1 Sept 30

Make all the homotopy equivalences invertible.

Nisnvictch topology on smooth schemes Nis_S which send all

No tubular neighborhood theorem.

Hilbert spaces of infinite affine spaces: show there are $\mathbb{A}^1$-equivalences,

$$\mathrm{Hilb}_d(\mathbb{A}^\infty) \xrightarrow{\sim} \mathrm{FFlat}_d \xrightarrow{\sim} \mathrm{Vect}_{d-1} \xrightarrow{\sim} \mathrm{Gr}_{d-1}(\mathbb{A}^\infty)$$

by sending,

$$\{X \to \mathbb{A}^\infty_S\} \mapsto (X \to S) \mapsto \text{etc}$$

for any $X \to S$ there is, up to homotopy, a unique embedding into infinite affine space. In the case $X \to S$ is $A \to B$ finite flat map then we send

Motivic homotopy theory, algebraic cobordism, other theories.

Universal six functor formalism given by motivic homology theory. (VERY COOL)

analogue of how Cohomology theories factor throught stable homotopy category.

Primer for unstable motivic homotopy theory.

Purity is replacement for tubular neighborhood theorem (hmmm!)

2 Oct 7

The data of an ∞-category $\mathcal{C}$,

(a) $X, Y \in \mathcal{C}$ a space $\mathrm{Map}_{\mathcal{C}}(X, Y)$

(b) replace strict composition with a homotopy class of composition maps,

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$

Definition 2.0.1. $h\mathcal{C}$ is the homotopy category of $\mathcal{C}$ whose objects are the same but,

$$\mathrm{Map}_{h\mathcal{C}}(X, Y) = \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y)) = [X, Y]_{\mathcal{C}}$$

Example 2.0.2. (a) 1-categories are discrete ∞-categories

(b) Spaces

(c) ∞-category of ∞-categories

(d) $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and there is a morphism,

$$h\mathrm{Fun}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(h\mathcal{C}, h\mathcal{D})$$

Proposition 2.0.3. (a) $\mathrm{Fun}(*, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}$

(b) a morphism $\alpha \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence iff its image in $\mathrm{Fun}(h\mathcal{C}, h\mathcal{D})$ is an equivalence iff α_X si an equivalence in $\mathcal{D}$ for all $X \in \mathcal{C}$.

Definition 2.0.4. Let $F : \mathcal{C} \to \mathcal{D}$ be an ∞-functor. A colimit of F is an object $d \in \mathcal{D}$ and a morphism $\alpha : F \to d*$ that induces an equivalence,

$$\mathrm{Map}_{\mathcal{D}}(d, X) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(d^*, X^*) \xrightarrow{\circ\alpha} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, X^*)$$

is an equivalence where d^* is the constant map $\mathcal{C} \to * \to \mathcal{D}$ at d.

Remark. Here, uniqueness up to unique isomorphism is replaced with "the space of colimits is contractible" capturing the higher uniqueness.

2.1 Commutative Monoid OBjects in $\mathcal{C}$

How do we capture the notion of a group object in an ∞-category? The associativity and inverse diagrams need higher coherences.

Definition 2.1.1. Let Fin_* be the 1-category of finite pointed sets. Let $[\underline{n}] = \{*, 1, \dots, n\}$ where $*$ is the base point. There are n maps $p_k[\underline{n}] \to [\underline{1}]$ sending $i \mapsto 1$ and the rest to $*$.

Definition 2.1.2. Let $\mathcal{C}$ be an ∞-category with finite products. Define $\text{CMon}\left(\mathcal{C}\right) \subset \text{Fun}\left(\text{Fin}_*, \mathcal{C}\right)$ to be the functors such that for each $n \geq 0$ the map,

$$\prod_{k=1}^{n} (p_k)_* : M([\underline{n}]) \to M([\underline{1}])^n$$

is an equivalence.

Remark. Think of $M([\underline{0}]) \cong *$ and $M([\underline{1}]) = M$ the "underlying object". Then we obtain an addition,

$$M([\underline{1}]) \times M([\underline{1}]) \xrightarrow{\sim} M([\underline{2}]) \to M([\underline{1}])$$

where the second map $[\underline{2}] \to [\underline{1}]$ sending $1, 2 \mapsto 1$. Furthermore, there is a unit,

$$M([\underline{0}]) \to M([\underline{1}])$$

given by the unique map $[\underline{0}] \to [\underline{1}]$.

Futhermore, in Fin_* the map $[\underline{2}] \to [\underline{1}]$ is invariant under swapping which gives homotopy coherent commutativity. Homotopy coherent associativity holds similarly.

Exercise 2.1.3. If $\mathcal{C}$ is discrete then this is the usual definition.

Remark. If $\mathcal{C} = \textbf{Space}$ and M is in $\text{CMon}\left(\textbf{Space}\right)$ then $\pi_0(M)$ is a commutative monoid in **Set** meaning a usual commutative monoid.

Definition 2.1.4. A cmonoid $M \in \text{CMon}\left(\textbf{Space}\right)$ is *group-like* if $\pi_0(M)$ is a group. Consider the full subcategory,

$$\text{CMon}\left(\textbf{Space}\right)^{\text{gp}} \subset \text{CMon}\left(\textbf{Space}\right)$$

Definition 2.1.5. The group completion of $M \in \text{CMon}\left(\textbf{Space}\right)$ is an object $M^{\text{gp}} \in \text{CMon}\left(\textbf{Space}\right)^{\text{gp}}$ along with a map $M \to M^{\text{gp}}$ which is initial in the sense,

$$\text{Map}_{\text{CMon}(\textbf{Space})^{\text{gp}}}\left(M^{\text{gp}}, N\right) \xrightarrow{\sim} \text{Map}_{\text{CMon}(\textbf{Space})}\left(M, N\right)$$

Remark. Given a symmetric monoidal 1-category $(\mathcal{C}, \otimes, 1)$ there is a functor $\text{Fin}_* \to 1\text{-Cat}$ defining this symmetric monoidal structure.

Given a 1-groupoid G we can take the classifying space BG and get a functor $B : \text{Grpd}_1 \to \textbf{Space}$. If $\mathcal{C}$ is a symmetric monoidal 1-category then we apply maximal groupoid functor $\mathcal{C} \to \mathcal{C}^{\simeq}$ compose with B to get $B\mathcal{C}^{\simeq} \in \text{CMon}\left(\textbf{Space}\right)$.

3 Oct 14

3.1 Nisnevich Sheaves

Let $\mathrm{PSh}(\mathrm{Sm}_S) = \mathrm{Fun}\left(\mathrm{Sm}_S^{\mathrm{op}}, \mathrm{Spc}\right)$

For the Zariski site Gerstein property, a presheaf $\mathscr{F}$ is a zariski sheaf iff,

(a) $\mathscr{F}(\varnothing)$ is contractible

(b) for all opens $U, V \subset U \cup V$ we have,

$$\begin{tikzcd} \mathscr{F}(U \cup V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{F}(V) \arrow[d] \\ \mathscr{F}(U) \arrow[r] & \mathscr{F}(U \cap V) \end{tikzcd}$$

is a homotopy pullback diagram.

3.1.1 Why not Take the Zariski Site

We want local models, for $x \in U$

$$X/(X \backslash \{x\}) \simeq U/(U \backslash \{x\})$$

this should be good. But moreover, for small enough U we want,

$$U/(U \backslash \{x\}) \simeq \mathbb{A}^n/(\mathbb{A}^n \backslash \{0\})$$

This works in the étale topology because smooth schemes are étale locally affine space,

$$\begin{tikzcd} X \arrow[d] & U \arrow[l, hook'] \arrow[d, "\text{ét}"] \\ \mathrm{Spec}\,(k) & \mathbb{A}^n \arrow[l] \end{tikzcd}$$

Therefore, this property should work in the étale topology. It will also work in the coarser topology called the Nisnevich topology.

3.1.2 Why not Take the étale Site

K-theory does not satisfy étale descent.

3.1.3 Criterion for a presheaf of spaces to be a Nisnevich sheaf

Definition 3.1.1. Let X be a smooth scheme over S. Nisnevich squares are diagrams of the form,

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\text{ét}"] & V \backslash W \arrow[l, hook'] \arrow[d, "\sim"] \\ U \arrow[r, hook, "\iota"] & X & X \backslash U \arrow[l, hook', "j"'] \end{tikzcd}$$

where ι is an open embedding and j is the closed embedding.

Example 3.1.2. (a) The Gerstein square gives a Nisnevich square,

$$\begin{tikzcd}
U \cap V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\text{ét}"] & V \backslash (U \cap V) \arrow[l, hook'] \arrow[d, "\sim"] \\
U \arrow[r, hook, "\iota"] & U \cup V & V \backslash U \arrow[l, hook', "j"']
\end{tikzcd}$$

(b)
$$\begin{tikzcd}
U \cap \{x\} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\text{ét}"] & V \backslash (U \cap V) \arrow[l, hook'] \arrow[d, "\sim"] \\
U \arrow[r, hook, "\iota"] & U \cup V & V \backslash U \arrow[l, hook', "j"']
\end{tikzcd}$$

(c)

Definition 3.1.3. $\mathscr{F} \in \mathrm{PSh}(\mathrm{Sm}_S)$ is a Nisnevich sheaf if,

(a) $\mathscr{F}(\varnothing)$ is contractible

(b) for any Nisnevich square,

$$\begin{tikzcd}
\mathscr{F}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{F}(U) \arrow[d] \\
\mathscr{F}(V) \arrow[r] & \mathscr{F}(W)
\end{tikzcd}$$

is a homotopy pullback.

