

1 Introduction

An affine scheme is the basic object of algebraic geometry. Associated to any ring A there is a scheme $\text{Spec}(A)$. If you haven't seen this before you should just think of it as a gadget which has the following data,

- (a) a topological space (also called $\text{Spec}(A)$ with the Zariski topology) whose points are prime ideal $\mathfrak{p} \subset A$ and whose closed sets are of the form,

$$V(I) = \{\mathfrak{p} \subset A \mid \mathfrak{p} \supset I\}$$

where $I \subset A$ is an ideal of A . Equivalently, a basis of open sets is given by,

$$D(f) = \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}$$

where $f \in A$ ranges over elements of A

- (b) which remembers the ring A thought of as the collection of “algebraic functions” on the topological space $\text{Spec}(A)$. We think of $a \in A$ as the function whose “value” at a point $\mathfrak{p} \in \text{Spec}(A)$ is the element $\bar{a} \in \kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ which is a field.

The most important property of affine schemes is that they are determined by their ring of global functions.

Definition 1.0.1. If A is a k -algebra then the set of k -points of $\text{Spec}(A)$ is the set of points $\mathfrak{p} \in \text{Spec}(A)$ such that the natural map $k \rightarrow \kappa(\mathfrak{p})$ is an isomorphism.

Example 1.0.2. $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[x, y])$ is the affine plane. The points are prime ideal $\mathfrak{p} \subset \mathbb{C}[x, y]$ the \mathbb{C} -points correspond to the maximal ideal which are of the form $\mathfrak{p} = (x - a, y - b)$. Therefore, the set of \mathbb{C} -points is \mathbb{C}^2 . We can describe the Zariski topology as the topology whose closed sets are the vanishing locus of polynomials on \mathbb{C}^2 .

2 Smooth Manifolds

Associated to a smooth manifold M is a natural ring, the ring of smooth functions $C^\infty(M)$. We will show that M is an affine scheme in the sense that it can be recovered from the ring $C^\infty(M)$ of smooth functions. However, the correspondence goes far deeper. Indeed, M as a topological space is the set of \mathbb{R} -points of $\text{Spec}(C^\infty(M))$ with the Zariski topology. We will see even more is true.

Lemma 2.0.1. M has the Zariski topology, meaning that every closed set $Z \subset M$ is the zero locus of a smooth function.

Proof. Since M is separable, there exists a countable open cover $\{U_k\}$ of $M \setminus Z$ by balls so there are smooth functions $\rho_k : M \rightarrow \mathbb{R}$ such that $0 \leq \rho_k(x) \leq 1$ and $\rho_k^{-1}(0) = M \setminus U_k$. Then consider,

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} \rho_k(x)$$

Since $|2^{-k} \rho_k| < 2^{-k}$ the series converges uniformly and absolutely so f is smooth. Furthermore,

$$f(x) = 0 \iff \forall k : \rho_k(x) = 0 \iff \forall k : x \notin U_k \iff x \in Z$$

□

Remark. Some related mathoverflow posts,

- (a) [smooth function with given zero set](#)
- (b) [separating open and closed set](#)
- (c) [function whose zero set is the Cantor set](#)
- (d) [closed set is set of zeros](#)
- (e) [manifold as zero locus](#)

2.1 Comparison map between M and $\text{Spec}(C^\infty(M))$

Let X be a topological space and $A \subset C^0(X)$ be a subring of the continuous (real valued) functions. Consider the map $\varphi : X \rightarrow X^{\text{aff}} = \text{Spec}(A)$ given by sending $x \mapsto \mathfrak{m}_x$ where,

$$\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}$$

which is a maximal ideal since it is the kernel of the evaluation map $\text{ev}_x : A \rightarrow \mathbb{R}$. We will later specialize to A being the ring of smooth functions but really all we will use about A is,

- (a) A separates points meaning for all $x, y \in X$ with $x \neq y$ there exists $f_x \in A$ with $f_x(x) = 1$ and $f_x(y) = 0$
- (b) A generates the topology of M meaning for every closed set $Z \subset M$ there is $f_Z \in A$ with $Z = f_Z^{-1}(0)$
- (c) if $f \in A$ and $f > 0$ then $\sqrt{f} \in A$ and $\frac{1}{f} \in A$

Proposition 2.1.1. The map $\Phi : X \rightarrow X^{\text{aff}}$ is continuous and injective if A separates points.

