Physics GR6037 Quantum Mechanics I
Assignment # 4

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December 7, 2017

Problem 12.

Let the Hamiltonian be given by,

$$H(\vec{r},\vec{p}) = \frac{\left(\vec{p} - \frac{p}{c}\vec{A}(\vec{r})\right)^2}{2m}$$

(a). Applying Hamilton's Equations:

$$\frac{\mathrm{d}H}{\mathrm{d}r_i} = \frac{1}{m}\left(p_j - \frac{q}{c}A_j\right)\left(-\frac{q}{c}\partial_i A_j\right) = -\dot{p}_i$$
$$\frac{\mathrm{d}H}{\mathrm{d}p_i} = \frac{1}{m}\left(p_i - \frac{q}{c}A_i\right) = \dot{r}_i$$

(b). Differentiating,

$$\ddot{r}_i = \frac{1}{m}\left(\dot{p}_i - \frac{q}{c}\frac{\mathrm{d}}{\mathrm{d}t}A(\vec{r})\right) = \frac{1}{m}\left(\dot{p}_i - \frac{q}{c}\dot{r}_j\partial_j A_i\right)$$

Now rewriting the first Hamilton equation as $\dot{p}_i = \frac{q}{c}\dot{r}_j\partial_i A_j$ and plugging in,

$$\ddot{r}_i = \frac{1}{m}\left(\frac{q}{c}\dot{r}_j\partial_i A_j - \frac{q}{c}\dot{r}_j\partial_j A_i\right) = \frac{q}{mc}\dot{r}_i\left(\partial_i A_j - \partial_j A_i\right) = \frac{q}{mc}\dot{r}_j F_{ij}$$

The space-space components of the Faraday tensor are $F_{ij} = \epsilon_{ijk}\left(\nabla \times \vec{A}\right)_k$ so,

$$\ddot{r}_i = \frac{q}{mc}\epsilon_{ijk}\dot{r}_j\left(\nabla\times\vec{A}\right)_k = \frac{q}{mc}\left(\dot{\vec{r}}\times\left(\nabla\times\vec{A}\right)\right)_i$$

Thus,

$$m\ddot{\vec{r}} = \frac{q}{c}\,\dot{\vec{r}}\times\vec{B}$$

Problem 13.

(a). In cylindrical coordinates, the time independent Schrodinger Equation becomes,

$$\begin{aligned} E\psi(\rho,\theta,z) &= -\frac{\hbar^2}{2m}\nabla^2\psi(\rho,\theta,z) + +V_\rho(\rho)\psi(\rho,\theta,z) + V_z(z)\psi(\rho,\theta,z) \\ &= -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V_\rho(\rho)\psi(\rho,\theta,z) + V_z(z)\psi(\rho,\theta,z)\end{aligned}$$

Making a seperation of variables, $\psi(\rho,\theta,z) = \psi_\rho(\rho)\psi_\theta(\theta)\psi_z(z)$ we get,

$$E = -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_\rho}{\partial\rho}\right)\frac{1}{\psi_\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}\frac{1}{\psi_\theta} + \frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z}\right) + V_\rho(\rho) + V_z(z)$$

This can be partitioned into terms which depend only on disjoint variables,

$$\left(E + \frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_\rho}{\partial\rho}\right)\frac{1}{\psi_\rho} + \frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z}\right) - V_\rho(\rho) - V_z(z)\right)\frac{\rho^2}{R^2} = -\frac{\hbar^2}{2mR^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}\frac{1}{\psi_\theta}$$

Both sides must be constant because the LHS does not depend on θ but the RHS depends on θ alone. Thus,

$$\mathcal{E} = -\frac{\hbar^2}{2mR^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}\frac{1}{\psi_\theta}$$
$$E = -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_\rho}{\partial\rho}\right)\frac{1}{\psi_\rho} + \frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z}\right) + \mathcal{E}\frac{R^2}{\rho^2} + V_\rho(\rho) + V_z(z)$$

The first equation is trivially solved by

$$\psi_\theta(\theta) = A\cos k\theta + B\sin k\theta \quad \text{with} \quad k = \sqrt{\frac{2mR^2\mathcal{E}}{\hbar^2}}$$

We can ignore negative $\mathcal{E}$ solutions because rising and falling exponentials cannot meet the periodic boundary conditions. The periodic boundary conditions give:
$\psi_\theta(\theta + 2\pi) = \psi_\theta(\theta)$ so $k = n \in \mathbb{Z}$ Thus,

$$\mathcal{E} = \frac{\hbar^2 n^2}{2mR^2}$$

Now we make the approximation that V_ρ and V_z tightly bind the particle about $(R,\theta,0)$ and change much more rapidly than $\frac{\rho^2}{R^2}$. Thus, we make the approximation,

$$E = -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_\rho}{\partial\rho}\right)\frac{1}{\psi_\rho} + \frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z}\right) + \mathcal{E} + V_\rho(\rho) + V_z(z)$$

Also, if the potentials are very tightly binding, then the exicted states in ρ, z coordinates will be on a much higher energy scale than motion about R. Let E_0 be the ground state energy of,

$$E = -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_\rho}{\partial\rho}\right)\frac{1}{\psi_\rho} + \frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z}\right) + V_\rho(\rho) + V_z(z)$$

