Mathematics GU4053 Algebraic Topology
Assignment # 11

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Problem 1.

(a). Let $X = S^2$ and let $A \subset X$ be the set containing the north and south poles of S^2. Then, (X, A) is a good pair so we have a long exact sequence of reduced homology,

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \tilde{H}_{n-1}(X) \longrightarrow \cdots$$

Since A is a discrete space on two points,

$$\tilde{H}_n(A) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

Furthermore, we know the reduced homology of spheres so,

$$\tilde{H}_n(X) = \tilde{H}_n(S^2) \cong \begin{cases} \mathbb{Z} & n = 2 \\ 0 & n \neq 2 \end{cases}$$

Therefore, at $n = 0$ the long exact sequence gives,

$$\cdots \longrightarrow \tilde{H}_0(A) \longrightarrow 0 \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0$$

and thus $\tilde{H}_0(X/A) = 0$ which we knew since X/A is path-connected. Furthermore, at $n = 1$, the long exact sequence gives,

$$\cdots \longrightarrow \tilde{H}_1(A) \longrightarrow 0 \longrightarrow \tilde{H}_1(X/A) \longrightarrow \tilde{H}_0(A) \longrightarrow 0$$

which implies that $\tilde{H}_1(X/A) \cong \tilde{H}_0(A) \cong \mathbb{Z}$. Finally, whenever $n > 1$ since A is the disjoint union of contractible spaces, $\tilde{H}_n(A) = \tilde{H}_{n-1}(A) = 0$ so from the long exact sequence,

$$0 \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A) \longrightarrow 0$$

and thus $\tilde{H}_n(X/A) \cong \tilde{H}_n(X)$. Putting these together and using the fact that $\tilde{H}_n(X) \cong H_n(X)$ for $n > 0$ and $\tilde{H}_0(X) \cong H_0(X) \oplus \mathbb{Z}$ we find that the homology of X/A is,

$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0, 1, 2 \\ 0 & n > 2 \end{cases}$$

(b). Let $X = S^1 \times (S^1 \vee S^1)$. Since X is path-connected, $H_1(X) \cong \mathbb{Z}$. Computing the fundamental group,

$$\pi_1(X) \cong \pi_1(S^1) \times \pi_1(S^1 \vee S^1) \cong \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})$$

Therefore, since X is 0-connected, by Hurewicz's theorem,

$$H_1(X) \cong \pi_1(X)^{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

Next, we can decompose $X = A \cup B$ where A and B are the two tori which are glued together to form $S^1 \times (S^1 \vee S^1)$ such that $A \cap B \cong S^1$. Applying the Mayer-Vietoris sequence,

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$$

Since $A \cap B = S^1$ and $H_{n-1}(S^1) = 0$ for $n > 2$ we have,

$$H_n(X) \cong H_n(A) \oplus H_n(B) = 0$$

since the homology of a torus vanishes for $n > 2$. For $n = 2$ the map $H_{n-1}(A \cap B) \to H_{n-1}(A) \oplus H_{n-1}(B)$ so the map $H_n(X) \to H_{n-1}(A \cap B)$ is zero.

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_2(X) \longrightarrow 0$$

where $H_2(A) \cong H_2(B) \cong \mathbb{Z}$. Thus, $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. Putting everything together,

$$H_n(X) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & n > 2 \end{cases}$$

(c).

(d). The identification space of a torus where points which differ by a rotation of $2\pi/m$ or $2\pi/m$ about the two principal directions simply gives a single square identified in the same way as a torus. Therefore, this space is homeomorphic to a torus so it has homology,

$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 1 \\ 0 & n > 2 \end{cases}$$

