Mathematics GU6308 Algebraic Topology
Assignment # 3

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1 Fomenko-Fuchs Chapter 18

1.1 9

Let X be a based CW complex (at x_0) with $\pi_0(X) = \pi_1(X) = \pi_2(X) = \cdots = \pi_{n-1}(X) = 0$ and $\pi_n(X) \neq 0$. Then consider the Serre fibration $EX \to X$ which has fiber ΩX. Recall there is a canonical isomorphism $\pi_i(\Omega X) = \pi_{i+1}(X)$ and thus we see,

$$\pi_0(\Omega X) = \pi_1(\Omega X) = \cdots = \pi_{n-2}(\Omega X) = 0$$

and $\pi_{n-1}(\Omega X) \neq 0$ so the primary obstruction of $EX \to X$ is given by a section $s : X^{n-1} \to EX$ and lies in $H^n(X; \pi_{n-1}(\Omega X)) = H^n(X; \pi_n(X))$. Note, that because X is $(n-1)$-connected we can replace (up to homotopy equivalence) X with a CW complex with one 0-cell and no k-cells for $k = 1, \ldots, n-1$. Then X^{n-1} is a point so the section $s : X^{n-1} \to EX$ is simply sending the base point to the trivial loop at the base-point. The primary obstruction of $EX \to X$ is then,

$$O_s \in H^n(X; \pi_n(X))$$

defined as follows. For each n-cell D^n with attaching map $f : S^{n-1} \to X^{n-1}$ (which must be the constant map) and we get maps $h : D^n \to X$ including each n-cell. Now, consider the pullback h^*EX over $h : D^n \to X$ and the pullback of the section $s : X^{n-1} \to EX$ gives a section,

$$h^*s : \partial D^n \to h^*EX$$

First, we need to trivialize h^*EX. Choose some base point $\tilde{x}_0 \in \partial D^n$ then for any $x \in D^n$ let γ_x denote the linear path in D^n from x to $\tilde{x}_0$ which gives $h \circ \gamma_x$ a canonical path on $S^n \subset X$ from each point on S^n to the base-point x_0 (note that these paths are continuous in $x \in D^n$ but not (and in fact multi-valued) as a function of $x \in S^n$).

Such choices gives an isomorphism $h^*EX \cong D^n \times \Omega X$ as follows. Note that,

$$h^*EX = \{(x, \gamma) \mid x \in D^n \text{ and } \gamma : I \to X \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) = x\}$$

Now consider the map $(x, \gamma) \mapsto (x, (h \circ \gamma_x) * \gamma)$ giving a loop at x_0, $(h \circ \gamma_x) * \gamma \in \Omega X$. In particular, consider the section $h^*s : \partial D^n \to h^*EX$ which sends $x \mapsto (x, e_{x_0})$ where e_{x_0} is the constant path at x_0. Under the isomorphism we get $h^*s : \partial D^n \to D^n \times \Omega X$ given by $x \mapsto (x, (h \circ \gamma_x) * e_{x_0}) = (x, h \circ \gamma_x)$. Using $\partial D^n = S^{n-1}$, the section $h^*s : S^{n-1} \to D^n \times \Omega X$ defines a class $[h^*s] \in \pi_{n-1}(D^n \times \Omega X) = \pi_{n-1}(\Omega X)$ via $S^{n-1} \to \Omega X$ sending $x \mapsto h \circ \gamma_x$. This map is homotopic to the adjunction $S^{n-1} \to \Omega X$

of the inclusion $h : S^n \to X$. Therefore, $[h^* s] \in \pi_{n-1}(\Omega X) = [h] \in \pi_n(X)$. Therefore, the obstruction class,

$$O_s \in H^n(X; \pi_n(X))$$

is the class sending each cell $h : D^n \to X$ to $[h] \in \pi_n(X)$ in particular, it sends the generators $[h] \in H_n(X; \mathbb{Z})$ to $[h] \in \pi_n(X)$ (the inverse of the Hurewicz isomorphism $h_n : \pi_n(X) \to H_n(X; \mathbb{Z})$) thus $O_s = [X]$ the fundamental class of X.

2 Fomenko-Fuchs Chapter 19

2.1 1

First, note that if $F_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$ for a real bundle F then the complex conjugation map $\mathbb{C} \to \overline{\mathbb{C}}$ which is an isomorphism of $\mathbb{C}$-vector spaces induces a $\mathbb{C}$-isomorphism $F \otimes_{\mathbb{R}} \mathbb{C} \to F \otimes_{\mathbb{R}} \overline{\mathbb{C}}$ i.e. $F_{\mathbb{C}} \cong \overline{F_{\mathbb{C}}}$ as complex vector bundles.

Now suppose that E is a rank n complex vector bundle with $\sigma : E \xrightarrow{\sim} \overline{E}$. I claim that we may assume that σ is involutive i.e. $E \xrightarrow{\sigma} \overline{E} \xrightarrow{\sigma} \overline{\overline{E}} = E$ is the identity (I will justify this at the end). Now recall that $\overline{E} = E$ as real bundles but the $\mathbb{C}$-action is conjugate. Thus, consider the real sub-bundle $F = \{x \in E \mid \sigma(x) = \mathrm{id}(x)\}$ thinking of $\mathrm{id} : E \to \overline{E}$ as the $\mathbb{R}$-linear identity. Note this is not a complex sub-bundle because if $x \in F$ then $\sigma(\lambda x) = \lambda \cdot \sigma(x) = \lambda \cdot x = \bar{\lambda} x$ which is not, in general, equal to λx unless $\lambda \in \mathbb{R}$.

