1 Quasi-Coherent Sheaves

Recall that for a DM-stack we defined the small étale site:

Definition 1.0.1. Let $\mathscr{X}$ be a DM-stack. Then the *small étale site* $\mathscr{X}_{\text{ét}}$ of $\mathscr{X}$ is the category of schemes equipped with an étale map $U \to \mathscr{X}$. A covering is $\{U_i \to U\}$ over $\mathscr{X}$ such that $\sqcup_i U_i \to U$ is surjective.

Then for a sheaf $\mathscr{F}$ on $\mathscr{X}_{\text{ét}}$ we defined its global sections,

$$\Gamma(\mathscr{X}, \mathscr{F}) := \mathrm{Hom}_{\mathfrak{Sh}(\mathscr{X}_{\text{ét}})}(1, \mathscr{F})$$

where 1 is the terminal sheaf (the sheafification of $U \mapsto *$).

Remark. This definition works nicely for $\mathscr{X}$ DM and naturally generalizes the étale site $X_{\text{ét}}$ of a scheme. However, there is a glaring flaw if we attempt to extend this definition to Artin stacks there is a catastrophic failure: $\mathscr{X}_{\text{ét}}$ could be empty! For example, $(B\mathbb{G}_m)_{\text{ét}}$ is empty. Indeed, DM-stacks are exactly those stacks with schemes as étale neighborhoods. To remedy this we could take the smooth site of $\mathscr{X}$. To stay in the world of étale cohomology we consider a hybrid site where the schemes are smooth over $\mathscr{X}$ but the covers are all étale.

Definition 1.0.2. Let $\mathscr{X}$ be an algebraic stack. Then the *lisse-étale site* $\mathscr{X}_{\ell-\text{ét}}$ is the category of schemes smooth over $\mathscr{X}$ with *arbitrary* maps of schemes over $\mathscr{X}$. A covering $\{U_i \to U\}$ is a collection of morphisms such that $\sqcup_i U_i \to U$ is surjective or étale.

Definition 1.0.3. Let $\mathscr{F}$ be a sheaf on $\mathscr{X}_{\ell-\text{ét}}$ then,

$$\Gamma(\mathcal{U}, \mathscr{F}) = \mathrm{Hom}_{\mathfrak{Sh}(\mathcal{U}_{\ell-\text{ét}})}\left(1_{\mathfrak{U}}, \mathscr{F}|_{\mathcal{U}_{\ell-\text{ét}}}\right)$$

where $1_{\mathcal{U}}$ is the *indicator sheaf* of the smooth $\mathscr{X}$-stack $\mathcal{U} \to \mathscr{X}$ the sheafification of the constant sheaf $\underline{*}$. This is the terminal object of $\mathcal{U}_{\ell-\text{ét}}$. This can be computed by choosing a smooth presentation,

$$R \rightrightarrows U \to \mathcal{U}$$

and setting,

$$\Gamma(\mathfrak{U}, \mathscr{F}) = \mathrm{eq}\left(\mathscr{F}(U) \rightrightarrows \mathscr{F}(R)\right)$$

Definition 1.0.4. The structure sheaf $\mathcal{O}_{\mathscr{X}}$ is defined via,

$$\mathcal{O}_{\mathscr{X}}(U) = \Gamma(U, \mathcal{O}_U)$$

is a ring object in the abelian category $\mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}})$. We therefore define the abelian category $\mathbf{Mod}_{\mathcal{O}_{\mathscr{X}}}$. Given a morphism $f : \mathscr{X} \to \mathcal{Y}$ of algebraic stacks there are morphisms of topoi,

$$\begin{tikzcd} \mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}}) \arrow[r, bend left, "f_*"] & \mathbf{Ab}(\mathcal{Y}_{\ell-\text{ét}}) \arrow[l, bend left, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mod}_{\mathcal{O}_{\mathscr{X}}} \arrow[r, bend left, "f_*"] & \mathbf{Mod}_{\mathcal{O}_{\mathcal{Y}}} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Given two $\mathcal{O}_{\mathscr{X}}$-modules $\mathscr{F}$ and $\mathscr{G}$ we define the tensor product $\mathscr{F} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathscr{G}$ as the sheafification of,

$$U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_{\mathscr{X}}(U)} \mathscr{G}(U)$$

and the Hom sheaf $\mathscr{H}\!om_{\mathcal{O}_{\mathscr{X}}}(\mathscr{F}, \mathscr{G})$ as the sheaf,

