# Contents

# 1 Chapter 1

# 2 Chapter 2

## 2.1 Section 2.1

### 2.1.1 2.2.2

Given an exact sequence of vector bundles,

$$0 \longrightarrow L \xrightarrow{\varphi} E \xrightarrow{\psi} F \longrightarrow 0$$

where $L$ is a line bundle. Consider the exact sequence,

$$0 \longrightarrow L \otimes \bigwedge^{i-1} F \longrightarrow \bigwedge^{i} E \longrightarrow \bigwedge^{i} F \longrightarrow 0$$

where the first map is defined by taking sections $s_1 \otimes (f_2 \wedge \cdots \wedge f_i)$ and lifing each $f_i$ to a section $e_j$ of $E$ up to a section $s_j$ of $L$ and then map $s_1 \otimes (f_2 \wedge \cdots \wedge f_i) \mapsto s_1 \wedge e_2 \wedge \cdots \wedge e_i$. This map is well-defined because,

$$\begin{aligned} s_1 \wedge e_2' \wedge \cdots \wedge e_i' &= s_1 \wedge (e_2 + \varphi(s_2)) \wedge \cdots \wedge (e_i + \varphi(s_i)) = s_1 \wedge e_2 \wedge \cdots \wedge e_n + s_1 \wedge s_2 \wedge \cdots \wedge e_n + \cdots \\ &= s_1 \wedge e_2 \wedge \cdots \wedge e_n \end{aligned}$$

because $s_1 \wedge s_j = 0$ since $L$ is a line bundle. Thus this map is well-defined and clearly $\ker \wedge^i \psi$ is the image of this map because $L \subset E$ maps to zero under $\psi$ thus the kernel is exterior products where one factor is in $L$. Furthermore, dualizing if we have an exact sequence of vector bundles,

$$0 \longrightarrow F \longrightarrow E \longrightarrow L \longrightarrow 0$$

where $L$ is a line bundle then there is an exact sequence,

$$0 \longrightarrow \bigwedge^{i} F \longrightarrow \bigwedge^{i} E \longrightarrow L \otimes \bigwedge^{i-1} F \longrightarrow 0$$

### 2.1.2 2.2.3

Let $E$ be a holomorphic vector bundle $E$ of rank $r$ there exists a non-degenerate pairing,

$$\bigwedge^k E \times \bigwedge^{r-k} E \to \det E$$

via $(e_1 \wedge \cdots \wedge e_k, e_{k+1} \wedge \cdots \wedge e_r) \mapsto e_1 \wedge \cdots \wedge e_r$. Locally $E \cong \mathcal{O}_X^{\oplus r}$ and thus the pairing is nondegenerate because we can take $e_i$ to be the standard basis of $\mathcal{O}_X^{\oplus r}$. Then $\det E \cong \mathcal{O}_X$ and $(e_1 \wedge \cdots \wedge e_k, e_{k+1} \wedge \cdots \wedge e_r) \mapsto e_1 \wedge \cdots \wedge e_r$ is a generator of $\det E \cong \mathcal{O}_X$.

Since the above pairing is nondegenerate, we get an isomorphism $\bigwedge^k E \xrightarrow{\sim} \bigwedge^{r-k} E^* \otimes \det E$.

### 2.1.3 2.2.5

Let $L, L^*$ be holomorphic line bundles on a compact complex manifold $X$. Suppose that $L$ and $L^*$ admit nonzero global holomorphic sections $s, s'$. Then consider $s \otimes s'$ a global section of $L \otimes L^* \cong \mathcal{O}_X$. However, all nonzero sections of $\mathcal{O}_X$ are nonvanishing because $X$ is compact and thus $H^0(X, \mathcal{O}_X) = \mathbb{C}$. Therefore, $s$ and $s'$ are nonvanishing meaning that $L \cong L^* \cong \mathcal{O}_X$.

### 2.1.4 2.2.6 DO!!

## 2.2 Section 2.6

### 2.2.1 2.6.1

I think $f$ is holomorphic iff $\mathrm{d}f(Iv) = i\mathrm{d}f(v)$

### 2.2.2 2.6.2 DO!!

### 2.2.3 2.6.3 DO!!

### 2.2.4 2.6.4 CHECK!!

Let $f : X \to Y$ be a surjective holomorphic map between connected xomplex manifolds. We want to look at the smooth locus of $f$.

I claim the following is true: for a morphism of vector budles (not necessarily constant rank) $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ then $\phi$ has full rank $k = \min\{m, n\}$ iff the morphism $\phi' : \bigwedge^k \mathcal{E}_1 \to \bigwedge^k \mathcal{E}_2$ is nonzero (is this true).

Therefore, the locus where $\phi$ is not full rank is the vanishing the section

$$\phi' \in \mathcal{HOM}_{\mathcal{O}_X}\left(\bigwedge^k \mathcal{E}, \bigwedge^k \mathcal{E}_2\right)$$

Now apply this to the map $f^*\Omega_Y \to \Omega_X$ to get the nonsmooth locus.

### 2.2.5 2.6.5 CHECK!!

The cousins' problem has a solution because $H^1(X, \mathcal{O}_X) = 0$. Question: why is every hypersurface defined by a $H^0(K^\times/\mathcal{O}_X^\times)$. Question: how are we supposed to use the poincare lemma.

#### 2.2.6 2.6.5 DO!!

#### 2.2.7 2.6.7

We define,

$$H_{\mathrm{BC}}^{p,q}(X) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) \mid \mathrm{d}\alpha = 0\}}{\partial\bar{\partial}\mathcal{A}^{p-1,q-1}(X)}$$

This makes sense because if $\alpha = \partial\bar{\partial}\gamma$ then

$$\mathrm{d}\alpha = \partial^2\bar{\partial}\gamma - \bar{\partial}^2\partial\gamma = 0$$

Now, the inclusion of d-closed forms into $\bar{\partial}$-closed forms induces a map,

$$H_{\mathrm{BC}}^{p,q}(X) \to H^{p,q}(X)$$

which is well-defined because if $\alpha = \partial\bar{\partial}\gamma$ then $\alpha = -\bar{\partial}\partial\gamma$ and is thus $\bar{\partial}$-exact. If $X$ is furthermore compact Kahler then by the $\partial\bar{\partial}$-lemma we see if $\alpha$ maps to zero i.e. $\alpha = \partial\bar{\partial}\beta$ and $\mathrm{d}\alpha = 0$ then $\alpha = \mathrm{d}\gamma$ so the map is injective. Furthermore, by the Hodge decomposition, $H^{p,q}(X)$ can be represented by Harmonic forms which are d-closed and thus this map is surjective as well.

#### 2.2.8 2.6.8 ASK RON!!

Is this just because we can take $M \to M$ via complex conjugation.