Viewing E as a real bundle with a complex structure $J : E \to E$ (i.e. a bundle automorphism J with $J^2 = -\mathrm{id}$) we know that the complex structure, $\bar{J} : \overline{E} \to \overline{E}$ defining $\overline{E}$ is $-J : E \to E$. Now, $\sigma : E \to \overline{E}$ is $\mathbb{C}$-linear meaning $\sigma \circ J = \bar{J} \circ \sigma$ and thus $\sigma \circ J = -J \circ \sigma$.

Notice, on the fiber that for any $v \in E_x$ we can write $v = v^+ + v^-$ with $v^{\pm} = \frac{1}{2}(v \pm \sigma v)$ where $\sigma v^{\pm} = \pm v$ since σ is involutive (the same holds for sections). Thus we get a decomposition,

$$E_x = E_x^+ \oplus E_x^-$$

into eigenspaces of σ and $F_x = E_x^+$ is the $+1$-eigenspace of σ. Since $\sigma \circ J = -J \circ \sigma$, we see that J swaps the sign of σ-eigenvalues so $J : E \to E$ acts on the fiber-wise decomposition as $J : E_x^{\pm} \to E_x^{\mp}$. In particular, since $J^2 = -\mathrm{id}$ it is invertible so $\dim_{\mathbb{R}} E_x^+ = \dim_{\mathbb{R}} E_x^- = \frac{1}{2} \dim_{\mathbb{R}} E_x = n$. Therefore, $F \subset E$ is a rank n *real* sub-bundle.

Consider the map $F \otimes_{\mathbb{R}} \mathbb{C} \to E$ given by $v \otimes \lambda \mapsto \lambda v$. On the fibers this gives $F_x \otimes_{\mathbb{R}} \mathbb{C} \to E = E_x^+ \oplus E_x^-$ and $F_x = E_x^+$ so for any $v \in E_x^+$ then $v \otimes 1 \mapsto v$. Furthermore, since $J : E_x^+ \to E_x^-$ is an isomorphism, we can write any $v' \in E_x^-$ as iv for $v \in E_x^+$ and thus $v \otimes i \mapsto iv = v'$ so $F_x \otimes_{\mathbb{R}} \mathbb{C} \to E$ is surjective. Since $\dim_{\mathbb{C}}(F \otimes_{\mathbb{R}} \mathbb{C}) = n$ ($\dim_{\mathbb{R}}(F) = n$) these bundles have equal rank over $\mathbb{C}$ so any fiber-wise $\mathbb{C}$-linear surjection is an isomorphism. Thus $E \cong F \otimes_{\mathbb{R}} \mathbb{C}$ for the real sub-bundle $F \subset E$.

Now I justify why $\sigma : E \to \overline{E}$ may be chosen to be involutive. Given any isomorphism $\varphi : E \to \overline{E}$ we can consider $\varphi \circ \varphi : E \to \overline{E} \to \overline{\overline{E}} = E$. It suffices to show that φ^2 has a $\mathbb{C}$-linear square root $\xi : E \to E$ commuting with φ since then we can take $\sigma = \varphi \circ \xi^{-1}$ and,

$$\sigma^2 = \varphi \circ \xi^{-1} \circ \varphi \circ \xi^{-1} = \varphi^2 \circ \xi^{-2} = \mathrm{id}$$

Choose a Hermitian metric on E. Now, on each fiber E_x we can choose square roots ξ_x of $\varphi_x^2 : E_x \to E_x$ (one construction uses the surjectivity of the exponential map $\exp : \mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ writing $\varphi_x^2 = e^M$ then take $\xi_x = e^{\frac{1}{2}M}$). Choosing some isomorphism $E_x \cong \mathbb{C}^n$ compatible with the Hermitian metric we can always choose a square root such that $\varphi_x(z) = \overline{\xi_x(z)}$ (where this complex conjugation is non-canonical it is induced by the choice of isomorphism $E_x \cong \mathbb{C}^n$) and thus $\langle \varphi_x(z), \xi_x(z) \rangle = \left\langle \overline{\xi_x(z)}, \xi_x(z) \right\rangle = |\xi_x(z)|^2 \geq 0$ so the quadratic form $\langle \varphi(-), \xi(-) \rangle$ is positive-definite on each fiber and, in fact, there is a unique choice of square root making the form positive-definite (since any other square root would negate some direction relative to φ_x). This gives a consistent choice of square roots on the fibers, and these ξ_x are clearly continuous on local charts so they glue to give a global automorphism $\xi : E \to E$ satisfying $\xi^2 = \varphi^2$. Furthermore, $\xi \circ \varphi = \varphi \circ \xi$ since the equality $\xi_x \circ \varphi_x = \varphi_x \circ \xi_x$ holds on fibers because, up to some choice of an isomorphism $E_x \cong \mathbb{C}^n$ the maps ξ_x and φ_x differ only by complex conjugation.

