# Contents



# 1 Irreducible Spaces

## 1.1 Irreducibility

**Definition 1.1.1.** A topological space $X$ is *irreducible* if $X$ is nonempty and whenever $X = Z_1 \cup Z_2$ for closed subsets $Z_1, Z_2 \subset X$ then either $Z_1 = X$ or $Z_2 = X$.

**Lemma 1.1.2.** Let $X$ be a topological space. The following are equivalent,

(a) $X$ is irreducible

(b) every nomepty open $U \subset X$ is dense

(c) any two nonempty opens $U_1, U_2 \subset X$ have nonempty intersection $U_1 \cap U_2$.

*Proof.* Let $X$ be irreducible and suppose $U \subset X$ is open. Then $\overline{U} \cup U^C = X$ so either $\overline{U} = X$ or $U^C = X$ because both $\overline{U}, U^C \subset X$ are closed. Thus, if $U$ is nonempty then $\overline{U} = X$.

Conversely, let $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed. Then $Z_1^C \subset Z_2$ so either $Z_1^C$ is empty or dense so $Z_2 = X$ thus either $Z_1 = X$ or $Z_2 = X$ so $X$ is irreducible.

Now (a) and (c) are equivalent because,

$$U_1 \cap U_2 = \varnothing \iff (U_1 \cap U_2)^C = X \iff U_1^C \cup U_2^C = X$$

So,

$$\begin{aligned}[U_1, U_2 \neq \varnothing \implies U_1 \cap U_2 \neq \varnothing] &\iff [U_1 \cap U_2 = \varnothing \implies U_1 = \varnothing \text{ or } U_2 = \varnothing] \\ &\iff [U_1^C \cup U_2^C = X \implies U_1^C = X \text{ or } U_2^C = X]\end{aligned}$$

□

**Lemma 1.1.3.** Let $S \subset X$ be a subspace with the subspace topology. Then $S$ is irreducible iff for any closed $Z_1, Z_2 \subset X$ such that $S \subset Z_1 \cup Z_2$ then either $S \subset Z_1$ or $S \subset Z_2$.

*Proof.* Suppose that $S$ is irreducible. Then $\tilde{Z}_i = Z_i \cap S$ are closed in $S$ and $S = \tilde{Z}_1 \cup \tilde{Z}_2$ so $S = \tilde{Z}_i$ i.e. $S \subset Z_i$ for some $i$.

Conversely, let $\tilde{Z}_1, \tilde{Z}_2 \subset S$ be closed such that $S = \tilde{Z}_1 \cup \tilde{Z}_2$. Then $\tilde{Z}_i = Z_i \cap S$ for some closed $Z_i \subset X$ because $S$ has the subspace topology. Then $S \subset Z_1 \cup Z_2$ so $S \subset Z_1$ or $S \subset Z_2$ and thus $S = \tilde{Z}_1$ or $S = \tilde{Z}_2$ so $S$ is irreducible. □

*Remark.* If $S \subset Y \subset X$ with the subspace topologies then,

$$S \text{ is "irreducible in } Y\text{"} \iff S \text{ is "irreducible in } X\text{"}$$

because irreducibility is an absolute property.

Explicitly, if $S$ is "irreducible in $Y$" and $S \subset Z_1 \cup Z_2$ for $Z_1, Z_2 \subset X$ closed then $Z_1 \cap Y, Z_2 \cap Y \subset Y$ are closed and $S \subset (Z_1 \cap Y) \cup (Z_2 \cap Y)$ so $S \subset Z_1 \cap Y$ or $S \subset Z_2 \cap Y$ so $S \subset Z_1$ or $S \subset Z_2$ menaing $S$ is "irreducible in $X$". Conversely, if $S$ is "irreducible in $X$" then if $S \subset Z_1 \cup Z_2$ for closed $Z_1, Z_2 \subset Y$ then there exist closed $Z_i' \subset X$ such that $Z_i = Z_i' \cap Y$ and $S \subset Z_1' \cup Z_2'$ so $S \subset Z_1'$ or $S \subset Z_2'$ and thus $S \subset Z_1$ or $S \subset Z_2$ showing that $S$ is "irreducible in $Y$".

**Lemma 1.1.4.** Let $U \subset X$ be open and $Z \subset X$ irreducible. Then $Z \cap U$ is irreducible iff $Z \cap U \neq \varnothing$.

*Proof.* If $Z \cap U = \varnothing$ then it is not irreducible by definition. Otherwise, assume $Z \cap U \neq \varnothing$ and suppose $Z \cap U \subset Z_1 \cup Z_2$ for closed subsets $Z_1, Z_2 \subset X$. Then $Z \subset Z_1 \cup Z_2 \cup U^C$ so $Z \subset Z_1$ or $Z \subset Z_2$ or $Z \subset U^C$ by irreducibility of $Z$ and the previous lemma. However, $Z \not\subset U^C$ because $Z \cap U \neq \varnothing$ so $Z \subset Z_1$ or $Z \subset Z_2$ so by the above lemma $Z \cap U$ is irreducible. □

**Lemma 1.1.5.** Let $Z \subset X$ be irreducible. Then $\overline{Z} \subset X$ is irreducible.

*Proof.* Suppose that $\overline{Z} = Z_1 \cup Z_2$ with $Z_1$ and $Z_2$ closed. Then $Z \subset Z_1 \cup Z_2$ so either $Z \subset Z_1$ or $Z \subset Z_2$. But since $Z_1$ and $Z_2$ are closed, we get $\overline{Z} = Z_1$ or $\overline{Z} = Z_2$. □

## 1.2 Irreducible Components

**Lemma 1.2.1.** Increasing unions of irreducible subsets are irreducible.

*Proof.* Consider a chain $T$ of irreducible subsets and consider,

$$U = \bigcup_{S \in T} S$$

Suppose $U = Z_1 \cup Z_2$ for closed subsets $Z_1$ and $Z_2$ of $U$. Then for each $S \in T$ we have $S \subset Z_1$ or $S \subset Z_2$. If for some $S_0 \in T$ we have $S_0 \not\subset Z_2$ (otherwise $Z_2 \supset U$ and we are done) then $S_0 \subset Z_1$ and for any $S \in T$ with $S \supset S_0$ we cannot have $S \subset Z_2$ else $S_0 \subset Z_2$. Therefore, $S \subset Z_1$. For any $S \in T$, since $T$ is totally ordered, either $S \subset S_0$ in which case $S \subset Z_1$ or $S \supset S_0$ in which case $S \subset Z_1$ (as we have just shown). Therefore, $U \subset Z_1$ so $U$ is irreducible. □

**Definition 1.2.2.** Let $X$ be a topological space then its irreducible components are the maximal irreducible subsets of $X$.

