Artin's Axioms and Formal Deformation Theory

Stanford-Berkeley Number Theory Learning Seminar

January 13, 2023

1 Definitions

Definition 1.0.1. An *algebraic space* is a functor $X : (\mathbf{Sch}_S)^{\text{op}}_{\text{fppf}} \to \text{Set}$ such that,

(a) F is a sheaf in the fppf topology

(b) the diagonal $\Delta_{X/S} : X \to X \times_S X$ is representable by schemes

(c) there is a scheme U and an étale surjection $U \twoheadrightarrow X$.

Definition 1.0.2. An *algebraic stack* is a category fibered in groupoids $\mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$ such that,

(a) $\mathcal{X}$ is a stack in the fppf topology

(b) $\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces

(c) there is an algebraic space U and an étale surjection $U \twoheadrightarrow \mathcal{X}$.

Remark. The map $U \to \mathcal{X}$ is only necessarily representable by algebraic spaces so to express the property of being an étale surjection consider any map from a scheme $T \to \mathcal{X}$ and an étale cover from a scheme $V \to U \times_{\mathcal{X}} T$ in the diagram,

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend left, "\text{ét surj}"] & U \times_{\mathcal{X}} T \arrow[r] \arrow[d] & T \arrow[d] \\ & U \arrow[r] & \mathcal{X} \end{tikzcd}$$

This property is independent of the choice of étale cover $V \to U \times_{\mathcal{X}} T$ by étale descent for étale surjective morphisms.

Remark. Why do we only require that $\mathcal{X}$ be smooth locally an algebraic space and its diagonal be representable by only algebraic spaces? The diagonal is closely related to the automorphism groups of objects $\mathcal{X}$ parametrizes. When $\pi : X \to S$ a proper finitely presented map of schemes, $\text{Hilb}_{X/S}$ is representable by an algebraic space but not generally by a scheme unless π is projective. This shows that $\text{Isom}_S(X, Y)$ between two proper S-schemes is usually only representable by an algebraic space. Therefore, we want to allow for Δ to be representable by algebraic spaces not just schemes to capture moduli of proper non-projective objects.

Definition 1.0.3. Consider $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ morphisms of categories fibered in groupoids. Then the 2-fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is defined as the category fibered in groupoids,

(a) objects are (x, y, γ) with $p(x) = p(y)$ and $\gamma : f(x) \to g(y)$ a morphism over id

(b) morphisms are $\varphi : (x, y, \gamma) \to (x', y', \gamma')$ are given by pairs $(\varphi_x : x \to x', \varphi_y : y \to y')$ such that the diagram,

$$\begin{tikzcd} f(x) \arrow[r, "\gamma"] \arrow[d, "\varphi_x"] & g(x) \arrow[d, "\varphi_y"] \\ f(x') \arrow[r, "\gamma'"] & g(x') \end{tikzcd}$$

commutes.

Proposition 1.0.4. For any objects $x, y \in \mathcal{X}(U)$. There is a 2-fiber product diagram,

$$\begin{tikzcd} \mathrm{Isom}(x, y) \arrow[r] \arrow[d] & U \arrow[d, "y"] \\ U \arrow[r, "x"'] & \mathcal{X} \end{tikzcd}$$

Definition 1.0.5. The *inertia stack* of $\mathcal{X}$ is the category fibered in groupoids $\mathscr{I}_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$.

Proposition 1.0.6. For any $x \in \mathcal{X}(U)$ there is a 2-fiber product diagram,

$$\begin{tikzcd} \mathrm{Isom}(x, x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{I}_X \arrow[d] \\ U \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

2 Presentations

Proposition 2.0.1. Let X be an algebraic space over S and $f : U \twoheadrightarrow X$ an étale surjection from a scheme U. Set $R = U \times_X U$ in the pullback diagram,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

then we have,

(a) $j : R \to U \times_S U$ is a monomorphism and $R(T) \subset U(T) \times U(T)$ is an equivalence relation for all $T \to S$

(b) the projections $s, t : R \to U$ are étale

(c) the diagram,

$$\begin{tikzcd} R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U \arrow[r] & X \end{tikzcd}$$

is a coequalizer in $\mathrm{Sh}((\mathbf{Sch}_S)_{\mathrm{fppf}})$.

