1 Feb 11

1.1 Line Bundles

There exists a map,

$$\Gamma(X, \mathcal{L}^{\otimes a}) \otimes \Gamma(X, \mathcal{L}^{\otimes b}) \to \Gamma(X, \mathcal{L}^{\otimes ab})$$

since we have an isomorphism $\mathcal{L}^{\otimes a} \otimes \mathcal{L}^{\otimes b} = \mathcal{L}^{\otimes ab}$. Furthermore, since $\mathcal{L}$ is rank 1 this map is commutative since $s \times s' = s' \otimes s$ since they only differ by a section of $\mathcal{O}_X$. This alows us to define the following graded ring structure.

Definition 1.1.1. Let $\mathcal{L}$ be an invertable $\mathcal{O}_X$-module, $\mathscr{F}$ any $\mathcal{O}_X$-module and $s \in \mathcal{L}(X)$ a global section. Then we define the following graded ring.

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

and then the following module,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which is a graded $\Gamma_*(X, \mathcal{L})$-module. Furthermore, there is a map,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F})_{(s)} \to \mathscr{F}(X_s) = \Gamma(X_s, \mathscr{F})$$

sending $\frac{t}{s^n} \mapsto t|_{X_s} \otimes (s|_{X_s})^{\otimes -n}$.

Proposition 1.1.2. Let X be a quasi-compact, quasi-seperated scheme and $\mathscr{F}$ be quasi-coherent. Then the above map is an isomorphism.

Proof. Tag OB5K. (Compare with that Hartshorne Excercise 2.16). $\square$

Example 1.1.3. Let A be a graded ring such that A is generated by A_1 as a A_0-algebra (e.g. $A = k[X_0, \dots, X_n]$). Let $X = \text{Proj}\,(A)$ and consider the graded module $M = A(n)$ which is the graded module $M_k = A_{k+n}$. Then we can construct the Serre twists,

$$\mathcal{O}_X(n) = \widetilde{M} = \widetilde{A(n)}$$

which is an invertable $\mathcal{O}_X$-module. Furthermore,

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$$

Remark. This will not be invertible and these maps will not be isomorphisms in general when A does not satisfy the required conditions.

Proof. We can decompose,

$$X = \bigcup_{f \in A_1} D_+(f) = \bigcup_{f \in A_1} \text{Spec}\,\left(A_{(f)}\right)$$

via the given assumptions. We know that,

$$\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{A(n)}|_{D_+(f)} = \widetilde{A[f^{-1}]_n}$$

However it is clear that $A[f^{-1}]_n = A[f^{-1}]_0 \cdot f^n$ so this sheaf is free of rank 1. $\square$

Remark. For $n = 1$ any element $f \in A_1$ gives a global section $f \in \Gamma(X, \mathcal{O}_X(1))$ such that $D_+(f) = X_s$ and hence,

$$\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(1)|_{X_s}$$

Corollary 1.1.4. In the setting above, further assume that A is generated by finitely many $f \in A_1$ as an A_0-algebra. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ if we set,

$$M = \Gamma_*(X, \mathcal{O}_X(1), \mathscr{F})$$

as a graded A-module via the map,

$$A \to \Gamma_*(X, \mathcal{O}_X(1)) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

Then we get, $\mathscr{F} = \widetilde{M}$.

Proof. Tag □

2 Feb. 13

Definition 2.0.1. Let X be a scheme and $\mathcal{L}$ an invertivle $\mathcal{O}_X$-module . We say $\mathcal{L}$ is *ample* if X is quasi-compact and $\forall x \in X : \exists n > 0 : s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $x \in X_s$.

Example 2.0.2. Let $X = \text{Proj}\,(A)$ where A is generated by A_1 as a A_0-algebra and $A_1 = f_1 A_0 + \cdots + f_r A_0$. Then $\mathcal{O}_X(1)$ is invertible and X is covered by $D_+(f_i)$ and is quasi-compact, and $D_+(f_i) = X_{s_i}$ where $s_i \in \Gamma(X, \mathcal{O}_X(1))$ is a section corresponding to f_i.

Proposition 2.0.3. Let X be quasi-compact and quasi-seperated for $\mathcal{L} \in \text{Pic}\,(X)$ the following are equivalent,

(a) $\mathcal{L}$ is ample

(b) for all $\mathcal{O}_X$-modules $\mathscr{F}$ locally of finite type there exists $n > 0$ s.t. $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections.

Proof. TAG 01Q3. □

Lemma 2.0.4. $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is ample for any $n > 0$.

Lemma 2.0.5. If X is affine, and $\mathcal{L}$ is invertible, and $s \in \Gamma(X, \mathcal{L})$ then X_s is affine.

Definition 2.0.6. A scheme is noetherian if it has a finite open cover by spectra of noetherian rings.

Remark. It is equivalent to require that X is quasi-compact and $\mathcal{O}_X(U)$ is noetherian for each affine open.

Lemma 2.0.7. A locally noetherian scheme is quasi-seperated.

Proof. If U, V are affines then $U \cap V$ is quasi-compact since every subspace of a noetherian space is quasi-compact. □

Definition 2.0.8. Let X be a neotherian scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is *coherent* if it is quasi-coherent and locally of finite type.

Remark. It is equivalent to require that locally on affine opens $\mathscr{F}|_U = \widetilde{M}$ for a finitely-generated module M.

Remark. The inclusion functors,

$$\mathfrak{Coh}(\mathcal{O}_X) \subset \mathfrak{QCoh}(\mathcal{O}_X) \subset \mathscr{Mod}(\mathcal{O}_X)$$

are exact and preserved under extensions i.e. given a short exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

if $\mathscr{F}_1, \mathscr{F}_2$ are (quasi)-coherent then $\mathscr{F}_2$ is also (quasi)-coherent.

Lemma 2.0.9. A scheme of finite type over a noetherian scheme is noetherian.

Proof. Since $f : X \to Y$ is finite type f is quasi-compact but Y is quasi-compact open so its preimage X is also quasi-compact. Furthermore, for any affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(B) = V \subset Y$ such that $f(U) \subset V$ we get a ring map $B \to A$ of finite type so $B[x_1, \dots, x_n] \twoheadrightarrow A$ and since B is noetherian we see that A is noetherian so X is quasi-compact and covered by $\text{Spec}(A)$ for noetherian rings A. $\square$

Remark. We want to prove the following theorem. Let R be a noetherian ring, X a projective (or proper) scheme over R (then X is noetherian), and $\mathscr{F}$ a coherent sheaf on X, then,

$$H^i(X, \mathscr{F})$$

is a finite R-module for any i and $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$.

3 Feb 18

Definition 3.0.1. An immersion $j : X \to Y$ is a morphism which may be factored as $X \to U \to Y$ where $X \to U$ is a closed immersion and $U \to Y$ is an open immersion.

Definition 3.0.2. Let R be a ring, and X a scheme over R. We say X is *quasi-projective over R* iff there exists a quasi-compact immersion $j : X \to \mathbb{P}^n_R$ over R.

Remark. If X is proper over R (or just universally closed) then j is automatically a closed immersion since $\mathbb{P}^n_R \to \text{Spec}(R)$ is seperated and $X \to \text{Spec}(R)$ is universally closed implies that $j : X \to \mathbb{P}^n_R$ is universally closed and in particular topologically closed and thus closed as an immersion. This gives the following lemma.

Lemma 3.0.3. X is projective over R iff X is quasi-projective and proper over R.

Theorem 3.0.4. Let R be a ring and X a scheme over R. The TFAE,

(a) X is quasi-projective over R

(b) X is of finite type over R and X has an ample invertible module $\mathcal{L}$

(c) there exists a quasi-compact open immersion $X \hookrightarrow X'$ with X' projective over R.

Lemma 3.0.5. Let $j : X \to Y$ be a quasi-compact immersion and $\mathcal{L}$ an ample line bundle on Y. Then $j^*\mathcal{L}$ is an ample line bundle on Y.

Proof. (DO THIS!!) $\square$

Lemma 3.0.6. Let $j : X \to Y$ be a quasi-compact immersion and X' is scheme-theoretic image. Then $j : X \to X'$ is an open immersion.

Proof. Since j is qc and qs (immersions are seperated) then $j_*\mathcal{O}_X$ is quasi-coherent and thus $\mathscr{I} = \ker(\mathcal{O}_Y \to \mathcal{O}_X)$ is quasi-coherent so we find $X' = V(\mathscr{I})$ (FINSIH THIS) $\square$

Example 3.0.7. $\text{Spec}(k[[x]]) \to \text{Spec}(k[x])$ has scheme theoretic image $\text{Spec}(k[x])$ since its image contains the generic point. However, its set theoretic image is two points.