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_U}\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$$

where $\mathscr{F}|_U$ means the restriction to the site $U_{\ell-\text{ét}}$ (note this is much more data than the restriction to U_{Zar}).

1.1 Quasi-Coherent Sheaves

As above we denote by $\mathscr{F}|_U$ the restriction of $\mathscr{F}$ to $U_{\ell-\text{ét}}$ and $\mathscr{F}|_{U_{\text{Zar}}}$ the restriction to U_{Zar}. Then we define,

Definition 1.1.1. Let $\mathscr{X}$ be an algebraic stack. A $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{F}$ is *quasi-coherent* if:

(a) for every smooth $U \to \mathscr{X}$ the restriction $\mathscr{F}|_{U_{\text{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\text{Zar}}}$-module

(b) for every morphism $f : V \to U$ of smooth $\mathscr{X}$-schemes, the induced morphism,

$$f^*(\mathscr{F}|_{U_{\text{Zar}}}) \to \mathscr{F}_{V_{\text{Zar}}}$$

is an isomorphism.

Remark. The above definition can be made in any site which refines the Zariski topology on each of its opens. However, in this generality such an object is usually called a *crystal in quasi-coherent sheaves* and the term *quasi-coherent* in an arbitrary site is reserved for the notion developed below. However, in most sites the two notions agree.

Definition 1.1.2. In an arbitrary ringed site $(\mathcal{C}, \mathcal{O})$ (or even an arbitrary ringed topos) a $\mathcal{O}$-module $\mathscr{F}$ is *quasi-coherent* if for each object $U \in \mathcal{C}$ there exists a cover $\{U_i \to U\}$ such that $\mathscr{F}|_{\mathcal{C}/U_i}$ is a *presentable* $\mathcal{O}$-module meaning there exists a presentation,

$$\bigoplus_J \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus_I \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \mathscr{F}|_{\mathcal{C}/U_i} \longrightarrow 0$$

We call the abelian subcategory of such sheaves $\text{QCoh}(\mathcal{C}) \subset \mathbf{Mod}_{\mathcal{O}_{\mathcal{C}}}$.

Definition 1.1.3. Let S be a scheme and $\mathcal{C} \subset \mathbf{Sch}_S$ a subcategory. Consider the presheaf of rings,

$$\mathcal{O} : \mathcal{C}^{\text{op}} \to \text{Ring}$$
$$(T \to S) \mapsto \Gamma(T, \mathcal{O}_T)$$

This is a sheaf for the fpqc topology. Furthermore, for any sheaf $\mathscr{F}$ on S_{Zar} there is a presheaf,

$$\mathcal{O} : \mathcal{C}^{\text{op}} \to \text{Ab}$$
$$(f : T \to S) \mapsto \Gamma(T, f^*\mathscr{F})$$

which is a $\mathcal{O}$-module. Furthermore, if $\mathscr{F}$ is quasi-coherent then $\mathscr{F}^a$ is a fpqc sheaf by descent.

