1 Remedial Curve Theory

1.1 Geometrically Irreducible

Lemma 1.1.1 (Tag 0553)**.** Let $f : X \to Y$ be a morphism of schemes. Assume,

(a) Y is irreducible with generic point η,

(b) X_η is geometrically irreducible

(c) f is of finite type

then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibers.

Lemma 1.1.2. Let $f : X \to Y$ be a morphisms of schemes. Suppose that,

(a) X and Y are integral

(b) X is normal

(c) the fibers of f are geometrically connected (e.g. $f_* \mathcal{O}_X = \mathcal{O}_Y$)

then the generic fiber $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically irreducible.

Proof. $X_\eta / \kappa(\eta)$ is geometrically irreducible iff $\kappa(\eta)$ is separably closed in $\kappa(\xi)$. This follows by Tag 054Q and Tag 0G33. Let $\alpha \in \kappa(\xi)$ be separably algebraic over $\kappa(\eta)$ i.e. a root of a separable polynomial $p \in \kappa(\eta)[x]$. There is a coordinate ring A of Y where all the denominators in p are invertible. We claim that $A[\alpha] \subset B$ where B is any coordinate ring of X containig A. Indeed, α is integral over A and hence over B so by normality $\alpha \in B$ so we get morphisms,

$$X_A \to \mathrm{Spec}\,(A[\alpha]) \to \mathrm{Spec}\,(A)$$

but the fibers of $X_A \to \mathrm{Spec}\,(A)$ are geometrically connected so we must have $\alpha \in A$ since otherwise the fibers of $\mathrm{Spec}\,(A[\alpha]) \to \mathrm{Spec}\,(A)$ are not geometrically irreducible. $\square$

Remark. If we only assumed that X/k is geometrically irreducible (which is weaker than X being normal) the result would not follow. Indeed, consider,

$$X = \mathrm{Proj}\left(k[t][X,Y,Z]/(X^2 - tY^2)\right) \to \mathrm{Spec}\,(k[t]) = Y$$

where k is algebraically closed. Then X, Y are geometrically integral since they are integral. Indeed, we need to check that the polynomials on the charts,

$$\left(\frac{X}{Z}\right)^2 - t\left(\frac{X}{Y}\right)^2 \qquad \left(\frac{X}{Y}\right)^2 - t \qquad 1 - t\left(\frac{Y}{X}\right)^2$$

are irreducible. They are since t does not admit a square root. However, the generic fiber is,

$$X = \mathrm{Proj}\left(k(t)[X,Y,Z]/(X^2 - tY^2)\right) \to \mathrm{Spec}\,(k(t))$$

is not geometrically irreducible since after the extension $k(t^{\frac{1}{2}})/k(t)$ we can split the polynomial. However, X is not normal since $t^{\frac{1}{2}}$ is in the fraction field (look at the second chart) but not in every chart since $H^0(X, \mathcal{O}_X) = k[t]$ and this does not contain $t^{\frac{1}{2}}$. The normalization of X is $\mathbb{P}^1 \times \mathrm{Spec}\left(k[t^{\frac{1}{2}}]\right)$ with the map,

$$[T_0, T_1] \mapsto [t^{\frac{1}{2}} T_0, T_0, T_1]$$

This "hits both branches" since $t^{\frac{1}{2}}$ "remembers which branch of the square root it is on" while still making $\widetilde{X}$ be an integral scheme as it must since it is the normalization of an integral scheme.

(IN THE ABOVE CASE IT IS ALSO GEOMETRICALLY INTEGRAL SINCE REDUCEDNESS IS A GENERIC PHENOMENON) NO ONLY WORKS WHEN Y HAS DIM = 1!!!!

Proposition 1.1.3. Let $f : X \to Y$ be a proper morphism of schemes. Let X, Y be integral and finite type over a perfect field k. If X is normal and $\dim Y = 1$ then the following are equivalent,

(a) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically integral

(b) $\kappa(\eta)$ is algebraically closed in $\kappa(\xi)$

(c) $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism.

