

Definition 0.0.1. Let k be algebraically closed and F characteristic zero. Let \mathcal{C} be the category of smooth projective varieties over k . Let GV be the category of graded F -vectorspaces.

Definition 0.0.2. Let k be algebraically closed and F characteristic zero. Let \mathcal{C} be the category of smooth projective varieties over k . Then a Weil Cohomology theory is a functor,

$$H^* : \mathcal{C} \rightarrow \text{Vec}_F$$

with a linear map $\gamma : \text{CH}^i(X) \rightarrow H^{2i}(X)$ and dual map,

$$\int_X : H^{2\dim X}(X) \rightarrow F$$

satisfying the axioms,

(a) Poincare Duality

- (a) $\dim_F H^i(X) < \infty$
- (b) $H^i(X) \times H^{\dim X - i}(X) \rightarrow H^{\dim X} \rightarrow F$ is a perfect pairing
- (c) $H^i(X) = 0$ for $i \notin [0, 2\dim X]$
- (d) $\dim_F H^0(X) = 1$

(b) Kunneth formula,

$$H^\bullet(X \times Y) = H^\bullet(X) \otimes_F H^\bullet(Y)$$

(c) compatibility of γ with $f^*H^*(Y) \rightarrow H^*(X)$ and $\int_X : H^{2\dim X} \rightarrow F$

- (a) $\gamma(f^*\beta) = f^*\gamma(\beta)$
- (b) $\gamma(f_*\alpha) = f_*\gamma(\alpha)$
- (c) $\gamma(\alpha \cdot \beta) = \gamma\alpha \smile \gamma\beta$
- (d) $\int_{\text{Spec}(k)} \gamma([\text{Spec}(k)]) = 1$

Theorem 0.0.3. Given H^* and γ and \int with $A(a)$, $A(b)$ (B) and (C) we get,

$$G : M_k \rightarrow \text{GV}$$

such that $G(!(1)) = F[i]$ and vice versa. Therefore,

$$G(h(x)) = H^*(X)$$

Definition 0.0.4. Given a Weil Cohomology theory the Betti numbers,

$$\beta_i(X) = \dim_F H^i(X)$$

Remark. Open question: are the Betti numbers the same for all Weil cohomology theories?

Theorem 0.0.5. $H_{\text{ét}}^*(-, \mathbb{Q}_\ell)$ is a Weil cohomology theory if $\ell \neq \text{ch}k$.

Theorem 0.0.6. The étale betti numbers do not depend on ℓ as long as $\ell \neq \text{ch}k$.

Proposition 0.0.7. If we drop A(c) and A(d) then there exists counterexamples to the open question.

Proof. Suppose we have WCT H^* with additional gradings,

$$H^*(X) = \bigoplus_{i \in \mathbb{Z}} H_i^*(X)$$

which is,

- (a) compaible with pullback
- (b) compatible with Junneth and γ
- (c) $\gamma(\alpha) \in H_0^*(X)$
- (d) \int_X factors through $H^{2 \dim X}(X) \rightarrow H_0^{2 \dim X}(X)$

Then I can twist the cohomology theory by setting,

$$H_{\text{new}}^n(X) = \bigoplus_{i \in \mathbb{Z}} H_i^{n+2i}(X)$$

We need to add $2i$ because we need things in even degree to stay in even degree.

For example, let $k = \mathcal{C}$ then H_{dR}^* is a WCT (we will prove this) and there is a grading,

$$H_{\text{dR}}^n(X) = \bigoplus_{p+q=n} H^{p,q} = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{\substack{p+q=n \\ q-p=i}} H^{p,q}$$

Then,

$$H_{\text{new}}^n(X) = \bigoplus_{i \in \mathbb{Z}} H_i^{n+2i}(X) = \bigoplus_{3p-q=n} H^{p,q}$$

Example, if E is an elliptic curve over \mathcal{C} then,

$$\begin{aligned} H_{\text{new}}^{-1}(E) &= H^{0,1}(E) = H^1(E, \mathcal{O}_E) = g = 1 \\ H_{\text{new}}^0(E) &= H^{0,0} = H^0(E, \mathcal{O}_E) = 1 \\ H_{\text{new}}^1(E) &= 0 \\ H_{\text{new}}^2(E) &= H^{1,1} = H^1(E, \Omega_E^1) \\ H_{\text{new}}^2(E) &= H^{1,0} = H^0(E, \Omega_E^1) = g = 1 \end{aligned}$$

Another example, let k_0 be a global field and $k = \overline{k_0}$. Let v be a finite place of k and a Frobenius $\sigma \in D$ the decomposition group of v inside $\text{Gal}(k/k_0)$. Then for X/k_0 we have $\sigma \in \text{Gal}(k/k_0)$ acting on $H_{\text{ét}}^n(X_k, \mathbb{Q}_\ell)$. We can decompose,

$$H_{\text{ét}}^n(X_k, \mathbb{Q}_\ell) = \bigoplus_{i \in \mathbb{Z}} H_i^n(X)$$

where $H_i^n(X)$ is the eigenspace of σ with eigenvalues being q -Weil numbers of weight $n+i$. Observation: if X has good reduction at v then $H^n(X) = H_0^n(X)$.