2.2 7

Consider a complex vector bundle E which we may view as a real vector bundle $E_\mathbb{R}$ of double the rank. Consider the map,

$$\mathbb{C} \otimes_\mathbb{R} E_\mathbb{R} \to E \oplus \overline{E}$$

as the sum of $\lambda \otimes x \mapsto \lambda x$ and $\lambda \otimes x \mapsto \lambda \cdot x = \bar{\lambda} x$ in $\overline{E}$. I claim this map is an isomorphism. First, note that it is clearly $\mathbb{C}$-linear (using the fact that $\overline{E}$ has conjugate $\mathbb{C}$-linear structure). Now, this map is surjective because for $(x, y) \in E \oplus \overline{E}$ consider,

$$\tfrac{1}{2} \otimes x + \tfrac{1}{2} i \otimes (-ix) + \tfrac{1}{2} \otimes y - \tfrac{i}{2} i \otimes (-iy) \mapsto (\tfrac{1}{2}x + \tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{2}y, \tfrac{1}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}y + \tfrac{1}{2}y) = (x, y)$$

Thus, the map $\mathbb{C} \otimes_\mathbb{R} E_\mathbb{R} \to E \oplus \overline{E}$ is surjective and by construction it is $\mathbb{C}$-linear of fibers. Since both sides are rank $2n$ complex vector bundles this map is an isomorphism since it is given fiber-wise by an invertible linear map.

2.3 9

Let $\iota : X \to Y$ be an immersed (or embedded) sub-manifold and TY be the tangent bundle of Y. Since $\iota : X \to Y$ is an immersion, we can an injection $\mathrm{d}\iota : TX \to \iota^* TY$ whose quotient is a vector bundle $N_Y X$ called the normal bundle so the canonical exact sequence,

$$0 \longrightarrow TX \longrightarrow \iota^* TY \longrightarrow N_Y X \longrightarrow 0$$

splits to give $\iota^* TY = TX \oplus N_Y X$.

In particular, consider the case that $Y = \mathbb{R}^m$ for some m i.e. ι gives an immersion (or embedding) into Euclidean space. Then TY is a trivial bundle so we find, $\iota^* TY = \underline{\mathbb{R}}^m$ is a trivial bundle on X. Therefore, $N_Y X \oplus TX = \underline{\mathbb{R}}^m$. There exists a perpendicular bundle E to TX such that $E \oplus TX = \underline{\mathbb{R}}^{2n}$ so we find,

$$N_Y X \oplus \underline{\mathbb{R}}^{2n} = E \oplus \underline{\mathbb{R}}^m$$

Therefore, $N_Y X$ is stably equivalent to E which is some bundle defined intrinsically on X (it is the complement to the tangent bundle TX) so for any choice of immersion $\iota : X \to Y$ into a Euclidean space the normal bundles $N_Y X$ are stably equivalent.

2.4 10

Let X be a n-dimensional oriented surface and $\iota : X \to \mathbb{R}^{n+1}$ an immersion. We know that $TX \oplus NX = \underline{\mathbb{R}}^{n+1}$. First, note that the first Stiefel-Whitney class w_1 is actually additive since,

$$w_1(E_1 \oplus E_2) = \sum_{p+q} w_p(E_1) \smile w_q(E_2) = w_1(E_1) \smile 1 + 1 \smile w_1(E_2) = w_1(E_1) + w_2(E_2)$$

Furthermore, recall that $w_1(E) = 0$ if and only if E is orientable. Since X is oriented $w_1(TX) = 0$ and clearly $w_1(\underline{\mathbb{R}}^{n+1}) = 0$ so we have,

$$w_1(TX \oplus NX) = w_1(TX) + w_1(NX) = w_1(\underline{\mathbb{R}}^{n+1}) = 0$$

but $w_1(TX) = 0$ so $w_1(NX) = 0$ proving that NX is orientable. However, $\text{rank}(TX) = n$ so NX is a line bundle since $\text{rank}(TX \oplus NX) = \text{rank}\underline{\mathbb{R}}^{n+1} = n + 1$. Then we conclude that NX is trivial since it is an orientable real line bundle (Lemma 4.0.1).

Finally, we have $TX \oplus NS = \underline{\mathbb{R}}^{n+1}$ but we have shown the normal bundle is trivial, $NS \cong \underline{\mathbb{R}}$ so we find,

$$TX \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}$$

and thus TX is stably trivial so X is stably parallelizable.

2.5 12

First, we prove that $w(E) = 0$ iff E is orientable iff E is trivial for bundles on $S^1 = \mathbb{RP}^1$. Vector bundles on $I = [0, 1]$ are trivial since I is contractible so, for a bundle $E \to S^1$ we have,

$$\begin{array}{ccc} I \times \mathbb{R}^n & \longrightarrow & E \\ \downarrow & & \downarrow \\ I & \longrightarrow & S^1 \end{array}$$

and thus E is determined by a gluing function $\phi \in \text{GL}(n, \mathbb{R})$ identifying fibers across the glued point. Now, a path $\gamma : I \to \text{GL}(n, \mathbb{R})$ gives a map $(t, x) \mapsto (t, \gamma(t)x)$ from $E_{\gamma(0)}$ to $E_{\gamma(1)}$ which is well-defined since $(0, x) \sim (1, \gamma(0)x)$ which map to $(0, \gamma(0)x)$ and $(1, \gamma(1)\gamma(0)x)$ which are equivalent in $E_{\gamma(1)}$ so this is a well-defined map. An inverse path gives the inverse so we see that the bundle is defined by the path-component of $\phi \in \text{GL}(n, \mathbb{R})$. Therefore, there are only two isomorphism classes for rank n bundles, those with positive determinant and those with negative determinant. The first class is trivial $E = \underline{\mathbb{R}}^n$ (which are clearly orientable) since we can take $\phi = \text{id}$. For the second class, we can take,

$$\phi = \text{diag}(-1, 1, \dots, 1)$$

and thus $E = \gamma \oplus \underline{\mathbb{R}}^{n-1}$ with γ the Möbius bundle. Thus E is non-orientable and has no non-vanishing sections. Clearly, $w(\underline{\mathbb{R}}^n) = 0$ and, for the second class, since E has no non-vanishing sections, there must be an obstruction on the 1-skeleton S^1 so $w(E) \neq 0$ proving the claim for S^1.

