Classical Mechanics from the Symplectic Viewpoint

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February 4, 2023

Contents

1 Introduction

1.1 Invitation

Consider a Hamiltonian system on $\mathbb{R}^n$ giving a phase space $\mathbb{R}^{2n}$ with coordinates, $q^1, \ldots, q^n, p_1, \ldots, p_n$ and a Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$. In these coordinates, Hamilton's equations of motion are,

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

Part of the power of the Hamiltonian framework is the greater freedom to reparametrize the problem beyond a simple coordinate change of phase space in the Lagrangian framework. Such reparametrizations are given by so called *canonical* transformations which are reparametrizations that preserve the "form" of Hamilton's equations. The desire to formalize this notion leads us to symplectic geomery.

The first step will be to put Hamilton's equation is a coordinate independent form in which canonical transformations will be ellucidated. Notice that the 1-form,

$$\mathrm{d}H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q^i} \mathrm{d}q_i + \frac{\partial H}{\partial p_i} \mathrm{d}p_i \right)$$

and the vector field,

$$X_H = \frac{\mathrm{d}}{\mathrm{d}t}(q, p)(t) = \sum_{i=1}^{n} \left(\dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} \right)$$

are related by Hamilton's equations. To relate 1-forms and vector fields we need a 2-form,

$$\omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i$$

which we call the symplectic form. Notice that,

$$\omega(X_H, -) = \sum_{i=1}^{n} \left(\dot{q}_i \mathrm{d}p_i - \dot{p}_i \mathrm{d}q^i \right)$$

and therefore Hamilton's equations may be rewritten as,

$$\iota_{X_H} \omega = \mathrm{d}H$$

1.2 More General Configuration Spaces

There is no reason to restrict ourselves to Euclidean configuration space. In fact, a natural symplectic form

$$\omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i$$

arises on the phase space T^*Q of any configuration manifold Q. To see how this happens, we first construct the tautological 1-form θ on T^*Q. Let $X = T^*Q$ and $\pi : X \to Q$ be the fiber bundle projection. Then $\mathrm{d}\pi : TX \to TQ$ is induced. A point $x \in X$ corresponds to some point $q \in Q$ and map $\varphi : T_q Q \to \mathbb{R}$. Then define,

$$\theta_x = \varphi \circ \mathrm{d}\pi_x$$

Thus $\theta_x : T_x X \to \mathbb{R}$ is linear giving a section $\theta : X \to T^* X$.

If we choose a chart (U, ψ) of Q with local coordinate functions $q^1, \dots, q^n$ (where $q^i = x^i \circ \psi$ for $\psi : U \to \mathbb{R}^n$ and $x^i : \mathbb{R}^n \to \mathbb{R}$ are the standard coordinates) then there is an induced chart $(\tilde{U}, \tilde{\psi})$ of X defined as $\tilde{U} = \pi^{-1}(U)$ with $\tilde{\psi} : \tilde{U} \to \mathbb{R}^n \times \mathbb{R}^n$ via,

$$\tilde{\psi}(q, p_i \mathrm{d}q^i) = (\psi(q), p_1, \dots, p_n)$$

where $\mathrm{d}q^i$ are derivates of the coordinate functions $q^i : U \to \mathbb{R}$. Notice that,

$$\mathrm{d}q^i = \mathrm{d}(x^i \circ \psi) = \mathrm{d}x_i \circ \mathrm{d}\psi = \psi^* \mathrm{d}x^i$$

Then let $p_i : \tilde{U} \to \mathbb{R}$ be the coordinate functions of the second projection $\tilde{U} \to \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Abusing notation, we write q^i for the pull back of q^i to Q, explicitly $q^i : \tilde{U} \xrightarrow{\pi} U \xrightarrow{q^i} \mathbb{R}$. Now we compute θ_x on the local vector fields $\frac{\partial}{\partial q^i}$ and $\frac{\partial}{\partial p_i}$. For the point $x = (q, \varphi)$ we have,

$$\theta_x \left(\frac{\partial}{\partial q^i} \right) = \varphi \left(\frac{\partial}{\partial q^i} \right) = p_i$$

where $\varphi = p_i \mathrm{d}q^i$ since $\mathrm{d}\pi \left(\frac{\partial}{\partial q^i} \right) = \frac{\partial}{\partial q^i}$ using that the first q^i is really $q^i \circ \pi$. Furthermore, clearly $\mathrm{d}\pi \left(\frac{\partial}{\partial p_i} \right) = 0$ since on the chart side $\tilde{U} \to U$ corresponds to $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ via the first projection. Thus,

$$\theta_x \left(\frac{\partial}{\partial p_i} \right) = 0$$

Since these vector fields form a local frame of TX we find that,

$$\theta = \sum_{i=1}^{n} p_i \mathrm{d}q^i$$

Therefore, the symplectic 2-form $\omega = -\mathrm{d}\theta$ is given in local coordinates as,

$$\omega = -\mathrm{d}\theta = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i$$

Therefore out “natural” symplectic form for doing Hamiltonian mechanics actually arises quite canonically on the cotangent space of any manifold or as the physicists would say: on the phase space induced by any configuration space.

2 Symplectic Geometry

Definition 2.0.1. Let V be a finite k-vectorspace and $\omega \in \bigwedge^2 V^*$ a 2-form. We say that ω is *nondegenerate* if for all nonzero $v \in V$ the map $\omega(v, -) \in V^*$ is nonzero. Equivalently, ω is nondegenerate exactly when the map $V \to V^*$ defined by $v \mapsto \omega(v, -)$ is an isomorphism.

Lemma 2.0.2. If ω is a nondegenerate 2-form on V then $\dim V = 2n$ is even.

Proof. Choose a basis $e_1, \dots, e_k$ of V. Then we have a matrix $M_{ij} = \omega(e_i, e_j)$ which is antisymmetric. Then ω is nondegenerate implies that $\det M \neq 0$. However, $M^\top = -M$ so we must have,

$$\det M = \det(-M) = (-1)^{\dim V} \det M$$

Thus $\dim V = 2n$ is even. $\square$

Definition 2.0.3. Let M be a smooth $2n$-manifold. A *symplectic form* ω on M is a closed non-degenerate 2-form. We say that the pair (M, ω) is a *symplectic manifold*. A *symplectomorphism* $f : (M, \omega_M) \to (N, \omega_N)$ is a smooth map $f : M \to N$ such that $f^*\omega_N = \omega_M$.

Remark. Consider a vector field X on M. Such a vector field defines a flow $\phi_t : M \to M$. We consider when this flow preserves the symplectic structure. This occurs when ϕ_t is a symplectomorphism i.e. when $\phi_t^*\omega = \omega$. Now, recall that, the Lie derivative is defined via,

$$\mathcal{L}_X\omega = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(\phi_t^*\omega\right)$$

Therefore $\phi_t : M \to M$ is symplectic iff $\mathcal{L}_X\omega = 0$.

Definition 2.0.4. We say a vector field X on M is *symplectic* if $\mathcal{L}_X\omega = 0$.

Definition 2.0.5. We say a vector field X on M is *Hamiltonian* if there exists a smooth function $H : M \to \mathbb{R}$ such that $\iota_X\omega = \mathrm{d}H$.

Lemma 2.0.6. Hamiltonain vector fields are symplectic.

Proof. Let X be Hamiltonian such that $\iota_X\omega = \mathrm{d}H$. Then, we use Cartan's magic formula,

$$\mathcal{L}_X\omega = \mathrm{d}(\iota_X\omega) + \iota_X\mathrm{d}\omega$$

Applying $\iota_X\omega = \mathrm{d}H$ and using $\mathrm{d}\omega = 0$ we find,

$$\mathcal{L}_X\omega = \mathrm{d}(\mathrm{d}H) = 0$$

□

3 Symptectic Geometry

Definition 3.0.1. A *symplectic form* on M is a closed non-degenerate 2-form ω. We say that (M, ω) is a *symplectic manifold*. A *symplectomorphism* $f : (M, \omega_M) \to (N, \omega_N)$ is a smooth map $f : M \to N$ such that $f^*\omega_N = \omega_M$.

Lemma 3.0.2. Symplectic forms can only exist on even-dimensional manifolds.

Proof. Locally, a symplectic form ω is a nondegenerate anti-symmetric bilinear form $S : T_pM \times T_pM \to \mathbb{R}$. So we have $S^\top = -S$ and $\det S \neq 0$. However,

$$\det S = \det S^\top = \det(-S) = (-1)^n \det S$$

since $\det S \neq 0$ we must have $(-1)^n = 1$ i.e. n is even. □

3.1 Symplectic Vector Fields

Definition 3.1.1. We say that a vector field X on (M, ω) is symplectic if $\mathcal{L}_X\omega = 0$.

Remark. We see that the condition $\mathcal{L}_X\omega = 0$ that a vector field be symplectic is equivalent to the condition that its flows $\phi_t : M \to M$ be symplectomorphisms since,

$$\mathcal{L}_X\omega = \frac{\mathrm{d}}{\mathrm{d}t}((\phi_t)^*\omega) = 0$$

Thus, symplectic vector fields are fields whose flows preserve the symplectic structure.

Lemma 3.1.2. Let (M, ω) be symplectic. A vector field X is symplectic iff $\iota_X\omega$ is closed.

Proof. From Cartan,

$$\mathcal{L}_X\omega = \iota_X \mathrm{d}\omega + \mathrm{d}\iota_X\omega = \mathrm{d}\iota_X\omega$$

because $\mathrm{d}\omega = 0$. Therefore,

$$\mathcal{L}_X\omega = 0 \iff \mathrm{d}(\iota_X\omega) = 0$$

□

Definition 3.1.3. We say that a vector field X on (M, ω) is Hamiltonian if the form $\iota_X\omega \in \Omega^1(M)$ is exact i.e. there exists a smooth function $H : M \to \mathbb{R}$ such that,

$$\iota_X\omega = \mathrm{d}H$$

Remark. Note that since ω is non-degenerate, the map $\omega : TM \to \Omega^1(M)$ via $X \mapsto \iota_X\omega$ is an isomorphism and thus we can consider $\omega^{-1} : \Omega^1(M) \to TM$. Then the above condition is that,

$$X = \omega^{-1}(\mathrm{d}H)$$

Lemma 3.1.4. Hamiltonian vector fields are symplectic.