Proof. The first two are immediate The last holds in any category of sheaves given that $U \to X$ is surjective. $\square$

Definition 2.0.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. The quotient stack,

$$p : [U/R] \to (\mathbf{Sch}_S)_{\text{fppf}}$$

is the stackification of the category fibered in groupoids,

$$(T \to S) \mapsto (U(T), R(T), s, t, c)$$

Proposition 2.0.3 (04T5)**.** Given an algebraic stack $\mathcal{X}$ there is a smooth morphism $U \to \mathcal{X}$ from a scheme. We recover the groupoid presentation by taking the 2-fiber product,

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\ U \arrow[r] & \mathcal{X} \end{tikzcd}$$

and R is an algebraic space because we assumed that $\Delta_{\mathcal{X}}$ is representable by algebraic spaces. Then there is a natural equivalence $[U/R] \xrightarrow{\sim} \mathcal{X}$.

3 Infinitesimal Deformation Theory

Remark. First we recall how to apply infinitesimal deformation theory in the relative setting. In the basic case, we want to probe properties of a morphisms of schemes $f : X \to S$ near a finite type point $x : \operatorname{Spec}(k) \to S$. There is some affine open $\operatorname{Spec}(\Lambda) \subset X$ containing x. Then we need to consider Artinian local rings A and diagrams,

$$\begin{tikzcd} & & & X \arrow[d, "f"] \\ \operatorname{Spec}(k) \arrow[r] & \operatorname{Spec}(A) \arrow[r] \arrow[urr, dashed] & \operatorname{Spec}(\Lambda) \arrow[r, hook] & S \end{tikzcd}$$

and consider the set of dashed arrows. This means our base category should be the category of local Artinian Λ-algebras with residue field k.

Definition 3.0.1. Let Λ be a noetherian ring and $\Lambda \to k$ a finite ring map with k a field. Let $\mathcal{C}_\Lambda$ be the category of,

(a) (A, φ) where A is an Artinian local Λ-algebra and $\varphi : A/\mathfrak{m}_A \to k$ a Λ-algebra isomorphism

(b) morphisms $f : (B, \psi) \to (A, \varphi)$ are local Λ-algebra maps such that $\varphi \circ (f \bmod \mathfrak{m}_A) = \psi$

Remark. As in the absolute case (which corresponds to $\Lambda = k$) we can factor any extension $B \twoheadrightarrow A$ into *small* extensions $\varphi : B' \twoheadrightarrow A$ where $\ker \varphi$ is principal and annihilated by $\mathfrak{m}_B$.

Definition 3.0.2. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. Define the category $\widehat{\mathcal{C}}_\Lambda$ of,

(a) pairs (R, φ) where R is a Noetherian complete local Λ-algebra and $\varphi : R/\mathfrak{m}_R \to k$ is a Λ-algebra isomorphism,

(b) morphisms $f : (S, \psi) \to (R, \varphi)$ are local Λ-algebra map such that $\varphi \circ (f \bmod \mathfrak{m}_S) = \psi$.

Remark. Then $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is naturally a full subcategory.

3.1 Deformation Functors

Definition 3.1.1. A *predeformation functor* is a functor $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ such that $F(k) = \{*\}$.

Remark. The condition $F(k) = \{*\}$ corresponds to choosing a fixed base object for the deformations.

Definition 3.1.2. Given a predeformation functor $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ we extend it to $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to \mathrm{Set}$ via,

$$\widehat{F}(R) = \varprojlim_n F(R/\mathfrak{m}_R^n)$$

A functor F is *pro-representable* if $\widehat{F}$ is representable.