Alternatively, if the question is asking to fix a base point x_0 and mod out by points on the *fixed* circles $S^1 \times \{x_0\}$ and $\{x_0\} \times S^1$ then we can use the relative homology of $S^1 \times S^1$ with A a finite number (in particular k) of points,

$$H_r(S^1 \times S^1, A) \cong \begin{cases} \mathbb{Z} & r = 2 \\ \mathbb{Z}^{k+1} & r = 1 \\ 0 & r \neq 1, 2 \end{cases}$$

which we calculated on assignment 9. In this case, we have $m + n - 1$ points on the boundary of the identification square. Thus, since (X, A) is a good pair,

$$\tilde{H}_r(S^1 \times S^1/A) \cong \tilde{H}_r(S^1 \times S^1, A) \cong \begin{cases} \mathbb{Z} & r = 2 \\ \mathbb{Z}^{n+m} & r = 1 \\ 0 & r \neq 1, 2 \end{cases}$$

Thus,

$$H_r(S^1 \times S^1/A) \cong \begin{cases} \mathbb{Z} & r = 2 \\ \mathbb{Z}^{n+m} & r = 1 \\ \mathbb{Z} & r = 0 \\ 0 & r > 2 \end{cases}$$

Problem 2.

Consider the commutative diagram formed from portions of the long exact sequences for the pairs (X^{n+1}, X^n), and (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}).

$$
\begin{tikzcd}
 & & & H_n(X^{n+1}, X^n) \cong 0 & \\
H_n(X^{n-1}) \cong 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & & \\
 & H_n(X^n) \arrow[ur, "\iota"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\delta_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\delta_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & \\
 & & H_{n-1}(X^{n-2}) \cong 0 \arrow[ur] & &
\end{tikzcd}
$$

The maps d_{n+1} and d_n are defined such as $d_{n+1} = j_n \circ \delta_{n+1}$ and $d_n = j_{n-1} \circ \delta_n$ such that the diagram commutes. By exactness, we have that $\mathrm{Im}(j_n) \cong \ker \delta_n$ but $\ker \delta_n = \ker d_n$ because j_{n-1} is injective by exactness. Therefore, $\ker n_n = \mathrm{Im}(j_n)$. However, by exactness, $\mathrm{Im}(j_n)$ is injective so it is an isomorphism onto its image. Therefore, $j_n : H_n(X^n) \xrightarrow{\sim} \ker d_n \subset H_n(X^n, X^{n-1})$. However, the group $H_n(X^n, X^{n-1})$ is the free abelian group on the n-cells of X. Thus, since $H_n(X^n)$ is isomorphic to a subgroup of a free abelian group, the group $H_n(X^n)$ is itself a free abelian group.

Problem 3.

Let $X = A_1 \cup A_2 \cup \cdots \cup A_n$ such that all intersections are either empty or have trivial reduced homology. Define the sequence of topological spaces,

$$Y_k = A_1 \cup A_2 \cup \cdots \cap A_k$$

and likewise,

$$Z_k = A_k \cap A_{k+1} \cap \cdots \cap A_r$$

Using the Mayer-Vietoris sequence we will prove by induction that $\tilde{H}_r(Y_k \cap Z_{k+1}) = 0$ for $r \geq k-1$. Consider the base case, $k = 1$. We have $Y_1 = A_1$ and $Z_2 = A_2 \cap A_3 \cap \cdots \cap A_r$ so $Y_1 \cap Z_2 = A_1 \cap A_2 \cap \cdots \cap A_r$. Therefore,

$$\tilde{H}_r(X_1 \cap Z_2) = \tilde{H}_r(A_1 \cap A_2 \cap \cdots \cap A_r) = 0$$

for all r since the intersections have trivial reduced homology. Now take the induction hypothesis, $\tilde{H}_r(Y_k \cap Z_{k+1}) = 0$ for $r \geq k-1$. We need to write the term $Y_{k+1} \cap Z_{k+2}$ in terms of the spaces we know more about.

$$Y_{k+1} \cap Z_{k+2} = (A_1 \cap Z_{k+2}) \cup (A_2 \cap Z_{k+2}) \cup \cdots \cup (A_{k+1} \cap Z_{k+2}) = (Y_k \cap Z_{k+1}) \cup Z_k$$

