1 Coherent Sheaves

2 Analytic Spaces

(DO DEFINITION!!!)

Definition 2.1. An *affine analytic space* X is

Definition 2.2. An *analytic space* X is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to affine analytic spaces.

Theorem 2.3. Let X be an analytic space. Then $\mathcal{O}_X$ is coherent.

3 Analytification

Definition 3.1. Analytification is the unique functor $(-)^{\text{an}} : \mathbf{Var}_k \to \mathbf{AnSpa}_k$ along with a map $a : X^{\text{an}} \to X$ of locally ringed spaces satisfying: (DO THIS!!!!)

Lemma 3.2. The canonical map $a : X^{\text{an}} \to X$ of locally ringed spaces is:

(a). flat

(b). open

(c).

Definition 3.3. Let $\mathscr{F}$ be a $\mathcal{O}_X$-module on a k-variety X. Then the analytification of $\mathscr{F}$ is the $\mathcal{O}_X^{\text{an}}$-module $\mathscr{F}^{\text{an}} = a^* \mathscr{F}$ on X^{an} where a is the canonical map $a : X^{\text{an}} \to X$.

Proposition 3.4. If $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module. Then $\mathscr{F}^{\text{an}}$ is a coherent $\mathcal{O}_X^{\text{an}}$-module.

4 The Main Theorems

4.1 The Equivalence of Cohomology

Theorem 4.1. Let X be a projective variety over k and $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then the natural map,

$$H^q(X, \mathscr{F}) \xrightarrow{\sim} H^q(X^{\text{an}}, \mathscr{F}^{\text{an}})$$

is an isomorphism.

4.2 The Equivalence of Morphisms

Lemma 4.2. Let M, N be A-modules and B an A-algebra. Then the natural map,

$$\text{Hom}_A (M, N) \otimes_A B \to \text{Hom}_A (M, N \otimes_A B) = \text{Hom}_B (M \otimes_A B, N \otimes_A B)$$

is an isomorphism when M is finite projective or B is A-flat.

Proof. (DO THIS!!!!) $\square$

Lemma 4.3. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules. Then the natural map,

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

is an isomorphism when $\mathscr{F}$ is finitely-presented.

Proof. Gortz, T. Wedhorn: Algebraic Geometry 1, Chapter 7, Prop: 7.27
EGA I, chapter 0, 6.7.6 □

Lemma 4.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_Y$-modules. Then the natural map,

$$f^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G}) \to \mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{F}, f^*\mathscr{G})$$

is an isomorphism when $\mathscr{F}$ is finite locally free or $\mathscr{F}$ is finitely presented and $f : X \to Y$ is flat.

Proof. □

Theorem 4.5 (GAGA II)**.** Let X be a (projective?) variety over k (hypotheses?). Then, for coherent $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$ the natural map,

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})^{\mathrm{an}} \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X^{\mathrm{an}}}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}})$$

is an isomorphism.

Proof. This is simply an application of the previous lemma to the map $f : X^{\mathrm{an}} \to X$ of locally ringed spaces which is flat. Since $\mathscr{F}$ and $\mathscr{G}$ are coherent they are finitely presented. □

Corollary 4.6. Let X be a projective variety over k and let $\mathscr{F}, \mathscr{G}$ be coherent $\mathcal{O}_X$-modules. Then,

$$\mathrm{Hom}_{\mathcal{O}_X^{\mathrm{an}}}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$$

Proof. Use GAGA II to give $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})^{\mathrm{an}} = \mathscr{H}om_{\mathcal{O}_X^{\mathrm{an}}}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}})$ and GAGA I to give,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) &= H^0(X, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})) = H^0(X^{\mathrm{an}}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})^{\mathrm{an}}) \\ &= H^0(X^{\mathrm{an}}, \mathscr{H}om_{\mathcal{O}_X^{\mathrm{an}}}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}})) = \mathrm{Hom}_{\mathcal{O}_X^{\mathrm{an}}}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}}) \end{aligned}$$

□

4.3 The Equivalence of ...

5 Other Fields