Definition 3.1.3. We say a morphism $\varphi : F \to G$ of functors on $\mathcal{C}_\Lambda$ is *smooth* the map,

$$F(B) \to F(A) \times_{G(A)} G(B)$$

induced by an extension $B \twoheadrightarrow A$ is surjective.

Definition 3.1.4. Let $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ be a predeformation functor. The *tangent space* of F is the set $TF = F(k[\epsilon])$. We will see under some assumptions this set is naturally a k-vectorspace.

Definition 3.1.5. Let $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ be a predeformation functor. A *hull*[1] for F is a pair (R, η) where $R \in \widehat{\mathcal{C}}_\Lambda$ and $\eta \in \widehat{F}(R)$ such that $h_R \to F$ is formally smooth and bijective on tangent spaces.

Remark. Let $k[\epsilon]$ be the ring $k[\epsilon]/(\epsilon^2)$ with the trivial Λ-algebra structure.

Definition 3.1.6. Let $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ be a predeformation functor. If $A' \to A$ and $A'' \to A$ are morphisms in $\mathcal{C}_\Lambda$ there is a natural map,

$$(*) \quad F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

Then Schlessinger's conditions are as follows,

(H1) if $A'' \twoheadrightarrow A$ is a small thickening then $(*)$ is surjective

(H2) if $A = k$ and $A'' = k[\epsilon]$ then $(*)$ is bijective

(H3) $TF = F(k[\epsilon])$ is finite-dimensional

(H4) if $A'' = A'$ and $A' \twoheadrightarrow A$ is a small thickening, then $(*)$ is bijective.

Remark. $TF = F(k[\epsilon])$ has a canonical vectorspace structure when F satisfies (H2) since we get,

$$F(k[\epsilon]) \times F(k[\epsilon]) \xrightarrow{\sim} F(k[\epsilon_1, \epsilon_2]) \to F(k[\epsilon])$$

using the map $k[\epsilon_1, \epsilon_2] \to k[\epsilon]$ via $\epsilon_1 \mapsto \epsilon$ and $\epsilon_2 \mapsto \epsilon$. The scalar multiplication is defined by $F(k[\epsilon]) \to F(k[\epsilon])$ induced by the map $\epsilon \mapsto c\epsilon$.

We cannot give TF a vectorspace structure without (H2) so it is more correct to group the Schlessinger conditions into pairs (H1) + (H2) and (H3) + (H4) as we do in the sequel.

[1]Some authors use the terminology *miniversal* formal object. However, in the deformation category setting, a minimal versal object may not induce an isomorphism of the tangent space so we reserve the term *miniversal* for a minimal versal object see Tag 06T0.

Definition 3.1.7. A pre-deformation functor $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ is a *deformation functor* if it satisfies (H1) and (H2).

Theorem 3.1.8 (Schlessinger)**.** Let $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ be a deformation functor. Then,

(a) F admits a hull if and only if it satisfies (H3)

(b) F is pro-representable if and only if it satisfies (H3) and (H4).

Example 3.1.9. Let X be a k-scheme, the functor $\mathrm{Def}_X : \mathcal{C}_k \to \mathrm{Set}$ defined by,

$$\mathrm{Def}_X : A \mapsto \{(X', \psi) \mid X' \text{ flat } A\text{-scheme with } \psi : X' \otimes_A k' \xrightarrow{\sim} X\}/\cong$$

is a deformation functor.

Example 3.1.10. Let $X = \mathrm{Spec}\,(k[x,y]/(xy))$ and $F = \mathrm{Def}_X$. If A is a finite type k-algebra and $P \twoheadrightarrow A$ is a presentation from a polynomial ring with kernel K then [H, Ex. 9.8] shows that,

$$\mathrm{Hom}_A\left(\Omega_{P/k} \otimes_k A, A\right) \longrightarrow \mathrm{Hom}_A\left(J/J^2, A\right) \longrightarrow T\mathrm{Def}_A \longrightarrow 0$$

arising from the conormal exact sequence,

$$J/J^2 \longrightarrow \Omega_{P/k} \otimes_P A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

