1 Homology

1.1 Introduction

Define a standard (unfilled) triangle with vertices α, β, γ and edges a, b, c. We will cook up some free abelian groups, $C_0 = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma$ the free group on the vertices and $C_1 = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$ the free group on the edges. Now define the boundary map $\partial : C_1 \to C_0$ by $\partial b = \alpha - \gamma$ and $\partial a = \gamma - \beta$ and $\partial c = \alpha - \beta$. Then the diagram,

$$0 \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

is a complex meaning that the composition of any two maps is the zero map. Consider the kernel of ∂. Which is the set,

$$\{ta \oplus ub \oplus vc \mid t(\gamma - \beta) + u(\alpha - \gamma) + v(\alpha - \beta) = 0\}$$

which has solutions, $t = u = -v$ which is the set $\{(1, 1, -1) \cdot t \mid t \in \mathbb{Z}\} \cong \mathbb{Z}$. We call this $H_1(C) = \ker \partial \cong \mathbb{Z}$ the first Homology group.

Now consider the filled triangle labeled in the same way. Now we have a 2-cell called A representing the filled triangle so $C_2 = \mathbb{Z}A$. Now define the boundary map $\partial_2 : C_2 \to C_1$ defined by $\partial_2 A = a+b-c$ (with some choice of orientation). Now, $H_1(C) = \ker \partial_1 \mathrm{Im}(\partial_2) \cong (1, 1, -1)\mathbb{Z}/(1, 1, -1)\mathbb{Z} = 0$.

1.2 Basic Definitions

Definition: A complex is any diagram such that the composition of any two maps (if it exists) is the zero map. In particular,

$$\cdots \xrightarrow{\partial_7} C_6 \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is a complex if $\mathrm{Im}(\partial_{i+1}) \subset \ker \partial_i$. We call the C_i chains and the $\ker \partial_i$ cycles and the $\mathrm{Im}(\partial_{i+1})$ boundaries.

Definition: Given a complex as above, the i^{th} homology group is given by,

$$H_i(C) = \ker \partial_i / \mathrm{Im}(\partial_{i+1})$$

Lemma 1.1. A sequence is exact if and only if it is a complex with trivial Homology groups.

1.3 Simplicial Homology

Definition: The standard n-simplex is the subset,

$$\Delta^n = \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \quad \text{and} \quad \forall i : t_i \geq 0 \right\}$$

We give Δ^n an orientation by ordering the vertices in the sequence defined by the order of the standard basis of $\mathbb{R}^{n+1}$,

$$(1, 0, \cdots, 0), \quad (0, 1, \cdots, 0), \quad \cdots \quad (0, 0, \cdots, 1)$$

Definition: An n-simplex is the convex hull of $n+1$ points in $\mathbb{R}^m$ that do not lie in any n-dimensional hyperplane.

Definition: The faces of an n-simplex are the convex hulls of any subset with n points of the simplex. There are $n+1$ faces each of which is an $n-1$-simplex.

Definition: A Δ-complex X is a topological space along with a collection of maps $\sigma_\alpha : \Delta^n \to X$ (where n can depend on α) subject to the constraints,

(a). $\sigma_\alpha|_{(\Delta^n)^\circ}$ is injective and if $\alpha \neq \beta$ then $\mathrm{Im}(\sigma_\alpha|_{(\Delta^n)^\circ}) \cap \mathrm{Im}(\sigma_\beta|_{(\Delta^n)^\circ}) = \varnothing$

(b). σ_α restricted to a face of Δ^n is equal to some σ_β up to homoeomorphism of the domains.

(c). A set $U \subset X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open for every α.

Definition: Given a Δ-complex X define $C_n(X)$ to be the free abelian group on all $\sigma_\alpha : \Delta^n \to X$ with n fixed and define the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ by,

$$\partial(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{i^{\text{th}}-\text{face}}$$

Lemma 1.2. Given a Δ-complex X the sequence $C(X)$ given by,

$$\cdots \xrightarrow{\partial_7} C_6 \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is a complex.

Proof.

$$\begin{aligned}
\partial_{n-1} \circ \partial_n(\sigma_\alpha) &= \sum_{i>j} (-1)^{j+i} (\sigma_\alpha|_{i^{\text{th}}-\text{face}})|_{j^{\text{th}}-\text{face}} + \sum_{i<j} (-1)^{j+i} (\sigma_\alpha|_{i^{\text{th}}-\text{face}})|_{(j-1)^{\text{th}}-\text{face}} \\
&= \sum_{i>j} (-1)^{j+i} (\sigma_\alpha|_{i^{\text{th}}-\text{face}})|_{j^{\text{th}}-\text{face}} + \sum_{i<j} (-1)^{j+1+i} (\sigma_\alpha|_{i^{\text{th}}-\text{face}})|_{j^{\text{th}}-\text{face}} \\
&= \sum_{i>j} (-1)^{j+i} (\sigma_\alpha|_{i^{\text{th}}-\text{face}})|_{j^{\text{th}}-\text{face}} - \sum_{i<j} (-1)^{j+i} (\sigma_\alpha|_{i^{\text{th}}-\text{face}})|_{j^{\text{th}}-\text{face}} = 0
\end{aligned}$$

□

Definition: Let X be a Δ-complex then the n^{th} homology group is,

$$H_n(X) = \ker \partial_n / \mathrm{Im}(\delta_{n+1})$$

which is the homology of the complex $C(X)$.