Mathematics GR6657 Algebraic Number Theory
Assignment # 4

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1. Consider the standard free resolution of $\mathbb{Z}$ given by,

$$\cdots \longrightarrow \Lambda_i \xrightarrow{\delta_i} \Lambda_{i-1} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_2} \Lambda_1 \xrightarrow{\delta_1} \mathbb{Z} \longrightarrow 0$$

where $\Lambda_i = \mathbb{Z}[G^{i+1}]$ and $\delta_i : \Lambda_i \to \Lambda_{i-1}$ is given by,

$$\delta_i(g_0, g_1, \dots, g_i) = \sum_{j=0}^{i} (-1)^j (g_0, \dots, \hat{g}_j, \dots, g_i)$$

First we need to show that this sequence is a complex. Take any $(g_0, \cdots, g_i) \in \Lambda_i$ then consider the composition $\delta_{i-1} \circ \delta_i(g_0, \dots, g_i) = 0$, because the composition of the two boundary maps will remove each pair of positions but in two different ways. Either, δ_i removes the first of the pair or δ_{i-1} removes the first. These two options form the same term but with opposite sign because if δ_i removes the first of the pair then the index of the second is shifted by one so there is a relative minus sign between the two terms. Therefore, the sum is zero. Explicitly,

$$\begin{aligned}
\delta_{i-1} \circ \delta_i(g_0, \dots, g_i) &= \sum_{k<j} (-1)^{j+k} (g_0, \dots, \hat{g}_k, \dots, \hat{g}_j, \dots, g_i) + \sum_{k \ge j} (-1)^{j+k} (g_0, \dots, \hat{g}_j, \dots, \hat{g}_{k+1}, \dots, g_i) \\
&= \sum_{k<j} (-1)^{j+k} (g_0, \dots, \hat{g}_k, \dots, \hat{g}_j, \dots, g_i) - \sum_{k' \ge j} (-1)^{j+k'} (g_0, \dots, \hat{g}_j, \dots, \hat{g}_{k'}, \dots, g_i) \\
&= 0
\end{aligned}$$

Therefore $\mathrm{Im}(\delta_i) \subset \ker \delta_{i-1}$ and the resolution forms a complex. Furhermore, define the chain homotopy, $h_i : \Lambda_{i-1} \to \Lambda_i$ by the map,

$$h_i(g_1, \dots, g_i) = (1, g_1, \dots, g_i)$$

Now, consider the composition,

$$\begin{aligned}
\delta_i \circ h_i(g_1, \dots, g_i) &= (g_1, \dots, g_i) + \sum_{j=1}^{i} (-1)^j (1, g_1, \dots, \hat{g}_j, \dots, g_i) \\
&= (g_1, \dots, g_i) + \sum_{j=1}^{i} (-1)^j h_{i-1}(g_1, \dots, \hat{g}_j, \dots, g_i) \\
&= (g_1, \dots, g_i) - h_{i-1} \left(\sum_{j'=0}^{i} (-1)^{j'} (g_1, \dots, \hat{g}_{j'+1}, \dots, g_i) \right) \\
&= (\mathrm{id} - h_{i-1} \circ \delta_{i-1})(g_1, \dots, g_i)
\end{aligned}$$

Therefore, take any cycle $X \in \ker \delta_{i-1}$ then $\delta_i \circ h_i(X) = X - h_{i-1} \circ \delta_{i-1}(X) = X$ so $X \in \mathrm{Im}(\delta_i)$ therefore X is a boundary. Thus, $\ker \delta_i \subset \mathrm{Im}(\delta_i)$ so in total $\ker \delta_i = \mathrm{Im}(\delta_i)$. Therefore, the free resolution is exact.

2. Let G be a group and A a G-module. Define the group of homogeneous i-cochains,

$$C^i_{\text{hom}}(G, A) = \{f : G^{i+1} \to A \mid f(gX) = gf(X) \forall X \in G^{i+1}\} \cong \mathrm{Hom}_G(\Lambda_i, A)$$

and the group of inhomogeneous i-cochains,

$$C^i_{\text{in}}(G, A) = \{f : G^i \to A\}$$

There is a bijection $F_i : C^i_{\text{hom}}(G, A) \to C^i_{\text{in}}(G, A)$ given by, $F_i : f \mapsto \phi$ such that