Remark. The diagram,

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & U \arrow[d] \\
V \arrow[r] & U \sqcup V
\end{tikzcd}$$

shows that,

$$\mathscr{F}(U \sqcup V) \simeq \mathscr{F}(U) \times \mathscr{F}(V)$$

3.2 $\mathbb{A}^1$-invariance

Definition 3.2.1. $F \in \mathrm{PSh}(\mathrm{Sm}_S)$ is $\mathbb{A}^1$-invariant if for all $X \in \mathrm{Sm}_S$ the maps,

$$\mathscr{F}(\mathrm{pr}_1) : \mathscr{F}(X) \to \mathscr{F}(X \times \mathbb{A}^1)$$

are homotopy equivalences.

Remark. We use the notation,

(a) $\mathcal{L}_{\mathrm{Nis}}(\mathrm{PSh}(\mathrm{Sm}_S)) \hookrightarrow \mathrm{PSh}(\mathrm{Sm}_S)$

(b) $\mathcal{L}_{\mathbb{A}^1}(\mathrm{PSh}(\mathrm{Sm}_S)) \hookrightarrow \mathrm{PSh}(\mathrm{Sm}_S)$

for the full subcategory of Nisnevich sheaves and $\mathbb{A}^1$-invariant presheaves respectively. Why do we use this notation? This is because there is a left adjoint to the inclusions which we denote by $\mathcal{L}$.

Definition 3.2.2. The unstable motivic homotopy category is,

$$\mathrm{Spc}(S) = \mathcal{L}_{\mathrm{Nis}}(\mathrm{PSh}(\mathrm{Sm}_S)) \cap \mathcal{L}_{\mathbb{A}^1}(\mathrm{PSh}(\mathrm{Sm}_S))$$

Definition 3.2.3. Let $R : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. A left adjoint for R is a pair $(L : \mathcal{C} \to \mathcal{D}, \alpha : L \circ R \to \mathrm{id}_{\mathcal{D}})$ such that for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ the natural map,

$$\mathrm{Hom}_{\mathcal{C}}(X, R(Y)) \xrightarrow{L} \mathrm{Hom}_{\mathcal{D}}(L(X), LR(Y)) \xrightarrow{\alpha_Y \circ -} \mathrm{Hom}_{\mathcal{D}}(L(X), Y)$$

is an equivalence.

Remark. There is a notion of uniqueness (HTT 5.2.6.2)

Remark. In the ∞-category setting: left adjoint gives colimits right adjoints are limits.

Definition 3.2.4. An ∞-category $\mathcal{C}$ is presentable if,

(a) $\mathcal{C}$ is locally small

(b) $\mathcal{C}$ has small colimits

(c) there exists regular cardinal κ and $S \subset \mathrm{Obj}(\mathcal{C})$ such that for all $X \in S$ is κ-compact (meaning $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves κ-filerted colimit) and every $X \in \mathrm{Obj}(\mathcal{C})$ can be presented as a colimits $\mathscr{F} : \langle S \rangle \to \mathcal{C}$.

Proposition 3.2.5. Let $\mathcal{C}$ be a presentable ∞-category,

(a) $\mathcal{C}$ has colimits [HTT, 5.5.2.9]

Proposition 3.2.6. The following ∞-categories are presentable,

(a) Spc is presentable [HTT 5.5.18]

(b) if $\mathcal{C}$ is small then $\mathfrak{Sh}(\mathcal{C})$ is presentable

(c) $\mathrm{Spc}(S)$ is presentable

(d) $\mathcal{L}_{\mathrm{Nis}}(\mathrm{PSh}(\mathrm{Sm}_S))$ and $\mathcal{L}_{\mathbb{A}^1}(\mathrm{PSh}(\mathrm{Sm}_S))$ are presentable.

Theorem 3.2.7. Let $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ be a functor of presentable ∞-categories. Then,

(a) $\mathscr{F}$ has a right adjoint iff $\mathscr{F}$ preserves small colimits

(b) $\mathscr{F}$ has a left adjoint iff $\mathscr{F}$ preserves small limits and κ-filtered colimits

HTT 5.5.2.9. □

3.3 Explicit Construction of $\mathcal{L}_{\mathbb{A}^1}$

Definition 3.3.1. Standard cosimplicial schemes,

$$\Delta^\bullet : \Delta \to \mathrm{Sm}_S$$

given by

$$[n] \mapsto \Delta^n = V(T_0 + \cdots + T_N = 1) \subset \mathbb{A}^{n+1}_S$$

and given a monotone map $f : [m] \to [n]$ send it to the restriction of,

$$T_i \mapsto T_{f(j)=i} T'_j$$

giving a map,

$$\mathbb{A}^{m+1}_S \to \mathbb{A}^{n+1}_S$$

with coordinates T'_j on the first and T_i on the second.

Theorem 3.3.2 (A25 of Primer)**.** Let $\mathscr{F} \in \mathrm{PSh}(\mathrm{Sm}_S)$ and $X \in \mathrm{Sm}_S$ then,

$$(\mathcal{L}_{\mathbb{A}^1}\mathscr{F})(X) = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathscr{F}(X \times \Delta^n)$$

We can also set,

$$\mathscr{F}_{\mathbb{A}^1} : \Delta^{\mathrm{op}} \to \mathrm{PSh}(\mathrm{Sm}_S) \quad [n] \mapsto \mathscr{F}(- \times \Delta^n)$$

Definition 3.3.3. A morphism $\alpha : \mathscr{F} \to \mathscr{G}$ in $\mathrm{PSh}(\mathrm{Sm}_S)$ is an $\mathbb{A}^1$-equivalence if $\mathcal{L}_{\mathbb{A}^1}(f)$ is an equivalence.

Remark. For $\pi_X : X \times \mathbb{A}^1 \to X$ then $h^{\mathrm{pr}_X} : h^{X\times\mathbb{A}^1} \to h^X$ is an $\mathbb{A}^1$-equivalence.

Definition 3.3.4. $\alpha, \beta : F \to G$ are $\mathbb{A}^1$-homotopic if ther eixsts,

$$\begin{tikzcd}
F \times h^{\{0\}} \arrow[d] \arrow[dr, "\alpha"] & \\
F \times h^{\mathbb{A}^1} \arrow[r] & \mathscr{G} \\
F \times h^{\{1\}} \arrow[u] \arrow[ur, "\beta"] &
\end{tikzcd}$$

Theorem 3.3.5. Let S be a regular noetherian scheem of finite dimension. There exists $K \in \mathrm{Spc}(S)$ such that for all $X = \mathrm{Spec}\,(A) \in \mathrm{Sm}_S$ we have $K(X) \simeq K(A)$ where $K(A)$ is the algebraic K-theory space of the ring A.

Remark. on affine $A \mapsto K(A)$ is an $\mathbb{A}^1$-invariant Nisevich sheaf. In particular,

$$K(A) \simeq K(A[t])$$

For general $X \in \mathrm{Sm}_S$ not affine then $K(X)$ is the Thomason-Traubagh K-theory.

4 Oct 21

4.1 Infinite dimensional Hilbert schemes

Fix S noetherian scheme. Let $\mathrm{PSh}(\mathbf{Sch}_S)$ be presheaes over schemes. We will be proving that some presheaves are either motivically equivalent or $\mathbb{A}^1$-equivalent.

Definition 4.1.1. Let FFlat_S be the stack of finite flat schemes over S,

$$T \to S \mapsto \{\text{groupoid of finite finitely presented flat morphisms } X \to T\}$$

Remark. A finite flat map $\varphi : X \to S$ is finitely presented iff $\varphi_* \mathcal{O}_X$ is a finitely presented $\mathcal{O}_T$-module. (This is like saying that a finite algebra is finitely presented as an algebra iff finitely presented as a module).

Remark. Then we have the decomposition,

$$\mathrm{FFlat} = \sqcup_{d \in \mathbb{N}} \mathrm{FFlat}_d$$

where FFlat_d *subset*FFlat is the open and closed substack where $\varphi_* \mathcal{O}_X$ is a vector bundle of rank d.

Definition 4.1.2. Consider the stack,

$$\mathrm{Vect}_S = \sqcup_{d \in \mathbb{N}\mathbb{N}} \mathrm{Vect}_d$$

which is the stack of vector bundles over S. There is a forgetful morphism,

$$\mathrm{FFlat} \to \mathrm{Vect}$$

sending $(X \to T) \mapsto \varphi_* \mathcal{O}_X$ which is degree preserving meaning it induces,

$$\mathrm{FFlat}_d \to \mathrm{Vect}_d$$

We refine this forgetful morphism (using the local) criterion for flatness,

$$\nu : \mathrm{FFlat}_d \to \mathrm{Vect}_{d-1}$$

given by,

$$(X \to T) \mapsto \varphi_*(\mathcal{O}_X)/(\mathcal{O}_T \cdot 1)$$

Remark. Locally $T = \mathrm{Spec}\,(R)$ and $X = \mathrm{Spec}\,(A)$ then the inclusion $R \to A$ gives a sequence,

$$0 \longrightarrow R \longrightarrow A \longrightarrow A/R \longrightarrow 0$$

Since A/R is finitely presented, it suffices to prove it is R-flat. By the local criterion for flatness we may take $(R, \mathfrak{m}, \kappa)$ to be local and we just need to show that $\mathrm{Tor}^1_R\,(\kappa, A/R) = 0$. We have a sequence,

$$0 \longrightarrow \mathrm{Tor}^1_R\,(\kappa, A) \to \mathrm{Tor}^1_R\,(\kappa, A/R) \longrightarrow \kappa \otimes R \longrightarrow \kappa \otimes A \longrightarrow \kappa \otimes (A/R)$$

but then $\kappa \otimes R \to \kappa \otimes A$ is injective because it is the inclusion of κ into a κ-algebra and $\mathrm{Tor}^1_R\,(\kappa, A) = 0$ since A is R-flat giving the required vanishing.