Proof. of Theorem (2) $\implies$ (1). Choose $r \geq 0$ and $n \geq 1$ and $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.t.

$$X = \bigcup_{i=0}^{r} X_{s_i}$$

and X_{s_i} affine. Write $X_{s_i} = \text{Spec}(A_i)$. Now R is finite type over R so A_i is finite type over R so we may take $a_{i1}, \dots, a_{iN_i} \in A_i$ which generate A_i as an R-algebra. Choose $m \geq 1$ and $s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ such that $a_{ij} = s_{ij} \cdot s_i^{\otimes -m}|_{X_{s_i}}$. Therefore, $s_0^{\otimes m}, \dots, s_r^{\otimes m}, s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ generate $\mathcal{L}^{\otimes mn}$ and therefore define a morphism $\varphi : X \to \mathbb{P}_R^{r + \sum N_i}$. It suffices to check that $X_{s_i} \to D_+(T_i)$ are a closed immersion. This holds because it is given by the ring map,

$$R[\tfrac{T_0}{T_1}, \dots, \tfrac{T_r}{T_i}, \tfrac{T_{ij}}{T_i}] \to A_i = \mathcal{O}_X(X_{s_i})$$

given by $\frac{T_{ij}}{T_i} \to a_{ij}$ which is clearly surjective so $X_{s_i} \to D_+(T_i)$ is a closed immersion. $\square$

Remark. If we had checked that $X_{s_{ij}} \to D(T_{ij})$ we also a closed immersion with $X_{s_{ij}}$ affine then $\varphi : X \to \mathbb{P}_R^N$ would be a *closed* immersion. We checked only that it is locally a closed immersion on X

3.1 Functorial Characterization of $\mathbb{P}_R^n$

Consider the functor, $F : \mathfrak{Sch}_R \to \mathfrak{Set}$ via,

$$T \mapsto \{(\mathcal{L}, s_0, \dots, s_n) \mid \mathcal{L} \in \text{Pic}(T)\ \ \mathcal{O}_T^{n+1} \xrightarrow{(s_0, \dots, s_n)} \mathcal{L} \text{ i.e. } s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \text{ generate}\}/\cong$$

where $(\mathcal{L}, s_0, \cdots, s_n) \cong (\mathcal{L}', s_0', \dots, s_n')$ if there is an isomorphism $\alpha : \mathcal{L} \to \mathcal{L}'$ with $\alpha(s_i) = s_i'$.

Theorem 3.1.1. $\mathbb{P}_R^n$ represents this functor, $\text{Hom}_{\mathfrak{Sch}_R}(T, \mathbb{P}_R^n) = F(T)$.

Proof. Given $\varphi : T \to \mathbb{P}_R^n$ we get $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}_R^n}(1)$ and $s_i = \varphi^*(T_i)$.

Conversely, given $(\mathcal{L}, s_0, \dots, s_n)$ and $U \subset T$ and $\square$

Theorem 3.1.2. If R is Noetherian and X is proper over R and $\mathcal{L}$ is ample on X then,

$$X \cong \text{Proj}(\Gamma_*(X, \mathcal{L}))$$

and $\Gamma_*(X, \mathcal{L})$ is a finitely-generated graded R-algebra whose degree zero part is a finite R-module.

Remark. We will prove this using cohomology.

4 Cohomology

Theorem 4.0.1. $\mathbf{Mod}_{\mathcal{O}_X}$ is a Grothendieck abelian category so there are enough injectives.

Definition 4.0.2. Therefore, we can produce the right-derived functors $H^i(X, -)$ of the global sections functor,

$$\Gamma(X, -) : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\Gamma(X, \mathcal{O}_X)}$$

where $(X, \mathcal{O}_X)$ is a ringed space. Since this is right-exact we find $H^0(X, -) = \Gamma(X, -)$.

Definition 4.0.3. Furthermore, given a morphism $f : X \to Y$ we can produce $R^i f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$ the right-derived functors of $f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$.

Remark. $\mathbf{Ab}(X) = \mathbf{Mod}_{\underline{\mathbb{Z}}}$ so we may apply the theory of cohomology of $\mathcal{O}_X$-modules to the ringed space $(X, \underline{\mathbb{Z}})$ to get a cohomology theory for abelian sheaves.

Lemma 4.0.4 (locality of cohomology)**.** Given $\xi \in H^p(X, \mathscr{F})$ with $p > 0$ there exists an open covering,

$$X = \bigcup_{i \in I} U_i$$

s.t. $\xi|_{U_i} = 0$ for each $i \in I$.

Proof. $\square$

Remark. The pullback is defined as follows,

5 Feb 20

5.1 Cech Cohomology

For any open covering $\mathfrak{U}$ of a space X and a sheaf $\mathscr{F}$ there is a simplicial abelian group,

$$\textstyle\prod_{i_0 \in I} \mathscr{F}(U_{i_0})$$

Then $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is the complex associated to the cosimplicial object.

Example 5.1.1. Given an exact sequence of sheaves,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

An obstruction to lifting a section $s \in \Gamma(X, \mathscr{H})$ is a cocycle in $\check{C}^1(\mathfrak{U}, \mathscr{F})$.

Lemma 5.1.2. Cech cohomology vanishes on injective objects in the category of presheaves.

Corollary 5.1.3. As a functor ON THE CATEGORY OF PRESHEAVES $\check{H}^i(\mathfrak{U}, -)$ are the right-derived functors of $\check{H}^0(\mathfrak{U}, -)$.

Lemma 5.1.4. Given a ringed space, $(X, \mathcal{O}_X)$ and B is a basis of top and Cov a set of coverings s.t.

(a) $\mathfrak{U}$ in cov implies that its union and all finite intersections are in B

(b) for U basis the coverings of U in Cov are cofinal

If $\mathscr{F} \in \mathscr{Mod}(\mathcal{O}_X)$ and

$$(*) \forall \mathfrak{U} \in Cov : \; \check{H}^p(\mathfrak{U}, \mathscr{F}) = 0$$

Then $H^p(\mathfrak{U}, \mathscr{F}) = 0$ for any U in the basis.

6 Feb 25

Lemma 6.0.1. Let $\mathfrak{U}$ be an open covering of X and $\mathscr{F} \in \mathscr{Mod}(\mathcal{O}_X)$ s.t. $H^p(U_{i_1} \cap \cdots U_{i_n}, \mathscr{F}) = 0$ for all finite intersections. Then $H^p(X, \mathscr{F}) = \check{H}^p(X, \mathscr{F})$ for all $p \geq 0$.

Proof. See proof in Hartshorne Ex. It goes as follows,

(a) Use an exact sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{G} \longrightarrow 0$$

with $\mathscr{I}$ injective.

(b) Show for any sheaf $\check{H}^0(X, \mathscr{F}) = H^0(X, \mathscr{F})$ just by the sheaf property.

(c) By the assumptions, there is an exact sequence on check complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{I}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathscr{G}) \longrightarrow 0$$

(d) this gives a long exact sequence of Cech cohomology

(e) use this exact sequence plus $\check{H}^p(\mathfrak{U}, \mathscr{I}) = 0$ for $p > 0$ (since flasque) to show that $\check{H}^p(\mathfrak{U}, \mathscr{G}) = \check{H}^{p+1}(\mathfrak{U}, \mathscr{F})$ and $\check{H}^1(\mathfrak{U}, \mathscr{F}) = \operatorname{coker} \check{H}^0(\mathfrak{U}, \mathscr{I}) \to \check{H}^0(\mathfrak{U}, \mathscr{G})$

(f) use long exact sequence of $H^p(U_{i_0, \dots, i_n}, -)$ to show that $\mathscr{G}$ also satisfies the hypotheses.

(g) use long exact sequence of $H^p(X, -)$ to show that the above hold for usual cohomology.

(h) then by induction we get $\check{H}^{p+1}(\mathfrak{U}, \mathscr{F}) = \check{H}^p(\mathfrak{U}, \mathscr{G}) = H^p(X, \mathscr{G}) = H^{p+1}(X, \mathscr{F})$ and the base case holds since they are both kernels.

□

Corollary 6.0.2. Let X be a scheme whose diagonal is affine (for example a separated scheme). Let $\mathfrak{U}$ be a covering of affine opens and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then,

$$H^p(X, \mathscr{F}) = \check{H}^p(X, \mathscr{F})$$

Remark. There is a Cech to cohomology spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \underline{H}^q(\mathscr{F})) \implies H^{p+q}(X, \mathscr{F})$$

Corollary 6.0.3. Let $f : X \to Y$ be a squasi-compact quasi-separated morphism of schemes. Then $R^i f_*$ sends quasi-coherent mofules to quasi-coherent modules.

Lemma 6.0.4. Let $f : X \to Y$, $F \in \mathscr{Mod}(\mathcal{O}_X)$ then $R^p f_* \mathscr{F}$ is the sheaf associated to the presheaf,

$$V \mapsto H^i(f^{-1}(V), \mathscr{F})$$

Proposition 6.0.5. We define the following modifications to the Cech complex,

$\check{C}^\bullet_{\text{alt}}$ is elements of the form $(s_{i_0\dots i_p})$ which are antisymmetric and vanish if any two indicies agree and the ordered check complex for a total order $<$ on I,

$$\check{C}^p_{\text{ord}} = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0 \cdots i_p})$$

There are the following relations between Cech complexes,

$$\begin{tikzcd} \check{C}^\bullet_{\text{alt}}(\mathfrak{U}, \mathscr{F}) \arrow[r, "\text{include}"] \arrow[rr, bend left] & \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \arrow[r, "\text{project}"] & \check{C}^\bullet_{\text{ord}}(\mathfrak{U}, \mathscr{F}) \end{tikzcd}$$

the curves arrow is an isomorphism of complexes and the horizontal arrows are homotopy equivalences.