*Remark.* The irreducible subsets of $X$ form a poset under inclusion. Furthermore, since chains have a maximum, by Zorn's lemma $X$ always has some irreducible component.

**Lemma 1.2.3.** Let $X$ be a topological space. The following hold,

(a) irreducible components are closed

(b) every irreducible subset of $X$ is contained in some irreducible component

(c) the irreducible components of $X$ cover $X$.

*Proof.* Let $C \subset X$ be an irreducible component. Then $\overline{C}$ is irreducible and $S \subset \overline{C}$ so $\overline{C} = C$ by maximality. Thus, $C$ is closed. For any irreducible set $S \subset X$, Zorn's Lemma gives a maximal element in the irreducible components above $S$ i.e. $S \subset C$ is contained in some irreducible component. In particular, since any point $x \in X$ is irreducible so $x \in C$ is contained in some irreducible component. Thus the irreducible components cover $X$. ☐

**Lemma 1.2.4.** Noetherian spaces have finitly many irreducible components.

*Proof.* Let $S$ be the poset of closed subspaces with infinitely many components ordered by inclusion. By the Noetherian hypothesis, descending chains in $S$ have minima so, by Zorn's lemma, $S$ has a minimum $Z$ which has infinitely many irreducible components. Clearly, $Z$ cannot be irreducible so we can write $Z = Z_1 \cup Z_2$ with $Z_1, Z_2 \subsetneq Z$ are proper closed subsets. By minimality, $Z_1, Z_2 \notin S$ and thus $Z_1, Z_2$ have finitely many irreducible components. Thus, $Z = Z_1 \cup Z_2$ has finitely many irreducible components so $S$ is empty. ☐

# 2 Quasi-Compactness and Noetherian Spaces

## 2.1 Noetherian Spaces

**Definition 2.1.1.** A topological space $X$ is Noetherian if every descending chain of closed sets stabilizes.

**Lemma 2.1.2.** Subspaces of Noetherian subspaces are Noetherian.

*Proof.* Let $S \subset X$ with $X$ noetherian. Then the closed sets of $S$ are exactly $S \cap Z$ for $Z \subset X$ closed. Thus descending chains of closed sets in $S$ stabilize. ☐

**Definition 2.1.3.** A space is quasi-compact if every open cover has a finite subcover.

**Lemma 2.1.4.** Noetherian spaces are quasi-compact.

*Proof.* Let $U_\alpha$ be an open cover of $X$ which is Noetherian. Then consider the poset $A$ under inclusion of finite unions of the $U_\alpha$ all of which are open sets of $X$. Since $X$ is Noetherian any ascending chain of opens must stabilize so any chain in $A$ has a maximum. Then by Zorn's lemma $A$ has a maximal element which must be $X$ since the $U_\alpha$ form a cover. Therefore there exists a finite subcover. ☐

**Corollary 2.1.5.** Every subset of a noetherian topological space is quasi-compact.

**Definition 2.1.6.** A continuous map $f : X \to Y$ is quasi-compact if for each quasi-compact open $U \subset Y$ then $f^{-1}(U)$ is quasi-compact open.

## 2.2 The Case for Schemes

**Lemma 2.2.1.** Affine schemes are quasi-compact.

*Proof.* Let $U_i$ be an open cover of $\mathrm{Spec}\,(A_i)$. Since $D(f)$ for $f \in A$ forms a basis of the topology on $\mathrm{Spec}\,(A_i)$ we can shrink to the case $U_i = D(f_i)$. Then.

$$X = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D((\{f_i \mid i \in I\}))$$

And thus the ideal $I = (\{f_i \mid i \in I\})$ is not contained in any maximal ideal so $I = (1)$. Therefore, there are $f_1, \dots, f_n$ such that $a_1 f_1 + \cdots a_n f_n = 1$ and thus $(f_1, \dots, f_n) = (1)$ which implies that,

$$X = D((f_1, \dots, f_n)) = \bigcup_{i=1}^{n} D(f_i)$$

so $X$ is quasi-compact. $\square$

**Definition 2.2.2.** A scheme $X$ is *locally Noetherian* if for every affine open $U$ the ring $\mathcal{O}_X(U)$ is Noetherian. $X$ is *Noetherian* if it is quasi-compact and locally-Noetherian.

**Lemma 2.2.3.** If $(f_1, \dots, f_n) = A$ and $A_{f_i}$ is Noetherian then $A$ is Noetherian.

*Proof.* For any ideal $I \subset A$ we know $I_{f_i} \subset A_{f_i}$ is finitely generated. Clearing denominators and collecting the finite union of these finite generators gives a map $A^N \to I$ which is surjective when localized $A^N_{f_i} \twoheadrightarrow I_{f_i}$. Consider the $A$-module $K = \mathrm{coker}\,(A^N \to I)$ then for any $x \in K$ we have $f_i^{n_i} \cdot x = 0$ for each $i$ but $f_i^{n_i}$ generate the unit ideal (since $D(f_i^{n_i}) = D(f_i)$ which cover $\mathrm{Spec}\,(A)$) so $x = 0$ to $A^N \twoheadrightarrow I$ so $I$ is finitely generated showing that $A$ is Noetherian. $\square$

**Lemma 2.2.4.** If $X$ has an open affine cover $U_i = \mathrm{Spec}\,(A_i)$ with $A_i$ noetherian then $X$ is locally noetherian. Moreover, if the cover can be made finite then $X$ is noetherian.

