Physics GR8040 General Relativity
Assignment # 1

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1.

(a)

Let a^i_j and b^i_j be $(1,1)$-tensors. Define the coefficients $c^i_j = a^i_j + b^i_j$ in all bases. We need to show that such an object has the tensor property. Under a transformation,

$$c^{i'}_{j'} = a^{i'}_{j'} + b^{i'}_{j'} = \Lambda^i_{i'}\Lambda^{j'}_j a^i_j + \Lambda^i_{i'}\Lambda^{j'}_j b^i_j = \Lambda^i_{i'}\Lambda^{j'}_j (a^i_j + b^i_j) = \Lambda^i_{i'}\Lambda^{j'}_j c^i_j$$

Therefore c transforms as a $(1,1)$-tensor.

(b)

Let a_{ij} and b^k be tensors and define the coefficients $c^k_{ij} = a_{ij}b^k$ in all bases. We need to show that c^k_{ij} has the tensor property. Under a change of basis,

$$c^{k'}_{i'j'} = a_{i'j'}b^{k'} = \Lambda^i_{i'}\Lambda^j_{j'}a_{ij}\Lambda^{k'}_k b^k = \Lambda^i_{i'}\Lambda^j_{j'}\Lambda^{k'} c^k_{ij}$$

so c transforms as a $(1,2)$-tensor.

(c)

Let a^i_{jk} be a $(1,2)$-tensor and define the coefficients $c_k = a^i_{ik}$ in all bases. We need to show that c_k has the tensor property. Under a change of basis,

$$c_{k'} = a^{i'}_{i'k'} = \Lambda^{i'}_i \Lambda^j_{i'}\Lambda^k_{k'}a^i_{jk} = \delta^j_i \Lambda^k_{k'}a^i_{jk} = \Lambda^k_{k'}a^i_{ik} = \Lambda^k_{k'}c_k$$

so c transforms as a $(0,1)$-tensor.

2.

Let $\phi : V \times V \to \mathbb{R}$ be a bilinear function on a real vectorspace $\mathbb{R}$. Let V have a basis $\mathbf{e}_i$. The map ϕ defines components $\phi_{ij} = \phi(\mathbf{e}_i, \mathbf{e}_j)$. Consider these components under a change of basis $\mathbf{e}_{i'} = \Lambda^i_{i'}\mathbf{e}_i$. Then, using bilinearity, we have,

$$\phi_{i'j'} = \phi(\mathbf{e}_{i'}, \mathbf{e}_{j'}) = \phi(\Lambda^i_{i'}\mathbf{e}_i, \Lambda^j_{j'}\mathbf{e}_j) = \Lambda^i_{i'}\Lambda^j_{j'}\phi(\mathbf{e}_i, \mathbf{e}_j) = \Lambda^i_{i'}\Lambda^j_{j'}\phi_{ij}$$

Thus, ϕ_{ij} transform as a $(0,2)$-tensor.

3.

Suppose that a $(0, m)$-tensor $a_{i_1,\cdots,i_m}$ is symmetric in some basis. Consider this object transformed to another basis which we can express in terms of the original basis as,

$$a_{i'_1,\cdots,i'_m} = \Lambda^{i_1}_{i'_1} \cdots \Lambda^{i_m}_{i'_m} a_{i_1,\ldots,i_m}$$

Therefore, swapping index i_a and i_b with $a < b$ we find,

$$a_{i'_1,\ldots,i'_b,\ldots,i'_a,\ldots i'_m} = \Lambda^{i_1}_{i'_1} \ldots \Lambda^{i_a}_{i'_b} \Lambda^{i_b}_{i'_a} \cdots \Lambda^{i_m}_{i'_m} a_{i_1,\ldots,i_a,\ldots,i_b,\cdots i_m} = \Lambda^{i_1}_{i'_1} \ldots \Lambda^{i_a}_{i'_b} \cdots \Lambda^{i_b}_{i'_a} \ldots \Lambda^{i_m}_{i'_m} a_{i_1,\ldots,i_b,\ldots,i_a,\ldots i_m}$$

