1 Cohomological Non Abelian Theorems in Char p

1.1 The Situation over $\mathbb{C}$

We have three cohomology spaces,

$$\mathcal{M}_{\mathrm{Dol}}(C) \sim \mathcal{M}_{\mathrm{dR}}(C) \sim \mathcal{M}_B(C)$$

the first is the stack of Higgs bundles (E, ϕ) with $\phi : E \to E \otimes \Omega^1$ which is $\mathcal{O}$-linear. The second is the space of flat connections and the third is the space of representations of $\pi_1(C)$. The first equivalence is by Hitchen the second is by Riemann-Hilbert.

(a) $h : \mathcal{M}_{\mathrm{Dol}}(C) \to A(C) = \bigoplus_{i=1}^r H^0(C, \omega_C^{\otimes r})$ sending $(E, \phi) \mapsto \mathrm{char}(\phi)$ the characteristic polynomial. This is the Hitchin morphism.

(b) $\mathcal{M}_{\mathrm{Hol}}(C)$ is the space of t-connections,

$$\nabla_t(fe) = t\mathrm{d}f + f\nabla_t(e)$$

There is a map $\mathcal{M}_{\mathrm{Hol}}(C) \to \mathbb{A}^1$ given by t and the 1-fiber is $\mathcal{M}_{\mathrm{dR}}$ and the 0-fiber is $\mathcal{M}_{\mathrm{Dol}}$.

1.2 Characteristic p

Let $k = \bar{k}$ and $\mathrm{char}\, k = p$. Let $\partial \in \mathrm{Der}(C/k)$ then $\partial^p = \partial \circ \cdots \circ \partial$ is a derivation. Then the p-curvature of ∇_t is,

$$\Psi(\nabla_t)(\partial) := \nabla_t(\partial)^p - t^{p-1}\nabla_t(\partial^p) \in \mathrm{End}\,(\mathcal{O}_C)\,E$$

is $\mathcal{O}_C$-linear. However,

$$\Psi(\nabla_t)(f\partial) = f^p\Psi(\nabla_t)(\partial)$$

and therefore defines a map,

$$\Psi(\nabla_t) : TC \to \mathrm{Frob}_*\mathrm{End}\,(\mathcal{O}_C)\,E$$

Remark. The map $\mathrm{Frob} : C \to C^{(1)}$ is the relative Frobenius over the field k.