1 Topics

(a). Basic homotopy theory

(b). Obstruction theory

(c). Characteristic Classes

(d). The Serre spectral sequence

(e). The Steenrod operations

(f). K-theory

References: Fuchs - Fomenko: homotopical topology, Hatcher's books
Six homeworks (one per topic)

2 Homotopy Theory

Basic Questions:

(a). given maps $f, g : X \to Y$ are they homotopy equivalent?

(b). given spaces X and Y are they homotopy equivalent?

Remark. All spaces will be connected and locally connected.

Definition The set $[X, Y] = \mathrm{Hom}\,(\mathbf{hTop}, X)\, Y$. Given based spaces X, Y we define $\langle X, Y\rangle = \mathrm{Hom}\,(\mathbf{hTop}_\bullet, X)\, Y$ where morphisms in $\mathbf{hTop}_\bullet$ are continuous maps preserving the basepoint up to homotopy. Note that homotopies in $\mathbf{Top}_\bullet$ are basepoint preserving.

Example 2.1. Consider S^n. Given $f : S^n \to X$ we can construct, $X \sqcup_f D^{n+1}$ by gluing along f. This is the coproduct,

$$\begin{tikzcd} D^{n+1} \arrow[r] & X \sqcup_f D^{n+1} \\ S^n \arrow[u] \arrow[r, "f"] & X \arrow[u] \end{tikzcd}$$

Now if $f \sim f'$ then $X \sqcup_f D^{n+1} \sim X \sqcup_f D^{n+1}$.

Definition Given a based space (X, x_0) we define the n^{th} homotopy group,

$$\pi_n(X, x_0) = \langle (S^n, p_0), (X, x_0)\rangle$$

The group structure is given by the equator squeezing map $s : S^n \to S^n \vee S^n$. Then we define $f * g = (f \vee g) \circ s$.

Proposition 2.2. $\pi_n(X, x_0)$ is abelian for $n \geq 2$.

Theorem 2.3. $\pi_n(S^m) = 0$ if $n < m$.

Theorem 2.4. $\pi_n(S^n) = \mathbb{Z}$

Theorem 2.5. $\pi_3(S^2) = \mathbb{Z}$ generated by the Hopf fibration $\eta : S^3 \to S^2$.

Theorem 2.6. For sufficiently large n,

$$\pi_{n+1}(S^n) = \mathbb{Z}/2\mathbb{Z} \qquad \pi_{n+2}(S^n) = \mathbb{Z}/2\mathbb{Z} \qquad \pi_{n+3}(S^3) = \mathbb{Z}/24\mathbb{Z}$$

Remark. Given $f : X \to Y$ we get $f_* : \pi_n(X) \to \pi_n(Y)$.

Theorem 2.7. Given a path $\gamma : x_1 \to x_2$ in X we get a map,

$$\gamma_\# : \pi_n(X, x_1) \to \pi_n(X, x_2)$$

depending only on the homotopy class of γ. In particular we have a $\pi_1(X, x_0)$-action on $\pi_n(X, x_0)$.

Remark. In the case $n = 1$ this is the conjugation action of $\pi_1(X, x_0)$ on itself.

Proposition 2.8. Given the previous proposition, we have,

$$[S^n, X] = \pi_n(X, x_0)/\pi_1(X, x_0)$$

Proposition 2.9. If $p : \tilde{X} \to X$ is a covering map then for $n \geq 2$ the induced map,

$$p_* : \pi_n(\tilde{X}) \to \pi_1(X)$$

is an isomorphism.

Proof. Injectivity is the homotopy lifting property. Furthermore given $f : S^n \to X$ we can lift it to $\tilde{f} : S^n \to \tilde{X}$ provided that $f_*(\pi_1(S^n)) \subset p_*(\pi_1(\tilde{X}))$. In the case $n \geq 2$, we have $\pi_1(S^n)$ thus such a lift always exists proving surjectivity. □

Example 2.10. Let Σ_g be a genus g surface. For $g \geq 1$ then Σ_g has universal cover $\mathbb{R}^2$ which is contractible and thus $\pi_n(\Sigma_g) = \pi_n(\mathbb{R}^2) = 0$ for $n \geq 2$.

