Mathematics W4043 Algebraic Number Theory
Assignment # 11

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6.1 I will prove that $p_k \leq 2^{k-1}$. The argument proceeds by induction. For $k = 1$, we compare $p_1 = 2$ and $2^{2^{k-1}} = 2^1 = 2$ so $p_1 \leq 2^{2^0}$. Now, assume that, for $i \leq k$, we have $p_i \leq 2^{2^{i-1}}$ then consider,

$$N = p_1 p_2 \cdots p_k - 1 \leq 2^{2^0} \cdot 2^{2^1} \cdot 2^{2^2} \cdots 2^{2^{2^{k-1}}} = 2^{2^0+2^1+2^2+\cdots+2^{k-1}} = 2^{2^k-1} \leq 2^{2^k}$$

By the fundamental theorem of arithmetic, since $N > 1$, there is some prime p_r such that $p_r \mid N$ but if $r \leq k$ then $p_r \mid p_1 \cdots p_k$. In that case, $p_r \mid p_1 \cdots p_k - N = 1$ which is impossible. Thus, $r \geq k+1$ so $p_{k+1} \leq p_r \leq N \leq 2^{2^k}$. Therefore, by induction, $p_k \leq 2^{2^{k-1}}$ for all k.

For $x > 2$ let k be the least integer such that $2^{2^k} \geq x$ then $2^{2^k} \geq x \geq 2^{2^{k-1}} \geq p_k$. We choose $x > 2$ such that $k \geq 1$. Therefore, the first k primes are all less than x and therefore,

$$\pi(x) \geq k = \log \log 2^{2^k} \geq \log_2 \log_2 x \geq \log \log x$$

6.4 Let 1 be the function $1(x) = 1$. Then, consider the function $\mu * 1$. Because both μ and 1 are multiplicative, thus tthe function $\mu * 1$ is as well. Therefore, we must only consider its values at prime powers. Because the only square-free divisors of p^k are 1 and p (for $k > 0$) then,

$$(\mu * 1)(p^k) = \sum_{d \mid p^k} \mu(d) 1(\tfrac{p^k}{d}) = \mu(1) + \mu(p) = 1 - 1 = 0$$

and likewise, the only divisor of 1 is 1 so $(\mu * 1)(1) = \mu(1) = 1$. Thus, if $n > 1$ then n is divisible by some prime power (with $k > 0$) so we can write $n = p^k m$ with $(p^k, m) = 1$. Thus, $(\mu * 1)(n) = (\mu * 1)(p^k) \cdot (\mu * 1)(m) = 0$. Therefore $(\mu * 1)(n) = 0$ for $n > 1$ and $(\mu * 1)(1) = 1$ so $\mu * 1 = \delta$.

Next, we show that $*$ is an assoicative operation. Let $D_3(n) = \{(a, b, c) \in \mathbb{N}^3 \mid abc = n\}$. Consider,

$$((f * g) * h)(n) = \sum_{d \mid n} (f * g)(d) h(\tfrac{n}{d}) = \sum_{d \mid n} \sum_{d' \mid d} f(d') g(\tfrac{d}{d'}) h(\tfrac{n}{d}) = \sum_{(a,b,c) \in D_3(n)} f(a) g(b) h(c)$$

because $d' \cdot \frac{d}{d'} \cdot \frac{n}{d} = n$ so $(d', \frac{d}{d'}, \frac{n}{d}) \in D_3(n)$ and given $(a,b,c) \in D_3(n)$ we let $d' = a$ and $d = ab$ so $b = \frac{d}{d'}$ and $c = \frac{n}{ab} = \frac{n}{d}$. Likewise,

$$(f * (g * h))(n) = \sum_{d|n} f(d)(g * h)(\tfrac{n}{d}) = \sum_{d|n} \sum_{d'|d} f(d)g(d')h(\tfrac{n}{d' \cdot d}) = \sum_{(a,b,c) \in D_3(n)} f(a)g(b)h(c)$$

because $d \cdot d' \frac{n}{d' \cdot d} = n$ so $(d, d', \frac{n}{d' \cdot d}) \in D_3(n)$ and given $(a,b,c) \in D_3(n)$ we let $d = a$ and $d' = b$ so $c = \frac{n}{ab} = \frac{n}{d' \cdot d}$. Therefore, $(f * g) * h = f * (g * h)$.