At energies above E_0 which are small compared to the potentials,

$$E - \mathcal{E} = -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_\rho}{\partial\rho}\right)\frac{1}{\psi_\rho} + \frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z}\right) + V_\rho(\rho) + V_z(z)$$

can only be in the ground state so

$$E = E_0 + \frac{\hbar^2 n^2}{2mR^2}$$

(b). In cartesian coordinates, the time independent Schrodinger Equation becomes,

$$\begin{aligned}
E\psi(x,y,z) &= -\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) + V(x,y,z)\psi(x,y,z) \\
&= -\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V(x,y,z)\psi(x,y,z)
\end{aligned}$$

Making a seperation of variables, $\psi(x,y,z) = \psi_x(x)\psi_y(y)\psi_z(z)$ we get,

$$E = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi_x}{\partial x^2}\frac{1}{\psi_x} + \frac{\partial^2\psi_y}{\partial y^2}\frac{1}{\psi_\theta} + \frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z}\right) + V(x,y,z)$$

Inside the box, $V(x,y,z) = 0$ so the equation is totally seperated,

$$\begin{aligned}
k_x^2 &= -\frac{\partial^2\psi_x}{\partial x^2}\frac{1}{\psi_x} \\
k_y^2 &= -\frac{\partial^2\psi_y}{\partial y^2}\frac{1}{\psi_y} \\
k_z^2 &= -\frac{\partial^2\psi_z}{\partial z^2}\frac{1}{\psi_z} \\
E &= \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)
\end{aligned}$$

Each equation is easily solved by

$$\psi_i(r_i) = A_i\cos k_i r_i + B_i \sin k_i r_i$$

but in each coordinate, $\psi_i(0) = \psi_i(L) = 0$ by boundary conditions. We can ignore imaginary k_i solutions because rising and falling exponentials cannot be zero at more than one point which violates the boundary conditions. Thus, $k_i = \frac{n_i\pi}{L}$ for $n \in \mathbb{Z}^+$ and $A_i = 0$. Then,

$$E = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

(c). Consider a torus embedded in $\mathbb{R}^3$ with the parametrization:

$$\begin{aligned}
x &= (R + r\cos\phi)\cos\theta \\
y &= (R + r\cos\phi)\sin\theta \\
z &= r\sin\phi
\end{aligned}$$

We calculate the basis vectors in the surface via:

$$\begin{aligned}
\vec{e}_\theta &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}\theta} = -(R + r\cos\phi)\sin\theta\,\hat{\imath} + (R + r\cos\phi)\cos\theta\,\hat{\jmath} \\
\vec{e}_\phi &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}\phi} = -r\sin\phi\cos\theta\,\hat{\imath} - r\sin\phi\sin\theta\,\hat{\jmath} + r\cos\phi\,\hat{k}
\end{aligned}$$

Then the metric is calculated from the dot products of basis vectors:

$$
\begin{aligned}
g_{\theta\theta} &= \vec{e}_\theta \cdot \vec{e}_\theta = (R + r\cos\phi)^2 \cdot (\sin^2\theta + \cos^2\theta) = (R + r\cos\phi)^2 \\
g_{\theta\phi} &= g_{\phi\theta} = \vec{e}_\theta \cdot \vec{e}_\phi = r(R + r\cos\phi)\sin\theta\sin\phi\cos\theta - r(R + r\cos\phi)\cos\theta\sin\phi\sin\theta = 0 \\
g_{\phi\phi} &= \vec{e}_\phi \cdot \vec{e}_\phi = r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi = r^2
\end{aligned}
$$

So the metric is,

$$
\mathbf{g} = \begin{pmatrix} (R + r\cos\phi)^2 & 0 \\ 0 & r^2 \end{pmatrix}
$$

Which is (thank the heavens) diagonal and has determinant, $g = \det\mathbf{g} = r^2(R + r\cos\phi)^2$. Applying the Voss-Weyl formula,

$$
\nabla^2\psi = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\, g^{ij}\, \partial_j\psi\right)
$$

where $g^{ij} = (g^{-1})_{ij}$ we arive at,

$$
\begin{aligned}
\nabla^2\psi &= \frac{1}{r(R + r\cos\phi)}\left[\frac{\partial}{\partial\theta}\left(r(R + r\cos\phi)(R + r\cos\phi)^{-2}\frac{\partial}{\partial\theta}\psi\right) + \frac{\partial}{\partial\phi}\left(r(R + r\cos\phi)r^{-2}\frac{\partial}{\partial\phi}\psi\right)\right] \\
&= \frac{1}{(R + r\cos\phi)^2}\frac{\partial^2}{\partial\theta^2}\psi + \frac{1}{r^2(R + r\cos\phi)}\frac{\partial}{\partial\phi}\left((R + r\cos\phi)\frac{\partial}{\partial\phi}\psi\right)
\end{aligned}
$$

Thus, on the surface of the torus, the time independent Schrodinger Equation becomes,

$$
\begin{aligned}
E\psi(\theta,\phi) &= -\frac{\hbar^2}{2m}\nabla^2\psi(\theta,\phi) \\
&= -\frac{\hbar^2}{2m}\left[\frac{1}{(R + r\cos\phi)^2}\frac{\partial^2}{\partial\theta^2}\psi + \frac{1}{r^2(R + r\cos\phi)}\frac{\partial}{\partial\phi}\left((R + r\cos\phi)\frac{\partial}{\partial\phi}\psi\right)\right]
\end{aligned}
$$

Now, let us take $r \ll R$ so that we may drop $r\cos\phi$ compared with R,

$$
E\psi(\theta,\phi) = -\frac{\hbar^2}{2m}\left(\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)\psi(\theta,\phi)
$$

