1 Angle Ranks of Abelian Varieties

1.1 Abelian Varieties over Finite Fields

Let A be an abelian variety over $\mathbb{F}_q$ then it has a Weil polynomial which is the char poly of $\mathrm{Frob}_q \circlearrowleft H^1_{\text{ét}}(A, \mathbb{Q}_\ell)$. It is monic of degree $2g$,

$$p = T^{2g} + a_1 T^{2g-1} + \cdots + a_g T^g + q a_{g-1} T^{g-1} + \cdots + q^g$$

and its roots in $\mathbb{C}$ are $\alpha_1, \dots, \alpha_{2g}$ where $\alpha_{g+1} = \overline{\alpha_i}$ and $|\alpha_i| = q^{\frac{1}{2}}$.

Theorem 1.1.1 (Honda-Tate)**.** There is basically a 1-to-1 correspondence between isogeny classes of abelian varieties over $\mathbb{F}_q$ and q-Weil polynomials with integral coeffients.

Remark. "Basically" because sometimes a q-Weil polynomial only gives an isogeny class over some field extension.

The newton polygon is for p-adic valuation normalized so that $v(q) = 1$.

Remark. Degenerate case straight line is supersingular. Supersingular abelian variety is always product of supersingular elliptic curves (DOESNT THIS MEAN THERE IS ONLY ONE? DOES HE MEAN ISOGENOUS TO?)

1.2 Angle Rank

Consider $\alpha_1, \dots, \alpha_{2g} \in \mathbb{C}$ Frobenius eigenvalues. We are looking for polynomial relations. For example $\alpha_i \alpha_{g+i} = q$. Then angle rank is,

$$\mathrm{rank}_{\mathbb{Z}} \frac{\alpha_1^{\mathbb{Z}} \alpha_2^{\mathbb{Z}} \dots \alpha_{2g}^{\mathbb{Z}}}{q^{\mathbb{Z}}} = \mathrm{rank}_{\mathbb{Z}}(\mathbb{Z} \arg(\alpha_1) + \mathbb{Z} \arg(\alpha_2) + \cdots + \mathbb{Z} \arg(\alpha_{2g}))/\mathbb{Z} 2\pi$$

1.3 Angle Rank and the Tate Conjecture

The Tate conjecture: eigenvalue q^i on $H^{2i}(A)$ is entirely explained by cycle classes menaing everything in the eigenspace is spanned by cycle classes (equivalent to twisting by i and considering invariants). This is true for $i = 1$ by Tate (similar to Lefschetz $(1,1)$-theorem for case $i = 1$ of Hodge conjecture). Also true for any A for which all q^i-eigenvalues are generated in codimension 1 which happends iff angle rank $= g$ (generic).

Example 1.3.1. A supersingular iff angle rank $= 0$ (boils down to alg integers with conjugates on unit circle).

1.4 A Theorem of Tankeev

Let $g = \dim A$ and consider A absolutely irreducible (meaning not isogenous to a product)

Theorem 1.4.1 (Tankeev, 1984)**.** If g is prime then angle rank of A is in $\{1, g-1, g\}$ and all occur.

WHY DO ABELIAN VAR OVER FIN FIELDS CORRESPOND TO CM ABELIAN VARIETIES??

Definition 1.4.2. An abelain variety A is almost ordinary if its newton polygon is $(0,0) \to (g-1, 0) \to (g+1, 1) \to (2g, g)$. This is codimension 1 in moduli.

Theorem 1.4.3 (LEnstra-Zarhin)**.** If A is almost ordinary then,

(a) if g is even then angle rank $= g$

(b) if g is odd then angle rank $\geq g - 1$.

Remark. This is also true if the newton slopes look like this 2-adically e.g. $1/3, 1/3, 1/3, 1/2, 1/2, 2/3, 2/3, 2/3$ only 2 slopes.

1.5 Slope Vectors and the angle rank.

Let $V \subset \mathbb{Q}^g$ be the subspace spanned by slope vectors. Let $\beta_i = \frac{\alpha_i}{\bar{\alpha}_i}$ then $(v(\beta_1), \ldots, v(\beta_g))$ for each valuation of $\mathbb{Q}(\beta_1, \ldots, \beta_g)$ above p. Then $\dim V =$ angle rank which is a $\mathbb{Q}$-representation of some finite group. Let $G = \operatorname{Gal}(\mathbb{Q}(\alpha_1, \ldots, \alpha_{2g})/\mathbb{Q})$ then get a sequence,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & \bar{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mathbb{Z}_2^g & \longrightarrow & \mathbb{Z}_2^g \rtimes S_g & \longrightarrow & S_g & \longrightarrow & 1 \end{array}$$

C is the code of A in the sense of "binary linear code". Then $G \circlearrowright V$ and the constracts on dimension of G-reps give constaints on the angle ranks (e.g. Tankeev).

1.6 Effects of the Code on the angle rank

Theorem 1.6.1. Suppose $\bar{G}$ acts primitively on $\langle 1, \ldots, g \rangle$ (meaning no nontrivial partition which is acted upon by the group). Notice that $(1, \ldots, 1) \in C$ corresponding to complex conjugation is C is not generated by this element then A has maximal angle rank (meaning $= g$).

Theorem 1.6.2 (Effective Zarhin)**.** Let A be abs. simple AB over $\mathbb{F}_q$ and $\dim A = g$. Let $\alpha_1, \ldots, \alpha_{2g}$ be the Frob eigenvalues and $G = \operatorname{Gal}(\mathbb{Q}(\alpha_1, \ldots, \alpha_{2g})/\mathbb{Q})$ and δ the angle rank. Then the vectors $(e_1, \ldots, e_{2g}) \in \mathbb{Z}^{2g}$ for which $\alpha_1^{e_1} \cdots \alpha_{2g}^{e_{2g}} \in q^{\mathbb{Z}}$ is generated by vectors of weight at most,

$$|G|(|G| - \delta)^3 (g\delta)^\delta$$

and we know $|G| \leq 2^g g!$.

$G = \operatorname{Gal}(\mathbb{Q}(\alpha_1, \ldots, \alpha_{2g})) \circlearrowright V$ by signed permutations because the second half of the α_j are conjugate to the first half up to multplies of q so we get only have the β are interesting and elements of Galois group exchaning α_i and α_j might change β to β^{-1}.

Theorem 1.6.3. Hodge conj for all CM AB ove $\mathbb{C}$ implies Tate for all AB over finite fields.