Proof. LEMMA 7.2 BADESCU □

Example 1.1.4. If the base has dimension > 1 this is false. For example,

$$X = \mathrm{Proj}\,(\mathbb{F}_p[s,t][X,Y,Z]/(X^p + sY^p + tZ^p)) \to \mathrm{Spec}\,(\mathbb{F}_p[s,t])$$

satisfies $f_* \mathcal{O}_X = \mathcal{O}_Y$ and X is normal but the generic fiber,

$$X = \mathrm{Proj}\,(\mathbb{F}_p(s,t)[X,Y,Z]/(X^p + sY^p + tZ^p)) \to \mathrm{Spec}\,(\mathbb{F}_p(s,t))$$

is not geometrically reduced. Indeed, although $\mathbb{F}_p(s,t)$ is algebraically closed in $\mathrm{Frac}\,(\mathbb{F}_p(s,t)[x,y]/(x^p + sy^p + t))$ is it not separable since separability implies reducedness of basechange by the field extension $\mathbb{F}_p(s^{\frac{1}{p}}, t^{\frac{1}{p}})$.

Remark. Therefore, we can make a fibration of a normal variety have geomtrically integral fibers by Stein factorizing.

Example 1.1.5. Notice that if $f : X \to Y$ is a proper map of smooth varities with $f_* \mathcal{O}_X = \mathcal{O}_Y$ the fibers do not all need to be geomtrically irreducible or reduced (even the integral ones) although the generic fiber is geometrically integral. Of course every fiber is geometrically connected. Consider,

$$X = \mathrm{Proj}\left(\mathbb{R}[t][X,Y,Z]/(X^2 + Y^2 + tZ^2)\right) \to \mathrm{Spec}\,(\mathbb{R}[t])$$

Then it is easy to see that X is smooth (since $X_{\mathbb{C}}$ is smooth using the Jacobian criterion). However, X_0 is integral but not geometrically connected. Likewise, consider,

$$X = \mathrm{Proj}\,(k[t][X,Y,Z]/(Z^p + aX^p + bY^p + tf(X,Y,Z)) \to \mathrm{Spec}\,(k[t])$$

where f is some polynomial such that X is smooth and k is a field of characteristic p with elements $a, b \in k$ which are not p-powers. Then the generic fiber is geometrically integral but X_0 is geometrically nonreduced.

Remark. Note that if X is any of,

(a) integral

(b) normal

(c) regular

then the same is true of X_η for any map $f : X \to Y$ by localization. However, unlike the case for irreducible above, the corresponding geometric versions do *not* hold as the following and previous examples shows.

Example 1.1.6. Quasi-elliptic fibration of $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ not geometricially normal, or regular.

Theorem 1.1.7 (Fujita 1982)**.** Let $f : X \to Y$ be a proper dominant morphism of integral locally noetherian schemes. Consider the following conditions,

(a) $\kappa(\xi_Y)$ is algebraically closed in $\kappa(\xi_X)$

(b) $\mathrm{rank}_Y(f_*\mathcal{O}_X) = 1$

(c) the general fiber satisfies $h^0(X_y, \mathcal{O}_{X_y}) = 1$

(d) $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

Then we have the following implications,

$$\begin{tikzcd} (a) \arrow[r] & (b) \arrow[l, bend right, "X \text{ normal}"'] \arrow[r, bend left, "Y \text{ normal}"] \arrow[d, leftrightarrow] & (d) \arrow[l] \\ & (c) & \end{tikzcd}$$

Proof. DO IT $\square$

Example 1.1.8. Consider,

$$X = \mathrm{Proj}\,(k[t][X,Y,Z]/(X^p + sY^p + tZ^p)) \to \mathrm{Spec}\,(k[t])$$

where $c \in k$ is not a p^{th}-power. Then X_η is a smooth genus $\frac{(p-1)(p-2)}{2}$ curve but X_s is integral but not geometrically reduced although $H^0(X_s, \mathcal{O}_{X_s}) = k$. The arithmetic genus is still constant but the geometric genus drops to zero.

1.2 Morphisms of Curves

Definition 1.2.1. A curve C over k is a separated finite type scheme over k of pure dimension 1.

Proposition 1.2.2.

Definition 1.2.3. Let X be a proper k-scheme. The *arithmetic-genus* of X is,

$$p_a(X/k) := \dim_k H^1(X, \mathcal{O}_X)$$

If $H^0(X, \mathcal{O}_X) = K$ is a field then we write,

$$p_a(X) := \dim_K H^1(X, \mathcal{O}_X)$$

Remark. The arithmetic genus is stable under field extension by flat base change. The point of the second definiiton is that the base field is unambiguous when it is defined.