In our case, let $P = k[x,y]$ and $J = (xy)$. Then we have,

$$A\partial_x \oplus A\partial_y \longrightarrow A \longrightarrow T\mathrm{Def}_A \longrightarrow 0$$

and therefore $T\mathrm{Def}_A = A/(x,y) = k$. Thus Def_X satisfies (H3) so it should have a hull. Indeed,

$$(k[\![t]\!], \mathrm{Spf}(k[\![t]\!][x,y]/(xy - t)))$$

is a hull (note the formal object is effective). Let's first understand why this hull is not a pro-representing object. For any map, $\varphi : k[\![t]\!] \to A$ the induced object,

$$\varphi_*(\mathrm{Spf}(k[\![t]\!][x,y]/(xy - t))) = \mathrm{Spec}\,(A[x,y]/(xy - \varphi(t)))$$

is unchanged (in isomorphism class) if we replace φ buy $\varphi' = u\varphi$ for any unit $u \in A$ since then we can scale x or y to remove u. However, recall that a deformation X' is equipped with a distinguished isomorphism $\varphi : X' \otimes_A k \xrightarrow{\sim} X$ with which isomorphisms of deformations must be compatible. Therefore, $\varphi' = u\varphi$ and φ define the same deformation if $u \in A^\times$ is a unit and $u \equiv 1 \mod \mathfrak{m}_A$. Therefore, the map, $h_R \to \mathrm{Def}_X$ is not injective for general A but is injective for $A = k[\epsilon]$ (since $(1 + a\epsilon) \cdot \epsilon = \epsilon$ so multiplication by such a does nothing) as must be true for a hull.

However Def_X is not pro-representable since it does not satisfy (H4). Indeed, consider $A = k[\epsilon]/(\epsilon^3)$ and consider,

$$\mathrm{Def}_X(A \times_k A) \to \mathrm{Def}_X(A) \times \mathrm{Def}_X(A)$$

I claim this is not injective. Indeed, $t = \epsilon_1 + \epsilon_2$ and $t = \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$ map to the same pair of deformations but I claim they are not related by such a unit. Write,

$$u = 1 + a\epsilon_1 + b\epsilon_2 + O(\epsilon^2)$$

then,

$$u(\epsilon_1 + \epsilon_2) = \epsilon_1 + \epsilon_2 + a\epsilon_1^2 + (a+b)\epsilon_1\epsilon_2 + b\epsilon_2^2 + O(\epsilon^3)$$

and we cannot have $a = b = 0$ but $a + b = 1$.

Remark. The above illustrates why it is necessary to define deformations of a scheme as equipped with a distinguished isomorphism $\varphi : X' \otimes_A k \xrightarrow{\sim} X$ otherwise Def_X will not be a deformation functor. Indeed, let Def'_X be the pre-deformation functor,

$$\mathrm{Def}'_X : A \mapsto \{X' \mid X' \text{ flat } A\text{-schemes such that } X' \otimes_A k' \cong X\}/\cong$$

but forgetting the isomorphism. Then for $X = \mathrm{Spec}\,(k[x,y]/(xy))$

$$\mathrm{Def}'_X(k[\epsilon_1, \epsilon_2]) \to \mathrm{Def}'_X(k[\epsilon]) \times \mathrm{Def}'_X(k[\epsilon])$$

is not injective. Indeed,

$$\mathrm{Spec}\,(k[\epsilon_1, \epsilon_2][x,y]/(xy + \epsilon_1 + \epsilon_2) \quad \text{and} \quad \mathrm{Spec}\,(k[\epsilon_1, \epsilon_2][x,y]/(xy + \epsilon_1 + 2\epsilon_2))$$

have the same image but are not isomorphic.

3.2 Deformation Categories

Definition 3.2.1. A *predeformation category* is a category cofibered in groupoids $\mathcal{F} \to \mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to the trivial category.