7 Feb. 27

Proposition 7.0.1. Let R be a Noetherian ring and $\mathscr{F}$ a coherent sheaf on $\mathbb{P}^n_R$. Then,

(a) $\exists r \geq 0 : \exists m \in \mathbb{Z}$ and a surjection $\mathcal{O}_X(m)^{\oplus r} \twoheadrightarrow \mathscr{F}$

(b) $H^i(\mathbb{P}^n_R, \mathscr{F}) = 0$ for $i \notin [0, n]$

(c) $H^i(\mathbb{P}^n_R, \mathscr{F})$ is a finite R-module

(d) for $i > 0$, $H^i(\mathbb{P}^n_R, \mathscr{F}(d)) = 0$ for any $d \geq d_0(\mathscr{F})$

(e) $\bigoplus_{d \geq 0} H^0(\mathbb{P}^n_R, \mathscr{F}(d))$ is a finite $P = R[T_0, \dots, T_n]$-module.

Proof. Recall that $\mathcal{O}_X(1)$ is ample so $\mathscr{F} \otimes \mathcal{O}_X(d)$ is generated by global sections for sufficiently large d and thus we get $\mathcal{O}_X^{\oplus r} \twoheadrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ and thus $\mathcal{O}_X(-d)^{\oplus r} \twoheadrightarrow \mathscr{F}$.

Note that $\mathbb{P}^n_R = \bigcup_i D_+(T_i)$ which is an open cover of $n+1$ affines so by Cech cohomology, cohomology vanishes above n.

Now we apply descending induction since $H^{n+1}(\mathbb{P}^n_R, \mathscr{F}) = 0$. Now we assume (3) and (4) for degree $k+1$. For a coherent sheaf $\mathscr{F}$ consider the exact sequence,

$$0 \longrightarrow \mathscr{G}(d) \longrightarrow \mathcal{O}_X(m+d)^{\oplus n} \longrightarrow \mathscr{F}(d) \longrightarrow 0$$

then, from the LES we get,

$$H^k(\mathbb{P}^n_R, \mathcal{O}_X(m+d)^{\oplus r}) \longrightarrow H^k(\mathbb{P}^n_R, \mathscr{F}(d)) \longrightarrow H^{k+1}(\mathbb{P}^n_R, \mathscr{G}(d))$$

For the case $d = 0$ we assume that $H^{k+1}(\mathbb{P}^n_R, \mathscr{G})$ is a finite R-module and, by computation, so is $H^k(\mathbb{P}^n_R, \mathcal{O}_X(m)^{\oplus r})$ and thus $H^k(\mathbb{P}^n_R, \mathscr{F})$ is a finite R-module. For $d \gg 0$ then we assume that $H^{k+1}(\mathbb{P}^n_R, \mathscr{G}(d)) = 0$ for sufficiently large d. Futhermore, for $k > 0$ we computed that $H^k(\mathbb{P}^n_R, \mathcal{O}_X(m)^{\oplus r}) = 0$ for $d \geq m$ and thus we see that $H^k(\mathbb{P}^n_R, \mathscr{F}(d)) = 0$ for sufficiently large d proving (3) and (4).

Finally, we also use descending induction and conisder the exact sequence,

$$\bigoplus_{d \geq 0} H^0(\mathbb{P}^n_R, \mathcal{O}_X(m+d)^{\oplus r}) \longrightarrow \bigoplus_{d \geq 0} H^0(\mathbb{P}^n_R, \mathscr{F}(d)) \longrightarrow \bigoplus_{d \geq 0} H^0(\mathbb{P}^n_R, \mathscr{G}(d))$$

By computation, the first term is a submodule of a finite P-module and the last term is zero is sufficiently large degrees. Thus the middle term M has a f.g. P-submodule M' such that M/M' is finite as an R-module so M is a f.g. P-module. □

Lemma 7.0.2. Let $f : X \to Y$ be an affine morphism of schemes. Then $H^p(X, \mathscr{F}) = H^p(Y, f_* \mathscr{F})$ for $\mathscr{F}$ quasi-coherent.

Proof. We use the Grothendieck spectral sequence and not that for $f : X \to Y$ affine and $\mathscr{F}$ quasi-coherent we have $R^p f_* \mathscr{F} = 0$ for $p > 0$ since quasi-coherent higher cohomology vanishes on affine schemes. □

Example 7.0.3. If X is a projective scheme over a Noetherian ring R. For closed immersion $X \hookrightarrow \mathbb{P}^n_R$,

$$H^i(X, \mathscr{F}) = H^i(\mathbb{P}^n_R, j_* \mathscr{F})$$

for quasi-coherent $\mathcal{O}_X$-modules.

Lemma 7.0.4. If $\mathscr{F} : X \to Y$ is finite and X and Y are Noetherian then f_* preserves coherent sheaves.

Proof. Since f is affine it preserves quasi-coherent modules. Since the morphism is additionally finite on rings so it changes finite modules to finite modules on the affine open level. □

Corollary 7.0.5. For any coherent $\mathscr{F}$ on a scheme X projective over Noetherian R then the above proposition holds with $\mathscr{F}(d) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ where $\mathcal{L}$ is an ample line bundle.

Remark. Let X be Noetherian over Noetherian R then let $n = \max\{\dim X_s \mid s \in \mathrm{Spec}\,(R)\}$ then $H^i(X, \mathscr{F}) = 0$ for $i > n$. Warning, this is not true for quasi-projective X over a Noetherian ring. For example, consider $X = \mathbb{A}^2_{\mathbb{Q}} \setminus \{0\} \to \mathbb{A}^2_{\mathbb{Q}}$ is quasi-projective over $R = \mathbb{Q}[x, y]$ but X does not have finitely generated cohomology.

Lemma 7.0.6. Let X be projective over a field k then X has an open cover by $\dim X + 1$ affines.

Proof. Choose $X \hookrightarrow \mathbb{P}^n_k$ show that we can find $F \in k[T_0, \ldots, T_n]_d$ s.t. $\dim\,(X \cap V(F)) < \dim X$. Namely, choose F not vanishing at the generic points of X by graded prime avoidance. Then we can repeat to get,

$$X \cap V(F_1) \cap \cdots \cap V(F_{\dim X + 1}) = \varnothing$$

and thus,

$$X = (X \cap D_+(F)) \cup \cdots \cup (X \cap D_+(F_{\dim X + 1})$$

where these factors are affine. □

Corollary 7.0.7. $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$ for $\mathscr{F}$ quasi-coherent on X projective over a field.

Theorem 7.0.8 (Grothendieck)**.** If $(X, \mathcal{O}_X)$ is a Noetherian ringed space then $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$ and any $\mathcal{O}_X$-module $\mathscr{F}$.

Remark. Since we can always choose $\mathcal{O}_X = \underline{\mathbb{Z}}$ in the above theorem applies to all abelian sheaves.

Lemma 7.0.9. If X is qc and qs then for $\mathscr{F}_i$ quasi-coherent and I an *arbitrary* index set,

$$H^p(X, \bigoplus_{i \in I} \mathscr{F}_i) = \bigoplus_{i \in I} H^p(X, \mathscr{F})$$

Remark. The above is always true in general for *finite* I since biproducts preserve exact sequences and injectives.

Proof. It is enough to show the above for Cech cohomology for finite affine open covers. Thus, iti s enough to show that,

$$\left(\bigoplus_{i\in I} \mathscr{F}_i\right)(U) = \bigoplus_{i\in I} \mathscr{F}_i(U)$$

If X is affine open in X (WAIT WHAT??) $\square$

7.1 Duality

Lemma 7.1.1. Let R be a ring, M an R-module, and X qc + sep over R. And some $n \geq 0$ such that $H^{n+1}(X, \mathscr{F})$ for all $\mathscr{F}$ quasi-coherent. Then, the functor $F : \mathfrak{QCoh}(\mathcal{O}_X) \to \mathbf{Mod}_R$ via $\mathscr{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathscr{F}), N)$ is representable by some $\omega_{X/R,M} \in \mathfrak{QCoh}(\mathcal{O}_X)$. That is,

$$F(-) = \mathrm{Hom}_{\mathcal{O}_X}\left(-, \omega_{X/R,M}\right)$$

Example 7.1.2. For $X = \mathrm{Spec}(A)$ then we have $\widetilde{N} \mapsto \mathrm{Hom}_R(N_R, M)$. Then,

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_A(N, \mathrm{Hom}_R(A, M))$$

so we would have $\omega_{A/R,M} = \widetilde{\mathrm{Hom}_{\mathrm{R}}(\mathrm{A}, \mathrm{M})}$.

Proof. First note that F acts on direct sums as,

$$F\left(\bigoplus_{i\in I} \mathscr{F}_i\right) = \mathrm{Hom}_R\left(H^n(X, \bigoplus_{i\in I} \mathscr{F}_i), M\right) = \mathrm{Hom}_R\left(\bigoplus_{i\in I} H^n(X, \mathscr{F}_i), M\right) = \prod_{i\in I} \mathrm{Hom}_R(H^n(X, \mathscr{F}_i), M)$$

Furthermore, F takes epis to monos since given an exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \twoheadrightarrow \mathscr{F}_3 \longrightarrow 0$$

then we get,

$$H^n(X, \mathscr{F}_2) \twoheadrightarrow H^n(X, \mathscr{F}_3) \longrightarrow H^{n+1}(X, \mathscr{F}_1) = 0$$

These together shows that F takes all small colimits to products. Then if F satisfies some mild set-theoretic condition then the adjoint functor theorem gives $\omega_{X/R,M}$ as a funtor on M. The ideal goes as follows. We take,

$$\omega_{X/R,M} = \mathrm{colim}_{\mathcal{C}}\, \mathscr{F}$$

where $\mathcal{C}$ is a category of pairs $(\mathscr{F}, \alpha)$ where $\mathscr{F}$ is a quasi-coherent sheaf and $\alpha \in F(\mathscr{F})$ and $\mathrm{Hom}_{\mathcal{C}}((\mathscr{F}, \alpha), \mathscr{G}, \beta)) = \varphi : \mathscr{F} \to \mathscr{G}$ and $\varphi^*\beta = \alpha$. However, this category is big so we cannot take a total colimit over it. We must resolve this set-theoretic issue.