Next, introduce a seperation of variables, $\psi(\theta,\phi) = \psi_\theta(\theta)\psi_\phi(\phi)$ then,

$$
E = -\frac{\hbar^2}{2m}\left(\frac{1}{R^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}\frac{1}{\psi_\theta} + \frac{1}{r^2}\frac{\partial^2\psi_\phi}{\partial\phi^2}\frac{1}{\psi_\phi}\right)
$$

The two terms contain only disjoint variables so they must both be constants. Let,

$$
\begin{aligned}
\frac{\partial^2\psi_\theta}{\partial\theta^2}\frac{1}{\psi_\theta} &= -k_\theta^2 \\
\frac{\partial^2\psi_\phi}{\partial\phi^2}\frac{1}{\psi_\phi} &= -k_\phi^2 \\
E &= \frac{\hbar^2}{2m}\left(\frac{k_\theta^2}{R^2} + \frac{k_\phi^2}{r^2}\right)
\end{aligned}
$$

These equations are eaily solved by,

$$\psi_\theta(\theta) = A_\theta \cos k_\theta \theta + B_\theta \sin k_\theta \theta$$
$$\psi_\phi(\phi) = A_\phi \cos k_\phi \phi + B_\phi \sin k_\phi \phi$$

We can ignore negative k solutions because rising and falling exponentials cannot meet the periodic boundary conditions. The periodic boundary conditions give $\psi_\theta(\theta + 2\pi) = \psi_\phi(\phi)$ so $k_\theta = n_\theta \in \mathbb{N}$ and $\psi_\phi(\phi + 2\pi) = \psi_\theta(\theta)$ so $k_\phi = n_\phi \in \mathbb{N}$ Thus,

$$E = \frac{\hbar^2}{2m}\left(\frac{n_\theta^2}{R^2} + \frac{n_\phi^2}{r^2}\right)$$

Problem 14.

Let the Hamiltonian be given by,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - qE\hat{x}$$

(a). We proceede by completing the square in $\hat{x}$,

$$\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x}^2 - \frac{2qE}{m\omega^2}\hat{x} + \left(\frac{qE}{m\omega^2}\right)^2 - \left(\frac{qE}{m\omega^2}\right)^2\right) \\ &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x} - \frac{qE}{m\omega^2}\right)^2 - \left(\frac{q^2E^2}{2m\omega^2}\right)\end{aligned}$$

Introduce rasing and lowering operators,

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{qE}{m\omega^2} - \frac{i}{m\omega}\hat{p}\right)$$
$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{qE}{m\omega^2} + \frac{i}{m\omega}\hat{p}\right)$$

Then,

$$\begin{aligned}\hat{a}^\dagger\hat{a} &= \frac{m\omega}{2\hbar}\left\{\left(\hat{x} - \frac{qE}{m\omega^2}\right)^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}\left[\left(\hat{x} - \frac{qE}{m\omega^2}\right)\hat{p} - \hat{p}\left(\hat{x} - \frac{qE}{m\omega^2}\right)\right]\right\} \\ &= \frac{1}{\hbar\omega}\left[\frac{1}{2}m\omega^2\left(\hat{x} - \frac{qE}{m\omega^2}\right)^2 + \frac{\hat{p}^2}{2m} - \frac{1}{2}\hbar\omega\right]\end{aligned}$$

because $[\hat{x} - c, \hat{p}] = i\hbar$ for any constant c. Pluggin into $\hat{H}$,

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega - \left(\frac{q^2E^2}{2m\omega^2}\right)$$

This is a standard quantum harmonic oscillator shifted by a constant energy. Now,

$$\begin{aligned}\hat{a}\hat{a}^\dagger &= \frac{m\omega}{2\hbar}\left\{\left(\hat{x} - \frac{qE}{m\omega^2}\right)^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{i}{m\omega}\left[\left(\hat{x} - \frac{qE}{m\omega^2}\right)\hat{p} - \hat{p}\left(\hat{x} - \frac{qE}{m\omega^2}\right)\right]\right\} \\ &= \frac{1}{\hbar\omega}\left[\frac{1}{2}m\omega^2\left(\hat{x} - \frac{qE}{m\omega^2}\right)^2 + \frac{\hat{p}^2}{2m} + \frac{1}{2}\hbar\omega\right]\end{aligned}$$

Thus, $[\hat{a}, \hat{a}^\dagger] = 1$ so $[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger$ and $[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}$. Thus, standard quantum harmonic oscilator results apply. In particular, there exists $|0\rangle$ such that $\hat{a}\,|0\rangle = 0$ and every energy eigenstate is some $|n\rangle = \frac{1}{\sqrt{n}}(\hat{a}^\dagger)^n\,|0\rangle$ with $\hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle$ and $\langle m|n\rangle = \delta_{mn}$.

Now,

$$\hat{H}\,|n\rangle = \left[\hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega - \frac{q^2E^2}{2m\omega^2}\right]|n\rangle = \left[\hbar\omega\left(n + \frac{1}{2}\right) - \frac{q^2E^2}{2m\omega^2}\right]|n\rangle$$

Thus the energy spectrum is given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{q^2E^2}{2m\omega^2}$$

Furthermore,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) + \frac{qE}{m\omega^2}$$