Finally, suppose that,

$$g(n) = \sum_{d|n} f(d) = (1 * f)(n)$$

Then, $\mu * g = \mu * (1 * f) = (\mu * 1) * f = \delta * f$ and

$$(\delta * f)(n) = \sum_{d|n} \delta(d) f(\tfrac{n}{d}) = \delta(1) f(n) = f(n)$$

therefore,

$$(\mu * g)(n) = \sum_{d|n} \mu(d) g(\tfrac{n}{d}) = (\delta * f)(n) = f(n)$$

6.6 Let χ be a nontrivial Dirichlet character modulo N.

(a) We make use of the taylor series for the logarithm,

$$-\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

which converges on the disk minus one point, $|x| \leq 1$ and $x \neq 1$. Thus, take $x = e^{i\theta}$ for $\theta \in (0, 2\pi)$ so that $|x| \leq 1$ and $x \neq 1$. Therefore,

$$-\log(1 - e^{i\theta}) = \sum_{n=1}^{\infty} \frac{(e^{i\theta})^n}{n} = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} = L(\theta)$$

We can rewrite this expression as,

$$\begin{aligned} L(\theta) &= -\log(1 - e^{i\theta}) = -\log(-2ie^{i\theta/2} \sin(\theta/2)) = -\log(2\sin(\theta/2)) - \log(-ie^{i\theta/2}) \\ &= -\log(2\sin(\theta/2)) - \log(e^{-i\frac{\pi}{2} + i\frac{\theta}{2}}) = -\log(2\sin(\theta/2)) + i\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \end{aligned}$$

(b) The formula given,

$$\chi(a) = G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \exp\left(\frac{2\pi i a x}{N}\right)$$

where the Gauss sum G of χ is given by,

$$G(\chi) = \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \chi(x) \exp\left(\frac{2\pi i x}{N}\right)$$

only holds in general if $(a, N) = 1$. Suppose that $(a, N) = 1$ then $a \in (\mathbb{Z}/N\mathbb{Z})^\times$. Because $\chi(a) = 0$ if and only if $a \notin (\mathbb{Z}/N\mathbb{Z})^\times$ then we can replace the sum over $\mathbb{Z}/N\mathbb{Z}$ with a sum over $(\mathbb{Z}/N\mathbb{Z})^\times$ in the Gauss sum. That is,

$$\begin{aligned}\chi(a)G(\bar{\chi}) = \chi(a) \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(x) \exp\left(\frac{2\pi i x}{N}\right) &= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(a^{-1})\bar{\chi}(x) \exp\left(\frac{2\pi i x}{N}\right) \\ = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(a^{-1}x) \exp\left(\frac{2\pi i x}{N}\right) &= \sum_{y \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(y) \exp\left(\frac{2\pi i a y}{N}\right)\end{aligned}$$

where $y = a^{-1}x$ and the sum runs over all $(\mathbb{Z}/N\mathbb{Z})^\times$ because $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ so multiplication by a^{-1} is simply a permutation of the group. Relabeling the summation variable,

$$\chi(a) = G(\bar{\chi})^{-1} \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(x) \exp\left(\frac{2\pi i a x}{N}\right) = G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \exp\left(\frac{2\pi i a x}{N}\right)$$

If N is prime then this formula holds for all $a \in \mathbb{Z}/N\mathbb{Z}$ because the only nonunit is $a = 0$ for which the formula reduces to,

$$\chi(a) = G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) = 0$$

which is in fact true because $\chi(a) = 0$ for all nonunits of $\mathbb{Z}/N\mathbb{Z}$.

(c) Now consider the Dirichlet L function $L(s, \chi)$ evaluated at $s = 1$,

$$L(s = 1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

Now applying the Gauss sum formula from above,

$$L(1, \chi) = \sum_{n=1}^{\infty} n^{-1} G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \exp\left(\frac{2\pi i n x}{N}\right) = G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \sum_{n=1}^{\infty} n^{-1} \exp\left(\frac{2\pi i n x}{N}\right)$$

Due to the subtlety of part (b) only generally holding for nonunits when N is prime, this formula also only holds in general for prime N. Now applying part (a),

$$\begin{aligned}L(1, \chi) &= G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \left[-\log\left(2 \sin\left(\frac{\pi x}{N}\right)\right) + i\left(\frac{\pi}{2} - \left(\frac{\pi x}{N}\right)\right)\right] \\ &= -G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \left[\log\left(\sin\left(\frac{\pi x}{N}\right)\right) + \log 2\right] + iG(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \left(\frac{\pi}{2} - \left(\frac{\pi x}{N}\right)\right)\end{aligned}$$

using the fact that any nontrivial character satsifeis, $\sum_{x \in \mathbb{Z}/N\mathbb{Z}} \chi(x) = 0$, we conclude that,

$$L(1, \chi) = -G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \log\left(\sin\left(\frac{\pi x}{N}\right)\right) - \frac{i\pi}{NG(\bar{\chi})} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) x$$

