1 Sep. 04 - Review of Hodge Theory

Varities are over $\mathbb{C}$.

Theorem 1.0.1 (Hodge Decomposition)**.** X compact Kahler manifold then the singular cohomology groups have a decomposition

$$H^i(X,\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q} = \bigoplus_{p+q=i} H^{p,q}(X) = \bigoplus_{p+q=i} \mathcal{H}^{p,q}$$

where $H^{p,q}(X)$ is the cohomology of (p,q)-forms with respect to $\bar{\partial}$ and $\mathcal{H}^{p,q}$ are the (p,q)-Harmonic forms with respect to some choice of Kahler metric. This splitting does not depend on the choice of Kahler metric. We can say $H^{p,q}$ is the space of d-closed forms containiing a class of a (p,q)-form. Such that,

(a) $H^{p,q}(X) = \overline{H^{q,p}(X)}$ using the structure $H^i(X,\mathbb{C}) = H^i(X,\mathbb{R})$

(b) Dolbeaut Theorem: $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$

Consequences:

(a) Betti numbers: $b_i(X) := \dim_{\mathbb{C}} H^i(X,\mathbb{C})$ is even for i odd

(b) Hodge numbers: $b_i(X) = \sum_{p+q=i} h^{p,q}(X)$ where $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$

Serre duality: if E is a vector bundle then $H^q(X,E) \cong H^{n-q}(X, E^\vee \otimes \omega_X)^\vee$. In particular, for Ω_X^p we get

$$H^q(X,\Omega_X^p) \cong H^{n-q}(X,\Omega_X^{n-p})^\vee$$

and hence $h^{p,q}(X) = h^{n-p,n-q}(X)$.

Moreover, assume X is projective. Then every subvarity contributes to $H^{p,p}(X)$. Say $Z \subset X$ is codimenion p then $\eta_X = PD[Z] \in H^{2p}(X,\mathbb{Z}) \cap H^{p,p}(X)$. Therefore $b_{2p}(X) \geq h^{p,p}(X) > 0$.

Example 1.0.2. The Hodge diamond with 1 on the middle column and 0 elsewhere exists and is $\mathbb{P}^n$.

If X is projective, then $H^(X,\mathbb{Z})$ caries a polarization so we get a polarized Pure hodge structure of weight i. Let $H_{\mathbb{Z}} := H^i(X,\mathbb{Z})$ is a finitely generated free abelian group. Such that

(a) there is a decomposition

$$H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}$$

(b) $H^{p,q} = \overline{H^{q,p}}$

And there is a pairing:

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$$

such that

(a) Q is symmetric when i is even, anti-symmetric when i is odd

(b) decomposition is orthogonal with respect to $(\sqrt{-1})^i S(\alpha,\beta)$ where $S(\alpha,\beta) = Q_{\mathbb{C}}(\alpha,\bar{\beta})$.

(c) $(\sqrt{-1})^{i(i+1)/2} S(\alpha,\alpha) > 0$

When it is only compact Kahler we can do this over $\mathbb{R}$ but not over $\mathbb{Q}$ or $\mathbb{Z}$.

1.1 Filtration

Alternatively consider the decreasing filtration:

$$F^\ell H_{\mathbb{C}} = \bigoplus_{p \geq \ell} H^{p,q}$$

This satisfies

$$F^\ell \cap \overline{F^{i-\ell+1}} = 0$$

and thus

$$F^\ell \oplus \overline{F^{i-\ell+1}} = H_{\mathbb{C}}$$

and these properties determine the Hodge structure.

1.2 Degeneration of the Hodge-to-De Rham Spectral sequence

Somewhat weaker than the full Hodge decomposition but works algebraically so has arithmetic information.

On the complex $\Omega_X^\bullet$ there is a decreasing Filtration

$$F^p \Omega_X^\bullet := \Omega_X^{\geq p} := [0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^n \to 0]$$

The associated graded is

$$\mathrm{gr}_F^p \Omega_X^\bullet = \Omega_X^p[-p]$$

the sheaf Ω_X^p supported in degree p. The assocated spectral sequence of the Filtered complex

$$E_1^{p,q} := \mathbb{H}^{p+q}(X, \mathrm{gr}_F^p \Omega_X^\bullet) = H^q(X, \Omega_X^p) \implies \mathbb{H}^{p+q}(X, \Omega_X^\bullet) \cong H^{p+q}(X, \mathbb{C})$$

Theorem 1.2.1. The spectral sequence degenerates at E_1.