Now, we use the following fact: a bundle E on X is orientable if and only if its restriction to any loop f^*E for $f : S^1 \to X$ is trivial (Lemma 4.0.2). For a bundle E on X we know that, for any $f : S^1 \to X$ we have $f^*w(E) = w(f^*E)$ with $f^* : H^*(X, \mathbb{Z}/2\mathbb{Z}) \to H^*(S^1; \mathbb{Z}/2\mathbb{Z})$.

If $w(E) = 0$ then $w(f^*E) = 0$ so we have f^*E is trivial for each loop $f : S^1 \to X$ so E is orientable.

Conversely, if E is orientable then we use the fact that the Hurewicz map $h_1 : \pi_1(X) \to H_1(X;\mathbb{Z})$ is the abelianization (in particular surjective) so, using the universal coefficient theorem,

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(X;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_1(X;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

we see $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$ and thus,

$$H^1(X;\mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_1(X;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}/2\mathbb{Z})$$

where the map sends a cohomology class c to the map $[f : S^1 \to X] \mapsto c(f_*[S^1])$. Now, for any $f : S^1 \to X$ we have $w(f^*E) = 0$ since E is orientable so $f^*w(E) = 0$ and thus $w(E)(f_*[S^1]) = 0$ for any loop so the class $w(E) = 0$ by the above isomorphism.

2.6 13

First, let E be a rank n complex vector bundle on B. First, we construct the Euler class of $E_{\mathbb{R}}$. Let F be the unit bundle of $E_{\mathbb{R}}$ (note F is also the unit bundle of E as a complex vector bundle since the complex norm and real norm coincide). Then $e(E_{\mathbb{R}})$ is the primary obstruction class of F constructed as follows. The fiber of F is S^{2n-1} so we have no obstruction to giving a section $s : B^{2n-1} \to F$ which gives an obstruction class for extending this section to B^{2n},

$$e(E_{\mathbb{R}}) = O_s \in H^{2n}(B; \pi_{2n-1}(S^{2n-1})) = H^{2n}(B;\mathbb{Z})$$

where the isomorphism $\pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$ is given by the orientation on E (and thus F) induced by the complex structure. The Chern class $c_n(E)$ is defined by the primary obstruction of the bundle $E_1 = F$ the unit bundle (i.e. the bundle of unitary 1-frames),

$$c_n(E) = O_s \in H^{2n}(B; \pi_{2n-1}(V_{\mathbb{C}}(n,1))) = H^{2n}(B;\mathbb{Z})$$

where the isomorphism $\pi_{2n-1}(V_{\mathbb{C}}(n,1)) = \pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$ is fixed by the orientation given by the complex structure. Thus we immediately see that $e(E_{\mathbb{R}}) = c_n(E)$.

Now we consider the Stiefel-Whitney class of $E_{\mathbb{R}}$. First, we just seen that,

$$w_{2n}(E_{\mathbb{R}}) = \rho_2 e(E_{\mathbb{R}}) = \rho_2 c_n(E)$$

using the previous equalities. Now, let F_k be the bundle of orthonormal k-frames of $E_{\mathbb{R}}$. We need to consider the relation between the bundle F_k and the bundle of unitary orthonormal k-frames E_k. There is an inclusion map $E_k \hookrightarrow F_{2k}$ because we can choose the Euclidean metric on $E_{\mathbb{R}}$ to be the real part of the Hermitian metric on E and thus unitary orthonormal frames are real orthogonal (although not vice-versa) and each unitary orthonormal k-frame defines a real orthonormal $2k$-frame via sending each e_j to e_j, ie_j. Recall that these bundles have fibers $V(2n,2k)$ and $V_{\mathbb{C}}(n,k)$ respectively with,

$$\begin{aligned} \pi_i(V(2n,2k)) = 0 \quad &\text{for} \quad i < 2(n-k) \\ \pi_i(V_{\mathbb{C}}(n,k)) = 0 \quad &\text{for} \quad i < 2(n-k)+1 \end{aligned}$$

Therefore, there is no obstruction to giving a section $s : B^{2(n-k)+1} \to E_k$ which will define an obstruction class,

$$c_{n-k+1}(E) = O_s \in H^{2(n-k+1)}(B; \pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k)))$$

Via the inclusion $E_k \hookrightarrow F_{2k}$, we get a section $s : B^{2(n-k)+1} \to F_{2k}$ but the primary obstruction of the bundle F_{2k} occurs on the $2(n-k)$-skeleton since $\pi_{2(n-k)}(V(2n,2k)) \neq 0$ so the primary obstruction vanishes since we have produced an extension to the $2(n-k)+1$-skeleton. Recall that $w_j(E_{\mathbb{R}})$ is defined as follows: consider the bundle F_ℓ with $j = 2n - \ell + 1$ and its primary obstruction occurs for a section $s : B^{2n-\ell} \to F_\ell$ which is the class,

$$w_j(E_{\mathbb{R}}) = O_s \in H^{2n-\ell+1}(B; \mathbb{Z}/2\mathbb{Z})$$

which is the obstruction to finding a section $s' : B^{2n-\ell+1} \to F_\ell$. Therefore, $w_{2(n-k)+1}(E_{\mathbb{R}}) = 0$ since then $\ell = 2k$ and F_{2k} admits a section $B^{2(n-k)+1} \to F_{2k}$ as demonstrated above so the odd Stiefel-Whitney classes vanish for a complex vector bundle.

