1 Basic Definitions and Examples

1.1 Genera

1.2 Riemann-Roch

1.3 Riemann–Hurwitz

2 Hyperelliptic Curves

Definition 2.0.1. A curve C is *hyperelliptic* if there exists a degree two map $f : C \to \mathbb{P}^1$.

Lemma 2.0.2. A curve C is hyperelliptic iff Ω^1_C is not very ample.

Proof. (DO THIS) $\square$

Proposition 2.0.3. Plane curves with $g > 1$ cannot be hyperelliptic.

Proof. Let $\iota : C \hookrightarrow \mathbb{P}^2$ be a plane curve. Then $\Omega^1_C = \iota^* \mathcal{O}_{\mathbb{P}^2}(d-3)$ where d is the degree of C. Since $g > 1$ we must have $d > 3$ and thus $\mathcal{O}_{\mathbb{P}^2}(d-3)$ is very ample defining the Veronese embedding $v : \mathbb{P}^2 \to \mathbb{P}^N$ s.t. $\mathcal{O}_{\mathbb{P}^2}(d-3) = v^* \mathcal{O}_{\mathbb{P}^N}(1)$. Then $v \circ \iota : C \to \mathbb{P}^N$ is an embedding such that $(v \circ \iota)^* \mathcal{O}_{\mathbb{P}^N}(1) = \Omega^1_C$. Thus Ω^1_C is very ample so C cannot be hyperelliptic. $\square$

Lemma 2.0.4. Let C have a $\mathfrak{g}^1_2$ then C is either hyperelliptic or rational.

Proof. Let D be a $\mathfrak{g}^1_2$ then $|D|$ defines a rational map $C \dashrightarrow \mathbb{P}^1$ of degree two. Suppose P were a basepoint of $|D|$ then $\dim |D - P| = 1$ which implies that C is rational because there is a rational degree one map $C \dashrightarrow \mathbb{P}^1$. $\square$

Proposition 2.0.5. Any genus 2 curve is hyperelliptic.

Proof. Consider the canonical divisor K_X which has $\deg K_X = 2g - 2 = 2$ and $\dim |K_X| = g - 1 = 1$ and thus gives a $\mathfrak{g}^1_2$. $\square$

3 Tangent Space

Definition 3.0.1. Let X be a scheme and $x \in X$ a point. Then we define:

(a) the geometric tangent space $T_x X = \text{Spec}\left(\text{Sym}_{\kappa(x)}\left(\mathfrak{m}_x / \mathfrak{m}_x^2\right)\right)$

(b) the projectiveized tangent space $\mathbb{P}(T_x X) = \text{Proj}\left(\text{Sym}_{\kappa(x)}\left(\mathfrak{m}_x / \mathfrak{m}_x^2\right)\right)$

(c) the geometric tangent cone $C_x X = \text{Spec}\left(\mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)\right)$ where,

$$\mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right) = \bigoplus_{n=0}^{\infty} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}$$

(d) the projectiveized tangent cone $\mathbb{P}(C_x X) = \text{Proj}\left(\mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)\right)$.

Remark. In particular, blowing up X at the sheaf of ideals $\mathscr{I}_x$ (defined as the subsheaf of $\mathcal{O}_X$ where evaluation in $\kappa(x)$ gives zero) gives the following,

$$\tilde{X} = \mathbf{Proj}_X \left(\bigoplus_{n=0}^{\infty} \mathscr{I}_x^n \right)$$

Choose an affine open neighborhood $x \in \mathrm{Spec}\,(A) = U \subset X$ then we see $\mathscr{I}_x|_U = \widetilde{\mathfrak{p}} \subset A$ is the prime corresponding to $x \in \mathrm{Spec}\,(A)$ and $\mathfrak{m}_x = \mathfrak{p} A_{\mathfrak{p}}$. Therefore, restricting $\pi : \tilde{X} \to X$ over U gives,

$$\mathrm{Proj} \left(\bigoplus_{n=0}^{\infty} \mathfrak{p}^n \right) \to \mathrm{Spec}\,(A)$$

and,

$$\mathrm{Bl}_{\mathfrak{p}}\,(A) = \bigoplus_{n=0}^{\infty} \mathfrak{p}^n$$

is the blowup algebra which is a graded A-algebra. Consider the fiber over x,

$$\mathrm{Proj}\,(\mathrm{Bl}_{\mathfrak{p}}\,(A)\,/\mathfrak{p}\mathrm{Bl}_{\mathfrak{p}}\,(A)) \to \mathrm{Spec}\,(\kappa(x))$$

where we see,

$$\mathrm{Bl}_{\mathfrak{p}}\,(A)\,/\mathfrak{p}\mathrm{Bl}_{\mathfrak{p}}\,(A) = \bigoplus_{n=0}^{\infty} \mathfrak{p}^n = \mathbf{gr}_{\mathfrak{p}}\,(A)$$

and therefore $\tilde{X}_x \to \mathrm{Spec}\,(\kappa(x))$ is $\mathrm{Proj}\,\big(\mathbf{gr}_{\mathfrak{p}}\,(A)\big) \to \mathrm{Spec}\,(\kappa(x))$. In particular, the tangent cone is the fiber over x in the blowup.