In the case R is Noetherian and X is finite type over R then any quasi-coherent $\mathscr{F}$ can be writen as a filtered colimit,

$$\mathscr{F} = \mathrm{colim}_{i\in I}\, \mathscr{F}_i$$

with $\mathscr{F}_i$ coherent. This means that in the colimit defining $\omega_{X/R,M}$ we can restrict to only coherent $\mathscr{F}$ and there is a set of isomorphism classes of coherent sheaves. $\square$

8 Mar 3

Remark. Here X will be a Noetherian scheme.

Lemma 8.0.1. Let X be a Noetherian scheme. Any presheaf on $\mathfrak{QCoh}(\mathcal{O}_X)$ which transforms colimits into limits is representable.

Lemma 8.0.2. Any quasi-coherent module $\mathscr{F}$ on X is a filtered colimit of coherent $\mathcal{O}_X$-modules. (In fact $\mathscr{F}$ is the rising union of its coherent submodules).

Corollary 8.0.3. For any $\mathscr{F} \in \mathfrak{QCoh}(\mathcal{O}_X)$ there exists an exact sequence,

$$\bigoplus_{j \in J} \mathscr{G}_j \longrightarrow \bigoplus_{i \in I} \mathscr{F}_i \longrightarrow 0$$

where $\mathscr{F}_i$ and $\mathscr{G}_j$ are coherent.

Lemma 8.0.4. There is a set of isomorphism classes of coherent $\mathcal{O}_X$-modules.

Proposition 8.0.5. Let X be finite type over R Noetherian. Let n be an integer s.t. $H^{n+1}(X, \mathscr{F}) = 0$ for any $\mathscr{F} \in \mathfrak{QCoh}(\mathcal{O}_X)$. Then, for any R-module M, the functor,

$$\mathscr{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathscr{F}), M)$$

is representable by $\omega_{X/R,M,n} \in \mathfrak{QCoh}(\mathcal{O}_X)$ i.e.

$$\mathrm{Hom}_R(H^n(X, \mathscr{F}), M) = \mathrm{Hom}_{\mathcal{O}_X}\left(\mathscr{F}, \omega_{X/R,M,n}\right)$$

functorially in $\mathscr{F} \in \mathfrak{QCoh}(\mathcal{O}_X)$.

Remark. For any integer p and $\mathscr{F} = \mathrm{colim}\, \mathscr{F}_i$ is a filted colimit of $\mathcal{O}_X$-modules on a Noetherian scheme (or qcqs scheme) we have,

$$H^p(X, \mathscr{F}) = \mathrm{colim}\, H^p(X, \mathscr{F}_i)$$

Theorem 8.0.6. If k is a field and $n \geq 0$. Then $\omega_{\mathbb{P}^n_k/k,k,n} = \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$. In particular,

$$H^n(\mathbb{P}^n_k, \mathscr{F})^\vee = \mathrm{Hom}_{\mathcal{O}_X}\left(\mathscr{F}, \mathcal{O}_{\mathbb{P}^n_k}(-n-1)\right)$$

functorially in $\mathscr{F} \in \mathfrak{QCoh}\left(\mathcal{O}_{\mathbb{P}^n_k}\right)$.

Proof. It suffices to show for $\mathscr{F}$ coherent. Pick a resolution,

$$\bigoplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^k_k}(e_j) \longrightarrow \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^n_k}(d_i) \longrightarrow \mathscr{F} \longrightarrow 0$$

Since $H^n(\mathbb{P}^n_k, -)$ is right exact (by dimension vanishing) we get,

$$\bigoplus_{j=1}^{s} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^k_k}(e_j)) \longrightarrow \bigoplus_{j=1}^{r} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_i)) \longrightarrow H^n(\mathbb{P}^n_k, \mathscr{F}) \longrightarrow 0$$

Then taking k-linear duals,

$$\begin{tikzcd}
\bigoplus_{j=1}^{s} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^k_k}(e_j))^\vee \arrow[d, equal] & \bigoplus_{j=1}^{r} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_i))^\vee \arrow[l] \arrow[d, equal] & H^n(\mathbb{P}^n_k, \mathscr{F}) \arrow[l] \arrow[d, equal, dashed] & 0 \arrow[l] \\
\bigoplus_{j=1}^{s} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^k_k}(-e_j - n - 1)) & \bigoplus_{j=1}^{r} H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-d_i - n - 1)) \arrow[l, "t"] & \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}\left(\mathscr{F}, \mathcal{O}_{\mathbb{P}^n_k}(-n-1)\right) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Note that,

$$\mathcal{O}_{\mathbb{P}^n_k}(-d - n - 1) = \mathscr{H}\!om_{\mathcal{O}_{\mathbb{P}^n_k}}\big(\mathcal{O}_{\mathbb{P}^n_k}(d), \mathcal{O}_{\mathbb{P}^n_k}(-n-1)\big)$$

gives the above "transpose" map t above by functoriality in the first argument along with the fact,

$$H^0(\mathbb{P}^n_k, \mathscr{H}\!om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathscr{F}, \mathscr{G})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$$

□

9 March 5

9.1 Serre Duality for $\mathbb{P}^n_k$ Continued.

Write $\omega = \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$ and $t : H^n(\mathbb{P}^n_k, \omega) \to k$ via the Check class,

$$\frac{1}{T_0 \cdots T_n} \mapsto 1$$

Then we know that ω represents the functor,

$$\mathscr{F} \mapsto H^n(\mathbb{P}^n_k, \mathscr{F})^\vee$$

on $\mathfrak{QCoh}(\mathcal{O}_X)$ with universal object t.

Theorem 9.1.1. For coherent modules $\mathscr{F}$, there is an isomorphism,

$$H^{n-i}(\mathbb{P}^n_k, \mathscr{F})^\vee = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \omega)$$

Proof. Both sides are contravariant δ-functors in $\mathscr{F}$ so it suffices to show that both are universal for which it suffices to show that both are coeffecable. For any coherent sheaf $\mathscr{F}$ we can find,

$$\mathcal{O}_{\mathbb{P}^n_k}(-q)^{\bigoplus r} \twoheadrightarrow \mathscr{F}$$

and then for $i > 0$ we know,

$$H^{n-i}(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-q)) = 0 \quad \text{and} \quad \mathrm{Ext}^i_{\mathcal{O}_X}\left(\mathcal{O}_{\mathbb{P}^n_k}(-q), \omega\right) = H^i(\mathbb{P}^n_k, \omega(q)) = H^i(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-n-1+q)) = 0$$

for sufficiently large $q \gg 0$ using our Cech calculations. □

Lemma 9.1.2. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module. Then.

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E}, \mathscr{G}) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{G})$$

where $\mathcal{E}^\vee = \mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

Proof. Choose an injective resolution $\mathscr{G} \to \mathscr{I}^\bullet$ then,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathscr{I}^\bullet) = \Gamma(X, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathscr{I}^\bullet)) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{I}^\bullet)$$

Now I claim that $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{I}^\bullet$ is an injective resolution over $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{G}$. To see this, we use,

$$\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{I}^\bullet) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} -, \mathscr{I}^\bullet)$$

but $I^\bullet$ is injective and $\mathcal{E}$ is flat so this is an exact functor. Taking cohomology of the first equality proves the lemma. ☐

Remark. We could also just say, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, -) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$ so taking their derived functors gives the same thing. However, $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} -$ is exact so taking derived functors of $\Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$.

Remark. The perfect pairings,

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathscr{F}, \omega) \times H^{n-i}(\mathbb{P}^n_k, \mathscr{F}) \to H^n(\mathbb{P}^n_k, \omega) \xrightarrow{t} k$$

factors through $H^n(\mathbb{P}^n_k, \omega)$. The first map can be realized as composition of ext classes or a cup product.

Remark. If $\mathscr{F}$ is locally free then we have a diagram,

$$\begin{tikzcd} H^i(\mathbb{P}^n_k, \mathscr{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \omega) \times H^{n-i}(\mathbb{P}^n_i, \mathscr{F}) \arrow[r] \arrow[d] & k \\ H^n(\mathbb{P}^n_k, \mathscr{F} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathscr{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \omega) \arrow[r] & H^n(\mathbb{P}^n_k, \omega) \arrow[u, "t"] \end{tikzcd}$$

which gives the same pairing using the unique evaluation pairing,

$$\mathscr{F} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathscr{F}^\vee = \mathscr{F} \otimes_{\mathbb{P}^n_k} \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathscr{F}, \mathcal{O}_{\mathbb{P}^n_k}) \to \mathcal{O}_{\mathbb{P}^n_k}$$

9.2 Dualizing Sheaves in General

Definition 9.2.1. Let X be proper over k and $\dim X = n$. A *dualizing sheaf* (ω_X, t) is a pair consisting of a coherent $\mathcal{O}_X$-module ω_X and a map $t : H^n(X, \omega_X) \to k$ which represents the functor,

$$\mathscr{F} \mapsto H^n(X, \mathscr{F})^\vee$$

Remark. We have proven, by abstract nonsense, that such a *quasi-coherent* dualizing sheaf exists but now we want to know when such a module is actually *coherent*.