Now we compute the even classes, $w_{2(n-k+1)}$ which are defined by the obstruction class of F_{2k-1}. There is a map $E_k \hookrightarrow F_{2k} \to F_{2k-1}$ by throwing out some element of the basis. There is no obstruction to finding a section $s : B^{2(n-k)+1} \to E_k$ (using the vanishing of the homology groups above) which gives an obstruction class,

$$c_{n-k+1}(E) = O_s \in H^{2(n-k+1)}(B; \pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k))) = H^{2(n-k+1)}(B; \mathbb{Z})$$

Now, via the map $E_k \to F_{2k-1}$, this gives a section $\tilde{s} : B^{2(n-k)+1} \to F_{2k-1}$. Since,

$$\pi_{(2n-k)+1}(V(2n, 2k-1)) \neq 0$$

the obstruction class of this section gives the primary obstruction of the bundle F_{2k-1},

$$w_{2(n-k+1)} = O_{\tilde{s}} \in H^{2(n-k+1)}(B; \mathbb{Z}/2\mathbb{Z})$$

Furthermore, the map $E_k \to F_{2k-1}$ on the fiber induces,

$$\pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k)) \to \pi_{2(n-k)+1}(V(2n, 2k-1))$$

I claim this map is nontrivial so it is the unique nonzero map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, namely ρ_2, reduction modulo 2. Therefore,

$$w_{2(n-k+1)}(E_{\mathbb{R}}) = O_{\tilde{s}} = \rho_2 O_s = \rho_2 c_{n-k+1}(E)$$

proving that $w_{2j}(E_{\mathbb{R}}) = \rho_2 c_{2j}(E)$.

Now I justify that the map,

$$\pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k)) \to \pi_{2(n-k)+1}(V(2n, 2k-1))$$

is nontrivial. Consider the diagram,

$$\begin{tikzcd}
V_{\mathbb{C}}(n,k) \arrow[r] & V(2n, 2k-1) \\
V_{\mathbb{C}}(n-k+1, 1) \arrow[r] \arrow[u] & V(2(n-k+1), 1) \arrow[u] \\
S^{2(n-k)+1} \arrow[r, "\mathrm{id}"] \arrow[u, equal] & S^{2(n-k)+1} \arrow[u, equal]
\end{tikzcd}$$

where the top map is the fiber map of $E_k \to F_{2k-1}$ the middle map is given by considering a unit vector $u \in \mathbb{C}^{n-k+1}$ and sending it to the orthonormal 2-frame u, iu of $\mathbb{R}^{2(n-k+1)}$ and then forgetting the second vector to get a 1-frame i.e. $u \mapsto u$ so the identity on $S^{2(n-k)+1} \to S^{2(n-k)+1}$. Therefore, taking homotopy groups we get,

$$\begin{tikzcd}
\pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k)) \arrow[r] & \pi_{2(n-k)+1}(V(2n,2k-1)) \\
\pi_{2(n-k)+1}(V_{\mathbb{C}}(n-k+1,1)) \arrow[r] \arrow[u] \arrow[d, equal] & \pi_{2(n-k)+1}(V(2(n-k+1),1)) \arrow[u] \arrow[d, equal] \\
\pi_{2(n-k)+1}(S^{2(n-k)+1}) \arrow[r, "\mathrm{id}"] & \pi_{2(n-k)+1}(S^{2(n-k)+1})
\end{tikzcd}$$

The bottom map is the identity $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$. The left-hand upward map $\pi_{2(n-k)+1}(V_{\mathbb{C}}(n-k+1,1)) \to \pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k))$ is an isomorphism from repeated application of the LES of the fibration

$$V_{\mathbb{C}}(n-1,k-1) \hookrightarrow V_{\mathbb{C}}(n,k) \to S^{2n-1}$$

(and using induction, see the next problem for details). Finally, the right-hand upward map is a surjection from repeated application of the LES of the fibration,

$$V(n-1,k-1) \hookrightarrow V(n,k) \to S^{n-1}$$

Explicitly, we get from the LES,

$$\pi_{i+1}(S^{n-1}) \longrightarrow \pi_i(V(n-1,k-1)) \longrightarrow \pi_i(V(n,k)) \longrightarrow \pi_i(S^{n-1})$$

Thus, for $i+1 < n-1$ we get an isomorphism $\pi_i(V(n-1,k-1)) \xrightarrow{\sim} \pi_i(V(n,k))$. In particular, for $i = n-k$ this works when $k > 2$ so we have $\pi_{n-k}(V(n-k+2,2)) = \pi_{n-k}(V(n,k))$. Now, the last step would use a fibration $V(j+1,1) \hookrightarrow V(j+2,2) \to S^{j+1}$ where $j = n-k$ so, applying the LES gives,

$$\begin{tikzcd}
\pi_{j+1}(S^{j+1}) \arrow[r] \arrow[d, equal] & \pi_j(V(j+1,1)) \arrow[r] & \pi_j(V(j+2,2)) \arrow[r] & \pi_j(S^{j+1}) \arrow[d, equal] \\
\mathbb{Z} & & & 0
\end{tikzcd}$$

showing that $\pi_j(V(j+1,1)) \to \pi_j(V(j+2,2))$ is surjective and thus, in combination with the isomorphism above, $\pi_{n-k}(V(n-k+1,1)) \to \pi_{n-k}(V(n-k+2,2)) \to \pi_{n-k}(V(n,k))$ is a surjection. In the case we were interested in, we had $n \mapsto 2n$ and $k \mapsto 2k-1$ giving a surjection,

$$\pi_{2(n-k)+1}(V(2(n-k+1),1)) \twoheadrightarrow \pi_{2(n-k)+1}(V(2n,2k-1))$$

as desired.