Q: can I have an elliptic curve degenerate to an elliptic curve over a field extension? I can degenerate to a multiple elliptic curve so maybe.

Maybe but only with an inseparable extension since automatically O-connected over dvr and then geometrically connected fibers.

IS THERE AN EXAMPLE OF O-CONN BUT THE H0 JUMPS UP TO AN INSEP EXTENSION. CAN'T HAVE IT JUMP UP TO SEP EXTENSION B/C MUST BE GEOMETRICALLY CONNECTED FIBERS. YES RAYNAUD EXAMPLE. If $k' = H^0(X_s, \mathcal{O}_{X_s})$ then there's a map $H^0(\hat{X}_s, \mathcal{O}_{\hat{X}_s}) \to k'$ but by theorem on formal functions the first thing is k. This doesn't actually work.

TOTAL SPACE INTEGRAL OVER A DVR WITH $H^0(X_\eta, \mathcal{O}_{X_\eta}) = K$ then automatically $\mathcal{O}$-connected and hence no field extensions in the fibers.

RIGHT?? CAN ONLY ADD NONREDUCEDNESS THIS IS THE ONLY WAY.

BAD DEFN!!

Definition 1.2.4. Let X be a curve which is a disjoint union of finitely many smooth curves C_i over an algebraically closed field k. Then the *geometric genus* (or just *genus*) of X is,

$$g(X) := p_a(X/k) = \sum_{i=1}^{n} p_a(C_i/k)$$

Definition 1.2.5. Let X be a curve over a field k. Consider $\widetilde{X}$ which is the normalization of $(X_{\bar{k}})_{\text{red}}$. This is a disjoint union of finitely many smooth curves C_i over $\bar{k}$. Thus we can define,

$$g(X/k) := g(\widetilde{X})$$

Remark. The geometric genus is stable under field extension by definition. However, notice that $g(X/k)$ depends on the base field. If we view X as over an extension k'/k then $g(X/k) = [k' : k]g(X/k')$.

UNDERSTAND RELATION BETWEEN ARITH AND GEOM GENUS

Lemma 1.2.6. Let $f : X \to Y$ is a nonconstant map of proper regular curves over an algebraically closed field k then $g(X) \geq g(Y)$.

Proof. Riemann-Hurwitz and Frobenius tricks. CITE HARTSHORNE □

Proposition 1.2.7. Let $f : X \to Y$ be a surjective map of proper curves over a field k. Then $g(X/k) \geq g(Y/k)$.

Proof. By definition, let $\widetilde{X}$ be the normalization of $(X_{\bar{k}})_{\text{red}}$ then $g(X/k) = g(\widetilde{X})$. Then the map $f : \widetilde{X} \to \widetilde{Y}$ is surjective since it is dominant and finite so we see that $g(X/k) \geq g(Y/k)$. □

Example 1.2.8. Say $E = \text{Proj}\left(\mathbb{R}[X,Y,Z]/(Y^2Z - X^3 - XZ^2)\right)$ is an elliptic curve. It is important that we consider the genus of $E_{\mathbb{C}}$ (as a curve over $\mathbb{R}$) as 2 not 1 because,

$$\text{Proj}\left(\mathbb{R}[X,Y,Z]/((Y^2Z - X^3)^2 + (XZ^2)^2)\right)$$

has normalization $E_{\mathbb{C}}$ but has genus 2 according to our definition since it has two components when passing to $\mathbb{C}$.

Proposition 1.2.9. Let $f : X \to Y$ be a dominant map of proper curves over k. Then $g(X) \geq g(Y)$.

WE CAN ASSUME ALL THE CURVES ARE GEOMETRICALLY INTEGRAL OVER THEIR H^0 FIELD BECAUSE THEY ARE THE GENERIC FIBERS OF A NORMAL SURFACE AND HENCE NORMAL. MAKE THIS WORK!!

1.3 Degenerations of Curves

Definition 1.3.1. A *degeneration of curves* is a proper flat family $X \to S = \text{Spec}\,(R)$ over a dvr R where X_η is an integral normal projective curve over $K = \text{Frac}\,(R)$. If X is normal we say that X is a *model* of X_η over R.

Lemma 1.3.2. The total space X of a degeneration of curves is integral.

