1 Toric Construction of Models

In this section we discuss the results of [Tim, Models] (2018) which gives a method of explicitly constructing a regular normal crossings model of a curve and explicilty describing its special fiber using the preceeding methods characterizing curves on toric surfaces.

Remark. In [Tim, Models], Dokchitser uses "curve" to refer to an integral separated *geometrically connected* scheme of finite type over a field. To mitigate any confusion, we render any results quoted from his work with "curve" replaced by "geometrically connected curve" when necessary. (OKAY THIS MAY BE WRONG!!)

1.1 Notations and Definitions

We work in the case of a discretely definition valued field K with valuation $\nu : K^\times \twoheadrightarrow \mathbb{Z}$, valuation ring R, uniformizer ϖ, and residue field κ. Given a smooth projective geometrically connected curve over K our goal will be to construct a regular normal crossings model over R. First we need to fix some notation.

Definition 1.1. Given a Laruent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$ recall the Newton polygon is,

$$\Delta(f) = \operatorname{Conv}\left(\{(i,j) \in \mathbb{Z}^2 \mid a_{ij} \neq 0\}\right) \subset \mathbb{R}^2$$

We will assume througout that $\operatorname{vol}(\Delta) > 0$. Now we refine the Newton polygon with respect to the valuation $\nu : K^\times \twoheadrightarrow \mathbb{Z}$,

$$\Delta_\nu(f) = \operatorname{LowerConvHull}\left(\{(i, j, v(a_{ij})) \mid (i,j) \in \Delta(f) \cap \mathbb{Z}^2\}\right) \subset \mathbb{R}^2 \times \mathbb{R}$$

The projection $\pi : \Delta_\nu \to \Delta$ is a homeomorphism. Thus, for each point $P \in \Delta$ there is a unique point $(P, \nu(P)) \in \Delta_\nu$ which defines a piecewise affine function $v : \Delta \to \mathbb{R}$ extending the valuation.

The bijection $\pi : \Delta_\nu \to \Delta$ pushes the polyhedral structure on Δ_ν onto Δ. Because Δ_ν is the lower convex hull of finitely many points in $\mathbb{R}^2 \times \mathbb{R}$ it decompses into faces of dimension $0, 1, 2$. Under the projection $\pi : \Delta_\nu \to \Delta$ the *v-vertices* P of Δ are the images of the 0-faces, the *v-edges* L are the images of the 1-faces, the *v-faces* F are the images of the 2-faces. These define a polygonal partition of Δ.

Definition 1.2. For each edge L and face F there is an associated integer δ_λ (with $\lambda = L$ or F), the *denominator*, defined as smallest positive m such that $\nu(P) \in \frac{1}{m}\mathbb{Z}$ for each $P \in \lambda \cap \mathbb{Z}^2$.

Remark. We now consider how to restrict a polynomial with respect to the ν-partition to form a Laurent polynomial supported on the faces and vertices. First, following the Notation of [Tim, Models] we define how to restrict the polynomial to some subset of a lattice.

Definition 1.3 (Restriction)**.** Let $S \subset \mathbb{Z}^n$ be a nonempty subset of a lattice and take Λ to be the smallest affine lattice $S \subset \Lambda \subset \mathbb{Z}^n$ containing S. Let Λ have rank r and choose an isomorphism $\phi : \mathbb{Z}^r \to \Lambda$. Then for a Laurent polynomial $g \in K[\mathbf{x}^{\pm 1}]$ we define the restriction $g|_S \in K[\mathbf{y}^{\pm 1}]$,

$$g|_S = \sum_{\mathbf{i} \in \phi^{-1}(S)} c_{\phi(\mathbf{i})} \mathbf{y}^{\mathbf{i}} \in K[\mathbf{y}^{\pm 1}] \quad \text{for} \quad g = \sum_{\mathbf{i} \in \mathbb{Z}^n} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$$

Note that different choices of an isomorphism $\Lambda \to \mathbb{Z}^r$ are related by an automorphism in $\operatorname{GL}_r(\mathbb{Z})$ acting on the variables $\mathbf{y}$.