Proof. Let X be Hamiltonian then $\iota_X\omega$ is exact and thus closed so X is symplectic. Explicitly,

$$\mathcal{L}_X\omega = \iota_X \mathrm{d}\omega + \mathrm{d}\iota_X\omega$$

Since ω is a symplectic form $\mathrm{d}\omega = 0$ and since X is Hamiltonainm $\iota_X\omega$ is exact and thus closed so $\mathrm{d}\iota_X\omega = 0$. Therefore,

$$\mathcal{L}_X\omega = 0$$

so X is symplectic. □

Lemma 3.1.5. Symplectic and Hamiltonian vector fields form Lie subalgebras. Furthermore,

$$[\mathfrak{sym}, \mathfrak{sym}] \subset \mathfrak{ham}$$

where we explicitly see that if X, Y are symplectic then $[X, Y]$ is Hamiltonian with Hamiltonian function $\iota_X\iota_Y\omega = \omega(Y, X)$ meaning that,

$$\iota_{[X,Y]}\omega = \mathrm{d}(\iota_X\iota_Y\omega) = \mathrm{d}(\omega(Y, X))$$

Proof. We know that,

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega$$

so if X, Y are symplectic then so is $[X, Y]$. Furthermore,

$$\iota_{[X,Y]}\omega = \mathcal{L}_X\iota_Y\omega - \iota_Y\mathcal{L}_X\omega$$

However, $\mathcal{L}_X \omega = 0$ since X is symplectic. Furthermore, by Cartan's formula,

$$\iota_{[X,Y]}\omega = \mathcal{L}_X \iota_Y \omega = \iota_X(\mathrm{d}\iota_Y \omega) + \mathrm{d}(\iota_X \iota_Y \omega)$$

However, since Y is symplectic, $\iota_Y \omega$ is closed and thus,

$$\iota_{[X,Y]}\omega = \mathrm{d}(\iota_X \iota_Y \omega) = \mathrm{d}(\omega(Y, X))$$

which is exact so $[X, Y]$ is Hamiltonian. $\square$

Remark. We have $\mathcal{L}_X \mathrm{d}\omega = \mathrm{d}(\mathcal{L}_X \omega)$ because d is a natural transformation in the sense that $f^*\mathrm{d} = \mathrm{d}f^*$ for any smooth map and, in particular, for the flow of X.

Proposition 3.1.6. Let (M, ω) be a symplectic manifold. Then,

$$H^1_{\mathrm{dR}}(M) \cong \mathfrak{sym}/\mathfrak{ham}$$

Indeed, there are exact sequences,

$$0 \longrightarrow H^0_{\mathrm{dR}}(M) \longrightarrow C^\infty(M) \xrightarrow{f \mapsto X_f} \mathfrak{ham} \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{ham} \longrightarrow \mathfrak{sym} \xrightarrow{X \mapsto \iota_X \omega} H^1_{\mathrm{dR}}(M) \longrightarrow 0$$

Proof. Obvious from the correspondences between $\mathfrak{sym}$ and closed forms and $\mathfrak{ham}$ and exact forms. $\square$

3.2 Coisotropic Bundles

Definition 3.2.1. Let (V, ω) be a symplectic vector space. Let $W \subset V$ be a subspace. Then the *symplectic dual* of W is,

$$W^\omega = \{v \in V \mid \forall w \in W : \omega(v, w) = 0\}$$

We say a subspace $W \subset V$ is,

(a) *isotropic* if $W \subset W^\omega$ or equivalently $\omega|_W = 0$

(b) *Lagrangian* if $W = W^\omega$

(c) *coisotropic* if $W \supset W^\omega$

(d) *symplectic* if $W \cap W^\omega = 0$ or equivalently $\omega|_W$ is nondegenerate.

Remark. In what follows, let (V, ω) be a fixed symplectic vector space.

Proposition 3.2.2. For any $W \subset V$ we have,

$$\dim W + \dim W^\omega = \dim V$$

Therefore, if $\dim V = 2n$ then,

(a) if W is isotropic then $\dim W \leq n$

(b) if W is Lagrangian then $\dim W = n$

(c) if W is coisotropic then $\dim W \geq n$

Proof. Consider the map $\varphi : V \to V^* \to W^*$ given by $v \mapsto \omega(v, -)|_W$. By definition $W^\omega = \ker \varphi$. Furthermore, φ is surjective because each map $V \to V^* \to W^*$ is surjective since ω is nondegenerate. Therefore, by the rank-nullity theorem,

$$\dim V = \dim W^* + \dim W^\omega = \dim W + \dim W^\omega$$

Then the rest follows immediately. □

Proposition 3.2.3. We have the following properties,

(a) if $W_1 \subset W_2 \subset V$ then $W_2^\omega \subset W_1^\omega$

(b) if $W \subset V$ then $(W^\omega)^\omega = W$

(c) W is isotropic $\iff$ W^ω is coisotropic

(d) W is symplectic $\iff$ W^ω is symplectic

Proof. The first is clear since if $v \in W_2^\omega$ then for any $w \in W_1 \subset W_2$ we have $\omega(v, w) = 0$ so $v \in W_1^\omega$. If $w \in W$ and $v \in W^\omega$ then by definition $\omega(w, v) = 0$ so $W \subset (W^\omega)^\omega$. Furthermore, they have the same dimension by the previous result so we conclude $W = (W^\omega)^\omega$. From this the next two properties are obvious. □

Definition 3.2.4. Let $\tau \in \{\text{isotropic}, \text{Lagrangian}, \text{coisotropic}, \text{symplectic}\}$ be a property. Then define,

(a) let (E, ω) be a symplectic vector bundle on an a manifold M. We say that a subbundle $E' \subset E$ has τ if each $E'_x \subset E_x$ in the symplectic vector space (E_x, ω_x) has τ for all $x \in M$

(b) if (M, ω) is a symplectic manifold, a τ vector bundle is a subbundle $E \subset TM$ which has τ with respect to the symplectic structure (TM, ω),

(c) Let $N \subset M$ be a submanifold. We say that N is τ if $TN \subset TM|_N$ has τ with respect to the symplectic structure $(TM|_N, \omega)$.

Proposition 3.2.5. Let (M, ω) be a symplectic manifold and $N \subset M$ a coisotropic submanifold. Then $(TN)^\omega$ is an integrable foliation on N whose leaves are isotropic submanifolds of M.

Proof. We need to show that $(TN)^\omega$ is involutive to apply Forbenius' theorem. If $X, Y \in \Gamma(N, (TN)^\omega)$ then we want to show that $[X, Y] \in \Gamma(N, (TN)^\omega)$. This is equivalent to showing that for any $Z \in \Gamma(N, TN)$ we have $\omega([X, Y], Z) = 0$. Indeed,

$$\mathrm{d}\omega(X, Y, Z) = X(\omega(Y, Z)) - Y(\omega(X, Z)) + Z(\omega(X, Y)) - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X)$$

However, X, Y are sections of $(TN)^\omega \subset TN$ and Z is a section of TN so $\omega(X, Z) = \omega(Y, Z) = \omega(X, Y) = 0$. Furthermore, since N is coisotropic $(TN)^\omega \subset TN$ and TN is involutive so $\omega([X, Z], Y) = \omega([Y, Z], X) = 0$ because $[X, Z], [Y, Z]$ are sections of TN and X, Y are sections of $(TN)^\omega$. Therefore,

$$\mathrm{d}\omega(X, Y, Z) = \omega([X, Y], Z)$$

but $\mathrm{d}\omega = 0$ so we see that $[X, Y] \in \Gamma(N, (TN)^\omega)$. Now let $L \subset N$ be a leaf of the foliation. This means that $TL = (TN)^\omega|_L$. We need to show this is isotropic inside $(TM|_L, \omega)$. However, $(TN)|_L \subset (TM)|_L$ is coisotropic so its dual is isotropic completing the proof. □

3.3 Poisson Bracket

Definition 3.3.1. Let $f, g : M \to \mathbb{R}$ be functions and let $X_f = \omega^{-1}(\mathrm{d}f)$ and $X_g = \omega^{-1}(\mathrm{d}g)$ be the associated Hamiltonian vector fields. Then we define the *Poisson bracket* via,

$$\{f, g\} = \omega(X_f, X_g)$$

Remark. From the definitions of X_f and X_g,

$$\begin{aligned}\{f, g\} &= \omega(X_f, X_g) = \mathrm{d}f(X_g) = X_g(f) = \mathcal{L}_{X_g} f \\ &= -\omega(X_g, X_f) = -\mathrm{d}g(X_f) = -X_f(g) = -\mathcal{L}_{X_f} g\end{aligned}$$

So $\{f, g\}$ represents the flow of f along the vector field generated by g.

Lemma 3.3.2. $[X_f, X_g] = -X_{\{f,g\}}$

Proof. We have shown that if X and Y are symplectic then,

$$\iota_{[X,Y]}\omega = \mathrm{d}(\omega(Y, X))$$

Therefore,

$$X_{\omega(Y,X)} = \omega^{-1}(\mathrm{d}(\omega(Y, X))) = [X, Y]$$

Now applying this to X_f and X_g we find,

$$[X_f, X_g] = \omega^{-1}(\mathrm{d}(\omega(X_g, X_f)) = -\omega^{-1}(\mathrm{d}\{f, g\}) = -X_{\{f,g\}}$$

$\square$

Proposition 3.3.3. The Poisson bracket on smooth functions forms a Lie algebra.

Proof. Clearly the Poisson bracket is bilinear. Furthermore, it is antisymmetric because,

$$\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\}$$

The Jacobi identity is equivalent to the fact that $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ via $\xi \mapsto [\xi, -]$ is a Lie algebra homomorphism.

In the current case, $\mathrm{ad}_f(g) = \{f, g\} = -X_f(g)$ so $\mathrm{ad}_f = -X_f$ as a derivation. Then we know that,

$$[\mathrm{ad}_f, \mathrm{ad}_g] = [-X_f, -X_g] = -X_{\{f,g\}} = \mathrm{ad}_{\{f,g\}}$$

since the commutator of vector fields is their comutator as differential operators. $\square$

Proposition 3.3.4. The map $\varphi : C^\infty(M) \to \mathfrak{ham}$ defined by $f \mapsto -X_f = -\omega^{-1}(\mathrm{d}f)$ is a morphism of Lie algebras.

