

# 1 Affine Curves with No Immersions

**Exercise 1.0.1.** Find a smooth affine curve  $C$  with no immersion  $C \hookrightarrow \mathbb{P}_k^2$ . Unlike in plane case,  $K_X \neq 0$  is not an obstruction ( $g = 3$  example).

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**Exercise 1.0.2.** An easier version, find a curve with no immersion  $C \hookrightarrow \mathbb{A}_k^2$ . My guess is that here  $K_X \neq 0$  is an obstruction but I can't prove it.

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# 2 Curves with Intersection at One Point in Ambient Space

**Exercise 2.0.1.** First question: given a curve  $C$  and a closed point  $P \in C$  when is there a function  $f \in \Gamma(C, \mathcal{O}_C)$  such that  $V(f) = \{P\}$  (topologically, it may be nonreduced that is fine).

For affine curves: the previous question is asking: given a one-dimensional Noetherian domain  $A$  and a maximal ideal  $\mathfrak{m}_0 \in \text{Spec}(A)$  when is there  $f \in A$  such that  $\sqrt{(f)} = \mathfrak{m}_0$ .

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Suppose that  $A$  is a Dedekind domain. Then by unique factorization,  $\sqrt{(f)} = \mathfrak{m}_0$  iff  $(f) = \mathfrak{m}_0^n$  meaning that this is possible iff  $\mathfrak{m}_0$  is torsion in the ideal class group. Choosing any nontorsion point of an elliptic curve is then a counter example.

Is the class group not being torsion the only obstruction?

**Exercise 2.0.2.** Given a closed immersion  $\iota : C \hookrightarrow S$  with  $S$  a surface, given a point  $P \in C$  when does there exist a closed curve  $Z \subset S$  such that  $Z \cap C = \{P\}$ ?

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First, an example. Consider the surface  $S = C \times_k \mathbb{P}_k^1$  with  $C \hookrightarrow S$  via the closed point  $0 \in \mathbb{P}_k^1$ . Then for any closed point  $P \in C$  we can consider the curve  $\{P\} \times_k \mathbb{P}_k^1 \subset C \times_k \mathbb{P}_k^1$  and  $\{P\} \times_k \mathbb{P}_k^1 \cap C = \{P\}$ . Therefore, this is certainly possible for an arbitrary curve. Perhaps we need to fix

The affine curve  $Z \subset X$  is defined by some sheaf of ideals  $\mathcal{J} \subset \mathcal{O}_S$ . Then  $\iota^{-1}\mathcal{J} \cdot \mathcal{O}_C$  gives the sheaf of ideals for  $\iota^{-1}(Z)$ .

First, consider  $S = \mathbb{A}_k^2$  then we have  $C = V(f)$  for some  $f \in k[x, y]$  and  $Z = V(g)$  for some  $g \in k[x, y]$  we can assume these are irreducible so  $C$  and  $Z$  are integral curves. Then, under the map  $k[x, y] \rightarrow k[x, y]/(f)$  we get  $\iota^{-1}\mathcal{J} \cdot \mathcal{O}_C = g \cdot k[x, y]/(f) = (g)$  and thus we reduce exactly to the previous problem for the affine curve  $C = \text{Spec}(k[x, y]/(f))$ .

Likewise, consider  $S = \mathbb{P}_k^2$  then we have  $Z = V(I)$  for a height one homogeneous prime ideal  $I$  (CHECK THIS). I am guessing that we can take  $I = (f)$  since  $k[X_0, X_1, X_2]$  is a UFD so height one primes are principal (MAKE SURE  $f$  is HOMOGENEOUS). Then  $Z = \text{Proj}(k[X_0, X_1, X_2]/(g))$  and I think  $\iota^{-1}(Z) = \text{Proj}(k[X_0, X_1, X_2]/(f, g))$  or equivalently  $V(g)$  in  $C = \text{Proj}(k[X_0, X_1, X_2]/(f))$ . So now reduce to when  $V(g)$  on a projective curve have support at only one point.

# 3 Weakly but not Strongly Toric Curve

Find a curve which is toric (i.e. weakly  $\Delta$ -nondegenerate for some  $\Delta$ ) but is never nondegenerate i.e. (never  $\Delta$ -nondegenerate for any  $\Delta$ ).

## 4 Affines in the Plane

**Exercise 4.0.1.** Given an example of an immersed curve  $C \hookrightarrow \mathbb{A}_k^2$  such that  $C$  is not (closed) embedded in any affine open of  $\mathbb{A}_k^2$ . Likewise, give an example of an immersed curve  $C \hookrightarrow \mathbb{P}_k^2$  such that  $C$  is not (closed) embedded in any affine open of  $\mathbb{P}_k^2$ .

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## 5 Why is this so hard

**Proposition 5.0.1.** There exists a smooth affine curve  $C$  over  $k$  with no immersion  $C \hookrightarrow \mathbb{A}_k^2$  and, in particular, no immersion  $C \hookrightarrow \mathbb{G}_m^2$ . Thus, there are smooth affine curves which are not affine plane curves.