Proof. We need to show that $\Phi^{-1}(D(f))$ is open and note that,

$$\Phi(x) \in D(f) \iff f \notin \mathfrak{m}_x \iff f(x) \neq 0$$

which is open since f is continuous. Since A separates points, $\mathfrak{m}_x \neq \mathfrak{m}_y$ for $x \neq y$ so Φ is injective. \square

Remark. This affinification is really a much more general construction. For any locally-ringed space X there is an affinification $X^{\text{aff}} = \text{Spec}(\Gamma(X, \mathcal{O}_X))$ which is left-adjoint to the inclusion $\mathbf{AffSch} \hookrightarrow \mathbf{LRS}$. Then the unit $X \rightarrow X^{\text{aff}}$ is a morphism of locally-ringed spaces and therefore is a continuous map.

Remark. First let's make some comments about the ring $C = C^\infty(M)$.

- (a) First, there is a unique ring map $\mathbb{R} \rightarrow C$. Indeed, there is a unique map $\mathbb{Q} \rightarrow C$ since the nonzero images of the unique map $\mathbb{Z} \rightarrow C$ are invertible
- (b) we define a partial order on C where $f \leq g$ iff there exists $h \in A$ such that $f + h^2 = g$. Consider the set of functions r with the property that for any $\epsilon > 0$ there exists $p, q \in \mathbb{Q}$ such that $|p - q| \leq \epsilon$ and $p \leq r \leq q$. This recovers precisely the copy of \mathbb{R} inside C .

- (c) Since any elements in the image of a map $\varphi : \mathbb{R} \rightarrow C$ satisfies this property so there is a unique map $\varphi : \mathbb{R} \rightarrow C$. Indeed if $x \in \mathbb{R}$ then for all $\epsilon > 0$ there are $p, q \in \mathbb{Q}$ with $|p - q| < \epsilon$ and $p \leq x \leq q$. Thus $p \leq \varphi(x) \leq q$ because $x \leq y$ meaning $x + z^2 = y$ for some $z \in \mathbb{R}$ and thus $\varphi(x) + \varphi(z)^2 = \varphi(y)$ meaning $\varphi(x) \leq \varphi(y)$.

Furthermore, the ring C knows the sup norm. Indeed let,

$$\sigma(f) = \{\lambda \in \mathbb{R} \mid f - \lambda \text{ is not invertible}\}$$

then $\sigma(f) = \text{im } f$ (but we didn't need to know that f represented a function) and hence the spectral norm,

$$\|f\| = \sup_{\lambda \in \sigma(f)} |\lambda|$$

which recovers the sup norm since $\sigma(f) = \text{im } f$. Therefore, we recover $C^\infty(M)$ as a normed \mathbb{R} -vectorspace.

Remark. The \mathbb{R} -points of M^{aff} correspond to ring maps $\varphi : C^\infty(M) \rightarrow \mathbb{R}$. Note that such maps are automatically \mathbb{R} -algebra morphisms since there is a unique ring map $\mathbb{R} \rightarrow C^\infty(M)$ and the composition $\mathbb{R} \rightarrow \mathbb{R}$ must be the identity since there is a unique ring map. We now want to classify these \mathbb{R} -algebra maps $\varphi : C^\infty(M) \rightarrow \mathbb{R}$.

Notice that if $f > 0$ then f is invertible so $\varphi(f) \neq 0$. Furthermore, \sqrt{f} is smooth so $\varphi(f) = \varphi(\sqrt{f})^2 \geq 0$ and therefore $\varphi(f) > 0$.

Now we return to the map, $\Phi : M \rightarrow M^{\text{aff}} = \text{Spec}(C^\infty(M))$ given by sending $x \mapsto \mathfrak{m}_x$ where,

$$\mathfrak{m}_x = \{f \in C^\infty(M) \mid f(x) = 0\}$$

Proposition 2.1.2. $\varphi : M \rightarrow M^{\text{aff}}$ is continuous, and is an isomorphism onto the \mathbb{R} -points of M^{aff} .

Proof. We have seen that Φ is continuous and injective into $M^{\text{aff}}(\mathbb{R})$. We need to show that $\Phi : M \rightarrow M^{\text{aff}}(\mathbb{R})$ is surjective and closed.

Let $\varphi : C^\infty(M) \rightarrow \mathbb{R}$ be a ring map (hence an \mathbb{R} -algebra map) and consider,

$$Z(\varphi) = \bigcap_{f \in \ker \varphi} Z(f) = \{x \in M \mid \ker \varphi \subset \mathfrak{m}_x\}$$

Since φ is surjective $\ker \varphi$ is maximal and since the map $x \mapsto \mathfrak{m}_x$ is injective $\ker \varphi$ is contained in at most one \mathfrak{m}_x so $Z(\varphi)$ has at most one point.

