Homework 1 Solutions

September 25, 2021

1

Suppose that $\sqrt{3}$ were rational. Then we could write $\sqrt{3}$ as a rational number in reduced form meaning,

$$\frac{a}{b} = \sqrt{3}$$

for some integers a, b (with $b \neq 0$) sharing no common factors. Therefore,

$$a^2 = 3b^2$$

Thus, a^2 is divisible by 3 so a is divisible by 3 (think about why). This means we can write $a = 3r$ for some integer r. Plugging in and diving by 3 gives,

$$b^2 = 3r^2$$

so by the same argument b is divisible by 3. Therefore, a and b share the factor 3 contradicting our assumption that $\frac{a}{b}$ is in reduced form. This contradicts our assumption that $\sqrt{3}$ is rational proving that $\sqrt{3}$ is not rational.

Notice that this argument only relies on 3 being prime and the following fact: if p is prime and p divides a^2 then p divides a (see if you can prove this in general).

Since 6 is not prime, it might seem that the above argument does not go through. However, 6 is square-free, meaning that it is not divisible by any square (besides 1). Furthermore, I claim that if r is square-free and a^2 is divisible by r then a is divisible by r (how would you prove this?). Therefore, the exact same argument works to show that $\sqrt{6}$ is irrational.

2

Let $A, B \subset S$.

(a)

If $x \in (A \cap B)^c$ then $x \notin A \cap B$. However, if $x \in A$ then $x \notin B$ (else $x \in A \cap B$ which it is not) and likewise if $x \in B$ then $x \notin B$ (if you are familiar with DeMorgan's law in logic, identify how it is used in this sentence). Therefore, $x \notin A$ or $x \notin B$ so by definition $x \in A^c$ or $x \in B^c$. By the definition of the union, this implies that $x \in A^c \cup B^c$. Therefore,

$$(A \cap B)^c \subset A^c \cup B^c$$

(b)

If $x \in A^c \cup B^c$ then $x \notin A$ or $x \notin B$. Suppose that $x \in (A \cap B)$ then both $x \in A$ and $x \in B$ which we just saw cannot be true. Therefore $x \notin (A \cap B)$ so,

$$A^c \cup B^c \subset (A \cap B)^c$$

3

Let $f : X \to Y$ be a function and $A, B \subset X$.

(a)

If $y \in f(A \cap B)$ then by definition there is some $x \in A \cap B$ such that $f(x) = y$. However, $x \in A$ so $y = f(x) \in f(A)$. Likewise $x \in B$ so $y = f(x) \in f(B)$. Thus $y \in f(A) \cap f(B)$ so we see that,

$$f(A \cap B) \subset f(A) \cap f(B)$$

(b)

Let $y \in f(A \cup B)$ then by definition there is some $x \in A \cup B$ such that $f(x) = y$. Now $x \in A$ or $x \in B$ and if $x \in A$ then $y = f(x) \in f(A) \subset f(A) \cup f(B)$. and if $x \in B$ then $y = f(x) \in f(B) \subset f(A) \cup f(B)$ so in either case $y \in f(A) \cup f(B)$ and thus,

$$f(A \cup B) \subset f(A) \cup f(B)$$

Conversely, since $A, B \subset A \cup B$ we see that $f(A), f(B) \subset f(A \cup B)$ and therefore,

$$f(A) \cup f(B) \subset f(A \cup B)$$

because a set that contains two subsets also contains their union. Therefore,

$$f(A \cup B) = f(A) \cup f(B)$$

(c)

A good example is when $f(A) = f(B)$ is some nonempty set but A and B are disjoint ($A \cap B = \varnothing$). For example, take $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. Then let $A = \{+1\}$ and $B = \{-1\}$. Then $f(A) = f(B) = \{+1\}$ which implies that $f(A) \cap f(B) = \{+1\}$. However, $A \cap B = \varnothing$ so $f(A \cap B) = \varnothing$ so $f(A \cap B)$ and $f(A) \cap f(B)$ are not equal.

4

Let $f : X \to Y$ be a function and $A, B \subset Y$.

(a)

Take $x \in f^{-1}(A \cap B)$ then by definition $f(x) \in A \cap B$ so $f(x) \in A$ and $f(x) \in B$. This means that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$ so $x \in f^{-1}(A) \cap f^{-1}(B)$ proving that,

$$f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$$

Conversely, if $x \in f^{-1}(A) \cap f^{-1}(B)$ then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$ so $f(x) \in A$ and $f(x) \in B$ so $f(x) \in A \cap B$ meaning that $x \in f^{-1}(A \cap B)$. Therefore,

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

(b)

Take $x \in f^{-1}(A \cup B)$ then by definition $f(x) \in A \cup B$ so $f(x) \in A$ or $f(x) \in B$. This means that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ so $x \in f^{-1}(A) \cup f^{-1}(B)$ proving that,

$$f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$$

Conversely, if $x \in f^{-1}(A) \cup f^{-1}(B)$ then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ so $f(x) \in A$ or $f(x) \in B$ so $f(x) \in A \cup B$ meaning that $x \in f^{-1}(A \cup B)$. Therefore,

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$