Proof. Immediate from $-X_{\{f,g\}} = [X_f, X_g] = [-X_f, -X_g]$. $\square$

Remark. Unfortunately the physicists convention for Hamilton's equations plus the definition of the Poisson bracket (mathematicians might have defined the Poisson bracket with a minus sign to agree with the convention of Lie brackets $[X, Y] = \mathcal{L}_X Y$ where as $\{f, g\} = \mathrm{d}f(X_g) = \mathcal{L}_{X_g} f$ explaining the sign difference between the two brackets) do not permit the map $f \mapsto X_f$ to be a Lie algebra homomorphism. One might attempt to remedy this by replacing X_f by $-X_f$ however this messes up the form of Hamilton's equations unless simultaneously ω is replaced by $-\omega$ which then messes up the sign of $\{-, -\}$. Thus the only true remedy is reversing either the Poisson bracket or the Lie bracket. However, $\mathfrak{X}(M)$ is sometimes given the opposite Lie algebra structue, remedying our conundrum, because this is the induced Lie bracket on $\mathfrak{X}(M) = \mathrm{Lie}(\mathrm{Diff}(M))$. Here we will take the opposite convention

3.4 Conserved Quantities

Suppose we have a symplectic manifold (M, ω) and a time-dependent function $H_- : M \times \mathbb{R} \to \mathbb{R}$. This represents a dynamical system generated by the Hamiltonian vector field $X_t = X_{H_t}$. Then a function $f : M \to \mathbb{R}$ is a *conserved quantity* if $\{f, H_t\} = 0$ for all t. This is because,

$$\mathcal{L}_{X_{H_t}} f = \mathrm{d}f(X_{H_t}) = \omega(X_f, X_{H_t}) = \{f, H_t\} = 0$$

meaning that f is preserved along the flow generated by H_t. However, we can turn this around and say that,

$$\mathcal{L}_{X_f} H_t = \mathrm{d}H_t(X_f) = \omega(X_{H_t}, X_f) = \{H_t, f\} = 0$$

so conserved quantities are equivalently functions which generate transformations (flows defined infinitesimally by the vector field X_f) which preserve H_t meaning H_t is symmetric under the flow X_f. Finding conserved quantities is what makes the dynamical system *integrable*. The best conserved quantities are those which Poisson commute. A fully integrable system will be one that admitts a "full" set of commuting conserved quantities. We will see that if $\dim M = 2n$ there can be at most n independent conserved quantities (where H_t is included in the set). This full set of quantities determines the trajectory to lie in some Lagrangian leaves.

Proposition 3.4.1. Let $f : M \to \mathbb{R}^k$ be a smooth map with $f = (f_1, \dots, f_k)$ such that $\{f_i, f_j\} = 0$. Let N be a regular level set of f. Then N is coisotropic and the X_{f_i} span its isotropic distribution.

Proof. A vector field $X \in \Gamma(N, TM|_N)$ is a section of TN iff $\mathrm{d}f_i(X) = 0$ for each i. This is equivalent to $\omega(X_{f_i}, X) = 0$ for all i. In partcular,

$$\mathrm{d}f_i(X_{f_j}) = \omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0$$

and therefore X_{f_j} are sections of TN so are parallel to N. Thus, if X is a section of $(TN)^\omega$ then,

$$\mathrm{d}f_j(X) = \omega(X_{f_j}, X) = 0$$

so X is a section of TN so $(TN)^\omega \subset TN$ proving that N is coisotropic. Furthermore, every section X of TN satisfies,

$$\omega(X_{f_i}, X) = \mathrm{d}f_i(X) = 0$$

and thus the X_{f_i} are sections of $(TN)^\omega$ so lie in the isotropic foliation. Furthermore, N is a regular level set so the $\mathrm{d}f_i$ are everywhere independent in $(T^*M)|_N$ and thus since $\omega^\# : TM \to T^*M$ is an isomrophism the X_{f_i} are independent. Since,

$$\operatorname{rank}(TN)^\omega = \dim M - \operatorname{rank} TN = \dim M - (\dim M - k) = k$$

and the X_{f_j} are sections of $(TN)^\omega$ which are everywhere independent on N so these span the isotropic distribution which comprises the tangent spaces of the leaves of its isotropic foliation. $\square$

Corollary 3.4.2. If $f : M \to \mathbb{R}^k$ are generically independent Poisson commuting functions, meaning f has a regular value, (in particular if f is a submersion) then $k \le \frac{1}{2} \dim M$.

Proof. Indeed, $\operatorname{rank}(TN)^\omega = k$ and $(TN)^\omega \subset (TM)|_N$ is an isotropic subspace so $k \le \frac{1}{2} \dim M$. $\square$

4 Hamiltonian Actions

(THERE IS A PROBLEM HERE WITH LEFT VS RIGHT INVT VECTOR FIELDS $\mathfrak{X}(M)$ NEEDS OPPOSITE LIE BRACKET)

Lemma 4.0.1. Let $\rho : G \times M \to M$ be a smooth action of a Lie group on a smooth manifold. Then there is a Lie algebra map $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ given by $\rho(\xi)_m = \mathrm{d}\rho_{(e,m)}(\xi, 0)$.

Proof. We need to show that $\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)]$. Let X_ξ denote the left-invariant vector field on G with $X_\xi(e) = \xi$. Then I claim that $\rho_*(X_\xi) = \rho(\xi)$. To see this, note that,

$$\mathrm{d}\rho_{(g,m)}(X_\xi(g)) = \mathrm{d}\rho_{(g,m)}(\mathrm{d}L_g(\xi)) = \mathrm{d}(\rho(g, -) \circ \rho)_{(e,m)}(\xi) = \mathrm{d}\rho(g, -)(\rho(\xi)_m)$$

(DO THISS!) □

Definition 4.0.2. A Lie group action $G \circlearrowright M$ on a symplectic manifold (M, ω) is *symplectic* if G acts through symplectomorphisms i.e. for each $g \in G$ the map $g : M \to M$ satisfies $g^*\omega = \omega$.

Remark. In this case, for each $\xi \in \mathfrak{g}$ the vector field $\rho(\xi)$ is symplectic.

Remark. We want a Hamiltonian action to be one that acts through Hamiltonian vector fields meaning $\rho(\xi) \in \mathfrak{ham}$ for each $\xi \in \mathfrak{ham}$. This means we know that $\iota_{\rho(\xi)}\omega$ is exact so $\iota_{\rho(\xi)}\omega = \mathrm{d}H_\xi$ for some choice of function $H_\xi : M \to \mathbb{R}$. However, we want to package the functions H_ξ together so they vary in a coherent way. This is formalized as follows.

Definition 4.0.3. Given a symplectic action $G \circlearrowright M$, a moment map is a smooth map $\mu : M \to \mathfrak{g}^*$ such that,

(a) $\mathrm{d}\langle \mu, \xi \rangle = \iota_{\rho(\xi)}\omega$

(b) $\mu : M \to \mathfrak{g}^*$ is G-equivariant where $G \circlearrowright \mathfrak{g}^*$ via the coadjoint action.

Definition 4.0.4. A *Hamiltonian action* $G \circlearrowright M$ is a symplectic action along with a choice of moment map $\mu : M \to \mathfrak{g}^*$.

Example 4.0.5. The translation action $\mathbb{R}^2 \circlearrowright \mathbb{R}^2$ clearly acts through Hamiltonian vector fields however is not Hamiltonian. To so see this, suppose there is a moment map $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ which is equivariant but $\mathbb{R}^2$ acts on the first copy by translation and on the section trivially so μ must be constant contradicting the first property.

Lemma 4.0.6. If G is connected, a moment map $\mu : M \to \mathfrak{g}^*$ is equivalent to a comoment map, a morphism of Lie algebras $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ such that $\mathrm{d}\tilde{\mu}(\xi) = \iota_{\rho(\xi)}\omega$.

Proof. Consider the natural correspondence between smooth functions $\mu : M \to \mathfrak{g}^*$ and linear maps $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$. Indeed, we define $\tilde{\mu}(\xi) = \langle \mu(-), \xi \rangle$ and $\mu(x) = \tilde{\mu}(-)(x)$. It is clear that $\mu : M \to \mathfrak{g}^*$ is G-equivariant iff $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ is G-equivariant where $G \circlearrowright C^\infty(M)$ via $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Indeed,

$$\tilde{\mu}(\mathrm{Ad}_g \cdot \xi) = \langle \mu(-), \mathrm{Ad}_g \cdot \xi \rangle = \langle \mathrm{Ad}_{g^{-1}} \cdot \mu(-), \xi \rangle = \left\langle \mu(g^{-1} \cdot -), \xi \right\rangle = g \cdot \langle \mu(-), \xi \rangle = g \cdot \tilde{\mu}(\xi)$$

and likewise,

$$\mu(g \cdot x) = \tilde{\mu}(-)(g \cdot x) = (g^{-1} \cdot \tilde{\mu}(-))(x) = \tilde{\mu}(\mathrm{Ad}_{g^{-1}} -)(x) = \mathrm{Ad}_g^* \cdot \tilde{\mu}(-)(x) = \mathrm{Ad}_g^* \cdot \mu(x)$$

Therefore, it suffices to show that G-equivariance of $\tilde{\mu}$ corresponds to $\tilde{\mu}$ being a map of Lie algebras. If $\tilde{\mu}$ is G-equivariant then differentiating $\tilde{\mu}(\mathrm{Ad}_g \cdot \eta) = g \cdot \tilde{\mu}(\eta)$ we see that,

$$\tilde{\mu}([\xi, \eta]) = -\rho(\xi)(\tilde{\mu}(\eta)) = -\omega^{-1}(\mathrm{d}\tilde{\mu}(\xi))(\tilde{\mu}(\eta)) = -X_{\tilde{\mu}(\xi)}(\tilde{\mu}(\eta)) = \{\tilde{\mu}(\xi), \tilde{\mu}(\eta)\}$$

Alternatively, if $\tilde{\mu}$ is a map of Lie algebras we need to integrate to find the G-action. Explicitly, we have shown that the derivative of,

$$\tilde{\mu}(\mathrm{Ad}_g \cdot \xi) - g \cdot \tilde{\mu}(\xi)$$

is zero at $g = e$ this suffices by the following lemma. ☐

Lemma 4.0.7. Let G be a connected Lie group with actions $G \circlearrowright M_1$ and $G \circlearrowright M_2$. Suppose that $f : M_1 \to M_2$ is a smooth map such that $\mathrm{d}f(\rho_1(\xi)) = \rho_2(\xi) \circ f$ for all $\xi \in \mathfrak{g}$. Then f is equivariant.