Thus it suffices to show that $Z \neq \emptyset$. Otherwise for each $x \in M$ we could find f_x such that $f_x \in \ker \varphi$ and $f_x(x) = 1$. Replacing f_x by f_x^2 we assume $f_x \geq 0$. The open sets $U_x = f_x^{-1}((0, \infty))$ are neighborhoods of x . Hence for any compact K we can find a finite set x_1, \dots, x_n such that the U_{x_i} cover K . Then consider,

$$f_K = f_{x_1} + \dots + f_{x_n}$$

and we see that $f_K > 0$ on K but $\varphi(f_K) = 0$. If $K = M$ we would be done by the remark that f_K is invertible. Otherwise, choose a positive exhaustion function $f : M \rightarrow \mathbb{R}$ meaning that $f^{-1}([0, c])$ is compact for all $c \in \mathbb{R}$. Let $\lambda = \varphi(f)$ and $K = f^{-1}([0, 2\lambda])$. Then we get f_K and there is some $\epsilon > 0$ with $f_K > \epsilon$ on K . Then consider,

$$h = f + \epsilon^{-1} \lambda f_K - \lambda$$

we see that $h > 0$ because for $x \notin K$ we have $f(x) > 2\lambda$ and for $x \in K$ we have $f_K > \epsilon$ and $f \geq 0$. Therefore h is invertible but,

$$\varphi(h) = \varphi(f) - \lambda = 0$$

giving a contradiction. Therefore $Z(\varphi) = \{x\}$ for some $x \in M$ and thus $\ker \varphi \subset \mathfrak{m}_x$ but $\ker \varphi$ is maximal so $\ker \varphi = \mathfrak{m}_x$.

Finally, Φ is closed since if $Z \subset M$ is a closed set there exists $f \in C^\infty(M)$ with $f^{-1}(0) = Z$ and then,

$$\Phi(Z) = \{\mathfrak{m}_x \mid x \in Z\}$$

however,

$$x \in Z \iff f(x) = 0 \iff f \in \mathfrak{m}_x$$

and therefore,

$$\varphi(Z) = \{\mathfrak{m}_x \mid f \in \mathfrak{m}_x\} = V(f) \cap M^{\text{aff}}(\mathbb{R})$$

is the closed in the subspace topology and is the “vanishing locus” of the ideal $(f) \subset C^\infty(M)$. \square

Remark. Notice that if M is compact, the proof that $Z(\varphi) \neq \emptyset$ did not use anything about the target field. Therefore, we have also shown that if M is compact then every map $\varphi : C^\infty(M) \rightarrow K$ for a field K must have $Z(\varphi) \neq \emptyset$.

3 What are the other points?

Proposition 3.0.1. $M_{\text{closed}}^{\text{aff}}$ is larger than $M^{\text{aff}}(\mathbb{R})$ if and only if M is non-compact in which case it contains “hyperreal valuations”.

Proof. Consider a sequence $s = \{z_i \in M\}_{i \in I}$ then we get a map,

$$\text{ev}_s : C^\infty(M) \rightarrow \prod_{i \in I} \mathbb{R}$$

by evaluating at the sequence. Choosing an ultrafilter \mathcal{U} on I we get,

$$\text{ev}_{s, \mathcal{U}} : C^\infty(M) \rightarrow \prod_{i \in I} \mathbb{R} \rightarrow \left(\prod_{i \in I} \mathbb{R} \right) / \mathcal{U} = \mathbb{R}^{\mathcal{U}}$$

which is the field of hyperreal numbers (if \mathcal{U} is non-principal). If \mathcal{U} is principal on the index i then $\text{ev}_{\mathcal{U}} = \text{ev}_{z_i}$ is just ordinary evaluation at a point. Likewise, if the sequence is constant at z then we get

$$C^\infty(M) \xrightarrow{\text{ev}_z} \mathbb{R} \hookrightarrow \mathbb{R}^{\mathcal{U}}$$

via the constant embedding which does not give a new point. However, otherwise we can get new interesting points.

