Mathematics GR6657 Algebraic Number Theory
Assignment # 5

Benjamin Church

April 6, 2018

1. Let G be a group. Consider the map $\Psi : \mathbb{Z}[G] \to \mathbb{Z}$ given by,

$$\Psi : \sum_{g \in G} n_g \, g \mapsto \sum_{g \in G} n_g$$

Take $I_G = \ker \Psi$. Consider the map,

$$\Phi : G \to I_G / I_G^2 \quad \text{given by} \quad g \mapsto (g-1) \pmod{I_G^2}$$

First, we need to show that Φ is a homomorphism,

$$g_1 g_2 \mapsto \; (g_1 g_2 - 1) = (g_1 - 1)(g_2 - 1) + (g_1 - 1) + (g_2 - 1) \equiv (g_1 - 1) + (g_2 - 1) \pmod{I_G^2}$$

since $(g_1 - 1)(g_2 - 1) \in I_G^2$. Given an element,

$$\sum_{i=1}^{r} n_i \, g_i \quad \text{such that} \quad \sum_{i=1}^{r} n_i = 0$$

we can write,

$$\sum_{i=1}^{r} n_i \, g_i = \sum_{i=1}^{r} (n_i \, g_i - 1) + \sum_{i=1}^{r} n_i = \sum_{i=1}^{r} (n_i \, g_i - 1)$$

and we know that,

$$\Phi : \prod_{i=1}^{r} g_i^{n_i} \mapsto \sum_{i=1}^{r} n_i \, (g_i - 1)$$

so Φ is surjective. Furthermore, I_G / I_G^2 is an abelian group so the map $\Phi : G \to I_G / I_G^2$ factors through $G^{\text{ab}} = G/[G, G]$. Take the map $\Phi^{\text{ab}} : G^{\text{ab}} \to I_G / I_G^2$. We construct an inverse map by $\Xi : I_G \to G^{\text{ab}}$ given by $\Xi : (g - 1) \mapsto g$ which is well defined because both groups are abelian so the map is invariant under reordering. Consider any product,

$$(g_1 - 1)(g_2 - 1) = (g_1 g_2 - 1) - (g_1 - 1) - (g_2 - 1) \mapsto g_1 g_2 g_1^{-1} g_2^{-1} \in [G, G]$$

so Ξ is trivial on I_G^2. Equivalently, we see that the kernel of Φ sending elements into I_G^2 is generated by commutators so Φ^{ab} is injective. Continuing, we find that Ξ factors through the quotient as a map,

$$\Xi : I_G / I_G^2 \mapsto G^{\text{ab}} \quad \text{acting as} \quad \Xi : (g - 1) \mapsto g$$

which is clearly and inverse of Φ^{ab}. Therefore, Φ^{ab} is an isomorphism.

2. Let G be a finite group. Set $\Lambda = \mathbb{Z}[G]$ and consider the map,

$$\Psi : \mathrm{Hom}_{\mathbb{Z}}(\Lambda, B) \to \Lambda \otimes_{\mathbb{Z}} B \quad \text{given by} \quad \Psi : \varphi \mapsto \sum_{g \in G} g \otimes \varphi(g)$$

First, suppose that,

$$\Psi(\varphi) = \sum_{g \in G} g \otimes \varphi(g) = 0$$

then each term is zero and thus $\varphi(g) = 0$ which means that $\varphi = 0$. Therefore, Φ is injective. Furthermore, for $g \in G$ and $h \in B$, consider the map,

$$\delta_g^h(x) = \begin{cases} h & x = g \\ 0 & x \neq g \end{cases}$$

This is a morphism of $\mathbb{Z}$-modules $\mathbb{Z}[G] \to B$. Furthermore,

$$\Psi : \delta_g^h \mapsto \sum_{x \in G} g \otimes \delta_g^h(x) = g \otimes h$$

which implies that,

$$\Psi : \sum_{g,h} n_{g,h} \delta_g^h \mapsto \sum_{g,h} n_{g,h} g \otimes h$$

and thus Ψ is surjective. It remains to check that Φ is a morphism of Λ-modules.

$$\Psi(\varphi_1 + \varphi_2) = \sum_{g \in G} g \otimes (\varphi_1(g) + \varphi_2(g)) = \sum_{g \in G} g \otimes \varphi_1(g) + \sum_{g \in G} g \otimes \varphi_2(g) = \Psi(\varphi_1) + \Psi(\varphi_2)$$

Furthermore,

$$\Phi(h \cdot \varphi) = \sum_{g \in G} g \otimes \varphi(h^{-1} g) = \sum_{g' \in G} hg' \otimes \varphi(g') = h \cdot \sum_{g' \in G} g' \otimes \varphi(g') = h \cdot \Phi(\varphi)$$

so Φ is an isomorphism of Λ modules.