Therefore, $\pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k)) \to \pi_{2(n-k)+1}(V(2n,2k-1))$ must be surjective since it is part of a commutative square of surjective maps. This completes the proof.

2.7 15

Consider a complex rank n vector bundle E on B. Recall that the Chern class is defined by considering the bundle of unitary k-frames E_k which has fiber $V_{\mathbb{C}}(n,k)$. Then $\pi_i(V_{\mathbb{C}}(n,k)) = 0$ for $i < 2(n-k)+1$ and thus we have no obstruction to finding a section $s : B^{2(n-k)+1} \to E_k$ the obstruction is in extending this section to the $2(n-k+1)$-skeleton which gives a class,

$$c_{n-k+1}(E) = O_s \in H^{2(n-k+1)}(B; \pi_{2(n-k)+1} V_{\mathbb{C}}(n,k))$$

We have an isomorphism, $\pi_{2(n-k+1)}(V_{\mathbb{C}}(n,k)) \cong \mathbb{Z}$ which I claim is canonically given by orientation of the complex structure. Note that, for the conjugate bundle $\overline{E}$, we have the bundle $\overline{E}_k$ which has fiber $\overline{V_{\mathbb{C}}(n,k)}$. Note that the underlying real vector bundle of $\overline{E}$ and $\overline{E}_k$ agree with E and E_k so we may choose the same section $\bar{s} = s : B^{2(n-k)+1} \to \overline{E}_k$. Therefore,

$$c_{n-k+1}(\overline{E}) = O_{\bar{s}} \in H^{2(n-k+1)}(B; \pi_{2(n-k)+1} \overline{V_{\mathbb{C}}(n,k)})$$

is an identical class but there may be a different canonical generator of $\pi_{2(n-k)+1}\overline{V_{\mathbb{C}}(n,k)}$.

There is a fibration $V_{\mathbb{C}}(n-1,k-1) \hookrightarrow V_{\mathbb{C}}(n,k) \to S^{2n-1} \subset \mathbb{C}^n$ given by sending an orthonormal frame to its first unit vector. Then, from the LES we get,

$$\pi_{i+1}(S^{2n-1}) \longrightarrow \pi_i(V_{\mathbb{C}}(n-1,k-1)) \longrightarrow \pi_i(V_{\mathbb{C}}(n,k)) \longrightarrow \pi_i(S^{2n-1})$$

When $k > 1$ take $i = 2(n-k)+1$ we get an isomorphism,

$$\pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k)) = \pi_{2(n-k)+1}(V_{\mathbb{C}}(n-1,k-1))$$

inductively, we get,

$$\pi_{2(n-k)+1}(V_{\mathbb{C}}(n,k)) = \pi_{2(n-k)+1}(V_{\mathbb{C}}(n-k+1,1)) = \pi_{2(n-k)+1}(S^{2(n-k)+1}) \cong \mathbb{Z}$$

All the isomorphisms are canonical except for the last $\pi_{2(n-k)+1}(S^{2(n-k)+1}) \cong \mathbb{Z}$ which depends on the orientation. There is a canonical orientation on $\mathbb{C}^{n-k+1}$ which induces a canonical generator of $\pi_{2(n-k)+1}(S^{2(n-k)+1})$. Conjugating the complex structure on $\mathbb{C}^{n-k+1}$ induces a factor of $(-1)^{n-k+1}$ on the orientation (since it corresponds to inverting the $n-k+1$ complex directions) and thus our oriented choice of isomorphism gives $\pi_{2(n-k)+1}(\overline{V_{\mathbb{C}}(n,k)}) = (-1)^{n-k+1}\mathbb{Z}$. Therefore, we have,

$$c_{n-k+1}(\overline{E}) = O_{\bar{s}} \in H^{2(n-k+1)}(B; (-1)^{n-k+1}\mathbb{Z})$$

Notice that $O_{\bar{s}} \mapsto O_s$ under the isomorphism induced by $V_{\mathbb{C}}(n,k) = \overline{V_{\mathbb{C}}(n,k)}$ (ignoring the complex orientation) so once we introduce the orientation which may give opposite signs to the generators of the homotopies of the above two identified spaces we get,

$$c_{n-k+1}(\overline{E}) = O_{\bar{s}} = (-1)^{n-k+1} O_s \in H^{2(n-k+1)}(B; \mathbb{Z})$$

therefore,

$$c_j(\overline{E}) = (-1)^j c_j(E) \in H^{2j}(B; \mathbb{Z})$$

Finally, if E is a rank n real line bundle. Then $E \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to its dual and thus,

$$2c_{2j+1}(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$$

2.8 18

We will use the splitting principle to prove these statements which says it suffices to check $\mathbb{Z}/2\mathbb{Z}$-characteristic class relations on bundles of the form $E = \zeta \oplus \cdots \oplus \zeta$ on $X = \mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty$. This holds because $\mathbb{Z}/2\mathbb{Z}$-characteristic classes are polynomials in the Stiefel-Whitney classes and $w_i(\zeta \oplus \cdots \oplus \zeta) = e_i(x_1, \dots, x_n)$ so a polynomial in w_i can only vanish on these bundles if it is the zero polynomial. Similar statements hold for the tautological bundle on $\mathbb{CP}^\infty$ for Chern classes.