Example 2.11. For $n \geq 2$ we have $\pi_n(\mathbb{RP}^k) = \pi_n(S^k)$.

2.1 Basic Operations on Spaces

Definition The suspension of X is $\Sigma X = X \vee S^1$.

Definition The loops space of X is $\Omega X = \mathrm{Hom}\left(\mathbf{Top}_\bullet, S^1\right) X$ with the compact-open topology.

Theorem 2.12 (Adjunction)**.**

$$\langle \Sigma X, Y \rangle = \langle X, \Omega Y \rangle$$

Example 2.13. $\Sigma S^n = S^{n+1}$

Proposition 2.14. $\pi_{n+1}(Y) = \langle S^{n+1}, Y \rangle = \langle \Sigma S^n, Y \rangle = \langle S^n, \Omega Y \rangle = \pi_n(\Omega Y)$

Proposition 2.15. The space ΩX is a group object in the category $\mathbf{hTop}_\bullet$.

Remark. The following definition is due to Hatcher.

Definition A pointed space (X, e, μ) is an H-space is there is a map $\mu : X \times X \to X$ such that $\mu(-, e) \sim \mathrm{id}$ and $\mu(e, -) \sim \mathrm{id}$ as pointed maps (relative to the basepoint).

Remark. Any topological group (group object in **Top**) is an H-space (pointed at the identity element).

Remark. Loop spaces are H-spaces since they are group objects in $\mathbf{hTop}_\bullet$.

Theorem 2.16 (Adams)**.** The spheres S^n admitting an H-space structure are exactly S^0, S^1, S^3, S^7.

Corollary 2.17. $\mathbb{R}^n$ has a unital division $\mathbb{R}$-algebra structure iff $n = 1, 2, 4, 8$.

Proof. Consider the unit length elements $U = S^{n-1}$. Then a division algebra on $\mathbb{R}^n$ gives a multiplication $U \times U \to U$ (well defined since $xy = 0 \implies x = 0$ or $y = 0$ and thus the result can be scalled to lie in U). □

3 Relative Groups

Definition Given a space X a subspace $A \subset X$ and a point $x_0 \in A$ we denote the pointed pair as (X, A, x_0).

Definition For a pointed pair (X, A, x_0) we define $\pi_n(X, A, x_0)$ as maps,

$$f : (D^n, S^{n-1}, p_0) \to (X, A, x_0)$$

modulo homotopy through maps of this form.

Remark. Suppose $[f] \in \pi_n(X, A, x_0)$ is zero if it is homotopic to a map with image inside A. In fact if this is the case then f may be homotoped relative to the boundary. Compression Lemma.

Theorem 3.1. There is a long exact sequence for the pointed pair (X, A, x_0),

$$\cdots \longrightarrow \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0) \longrightarrow \pi_{n-1}(X, x_0) \longrightarrow \pi_{n-1}(X, A, x_0) \longrightarrow \cdots$$

4 Results on CW Complexes

Definition A CW pair is a CW complex X with a subcomplex $A \subset X$ (a closed subset which is a cunion of cells e.g.X^k the k-skelleton).

Theorem 4.1 (homotopy extension)**.** Let (X, A) be a CW pair. Then (X, A) has the homotopy extension property i.e. $\iota : A \to X$ is a cofibration.

Proof. Working cell-by-cell we can reduce to the case $(X, A) = (D^n, S^{n-1})$. In this case we are given a map on $D^n \times \{0\} \cup S^{n-1} \times I$ which is a deformation retract of $D^n \times I$ so any map can be extended. □

Definition A map $f : X \to Y$ between CW complexes is *cellular* if $f(X^k) \subset Y^k$.

Theorem 4.2 (cellular approximation)**.** Any map $f : X \to Y$ of CW complexes is homotopic to a cellular map.

Corollary 4.3. If $n < m$ then $\pi_n(S^m) = 0$.

Theorem 4.4. If $\pi_i(X, x_0) = 0$ for $i \leq n$ (i.e. X is n-connected) then X is homotopic to a CW complex with a single zero 0-cell and no i-cells for $1 \leq i \leq n$.