1.3 Lefschetz Theorems

Theorem 1.3.1 (Hard Lefschetz)**.** (X, ω) compact Kahler manifold. Let $L = \omega \wedge -$ be the Lefschetz operator. It acts on cohomology by the de Rham theorem. Then for all $i \leq n$

$$L^{n-i} : H^i(X, \mathbb{C}) \to H^{2n-i}(X, \mathbb{C})$$

is an isomorphism.

Poincare dualiy: nondegenerate pairing

$$H^i(X, \mathbb{C}) \times H^{2n-i}(X, \mathbb{C}) \to \mathcal{C} \qquad (\alpha, \beta) \mapsto \int_X \alpha \smile \beta$$

Theorem 1.3.2 (Weak Lefschetz)**.** If X is smooth projective of dimension n and D is any ample effective divisor on X then

$$H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$$

is an isomorphism for $i < n-1$ and injective for $i = n-1$.

One approach: when D is smooth it follows from Kodaira vanishing. Consider $U = X \backslash D$ which is an affine variety then our theorem is just about the cohomology of a affine.

Theorem 1.3.3 (Andreotti-Frankel)**.** if Y is a closed subvariety of $\mathbb{C}^n$ then $H_j(Y, \mathbb{Z}) = H^j(Y, \mathbb{Z}) = 0$ for $j > n$. In fact, Y has the homotopy type of a CW complex of real dimension n.

Now we can prove Lefschetz: using the relative cohomology exact sequence

$$H^i(X, D; \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z}) \to H^{i+1}(X, D; \mathbb{Z})$$

Then by Alexander duality

$$H^i(X, D; \mathbb{Z}) \cong H_i(U, \mathbb{Z}) = 0$$

so we win by Andreotti-Frankel. Notice if D is smooth then $H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$ is a map of pure hodge structures.

Corollary 1.3.4. $H^{p,q}(X) \to H^{p,q}(D)$ are isomorphisms for $p + q < n - 1$ and injections for $p + q = n - 1$.

Theorem 1.3.5 (Lefschetz (1,1))**.** If X is smooth projective then every integral $(1,1)$-class $\alpha \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ is $c_1(\mathcal{L})$ for some line bundle and hence is a $\mathbb{Z}$-linear combination of classes of effective divisors.

Where does the c_1 map come from. The exponential map:

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

is exact (where we look at X as an analytic variety) so there is an exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

then the boundary map is c_1 viewing $H^1(X, \mathcal{O}_X^\times) = \text{Pic}\,(X)$ and its kernel is

$$\text{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$$

Theorem 1.3.6. If $f : X \to B$ is a proper submersion with smooth compact Kahler fibers then the Hodge numbers $h^{p,q}(X_b)$ are constant for $b \in B$.

Proof. By Ehresmann's theorem, it is C^∞-locally trivial and hence $b_i(X_b)$ is constant. However

$$b_i(X_b) = \sum_{p+q=i} h^{p,q}(X_b)$$

and the $h^{p,q}(X_b)$ are upper semi-continuous since they are cohomology groups of coherent sheaves. Therefore they must be constant. □

2 Sep. 09 - Rational Singularities

2.1 Resolution

Definition 2.1.1. Let X be a smooth variety, a *simple normal crossings (SNC)* divisor $E \subset X$ is a divisor of the form

$$E = \sum_{i=1}^{m} E_i$$

where E_i is a smooth irreducible hypersurface in X and such that the intersections are transverse i.e. analytically locally E is cut out by $x_1 \dots x_s = 0$ for some $s \leq n := \dim X$.

Example 2.1.2. A nodal curve in $\mathbb{P}^2$ is not SNC since its components are not smooth. However it is a normal crossings divisor (NC) which is just the local condition.