Consider, the class,

$$w_n(\xi \otimes \zeta) - \sum_{i=0}^{n} w_i(\xi) \times x^{n-i} \in H^n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$$

where $x \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$ is the generator in the cohomology ring $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[x]$. Using the splitting principle and naturality, it suffices to check this vanishes for $X = \mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty$ and $\xi = \zeta \oplus \cdots \oplus \zeta$. We write y_i for the generators of the cohomology $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$ in the left-hand factor. Now,

$$\begin{aligned} w_n\left((\zeta \oplus \cdots \oplus \zeta) \otimes \zeta\right) &= w_n(\zeta \otimes \zeta \oplus \cdots \oplus \zeta \otimes \zeta) \\ &= \big(w(\zeta \otimes \zeta) \cdots w(\zeta \otimes \zeta)\big)_n \\ &= \big((1 + y_1 + x) \cdots (1 + y_n + x)\big)_n \end{aligned}$$

using $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$ for line bundles and thus $w(L_1 \otimes L_2) = 1 + w_1(L_1) + w_1(L_2)$. Therefore,

$$\begin{aligned} w_n\left((\zeta \oplus \cdots \oplus \zeta) \otimes \zeta\right) &= \big((1 + y_1 + x) \cdots (1 + y_n + x)\big)_n \\ &= (y_1 + x) \cdots (y_n + x) = \sum_{i=0}^{n} e_i(y_1, \dots, y_n) x^{n-i} \\ &= \sum_{i=0}^{n} w_i(\zeta \oplus \cdots \oplus \zeta) \times x^{n-i} \end{aligned}$$

proving the formula. Therefore, for any rank n bundle ξ on X we find,

$$\rho_2 e(\xi \otimes \zeta) = w_n(\xi \otimes \zeta) = \sum_{i=0}^{n} w_i(\xi) \times x^{n-i} \in H^n(X \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$$

Now we prove the corresponding formula for Chern classes. Let $\zeta_{\mathbb{C}}$ be the tautological bundle on $\mathbb{CP}^\infty$ and let $x \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ be the generator of $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[x]$ (with x having degree 2). For a rank n complex vector bundle ξ on X, consider the class,

$$c_n(\xi \otimes \zeta_{\mathbb{C}}) - \sum_{i=0}^{n} c_i(\xi) \times x^{n-i}$$

It suffices to check that this class vanishes on bundles $\xi = \zeta_{\mathbb{C}} \oplus \cdots \oplus \zeta_{\mathbb{C}}$ on $X = \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$. We write y_i for the generators $y_i \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$. Now consider,

$$\begin{aligned}
c_n((\zeta_{\mathbb{C}} \oplus \cdots \oplus \zeta_{\mathbb{C}}) \otimes \zeta_{\mathbb{C}}) &= c_n(\zeta_{\mathbb{C}} \otimes \zeta_{\mathbb{C}} \oplus \cdots \oplus \zeta_{\mathbb{C}} \otimes \zeta_{\mathbb{C}}) \\
&= \big(c(\zeta_{\mathbb{C}} \otimes \zeta_{\mathbb{C}}) \cdots c(\zeta_{\mathbb{C}} \otimes \zeta_{\mathbb{C}})\big)_n \\
&= \big((1 + y_1 + x) \cdots (1 + y_n + x)\big)_n \\
&= (y_1 + x) \cdots (y_n + x) \\
&= \sum_{i=0}^{n} e_i(y_1, \dots, y_n) \cdot x^{n-i} \\
&= \sum_{i=0}^{n} c_i(\zeta_{\mathbb{C}} \oplus \cdots \oplus \zeta_{\mathbb{C}}) \times x^{n-i}
\end{aligned}$$

so the class vanishes on bundles of the form $\xi = \zeta_{\mathbb{C}} \oplus \cdots \oplus \zeta_{\mathbb{C}}$. Therefore, for any rank n complex vector bundle ξ on X we find,

$$e(\xi \otimes \zeta_{\mathbb{C}}) = c_n(\xi \otimes \zeta_{\mathbb{C}}) = \sum_{i=0}^{n} c_i(\xi) \times x^{n-i} \in H^{2n}(X \times \mathbb{CP}^\infty; \mathbb{Z})$$

3 Milnor-Stasheff

3.1 4C

Let $\xi \subset T\mathbb{RP}^n$ be a rank-2 sub-bundle (we will restrict the case to $n = 4, 6$). Since $T\mathbb{RP}^n$ is a real bundle on a compact manifold, we may give it a metric and thus the sub-bundle $\xi \subset T\mathbb{RP}^n$ admits a orthogonal complement which decomposes $T\mathbb{RP}^n = \xi \oplus \xi^\perp$. Therefore,

$$w(T\mathbb{RP}^n) = w(\xi) \cdot w(\xi^\perp)$$

We have already computed,

$$w(T\mathbb{RP}^n) \in H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[\alpha]/(\alpha^{n+1})$$

to be,

$$w(T\mathbb{RP}^n) = \sum_{k=0}^{n+1} \binom{n+1}{k} \alpha^k \mod 2$$

Furthermore,

$$w(\xi) = 1 + a_1\alpha + a_2\alpha^2$$

Now, first restrict to the case $n = 4$ we see,

$$w(T\mathbb{RP}^4) = 1 + \alpha + \alpha^4$$

and since $\xi^\perp$ is also of rank 2 we gave,

$$w(\xi^\perp) = 1 + b_1\alpha + b_2\alpha^2$$

We need,

$$(1 + a_1\alpha + b_1\alpha^2) \cdot (1 + b_1\alpha + b_2\alpha^2) = 1 + \alpha + \alpha^4$$

However, expanding,

$$1 + (a_1 + a_2)\alpha + (a_1 b_1 + a_2 + b_2)\alpha^2 + (a_1 b_2 + a_2 b_1)\alpha^3 + a_2 b_2 \alpha^4$$

with $a_1, a_2, b_1, b_2 \in \mathbb{F}_2$. Thus we need $a_2, b_2 = 1$ since $a_2 b_2 = 1$ and $a_1 + b_1 = 1$ so $a_1 b_2 + a_2 b_1 = a_1 + b_1 = 1$ so the coefficient of α^3 is nonzero showing that the above factorization is impossible and thus no such ξ exists.

