Math GR6262 Algebraic Geometry
Assignment # 9

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August 27, 2020

1 Problem 1

Let X be a scheme over a field k and $x \in X$ have residue field k in the sense that the map $X \to \mathrm{Spec}\,(k)$ induces the identity at the stalk $\mathcal{O}_{\mathrm{Spec}(k),(0)} \to \mathcal{O}_{X,x} \to k(x)$.

Let $U \subset X$ be any affine open neighborhood $U = \mathrm{Spec}\,(A)$ and $x \in U$ corresponds to $\mathfrak{p} \subset A$ then $k(x) = k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = (A/\mathfrak{p})_{\mathfrak{p}}$. Furthermore, the map $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(k)$ makes A a k-algebra compatibly with the isomorphism $k(x) = k$ i.e. the diagram commutes,

$$\begin{tikzcd} A \arrow[rr] & & k(x) \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We may factor this map via,

$$k \hookrightarrow A \longrightarrow A/\mathfrak{p} \hookrightarrow (A/\mathfrak{p})_{\mathfrak{p}} \xrightarrow{\sim} k(x)$$

which composes the the identity. Because $A/\mathfrak{p}$ is a domain, the map $A/\mathfrak{p} \hookrightarrow (A/\mathfrak{p})_{\mathfrak{p}}$ is injective. Therefore, the tower of inclusions collapses showing $A/\mathfrak{p} = k(x) = k$ which implies that $\mathfrak{p}$ is maximal since k is a field. Thus $\mathfrak{p} \in \mathrm{Spec}\,(A)$ is a closed point. Therefore, $x \in U$ is closed for each affine open neighborhood. Therefore there exists a closed $C \subset X$ such that $C \cap U = \{x\}$ and thus

$$U^C \cup \{x\} = (U \setminus \{x\})^C = (C^C \cap U)^C = C \cup U^C$$

is closed. Now let $\{U_\alpha\}$ be an affine cover of X. If $x \in U_\alpha$ then we have shown that $U_\alpha^C \cup \{x\}$ is closed otherwise $x \in U_\alpha^C$ so $U_\alpha^C \cup \{x\}$ is closed. Therefore, using the fact that U_α cover X, the set

$$\bigcap_\alpha U_\alpha^C \cup \{x\} = \left(\bigcap_\alpha U_\alpha \right) \cup \{x\} = \varnothing \cup \{x\} = \{x\}$$

is closed.

2 Tag: 029E

Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point and $s = f(x)$. Note that $\mathrm{Spec}\,(k(x)[\epsilon]) = \{(\epsilon)\}$ and $\epsilon^2 = 0$. Consider the commutative diagram,

$$\begin{tikzcd}
\mathrm{Spec}\,(k(x)) \arrow[r] \arrow[rr, bend left] \arrow[dr] & \mathrm{Spec}\,(k(x)[\epsilon]) \arrow[r, dashed, "q"] \arrow[d] & X \arrow[d, "f"] \\
& \mathrm{Spec}\,(k(s)) \arrow[r] & S
\end{tikzcd}$$

where $\mathrm{Spec}\,(k(x)) \to \mathrm{Spec}\,(k(x)[\epsilon])$ is induced by the quotient map $k(x)[\epsilon] \to k(x)[\epsilon]/(\epsilon) = k(x)$ and $\mathrm{Spec}\,(k(x)[\epsilon]) \to \mathrm{Spec}\,(k(s))$ is induced by the inclusion $k(s) \to k(x)[\epsilon]$ and the maps $\mathrm{Spec}\,(k(x)) \to X$ and $\mathrm{Spec}\,(k(s)) \to S$ are the canonical maps inducing the identity at the residue field.

Given a morphism $q : \mathrm{Spec}\,(k(x)[\epsilon]) \to X$ making the diagram commute we may consider the corresponding maps at stalks,

$$\begin{tikzcd}
k(x) & k(x)[\epsilon] \arrow[l] & \mathcal{O}_{X,x} \arrow[l, "q^{\#}"'] \arrow[ll, bend right] \\
& k(s) \arrow[ul] \arrow[u] & \mathcal{O}_{S,s} \arrow[l] \arrow[u, "f^{\#}"']
\end{tikzcd}$$

