1 Locally Free Sheaves

2 Algebraic Vector Bundles

3 Derivations

4 Connections

Remark. Here we have a locally ringed space $X \to S$ over S. We write $\Omega_X = \Omega_{X/S}$ and

Lemma 4.1. Suppose that $\nabla_1, \nabla_2 : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ are connections. Then $\nabla_1 - \nabla_2 : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ is a $\mathcal{O}_X$-module map.

Proof. $(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 s - \nabla_2 s) + \mathrm{d}f \otimes s - \mathrm{d}f \otimes s = f(\nabla_1 - \nabla_2)s$. □

Remark. Therefore, the space of connections is a affine subspace of $\mathrm{Hom}(\mathcal{E}, \Omega^1_X \mathcal{E})$. Then if $\mathcal{E}$ is finite locally free,

$$\mathrm{Hom}\big(\mathcal{E}, \Omega^1_X \mathcal{E}\big) = H^0(X, \Omega^1_X \otimes_{\mathcal{O}_X} \mathscr{E}nd_{\mathcal{O}_S}(\mathcal{E}))$$

Definition 4.2. The first Chern class $c_1 : \mathrm{Pic}\,(X) \to H^1(X, \Omega^1) \subset H^2_{\mathrm{dR}}(X)$ is defined by $H^1(X, -)$ applied to the map $\mathrm{dlog} : \mathcal{O}_X^\times \to \Omega^1_X$ defined as $\mathrm{dlog}(f) = f^{-1}\mathrm{d}f$.

Proposition 4.3. A line bundle $\mathcal{L}$ admits a connection $\nabla : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$ if and only if $c_1(\mathcal{L}) = 0$.

Proof. A line bundle $\mathcal{L}$ is represented by a Cech cocycle $(U_i, f_{ij}) \in H^1(X, \mathcal{O}_X^\times)$. Then a connection on a line bundle is represented by (U_i, ω_i) with $\omega_i \in \Omega^1_X(U_i)$ where (U_i, s_i) is a trivialization of $\mathcal{L}$ with $\mathcal{O}_{U_i} \xrightarrow{s_i} \mathcal{L}|_{U_i}$ then $s_i|_{U_i \cap U_j} = f_{ij} s_j|_{U_i \cap U_j}$ and $\nabla s_i = \omega_i \otimes s_i$. However, we must have on $U_i \cap U_j$,

$$\nabla s_i = \nabla f_{ij} s_j = f_{ij} \nabla s_j + \mathrm{d}f_{ij} \otimes s_j$$

Therefore,

$$\omega_i \otimes f_{ij} s_j = f_{ij} \omega_j \otimes s_j + \mathrm{d}f_{ij} \otimes s_j$$

and thus,

$$(\omega_i - \omega_j)|_{U_i \cap U_j} = \mathrm{dlog}(f_{ij})$$

Consider the Cech differential $\mathrm{d} : \check{C}^0(\mathfrak{U}, \Omega^1_X) \to \check{C}^1(\mathfrak{U}, \Omega^1_X)$ which takes the sections (ω_i) to the coboundary $(\omega_i - \omega_j)|_{U_{ij}}$. Therefore, such a connection i.e. such a class exists iff the class,

$$c_1(\mathcal{L}) = [\mathrm{dlog}(f_{ij})] \in \check{H}^1(X, \Omega^1_X)$$

is trivial since it is a coboundary. □

5 Riemann-Hilbert Correspondence

6 Differential Operators

7 Sheaves of Jets

8 The Atiyah Class