$$\phi(g_1, \cdots, g_i) = f(1, g_1, g_1 g_2, \ldots, g_1 \cdots g_i)$$

Now, consider the homogeneous differential,

$$d^i_{\text{hom}} : C^i_{\text{hom}}(G, A) \to C^{i+1}_{\text{hom}}(G, A)$$

given by sending $d^i_{\text{hom}} : f \mapsto f \circ \delta_{i+1}$. Consider, $F_{i+1} \circ d^i_{\text{hom}}(f) = F_{i+1}(f \circ \delta_{i+1})$. This map is a inhomogeneous $i+1$-cochain acting as,

$$\begin{aligned}
F_{i+1}(f \circ \delta_{i+1})(g_1, \ldots, g_{i+1}) &= (f \circ \delta_{i+1})(1, g_1, g_1 g_2, \ldots, g_1 \cdots g_{i+1}) \\
&= \sum_{j=0}^{i+1} (-1)^j f(1, g_1, g_1 g_2, \ldots, \widehat{g_1 g_2 \cdots g_j}, \ldots, g_1 \cdots g_{i+1}) \\
&= f(g_1, g_1 g_2, \ldots, g_1 \cdots g_{i+1}) \\
&\qquad + \sum_{j=1}^{i+1} (-1)^j f(1, g_1, g_1 g_2, \ldots, \widehat{g_1 g_2 \cdots g_j}, \ldots, g_1 \cdots g_{i+1})
\end{aligned}$$

However, for $j < i+1$,

$$\begin{aligned}
f(1, g_1, g_1 g_2, \ldots, &\widehat{g_1 g_2 \cdots g_j}, \ldots, g_1 \cdots g_{i+1}) \\
&= f(1, g_1, g_1 g_2, \ldots, g_1 \ldots g_{j-1}, g_1 \ldots g_{j-1}(g_k g_{j+1}), \ldots, g_1 \cdots g_{i+1}) \\
&= \phi(g_1, g_2, \ldots, g_j g_{j+1}, \ldots g_{i+1})
\end{aligned}$$

Thus,

$$\begin{aligned}
F_{i+1}(f \circ \delta_{i+1})(g_1, \ldots, g_{i+1}) &= g_1 \cdot f(1, g_2, \ldots, g_2 \cdots g_{i+1}) + \sum_{j=1}^{i} (-1)^j \phi(g_1, g_2, \ldots, g_j g_{j+1}, \ldots g_{i+1}) \\
&\quad + (-1)^{i+1} f(1, g_1, g_1 g_2, \ldots, g_1 \ldots g_i) \\
&= g_1 \cdot \phi(g_2, \ldots, g_{i+1}) \\
&\qquad + \sum_{j=1}^{i} (-1)^j \phi(g_1, g_2, \ldots, g_j g_{j+1}, \ldots g_{i+1}) + (-1)^{i+1} \phi(g_1, \ldots, g_i)
\end{aligned}$$

which is the formula for the inhomogeneous differential,

$$d^i : C^i_{\text{in}}(G, A) \to C^{i+1}_{\text{in}}(G, A)$$

Therefore, $d^i_{\text{in}} = F_{i+1} \circ d^i_{\text{hom}}$

3. Let G be the trivial group. We know that $(-)^G$ is an equivalent functor to $\operatorname{Hom}_G(\mathbb{Z}, -)$ which is left exact. Furthermore, for any G-module A, the cohomology $H^i(G, -)$ is isomorphic to the derived functor of $\operatorname{Hom}_G(\mathbb{Z}, -)$. However, since G is trivial, $M^G \cong M$ and therefore $\operatorname{Hom}_G(\mathbb{Z}, -)$ is the identity functor. Therefore, choosing any injective resolution of A,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

the complex obtained by applying the functor $\operatorname{Hom}_G(\mathbb{Z}, -)$,

$$0 \longrightarrow \operatorname{Hom}_G(\mathbb{Z}, A) \longrightarrow \operatorname{Hom}_G(\mathbb{Z}, I^0) \longrightarrow \operatorname{Hom}_G(\mathbb{Z}, I^1) \longrightarrow \operatorname{Hom}_G(\mathbb{Z}, I^2) \longrightarrow \cdots$$

remains exact because we have done nothing by applying the identity functor. Therefore, the derived functor of $\operatorname{Hom}_G(\mathbb{Z}, -)$ is trivial because it is the cohomology of an exact sequence.