Proof. We need to show that every affine open $\text{Spec}\,(A) = U \subset X$ has A a domain. Indeed, $R \to A$ is flat so $A \hookrightarrow A_K$ is injective but A_K is an affine open of X_K which is integral so A_K and A is a domain. □

Lemma 1.3.3. Let $X \to S$ be a proper flat map from an integral scheme to a dvr. Then the following are equivalent,

(a) $f_* \mathcal{O}_X = \mathcal{O}_S$

(b) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = K$

Proof. Indeed, $R' = H^0(X, \mathcal{O}_X)$ is a finite R-algebra and since X is integral it is a domain. Also there are maps $R \to R' to A$ for any affine open $U = \text{Spec}\,(A)$ of X but $R \to A$ is flat and hence injective so $R \to R'$ is injective. Furthermore, by flat base change,

$$H^0(X_K, \mathcal{O}_{X_K}) = R' \otimes_R K$$

so if $H^0(X_K, \mathcal{O}_{X_K}) = K$ then $\text{Frac}\,(R') = \text{Frac}\,(R)$ under $R \hookrightarrow R'$ but $R \to R'$ is integral and R is normal so $R' = R$. The other direction is obvious. □

Corollary 1.3.4. Let $X \to S$ be a degeneration of curves. Consider the following properties,

(a) $X_\eta \to \text{Spec}\,(K)$ is geometrically integral

(b) $f_* \mathcal{O}_X = \mathcal{O}_S$

(c) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = K$

(d) $X_\eta \to \text{Spec}\,(K)$ is geometrically connected.

Then we have,

$$(a) \implies (b) \implies (c) \implies (d)$$

Proof. Use the above and Tag 0BUG (8). □

Remark. Even if $f_* \mathcal{O}_X = \mathcal{O}_S$ we don't necessarily have that X_η is geometrically reduced e.g. Example 1.1.4.

Proposition 1.3.5. Let $X \to \text{Spec}\,(R)$ be a degeneration of curves. Then each irreducible component $\Gamma_i \subset X_s$ is a proper integral curve over $\kappa_i = H^0(\Gamma_i, \mathcal{O}_{\Gamma_i})$ and,

$$g(\Gamma_i/\kappa_i) \le g(X_\eta/K)$$

Proof. Flatness implies that X_s has pure dimension 1 and X_s is proper over $\kappa(s)$. Since Γ_i is integral we see that κ_i is a field. GO TO MINIMAL MODEL AND USE DOMINATION MAKES GENUS GO DOWN. □

Remark. The degeneration of a twisted cubic to a nodal cubic shows that the corresponding statement for arithmetic genera is false. However, if we assume cohomological flatness we can conclude rhe following. In the case that we have some control over the singularities then Raynaud has proved that $X \to \mathrm{Spec}\,(R)$ satisfies cohomological flatness.

Proposition 1.3.6. Let $X \to \mathrm{Spec}\,(R)$ be a degeneration of curves with $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$ (e.g. if $\mathcal{O}_X$ is cohomologically flat). Then for the irreducible components $\Gamma_i \subset X_s$ we have,

$$\sum_{i=1}^{n} p_a(\Gamma_i/\kappa(s)) \le p_a(X_\eta/K)$$

In particular, for each i,

$$p_a(\Gamma_i/\kappa(s)) \le p_a(X_\eta/K)$$

Proof. By upper semicontinuity this implies that $H^0(X_K, \mathcal{O}_{X_K}) = K$. From constancy of the Euler characteristic and the assumption we see that,

$$p_a(X_\eta/K) = p_a(X_s/\kappa(s)) \ge \sum_{i=1}^{n} p_a(\Gamma_i/\kappa(s))$$

so the result follows. First note that, if we set $Y = (X_s)_{\mathrm{red}}$ then,

$$H^1(Y, \mathcal{O}_{X_s}) \twoheadrightarrow H^1(Y, \mathcal{O}_Y)$$

shows that,

$$\dim_{\kappa(s)} H^1(Y, \mathcal{O}_{X_s}) \ge \dim_{\kappa(s)} H^1(Y, \mathcal{O}_Y)$$