#### 2.2.9 2.6.9 DO!!

#### 2.2.10 2.6.10 DO!!

#### 2.2.11 2.6.11 ASK RON!!

# 3 Chapter 3

## 3.1 Section 3.1

#### 3.1.1 3.1.1 DO!!

Let $X$ be a complex manifold with an almost complex structure $(M, I)$. We need to find a Riemannian structure $g$ on $M$ such that $g$ is compatible with $I$ meaning $g(I-, I-) = g(-,-)$. (FINISH)

#### 3.1.2 3.1.2

Let $X$ be a connected complex manifold of dimension $n > 1$ and let $g$ be a Kähler metric. Suppose that $g' = e^f \cdot g$ for some real smooth function $f \in \mathcal{A}^0(X)$ is also a Kähler metric. Then the associated Kähler forms satisfy $\omega' = e^f \cdot \omega$. Since both are Kähler forms, we must have $\mathrm{d}\omega' = 0$ and $\mathrm{d}\omega = 0$. However,

$$\mathrm{d}\omega' = e^f \mathrm{d}f \wedge \omega + e^f \mathrm{d}\omega = e^f \mathrm{d}f \wedge \omega$$

Therefore, $\mathrm{d}f \wedge \omega = 0$ and thus $L(\mathrm{d}f) = 0$. However, since $n > 1$ the Lefschetz operator is injective on $k$-forms for $k < n$ and thus $\mathrm{d}f = 0$. Since $X$ is connected then $f$ is constant so $\omega' = c\omega$.

### 3.1.3 3.1.3 DO!!

### 3.1.4 3.1.4 DO!!

### 3.1.5 3.1.5 DO!!

### 3.1.6 3.1.7 DO!!

### 3.1.7 3.1.8 DO!!

### 3.1.8 3.1.12 DO!!

### 3.1.9 3.1.13 DO!!

## 3.2 Section 3.2

### 3.2.1 3.2.1

### 3.2.2 3.2.2

### 3.2.3 3.2.4

What does this really mean?? Ask Ron.

### 3.2.4 3.2.5 DO!!

### 3.2.5 3.2.6

Let $X$ be a compact Kähler manifold. Then,

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

Furthermore, $H^{q,p} = \overline{H^{p,q}}$. Therefore,

$$b_{2k+1} = \sum_{p+q=2k+1} h^{p,q} = \sum_{i=0}^{k} (h^{2k+1-i,i} + h^{i,2k+1-i}) = 2\sum_{i=0}^{k} h^{2k+1-i,i}$$

is even.

### 3.2.6 3.2.7 DO!!

No! (PROVE IT)

### 3.2.7 3.2.8

Let $X$ be a compact Kähler manifold. Let $\omega \in H^0(X, \Omega_X^p)$. Clearly, $\bar{\partial}\omega = 0$ since $\omega$ is a holomorphic $(p,0)$-form. Furthermore,

$$\bar{\partial}^* \omega = -(\bar{\star} \circ \bar{\partial} \circ \bar{\star})\,\omega$$

but $\bar{\star}\omega$ is a $(n-p,n)$-form and thus $\bar{\partial}\bar{\star}\omega = 0$. Therefore, $\bar{\partial}\omega = 0$ and $\bar{\partial}^*\omega = 0$ and thus $\Delta_{\bar{\partial}}\,\omega = 0$.

### 3.2.8 3.2.9 DO!!

### 3.2.9 3.2.10 CHECK!!

Let $(X, g)$ be a compact hermitian manifold. Show that any d-harmonic $(p, q)$-form is also $\bar{\partial}$-harmonic.

Since $\alpha$ is d-harmonic, we have $\mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$. Therefore, $\partial\alpha = 0$ and $\bar{\partial}\alpha = 0$ and $\partial^*\alpha = 0$ and $\bar{\partial}^*\alpha = 0$. Therefore, $\Delta_{\partial}(\alpha) = 0$ and $\Delta_{\bar{\partial}}(\alpha) = 0$.

### 3.2.10 3.2.11

Let $X = \mathbb{P}^n$. Consider the Euler sequence,

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{O}_X(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Then we may apply exterior powers to get the following sequence,

$$0 \longrightarrow \Omega_X^p \longrightarrow \textstyle\bigwedge^p \mathcal{O}_X(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega_X^{p-1} \longrightarrow 0$$

However,

$$\bigwedge^p \mathcal{O}_X(-1)^{\oplus(n+1)} = \mathcal{O}_X(-p)^{\oplus\binom{n+1}{p}}$$

and thus we have the sequence,

$$0 \longrightarrow \Omega_X^p \longrightarrow \mathcal{O}_X(-p)^{\oplus\binom{n+1}{p}} \longrightarrow \Omega_X^{p-1} \longrightarrow 0$$

Now applying the cohomology sequence we find,

$$H^{q-1}(X, \mathcal{O}_X(-p))^{\oplus\binom{n+1}{p}} \longrightarrow H^{q-1}(X, \Omega_X^{p-1}) \longrightarrow H^q(X, \Omega_X^p) \longrightarrow H^q(X, \mathcal{O}_X(-p))^{\oplus\binom{n+1}{p}}$$

Therefore, if $0 < q < n$ and $p > 0$ then $H^{q-1}(X, \Omega_X^{p-1}) \xrightarrow{\sim} H^q(X, \Omega_X^p)$. Furthermore, if $q = 0$ and $p > 0$ then $H^0(X, \Omega_X^p) = 0$ because we get an exact sequence,

$$0 \longrightarrow H^0(X, \Omega_X^p) \longrightarrow H^0(X, \mathcal{O}_X(-p))^{\oplus\binom{n+1}{p}}$$

and $H^0(X, \mathcal{O}_X(-p)) = 0$. Finally, if $q = n$ and $p < n + 1$ (which it must) then $H^{q-1}(X, \Omega_X^{p-1}) \xrightarrow{\sim} H^q(X, \Omega_X^p)$ because $H^n(X, \mathcal{O}_X(-p)) = H^0(X, \mathcal{O}_X(p - n - 1))$ by Serre duality.

