**Mathematics GU6308 Algebraic Topology**
**Assignment # 4**

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# 1 Maps of Hopf Invariant Two

Recall that the Hopf invariant is a integer $h(f) \in \mathbb{Z}$ defined for maps $f : S^{2n-1} \to S^n$ as follows.

**Definition 1.0.1.** Let $f : S^{2n-1} \to S^n$ be a continuous map. Then consider $C_f = D^{2n} \cup_f S^n$. Choosing generators we have $H^n(C_f; \mathbb{Z}) = \alpha\mathbb{Z}$ and $H^{2n}(C_f; \mathbb{Z}) = \beta\mathbb{Z}$. Then,

$$\alpha^2 \in H^{2n}(C_f; \mathbb{Z}) \implies \alpha^2 = h(f)\beta$$

*Remark.* Notice that when $n$ is odd $\alpha^2 = \alpha \smile \alpha = 0$ since $\alpha$ has odd degree. Therefore, we may restrict our consideration to maps $f : S^{4n-1} \to S^{2n}$.

**Proposition 1.0.2.** The Hopf invariant gives a homomorphism $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$ with the following properties,

(a) if $n$ is odd then $h = 0$ (since $\alpha \smile \alpha = 0$ in odd $n$).

(b) for the Hopf fibration $H : S^3 \to S^2$ then $C_f = S^2 \cup_H D^4 = \mathbb{CP}^2$ and $H^*(\mathbb{CP}^2; \mathbb{Z}) = \mathbb{Z}[x]/(x^3)$ so the generator of $H^2(\mathbb{CP}^2; \mathbb{Z})$ squares to the generator of $H^4(\mathbb{CP}^2; \mathbb{Z})$ which implies that $h(H) = 1$. In particular, $h : \pi_3(S^2) \xrightarrow{\sim} \mathbb{Z}$ sending $H \mapsto 1$.

Our main result is the following.

**Theorem 1.0.3.** For all $n$, there exists a map $f : S^{4n-1} \to S^{2n}$ with Hopf invariant: $h(f) = 2$.

To prove this theorem, we consider the following spaces.

## 1.1 The James Restricted Product

**Definition 1.1.1.** Let $(X, e)$ be a based topological space. Define the *James restricted product* as the following quotient space,

$$J_k(X) = X^k / \sim$$

where we identify $(x_1, \dots, x_i, e, \dots, x_k) \sim (x_1, \dots, e, x_i, \dots, x_k)$. Furthermore, we can define the total James space, $J(X) = \varinjlim J_m(X)$.

**Example 1.1.2.** We have $J_1(X) = X$ and $J_2(X) = X \times X/(x, e) \sim (e, x)$.

When $X$ is a CW complex, $J_m(X)$ inherits a CW complex structure from the product CW structure on $X$. Explicitly, we glue together the sub-complexes with one coordinate fixed at $e$. These James restricted products are especially interesting for us in the case of spheres in which case the cohomology is particularly easy to understand.

**Theorem 1.1.3.** Fix even $n > 0$. Then,

$$H^q(J(S^n); \mathbb{Z}) = \begin{cases} \mathbb{Z} & n \mid q \\ 0 & \text{else} \end{cases}$$

Let $\alpha_k \in H^{nk}(J(S^n); \mathbb{Z})$ be a generator. If $n$ is even then for each $k \geq 1$ we have $\alpha_1^k = \pm k! \cdot \alpha_k$.

*Proof.* Let $S^n$ have its usual CW structure $e^0 \cup e^n$. Then we get a product-quotient CW structure on $J(S^n)$ which is $e^0 \cup e^n \cup e^{2n} \cup e^{3n} \cup \cdots$. Therefore, we immediately see that $H^q(J(S^n); \mathbb{Z}) = 0$ whenever $n \not| q$. Furthermore, assuming $n > 1$ (we only need this case) the cellular chain complex is $C_{nk} = \mathbb{Z}$ and otherwise $C_q = 0$ so the complex has segments,

$$\cdots \longrightarrow 0 \longrightarrow C_{nk} \longrightarrow 0 \longrightarrow \cdots$$

Therefore, $H^{nk}(J(S^n); \mathbb{Z}) = \alpha_k \mathbb{Z}$ generated by $\alpha_k$ which is dual to the $nk$-cell $e^{nk}$. It remains to compute the cup product structure.