There is also a morphism in the other direction,

$$\delta : \mathrm{Vect}_{d-1} \to \mathrm{FFlat}_d$$

given by sending,

$$\mathcal{E} \mapsto \mathrm{Sym}_{\mathcal{O}_T}(\mathcal{E}) / (\mathcal{E}^{\otimes 2}) = \mathcal{O}_T \oplus \mathcal{E}$$

where the multiplication sends $\mathcal{E} \otimes \mathcal{E} \to 0$.

Proposition 4.1.3. The morphisms ν and δ are $\mathbb{A}^1$-homotopy inverses.

Proof. We need to show that $\nu \circ \delta \simeq \mathrm{id}_{\mathrm{Vect}_{d-1}}$. Actually, they are canonically isomorphic because of the trivial sequence,

$$0 \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_T \oplus \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow 0$$

The other direction $\delta \circ \nu \simeq \mathrm{id}_{\mathrm{FFlat}_d}$ follows from the Rees construction. We want a map,

$$h : \mathbb{A}^1_S \times_S \mathrm{FFlat}_d \to \mathrm{FFlat}_d$$

such that $h(0, -) \cong \delta \circ \nu$ and $h(1, -) \cong \mathrm{id}_{\mathrm{FFlat}_d}$.

A map $\mathbb{A}^1_S \times_S \mathrm{FFlat}_d \to \mathrm{FFlat}_d$ amounts to a finite flat algebra on $\mathbb{A}^1_S \times_S \mathrm{FFlat}_d$ or equivalently a finite flat $\mathcal{O}_{\mathbb{A}^1_S} \otimes \mathcal{O}_{\mathrm{FFlat}_d}$-algebra,

$$\mathcal{A}_{\mathrm{Rees}} = \bigoplus_{i=0}^{\infty} \mathcal{E}_i t^i \quad \text{where} \quad \mathcal{E}_i = \begin{cases} \mathcal{O}_{\mathrm{FFlat}_d} & i = 0 \\ \mathcal{A}_{\mathrm{univ}} & i \geq 1 \end{cases}$$

where $\mathbb{A}^1_S = \mathbf{Spec}_S(\mathcal{O}_S[t])$ and $\mathcal{A}_{\mathrm{univ}}$ is the universal flat algebra on FFlat. Claim: this is a finite flat $\mathcal{O}_T[t] \otimes \mathcal{O}_{\mathrm{FFlat}_d}$-algebra. The restriction to $\{1\} \times \mathrm{FFlat}_d$ yields the map,

$$\mathrm{FFlat}_d \to \mathrm{FFlat}_d$$

given by,

$$\mathrm{colim}_i\, \mathcal{E}_i = \mathcal{A}_{\mathrm{univ}}$$

as a flat $\mathcal{O}_{\mathrm{FFlat}_d}$-algebra which thus gives the identity $\mathrm{id} : \mathrm{FFlat}_d \to \mathrm{FFlat}_d$. The restriction to $\{0\} \times \mathrm{FFlat}_d$ yields the associated graded,

$$\mathcal{A}_{\mathrm{Rees}}/(t) = \bigoplus_{i=0}^{\infty} (\mathcal{E}_i / \mathcal{E}_{i-1}) \cong \mathcal{O}_{\mathrm{FFlat}_d} \oplus (\mathcal{A}_{\mathrm{univ}} / \mathcal{O}_{\mathrm{FFlat}_d})$$

which by definition is the algebra corresponding to $\delta \circ \nu$. $\square$

Proposition 4.1.4. Let $f : X \to Y$ be a morphism of presheaves on Aff_S satisfying the closed gluing condition. Then if for all algebras $\mathcal{A}$ over S and finitely generated ideals $\mathscr{I} \subset \mathcal{A}$ and diagrams,

$$\begin{tikzcd} \mathrm{Spec}\,(A/I) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a lift up to homotopy meaning the morphism,

$$\pi_0(X(\mathrm{Spec}\,(A))) \to \pi_0(X(\mathrm{Spec}\,(A/I)) \times_{Y(\mathrm{Spec}(A/I))} Y(\mathrm{Spec}\,(A)))$$

is surjective then f is a universal $\mathbb{A}^1$-equivalence on Aff_S.

Remark. $\mathcal{L}_{\mathbb{A}^1}$ does not generally commute with base change so being a universal $\mathbb{A}^1$-equivalence is strictly stronger than being an $\mathbb{A}^1$-equivalence.

Remark. Note that π_0 does not commute with homotopy pullbacks. Consider,

$$\begin{tikzcd} \Omega S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$

then $\pi_0(\Omega S^1) = \mathbb{Z}$ but $\pi_0(*) = 1$.

Definition 4.1.5. A morphism $f : X \to Y$ satisfies the *closed gluing condition* if for all schemes T over S and morphisms $T \to Y$ the fiber product $X \times_Y T$ satisfies

(a) sends the empty scheme to a point

(b) sends pushouts of closed immersions of schemes to pullbacks.

Definition 4.1.6. The infinite Hilbert scheme $\mathrm{Hilb}_d(\mathbb{A}^\infty_S)$ is te functor from Aff_S given by,

$$(T \to S) \mapsto \mathrm{colim}_{n\to\infty} \mathrm{Hilb}_d(\mathbb{A}^n_S)(T \to S)$$

where the inclusions are induced by the sequence,

$$\mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \hookrightarrow \mathbb{A}^3 \hookrightarrow \cdots$$

Remark. This functor classified finite flat finitely presented S-schemes of degree d along with a closed embedding into affine space of S of *some* dimension.

Definition 4.1.7. The funcor $\mathrm{Hilb}_d(\mathbb{A}^{\mathbb{N}}_S)$ is the functor that sends,

$$(T \to S) \mapsto \{Z \hookrightarrow \mathbf{Spec}_T\,(\mathcal{O}_T[t_n]_{n\in\mathbb{NN}})\}$$

the usual definition of the Hilbert functor of the actual scheme $\mathbb{A}^{\mathbb{NN}}_S$ without taking a colimit.

Proposition 4.1.8. The forgetfull functors,

(a) $\mathrm{Hilb}_d(\mathbb{A}^\infty_S) \to \mathrm{FFlat}_d$

(b) $\mathrm{Hilb}_d(\mathbb{A}^{\mathbb{NN}}_S) \to \mathrm{FFlat}_d$

are universal $\mathbb{A}^1$-equivalences.

Proof. First, these morphisms satisfy the closed gluing condition □

5 Oct 28

Recall we defined,

$$\begin{tikzcd} \mathrm{Hilb}_d(\mathbb{A}^\infty) \arrow[r] \arrow[d, hook] & \mathrm{FFlat}_d \arrow[r] & \mathrm{Vect}_{d-1} & \mathrm{Gr}_{d-1}(\mathbb{A}^\infty \arrow[l] \\ \mathrm{Hilb}_d(\mathbb{A}^{\mathbb{N}}) \arrow[ur] \end{tikzcd}$$

All of these maps will turn out to be $\mathbb{A}^1$-equivalences.

Definition 5.0.1. A presheaf $\mathscr{F} \in \mathrm{PSh}(\mathbf{Sch}_S)$ satisfies the *closed gluing condition* if,

(a) $\mathscr{F}(\varnothing) \cong *$

(b) for every pushout diagram,

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\ Y \arrow[r, hook] & W \end{tikzcd}$$

comprised of closed embeddings we have,

$$\mathscr{F}(W) \cong \mathscr{F}(X) \times_{\mathscr{F}(Z)} \mathscr{F}(Y)$$

Proposition 5.0.2. Let $\phi : Z \hookrightarrow Y$ and $\psi : Z \hookrightarrow X$ be closed immersion of schemes. Then the pushout $W = X \sqcup_Z Y$,

$$\begin{tikzcd} Z \arrow[r, hook, "\phi"] \arrow[d, hook, "\psi"] & Y \arrow[d, "b"] \\ X \arrow[r, "a"] & W \end{tikzcd}$$

exists in the category of schemes and,

(a) a and b are closed immersions

(b) $\mathcal{O}_W = \ker(a_* \mathcal{O}_X \oplus b_* \mathcal{O}_Y \to \iota_* \mathcal{O}_Z)$.