Proof. Since G is connected, it is generated by the image of $\exp : \mathfrak{g} \to G$. Therefore, it suffices to show that $f(\exp(tX) \cdot m) = \exp(tX) \cdot f(m)$. However, $\exp(tX) \cdot m$ and $\exp(tX) \cdot f(m)$ is given by the flows along the vector fields $\rho_1(X)$ and $\rho_2(X)$ to time t. Then $f(\exp(tX) \cdot m)$ is an integral curve at $f(m)$ to the field $\mathrm{d}f(\rho_1(X)) = \rho_2(X) \circ f$ and this integral curve is unique. ☐

Lemma 4.0.8. Let $G \circlearrowright M$ be a Hamiltonian action with moment map $\mu : M \to \mathfrak{g}^*$. Then the derivative $\mathrm{d}\mu : TM \to \mathfrak{g}^*$ is given by $X \mapsto \omega(\rho(-), X)$.

Proof. We know that $\mathrm{d}\langle \mu, \xi \rangle = \iota_{\rho(\xi)}\omega$. Thus for $X \in \Gamma(M, TM)$ viewing $\xi \in \mathfrak{g}$ as a function on $\mathfrak{g}^*$,

$$\mathrm{d}\mu(X)(\xi) = X(\xi \circ \mu) = X(\langle \mu, \xi \rangle) = \mathrm{d}\langle \mu, \xi \rangle(X) = \omega(\rho(\xi), X)$$

☐

Definition 4.0.9. Let $G \circlearrowright M$ be a symplectic action. Then consider the pullback of Lie algebras,

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^\infty(M) \arrow[d, "\varphi"] \\ \mathfrak{g} \arrow[r, "\rho"'] & \mathfrak{sym}^{\mathrm{op}} \end{tikzcd}$$

Explicitly,

$$\tilde{\mathfrak{g}} = \{(\xi, f) \in \mathfrak{g} \oplus C^\infty(M) \mid \rho(\xi) = X_f\}$$

Remark. The flipped bracket on $\mathfrak{sym}^{\mathrm{op}}$ is necessary to make both maps Lie algebra maps. We have discussed why $f \mapsto X_f$ requires flipping the bracket. However, $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ *also* requires a bracket flip in our conventions. This is because $[-,-]$ on $\mathfrak{g}$ is defined in terms of *left*-invariant vector fields to be consitent with the commutator on $\mathfrak{gl}_n$. However, for a left action, say $\ell : G \to G$, the vector fields $\rho_*(\ell)$ are naturally *right*-invariant and hence have the opposite Lie bracket. In general if $\rho : G \times M \to M$ is a right action then $\rho_* : \mathfrak{g} \to \mathfrak{X}(M)^{\mathrm{op}}$ is a map of Lie algebras.

Remark. I claim that the map $\varphi : C^\infty(M) \to \mathfrak{sym}$ is G-equivariant. Consider $\varphi(g \cdot f) = -X_{g \cdot f}$. First,

$$\mathrm{d}(g \cdot f) = \mathrm{d}f \circ \mathrm{d}g^{-1}$$

However, because the action is symplectic,

$$\omega(\mathrm{d}g(X_f), Y) = \omega(X_f, \mathrm{d}g^{-1}(Y))$$

and therefore $\mathrm{d}(g \cdot f)(Y) = \mathrm{d}f \circ \mathrm{d}g^{-1}(Y) = \omega(\mathrm{d}g(X_f), Y)$ which shows that,

$$X_{g \cdot f} = \mathrm{d}g(X_f)$$

Therefore, the above diagram is in the category of G-equivariant Lie algebras. Explicilty,

$$[\mathrm{Ad}(g) \cdot \xi_1, \mathrm{Ad}(g) \cdot \xi_2] = \mathrm{Ad}(g) \cdot [\xi_1, \xi_2]$$

and likewise,

$$\{g \cdot f_1, g \cdot f_2\} = \omega(X_{g \cdot f_1}, X_{g \cdot f_2}) = \omega(\mathrm{d}g(X_{f_1}), \mathrm{d}g(X_{f_2})) = \omega(X_{f_1}, X_{f_2}) \circ g^{-1}$$

meaning that $\{g \cdot f_1, g \cdot f_2\} = g \cdot \{f_1, f_2\}$.

Proposition 4.0.10. Let $G \circlearrowright M$ be a symplectic action such that $\rho(\xi) \in \mathfrak{ham}$. Then there is a central extension of Lie algebras,

$$0 \longrightarrow \mathbb{R}^{\pi_0(M)} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

Proof. It is clear that $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is surjective because $C^\infty(M) \twoheadrightarrow \mathfrak{ham}$ is surjective and $\rho : \mathfrak{g} \to \mathfrak{sym}$ lands inside $\mathfrak{ham}$. Then consider,

$$\ker(\tilde{\mathfrak{g}} \to \mathfrak{g}) = \{f \in C^\infty(M) \mid X_f = 0\}$$

However, $\omega(X_f, -) = \mathrm{d}f$ and thus $\mathrm{d}f = 0$ so f is locally constant. Furthermore, for any element $(0, f) \in \ker(\tilde{\mathfrak{g}} \to \mathfrak{g})$ we know $X_f = 0$ so $\{f, g\} = \omega(X_f, X_g) = 0$ so the extension is central. □

Proposition 4.0.11. Let $G \circlearrowright M$ be a symplectic action such that $\rho(\xi) \in \mathfrak{ham}$. Then moment maps $\mu : M \to \mathfrak{g}^*$ correspond to splittings in the category of G-representations of,

$$0 \longrightarrow \mathbb{R}^{\pi_0(M)} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

giving an obstruction class $\delta \in \mathrm{Ext}^1_{\mathbb{R}[G]}(\mathfrak{g}, \mathbb{R})^{\pi_0(M)} = H^1(G, \mathfrak{g}^*)^{\pi_0(M)}$ to the action being Hamiltonian.

Proof. There is a canonical map $\tilde{\mu} : M \to \tilde{\mathfrak{g}}^*$ defined by $\langle \tilde{\mu}(x), (\xi, f) \rangle = f(x)$ which is G-equivariant because,

$$\langle \tilde{\mu}(g \cdot x), (\xi, f) \rangle = f(g \cdot x) = \left\langle \tilde{\mu}(x), (\mathrm{Ad}(g^{-1}) \cdot \xi, g^{-1} \cdot f) \right\rangle$$

and thus $\tilde{\mu}(g \cdot x) = g \cdot \tilde{\mu}(x)$ for the dual of the (coadjoint, translation) action on $\tilde{\mathfrak{g}}$. Therefore,

$$\mathrm{d}\langle \tilde{\mu}, (\xi, f) \rangle = \mathrm{d}f = \iota_{X_f} \omega = \iota_{\rho(\xi)} \omega$$

Then suppose that $s : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a section. Then consider $\mu = s^* \circ \tilde{\mu}$. Then,

$$\mathrm{d}\langle \mu, \xi \rangle = \mathrm{d}\langle s^* \circ \tilde{\mu}, \xi \rangle = \mathrm{d}\langle \tilde{\mu}, s(\xi) \rangle = \iota_{\rho(\xi)} \omega$$

because $s(\xi) = (\xi, f)$ for some f such that $X_f = \rho(\xi)$. Therefore, since $s^* \circ \tilde{\mu}$ is G-equivariant, $s^* \circ \tilde{\mu}$ is a moment map. Conversely, given a moment map $\mu : M \to \mathfrak{g}^*$ then $q : \xi \mapsto \langle \mu, \xi \rangle$ gives a G-equivariant map $\mathfrak{g} \to C^\infty(M)$ such that the diagram of G-representations,

$$\begin{tikzcd} & C^\infty(M) \arrow[d, "\varphi"] \\ \mathfrak{g} \arrow[ur, "q"] \arrow[r, "\rho"'] & \mathfrak{sym}^{\mathrm{op}} \end{tikzcd}$$

commutes because $\mathrm{d}\langle \mu, \xi \rangle = \iota_{\rho(\xi)}\omega$ so $\varphi(\langle \mu, \xi \rangle) = \rho(\xi)$ and is G-invariant since

$$(g \cdot \langle \mu, \xi \rangle)(x) = \left\langle \mu(g^{-1} \cdot x), \xi \right\rangle = \left\langle \mathrm{Ad}^*_{g^{-1}} \mu, \xi \right\rangle = \langle \mu, \mathrm{Ad}_g \xi \rangle$$

and therefore we get a G-section $s : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $s^* \circ \tilde{\mu} = \mu$ because,

$$\langle s^* \circ \tilde{\mu}, \xi \rangle = \langle \tilde{\mu}, (\xi, q(\xi) \rangle = q(\xi) = \langle \mu, \xi \rangle$$

Finally, given a G-section $s : \mathfrak{g} \to \tilde{\mathfrak{g}}$ or equivalently a G-map $q : \mathfrak{g} \to C^\infty(M)$ then for the moment map $\mu = s^* \circ \tilde{\mu}$ consider $q'(\xi) = \langle s^* \circ \tilde{\mu}, \xi \rangle = - \langle \tilde{\mu}, (\xi, q(\xi)) \rangle = q(\xi)$ and thus our proceedure recovers the section s so this is a bijective correspondence. ☐

Remark. If G is compact, the category of G-representations is semi-simple so any exact sequence splits. Thus, any symplectic action $G \circlearrowright M$ such that $\rho(\xi) \in \mathfrak{ham}$ for all $g \in G$ is Hamiltonian.

Remark. If G is noncompact we have to be careful with this extension. For example, if $G = \mathbb{R}^{2n}$ acting on itself by translation, then $\mathfrak{g} = \mathbb{R}^{2n}$ with the trivial action but $\tilde{\mathfrak{g}}$ is the space of affine-linear maps $\mathbb{R}^{2n} \to \mathbb{R}$ i.e. functions of the form $f(\vec{x}) = \vec{x} \cdot \vec{y} + a$ for fixed $\vec{y}$ and a. The action of G is by translation $(\vec{v} \cdot f)(\vec{x}) = f(\vec{x} - \vec{v}) = \vec{x} \cdot \vec{y} + (a - \vec{v} \cdot \vec{y})$ which for nonzero $\vec{y}$ modifies $f(0)$. In fact, $\tilde{\mathfrak{g}}$ is the Heisenberg algebra $\mathfrak{h}_{2n+1}$ with the conjugation action by $\mathbb{R}^{2n} \subset H_{2n+1}$. Therefore, $\tilde{\mathfrak{g}}$ is a nontrivial G-representation but fits in an extension,

$$0 \longrightarrow \mathbb{R} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathbb{R}^{2n} \longrightarrow 0$$

of trivial representations. In particular, the sequence is non-split. Indeed, we have seen that this action by Hamiltonian vector fields is *not* Hamiltonian.