All these spaces are open so we can consider the Mayer-Vietoris sequence,

$$\tilde{H}_r((Y_k \cap Z_{k+1}) \cap Z_k) \longrightarrow \tilde{H}_r(Y_k \cap Z_{k+1}) \oplus \tilde{H}_r(Z_k) \longrightarrow \tilde{H}_r((Y_k \cap Z_{k+1}) \cup Z_k) \longrightarrow \tilde{H}_{r-1}((Y_k \cap Z_{k+1}) \cap Z_k)$$

If we take $r \geq k-1$ then $\tilde{H}_r(Y_k \cap Z_{k+1}) = 0$ by hypothesis and $\tilde{H}_r(Z_k) = 0$ because Z_k is an intersection. Furthermore, if $r \geq k$ then $r-1 \geq k-1$ so again by the induction hypothesis,

$$\tilde{H}_{r-1}((Y_k \cap Z_{k+1}) \cap Z_k) = \tilde{H}_{r-1}(Y_k \cap Z_{k+1}) = 0$$

Therefore, if $r \geq (k+1)-1$ the above long exact sequence is zero except for the middle left which is then forced to be zero,

$$\tilde{H}_r(Y_{k+1} \cap Z_{k+2}) = \tilde{H}_r((Y_k \cap Z_{k+1}) \cup Z_k) = 0$$

so the claim holds by induction. Therefore, the claim holds for $k = n$. Thus, for $r \geq n-1$ we have that $\tilde{H}_r(X) = \tilde{H}_r(Y_n \cap Z_{n+1}) = 0$.

Furthermore, we know that $\tilde{H}_n(S^n) \cong \mathbb{Z}$. However, we can decompose S^n into $n+2$ open sets with homologically trivial intersections. To do this, view the sphere S^n as the boundary of an $n+1$-simplex $\partial\Delta^{n+1}$. Take the open sets to be the faces of $\partial\Delta^{n+1}$ of which there are $n+2$. [1] Furthermore, the intersections of these faces are lower dimensional (solid) simplices which are all contractible. Therefore, the theorem requires that $\tilde{H}_r(S^n) = 0$ for $r \geq (n+2)-1 = n+1$ which is strict because $\tilde{H}_n(S^n) \neq 0$.

Problem 4.

Let A, B be abelian groups and let F be a free abelian group. Consider the diagram in **AbGrp**,

$$\begin{tikzcd} & F \arrow[d, "f"] \\ A \arrow[r, two heads, "g"] & B \end{tikzcd}$$

where $g : A \to B$ is surjective. Since F is a free abelian group it is in the image of the free functor **Free** : **Set** $\to$ **AbGrp**. Let $F = \mathbf{Free}(S)$ for some set S. Furthermore, **Free** is right adjoint to the forgetful functor **Forget** : **AbGrp** $\to$ **Set**. Consider the diagram in **Set**,

$$\begin{tikzcd} & S \arrow[dl, "h'"'] \arrow[d, "f'"] \\ \mathbf{Forget}(A) \arrow[r, two heads, "\mathbf{Forget}(g)"] & \mathbf{Forget}(B) \end{tikzcd}$$

where the map $f' : S \to \mathbf{Forget}(B)$ is defined by $f'(s) = f(s) \in B$. Clearly, the map $\mathbf{Forget}(f)$ is still surjective because it acts on elements of the underlying sets identically to g. Therefore, there is a map $h' : S \to \mathbf{Forget}(A)$ in the category **Set** such that the diagram commutes because $\mathbf{Forget}(f)$

[1] These sets are not actually open. However, we can take open sets which are ϵ-neighborhoods of the faces which deformation retract onto the faces.