Proof. The pushout exists in the category of ringed spaces and has the above description with $|W| = |X| \sqcup_{|Z|} |Y|$ since then the ring structure is universal. Claim: if W is a scheme then it is a pushout in **Sch**. Then,

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Sch}}(X,S) \times_{\mathrm{Hom}_{\mathbf{Sch}}(Z,S)} \mathrm{Hom}_{\mathbf{Sch}}(Y,S) \arrow[d, hook] & \mathrm{Hom}_{\mathbf{Sch}}(W,S) \arrow[l] \arrow[d, hook] \\ \mathrm{Hom}_{\mathrm{RS}}(X,S) \times_{\mathrm{Hom}_{\mathrm{RS}}(Z,S)} \mathrm{Hom}_{\mathrm{RS}}(Y,S) & \mathrm{Hom}_{\mathrm{RS}}(W,S) \arrow[l, "\sim"'] \end{tikzcd}$$

We want to show that the top map is an isomorphism. Via the diagram it is an injection. To check that the map is surjective we can work locally on S. Assume $S = \mathrm{Spec}(A)$ then get the diagram,

$$
\begin{tikzcd}
\mathrm{Hom}\left(A, H^0(\mathcal{O}_X)\right) \times_{\mathrm{Hom}(A, H^0(\mathcal{O}_Z))} \mathrm{Hom}\left(A, H^0(\mathcal{O}_Z)\right) \arrow[d, hook] & \mathrm{Hom}\left(A, H^0(\mathcal{O}_W)\right) \arrow[l] \arrow[d, hook] \\
\mathrm{Hom}_{\mathrm{RS}}\left(X, S\right) \times_{\mathrm{Hom}_{\mathrm{RS}}(Z,S)} \mathrm{Hom}_{\mathrm{RS}}\left(Y, S\right) & \mathrm{Hom}_{\mathrm{RS}}\left(W, S\right) \arrow[l, "\sim"']
\end{tikzcd}
$$

An element on the left is the same data as a map,

$$
\begin{aligned}
A \to H^0(\mathcal{O}_X) \times_{H^0(\mathcal{O}_Z)} H^0(\mathcal{O}_Y) &= \ker\left(H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_Z)\right) \\
&= H^0(\ker\left(a_* \mathcal{O}_X \oplus a_* \mathcal{O}_Y \to \iota_* \mathcal{O}_Z\right) = H^0(\mathcal{O}_W)
\end{aligned}
$$

so the map is surjective. Now we need to show that W is a scheme and that a and b are closed immersions. Away from $\iota(Z)$ this is clear because then we are taking disjoint union. Choose $z \in Z$ we need an open affine neighborhood of $\iota(z) \in W$. Claim, there are $U \subset X$ and $V \subset Y$ such that $\phi^{-1}(U) = \psi^{-1}(V)$ then $L = U \sqcup_{\phi^{-1}(U)} V$ is an open of W. Therefore, we reduce to the case that all X, Y, Z are affine. Consider the pushout in the category of affine schemes,

$$
W_{\mathrm{Aff}} = \mathrm{Spec}\left(H^0(\mathcal{O}_X) \times_{H^0(\mathcal{O}_Z)} H^0(\mathcal{O}_Y)\right)
$$

By the universal property, there is a map $W \to W_{\mathrm{Aff}}$. It suffices to show that this is a homeomorphism since then the rings argree by the definition of W. This is elementary commutative algebra [Schwede, Thm 3.4]. $\square$

Lemma 5.0.3. If $f : \mathscr{F} \to \mathscr{G}$ is a morphism in $\mathrm{PSh}(\mathbf{Sch}_S)$ such that f satisfies the closed gluing condition (meaning the fibers over schemes do) and for any ring A over S and finitely generated ideal $I \subset A$ the morphism,

$$
\mathscr{F}(A) \to \mathscr{F}(A/I) \times_{\mathscr{G}(A/I)} \mathscr{G}(A)
$$

is surjective on π_0. Then f is a universal $\mathbb{A}^1$-equivalence on affine schemes.

Remark. A morphism $f : \mathscr{F} \to \mathscr{G}$ is an $\mathbb{A}^1$-equivalent on affine schemes if $\mathcal{L}_{\mathbb{A}^1}(f)_R$ is an equivalence for any ring R.

Remark. $\mathcal{L}_{\mathbb{A}^1}$ does not preserve fiber products so being a universal $\mathbb{A}^1$-equivalent is important. For example consider a family $f : X \to \mathbb{A}^1$. Then the fibers X_t are base changes of f but $\mathcal{L}_{\mathbb{A}^1}(*) \times_{\mathcal{L}_{\mathbb{A}^1}\mathbb{A}^1} \mathcal{L}_{\mathbb{A}^1} X$ are all the same since $\mathcal{L}_{\mathbb{A}^1}\mathbb{A}^1 = *$. Therefore, $\mathcal{L}_{\mathbb{A}^1}(X_t)$ would all be equivalent. This is not true. Consider a map $f : \mathbb{A}^1 \to \mathbb{A}^1$ with different numbers of points in the fibers. Since a finite collection of points is $\mathbb{A}^1$-invariant and they are not equivalent as presheaves they are not $\mathbb{A}^1$-equivalent.

Proof. We proved if $[Z = \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)] \in \mathrm{FFlat}_d(A)$ and $Z_{A/I} \hookrightarrow \mathbb{A}^n_{A/I}$ for some n then there is a lift $Z \hookrightarrow \mathbb{A}^{n+m}_A$.

Let $Z_{A/I} \hookrightarrow \mathbb{A}^{\mathbb{N}}_{A/I}$ equivalently a collection $(\bar{b}_i)_{i \in \mathbb{N}}$ in B/IB such that,

$$
\mathrm{Sym}_{\bullet}\left(\bigoplus_{i=1}^{\infty} A b_i\right) \twoheadrightarrow B/IB
$$

First WLOG can take $\bar{b}_1, \ldots, \bar{b}_r$ generate B/I as an A/I-algebra. For every other $\bar{b}_i$ with $i > r$ we can write,

$$
\bar{b}_i = \bar{p}_i(\bar{b}_1, \ldots, \bar{b}_r)
$$

Lift to elements $b_i \in B$ for $i \leq r$ and lift $p_i \in A[x_1, \dots, x_N]$ and make the definition. The problem is the lifted symmetric algebra may not surject. However, we do know,

$$\mathrm{Sym}_\bullet\left(\bigoplus_{i=1}^{r} Ab_i\right) \twoheadrightarrow B/IB$$

is surjective. Take some $h_1, \dots, h_m$ generating IB as an A-module. set,

$$b_i = \begin{cases} b_i & i \leq r \\ p_i(b_1, \dots, b_R) + h_{i-m} & r+1 \leq i \leq r+m \\ p_i(b_1, \dots, b_r) & i > m+r \end{cases}$$

and we can check that this works.

Closed gluing: consider the diagram for some test scheme T,

$$\begin{tikzcd} F_T \arrow[r] & \mathrm{Hilb}_d(\mathbb{A}^n) \arrow[d] \\ T \arrow[r] & \mathrm{FFlat}_d \end{tikzcd}$$

we need to show that $F_T : \mathbf{Sch}_T^{\mathrm{op}} \to \mathbf{Set} \to \mathrm{Spc}$ satisfies closed gluing. This is the functor,

$$[W \to T] \mapsto \mathrm{Hom}_{W\text{-closed}}\left(B_W, A_W\right)$$

where $B \to T$ is the finite flat scheme defined by $T \to \mathrm{FFlat}_d$ and A is the affine space. Thus we need to show that,

$$\mathrm{Hom}_{\mathrm{cl}}\left(B_W, A_W\right) \to \mathrm{Hom}_{\mathrm{cl}}\left(B_X, A_X\right) \times_{\mathrm{Hom}_{\mathrm{cl}}(B_Z, A_Z)} \mathrm{Hom}_{\mathrm{cl}}\left(B_Y, A_Y\right)$$

is an isomorphism. Notice that base change preserves the pushout squares of closed embeddings and thus if we remove the closed conditions the above is an isomorphism. Therefore, it suffices to show that closed embeddings on the right glue to a morphism $B_W \to A_W$ which is a closed embedding. Topologically, this is clear. Properness is clear, radicial can be checked using functor of points (DO THIS!!) $\square$

Definition 5.0.4. $\mathrm{Gr}_d(\mathbb{A}^\infty) : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set} \to \mathrm{Spc}$ is the functor,

$$\mathrm{Gr}_D(\mathbb{A}^\infty) : (T \to S) \mapsto \mathrm{colim}_n\, \mathrm{Gr}_{n,d}(T)$$

Definition 5.0.5. There is a forgetful functor,

$$f : \mathrm{Gr}_d(\mathbb{A}^\infty) \to \mathrm{Vect}_d$$

given by sending $[\mathcal{O}_T^{\oplus n} \twoheadrightarrow \mathscr{F}] \mapsto \mathscr{F}$.

Proposition 5.0.6. f is an $\mathbb{A}^1$-equivalent.

Proof. Using the lemma, we need to check closed gluing and lifts. For lifting, let A be a ring over S and fix a finitely generated ideal $I \subset A$. Let $\mathscr{F} = \widetilde{M}$ be a rank d vector bundle on $\mathrm{Spec}\,(A)$. Want to show that given $(A/I)^{\oplus n} \twoheadrightarrow M/IM$ we can lift to a surjection $A^{\oplus(n+m)} \twoheadrightarrow M$. This surjection corresponds to elements $\bar{m}_i \in M/IM$. Lift these to $m_i \in M$ then the map $A^{\oplus n} \to M$ may not be surjective but taking generators $h_1, \dots, h_m \in IM$ the map $A^{\oplus(n+m)} \twoheadrightarrow M$ is surjective. $\square$

6 Integral Crystalline-Tate Conjecture

Definition 6.0.1 (Tate)**.** Let X be a smooth projective variety over $\mathbb{C}$ and $z \in H^{2r}(X, \mathbb{Z})$ and also Hodge ($Z \to H^{2r}_{\mathrm{dR}}(X)$ lies in $H^{r,r}(X)$ is an algebraic cycle.