Theorem 1.1.4 (Tag 03OJ)**.** Let S be a scheme. Let $\mathcal{C}$ be a site such that,

(a) $\mathcal{C}$ is a full subcategory of $\mathbf{Sch}_S$

(b) any Zariski covering of $T \in \mathcal{C}$ can be refined by a covering of $\mathcal{C}$

(c) $\text{id} : S \to S$ is an object of $\mathcal{C}$ (so it particular $\mathcal{C}$ has a terminal object)

(d) every covering of $\mathcal{C}$ is an fpqc covering of schemes

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and there is an equivalence of categories,

$$\text{QCoh}(S) \xrightarrow{\sim} \text{QCoh}(\mathcal{C})$$
$$\mathscr{F} \mapsto \mathscr{F}^a$$

Proof. This is basically a rephrasing of fpqc descent. $\square$

Proposition 1.1.5. Let $\mathscr{F}$ be a $\mathcal{O}_{\mathscr{X}_{\ell-\text{ét}}}$-module. Then the following are equivalent,

(a) $\mathscr{F}$ is quasi-coherent in the general sense

(b) $\mathscr{F}$ is quasi-coherent in the crystal sense.

Proof. C.f. 06WK. Let $\mathcal{C} = \mathscr{X}_{\ell-\text{ét}}$. Suppose that $\mathscr{F}$ satisfies (a). Then the restriction of $\mathscr{F}$ is quasi-coherent on $\mathcal{C}_{/U}$ and thus by the previous lemma $\mathscr{F}|_{\mathcal{C}} = (\mathscr{F}|_{U_{\text{Zar}}})^a$ and therefore satisfies (b). Given (b) take any $U \to \mathscr{X}$ smooth. Then we know $\mathscr{F}|_{U_{\text{Zar}}}$ is quasi-coherent so there is an affine Zariski open cover $\{U_i \to U\}$ such that $\mathscr{F}|_{(U_i)_{\text{Zar}}}$ is presented. Then the claim is that $\mathscr{F}|_{\mathcal{C}/U_i}$ is also presented. Indeed, the comparison map induced by $f : V \to U$ is an isomorphism the presentation pulls back to give a presentation of $\mathscr{F}|_{\mathcal{C}/U_i}$. $\square$

1.2 Descent Data

Definition 1.2.1. Let (U, R, s, t, c, e) be a groupoid scheme over S where $s, t : R \rightrightarrows U$ are the source and target maps and $c : R \times_{s,U,t} R \to R$ is the composition and $e : U \to R$ is the identity. Then the category of *descent data* consists of the category of pairs $(\mathscr{F}, \varphi)$ where $\mathscr{F}$ is a sheaf on U and φ is an isomorphism,

$$\varphi : t^*\mathscr{F} \xrightarrow{\sim} s^*\mathscr{F}$$

such that $e^*\varphi = \text{id}$ and satisfying the cocycle condition,

$$c^*\varphi = \pi_2^*\varphi \circ \pi_1^*\varphi$$

as morphisms of sheaves on $R \times_{s,U,t} R$.

Example 1.2.2. For any cover $U \to X$ we can form the "Cech groupoid" $U \times_X U \rightrightarrows U$ whose composition is given by projection,

$$(U \times_X U) \times_{\pi_1, U, \pi_2} (U \times_X U) = U \times_X U \times_X U \to U \times_X U \qquad ((a, b), (c, a)) \mapsto (c, a, b) \mapsto (c, b)$$

For this we recover the ordinary notion of a descent datum.

Example 1.2.3. Let $G \circlearrowright X$ be an action of an algebraic group on a scheme. Then there a groupoid $G \times X \rightrightarrows X$ whose composition $G \times G \times X \to G \times X$ is given by multiplication in the group.

For this we will recover the notion of G-equivariance.

Proposition 1.2.4. Let $R \rightrightarrows U$ be a smooth presentation of an algebraic stack $\mathscr{X}$ by schemes. There is an equivalence of categories,

$$\text{QCoh}(\mathscr{X}) \to \text{DD}_{\text{QCoh}}(U/R) \quad \mathscr{F} \mapsto (\mathscr{F}|_{U_{\text{Zar}}}, \varphi)$$

where $\text{DD}_{\text{QCoh}}(U/R)$ is the category of descent data for quasi-coherent sheaves along the groupoid $R \rightrightarrows U$.