is surjective so there is a right inverse $i : \mathbf{Forget}(B) \to \mathbf{Forget}(A)$ and we can take $h' = i \circ f'$ such that $\mathbf{Forget}(g) \circ h' = (\mathbf{Forget}(g) \circ i) \circ f' = f'$. Becauset **Forget** and **Free** are adjoints,

$$\mathrm{Hom}_{\mathbf{AbGrp}}(\mathbf{Free}(S), A) \cong \mathrm{Hom}_{\mathbf{Set}}(S, \mathbf{Forget}(A))$$

naturally. Therefore, the map $h' \in \mathrm{Hom}_{\mathbf{Set}}(S, \mathbf{Forget}(A))$ corresponds to $h \in \mathrm{Hom}_{\mathbf{AbGrp}}(\mathbf{Free}(S), A)$ such that the diagram,

$$\begin{tikzcd} & \mathbf{Free}(S) \arrow[dl, "h"'] \arrow[d, "f"] \\ A \arrow[r, two heads, "g"] & B \end{tikzcd}$$

commutes because the maps g and f correspond to $\mathbf{Forget}(g)$ and f' under the natural adjointness relation. Therefore, $F = \mathbf{Free}(S)$ is a projective object in **AbGrp**.

Problem 5.

Let $f : A \to F$ be a surjective map of abelian groups where F is a free abelian group. By the previous problem, F is a projective object so we have a commutative diagram which I have extended to an exact sequence,

$$\begin{tikzcd} & & & F \arrow[dl, dashed, "h"'] \arrow[d, "\mathrm{id}_F"] & \\ 0 \arrow[r] & \ker f \arrow[r, hook] & A \arrow[r, two heads, "f"] & F \arrow[r] & 0 \end{tikzcd}$$

However, $f \circ h = \mathrm{id}_F$ so the exact sequence splits on the right. Therefore, $A \cong \ker f \oplus F$.

Problem 6.

Let (C, d) be a chain complex of abelian groups such that C_n is free. For each n, consider the exact sequence,

$$0 \longrightarrow \ker d_n \xhookrightarrow{\iota} C_n \xrightarrow{d_n} \mathrm{Im}(d_n) \longrightarrow 0$$

Since $\mathrm{Im}(d_n) \subset C_{n-1}$ which is a free group, we know that $\mathrm{Im}(d_n)$ is free. Therefore, by the above problem, $C_n \cong \ker d_n \oplus \mathrm{Im}(d_n)$ since $\mathrm{Im}(d_n)$ is free and $d_n : C_n \to \mathrm{Im}(d_n)$ is surjective. Define the map, $f_n : C_n \to H_n(C)$ by the composition,

$$C_n \xrightarrow{\sim} \ker d_n \oplus \mathrm{Im}(d_n) \xrightarrow{\pi_1} \ker d_n \xrightarrow{\pi_{H_n}} H_n(C)$$

We need to show that $f : C \to H$ is a chain map and a quasi-isomorphism where H is the chain complex $(H_n(C), 0)$. Consider the diagram,

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & H_{n+1} \arrow[r, "0"] & H_n \arrow[r, "0"] & H_{n-1} \arrow[r, "0"] & \cdots \end{tikzcd}$$

However, $f_n \circ d_{n+1} = 0 = 0 \circ f_{n+1}$ since $\mathrm{Im}(d_{n+1}) \subset \ker \pi_{H_n} \subset \ker f_n$. Therefore, f_n is a chain map. Consider the induced map,

$$f_* : H_n(C) \to H_n(H) = H_n(C)$$

where $H_n(H) = \ker 0_n / \mathrm{Im}(0_{n+1}) = H_n = H_n(C)$. The induced map acts on $a \in \ker d_n$ via,

$$f_*(a + \mathrm{Im}(d_n)) = f(a) + \mathrm{Im}(0_n)$$

However, $a \in \ker d_n$ so $\pi_1(a) = a$ and thus $f(a) = \pi_{H_n}(a) = a + \mathrm{Im}(d_n)$. Therefore,

$$f_*(a + \mathrm{Im}(d_n)) = a + \mathrm{Im}(d_n)$$

so f_* is the identity map on $H_n(C) \to H_n(H)$ under the identification $H_n(H) \cong H_n(C)$. Thus, f is a quasi-isomorphism.