And,

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

Therefore,

$$\begin{aligned}
\langle\hat{x}\rangle = \langle n|\,\hat{x}\,|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|\left(\hat{a} + \hat{a}^\dagger\right)|n\rangle + \frac{qE}{m\omega^2}\,\langle n|n\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\langle n|n+1\rangle + \sqrt{n}\,\langle n|n-1\rangle\right) + \frac{qE}{m\omega^2} = +\frac{qE}{m\omega^2} \\
\langle\hat{p}\rangle = \langle n|\,\hat{p}\,|n\rangle &= i\sqrt{\frac{m\hbar\omega}{2}}\,\langle n|\left(\hat{a}^\dagger - \hat{a}\right)|n\rangle \\
&= i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n+1}\,\langle n|n+1\rangle - \sqrt{n}\,\langle n|n-1\rangle\right) = 0 \\
\left\langle\hat{p}^2\right\rangle = \langle n|\,\hat{p}^2\,|n\rangle &= -\frac{m\hbar\omega}{2}\,\langle n|\left(\hat{a}^\dagger - \hat{a}\right)^2|n\rangle \\
&= -\frac{m\hbar\omega}{2}\,\langle n|\left[(\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + (\hat{a})^2\right]|n\rangle = \frac{m\hbar\omega}{2}\,(2n+1)
\end{aligned}$$

(b). When the field is switched off, the new Hamiltonian becomes exactly the centered quantum harmonic oscilator. Let $|\psi_0\rangle = |0_{old}\rangle$ and then the new eigenstates are the standard quantum harmonic oscillator states: $|n_{new}\rangle$. Now the probability for the particle to be found in the new ground state is $P_0(t) = |\,\langle 0_{new}|\psi(t)\rangle\,|^2$. However,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle 0_{new}|\psi(t)\rangle = \langle 0_{new}|\,\frac{1}{i\hbar}\hat{H}\,|\psi(t)\rangle = \langle\psi(t)|\,\frac{-1}{i\hbar}\hat{H}\,|0_{new}\rangle^* = \frac{\omega}{2i}\,\langle\psi(t)|0_{new}\rangle^* = \frac{\omega}{2i}\,\langle 0_{new}|\psi(t)\rangle$$

Thus,

$$\langle 0_{new}|\psi(t)\rangle = \langle 0_{new}|\psi(0)\rangle\,e^{-i\omega t/2}$$

so,

$$P_0(t) = |\,\langle 0_{new}|\psi(t)\rangle\,|^2 = |\,\langle 0_{new}|\psi(0)\rangle\,|^2 = P_0(0) = |\,\langle 0_{new}|0_{old}\rangle\,|^2$$

The old and new ground states are easily found by anihilating them with thier respective lowering operators,

$$\langle x|\,\hat{a}_{old}\,|0_{old}\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{qE}{m\omega^2} + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi_0^{old}(x) = 0$$

which gives a normalized wavefunction:

$$\psi_0^{old}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\left(\frac{m\omega}{2\hbar}\right)\left(x - \frac{qE}{m\omega^2}\right)^2\right]$$

Similarly, the equation,

$$\langle x|\,\hat{a}_{new}\,|0_{new}\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi_0^{new}(x) = 0$$

has a normalized solution:

$$\psi_0^{new}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega x^2}{2\hbar}\right]$$

Therefore,

$$\begin{aligned}
\langle 0_{new}|0_{old}\rangle &= \int_{-\infty}^{\infty}\psi_0^{new}(x)^*\psi_0^{old}(x)\mathrm{d}x \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left\{-\frac{m\omega x^2}{2\hbar}\left[\left(x - \frac{qE}{m\omega^2}\right)^2 + x^2\right]\right\}\mathrm{d}x
\end{aligned}$$

Expanding the exponent,

$$\begin{aligned}
-\frac{m\omega}{2\hbar}\left[\left(x - \frac{qE}{m\omega^2}\right)^2 + x^2\right] &= -\frac{m\omega x^2}{2\hbar}\left(2x^2 - \frac{2qE}{m\omega^2}x + \left(\frac{qE}{m\omega^2}\right)^2\right) \\
= -\frac{m\omega}{\hbar}\left(x^2 - \frac{qE}{m\omega^2}x + \left(\frac{qE}{2m\omega^2}\right)^2 + \frac{1}{4}\left(\frac{qE}{m\omega^2}\right)^2\right) \\
= -\frac{m\omega}{\hbar}\left(x - \frac{qE}{2m\omega^2}\right)^2 - \frac{1}{4}\left(\frac{q^2E^2}{m\hbar\omega^3}\right)
\end{aligned}$$

Then pluggin back into the integral,

$$\begin{aligned}
\langle 0_{new}|0_{old}\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left[-\frac{m\omega}{\hbar}\left(x - \frac{qE}{2m\omega^2}\right)^2 - \frac{1}{4}\left(\frac{q^2E^2}{m\hbar\omega^3}\right)\right]\mathrm{d}x \\
&= \exp\left[-\frac{1}{4}\left(\frac{q^2E^2}{m\hbar\omega^3}\right)\right]
\end{aligned}$$

Thus,

$$P_0(t) = \exp\left[-\frac{1}{2}\left(\frac{q^2E^2}{m\hbar\omega^3}\right)\right]$$

Problem 15.

Classically, the Hamiltonian for a particle of mass m constrained on a sphere of radius r is $H = \frac{p^2}{2m} + mgr(1-\cos\theta)$ in shperical coordinates with the $+\hat{z}$ direction aligned with $\vec{g}$.