If E is an elliptic curve with semistable and bad reduction at v then, as before,

$$\dim_{\mathbb{Q}_\ell} H_{\text{new}}^n(E) = \begin{cases} 1 & n = -1, 0, 2, 3 \\ 0 & \text{else} \end{cases}$$

□

Remark. To get a WCT you need for any X that,

$$\begin{aligned} n - 2i < 0 &\implies H_{-i}^n(X) = 0 \\ i \neq 0 &\implies H_i^{2i}(X) = 0 \end{aligned}$$

1 Chern Classes in Cohomology

Definition 1.0.1. Let $\text{Vect}X$ be the category of finite locally free \mathcal{O}_X -modules on X .

Remark. Let \mathcal{S} be some not full category of schemes such that for ally $X \in \mathcal{S}$ we have,

- (a) X is quasi-compact and quasi-separated
- (b) if $U \subset X$ is open and closed then $U \rightarrow X$ is a morphism of \mathcal{S} and if $Y \rightarrow U$ is given with $Y \rightarrow X$ in \mathcal{S} then $Y \rightarrow U$ in \mathcal{S}
- (c) If $\mathcal{E} \in \vec{X}$ then $\mathbb{P}(\mathcal{E}) \rightarrow X$ is in \mathcal{S} and if $f : Y \rightarrow X$ is in \mathcal{S} then $\mathbb{P}(f^*\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$ is in \mathcal{S} and if $\mathcal{E} \rightarrow \mathcal{F}$ is epic with $\mathcal{F} \in \text{Vect}X$ then $\mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{E})$ is in \mathcal{S} .

Furthermore let $A^* : \mathcal{S}^{\text{op}} \rightarrow$ nonnegatively graded \mathbb{Q} – algebras and a transformation of functors

$$c_1^A : \text{Pic}(-) \rightarrow A^1(-)$$

such that

- (a) $c_1^A(\mathcal{L})$ is in the center of $A^1(X)$
- (b) $A^*(U \amalg V) = A^*(U) \times A^*(V)$
- (c) $f^*c_1^A(\mathcal{L}) = c_1^A(f^*\mathcal{L})$ (a natural transformation)
- (d) projective space bundle formula, given $\mathbb{P}(\mathcal{E}) \rightarrow X$,

$$\bigoplus_{i=0}^{r-1} A^*(X) \longrightarrow A^*(\mathbb{P}(\mathcal{E}))$$

the map given is,

$$\bigoplus_{i=0}^{r-1} c_1^A(\mathcal{O}_{\mathbb{P}(\mathcal{E})})(1) \circ p^*$$

- (e) If $\iota : Y \rightarrow X$ is a morphism in \mathcal{S} and is the inclusion of an effective Cartier divisor then,

$$\forall a \in A^*(X) : \iota^*(a) = 0 \implies c_1^A(\mathcal{O}_X(Y)) \smile a = 0$$

Theorem 1.0.2. Given the above situation, there is a unique canonical map,

$$\text{ch}^A : K_0(\text{Vect} X) \rightarrow \prod_{i \in \mathbb{N}} A^i(X)$$

for $X \in \mathcal{S}$ s.t

- (a) ch^A is a ring map
- (b) $f : Y \rightarrow X$ is a map in \mathcal{S} then $\text{ch}^A \circ f^* = f^* \circ \text{ch}^A$
- (c) $\text{ch}^A(\mathcal{L}) = \exp(c_1^A(\mathcal{L}))$

2 A Long Theorem

See handout

Proof. We need to make the cycle class maps. (A1) - (A4) define maps $K_0(\text{Vect}(X)) \rightarrow \bigoplus H^{2i}(X)$ compatible with pullbacks. Then we get maps,

$$\begin{array}{ccc} CH^*(X) \otimes \mathbb{Q} & \xrightarrow{\quad \gamma \quad} & \bigoplus_{i \geq 0} H^{2i}(X) \\ & \swarrow \text{ch} \quad \searrow \text{ch}^H & \\ & K_0(\text{Vect}(X)) & \end{array}$$

γ is then compatible with grading. To show this, note that,

$$\text{ch}_i^H(\phi^2(\alpha)) = 2^i \text{ch}_i^H(\alpha)$$

and then conclude.

It is automatic from the construction (Ca) and (Cc). Now we need to show (A1) - (A7) imply (Aa) and (Ab)

Consider,

$$\begin{aligned} \eta : F &\xrightarrow{\gamma([\Delta])} H^*(X) \otimes_F H^*(X) = H^*(X \times X) \\ \varepsilon : H^*(X \times X) &= H^*(X) \otimes_F H^*(X) \xrightarrow{\Delta^*} H^*(X \times X) \xrightarrow{\lambda} F \end{aligned}$$

It suffices to show that,

$$H^*(X) \xrightarrow{\eta \otimes 1} H^*(X) \otimes H^*(X) \otimes H^*(X) \xrightarrow{1 \otimes \varepsilon} H^*(X)$$

is the identity then the functor has a left-adjoint. I claim,

$$\begin{aligned} ((1 \otimes \varepsilon) \circ (\eta \otimes 1))(a) &= (1 \otimes \lambda)(\gamma([\Delta])) \smile p_2^* a \\ &= (1 \otimes \lambda)(\gamma([\Delta])) \smile p_1^* a \\ &= a \cdot \lambda(\gamma([\Delta])) \\ &= a \cdot \text{id} \end{aligned}$$