Consider the restriction $q^{\#} : \mathfrak{m}_x \to (\epsilon) \subset k(x)[\epsilon]$ since this map is local its image lies in (ϵ) the maximal ideal of $k(x)[\epsilon]$. Then $q^{\#}(\mathfrak{m}_x^2) \subset (\epsilon^2) = 0$ and thus $\mathfrak{m}_x^2 \subset \ker q^{\#}$. Furthermore, by the commutativity of the diagram, the map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \xrightarrow{q^{\#}} k(x)[\epsilon]$ factors through $k(s)$ and thus $q^{\#}(\mathfrak{m}_s \mathcal{O}_{X,x}) = 0$ so $\mathfrak{m}_s \mathcal{O}_{X,x} \subset \ker q^{\#}$. Thus we may factor,

$$\begin{tikzcd}
\mathfrak{m}_x \arrow[rr, "q^{\#}"] \arrow[dr] & & (\epsilon) \cong k(x) \\
& \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \arrow[ur] &
\end{tikzcd}$$

Furthermore, $\mathcal{O}_{X,x} \to k(x)[\epsilon] \to k(x)$ is the identity so the induced map,

$$\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to (\epsilon)$$

is $k(x)$-linear.

Conversely, suppose that $k(x) = k(s)$. Given the diagram, the doted morphism is uniquely determined on the underlying topological spaces since it must send the unique point of $\mathrm{Spec}\,(k(x)[\epsilon])$ to x. Therefore it suffices to show that a local stalk map $q^{\#} : \mathcal{O}_{X,x} \to k(x)[\epsilon]$ is uniquely determined by a $k(x)$-linear map,

$$z : \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to k(x)$$

First, note that since $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is local we have maps,

$$\mathcal{O}_{S,s}/\mathfrak{m}_s \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x$$

whose composition gives the natural map $k(s) \to k(x)$ which we assume to be an isomorphism. Denote $k(s) = k(x) = k$ then the above maps give $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ a natural k-algebra structure. The projection map (defined since $\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x} \subset \mathfrak{m}_x$),

$$\frac{(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x})}{\mathfrak{m}_x^2} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k$$

has kernel $\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x})$ giving a canonical decomposition as k-modules,

$$\frac{(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x})}{\mathfrak{m}_x^2} = \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}} = k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}}$$

Therefore, we get a map $q : \mathcal{O}_{X,x} \to k[\epsilon]$ via,

$$\mathcal{O}_{X,x} \to \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}} \to k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}} \xrightarrow{\mathrm{id} \oplus \epsilon z} k[\epsilon]$$

where the last map sends $(a, m) \mapsto a + z(m)\epsilon$. I claim that this map makes the diagram commute and is unique. First, it is clear that restructing q to $\mathfrak{m}_x$ recovers the map z with image embdded as $k\epsilon \subset k[\epsilon]$. The constructed map $q : \mathcal{O}_{X,x} \to k[\epsilon]$ is a priori k-linear when it desends to a map $\mathcal{O}_{X,x}/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}) \to k[\epsilon]$ but we additionally need to show that q is a ring map. For $a, b \in \mathcal{O}_{X,x}$ then we write $a = \bar{a} + m_a$ and $b = \bar{b} + m_b$ in $\mathcal{O}_{X,x}/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x})$ where $m_a, m_b \in \mathfrak{m}_x$ and $\bar{a}, \bar{b} \in k$. Then,

$$ab = (\bar{a} + m_a)(\bar{b} + m_b) = \bar{a}\bar{b} + \bar{a}m_b + \bar{b}m_a + m_a m_b = \bar{a}\bar{b} + \bar{a}m_b + \bar{b}m_a$$

since $m_a m_b \in \mathfrak{m}_x^2$. Then applying q we get,

$$q(ab) = \bar{a}\bar{b} + \bar{a}z(m_b)\epsilon + \bar{b}z(m_a)\epsilon = \bar{a}\bar{b} + (\bar{a}z(m_b) + \bar{b}z(m_a))\epsilon$$

and furthermore,

$$\begin{aligned} q(a)q(b) &= (\bar{a} + z(m_a)\epsilon)(\bar{b} + z(m_b)\epsilon) = \bar{a}\bar{b} + (\bar{a}z(m_b) + \bar{b}z(m_a))\epsilon + z(m_a)z(m_b)\epsilon^2 \\ &= \bar{a}\bar{b} + (\bar{a}z(m_b) + \bar{b}z(m_a))\epsilon \\ &= q(ab) \end{aligned}$$

and thus $q : \mathcal{O}_{X,x} \to k[\epsilon]$ is a ring map. Notice that,

$$\mathrm{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, k[\epsilon]) = \mathrm{Der}_{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, k) = \mathrm{Hom}_k\left(\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}), k\right)$$