Lemma 4.5. If (X, A) is a CW-pair and A is contractible then $X \to X/A$ is a homotopy equivalence.

5 More Results on CW Complexes (01/29)

Theorem 5.1 (Whitehead)**.** Let $f : X \to Y$ be a map of CW complexes such that $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for each n then f is a homotopy equivalence.

Example 5.2. If $\pi_n(X, x_0) = 0$ for all $n \geq 0$ and X is a CW complex then X is contractible. To see this consider the constant map $X \to *$.

Example 5.3. Consider $S^\infty = \varinjlim S^n$ where we consider $S^n \subset S^{n+1}$ as the equator. Then $\pi_n(S^\infty) = 0$ since any map $S^n \to S^\infty$ can be deformed to a point using the copy of S^{n+1}. Thus S^∞ is contractible.

Remark. In Whitehead's theorem, simply knowing $\pi_n(X) \cong \pi_n(Y)$ for each $n \geq 0$ does not imply $X \sim Y$ we need these isomorphisms to be induced by a single topological map $f : X \to Y$.

Example 5.4. Quotienting by the natural involution on S^∞ we get a double cover $p : S^\infty \to \mathbb{RP}^\infty$. Using covering theory we find,

$$\pi_n(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & n > 1 \end{cases}$$

Furthermore, consider $X = S^2 \times \mathbb{RP}^\infty$ whose universal cover is $\tilde{X} = S^2 \times S^\infty \sim S^2$ and thus,

$$\pi_n(X) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n > 1 \end{cases}$$

This has exactly the same homotopy groups as $Y = \mathbb{RP}^2$ whose universal vover is also $\tilde{X} = S^2$ and also has a two-fold cover. However, $H_*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$ is finite dimensional and $H_*(S^2 \times \mathbb{RP}^\infty, \mathbb{Z}/2\mathbb{Z})$ is infinite dimensional so they cannot be homotopy equivalent.

Definition The mapping cylinder of a morphism $f : X \to Y$ is the pushout,

$$Mf = Y \coprod_f (X \times I)$$

There is a natural inclusion $\iota : X \hookrightarrow Mf$ and a deformation retract $j : Mf \to Y$.

Remark. If X and Y are CW complexes then we may homotope $f : X \to Y$ to a cellular map in which case Mf is a CW complex and $\iota : X \hookrightarrow M(f)$ makes (Mf, X) a CW pair.

Definition If X and Y are any spaces $f : X \to Y$ is a *weak homotopy equivalence* if $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n \geq 0$.

Theorem 5.5. Any space is weakly homotopy equivalent to a CW complex.

Remark. Suspension is a functor: given $f : X \to Y$ we get $\Sigma f : \Sigma X \to \Sigma Y$ given by $\Sigma f(t, x) = (t, f(x))$.

Remark. The unit of the suspension-looping adjunction gives a map $X \to \Omega\Sigma X$ given by $x \mapsto (t \mapsto (t, x))$. Applying the functor π_n gives the Freudenthal map $\sigma_n : \pi_n(X) \to \pi_{n+1}(\Sigma X)$.

Theorem 5.6 (Freudenthal Suspension)**.** Let X be an n-connected pointed space. Then the Freudenthal map $\Sigma_k : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism if $k \leq 2n$ and an epimorphism if $k = 2n + 1$.

Corollary 5.7. $\pi_n(S^n) = \mathbb{Z}$.

Proof. We show this by induction. For $n = 1$ the result $\pi_1(S^1) = \mathbb{Z}$ is a simple application of covering space theory. Now we assume the result for S^n. Then since S^n is $(n-1)$-connected, by the Fruedenthal suspension theorem we get an isomorphism $\pi_k(S^n) \xrightarrow{\sim} \pi_{k+1}(S^{n+1})$ for $k < 2n - 1$. Setting $k = n$ we see that $\pi_{n+1}(S^{n+1}) \cong \pi_n(S^n)$ for $n > 1$. However, for the case $n = 1$ we only get an epimorphism $\pi_1(S^1) \to \pi_2(S^2)$ since $1 = 2 - 1$. However, there is a surjective degree map $\pi_2(S^2) \to \mathbb{Z}$ and thus $\pi_2(S^2) = \mathbb{Z}$. □

6 Spectra

Definition A spectrum is a sequence X_n of CW complexes along with structure maps $s_n : \Sigma X_n \to X_{n+1}$.