Consider the quotient map $q : (S^n)^k = S^n \times \cdots \times S^n \to J_k(S^n)$. Now consider $H^n((S^n)^k; \mathbb{Z})$. By Kunneth,

$$H^n((S^n)^k; \mathbb{Z}) = \bigoplus_{i=1}^{k} x_i \cdot H^n(S^n; \mathbb{Z}) = x_1 \mathbb{Z} \oplus \cdots \oplus x_k \mathbb{Z}$$

where $x_i \in H^n((S^n)^k; \mathbb{Z})$ is the generator dual to the $n$-cell of $(S^n)^k$ corresponding to each $S^n$. Since the map $q : (S^n)^k \to J_k(S^n)$ glues these $n$-cells to form the singular $n$-cell $e^n$ we find that,

$$q^*(\alpha_1) = x_1 + \cdots + x_k$$

Furthermore, by the Kunneth formula,

$$H^*((S^n)^k; \mathbb{Z}) = \bigotimes_{i=1}^{k} H^*(S^n; \mathbb{Z}) = \bigotimes_{i=1}^{k} \mathbb{Z}[\alpha_i]/(\alpha_i^2) = \mathbb{Z}[\alpha_1, \dots, \alpha_k]/(\alpha_1^2, \dots, \alpha_k^2)$$

which. when $n$ is even, is a commutative ring (we need not worry about factors of $-1$ in definition of the product on tensors) with $\alpha_i$ in degree $n$. Therefore, for $1 \leq \ell \leq k$ we find,

$$q^*(\alpha_k) = \sum_{i_1 < \cdots < i_\ell} x_{i_1} \smile \cdots \smile x_{i_\ell}$$

because the unique $n\ell$-cell $e^{n\ell}$ of $J_k(S^n)$ is the gluing of the $nk$-cells of $(S^n)^k$,

$$\{e^n_{i_1} \times \cdots \times e^n_{i_\ell} \mid i_1 < \cdots < i_\ell\}$$

where the other factors are $e^0$. Furthermore,

$$q^*(\alpha_1^k) = (x_1 + \cdots + x_k)^k = k! \cdot x_1 \smile \cdots \smile x_k = k! \cdot q^*(\alpha_k)$$

The map $(S^n)^k \to J_k(S^n) \hookrightarrow J(S^n)$ induces an isomorphism $q^* : H^{nk}(J(S^n); \mathbb{Z}) \xrightarrow{\sim} H^{nk}((S^n)^k; \mathbb{Z})$ because $H^{nk}(J(S^n); \mathbb{Z}) = \alpha_k \mathbb{Z}$ and $q^*(\alpha_k) = \alpha_1 \smile \cdots \smile \alpha_k$ which generates $H^{nk}((S^n)^k; \mathbb{Z})$. Therefore,

$$\alpha_1^k = k! \cdot \alpha_k$$

$\square$

### 1.2 Proof of the Main Theorem

Let $n > 0$ be an even number. We wish to construct a map $f : S^{2n-1} \to S^n$ with $h(f) = \pm 2$. We consider, explicitly, the space $J_2(S^n) = S^n \times S^n/(x,e) \sim (e,x)$. Consider the cell structure,

$$S^n = \{e\} \cup D^n$$

Then we get a cell decomposition,

$$J_2(S^n) = \{e\} \cup D^n \cup D^{2n} = S^n \cup D^{2n}$$

since the product cells $\{e\} \times D^n$ and $D^n \times \{e\}$ are glued together. Therefore, the map,

$$f : S^{2n-1} = \partial D^{2n} \to J_2(S^n) \to (J_2(S^n))^{2n-1} = S^n$$

gives a presentation $J_s(S^n) = C_f = S^n \cup_f D^{2n}$. I claim that $h(f) = \pm 2$. Indeed, consider a generator $\alpha_1 \in H^n(C_f; \mathbb{Z}) = H^n(J_2(S^n); \mathbb{Z})$ and a generator $\alpha_2 \in H^{2n}(C_f; \mathbb{Z}) = H^{2n}(J_2(S^n); \mathbb{Z})$. Then by Theorem 1.1.3, we have $\alpha_1 \smile \alpha_1 = \pm 2\alpha_1$ showing that $h(f) = \pm 2$.