To finish the base case $p = 0$ we know,

$$H^q(X, \mathcal{O}_X) = \begin{cases} \mathbb{C} & q = 0 \\ 0 & q > 0 \end{cases}$$

Therefore, by induction, $H^p(X, \Omega_X^p) = H^{p-1}(X, \Omega_X^{p-1}) = \mathbb{C}$ for $p \le n$. Furthermore, if $p \neq q$ then reducing via $H^{q-1}(X, \Omega_X^{p-1}) \xrightarrow{\sim} H^q(X, \Omega_X^p)$ we get to either $H^q(X, \Omega_X^0) = 0$ with $q > 0$ or $H^0(X, \Omega_X^p) = 0$ with $p > 0$. Therefore,

$$H^q(X, \Omega_X^p) = \begin{cases} \mathbb{C} & p = q \le n \\ 0 & p \neq q \end{cases}$$

Now consider the exponential sequence,

$$H^0(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X)$$

but $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ and $H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}\,(X)$. Therefore, the exponential sequence defines an isomorphism $\mathrm{Pic}\,(X) \xrightarrow{\sim} H^2(X, \mathbb{Z})$. By the Kähler decomposition $H^2(X, \mathbb{C}) = H^{1,1}(X) = H^1(X, \Omega_X^1)$ and thus $\dim_{\mathbb{C}} H^2(X, \mathbb{C}) = 1$. Therefore, $H^2(X, \mathbb{Z}) = \mathbb{Z}$ because $H^2(X, \mathbb{Z})$ is torsion-free (WHY?).

### 3.2.11 3.2.12 DO!!

### 3.2.12 3.2.13

Let $X$ be a compact Kähler manifold and $\alpha \in \mathcal{A}^k(X)$ which is d-closed and d$^c$-exact where $\mathrm{d}^c = i(\bar{\partial} - \partial)$. Notice that $\mathrm{dd}^c = 2i\partial\bar{\partial}$. Write $\alpha = \alpha^{k,0} + \cdots + \alpha^{0,k}$. Since $\mathrm{d}\alpha = 0$ and $\mathrm{d}^c\alpha = 0$ we see that $\partial\alpha = 0$ and $\bar{\partial}\alpha = 0$ and this must be true for each $\alpha^{p,q}$ because $\Pi^{p+1,q}\partial\alpha = \partial\alpha^{p,q}$ etc. Then by the $\partial\bar{\partial}$-lemma we know $\alpha^{p,q} = \partial\bar{\partial}\beta$ for $\beta \in \mathcal{A}^{p-1,q-1}(X)$. Therefore,

$$\alpha = \alpha^{k,0} + \cdots + \alpha^{0,k} = \partial\bar{\partial}(\beta^{k,0} + \cdots + \beta^{0,k}) = -\frac{i}{2}\mathrm{dd}^c\beta$$

where $\beta = \beta^{k,0} + \cdots + \beta^{0,k}$.

### 3.2.13 3.2.14 DO!!

DO! LOOK AT PREVIOUS BC PROBLEM AND CORRESPOND

### 3.2.14 3.2.15

Let $(X, g)$ be a compact hermitian manifold and let $[\alpha] \in H^{p,q}(X)$ be a cohomology class. If we deform $\alpha' = \alpha + t\bar{\partial}\beta$ by an exact form such that $[\alpha'] = [\alpha]$. Then consider the norm,

$$||\alpha'||^2 = \left\langle \alpha + t\bar{\partial}\beta, \alpha + t\bar{\partial}\beta \right\rangle = ||\alpha||^2 + 2t\Re\left\langle \alpha, \bar{\partial}\beta \right\rangle + t^2||\bar{\partial}\beta||^2$$

Suppose that $\alpha$ has minimal norm then the linear term must be zero so $\Re\left\langle \alpha, \bar{\partial}\beta \right\rangle = 0$. Likewise, replacing $\beta$ by $i\beta$ then $\Re\left\langle \alpha, \bar{\partial}i\beta \right\rangle = \mathrm{Im}\left(\left\langle \alpha, \bar{\partial}\beta \right\rangle\right)$ so we must have $\left\langle \alpha, \bar{\partial}\beta \right\rangle = 0$. However, $\left\langle \alpha, \bar{\partial}\beta \right\rangle = \left\langle \bar{\partial}^*\alpha, \beta \right\rangle = 0$ for arbitrary $(p, q-1)$-forms $\beta$. In particular, we can take,

$$||\bar{\partial}^*\alpha||^2 = \left\langle \bar{\partial}^*\alpha, \bar{\partial}^*\alpha \right\rangle = 0$$

so $\bar{\partial}^*\alpha = 0$. Furthermore $\bar{\partial}\alpha = 0$ since it defines a cohomology class. Thus $\Delta_{\bar{\partial}}\alpha = 0$ so the representatives of minimal norm are exactly the harmonic representatives.

### 3.2.15 3.2.16

Let $X$ be a compact Kähler manifold. Let $\omega$ and $\omega'$ be Kähler forms such that $[\omega] = [\omega'] \in H^2(X, \mathbb{R})$. Then $\eta = \omega - \omega' = \mathrm{d}\alpha$ for some real 1-form $\alpha$. Thus $\eta$ is a closed real $(1,1)$-form which is d-exact and thus by the $\partial\bar{\partial}$-lemma $\eta = i\partial\bar{\partial}f$ for some $f \in \mathcal{A}^{0,0}$. Notice,

$$\bar{\eta} = -i\bar{\partial}\partial\bar{f} = i\partial\bar{\partial}\bar{f}$$

however $\eta$ is real so $\bar{\eta} = \eta$ and thus $\bar{f} = f$ so $f \in \mathcal{A}^0_{\mathbb{R}}$ is a real function and,

$$\omega = \omega' + i\partial\bar{\partial}f$$

## 3.3 Section 3.3

### 3.3.1 3.3.1 DO!!

### 3.3.2 3.3.2 DO!!

### 3.3.3 3.3.3 DO!!

# 4 Chapter 4

## 4.1 Section 4.1

### 4.1.1 4.1.1

Let $L$ be a holomorphic line bundle globally generated by sections $s_1, \dots, s_k \in H^0(X, L)$. Then $L$ admits a canonical hermitian structure $h$ defined by locally choosing a trivialization, $\phi : L|_U \xrightarrow{\sim} \mathcal{O}_U$ then for any local sections $\alpha, \beta \in \mathcal{L}(U)$,

$$h(\alpha, \beta) = \frac{\psi(\alpha) \cdot \overline{\psi(\beta)}}{\sum_i |\psi(s_i)|^2}$$

This is well-defined because for any other choice of local trivialization $\psi' : L|_u \xrightarrow{\sim} \mathcal{O}_U$ gives a transition function $t = \psi' \circ \psi^{-1} : \mathcal{O}_U \to \mathcal{O}_U$ which is a holomorphic function on $U$. Then $\psi' = (\psi' \circ \psi^{-1}) \circ \psi = t\psi$ and therefore,

$$\frac{\psi'(\alpha) \cdot \overline{\psi'(\beta)}}{\sum_i |\psi'(s_i)|^2} = \frac{t\psi(\alpha) \cdot \overline{t\psi(\beta)}}{\sum_i |t\psi(s_i)|^2} = \frac{|t|^2 \psi(\alpha \cdot \overline{\psi(\beta)}}{|t|^2 \sum_i |\psi(s_i)|^2} = \frac{\psi(\alpha) \cdot \overline{\psi(\beta)}}{\sum_i |\psi(s_i)|^2}$$