Definition Let X be a spectrum then we define the homotopy groups of X via,

$$\pi_k(X) = \varinjlim_n \pi_{k+n}(X_n)$$

where the maps $\Sigma X_n \to X_{n+1}$ induce $\pi_{k+n}(X_n) \to \pi_{k+n+1}(X_{n+1})$ by adjunction making the groups $\pi_{k+n}(X_n)$ a directed system.

Remark. Spectra may have homotopy in negative dimension i.e. $\pi_k(X) \neq 0$ for $k \leq 0$ in general.

Definition We say a spectrum is stable if the structure maps are eventually all weak homotopy equivalences.

Example 6.1. Given a CW complex X we can form the suspension specturm $X_n = \Sigma^n X = S^n \wedge X$ with identity maps $\Sigma X_n \to X_{n+1}$. This is clearly a stable spectrum.

Example 6.2. The suspension spectrum of S^0 is the sphere spectrum $\mathbf{S}$ given by $\mathbf{S}_n = S^n$ with the natural homeomorphisms $\Sigma S^n \to S^{n+1}$.

Definition An Ω-spectrum is a specturm X such that the adjunction of the structue map $X_n \to \Omega X_{n+1}$ is a weak homotopy equivalence.

7 Feb 12

Theorem 7.1. Two CW complexes of type $K(G, n)$ are homotopy equivalent.

Proof. Let X, Y be CW complexes. Assume that X has no $1, \dots, (n-1)$-cells (since it is $(n-1)$-connected) and one 0-cell (since it is connected). Then,

$$X^n = \bigvee_{i \in I} S^n$$

each of these spheres represents an element $\pi_n(X) = G$. Construct $f_n : X^n \to Y$ by sending each S^n to the corresponding element in $\pi_n(Y) = G$. Next construct $f_{n+1} : X^{n+1} \to Y$ so that each $\partial D^{n+1} = S^n \xrightarrow{f_n} Y$ represents $0 \in \pi_n(Y)$ (since the $(n+1)$-cells give the relations on G) then $\partial D^{n+2} = S^{n+1} \xrightarrow{f_{n+1}} Y$ is nullhomotopic because $\pi_{n+1}(Y) = 0$. Repeating, we can extend to all X. □

Remark. Key point: $\pi_n(X)$ is generated by n-cells and has relations by $(n+1)$-cells. This is a first glimpse of obstruction theory. We ask the following questions:

Q1 Given a CW pair (X, A) and $f : A \to Y$ can we extend this to $\tilde{f} : X \to Y$?

Q2 Given a giber bundle $p : E \to B$ and a map $f : X \to B$ can we lift it to $\tilde{f} : X \to E$?

For Q1, assume that $\pi_1(Y) \circlearrowright \pi_n(Y)$ trivially (i.e. Y is simple so we need not worry about base-points!). Given $f : X^n \to Y$ can we extend it to X^{n+1}? Gluing a disk D^{n+1} then f extends to D^{n+1} iff $f|_{S^n} : S^n \to Y$ is nullhomotopic i.e. is zero in $\pi_n(Y)$. In general, to each $(n+1)$-cell e, $[f_e] \in \pi_n(Y)$ then we can construct $c_f \in C^{n+1}(X, \pi_n(Y))$ a cellular cochain called the obstruction cochain. Then f extends to $X^{n+1} \iff c_f = 0$.

Lemma 7.2. $\delta c_f = 0$ i.e. c_f is a cocycle. Therefore, $O_f := [c_f] \in H^{n+1}(X; \pi_n(Y))$ is the obstructuon class.

Theorem 7.3. $f|_{X^{n-1}}$ extends to X^{n+1} iff $O_f = 0$.