Remark. Let $\mathcal{F}$ be a predeformation category and $x_0 \in \mathcal{F}(k)$. Then for any $x \in \mathcal{F}$ over A let $q : A \to k$ then there is a pushforward $x \to q_* x$ and $q_* x \in \mathcal{F}(k)$ so there is a unique isomorphism $q_* x \xrightarrow{\sim} x_0$ and hence there is a canonical morphism $x \to x_0$ in $\mathcal{F}$.

Remark. If $F : \mathcal{C}_\Lambda \to \mathrm{Set}$ is a predeformation functor then the associated cofibered set $\mathcal{F}_F \to \mathcal{C}_\Lambda$ is a predeformation category. Likewise, if $\mathcal{F} \to \mathcal{C}_\Lambda$ is a predeformation category then the functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \mathrm{Set}$ is a predeformation functor.

Definition 3.2.2. Let $\mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The *category of formal objects of* $\widehat{\mathcal{F}}$ is the category of,

(a) formal objects (R, ξ_n, f_n) consists of an object $R \in \widehat{\mathcal{C}}_\Lambda$, and objects $\xi_n \in \mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ over the projection $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$

(b) morphisms $a : (R, \xi_n, f_n) \to (S, \eta_n, g_n)$ consists of a map $a_0 : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \to \eta_n$ of morphisms in $\mathcal{F}$ lying over $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ such that the diagrams,

$$\begin{tikzcd} \xi_{n+1} \arrow[r, "f_n"] \arrow[d, "a_{n+1}"] & \xi_n \arrow[d, "a_n"] \\ \eta_{n+1} \arrow[r, "g_n"] & \eta_n \end{tikzcd}$$

commute for each $n \in \mathbb{N}$.

Proposition 3.2.3 (06H4)**.** The formal objects forms a category cofibered in groupoids $\hat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$.

Definition 3.2.4. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We say that $\mathcal{F}$ satisfies the *Rim-Schlessinger (RS) condition* if for all $A_1 \to A$ and $A_2 \to A$ in $\mathcal{C}_\Lambda$ with $A_2 \twoheadrightarrow A$ surjective,

$$\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence. A *deformation category* is a predeformation category $\mathcal{F}$ satisfying (RS).

Lemma 3.2.5 (06J5)**.** Condition (RS) is equivalent to: for every diagram in $\mathscr{F}$,

$$\begin{tikzcd} & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, there exists a fiber product $x_1 \times_x x_2$ in $\mathscr{F}$ such that the diagram,

$$\begin{tikzcd} x_1 \times_x x_2 \arrow[r] \arrow[d] & x_2 \arrow[d] \\ x_1 \arrow[r] & x \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A \end{tikzcd}$$

Lemma 3.2.6 (07WQ)**.** If $\mathcal{X} \to S$ is an algebraic stack then for any $\mathrm{Spec}\,(k) \to S$ and $x_0 \in \mathcal{X}(k)$ the deformation category $\mathscr{F}_{\mathcal{X},k,x_0}$ satisfies (RS).

Remark. By Schlessinger's theorem, this is telling us that a deformation functor $F = D_{X,x_0}$ represented by some pointed finite-type quasi-separated[2] algebraic space $x_0 \in X$ over a noetherian scheme S is pro-representable. So even though X does not have a canonical local ring it does have a formal local ring $\widehat{\mathcal{O}_{X,x_0}}$. We can calculate it from the formal local ring of any étale cover $U \to X$. This is well-defined because for two étale covers $U_1 \to X$ and $U_2 \to X$ we have $U_1 \times_X U_2$ is an étale cover of both and these maps identify the formal local rings. There is a subtly here about the residue field of the preimage of x_0 in these étale covers meaning that the complete local rings will not be isomorphic until after a field extension. The technical assumptions ensure that X is decent and then the discussion of Tag 0EMV applies.

3.3 Versality

Remark. A versal object is a universal object without the "uni" i.e. without the uniqueness.