Corollary 4.0.12. If one exists, the space of moment maps is isomorphic to $\mathrm{Hom}_G(\mathfrak{g}, \mathbb{R})^{\pi_0(M)}$ which are $\pi_0(M)$ choices of G-invariant elements of $\mathfrak{g}^*$ representing an additive constant shift for $\mu : M \to \mathfrak{g}^*$ on each connected component of M.

Proof. This follows directly from the correspondence between moment maps and splittings of,

$$0 \longrightarrow \mathbb{R}^{\pi_0(M)} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

as G-representations which form a $\mathrm{Hom}_G(\mathfrak{g}, \mathbb{R})^{\pi_0(M)}$-torsor. ☐

Proposition 4.0.13. If G is reductive then any symplectic action $G \circlearrowright M$ with $\rho(\xi) \in \mathfrak{ham}$ is hamiltonian. Additionally, if $Z(\mathfrak{g}^*)$ is trivial then the moment map is unique.

Proof. The category of representations of a reductive group is semi-simple. Therefore, exact sequences of G-representations always split. ☐

Corollary 4.0.14. Let $\mathbb{T}^n = (S^1)^n$ be the torus group and (M, ω) a simply-connected symplectic manifold. Then any symplectic action $\mathbb{T}^n \circlearrowright M$ is Hamiltonian and the space of moment maps is affine over $\mathfrak{t}^* \cong \mathbb{R}^n$.

Proof. Since $H^1_{\mathrm{dR}}(M) = 0$ we see that $\mathfrak{sym} = \mathfrak{ham}$ so $\mathbb{T}^n$ acts via Hamiltonian vector fields. Since $\mathbb{T}^n$ is abelian $\widetilde{\mathfrak{t}} \to \mathfrak{t}$ has the trivial action so it is split and the space of splittings is affine over $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$ since $\pi_0(M) = *$. ☐

Example 4.0.15. In the above corollary, M being simply-connected is necessary. For example consider $M = S^1 \times S^1$ with the symplectic structure $\omega = \mathrm{d}x \wedge \mathrm{d}y$ where x and y are the coordinates on the two factors. Let $S^1 \circlearrowright M$ via left translation on the first factor. Since ω is constant (this makes sense since the tangent bundle is trivial), this is clearly a symplectic action. However, translation is not a Hamiltonian vector field because $\omega(\frac{\partial}{\partial x}, -) = \mathrm{d}y$ which is not closed since it has a nonvanishing integral along the curve $\{*\} \times S^1 \subset S^1 \times S^1$. Therefore, this action cannot be Hamiltonian.

Proposition 4.0.16. Let G be a Lie group acting smoothly on a manifold $G \circlearrowright Q$. Then there is an induced action $G \circlearrowright T^*Q$ which is automatically Hamiltonian for the standard symplectic structure on T^*Q. The moment map is (FINISH)

Proof. The action is defined as $g \cdot (q, p) = (g \cdot q, (\mathrm{d}g^{-1})^* p)$. Notice that $\pi : T^*Q \to Q$ is by definition G-equivariant. The tautological 1-form θ has the defining property that for any 1-form $\beta : Q \to T^*Q$ we have $\beta^*\theta = \beta$. Then consider the form $\tilde{g}^*\theta$ for $\tilde{g} : T^*Q \to T^*Q$. We have,

$$\beta^* \tilde{g}^* \theta = (\tilde{g} \circ \beta)^* \theta = (\beta \circ g)^* \theta = g^* \beta^* \theta = g^* \beta$$

Then for $\xi \in \mathfrak{g}$ the vector field $\rho(\xi)$ is

We define $\mu = \iota_{\rho(-)}\theta$.

(FINISH THIS!!) ☐

Corollary 4.0.17. If G is semi-simple then there exists a unique moment map for any symplectic action $G \circlearrowright M$ with $\rho(\xi) \in \mathfrak{ham}$.

Proof. Uniqueness follows from the fact that $\mathfrak{g}$ is semi-simple and thus has a trivial center. Furthermore, since the category of representations of a semi-simple Lie group is semi-simple, the sequence is split uniquely giving a unique choice of moment map. ☐

4.1 Studying the Extension Class

Proposition 4.1.1. Any G-equivariant section $\mathfrak{g} \to \tilde{\mathfrak{g}}$ is automatically a Lie algebra map. If G is connected, Lie algebra sections and G-equivariant sections coincide.

Proof. By the following lemmas, it suffices to show that any section is compatible in the sense of the following definition. We need to show that if $s : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a section then $\mathrm{ad}_{s(\xi)} = \rho_*(\xi)$ for the action $G \circlearrowright \tilde{\mathfrak{g}}$. Indeed,

$$[s(\xi), (\eta, f)] = ([\xi, \eta], \{s(\xi), f\})$$

and by definition of the section $\varphi(s(\xi)) = X_{s(\xi)} = \rho_M(\xi)$. Thus,

$$\{s(\xi), f\} = -\mathrm{d}f(X_{s(\xi)}) = -\mathrm{d}f(\rho_M(\xi)) = -\rho_M(\xi)(f)$$

proving the claim since $\rho_*(\xi)$ acts on $\tilde{\mathfrak{g}}$ via $([\xi, -], -\rho_M(\xi)(-))$ ☐

Definition 4.1.2. Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$. We say a G-action $\rho : G \to \mathrm{Aut}(\mathfrak{h})$ and a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ are compatible if $\rho_*(\xi) = \mathrm{ad}_{\varphi(\xi)}$ for all $\xi \in \mathfrak{g}$.

Remark. We equip $\mathfrak{g}$ with the adjoint action and consider when the map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is G-equivariant.

Lemma 4.1.3. If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is compatible with a G-action on $\mathfrak{h}$ then,

(a) if φ is G-equivariant then φ is a Lie algebra map

(b) if φ is a Lie algebra map and G is connected then φ is G-equivariant.

Proof. φ being a Lie algebra map is equivalent to $\varphi \circ \mathrm{ad}_\xi = \mathrm{ad}_{\varphi(\xi)} \circ \varphi$. Since $\mathrm{ad}_{\varphi(\xi)} = \rho_*(\xi)$ the condition of being a Lie algebra map is equivalent to,

$$\varphi \circ \mathrm{ad}_\xi = \rho_*(\xi) \circ \varphi$$

This is the derivative of the G-equivariance condition giving the first statement. Then we have shown that if G is connected, preserving the flow implies G-equivariance giving the second. $\square$

Corollary 4.1.4. Let $G \circlearrowright M$ via Hamiltonian vector fields. Then there exists an obstruction class $\delta \in H^2(\mathfrak{g}, \mathbb{R})^{\pi_0(M)}$ to the existence of a moment map.

4.2 Obsolete Lemmas

Lemma 4.2.1. Given a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ with a compatible action and $Z(\mathfrak{h}) = 0$ then φ is automatically a Lie algebra map and, if G is connected, a G-equivariant map.

Proof. It is clear that,

$$\mathrm{ad}_{\varphi([\xi,\eta])} = [\varphi([\xi,\eta]), -] = \rho_*([\xi,\eta]) = [\rho_*(\xi), \rho_*(\eta)] = [\mathrm{ad}_{\varphi(\xi)}, \mathrm{ad}_{\varphi(\eta)}]$$

Therefore φ is a Lie algebra map by the previous lemma. Furthermore, if G is connected then to show that φ is G-equivariant it suffices to show that φ is compatible with the vector fields induced by the G-action i.e.

$$\varphi \circ \mathrm{ad}_\xi = \mathrm{ad}_{\varphi(\xi)} \circ \varphi$$

which is nothing other than a restatement of the fact that φ is a Lie algebra map,

$$\varphi([\xi, -]) = [\varphi(\xi), \varphi(-)]$$

$\square$

Lemma 4.2.2. Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a linear map between Lie algebras such that,

$$\mathrm{ad}_{\varphi([\xi,\eta])} = [\mathrm{ad}_{\varphi(\xi)}, \mathrm{ad}_{\varphi(\eta)}]$$

If $Z(\mathfrak{h}) = 0$ then φ is a Lie algebra homomorphism.

Proof. By assumption $X = \varphi([\xi,\eta]) - [\varphi(\xi), \varphi(\eta)]$ has $\mathrm{ad}_X = 0$ so $X \in Z(\mathfrak{h})$ so $X = 0$. $\square$

Remark. The assumption that $Z(\mathfrak{h}) = 0$ is necessary. Indeed consider any linear isomorphism $\mathfrak{sl}_2 \xrightarrow{\sim} \mathbb{R}^3$ this cannot be a map of Lie algebras since $\mathfrak{sl}_2$ is nontrivial. However, $\mathbb{R}^3$ is abelian so the adjoint condition is trivially satisfied.

4.3 Full Examples

Example 4.3.1. Consider the standard action $\mathrm{SL}_2 \circlearrowright \mathbb{R}^2$ where $\mathbb{R}^2$ is equipped with the invariant symplectic form,

$$\omega = \mathrm{d}x \wedge \mathrm{d}y$$

Our results show this action is Hamiltonian with a unique moment map. Consider the generators of $\mathfrak{sl}_2$,

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where we have,

$$[e, f] = h$$
$$[h, e] = 2e$$
$$[h, f] = -2f$$

and the map ρ_* is,

$$h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$
$$e \mapsto y\frac{\partial}{\partial x}$$
$$f \mapsto x\frac{\partial}{\partial y}$$

notice that this reverses the Lie bracket. These vector fields are Hamiltonian. Indeed, consider the corresponding functions,

$$H(x, y) = xy$$
$$E(x, y) = \tfrac{1}{2}y^2$$
$$F(x, y) = -\tfrac{1}{2}x^2$$

then we have,

$$\omega(\rho(h), -) = x\mathrm{d}y + y\mathrm{d}x = \mathrm{d}H$$
$$\omega(\rho(e), -) = y\mathrm{d}y = \mathrm{d}E$$
$$\omega(\rho(f), -) = -x\mathrm{d}x = \mathrm{d}F$$

Then we compute the Poisson brackets,

$$\{E, F\} = H$$
$$\{H, E\} = 2E$$
$$\{H, F\} = -2F$$

which has the same Lie algebra structure as the generators of $\mathfrak{sl}_2$. These are the only choices of H, E, F that work since other choices must differ by a constant but these constants will not be preserved by the Poisson bracket. Therefore $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is a trivial central extension by $\tilde{\mathbb{R}}$ and is canonically split as expected.