Proof. First we prove the Lemma. Consider the diagram,

$$\begin{tikzcd}
C_{n+2}(X) \arrow[r, equal] \arrow[dd, "\partial"] & H_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, "h^{-1}"] & \pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[d, "\partial"] \\
& & \pi_{n+1}(X^{n+1}) \arrow[d, "\iota"] \\
C_{n+1}(X) \arrow[r, equal] \arrow[ddrr, "c_f"] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "h^{-1}"] & \pi_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial"] \\
& & \pi_n(X^n) \arrow[d, "f_*"] \\
& & \pi_n(Y)
\end{tikzcd}$$

The piece of the LES,

$$\pi_{n+1}(X^{n+1}) \longrightarrow \pi_{n+1}(X^{n+1}, X^n) \longrightarrow \pi_n(X^n)$$

composes to zero so by the commutativity of the above diagram $c_f \circ \partial = 0$. $\square$

Definition Suppose there are two maps $f, g : X^n \to Y$ that agree on X^{n-1} then for each n-cell D^n if we glue two D^n along the boundary on which f, g agree then we get a map $(f, g) : S^n \to Y$ and thus an element $\pi_n(Y)$ for each n-cell. This gives a difference cochain $d_{f,g} \in C^n(X; \pi_n(Y))$ and $d_{f,g} = 0$ iff $f, g : X^n \to Y$ are homotopic relative to X^{n+1}.

Lemma 7.4. $\delta d_{f,g} = c_g - c_f$.

Lemma 7.5. Given $f : X^n \to Y$ for any $d \in C^n(X; \pi_n(Y))$ there is $g : X^n \to Y$ with $f|_{X^{n-1}} = g|_{X^{n-1}}$ s.t. $d_{f,g} = d$.

Proof. For $d \in C^n(X; \pi_n(Y))$ then for an n-cell e we have $d(e) \in \pi_n(Y)$ then consider the sum of maps f and $d(e)$ using the sum structure on e contracting the equator. $\square$

Proof. Now we prove the theorem. Suppose that $O_f = 0$ then $c_f = \delta d$ for some $d \in C^n(X; \pi_n(Y))$. Now there exists $g : X^n \to Y$ with $f|_{X^{n-1}} = f|_{X^{n-1}}$ and $d_{f,g} = -d$. Also, $\delta d_{f,g} = c_g - c_f$ and thus $c_g = c_f + \delta d_{f,g} = c_f - \delta d = 0$ therefore $c_g = 0$ so g can extend to X^{n+1} and $f|_{X^{n-1}} = g|_{X^{n-1}}$. $\square$

Theorem 7.6. Let $f, g : X^n \to Y$ be maps with $f|_{X^{n-2}} = g|_{X^{n-2}}$. Then $[d_{f,g}] = 0$ iff they are homotopic relative to X^{n-2}.

7.1 Cohomology of $K(G, n)$

Let $n \geq 2$ and G abelian. Consider a map $f : X \to K(G, n)$. By Hurewicz, $H_n(K(G, n), \mathbb{Z}) = \pi_n(K(G, n)) = G$ and $H_{n-1}(K(G, n), \mathbb{Z}) = 0$. Now, by the universal coefficient theorem,

$$H^n(K(G, n), G) = \mathrm{Hom}\,(, H_n(K(G, n), \mathbb{Z}))\, G = \mathrm{Hom}\,(, G)\, G$$

Therefore, there is a canonical element $\mathbb{1} \in H^n(K(G, n), G)$ which is the class of $\mathrm{id} : G \to G$.

Also, via $f : X \to K(G, n)$, we also get $f^*(\mathbb{1}) \in H^n(X; G)$, which depends only on the homotopy class of f.

Theorem 7.7. The map $[X, K(G, n)] \to H^n(X, G)$ sending $[f] \mapsto f^\times(\mathbb{1})$ is an isomorphism.

Remark. We say that $K(G, n)$ classifies $H^n(-, G)$ meaning that the functor,

$$H^n(-, G) : \{\text{CW-complexes}\} \to \mathbf{Set}$$

is represented by $[-, K(G, n)]$.

Definition Given a contravariant functor $h : \{\text{CW-complexes}\} \to \mathbf{Set}$ we say that C classifies h if there is a natural isomorphism $h \cong [-, C]$ in this case we say that h is representable and the pair $(C, \mathrm{id} \in h(C))$ is a representation of h.