*Proof.* Let $V = \mathrm{Spec}\,(B) \subset X$ be an affine open, Then $V \cap U_i \subset V$ is open so it may be covered by principal opens $D(f_{ij}) \subset V \cap U_i$ for $f_{ij} \in B$. Since $V$ is quasi-compact we may find a finite subcover. We need to show that $B_{f_{ij}}$ is Noetherian then since $D(f_{ij})$ cover $V$ we use the lemma to conclude that $B$ is Noetherian. However, $D(f_{ij}) \subset V \cap U_i$ can be covered by principal opens (of $U_i = \mathrm{Spec}\,(A_i)$) $W_{ijk} \subset D(f_{ij}) \subset U_i = \mathrm{Spec}\,(A_i)$ and each $(A_i)_{f_{ijk}}$ is Noetherian since $A_i$ is, so using the same lemma we find that $B_{f_{ij}}$ is Noetherian.

Now suppose the cover is finite and let $V_j$ be any open cover of $X$. We need to show $X$ is quasi-compact so we must show that $V_i$ has a finite subcover. Consider $U_i \cap V_j$ which is open in the affine $U_i = \mathrm{Spec}\,(A_i)$ so it may be covered by principal opens $D(f_{ijk}) \subset U_i \cap V_j$. Now,

$$U_i = \bigcup_{j,k} D(f_{ijk})$$

but $U_i$ is affine and thus quasi-compact so we may find an finite subcover which only uses finitely many $V_i$ but the cover $U_i$ of $X$ is also finite so only finitely many $V_i$ are needed to cover $X$. $\square$

**Corollary 2.2.5.** $X = \mathrm{Spec}\,(A)$ is Noetherian iff $A$ is a Noetherian ring.

*Proof.* If $X$ is Noetherian then $\mathcal{O}_X(X) = A$ is a Noetherian ring ($X$ is affine and thus quasi-compact). Conversely $\mathrm{Spec}\,(A)$ is a finite Noetherian affine cover so $X$ is Noetherian. $\square$

*Remark.* It is not the case that for a Noetherian scheme we must have $\mathcal{O}_X(X)$ a noetherian ring even for varieties. See http://sma.epfl.ch/ ojangure/nichtnoethersch.pdf.

**Corollary 2.2.6.** A Noetherian ring has finitely many minimal primes.

*Proof.* Let $A$ be Noetherian then primes $\mathfrak{p} \in \mathrm{Spec}\,(A)$ correspond to irreducible closed subsets $V(\mathfrak{p})$ and thus minimal primes correspond to irreducible components of $\mathrm{Spec}\,(A)$. Therefore, since $\mathrm{Spec}\,(A)$ is Noetherian, we see that $\mathrm{Spec}\,(A)$ has finitely many irreducible components and thus finitely many minimal primes. $\square$

**Lemma 2.2.7.** If $A$ is Noetherian then $\mathrm{Spec}\,(A)$ is a Noetherian topological space.

*Proof.* Every descending chain of subsets is of the form $V(I_1) \supsetneq V(I_2) \supsetneq V(I_3) \supsetneq \cdots$ but the ideals,

$$\sqrt{I_1} \subsetneq \sqrt{I_2} \subsetneq \sqrt{I_3} \subsetneq \cdots$$

satbilize since $A$ is Noetherian and thus so does the chain of closed subsets. $\square$

**Lemma 2.2.8.** If $X$ is a Noetherian scheme then its underlying topological space is Noetherian.

*Proof.* Choose a finite covering $U_i = \mathrm{Spec}\,(A_i)$ by Noetherian rings. Then for any descending chain of closed subsets $Z_1 \supsetneq Z_2 \supsetneq Z_3 \supsetneq \cdots$ we know $Z \cap U_i$ stabilizes at $n_i$ since $\mathrm{Spec}\,(A_i)$ is a Noetherian space. Thus, $Z$ satibilizes at $\max n_i$ which exists since the cover is finite. $\square$

*Remark.* The converses of the above are false and so is $X$ Noetherian. Let $R$ be a non-Noetherian valuation ring. Then $\mathrm{Spec}\,(R)$ has two points and thus is Noetherian as a topological space but not as a scheme since $R$ is not a Noetherian ring.

**Lemma 2.2.9.** If $X$ is locally Noetherian then any immersion $\iota : Z \hookrightarrow X$ is quasi-compact.

*Proof.* Closed immersions are affine and thus quasi-compact so it suffices okay to show that open immersions are quasi-compact. Let $j : U \to X$ be an open immersion. It suffices to check that $j^{-1}(U_i)$ is quasi-compact on an affine open cover $U_i = \mathrm{Spec}\,(A_i)$ with $A_i$ Noetherian. But $j : j^{-1}(U_i) \to U_i \cap U$ is a homeomorphism and $\mathrm{Spec}\,(A_i)$ is a Noetherian topological space so every subset is quasi-compact and, in particular, $U_i \cap U$ is quasi-compact so $j^{-1}(U_i)$ is also. $\square$

*Remark.* When $X$ is Noetherian then it is a Noetherian space so any inclusion map $\iota : Z \hookrightarrow X$ for *any* subset $Z \subset X$ is quasi-compact since every subset is quasi-compact. In particular, every subset of $X$ is retrocompact.

## 2.3 Quasi-Compact Morphisms

**Lemma 2.3.1.** A morphism $f : X \to Y$ is quasi-compact iff $Y$ has a cover by affine opens $V_i$ such that $f^{-1}(V_i)$ is quasi-compact.