However, for the above to hold, we need the fact that,

$$\gamma([\Delta]) \smile (p_1^*a - p_2^*a) = 0$$

Since this class pulls back to zero on X we know this by (A4) for the case of X a divisor on $X \times X$. However, this is only true for $\dim X = 1$. We need to blow up,

$$\begin{array}{ccc} E & \xrightarrow{\bar{j}} & \text{Bl}_\Delta(X \times X) \\ \downarrow \pi & & \downarrow b \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

Now we get,

$$j^*b^*(p_1^*a - p_2^*a) = 0 \implies b^*(p_1^*a - p_2^*a) \smile j(E) = 0$$

This implies that,

$$b^*(p_1^*a - p_2^*a) \smile \gamma(b^*[\Delta]) = 0$$

However, γ is compatible with pullback and thus,

$$b^*(p_1^*a - p_2^*a) \smile \gamma([\Delta]) = 0$$

which implies that,

$$(p_1^*a - p_2^*a) \smile \gamma([\Delta]) = 0$$

Lemma 2.0.1. Let X be a smooth projective variety and $Z \subset X$ a smooth projective subvariety. We blowup to give,

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ \downarrow \pi & & \downarrow b \\ Z & \xrightarrow{i} & X \end{array}$$

Assume there exists a $\alpha \in K_0(\text{Vect}(X))$ with $\iota^*\alpha$ is the class of the conormal sheaf $C_{Z/X}$ of Z in X . Then there exists a $\theta \in CH^*(X) \otimes \mathbb{Q}$ such that,

$$(a) \quad b^*[Z] = [E] \cdot \theta \text{ in } CH^*(X') \otimes \mathbb{Q}$$

$$(b) \quad \pi_*j^*\theta = [Z] \text{ in } CH^*(Z) \otimes \mathbb{Q}$$

Proof. Suppose that $Z = V(s)$ for some $s \in \Gamma(X, \mathcal{E})$ for a vector bundle of rank $r = \text{codim} (Z, X)$. Then,

$$[Z] = c_r(\mathcal{E}) \frown [X] \implies b^*[Z] = c_r(b^*\mathcal{E}) \frown [X']$$

Fact, s gives a nowhere vanishing section of $b^*\mathcal{E}(-E)$. Thus,

$$0 = c_r(b^*\mathcal{E}(-E)) = b^*c_r(\mathcal{E}) - b^*c_{r-1}(\mathcal{E}) \cdot c_1(\mathcal{O}_{X'}(E)) + \cdots + (-1)^r c_1(\mathcal{O}_{X'}(E))^r$$

Therefore, we take,

$$\theta = (b^*c_{r-1}(\mathcal{E}) - \cdots - (-1)^r c_1(\mathcal{O}_{X'}(E))^{r-1}) \frown [X']$$

Then the second relation is automatically true. Then,

$$\pi_*j^*\theta = \pi_*((-1)^{r-1} c_1(\mathcal{O}_{X'}(E))^{r-1} \frown [E]) = [Z]$$

by the projective bundle formula noting that $\mathcal{O}_{X'}(E)|_E = \mathcal{O}_E(-1)$. □

The assumption holds for $\Delta : X \rightarrow X \times X$ because $C_{\Delta/X \times X} = \Omega_X^1$. □

3 dr Rham Cohomology

Definition 3.0.1. Let $f : X \rightarrow S$ be a morphism of schemes then we have the de Rham complex $\Omega_{X/S}^\bullet$. Whenever we have a diagram,

$$\begin{array}{ccc} X & \longleftarrow & U = \operatorname{Spec}(A) \\ \downarrow f & & \downarrow \\ S & \longleftarrow & V = \operatorname{Spec}(R) \end{array}$$

Then $\Gamma(U, \Omega_{X/S}^\bullet) = \Omega_{A/R}^\bullet$. Then $\Omega_{X/S}^\bullet$ is a sheaf of strictly commutative differential graded $f^{-1}\mathcal{O}_S$ -modules.

Definition 3.0.2. We define, $\Omega_{A/R}^0 = A$ and,

$$\Omega_{A/R}^1 = (\bigoplus_{a \in A} da) / (\{dr, d(aa') - ada' - a'da, d(a + a') - da - da' \mid r \in R, a, a' \in A\})$$

Then,

$$\Omega_{A/R}^i = \bigwedge_A^i (\Omega_{A/R}^1)$$

Furthermore, we define,

$$d(a_0 da_1 \wedge da_2 \wedge \cdots \wedge da_n) = da_0 \wedge da_1 \wedge \cdots \wedge da_n$$

Definition 3.0.3. The de Rham cohomology is defined by hypercohomology,

$$\begin{aligned} H_{\text{dR}}^*(X/S) &= H^*(R\Gamma(X, \Omega_{X/S}^\bullet)) \\ &= \mathbb{H}^*(X, \Omega_{X/S}^\bullet) \end{aligned}$$

Proposition 3.0.4. Suppose the cartesian diagram,

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow f' & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

Then we get a map $h_* \Omega_{X'/S'}^\bullet \leftarrow \Omega_{X/S}^\bullet$ gives a map,

$$H_{\text{dR}}^*(X'/S') \leftarrow H_{\text{dR}}^*(X/S)$$

3.1 Cup Product

Remark. Assume that X has affine diagonal. Then choose an affine open covering \mathfrak{U} ,

$$X = \bigcup_{i \in I} U_i$$

Then affineness of Δ_X implies that,

$$U_{i_0} \cap \cdots \cap U_{i_p}$$

is affine. Then consider the double complex,

$$\mathcal{C}^{p,q} = \mathcal{C}^p(\mathfrak{U}, \Omega_{X/S}^q)$$

Then we make take its total complex $T = \operatorname{Tot}(\mathcal{C}^{\bullet, \bullet})$.