where I used the symmetry of a to swap i_a and i_b. Now renaming $i_a \mapsto i_b$ and $i_b \mapsto i_a$ we find,

$$a_{i'_1,\ldots,i'_b,\ldots,i'_a,\ldots i'_m} = \Lambda^{i_1}_{i'_1} \ldots \Lambda^{i_a}_{i'_b} \Lambda^{i_b}_{i'_a} \cdots \Lambda^{i_m}_{i'_m} a_{i_1,\ldots,i_a,\ldots,i_b,\cdots i_m} = \Lambda^{i_1}_{i'_1} \ldots \Lambda^{i_b}_{i'_b} \cdots \Lambda^{i_a}_{i'_a} \ldots \Lambda^{i_m}_{i'_m} a_{i_1,\ldots,i_a,\ldots,i_b,\ldots i_m} = a_{i'_1,\ldots,i'_a,\ldots,i'_b,\ldots i'_m}$$

and thus the symmetry is preserved.

4.

Let a_{ij} be a $(0, 2)$-tensor. Define $b_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $c_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$. By the first problem b_{ij} and c_{ij} are tensors. Furthermore,

$$\begin{aligned} b_{ji} &= \tfrac{1}{2}(a_{ji} + a_{ij}) = \tfrac{1}{2}(a_{ij} + a_{ji}) = b_{ij} \\ c_{ji} &= \tfrac{1}{2}(a_{ji} - a_{ij}) = -\tfrac{1}{2}(a_{ij} - a_{ji}) = -c_{ji} \end{aligned}$$

so b_{ij} is symmetric and c_{ij} is antisymmetry. Finally, it is clear that $a_{ij} = b_{ij} + c_{ij}$.

5.

(a)

The object δ_{ij} satisfies the tensor property with respect to a transformation Λ exactly when,

$$\delta_{i'j'} = \Lambda^i_{i'} \Lambda^j_{j'} \delta_{ij} = \Lambda^i_{i'} \Lambda^i_{j'}$$

which is neatly summarized by the equivalent condition on Λ as a matrix that $\Lambda^\top \Lambda = I$. Therefore δ_{ij} is a tensor with respect to exactly the orthogonal transformations. In particular, reflections are orthogonal because they preserve dot products so reflections also preserve δ_{ij}.

(b)

The easiest way to see the failure of ϵ_{ijk} to be a tensor is to consider the cross product of $(1, 0)$-tensors a and b i.e. $c_i = \epsilon_{ijk} a^j b^k$. Under a full parity inversion, (which, in 3D, is a rotation plus a single reflection), we find $c_{i'} = \epsilon_{ijk} a^{j'} b^{k'} = \epsilon_{ijk}(-a^j)(-b^k) = \epsilon_{ijk} a^j b^k = c_i$. However, if c_i were a type $(0, 1)$-tensor then it would transform as $c_i \mapsto -c_i$ under a parity inversion. Since it does not, we know that ϵ_{ijk} cannot satisfy the tensor property for such transformations otherwise its contraction with other tensors must also be tensorial.

6.

A linear map $\mathbf{B} : V \to V$ can be expressed in a basis $\mathbf{e}_i$ via the rule $\mathbf{B}(\mathbf{e}_j) = B^i_j \mathbf{e}_i$. Therefore, $B^i_j = \mathbf{e}^i(\mathbf{B}(\mathbf{e}_j))$. Consider these coefficients under a transformation,

$$B^{i'}_{j'} = \mathbf{e}^{i'}(\mathbf{B}(\mathbf{e}_{j'})) = \Lambda^{i'}_i \mathbf{e}^i(\mathbf{B}(\Lambda^j_{j'}\mathbf{e}_j)) = \Lambda^{i'}_i \Lambda^j_{j'} \mathbf{e}^i(\mathbf{B}(\mathbf{e}_j))\Lambda^{i'}_i \Lambda^j_{j'} B^i_j$$

using linearity. Therefore, B^i_j transforms as a $(1,1)$-tensor.