Then the dual bundle $L^*$ obtains a natural hermitian structure $h^*$ defined on $\alpha, \beta \in L^*(U)$ i.e. maps $L|_U \to \mathcal{O}_U$ via,

$$h^*(\alpha, \beta) = \alpha(h^{-1}(\beta))$$

viewing $h$ as an $\mathbb{C}$-antilinear isomorphism $h : \mathcal{L} \xrightarrow{\sim} \mathcal{L}^*$. Furthermore, there is an inclusion $L^* \hookrightarrow \mathcal{O}_X^{\oplus k}$ dual to $\mathcal{O}_X^{\oplus k} \twoheadrightarrow \mathcal{L}$ defined by applying $\alpha \in L^*(U)$ to $s_1, \dots, s_n$. By restricting the standard hermitian structure on $\mathcal{O}_X^{\oplus k}$ to $L^*$ we get a hermitian structure $h'$ on $\mathcal{L}$. Explicitly, for $\alpha, \beta \in L^*(U)$,

$$h'(\alpha, \beta) = \sum_i \alpha(s_i) \overline{\beta(s_i)}$$

However, chooseing a local trivialization $\psi : \mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U$ then $\psi^* : \mathcal{O}_U \xrightarrow{\sim} \mathcal{L}^*|_U$ given by $\psi^*(f) = f\psi$ is an isomorphism. We explicity find,

$$h^{-1}(\psi^*(f)) = \bar{f} h^{-1}(\psi) = f \sum_i s_i \cdot \overline{\psi(s_i)}$$

Therefore,

$$h^*(\psi^*(f), \psi^*(g)) = f\bar{g}\psi(h^{-1}(\beta)) = f\bar{g} \sum_i |\psi(s_i)|^2 = h'(\psi^*(f), \psi^*(g))$$

Thus $h^* = h'$.

#### 4.1.2 4.1.2

Let $L$ be a holomorphic line bundle of degree $d > 2g(C) - 2$ on a curve $C$ where $\deg K_C = 2g(C) - 2$. By Serre duality $H^1(C, L) = H^0(X, L^* \otimes K_C) = 0$ since $\deg(L^* \otimes K_C) = 2g(C) - 2 - d < 0$. In particular if $L$ is a line bundle with $\deg L > 0$ then $H^1(C, K_C \otimes L) = 0$.

#### 4.1.3 4.1.3 DO!!

Let $X = \mathbb{P}^n$. To compute the cohomology $H^q(X, \mathcal{O}_X(k))$ consider the

#### 4.1.4 4.1.4

Let $E$ be a hermitian holomorhic vector bundle on a compact Kähler manifold $X$ Let $s \in H^0(X, \Omega^p \otimes E)$ be a global section. Then we know,

$$\Delta_E(s) = 0 \iff \bar{\partial}_E s = 0 \text{ and } \bar{\partial}_E^* s = 0$$

Notice that $\Omega^p \otimes E$ is the kernel of $\bar{\partial}_E : \mathcal{A}^{p,0}(X, E) \to \mathcal{A}^{p,1}(X, E)$ therefore $\bar{\partial}_E s = 0$ automatically. Furthermore,

$$\bar{\partial}_E^* s = -\bar{\star}_E \bar{\partial}_{E^*} \bar{\star}_E s$$

However $\bar{\star}_E s \in \mathcal{A}^{n-p,n}(X, E^*)$ and thus $\bar{\partial}_{E^*} \bar{\star}_E s \in \mathcal{A}^{n-p,n+1}(X, E^*) = 0$. Therefore, $\bar{\partial}_E^* s = 0$ so $\nabla_E s = 0$ and thus $s$ is Harmonic.

#### 4.1.5 4.1.5 DO!!

### 4.2 Section 4.1

#### 4.2.1 4.2.1 DO!!

#### 4.2.2 4.2.2 DO!!

#### 4.2.3 4.2.3 DO!!

#### 4.2.4 4.2.4 DO!!

#### 4.2.5 4.2.5 DO!!

Let $(E, h)$ be a hermitian vector bundle and suppose $E = E_1 \oplus E_2$. Then $E_1, E_2$ are hermitian with hermitian structures $h_1$ and $h_2$ induced by the inclusions. Now let $\nabla_1$ and $\nabla_2$ be the induced connections.

#### 4.2.6 4.2.6

Let $E$ be a vector bundle and $\nabla$ a connection on $E$. Then there is an induced connection $\nabla' : \bigwedge^2 E \to \Omega^1_X \otimes \bigwedge^2$ as the quotient of the induced connection on $E^{\otimes 2}$. Explicitly,

$$\nabla'(s_1 \wedge s_2) = \nabla s_1 \wedge s_2 + s_1 \wedge \nabla s_2$$

This is well-defined because,

$$\nabla'(s \wedge s) = \nabla s \wedge s + s \wedge \nabla s = 0$$

Furthermore, there is an induced connection $\nabla' : \det E \to \Omega^1_X \otimes \det E$ as the quotient of the induced connection on $E^{\otimes n}$. Explicitly,

$$\nabla'(s_1 \wedge \cdots \wedge s_n) = \nabla s_1 \wedge \cdots \wedge s_n + \cdots + s_1 \wedge \cdots \wedge \nabla s_n$$

which is well-defined as above. Let $e_i$ be a local frame of $E$ then we can write $\nabla = \mathrm{d} + A$ where $A$ is a matrix of 1-forms which really means,

$$\nabla(f_j e_j) = \mathrm{d}f_j \otimes e_j + f_j \nabla e_j = \mathrm{d}f_j \otimes e_j + f_j A_{ij} \otimes e_i = (\mathrm{d}f_i + A_{ij} f_j) \otimes e_i$$

where $A_{ij} \otimes e_i = \nabla e_j$. Then we find that,

$$\begin{aligned}\nabla'(e_1 \wedge \cdots \wedge e_n) &= A_{1k} \otimes e_k \wedge \cdots \wedge e_n + \cdots + e_1 \wedge \cdots \wedge A_{nk} \otimes e_k = (A_{11} + \cdots + A_{nn}) \otimes (e_1 \wedge \cdots \wedge e_n) \\ &= \mathrm{tr}\,(A) \otimes (e_1 \wedge \cdots \wedge e_n)\end{aligned}$$

Therefore, the connection $\nabla' : \det E \to \Omega^1_X \otimes \det E$ on the line bundle $\det E$ is locally given by the 1-form $\mathrm{tr}\,(A)$.