Lemma 3.1.1. $R(X, \Omega_{X/S}^\bullet) = \text{Tot}(\check{\mathcal{C}}^{\bullet, \bullet})$ in $D(\Gamma(S, \mathcal{O}_S))$.

Remark. Let $\alpha = \{\alpha_{i_0, \dots, i_p}\} \in \text{Tot}(\check{\mathcal{C}}(\mathfrak{U}, \Omega_{X/S}^\bullet))^n$ then,

$$d(\alpha)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \dots \hat{i}_j \dots i_{p+1}} |_{U_{i_0 \dots i_p}} + (-1)^{p+1} d_{\Omega^\bullet}(\alpha_{i_0 \dots i_{p+1}})$$

Proposition 3.1.2. There is a map of complexes,

$$\text{Tot}(\text{Tot}(\check{\mathcal{C}}^{\bullet, \bullet}) \otimes \text{Tot}(\check{\mathcal{C}}^{\bullet, \bullet})) \xrightarrow{\sim} \text{Tot}(\check{\mathcal{C}}^{\bullet, \bullet})$$

which is defined y,

$$(\alpha \smile \beta)_{i_0 \dots i_p} = \sum_{r=0}^p (-1)^{(p+r) \deg \alpha + r p + r} \alpha_{i_0 \dots i_r} \wedge \beta_{i_r \dots i_p}$$

where $\deg \alpha$ is the degree in the total complex (i.e. α is a sum of check cycles of forms whose degrees each sum to $\deg \alpha$). Furthrmore, this map is associative and is graded commutative up to homotopy.

Proposition 3.1.3. The construction above defines a cup,

$$\smile: H_{\text{dR}}^p(X/S) \times H_{\text{dR}}^q(X/S) \rightarrow H_{\text{dR}}^{p+q}(X/S)$$

which is graded commutative.

3.2 Hodge Cohomology

Definition 3.2.1. Th Hodge cohomology is,

$$H_{\text{hdg}}^n(X/S) = \bigoplus_{p+q=n} H^q(X, \Omega_{X/S}^p)$$

which is functorial and has a cup product.

Definition 3.2.2. The Hodge filtration on de Rham cohomology is,

$$F^p H_{\text{dR}}^*(X/S) = \text{Im}((H^*(X, \sigma_{\geq p} \Omega_{X/S}^\bullet) \rightarrow H_{\text{dR}}^*(X/S))$$

for the filtration,

$$\Omega_{X/S}^\bullet \supset \sigma_{\geq 1} \Omega_{X/S}^\bullet \supset \sigma_{\geq 2} \Omega_{X/S}^\bullet \supset \dots$$

Remark. Any such filtration defines a spectral sequence.

Proposition 3.2.3. There is a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_{X/S}^p) \implies H_{\text{dR}}^{p+q}(X/S)$$

with E_1 -page,

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \Omega_{X/S}^1) \longrightarrow H^1(X, \Omega_{X/S}^2) \longrightarrow H^1(X, \Omega_{X/S}^3) \longrightarrow \dots$$

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \Omega_{X/S}^1) \longrightarrow H^1(X, \Omega_{X/S}^2) \longrightarrow H^1(X, \Omega_{X/S}^3) \longrightarrow \dots$$

$$H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \Omega_{X/S}^1) \longrightarrow H^0(X, \Omega_{X/S}^2) \longrightarrow H^0(X, \Omega_{X/S}^3) \longrightarrow \dots$$

Proposition 3.2.4. If $S = \text{Spec}(k)$ then the Hodge-to-de Rham spectral sequence degenerates at E_1 iff $\dim_k H_{\text{dR}}^n = \dim_k H_{\text{hdg}}^n$ for each n .

3.3 Künneth Formula

Proposition 3.3.1. Let $S = \operatorname{Spec}(A)$ and $X \rightarrow S, Y \rightarrow S$ be proper and smooth. Then we get,

$$R\Gamma(X, \Omega_{X/S}^\bullet) \otimes_A^\mathbb{L} R\Gamma(Y, \Omega_{Y/S}^\bullet) \cong R\Gamma(X \times_S Y, \Omega_{X \times_S Y/S}^\bullet)$$

from an isomorphism,

$$p_X^{-1} \Omega_{X/S}^\bullet \otimes_{\mathcal{O}_S} p_Y^{-1} (\Omega_{Y/S}^\bullet) \cong \Omega_{X \times_S Y/S}^\bullet$$

To see this isomorphism, we use the fact that, via smoothness,

$$\Omega_{X \times_S Y/S}^n = \bigoplus_{p+q=n} p_X^* \Omega_{X/S}^p \otimes_{\mathcal{O}_{X \times_S Y}} p_Y^* \Omega_{Y/S}^q \cong p_X^{-1} \Omega_{X/S}^p \otimes_{\mathcal{O}_S} p_Y^{-1} \Omega_{Y/S}^q$$