Let $z_i \rightarrow z$ be a convergent sequence. Then if $\text{ev}_{s, \mathcal{U}}(f) = 0$ we see that $\{i \in I \mid f(z_i) = 0\} \in \mathcal{U}$ and is, in particular, infinite. Thus by continuity $f(z) = 0$. Therefore, $\ker \text{ev}_{s, \mathcal{U}} \subsetneq \mathfrak{m}_z$ so we find nontrivial prime ideals inside \mathfrak{m}_z . These are not closed points.

If M is not compact then we can choose a sequence $s = \{z_i\}_{i \in I}$ with no limit points. Then I claim that $\text{ev}_{s, \mathcal{U}}$ is surjective meaning $\ker \text{ev}_{s, \mathcal{U}}$ is a new maximal ideal. Indeed, there is a countable cover

$\{U_i\}_{i \in I}$ such that $z_i \in U_i$ is the only element of the sequence. We may refine this cover so that it is locally finite and use a partition of unity to construct a function f having any specified sequence of values $(a_i)_{i \in I}$ at the points z_i and thus ev_s is already surjective.

The most interesting case is when M is compact. We have seen that any map $\varphi : C^\infty(M) \rightarrow K$ satisfies $\ker \varphi = \mathfrak{m}_x$ for some x since both are maximal ideal. Why does the previous construction not work? Any ultrafilter \mathcal{U} on I pushes forward to M and has a unique limit point $\mathcal{U} \rightarrow z$. This is the extension from the Stone-Cech compactification,

$$\begin{array}{ccc} I & \longrightarrow & \beta I \\ & \searrow & \downarrow \\ & & M \end{array}$$

By definition we say that $\mathcal{U} \rightarrow z$ if every neighborhood of z contains an element of \mathcal{U} . Suppose that $f \in \ker \text{ev}_{s, \mathcal{U}}$ then for every neighborhood V of z there is an infinite index set $J \in \mathcal{U}$ so that $z_i \in V$ thus $f(z_i) = 0$ for some element of V so by continuity $f(z) = 0$. Hence $\ker \text{ev}_{s, \mathcal{U}} \subset \mathfrak{m}_{\lim \mathcal{U}}$ so these do not contribute new closed points. \square

4 Recovering the Smooth Structure

The smooth structure on a manifold is determined by its ring of smooth functions (as these determine the sheaf of smooth functions). Therefore, we can see that the topological space $M^{\text{aff}}(\mathbb{R})$ inherits a natural smooth structure from the ring $C^\infty(M)$ which is the ring of “algebraic functions” on this scheme. Furthermore, this is not just an abstract ring, its elements are canonically functions on $M^{\text{aff}}(\mathbb{R})$ in the following way. For any $x \in M^{\text{aff}}(\mathbb{R})$ this corresponds to an ideal \mathfrak{m}_x and a function $\varphi_x : C^\infty(\mathbb{R}) \rightarrow \mathbb{R}$ which we call evaluation at x . Then f becomes a function where $f(x) = \varphi_x(f)$.

Using this definition, the map $M \rightarrow M^{\text{aff}}(\mathbb{R})$ is a diffeomorphism. Indeed, it is a homeomorphism and is an isomorphism on rings of smooth functions. This is a general fact, for any locally-ringed space, the map $X \rightarrow X^{\text{aff}}$ is an isomorphism on global functions by definition.

Proposition 4.0.1. The functor $M \mapsto M^{\text{aff}}$ gives a fully faithful embedding $\text{SmMfd} \hookrightarrow \text{AffSch}$.

Proof. Concretely this means the map,

$$\{\text{smooth map } f : M \rightarrow N\} \rightarrow \{\text{ring maps } C^\infty(N) \rightarrow C^\infty(M)\}$$

via sending,

$$f \mapsto f^* \text{ where } f^* : \varphi \mapsto \varphi \circ f$$

is a bijection. This is well-known but here we can give a slick proof. The inverse is given as follows. A ring map $g : C^\infty(N) \rightarrow C^\infty(M)$ gives a morphism of schemes $g : M^{\text{aff}} \rightarrow N^{\text{aff}}$ and therefore we get a continuous map,

$$\begin{array}{ccc} M^{\text{aff}}(\mathbb{R}) & \xrightarrow{g} & N^{\text{aff}}(\mathbb{R}) \\ \sim \uparrow & & \sim \uparrow \\ M & \dashrightarrow & N \end{array}$$

giving a continuous map $f : M \rightarrow N$ such that $\mathbf{m}_{f(x)} = g^{-1}(\mathbf{m}_x)$ therefore, $\varphi_{f(x)} = \varphi_x \circ g$ (because any ring map g is an \mathbb{R} -algebra map since there is a unique map $\mathbb{R} \rightarrow C^\infty(M)$) meaning for any $h \in C^\infty(N)$ then,

$$(f^*h)(x) = h(f(x)) = \varphi_{f(x)}(h) = \varphi_x(g(h)) = (g(h))(x)$$

and therefore $g = f^*$. □

5 Kahler Differentials

Remark. In this section we critically use that $A = C^\infty(M)$.

