Math GR6262 Algebraic Geometry
Assignment # 8

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1 Problem 1

Consider the polynomial,

$$F = X_0^5 + X_1^5 + X_2^5$$

Now suppose that,

$$F = \frac{\partial F}{\partial X_i} = 0$$

simultaneously. Then,

$$5X_0^4 = 5X_1^4 = 5X_2^4 = 0$$

over characteristic zero this implies that $X_0 = X_1 = X_2 = 0$. Thus, if we consider the scheme,

$$X = \text{Proj}\left(\mathbb{Q}[X_0, X_1, X_2]/(F)\right)$$

then X is a curve over $\mathbb{Q}$ of genus,

$$g = \frac{(d-1)(d-2)}{2} = 6$$

because $d = 5$. Furthermore, X is smooth because,

$$X_{\overline{\mathbb{Q}}} = X \times_{\text{Spec}(\mathbb{Q})} \text{Spec}\left(\overline{\mathbb{Q}}\right) = \text{Proj}\left(\overline{\mathbb{Q}}[X_0, X_1, X_2]/(F)\right)$$

is smooth since we need only check this on geometric points as we have already done. Equivalently, consider the ideal,

$$D = \left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \frac{\partial F}{\partial X_2}\right) = (5X_0^4, 5X_1^4, 5X_2^4)$$

Thus as long as the characteristic is not 5 then $D = (X_0^4, X_1^4, X_2^4)$ and therefore, letting,

$$I = (F, D)$$

we see that $(X_0, X_1, X_2) \subset \sqrt{I}$ so I is an irrelevant ideal i.e. no relevant primes lie above I so X is smooth since its singular locus is empty.

2 Problem 2

Let $k = \mathbb{F}_2$ and consider the quartic surface in $\mathbb{P}^3_k$ defined by,

$$F = X_0^4 + X_0 X_1^3 + X_1 X_2^3 + X_2 X_3^3$$

Consider the derivatives,

$$\begin{aligned} F_0 &= \frac{\partial F}{\partial X_0} = X_1^3 \\ F_1 &= \frac{\partial F}{\partial X_1} = X_0 X_1^2 + X_2^3 \\ F_2 &= \frac{\partial F}{\partial X_2} = X_1 X_2^2 + X_3^3 \\ F_3 &= \frac{\partial F}{\partial X_3} = X_2 X_3^2 \end{aligned}$$

which are true because the characteristic of k is 2. Consider the homogeneous ideal $I = (F, F_0, F_1, F_2, F_3)$. Now, $X_1^3 \in I$ and thus $X_1 \in \sqrt{I}$. Furthermore, $F_1 \in \sqrt{I}$ and thus,

$$X_2^3 = F_1 - X_0 X_1^2 \in \sqrt{I} \implies X_2 \in \sqrt{I}$$

Furthermore, $F_2 \in I$ and thus,

$$X_3^3 = F_2 - X_1 X_2^2 \in \sqrt{I}$$

Finally

$$X_0^4 = F - X_1 F_1 - X_3 F_3 \in I \implies X_0 \in \sqrt{I}$$

Therefore,

$$(X_0, X_1, X_2, X_3) \subset \sqrt{I}$$

and thus I is an irrelevant ideal meaning that $\text{Proj}\,(k[X_0, X_1, X_2, X_3]/I)$ is empty since every prime above I is irrelevant i.e. contains the ideal (X_0, X_1, X_2, X_3). Thus the variety $\text{Proj}\,(k[X_0, X_1, X_2, X_3]/(F))$ is a smooth degree four surface in $\mathbb{P}^3_k$.