# 2 K-Theory of Projective Space

### 2.1 K-Theory

Recall that for a (paracompact) space $X$, we define the $K$-theory of $X$, $K(X)$ to be the Grothendieck group of the exact category of complex vector bundles on $X$ with short exact sequences (which automatically split). Then $K(X)$ becomes a ring under the tensor product operation. Then $K$ becomes a contravariant functor from spaces to rings. We make the following definitions of the $K$-groups.

**Definition 2.1.1.** For a (connected paracompact) space $X$, define,

(a) $\tilde{K}(X) = \ker(K(X) \to K(*) = \mathbb{Z})$

(b) $\tilde{K}^{-q}(X) = \tilde{K}(\Sigma^q X)$

(c) $K^{-q}(X) = \tilde{K}^{-q}(X \sqcup *)$

(d) $K(X, A) = \tilde{K}(X/A)$

Then we have the following important results about $K$-theory.

**Proposition 2.1.2.** Let $(X, A)$ be a CW pair with $A \xrightarrow{\iota} X \xrightarrow{q} X/A$. Then there is an associated long exact sequence of $K$-theory,

$$\cdots \longrightarrow K^{-n}(X,A) \xrightarrow{q^*} K^{-n}(X) \xrightarrow{\iota^*} K^{-n}(A) \longrightarrow K^{-n+1}(X,A) \xrightarrow{q^*} K^{-n+1}(X) \longrightarrow \cdots$$

**Theorem 2.1.3** (Bott)**.** There is a periodicity of $K$-theory, $\tilde{K}(X) \xrightarrow{\sim} \tilde{K}(\Sigma^2 X) = \tilde{K}^{-2}(X)$.

*Remark.* This periodicity allows us to define $\tilde{K}^q(X) = \tilde{K}^{q-2k}(X)$ for $2k > q$. Furthermore, by periodicity, only $\tilde{K}^0(X)$ and $\tilde{K}^1(X)$ are important thus motivating the following definition.

**Definition 2.1.4.** $K^*(X) = K^0(X) \oplus K^1(X)$. Furthermore, we can give $K^*(X)$ a $K^0(X)$-algebra structure. Then $K^*(X) \cong K(X \times S^1)$.

**Proposition 2.1.5.** We have the following explicit computations,

(a) $K(S^{2n}) = \mathbb{Z}[H]/(H-1)^2$ so $\tilde{K}^0(S^{2n}) = \mathbb{Z}$

(b) $K(S^{2n+1}) = \mathbb{Z}$ so $\tilde{K}^0(S^{2n+1}) = 0$.

## 2.2 $G$-Spaces

**Definition 2.2.1.** Let $G$ be a topological group. A *$G$-space* is a topological space along with a continuous action $\rho : G \times X \to X$. A *morphism* of $G$-spaces is a continuous map $f : X \to Y$ which commutes with the $G$-action. We say a vector bundle $\pi : E \to X$ is a *$G$-bundle* if $E$ is a $G$-space with a linear action and $\pi : E \to X$ is a morphism of $G$-spaces.

**Proposition 2.2.2.** Suppose that $G \circlearrowright X$ freely. Then there is an equivalence of categories between the category of $G$-vector bundles on $X$ and the category of vector bundles on $X/G$.

*Proof.* We give a sketch. Given a $G$-vector bundle $E \to X$ the projection is $G$-equivariant and thus we get a quotient $E/G \to X/G$ which is a vector bundle since $G$ acts freely so $E/G \to X/G$ is locally isomorphic to $E \to X$. Conversely, given a vector bundle $V \to X/G$ consider the map $\pi : X \to X/G$ and take the vector bundle $\pi^* V \to X$. However, $\pi^* \hookrightarrow X \times V$ and $X \times V$ has a natural $G$-action via $g \cdot (v, x) = (v, g \cdot x)$ giving an action of $\pi^* V$ compatible with the projection $\pi^* V \to X$. These constructions are inverse. □

**Definition 2.2.3.** Let $G$ be a finite discrete group and $X$ a $G$-space. Let $\mathrm{Vect}_G(X)$ denote the category of $G$-vector bundles on $X$. The set of isomorphism classes forms a commutative monoid under $\oplus$. Then let $K_G(X)$ be the group completion which is a ring under $\otimes$.

**Example 2.2.4.** If $G = 1$ then $K_G(X) = K(X)$.

**Example 2.2.5.** If $X = *$ then $\mathrm{Vect}_G(X)$ is the category of finite dimensional $G$-representations. Then $K_G(X) = R(G)$ which is the Grothendieck group of $G$-representations.