Remark. The notational complexity of the above definition derives from making the polynomial $g|_S$ an element of a standard Laurent polynomial ring $k[\mathbf{y}^{\pm 1}]$. We can simplify the above notation using our previous abstract notation used for the toric constructions. Given a lattice M and a Laurent polynomial $g \in K[M]$ and a subset $S \subset M$ we define the restriction,

$$g|_S = \sum_{m \in S} c_m \chi^m \in K[\langle S \rangle] \quad \text{where} \quad g = \sum_{m \in M} c_m \chi^m$$

where $\langle S \rangle$ is the sublattice of M generated by S. The above definition is recovered choosing some isomorphism $\phi : \langle S \rangle \to \mathbb{Z}^r$ giving an isomorphism $K[\langle S \rangle] \cong K[\mathbf{y}]$.

Definition 1.4 (Reduction). For a Laurent polynomial $h \in K[x^{\pm 1}, y^{\pm 1}]$, there exist integers, $c, m, n \in \mathbb{Z}$ such that $\tilde{h}(x, y) = \varpi^c h(\varpi^m x, \varpi^n y)$ has coefficients in R and $\tilde{h} \mod \varpi \in \kappa[x, y]$ has the same Newton polygon as h. Then we say that $\overline{h} = \tilde{h} \mod \varpi$ is *reduction* of h.

Example 1.5. (DO THIS!!!!!!!)

Definition 1.6. In paticular for λ an edge L or face F we define the restriction $f|_\lambda = f|_S$ for the set, $S = \{P \in \lambda \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}$. Note, S contains the vertices of L or F.

Remark. Reduction gives, for each edge L and face F, polynomials $\overline{f|_L} \in \kappa[t]$ and $\overline{f|_F} \in \kappa[x, y]$. This gives affine curves over κ on each edge and face which we complete in a toric compactification as follows.

Definition 1.7 (Components). We define the following schemes over κ:

(a). $X_L = V(\overline{f_L}) \subset \mathbb{G}_{m,\kappa}$

(b). $X_F = V(\overline{f_F}) \subset \mathbb{G}_{m,\kappa}^2$

(c). $\overline{X}_F = X_F^\Delta$ is the completion of X_F with respect to its Newton polygon F i.e. the closure of the immersion $X_F \hookrightarrow \mathbb{G}_{m,\kappa}^2 \hookrightarrow \mathbb{T}_F$. By Theorem **??**, $\overline{X}_F$ is connected and, in fact, the Theorem applies for any finite extension $\kappa' \supset \kappa$ showing that $\overline{X}_F$ is geometrically connected.

Example 1.8. (DO EXAMPLE!!!!!!)

Definition 1.9. We say that $f \in k[x^{\pm 1}, y^{\pm 1}]$ is *strictly Δ_ν-regular* if all X_F and X_L are smooth over κ.

Remark. The condition that all X_L are smooth implies that f is nondegenerate with respect to its Newton polgon since it implies that f restricted to each edge is smooth.

Definition 1.10. A Laurent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$ is *Δ_ν-regular* if each X_F is smooth and for the interior edges L and edges $L \subset \partial F$ with $\delta_L \neq \delta_F$ we require X_L is smooth and otherwise we require $\overline{X}_F$ is *outer-regular* i.e. smooth at the points corresponding to L via,

$$\overline{X}_F(\bar{\kappa}) \setminus X_K(\bar{\kappa}) \longleftrightarrow \coprod_{L \supset \partial F} X_L(\bar{\kappa})$$

Remark. As with toric regularity, we have defined the notion of Δ_ν-regular with respect to a given Lauret polynomial i.e. to a given affine model C_0 of a curve. As before, we extend this definition to an arbitrary curve in the obvious way.

Definition 1.11. A curve C over K is (strictly) Δ_ν-regular if C is birationally equivalent to some affine $U_f \subset \mathbb{G}_{m,K}^2$ for some (strictly) Δ_ν-regular Laruent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$.

Remark. We need one more notion in order to describe the model of C which is a combinatorial connectivity between two adjacent faces F_1, F_2 sharing an edge L.