Suppose χ is an even character then clearly $\bar{\chi}$ is also an even character. Then, $\chi(x)x$ is an odd function so the terms $\bar{\chi}(x)x$ and $\bar{\chi}(-x)(-x)$ cancel. Futhermore, because N is a prime and therefore odd (the case $N = 2$ has no nontrivial characters), only 0 is its own additive inverse since $x \equiv -x \pmod{p} \implies 2x \equiv 0 \pmod{p} \implies p \mid 2x \implies p \mid x$ so the sum,

$$\sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) x = 0$$

Therefore, for an even character χ,

$$L(1, \chi) = -G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \log\left(\sin\left(\frac{\pi x}{N}\right)\right)$$

If χ is an odd character (so $\bar{\chi}$ is odd), then since the sin function is even about $\frac{\pi}{2}$, we can pair the terms in the first sum. The term, $\log\left(\sin\left(\frac{\pi x}{N}\right)\right) = \log\left(\sin\left(\frac{\pi(N-x)}{N}\right)\right)$ and $\bar{\chi}(N - x) = \bar{\chi}(-x) = -\bar{\chi}(x)$. Therefore,

$$\bar{\chi}(x) \log\left(\sin\left(\frac{\pi x}{N}\right)\right) + \bar{\chi}(N-x) \log\left(\sin\left(\frac{\pi(N-x)}{N}\right)\right) = 0$$

again, because N is odd, I can pair the terms in the sum like this without double counting (because $N - x \neq x$). Therefore,

$$\sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \log\left(\sin\left(\frac{\pi x}{N}\right)\right) = 0$$

so for an odd character, the entire expression for $L(1, \chi)$ reduces to,

$$L(1, \chi) = -\frac{i\pi}{NG(\bar{\chi})} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) x$$

(d) First of all, for a character modulo 4 the above expressions do not necessarily hold because 4 is not prime and the above results rely upon N being prime. Second of all, for the character modulo 4 such that $\chi(-1) = -1$ the given value is simply wrong,

$$L(1, \chi) \neq \frac{\pi}{2\sqrt{2}}$$

this is easily checked directly from the definition because,

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan(1) = \frac{\pi}{4}$$

since $\chi(n) = 0$ for odd n and $\chi(2k + 1) = (-1)^k$.

Next, let χ' be an even nontrivial character modulo 5 such that $\chi'(2) = \chi'(3) = -1$. Because 5 is prime we can apply the above results to conclude that,

$$\begin{aligned}
L(1, \chi) &= -G(\bar{\chi})^{-1} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \log\left(\sin\left(\frac{\pi x}{N}\right)\right) \\
&= -G(\bar{\chi})^{-1} \left[\log\left(\sin\left(\frac{\pi}{5}\right)\right) - \log\left(\sin\left(\frac{2\pi}{5}\right)\right) - \log\left(\sin\left(\frac{3\pi}{5}\right)\right) + \log\left(\sin\left(\frac{4\pi}{5}\right)\right)\right] \\
&= -G(\bar{\chi})^{-1} \log\left(\frac{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{4\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)}\right)
\end{aligned}$$

Also,

$$G(\bar{\chi}) = \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \bar{\chi}(x) \exp\left(\frac{2\pi i x}{N}\right) = e^{i\frac{2\pi}{5}} - e^{i\frac{4\pi}{5}} - e^{i\frac{6\pi}{5}} + e^{i\frac{8\pi}{5}} = \sqrt{5}$$

So our final answer is,

$$L(1,\chi) = -\frac{1}{\sqrt{5}} \log\left(\frac{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{4\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)}\right) = \frac{1}{\sqrt{5}} \log\left(\frac{\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{4\pi}{5}\right)}\right)$$

By the way,

$$\eta = \frac{\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{4\pi}{5}\right)} = \frac{3+\sqrt{5}}{2} \neq \frac{1+\sqrt{5}}{2}$$

but whatever.

Part II

Let K be a number field and $\chi : CL(K) \to \mathbb{C}^\times$ a homomorphism.

1. Since $Cl(K)$ is a finite group its image under χ is also a finite group in $\mathbb{C}^\times$. Thus, by Lagrange's theorem, $\chi(Cl(K))$ has an exponent. Therefore, for some $n \in \mathbb{N}$ we must have $\chi(\mathfrak{a})^n = 1$ so $|\chi(\mathfrak{a})| = 1$.

2. Define,

$$L(s,\chi) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}$$

Then, consider the sum of the absolute values with $\sigma = \mathrm{Re}(s)$,

$$\sum_{\mathfrak{a} \subset \mathcal{O}_K} \left|\frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}\right| = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{|\chi(\mathfrak{a})|}{\mathrm{N}(\mathfrak{a})^\sigma} = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{\mathrm{N}(\mathfrak{a})^\sigma} = \zeta_K(\sigma)$$

which we proved converges for $\mathrm{Re}(s) = \sigma > 1$. Therefore, $L(s,\chi)$ converges absolutly for $\mathrm{Re}(s) > 1$.