*Proof.* First, we show that if  $j : C \hookrightarrow \mathbb{A}_k^2$  is an immersion then  $\Omega_{C/k} \cong \mathcal{O}_C$  is trivial. We can factor  $j$  as  $C \hookrightarrow \bar{C} \hookrightarrow \mathbb{A}_k^2$  into an open immersion followed by a closed immersion [Stacks, Tag 03DQ]. Then  $\bar{C}$  □

Take an algebraically closed field  $k$ . Show the following,

- (a) if  $C \hookrightarrow \mathbb{P}_k^2$  is an immersion then there is a plane curve  $\bar{C} \subset \mathbb{A}_k^2$  (closed immersion) and an open immersion  $C \hookrightarrow \bar{C}$ .
- (b) for any closed curve  $\bar{C} \subset \mathbb{A}_k^2$  we have  $\Omega_{\bar{C}/k} = \mathcal{O}_{\bar{C}}$
- (c) thus, since  $C \hookrightarrow \bar{C}$  is étale we have  $\Omega_{C/k} = \mathcal{O}_C$  so it suffices to construct a smooth affine curve with nontrivial canonical bundle  $\Omega_{C/k}$ .
- (d) Choose a curve  $C$  with genus  $g(C) \geq 2$  then  $\deg \Omega_{C/k} \geq 2$  and choose a point  $P \in C$  such that  $K_X \not\sim (2g-2)[P]$  for any  $k \in \mathbb{Z}$ .
- (e) Show that  $U = C \setminus \{P\}$  is affine,
- (f) Then  $U \hookrightarrow C$  is étale so  $\Omega_{U/k} = f^* \Omega_{C/k}$  so  $K_U \sim [P]$ .
- (g) Show that this is nontrivial using the exact sequence,

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(C) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0$$

the first map sending  $1 \mapsto [P]$  so we need to show that  $K_X \not\sim (2g-2)[P]$  for any  $k$ .

## 6 Supersingular Stuff Here

## 7 TODO

Milne's Notes on Etale Cohomology

Write notes on GAGA and analytification of a Scheme

Finish K-book

Finish Milne's Notes on Shimura Varieties

## 7.1 Curves Covering Eachother

**Exercise 7.1.1.** Fix a smooth projective curve  $C$ . When does a degree  $n$  and ramification data of points on  $C$  along with a partition of  $n$  at each point (uniquely) determine a branched cover  $f : \tilde{C} \rightarrow C$  with degree  $n$  and the given ramification data?

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For example when  $C = \mathbb{P}^1$  and  $n = 2$  then the ramification data uniquely defines a hyperelliptic curve and such a hyperelliptic curve with given ramification always exists.

In general this is clearly false. E.g. take trivial ramification then we are asking does  $C$  have multiple nonisomorphism étale covers of degree  $n$ . Generically the answer will be yes there are many (prove this??).

Is this true when  $C = \mathbb{P}^1$  in general? Existence? Uniqueness?

## 8 Questions For Johan

**Exercise 8.0.1.** Which hypersurfaces  $X \subset \mathbb{P}_k^n$  are rational. I managed to prove that only hyperplanes and conics  $X \subset \mathbb{P}_k^2$  are isomorphic to  $\mathbb{P}_k^{n-1}$  but which are actually rational?

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**Exercise 8.0.2.** Which hypersurfaces  $X \subset \mathbb{P}_k^n$  are of the form  $\mathbb{P}^{a_1} \times_k \cdots \times_k \mathbb{P}^{a_r}$ ?

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This might be trivial since in projective space all lines intersect but this is not true of these schemes.

**Exercise 8.0.3.** Is there a splitting principal in algebraic geometry?

How difficult is it to compute the Atyiah class for a vector bundle? What tools are there to do this computation?

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**Exercise 8.0.4.** What is the “correct” definition of the Chern classes in algebraic geometry?

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**Exercise 8.0.5.** Ravi says: Having made a spirited case for smoothness, we should be clear that regularity is still very useful. For example, it is the only concept which makes sense in mixed characteristic. In particular,  $Z$  is regular at its points (it is a regular ring), and more generally, discrete valuation rings are incredibly useful examples of regular rings.

Why does smoothness not make sense in mixed characteristic?

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**Exercise 8.0.6.** what is the difference between the neron model and the minimal model?

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**Exercise 8.0.7.** When is a smooth hypersurface  $X \subset \mathbb{P}_k^n$  of degree  $d$  uniruled. My guess is exactly when  $d < n + 1$  at least for  $k = \mathbb{C}$ . I think I can prove this for small  $n$ . Is it true in general? What about other fields?

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**Exercise 8.0.8.** Why do people talk so much about supersingular K3 surfaces in particular? What is it about K3 surfaces which make the notion of supersingularity so interesting.

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**Exercise 8.0.9.** In the section on Etale cohomology, you make a distinction between the big and small pushforward of a sheaf on a site. Why? These seem to me to basically be the same functor. I understand that their adjoint depends on the site, is this why we distinguish them? On the same note, why do you only consider small sites for the Zariski and Etale topologies and not for fppf and other topologies?

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**Exercise 8.0.10.** Why is unramified defined as locally finite type as opposed to locally of finite presentation the way smooth and étale are?

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Because we want to include all closed immersions as unramified.

**Exercise 8.0.11.** How to define  $\pi_n^{\text{ét}}(X)$ .

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**Exercise 8.0.12.** Why does Ravi define the domain of a rational function to contain the associated points?

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This is a different concept which EGA calls a pseudo-morphism.

**Exercise 8.0.13.** Why is cohomology defined as  $\mathcal{O}_X$ -modules not as abelian sheaves, Isn't functoriality easier to prove then because  $f^{-1}$  is exact so you don't have to invert a quiver. Of course it gives the same theory, you even get the  $\mathcal{O}_X(X)$ -module structure because you can phrase being a  $\mathcal{O}_X$ -module as categorically being an action of the monoid object  $\mathcal{O}_X$  on  $\mathcal{F}$  and this is preserved by the functors  $H_{\text{Ab}}^n(X, \mathcal{F})$ .

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## 9 Questions For Stack Exchange

**Exercise 9.0.1.** The thing about a function which only cuts out one point.

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**Exercise 9.0.2.** What does supersingularity actually mean.

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**Exercise 9.0.3.** Soft question. Non mathematical questions with mathematical answers.

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