There is an algebraic description of the differential forms of an \mathbb{R} -algebra (or any ring) called the Kahler differentials. Defined by choosing a surjection $\mathbb{R}[S] \twoheadrightarrow A$ from a free algebra. Then let J be the kernel and define,

$$\Omega_{A/\mathbb{R}} := \left(\bigoplus_{s \in S} A ds \right) / \left(\sum_{s \in S} \frac{\partial f}{\partial s} ds \right)_{f \in J}$$

This satisfies the universal property, for any A -module M ,

$$\text{Der}_{\mathbb{R}}(A, M) = \text{Hom}_A(\Omega_{A/\mathbb{R}}, M)$$

Therefore the universal derivation $d : C^\infty(M) \rightarrow \Omega^1(M)$ to global 1-forms defines a comparison morphism,

$$a : \Omega_{C^\infty(M)/\mathbb{R}}^1 \rightarrow \Omega^1(M)$$

I claim that a is surjective but not injective. Surjectivity is clear from the construction of Kahler differentials.

Proposition 5.0.1. The map $a : \Omega_{C^\infty(M)/\mathbb{R}}^1 \rightarrow \Omega^1(M)$ is not injective.

Proof. Consider a map,

$$C^\infty(M) \xrightarrow{\pi} C^\infty(I) \xrightarrow{q} \mathbb{R}[[t]]$$

where the first map is restriction to some interval I in a coordinate chart of M and q is the map taking a smooth function on I to its Taylor series at 0. Note that an amazing theorem is that q is surjective. Now I claim that in $\Omega_{\mathbb{R}[[t]]/\mathbb{R}}$ we have $de^x \neq e^x dx$. Indeed, consider the map,

$$\Omega_{\mathbb{R}[[t]]/\mathbb{R}} \rightarrow \Omega_{\mathbb{R}((t))/\mathbb{R}}$$

then we see that if L/K is a field extension of characteristic zero fields and $\alpha, \beta \in L$ are transcendently independent over K then $d\alpha, d\beta \in \Omega_{L/K}$ are L -independent. Then if A is a K -algebra domain and $\alpha, \beta \in A$ transcendently independent then there are derivations,

$$A \rightarrow L \rightarrow L$$

which send $\alpha \mapsto 0$ or $\beta \mapsto 0$. In particular we apply this to $A = \mathbb{R}[[t]]$ with t, e^t . Suppose these satisfy a polynomial relation $f \in \mathbb{R}[X, Y]$ so $f(t, e^t) = 0$ but since $f \neq 0$ there is some integer n such that $f(n, Y)$ is a nonzero polynomial and $f(n, e^n) = 0$ contradicting the transcendence of e . □

The problem, as seen from the definition, is that Kahler differentials are a very “free” construction. It captures all derivations not just the “smooth” ones we want to consider. Really, $C^\infty(M)$ is a topological ring and we should consider the topology in the set of differentials. This makes us consider the category of smooth algebras also called C^∞ -rings. However, what is really interesting is that when we take the dual, we get the right thing. The following is standard but is very surprising from this perspective.

Proposition 5.0.2. Let $A = C^\infty(M)$ then the natural map,

$$\mathcal{X}(M) = \text{Hom}_A(\Omega^1(M), A) \rightarrow \text{Hom}_A(\Omega_{A/\mathbb{R}}, A) = \text{Der}_{\mathbb{R}}(A, A)$$

is an isomorphism.