### 4.2.7 4.2.7 DO!!

### 4.2.8 4.2.8

First note that if $\nabla_1$ and $\nabla_2$ are connections on $E_1$ and $E_2$ we define a connection $\nabla : E_1 \otimes E_2 \to \Omega \otimes (E_1 \otimes E_2)$ via $\nabla(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$ and also $\nabla : \mathrm{Hom}_{\mathcal{O}_X}(E_1, E_2) \to \mathrm{Hom}_{\mathcal{O}_X}(E_1, E_2)$ via $\varphi \mapsto \nabla \varphi$ such that $(\nabla \varphi)(s) = \nabla_2 \varphi(s) - \varphi(\nabla_1(s))$ for any $s \in \Gamma(U, E)$.

Therefore, if $\nabla$ is a connection on $E$ define the dual connection $\nabla^* : E^* \to \Omega^1_X \otimes E^*$ on $E^*$ via sending $\varphi : E \to \mathcal{O}_X$ so $(\nabla^* \varphi)(s) = \mathrm{d}\varphi(s) - \varphi(\nabla s)$.

Consider the induced connection $\nabla$ on $(E \otimes \overline{E})^*$. Then for any section $h \in \Gamma(U, (E \otimes \overline{E})^*)$, for instance a hermitian metric, we get $\nabla h$ such that for any $s_1, s_2 \in E$,

$$\begin{aligned}(\nabla h)(s_1 \otimes \bar{s}_2) &= \mathrm{d}h(s_1 \otimes \bar{s}_2) - h(\nabla(s_1 \otimes \bar{s}_2)) = \mathrm{d}h(s_1 \otimes \bar{s}_2) - h(\nabla s_1 \otimes \bar{s}_2) - h(s_1 \otimes \bar{\nabla} \bar{s}_2) \\ &= \mathrm{d}h(s_1 \otimes \bar{s}_2) - h(\nabla s_1 \otimes \bar{s}_2) - h(s_1 \otimes \overline{\nabla s_2})\end{aligned}$$

Therefore, as a metric,

$$(\nabla h)(s_1, s_2) = \mathrm{d}h(s_1, s_2) - h(\nabla s_1, s_2) - h(s_1, \nabla s_2)$$

Therefore, $\nabla$ is hermitian with respect to the hermitian structure $(E, h)$ iff $\nabla h = 0$.

### 4.2.9 4.2.9 DO!!

## 4.3 Section 4.3

### 4.3.1 4.3.1 DO!!

Consider $\nabla^2 : \mathcal{A}^k(E) \to \mathcal{A}^{k+2}(E)$ on $\omega \otimes s$ where $\omega$ is a $k$-form and $s$ is a section of $E$. Then,

$$\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla(s)$$

and thus,

$$\begin{aligned}\nabla^2(\omega \otimes s) &= \mathrm{d}\mathrm{d}\omega \otimes s + (-1)^{k+1}\mathrm{d}\omega \wedge \nabla s + (-1)^k \mathrm{d}\omega \wedge \nabla(s) + (-1)^{2k}\omega \wedge \nabla^2(s) \\ &= \omega \wedge \nabla^2(s)\end{aligned}$$

since the $\mathrm{d}\mathrm{d}\omega = 0$ and the middle terms cancel. Here we use the generalized Leibniz formula,

$$\nabla(\omega \wedge \alpha) = \mathrm{d}\omega \wedge \alpha + (-1)^k \omega \wedge \nabla\alpha$$

where $\omega$ is a $k$-form and $\alpha \in \mathcal{A}^\ell(E)$. Furthermore $\nabla^2(s) = F_\nabla(s)$ so we find,

$$\nabla^2(\omega \otimes s) = \omega \wedge F_\nabla(s)$$

and therefore for a general $E$-valued $k$-form $\alpha$ we see that $\nabla^2(\alpha) = \mathrm{tr}\,(\omega \wedge F_\nabla)$ where we view $\omega \in H^0(X, \Omega_X^k \otimes E)$ and $F_\nabla \in H^0(X, \Omega_X^2 \otimes E \otimes E^*)$ and taking the map $\wedge : \Omega_X^k \otimes \Omega_X^2 \to \Omega_X^{k+2}$ and contracting the $E^*$ from $F_\nabla$ with the $E$ from $\omega$.

### 4.3.2 4.3.2 DO!!

### 4.3.3 4.3.3 DO!!

Let $(E_1, h_1)$ and $(E_2, h_2)$ be two hermitian holomorphic vector bundles endowed with hermitian connections $\nabla_1, \nabla_2$ such that the curvature of both is (semi)-positive.

(a) The curvature of $\nabla^*$ on $E^*$ is $F_{\nabla^*} = -F_\nabla^\top$ which is (semi)-negative since $F_\nabla$ is (semi)-positive.

(b) The curvature of $\nabla$ on $E_1 \otimes E_2$ is $F_\nabla = F_{\nabla_1} \otimes \mathrm{id} + \mathrm{id} \otimes F_{\nabla_2}$ which is (semi)-positive since $F_{\nabla_1}$ and $F_{\nabla_2}$ are. Furthermore if one of $F_{\nabla_i}$ is positive then $F_\nabla$ is positive since the other term is nonegative.

(c) The curvature of $\nabla$ on $E_1 \oplus E_2$ is $F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}$ which is (semi)-positive since $F_{\nabla_1}$ and $F_{\nabla_2}$ are.

### 4.3.4 4.3.4 DO!!

### 4.3.5 4.3.5 CHECK!!

Let $X$ be complex manifold. Let $L$ be a holomorphic line bundle with a hermitian structure $h$ whose Chern connection has positive curvature. Then $F_\nabla \in \mathcal{A}^{1,1}(X)$ is an imaginary $(1,1)$-form. Furthermore, note that $F_\nabla = \bar{\partial}\partial \log h$ and thus,

$$\mathrm{d}F_\nabla = (\partial + \bar{\partial})\bar{\partial}\partial \log h = 0$$

because $\bar{\partial}^2 = 0$ and $\partial\bar{\partial}\partial = -\partial^2\bar{\partial} = 0$. Since $\omega = iF_\nabla$ is positive, it is a Kähler form. Furthermore if $X$ is compact then,

$$\int_X A(L)^n = \int_X F_\nabla^n = \int_X \omega^n = n! \int_X \mathrm{vol}_\omega > 0$$

(CHECK THIS! FACTORS OF I)

### 4.3.6 4.3.6

Let $X = \mathbb{P}^n$ and $\omega_X = \mathcal{O}_X(-n-1)$ be the canonical bundle. The sections $x_0, \dots, x_n \in \Gamma(X, \mathcal{O}_X(1))$ define a canonical hermitian structue $h$ on $\mathcal{O}_X(1)$ which has the property that $\frac{i}{2\pi} F_\nabla = \omega_{\text{FS}}$ which is positive. Then $\omega_X = \mathcal{O}_X(-n-1)$ has a canonical hermitian structure $(h^*)^{\otimes n+1}$ which has curvature $\frac{i}{2\pi} F_{\nabla'} = -(n+1)\omega_{\text{FS}}$ which is therefore negative.