*Proof.* Clearly if $f$ is quasi-compact then any affine open cover $V_i$ of $Y$ satisfies $f^{-1}(V_i)$ is quasi-compact since $V_i$ is a quasi-compact open by virtue of being affine open.

Now assume that such a cover exists. Let $U \subset Y$ be a quasi-compact open. Then $U$ is covered by finitely may $V_1, \dots, V_n$. Then $U \cap V_i$ is open in $V_i$ which is affine so it is covered by standard opens

$W_{ij}$. Since $U$ is quasi-compact then we can choose finitely many $W_{ij}$. Now $f^{-1}(V_i)$ is quasi-compact by assumption so it has a finite cover by affine opens,

$$f^{-1}(V_i) = \bigcup_{j=1}^{N} \tilde{V}_{ij}$$

Then $f : \tilde{V}_{ik} \to V_i$ is a morphism of affine schemes so $f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$ is a principal affine. Therefore,

$$f^{-1}(U) = \bigcup_{i=1}^{n} f^{-1}(V_i \cap U) = \bigcup_{i,j} f^{-1}(W_{ij}) = \bigcup_{i,j,k} f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$$

is a finite union of principal affines each of which is quasi-compact so $f^{-1}(U)$ is quasi-compact. □

**Proposition 2.3.2.** $X$ is quasi-compact iff any morphism $X \to T$ for some affine scheme $T$ is quasi-compact.

*Proof.* If $X$ is quasi-compact then $f : X \to T$ is quasi-compact since $T$ is an affine open cover of itself and $f^{-1}(T)$ is quasi-compact. Conversely, if $f : X \to T$ is quasi-compact with $T$ affine then $T$ is quasi-compact open in $T$ so $X = f^{-1}(T)$ is quasi-compact. □

**Lemma 2.3.3.** The base change of a quasi-compact morphism is quasi-compact.

*Proof.* (DO THIS) □

## 2.4 Affine Morphisms

**Definition 2.4.1.** A morphism $f : X \to Y$ is *affine* if the preimage of every affine open is affine.

**Lemma 2.4.2.** Every morphism of affine schemes is affine and thus quasi-compact.

*Proof.* Let $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ and $f : X \to Y$ be a morphism of affine schemes given by a ring map $\varphi : B \to A$. Then, any affine open $\mathrm{Spec}\,(C) = V \subset Y$ can be covered by principal opens $D(f_i)$ for $f_i \in B$. Note that under $\psi : B \to C$ we see that $D(f_i) = D(\psi(f_i))$ since $D(f_i) \subset \mathrm{Spec}\,(C)$. Since $D(\psi(f_i))$ cover $\mathrm{Spec}\,(C)$ then $\psi(f_i) \in C$ generate the unit ideal. Then we have $f^{-1}(D(f_i)) = D(\varphi(f_i))$ which is affine and $\varphi(f_i)$ generate the unit ideal of $\Gamma(f^{-1}(V), \mathcal{O}_X)$ so $f^{-1}$ is affine. □

*Remark.* An alternative proof goes as follows. Consider the pullback diagram,

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ X \arrow[r] & Y \end{tikzcd}$$

then open immersions are stable under base change so $f^{-1}(U) = U \times_Y X = \mathrm{Spec}\,(C \otimes_B A)$ if affine.

*Remark.* In fact, by Tag 01S8, a morphism $f : X \to S$ is affine iff $X$ is relatively affine over $S$ meaning $X = \mathbf{Spec}_S\,(\mathcal{A})$ for some quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$.

**Lemma 2.4.3.** Let $f : X \to Y$ be a morphism and $W_i$ an affine open cover of $Y$ such that $f^{-1}(W_i)$ is affine. Then $f$ is affine.

*Proof.* Let $\operatorname{Spec}(A) = V \subset Y$ be affine open. Then $V_i = V \cap W_i$ is open in the affine open $V = \operatorname{Spec}(A)$ so it can be covered by principal opens $D(f_{ij}) \subset V \cap W_i$ for $f_{ij} \in A$. Since $f : f^{-1}(W_i) \to W_i$ is a morphism of affine schemes, the preimage of the affine open $D(f_{ij}) \subset V \cap W_i$ is affine $f^{-1}(D(f_{ij}))$ (note that $D(f_{ij}) \subset V \cap W_i$ is not necessarily a prinicpal affine open of $W_i$). But since $D(f_{ij})$ cover $\operatorname{Spec}(A)$ the $f_{ij} \in A$ generate the unit ideal and thus $f^{\#}(f_{ij}) \in \Gamma(f^{-1}(V), \mathcal{O}_X)$ generate the unit ideal and $(f^{-1}(V))_{f_{ij}} = f^{-1}(D(f_{ij}))$ is affine so $f^{-1}(V)$ is affine. $\square$

**Lemma 2.4.4.** The base change of an affine morphism is affine.

*Proof.* (DO THIS) $\square$

**Lemma 2.4.5.** Affine morphisms are quasi-compact.

*Proof.* If $f : X \to Y$ is affine then any affine open cover $V_i$ of $Y$ gives $f^{-1}(V_i)$ is affine and thus quasi-compact so $f$ is quasi-compact. $\square$

## 2.5 Separatedness

**Definition 2.5.1.** A morphism $f : X \to Y$ with diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is,

(a) *separated* if the diagonal $\Delta_{X/Y}$ is a closed immersion

(b) *affine-separated* if the diagonal $\Delta_{X/Y}$ is affine

(c) *quasi-separated* if the diagonal $\Delta_{X/Y}$ is quasi-compact

**Lemma 2.5.2.** Any morphism of affine schemes is separated. Furthermore, affine morphisms are separated.

*Proof.* For a map $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ the diagonal is $\operatorname{Spec}(A) \to \operatorname{Spec}(A \otimes_B A)$ given by $A \otimes_B A \to A$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ which is surjective so the diagonal is a closed immersion. The second fact is Tag 01S7. $\square$