## 2.3 Thom Isomorphism

**Definition 2.3.1.** Let $E \to X$ be a vector bundle. Then we define the unit sphere bundle $S(E)$ and the unit ball bundle $B(E)$. Then the *Thom space* is $X^E = B(E)/S(E)$. Note that,

$$K(B(E), S(E)) = \tilde{K}(X^E)$$

Furthermore, the exterior bundle $\Lambda^*(E)$ defines a vector bundle $\lambda_E \in \tilde{K}(X^E)$.

**Proposition 2.3.2.** Let $E$ be a decomposable vector bundle over $X$. Then $\tilde{K}^*_G(X^E)$ is a free $K^*_G(X)$-module with $\lambda_E$ as generator.

*Proof.* Atiyah Proposition 2.7.2. □

**Theorem 2.3.3.** Let $X$ be a $G$-space such that $K^1_G(X) = 0$ and $E$ be a decomposable $G$-vector bundle. Let $S(E)$ be the associated sphere bundle then there is an exact sequence,

$$0 \longrightarrow K^1_G(S(E)) \longrightarrow K^0_G(X) \xrightarrow{\varphi} K^0_G(X) \longrightarrow K^0_G(S(E)) \longrightarrow 0$$

where $\varphi$ is multiplication by,

$$\lambda_E^{-1} = \sum (-1)^i [\Lambda^i E]$$

*Proof.* Consider the pair $(B(E), S(E))$ where $B(E)$ is the unit ball bundle. Then there is a long exact sequence in $K$-theory,

$$
\begin{tikzcd}
\cdots \arrow[r] & K^{-1}(B(E), S(E)) \arrow[r] & K_G^{-1}(B(E)) \arrow[r] & K_G^{-1}(S(E)) \arrow[dll, rounded corners, to path={}] \\
& K_G^0(B(E), S(E)) \arrow[r] & K_G^0(B(E)) \arrow[r] & K_G^0(S(E)) \\
& K_G^1(B(E), S(E)) \arrow[r] & K_G^1(B(E)) \arrow[r] & \cdots
\end{tikzcd}
$$

but $B(E)$ is homotopy equivalent to $X$. Therefore, we get,

$$
\begin{aligned}
K_G^1(B(E)) &= K_G^1(X) = 0 \\
K_G^0(B(E)) &= K_G^0(X)
\end{aligned}
$$

which gives an exact sequence (using Bott periodicity),

$$
0 \longrightarrow K_G^1(S(E)) \longrightarrow K_G^0(B(E), S(E)) \longrightarrow K_G^0(X) \longrightarrow K_G^0(S(E)) \longrightarrow K_G^1(B(E), S(E)) \longrightarrow 0
$$

However, the Thom space $X^E = B(E)/S(E)$ gives $K^*(B(E), S(E)) = \tilde{K}^*(X^E)$ which we have shown is a graded free $K^*(X)$-module with $\lambda_E$ generating. Therefore,

$$
\begin{aligned}
K_G^0(B(E), S(E)) &= \lambda_E \cdot K_G^0(X) \\
K_G^1(B(E), S(E)) &= 0
\end{aligned}
$$

so we get the required exact sequence,

$$
0 \longrightarrow K_G^1(S(E)) \longrightarrow K_G^0(X) \xrightarrow{\lambda_E^{-1}} K_G^0(X) \longrightarrow K_G^0(S(E)) \longrightarrow 0
$$

□

**Lemma 2.3.4.** Let $X$ be a point then $K_G^1(X) = 0$.

*Proof.* Since $K_G^*(X) = K_G(X \times S^1)$ it suffices to show that the map $K_G(S^1) \to K_G(*)$ is an isomorphism where $S^1$ is given a trivial $G$-action. Then.

$$
K_G(S^1) \cong K(S^1) \otimes R(G) \cong K(*) \otimes R(G) \cong K_G(*)
$$

where we used $K(S^1) \cong K(*) = \mathbb{Z}$. □

**Corollary 2.3.5.** Let $G$ be a cyclic group and $E$ a $G$-module with $S(E)$ having a free $G$-action. Then there is an exact sequence,

$$
0 \longrightarrow K^1(S(E)/G) \longrightarrow R(G) \longrightarrow R(G) \longrightarrow K^0(S(E)/G) \longrightarrow 0
$$