Proof. For any smooth map $V \to \mathscr{X}$ there is a further smooth refinement $V' \to V$ such that $V' \to \mathscr{X}$ factors through $U \to \mathscr{X}$. Hence, applying descent to $V' \to V$, any quasi-coherent sheaf $\mathscr{F}$ on $\mathscr{X}_{\ell-\text{ét}}$ is determined by its descent data over $R \rightrightarrows U$. $\square$

Definition 1.2.5. Let $G \circlearrowright X$ be an action of a group scheme on a scheme (or algebraic space). The category of G-equivariant sheaves is defined as the category of descent data for the groupoid $G \times X \rightrightarrows X$.

Remark. Some standard diagram chasing shows that this is formally the same as the usual definition of a G-equivariant sheaf in [Stacks]. In the case that G is a finite constant group it is easy to check that this agrees with the naive notion in terms of compatible isomorphisms between the pullbacks along the action by elements $g \in G$.

Proposition 1.2.6. There is an equivalence of categories,

$$\mathrm{QCoh}([X/G]) \to \mathrm{QCoh}_G(X)$$

Proof. This is a special case of the previous proposition. $\square$

1.3 Examples

Example 1.3.1. Let $\mathscr{X} \to S$ be a DM-stack. Then the sheaf,

$$\Omega_{\mathscr{X}/S} : (U \to \mathscr{X}) \mapsto \Gamma(U, \Omega_{U/S})$$

is quasi-coherent since any morphism $f : V \to U$ in $\mathscr{X}_{\text{ét}}$ is étale so the map,

$$f^*\Omega_{U/S} \xrightarrow{\sim} \Omega_{V/S}$$

is an isomorphism. However, if $\mathscr{X} \to S$ is not DM we don't have access to $\mathscr{X}_{\text{ét}}$ nor can we define $(\Omega_{X/S})^a$ on X_{fppf} as we can for a scheme since there is no Zariski or étale site to define this sheaf over for a bootstrap. There is still a sheaf of $\mathcal{O}_{\mathscr{X}_{\ell-\text{ét}}}$-modules,

$$\Omega_{\mathscr{X}/S} : (U \to \mathscr{X}) \mapsto \Gamma(U, \Omega_{U/S})$$

but it is not quasi-coherent. This is the sort of sheaf the stacks project calls *locally quasi-coherent* meaning that it is quasi-coherent when restricted to $U_{\text{ét}}$ for any $U \to \mathscr{X}$.

Remark. Indeed, it is not clear that an Artin stack $\mathscr{X} \to S$ should have any good notion of a cotangent bundle $\Omega_{\mathscr{X}/S}$. For example, consider $\mathscr{X} = \mathbf{B}\mathbb{G}_m$ which is smooth of relative dimension -1 so what should $\Omega_{\mathscr{X}/S}$ even be? It can't be a vector bundle of rank -1 can it! To fix this conundrum, we either work with $\Omega_{\mathscr{X}/S}$ as defined above which is not quasi-coherent and hence does not have a well-defined rank or we define the cotangent complex $\mathbb{L}_{\mathscr{X}/S} \in D^{\le 1}_{\text{QCoh}}(\mathscr{X})$ (technically it's an ind-object in this generality) [Champs Algebriques, Chapter 17] which encodes the deformation theory of $\mathscr{X}$. Note that unlike for a scheme, $\mathbb{L}_{\mathscr{X}/S}$ can be supported in degree 1. In fact, the following are equivalent,

(a) $\mathscr{X} \to S$ is DM

(b) $\mathcal{H}^1(\mathbb{L}_{\mathscr{X}/S}) = 0$

Proof: [Champs Algebriques, Cor. 17.9.2].

1.4 Picard Groups

Let $\mathscr{X}$ be an algebraic stack. Then $\mathrm{Pic}(\mathscr{X})$ denotes the set of isomorphism classes of line bundles on $\mathscr{X}$ which becomes an abelian group under $\otimes$.