7.

Consider the metric $(0,2)$- tensor g_{ij}. We define its inverse by the equation $g_{ij}g^{jk} = \delta^k_i$. Under a transformation, the transformed version of g^{jk} must still satisfy the transformed equation,

$$g_{i'j'}g^{j'k'} = \delta^{k'}_{i'} \implies \Lambda^i_{i'}\Lambda^j_{j'}g_{ij}g^{j'k'} = \delta^{k'}_{i'}$$

This equation is most easily manipulated in matrix form,

$$\Lambda^\top g \Lambda g'^{-1} = I$$

which gives,

$$g'^{-1} = \Lambda^{-1}g^{-1}(\Lambda^{-1})^\top$$

Rewriting this in components,

$$g^{i'j'} = \Lambda^{i'}_i \Lambda^{j'}_j g^{ij}$$

which shows that g^{ij} transforms as a $(2,0)$-tensor.

8.

Consider the tensor $\epsilon^{ijk}\epsilon_{jkm}$. When $i \neq m$ then there do not exist values for j and k such that both ijk and jkm are permutations of 123 since all four of $ijkm$ must be different but take on at most three values. Thus, for $i \neq m$ we have $\epsilon^{ikj}\epsilon_{jkm} = 1$. Furthermore, when $i = m$, then the only nonzero terms come from i, j, k all different. Given a fixed i there are two such terms. Then, $\epsilon^{ijk} = \epsilon_{jki} = \epsilon_{jkm}$ so these two terms are both $+1$. Therefore,

$$\epsilon^{ijk}\epsilon_{jkm} = 2$$

In summary,

$$\epsilon^{ijk}\epsilon_{jkm} = 2\delta^i_m$$

9.

Assume the metric signature $\eta = \mathrm{diag}(-,+,+,+)$. We have the tensors in a given basis,

$$X^\mu{}_\nu = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} \implies X_\mu{}^\nu = \eta_{\mu\alpha}\eta^{\nu\beta}X^\alpha{}_\beta = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$$

Furthermore,

$$X_{[\mu,\nu]} = \frac{1}{2}\begin{pmatrix} -2 & -1 & 0 & -3 \\ -1 & 0 & 4 & 3 \\ 0 & 4 & 0 & 0 \\ -3 & 3 & 1 & -4 \end{pmatrix} \qquad X^{(\mu,\nu)} = \frac{1}{2}\begin{pmatrix} 0 & -1 & -2 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & -2 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

and likewise,

$$X^{\lambda}_{\lambda} = -4$$

Next consider the vector,

$$V^{\mu} = (-1, 2, 0, -2)$$

which gives,

$$V^{\mu}V_{\mu} = 7 \qquad V_{\mu}X^{\mu\nu} = (4, -2, 5, 7)$$

10.

(a)

Suppose that a particle moves with constant acceleration $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ with respect to an inertial frame (t, x, y, x). Then the proper time satisfies,

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \frac{a^2t^2}{c^2}}$$

Integrating this equation gives,

$$\tau = \frac{c}{2a}\left(\frac{at}{c}\sqrt{1 - \frac{a^2t^2}{c^2}} + \sin^{-1}\left(\frac{at}{c}\right)\right)$$

(b)

Now suppose that the particle experiences constant *proper* acceleration i.e. $\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}\tau}$ is constant in the rest frame of the particle. In this frame,

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}\tau} = (0, a, 0, 0)$$

where a is the coordinate acceleration in the rest frame (normalized by factors of c in the following) and thus also the proper acceleration. A Lorentz transformation to the frame (t, x, y, z) then gives,

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}\tau} = (\gamma\beta a, \gamma a, 0, 0)$$

