Deligne's Theory of Absolute Hodge Cycles

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1 Sept. 26 CM Abelian Varities

Proposition 1.0.1. Let F be a number field and c a field automorphism of F with $c^2 = 1$. Then the following are equivalent,

(a) $\forall \tau : F \hookrightarrow \mathbb{C}$ then $\tau \circ c = \bar{\tau}$

(b) $\mathrm{tr}_{\mathscr{F}/\mathbb{Q}}\, ac(a) > 0$ for all $a \in F^\times$

(c) $F^+ = F^\tau$ is a totally real field and either $F = F^+$ or F is totally imaginary.

Then (F, c) is a CM pair and for each F there is at most one c making (F, c) a CM pair and if it exists then F is CM.

Example 1.0.2. $\mathbb{Q}, \mathbb{Q}(i), \mathbb{Q}(\sqrt{\sqrt{2} - 2})$ are all CM.

Let L be a field of characteristic 0.

Definition 1.0.3. An abelian variety A/L is a smooth, projective, geometrically connected, abelian group scheme A/L.

Remark. Given A/L we have the invariants,

(a) The Lie algebra $(A) = T_e A$ is an L-vectorspace and we get,
$$(A)^\vee = \Omega^1(A)$$

(b) The Tate module,
$$TA = \varprojlim_N A[N](\tau) \cong \hat{\mathbb{Z}}^{2 \dim A}$$
which is a $\mathrm{Gal}\left(\bar{L}/L\right)$-module. Then write,
$$VA = TA \otimes_{\hat{\mathbb{Z}}} \mathbb{A}^\infty$$

(c) $\mathrm{Hom}\,(A, B)$ is a fg free abelian group with a Galois action

(d) $\mathrm{Hom}_\circ\,(A, B) = \mathrm{Hom}\,(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then an element $f \in \mathrm{Hom}_\circ\,(A, B)$ is called *quasi-isogeny* if there exists $f^{-1} \in \mathrm{Hom}_\circ\,(A, B)$ a two-sided rational inverse

(e) morphisms which are quasi-isogenies are called isogenies and are exactly the finite flat group maps. Then $\deg f = TB/f_* TA$.

(f) End $(\circ)\, A$ is a semi-simple $\mathbb{Q}$-algebra and hence a sum of matrix algebras over division algebras D_i with $Z(D_i) = F_i$ fields over L so we write,

$$\mathrm{End}\,(\circ)\, A \cong \bigoplus_i M_{n_i}(D_i)$$

then we get the bound,

$$\sum n_i \operatorname{rank}_{F_i}(D_i)[F_i : \mathbb{Q}] \le 2 \dim A$$

Note: the Hom and End are taken over $\overline{L}$ and given a Galois action.

Definition 1.0.4. We call A *potentially CM* if the previous inequality is an equality.

Proposition 1.0.5. The following are equivalent,

(a) A is potentially CM

(b) there exists $F \subset \mathrm{End}\,(\circ)\, A$ which is a product of fields with $[F : \mathbb{Q}] = 2 \dim A$

(c) there exists $F \subset \mathrm{End}\,(\circ)\, A$ a product of imaginary CM fields with $[F : \mathbb{Q}] = 2 \dim A$

(d) the F_i in the previous remark are CM fields.

Definition 1.0.6. If F is an imaginary CM field then a F-CM abelian variety over L is a pair (A, ι) where A/L is an abelian variety of dimension $[F : \mathbb{Q}]/L$ and $\iota : F \hookrightarrow \mathrm{End}\,(\circ)\, A^{\mathrm{Gal}\left(\overline{L}/L\right)}$.

Example 1.0.7. Let $y^2 z = x^3 - xz^2$ over $\mathbb{Q}(i)$ is a $\mathbb{Q}(i)$-CM abelian variety where,

$$[i] \cdot [x : y : z] = (-x : iy : z]$$

In this case it is a coincidence that the extension of $L = \mathbb{Q}$ over which the CM is realized equals the CM field.

We have $A^\vee$ is the moduli space of homologically trivial line bundles on A with universal (Poincare bundle) on $A \times A^\vee$ and $A^{\vee\vee} = A$. For $f : A \to B$ then get $f^\vee : B^\vee \to A^\vee$. Then there is a pairing,

$$TA \times TA^\vee \to T\mathbb{G}_m = \hat{\mathbb{Z}}(1)$$

Definition 1.0.8. A *polarization* of A is a map $\lambda : A \to A^\vee$ such that $\lambda^\vee = \lambda$ and $(1 \times \lambda)^*$ is ample.

