1 The Formal Immersion Step (the new hotness on tiktak)

Theorem 1.0.1. Let N be a prime, either 11 or ≥ 17 (ensuring that $X_0(N)$ has genus > 0) then there are no elliptic curves over $\mathbb{Q}$ with a torsion point of order N.

Kep points,

(a) if E has good reduction at 3 then $E[N](\mathbb{Q}) \hookrightarrow \overline{E}(\mathbb{F}_3)$ which has order at most 9 by Hasse so $N < 9$.

(b) if E has multiplicative reduction we can get crazy polygons so no control on N

(c) if E has additive reduction: what can the special fiber of the minimal regular proper model be? From Kodaia classification, there are a bounded number of components and hence a bound on $\#\overline{E}(\mathbb{F}_3) \leq 12$.

Assume from now on that $N = 11$ or $N > 17$.

Proposition 1.0.2. If (E, C) is a pair of an elliptic curve over $\mathbb{Q}$ and a cyclic subgroup scheme $C \subset E$ of order N. Then E has potentially good reduction away from $2N$.

Remark. This implies you can't have multiplicative reduction because potentially good reduction means the semistable reduction is good but multiplicative reduction is also semistable.

Remark. Recall that,

$$\text{good reduction} \iff T_\ell E \text{ is unramified}$$

$$\text{mult. reduction} \iff I \to \mathrm{GL}(V_\ell E) \text{ is (nontrivial) unipotent}$$

Proposition 1.0.3. Let $\mathcal{A}$ be the Neron model over $\mathbb{Z}[1/2N]$ of the Eisenstein quotient A of $J = \mathrm{Jac}\,(X_0(N))$. Define,

$$X_0(N)_{\mathbb{Q}} \longrightarrow J \longrightarrow A$$

$f : X_0(N) \to \mathcal{A}$ over $\mathbb{Z}[1/2N]$ sends $\infty \mapsto 0$. Then if $p \not| \, 2N$ then $\infty \in X_0(N)(\mathbb{Z}_{(p)})$ is the only $\mathbb{Z}_{(p)}$-point of $X_0(N)$ mapping to $0 \in \mathcal{A}(\mathbb{Z}_{(p)})$ which reduces to $\infty \in X_0(N)(\mathbb{F}_p)$.

Definition 1.0.4. Let $f : Y \to Z$ is lft and Y, Z are locally noetherian. If $y \, inU$ say f is a *formal immersion at y* if $\mathcal{O}^\wedge_{Z,f(y)} \twoheadrightarrow \mathcal{O}^\wedge_{Y,y}$ is surjective.

Definition 1.0.5. Y, Z are ft + sep over a locally noetherian base S. If f is an S-morphism and $y \in Y(S)$ is a section then f is a *formal immersion along y* if,

(a) f is a formal immersion along all points of y

(b) f_s is a formal immersion at y_s for all $s \in S$.

Remark. This is supposed to be equivalent to $\widehat{Y}_y \hookrightarrow \widehat{Z}_{f(y)}$.

Lemma 1.0.6. Let A, B be complete noeth. local rings and $f : A \to B$ is a local map such that $f : A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ and $f : \mathfrak{m}_A/\mathfrak{m}_A^2 \twoheadrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.

Proof. Approximate. $\square$

Proposition 1.0.7. Let Y be separated and $f : Y \to Z$ be a formal immersion at $y \in Y$. Let T be an integral noetherian scheme with $p_1, p_@ \in Y(T)$ are s.t. $y = p_1(t) = p_2(t)$ at some $t \in T$ and $f \circ p_1 = f \circ p_2$ then $p_1 = p_2$.

Lemma 1.0.8. Let A, B be complete noetherian local rings flat over a dvr (R, π) with a map $A \to B$ such that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A$ is an isomorphism. Then $A \to B$ is surjective iff $A/\pi \to B/\pi$ is surjective.

Proof. This follows from the fact that $\mathfrak{m}_A/(\mathfrak{m}_A^2 + \pi A) \twoheadrightarrow \mathfrak{m}_B/(\mathfrak{m}_B^2 + \pi B)$ being surjective implies that it was surjective before moding by π. □

Corollary 1.0.9. We can check formal immersions at the special fiber of a DVR.