(a). Canonical quantization gives:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + mgr(1-\cos\theta)$$

where ∇^2 constrained to a sphere in spherical coordinates is:

$$\nabla^2 = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Thus, the time independent Schrodinger Equation becomes:

$$E\psi(\theta,\phi) = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(\theta,\phi) + mgr(1-\cos\theta)\psi(\theta,\phi)$$

This equation is seperable. It can be written as:

$$r^2\sin^2\theta\left(E + \frac{\hbar^2}{2m}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] - mgr(1-\cos\theta)\right)\psi(\theta,\phi) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\phi^2}\psi(\theta,\phi)$$

Thus if we perform a seperation of variables, $\psi(\theta,\phi) = \psi_\theta(\theta)\psi_\phi(\phi)$ then,

$$r^2\sin^2\theta\left(E + \frac{\hbar^2}{2m}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_\theta}{\partial\theta}\right)\frac{1}{\psi_\theta}\right] - mgr(1-\cos\theta)\right) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi_\phi}{\partial\phi^2}\frac{1}{\psi_\phi}$$

Since the two sides of this equation depend only on disjoint variables, both sides must equal a constant. In particular,

$$\frac{\partial^2\psi_\phi}{\partial\phi^2}\frac{1}{\psi_\phi} = -k_\phi^2$$

$$E\psi_\theta(\theta) = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\psi_\theta(\theta) + \frac{\hbar^2k_\phi^2}{2mr^2\sin^2\theta}\psi_\theta(\theta) + mgr(1-\cos\theta)\psi_\theta(\theta)$$

As before, periodic boundary conditions on ϕ require that k_ϕ is real with $k_\phi = m_\phi \in \mathbb{N}$. Now, to make the differential equation tractable, we introduce a small angle approximation to second order. The Schrodinger Equation becomes:

$$E\psi_\theta(\theta) = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\right]\psi_\theta(\theta) + \left[\frac{\hbar^2m_\phi^2}{2mr^2\theta^2} + \frac{1}{2}mgr\theta^2\right]\psi_\theta(\theta)$$

Let's make this equation a bit less nasty by introducing a characteristic angluar frequency $\omega = \sqrt{\frac{g}{r}}$ and angular scale $\alpha = \sqrt{\frac{\hbar}{m\omega r^2}}$ then write $x = \frac{\theta}{\alpha}$. Now setting $\mathcal{E} = \frac{E}{\hbar\omega}$, we rewrite:

$$\frac{E}{\hbar\omega}\psi_\theta(\theta) = -\frac{\hbar}{2m\omega r^2}\left[\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\right]\psi_\theta(\theta) + \left[\frac{\hbar m_\phi^2}{2m\omega r^2\theta^2} + \frac{mgr}{2\hbar\omega}\theta^2\right]\psi_\theta(\theta)$$

$$\frac{E}{\hbar\omega}\psi_\theta(\theta) = -\frac{\hbar}{2m\omega r^2}\left[\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\right]\psi_\theta(\theta) + \left[\frac{\hbar m_\phi^2}{2m\omega r^2\theta^2} + \frac{m\omega r^2}{2\hbar}\theta^2\right]\psi_\theta(\theta)$$

$$2\mathcal{E}\psi_\theta(\theta) = -\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)\right]\psi_\theta(\theta) + \left[\frac{m_\phi^2}{x^2} + x^2\right]\psi_\theta(\theta)$$

Now, as $x \to \infty$, the equation becomes,

$$-\frac{\partial^2 \psi_\theta}{\partial x^2} + x^2 \psi_\theta = 0$$

Thus for large x, $\psi_\theta(x) \propto e^{-\frac{1}{2}x^2}$ so write $\psi_\theta(x) = u(x)e^{-\frac{1}{2}x^2}$. Plugging in,

$$\begin{aligned}
2\mathcal{E}ue^{-\frac{1}{2}x^2} &= -\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)u(x)e^{-\frac{1}{2}x^2} + \left[\frac{m_\phi^2}{x^2} + x^2\right]u(x)e^{-\frac{1}{2}x^2} \\
2\mathcal{E}ue^{-\frac{1}{2}x^2} &= -\frac{1}{x}\frac{\partial}{\partial x}\left(xu'e^{-\frac{1}{2}x^2} - x^2ue^{-\frac{1}{2}x^2}\right) + \left[\frac{m_\phi^2}{x^2} + x^2\right]ue^{-\frac{1}{2}x^2} \\
2\mathcal{E}ue^{-\frac{1}{2}x^2} &= \left(-\frac{u'}{x} + 2u - u'' + 2xu' - x^2u\right)e^{-\frac{1}{2}x^2} + \left[\frac{m_\phi^2}{x^2} + x^2\right]ue^{-\frac{1}{2}x^2} \\
2\mathcal{E}u &= -\frac{u'}{x} + 2u - u'' + 2xu' + \frac{m_\phi^2}{x^2}u
\end{aligned}$$

Thus collecting terms of equal power in x,

$$2(\mathcal{E}-1)u - 2xu' = -\frac{u'}{x} - u'' + \frac{m_\phi^2}{x^2}u$$

Now, we apply the series expansion $u = \sum_{i=0}^{\infty} c_i x^i$:

$$\sum_{i=0}^{\infty}\left[2(\mathcal{E}-1) - 2i\right]c_i x^i = \sum_{i=0}^{\infty}\left[-i - i(i-1) + m_\phi^2\right]c_i x^{i-2} = \sum_{i=0}^{\infty}\left[m_\phi^2 - i^2\right]c_i x^{i-2}$$

Reparametrizing the second sum,

$$\sum_{i=0}^{\infty}\left[2(\mathcal{E}-1) - 2i\right]c_i x^i = \sum_{i=0}^{\infty}\left[m_\phi^2 - (i+2)^2\right]c_{i+2}x^i + \frac{m_\phi^2}{x^2}c_0 + \frac{m_\phi^2 - 1}{x}c_1$$