### 4.3.7 4.3.7 DO!!

### 4.3.8 4.3.8 DO!!

### 4.3.9 4.3.9

Let $X$ be a compact Kähler manifold with $b_1(X) = 0$. Suppose that $\nabla$ is a flat connection on $\mathcal{O}_X$ with $\nabla^{0,1} = \bar{\partial}$. Then $\nabla = \mathrm{d} + \omega$ where $\omega : \mathcal{A}^0(X) \to \mathcal{A}^1(X)$ is $\mathcal{A}^0(X)$-linear and thus $\omega \in \mathcal{A}^1(X)$. Futhermore, $\nabla^{0,1} = \bar{\partial}$ so $\omega$ is a smooth $(1,0)$-form. Now consider the curvature,

$$F_\nabla = \nabla \circ \nabla(1) = \nabla(\omega \otimes 1) = \mathrm{d}\omega \otimes 1 - \omega \wedge \nabla(1) = \mathrm{d}\omega \otimes 1 - \omega \wedge \omega \otimes 1 = \mathrm{d}\omega$$

Since $\nabla$ is flat we must have $\mathrm{d}\omega = 0$. Thus $\omega$ defines a de Rham cohomology class $[\omega] \in H^1(X, \mathbb{C})$ but $b_1(X) = 0$ so $\omega$ is exact. Take $\omega = \mathrm{d}f$ for some smooth function $f$. However, $\omega$ is a $(1,0)$-form so $f$ is holomorphic. But $X$ is compact so $f$ is constant and thus $\omega = 0$ showing that $\nabla = \mathrm{d}$.

Now suppose that $L$ is a line bundle on $X$ with $c_1(L) = 0$. From the exponential sequence,

$$H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}\,(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

and thus $\ker c_1 = \mathrm{Im}\,(H^1(X, \mathcal{O}_X) \to \mathrm{Pic}\,(X))$. However, $b_1(X) = 0$ so by the Kähler decomposition, $H^1(X, \mathcal{O}_X) = 0$. Therefore, $\ker c_1$ is trivial so $L = \mathcal{O}_X$.

### 4.3.10 4.3.10 DO!!

Let $\nabla$ be a connection on a complex vector bundle $E$. We want to show that $E$ locally has parallel frames iff $F_\nabla = 0$.

Suppose that $E$ has a local frame $e_1, \dots, e_n$ of parallel sections over $U$ i.e. $\nabla e_i = 0$ and these are independent on each fiber. Since the curvature form $\omega_\nabla(s) = \nabla_1 \circ \nabla(s)$ is $\mathcal{O}_X$-linear, writing $s = f_i e_i$ we get,

$$\omega_\nabla(f_i e_i) = f_i \omega_\nabla(e_i) = f_i \nabla_1 \circ \nabla e_i = 0$$

Therefore, $\omega_\nabla = 0$ so $\nabla$ must be flat.

Locally write $E|_U \cong \mathcal{O}_U^{\oplus n}$ write $e_i$ for a local frame of $E|_U$. Now write $\nabla e_j = \omega_{ij} \otimes e_i$ thus we see,

$$\nabla(f_j e_j) = \mathrm{d}f_j \otimes e_j + \omega_{ij} f_j \otimes e_i = (\mathrm{d}f_i + \omega_{ij} f_j) \otimes e_i$$

Now, applying $\nabla_1 : \Omega^1_X \otimes E \to \Omega^2_X \otimes E$ we get,

$$\begin{aligned} \nabla_1 \circ \nabla(f_j e_j) &= \nabla_1(\mathrm{d}f_i + \omega_{ij} f_j) \otimes e_i = \mathrm{d}\mathrm{d}f_i \otimes e_i + \mathrm{d}(\omega_{ij} f_j) \otimes e_i - (\mathrm{d}f_i + \omega_{ij} f_j) \wedge \nabla e_i \\ &= (\mathrm{d}\omega_{ij}\, f_j - \omega_{ij} \wedge \mathrm{d}f_j) \otimes e_i - (\mathrm{d}f_i + \omega_{ij} f_j) \wedge \omega_{ki} \otimes e_k \\ &= \mathrm{d}\omega_{ij} f_j \otimes e_i + \mathrm{d}f_j \wedge \omega_{ij} \otimes e_i - \mathrm{d}f_i \wedge \omega_{ki} \otimes e_k + \omega_{ki} \wedge \omega_{ij} f_j \otimes e_k \\ &= (\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj}) f_j \otimes e_i \end{aligned}$$

Therefore,

$$\omega_\nabla(f_j e_j) = (\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj}) f_j \otimes e_i$$

is linear as it should be. Now assume $\nabla$ is flat i.e. $\omega_\nabla = 0$. Thus,

$$\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj} = 0$$

First, in the case $n = 1$ the connection is given by a 1-form $\omega$. Then $\omega_\nabla = 0 \iff \mathrm{d}\omega = 0$ in which case locally $\omega = -\mathrm{d}f$ and thus $\nabla(fe) = \mathrm{d}f \otimes e + \omega \otimes e = 0$ so we get a frame of parallel sections.

Now we proceed by induction for the general case. First, using a $\mathrm{GL}_((n), \mathbb{C})$ transformation we can
Assume we can find a frame $e_1, \dots, e_{n-1}, s$ such that $\nabla e_i = 0$. (FINISH)

## 4.4 Section 4.4

### 4.4.1 4.4.1 DO!!

### 4.4.2 4.4.2

Let $X$ be a compact complex manifold and $L$ a basepoint-free line bundle. Then $L$ defines a map $f : X \to \mathbb{P}^N$ such that $f^* \mathcal{O}_{\mathbb{P}^N}(1) = L$. Let $h$ be the standard hermitian structure on $\mathcal{O}_{\mathbb{P}^N}(1)$ so $f^* h$ gives a hermitian structure on $L$. Taking the Chern connections $\nabla_{f^*h} = f^* \nabla_h$ and thus,

$$F(L, f^*h) = F(f^* \mathcal{O}_{\mathbb{P}^N}(1), f^* h) = f^* F(\mathcal{O}_{\mathbb{P}^N}(1), h) = f^* \omega_{\mathrm{FS}}$$

which is a positive form. Therefore,

$$c_1(L) = f^* [\omega_{\mathrm{FS}}]$$

so we see that,

$$\int_X c_1(L)^n = \int_X (f^* \omega_{\mathrm{FS}})^n = \int_X f^* \omega_{\mathrm{FS}}^n \geq 0$$

### 4.4.3 4.4.3 ASK RON!!

### 4.4.4 4.4.4 ASK RON!!

Ask Ron about interpretation!!