3.4 First Chern Class

Proposition 3.4.1. For $X \rightarrow S$ we have a map of complexes $\mathrm{dlog} : \mathcal{O}_X^\times[-1] \rightarrow \Omega_{X/S}^\bullet$ defined by,

$$u \mapsto \mathrm{dlog} u = \frac{du}{u}$$

which is a map of complexes because $\mathrm{ddlog} = 0$. This gives a map,

$$c_1^{\mathrm{dR}} : \operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^\times) = \mathbb{H}^2(X, \mathcal{O}_X^\times[-1]) \xrightarrow{\mathrm{dlog}} H_{\mathrm{dR}}^2(X/S)$$

similarly,

$$c_1^{\mathrm{hdg}} : \operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^\times) \xrightarrow{\mathrm{dlog}} H^1(X, \Omega_{X/S}^1) \hookrightarrow H_{\mathrm{hdg}}^2(X/S)$$

These maps are compatible with pullbacks,

$$\begin{array}{ccccc} & \operatorname{Pic}(X) & & & \\ & \swarrow c_1^{\mathrm{hdg}} & \downarrow c_1 & \searrow c_1^{\mathrm{dR}} & \\ H_{\mathrm{hdg}}^2(X/S) & \longleftarrow & H^2(X, \sigma_{\geq 1} \Omega_{X/S}^\bullet) & \longrightarrow & H_{\mathrm{dR}}^2(X/S) \end{array}$$

Proposition 3.4.2. Let $h = c_1^{\mathrm{dR}}(\mathcal{O}_X(1))$ we have,

$$H_{\mathrm{dR}}^*(\mathbb{P}_A^n/A) = A[h]/(h^{n+1})$$

and,

$$H_{\mathrm{hdg}}^*(\mathbb{P}_A^n/A) = A[h']/(h'^{n+1})$$

where $h' = c_1^{\mathrm{hdg}}(\mathcal{O}_X(1))$.

Proof. For Hdodge cohomology, use coherent cohomology and the short exact sequence,

$$0 \longrightarrow \Omega^p \longrightarrow \bigwedge^p(\mathcal{O}_X(-1)^{\oplus n+1}) \longrightarrow \Omega^{p-1} \longrightarrow 0$$

and we used induction plus computing the class $(h')^p$.

Then the spectral sequence shows that $H_{\mathrm{dR}}^i(X/S)$ is zero except for $i = 2p$ for $0 \leq p \leq n$ in which case it is a free module. then use that in degree $2n$ we have,

$$H_{\mathrm{dR}}^{2n}(X/S) = H_{\mathrm{hdg}}^{2n}(X/S)$$

because this is the cohomological dimension so there are not other terms which can enter. \square

Proposition 3.4.3 (Projective Space Bundle Formula). For a projective bundle P over S ,

$$H_{\text{dR}}^*(P/S) = H_{\text{dR}}^*(X)[h]/(h^r)$$

where $P = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ for some \mathcal{E} finite locally free sheaf of rank r (i.e. rank r vector bundle). Furthermore, we set $h = c_1^{\text{dR}}(\mathcal{O}_P(1))$.

Proof. Using a Leray spectral sequence we can reduce to the case $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$ and thus $P = \mathbb{P}^{n-1}$. Suppose that X is smooth over affine S . Then $P = \mathbb{P}_X^{r-1} = \mathbb{P}_S^{n-1} \times_S X$ and thus we may conclude using Künneth. \square

4 Nov. 15

Proposition 4.0.1. For all finite syntomic morphisms of schemes $f : X \rightarrow Y$ there is a canonical map,

$$\Theta_{Y/X} : f_* \Omega_{Y/\mathbb{Z}}^\bullet \rightarrow \Omega_{X/\mathbb{Z}}^\bullet$$

uniquely determined by the following properties,

- (a) in degree zero we get the usual trace
- (b) $\Theta_{Y/X}(f^* \omega \wedge \eta) = \omega \wedge \Theta_{Y/X}(\eta)$
- (c) if f is a morphism over S we also obtain,

$$\Theta_{Y/X} : f_* \Omega_{Y/S}^\bullet \rightarrow \Omega_{X/S}^\bullet$$

- (d) compatible with base change.

Definition 4.0.2. $f : X \rightarrow Y$ is syntomic if f is flat and locally of finite presentation and its fibres are local complete intersections.

Syntomic is equivalent to f flat and a local complete intersection

equivalently flat and locally of finite presentation and the cotangent complex is perfect in $[-1, 0]$.

Remark. If $f : X \rightarrow Y$ is syntomic then f is locally quasi-finite (i.e. fibres are finite) iff the rank of the naive cotangent complex $NL_{Y/X}$ is zero.

Remark. Being local complete intersection is not preserved under base change. However it is preserved under flat base change if the morphism is also flat. Therefore, syntomic morphisms are preserved under flat base change.

