1 Cartier Divisors

1.1 Regular Sections

Definition 1.1. A section $f \in \Gamma(U, \mathcal{O}_X)$ is *regular* if $\mathcal{O}_X|_U \xrightarrow{f} \mathcal{O}_X|_U$ is injective.

Lemma 1.2. A section $f \in \Gamma(U, \mathcal{O}_X)$ is regular iff $f_x \in \mathcal{O}_{X,x}$ is a nonzero divisor for each $x \in U$.

Proof. f is regular when for any open $V \subset U$ and $g \in \Gamma(V, \mathcal{O}_X)$ we have $f|_V g = 0 \implies g = 0$ which is exactly the condition that $f_x \in \mathcal{O}_{X,x}$ is a nonzero divisor for each $x \in U$ since $f_x \in \mathcal{O}_{X,x}$ is a zero divisor if there is some neighborhood $x \in V$ and nonzero $g \in \Gamma(V, \mathcal{O}_X)$ with $f|_V g = 0$. $\square$

Definition 1.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Then define the sheaf of regular sections S_X via,

$$S_X(U) = \{f \in \Gamma(U, \mathcal{O}_X) \mid \text{regular}\}$$

Then S_X is a sheaf because a section is regular exactly if it is regular on a cover.

Definition 1.4. Let $(X, \mathcal{O}_X)$ be a ringed space. The sheaf $\mathscr{K}_X$ of *meromorphic functions* on X is the $\mathcal{O}_X$-module associated to the presheaf,

$$U \mapsto S_X(U)^{-1}\mathcal{O}_X(U)$$

Lemma 1.5. Let X be an integral scheme X with generic point $\xi \in X$. Then for any open $U \subset X$, the map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\xi}$ is injective.

Proof. Choose an open cover $U_i = \mathrm{Spec}\,(A_i) \subset X$ where A_i is a domain then $K(X) = \mathcal{O}_{X,\xi} = \mathrm{Frac}\,(A_i)$ since $\xi \in \mathrm{Spec}\,(A_i)$ is the generic point. Thus, $\mathcal{O}_X(U) \to \mathcal{O}_{X,\xi}$ is an injection because, if $f_\xi = 0$ then consider $f|_{U \cap U_i} \in A_i$ but A_i is a domain so if $f_\xi \in \mathrm{Frac}\,(A_i)$ is zero then $f|_{U \cap U_i} = 0$ for each U_i so $f = 0$. $\square$

Remark. The above lemma alows us to view all functions on X as elements of $K(X)$. In fact, the meromorphic functions on X are exactly $K(X)$.

Proposition 1.6. Let X be a integral scheme. Then $\mathscr{K}_X = \underline{K(X)}$.

Proof. Let $\xi \in X$ be the generic point and $U \subset X$ an open set. Consider the presheaf map $S_X(U)^{-1}\mathcal{O}_X(U) \to K(X)$ sending $f \mapsto f_\xi$ which is well-defined because regular sections have $f_\xi \neq 0$ and $K(X)$ is a field so regular sections are invertible in $K(X)$. Sheafifying, gives a map $\mathscr{K}_X \to \underline{K(X)}$. To show this map is an isomorphism it suffices to check on the stalks which can be computed from the above presheaves. By above, the map $S_X(U)^{-1}\mathcal{O}_{X,(}U) \to K(X)$ is always injective. Furthermore, for any $x \in X$ choose an affine open neighborhood $U = \mathrm{Spec}\,(A)$ with A a domain. Then $S_X(U) = A \setminus \{0\}$ since $A \to A_\mathfrak{p}$ is injective and $A_\mathfrak{p}$ is a domain for each prime $\mathfrak{p}$ so evey nonzero $f \in A$ is regular. Thus, $S_X(U)^{-1}\mathcal{O}_X(U) = \mathrm{Frac}\,(A)$ and the map $S_X(U)^{-1}\mathcal{O}_X(U) \to K(X) = A_{(0)} = \mathrm{Frac}\,(A)$ is an isomorphism. $\square$

1.2 Cartier Divisors

Definition 1.7. Let X be a ringed space. The *sheaf of Cartier divisors* on X is $\mathfrak{Div}_X = \mathscr{K}_X^\times / \mathcal{O}_X^\times$. The group of Cartier divisors is $\mathrm{Ca}\,(X) = H^0(X, \mathfrak{Div}_X)$ and the Cartier class group is,

$$\mathrm{CaCl}\,(X) = \mathrm{coker}\,(H^0(X, \mathscr{K}_X^\times) \to H^0(X, \mathfrak{Div}_X))$$

Proposition 1.8. There is a natural embedding $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X)$ which is an isomorphism when $H^1(X, \mathscr{K}_X^\times) = 0$.

Proof. Consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathscr{K}_X^\times \longrightarrow \mathfrak{Div}_X \longrightarrow 0$$

Taking cohomology gives,

$$0 \longrightarrow H^0(X, \mathcal{O}_X^\times) \longrightarrow H^0(X, \mathscr{K}_X^\times) \longrightarrow H^0(X, \mathfrak{Div}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathscr{K}_X^\times)$$

But $H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X)$ and by exactness, we get an exact sequence,

$$0 \longrightarrow \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^1(X, \mathscr{K}_X^\times)$$

□

Remark. The condition $H^1(X, \mathscr{K}_X^\times) = 0$ occurs when X is an integral scheme. Then $\mathscr{K}_X^\times = \underline{K(X)^\times}$ is a constant sheaf and X is irreducible so its higher cohomology vanishes.

2 Effective Cartier Divisors

2.1 Closed Subschemes

2.2 Effective Cartier Divisors as Closed Subschemes

2.3 Relationship to the Previous Definition

3 Weil Divisors

We only consider Weil divisors for sufficiently nice schemes. (DEFINE)

3.1 The Sheaf Associated to a Weil Divisor

3.2 The Relationship between Weil Divisors and Cartier Divisors

4 Reflexive Sheaves

5 The Chow Ring