### 4.4.5 4.4.5 DO!!

### 4.4.6 4.4.6 DO!!

### 4.4.7 4.4.7 DO!!

### 4.4.8 4.4.8 DO!!

### 4.4.9 4.4.9

Note that $\mathrm{End}\,(E) \cong E^* \otimes E$ then,

$$c_k(\mathrm{End}\,(E)) = \sum_{i+j=k} c_i(E^*) \cdot c_j(E) = \sum_{i+j} (-1)^i c_i(E) \cdot c_j(E)$$

In particular,

$$c_1(\mathrm{End}\,(E)) = c_0(E) \cdot c_1(E) - c_1(E) \cdot c_0(E) = 0$$

and likewise,

$$c_2(\mathrm{End}\,(E)) = c_0(E)\cdot c_2(E) - c_1(E)\cdot c_1(E) + c_2(E)\cdot c_0(E) = 2c_2(E) - c_1(E)^2$$

Then if $E = L \oplus L$ where $L$ is a line bundle we have,

$$c(L) = 1 + c_1(L)$$

and thus,

$$c_1(E) = 2c_1(L) \quad \text{and} \quad c_2(E) = c_1(L)^2$$

Therefore, we see that,

$$(4c_2 - c_1^2)(E) = 4c_1(E)^2 - (4c_1(E))^2 = 0$$

Furthermore, if $E \cong E^*$ then $c_{2k+1}(E) = c_{2k+1}(E^*) = (-1)^{2k+1}c_{2k+1}(E) = -c_{2k+1}(E)$ and thus $c_{2k+1}(E) = 0$.

### 4.4.10 4.4.10

Let $L$ be a holomorphic line bundle on $X$ a compact Kähler manifold. Suppose that $c_1(L) = [\alpha]$ where $\alpha$ is closed a real $(1,1)$-form. Let $h_0$ be a Hermitian structure on $L$ then,

$$c_1(L, h_0) = \frac{i}{2\pi}\bar{\partial}\partial \log h_0$$

Now consider,

$$\eta = \alpha - c_1(L, h_0)$$

is a real $(1,1)$-form and since $[\alpha] = [c_1(L, h_0)]$ also $\eta$ is d-exact. Thus, by the $\partial\bar{\partial}$-lemma, we know,

$$\eta = -\frac{i}{2\pi}\partial\bar{\partial} f$$

for $f \in \mathcal{A}^{0,0}_{\mathbb{R}}(X)$ i.e. $f$ is a real smooth function. Therefore,

$$\alpha = \frac{i}{2\pi}\bar{\partial}\partial\,[f + \log h_0] = \frac{i}{2\pi}\bar{\partial}\partial \log e^f h_0$$

Therefore, let $h = e^f h_0$ be annother Hermitian structure (since $f$ is real) then we see $c_1(L, h) = \alpha$.

### 4.4.11 4.4.11

Let $X$ be compact Kähler and $E$ a vector bundle with a Chern connection $\nabla$. If we let,

$$\sum_{i=0}^{r} \tilde{P}_i(B) = \mathrm{tr}\left(e^B\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{tr}\,(B^n)$$

so,

$$\tilde{P}_k(B_1, \dots, B_k) = \frac{1}{k!}\mathrm{tr}\,(B_1 \cdots B_k)$$

and then define,

$$\mathrm{ch}_k(E, \nabla) := \tilde{P}_k\left(\frac{i}{2\pi}F_\nabla\right) \in \mathcal{A}^{2k}_{\mathbb{C}}(M)$$

where $\tilde{P}_k$ acts on $\text{End}\,(E)$-valued 2-forms via,

$$\tilde{P}_k(\alpha_1 \otimes \varphi_1, \dots, \alpha_k \otimes \varphi_k) = (\alpha_1 \wedge \dots \wedge \alpha_k)\, \tilde{P}_k(\varphi_1, \dots, \varphi_k) = (\alpha_1 \wedge \dots \wedge \alpha_k)\, \frac{1}{k!} \text{tr}\,(\varphi_1 \cdots \varphi_k)$$

This is the composition of $(\Omega_X^2)^{\otimes k} \to \Omega_X^{2k}$ via exterior product and $\text{End}\,(E)^{\otimes k} \to \text{End}\,(E)$ via composition and finally taking trace. We see that,

$$\text{ch}_k(E, \nabla) = \frac{1}{k!} \left( \frac{i}{2\pi} \right)^k \text{tr}\left( F_\nabla^{\otimes k} \right)$$

where $F_\nabla^{\otimes k}$ is the image under $(\Omega_X^2 \otimes \text{End}\,(E))^{\otimes k} \to \Omega_X^{2k} \otimes \text{End}\,(E)$. Now taking Dolbeault cohomology classes via $\mathcal{A}_{\mathbb{C}}^{k,k}(\text{End}\,(E)) \to H^k(X, \Omega^k \otimes \text{End}\,(E))$,

$$\text{ch}_k(E) = \frac{1}{k!} \left( \frac{i}{2\pi} \right)^k \text{tr}\left( [F_\nabla]^{\otimes k} \right)$$

where $[F_\nabla]^{\otimes k}$ is the image under the map,

$$H^1(X, \Omega_X^1 \otimes \text{End}\,(E)) \times \dots \times H^1(X, \Omega_X^1 \otimes \text{End}\,(E)) \to H^k(X, \Omega^k \otimes \text{End}\,(E))$$

Furthermore $[F_\nabla] = A(E)$ so we get,

$$\text{ch}_k(E) = \frac{1}{k!} \left( \frac{i}{2\pi} \right)^k \text{tr}\left( A(E)^{\otimes k} \right)$$

as a class under the map $H^k(X, \Omega^k \otimes \text{End}\,(E)) \xrightarrow{\text{tr}} H^k(X, \Omega_X^k) \subset H^{2k}(X, \mathbb{C})$.

#### 4.4.12 4.4.12

Let $X$ be compact Kähler and $E$ a holomorphic vector bundle admitting a holomorphic connection. Then $A(E) = 0$ and therefore $c_k(E) = 0$.

# 5 Chapter 5

## 5.1 Section 5.1

### 5.1.1 5.1.1 DO!!

## 5.2 Section 5.2

### 5.2.1 5.2.1 DO!!

## 5.3 Section 5.3

### 5.3.1 5.3.1 DO!!

# 6 Chapter 6

## 6.1 Section 6.1

### 6.1.1 6.1.1

(a) Let $X = \mathbb{P}^n$ then by the Euler sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus(n+1)} \longrightarrow \mathcal{T}_X \longrightarrow 0$$

giving a long exact sequence,

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X(1))^{\oplus(n+1)} \longrightarrow H^1(X, \mathcal{T}_X) \longrightarrow H^2(X, \mathcal{O}_X)$$

However, $H^i(X, \mathcal{O}_X(k)) = 0$ for $i > 0$ when $k \geq 0$ and thus we find $H^1(X, \mathcal{T}_X) = 0$.