Definition 1.12 (Slopes)**.** Edges are either *inner/interior* meaning they form the boundary between two ν-faces F_1 and F_2 or *outer/exterior* on the boundary of Δ. For an edge L there is a unique affine function $L_{(F_1)}^* : \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ with $L_{(F_1)}^*|_L = 0$ and $L_{(F_1)}^*|_{F_1} \geq 0$. Then the edge has two corresponding integers called the *slopes* Defined as follows. Choose $P_0, P_1 \in \mathbb{Z}^2$ with $L_{(F_1)}^*(P_0) = 0$ and $L_{(F_1)}^*(P_1) = 1$. Then,

$$s_1^L = \delta_L(\nu_1(P_1) - \nu_1(P_0)) \qquad s_2^L = \begin{cases} \delta_L(\nu_2(P_1) - \nu_2(P_0)) & L \text{ inner} \\ \lfloor s_1^L - 1 \rfloor & L \text{ outer} \end{cases}$$

where $\nu_i : \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ is the unique affine function which agrees with ν on F_i (recall that the faces are defined such that ν is affine when restricted to each face. Given the slopes, we may consider a sequence of rational numbers $\frac{m_i}{d_i} \in \mathbb{Q}$ such that,

$$s_1^L = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \frac{m_2}{d_2} > \cdots > \frac{m_r}{d_r} > \frac{m_{r+1}}{d_{r+1}} \quad \text{and} \quad \begin{vmatrix} m_i & m_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = m_i d_{i+1} - m_{i+1} d_i = 1$$

Then $r(L)$, the minimal length of this sequence, and the denominators d_i of this minimal sequence, are important combinatorial parameter of the edge L. It turns out such a minimal sequence is unique.

Remark. The existence of such a sequence needs some consideration. Take all rational numbers in $[s_1^L, s_2^L] \cap \mathbb{Q}$ with denominators bounded by the largest denominator of s_1^L and s_2^L. This is a shifted Farey series. We define the Farey series F^n to be the ordered sequence of rational numbers with denominator less than or equal to n put in lowest terms. Then, if $\frac{a}{b} < \frac{c}{d}$ are consecutive terms in the Farey series then $\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}$. Therefore,

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc = 1$$

[Tim, Remark 3.15] [HW, Ch. III, Thm. 28, Thm. 29] Furthermore, if the sequence in $[s_1^L, s_2^L] \cap \mathbb{Q}$ with bounded denominators contains consecutive terms,

$$\frac{a}{b} > \frac{a+c}{b+d} > \frac{b}{d}$$

then we must have,

$$a(b+d) - b(a+c) = ab + ad - ab - bc = ad - bc = 1$$

meaning that $\frac{a}{b} > \frac{b}{d}$ have the required adjacency property and thus $\frac{a+c}{b+d}$ may be removed from the sequence. We will reinterpret this as a blowdown of regular normal crossings models after we state the main theorems describing the structure of such models from the above combinatorial data.

1.2 Main Theorems

We describe the properties of the model $\mathcal{C}_\Delta$ over R associated to some polygonal parition of Δ defined by a Laurent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$.

Theorem 1.13 (DK, Thm. 3.13)**.** Let C be a smooth projective Δ_ν-regular curve birational to U_f for a Δ_ν-regular Laurent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$. Then $\mathcal{C}_\Delta / R$ is a regular normal crossing model of C and the special fiber $\mathcal{C}_\kappa$ geometrically decomposes into components as follows:

(a). each ν-face F of Δ gives a smooth complete curve $\overline{X}_F \times_\kappa \kappa^{\text{sep}}$ (CORRECT?) over κ^{sep} with multiplicity δ_F and genus $g_F = |\{P \in F^\circ \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}|$

(b). each ν-edge L with sequence $\frac{m_i}{d_i} \in \mathbb{Q}$ $(0 \le i \le r+1)$ gives $|X_L(\kappa^{\text{sep}})|$ chains of length r of closed subschemes intersecting transversally each isomorphic to $\mathbb{P}^1_{\kappa^{\text{sep}}}$ with multiplicities in $\mathcal{C}_\kappa$ given by $\delta_L d_1, \ldots, \delta_L d_r$.

Furthermore, the $\text{Gal}\,(\kappa^{\text{sep}}/\kappa)$-action on $\mathcal{C}_\kappa \times_\kappa \kappa^{\text{sep}}$ is given by acting on each component $X_F \times_\kappa \kappa^{\text{sep}}$ and permuting the $\mathbb{P}^1_{\kappa^{\text{sep}}}$ chains via the natural action of $\text{Gal}\,(\kappa^{\text{sep}}/\kappa)$ on $|X_L(\kappa^{\text{sep}})|$.