Remark. Consider the case that X is the disjoint union of a curve and a surface. Then $H^2(X, -)$ ignores cohomology on the curve since it vanishes above $H^1(X, -)$. Thus the dualizing sheaf will be zero on the curve. To fix this one looks for a dualizing complex,

$$\omega_X^\bullet \in D^b(\mathfrak{QCoh}(\mathcal{O}_X))$$

such that $H^i(X, \mathscr{F})$ is dual to $\mathrm{Ext}^{-i}_{\mathcal{O}_X}(\mathscr{F}, \omega_X^\bullet)$.

Theorem 9.2.2. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}^n_k$,

$$\iota_* \omega_X \cong \mathscr{E}xt^c_{\mathcal{O}_X}\big(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k}\big)$$

where $c = n - \dim X$ is the codimension.

Lemma 9.2.3. Let $\iota : X \to Y$ be a clsoed immersion of schemes then $\iota_* : \mathfrak{QCoh}\,(\mathcal{O}_X) \to \mathfrak{QCoh}\,(\mathcal{O}_X)$ defines an equivalence of categories onto its image which is the full subcategory of quasi-coherent $\mathcal{O}_Y$-modules $\mathscr{F}$ such that $\mathscr{I} \cdot \mathscr{F} = 0$ for $\mathscr{I} = \ker(\mathcal{O}_Y \to \iota_* \mathcal{O}_X)$.

Remark. If X and Y are Noetherian schemes, then the above holds also for coherent modules.

Remark. If $f : X \to Y$ is an affine morphism, $\mathfrak{QCoh}\,(\mathcal{O}_X)$ is the category of pairs $(\mathscr{F}, \gamma)$ with $\mathscr{F} \in \mathfrak{QCoh}\,(\mathcal{O}_Y)$ and $\gamma : f_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{F}$ gives $\mathscr{F}$ a $f_* \mathcal{O}_X$-module structure meaning $f_* \mathcal{O}_X$ is a monoid object and $f_* \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{F}$ is an action of a monoid object.

10 Mar. 12

Lemma 10.0.1. Let A be Noetherian and M, N be finite-presentation A-modules and $X = \mathrm{Spec}\,(A)$. Then,

$$\mathscr{H}om_{\mathcal{O}_X}\Big(\widetilde{M}, \widetilde{N}\Big) = \widetilde{\mathrm{Hom_A\,(M, N)}}$$

Proof. The isomorphism,

$$\mathrm{Hom}_A\,(M, N)_f = \mathrm{Hom}_{A_f}\,(M_f, N_f)$$

for finitely-presented modules patch together on the open sets $D(f)$ to give an isomorphism,

$$\widetilde{\mathrm{Hom_A\,(M, N)}} = \mathscr{H}om_{\mathcal{O}_X}\Big(\widetilde{M}, \widetilde{N}\Big)$$

$\square$

Lemma 10.0.2. Let A be Noetherian and M, N be finite A-modules and $X = \mathrm{Spec}\,(A)$. Then,

$$\mathscr{E}xt^i_{\mathcal{O}_X}\Big(\widetilde{M}, \widetilde{N}\Big) = \widetilde{\mathrm{Ext^i_A\,(M, N)}}$$

Proof. This holds for $i = 0$ by the above. Then we apply dimension-shifting to prove this in general. Given a $\square$

Lemma 10.0.3. For $p < \dim P - \dim X$ we have,

$$\mathscr{E}xt^p_{\mathcal{O}_X}(\iota_* \mathcal{O}_X, \omega_P) = 0$$

Proof. This reduced to the algebra question, given $B = k[x_1, \dots, x_n] \twoheadrightarrow A$ then,

$$\mathrm{Ext}^p_B\,(A, B) = 0$$

for $p < \dim B - \dim A$. To see this, recall we have $\iota : X \hookrightarrow P = \mathbb{P}^n_k$ then $X \cap D_+(T_i) \subset X$ and $D_+(T_i) = \mathrm{Spec}\,(B)$. Then, $\omega_P|_{D_+(T_i)} = \mathcal{O}_X|_{D_+(T_i)} = \widetilde{B}$. Furthermore, $\iota : X \hookrightarrow P$ is affine (closed immersion) so $X \cap D_+(T_i) = \mathrm{Spec}\,(A)$ for $A = B/I$.

Since B is Cohen-Macaullay we have vanishing for,

$$\mathrm{depth}_I(A) \geq \dim B - \dim A$$

$\square$

Proof. ☐

Theorem 10.0.4. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}^n_k$,

$$\iota_* \omega_X \cong \mathscr{E}xt^c_{\mathcal{O}_X}\big(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k}\big)$$

where $c = n - \dim X$ is the codimension.

Proposition 10.0.5. If $\iota : X \to Y$ is a closed immersion then $\iota^* \iota_* \mathscr{F} = \mathscr{F}$ for any $\mathcal{O}_X$-module $\mathscr{F}$ and if $\mathscr{I} \cdot \mathscr{G} = 0$ for some $\mathcal{O}_Y$-module $\mathscr{F}$ then $\mathscr{G} = \iota_* \iota^* \mathscr{G}$ where $\mathscr{I} = \ker\left(\mathcal{O}_Y \to \iota_* \mathcal{O}_X\right)$.

11 Local Property

Definition 11.0.1. A property P of ring maps is *local* if,

(a) $P(R \to A) \implies P(R_f \to A_f)$ for all $f \in R$

(b) $P(R_f \to A)$ for some $f \in R$ then $P(R \to A_a)$ for any $a \in A$

(c) if $P(R \to A_{a_i})$ for $(a_1, \dots, a_r) = A$ then $P(R \to A)$.

Definition 11.0.2. We say a morphism of schemes $f : X \to Y$ is locally P for some local property P if for each $x \in X$ there is an affine open $U = \mathrm{Spec}\,(A)$ with $x \in U \subset X$ and $V = \mathrm{Spec}\,(R)$ with $V \subset Y$ with $f(U) \subset V$ such that $P(R \to A)$.

Lemma 11.0.3. If $f : X \to Y$ is locally P then for any affine opens $U \subset X$ and $V \subset Y$ with $f(U) \to V$ then $P(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$.

Remark.

12 Smooth Maps

Definition 12.0.1. A ring map $R \to A$ is *smooth* if it is of finite presentation,

$$A \cong R[x_1, \dots, x_n]/I$$

where I is finitely generated. Then consider,

$$I/I^2 \xrightarrow{\mathrm{d}} \bigoplus_{i=1}^{n} A\mathrm{d}x_i$$

given by,

$$f \mapsto \mathrm{d}f = \sum \frac{\partial f}{\partial x_i} \mathrm{d}x_i$$

Then d is injective and its cokernel is a projective A-module. Since $K = \mathrm{coker}\,\mathrm{d}$ is projective an finitely generated then it is locally free so it has a rank function. We say that $R \to A$ is *smooth of relative dimension* n if K is of constant rank n.

Remark. Smoothness satisfies the following,

(a) local

(b) preserved under composition

(c) preserved uncer base change

Example 12.0.2. Take $R \to R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ such that,

$$\det \left(\frac{\partial f_j}{\partial x_i} \right)_{\substack{i=1,\dots,c \\ j=1,\dots,c}}$$

maps to an invertible element of A under $R[x_1, \dots, x_n] \to A$. Then,

$$\frac{(f_1, \dots, f_c)}{(f_1, \dots, f_c)^2} \to \bigoplus_{i=1}^{n} A \mathrm{d}x_i \to \operatorname{coker} \mathrm{d} \to 0$$

makes coker projective since the matrix for the map,

$$f_i \mapsto \frac{\partial f_i}{\partial x_j} \mathrm{d}x_j$$

in the basis $\{f_i\}$ and $\{\mathrm{d}x_j\}$ is invertible. Therefore, the cokernel is locally free. We think of this situation as $f = (f_1, \dots, f_c)$ defining a map $R[y_1, \dots, y_c] \to R[x_1, \dots, x_n]$ via $y_i \mapsto f_i$ thus we get a morphism $f : \mathbb{A}^n_R \to \mathbb{A}^c_R$ then $\operatorname{Spec}(R[x_1, \cdots x_n]/(f_1, \dots, f_c))$ is the fiber above the point zero $(y_1, \dots, y_c)$.

In differential geometry, such a map $f : \mathbb{C}^n \to \mathbb{C}^c$ is a submersion since the Jacobian matrix has full rank. Therefore,

$$\begin{tikzcd} f^{-1}(\{0\}) \arrow[r, hook] \arrow[d] & \mathbb{C}^n \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & \mathbb{C}^c \end{tikzcd}$$

Then $f^{-1}(\{0\})$ is a smooth manifold by the implicit function theorem. We call this situation standard smooth.

Lemma 12.0.3. A map $R \to A$ is smooth if and only if there exist a_i s.t. $(a_1, \dots, a_r) = A$ and $R \to A_{a_i}$ is standard smooth.

Definition 12.0.4. For a standard smooth ring map, $R \to A$ we can factor,

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R[x_{c+1}, \dots, x_n] \arrow[d] \\ & R[x_1, \dots, x_n]/(f_1, \dots, f_c) \end{tikzcd}$$

Then the downward map is étale.