Definition 2.1.3. A resolution of singularities of a variety X is a proper birational map $f : \widetilde{X} \to X$ with $\widetilde{X}$ smooth

(a) a log resolution of X is a resolution $f : \widetilde{X} \to X$ such that (the support of) $\operatorname{Exc} f$ is an SNC divisor

(b) more generally, for any effective $\mathbb{Q}$-divisor D on X a log resolution (X, D) is a resolution $f : \widetilde{X} \to X$ with $\widetilde{D} = f^{-1}(D) \cup \operatorname{Exc} f$ has SNC support.

Theorem 2.1.4 (Hironaka)**.** For any X (or pair (X, D)) in characteristic zero, log resolutions exist. Moreover, they can be chosen with the properties

(a) f is an isomorphism away from the singular locus of X (or the singular locus of X union the non SNC locus of (X, D))

(b) f is a composition of blowups with smooth centers.

Example 2.1.5. Let $Y \subset \mathbb{P}^n$ be a smooth subvariety. Then there is an affine cone $X := C(Y)$ given by taking the homogeneous equations and viewing them in $\mathbb{A}^{n+1}$. To resolve, we blowup the origin of $\mathbb{A}^{n+1}$. The strict transform of E intersects $E \cong \mathbb{P}^n$ at the projectivization of the tangent cone which is again isomorphic to Y making X the total space of a vector bundle over Y. This gives a log resolution of $(\mathbb{A}^{n+1}, X)$.

Remark. An "ordinary singularity" here was a singularity whose projectivization of its tangent cone is smooth.

2.2 Rational Singularities

Definition 2.2.1. A variety $X/\mathbb{C}$ has *rational singularities* if for any resolution of singularities $f : \widetilde{X} \to X$ we have $Rf_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$.

Remark. We split this into two conditions:

(a) $f_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ i.e. X is normal

(b) $R^i f_* \mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$ which is the real condition.

Exercise 2.2.2. If $f : Y \to X$ is a proper birational map of smooth varities then $\mathrm{R}^i f_* \mathcal{O}_Y = 0$ for $i > 0$.

Remark. The above is also true in characteristic p but is much harder to show.

Exercise 2.2.3. Grauert-Riemenschneider Theorem: if $f : Y \to X$ is a generically finite surjective morphism (in characteristic zero) and Y is smooth then $R^i f_* \omega_Y = 0$ for $i > 0$.

Proposition 2.2.4. Let $f : \widetilde{X} \to X$ be a resolution of singularities. Then the following are equivalent:

(a) $\mathrm{R} f_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$

(b) X is Cohen-Macaualy and $f_* \omega_{\widetilde{X}} \cong \omega_X$ where ω_X is the dualizing sheaf of X (which exists for X CM and irreducible)

Remark. There is a less fancy duality theory proof in Kollar-Mori section 5.1.

Proposition 2.2.5. every variety X has a dualizing complex $\omega_X^\bullet \in D^\flat(X)$. One way to get it is to embed $X \hookrightarrow Y$ in a smooth variety Y (we can always do this locally) then $\omega_X^\bullet := \mathrm{RHom}_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y)[\dim Y]$ note that $\omega_Y[\dim Y] = \omega_Y^\bullet$ since Y is smooth.

Remark. Always we let $\omega_X := \mathcal{H}^{-\dim X}(\omega_X^\bullet)$ is a sheaf but the issue is that there may be terms in larger degree. The commutative algbera problem we are solving is (recall ω_Y is a line bundle)

$$\mathrm{Ext}_i^R(R/I, R) = 0$$

for R a regular ring and the smallest i for which this does not hold is $\mathrm{depth}_R(I, R)$ and R/I is CM iff $\mathrm{depth}_R(I, R) = \mathbf{ht}(I)$ i.e. only one cohomology group so the dualizing complex is a dualizing sheaf.

Lemma 2.2.6. X is CM iff $\omega_X^\bullet = \omega_X[\dim X]$. Note that ω_X is a line bundle iff X is Gorenstein.