Definition 1.0.9. A *quasi-polarization* is a $\lambda \in \mathrm{Hom}_\circ\,(A, A^\vee)$ such that $n\lambda$ is a polarization for some positive integer n. Then $\langle -, \lambda - \rangle_A : VA \times VA \to \mathbb{A}^\infty(1)$ is alternation. The Rosati involution is defined by,

$$f \mapsto f^{*\lambda} = \lambda^{-1} \circ f^\vee \circ \lambda \in \mathrm{End}\,(\circ)\, A$$

If A is a polarized CM and λ is a quasi-polarizarion there exists $F \subset \mathrm{End}\,(\circ)\, A$ a product of CM fields with $[F : \mathbb{Q}] = 2 \dim A$ and F is preserved by $*_\lambda$ menaing $*_{\lambda|_F = c}$.

Definition 1.0.10. By a *polarized F-CM abelian variety* we mean (A, ι, λ) where (A, ι) is a F-CM abelian variety and $\lambda : A \to A^\vee$is a quasi-polarization such that $*_\lambda|_F = c$.

Definition 1.0.11. Let,

$$\Phi(A, \iota) = \{(\sigma_i : F_i \hookrightarrow \mathbb{C})\}$$

1.1 $L = \mathbb{C}$

Over $L = \mathbb{C}$ we have,

$$A(\mathbb{C}) \cong (A)/H_1(A, \mathbb{Z})$$

and $VA = H_1(A(\mathbb{C}), \mathbb{Z}) \otimes \text{Å}^{\infty}$.

Proposition 1.1.1. Polarizations on A correspond to Riemann forms:

$$E : H_1(A(\mathbb{C}), \mathbb{Z})^2 \to \mathbb{Z}$$

alternating and non-degenerate where,

$$E_{\mathbb{R}}(ix, iy) = E_{\mathbb{R}}(x, y)$$

and $E_{\mathbb{R}}(ix, x) > 0$ for all $x \neq 0$.

Proposition 1.1.2. For W a fd $\mathbb{C}$-vectorspace and $\mathbb{Z} \subset W$ a $\mathbb{Z}$-lattice then W/Λ is the complex manifold underlying an abelian variety if and only if Λ has a Riemann form.

Lemma 1.1.3. If $(A, \iota)/L$ is an F-CM abelian variety then,

$$\Phi(A, \iota) \sqcup \Phi(A, \iota) \circ c = \text{Hom}\left(F, \overline{L}\right)$$

Proof. Reduce to $L = \mathbb{C}$ then we check this is $(A) \otimes_{\mathbb{R}} \mathbb{C} = H_1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. Then there is an action of F on the RHS but these have the same $\mathbb{Q}$-dimension so the RHS equals F since the only representation of a field which has the same dimension over the perfect field is just the field F. $\square$

Construction of CM-abelian varities:

Definition 1.1.4. Polarized F-CM data: $(F, \mathfrak{a}, \Phi, \xi)$ where

(a) F is an imaginary CM-field

(b) $\mathfrak{a} \subset F$ is a $\mathbb{Z}$-lattice

(c) $\xi \in F^{\times}$ with $\xi^2 \in F^{+}_{\ll 0}$

(d) $\Phi = \{\varphi : F \hookrightarrow \mathbb{C} \mid \text{im}\, \varphi(\xi) > 0\}$ which is determined by the previous data.

Given this data, we can associate a polarized F-CM abelian variety,

$$A(\sigma) = \mathbb{C}^{\Phi}/\Phi(\mathfrak{a})$$

where,

$$\Phi(\mathfrak{a}) = \{(\varphi(a))_{\varphi \in \Phi} \mid a \in \mathfrak{a}\}$$

The action of F on $A(\sigma)$ is given by,

$$\iota(a) \cdot (x_{\varphi})_{\varphi \in \Phi} = (\varphi(a) x_{\varphi})_{\varphi \in \Phi}$$

which makes sense for all $a \in \mathfrak{a}$ preserving the lattice under this action which is an order. Therefore, all of $\mathfrak{a}$ acts by quasi-isogenies. Furthermore, we get an F-linear isomorphism,

$$\eta^{\text{can}} : \text{Å}^{\infty}_{F} \xrightarrow{\sim} VA$$

defined by,

$$\phi : F/\mathfrak{a} \to A(\sigma)$$

Furthermore,

(a) any polarized F-CM abelian variety arises in this way

(b) the maps between these examples are,

$$\mathrm{Hom}_F\left(\mathbf{Ab}(F, \mathfrak{a}, \Phi, \xi), \mathbf{Ab}(F, \mathfrak{a}', \Phi', \xi')\right) = \begin{cases} 0 & \Phi \neq \Phi' \\ \{a \in F a\mathfrak{a} \subset \mathfrak{a}' & \Phi = \Phi' \end{cases}$$