Definition 12.0.5. A smooth morphism of schemes is a morphism which is locally smooth.

Remark. Using the previous lemma, for any smooth morphism of schemes $X \to Y$ it is locally standard smooth so we can factor,

$$\begin{tikzcd}
X & U \arrow[l, hook'] \arrow[d] \arrow[dr, "\text{etale}"] & \\
 & & \mathbb{A}^d_V \arrow[dl, "\text{projection}"] \\
Y \arrow[from=uu, "f"] & V \arrow[l, hook'] &
\end{tikzcd}$$

Definition 12.0.6. A variety X over k is smooth iff $X \to \mathrm{Spec}\,(k)$ is smooth.

Definition 12.0.7. A locally noetherian scheme X is *regular* or *nonsingular* iff $\mathcal{O}_{X,x}$ is regular at each $x \in X$.

Remark. For locally Noetherian schemes it suffices to check regularity on the closed points.

Theorem 12.0.8. If $X \to \mathrm{Spec}\,(k)$ is smooth then X is regular.

Theorem 12.0.9. If k is perfect then a variety is smooth iff it is regular.

Example 12.0.10. Let $k = \mathbb{F}_p(t)$ then take $\mathrm{Spec}\,(k[x]/(y^2 - (x^p - t)))$ which is regular but not smooth. Consider,

$$\mathrm{d}(y^2 - x^p + t) = 2y\mathrm{d}y + 0$$

and thus we have,

$$(f)/(f^2) \mapsto A\mathrm{d}x \oplus A\mathrm{d}y$$

which is injective but the cokernel is $A \oplus A/yA$ but A/yA has torsion so cannot be projective and thus not smooth.

However, we just need to check regularity at $(y, x^p - t) = (y) \subset A$ which is a height one ideal and generated by one element so $A_{(y)}$ is regular.

13 Differentials

Remark. See Tags O8RL, O8RT.

Definition 13.0.1. For a ring map $\varphi : R \to A$ the A-module of differentials $\Omega_{A/R}$ is generated by the symbols $\mathrm{d}a$ for $a \in A$ such that,

(a) $\mathrm{d}(a_1 + a_2) = \mathrm{d}a_1 + \mathrm{d}a_2$

(b) $\mathrm{d}a_1 a_2 = \mathrm{d}a_1 \cdot a_2 + a_1 \cdot \mathrm{d}a_2$

(c) $\mathrm{d}r = 0$ for $r \in R$

Then $\mathrm{d}_{R/A} : R \to \Omega_{R/A}$ is the universal derivation meaning that $\Omega_{R/A}$ represents the functor $\mathrm{Der}_R\,(A, -)$ i.e.

$$\mathrm{Hom}_A\left(\Omega_{A/R}, M\right) \cong \mathrm{Der}_R\,(A, M)$$

via $(f : \Omega_{A/R} \to M) \mapsto f \circ \mathrm{d}_{A/R}$

Definition 13.0.2. Given a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ then there is a universal derivation $\mathrm{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$ s.t.

$$\mathrm{Hom}_{\mathcal{O}_X}\left(\Omega_{X/Y}, \mathscr{F}\right) \cong \mathrm{Der}_{f^{-1}\mathcal{O}_Y}\left(\mathcal{O}_X, \mathscr{F}\right)$$

Where a derivation $\varphi : \mathcal{O}_X \to \mathscr{F}$ is an abelian map such that $\varphi(fs) = f\varphi(s) + \varphi(f)s$ and under $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ we send $s \in \mathcal{O}_Y(U)$ to $\varphi(s) = 0$.

Lemma 13.0.3. For a morphism of schemes $f : X \to Y$ we have,

$$\begin{tikzcd} X \arrow[d, "f"] & U = \mathrm{Spec}\,(A) \arrow[l] \arrow[d] \\ Y & V = \mathrm{Spec}\,(R) \arrow[l] \end{tikzcd}$$

Then we have $\Omega_{X/Y}|_U = \widetilde{\Omega_{A/R}}$.

13.1 The Diagonal

(Tag O1R1) Consider $R \to A$ then consider the map,

$$\Omega_{A/R} \xrightarrow{\sim} J/J^2$$

via $\mathrm{d}a \mapsto a \otimes 1 - 1 \otimes a$ where $J = \ker\left(A \otimes_R A \to A\right)$ via $a \otimes b \mapsto ab$. This situation generalizes to Schemes in which,

$$\Omega_{X/Y} = \Delta^*_{X/Y}(\mathscr{J})$$

where $\mathscr{J}$ is the sheaf of ideals of $\Delta : X \to X \times_Y X$ i.e. $\mathscr{J} = \ker\left(\mathcal{O}_{X \times_Y X} \to \Delta^*_{X/Y}\mathcal{O}_X\right)$. This is the conormal sheaf of $\Delta_{X/Y} : X \to X \times_Y X$.

14 Mar. 31

14.1 Conormal Sheaf

Definition 14.1.1. Let $\iota : Z \hookrightarrow X$ be a closed immersion with $\mathscr{I} = \ker\left(\mathcal{O}_X \to \iota_*\mathcal{O}_Z\right)$. Then the conormal sheaf is $\mathcal{C}_{Z/X} = \iota^*\mathscr{I}$ and thus $\iota_*\mathcal{C}_{Z/X} = \mathscr{I}/\mathscr{I}^2$.

Remark. Affine locally $X = \mathrm{Spec}\,(A)$ and $Z = \mathrm{Spec}\,(A/I)$. Then we get the A/I-module $I \otimes_A A/I = I/I^2$ pushing forward to the A-module I/I^2.

Proposition 14.1.2. Let $\iota : Z \hookrightarrow X$ is a closed immersion over S then there is an exact sequence,

$$\mathcal{C}_{Z/X} \longrightarrow \iota^*\Omega_{X/S} \longrightarrow \Omega_{Z/S} \longrightarrow 0$$

If $Z \to S$ is smooth then it is short exact.

Proof. This follows from the exact sequence, with $R \to A$ and $B = A/I$,

$$I/I^2 \xrightarrow{\alpha} \Omega_{A/R} \otimes_A B \xrightarrow{\beta} \Omega_{B/R} \longrightarrow 0$$

given by $f \mapsto \mathrm{d}f \otimes 1$ and $\mathrm{d}a \otimes a \mapsto \mathrm{d}\bar{a}$. The second, β, is clearly surjective. Furthermore, given $B \to \Omega_{A/R} \otimes_A B/\mathrm{Im}(\alpha)$ via $\bar{a} \mapsto \mathrm{d}a \otimes 1$ is a well-defined R-derivation and thus factors through $\Omega_{B/R}$ giving the required isomorphism. $\square$

Proposition 14.1.3. Given

Remark. The complex,

$$I/I^2 \to \bigoplus_{i=1}^{n} A\mathrm{d}x_i$$

for $A = R[x_1, \dots, x_n]/I$. This is a truncated version of $NL_{A/R}$, the naive cotangent complex of A/R and $\operatorname{coker} \mathrm{d} = \Omega_{A/R}$ and $H^{-1}(NL_{A/R}) = H^{-1}(L_{A/R})$.

Remark. In the case $A = R[x_1, \dots, x_n]/I$, we get,

$$I/I^2 \longrightarrow \Omega_{R[x_1,\dots,x_n]/R} \otimes A \longrightarrow \Omega_{A/R} \longrightarrow 0$$

and thus we see $\operatorname{coker} \mathrm{d} = \Omega_{A/R}$ as reuqired above.

Definition 14.1.4. Consider the presentation $R[A] \to A$ and $J = \ker(R[A] \to A)$. Then we can define the naive cotangent complex is $NL_{A/R} := (J/J^2 \xrightarrow{\mathrm{d}} \Omega_{R[A]/R} \otimes_{R[A]} A)$ with J/J^2 in degree -1.

Remark. The naive cotangent complex is homotopy equivalent to the complex above for a choice of presentation therefore we see the following.

Proposition 14.1.5. A ring map $R \to A$ is smooth if the naive cotangent complex $NL_{A/R}$ is quasi-isomorphic to a projective module in degree zero. Explicitly $H^{-1}(NL_{A/R}) = 0$ and $H^0(NL_{A/R}) = \Omega_{A/R}$ is projective.

15 April 2

Theorem 15.0.1. Let $f : X \to S$ be a morphism of schemes then the FAE,

(a) f is smooth

(b) f is locally finite presentation, flat, and the fibers $X_s \to \operatorname{Spec}(\kappa(s))$ are smooth.

(c) f is locally finitely presentation and f is formally smooth.

Definition 15.0.2. A morphism $f : X \to S$ is formally smooth iff every diagram of the form,

$$\begin{tikzcd} \operatorname{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(A/I^2) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

gives a map $\operatorname{Spec}(A/I^2) \to X$ making it commute.

15.1 Smoothness over Fields

Let k be a field and $S = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ with $\mathfrak{q} \subset S$ a prime ideal. Let $d = \dim S_{\mathfrak{q}} + \text{trdeg}_k(\kappa(\mathfrak{q}))$.