Remark. The exact same construction shows that given a ring A and ideal $I \subset A$ the blowup $\mathrm{Proj}\,(\mathrm{Bl}_I\,(A)) \to \mathrm{Spec}\,(A)$ where,

$$\mathrm{Bl}_I\,(A) = \bigoplus_{n=0}^{\infty} I^n$$

is the blowup algebra, has fiber over the closed subscheme $V(I)$ equal to,

$$\mathrm{Proj}\,(\mathbf{gr}_I\,(A)) \to \mathrm{Spec}\,(A/I)$$

which is the projectized tangent cone of I.

Remark. We can generalize this further. For a sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ we can form the blowup,

$$\tilde{X} = \mathbf{Proj}_X \left(\bigoplus_{n=0}^{\infty} \mathscr{I}^n \right) \to X$$

Restricting to the closed subscheme $Z = V(\mathscr{I}) \subset X$ we find,

$$\mathbf{Proj}_Z \left(\bigoplus_{n=0}^{\infty} \mathscr{I}^n / \mathscr{I}^{n+1} \right) \to Z$$

but notice that the graded algebra,

$$(\mathcal{O}_X/\mathscr{I}) \otimes_{\mathcal{O}_X} \bigoplus_{n=0}^{\infty} \mathscr{I}^n = \bigoplus_{n=0}^{\infty} \mathscr{I}^n/\mathscr{I}^{n+1} = \bigoplus_{n=0} (\mathscr{I}/\mathscr{I}^2)^{\otimes n}/K = \mathrm{Sym}_{\mathcal{O}_Z}\left(\mathscr{I}/\mathscr{I}^2\right)$$

and $\mathcal{C}_{Z/X} = \mathscr{I}/\mathscr{I}^2$ is the conormal bundle (sheaf) so we find a pullback diagram,

$$\begin{array}{ccc} \mathbb{P}(\mathcal{C}_{Z/X}) & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \longrightarrow & X \end{array}$$

and thus $\tilde{X} \to X$ is a projective bundle over Z and an isomorphism over $X \setminus Z$. We call $\mathbb{P}(\mathcal{C}_{Z/X})$ the projectiveized tangent cone of Z.

Proposition 3.0.2. If $x \in X$ is a regular point then $C_x X = T_x X$.

Proof. When $\mathcal{O}_{X,x}$ is regular then $\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \cong \kappa(x)[x_1, \ldots, x_r]$ where $x_1, \ldots, x_r \in \mathfrak{m}_x$ are a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $\kappa(x)$-vectorspace. Therefore, $\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) = \mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)$ as graded rings and thus $C_x X = T_x X$ as well as the projective versons. $\square$

Proposition 3.0.3. Let X be finite type over k and $x \in X$ be a closed point. There is a canonical map,

$$\widehat{T_x X} \leftarrow \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right) \to \mathrm{Spec}\left(\mathcal{O}_{X,x}\right) \to X$$

which is an isomorphism exactly when $x \in X$ is regular.

Proof. By the Cohen structure theorem $\widehat{\mathcal{O}_{X,x}} = k[[x_1, \ldots, x_r]]/I$ where $x_1, \ldots, x_r \in \mathfrak{m}_x$ are a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$ and $I = (0)$ exactly when $x \in X$ is regular proving that the canonical map $T_x X \to \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right)$ is $\square$

4 Formal Schemes

Definition 4.0.1. Let A be a ring and $I \subset A$ an ideal. Then the completion of A along I is,

$$\hat{A} = \varprojlim_n A/I^n$$

Furthermore, for any A-module M we can complete M along I to get,

$$\hat{M} = \varprojlim_n M/I^n M = \varprojlim_n (M \otimes_A A/I^n) = M \otimes_A \hat{A}$$

Proposition 4.0.2. Let A be a ring and $I \subset A$ an ideal and M an A-module. Then $\hat{M}$ satisfies the following universal property. Any map $\varphi : M \to N$ to a complete A-module N facrors uniquely as $M \to \hat{M} \xrightarrow{\hat{\varphi}} N$.