Now the inequality follows from the analysis of the map of sheaves,

$$\mathcal{O}_Y \to \prod_{i=1}^{n} \mathcal{O}_{\Gamma_i}$$

The cokernel of this map is supported in dimension 0 at the intersection points of the components. Splitting this sequence into,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_Y \longrightarrow \mathscr{I} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{I} \longrightarrow \prod \mathcal{O}_{\Gamma_i} \longrightarrow \mathcal{C} \longrightarrow 0$$

and thus since $\dim Y = 1$ and $\dim \mathrm{Supp}\,(\mathcal{C}) = 0$ we see that,

$$H^1(X, \mathcal{O}_Y) \twoheadrightarrow H^1(Y, \mathscr{I}) \qquad H^1(\mathscr{I}, \mathcal{O}_{\mathscr{I}}) \twoheadrightarrow \prod_{i=1}^{n} H^1(\Gamma_i, \mathcal{O}_{\Gamma_i})$$

Therefore,

$$H^1(X, \mathcal{O}_Y) \twoheadrightarrow \prod_{i=1}^{n} H^1(\Gamma_i, \mathcal{O}_{\Gamma_i})$$

so the result follows. $\square$

Theorem 1.3.7 (Raynaud)**.** Let $X \to \mathrm{Spec}\,(R)$ be a degeneration of curves with X normal and $f_*\mathcal{O}_X = \mathcal{O}_S$. Suppose that,

char$(\kappa(s))$ is coprime to the gcd of the multiplicites of the components of X_s

then $f_*\mathcal{O}_X = \mathcal{O}_S$ universally.

Remark. If $f_*\mathcal{O}_X = \mathcal{O}_S$ universally then in particular $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$ so we can conclude our arithmetic genus inequality.

2 Stuff About Models of Curves

Lemma 2.0.1. Let $X \to \operatorname{Spec}(R)$ be a degeneration of curves. Let $\Gamma \subset X_s$ be a (reduced) irreducible component of X_s and suppose that Γ is normal and X_s is reduced at the generic point $\eta \in \Gamma$. Let $\kappa = H^0(\Gamma, \mathcal{O}_\Gamma)$ which is a finite extension of $\kappa(s)$. Then,

$$p_a(\Gamma/\kappa(s)) \leq p_a(X_K/K)$$

Proof. Consider $\widetilde{X} \to X^\nu \to X$ which is a resolution of singularities of the normalization. PROVE USING LEMMA OF ZARISKI THAT X IS NOMAL at η. THUS MAP IS ISOMORPHSM OVER OPEN OF Γ SO WIN. □

TWISTED CUBIC DEGENERATE TO QUASI-ELLIPTIC WITH EMBEDDED POINT, CANT HAPPEN SINCE ITS NORMAL AT THAT POINT. COULD HAPPEN IF NONREDUCED EVERYWHERE MAYBE

COUNTEREXAMPLE WITH NOT NORMAL OR NOT GENERICALLY REDUCED:

(a) if Γ is not normal then a twisted cubic degenerating to a nodal cubic (with an embedded point) gives a counterexample

(b) if X_s is not reduced at η then (COUNTEREXAMPLE??)

3 Controlling Arithmetic Genus in Families

Remark. Notation, let $(R, \mathfrak{m}, \kappa)$ be a DVR with fraction field $K = \operatorname{Frac}(R)$. For $X \to \operatorname{Spec}(R)$ let $X_\eta = X_K$ be the generic fiber and $X_s = X_\kappa$ the special fiber.

3.1 Setup

Let $X \to \operatorname{Spec}(R)$ be a normal degeneration of curves. Then consider the following data. Let $\Gamma_i \subset X_s$ be the (reduced) irreducible components of the special fiber and the following rings,

(a) $A = H^0(X_s, \mathcal{O}_{X_s})$

(b) $\kappa' = H^0((X_s)_{\text{red}}, \mathcal{O}_{(X_s)_{\text{red}}})$

(c) $\kappa_i = H^0(\Gamma_i, \mathcal{O}_{\Gamma_i})$

where A is an Artin local κ-algebra and κ' and κ_i are finite field extensions of κ (Tag) since X_s is geometrically connected.

3.2 Conjectures

Here are three conjectures in decreasing order of strength.

Definition 3.2.1. Let X be a (nonprojective) curve over k. Then we define the arithmetic genus $p_a(X/k)$ as the smallest $p_a(\widetilde{X}/k)$ over all compactifications $X \hookrightarrow \widetilde{X}$ defined over k. Explicitly, these are proper k-schemes $\widetilde{X}$ equipped with an open embedding $X \hookrightarrow \widetilde{X}$.

3.3 Results

GENUS DROPS IN NORMAL NONPROPER FAMILIES.