QUESTION 1.14. In the paper, the decomposition is certianally geometric but he says the scheme is $\overline{X}_F$ rather than base changing over the algebraic closure. How should I write it? Also, is it the case that $\overline{X}_F \times_\kappa \kappa^{\text{sep}}$ is smooth over κ^{sep}? This is just the invariance of smoothness under base change right?

Remark. The genus of $\overline{X}_F$ is exactly the number of lattice points interior to the Newton polygon defining $X_F \subset \mathbb{G}^2_{m,\kappa}$ by Baker's theorem. Recall this Newton polygon is the the restriction of f to the set $S = \{P \in F \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}$ so the lattice generated by S only has lattice points where at points of $\mathbb{Z}^2$ where $\nu(P) \in \mathbb{Z}$ explaining the genus formula above.

Remark. (ASK JOHAN ABOUT THIS???) What about $f = \varpi x^2 + y^2 + 1$ which has $\nu(i,j) = \frac{1}{2}i$ and thus the single interior point P has $\nu(P) = \frac{1}{2}$. However, $g = \varpi^2 x^2 + y^2 + 1$ has $\nu(P) = 1$.

(GIVE THIS AS EXAMPLE)

Theorem 1.15 (DK, Thm. 3.14)**.** Let $f \in K[x^{\pm 1}, y^{\pm 1}]$ be any Laurent polynomial defining a 1-dimensional scheme $C_0 = U_f \subset \mathbb{G}^2_{m,K}$. Then $\mathcal{C}_\Delta / R$ is a proper flat model of the toric completion $C = C_0^\Delta$ with respect to the Newton polygon $\Delta = \Delta(f)$. The special fiber $\mathcal{C}_\kappa$ is a union of closed subschemes $\overline{X}_F$ indexed by ν-faces F and chains $X_L \times_\kappa \Gamma_L$ where Γ_L is a union of $\mathbb{P}^1_\kappa$ intersecting transversally as follows:

(a). for each ν-edge F: the scheme $\overline{X}_F$ has multiplicity δ_F and, via Theorem **??** are geometrically connected and have arithmetic genus $g_F = |\{P \in F^\circ \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}|$.

(b). for each ν-edge L choose a sequence $\frac{m_i}{d_i} \in \mathbb{Q}$ $(0 \le i \le r+1)$ then let $\Gamma_L = \Gamma^1_L \cup \cdots \cup \Gamma^r_L$ with each Γ^i_L isomorphic to $\mathbb{P}^1_k$ embedded with multiplicity $\delta_L d_i$ and meeting transversally where we identify $0 \in \Gamma^i_L$ with $\infty \in \Gamma^{i+1}_L$. If $r = 0$ then let $\Gamma_L = \text{Spec}\,(k)$.

(c). The subscheme $X_L \times \{0\} \subset X_L \times \Gamma^1_L$ is identified with $\overline{X}_{F_1} \setminus X_{F_1}$ for the ν-face F_1 boardering L and, when L is inner, likewise $X_L \times \{\infty\} \subset X_L \times \Gamma^r_L$ is identified with $\overline{X}_{F_2} \setminus X_{F_2}$ for the other ν-face F_2 boardering L. These intersections are transversal and, in fact, as a scheme the intersection is $V(\overline{f_L}^{\delta_L}) \subset X_L \subset \overline{X}_F$.

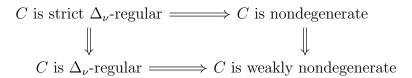Furthermore, the model $\mathcal{C}_\Delta$ is geometricall regular ar,

(a). the smooth locus of X_F, for each ν-face L

(b). the smooth locus of $X_L \times \Gamma_L$, for each ν-edge L

(c). the smooth points of $\overline{X}_F \setminus X_F$ corresponding to L when $L \subset \partial F$ is an outer edge with $\delta_L = \delta_F$ and $r = 0$.

Furthermore, if C_0 is Δ_ν-regular then $C = C_0^\Delta$ is smooth and thus the unique smooth proper curve birational to C_0 and $\mathcal{C}_\Delta/R$ is a regular normal corossings model of C.