Theorem 6.0.2 (Atiyah-Hirzebruch)**.** There exists X smooth projective variety over $\mathbb{C}$ and a class $z \in H^{2r}(X, \mathbb{Z})$ is Hodge but not algebraic.

Example 6.0.3. $X \sim B((Z/p)^{\times 3})$ and Z is p-torsion.

Definition 6.0.4. Let X be smooth projective geometrically connected over $\mathbb{F}_q$ and assume $z \in H^{2r}_{\mathrm{crys}}(X/W)$ which is Tate then,

$$\mathrm{Im}\left((H^r_{\mathrm{proet}}(X, W\Omega^r_{\log}) \to H^{2r}_{\mathrm{crys}}(X/W))\right)$$

is algebraic.

Theorem 6.0.5. The above conjecture is false. We can do this for $X \sim B\mu_p^{\times 3}$ with $p = \mathrm{char}(\mathbb{F}_q)$.

6.1 Atiyah-Hirzebruch counter examples

What A-H proved is that any cycle $z \in H^{2r}(X, \mathbb{Z})$ which is algebraic the AHSS $H^\bullet \implies KU^\bullet$ then $\mathrm{d}_r(z) = 0$. These give obstructions to being algebraic.

Proof. Consider motivic cohomology $H^i_{\mathrm{mot}}(X, \mathbb{Z}(j))$. Then it satisfies the following properties,

(a) $H^{2j}_{\mathrm{mot}}(X, \mathbb{Z}(j)) \cong \mathrm{CH}^j(X)$

(b) $H^p_{\mathrm{mot}}(X, \mathbb{Z}(q)) = 0$ for $p > 2q$

(c) there is a spectral sequence,

$$E_2^{i,j} = H^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \implies K_{-i-j}(X)$$

There is a cycle class map giving a morphism from the above spectral sequence to the spectral sequence,

$$H^{i-j}_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \implies KU_{-i-j}(X^{\mathrm{an}})$$

Then the differentials on the Chow part satisfy,

$$\mathrm{d}_r^{j,-j} : H^{2j}_{\mathrm{mot}}(X, \mathbb{Z}(j)) = \mathrm{CH}^j(X) \to H^{j+r-(-j-r+1}_{\mathrm{mot}}(X, \mathbb{Z}(j+r-1)) = 0$$

since $j+r-(-j-r+1) = 2j+2r-1$ is bigger than $2(j+r-1)$. Therefore, using the commutativity the topological d_r must also vanish for any algebraic cycle. $\square$

We want to understand $\mathrm{d}_r : H^\bullet \to H^\bullet$. Baby case $M \in D(\Lambda)$ we get a sequence,

$$H^{a-1}(M)[-q+1] \to \tau^{[q-1,q]}M \to H^q(M)[-q] \xrightarrow{\delta} H^{q-1}(M)[-q+2]$$

For simplicitly, weork with mod $p = 2$ coefficients. To provide counter example suffices to produce a class,

$$z \in H^{2r}(X, \mathbb{F}_p)$$

and z lifts to $\mathbb{Z}$ iff $\beta(z) = 0$ (bokstein) not in the image of $\mathrm{CH}^r(X)/p$. In topology,

$$(\mathbb{Z}/2)[2] \to \tau_{[2,0]} KU/2 \to \mathbb{Z}/2 \xrightarrow{Q_i} \mathbb{Z}/2[3]$$

where the maps Q_i are Milnor's operations:

$$Q_0 = \beta$$
$$Q_1 = \beta \mathrm{sq}^2 - \mathrm{sq}^2 \beta$$
$$\vdots$$

Lesson learned: cohomology operations are,

(a) Steenrod operations

(b) instructions to build cohomology theories

(c) formal group laws.

Let's explain the formal group laws:

$$E^*(\mathbb{CP}^\infty) = E^*(pt)[[x]]$$

and then the multiplication $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ gives the structure of a formal group law. Somehow,

$$Q_i \iff K(j)$$

Morava's K-theory

Proposition 6.1.1. For any cycle $z \in H^{2r}(X, \mathbb{Z}/2)$ which is algebraic then $Q_j z = 0$ for all $j \geq 0$.

Proof. Consider the diagram,

$$\begin{tikzcd} H^{2r,r}_{\mathrm{mot}}(X, \mathbb{Z}/2) \arrow[r] \arrow[d] & H^{2r}(X, \mathbb{Z}/2) \arrow[d] \\ 0 \arrow[r] & H^{2r+2,2r-1}(X, \mathbb{Z}/2) \end{tikzcd}$$

6.2 Characteristic p

For X a smooth projective scheme over $\mathbb{F}_q$ we have $H^r_{\mathrm{crys}}(X/W)$ with $W = W(\mathbb{F}_q)$ is a W-module. This is a good Weil cohomology theory in the absence of a lift of X to characteristic zero.

Construction of cycle class maps: a theorem of Geisser and LEvine:

$$\mathrm{CH}^r(X)/p^n = H^{2r}_{\mathrm{mot}}(X, \mathbb{Z}/p^n(r)) = H^r_{\mathrm{Zar}}(X, W_n \Omega^r_{\log})$$

where $W_n \Omega^r_{\log}/p = \Omega^r_{\log}$ is the abelian subsheaf of Ω^r generated by $\mathrm{dlog}\, f_i$ for units f_i. $\square$

Remark. $H^r_{\text{ét}}(X, W\Omega^r_{\log})[p^{-1}] \xrightarrow{\sim} H^{2r}_{\mathrm{crys}}(X/W)[p^{-1}]^{\varphi_X = p^r}$

Definition 6.2.1. X is *integrally crys-Tate* in degree r level n if the cycle class map surjective on to the image of,

$$H^r_{\text{ét}}(X, W_n \Omega^r_{\log}) \to H^{2r}_{\mathrm{crys}}(X/W_n)$$

Definition 6.2.2 (Tate)**.** $\mathrm{CH}^r(X)_{\mathbb{Q}_p} \twoheadrightarrow H^{2r}_{\mathrm{crys}}(X/W)[p^{-1}]^{\varphi_X = p^r}$

There is also of course the ℓ-adic Tate conjecture for Galois-fixed points of ℓ-adic cohomology.

Proposition 6.2.3. $\mathrm{Tate}^1_\ell \iff \mathrm{Tate}^1_p$

Theorem 6.2.4 (de Jong, Marrow)**.** $\mathrm{Tate}^1_\ell(X)$ for X a surface implies $\mathrm{Tate}^r_\ell(X)$ for all X and all r.

7 Nov 4

Definition 7.0.1.

$$\mathrm{FFlat}^{\mathrm{mrk}} : X \to \{(f, s) \mid f : Z \to X \text{ and } s : X \to Z \text{ section}\}$$

and similarly,

$$\mathrm{FFlat}^{\mathrm{nu}} = \text{ fflat but non-unital}$$

There are maps between these given by unitalization and taking the augmentation ideal $\ker(\mathcal{O}_Z \to \mathcal{O}_X)$.

Proposition 7.0.2. ν is an $\mathbb{A}^1$-equivalence.

Proof. $\pi \circ \nu = \mathrm{id}$. Define $\mathrm{FFlat}^{\mathrm{nu}} \to \mathrm{Hom}(\mathbb{A}^1, \mathrm{FFlat}^{\mathrm{nu}})$ sending $A \mapsto tA[t]$ □

Lemma 7.0.3. θ is an $\mathbb{A}^1$-equivalence.

Proof. Consider the diagram,

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\alpha"] \arrow[d, "\nu"] & \mathrm{FFlat}_{\geq 1} \\ \mathrm{FFlat}^{\mathrm{nu}} \arrow[r, "\sim"] & \mathrm{FFlat}^{\mathrm{mrk}} \arrow[u, "\theta"'] \end{tikzcd}$$

The 2-out-of-3 after $\mathcal{L}_{\mathbb{A}^1}$ shows that θ is an $\mathbb{A}^1$-equivalent. □

Remark. Both the map α from last time and the bottom unitalization equivalent increase the degree by 1.

Lemma 7.0.4.
$$\begin{tikzcd} Z \arrow[r, "\iota"] \arrow[d, "j"] & X \\ Y & \end{tikzcd}$$

Z, Y, X finite local free S-schemes then $X \sqcup_Z Y$ exists and is finite locally free over S.

Proof. Pushout exists by Stacks OE25 need to check if $M \twoheadrightarrow P$ and $N \twoheadrightarrow P$ are maps of finite loc free modules then $M \times_P N$ is finite locally free. From the sequence,

$$0 \longrightarrow N \longrightarrow M \times_P N \longrightarrow \ker(M \twoheadrightarrow P) \longrightarrow 0$$

this follows by splitting since these are projective. □

Theorem 7.0.5. η^{st} is an $\mathbb{A}^1$-equivalence.