Proof of Proposition. $A = \{x \in T \mid p_1(x) = p_2(x)\}$ then Y is separated implies $A \subset T$ closed and T is integral so suffices to show $\mathrm{Spec}\,(\mathcal{O}_{T,t}) \to T$ factors through $A \hookrightarrow T$. So assume T is local with closed point t. Can assume Y is local with closed point y.

$$\begin{tikzcd}
\mathcal{O}_{T,t} \arrow[r, hook] & \widehat{\mathcal{O}}_{T,t} & \\
\mathcal{O}_{Y,y} \arrow[u, shift left] \arrow[u, shift right] \arrow[r] & \widehat{\mathcal{O}}_{Y,y} \arrow[u, shift left] \arrow[u, shift right] & \widehat{\mathcal{O}}_{Z,f(y)} \arrow[l]
\end{tikzcd}$$

thus the maps must agree on the local rings since they agree after composing with the surjection. □

Goal show that if $T_{\mathbb{Q}} \twoheadrightarrow A$ is any surjection of abelian varities with connected kernel (what we call an optimal quotient) then $X_0(N) \to J \to \mathcal{A}$ over $\mathbb{Z}[1/2N]$ is a formal immerison.

Setup N is prime > 2 and $S = \mathrm{Spec}\,(\mathbb{Z}[1/2N])$ and $X = X_0(N)$ then $J = J_0(N)$ and $\mathbb{T} \hookrightarrow \mathrm{End}\,(J)$ the Hecke algebra.

Remark. all optimal quotients of J are of the form J/IJ where $I \subset \mathbb{T}$ is a *saturated* ideal ($\mathbb{T}/I$ is torsion-free). Then $J_{\mathbb{Q}} = J_0(N)^{\mathrm{new}}_{\mathbb{Q}}$ so everything in Daniel's talk applies. In particular,

$$J_{\mathbb{Q}} \sim \prod_{f \in C} A_f$$

with C Galois orbits of cusp forms. Also,

$$\mathrm{End}_{\mathbb{Q}}\,(A_f) = K_f = \mathrm{im}\,\mathbb{T}$$

with $[K_f : \mathbb{Q}] = \dim A_f$. Then any optimal quotient of $J_{\mathbb{Q}}$ is $\prod_{g \in C'} A_g$ with $C' \subset C$.

Theorem 1.0.10. The tangent space $T_0(F)$ is a free $\mathcal{T}_{\mathbb{Z}[1/2N]}$-module of rank 1 generated by $\frac{\mathrm{d}}{\mathrm{d}q}|_0$.

Remark. This is saying,

$$S_2(N)_R \cong H^0(J_R, \Omega^1_{J_R/R}) = T_0^*(J_R)$$

for any ring R. This is because level N cusp 2-forms are exactly given by forms on $X_0(N)$ and these are the same as forms on its Jacobian.

Corollary 1.0.11. If A is an optimal quotient of J then $X \to \mathcal{A}$ sending $\infty \mapsto 0$ is a formal immersion over S.

Proof. It suffices to show that $T_\infty X \hookrightarrow T_0\mathcal{A}$ over each prime. Then in the a sequence,

$$0 \longrightarrow B \longrightarrow J \longrightarrow A \longrightarrow 0$$

since J and A have good reduction so does B by Neron-Ogg-Shafarevich. Then Raynaud's theorem gives an exact sequence,

$$0 \longrightarrow T_0(\mathcal{B}) \longrightarrow T_0(J) \longrightarrow T_0(\mathcal{A}) \longrightarrow 0$$

(HMMM) $\square$

Reduction, $M' = T_0(T)/(\mathbb{T}_{\mathbb{Z}[1/2N]}\frac{\mathrm{d}}{\mathrm{d}q})$. But $T_0(J)$ is finite over $\mathbb{Z}[1/2N]$ hence also $\mathbb{T}_{\mathbb{Z}[1/2N]}$. Suffices to show that $M'/\mathfrak{m}M' = 0$ for all $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$ i.e. $\frac{\mathrm{d}}{\mathrm{d}q}$ generated $T_0(T)/\mathfrak{m}T_0(J)$.