2   Relationships Between Toric Notions of Regularity

We have discussed a number of regularity conditions on curves originating from their compatiblity in some sense with a certain set of toric embeddings. The utility of these conditions is the ability to verify them from the equation defining some affine model of the curve in $\mathbb{G}_{m,k}^2$. Although these notions are clearly related, we here show that they are, indeed, inequivalent. In this situation, we have a discrete valued field K with valuation ring R and residue field κ. On the special fiber, we will distinguish between the arithmetic (κ non-algebraically closed) and geometric (κ arithmetically closed) situations. The main result of this section is as follows.

Proposition 2.1. Let C be a smooth curve over K. Then we have the following implications,

$$C \text{ is strict } \Delta_\nu\text{-regular} \implies C \text{ is nondegenerate}$$
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$
$$C \text{ is } \Delta_\nu\text{-regular} \implies C \text{ is weakly nondegenerate}$$

Furthermore, no implication is reversible.

The fact that Δ-nondegeneracy implies weak Δ-nondegeneracy is simply an application of Baker's theorem (recall that this notion was created, by design, as a weaker form of Δ-nondegeneracy, hence the name). Likewise, strict Δ_ν-regularity implying Δ_ν-regularity is also a consequence of Baker's theorem since the outer-regular condition introduced in the definition of Δ_ν-regularity is satisfied when each X_F is smooth and X_L is smooth since these imply that $\overline{X}_F$ is smooth as well via Baker's theorem.

That Δ_ν-regularity implies weak nondegeneracy follows from the main theorem

3   Stuff

It does not suffice to take *any* affine open as the following example shows, we must indeed take a sufficiently small open so the notion of birationality here is actually necessary.

Proposition 3.1. There exists a smooth affine curve C over k with no closed immersion $C \hookrightarrow \mathbb{G}_{m,k}^2$.

Proof. We use the obstruction that any curve embedded in $\mathbb{G}_{m,k}^2$ must have a trivial canonical bundle (Lemma 3.2). Therefore, it sufficees to produce a smooth affine curve with a nontrivial canonical bundle. The affiness is easy to arrange since for any smooth complete curve $\overline{C}$ then removing a single point leaves an affine curve (Lemma 3.3). Setting $C = \overline{C} \setminus \{P\}$, the inclusion $j : C \to \overline{C}$ induces an exact sequence,

$$\mathbb{Z} \longrightarrow \text{Cl}\left(\overline{C}\right) \overset{j^*}{\longrightarrow} \text{Cl}\left(C\right) \longrightarrow 0$$

where the first map is $1 \mapsto [P]$. Therefore, a divisor class D is sent to zero under $f^*D \sim 0$ iff $D \sim \deg D \cdot [P]$. We need to show that the canonical divisor does not vanish $K_C \not\sim 0$ and thus that $K_{\overline{C}} \not\sim (2g-2) \cdot [P]$ since $j : C \hookrightarrow \overline{C}$ is étale so $\Omega_{C/k} = f^*\Omega_{\overline{C}/k}$. Therefore, it suffices to produce a curve $\overline{C}$ with a point $P \in \overline{C}$ such that $K_{\overline{C}} \not\sim (2g-2) \cdot [P]$. Note that because the $(2g-2)$-torsion in the Picard group for $g \geq 2$ is a finite group, all but finitely many choices for P on any smooth complete curve of genus $g \geq 2$ will work.

Specifically, take $\overline{C} = \text{Proj}\left(k[X,Y,Z]/(X^4 - X^2Z^2 + (Y-Z)^4 - Z^4)\right)$ which is easily verified to be smooth in characteristic $p \neq 2, 5$ and has genus $g = 3$ since it is a plane curve of degree 4 in $X = \mathbb{P}^2_k$. Then choose $P = [0 : 0 : 1]$. Notice that under $\iota : \overline{C} \hookrightarrow \mathbb{P}^2_k$ we have $\Omega_{\overline{C}/k} \cong \iota^*\mathcal{O}_X(1)$ by the adjunction formula. We need to check that $(2g-2) \cdot [P]$ is not one of the effective divisors linearly equivalent to $K_{\overline{C}}$. Such divisors are parametrized by sections $H^0(\overline{C}, \Omega_{\overline{C}/k}) = H^0(X, \iota_*\iota^*\mathcal{O}_X(1))$. By the projection formula $\iota_*\iota^*\mathcal{O}_X(1) = \iota_*\mathcal{O}_{\overline{C}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)$. To compute the sections of this coherent $\mathcal{O}_X$-module we apply the exact sequence of the Cartier divisor $\overline{C}$ twisted by $\mathcal{O}_X(1)$,