5 Connections on Principal Bundles

Definition 5.0.1. Let $\pi : P \to X$ be a principal G-bundle. Then conisder the exact sequence of vector bundles on P,

$$0 \longrightarrow \ker \mathrm{d}\pi \longrightarrow TP \xrightarrow{\mathrm{d}\pi} \pi^* TX \longrightarrow 0$$

A *connection* on P is a G-invariant splitting. Explicitly, a bundle map $\delta : \pi^* TX \to TP$ such that,

(a) $\mathrm{d}\pi \circ \delta = \mathrm{id}_{\pi^* TX}$

(b) for each $g \in G$ the diagram,

$$\begin{tikzcd}
\pi^* TX \arrow[r, "\delta"] \arrow[d, equal] & TP \arrow[d, "\mathrm{d}\ell_g"] \\
\ell_g^* \pi^* TX \arrow[r, "\ell_g^* \delta"] & \ell_g^* TP
\end{tikzcd}$$

commutes where $\ell_g : P \to P$ is the left action by $g \in G$. Note that $\pi \circ \ell_g = \pi$ so there is a natural isomorphism $\pi^* = \ell_g^* \pi^*$.

Remark. The equivariant condition is equivalent to δ being a morphism of descent data for the covering $\pi : P \to X$ or equivalently a morphism of G-equivariant bundles.

Remark. Such a splitting is equivalent to the choice of a G-equivariant complement to the vertical space $V = \ker \mathrm{d}\pi$. Explicitly this is a subbundle $H \subset TP$ such that $TP = H \oplus V$ and $\mathrm{d}\ell_g : TP \to \ell_g^* TP$ takes H to $\ell_g^* H$. This is a G-invariant Ehresmann connection on P.

Lemma 5.0.2. A connection on $\pi : P \to X$ is equivalent to the choice of a $\mathfrak{g}$-valued 1-form $\theta \in \Gamma(P, T^*P \otimes \mathfrak{g})$ such that,

(a) (FINISH!!)

Proof. A connection is a right splitting of the sequence,

$$0 \longrightarrow V \longrightarrow TP \xrightarrow{\mathrm{d}\pi} \pi^* TX \longrightarrow 0$$

which is equivalent to a choice of right splitting $\theta : TP \to V$ such that $\theta|_V = \mathrm{id}_V$. However, since P is a principal G-bundle, V is the trivial bundle $P \times \mathfrak{g}$ because $\xi \mapsto \rho(\xi)$ for the action $G \circlearrowright P$ is an isomorphism of vector bundles $P \times \mathfrak{g} \to V$. Therefore, $\theta : TP \to V \cong P \times \mathfrak{g}$ is equivalent to a form $\theta \in \Gamma(P, T^*P \otimes \mathfrak{g})$. (PROVE PROPERTIES!!!) □

(DO THIS Adjoint bundle AP!!)

Definition 5.0.3. The curvature of a connection $\delta : \pi^* TX \to TP$ is a $\mathcal{A}_P$-valued 2-form $F \in \Gamma(P, \wedge^2 T^*X \otimes \mathcal{A}_P)$ on X defined by,

$$F(X, Y) = [\delta(X), \delta(Y)] - \delta([X, Y])$$

6 Quaternionic Manifolds

6.1 First Attempts

6.2 Definition via G-Structues

Remark. We have the following setup. Let $V = \mathbb{H}^n$ be a $\mathbb{R}$-vector space and *left* $\mathbb{H}$-module. The $\mathbb{H}$-module structue is equivalent to a map $\mathbb{H} \to \mathrm{End}_{\mathbb{R}}(V)$ whose (faithful) image is a subalgebra $H \subset \mathrm{End}_{\mathbb{R}}(V)$ isomorphic to $\mathbb{H}$. The group $\mathbb{H}^\times \times \mathrm{GL}(n, \mathbb{H})$ acts on V via $(q, A) \cdot v = q \cdot v \cdot A^{-1}$ (which is well-defined because right and left actions commute). Notice that $\mathrm{GL}(n, \mathbb{H})$ acts via $\mathbb{H}$-linear maps while $\mathbb{H}^\times$ does not because $\mathbb{H}^\times$ is not abelian and acts on the left. Therefore, we get a map,

$$\mathbb{H}^\times \times \mathrm{GL}(n, \mathbb{H}) \to \mathrm{GL}(4n, \mathbb{R}) = \mathrm{Aut}_{\mathbb{R}}(V)$$

We denote its image by $G_{\mathbb{H}}$. Clearly, $G_{\mathbb{H}}$ is the product of $H^\times$ and $\mathrm{GL}(n, \mathbb{H})$ inside $\mathrm{GL}(4n, \mathbb{R})$,

$$G_{\mathbb{H}} = \mathbb{H}^\times \cdot \mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(4n, \mathbb{R})$$

Furthermore, because $\mathbb{H}^\times \cap \mathrm{GL}(n, \mathbb{H}) = \mathbb{R}^\times$ inside $\mathrm{GL}(4n, \mathbb{R})$, there is an isomorphism,

$$G_{\mathbb{H}} \cong (\mathbb{H}^\times \times \mathrm{GL}(n, \mathbb{H}))/\mathbb{R}^\times$$

Notice that, as it must given the embedding into $\mathrm{GL}(4n, \mathbb{R})$, that $G_{\mathbb{H}}$ acts on V because,

$$(\lambda q, \lambda A) \cdot v = (\lambda q) \cdot v \cdot (\lambda^{-1} A^{-1}) = q \cdot v \cdot A^{-1} = (q, A) \cdot v$$

for all $\lambda \in \mathbb{R}^\times$.

Lemma 6.2.1. $\mathrm{Aut}(\mathbb{H}) = \mathrm{Inn}(\mathbb{H}) \cong \mathrm{SO}(3)$.

Proof. For any unit imaginary quaternions $v, u, w \in S^2 \subset \mathrm{Im}(\mathbb{H})$ we know that $vu = -v \cdot u + v \times u$. Since any $\varphi \in \mathrm{Aut}(\mathbb{H})$ must preserve scalars we see that $\varphi(v) \cdot \varphi(u) = v \cdot u$. Furthermore, it preserves the scalar part of $v(uw) = -v \cdot (u \times w)$ meaning that $\mathrm{Aut}(\mathbb{H})$ preserves the metric and orientation form on $\mathbb{R}^3$ and fixes zero giving a map $\mathrm{Aut}(\mathbb{H}) \to \mathrm{SO}(3)$. Furthermore, because automorphisms fix the scalar part and respect scaling, such a transformation of the imaginary sphere determines the automorphism so $\mathrm{Aut}(\mathbb{H}) \xrightarrow{\sim} \mathrm{SO}(3)$. Furthermore, we know that all rotations of the imaginary sphere are realized through inner automorphisms. $\square$

Lemma 6.2.2. Since V is a $G_{\mathbb{H}}$-representation, we get a $G_{\mathbb{H}}$-action on $\mathrm{End}_{\mathbb{R}}(V)$. Then H is invariant under $G_{\mathbb{H}}$ and $G_{\mathbb{H}}$ is exactly the stabilizer of H under the inclusion $G_{\mathbb{H}} \subset \mathrm{GL}(4n, \mathbb{R})$,

$$G_{\mathbb{H}} = \mathrm{Stab}(H) = \{\varphi \in \mathrm{Aut}_{\mathbb{R}}(V) \mid \varphi \cdot H = H\}$$

Furthermore, the subgroup $\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(4n, \mathbb{R})$ is the *pointwise stabilizer*,

$$\mathrm{GL}(n, \mathbb{H}) = \mathrm{Stab}(\{H\}) = \{\varphi \in \mathrm{Aut}_{\mathbb{R}}(V) \mid \forall h \in H : \varphi \cdot h = h\}$$

Proof. By definition, $\varphi \cdot h = \varphi \circ h \circ \varphi^{-1}$ meaning that,

$$\varphi \in \mathrm{Stab}(\{H\}) \iff \forall h \in H : \varphi \cdot h = h \iff \forall h \in H : \varphi \circ h = h \circ \varphi$$

and thus $\mathrm{Stab}(\{H\})$ is the group of H-linear automorphisms of V which is exactly $\mathrm{GL}(n, \mathbb{H})$ acting on the right.

Now we consider the case that $\varphi \cdot h \in H$. Since $\mathrm{Aut}_{\mathbb{R}}(V)$ acts on $\mathrm{End}_{\mathbb{R}}(V)$ by algebra automorphism we know that $h \mapsto \varphi \cdot h = \varphi \circ h \circ \varphi^{-1}$ is an algebra automorphism. Since all automorphisms of $\mathbb{H}$ are inner, there exists some $q \in H^{\times}$ such that,

$$\varphi \circ h \circ \varphi^{-1} = q^{-1} \circ h \circ q$$

Therefore, $\varphi' = q \circ \varphi$ is $\mathbb{H}$-linear so $\varphi' \in \mathrm{GL}(n, \mathbb{H})$ and thus $\varphi \in q \circ \mathrm{GL}(n, \mathbb{H}) \subset G_{\mathbb{H}}$. Therefore, we conclude that $\mathrm{Stab}(H) = G_{\mathbb{H}}$. □

Proposition 6.2.3. Let V be a $4n$ dimensional $\mathbb{R}$-vectorspace. Then,

(a) the data of a $G_{\mathbb{H}}$-torsor of isomorphisms $V \to \mathbb{H}^n$ is equivalent to the data of a subalgebra $H \subset \mathrm{End}_{\mathbb{R}}(V)$ isomorphic to $\mathbb{H}$

(b) the data of a $\mathrm{GL}(n, \mathbb{H})$-torsor of isomorphisms $V \to \mathbb{H}^n$ is equivalent to the data of a subalgebra $H \subset \mathrm{End}_{\mathbb{R}}(V)$ and an algebra isomorphism $\varphi : \mathbb{H} \to H$.