**Lemma 2.5.3.** The composition of (quasi/affine)-separated morphisms are (quasi/affine)-separated.

*Proof.* (DO THIS) $\square$

**Lemma 2.5.4.** For any morphism $f : X \to Y$ the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is an immersion.

*Proof.* Let $V_i$ be an affine cover of $Y$ then choose an affine open cover $U_{ij}$ of $X$ with $f(U_{ij}) \subset V_i$. Then the diagonal of the affine map $U_{ij} \to V_j$ is $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ which is a closed immersion since it corresponds to $A_{ij} \otimes_{B_i} A_{ij} \to A_{ij}$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ is surjective. Therefore $f : X \to Y$ is locally on $X$ a closed immersion and thus an immersion. $\square$

*Remark.* Therefore, to show that $f : X \to Y$ is separated, it suffices to show that the diagonal is closed (here equivalently meaning that the map or its image is closed).

**Lemma 2.5.5.** If $X$ is Noetherian then every morphism $f : X \to S$ is quasi-compact and quasi-seperated.

*Proof.* Every subset of $X$ is quasi-compact since $X$ is (topologically) Noetherian. Then apply the first part to the diagonal $\Delta_{X/S} : X \to X \times_S X$ which is then quasi-compact and thus $f : X \to S$ is quasi-separated. $\square$

**Lemma 2.5.6.** Let $f : X \to S$ be affine-separated/quasi-separated with $S = \mathrm{Spec}\,(A)$ affine. Then for any two affine opens $U, V \subset X$ the intersection $U \cap V$ is affine/quasi-compact.

*Proof.* Consider the pullback diagram,

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d, hook] & U \times_S V \arrow[d, hook] \\ X \arrow[r, "\Delta_{X/S}"] & X \times_S X \end{tikzcd}$$

where $U \cap V = \Delta_{X/S}(U \times_S V)$ using the basechange of an open immersion is an open immersion. Then since $S$ is affine, $U \times_S V$ is affine and thus quasi-compact open of $X \times_S X$. Then if $f$ is affine-separated then $\Delta_{X/S}$ is affine so $U \cap V = \Delta_{X/S}(U \times_S V)$ is affine. If $f$ is quasi-separated then $\Delta_{X/S}$ is quasi-compact so $U \cap V = \Delta_{X/S}(U \times_S V)$ is quasi-compact. $\square$

*Remark.* In the separated case, we see that $U \cap V$ is affine and $\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.

*Remark.* Tag 01KO gives a generalization of this lemma. For the separated case see Tag 01KP.

**Lemma 2.5.7.** Let $f : X \to Y$ be quasi-compact and quasi-separated and $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module then $f_*\mathscr{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

*Proof.* Sinsce this is local on $Y$ we can restrict to the case that $Y$ is affine. Then $X = f^{-1}(Y)$ is quasi-compact (when $Y$ is not affine $f^{-1}(V)$ will be quasi-compact) so take a finite affine open cover $U_i$ and since $f : X \to Y$ is quasi-seperated over an affine then by the above lemma $U_i \cap U_j$ is quasi-compact so it has a finite affine open cover $U_{ijk}$. Then, by the sheaf property, there is an exact sequence of sheaves on $Y$

$$0 \longrightarrow f_*\mathscr{F} \longrightarrow \bigoplus_i f_*(\mathscr{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} f_*(\mathscr{F}|_{U_{ijk}})$$

which works because these are finite sums. However, $f : U_{ijk} \to Y$ is a morphism of affine schemes and since $\mathscr{F}$ is quasi-coherent we have $\mathscr{F}|_{U_{ijk}} = \widetilde{M_{ijk}}$ so $f_*(\mathscr{F}|_{U_{ijk}}) = \widetilde{M_{ijk}}$ as an $\mathcal{O}_Y(Y)$-module. Thus, $f_*\mathscr{F}$ is a kernel of quasi-coherent $\mathcal{O}_Y$-modules and thus is quasi-coherent. $\square$

*Remark.* If $X$ is Noetherian then $f : X \to Y$ is automatically quasi-compact and quasi-separated so there is no issue in the above lemma.

# 3 Sober Spaces

**Definition 3.0.1.** A topological space is $T_0$ if for each pair of distinct points there is a neighborhood of one that does not contain the other.

**Proposition 3.0.2.** All schemes are $T_0$.

*Proof.* Let $X$ be a scheme and $x, y \in X$ distinct points. If $x$ and $y$ lie in different affine opens then this is an open seperation. If $x, y$ lie in the same affine open $U = \mathrm{Spec}\,(A)$ then they correspond to distinct prime ideals $\mathfrak{p}, \mathfrak{q} \subset A$. Since $\mathfrak{p} \neq \mathfrak{q}$ there exists some element of one that is not in the other. Without loss of generality suppose that there is some $f \in \mathfrak{p}$ with $f \notin \mathfrak{q}$. Thus, $\mathfrak{q} \in D(f)$ and $\mathfrak{p} \notin D(f)$ so $x$ and $y$ are seperated by some open $D(f) \subset U \subset X$. $\square$

**Definition 3.0.3.** A *generic point* $\xi \in Z$ of a closed irreducible set $Z$ is such that $\overline{\{\xi\}} = Z$.

**Proposition 3.0.4.** Let $X$ be a toplogical space and $\xi \in X$ then $\overline{\{\xi\}}$ is a closed irreducible set with generic point $\xi$.