Example 1.4.1. If G is an affine algebraic k-group then $\mathrm{Pic}(\mathbf{B}G) = \mathrm{Hom}_{\mathrm{gp}}(G, \mathbb{G}_m)$ is the group of characters. For example,

(a) $\mathrm{Pic}(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$

(b) $\mathrm{Pic}(\mathbf{B}\mathrm{GL}_n) = \mathbb{Z}$

(c) $\mathrm{Pic}(\mathbf{B}\mathrm{PGL}_n) = \{0\}$.

This is because line bundles on $\mathbf{B}G$ are the same as line bundles on $\mathrm{Spec}(k)$ along with descent data i.e. a G-action. This is the same as a 1-dimensional G-representation.

Example 1.4.2. Consider the action, $\mathbb{G}_m \circlearrowright \mathbb{A}^n$ with weights $d_1, \dots, d_n$. Let the *weighted projective stack* be the DM-stack (at least if $p \not| d_i$),

$$\mathcal{P}(d_1, \dots, d_n) = [(\mathbb{A}^n \backslash \{0\})/\mathbb{G}_m]$$

Here let k be a field of characteristic not dividing any d_i or 2 or 3.

(a) The map $\mathrm{Pic}(\mathbf{B}\mathbb{G}_m) \to \mathrm{Pic}(\mathcal{P}(d_1, \dots, d_n))$ induced by the canonical $\mathbb{G}_m$-bundle is an isomorphism. Indeed, this reduces to classifying $\mathbb{G}_m$-actions on $\mathcal{O}_{\mathbb{A}^n \backslash \{0\}}$. By Hartogs' these correspond to $\mathbb{G}_m$-actions on $\mathcal{O}_{\mathbb{A}^n}$ and thus to different grading of $A = k[x_1, \dots, x_n]$ as an A-module with x_i given weight d_i. These are just overall shifts $A(d)$ i.e. putting 1 in degree d. This is the same as the pullback of the bundle over $\mathbf{B}\mathbb{G}_m$ corresponding to $\mathbb{G}_m \xrightarrow{z^d} \mathbb{G}_m$.

(b) Using Weierstrass models we get an isomorphism,

$$\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4, 6)$$

Therefore, $\mathrm{Pic}\left(\overline{\mathcal{M}}_{1,1}\right) = \mathbb{Z}\omega_{\mathcal{C}/\mathcal{M}}$

(c) Then it turns out that,

$$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z}\omega_{\mathcal{C}/\mathcal{M}}$$

This is because the discriminant Δ is a section of $\mathcal{O}(12)$ which is nowhere vanishing for smooth families.

1.5 Global Quotients and the Resolution Property

Definition 1.5.1. An algebraic stack $\mathscr{X}$ is a *global quotient stack* if there is an isomorphism $\mathscr{X} \cong [U/\mathrm{GL}_n]$ where U is an algebraic space. This is equivalent to asking for the existence of a GL_n-bundle $U \to \mathscr{X}$ where U is an algebraic space. By definition this is the same as a representable morphism $\mathscr{X} \to \mathbf{B}\mathrm{GL}_n$.

Proposition 1.5.2. Let $\mathscr{X} \to \mathcal{Y}$ be a surjective, flat, and projective morphism of noetherian algebraic stacks. If $\mathscr{X}$ is a quotient stack then $\mathcal{Y}$ is a quotient stack.

Definition 1.5.3. A noetherian algebraic stack has the *resolution property* if every coherent sheaf if a quotient of a vector bundle.

A smooth or quasi-projective scheme over a field has the resolution property. Any noetherian normal $\mathbb{Q}$-factorial scheme with affine diagonal also has the resolution property.

Proposition 1.5.4. *Let G be an affine algebraic k-group with an action $G \circlearrowright U$ on a quasi-projective k-scheme U. Assume that there is an ample line bundle $\mathcal{L}$ with a G-action. Then $[\mathrm{Spec}\,(A)\,/G]$ has the resolution property.*

Remark. While not every line bundle $\mathcal{L}$ on a normal k-scheme admits a G-action, it turns out there is always some positive power such that $\mathcal{L}^{\otimes n}$ has a G-action.