Definition 3.3.1. A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of categories cofibered in groupoids over $\mathcal{C}_\Lambda$ is *smooth* if for every extension $B \twoheadrightarrow A$ in $\mathcal{C}_\Lambda$ the map,

$$\mathscr{F}(B) \to \mathscr{F}(A) \times_{\mathscr{G}(A)} \mathscr{F}(B)$$

is essentially surjective.

Remark. This is basically the formal lifting criterion for formal smoothness. Indeed, if these deformation categories are induced by the representable functors for a morphism of schemes $f : X \to Y$ then we get that,

$$X(B) \to X(A) \times_{Y(A)} Y(B)$$

is surjective which is equivalent to there existing a dashed arrow in each lifting diagram,

$$\begin{tikzcd} \mathrm{Spec}\,(A) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathrm{Spec}\,(B) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

[2] I don't know if these are the *right* conditions but they make the discussion work.

Lemma 3.3.2. Smoothness of $\varphi : \mathcal{F} \to \mathcal{G}$ is equivalent to the following explicit condition. For every surjection $B \twoheadrightarrow A$ in $\mathcal{C}_\Lambda$ and $y \in \mathcal{G}(B)$ and $x \in \mathcal{F}(A)$ equipped with a map $y \to \varphi(x)$ over $B \twoheadrightarrow A$ there is $x' \in \mathcal{F}(B)$ and a morphism $x' \to x$ over $B \twoheadrightarrow A$ and a morphism $\varphi(x') \to y$ over $\mathrm{id} : B \to V$ such that,

$$\begin{tikzcd} \varphi(x') \arrow[r] \arrow[dr] & y \arrow[d] \\ & \varphi(x) \end{tikzcd}$$

Definition 3.3.3. Let $R \in \widehat{\mathcal{C}}_\Lambda$. We say $\xi \in \widehat{\mathcal{F}}(R)$ is *versal* if the morphism $\xi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ defined by ξ is smooth.

Remark. The morphism is defined as follows. For any $A \in \mathcal{C}_\Lambda$ and map $\varphi : R \to A$ it will factor as $\varphi_n : R/\mathfrak{m}^n \to A$ we send $(A, \varphi) \mapsto (\varphi_n)_* \xi_n$. The compatibility isomorphisms of the formal object ξ make this well-defined.

Remark. Let ξ be a formal object of $\mathcal{F}$. Versality of ξ is equivalent to: the existence of a dashed arrow for any diagram,

$$\begin{tikzcd} & y \arrow[d] \\ \xi \arrow[r] \arrow[ur, dashed] & x \end{tikzcd}$$

in $\widehat{\mathcal{F}}$ such that $y \to x$ lies over a surjective map $B \twoheadrightarrow A$ of Artinian rings.

Theorem 3.3.4 (Rim-Schlessinger)**.** A deformation category $\mathcal{F}$ such that $T\mathcal{F} = \overline{\mathcal{F}}(k[\epsilon])$ is finite dimensional admits a versal formal object.

Example 3.3.5. Let X be a k-scheme. The cofibered category of deformations $\mathcal{D}ef_X \to \mathcal{C}_k$ is a deformation category. If X is finite type and either proper or affine then $T\mathcal{D}ef_X = T\mathrm{Def}_X$ is finite dimensional so X admits a versal formal deformation $\mathcal{X} \to \mathrm{Spf}(R)$.

Definition 3.3.6. Given a category fibered in groupoids,

$$p : \mathcal{X} \to (\mathbf{Sch}_S)_{\mathrm{fppf}}$$

and a finite type point $\mathrm{Spec}\,(k) \to S$ and $x_0 \in \mathcal{X}(k)$. First factor $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(\Lambda) \hookrightarrow S$ through some affine open such that $\Lambda \to k$ is finite. The category $\mathcal{C}_\Lambda$, up to canonical equivalence, does not depend of the choice of affine open $\mathrm{Spec}\,(\Lambda) \subset S$. Note that $\mathcal{C}_\Lambda$ is equivalent to the opposite category of factorizations,

$$\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(A) \to S$$

such that A is Artin local and $A \to k$ identifies k with the residue field. Now let $\mathcal{F}_{\mathcal{X},k,x_0}$ be the category of,

(a) morphisms $x_0 \to x$ of $\mathcal{X}$ over $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(A)$ as S-map in $\mathcal{C}_\Lambda$,

(b) morphisms $(x_0 \to x) \to (x_0 \to x')$ are diagrams,

$$\begin{tikzcd} x & & x' \arrow[ll] \\ & x_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in $\mathcal{X}$ (notice the reversal of arrows).