Proof. Consider the following schemes,

$$R = \mathrm{Spec}\,(\mathbb{Z}[x, t]/((x - t)x))$$
$$R_0 = \mathrm{Spec}\left(\mathbb{Z}[x]/(x^2)\right)$$
$$R_1 = \mathrm{Spec}\,(\mathbb{Z}) \sqcup \mathrm{Spec}\,(\mathbb{Z})$$

Then we note,

$$\epsilon : (f : Z \to X, s : X \to Z) \mapsto Z \sqcup_S (R_0)_S$$

define,

$$H : \mathrm{FFlat}^{\mathrm{mrk}} \to \mathrm{Hom}\left(\mathbb{A}^1, \mathrm{FFlat}^{\mathrm{mrk}}\right)$$

as follows,

$$H : (f, s) \mapsto \mathbb{A}^1_Z \sqcup_{\mathbb{A}^1_S} R_S$$

Then $H_0 = \epsilon$ and $H_1 = \sigma^{\mathrm{mrk}}$. □

7.1 The Group Structures

$\oplus\otimes$ on Vect and $\sqcup, \times$ on FFlat give E_∞-semiring structures.

Theorem 7.1.1. $\eta^{\mathrm{gp}} : \mathrm{FFlat}^{\mathrm{gp}} \to \mathrm{Vect}^{\mathrm{gp}}$ is an $\mathbb{A}^1$-equivalence.

Remark. Now we prove the lemma that Andres keeps skipping.

Lemma 7.1.2 (4.1)**.** Let $f : \mathscr{F} \to \mathscr{G}$ be a map of presheaves on $\mathbf{Sch}_S$ such that the fibers ofver any scheme $T \to S$ satisfies closed gluing. And suppose that for anny affine $\mathrm{Spec}\,(A)$ with finite generated ideal $I \subset A$ the map,

$$\mathscr{F}(A) \to \mathscr{F}(A/I) \times_{\mathscr{G}(A/I)} \mathscr{G}(A)$$

is surjective on π_0. Then f is a universal $\mathbb{A}^1$-equivalence on affine schemes.

Proof. It suffices to consider $\mathscr{G} = *$ for some reason (WHY). Recall $\Delta^n_A = \mathrm{Spec}\,(A[t_0, \dots, t_n]/(t_0 + \cdots + t_n - 1$ and,

$$\mathcal{L}_{\mathbb{A}^1}\mathscr{F}(\mathrm{Spec}\,(A)) = \mathrm{colim}\,\mathscr{F}(\Delta^n_A)$$

We need to show that this is contractible. Then closed gluing implies,

$$\mathrm{Hom}\,(\partial\Delta^n_A, \mathscr{F}(\Delta^\bullet_A)) \cong \mathscr{F}(\partial\Delta^n_A)$$

where $\mathscr{F}(\Delta^\bullet_A)$ is the simplicial space $[n] \mapsto \mathscr{F}(\Delta^n_A)$. Then,

$$\mathrm{Hom}\,(\Delta^n, \mathscr{F}(\Delta^\bullet_A)) \to \mathrm{Hom}\,(\partial\Delta^n_A, \mathscr{F}(\Delta^\bullet_A))$$

is surjective on π_0 because it is the same as,

$$\mathscr{F}(\Delta^n_A) \to \mathscr{F}(\partial\Delta^n_A)$$

which is induced by the closed immersion $\partial\Delta^n_A \hookrightarrow \Delta^n_A$ and thus we apply the assumption. Therefore,

$$|\mathscr{F}(\Delta^\bullet_A)| = \mathrm{colim}_n\,\mathscr{F}(\Delta^n_A)$$

is contractible (WHY). □

8 My Talk

Theorem 8.0.1. Let $n \geq d \geq 0$ and k a field. The morphism,

$$\Sigma\mathrm{Hilb}_d(\mathbb{A}^n_k) \to \Sigma\mathrm{Hilb}_d(\mathbb{A}^\infty_k)$$

induced by the inclusion is $\mathbb{A}^1$-$(n - d + 1)$-connected.

8.1 Connectivity

Definition 8.1.1. We say a space X is n-connected if $\pi_i(X, x) = 0$ for $i \leq n$ and $x \in X$.

Definition 8.1.2. We say a morphism $f : X \to Y$ is n-connected if for each $x \in X$,

(a) $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ is an isomorphism for $i \leq n$

(b) $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ is an epimorphism for $i = n + 1$.

Remark. From the Puppe sequence, a morphism $f : X \to Y$ is n-connected iff the homotopy fiber F_f is n-connected.

Definition 8.1.3. Let $\mathscr{F} \in \mathrm{PSh}(\mathrm{Sm}_S)$ be a presheaf. We define the homotopy groups $\pi_i(F)$ as the Nisnevich sheafification of the presheaf,

$$U \mapsto \pi_i(\mathscr{F}(U))$$

Remark. Strictly we should consider pointed presheaves $* \to \mathscr{F}$.

Definition 8.1.4. We say a presheaf $\mathscr{F} \in \mathrm{PSh}(\mathrm{Sm}_S)$ is n-connected if $\pi_i(\mathscr{F}) = 0$ for $i \leq n$.

Definition 8.1.5. We say a morphism $f : \mathscr{F} \to \mathscr{G}$ in $\mathrm{PSh}(\mathrm{Sm}_S)$ is n-connected if

(a) $f_* : \pi_i(\mathscr{F}) \to \pi_i(\mathscr{G})$ is an isomorphism for $i \leq n$

(b) $f_* : \pi_i(\mathscr{F}) \to \pi_i(\mathscr{G})$ is an epimorphism for $i = n + 1$.

Definition 8.1.6. A presheaf $\mathscr{F} \in \mathrm{PSh}(\mathrm{Sm}_S)$ is $\mathbb{A}^1$-(n)-connected if $\mathcal{L}_{\mathrm{mot}}(\mathscr{F})$ is n-connected. A morphism $f : \mathscr{F} \to \mathscr{G}$ in $\mathrm{PSh}(\mathrm{Sm}_S)$ is $\mathbb{A}^1$-(n)-connected if $\mathcal{L}_{\mathrm{mot}}(f)$ is n-connected.

Remark. Equivalently, we can say a morphism $f : \mathscr{F} \to \mathscr{G}$ is n-connected if on each Nisevich stalk $f_x : \mathscr{F}_x \to \mathscr{G}_x$ is n-connected in the ordinary sense. Again f is $\mathbb{A}^1$-(n)-connected if $\mathcal{L}_{\mathrm{mot}}(f)$ is n-connected in this sense.

Remark. We can furthermore define,

$$\pi_i^{\mathbb{A}^1}(\mathscr{X}) := \pi_i(\mathcal{L}_{\mathrm{mot}}\mathscr{X})$$

and then $\mathbb{A}^1$-connectivity of a space or of a map corresponds to the expected definition in terms of $\pi_i^{\mathbb{A}^1}$.

Definition 8.1.7. Recall that we define the loop space and suspension of a space or presheaf as the pullback (pushout),

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

Remark. Since $\mathcal{L}_{\mathrm{mot}}$ preserves homotopy colimits $\mathcal{L}_{\mathrm{mot}}(\Sigma\mathscr{X}) = \Sigma\mathcal{L}_{\mathrm{mot}}(\mathscr{X})$. In particular, if $\mathscr{X}$ is motivic then $\Sigma\mathscr{X}$ is also motivic. However, $\mathcal{L}_{\mathrm{mot}}$ and Ω do not generally commute. This means although $\pi_i(\Omega\mathscr{X}) = \pi_{i+1}(\mathscr{X})$ is immediate from the definitions, $\pi_i^{\mathbb{A}^1}(\Omega\mathscr{X}) \neq \pi_{i+1}^{\mathbb{A}^1}(\mathscr{X})$ in general. However, if $\mathscr{X}$ is motivic then $\Omega\mathscr{X}$ is also motivic meaning that if $\mathscr{X}$ is motivic then,

$$\pi_i^{\mathbb{A}^1}(\Omega\mathscr{X}) = \pi_{i+1}^{\mathbb{A}^1}(\mathscr{X})$$

from the result from the unstabilized π_i.

Lemma 8.1.8. If $\mathscr{X} \in \mathrm{PSh}(\mathrm{Sm}_S)$ is motivic then $\Omega\mathscr{X}$ is motivic.