Proof. The G-line bundle $\mathcal{L}$ corresponds to a line bundle on $[U/G]$ which is ample which respect to the morphism $p : [U/G] \to \mathbf{B}G$ since relative ampleness can be checked after smooth covers (it can be reduced to a fiberwise condition). For a coherent sheaf $\mathscr{F}$ on $[U/G]$ the natural map,

$$\mathcal{L}^{-\otimes N} \otimes p^* p_*(\mathcal{L}^{\otimes N} \otimes \mathscr{F}) \twoheadrightarrow \mathscr{F}$$

is surjective for $N \gg 0$ since relative ampleness implies global generation of $\mathcal{L}^{\otimes N} \otimes \mathscr{F}$. The pushforward $p_*(\mathcal{L}^{\otimes N} \otimes \mathscr{F})$ is quasi-coherent on $\mathbf{B}G$ hence a G-representation. We can hence write it as an increasing union of finite-dimensional G-representations V_i and obtain,

$$\operatorname*{colim}_i (\mathcal{L}^{-\otimes N} \otimes p^* V_i) \twoheadrightarrow \mathscr{F}$$

since $\mathscr{F}$ is coherent, this stabilizes at some stage meaning,

$$\mathcal{L}^{-\otimes N} \otimes p^* V_i \twoheadrightarrow \mathscr{F}$$

at some finite stage i. □

Theorem 1.5.5 (Totaro-Gross)**.** *Let $\mathscr{X}$ be a quasi-separated normal algebraic stack of finite type over k. Assume that the stabilizer group at every closed point is smooth and affine. Then the following are equivalent:*

(a) $\mathscr{X}$ has the resolution property

(b) $\mathscr{X} \cong [U/\mathrm{GL}_n]$ with U quasi-affine

(c) $\mathscr{X} \cong [\mathrm{Spec}\,(A)\,/G]$ with G an affine algebraic group.

In particular, $\mathscr{X}$ has affine diagonal.

Remark. The normality hypothesis on $\mathscr{X}$ and smoothness hypothesis on the stabilizers are unnecessary. However, the affineness hypothesis on the stabilizers is necessary. For example, $\mathbf{B}E$ the classifying stack of an elliptic curve has the resolution property.

1.6 Sheaf Cohomology

Lemma 1.6.1. If $\mathscr{X}$ is an algebraic stack, the categories $\mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}})$ and $\mathbf{Mod}_{\mathscr{X}}$ have enough injective. If $\mathscr{X}$ is quasi-separated then $\mathrm{QCoh}(\mathscr{X})$ has enough injectives.

Definition 1.6.2. Let $\mathscr{X}$ be an algebraic stack and $\mathscr{F}$ a sheaf on $\mathscr{X}_{\ell-\text{ét}}$. The *cohomology groups* $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ are the derived functors of,

$$\Gamma(\mathscr{X}, -) : \mathbf{Ab}(\mathscr{X}_{\ell-\text{ét}}) \to \mathbf{Ab}$$

applied to $\mathscr{F}$,

$$H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F}) = R^i\Gamma(\mathscr{X}, \mathscr{F})$$

Definition 1.6.3. Given a smooth covering $\mathfrak{U} = \{\mathcal{U}_i \to \mathscr{X}\}_{i\in I}$ of algebraic stacks and an abelian presheaf $\mathscr{F}$ on $\mathscr{X}_{\ell-\text{ét}}$ the *Cech complex* $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ of $\mathfrak{U}$ with respect to $\mathfrak{U}$ is,

$$\check{C}^n(\mathfrak{U}, \mathscr{F}) = \prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathscr{F}(\mathcal{U}_{i_0} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathcal{U}_{i_n})$$

with differential,

$$\mathrm{d}^n : \check{C}^n(\mathfrak{U}, \mathscr{F}) \to \check{C}^{n+1}(\mathfrak{U}, \mathscr{F}) \quad (s_{i_0,\dots,i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_{\hat{k}}^* s_{i_0,\dots,\hat{i}_k,\dots,i_n} \right)_{i_0,\dots,i_{n+1}}$$

where the projection $p_{\hat{k}}$ forgets the t^{th} coordinate. The *Čech cohomology* of $\mathscr{F}$ with respect to $\mathfrak{U}$ is,

$$\check{H}^i(\mathfrak{U}, \mathscr{F}) := H^i(\check{C}^\bullet(\mathfrak{U}, \mathscr{F}))$$