Then $p : \mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{C}_\Lambda$ is a predeformation category. We say that a formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ is *versal* if ξ is versal as a formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$ with $k = R/\mathfrak{m}_R$ and $x_0 = \xi_1$. We say that $x \in \mathcal{X}(U)$ is versal at a finite type point $u_0 \in U$ if $\hat{x} \in \widehat{\mathcal{F}}_{\mathcal{X},\kappa(u_0),x_0}$ is versal where $x_0 : \mathrm{Spec}\,(k) \to U \to \mathcal{X}$ is the image.

Definition 3.3.7. Let S be a locally noetherian scheme and $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$ a category fibered in groupoids. We say $\mathcal{X}$ satisfies *openness of versality* if given a scheme U locally of finite type over S, an object $x \in \mathcal{X}(U)$, and a finite type point $u_0 \in U$ such that x is versal at u_0 then there is exists an open neighborhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

3.4 Effectivity

Definition 3.4.1. A formal object $\xi = (R, \xi_n, f_n) \in \widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$ is *effective* if it arises from $\tilde{\xi} \in \mathcal{X}(R)$.

Lemma 3.4.2 (07X3)**.** If $\mathcal{X} \to S$ is an algebraic stack over a locally noetherian scheme S then every formal object is effective.

Proof. First, if X is a scheme then for all local rings R factoring $\mathrm{Spec}\,(k) \to X$ the map corresponds to $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}) \to X$ so if R is complete,

$$X(R) = \mathrm{Hom}_{\mathrm{loc}}\,(\mathcal{O}_{X,x}, R) = \varprojlim_n \mathrm{Hom}_{\mathrm{loc}}\,(\mathcal{O}_{X,x}, R/\mathfrak{m}_R^n) = \varprojlim_n X(R/\mathfrak{m}_R^n)$$

The general case follows from an intricate descent argument. $\square$

4 Artin's Axioms

Theorem 4.0.1 (Artin Approximation)**.** Let S be a locally noetherian scheme and a category fibered in groupoids $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$. Let R be a Noetherian complete local ring with residue field k with $\mathrm{Spec}\,(R) \to S$ finite type and $x \in \mathcal{X}(R)$. Let $s \in S$ be the image of $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(R) \to S$. Assume that,

(a) $\mathcal{O}_{S,s}$ is a G-ring

(b) p is limit-preserving on objects.

Then for every $N \geq 1$ there exist,

(a) a finite type S-algebra A

(b) a maximal ideal $\mathfrak{m}_A \subset A$

(c) an object $x_A \in \mathcal{X}(A)$

(d) an S-isomorphism $R/\mathfrak{m}_R^N \xrightarrow{\sim} A/\mathfrak{m}_A^N$

(e) an isomorphism $x|_{R/\mathfrak{m}_R^N} \xrightarrow{\sim} x_A|_{A/\mathfrak{m}_A^N}$ over the previous map

(f) an isomorphism $\mathbf{gr}_{\mathfrak{m}_R}(R) \xrightarrow{\sim} \mathbf{gr}_{\mathfrak{m}_A}(A)$ of graded k-algebras.

Lemma 4.0.2. Let S be a locally noetherian scheme and $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$ a category fibered in groupoids. Let ξ be a formal object of $\mathcal{X}$ with $x_0 = \xi_1$ lying over $\text{Spec}(k) \to S$ with image $s \in S$ such that,

(a) ξ is versal

(b) ξ is effective

(c) $\mathcal{O}_{S,s}$ is a G-ring

(d) $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$ is limit-preserving

then there exists a finite type morphism $U \to S$, a finite type point $u_0 \in U$ with residue field k and $x \in \mathcal{X}(U)$ such that $x : U \to \mathcal{X}$ is versal at u_0 and $x|_{\text{Spec}(\mathcal{O}_{U,u_0})}$ induces ξ.