Remark. Let $f : Y \rightarrow X$ be a finite surjective morphism of varieties over $k \supset \mathbb{Q}$ with X smooth. By miracle flatness $f : Y \rightarrow X$ is flat when Y is Cohen-Macaulay. Then consider,

$$\begin{array}{ccc} f_* \Omega_{Y/k}^\bullet & \xrightarrow{\Theta} & \Omega_{X/k}^\bullet \\ & \searrow \nu^* & \nearrow \\ & f_*^\nu \Omega_{Y^\nu/k}^\bullet & \end{array}$$

where $\nu : Y^\nu \rightarrow Y$ is the normalization which for a variety is a finite map and $f^\nu = f \circ \nu$. So we may also assume that Y is normal (smooth in codimension one). Since $\text{ch} k = 0$ then f is étale over a dense open $U \subset X$ so $f^{-1}(U) \rightarrow U$ is finite étale. Therefore,

$$\Omega_{Y/k}^\bullet|_{f^{-1}(U)} \cong f^* \Omega_{X/k}^\bullet|_{f^{-1}(U)}$$

by étale. So over U we can use the identification,

$$f_* \Omega_{Y/k}^\bullet|_U \cong f_* f^* \Omega_{X/k}^\bullet|_U = \Omega_{U/k}^\bullet \otimes_{\mathcal{O}_U} f_* \mathcal{O}_{f^{-1}(U)}$$

Then we apply the usual trace map,

$$f_* \mathcal{O}_{f^{-1}(U)} \rightarrow \mathcal{O}_U$$

to give a map,

$$f_* \Omega_{Y/k}^\bullet|_U \cong \Omega_{U/k}^\bullet \otimes_{\mathcal{O}_U} f_* \mathcal{O}_{f^{-1}(U)} \rightarrow \Omega_{U/k}^\bullet \otimes_{\mathcal{O}_U} \mathcal{O}_U = \Omega_{U/k}^\bullet$$

I claim that this map is compatible with differentials and extends uniquely to $\Theta_{Y/X} : f_* \Omega_{Y/k}^\bullet \rightarrow \Omega_{X/k}^\bullet$.

Suppose that $f : f^{-1}(U) \rightarrow U$ is Galois with group G then,

$$\Theta|_U(\omega) = \sum_{\sigma \in G} \sigma^*(\omega)$$

and each σ^* is compatible with differentials.

Alternatively, you can do étale localization to reduce to $f^{-1}(U)$ is a finite product of isomorphic copies then our trace map is just summing.

Now, by Hartog's theorem it suffices to extend the map to codimension 1. Here we use that X/k is smooth so $\Omega_{X/k}^p$ is finite locally free, so reflexive (isomorphic to its double dual). Then, generically, in codimension one, we get étale locally that,

$$\begin{array}{ccc} Y & \xlongequal{\quad} & \text{Spec}(B) \\ \downarrow & & \downarrow \\ X & \xlongequal{\quad} & \text{Spec}(A) \end{array}$$

is of the form $A \rightarrow B = A[x]/(x^e - a)$ for $a \in A$. Then,

$$\Omega_{A/k}^1 = A da \oplus R \quad \text{and} \quad \Omega_{B/k}^1 = B dx \oplus (B \otimes_A R)$$

Then, for $g \in B$ and $\omega \in \Omega_A^\bullet$ we have $\text{Tr}(g\omega) = \text{Tr}(g)\omega$ and,

$$\text{Tr}(g dx \wedge \omega) = \text{Tr}(g dx) \wedge \omega$$

Thus it suffices to compute,

$$\text{tr}(x^i dx) = \begin{cases} 0 & 0 \leq i \leq e-2 \\ da & i = e-1 \end{cases}$$

Remark. The ring map $A \rightarrow A[x]/(x^p - a)$ is a local complete intersection when $\mathbb{F}p \subset A$. Then we still have,

$$\Theta_{B/A}(x^i dx) = \begin{cases} 0 & 0 \leq i \leq p-2 \\ da & i = p-1 \end{cases}$$

Furthermore, in degree zero we get $\text{Tr}_{B/A} : B \rightarrow A$ is always zero.

Remark. First idea of Gatel : de deformations we can reduce to the case X and Y are smooth over \mathbb{Z} and $f : Y \rightarrow X$ is étale over a dense open.

Remark. Now, we can construct a canonical isomorphism,

$$\begin{aligned} a_{Y/X} : \det(NL_{Y/X}) &\rightarrow \omega_{Y/X} \\ \delta(NL_{Y/X}) &\mapsto \tau_{Y/X} \end{aligned}$$

Recall that locally $NL_{Y/X} = (\mathcal{E}^{-1} \rightarrow \mathcal{E})$ both of rank t^r . Then.

$$\delta(NL_{Y/X}) = \det \alpha \in \det(NL_{Y/X}) = \bigwedge^r \mathcal{E}^0 \otimes \left(\bigwedge^r \mathcal{E}^{-1} \right)^\vee$$

Then $\omega_{Y/X}$ is the unique \mathcal{O}_X -module s.t. $f_* \omega_{Y/X} = \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X)$. Then,

$$\tau_{Y/X} = (\text{Tr}_f : f_* \mathcal{O}_Y \rightarrow \mathcal{O}_X)$$

Then construct a canonical morphism,

$$c_{Y/X}^p : \Omega_{Y/\mathbb{Z}}^p \rightarrow f^* \Omega_{X/\mathbb{Z}}^p \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

such that,

$$f^* \Omega_{X/\mathbb{Z}}^p \longrightarrow \Omega_{Y/\mathbb{Z}}^p \longrightarrow f^* \Omega_{X/\mathbb{Z}}^p \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})$$

Finally,

$$\begin{array}{ccc} f^* \Omega_{Y/\mathbb{Z}}^\bullet & \xrightarrow{\quad \Theta_{Y/X} \quad} & \Omega_{X/\mathbb{Z}}^\bullet \\ \downarrow f_* c_{Y/X}^\bullet & & \uparrow \text{eval}_1 \\ f_*(f^* \Omega_{X/\mathbb{Z}}^\bullet \otimes_{\mathcal{O}_Y} \det(NL_{Y/X})) & \xrightarrow{c_{Y/X}} & f_*(f^* \Omega_{X/\mathbb{Z}}^\bullet \otimes \omega_{Y/X}) = \Omega_{X/\mathbb{Z}}^\bullet \otimes \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X) \end{array}$$

where the equality is by the projection formula.