$$0 \longrightarrow \mathcal{O}_X(-3) \longrightarrow \mathcal{O}_X(1) \longrightarrow \iota_*\mathcal{O}_{\overline{C}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \longrightarrow 0$$

and applying cohomology,

$$H^0(X, \mathcal{O}_X(-3)) \longrightarrow H^0(X, \mathcal{O}_X(1)) \longrightarrow H^0(\overline{C}, \iota^*\mathcal{O}_X(1)) \longrightarrow H^1(X, \mathcal{O}_X(-3))$$

but $H^q(X, \mathcal{O}_X(-3)) = 0$ for $q \leq 1$ so the map $H^0(X, \mathcal{O}_X(1)) \twoheadrightarrow H^0(\overline{C}, \iota^*\mathcal{O}_X(1))$ given by pulling back sections, $s \mapsto \iota^*s$, is bijective. In particular, since any section $s \in H^0(\overline{C}, \Omega_{\overline{C}/k})$ is the pullback of some hyperplane equation $h \in H^0(X, \mathcal{O}_X(1))$, the divisor of zeros associated to s is the hyperplane section $\iota^{-1}(H) = \overline{C} \cap H$ with the hyperplane $H = V(h)$. However, I claim that any line passing through P intersects $\overline{C}$ in more than one point. To see this, consider the tangent line L to $\overline{C}$ at P is $\mathbb{P}^1_k \to X$ given by $[T_0 : T_1] \mapsto [T_0 : 0 : T_1]$ but $L \cap \overline{C} = \{[0:0:1], [1:0:1], [-1:0:1]\}$. Therefore, there cannot be any line passing through only P meaning that $\{P\}$ cannot be the support of any effective divisor in canonical linear system $H^0(\overline{C}, \Omega_{\overline{C}/k})$. Thus, $K_{\overline{C}} \not\sim (2g-2) \cdot [P]$ so $C = \overline{C} \setminus \{P\}$ has nontrivial canonical bundle yet is affine providing the required example. $\square$

Remark. Notice that although C is not an affine plane curve (in the sense of having a closed immersion $C \hookrightarrow D(q) \subset \mathbb{A}^2_k$ to some principal affine open) there is an immersion $C \hookrightarrow \mathbb{P}^2_k$ since $\overline{C} \hookrightarrow \mathbb{P}^2_k$ is a complete plane curve.

We conclude by providing proofs of the required lemmas.

Lemma 3.2. Let $C \hookrightarrow D(q) \subset \mathbb{A}^2_k$ be a smooth curve embedded in some standard open $D(q)$ in the affine plane. Then the canonical bundle $\Omega^1_{C/k} \cong \mathcal{O}_C$ is trivial.

Proof. Let $A = k[x, y, q^{-1}]$ so $D(q) = \text{Spec}\,(A)$. Note that $C = \text{Spec}\,(R)$ with $R = A/I$ where $I = \ker\left(A \to \Gamma(C, \mathcal{O}_C)\right)$. Furthermore, $I = (f)$ since $\dim C = 1$ thus $\mathbf{ht}\,(I) = 1$ but C is irreducible and thus I is prime and since A is a UFD, $I = (f)$ since each height one prime is principal. Furthermore, C is smooth so $(f, f_x, f_y) = A$ where f_x, f_y are the partial derivatives of f with respect to x and y. Therefore, we can choose $g, h \in A$ such that $gf_x + hf_y = 1$ in R. Now, consider the R-module of differentials, $\Omega_{R/k} = (R\mathrm{d}x \oplus R\mathrm{d}y)/(f_x\mathrm{d}x + f_y\mathrm{d}y)$.