Next, the diagram commutes because q composed with $k[\epsilon] \to k$ via $\epsilon \mapsto 0$ sends $a \mapsto \bar{a}$ and thus equals the projection $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k$. Furthermore, $\mathcal{O}_{S,s} \to k(s) \to k(x)[\epsilon]$ is exactly given by $\mathcal{O}_{S,s} \to \mathcal{O}_{S,s}/\mathfrak{m}_s \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x) \subset k(x)[\epsilon]$ which equals $q \circ f^{\#}$ because the image of $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is $k = \mathcal{O}_{S,s}/\mathfrak{m}_s \subset \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ so q sends the image of $\mathcal{O}_{S,s}$ to $k(s) = k(x) \subset k(x)[\epsilon]$ under the projection. Since the diagram commutes, it suffices to show that such a construction will recover the origional map $q^{\#} : \mathcal{O}_{X,x} \to k[\epsilon]$. The difference $\tilde{q} = q - q^{\#}$ is a map $\mathcal{O}_{X,s} \to k[\epsilon]$ which factors through,

$$\frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}} = k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}}$$

but is zero on each factor because q and $q^{\#}$ agree on $\mathcal{O}_{X,x} \to k$ and on $\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x})$ by construction. Thus $\tilde{q} = 0$ since it factors though the zero map on each factor of the quotient. Therefore, $q = q^{\#}$ proving the result.

3 Tag: 029G

Let K be a field then consider the diagram of schemes,

$$\begin{tikzcd}
\mathrm{Spec}\,(K) \arrow[r] \arrow[d] & \mathrm{Spec}\,(K[\epsilon_1]) \arrow[d] \\
\mathrm{Spec}\,(K[\epsilon_2]) \arrow[r] & \mathrm{Spec}\,(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2))
\end{tikzcd}$$

we are asked to show that this diagram is a pushout in the category of schemes. Let X be any scheme and consider a commutative diagram,

$$\begin{tikzcd}
\mathrm{Spec}\,(K) \arrow[r] \arrow[d] & \mathrm{Spec}\,(K[\epsilon_1]) \arrow[d] \arrow[ddr, bend left, "f"] & \\
\mathrm{Spec}\,(K[\epsilon_2]) \arrow[r] \arrow[drr, bend right, "g"'] & \mathrm{Spec}\,(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \arrow[dr, dashed] & \\
 & & X
\end{tikzcd}$$

Each affine scheme has one point so a map $\mathrm{Spec}\,(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \to X$ is given by choosing a point $x \in X$ and map $\mathcal{O}_{X,x} \to K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)$. We chose the point $x \in X$ as the image of f which equals the image of g. The sheaf maps (which on a one point space are equivalent to the maps on the stalk) must satisfy the diagram,

$$\begin{tikzcd}
K & K[\epsilon_1] \arrow[l] & \\
K[\epsilon_2] \arrow[u] & K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2) \arrow[l, "\pi_2"] \arrow[u, "\pi_1"'] & \\
 & & \mathcal{O}_{X,x} \arrow[ul, dashed] \arrow[uul, bend right, "f^\#"'] \arrow[ull, bend left, "g^\#"]
\end{tikzcd}$$

However, $K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2) = K[\epsilon_1] \times_K K[\epsilon_2]$ is the pullback in the category of rings and thus there exists a unique map $\mathcal{O}_{X,x} \to K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)$ making the diagram commute. Since the topological part is fixed this is equivalent to a giving a unqiue morphism of schemes $\mathrm{Spec}\,(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \to X$ such that the first diagram commutes. This proof works because $K[\epsilon_1] \times_K K[\epsilon_2]$ is the pullback in the category of rings making (by the antiequivalence of the Spec functor) the origional diagram a pushout in the category of affine schemes. However, any morphism $\mathrm{Spec}\,(k[\epsilon_i]) \to X$ factors through an open immersion of some affine patch because the image is a single point which must lie in some affine open. Therefore, this pushout diagram in the category of affine schemes is a pushout in the category of schemes.