Theorem 1.6.4. Let $\mathscr{X}$ be an algebraic stack and $\mathscr{F}$ a quasi-coherent sheaf on $\mathscr{X}_{\ell-\text{ét}}$. Then for any cover $\mathfrak{U} = \{\mathcal{U}_i \to \mathscr{X}\}_{i\in I}$ there exists a spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, H^q(-, \mathscr{F})) \implies H^{p+q}(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$$

where $H^q(-, \mathscr{F})$ is the presheaf $U \mapsto H^q(U_{\ell-\text{ét}}, \mathscr{F})$.

Proof. Consider the commutative diagram of functors,

$$\begin{tikzcd} \mathrm{Sh}(\mathscr{X}_{\ell-\text{ét}}) \arrow[r, hook, "a"] \arrow[dr, "\Gamma"'] & \mathrm{PSh}(\mathscr{X}_{\ell-\text{ét}}) \arrow[d, "\check{H}^0"] \\ & \mathbf{Ab} \end{tikzcd}$$

Notice that $\check{C}^\bullet(\mathfrak{U}, -)$ is exact in the category of presheaves which shows that $\check{H}^\bullet(\mathfrak{U}, -)$ forms a δ-functor. In fact, since $\check{H}^i(\mathfrak{U}, \mathscr{I}) = 0$ for $i > 0$ and any injective sheaf (this is a very general fact, see Tag 03OR) it is a universal δ-functor. Now the inclusion a takes injectives to injectives because sheaves form a reflexive subcategory (maps to a sheaf factors through the sheafification). Therefore, we apply the Grothendieck spectral sequence so it suffices to compute $R^q a(\mathscr{F})$ of a sheaf $\mathscr{F}$. Since the functor $(-) \mapsto \Gamma(U, -)$ is exact on presheaves we see that,

$$R^q a(\mathscr{F})(U) = R^q\Gamma(U, \mathscr{F}) = H^q(U, \mathscr{F})$$

so we conclude. ∎

Theorem 1.6.5. If X is an affine scheme and $\mathscr{F}$ is a quasi-coherent $\mathcal{O}_{\mathscr{X}_{\ell-\text{ét}}}$-module then,

$$H^i(X_{\ell-\text{ét}}, \mathscr{F}) = \begin{cases} \Gamma(X, \mathscr{F}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. We refine to affine coverings $\{\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)\}$ then $\mathscr{F}$ is quasi-coherent (in all equivalent notions) and hence arises from some A-module M. To show that $\check{H}^{>0} = 0$ for this covering we show that the Amistur complex,

$$0 \longrightarrow M \longrightarrow M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \otimes_A B \longrightarrow \cdots$$

is exact. Indeed, after applying $B \otimes_A -$ which is faithfully flat this complex obtains a nullhomotopy. Now to conclude, we can either apply Cartan's criterion (Tag 03F9) or use hypercoverings and the fact that hypercover Cech cohomology computes derived functor cohomology. □

Proposition 1.6.6. Let $\mathscr{X}$ be an algebraic stack with affine diagonal and $\mathscr{F}$ be a quasi-coherent sheaf. If $\mathfrak{U} = \{U_i \to \mathscr{X}\}$ is an étale covering with each U_i affine, then $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F}) = \check{H}^i(\mathfrak{U}, \mathscr{F})$.