Grothendieck Duality: for $f : Y \to X$ proper

$$\mathrm{R}f_* \mathrm{RHom}_Y(\mathscr{F}, \omega_Y^\bullet) \cong \mathrm{RHom}_X(\mathrm{R}f_* \mathscr{F}, \omega_X^\bullet)$$

Proof of proposition. Let $\mathscr{F} = \omega_{\widetilde{X}}[\dim X]$ and apply Grothendieck duality to $f : \widetilde{X} \to X$ then

$$\mathrm{R}f_* \mathcal{O}_{\widetilde{X}} = \mathrm{R}f_* \mathrm{RHom}_{\widetilde{X}}(\omega_{\widetilde{X}}[\dim X], \omega_{\widetilde{X}}[\dim X]) = \mathrm{RHom}_X(\mathrm{R}f_* \omega_{\widetilde{X}}[\dim X], \omega_X^\bullet)$$

But by Grauert-Riemenschneider: $\mathrm{R}f_* \omega_{\widetilde{X}} = f_* \omega_{\widetilde{X}}$ and thus, setting $n := \dim X$ we get,

$$\mathrm{R}f_* \mathcal{O}_{\widetilde{X}} = \mathrm{RHom}_X(f_* \omega_{\widetilde{X}}[n], \omega_X^\bullet)$$

Now applying the functor $\mathrm{RHom}_X(-, \omega_X^\bullet)$ we get

$$\mathrm{RHom}_X(\mathrm{R}f_* \mathcal{O}_{\widetilde{X}}, \omega_X^\bullet) \cong f_* \omega_{\widetilde{X}}[n]$$

Therefore $\mathrm{R}f_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ iff $\omega_X^\bullet \cong f_* \omega_{\widetilde{X}}[n]$. ∎

Corollary 2.2.7. The following are equivalent:

(a) X has rational singularities

(b) there exists a resolution $f : \widetilde{X} \to X$ such that $\mathrm{R}f_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$.

Proof. (a) implies (b) is obvious so assume (b). By the proposition, X is CM and $f_* \omega_{\widetilde{X}} \cong \omega_X$. Given any other resolution $g : Y \to X$ we can find a roof i.e. a resolution $h : Z \to X$ that dominates both. We need to show that $g_* \omega_Y = \omega_X$ but in the diagram

$$\begin{tikzcd} & Z \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] \arrow[dd, "h"] & \\ \widetilde{X} \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

since φ, ψ are birational maps of smooth varities we have $\varphi_* \omega_Z = \omega_{\widetilde{X}}$ and $\psi_* \omega_Z = \omega_Y$ and thus $h_* \omega_Z = f_* \omega_{\widetilde{X}} = \omega_X$ by assumption so $h_* \omega_Z = g_* \omega_Y$ so we win. □

Remark. We of course could have made the same argument with $\mathrm{R} f_* \mathcal{O}_X$ and it is just as easy except that we need to use composition of derived functors / Leray spectral sequence. But this way we can just work the level of sheaves. Also it will be useful to use the dualizing sheaf characterization of rational singularities.

Example 2.2.8. (a) If C is a curve, rational singularities implies C normal so C is smooth.

(b) $(x_1^2 + x_2^2 + x_3^2 = 0) \subset \mathbb{A}^3$ the quadric cone has rational singularities

(c) if $Y \subset \mathbb{P}^n$ is a smooth hypersurface of degree d then $X = C(Y)$ is a rational singularity iff $d \le n$. Indeed, consider the resolution $\widetilde{X} \to X$ given by blowing up the cone point. Since X is a hypersurface in $\mathbb{A}^{n+1}$ it is lci and hence Gorenstein so we just need to check if $f_* \omega_{\widetilde{X}} = \omega_X$.

3 Sept. 11 - Rational Singularities II

3.1 Some Advantages of Rational Singularities

Let X be a projective variety with only rational singularities then

(a) if $f : \widetilde{X} \to X$ is a resolution of singularities: $H^i(X, \mathcal{O}_X) = H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ for all i. Proof: both equal $H^i(X, Rf_* \mathcal{O}_{\widetilde{X}})$

(b) Kodaira vanishing holds: if L is an ample line bundle on X then $H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$. Proof: $\omega_X = \mathrm{R} f_* \omega_{\widetilde{X}}$ and therefore by the projection formula

$$H^i(\widetilde{X}, \omega_{\widetilde{X}} \otimes f^* L) = H^i(X, \omega_X \otimes L)$$

and the LHS is zero by Kawamata-Viehweg vanishing.