*Proof.* Note that finite $G$-representations are automatically semi-simple so the $G$-module $E$ is a decomposable bundle over a point. Then the result follows by applying the previous exact sequence to a point using that $K_G^1(X) = 0$. Furthermore, we use that $K_G^*(X) = K^*(X/G)$ when $G$ acts freely on $X$. □

## 2.4 Application to the Case of Projective Space

*Remark.* For $E = \mathbb{C}^n$ we have $S(E) = S^{2n-1}$. Let $G = \mathbb{Z}/2\mathbb{Z}$ which acts freely on $E$ via $x \mapsto -x$. Then $G$ acts on $S(E)$ freely via $x \mapsto -x$, the antipodal action. Therefore, $S(E)/G = \mathbb{RP}^{2n-1}$. This will allow us to apply the above sequence. First we need to understand the representation theory of $G$. First, recall that by Maschke's theorem, $G$-representations are semi-simple so need only understand irreducible representations.

**Theorem 2.4.1.** Let $G$ be a finite abelian group. Then all irreducible $G$-representations are one-dimensional i.e. are characters.

*Proof.* Let $\rho : G \to \mathrm{Aut}\,(V)$ be an irreducible $G$-representation. Then for any $g, h \in G$ we have,

$$\rho(g) \circ \rho(h) = \rho(gh) = \rho(hg) = \rho(h) \circ \rho(g)$$

Therefore, $\rho(g) : V \to V$ is a $G$-morphism. Since $V$ is irreducible, by Shur's Lemma, $\rho(g) = \lambda_g \mathrm{id}$ and thus $\rho : G \to \mathbb{C}^\times$ is a character. □

**Example 2.4.2.** Representations of $G = \mathbb{Z}/2\mathbb{Z}$ are thus direct sums of characters. The characters $\rho : G \to \mathbb{C}^\times$ are determined by the image of 1. We must have $\rho(1) = \pm 1$. These options are 1 the trivial character and $\rho$ the nontrivial character. Furthermore, $\rho \otimes \rho : G \to \mathbb{C}^\times$ is trivial since $(-1)^2 = 1$. Therefore, representations are sums,

$$n + m\rho := 1 \oplus \cdots 1 \oplus \rho \oplus \cdots \oplus \rho$$

for $n, m \geq 0$ with the relation $\rho^{\otimes 2} = 1$. Thus, taking the group completion we find,

$$R(G) = \mathbb{Z}[\rho]/(\rho^2 - 1)$$

Furthermore, the map $\lambda_{-1} : R(G) \to R(G)$ is given by multiplication by,

$$\lambda_{-1} = \sum (-1)^i \rho^i = (1 - \rho)^n$$

**Proposition 2.4.3.** We have $\tilde{K}^0(\mathbb{RP}^{2n-1}) = \mathbb{Z}/2^{n-1}\mathbb{Z}$ and $K^1(\mathbb{RP}^{2n-1}) = \mathbb{Z}$.

*Proof.* Applying the exact sequence,

$$0 \longrightarrow K^1(\mathbb{RP}^{2n-1}) \longrightarrow \mathbb{Z}[\rho]/(\rho^2 - 1) \longrightarrow \mathbb{Z}[\rho]/(\rho^2 - 1) \longrightarrow K^0(\mathbb{RP}^{2n-1}) \longrightarrow 0$$

We change variables $\rho = \sigma - 1$ then $\sigma^2 = 2\sigma$ and the map sends $1 \mapsto \sigma^n = 2^{n-1}\sigma$. Then the kernel is given by elements killed by $2^{n-1}\sigma$ which are of the form $(\sigma - 2)\mathbb{Z}$ and thus,

$$K^1(\mathbb{RP}^{2n-1}) \cong \mathbb{Z}$$

Finally, the cokernel is,

$$K^0(\mathbb{RP}^{2n-1}) = \mathbb{Z}[\sigma]/(\sigma^2 - 2\sigma, 2^{n-1}\sigma) = \mathbb{Z} \oplus \mathbb{Z}/2^{n-1}\mathbb{Z}$$

□

**Proposition 2.4.4.** We have $K^0(\mathbb{RP}^{2n}) = \mathbb{Z} \oplus \mathbb{Z}/2^n\mathbb{Z}$ and $K^1(\mathbb{RP}^{2n}) = 0$.