Remark. Now we give a local construction of $c_{Y/X}^p$ for $A \rightarrow B = A[x_1, \dots, x_n]/(f_1, \dots, f_n)$ global relative complete intersection. Then consider,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A/\mathbb{Z}} \otimes B & \longrightarrow & \Omega_{A[x]/\mathbb{Z}} \otimes B & \longrightarrow & \Omega_{A[x]/A} \otimes B \longrightarrow 0 \\ & & & & \uparrow & & \uparrow q \\ & & & & (f_1, \dots, f_n)/(f_1, \dots, f_n)^2 & = & (f_1, \dots, f_n)/(f_1, \dots, f_n)^2 \end{array}$$

Where,

$$\begin{aligned} \bigwedge^p(\Omega_{B/\mathbb{Z}}) &\rightarrow \bigwedge^p(\Omega_{A/\mathbb{Z}} \otimes B) \otimes_B \det_B(q) \\ \xi &\mapsto \eta \otimes \frac{dx_1 \wedge \cdots \wedge dx_n}{f_1 \wedge \cdots \wedge f_n} \end{aligned}$$

iff,

$$\eta \wedge dx_1 \wedge \cdots \wedge dx_n = \tilde{\xi} \wedge f_1 \wedge \cdots \wedge f_n \in \bigwedge^{p+n}(\Omega_{A[\underline{x}]/\mathbb{Z}} \otimes B)$$

Remark. Garel shows that,

$$\Theta_{Y/X}(\mathrm{dlog} u) = \mathrm{dlog}(\mathrm{Nm}_{Y/X}(u))$$

for any $u \in \mathcal{O}_Y^\times$ and,

$$\Theta_{Y/X}(c_1^{\mathrm{dR}}(\mathcal{N})) = c_1^{\mathrm{dR}}(\mathrm{Nm}_{Y/X}(\mathcal{N}))$$

5 Complex with Log Poles

Let $k = \bar{k}$ of characteristic zero. Let X be smooth projective over k and $Y \subset X$ a smooth divisor.

Proposition 5.0.1. There is a canonical short exact sequence of complexes,

$$0 \longrightarrow \Omega_X^\bullet \longrightarrow \Omega_X^\bullet(\log Y) \xrightarrow{\mathrm{res}} \Omega_Y^\bullet[-1] \longrightarrow 0 \quad \text{as modules of differentials over } k.$$

Proof. Etale locally on X there are coordinates x_1, \dots, x_d such that $Y = \{x_d = 0\}$ in X . Then you define $\Omega_X^1(\log Y)$ free on, $dx_1, \dots, dx_{d-1}, \frac{dx_d}{x_d}$ over \mathcal{O}_X . We can define $\Omega_X^1 \subset \Omega_X^1(Y)$. Now set,

$$\mathrm{res}\left(\frac{dx_d}{x_d}\right) = 1 \in \mathcal{O}_Y$$

to give,

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log Y) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

Now set $\Omega_X^p(\log Y) = \bigwedge^p(\Omega_X^1(\log Y))$ and,

$$\mathrm{res}\left(\frac{dx_d}{x_d} \wedge \omega\right) = \omega|_Y$$

□

Remark. If $k = \mathbb{C}$ we know $H_{\mathrm{dR}}^*(X/\mathbb{C}) = H_B^*(X^{\mathrm{an}}, \mathbb{C})$. Also,

$$H^*(X, \Omega_X^\bullet(\log Y)) = H_B^*(X^{\mathrm{an}} \setminus Y^{\mathrm{an}}, \mathbb{C})$$

Proposition 5.0.2. The boundar map $\partial : H_{\mathrm{dR}}^i(Y) \rightarrow H_{\mathrm{dR}}^{i+2}(X)$ from the short exact sequence above fis into a commutative diagram,

$$\begin{array}{ccc} H_{\mathrm{dR}}^i(X) & \xrightarrow{\sim c_1} & H_{\mathrm{dR}}^{i+2}(X) \\ \downarrow & \nearrow \partial & \downarrow \\ H_{\mathrm{dR}}^i(Y) & \xrightarrow{\sim c_1|_Y} & H_{\mathrm{dR}}^{i+2}(Y) \end{array}$$

where $c_1 = c_1^{\text{dR}}(\mathcal{O}_X(-Y))$.

Corollary 5.0.3. If $a|_Y = 0$ then $a \smile c_1(\mathcal{O}_X(Y)) = 0$ for $a \in H_{\text{dR}}^i(X)$.

Theorem 5.0.4. With k as above the functor $X \mapsto H_{\text{dR}}^*(X)$ is a (classical) Weil Cohomology Theory with coefficients in $F = k$.

Proof. It suffices to give data,

$$(D0) \quad F(1) = k$$

$$(D1) \quad H^*(X) = H_{\text{dR}}^*(X)$$

$$(D2') \quad c_1^{\text{dR}} : \text{Pic}(X) \rightarrow H_{\text{dR}}^2(X)$$

We need to check axioms (A1) - (A9).

(A1) de Rham cohomology is compatible with disjoint unions.