Proof. Since $\Omega\mathscr{X}$ is a sheaf (limits in sheaves and presheaves agree) it suffices to show that $\Omega\mathscr{X}$ is $\mathbb{A}^1$-invariant meaning,

$$\mathrm{Hom}\left(Y \times \mathbb{A}^1, \Omega\mathscr{X}\right) = \mathrm{Hom}\left(Y \times \mathbb{A}^1, \Omega\mathscr{X}\right)$$

Indeed,

$$\mathrm{Hom}\left(Y \times \mathbb{A}^1, \Omega\mathscr{X}\right) = \mathrm{Hom}\left(\Sigma(Y \times \mathbb{A}^1), \mathscr{X}\right) = \mathrm{Hom}\left(\mathcal{L}_{\mathrm{mot}}\Sigma(Y \times \mathbb{A}^1), \mathscr{X}\right)$$

but $\mathcal{L}_{\mathrm{mot}}\Sigma = \Sigma\mathcal{L}_{\mathrm{mot}}$ and hence $\mathcal{L}_{\mathrm{mot}}\Sigma(Y \times \mathbb{A}^1) = \Sigma\mathcal{L}_{\mathrm{mot}}(Y \times \mathbb{A}^1) = \Sigma\mathcal{L}_{\mathrm{mot}}Y$ and hence,

$$\mathrm{Hom}\left(Y \times \mathbb{A}^1, \Omega\mathscr{X}\right) = \mathrm{Hom}\left(\Sigma\mathcal{L}_{\mathrm{mot}}Y, \mathscr{X}\right) = \mathrm{Hom}\left(\mathcal{L}_{\mathrm{mot}}Y, \Omega\mathscr{X}\right) = \mathrm{Hom}\left(Y, \Omega\mathscr{X}\right)$$

□

Lemma 8.1.9 (8.8)**.** Let k be a perfect field, $f : Y \to X$ a morphism in $\mathrm{PSh}(\mathrm{Sm}_k)$ and $n \geq -1$. If f is $\mathbb{A}^1$-(n)-connected then $\mathrm{cofib}(f)$ is $\mathbb{A}^1$-$(n+1)$-connected. The coverse holds if X and Y are $\mathbb{A}^1$-(1)-connected.

Proof. Since $\mathcal{L}_{\mathrm{mot}}$ is a left-adjoint, it preserves homotopy colimits. Thus $\mathcal{L}_{\mathrm{mot}}(\mathrm{cofib}\, f) = \mathrm{cofib}(\mathcal{L}_{\mathrm{mot}} f)$ and hence we can assume that X, Y are motivic. Then it reduces to the corresponding lemma for ordinary spaces □

8.2 Connectivity Theorem

(FIX THIS COMPLETELY)

Theorem 8.2.1 (Mor12, 1.18)**.** Let $\mathscr{X}$ be a pointed presheaf and $n \geq 0$ an integer. If $\mathscr{X}$ is n-connected then it is $\mathbb{A}^1$-(n)-connected meaning $\pi_i^{\mathbb{A}^1}(\mathscr{X}) = \pi_i(\mathcal{L}_{\mathrm{mot}}(\mathscr{X})) = 0$ for $i \leq n$.

Theorem 8.2.2 (Mor12, 6.56)**.** Let $f : \mathscr{X} \to \mathcal{Y}$ be a morphism with $\mathcal{Y}$ pointed and 0-connected. Assume that the sheaf of groups $\pi_0^{\mathbb{A}^1}(\Omega^1(\mathcal{Y})) = \pi_0(\mathcal{L}_{\mathrm{mot}}(\Omega^1(\mathcal{Y})))$ is strongly $\mathbb{A}^1$-invariant (WHAT DOES IT MEAN). Let $n \geq 1$ be an integer and assume f is $(n-1)$-connected, then $\mathcal{L}_{\mathrm{mot}}(f)$ is also $(n-1)$-connected.

Remark. Notice that $\mathcal{L}_{\mathbb{A}^1}$ may not preserve Ω^1 because $\mathcal{L}_{\mathbb{A}^1}$ is a left adjoint but not a right adjoint. Therefore $\pi_i^{\mathbb{A}^1}(\Omega(\mathcal{Y}))$ need not equal $\pi_{i+1}(\mathcal{Y})$.

8.3 Purity

Definition 8.3.1. The Thom space of a vector bundle $\mathcal{E}$ over X is,

$$\mathrm{Th}(\mathcal{E}) = \mathbb{V}(\mathcal{E})/(\mathbb{V}(\mathcal{E})\backslash\iota(X))$$

where $\mathbb{V}(\mathcal{E})$ is the total space and $\iota : X \to \mathbb{V}(\mathcal{E})$ is the zero section.

Lemma 8.3.2. Consider the natural closed embedding $\mathbb{P}_X(\mathcal{E}) \to \mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X)$. Then the canonical morphism of pointed sheaves,

$$\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)/\mathbb{P}(\mathcal{E}) \to \mathrm{Th}(\mathcal{E})$$

is an $\mathbb{A}^1$-equivalence.

Proof. Consider the open covering, $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) = \mathbb{V}_X(\mathcal{E}) \cup (\mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \backslash X)$ which gives a gluing diagram,

$$\begin{tikzcd}
\mathbb{V}_X(\mathcal{E}) \arrow[r] & \mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \\
\mathbb{V}_X(\mathcal{E}) \backslash X \arrow[r] \arrow[u] & \mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \backslash X \arrow[u]
\end{tikzcd}$$

and hence a homotopy pushout. Therefore, we complete it to,

$$\begin{tikzcd}
\mathbb{V}_X(\mathcal{E}) \arrow[r] & \mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \arrow[r] & \mathrm{Th}(\mathcal{E}) \\
\mathbb{V}_X(\mathcal{E}) \backslash X \arrow[r] \arrow[u] & \mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \backslash X \arrow[u] \arrow[r] & * \arrow[u]
\end{tikzcd}$$

where each square is a homotopy pushout so,

$$\mathrm{Th}(\mathcal{E}) = \frac{\mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X)}{\mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \backslash X}$$

Therefore, since the map $\mathbb{P}_X(\mathcal{E}) \to \mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \to \mathrm{Th}(\mathcal{E})$ factors through $\mathbb{P}_X(\mathcal{E} \oplus \mathcal{O}_X) \backslash X$ giving a map,

$$\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) / \mathbb{P}(\mathcal{E}) \to \mathrm{Th}(\mathcal{E})$$

It suffices to show that $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \backslash X$ is an $\mathbb{A}^1$-equivalence. Indeed, it is the embedding of $\mathbb{P}(\mathcal{E})$ as the zero section of the total space of the tautological bundle on $\mathbb{P}(\mathcal{E})$ so it is a vector bundle over $\mathbb{P}_X(\mathcal{E})$ and hence an $\mathbb{A}^1$-equivalence. $\square$

Lemma 8.3.3. If $Z \subset X$ is a closed subscheme and Z, X are smooth k-schemes then,

$$X/(X \backslash Z) \cong_{\mathrm{mot}} \mathrm{Th}(\mathcal{N}_{Z|X})$$

Proof. DO THIS ONE!!! $\square$

Example 8.3.4. If $Z = \{p\}$ is a point then this shows that,

$$X/(X \backslash \{p\}) \cong_{\mathrm{mot}} \mathbb{A}^d / (\mathbb{A}^d \backslash \{0\})$$

where $d = \dim X$.

Lemma 8.3.5. Let $\mathcal{E}$ be a vector bundle of rank r. Then $\Sigma \mathrm{Th}(\mathcal{E})$ is $\mathbb{A}^1$-(r)-connected.

Proof. DO THIS!!! $\square$

Remark. If $U \to X$ is an open immersion of schemes then $\mathrm{cofib}(U \to X) = X/U$. Indeed this holds more generally ... (DO THIS).

Lemma 8.3.6 (8.9)**.** Let k be a perfect field, X a smooth k-scheme, and $Z \subset X$ a closed subscheem of codimension $\geq r$. Then $\Sigma(X/(X \backslash Z))$ is $\mathbb{A}^1$-(r)-connected.

Proof. X is a disjoint union of quasi-compact smooth schemes so we may assume that X is quasi-compact. If Z is smooth then $X/(X\backslash Z)$ is $\mathbb{A}^1$-$(r-1)$-connected by purity.

Notice that I can assume Z is smooth because the sheaf represented by Z and Z_{red} on Sm_k are the same since a morphism from a reduced scheme factors through the reduction.

Since k is perfect, generic smoothness gives a filtration,

$$\varnothing = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = Z$$

of closed subschemes such that $Z_j \backslash Z_{j-1}$ is smooth. We prove the result by induction on n. Consider the cofiber sequence,

$$\frac{X\backslash Z_{n-1}}{X\backslash Z} \to \frac{X}{X\backslash Z} \to \frac{X}{X\backslash Z_{n-1}}$$

Therefore because homotopy colimits commute we get a cofiber sequence,

$$\Sigma\left(\frac{X\backslash Z_{n-1}}{X\backslash Z}\right) \to \Sigma\left(\frac{X}{X\backslash Z}\right) \to \Sigma\left(\frac{X}{X\backslash Z_{n-1}}\right)$$

by the induction hypothesis $X/(X\backslash Z_{n-1})$ is $\mathbb{A}^1$-(r)-connected (indeed it should be $\mathbb{A}^1$-$(r+1)$-connected because $\operatorname{codim}(Z_{n-1}, X) \geq r+1$ but we don't need this) and hence by Lemma 8.8 the morphism,

$$\Sigma\left(\frac{X\backslash Z_{n-1}}{X\backslash Z}\right) \to \Sigma\left(\frac{X}{X\backslash Z}\right)$$

is $\mathbb{A}^1$-$(r-1)$-connected. Since, $X\backslash Z_{n-1}$ and $X\backslash Z$ are smooth, by the smooth case we know that $\Sigma((X\backslash Z_{n-1})/(X\backslash Z))$ is $\mathbb{A}^1$-(r)-connected and hence $\Sigma(X/(X\backslash Z))$ is $\mathbb{A}^1$-(r)-connected. $\square$

8.4 Lemmas

Definition 8.4.1. Let $\mathcal{Z}_d \to \mathrm{FFlat}_d$ be the universal finite locally free scheme of degree d and let $\mathcal{V}_d(\mathbb{A}^n)$ be the vector bundle over FFlat_d defined by,

$$\mathcal{V}_d(\mathbb{A}^n) = \mathrm{Hom}_{\mathrm{FFlat}_d}\left(\mathcal{Z}_d, \mathbb{A}^n_{\mathrm{FFlat}_d}\right)$$

Then, $\mathrm{Hilb}_d(\mathbb{A}^n) \hookrightarrow \mathcal{V}_d(\mathbb{A}^n)$ is the open substack of closed immersions $\mathcal{Z}_d \to \mathbb{A}^n_{\mathrm{FFlat}_d}$.