Proposition 15.1.1 (Jacobian Criterion)**.** The following are equivalent,

(a) S is smooth over k at $\mathfrak{q}$

(b) the rank over $\kappa(\mathfrak{q})$ of the matrix,

$$\left(\frac{\partial f}{\partial x_i} \right)_{\substack{j=1,\dots,m \\ i=1,\dots,n}} \quad \text{mod } \mathfrak{p}$$

is equal to $n - d$ (it is always at most $n - d$).

Remark. Smoothness at $\mathfrak{q}$ implies that S is regular at $\mathfrak{q}$ but this is not sufficient.

Remark. If S is equidimensional of dimension d then the nonsmooth locus of $\text{Spec}(S)$ is the vanishing locus of all $(n-d) \times (n-d)$ minors of the matrix,

$$\left(\frac{\partial f}{\partial x_i} \right)_{\substack{j=1,\dots,m \\ i=1,\dots,n}}$$

Example 15.1.2. Let $Z = V(F_1, \dots, F_c) \subset \mathbb{P}^n_k = X$ be a complete intersection of type $(d_1, \dots, d_c)$. We saw that,

$$\omega_Z \cong \mathscr{E}\!xt^i_{\mathcal{O}_X}(\mathcal{O}_Z, \omega_X) = \omega_X(d_1 + \cdots + d_c)|_Z$$

Theorem 15.1.3. If $Z \subset X = \mathbb{P}^n_k$ is a smooth projective scheme equidimensional of $\dim Z = n - c$ then,

$$\omega_Z = \mathscr{E}\!xt^c_{\mathcal{O}_X}(\mathcal{O}_Z, \omega_X) \cong \omega_X|_Z \otimes \bigwedge^c \mathcal{C}_{Z/X}^\vee$$

Proof. See HAR [III.7.11 + II.8.17] The exact sequence,

$$0 \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \Omega_{X/k}|_Z \longrightarrow \Omega_{Z/k} \longrightarrow 0$$

implies that $\mathcal{C}_{Z/X}$ is a vector bundles of rank c. Then Z is locally a complete intersection so we can conclude using our previous argument. $\square$

Lemma 15.1.4. Consider an exact sequence of locally free sheaves,

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow 0$$

then there is a canonical isomorphism of line bundles,

$$\bigwedge^{\text{top}} \mathscr{F}_1 \otimes_{\mathcal{O}_X} \bigwedge^{\text{top}} \mathscr{F}_3 = \bigwedge^{\text{top}} \mathscr{F}_2$$

Proposition 15.1.5. For $X = \mathbb{P}^n_k$ we know,

$$\omega_X \cong \mathcal{O}_X(-n-1) \cong \bigwedge^n \Omega_{X/k}$$

Proof. There is a short exact sequence of locally free sheaves,

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_X(-1)\mathrm{d}T_i \longrightarrow \mathcal{O}_X \longrightarrow 0$$

on $D_+(T_i)$ given by,

$$\mathrm{d}\left(\frac{T_i}{T_j}\right) = \frac{\mathrm{d}T_i}{T_j} - \frac{T_i \mathrm{d}T_j}{T_j^2}$$

and,

$$\frac{\mathrm{d}T_i}{T_j} \mapsto \frac{T_i}{T_j}$$

Then, taking top exterior powers gives.

$$\bigwedge^{n+1}\left(\bigoplus_{i=0}^{n} \mathcal{O}_X(-1)\right) = \mathcal{O}_X(-n-1)$$

□

Lemma 15.1.6 (Adjunction)**.** Let $Z \subset X$ be a smooth closed subscheme. Then,

$$\bigwedge^{\dim Z} \Omega_{Z/k} \cong \bigwedge^{\dim X} \Omega_{X/k}|_Z \otimes \bigwedge^{c} \mathcal{C}_{Z/X}^{\vee}$$

Proof. This is exactly the top exterior power of the sequence,

$$0 \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \Omega_{X/k}|_Z \longrightarrow \Omega_{Z/k} \longrightarrow 0$$

□

Remark. This implies that $K_Z = K_X|_Z + c_1(N_{Z/X})$.

Theorem 15.1.7. Let Z be a smooth projective variety over k. Then $\omega_Z \cong \bigwedge^{\dim Z} \Omega_{Z/k}$.

Proof. Choose an embedding $Z \hookrightarrow \mathbb{P}^n_k$ then we use adjunction for dualizing sheaves and for canonical sheaves which are the same to conclude. □

16 Varieties

Definition 16.0.1. A *variety* is an integral separated scheme with $X \to \mathrm{Spec}\,(k)$ finite type.

Remark. Problems: products of varieties are not varieties. E.g. $k = \mathbb{Q}$ and $X = \mathrm{Spec}\,(\mathbb{Q}(i))$ then $X \times_{\mathbb{Q}} X = \mathrm{Spec}\,(\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i))$ is not integral.

The problem is that varieties are not geometrically integral.

Definition 16.0.2. A scheme X over k is geometrically integral if $X_{\bar{k}} = X \times_k \bar{k}$ is integral.

Remark. There are not "enough" rational points of a general variety. For example, given,

$$X = V(x^2 + y^2 + z^2) \subset \mathbb{P}^3_{\mathbb{Q}}$$

then $X(\mathbb{Q}) = \varnothing$. We say $X(k) = \mathrm{Hom}_k\left(\mathrm{Spec}\,(k), X\right)$ is the set of k-rational points.

Remark. Alternative definition: we require that varieties are geometrically integral. Then products of varieties are varieties.

Definition 16.0.3. A *curve* is a variety of dimension one.

Lemma 16.0.4. A curve is regular iff it is normal.

Proof. A Noetherian local ring of dimension one is regular iff it is normal iff it is a DVR. □

Lemma 16.0.5. A curve is either affine or projective.

16.1 Rational Maps

Definition 16.1.1. Let X, Y be varieties over k. A rational map $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, f) with $U \subset X$ is a dense open and $f : U \to Y$ is a morphism on U such that $(U, f_U) \sim (V, f_V)$ iff there is $W \subset U \cap V$ with $f_U|_W = f_V|_W$. Since X, Y are reduced and separated this implies that $f_U|_{U\cap V} = f_V|_{U\cap V}$.

Remark. Rational maps cannot be composed e.g. const map to a point then projection away from that point is undefined. To fix this we ask that the rational maps be dominant.

Definition 16.1.2. A rational map $f : X \dashrightarrow Y$ is dominant if for any representative (f, U) then $f(U) \subset Y$ is dense.

Theorem 16.1.3. The category varieties over k with rational maps is anti-equivalent to the category of finitely generated extensions of k. Sending $(f : X \dashrightarrow Y) \mapsto f_\eta : K(Y) \to K(X)$.

Definition 16.1.4. The function field $K(X)$ of an integral scheme is $\mathcal{O}_{X,\xi}$ where $\xi \in X$ is the generic point. For any affine open $\mathrm{Spec}\,(A) \subset X$ then A is a domain and $(0) \subset A$ is the generic point of $\mathrm{Spec}\,(A)$ and also of X. Then, $\mathcal{O}_{X,\xi} = \mathrm{Frac}\,(A)$. If X is finite type over k then A is a finitely-generated k-algebra so $K(X) = \mathrm{Frac}\,(A)$ is a finitely-generated field extension of k.

Lemma 16.1.5. Let X, Y be varieties over k. Any k-map $\varphi : K(Y) \to K(X)$ induces a rational map $X \dashrightarrow Y$ acting by φ on the generic points.

Proof. Let $U = \mathrm{Spec}\,(A) \subset X$ be an open and $V = \mathrm{Spec}\,(B) \subset Y$ an open. Then $B = k[b_1, \dots, b_n]$ we can wrtie $\varphi(b_i) = \frac{a_i}{a_i'}$ for $a_i, a_i' \in A$ with $a_i' \neq 0$. Then replace U by $D(a_1', \dots, a_n') \subset \mathrm{Spec}\,(A) \subset X$. Then we get,

$$\begin{tikzcd} K(X) & K(Y) \arrow[l, hook', "\varphi"'] \\ A' \arrow[u, hook] & B \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

This is dominant since $\phi^{-1}(0) = (0)$ because ϕ is injective. □

Definition 16.1.6. Varieties X, Y over k are *birational* iff they are isomorphic in the rational category iff $K(X) = K(Y)$ as k-extensions.

17 Curves

Theorem 17.0.1. There is an anti-equivalence of categories between the category of normal projective curves over k and the category of trancendence degree one field extensions of k.

Proposition 17.0.2. The functor $C \mapsto K(C)$ is essentially surjective.

Proof. Given a transendence degree one field K/k we know $K = k(X)$ for some affine $X \subset \mathbb{A}^n_k$ then take the closure $\overline{X} \subset \mathbb{P}^n_k$. Since $X \subset \overline{X}$ is a dense open then $K(\overline{X}) = K(X) = K$. Then we normalize to get $\overline{X}^\nu \to \overline{X}$ which is birational so get get $K(\overline{X}^\nu) = K(X)$. $\square$

Lemma 17.0.3. Let $X \dashrightarrow Y$ be a rational map from a normal curve to a projective variety. Then it extends to $X \to Y$.