Consider the map $\phi : R \to \Omega_{R/l}$ sending $1 \mapsto h\mathrm{d}x - g\mathrm{d}y$. Note that,

$$\mathrm{d}x = gf_x\mathrm{d}x + hf_y\mathrm{d}x = hf_y\mathrm{d}x - gf_y\mathrm{d}y \implies f_y \mapsto \mathrm{d}x$$

$$dy = g f_x \mathrm{d}y + h f_y \mathrm{d}y = g f_x \mathrm{d}y - h f_x \mathrm{d}x \implies -f_x \mapsto \mathrm{d}y$$

so ϕ is surjective. Furthermore, suppose that $\phi(a) = 0$ then $\phi(f_x a) = \phi(f_y a) = 0$ so in $R\mathrm{d}x \oplus R\mathrm{d}y$ we have $a\mathrm{d}x, a\mathrm{d}y \in (f_x \mathrm{d}x + f_y \mathrm{d}y)$ meaning $a\mathrm{d}x = c_1(f_x \mathrm{d}x + f_y \mathrm{d}y)$ and $a\mathrm{d}y = c_2(f_x \mathrm{d}x + f_y \mathrm{d}y)$ giving $c_1 f_y = 0$ and $c_2 f_x = 0$ and $c_1 f_x = c_2 f_y = a$ since $R\mathrm{d}x \oplus R\mathrm{d}y$ is free. But then

$$a = g f_x a + h f_y a = g f_x c_2 f_y + h f_y c_1 f_x = 0$$

since $c_2 f_x = c_1 f_y = 0$ so ϕ is injective. Thus $\Omega_{R/k} \cong R$ and sheafifying gives, $\Omega_{C/k} \cong \mathcal{O}_C$. □

Lemma 3.3. Let C be a smooth proper curve and $P \in C$ a point. Then $C \setminus \{P\}$ is affine.

Proof. The divisor $D = \nu[P]$ is very ample for sufficiently large ν (in fact for $\nu \geq 2g+1$) [Har, IV.3.2]. Therefore, the linear system $|\nu[P]|$ defines a closed (C is proper) immersion $j : C \hookrightarrow \mathbb{P}_k^{\nu-1}$ such that $\mathcal{O}_C(D) = j^* \mathcal{O}_{\mathbb{P}_k^{\nu-1}}(1)$ with the hyperplane sections pulling back to a basis of $H^0(C, \mathcal{O}_C(D))$. Since D is effective, there is some section $s \in H^0(C, \mathcal{O}_C(D))$ with $V(s) = \{P\}$ and thus some hyperplane section $h \in H^0(\mathbb{P}_k^{\nu-1}, \mathcal{O}_{\mathbb{P}_k^{\nu-1}}(1))$ with $s = j^* h$ and thus $\{P\} = j^{-1}(H \cap j(C))$ where $H = V(h) \subset \mathbb{P}_k^{\nu-1}$. Finally, $C \setminus \{P\} = j^{-1}(\mathbb{P}_k^{\nu-1} \setminus H)$ which is affine since j is a closed immersion and thus affine and the complement of a hyperplane in projective space is a standard open. □

4 Example

In order to extend our argument to heigher genus curves, we need to apply Riemann-Roch. However, since this example only works in the arithmetic setting we require a slight modification to the standard statement of Riemann-Roch found in Hartshorne. To ensure there is no confusion, we will provide a proof here.

Theorem 4.1 (Riemann-Roch). Let X be a smooth proper curve over k with $g = \dim_k H^0(X, \omega_X)$ its genus. Then for any line bundle $\mathcal{L}$,

$$\chi(\mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g$$

Where $\deg \mathcal{L}$ is defined in the arithmetic case as follows. Choose a nonzero $s \in H^0(X, \mathcal{L})$ and a local trivialization $\{(U_i, s_i)\}$ with $s_i \in \mathcal{L}(U_i)$ such that $\mathcal{O}_{U_i} \xrightarrow{s_i} \mathcal{L}|_{U_i}$ is an isomorphism. Then define,

$$\deg \mathcal{L} = \sum_{P \in X} [\kappa(P) : k] \operatorname{ord}_P(s/s_i)$$

for some i with $P \in U_i$.

Proof. First, note that by Serre duality, $H^1(X, \mathcal{L}) \cong H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee)^\vee$ so,

$$\chi(\mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee)$$

Since X is smooth, every line bundle $\mathcal{L}$ is $\mathcal{O}_X(D)$ for some divisor D. However, □