Y$ i.e. $a' = \pi \circ \tilde{a} = f \circ a$ for some $a : T \to X$. By the universal property, such maps are canonically identified with maps $T \to X \times_Y \mathbf{Spec}_Y(\mathcal{A})$. Therefore, $X \times_Y \mathbf{Spec}_Y(\mathcal{A})$ represents the functor F for the pair $(X, f^*\mathcal{A})$ so by Yoneda,

$$X \times_Y \mathbf{Spec}_Y(\mathcal{A}) = \mathbf{Spec}_X(f^*\mathcal{A})$$

since these schemes both represent the same functor F. $\square$

Lemma 2.0.9. Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_Y$-module. The pullback bundle of the associated vector bundle is the associated vector bundle of the pullback sheaf,

$$f^*\mathbf{V}(\mathscr{F}) \cong \mathbf{V}(f^*\mathscr{F})$$

Proof.

$$\begin{aligned} f^*\mathbf{V}(\mathscr{F}) &= X \times_Y \mathbf{Spec}_Y\left(\mathrm{Sym}_{\mathcal{O}_Y}(\mathscr{F})\right) = \mathbf{Spec}_X\left(f^*\mathrm{Sym}_{\mathcal{O}_Y}(\mathscr{F})\right) \\ &= \mathbf{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X}(f^*\mathscr{F})\right) = \mathbf{V}(f^*\mathscr{F}) \end{aligned}$$

$\square$

Example 2.0.10. Let $X = \mathbb{P}^n_k = \mathrm{Proj}\,(k[X_0, \cdots, X_n])$ and consider the invertable sheaf $\mathcal{O}_X(-1)$ on X. This is known as the tautological bundle or rather its associated vector bundle $\mathbf{V}(\mathcal{O}_X(-1))$ is the tautological bundle. Topologically, it is the line bundle whose fiber above each point in $\mathbb{P}^n_k$ is the line in $\mathbb{A}^{n+1}_k$ it corresponds to. Furthermore, using our formula, the sections of the tautological bundle are exactly,

$$H^0(X, \mathcal{O}_X(-1)^\vee) = H^0(X, \mathcal{O}_X(1)) = k[X_0, \cdots, X_n]_{(0)}$$

These sections X_i correspond to the coordinates on $\mathbb{A}^{n+1}_k$.

3 Group Schemes Acting on Sheaves

Remark. It is easy to define an equivariant group scheme action in the category of vector bundles over a scheme. Our strategy to figure out how to act a group scheme on a quasi-coherent sheaf equivariantly is to use the anti-equivalence of quasi-coherent sheaves and vector bundles.

Definition 3.0.1. Let $\mathscr{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules and a group scheme G act on X. Then an G action on $\mathscr{F}$ is the same as a G-equivariant action on the associated vector bundle $\pi : \mathbf{V}(\mathscr{F}) \to X$ such that π is a morphism of G-schemes,

$$\begin{tikzcd}
G \times \mathbf{V}(\mathscr{F}) \arrow[r, "\rho_V"] \arrow[d, "\mathrm{id}\times\pi"'] & \mathbf{V}(\mathscr{F}) \arrow[d, "\pi"] \\
G \times X \arrow[r, "\rho"'] & X
\end{tikzcd}$$

and ρ_V is a morphism of vector bundles i.e. a bundle map over ρ.

Remark. We will now unwind this definition to recover a purely sheaf-theoretic notion of a G-equivariant sheaf action.

Proof. Let $p : G \times X \to X$ be the projection. Note that, canonically,

$$G \times \mathbf{V}(\mathscr{F}) \cong (G \times X) \times_X \mathbf{V}(\mathscr{F}) = p^*\mathbf{V}(\mathscr{F})$$

Furthermore, we have a diagram,

$$\begin{tikzcd}
G \times \mathbf{V}(\mathscr{F}) \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\rho_V"] \arrow[ddr, bend right, "\mathrm{id}\times\pi"'] & & \\
& \rho^*\mathbf{V}(\mathscr{F}) \arrow[r, "\rho_V"] \arrow[d, "\rho^*\pi"'] & \mathbf{V}(\mathscr{F}) \arrow[d, "\pi"] \\
& G \times X \arrow[r, "\rho"'] & X
\end{tikzcd}$$

commutes. This gives a bundle map $\varphi : G \times \mathbf{V}(\mathscr{F}) \to \rho^*\mathbf{V}(\mathscr{F})$. Therefore we have a morphism $\varphi : p^*\mathbf{V}(\mathscr{F}) \to \rho^*\mathbf{V}(\mathscr{F})$ of vector bundles over $G \times X$ and thus, by the lemma, a morphism $\varphi : \mathbf{V}(p^*\mathscr{F}) \to \mathbf{V}(\rho^*\mathscr{F})$. By the anti-equivalence of vector bundles and quasi-coherent sheaves, this is the same as giving a morphism $\varphi : \rho^*\mathscr{F} \to p^*\mathscr{F}$ of quasi-coherent sheaves on $G \times X$, this morphism will be the defining feature of a G-sheaf. Next, we will investigate what restrictions may be placed on such a morphism.