There are no terms on the left which can cancel the $\frac{1}{x}$ and $\frac{1}{x^2}$ divergences so if $m_\phi \neq 0$ then $c_0 = 0$ and if $m_\phi^2 \neq 1$ then $c_1 = 0$. However, all other terms of equal order must cancel so,

$$\frac{c_{i+2}}{c_i} = \frac{2(\mathcal{E} - 1 - i)}{m_\phi^2 - (i+2)^2}$$

If this series never terminates, then for large i the recurrence relation goes as,

$$\frac{c_{i+2}}{c_i} \approx \frac{2(i+1)}{(i+2)^2} \approx \frac{1}{i/2}$$

and therefore,

$$c_i \propto \frac{1}{(i/2)!} \quad \text{so} \quad u = \sum_{i=0}^{\infty}\frac{x^{2i}}{i!} = e^{x^2}$$

which diverges faster than the other factor so $\psi_\theta \to \infty$ which violates the small angle approximation let alone normalizability! Thus, the series must terminate at some step i_{max} so the numerator $2(\mathcal{E} - i_{max} - 1) = 0$ so $\mathcal{E} = i_{max} + 1$. Thus, the energy levels are

$$E = \hbar\omega(i_{max} + 1) = \hbar\sqrt{\frac{g}{r}}(i_{max} + 1)$$

where $i_{max}+1 \in \mathbb{Z}^+$. In particular, the ground state energy is $E_0 = \hbar\sqrt{\frac{g}{r}}$ with a wave function given by: $c_0 = N$ and all other terms are zero since $\mathcal{E} - 1 = 0$ and $c_1 = 0$ so that all odd terms are zero (else they will not terminante). Thus, $m_\phi = 0$ because $c_0 \neq 0$. In summary, $u(x) = N$ and therefore,

$$\psi_0(\theta, \phi) = Ne^{-\frac{m\omega}{2\hbar}(r\theta)^2}$$

In general, the ϕ-wavefunction is given by,

$$\psi_\phi(\phi) \propto e^{im_\phi \phi}$$

(b). We continue the series to higher order terms. Note that for $m_\phi \geq 2$ the denominator will blow up for $i_m = |m_\phi| - 2$ so in that case, we begin the series with $c_{i_m} = 0$, $c_{i_m+1} = 0$, and $c_{i_m+2} \neq 0$ so that $c_{i_m+2} \cdot (m_\phi^2 - (i_m+2)^2) = c_{i_m} \cdot 2(\mathcal{E} - i_m - 1)$ is satisfied. This means that the series must terminate at $i_{max} \geq i_m + 2$ with i_{max} and i_m having the same parity (since a zero term series gives $\psi = 0$ and the non-zero terms skip by two) so $\mathcal{E} \geq |m_\phi| + 1$ with $\mathcal{E} \equiv (|m_\phi|+1) \bmod 2$.

Thus, the next energy level corresponds to $\mathcal{E} = 2$ and $m_\phi = -1, +1$ These two cases correspond to series $u = Nx$ and a ϕ-wavefunction $\psi_\phi \propto e^{im_\phi\phi} = e^{\pm i\phi}$. Thus we have two states with $E = 2\hbar\sqrt{\frac{g}{r}}$,

$$\psi_{1,\pm 1}(\theta, \phi) = N\sqrt{\frac{m\omega}{\hbar}} e^{\pm i\phi}\, r\theta\, e^{-\frac{m\omega}{2\hbar}(r\theta)^2}$$

The next energy level corresponds to $\mathcal{E} = 3$ and $m_\phi = -2, 0, +2$ These three cases correspond to series $u = Nx^2$ for $m_\phi = \pm 2$ and $N(1 - x^2)$ for $m_\phi = 0$. Thus we have three states with $E = 3\hbar\sqrt{\frac{g}{r}}$,

$$\psi_{2,0}(\theta, \phi) = N\left(1 - \frac{m\omega}{\hbar}(r\theta)^2\right) e^{-\frac{m\omega}{2\hbar}(r\theta)^2}$$
$$\psi_{2,\pm 2}(\theta, \phi) = N\frac{m\omega}{\hbar} e^{\pm 2i\phi}\,(r\theta)^2\, e^{-\frac{m\omega}{2\hbar}(r\theta)^2}$$

(c). For the small angle approximation to be reasonable for the ground state, we require that the spread of the gaussian be much less than 1 rad. Thus,

$$\frac{\hbar}{m\omega r^2} \ll 1 \quad \text{thus} \quad \hbar^2 \ll m^2 g r^3$$

Alternatively: if one notices that, in the small angle approximation, this Hamiltonian corresponds to a 2D harmonic oscillator in polar coordinates then the problem can easily be solved by factorization. For the small angle approximation Hamiltonain,

$$\hat{H} = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + \frac{1}{2}mgr\theta^2$$

Define right and left circular ladder operators using the scale parameter $\alpha = \sqrt{\frac{\hbar}{m\omega r^2}}$,

$$\hat{a}_R = \frac{e^{-i\phi}}{2}\left[\frac{\theta}{\alpha} + \alpha\frac{\partial}{\partial\theta} - \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right]$$
$$\hat{a}_L = \frac{e^{i\phi}}{2}\left[\frac{\theta}{\alpha} + \alpha\frac{\partial}{\partial\theta} + \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right]$$

These operators have the expected commutation relations:

$$\begin{aligned}
[\hat{a}_R, \hat{a}_R^\dagger] &= 1 \qquad & [\hat{a}_L, \hat{a}_L^\dagger] &= 1 \\
[\hat{a}_R, \hat{a}_L] &= 0 \qquad & [\hat{a}_R^\dagger, \hat{a}_L^\dagger] &= 0 \\
[\hat{a}_R^\dagger, \hat{a}_L] &= 0 \qquad & [\hat{a}_R, \hat{a}_L^\dagger] &= 0
\end{aligned}$$

which are simple yet tedious to check.