(b) Let $X = \mathbb{C}^n/\Gamma$ be a complex torus. Then we know $\mathcal{T}_X = \mathcal{O}_X^{\oplus n}$ and we need to compute $H^1(X, \mathcal{T}_X) = H^1(X, \mathcal{O}_X)^{\oplus n}$. Then $h^{0,1} = H^1(X, \mathcal{O}_X)$ and $b_1 = h^{0,1} + h^{1,0} = 2h^{0,1}$ by Serre duality. However, $b_1 = 2n$ because $H^1(X, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ and $\Gamma$ has rank $2n$. Therefore $h^{0,1} = n$ and thus $H^1(X, \mathcal{T}_X) = H^1(X, \mathcal{O}_X)^{\oplus n}$ meaning that $\dim_{\mathbb{C}} H^1(X, \mathcal{T}_X) = n^n$.

(c) Let $X$ be a compact curve of genus $g$. Then $\mathcal{T}_X$ is a line bundle of degree $2-2g$. Then, by Serre duality, $H^1(X, \mathcal{T}_X) = H^0(X, \Omega_X^{\otimes 2})$. However if $g = 0$ then $\Omega_X$ is negative so $H^1(X, \mathcal{T}_X) = 0$. If $g = 1$ then $\mathcal{T}_X \cong \Omega_X \cong \mathcal{O}_X$ in which case $\dim_{\mathbb{C}} H^1(X, \mathcal{T}_X) = 1$. Finally, if $g > 1$ then $\Omega_X$ is positive and $\Omega_X^{\otimes 2}$ has degree $4g - 4 > 2g - 2$ so $H^1(X, \Omega_X^{\otimes 2}) = 0$ and thus, by Riemann-Roch,

$$\dim_{\mathbb{C}} H^1(X, \mathcal{T}_X) = \deg \Omega_X^{\otimes 2} + 1 - g = 3g - 3$$

### 6.1.2 6.1.2 DO!!

Let $X$ be a compact complex manifold (not necessarily Kähler) and $\sigma \in H^0(X, \Omega_X^2)$ an everywhere non-degenerate holomorphic two-form meaning the map $\sigma : \mathcal{T}_X \to \Omega_X$ is an isomorphism. Since $\sigma$ is nondegenerate fiberwise, we see that $\dim X = 2r$ is even. Furthermore,

$$\sigma^r = \sigma \wedge \cdots \wedge \sigma \in H^0(X, \Omega_X^{2r})$$

is everywhere nonvanishing and thus $\sigma^r \in H^0(X, K_X)$ trivializes the canonical bundle $K_X$ so $X$ is a Calabi-Yau. Therefore, applying the results of this section with $\Omega = \sigma^r$, if $v \in H^1(X, \mathcal{T}_X)$ is a cohomology class, then there exists a $\bar{\partial}$-closed lift $\phi_1 \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ representing $H^1(X, \mathcal{T}_X)$ such that the Maurer-Cartan equation admits a formal solution $\sum_i \phi_i t^i$ extending $\phi_1$.

### 6.1.3 6.1.3

Let $X$ be a compact complex manifold with $H^2(X, \mathcal{T}_X) = 0$. Take a cohomology class $v \in H^1(X, \mathcal{T}_X)$ and lift to some $\bar{\partial}$-closed $\phi_1 \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ such that $[\phi_1] = v$. Now for induction suppose we have classes $\phi_1, \ldots, \phi_n \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ satisfying the Maurer-Cartan equations up to degree $n$,

$$\begin{aligned} \bar{\partial}\phi_1 &= 0 \\ \bar{\partial}\phi_2 &= -\sum_{0<i<2} [\phi_i, \phi_{2-i}] \\ &\vdots \\ \bar{\partial}\phi_n &= -\sum_{0<i<n} [\phi_i, \phi_{n-i}] \end{aligned}$$

Now consider,

$$\omega = -\sum_{0<i<n+1} [\phi_i, \phi_{n+1-i}]$$

and we have,

$$\begin{aligned}
\bar{\partial}\omega &= -\sum_{0<i<n+1} \bar{\partial}[\phi_i, \phi_{n+1-i}] = -\sum_{0<i<n+1} \left([\bar{\partial}\phi_i, \phi_{n+1-i}] + [\phi_i, \bar{\partial}\phi_{n+1-i}]\right) \\
&= \sum_{0<i<n+1} \left( \sum_{0<j<i} [[\phi_j, \phi_{i-j}], \phi_{n+1-i}] + \sum_{0<j<n+1-i} [\phi_i, [\phi_j, \phi_{n+1-i-j}]] \right) \\
&= \sum_{0<i<n+1} \sum_{0<j<i} [[\phi_j, \phi_{i-j}], \phi_{n+1-j}] + \sum_{0<i<n+1} \sum_{0<j<i} [\phi_{n+1-i}, [\phi_j, \phi_i]] = 0
\end{aligned}$$

replacing $i$ by $n+1-i$ in the second sum. Vanishing follows from $[\alpha, \beta] = (-1)^{k\ell+1}[\beta, \alpha]$ for $\alpha \in \mathcal{A}^{0,k}(\mathcal{T}_X)$ and $\beta \in \mathcal{A}^{0,\ell}(\mathcal{T}_X)$ and in our case $k = 2$. Therefore $\omega \in \mathcal{A}^{0,2}(\mathcal{T}_X)$ is $\bar{\partial}$-closed but $H^2(X, \mathcal{T}_X) = 0$ so every $\bar{\partial}$-closed $\mathcal{T}_X$-valued 2-form is $\bar{\partial}$-exact meaning there exists $\phi_{n+1} \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ such that $\omega = \bar{\partial}\phi_{n+1}$. Explicitly,

$$\bar{\partial}\phi_{n+1} = -\sum_{0<i<n+1} [\phi_i, \phi_{n+1-i}]$$

solving the Maurer-Cartan equation recursively.

# 7 Extra Questions for Ron

### 7.0.1 1

Kodaira embedding says that every positive line bundle is ample in the sense of having some power very ample. Does the algebraic geometry definition work here? I.e. $L$ is ample iff for each bundle $Q$ we have $Q \otimes L^n$ generated by global sections for $n \gg 0$. Do we need $Q$ to be arbitrary coherent sheaf.

Yes, in fact we only need this for vector bundles because it then follows by resolution for all coherent sheaves.

### 7.0.2 2

If we have a big line bundle $H^0(X, L^{\otimes m}) \sim m^n$ then does it follow there is an ample line bundle i.e. $X$ is projective. I am guessing not. This is similar to asking if there are non algebraic examples of compact Moishezon manifolds $a(X) = \dim X$.