Proof. This is the dual of a surjective map and therefore injective. Thus it suffices to prove that any derivation $D : A \rightarrow A$ arises from a smooth vector field. We do this for $M = \mathbb{R}^n$ first. Using the Hadamard lemma, for any $f \in A$ and $p \in \mathbb{R}^n$ we write,

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) h_i(x)$$

where h_i is a function such that $h_i(p) = \left. \frac{df}{dx_i} \right|_{x=p}$. Then applying the derivation,

$$D(f) = \sum_{i=1}^n D((x^i - p^i)) h_i(x) + \sum_{i=1}^n (x^i - p^i) D(h_i)$$

then evaluating the output at $x = p$ show that,

$$D(f)(p) = \sum_{i=1}^n v^i \left. \frac{df}{dx_i} \right|_{x=p}$$

where $v^i(p) = D(x^i)(p)$ is a smooth function of p . Doing this for all p , we see that,

$$D = \sum_{i=1}^n v^i \frac{d}{dx_i}$$

which exactly means that D is a smooth vector field. We can apply the same argument to M using partitions of unity. \square

6 Vector Bundles

Theorem 6.0.1 (Serre-Swan). Let M be a smooth manifold. There is an equivalence of categories,

$$\{\text{vector bundles on } M\} \xrightarrow{\sim} \{\text{finite projective } C^\infty(M)\text{-modules}\}$$

given by,

$$E \mapsto \Gamma(M, E)$$

Remark. There is a similar result for any compact Hausdorff space X with the ring $C^0(X, \mathbb{R})$ of continuous real functions.

This amazing theorem tells us that $M \mapsto M^{\text{aff}}$ induces an equivalence of categories between smooth vector bundles on M and algebraic vector bundles (locally-free coherent $\mathcal{O}_{M^{\text{aff}}}$ -modules) on M^{aff} .

Corollary 6.0.2. We can recover a smooth manifold M from its category $\mathbf{Vect}(M)$ of smooth vector bundles.

Proof. By Serre-Swan $\mathbf{Vect}(M)$ is equivalent to $\mathbf{Mod}_{C^\infty(M)}$. If R is any commutative ring then $R \cong \text{End}(\text{id}_{\text{Proj}_R^{\text{fin}}})$ so we recover $C^\infty(M)$ as a commutative ring. From $C^\infty(M)$ we have seen we can recover M . \square

Even more amazingly this theorem can be applied to produce interesting commutative algebra examples leveraging our knowledge of topology.

Remark. A finite projective modules P admit a finite projective complement Q such that $P \oplus Q$ is free. However, it is tricky to find a nontrivial projective which is *stably-free* meaning that we can take Q to be free. The following example shows a remarkable application of Serre-Swan.

Corollary 6.0.3. Let $A = \mathbb{R}[x_0, \dots, x_n]/(x_0^2 + \dots + x_n^2 - 1)$. Let P be the A -module with generators s_0, \dots, s_n and relation,

$$\sum_i x_i s_i = 0$$

Then $P \oplus A$ is free but P is not free for $n \neq 1, 3, 7$ and indecomposable if n is even.

Proof. The $n \neq 1, 3, 7$ should make you immediately suspicious. Consider $A \subset C^\infty(S^n)$ as the subring of polynomial functions. Then $P \otimes_A C^\infty(S^n) \cong \Gamma(TS^n)$ which is trivial if and only if $n = 1, 3, 7$ by Adam's theorem and indecomposable if n is even. Therefore, P is not free if $n \neq 1, 3, 7$ and indecomposable if n is even. Furthermore, we know TS^n is stably trivial. This does not immediately descent to A -module results (we could tensor a non-free module and have it become free) but we can easily deduce the rest algebraically.

It remains to show that P is projective and stably trivial. Indeed, let $F = A^{\oplus n}$ be a free module with basis s_0, \dots, s_n and consider the map,

$$g : F \rightarrow F$$

via,

$$g(s_i) = x_i \sum_j x_j s_j$$

Then notice,

$$g^2(s_i) = x_i \sum_j x_j^2 \sum_k x_k s_k = x_i \sum_k x_k s_k$$

and thus g is idempotent. Thus g splits F (consider $1 - g$),

$$F \cong \ker g \oplus \text{im } g$$

so both $\ker g$ and $\text{im } g$ are projective. Moreover,

$$g\left(\sum_i x_i s_i\right) = \sum_i x_i^2 \sum_j x_j s_j = \sum_j x_j s_j$$

which clearly generates $\text{im } g$ so $\text{im } g \cong A$. Then $P = F / \text{im } g$ so via the splitting,

$$\ker g \cong P$$

and hence P is projective and stably free. \square

References:

- (a) The above example and more can be found in Swan's [original paper](#).
- (b) [can we recover a compact manifold from its rings of functions](#)
- (c) [smooth manifold determined by ring](#)
- (d) [can we recover a space from its continuous functions](#)
- (e) [manifolds from sheaves](#)
- (f) [recover a compact smooth manifold from its ring of smooth functions](#)
- (g) [ring of smooth functions vs continuous functions](#)
- (h)