The map $\rho : G \times \mathbf{V}(\mathscr{F}) \to \mathbf{V}(\mathscr{F})$ is an action and thus additionally must satisfy,

$$\begin{tikzcd}
G \times G \times \mathbf{V}(\mathscr{F}) \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\rho_V"'] & G \times \mathbf{V}(\mathscr{F}) \arrow[d, "\rho_V"] \\
G \times \mathbf{V}(\mathscr{F}) \arrow[r, "\rho_V"'] & \mathbf{V}(\mathscr{F})
\end{tikzcd}$$

The corresponding diagram for the G-action on X lets us consider the pullbacks of vector bundles on $G \times X$ over the maps $m \times \mathrm{id}_X$ and $\mathrm{id} \times \rho$. We have a morphism $\varphi : p^*\mathbf{V}(\mathscr{F}) \to \rho^*\mathbf{V}(\mathscr{F})$ of vector bundles over $G \times X$. Applying the pullback functors we get morphisms,

$$(m \times \mathrm{id}_X)^*\varphi : (m \times \mathrm{id}_X)^* p^*\mathbf{V}(\mathscr{F}) \to (m \times \mathrm{id}_X)^*\rho^*\mathbf{V}(\mathscr{F})$$
$$(\mathrm{id} \times \rho)^*\varphi : (\mathrm{id} \times \rho)^* p^*\mathbf{V}(\mathscr{F}) \to (\mathrm{id} \times \rho)^*\rho^*\mathbf{V}(\mathscr{F})$$

Note that $\rho\circ(\mathrm{id}\times\rho) = \rho\circ(m\times\mathrm{id}_X)$ by commutativity of the diagram and thus $(m\times\mathrm{id}_X)^*\rho^*\mathbf{V}(\mathscr{F}) = (\mathrm{id}\times\rho)^*\rho^*\mathbf{V}(\mathscr{F})$. Denote this bundle over $G\times G\times X$ as P. Also, $p\circ(m\times\mathrm{id}_X) = p\circ p_{23}$ the projection

$G \times G \times X \to X$ and $p \circ (\mathrm{id} \times \rho) = \rho \circ p_{23}$ the map $G \times G \times X \to X$ via $(g, h, x) \mapsto (h, x) \mapsto h \cdot x$. Then pulling back the bundle map $\varphi : p^* \mathbf{V}(\mathscr{F}) \to \rho^* \mathbf{V}(\mathscr{F})$ along $p_{23} : G \times G \times X \to G \times X$ gives a morphism,

$$p_{23}^* \varphi : p_{23}^* p^* \mathbf{V}(\mathscr{F}) \to p_{23}^* \rho^* \mathbf{V}(\mathscr{F})$$

of vector bundles over $G \times G \times X$ between the two domains of the previous maps. We need to be careful because there are two inequivalent bundle maps $P \to \rho^* \mathbf{V}(\mathscr{F})$ since P is realized as the pullback under two distinct maps. However, if we apply the bundle map down to $f_{\rho^*} : \rho^* \mathbf{V}(\mathscr{F}) \to \mathbf{V}(\mathscr{F})$ these become equal. Now we will apply the pullback lemma (see below) to show that maps between double pullbacks are uniquely determined by bundle maps to $\mathbf{V}(\mathscr{F})$ over the corresponding map $G \times G \times X \to X$. Thus, the commutative diagram above implies that the composition of bundle maps to $\mathbf{V}(\mathscr{F})$ are equal and thus the corresponding pullbacks are also equal,

$$(\mathrm{id} \times \rho)^* \varphi \circ p_{23}^* \varphi = (m \times \mathrm{id}_X)^* \varphi$$

Via the anti-equivalence between quasi-coherent sheaves and vector-bundles we find that φ must satisfy the commutative diagram of quasi-coherent $\mathcal{O}_{G \times G \times X}$-modules,

$$\begin{tikzcd}
(m \times \mathrm{id}_X)^* p^* \mathscr{F} & (\mathrm{id} \times \rho)^* \rho^* \mathscr{F} \arrow[l, "p_{23}^* \varphi"'] \\
(m \times \mathrm{id}_X)^* \mathscr{F} \arrow[u, "(m \times \mathrm{id}_X)^* \varphi"] \arrow[r, equal] & (\mathrm{id} \times \rho)^* \rho^* \mathscr{F} \arrow[u, "(\mathrm{id} \times \rho)^* \varphi"']
\end{tikzcd}$$