*Proof.* Clearly, $\overline{\{\xi\}}$ is closed. Suppose that $\overline{\{\xi\}} \subset Z_1 \cup Z_2$ then $\xi \subset Z_1$ or $\xi \subset Z_2$ and thus $\overline{\{\xi\}} \subset Z_1$ or $\overline{\{\xi\}} \subset Z_2$ so $\overline{\{\xi\}}$ is irreducible. Clearly, $\xi$ is a generic point of $\overline{\{\xi\}}$. $\square$

**Definition 3.0.5.** A topological space is *sober* if every irreducible closed set has a unique generic point.

**Proposition 3.0.6.** Any Hausdorff space is sober.

*Proof.* Let $Z$ be irreducible and closed. Suppose that $Z$ has more than one point. Take distinct $x, y \in Z$ and, using the Hausdorff property, open sets $x \in U$ and $y \in V$ such that $U \cap V = \varnothing$. Now consider $Z_1 = Z \cap U^C$ and $Z_2 = Z \cap V^C$ which are closed in $Z$ proper because $x \notin Z_1$ and $y \notin Z_2$. Furthermore, $Z_1 \cup Z_2 = Z \cap (U^C \cup V^C) = Z \cap (U \cap V)^C = Z$ so $Z$ cannot be irreducble. Thus, the only irreducible sets are points which clearly have a unique generic point because all points in a $T_2$ space are closed. $\square$

**Lemma 3.0.7.** Any prime $\mathfrak{p} \in \mathrm{Spec}\,(A)$ in an affine scheme satisfies $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

*Proof.* Any closed set in $\mathrm{Spec}\,(A)$ is of the form $V(I)$ for some ideal $I \subset A$. Consider the closed sets $\mathfrak{p} \in V(I)$ containing $\mathfrak{p}$ which correspond to $\mathfrak{p} \supset I$. Clearly, $\mathfrak{p} \in V(\mathfrak{p})$ and if $\mathfrak{p} \in V(I)$ then $V(\mathfrak{p}) \subset V(I)$ since $\mathfrak{p} \supset I$. Therefore $V(\mathfrak{p})$ is the closure of $\mathfrak{p}$. $\square$

**Lemma 3.0.8.** Every closed irreducible set of an affine scheme $\mathrm{Spec}\,(A)$ is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p} \subset A$.

*Proof.* First, all closed subsets of $\mathrm{Spec}\,(A)$ are of the form $V(I)$. First, if $I = \mathfrak{p}$ is prime and $V(\mathfrak{p}) \subset V(I_1) \cup V(I_2) = V(I_1 I_2)$ then $\mathfrak{p} \supset I_1 I_2$. However, since $\mathfrak{p}$ is prime we have either $\mathfrak{p} \supset I_1$ or $\mathfrak{p} \subset I_2$ so $V(\mathfrak{p}) \subset V(I_1)$ or $V(\mathfrak{p}) \subset V(I_2)$ proving that $V(\mathfrak{p})$ is irreducible. Conversely, if $V(I)$ is irreducible then take $x, y \in A$ such that $xy \in \sqrt{I}$ and thus,

$$\sqrt{(xy)} \subset \sqrt{I} \implies V(I) \subset V((xy)) = V((x)) \cup V((y))$$

Since $V(I)$ is irreducible we must have either $V(I) \subset V((x))$ or $V((y)) \subset V(I)$ which implies that $\sqrt{(x)} \subset \sqrt{I}$ or $\sqrt{(y)} \subset \sqrt{I}$. Therefore, $x \in \sqrt{I}$ or $y \in \sqrt{I}$ so $\sqrt{I}$ is prime and $V(I) = V(\sqrt{I})$. $\square$

**Proposition 3.0.9.** Any scheme is sober.

*Proof.* First consider the affine case $X = \mathrm{Spec}\,(A)$. Any irreducible closed set in $X$ is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p} \subset A$. Thus $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ is the unique generic point. Now let $X$ be any scheme and $Z \subset X$ a closed irreducible subset. $X$ has a cover by affine opens so take some affine open $U$ which intersects $Z$. Since $U$ is an affine scheme and $U \cap Z$ is a closed irreducible subet of $U$ there exists a unique generic point $\xi \in U \cap Z$. Because $Z$ is closed in $X$ we then have $Z \cap U \subset \overline{\{\xi\}} \subset Z$. However, $Z \cap U$ is open in $Z$ and $\overline{\{\xi\}}$ is closed in $Z$, an irreducible, which implies that either $U \cap Z$ is empty (which is false by assumption) or $\overline{\{\xi\}} = Z$. Thus $Z$ has a generic point $\xi$. Suppose that $\xi, \xi' \in Z$ were both generic points then both must be limit points of each other and thus have exactly the same open neighborhoods contradicting the fact that $Z \subset X$ is $T_0$. $\square$

## 3.1 Specialization

**Definition 3.1.1.** Let $X$ be a topological space and $\xi_1, \xi_2 \in X$. We write $\xi_1 \rightsquigarrow \xi_2$ if $\xi_2 \in \overline{\{\xi_1\}}$ i.e if $\xi_2$ is a limit point of $\xi_1$. We say $\xi_1$ is a *generalization* of $\xi_2$ and $\xi_2$ is a *specialization* of $\xi_1$.

# 4 Dimension Theory

## 4.1 Introduction

**Definition 4.1.1.** Let $X$ be a topological space. The *Krull dimension* or *combinatorial dimension* of $X$ is the maximal length of chains of irreducible closed subsets,

$$\dim(X) = \max\{n \in \mathbb{Z} \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ is a chain of closed irreducible subsets } Z_i \subset X\}$$

and $\dim X = \infty$ if there is no maximum and $\dim X = -\infty$ if $X$ is empty.

**Definition 4.1.2.** For $x \in X$ we define the dimension at $x$ as,

$$\dim_x(X) = \inf_{x \in U} \dim(U)$$

taken over open neighborhoods $U$ of $x$.