*Proof.* Consider the exact sequences of the pairs $(\mathbb{RP}^{2n}, \mathbb{RP}^{2n-1})$ and $(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n})$. First,

$$K^1(\mathbb{RP}^{2n}, \mathbb{RP}^{2n-1}) \longrightarrow K^1(\mathbb{RP}^{2n}) \longrightarrow K^1(\mathbb{RP}^{2n})$$

but $K^1(\mathbb{RP}^{2n}, \mathbb{RP}^{2n-1}) = \tilde{K}^1(\mathbb{RP}^{2n}/\mathbb{RP}^{2n-1}) = \tilde{K}^1(S^{2n}) = 0$ and thus $K^1(\mathbb{RP}^{2n}) \hookrightarrow K^1(\mathbb{RP}^{2n-1})$ is injective. Furthermore,

$$K^1(\mathbb{RP}^{2n+1}) \longrightarrow K^1(\mathbb{RP}^{2n}) \longrightarrow K^2(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n})$$

but $K^2(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n}) = \tilde{K}^0(\mathbb{RP}^{2n+1}/\mathbb{RP}^{2n}) = \tilde{K}^0(S^{2n+1}) = 0$ and thus $K^1(\mathbb{RP}^{2n+1}) \twoheadrightarrow K^1(\mathbb{RP}^{2n})$ is surjective. Furthermore, the composition $K^1(\mathbb{RP}^{2n+1}) \twoheadrightarrow K^1(\mathbb{RP}^{2n}) \hookrightarrow K^1(\mathbb{RP}^{2n-1})$ may be computed from the morphism $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n+1}$ applied to the previous exact sequence in the cases $n$ and $n+1$ to give a diagram,

$$\begin{tikzcd}
0 \arrow[r] & K^1(\mathbb{RP}^{2n+1}) \arrow[r] \arrow[d] & \mathbb{Z}[\rho]/(\rho^2-1) \arrow[r, "\sigma^{n+1}"] \arrow[d, "\sigma"] & \mathbb{Z}[\rho]/(\rho^2-1) \arrow[r] \arrow[d, "\mathrm{id}"] & K^0(\mathbb{RP}^{2n+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^1(\mathbb{RP}^{2n-1}) \arrow[r] & \mathbb{Z}[\rho]/(\rho^2-1) \arrow[r, "\sigma^{n}"] & \mathbb{Z}[\rho]/(\rho^2-1) \arrow[r] & K^0(\mathbb{RP}^{2n-1}) \arrow[r] & 0
\end{tikzcd}$$

However, $\ker \sigma^{n+1} = (\sigma - 2)\mathbb{Z}$ and thus $\sigma \ker \sigma^{n+1} = 0$ so the map $K^1(\mathbb{RP}^{2n+1}) \to K^1(\mathbb{RP}^{2n-1})$ is zero. Therefore, using the above factorization, $K^1(\mathbb{RP}^{2n}) = 0$. Furthermore, the exact sequence of the pair $(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n})$ gives,

$$K^0(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n}) \longrightarrow K^0(\mathbb{RP}^{2n+1}) \longrightarrow K^0(\mathbb{RP}^{2n}) \longrightarrow K^1(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n}) \longrightarrow K^1(\mathbb{RP}^{2n+1})$$

However, $K^0(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n}) = \tilde{K}^0(\mathbb{RP}^{2n+1}/\mathbb{RP}^{2n}) = \tilde{K}^0(S^{2n+1}) = 0$. Furthermore,

$$K^1(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n}) = \tilde{K}^1(\mathbb{RP}^{2n+1}/\mathbb{RP}^{2n}) = \tilde{K}^1(S^{2n+1}) = \mathbb{Z}$$

and we showed that $K^1(\mathbb{RP}^{2n+1}) = \mathbb{Z}$. Therefore, the sequence becomes,

$$0 \longrightarrow K^0(\mathbb{RP}^{2n+1}) \longrightarrow K^0(\mathbb{RP}^{2n}) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}$$

However, every map $\mathbb{Z} \to \mathbb{Z}$ is injective so $K^0(\mathbb{RP}^{2n}) \to \mathbb{Z}$ is zero. Thus, $K^0(\mathbb{RP}^{2n+1}) \xrightarrow{\sim} K^0(\mathbb{RP}^{2n})$ is an isomorphism showing that $K^0(\mathbb{RP}^{2n}) = \mathbb{Z} \oplus \mathbb{Z}/2^n\mathbb{Z}$.

□