However, in terms of coordinate variables,

$$u^{\alpha} = (\gamma, \gamma\beta, 0, 0)$$

and thus,

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}\tau} = \gamma(\dot{\gamma}, \dot{\gamma}\beta + \gamma\dot{\beta}, 0, 0) = (\gamma\dot{\gamma}, \gamma\dot{\gamma}\beta + \gamma^2\dot{\beta}, 0, 0)$$

where,

$$\dot{\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\sqrt{1-\beta^2}} = \frac{\beta\dot{\beta}}{(1-\beta^2)^{\frac{3}{2}}} = \gamma^3\beta\dot{\beta}$$

Comparing the two expressions for the four-acceleration we find that,

$$\dot{\beta} = \gamma^{-3}a$$

Thus,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}} a$$

which implies that,

$$v(t) = \frac{at}{\sqrt{1+\left(\frac{at}{c}\right)^2}}$$

Finally, the proper time is given by,

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1-\beta^2} = \frac{1}{\sqrt{1+\left(\frac{at}{c}\right)^2}}$$

which implies that,

$$\tau = \sinh^{-1}\left(\frac{at}{c}\right)$$

(c)

Finally, suppose that a^α is constant in the fixed frame (t,x,y,z). Then we have,

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = (0,a,0,0)$$

in this frame at all times. However, as before,

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = (\gamma^4\beta\dot{\beta}, \gamma^4\dot{\beta}, 0, 0)$$

This is inconsistent unless the time-component of a^α is allowed to vary. Making this assumption, we find,

$$u^x(\tau) = a\tau$$

Therefore,

$$\gamma\beta = a\tau \implies \gamma^2 - 1 = (a\tau)^2$$

which implies that

$$\gamma = \sqrt{1+(a\tau)^2}$$

and thus,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \sqrt{1+(a\tau)^2}$$

Integrating this we find,

$$t = \int_0^t \mathrm{d}\tau\, \sqrt{1+(a\tau)^2} = \tfrac{\tau}{2}\sqrt{1+(a\tau)^2} + \tfrac{1}{2a}\sinh^{-1}(a\tau)$$

This cannot be inverted explicitly.

11.

Consider a source emitting uniformly at a constant definite frequency ν with luminosity L in its rest frame.

(a)

In the rest frame of the particle S, the derivative of energy momentum vector has the form,

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = (-L, 0, 0, 0)$$

since proper time coincides with coordinate time and the radiation carries away zero total momentum. Therefore, boosting to the laboratory frame S' we find,

$$\frac{\mathrm{d}p'^\alpha}{\mathrm{d}\tau} = (-\gamma L, -\gamma\beta L, 0, 0)$$

Thus, in this frame,

$$\frac{\mathrm{d}E'}{\mathrm{d}\tau} = -\gamma L$$

implying that,

$$L' = -\frac{\mathrm{d}E'}{\mathrm{d}t'} = L$$

recovering the well-known fact that radiated power is a Lorentz invariant.

(b)

Consider a photon emitted in the frame S with angular variables (θ, ϕ) with the pole $\theta = 0$ oriented along the direction of motion. This photon will have a wavevector (ignoring factors of c for now),

$$k^\alpha = (\nu, \nu\cos\theta, \nu\sin\theta\cos\phi, \nu\sin\theta\sin\phi)$$

Under a Lorentz transformation, this wavevector becomes,

$$k^\alpha = (\gamma\nu(1 + \beta\cos\theta), \gamma\nu(\cos\theta + \beta), \nu\sin\theta\cos\phi, \nu\sin\theta\sin\phi)$$

which means that in S' the photon has frequency $\nu' = \gamma(1 + \beta\cos\theta)\nu$ and angular coordinates (θ', ϕ') with,

$$\cos\theta' = \frac{\cos\theta + \beta}{\beta\cos\theta + 1} \qquad \sin\theta' = \frac{\sin\theta}{\gamma(\beta\cos\theta + 1)}$$

and $\phi' = \phi$. The number of photons is conserved so the number of photons $N(\theta)$ in a solid angle from polar angle 0 to θ must satisfy $N'(\theta') = N(\theta)$. Thus, the angular distribution of photons in S' is computed as,