*Remark.* For any subset $S \subset X$, if $Z \subset S$ is closed irreducible then $\overline{Z} \subset X$ is closed irreducible so we get an inclusion of chains in $S$ to chains in $X$. Thus,

$$\dim S \leq \dim X$$

**Definition 4.1.3.** Let $Z \subset X$ be a closed irreducible subset. Then,

$$\operatorname{codim}(Z, X) = \{n \in \mathbb{Z} \mid Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ is a chain of closed irreducible subsets } Z_i \subset X\}$$

and for any closed subspace $Y \subset X$ we define,

$$\operatorname{codim}(Y, X) = \inf_{Z \subset Y} \operatorname{codim}(Z, X)$$

over $Z \subset Y \subset X$ closed irreducible subsets in $X$. Furthermore, for any subspace $S \subset X$ we may define,

$$\operatorname{codim}(S, X) = \operatorname{codim}\left(\overline{S}, X\right)$$

**Proposition 4.1.4.** For any subspace $Y \subset X$,

$$\dim(X) \geq \operatorname{codim}(Y, X) + \dim(Y)$$

*Proof.* Let $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ be a maximal chain of closed irreducible subset of $Y$ realizing $\dim(Y)$. Then taking closures gives a chain of irreducible closed subsets of $X$ conainted in $\overline{Y}$. Then choose a maximal chain $\tilde{Z}_i$ realizing $\operatorname{codim}\left(\overline{Z_n}, X\right)$ to give a chain,

$$\overline{Z_0} \subsetneq \cdots \subsetneq \overline{Z_n} = \tilde{Z}_0 \subsetneq \tilde{Z}_1 \subsetneq \cdots \subsetneq \tilde{Z}_k$$

Therefore, $n + k \leq \dim(X)$. However, $n = \dim(Y)$ and because $\overline{Z_n} \subset \overline{Y}$ we have,

$$k = \operatorname{codim}\left(\overline{Z_n}, X\right) \geq \operatorname{codim}(Y, X) = \operatorname{codim}(Y, X)$$

and thus,

$$\dim(X) \geq n + k \geq \operatorname{codim}(Y, X) + \dim(Y)$$

□

**Lemma 4.1.5.** If $Z \subset X$ is irreducible and $U$ is open and $U \cap Z \neq \varnothing$ then $Z \cap U$ is irreducible. Furthermore, if $Z \subset X$ is irreducible then $\overline{Z}$ is irreducible.

*Proof.* If we have closed $Z_1, Z_2 \subset X$ with $Z_1 \cup Z_2 \supset Z \cap U$ then $Z_1 \cup Z_2 \cup U^C \supset Z$ so one must cover $Z$ since it is irreducible but $Z \not\subset U^C$ so either $Z_1 \supset Z \cap U$ or $Z_2 \supset Z \cap U$.

Likewise, for closed $Z_1, Z_2 \subset X$ with $Z_1 \cup Z_2 \supset \overline{Z} \supset Z$ then by irreducibility $Z_1 \supset Z$ or $Z_1 \supset Z$ but these are closed so $Z_1 \supset \overline{Z}$ or $Z_2 \supset \overline{Z}$. $\square$

**Lemma 4.1.6.** Consider a closed subset $Y \subset X$ and an open $U \subset X$ with $U \cap Z \neq \varnothing$ for each irreducible component $Z \subset Y$. Then $\text{codim}\,(Y, X) = \text{codim}\,(Y \cap U, U)$.

*Proof.* Consider a chain of irreducibles $Z_i \supsetneq Z_{i+1}$ with $Z_0 \subset Y$. I claim that $Z_i \mapsto Z_i \cap U$ and $Z_i \mapsto \overline{Z_i}$ are inverse functions giving a bijection between closed irreducible chains in $X$ with final terms containined in $Y$ and closed irreducible chains in $U$ with final term contained in $Y \cap U$. Note, if $Z_i \subset Y \cap U$ then $\overline{Z_i} \subset Y$ since $Y$ is closed in $X$. Furthermore, $Z_i \mapsto Z_i \cap U$ remains irreducible if it is nonempty. The chain $Z_i$ realizing $\text{codim}\,(Y, X)$ must begin an irreducible component of $Y$ so we have indeed that $Z_i \cap U \neq \varnothing$.

First, $\overline{Z_i \cap U} \subset Z_i$ and is closed in $X$. Then $\overline{Z_i \cap U} \cup U^C \supset Z_i$ so because $Z_i$ is irreducible $\overline{Z_i \cap U} = Z_i$ since by assumption $Z_i \not\subset U^C$. Conversely, if $Z_i \subset U$ is a closed irreducible subset then $\overline{Z_i}$ is closed and irreducible in $X$ and $Z_i \subset \overline{Z_i} \cap U$ but $Z_i = C \cap U$ for closed $C \subset X$ so $Z_i \subset C$ and thus $\overline{Z_i} \subset C$ so $\overline{Z_i} \cap U \subset C \cap U = Z_i$ meaning $Z_i = \overline{Z_i} \cap U$. Thus we have shown these operations are inverse to eachother.

Finally, if $Z_i \cap U = Z_{i+1} \cap U$ then $\overline{Z_i \cap U} = \overline{Z_i \cap U}$ so $Z_i = Z_{i+1}$ so the chain does not degenerate. Likewise, if $\overline{Z_i} = \overline{Z_{i+1}}$ then $\overline{Z_i} \cap U = \overline{Z_{i+1}} \cap U$ so $Z_i = Z_{i+1}$. Therefore, we get a length-preserving bijection between the chains defining $\text{codim}\,(Y, X)$ and $\text{codim}\,(Y \cap U, U)$. $\square$

## 4.2 Equidimensionality

**Proposition 4.2.1.** Let $X$ be a topological space and $Z_i$ its irreducible components. Then,

$$\dim(X) = \sup_{i \in I} \dim(Z_i)$$

*Proof.* Clearly, $\dim(X) \geq \dim(Z_i)$. Furthermore, choose a maximal chain of closed irreducible subsets of $X$,

$$W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n$$

Since $W_n$ is irreducible, we must have $W_n \subset Z_i$ for some $i \in I$ so this is a chain in $Z_i$ showing that,

$$\dim(Z_i) \geq \dim(X)$$

$\square$

**Definition 4.2.2.** We say that $X$ is *equidimensional* if $\dim(Z) = \dim(X)$ for any irreducible component $Z \subset X$.