Remark. Since $\mathcal{Z}_d \to \mathrm{FFlat}_d$ and $\mathbb{A}^n_d \to \mathrm{FFlat}_d$ are both relative schemes $\mathrm{Hom}_{\mathrm{FFlat}_d}\left(\mathcal{Z}_d, \mathbb{A}^n_{\mathrm{FFlat}_d}\right)$ is automatically a stack.

Lemma 8.4.2. The closed complement of $\mathrm{Hilb}_d(\mathbb{A}^n)$ in $\mathcal{V}_d(\mathbb{A}^n)$ has codimension at least $n-d+2$ in every fiber over FFlat_d.

Proof. Let k be a field and $S = \mathrm{Spec}\,(R)$ for a finite k-scheme of degree d. We need to show that the closed cimplement of $\mathrm{Emb}_k(S, \mathbb{A}^n_k)$ in $\mathrm{Hom}_k\,(S, \mathbb{A}^n_k) \cong \mathbb{A}^{nd}_k$ has codimension at least $n-d+2$. A k-morphism $S \to \mathbb{A}^n_k$ can be viewed as a k-algebra map,

$$k[x_1, \dots, x_n] \to R$$

which is an embedding iff it is surjective. It suffices for the images of $x_1, \dots, x_n$ to generate $R/(k\cdot 1)$ as a k-module. This is the open set characterized by where a map $k^n \to k^r$ (with $r = d-1$) is surjective whose codimension (in the space of matrices) is $n-r+1 = n-d+2$. Indeed, being nonsurjective is equivalent to having its image lie in some hyperplane $H \in \mathbb{P}^{r-1}$ so has dimension $n(r-1)+(r-1) = nr-(n-r+1)$ corresponding to a surjective map $k^n \to H$ and a choice of H. $\square$

Remark. This bound is optimal when $n \geq d-2$. The worse case is for the square-zero extension algebra,

$$R = k[x_1, \dots, x_{d-1}]/(x_i x_j)$$

in which case a map $k[x_1, \dots, x_n] \to R$ is surjective iff $x_1, \dots, x_n$ generate $R/(k \cdot 1)$ as a k-module. Therefore, the non-surjective maps $k[x_1, \dots, x_n] \to R$ have codimension exactly $n - d + 2$.

Remark. For our purposes we say that a subpresheaf $U \subset X$ is *open* if the map $U \to X$ is representable by schemes and for each scheme $T \to X$ we have that $U \times_X T \to T$ is an open immersion of schemes.

Proposition 8.4.3 (8.10)**.** Let k be a perfect field and $X \in \mathrm{PSh}(\mathrm{Sm}_k)$ a presheaf. Let $V \to X$ a vector bundle, and $r \geq 0$. Let $U \subset V$ be an open subpresheaf such that, for each finite field extension k'/k and $\alpha \in X(k')$ the closed complement of $\alpha^*(U)$ in $\alpha^*(V)$ has codimension at least r. Then, $\Sigma^2 U \to \Sigma^2 X$ is $\mathbb{A}^1$-(r)-connected. If U and X are $\mathbb{A}^1$-connected, then the morphism $\Sigma U \to \Sigma X$ is $\mathbb{A}^1$-$(r-1)$-connected.

Proof. By Lemma 8.8, it suffices to show that $\Sigma\,\mathrm{cofib}(U \to X)$ is $\mathbb{A}^1$-(r)-connected. Colimits of pointed objects preserve connectivity (WHY), hence $\mathbb{A}^1$-connectivity by the $\mathbb{A}^1$-connectivity theorem, so we are reduced by universality of colimits to the case $X \in \mathrm{Sm}_k$ since every object is a colimit of representables. Since $V \to X$ is an $\mathbb{A}^1$-equivalence (because it is a vector bundle) and $V \backslash U$ has codimension $\geq r$ in V the result follows from Lemma 8.9. $\square$

Lemma 8.4.4 (8.11)**.** Let k be a field and $n \geq d-1 \geq 0$. Then $\mathrm{Hilb}_d(\mathbb{A}^n_k)$ is $\mathbb{A}^1$-connected.

Proof. It suffices to show that $\mathrm{Hilb}_d(\mathbb{A}^n)(k)$ is nonempty and that points are connected by $\mathbb{A}^1$-curves. Explicitly, for every separable finitely generated field extension F/k we say that $x, y \in \mathrm{Hilb}_d(\mathbb{A}^n)(F)$ are connected if there is a map $\mathbb{A}^1_F \to \mathrm{Hilb}_d(\mathbb{A}^n)_F$ such that $0 \mapsto x$ and $1 \mapsto y$. We need to show that all F-points are connected in a chain menaing equivalent by the equivalence relation generated by being connected. By some results of Morel this proves $\mathbb{A}^1$-connectedness (it certainlly does in a moral sense).

For $\mathbb{A}^n_F = \mathrm{Spec}\,(F[x_1, \dots, x_n])$. Consider an F-point $[A] \in \mathrm{Hilb}_d(\mathbb{A}^n)(F)$ corresponding to a surjection $\pi : F[x_1, \dots, x_n] \twoheadrightarrow A$. We claim that π can be connected by a chain to a surjection such that the images of $1, x_1, \dots, x_{d-1}$ are linearly independent. Indeed, otherwise there is some x_i in the span of $1, x_1, \dots, x_{i-1}$ then,

$$\pi|_i : F[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n] \twoheadrightarrow A$$

is a surjection so we consider the $F[t]$-algebra map $\rho : F[x_1, \dots, x_n, t] \to A[t]$ defined by $\rho(x_j) = \pi(x_j)$ for $j \neq i$ and $\rho(x_i) = ta + (1-t)\pi(x_i)$ where $a \in A$ is some element not in the span of $1, x_1, \dots, x_{i-1}$. This defines a homotopy between π and $\pi' = \rho(1)$ where the images of $1, x_1, \dots, x_i$ are linearly independent. Since $\dim_F(A) = d$ this will make $1, x_1, \dots, x_{d-1}$ span A and then we use another homotopy to set $\pi(x_j) = 0$ for $j \geq d$ just by scaling $\pi(x_j)$ by t.

Now we use the Rees algebra,

$$\mathrm{Rees}(F) := F \oplus tA[t] \subset A[t]$$

which is finite locally free over $F[t]$. Define an $F[t]$-algebra map $\tilde{\pi} : F[x_1, \dots, x_n, t] \to \mathrm{Rees}(F)$ by $\tilde{\pi}(x_i) = t\pi(x_i)$ for all i. Since we can assume $1, x_1, \dots, x_{d-1}$ span A as an F-module, te image of $\tilde{\pi}$ contains t and tA and so is surjective. Geometrically $\tilde{\pi}$ corresponds to the morphism

$$\tilde{\pi} : \mathbb{A}^1_F \to \mathrm{Hilb}_d(\mathbb{A}^n)$$

that links $[A] = \tilde{\pi}(1)$ with $\tilde{\pi}(0)$ and $\mathrm{Rees}(F)/(t) = F \oplus F^{d-1}$ is the square-zero extension equppied with the canonical map $\tilde{\pi}(0) : F[x_1, \ldots, x_n] \to \mathrm{Rees}(F)/(t)$ given by sending $x_1, \ldots, x_{d-1}$ to the standard basis of F^{d-1} and $x_j \mapsto 0$ for $j \geq d$. This is independent of $[A]$ so we conclude that all points are chain connected. $\square$

8.5 Proof of the Main Theorem

Since the coclusions are preserved by aessentially smooth base change (WHAT DOES THIS MEAN) we may replace k be a perfect subfield.

Recall that the forgetful map $\mathrm{Hilb}_d(\mathbb{A}^\infty) \to \mathrm{FFlat}_d$ is a motivic equivalence. Note that both $\mathrm{Hilb}_d(\mathbb{A}^n)$ and FFlat_d are $\mathbb{A}^1$-connected by Lemma 8.11. Then we apply Prop 8.10 to the inclusion, $\mathrm{Hilb}_d(\mathbb{A}^n) \hookrightarrow \mathcal{V}_d(\mathbb{A}^n) \to \mathrm{FFlat}_d$ to conclude that $\Sigma\mathrm{Hilb}_d(\mathbb{A}^n) \to \Sigma\mathrm{FFlat}_d$ is $\mathbb{A}^1$-$(n-d-1)$-connected since the codimension of the complement inside $\mathcal{V}_d(\mathbb{A}^n)$ is $n - d + 2$.

Remark. Note that the assumptions of Prop 8.7 and 8.10 are preserved by any base change $X' \to X$. For example, by considering the open substack $\mathrm{FSyn}_d \subset \mathrm{FFlat}_d$ of syntomic finite flat covers, we see that Theorem 8.2 holds with Hilb_d replaced by $\mathrm{Hilb}_d^{\mathrm{lci}}$.