$$\frac{\mathrm{d}N'}{\mathrm{d}\Omega'} = \frac{1}{2\pi}\frac{\mathrm{d}N'(\theta')}{\mathrm{d}\cos(\theta')} = \frac{1}{2\pi}\frac{\mathrm{d}\cos\theta}{\mathrm{d}\cos\theta'}\frac{\mathrm{d}N(\theta)}{\mathrm{d}\cos\theta} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}\cos\theta'}\frac{Lt}{4\pi h\nu}$$

because the angular distribution of photons emitted in a time t is,

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega} = \frac{Lt}{4\pi h\nu}$$

is uniform in the rest frame S. An identical argument running the opposite direction will show that,

$$\cos\theta = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta')}$$

Therefore,

$$\frac{\mathrm{d}\cos\theta}{\mathrm{d}\cos\theta'} = \frac{1}{1-\beta\cos\theta'} + \frac{\beta(\cos\theta' - \beta)}{(1-\beta\cos\theta')^2} = \frac{1}{\gamma^2(1-\beta\cos\theta')^2}$$

Therefore, the angular distribution of photons in the lab frame S' is given by,

$$\frac{\mathrm{d}N'}{\mathrm{d}\Omega'} = \frac{Lt}{4\pi h\nu}\frac{1}{\gamma^2(1-\beta\cos\theta')^2}$$

We can likewise compute the angular distribution of power by computing the energy flow in photons though a given solid angle. This is simply,

$$\frac{\mathrm{d}L'}{\mathrm{d}\Omega'} = \frac{\mathrm{d}N'}{\mathrm{d}\Omega'\mathrm{d}t'}h\nu'(\theta') = \frac{L}{4\pi}\frac{\mathrm{d}t}{\mathrm{d}t'}\frac{\nu'}{\nu}\frac{1}{\gamma^2(1-\beta\cos\theta')^2}$$

Now,

$$\frac{\nu'}{\nu} = \gamma(1+\beta\cos\theta) = \gamma\left(1 + \beta\frac{\cos\theta' - \beta}{1-\beta\cos\theta'}\right) = \frac{\gamma(1-\beta^2)}{1-\beta\cos\theta'} = \frac{1}{\gamma(1-\beta\cos\theta')}$$

Furthermore, for the time interval t in the rest frame during which the photons are emitted, we have $t' = \gamma t$ in the frame S'. Thus,

$$\frac{\mathrm{d}L'}{\mathrm{d}\Omega'} = \frac{L}{4\pi}\frac{1}{\gamma^4(1-\beta\cos\theta')^3}$$

A good check of our work,

$$L' = \int \frac{\mathrm{d}L'}{\mathrm{d}\Omega'}\mathrm{d}\Omega' = \frac{L}{4\pi\gamma^4}\int_{-1}^{1}\frac{2\pi\mathrm{d}(\cos\theta\ )}{(1-\beta\cos\theta')^3} = \frac{L}{4\pi\gamma^4}(4\pi\gamma^4) = L$$

which shows that the total radiated power L' in the frame S' satisfied $L' = L$ so the total radiated power is indeed Lorentz invariant.

(c)

We have computed the frequency ν' of photons in the frame S' at a given angular position (θ', ϕ') to be,

$$\frac{\nu'}{\nu} = \frac{1}{\gamma(1-\beta\cos\theta')}$$

Therefore, averaging over the angular distribution, we find,

$$\langle\nu'\rangle = \frac{1}{N}\int\frac{\mathrm{d}N'}{\mathrm{d}\Omega'}\nu'\mathrm{d}\Omega = \frac{\nu}{4\pi}\int\frac{1}{\gamma^3(1-\beta\cos\theta')^3} = \gamma\nu$$

so the average frequency is enhanced over the frequency of the emitted radiation in the source frame. Thus, the mean photon energy in S' is $\langle h\nu'\rangle = \gamma h\nu$. This is clear from the invariance of the radiated power L. The number of photons emitted during some interval is fixed and the energy emitted is larger by γ in the frame S' since it is a constant power over a time interval longer by a factor of γ. Therefore, the energy of each photon must also be larger by a factor of γ.