Proof. Given a $G_{\mathbb{H}}$ (or $\mathrm{GL}(n, \mathbb{H})$) torsor of isomorphism $V \to \mathbb{H}^n$ choose one such isomorphism $\psi : V \xrightarrow{\sim} \mathbb{H}^n$. Then $H = \psi^{-1} \circ \mathbb{H} \circ \psi \subset \mathrm{End}_{\mathbb{R}}(V)$ is a subalgebra isomorphic to $\mathbb{H}$ via

$$\varphi : q \mapsto \psi^{-1} \circ (q \cdot -) \circ \psi$$

Furthermore, any other isomorphism $\psi' : V \xrightarrow{\sim} \mathbb{H}^n$ is of the form $\psi' = g \circ \psi$. Then,

$$\psi'^{-1} \circ \mathbb{H} \circ \psi' = \psi^{-1} \circ (g^{-1} \circ \mathbb{H} \circ g) \circ \psi = \psi^{-1} \circ \mathbb{H} \circ \psi$$

because $G_{\mathbb{H}}$ stabilizes $\mathbb{H} \subset \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}^n)$ so $H \subset \mathrm{End}_{\mathbb{R}}(V)$ is well-defined. Furthermore, if we have a $\mathrm{GL}(n, \mathbb{H})$-torsor, then $\varphi' : q \mapsto \psi^{-1} \circ (g^{-1} \circ (q \cdot -) \circ g) \circ \psi = \psi^{-1} \circ (q \cdot -) \circ \psi$ because g stabilizes $\mathbb{H} \subset \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}^n)$ pointwise so we get a well-defined algebra isomorphism $\varphi : \mathbb{H} \to H$.

Conversely, given a subalgebra $H \subset \mathrm{End}_{\mathbb{R}}(V)$ Then define, $S \subset \mathrm{Iso}(V, \mathbb{H}^n)$ as the set of isomorphisms ψ such that $H = \psi^{-1} \circ \mathbb{H} \circ \psi$. For any pair, $\psi' \circ \psi^{-1}$ preserves $\mathbb{H} \subset \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}^n)$ so S is a $G_{\mathbb{H}}$-torsor. Additionally, given an algebra isomorphism $\varphi : \mathbb{H} \xrightarrow{\sim} H$, let $S' \subset \mathrm{Iso}(V, \mathbb{H}^n)$ be the subset such that for all $q \in \mathbb{H}$ we have $\psi^{-1} \circ (q \cdot -) \circ \psi = \varphi(q)$ or equivalently $(q \cdot -) \circ \psi = \psi \circ \varphi(q)$ i.e. those ψ that are $\mathbb{H}$-linear via the given action $\varphi : \mathbb{H} \to H \subset \mathrm{End}_{\mathbb{R}}(V)$. Then clearly $\psi' \circ \psi^{-1}$ preserves $\mathbb{H} \subset \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}^n)$ pointwise and thus S' forms a $\mathrm{GL}(n, \mathbb{H})$-torsor.

These constructions are inverse to eachother. □

Theorem 6.2.4. Let M be a smooth manifold of dimension $4n$. Then,

(a) a $G_{\mathbb{H}}$-structure on M is equivalent to an algebra subbundle $H \subset \mathrm{End}(TM)$ with $H_x \xrightarrow{\sim} \mathbb{H}$

(b) a $\mathrm{GL}(n, \mathbb{H})$-structure on M is equivalent to an algebra subbundle $H \subset \mathrm{End}(TM)$ with a global trivialization $\varphi : \mathbb{H} \times M \xrightarrow{\sim} H$. This is equivalent to a choice of $I, J, K \in \Gamma(M, \mathrm{End}(TM))$ satisfying the quaterion algebra relations: $I^2 = J^2 = K^2 = -\mathrm{id}$ and $IJK = -\mathrm{id}$.

Proof. A $G_{\mathbb{H}}$-structure on M means a reduction of structure group of the frame bundle $F(M)$ to $F_{\mathbb{H}}(M) \to F(M)$ and thus at each point we get a $G_{\mathbb{H}}$-torsor of isomorphisms $T_x M \xrightarrow{\sim} \mathbb{H}^n$ which is equivalent data to a subalgebra $H_x \subset \mathrm{End}(T_x M)$. We need to make sure these data vary smoothly. (DO THIS!!) We can write this down as $F_{\mathbb{H}}(M) \times_{G_{\mathbb{H}}} \mathbb{H} \hookrightarrow \mathrm{End}\,(TM)$ where the map is via $\mathbb{H} \hookrightarrow \mathrm{End}_{\mathbb{R}}\,(\mathbb{H}^n)$ and applying the associated bundle construction gives,

$$F_{\mathbb{H}}(M) \times_{G_{\mathbb{H}}} \mathbb{H} \hookrightarrow F_{\mathbb{H}}(M) \times_{G_{\mathbb{H}}} \mathrm{End}_{\mathbb{R}}\,(\mathbb{H}^n) = F(M) \times_{\mathrm{GL}(4n,\mathbb{R})} \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{4n}\right) = \mathrm{End}\,(TM)$$

Likewise, for a $\mathrm{GL}\,(n, \mathbb{H})$-struture, notice that $\mathrm{GL}\,(n, \mathbb{H})$ acts trivially on $\mathbb{H} \subset \mathrm{End}_{\mathbb{R}}\,(\mathbb{H}^n)$ and therefore the associated bundle,

$$F_{\mathbb{H}}(M) \times_{\mathrm{GL}(n,\mathbb{H})} \mathbb{H} \xrightarrow{\sim} M \times \mathbb{H}$$

is canonically trivialized.

Going in reverse, given H construct the bundle of quaternionic frames $\psi : T_x M \xrightarrow{\sim} \mathbb{H}^n$ compatible with H in the sense that $H_x = \psi^{-1} \circ \mathbb{H} \circ \psi$ inside $\mathrm{End}\,(T_x M)$. Given H and a trivialization φ or equivalently the bundle of $\mathbb{H}$-linear quaternionic frames $\psi : T_x M \xrightarrow{\sim} \mathbb{H}^n$. □

Remark. A $G_{\mathbb{H}}$-structure does not in general admit globally defined almost complex structures $I, J, K \in \mathrm{End}\,(TM)$ satisfying $IJK = -\mathrm{id}$. However, such always exist locally (although such choices are not canonical given the data in contrast to the global I, J, K from a $\mathrm{GL}\,(n, \mathbb{H})$-structure).

Remark. Let's unwind this story for almost complex structures. An almost complex structure is a choice of $I \in \Gamma(M, \mathrm{End}\,(TM))$ such that $I^2 = -\mathrm{id}$. This is the same as an algebra subbundle $C \subset \mathrm{End}\,(TM)$ along with a global trivialization $\varphi : M \times \mathbb{C} \xrightarrow{\sim} C$ (remember that $\mathrm{End}\,(TM)$ comes equiped with a canonical section id so id, I gives a global frame of C).

Given a complex vectorspace, say $V = \mathbb{C}^n$, then $\mathrm{GL}\,(n, \mathbb{C})$ is the subgroup preserving $\mathbb{C} \subset \mathrm{GL}\,(2n, \mathbb{R})$ pointwise (in the adjoint action so this just says that $\mathrm{GL}\,(n, \mathbb{C})$ are exactly the linear maps commuting with $\mathbb{C}$-multiplication). Therefore, for a real vectorspace of dimension $2n$, a $\mathrm{GL}\,(n, \mathbb{C})$-torsor of isomorphisms $V \xrightarrow{\sim} \mathbb{C}^n$ is equivalent to the data of an subalgebra $C \subset \mathrm{End}_{\mathbb{R}}\,(V)$ along with an algebra isomorphism $\varphi : \mathbb{C} \to C$.

Therefore, an almost complex structure on a $2n$ dimensional manifold M is equivalent to a choice of subbundle $C \subset \mathrm{End}\,(TM)$ along with a global trivialization $\varphi : M \times \mathbb{C} \xrightarrow{\sim} C$ which is equivalent to a $\mathrm{GL}\,(n, \mathbb{C})$-structure on M.

Now what is the equivalent notion of stabilizing the algebra $C \subset \mathrm{End}_{\mathbb{R}}\,(V)$. Consider,

$$\mathrm{Stab}\, C = \{\varphi \in \mathrm{Aut}_{\mathbb{R}}\,(V) \mid \varphi \cdot C = C\}$$

these are the matrices $A \in \mathrm{GL}\,(2n, \mathbb{R})$ such that $A^{-1} \circ (z \cdot -)A = z' \cdot -$ for some $z' \in \mathbb{C}$. However, since $\mathrm{Aut}_{\mathbb{R}}\,(V)$ acts on $\mathrm{End}_{\mathbb{R}}\,(V)$ by $\mathbb{R}$-algebra automorphisms, $\mathrm{Stab}\, C$ acts on C by $\mathbb{R}$-algebra automorphisms of which there is only $\{\mathrm{id}, \sigma\}$ with $\sigma \in \mathrm{End}_{\mathbb{R}}\,(V)$ realized by complex conjugation on V. Then $\sigma^k \circ \varphi$ preserves C so $\sigma^k \circ \varphi \in \mathrm{GL}\,(n, \mathbb{C})$ and thus $\mathrm{Stab}\, C = \langle \sigma \rangle \cdot \mathrm{GL}\,(n, \mathbb{C})$ (IS THIS A SEMI-DIRECT PRODUCT?)

Remark. Recall, smooth and complex linear derivative is the same as holomorphic. Therefore, a smooth map preserving the complex structure of the holomorphic tangent bundle (the one built from complex charts for a complex manifold) is exactly a holomorphic map. Therefore, to acess smooth things like fomrs, they cannot have complex linear derivatives and thus we need to take the complexified real tangent bundle if we want to acess them. There's an isomorphism of complex vector bundles $(TM, I) \xrightarrow{\sim} T^{1,0}M$ but we think of the first one as real valued vector fields while $T^{1,0}M$ is complex valued *complex-linear* vector fields. These are the same because if V is a $\mathbb{C}$-vectorspace then $\mathbb{R}$-linear maps $\mathbb{R} \to V$ are the same as $\mathbb{C}$-linear maps $\mathbb{C} \to V$ (SOMTHING LIKE THIS!!).

6.3 Integrability Conditions

6.4 Special Holonomy

7 Some Real Algebras

7.1 Algebra Basics

Remark. Rings are assumed to be unital but need not be commutative. Homomorphisms of rings must preserve the unit.

Definition 7.1.1. An *algebra* over a commutative ring R is a R-module A equiped with an R-bilinear map $B : A \times A \to A$ or equivalently an R-linear structure map $B : A \otimes_R A \to A$. A homomorphism of R-algebras $f : A \to A'$ is an R-linear map such that $f(B(x,y)) = B'(f(x), f(y))$.

Remark. We conventionally write xy or $x \cdot y$ for $B(x,y)$.