Remark. For a variety over $\mathbb{C}$ having rational singularities is the same notion in the algebraic or analytic categories. Indeed if $f : Y \to X$ is proper then $(R^i f_* \mathscr{F})^{\mathrm{an}} = R^i f_*^{\mathrm{an}} \mathscr{F}^{\mathrm{an}}$ for any coherent sheaf $\mathscr{F}$ on Y.

Example 3.1.1. We continue our calculation of the cone over a hypersurface from last time. Let $X = C(Y) \subset \mathbb{A}^{n+1}$ be the cone over a smooth projective variety $Y \subset \mathbb{P}^n$. Since X is a hypersurface, it is CM so we need only understand when $f_* \omega_{\widetilde{X}} = \omega_X$ where $f : \widetilde{X} \to X$ is the log resolution of $(\mathbb{A}^{n+1}, X)$ givien by blowing up the origin. Recall

$$K_{\widetilde{\mathbb{A}}^{n+1}} = f^* K_{\mathbb{A}^{n+1}} + nE$$

and likewise

$$f^* X = \widetilde{X} + dE$$

since the multiplicity of the point is d. Now we compute

$$K_{\widetilde{X}} = (K_{\widetilde{\mathbb{A}}^{n+1}} + \widetilde{X})|_{\widetilde{X}} = f^*(K_{\mathbb{A}^{n+1}} + X) + (n - d)E$$

Therefore, by the projection formula,

$$f_* \omega_{\widetilde{X}} = \omega_X \otimes f_* \mathcal{O}_{\widetilde{X}}((n-d)E|_X) = \begin{cases} \omega_X & d \leq n \\ \omega_X \otimes \mathscr{I}_0^{(d-n)} & d > n \end{cases}$$

where $\mathscr{I}_0$ is the ideal sheaf of the origin of the cone. Therefore, X has rational singularities if and only if $d \leq n$.

Example 3.1.2. Abstract cones: fix Y a variety and let L be an ample line bundle. Let,

$$X = \operatorname{Spec}\left(\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})\right) \to Y$$

If Y is normal and $L = \mathcal{O}_Y(1)$ for an embedding $Y \hookrightarrow \mathbb{P}^n$ then $X = C(Y, L)$ is the normalization of $C(Y)$. Exercise: $C(Y, L)$ has rational singularities if and only if $H^i(Y, L^{\otimes m}) = 0$ for all $i > 0$ and $m \geq 0$. In particular, this requires $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$. This fails for all Calabi-Yau but holds for Fano if we take $L = -K_Y$ by Kodaira vanishing:

$$H^i(Y, L^{\otimes m}) = H^i(Y, L^{\otimes(m+1)} \otimes K_Y) = 0$$

since $m + 1 > 0$ so $L^{\otimes(m+1)}$ is ample.

3.2 More Flexible Interpretation of Rational Singularities

Theorem 3.2.1 (Kovács)**.** Let $f : Y \to X$ be a proper morphism of complex varities such that Y has rational singularities (e.g. Y smooth) and the morphism $\varphi : \mathcal{O}_X \to \mathrm{R}f_* \mathcal{O}_Y$ has a splitting in $D^\flat(X)$ (i.e. $\exists : \psi : \mathrm{R}f_* \mathcal{O}_Y \to \mathcal{O}_X$ such that $\psi \circ \varphi$ is a quis) then X has rational singularities.