We can rewrite this function as,

$$L(s,\chi) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{Y(n)}{n^s}$$

where $Y(n) = \sum_{\mathrm{N}(\mathfrak{a})=n} \chi(\mathfrak{a})$ which exists because there are only a finite number of ideals of a given norm. This fact holds because any ideal of norm n is a product of prime ideas with norms dividing n. However, each of these primes lies above a rational prime dividing the norm of the prime ideal. Since there is a finite set of rational primes dividing n and each prime has a finite factorization in ideals, there are finitly many possible combinations of prime ideas to form an ideal of norm n. Now we apply Abel's summation formula to $L(s,\chi)$.

$$L(s,\chi,N) = \sum_{n=1}^{N} \frac{Y(n)}{n^s} = A(N)\frac{1}{N^s} - \int_1^N A(x)\left(\frac{1}{x^s}\right)' \mathrm{d}x = A(n)\frac{1}{n^s} + \int_1^N A(x)\frac{s}{x^{s+1}}\mathrm{d}x$$

where $A(x) = \sum_{n \leq x} Y(n)$. If $A(x)$ is a bounded function, then $L(s,\chi) = \lim_{N \to \infty} L(s,\chi,N)$ is convergent for $\mathrm{Re}(s) > 0$ because then the leading term goes to zero,

$$\lim_{N \to \infty} A(N) \tfrac{1}{N^s} = 0$$

because $A(N)$ is bounded and N^s is unbounded. Also, the integral term in $L(s,\chi)$ is also convergent since,

$$\int_1^\infty \frac{s}{x^{s+1}} \mathrm{d}x = -\left[\frac{1}{x^s}\right]_1^\infty = 1$$

is convergent and the function $A(x)$ is bounded. Therefore, for $L(s,\chi)$ to be convergent on the half-plane $\mathrm{Re}(s) > 0$ we require the function,

$$A(x) = \sum_{n \leq x} Y(n) = \sum_{n \leq x} \sum_{\mathrm{N}(\mathfrak{a}) = n} \chi(\mathfrak{a}) = \sum_{\mathrm{N}(\mathfrak{a}) \leq x} \chi(\mathfrak{a})$$

to be bounded. In particular, this requires that $\chi : Cl(K) \to \mathbb{C}$ be a nontrivial homomorphism which clearly requires $Cl(K)$ to be a nontrivial group. Therefore, it is a necessary condition that $\mathcal{O}_K$ have a nontrivial class group and therefore not be a PID.

3. Let I^S be the set of ideals not divisible by any ideal of S. Now define,

$$L^S(s,\chi) = \sum_{\mathfrak{a} \in I^S} \frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}$$

by Dedekind prime factorization,

$$\begin{aligned}
L^S(s,\chi) &= \sum_{r \geq 0} \sum_{(k_1, \cdots, k_r) \in (\mathbb{Z}^+)^r} \sum_{\mathfrak{p}_i \notin S} \frac{\chi(\mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r})}{\mathrm{N}(\mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r})^s} = \sum_{r \geq 0} \sum_{(k_1, \cdots, k_r) \in (\mathbb{Z}^+)^r} \sum_{\mathfrak{p}_i \notin S} \prod_{i=1}^{r} \frac{\chi(\mathfrak{p}_1)^{k_i}}{\mathrm{N}(\mathfrak{p}_1)^{s \cdot k_i}} \\
&= \prod_{\mathfrak{p} \notin S} \sum_{k=0}^{\infty} \frac{\chi(\mathfrak{p})^{k_i}}{\mathrm{N}(\mathfrak{p})^{s \cdot k_i}} = \prod_{\mathfrak{p} \notin S} \left(1 - \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}\right)^{-1} \\
&= \prod_{\mathfrak{p} \in S} \left(1 - \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}\right) \prod_{\mathfrak{p} \subset \mathcal{O}_K} \left(1 - \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}\right)^{-1} = \prod_{\mathfrak{p} \in S} \left(1 - \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}\right) L(s,\chi)
\end{aligned}$$

In the region on which $L(s,\chi)$ converges absolutly, $L(s,\chi)$ cannot be zero because convergence of the Euler product requires convergence of $\log L(s,\chi)$ which implies that $L(s,\chi)$ cannot be zero. Futhermore, $|\chi(\mathfrak{p})| = 1$ and $|\mathrm{N}(\mathfrak{p})^{-s}| < 1$ because $\mathrm{N}(\mathfrak{p}) > 1$. Therefore, $|\chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s}| < 1$ so $1 - \chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s} \neq 0$. This means that neither of the terms in the factorization of $L^S(s,\chi)$ can be zero so there are no obvious zeros for $\mathrm{Rs}(s) > 1$.