Now consider the combination,

$$\begin{aligned}
\hat{a}_R^\dagger\hat{a}_R + \hat{a}_L^\dagger\hat{a}_L &= \frac{e^{i\phi}}{2}\left[\frac{\theta}{\alpha} - \alpha\frac{\partial}{\partial\theta} - \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right]\frac{e^{-i\phi}}{2}\left[\frac{\theta}{\alpha} + \alpha\frac{\partial}{\partial\theta} - \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right] \\
&+ \frac{e^{-i\phi}}{2}\left[\frac{\theta}{\alpha} - \alpha\frac{\partial}{\partial\theta} + \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right]\frac{e^{i\phi}}{2}\left[\frac{\theta}{\alpha} + \alpha\frac{\partial}{\partial\theta} + \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right] \\
&= \frac{1}{4}\left[z^2 - \frac{\partial}{\partial z}z - 1 - i\frac{\partial}{\partial\phi} + z\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2}\right. \\
&\left. - \frac{1}{z}\frac{\partial}{\partial z} - \frac{i}{z}\frac{\partial}{\partial\phi}\frac{\partial}{\partial x} - i\frac{\partial}{\partial\phi} + i\frac{\partial}{\partial z}z\frac{\partial}{\partial\phi} + \frac{i}{z^2}\frac{\partial}{\partial\phi} - \frac{1}{z^2}\frac{\partial^2}{\partial\phi^2}\right] \\
&+ \frac{1}{4}\left[z^2 - \frac{\partial}{\partial z}z - 1 + i\frac{\partial}{\partial\phi} + z\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2}\right. \\
&\left. - \frac{1}{z}\frac{\partial}{\partial z} + \frac{i}{z}\frac{\partial}{\partial\phi}\frac{\partial}{\partial x} + i\frac{\partial}{\partial\phi} - i\frac{\partial}{\partial z}z\frac{\partial}{\partial\phi} - \frac{i}{z^2}\frac{\partial}{\partial\phi} - \frac{1}{z^2}\frac{\partial^2}{\partial\phi^2}\right] \\
&= \frac{1}{4}\left[z^2 - 2 - 2i\frac{\partial}{\partial\phi} - \frac{1}{z}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2} - \frac{1}{z^2}\frac{\partial^2}{\partial\phi^2}\right] \\
&+ \frac{1}{4}\left[z^2 - 2 + 2i\frac{\partial}{\partial\phi} - \frac{1}{z}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2} - \frac{1}{z^2}\frac{\partial^2}{\partial\phi^2}\right] \\
&= -\frac{1}{2}\left[\frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\right) + \frac{1}{z^2}\frac{\partial^2}{\partial\phi^2}\right] + \frac{1}{2}z^2 - 1 \\
&= \frac{\hbar}{2m\omega r^2}\left[\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\theta^2}\frac{\partial^2}{\partial\phi^2}\right] + \frac{1}{2}\frac{mgr}{\hbar\omega}\theta^2 - 1 = \frac{1}{\hbar\omega}\hat{H} - 1
\end{aligned}$$

Thus,

$$\hat{H} = \hbar\omega\left(\hat{a}_R^\dagger\hat{a}_R + \hat{a}_L^\dagger\hat{a}_L + 1\right)$$

Because $\langle\psi|\,\hat{a}^\dagger\hat{a}\,|\psi\rangle = \langle\hat{a}\psi|\hat{a}\psi\rangle \geq 0$ we immedietly see that the ground state is killed by $\hat{a}_L$ and $\hat{a}_R$ and thus has energy $\hbar\omega$. From the above commutation relations,

$$\begin{aligned}
[\hat{H}, \hat{a}_R^\dagger] &= \hbar\omega\hat{a}_R^\dagger \qquad & [\hat{H}, \hat{a}_R] &= -\hbar\omega\hat{a}_R \\
[\hat{H}, \hat{a}_L^\dagger] &= \hbar\omega\hat{a}_L^\dagger \qquad & [\hat{H}, \hat{a}_L] &= -\hbar\omega\hat{a}_L
\end{aligned}$$

Therefore, $\hat{a}_R^\dagger$ and $\hat{a}_L^\dagger$ increse the energy of a state by $\hbar\omega$ while $\hat{a}_R$ and $\hat{a}_L$ decrese the energy by $\hbar\omega$. Futhermore, by subtracting the second term in the above derivation, we see that:

$$\begin{aligned}
\hat{a}_R^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_L &= \frac{1}{4}\left[z^2 - 2 - 2i\frac{\partial}{\partial\phi} - \frac{1}{z}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2} - \frac{1}{z^2}\frac{\partial^2}{\partial\phi^2}\right] \\
&- \frac{1}{4}\left[z^2 - 2 + 2i\frac{\partial}{\partial\phi} - \frac{1}{z}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2} - \frac{1}{z^2}\frac{\partial^2}{\partial\phi^2}\right] \\
&= -i\frac{\partial}{\partial\theta} = \frac{1}{\hbar}\hat{L}_z
\end{aligned}$$