(A2) c_1^{dR} is compatible with pullbacks

(A3) the projective space bundle formula (which we proved)

(A4) is the corollary above.

Remark. The grothendieck argument shows that (A1) - (A4) gives a cycle class map $\text{ch}^{\text{dR}} : K_0(\text{Vect}(X)) \rightarrow H_{\text{dR}}^*(X)$. Then you define,

$$\gamma(\alpha) = \text{ch}^{\text{dR}}(\text{ch}^{-1}(\alpha))$$

with $\text{ch} : K_0(\text{Vect}(X))_{\mathbb{Q}} \xrightarrow{\sim} \text{CH}^*(X)_{\mathbb{Q}}$.

(A5) Kunneth formula.

(A6) We need a k -linear map,

$$\lambda : H_{\text{dR}}^{2d}(X) \rightarrow k$$

such that $(1 \otimes \lambda)(\gamma([\Delta])) = 1 \in H_{\text{dR}}^0(X)$. We can decompose, $H_{\text{dR}}^*(X \times X) = H_{\text{dR}}^*(X) \otimes H_{\text{dR}}^*(X)$ by Kunneth and write $\gamma(\Delta) = \gamma_0 + \cdots + \gamma_{2d}$. Then,

$$(1 \otimes \lambda)(\gamma([\Delta])) = (1 \otimes \lambda)(\gamma_0 + \cdots + \gamma_{2d})$$

Pick $x \in X$ closed,

$$\begin{array}{ccc} x & \longrightarrow & \Delta \\ \downarrow & & \downarrow \\ X & \xrightarrow{x \times 1_X} & X \times X \end{array}$$

Conclude that $H_{\text{dR}}^0 \otimes H_{\text{dR}}^{2d} \rightarrow H_{\text{dR}}^{2d}$ via $\gamma_0 \mapsto \gamma([x])$. We conclude that $\gamma([x])$ is independent of x and it is enough to show that $\gamma(\alpha) \neq 0$ for some zero cycle α on X . We pick a finite morphism $X \xrightarrow{f} \mathbb{P}_k^d$ and consider,

$$\begin{array}{ccccc} & & \deg f & & \\ & \nearrow & & \searrow & \\ \Omega_{\mathbb{P}^d}^\bullet & \longrightarrow & f_*\Omega_X^\bullet & \xrightarrow{\Theta} & \Omega_{\mathbb{P}^d}^\bullet \end{array}$$

So $f^* : H_{\text{dR}}^{2d}(\mathbb{R}^d) \rightarrow H_{\text{dR}}^{2d}(X)$ is injective and $\gamma \circ f^* = f^* \circ \gamma$ so it suffices to observe that,

$$\gamma([x] \in \mathbb{P}^d) = c_1(\mathcal{O}(1))^d \frown [\mathbb{P}^d] \neq 0$$

- (A7) If $b : X' \rightarrow X$ is a blowup with smooth center $Z \subset X$. Then $b^* : H^*_{\text{dR}}(X) \rightarrow H^*_{\text{dR}}(X')$ is injective. First you show there exists a distinguished triangle,

$$\Omega_X^\bullet \longrightarrow Rb_*\Omega_{X'}^\bullet \oplus \Omega_Z^\bullet \longrightarrow R\pi_*\Omega_E^\bullet \xrightarrow{+1} \dots$$

In $D(X)$ by a local calculation. This gives,

$$H_{\text{dR}}^*(X) \longrightarrow H_{\text{dR}}^*(X') \oplus H_{\text{dR}}^*(Z) \longrightarrow H_{\text{dR}}^*(E) = H_{\text{dR}}^*(Z)[\xi]/(\xi^n)$$

\longleftarrow

the last equality holds by the projective space bundle formula and the backways map is the Gysin map. Show using the Gysin map surjectivity for the second map.

- (A8) If X is a smooth projective variety then $H_{\text{dR}}^0(X) = k$.
- (A9) If $Y \subset X$ is a smooth divisor and X is a smooth projective (irreducible) variety of dimension d then we get a diagram,

$$\begin{array}{ccc} H_{\text{dR}}^{2\dim Y}(X) & \xrightarrow{-\smile c_1^{\text{dR}}(\mathcal{O}_X(Y))} & H_{\text{dR}}^{2\dim X}(X) \\ \downarrow & \nearrow -\partial & \downarrow \lambda_X \\ H_{\text{dR}}^{2\dim Y}(Y) & \xrightarrow{\lambda_Y} & k \end{array}$$

with the upper triangle commuting. We need to show that the lower triangle also commutes. It suffices to check commutativity for some $a \in H_{\text{dR}}^{2\dim Y}(X)$ with nonzero image in $H_{\text{dR}}^{2\dim Y}(Y)$ since this is one-dimensional (and the commutativity of the upper triangle implies that anything of image zero must commute). We can pick,

$$a = c_1^{\text{dR}}(\mathcal{L})^{\dim Y}$$

for a line bundle \mathcal{L} . Then,

$$\lambda_Y(a|_Y) = \deg_Y(c_1(\mathcal{L})^{\dim Y} \frown [Y])$$

Then,

$$\lambda_X(a \smile c_1^{\text{dR}}(\mathcal{O}_X(Y))) = \deg_X(c_1(\mathcal{L})^{\dim Y} \frown c_1(\mathcal{O}_X(Y)) \frown [X])$$

□