Proof. Follows immediately from the Cech-to-derived spectral sequence and the above. □

Remark. To remove the "affine diagonal" condition we need to use hypercovers. Indeed, if $U_\bullet \to \mathscr{X}$ is a simplicial hypercover in $\mathscr{X}_{\ell-\text{ét}}$ where each $U_\bullet$ is an affine scheme and $\mathscr{F}$ is quasi-coherent then,

$$H^i(\mathscr{X}, \mathscr{F}) = \check{H}^i(U_\bullet, \mathscr{F})$$

Proposition 1.6.7. Let X be a scheme (or a DM-stack with a sheaf on $\mathscr{X}_{\text{ét}}$) with affine diagonal[1] and $\mathscr{F}$ a quasi-coherent sheaf. Then,

$$H^i(X, \mathscr{F}) = H^i(X_{\ell-\text{ét}}, \mathscr{F}_{\ell-\text{ét}})$$

for all i where $\mathscr{F}_{\ell-\text{ét}}$ is the $\mathcal{O}_{X_{\ell-\text{ét}}}$-module defined by,

$$\mathscr{F}_{\ell-\text{ét}}(U) = \Gamma(U, f^* \mathscr{F})$$

for a smooth map $f : U \to X$. (In the stack case it is pullback under $f : \mathscr{X}_{\ell-\text{ét}} \to \mathscr{X}_{\text{ét}}$).

Proof. Choose an affine Zariski cover U of X (affine étale cover U of $\mathscr{X}$) by the assumption on the diagonal we see that,

$$H^i(X_{\ell-\text{ét}}, \mathscr{F}) = \check{H}^i(\mathrm{U}, \mathscr{F}) = H^i(X, \mathscr{F})$$

(and similarly for $\mathscr{X}$). The affine diagonal condition is to ensure that projects in the Cech complex are affine and hence have vanishing $H^{>0}$. However, this condition is not necessary. We can always choose a Zariski hypercover $U_\bullet \to X$ by affines and similar arguments show that,

$$H^i(X_{\ell-\text{ét}}, \mathscr{F}) = \check{H}^i(U_\bullet, \mathscr{F}) = H^i(X, \mathscr{F})$$

□

Proposition 1.6.8. Let $\mathscr{X}$ be an algebraic stack.

(a) $\mathscr{F}$ is an $\mathcal{O}_{\mathscr{X}}$-module then $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ agrees with $R^i\Gamma : \mathbf{Mod}_{\mathcal{O}_{\mathscr{X}}} \to \mathbf{Ab}$ computed in the category of $\mathcal{O}_{\mathscr{X}}$-modules.

[1]If we use hypercovers (see the discussion in the proof then we can remove this condition.

(b) If $\mathscr{X}$ has affine diagonal and $\mathscr{F}$ is a quasi-coherent sheaf on $\mathscr{X}$, then the cohomology $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ agrees with $R^i\Gamma : \mathrm{QCoh}(\mathscr{X}) \to \mathbf{Ab}$ computed in the category of quasi-coherent modules.

For a morphism $f : \mathscr{X} \to \mathcal{Y}$ of algebraic stacks (resp. quasi-compact morphism of algebraic stacks with affine diagonals) then (a) (resp. (b)) holds also for $R^i f_* \mathscr{F}$ of an $\mathcal{O}_{\mathscr{X}}$-module (resp. quasi-coherent sheaf).

Remark. If $\mathscr{X}$ does not have affine diagonal, then the sheaf cohomology $H^i(\mathscr{X}_{\ell-\text{ét}}, \mathscr{F})$ of a quasi-coherent sheaf may differ from the derived functor $R^i\Gamma(\mathscr{X}, -) : \mathrm{QCoh}(\mathscr{X}) \to \mathbf{Ab}$.

Proposition 1.6.9. If $\mathscr{X}$ is an algebraic stack and $\mathscr{F}_i$ is a directed system of abelian sheaves in $\mathscr{X}_{\ell-\text{ét}}$ then $\operatorname{colim}_i H^i(\mathscr{X}, \mathscr{F}_i) \to H^i(\mathscr{X}, \operatorname{colim}_i \mathscr{F}_i)$ is an isomorphism.