Furthermore,

$$\begin{tikzcd}
\mathbf{V}(\mathscr{F}) \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\mathrm{id}"] & G \times \mathbf{V}(\mathscr{F}) \arrow[d, "\rho_V"] \\
& \mathbf{V}(\mathscr{F})
\end{tikzcd}$$

This says we may factor the identity map as,

$$\begin{tikzcd}
\mathbf{V}(\mathscr{F}) \arrow[r, "e \times \mathrm{id}_V"] \arrow[d, "\pi"] & p^* \mathbf{V}(\mathscr{F}) \arrow[r, "\varphi"] \arrow[d, "p^* \pi"] & \rho^* \mathbf{V}(\mathscr{F}) \arrow[r, "\rho_V"] \arrow[d, "\rho^* \pi"] & \mathbf{V}(\mathscr{F}) \arrow[d, "\pi"] \\
X \arrow[r, "e \times \mathrm{id}_X"] & G \times X \arrow[r, "\mathrm{id}"] & G \times X \arrow[r, "\rho"] & X
\end{tikzcd}$$

meaning that $\mathbf{V}(\mathscr{F}) \to \rho^* \mathbf{V}(\mathscr{F})$ is the pullback over $e \times \mathrm{id}_X : X \to G \times X$ so $\mathrm{id} : \mathbf{V}(\mathscr{F}) \to \mathbf{V}(\mathscr{F})$ is the unique map which projects to $\pi : \mathbf{V}(\mathscr{F}) \to X$ and $\varphi \circ (e \times \mathrm{id}_V) : \mathbf{V}(\mathscr{F}) \to \rho^* \mathbf{V}(\mathscr{F})$. Therefore, applying the pullback functor on vector bundles, $(e \times \mathrm{id}_X)^* \varphi : \mathbf{V}(\mathscr{F}) \to \mathbf{V}(\mathscr{F})$ is the identity. Note that,

$$(e \times \mathrm{id}_X)^* p^* \mathbf{V}(\mathscr{F}) = (e \times \mathrm{id}_X)^* \rho^* \mathbf{V}(\mathscr{F}) = \mathbf{V}(\mathscr{F})$$

because $\rho \circ (e \times \mathrm{id}_X) = p \circ (e \times \mathrm{id}_X) = \mathrm{id}_X$. Thus applying the anti-equivalence we find the condition $(e \times \mathrm{id}_X)^* \varphi : \mathscr{F} \to \mathscr{F}$ is the identity morphism of $\mathcal{O}_X$-modules. $\square$

Remark. This derivation leads us to the following definition.

Definition 3.0.2. Let $\mathscr{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules and a group scheme G act on X. Then an G action on $\mathscr{F}$ making $\mathscr{F}$ a G-equivariant sheaf on X is a morphism $\varphi : \rho^*\mathscr{F} \to p^*\mathscr{F}$ of $\mathcal{O}_{G\times X}$-modules which satisfies the following coherence conditions. The diagram,

$$\begin{tikzcd}
(m \times \mathrm{id}_X)^* p^* \mathscr{F} & (\mathrm{id} \times \rho)^* \rho^* \mathscr{F} \arrow[l, "p_{23}^*\varphi"'] \\
(m \times \mathrm{id}_X)^* \mathscr{F} \arrow[u, "(m\times \mathrm{id}_X)^*\varphi"] \arrow[r, equal] & (\mathrm{id} \times \rho)^* \rho^* \mathscr{F} \arrow[u, "(\mathrm{id}\times\rho)^*\varphi"']
\end{tikzcd}$$

commutes in the category of $\mathcal{O}_{G\times G\times X}$-modules and $(e \times \mathrm{id}_X)^*\varphi : \mathscr{F} \to \mathscr{F}$ is the identity map of $\mathcal{O}_X$-modules.

Lemma 3.0.3 (Pullback)**.** Given two Cartesian squares,

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C'
\end{tikzcd}$$

the outer rectangle is Cartesian as well.

Example 3.0.4. For any group scheme action G on X the structure sheaf $\mathcal{O}_X$ is always G-equivariant with a trivial action because under $\rho : G \times X \to X$ we can pull back,

$$\rho^*\mathcal{O}_X = \rho^{-1}\mathcal{O}_X \otimes_{\rho^{-1}\mathcal{O}_X} \mathcal{O}_{G\times X} = \mathcal{O}_{G\times X} = p^*\mathcal{O}_X$$

Theorem 3.0.5. Let G be a group scheme and X a G-scheme. Let $\mathscr{F}$ be a quasi-coherent G-equivariant sheaf on X. Then there is a G-action on global sections making $\Gamma(X, \mathscr{F}^\vee)$ a G-module.