*Remark.* Equidimensionality is equivalent to: all irreducible components have the same dimension.

**Proposition 4.2.3.** Let $X$ be a topological space. Then,

$$\dim(X) = \sup_{x \in X} \dim_x(X)$$

*Proof.* Clearly $\dim(X) \geq \dim_x(X)$. Furthermore, choose a maximal chain of closed irreducibles,

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

and choose a point $x \in Z_0$. Then for any open neighborhood $x \in U$ we see that,

$$Z_0 \cap U \subsetneq Z_1 \cap U \subsetneq \cdots \subsetneq Z_n \cap U$$

is a chain of closed irreducible subsets of $U$ (since all are nonempty because they contain $x$). Thus $\dim_x(X) \geq \dim(X)$. $\square$

**Definition 4.2.4.** A space $X$ is *equicodimensional* if $\operatorname{codim}(x, X) = \dim(X)$ for every point $x \in X$.

**Definition 4.2.5.** A space $X$ is *biequidimensional* if every maximal chain of closed irreducible subsets has length $\dim(X)$.

*Remark.* If $X$ is biequidimensional this clearly implies $X$ is equidimensional, equicodimensional, and catenary but the converse is false in general. However, the converse holds if $X$ is finite dimensional and irreducible [Emerton and Gee, Lem. 2.32] (https://arxiv.org/pdf/1704.07654v2.pdf).

**Lemma 4.2.6.** If $X$ is biequidimensional then for any closed subset $Y \subset X$,

$$\dim(X) = \operatorname{codim}(Y, X) + \dim(Y)$$

*Proof.* Choose a chain of closed irreducibles achieving $\operatorname{codim}(Y, X)$ and thus terminating at some $Z \subset Y$. Then this chain may be extended to a maximal chain by adding irreducible closed subsets of $Y$ (since closed subsets of $Y$ are closed in $X$ since $Y$ is closed). By biequidimensionality, all such maximal chains have length $\dim(X)$ and thus,

$$\dim(X) \leq \operatorname{codim}(Y, X) + \dim(Y)$$

which along with the reverse innequality (which holds generally) proves the claim. $\square$

## 4.3 Catenary Spaces

**Definition 4.3.1.** A topological space $X$ is *catenary* if for every pair $Z \subset Z'$ of closed irreducible subsets,

(a) $\operatorname{codim}(Z, Z') < \infty$

(b) every maximal chain $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = Z'$ has the same length.

**Lemma 4.3.2.** Let $X$ be a topological space. Then the following are equivalent,

(a) $X$ is catenary

(b) for any triple of irreducible closed subsets $Z_1 \subset Z_2 \subset Z_3$,

$$\operatorname{codim}(Z_1, Z_3) = \operatorname{codim}(Z_1, Z_2) + \operatorname{codim}(Z_2, Z_3)$$

and $\operatorname{codim}(Z_1, Z_3)$ is finite.

## 4.4 Catenary Rings

**Definition 4.4.1.** We say a ring $A$ is *catenary* if $\mathrm{Spec}\,(A)$ is catenary as a a topological space. Explicitly, $A$ is catenary if for all pairs of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ all chains of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}'$$

can be extended to a maximal chain and all maximal chains have the same length.

**Definition 4.4.2.** A Noetherian ring $A$ is *universally catenary* if every finite type $A$-algebra is catenary.

**Proposition 4.4.3.** If $A$ is one of the following,

(a) a field

(b) a Dedekind domain

(c) a localization of a univerally catenary ring

then $A$ is universally catenary.

**Example 4.4.4.** There exist Noetherian rings of dimension two which are not universally catenary and thus there exist non catenary Noetherian rings. For an example see Tag 02JE.

## 4.5 Dimension Theory of Schemes

**Lemma 4.5.1.** Let $Z \subset X$ be a closed irreducible subset with generic point $\xi \in Z$. Then,

$$\mathrm{codim}\,(Z, X) = \dim \mathcal{O}_{X,\xi}$$

*Proof.* Take affine open neighborhood $\xi \in U = \mathrm{Spec}\,(A) \subset X$. Then for $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponding to $\xi$ we get $A_{\mathfrak{p}} = \mathcal{O}_{X,\xi}$. However, $\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U)$ and $Z \cap U = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Therefore,

$$\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U) = \mathbf{ht}\,(\mathfrak{p}) = \dim A_{\mathfrak{p}} = \dim \mathcal{O}_{X,\xi}$$

□

## 4.6 Dimension Theory for Finite Type $k$-Schemes

# 5 General Lemmata

**Lemma 5.0.1.** Let $X$ be a sober space. A closed subset $Y \subset X$ has positive codimension iff $Z$ does not contain the generic point of any irreducible component.

*Proof.* Suppose $Y$ satisfies the hypothesis. For any closed irreducible $Z \subset Y$ we know that $Z$ is contained in an irreducible component $Z'$. The inclusion $Z \subsetneq Z'$ must be proper since otherwise $Y$ would contain the generic point of $Z'$ and thus $\mathrm{codim}\,(Z, Z') \geq 1$ so $\mathrm{codim}\,(Y, X) \geq 1$. Conversely, suppose that $\xi$ is the generic point of some irreducible component and $\xi \in Y$ then $Z = \overline{\{\xi\}} \subset Y$ because $Y$ is closed. However, $\mathrm{codim}\,(Z, X) = 0$ because $Z$ is maximal with respect to closed irreducible subsets. Thus $\mathrm{codim}\,(Y, X) = 0$. □