Proof. The kernel of $M \to N/I^n N$ contains $I^n M$ and thus factors as $M \to M/I^n M \to N/I^n N$. Taking inverse limits gives $M \to \hat{M} \to N$. Uniqueness follows from the fact that a map $\hat{M} \to M$ is determined completely by $\hat{M} \to M/I^n M \to N/I^n N$. $\square$

Lemma 4.0.3. Let A be a ring and $I \subset A$ an ideal. Then the units of $\hat{A}$ are exactly those elements which map to units under $\hat{A} \to A/I$.

Proof. Suppose that $u \in \hat{A}$ is a unit. Then clearly its image under $\hat{A} \to A/I$ is a unit. Conversely, suppose that $u \mapsto u_1 \in A/I$ is a unit. Then there exists $v_1 \in A/I$ s.t. $u_1 v_1 = 1$ so lifting v_1 we get $u_2 \tilde{v}_1 = 1 + r$ for $r \in I$ so we can take $r u_2 \tilde{v}_1 = r + r^2 = r$ and thus $u_2(\tilde{v}_1 - r\tilde{v}_1) = 1$. Write $v_2 = \tilde{v}_1 - r\tilde{v}_1$ and we lift to see $u_3 \tilde{v}_2 = 1 + r'$ for $r' \in I^2$ etc giving by induction an element $v \in \hat{A}$ such that $uv = 1$ in each A/I^n and thus in $\hat{A}$. $\square$

Lemma 4.0.4. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then $\hat{A} = \widehat{A_\mathfrak{m}}$ is local.

Proof. Consider,

$$\widehat{A_\mathfrak{m}} = \varprojlim_n (A_\mathfrak{m}/\mathfrak{m}^n A_\mathfrak{m}) = \varprojlim_n (A/\mathfrak{m}^n)_\mathfrak{m}$$

However, since $A/\mathfrak{m}^n$ is local with maximal ideal $\mathfrak{m}$ we see that $(A/\mathfrak{m}^n)_\mathfrak{m} = A/\mathfrak{m}^n$ and thus,

$$\widehat{A_\mathfrak{m}} = \varprojlim_n (A/\mathfrak{m}^n)_\mathfrak{m} = \varprojlim_n A/\mathfrak{m}^n = \hat{A}$$

$\square$

Remark. Localization does not, in general, behave nicely with completion. For example, let $A = \mathbb{Z}_p[x]$ and $\mathfrak{p} = (x)$. Then $\hat{A}_\mathfrak{p} = \widehat{\mathbb{Q}_p[x]_{(x)}} = \mathbb{Q}_p[[x]]$. However, $\hat{A} = \mathbb{Z}_p[[x]]$ and $\hat{A}_{\hat{\mathfrak{p}}} = \mathbb{Z}_p[[x]]_{(x)}$ which is a proper subring of $\mathbb{Q}_p[[x]]$ because it does not contain $1 + p^{-1}x + p^{-2}x^2 + \cdots$ and is this, in particular, not complete.

Lemma 4.0.5. Suppose that $(A, \mathfrak{m})$ is a local ring. Then $\mathrm{Spec}\left(\hat{A}\right) \to \mathrm{Spec}\,(A)$ is a homeomorphism.

(THIS IS TOTALLY FALSE IMPLIES A IS UNABRANCH AT LEAST)

Proof. The units in $\hat{A}$ are everything except the preimage of zero under $\hat{A} \to A/\mathfrak{m} = \kappa$. Therefore $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$ is the unique maximal ideal of $\hat{A}$ making A local. I claim that $\mathfrak{p} \mapsto \mathfrak{p}\hat{A}$ is an inverse to $\mathrm{Spec}\left(\hat{A}\right) \to \mathrm{Spec}\,(A)$. (DO THIS!!) $\square$

Example 4.0.6. Consider $X = \mathrm{Spec}\,(k[x,y]/(y^2 - x^2(x-1))) \subset \mathbb{A}^2_k$. Take $p = (x, y)$. We know X is connected and thus $\mathcal{O}_{X,p} = A_\mathfrak{p}$ has a unique minimal prime. However,

$$\widehat{\mathcal{O}_{X,p}} = \hat{A} = k[[x,y]]/(y^2 - x^2(x+1)) \cong k[[x,y]]/(x^2 - y^2) = k[[x,y]]/((y-x)(x+y)) \cong k[[u]] \times k[[v]]$$

which has two minimal primes (branches).

5 Multiplicity of a Point

Definition 5.0.1. Let X be a curve and $x \in X$ a point. Then the multiplicity $m(x)$ is defined as:

$$m(x) = \lim_{n \to \infty} \dim_{\kappa(x)} \left(\mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}\right)$$