Proof. Consider a section $s : X \to \mathbf{V}(\mathscr{F})$ of the vector bundle $\pi : \mathbf{V}(\mathscr{F}) \to X$ associated to the sheaf $\mathscr{F}$. For fixed $g \in G$ we consider the map ι_g defined by $x \mapsto (g, g^{-1} \cdot x)$. (This may map be defined as follows. The maps $\mathrm{id} : X \to X$ and $X \to \{g^{-1}\} \subset G$ define $x \mapsto (g^{-1}, x)$ applying ρ gives $x \mapsto g^{-1}x$. Pair this with the constant map $X \to \{g\} \subset G$). Consider the diagram,

$$\begin{tikzcd}
G \times \mathbf{V}(\mathscr{F}) \arrow[r, "\rho_V"] \arrow[d, shift right, "\mathrm{id}\times\pi"'] & \mathbf{V}(\mathscr{F}) \arrow[d, "\pi"] \\
G \times X \arrow[u, bend right, "\mathrm{id}\times s"'] \arrow[r, "\rho"] & X \arrow[l, bend left, "\iota_g"]
\end{tikzcd}$$

Now define $g \cdot s = \rho_V \circ (\mathrm{id} \times s) \circ \iota_g$. I claim that $g \cdot s$ is a section of the bundle $\pi : \mathbf{V}(\mathscr{F}) \to X$. To see this,

$$\pi \circ (g \cdot s) = \pi \circ \rho_V \circ (\mathrm{id} \times s) \circ \iota_g = \rho \circ (\mathrm{id} \times \pi) \circ (\mathrm{id} \times s) \circ \iota_g = \rho \circ \iota_g = \mathrm{id}_X$$

The coherence conditions then imply that this is an action. This gives a G-action on the dual $\Gamma(X, \mathscr{F}^\vee)$. It is instructive to rephrase this action. We have seen how an equivariant action on a vector bundle induces an morphism of the two pullback bundles. The morphism $\varphi : p^*\mathbf{V}(\mathscr{F}) \to \rho^*\mathbf{V}(\mathscr{F})$ of bundles over $G \times X$ induces a map on their sections $\varphi : \Gamma(X, p^*\mathbf{V}(\mathscr{F})) \to \Gamma(X, \rho^*\mathbf{V}(\mathscr{F}))$ □

Proposition 3.0.6. In particular, if work in the category of schemes over a field k then we can form a dual G-action on $\mathscr{F}$ sections (rather than $s : X \to \mathbf{V}(\mathscr{F})$ sections which are $\mathscr{F}^\vee$ sections) giving $\Gamma(X, \mathscr{F})$ a G-representation structure over k.

Proof. Recall that we have a morphism of $\mathcal{O}_{G\times X}$-modules $\varphi : \rho^*\mathscr{F} \to p^*\mathscr{F}$. Furthermore, the action $\rho : G \times X \to X$ defines the pullback functor,

$$\rho^* : \mathfrak{QCoh}\,(\mathcal{O}_X) \to \mathfrak{QCoh}\,(\mathcal{O}_{G\times X})$$

Applying this to a $\mathcal{O}_Y$-module morphism $s : \mathcal{O}_Y \to \mathscr{F}$ gives $\rho^* s : \mathcal{O}_{G\times X} \to \rho^*\mathscr{F}$ (note for $f : X \to Y$ that $f^*\mathcal{O}_Y = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = \mathcal{O}_X$). Since $\mathcal{O}_X$-module maps $\mathcal{O}_X \to \mathscr{F}$ are exactly global sections $\Gamma(X, \mathscr{F})$ we have constructed the pullback map on sections $\rho^* : \Gamma(X, \mathscr{F}) \to \Gamma(G\times X, \rho^*\mathscr{F})$. Composing gives a morphism,

$$\Gamma(X, \mathscr{F}) \xrightarrow{\rho^*} \Gamma(G \times X, \rho^*\mathscr{F}) \xrightarrow{\varphi} \Gamma(G \times X, p^*X)$$

Since we are working in the category of schemes over k, we may now apply the Künneth formula,

$$H^0(G \times X, p^*\mathscr{F}) = H^0(G \times X, p_1^*\mathcal{O}_G \otimes_{\mathcal{O}_{G\times X}} p_2^*\mathscr{F}) = H^0(G, \mathcal{O}_G) \otimes_k H^0(X, \mathscr{F})$$

Therefore, we have a map,

$$\Gamma(X, \mathscr{F}) \to \Gamma(G, \mathcal{O}_G) \otimes_k \Gamma(X, \mathscr{F})$$

Since $\Gamma(G, \mathcal{O}_G) \cong \mathrm{Hom}_k\,(G, \mathbb{A}^1_k)$ the above map gives an *algebraic action* on the k-vectorspace $\Gamma(X, \mathscr{F})$. The coherence of the action follows from the coherence conditions on φ. □