Definition 7.1.2. Let A be an R-algebra. Then we say that A is:

(a) *unital* if there exists an element $1_A \in A$ such that $1_A \cdot x = x \cdot 1_A = x$ for all $x \in A$

(b) *associative* if for all $x, y, z \in A$ we have $(xy)z = x(yz)$

(c) *division* if for all $a, b \in A$ with $a \neq 0$ the equations $ax = b$ and $xa = b$ have unique solutions

(d) *zero-divisor free* if for all $a, b \in A$ such that $ab = 0$ either $a = 0$ or $b = 0$.

Proposition 7.1.3. A unital algebra has a unique unit.

Proof. Suppose that $1_A, 1'_A \in A$ are both units. Then $1_A = 1_A \cdot 1'_A = 1'_A$ by the unit properties of 1_A and $1'_A$. □

7.2 Division Algebras

Proposition 7.2.1. Let $R = K$ be a field. Then a finite dimensional K-algebra is zero-divisor free iff it is a divison algebra.

Proof. For any nonzero $a \in A$. The maps $B(a, -)$ and $B(-, a)$ are endomorphisms of finite dimensional K-vectorspaces and thus are injective iff bijective. Injectivity is equivalent to $ab = 0$ implies $b = 0$ and $ba = 0$ implies $b = 0$ which is equivalent to being zero-divisor free. Bijectivity is equivalent to A being a division ring. □

Proposition 7.2.2. If K is algebraically closed, then K is the only finite dimensional unital division algebra over K.

Proof. Let A be a finite dimensional unital division algebra over K. Since K is algebraically closed, for each $a \in A$ the map $\ell_a : A \to A$ has an eigenvector, that is a nonzero $v \in A$ and $\lambda \in K$ such that $av = \lambda v$ and thus $(a - \lambda \cdot 1_A) \cdot v = 0$. However, $v \neq 0$ so because A is a finite dimensional division algebra it is zero divisor free meaning that $a = \lambda \cdot 1_A$. Since $a \in A$ is arbitrary, we see that $K \to A$ is an isomorphism. □

Remark. The three hypotheses: finite dimensional, unital, division are each necessary. For example, over $K = \mathbb{C}$ we have $A = \mathbb{C}(x)$ which is a unital division algebra but not finite dimensional, $A = HMMMMMMMMMMMMMM$, and $A = \mathrm{M}_2(\mathbb{C})$ which is finite dimensional and unital but not division because of nilpotent matrices.

7.3 Properties of Subalgebras

Proposition 7.3.1. Let A be an R-algebra. Then the center $Z(A) = \{x \in A \mid \forall a \in A : ax = xa\}$ is a submodule. If A is associative then $Z(A)$ is a subalgebra.

Proof. Clearly, if $x, y \in Z(A)$ then $r(x + y) = rx + ry = xr + yr = (x + y)r$ so $x + y \in Z(A)$. Furthermore, for all $r \in R$ we know that $a(rx) = r \cdot (ax) = r \cdot (xa) = (rx) \cdot a$ so $rx \in Z(A)$. Thus $Z(A)$ is a submodule. Similarly, if A is associative,

$$a(xy) = (ax)y = (xa)y = x(ay) = x(ya) = (xy)a$$

and thus $xy \in Z(A)$ so $Z(A)$ is a subalgebra. □

Remark. The center of a unital associative algebra is a ring, thus motivating the following result.

Proposition 7.3.2. Unital associative R-algebras are equivalent to rings A with the additional data of a homomorphism of (commutative) rings $\varphi : R \to Z(A)$.

Proof. Let A be a unital associative R-algebra. Then $(A, +, \cdot)$ defines exactly the structure of a ring. Furthermore, there is a map $\varphi : R \to Z(A)$ given by $r \mapsto r \cdot 1_A$. To see why this lands in the center, notice that for all $x \in A$ we have

$$x(r \cdot 1_A) = r \cdot (x \cdot 1_A) = r \cdot x = r \cdot (1_A \cdot x) = (r \cdot 1_A) \cdot x$$

because the product is R-bilinear. Conversely, given a ring A and a map $\varphi : R \to Z(A)$ then A becomes an R-module via $r \cdot x = \varphi(r)x$. Furthermore, because $\varphi(r) \in Z(A)$ the product become bilinear since $x(r \cdot y) = x\varphi(r)y = \varphi(r)xy = r \cdot (xy)$ (linearity in the first factor and distributative laws follow directly from associativity). □

Corollary 7.3.3. Rings are exactly unital associative algebras over $\mathbb{Z}$.

Definition 7.3.4. Let A be an R-algebra. A *left (resp. right) ideal* is an R-submodule $I \subset A$ such that $A \cdot I \subset I$ (resp. $I \cdot A \subset I$). A *two-sided ideal* or simply an *ideal* is both a left and a right ideal.

Remark. Any left/right/two-sided ideal $I \subset A$ is a subalgebra of A.

Remark. If A is a ring, the algebra structure makes a right ideal into a right A-module, a left ideal into a left A-module, and an ideal into an A-bimodule.

Definition 7.3.5. Let A be a unital associative R-algebra. Then we say that A is:

(a) *central* if $Z(A) = R \cdot 1_A$

(b) *simple* if A has no nontrivial ideals.

Proposition 7.3.6. Let A be a simple ring (or unital associative R-algebra). Then $K = Z(A)$ is a field and A naturally has the structure of a central unital associative K-algebra.

Proof. First, note that a simple commutative ring is a field (because then (0) is maximal). I claim that if A is a simple ring then $Z(A)$ is simple. Suppose that $I \subset Z(A)$ is an ideal. Then xA is a two-sided ideal because $x \in Z(A)$. If $x \neq 0$ then $xA = A$ so $xa = 1$ for some $a \in A$. Forthermore, $ab = ab(xa) = axba = ba$ for all $b \in A$ so $a \in Z(A)$. Thus $1_A = xa \in I$ so $I = Z(A)$. Therefore, $Z(A)$ is a field and the identity map $K \to Z(A)$ makes A a unital associative K-algebra such that $Z(A) = K$ so A is central. □

7.4 Central Simple Algebras

Definition 7.4.1. Let K be a field. A *Brauer algebra* over K is a finite dimensional unital associative central simple algebra over K.

Proposition 7.4.2. Every Brauer algebra is isomorphic to a matrix algebra over a divison algebra over K.

Proof. DO!! $\square$

(DOOO THIS SECTION!!!)

7.5 Normed Algebras

Definition 7.5.1. Let V be a K-vectorspace. A *quadratic form* on K is a map $q : V \to K$ so that,

(a) $q(\lambda \cdot v) = \lambda^2 q(v)$ for each $\lambda \in K$ and $v \in V$

(b) $B(v, w) = q(v + w) - q(v) - q(w)$ is a bilinear form $B : V \times V \to K$.

NONDENGENERATE

Definition 7.5.2. A *composition algebra* over K is a finite dimensional K-algebra equiped with a nondegenerate quadratic form $N : A \to K$ such that $N(xy) = N(x)N(y)$ for all $x, y \in A$.

Theorem 7.5.3.

Remark. We want to define an algebra structure on $\mathbb{R}^n$. In analogy with the quaternions, we use a vectorspace splitting $\mathbb{R}^n = \mathbb{R} \cdot 1 \oplus \mathbb{R}^{n-1}$. We write elements as $x = a + \vec{v}$ with $a \in \mathbb{R}$ and define multiplication as follows,

$$xy = (a + \vec{v})(b + \vec{u}) = ab - \vec{v} \cdot \vec{u} + a\vec{u} + b\vec{v} + \vec{v} \times \vec{u}$$

where $\vec{v} \times \vec{u}$ is a bilinear "cross product" $V \times V \to V$. Clearly, this is a bilinear mulitiplication map. Furthermore, we have an involution $x \mapsto x^*$ via $a + \vec{v} \mapsto a - \vec{v}$. Then we want $(xy)^* = y^* x^*$ which is equivalent to $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$. We want to define a norm $N(x) = xx^*$. Notice that $xx^* = a^2 + \vec{v} \cdot \vec{v}$ is nongegenerate. To have multiplicativity of N we must have,

$$(xy)(xy)^* = (xy)(y^* x^*) = x(yy^*)y$$

Writing this out,

$$\begin{aligned}
(xy)(y^*x^*) &= (ab - \vec{v} \cdot \vec{u} + a\vec{u} + b\vec{v} + \vec{v} \times \vec{u})(ab - \vec{v} \cdot \vec{u} - a\vec{u} - b\vec{v} - \vec{v} \times \vec{u}) \\
&= (ab - \vec{v} \cdot \vec{u})^2 + (a^2||\vec{u}||^2 + b^2||\vec{v}||^2 + ||\vec{v} \times \vec{u}||^2 + 2ab\, \vec{v} \cdot \vec{u} + 2a\, \vec{u} \cdot (\vec{v} \times \vec{u}) + 2b\, \vec{v} \cdot (\vec{v} \times \vec{u})) \\
&= (a^2 + ||\vec{v}||^2)(b^2 + ||\vec{u}^2||) + ||\vec{v} \times \vec{u}||^2 + (\vec{v} \cdot \vec{u})^2 - ||\vec{v}||^2||\vec{u}||^2 + 2(a\vec{u} + b\vec{v}) \cdot (\vec{v} \times \vec{u}) \\
x(yy^*)x^* &= (a^2 + ||\vec{v}||^2)(b + ||\vec{u}||^2)
\end{aligned}$$

Therefore, for these to agree we must have,

$$||\vec{v} \times \vec{u}||^2 + (\vec{v} \cdot \vec{u})^2 - ||\vec{v}||^2||\vec{u}||^2 + 2(a\vec{u} + b\vec{v}) \cdot (\vec{v} \times \vec{u}) = 0$$

Taking $a = b = 0$ we see that,

$$||\vec{v} \times \vec{u}||^2 = ||\vec{v}||^2||\vec{u}||^2 - (\vec{v} \cdot \vec{u})^2$$

Taking $a = 0$ we see that $\vec{u} \cdot (\vec{v} \times \vec{u}) = 0$ and likewise for $b = 0$ we see that $\vec{v} \cdot (\vec{v} \times \vec{u}) = 0$. This justifies the following definition.

Definition 7.5.4. A *cross product* on an inner product space is a bilinear map $\times : V \times V \to V$ such that,

(a) $\vec{v} \cdot (\vec{v} \times \vec{u}) = 0$ and $(\vec{v} \times \vec{u}) \cdot \vec{u} = 0$

(b) $||\vec{v} \times \vec{u}||^2 = ||\vec{v}||^2 ||\vec{u}||^2 - (\vec{v} \cdot \vec{u})^2$.