3.

Theorem 0.1 (Tate)**.** *Let G be a finite group and let C be a G-module. Suppose that all (not necessarily proper) subgroups H of G satisfy $G^1(H, C) = 0$ and $H^2(H, C)$ is cyclic of order $|H|$. Then, there is an isomorphism,*

$$\hat{H}^r(G, \mathbb{Z}) \to \hat{H}^{r+2}(G, C)$$

depending only on the choice of a generator for $H^2(G, C)$.

Proof. I will follow Milne's notes on Class Field Theory (p. 81 - 82). First we choose a generator γ of the cyclic group $H^2(G, C)$ which when restricted to any subgroup H must also generate $H^2(H, C)$. Given a cocycle $\varphi \in C_2(G, C) = \mathrm{Hom}_G(\mathbb{Z}[G \times G], C)$ which goes to the generator of $H^2(G, C)$ when homology is taken we can define a G-module,

$$C(\varphi) = C \oplus \bigoplus_{\sigma \in G} [x_\sigma] \mathbb{Z}$$

Where G acts on the free group by its action on the basis symbols,

$$\sigma \cdot x_\tau = x_{\sigma\circ\tau} - x_\sigma + \varphi(\sigma,\tau)$$

An easy computation shows that this defines an action of G on $C(\varphi)$. We now need to show that,

$$H^1(H, C(\varphi)) = H^2(H, C(\varphi)) = 0$$

with the action defined above. We have a short exact sequence,

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

since I_G is the kernel of the map $\mathbb{Z}[G] \to \mathbb{Z}$ and is thus generated by elements of the form $\sigma - 1$ for $\sigma \in G\backslash\{1\}$. We know that $\mathbb{Z}[G]$ is an induced module and thus $H^r(G, \mathbb{Z}[G])$ for all r[1]. We can define a map $\alpha : C(\varphi) \to \mathbb{Z}[G]$ such that, $\alpha(c) = 0$ for $c \in C$ and $\alpha(x_\sigma) = \sigma - 1$. Which gives rise to a short exact sequence of G-modules,

$$0 \longrightarrow C \longrightarrow C(\varphi) \xrightarrow{\alpha} I_G \longrightarrow 0$$

This short exact sequence of G-modules gives rise to a long exact sequence of homology,

$$H^0(H,C) \to H^0(H,C(\varphi)) \longrightarrow H^0(H,I_G) \longrightarrow H^1(H,C) \to H^1(H,C(\varphi)) \to H^1(H,I_G) \to$$

$$\to H^2(H,C) \longrightarrow H^2(H,C(\varphi)) \to H^2(H,I_G) \longrightarrow \cdots$$

However, we know that $H^1(H,C) = 0$ and $H^2(H,I_G) \cong H^1(H,\mathbb{Z}) = 0$. Therefore, we have the exact sequence,

$$0 \longrightarrow H^1(H,C(\varphi)) \longrightarrow H^1(H,I_G) \longrightarrow H^2(H,C) \longrightarrow H^2(H,C(\varphi)) \longrightarrow 0$$

We use this sequence to argue that $H^1(G,C(\varphi)) = H^2(G,C(\varphi)) = 0$ and therefore that all the cohomology groups vanish because G is finite.

As we have already seen, there are exact sequences,

$$0 \longrightarrow C \longrightarrow C(\varphi) \xrightarrow{\alpha} I_G \longrightarrow 0$$

and,

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which together give a sequence,

$$0 \longrightarrow C \longrightarrow C(\varphi) \xrightarrow{\alpha} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

[1] $\mathbb{Z}[G] \cong \mathrm{Ind}_1^G(\mathbb{Z})$ so by Shapiro's Lemma, $H^r(G, \mathrm{Ind}_1^G(\mathbb{Z}) \cong H^r(1,\mathbb{Z}) = 0$

which remains exact by a homology computation. However, we know that $H^r(G, C(\varphi)) = 0$ and $H^r(G, \mathbb{Z}[G]) = 0$ so the map $H^r(G, \mathbb{Z}) \to H^{r+2}(G, C)$ is an isomorphism by proposition 1.13 in Milne.

$\square$