Proof. Consider compatible resolutions:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\tilde{f}"] \arrow[d, "\sigma"] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Get a diagram of complexes of sheaves,

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\varphi"] \arrow[d, "\alpha"] & \mathrm{R}f_* \mathcal{O}_Y \arrow[d, "\beta"] \\ \mathrm{R}f_* \mathcal{O}_{\widetilde{X}} \arrow[r, "\gamma"] & \mathrm{R}(f \circ \sigma)_* \mathcal{O}_{\widetilde{Y}} \end{tikzcd}$$

We are assuming β is an isomorphism and φ has a splitting ψ. Therefore $\gamma \circ \alpha$ has a splitting meaining $\psi \circ \gamma \circ \alpha = \mathrm{id}$ so α has a splitting. Therefore, we reduce to the same statement but applied to the resolution $\pi : \widetilde{X} \to X$.

Consider the splitting

$$\mathcal{O}_X \xrightarrow{\varphi} \mathrm{R}f_* \mathcal{O}_{\widetilde{X}} \xrightarrow{\psi} \mathcal{O}_X$$

composing to the identiy. Dualize to obtain

$$\omega_X^\bullet \to \mathrm{R}f_* \omega_Y[n] \to \omega_X^\bullet$$

and the composition is still a quasi-isomorphism. This is because

$$\mathrm{RHom}_Y\left(\mathrm{R}f_*\mathcal{O}_{\widetilde{X}}, \omega_X^\bullet\right) = \mathrm{R}f_*\mathrm{RHom}_X\left(\mathcal{O}_{\widetilde{X}}, \omega_{\widetilde{X}}[n]\right) = \mathrm{R}f_*\omega_{\widetilde{X}}[n]$$

since $\widetilde{X}$ is smooth. By G-S $\mathrm{R}f_*\omega_{\widetilde{X}}[n] = (f_*\omega_{\widetilde{X}})[n]$ and therefore taking cohomology sheaves we get splittings

$$\mathcal{H}^i(\omega_X^\bullet) \to R^{i+n}f_*\omega_{\widetilde{X}} \to \mathcal{H}^i(\omega_X^\bullet)$$

since the middle term is zero for $i \neq -n$ we get $\mathcal{H}^i(\omega_X^\bullet)$ is zero for $i \neq -n$ so X is CM. Furthermore, for $i = -n$ we get maps

$$\omega_X \to f_*\omega_{\widetilde{X}} \to \omega_X$$

the map $f_*\omega_{\widetilde{X}} \to \omega_X$ always exists (it is the trace map) and it is injective but this composition shows surjective and hence an isomormphism. Therefore we conclude X has rational singularities. ☐

Corollary 3.2.2. If $f : Y \to X$ is a finite surjective morphism of normal varities and Y has rational singularities then X has rational singularities.

Proof. Indeed, there is a trace map $f_*\mathcal{O}_Y \to \mathcal{O}_Y$ giving a splitting so the theorem applies (recall $Rf_*\mathcal{O}_Y = f_*\mathcal{O}_Y$ since f is finite). ☐

Corollary 3.2.3. Finite quotient singularities are rational.

Example 3.2.4. All quotients by reductive groups have rational singularities because there is a splitting $f_8\mathcal{O}_Y \to \mathcal{O}_X$ given by the Reynolds operator (defined via an averaging procedure)

It is important to know that quotient singularities are rational because then coarse moduli spaces of smooth Artin stacks are rational. In particular, the moduli spaces of curves or of vector bundles on a smooth curve.

Exercise 3.2.5. Kovác's paper: let $f : Y \to X$ be a surjective morphism with connected fibers of projective varities over $\mathbb{C}$ and Y has rational singularities. Then X has rational singularities $\iff R^{\dim X - \dim Y} f_*\omega_Y = \omega_X$.

Kollár noted that if $f : Y \to X$ is a map whose *general* fiber has dimenson k then $R^i f_*\omega_Y = 0$ for $i > k$. He also showed that if X is smooth then $R^k f_*\omega_Y = \omega_X$. The above exercise generalizes this.

Exercise 3.2.6. Other examples of rational singularities (references in the notes):

(a) toric varities

(b) generic determinantal varieties: fix integers n, m and $k = \min\{n, m\}$ then consider the vanishing of the $k \times k$ minors of a $m \times n$ matrix of indeterminants

(c) the previous implies that Θ-divisors of Jacobians have rational singularities.