Proof. Choose an object $x_R \in \mathcal{X}(R)$ whose completion is ξ. Apply Artin approximation with $N = 2$ to obtain $A, \mathfrak{m}_A, x_A \in \mathcal{X}(A)$ approximating ξ. Let η be the formal object completing $x_A|_{\text{Spec}(\hat{A})}$ (the completion of A at $\mathfrak{m}_A$). Then a lift for the diagram in $\widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$,

$$\begin{tikzcd} & \eta \arrow[d] \\ \xi \arrow[r] \arrow[ur, dashed] & \xi_2 = \eta_2 \end{tikzcd} \quad \text{lying over} \quad \begin{tikzcd} & \hat{A} \arrow[d] \\ R \arrow[r] \arrow[ur, dashed] & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{tikzcd}$$

exists because ξ is versal. Since the map $R \to \hat{A}$ induces an isomorphism on tangent spaces and by construction $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ we conclude that $R \to \hat{A}$ is an isomorphism. Hence $\eta \cong \xi$ is versal so the map $x_A : \text{Spec}(A) \to \mathcal{X}$ is versal at $\widehat{x_A|_{\text{Spec}(\hat{A})}} = \eta$. $\square$

Theorem 4.0.3. Let S be a locally Noetherian base scheme and consider a category fibered in groupoids $p : \mathcal{X} \to (\mathbf{Sch}_S)_{\text{fppf}}$. For each finite type morphism $\text{Spec}(k) \to S$ with k a field and $x_0 \in \mathcal{X}(\text{Spec}(k))$ assume that,

(a) $\mathcal{X}$ is a stack for the étale topology

(b) $\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces

(c) $\mathcal{X}$ is limit preserving (preserves filtered colimits)

(d) $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS)

(e) $T\mathcal{F}_{\mathcal{X},k,x_0}$ is finite dimensional for all k and all $x_0 \in \mathcal{F}(k)$

(f) every formal object of $\mathcal{X}$ is effective

(g) $\mathcal{X}$ satisfies openness of versality

(h) $\mathcal{O}_{S,s}$ is a G-ring for all finite type points $s \in S$

(i) a set theoretic condition

then $\mathcal{X}$ is an algebraic stack.

Proof. It suffices to show that for each finite type $\operatorname{Spec}(k) \to S$ and $x_0 \in \mathcal{X}(k)$ there is a finite type morphism $U \to S$ and a smooth map $U \to \mathcal{X}$ such that there is a finite type point $u_0 : \operatorname{Spec}(k) \to U$ such that $x|_{u_0} \cong x_0$.

By Rim-Schlessinger $\mathcal{F}_{\mathcal{X},k,x_0}$ admits a versal formal object ξ which is then effective. Artin approximation allows us to approximate an effective formal object by a finite type object $U \to \mathcal{X}$ which is versal at $u_0 \in U$. By openness of versality, we can shrink U such that $U \to \mathcal{X}$ is versal at every finite type point.

Finally, prove that a representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of limit preserving categories fibered in groupoids which is smooth on deformation categories is smooth (Tag 07XX). Indeed, for $T \to \mathcal{Y}$ the condition says that $f : \mathcal{X}_T \to T$ is a formally smooth map of algebraic spaces[3] and the limit-preserving condition gives finitely presented. $\square$

Remark. Usually most difficult to prove openness of versality. There a number of deformation-theoretic techniques for proving this but require effectivity of formal objects over more general formal schemes. There are also tangent-obstruction theory methods for proving openness of versality.

[3]There is a subtly here with changing fields that requires the full strength of (RS) where as proving that a versal object exists only requires (S1) and (S2) and finite-dimensionality of tangent spaces