Now consider the case $n = 6$. We have,

$$w(T\mathbb{RP}^6) = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6$$

Furthermore we have,

$$w(\xi) = 1 + a_1\alpha + a_2\alpha^2$$

and

$$w(\xi^\perp) = 1 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + b_4\alpha^4$$

Therefore,

$$\begin{aligned} w(\xi) \cdot w(\xi^\perp) &= (1 + a_1\alpha + a_2\alpha^2) \cdot (1 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + b_4\alpha^4) \\ &= 1 + (a_1 + b_1)\alpha + (a_1 b_1 + a_2 + b_2)\alpha^2 + (a_1 b_2 + a_2 b_1 + b_3)\alpha^3 + (a_2 b_2 + a_1 b_3 + b_4)\alpha^4 \\ &\quad + (a_1 b_4 + a_2 b_3)\alpha^5 + a_2 b_4 \alpha^6 \end{aligned}$$

Then

$$\begin{aligned} a_2 b_4 = 1 &\quad \implies a_2 = b_4 = 1 \\ a_1 b_4 + a_2 b_3 = 1 &\quad \implies a_1 + b_3 = 1 \\ a_1 + b_1 = 1 &\quad \implies a_1 = 1 \text{ or } b_1 = 1 \text{ and } a_1 b_1 = 0 \\ a_1 b_1 + a_2 + b_2 = 1 &\quad \implies b_2 = 0 \\ a_1 b_2 + a_2 b_1 + b_3 = 1 &\quad \implies b_1 + b_3 = 1 \\ a_2 b_2 + a_1 b_3 + b_4 = 1 &\quad \implies a_1 b_3 = 0 \end{aligned}$$

Thus we have,

$$\begin{aligned} a_1 + b_1 &= 1 \\ b_1 + b_3 &= 1 \\ a_1 + b_3 &= 1 \end{aligned}$$

Then $2a_1 + b_1 + b_3 = 0$ and $2a_1 = 0$ so $b_1 + b_3 = 0$ contradicting the middle equation giving a contradiction. Therefore, such a decomposition $\gamma \oplus \gamma^\perp = T\mathbb{RP}^6$ is impossible.

4 Lemmas

Proposition 4.0.1. Any orientable real line bundle is trivial.

Proof. In general, an orientation on E is a $\mathrm{GL}^+(n, \mathbb{R})$ structure on E but $\mathrm{SL}(n, \mathbb{R})$ is a deformation retract of $\mathrm{GL}^+(n, \mathbb{R})$ so we get an $\mathrm{SL}^+(n, \mathbb{R})$-structure on E which determines a non-vanishing section $\omega \in \Gamma(X, \bigwedge^n E)$ of the top exterior power. Restricting to line bundles, $\bigwedge^n E = E$ so we get a non-vanishing section of E which trivializes the line bundle.

Alternatively, note that w_1 gives a bijection from line bundles to $H^1(X; \mathbb{Z}/2\mathbb{Z})$ and $w_1(E) = 0$ if and only if E is orientable meaning that E is trivial if and only if it is orientable. $\square$

Proposition 4.0.2. A vector bundle E on X is orientable if and only if its restriction to any loop f^*E for $f : S^1 \to X$ is trivial

Proof. The orientability of E is equivalent to the the top exterior power $\bigwedge^n E$ being a trivial line bundle. Thus it suffices to show the equivalent statement for triviality or equivalently orientability of line bundles. One direction is clear, if L is a trivial line bundle on X then for any $f : S^1 \to X$ we have f^*L is a trivial line bundle. Conversely, suppose that f^*L is orientable for any loop $f : S^1 \to X$. We need to show that L is orientable. Consider a local trivialization U_i with index set I of L on which $L|_{U_i} \xrightarrow{\varphi_i} \mathbb{R} \times U_i$. Then consider the graph Γ on vertices I and an edge for each nonempty intersection $U_i \cap U_j \neq \varnothing$. Each edge is given a sign s_{ij} which is the sign of the determinant of the transition map $\varphi_i \circ \varphi_j^{-1}$ (note $\varphi_I \circ \varphi_j^{-1} : \mathbb{R} \times U_i \to \mathbb{R}$ gives a map $U_i \to \mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^\times$ and U_i is connected so the image has well-defined sign). Then an orientation of L is equivalent to a choice of sign (equivalent to a choice of fiberwise-linear isomorphism $L|_{U_i} \xrightarrow{\varphi_i} \mathbb{R} \times U_i$) for each U_i which is compatible with the signs of the edges. We can do this by choosing some orientation on some base U_0 and for each path in the graph choosing signs according to the edges in the path. This method only goes wrong when there are two paths from U_0 to U_i which disagree with the correct choice of sign on U_i. Such paths give a loop in Γ which cannot be given an orientation since the signs induced by the edges come back to disagree with the starting choice. Then, choosing a path $\gamma : S^1 \to X$ which induces the problematic path on Γ then γ^*L cannot be orientable since a non-vanishing section of γ^*L would give a consistent choice of signs on the U_i which γ passes through. Therefore, γ^*L is non-orientable for some loop $\gamma : S^1 \to L$. $\square$