1 Connections on Principle Bundles

Proposition 1.0.1. Let $\pi : G \to S$ be a group scheme. Then,

$$\Omega_{G/S} = \pi^* e^* \Omega_{G/S}$$

so if we set,

$$\omega_{G/S} = e^* \Omega_{G/S}$$

then,

$$\Omega_{G/S} = \pi^* \omega_{G/S}$$

Furthermore, by the projection form,

$$\pi_* \Omega_{G/S} = \omega_{G/S} \otimes_{\mathcal{O}_S} \mathcal{O}_G$$

and thus if π is $\mathcal{O}$-connected (e.g. for G/S an abelian scheme) then,

$$\pi_* \Omega_{G/S} = \omega_{G/S}$$

Proof. Consider the Cartesian diagram,

$$\begin{tikzcd}
G \times_S G \arrow[drr, bend left, "m"] \arrow[dr, dashed] \arrow[ddr, bend right, "\pi_1"'] & & \\
& G \times_S G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & G \arrow[d] \\
& G \arrow[r] & S
\end{tikzcd}$$

because the dashed arrow is an isomorphism, the outside square is Cartesian so $m^* \Omega_{G/S} = \pi_1^* \Omega_{G/S}$. Then,

$$\pi^* e^* \Omega_{G/S} = (e \circ \pi, \mathrm{id})^* \pi_1^* \Omega_{G/S} = (e \circ \pi, \mathrm{id})^* m^* \Omega_{G/S} = \mathrm{id}^* \Omega_{G/S} = \Omega_{G/S}$$

$\square$

Remark. If $S = k$ then $\omega_{G/S} = \mathfrak{g}$ is the Lie algebra.

Proposition 1.0.2. Let $G \to S$ be a smooth group scheme and $\pi : P \to X$ be a G-bundle. Consider the sequence,

$$0 \longrightarrow \pi^* \Omega_X \longrightarrow \Omega_P \longrightarrow \Omega_{P/X} \longrightarrow 0$$

This is an exact sequence of descent data and therefore arises as π^* of an exact sequence,

$$0 \longrightarrow \Omega_X \longrightarrow Q \longrightarrow \mathrm{ad}(P) \longrightarrow 0$$

where $\mathrm{ad}(P)$ is the adjoint bundle,

$$\mathrm{ad}(P) = P \times_G \mathfrak{g}$$