Proof. We have $X \dashrightarrow Y \subset \mathbb{P}^n_k$ since Y is closed it is enough to extend $X \to \mathbb{P}^n_k$. After replacing $\mathbb{P}^n_k$ by a smaller projective space we may assume the map is $[f_0 : \cdots : f_n]$ for not all zero $f_i \in k(X)$. For a closed point $x \in X$ the ring $\mathcal{O}_{X,x}$ is a DVR with uniformizer ϖ since X is a normal curve. Then $K(X) = \mathrm{Frac}\,(\mathcal{O}_{X,x})$ so we can write $f_i = u_i \varpi^{n_i}$ for $u_i \in \mathcal{O}_{X,x}^\times$ and $n_i \in \mathbb{Z}$. Let $N = \min n_i$. Now,

$$[f_0 : \cdots : f_n] = [\varpi^N f_0 : \cdots : \varpi^N f_n]$$

so take $g_i = \varpi^N f_i \in \mathcal{O}_{X,x}$ and there is some $g_i \in \mathcal{O}_{X,x}^\times$ so there exists $U \subset X$ open such that $g_i \in \mathcal{O}_X(U)$ and $g_j \in \mathcal{O}_X(U)^\times$. Therefore, can extend,

$$[g_0 : \cdots : g_n] : U \to D_+(T_i) \subset \mathbb{P}^n_k$$

$\square$

Proposition 17.0.4. The functor $C \mapsto K(C)$ is fully faithful.

Proof. $\square$

Proposition 17.0.5. If Y is projective then its normalization Y^ν is projective.

18 April 16

Definition 18.0.1. Let $T \subset X$ be a proper closed subscheme of X. Then,

$$[T]_{\dim X - 1} = \sum_{Z \subset T} m_{Z,T}[Z]$$

where Z is a prime divisor with generic point $\xi \in Z$ and,

$$m_{Z,T} = \ell_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{T,\xi'})$$

Definition 18.0.2. Let $f : X \to Y$ be a morphism of schemes. Then we get a morphism $f^* : \mathrm{Div}\,(Y) \to \mathrm{Div}\,(X)$ via,

$$[Z] \mapsto [f^{-1}(Z)]_{\dim X - 1}$$

where $f^{-1}(Z) = X \times_Y Z$.

Let $f : X \to Y$ be a flat morphism of varieties. We get a map $f^* : \mathrm{Div}\,(Y) \to \mathrm{Div}\,(X)$ and $f^{-1}(Z)$ has pure codimension one. Furthermore, we get the following.

Proposition 18.0.3. $f^* \mathrm{div}_Y(g) = \mathrm{div}_X(g \circ f)$

Therefore, we get $f^* : \mathrm{Cl}\,(Y) \to \mathrm{Cl}\,(X)$. Furthermore, there is a diagram,

$$\begin{array}{ccc} \mathrm{Pic}\,(Y) & \xrightarrow{f^*} & \mathrm{Pic}\,(X) \\ \downarrow & & \downarrow \\ \mathrm{Cl}\,(Y) & \longrightarrow & \mathrm{Cl}\,(X) \end{array}$$

Example 18.0.4. For nonsingular curves, any nonconstant map $f : X \to Y$ is flat.

18.1 Ramification

Let $f : X \to Y$ be a nonconstant map of nonsingular curves. Then for $x \in X$ we have $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ which is a map of DVRs. Then we define the ramification index e_x via $\varpi_{f(x)} \mapsto \varpi_x^{e_x}$.

Then we find,

$$f^*[y] = \sum_{x \in f^{-1}(y)} e_x[x]$$

To see this, consider $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Now, $y \in Y$ is the closed subscheme cut out by a sheaf of ideals $M_y \subset \mathcal{O}_Y$ such that $(M_y)_y = \mathfrak{m}_y \subset \mathcal{O}_{Y,y}$. Now, $f^{-1}(y) \subset X$ is the closed subscheme of X cut out by $f^* M_y \subset \mathcal{O}_X = f^* \mathcal{O}_Y$. If $\varpi \in \mathcal{O}_{Y,y}$ is uniformizer. Then $M_y \subset \mathcal{O}_Y$ is locally generated by π near y. Therefore, $f^* M_y$ is generated by pullback of ϖ_y. But under $\mathcal{O}_{Y,Y} \to \mathcal{O}_{X,x}$ we get $\varpi_y \mapsto \varpi_x^{e_x}$. Therefore,

$$\ell_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x}/\varpi_x^e \mathcal{O}_{X,x}\right) = e_x$$

18.2 Finite Pushforward

let $f : X \to Y$ be finite and dominant morphism of varieties. Let, $f_* : \mathrm{Div}\,(X) \to \mathrm{Div}\,(Y)$ via,

$$[Z] \mapsto \deg\left(Z/f(Z)\right)[f(Z)]$$

Since $\dim X = \dim Y$ and f finite we see that $f(Z) \subset Y$ is a prime divisor (it is closed since f is proper). So we define,

$$\deg\left(Z/f(Z)\right) = [K(Z) : K(f(Z))]$$

Definition 18.2.1. Let $f : X \dashrightarrow Y$ be a dominant rational map of varieties with $\dim X = \dim Y$. Then we define the *degree*,

$$\deg\left(f\right) = [K(X) : K(Y)]$$

which is finite since they are finitely-generated field extensions of k with the same transcendence degree.

Proposition 18.2.2. Pushforward is functorial with respect to finite dominant morphisms. Also, $f_* \mathrm{div}_X(g) = \mathrm{div}_Y(N_{X/Y}(g))$ where $N_{X/Y} : K(X)^\times \to K(Y)^\times$ is the norm of the finite extension $k(X)/K(Y)$. Therefore, we get an induced map,

$$f_* : \mathrm{Cl}\,(X) \to \mathrm{Cl}\,(Y)$$

Example 18.2.3. If $f : X \to Y$ is a finite morphism of nonsingular curves over k. Let $x \in X$ be a closed point and $y = f(x) \in Y$. Then,

$$f_*[x] = [\kappa(x) : \kappa(y)][y]$$

If k is algebraically closed then $f_*[x] = [y]$.

When $f : X \to Y$ is a finite flat map. Then there is a multiplicative norm map,

$$N_f : f_*\mathcal{O}_X \to \mathcal{O}_Y$$

of degree $d = \deg(X/Y)$ which induces a norm map,

$$N_f : \mathrm{Pic}\,(X) \to \mathrm{Pic}\,(Y)$$

as follows. Consider $N_f : f_*\mathcal{O}_X^\times \to \mathcal{O}_Y$ which is a homomorphism. This induces,

$$N_f : H^1(X, \mathcal{O}_X^\times) \to H^1(Y, \mathcal{O}_X^\times)$$

(See OBCX). There is a commutative diagram,

$$\begin{tikzcd}
\mathrm{Pic}\,(X) \arrow[r, "N_f"] \arrow[d] & \mathrm{Pic}\,(Y) \arrow[d] \\
\mathrm{Cl}\,(X) \arrow[r, "f_*"] & \mathrm{Cl}\,(Y)
\end{tikzcd}$$

Remark. We can show that,

$$N_f(\mathcal{L}) = \bigwedge^d (f_*\mathcal{L}) \otimes \bigwedge^d (f_*\mathcal{O}_X)^{\otimes -1}$$

Alternatively,

Lemma 18.2.4. In the case of curves. For $D \in \mathrm{Div}\,(X)$ we have,

$$\deg(D) = \deg(f_*D)$$

Proof. Suppose that $D = [x]$ and $y = f(x) \in Y$. Then $f_*[x] = [\kappa(x) : \kappa(y)][y]$ then,

$$\deg(f_*[x]) = [\kappa(x) : \kappa(y)][\kappa(y) : k] = [\kappa(x) : k] = \deg[x]$$

Then the lemma holds by additivity. □

Corollary 18.2.5. Let X be a projective curve then $\deg(\mathrm{div}_X(f)) = 0$ for any $f \in K(X)^\times$.

Proof. Suppose k is infinite. Choose $X \hookrightarrow \mathbb{P}_k^N$. Then project away from X to get,

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr] \arrow[dddr] & \mathbb{P}_k^N \arrow[d, dashed] \\
& \mathbb{P}_k^{N-1} \arrow[d] \\
& \vdots \arrow[d] \\
& \mathbb{P}_k^1
\end{tikzcd}$$

Thus we get $\varphi : X \to \mathbb{P}^1_k$ which is nonconstant and thus finite flat. So any projective curve is a finite cover of $\mathbb{P}^1_k$. Then we use,

$$f_*(\mathrm{div}_X(f)) = \mathrm{div}_{\mathbb{P}^1_k}(N_\varphi(f)) = 0$$

since we have proven the fact for $\mathbb{P}^1_k$. $\square$

Corollary 18.2.6. For a projective curve we get the following diagram,

$$\begin{tikzcd}
 & 0 & \\
\mathrm{Pic}\,(X) \arrow[r] & \mathrm{Cl}\,(X) \arrow[u] \arrow[r, "\deg"] & \mathbb{Z} \\
 & \mathrm{Div}\,(X) \arrow[u] \arrow[ur, "\deg"] & \\
 & K(X)^\times \arrow[u] \arrow[uur] &
\end{tikzcd}$$

which defines degree maps $\deg : \mathrm{Cl}\,(X) \to \mathbb{Z}$ and $\deg : \mathrm{Pic}\,(X) \to \mathbb{Z}$.

Theorem 18.2.7 (Riemann-Roch)**.** For X projective curve and $\mathcal{L} \in \mathrm{Pic}\,(X)$ we have,

$$\chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X) = \deg \mathcal{L}$$

where,

$$\chi(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L})$$

Remark. We will use the following notation,

$$h^p(X, \mathscr{F}) = \dim_k H^p(X, \mathscr{F})$$