Thus, $\hat{L}_z = \hbar\left(\hat{a}_R^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_L\right)$ so from the above commutation relations:

$$\begin{aligned}
[\hat{L}_z, \hat{a}_R^\dagger] &= \hbar\hat{a}_R^\dagger \qquad [\hat{L}_z, \hat{a}_R] = -\hbar\hat{a}_R \\
[\hat{L}_z, \hat{a}_L^\dagger] &= -\hbar\hat{a}_L^\dagger \qquad [\hat{L}_z, \hat{a}_L] = \hbar\hat{a}_L
\end{aligned}$$

Therefore, $\hat{a}_R^\dagger$ acts to add an energy mode with positive z-angluar momentum and $\hat{a}_L^\dagger$ acts to add an energy mode with negative z-angular momentum. For our seperated eigenstates,

$$\langle(\theta,\phi)|\,\hat{L}_z\,|\psi\rangle = -i\hbar\frac{\partial}{\partial\phi}\psi(\theta,\phi) = \hbar m\psi(\theta,\phi)$$

So $m = N_R - N_L$ and $n = N_R + N_L$ where N_R and N_L are the eigenstates of $\hat{a}_R^\dagger \hat{a}_R$ and $\hat{a}_L^\dagger \hat{a}_L$ respectively. This explains the spectrum found above because $\mathcal{E} = n + 1 = N_R + N_L + 1 \geq m + 1$ and $n - m = 2N_L$ so $n \equiv m \mod 2$. Acting on the ground state, there are two choices,

$$\begin{aligned}
|\psi_{1,+1}\rangle &= \hat{a}_R^\dagger\,|0\rangle \\
|\psi_{1,-1}\rangle &= \hat{a}_L^\dagger\,|0\rangle
\end{aligned}$$

one right circulating mode or one left circulating mode. For some reason we are missing the $m = 0$ state of a $l = 1$ multiplet. This has to do with the fact that we are in 2D and the $m = 0$ state in 3D protrudes perpendicular to the plane defined by $\hat{L}_z$. Continuing to the next energy level we get three possibilities (because $[\hat{a}_R^\dagger, \hat{a}_L^\dagger] = 0$) which are:

$$\begin{aligned}
|\psi_{2,+2}\rangle &= \frac{1}{\sqrt{2}}(\hat{a}_R^\dagger)^2\,|0\rangle \\
|\psi_{2,-2}\rangle &= \frac{1}{\sqrt{2}}(\hat{a}_L^\dagger)^2\,|0\rangle \\
|\psi_{2,0}\rangle &= \frac{1}{\sqrt{2}}\hat{a}_R^\dagger\hat{a}_L^\dagger\,|0\rangle
\end{aligned}$$

We can also get the explicit wavefunctions from these ladder operators. If both $\hat{a}_R$ and $\hat{a}_L$ kill $|0\rangle$ then so does any linear combination. In particular,

$$\langle(\theta,\phi)|\,(e^{i\phi}\hat{a}_R + e^{-i\phi}\hat{a}_L)\,|0\rangle = \left[\frac{\theta}{\alpha} + \alpha\frac{\partial}{\partial\theta}\right]\psi(\theta,\phi) = 0$$

therefore,

$$\psi_0(\theta,\phi) = f(\phi)e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2}$$

Similarly,

$$i \langle(\theta,\phi)| \left(e^{i\phi}\hat{a}_R - e^{-i\phi}\hat{a}_L\right)|0\rangle = \frac{\alpha}{\theta}\frac{\partial}{\partial\phi} f(\phi) e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2} = 0$$

and thus $f(\phi)$ is constant so,

$$\psi_0(\theta,\phi) = K e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2}$$

Now, acting with the rasing operators,

$$\psi_{1,+1} = \hat{a}_R^\dagger \psi_0 = \frac{e^{i\phi}}{2}\left[\frac{\theta}{\alpha} - \alpha\frac{\partial}{\partial\theta} - \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right] K e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2} = \frac{K}{\alpha} e^{i\phi}\,\theta\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2}$$

$$\psi_{1,-1} = \hat{a}_L^\dagger \psi_0 = \frac{e^{-i\phi}}{2}\left[\frac{\theta}{\alpha} - \alpha\frac{\partial}{\partial\theta} + \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right] K e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2} = \frac{K}{\alpha} e^{-i\phi}\,\theta\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2}$$

Similarly, we can obtain the second excited states:

$$\psi_{2,+2} = \hat{a}_R^\dagger \psi_{1,+1} = \frac{e^{i\phi}}{2}\left[\frac{\theta}{\alpha} - \alpha\frac{\partial}{\partial\theta} - \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right] \frac{K}{\alpha} e^{i\phi}\,\theta\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2} = \frac{K}{\alpha^2} e^{2i\phi}\,\theta^2\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2}$$

$$\psi_{2,-2} = \hat{a}_L^\dagger \psi_{1,-1} = \frac{e^{-i\phi}}{2}\left[\frac{\theta}{\alpha} - \alpha\frac{\partial}{\partial\theta} + \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right] \frac{K}{\alpha} e^{-i\phi}\,\theta\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2} = \frac{K}{\alpha^2} e^{-2i\phi}\,\theta^2\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2}$$

$$\psi_{2,0} = \hat{a}_L^\dagger \psi_{1,+1} = \frac{e^{-i\phi}}{2}\left[\frac{\theta}{\alpha} - \alpha\frac{\partial}{\partial\theta} + \alpha\frac{i}{\theta}\frac{\partial}{\partial\phi}\right] \frac{K}{\alpha} e^{i\phi}\,\theta\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2} = K\left(\frac{\theta^2}{\alpha^2} - 1\right)\, e^{-\frac{1}{2}\left(\frac{\theta}{\alpha}\right)^2}$$