Lemma 1.0.12. $S_2(N)^{\text{new}}_{\mathbb{Q}}$ is a free $\mathbb{T}_{\mathbb{Q}}$-module of rank 1 generated by $\frac{\mathrm{d}}{\mathrm{d}q}|_0$.

Lemma 1.0.13. For $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$ and $T_0(J)/\mathfrak{m}T_0(J) = 0$.

Proof. Finiteness of $T_0(J)$ and NAK and $T_0(J) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$. $\square$

Lemma 1.0.14. For $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$ then $\frac{\mathrm{d}}{\mathrm{d}q}$ has nonzero image in $T_0(J)/\mathfrak{m}T_0(J)$.

Proof. If $f \in S_2(N)_{\overline{\mathbb{F}}_\ell}$ has a q-expansion,

$$f = \sum_{n=1}^{\infty} a_n q^n$$

then $\frac{\mathrm{d}}{\mathrm{d}q}(f) = a_1$ and we win by showing that if f is an eigenform with $a_1 = 0$ then $f = 0$. This is because $\frac{\mathrm{d}}{\mathrm{d}q}(T_n f) = a_n$ so if $T_n f = \lambda f$ for $\lambda \neq 0$ then we also have all $a_n = 0$.

Let's do this in more detail. Let ℓ be the characteristic of $F = (\mathbb{T} \otimes \mathbb{Z}[1/2N])/\mathfrak{m}$ and $R = (\mathbb{T} \otimes \mathbb{Z}[1/2N]) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell$. And let $M = T_0(J) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell$. Then there is an exact sequence,

$$\begin{tikzcd}
\mathfrak{m} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell \arrow[r] & R \arrow[r] & F \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell \arrow[r] \arrow[d, equal] & 0 \\
& & \prod_{i \in I} \overline{\mathbb{F}}_\ell & \\
0 \arrow[r] & \mathfrak{m}R \arrow[r] & R \arrow[r] \arrow[u, two heads] & \overline{\mathbb{F}}_\ell \arrow[r] & 0 \\
& & & F \arrow[u, hook]
\end{tikzcd}$$

by tensoring the inclusion $F \hookrightarrow \overline{\mathbb{F}}_\ell$ we get $T_0(J)/\mathfrak{m}T_0(J) \hookrightarrow M/\mathfrak{m}M$. As R-modules,

$$(M/\mathfrak{m}M)^\vee \cong M^\vee[\mathfrak{m}] \cong H^0(X_{\overline{\mathbb{F}}_\ell}, \Omega^1_{X/\overline{\mathbb{F}}_\ell})[\mathfrak{m}]$$

$\square$

Theorem 1.0.15. if $f : X \to S$ is a smooth proper relative curve then $R^i f_* \Omega_{X/S}$ commutes with all base change.

Proof. If S is reduced this comes from Grauert. Otherwise use cohomology and base change. $\square$

In particular: if $f \in S_2(N)_{\overline{\mathbb{F}}_\ell}[\mathfrak{m}]$ is nonzero can lift to char 0 and then $\frac{\mathrm{d}}{\mathrm{d}q}(T_n f) = a_n(f)$ follows from analysis.

Lemma 1.0.16. For every $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$. Then $T_0(J)/\mathfrak{m}T_0(J)$ is free over $\mathbb{T}_{\mathbb{Z}[1/2N]}/\mathfrak{m}$ generated by $\frac{\mathrm{d}}{\mathrm{d}q}$.

Proof. $\dim_F T_0(J)/\mathfrak{m}T_0(J) = \dim_{\overline{\mathbb{F}}_\ell} M^\vee[\mathfrak{m}]$ then let a_n be the image of T_n in $R/\mathfrak{m} = \overline{\mathbb{F}}_\ell$ then if $f \in S_2(N)_{\overline{\mathbb{F}}_\ell}[\mathfrak{m}]$ and $T_n(f) = a_n(F)$ so f is a multiple of $q + a_2 q^2 + \cdots$. ☐