1 Motivation

Let X be a normal (quasi-projective) variety over a field k. Then,

$$H^1(X, \mathbb{G}_m) = \{\lambda_{ij} \in \mathbb{G}_m(U_{ij})\}/\{\delta_i \in \mathbb{G}_m(U_i)\}$$

is the group of line bundles.

Remark. For (quasi-projective, I think) X the Cech and etale cohmology groups agree. Therefore, when we write $H^i(X, -)$ we are going to mean $\check{H}^i(X, -)$ on the étale site.

Then,

$$H^2(X, \mathbb{G}_m) = \{\text{azumaya algebras}\}/\text{morita equivalence}$$

where Azumaya algebras are equivalent to PGL_r-bundles on X. These also classify $\mathbb{G}_m$-gerbes on X where a $\mathbb{G}_m$-gerbe is a stack locally isomorphic to $X \times B\mathbb{G}_m$. This alows us to write double intersection data in terms of single intersection data.

Then the transition functions,

$$U \times B\mathbb{G}_m|_{U \cap V} \xrightarrow{\varphi_{U \cap V}} V \times B\mathbb{G}_m|_{U \cap V}$$

This is given by a map $U \cap V \to B\mathbb{G}_m$ and therefore by a line bundle $\mathcal{L}_{U \cap V} \in \mathrm{Pic}\,(U \cap V)$. Therefore we have $\mathcal{L}_{ij}$ on each U_{ij} and then there is the data of an isomorphism,

$$\varphi : \mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ik}^{-1} \xrightarrow{\sim} \mathcal{O}_{U_{ijk}}$$

and therefore we get a $\mathbb{G}_m(U_{ijk})$-torsor of choices of φ. This defines a class $[\alpha] \in H^2(X, \mathbb{G}_m)$. Then we could say that,

$$H^2(X, \mathbb{G}_m) = H^1(X, \mathrm{Pic}\,(X))$$

where $\mathrm{Pic}\,(X)$ is the Picard stack.

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_r \longrightarrow \mathrm{PGL}_r \longrightarrow 0$$

Therefore we get a map $H^1(X, \mathrm{PGL}_r) \to H^2(X, \mathbb{G}_m)$ and as we vary over r these are jointly surjective (HARD). We want another perspective on Mortia equivalence which is somehow the same as taking the cokernel $H^1(X, \mathrm{GL}_r) \to H^1(X, \mathrm{PGL}_r)$.

Definition 1.0.1. Morita equivalence is $A \sim B$ if their category of modules are equivalent (probably in a was preserving the underlying abelian sheaf).

Theorem 1.0.2 (Krashen-T)**.** Let X be a smooth projective variety over k. Then,

$$H^3(X, \mathbb{G}_m) \xrightarrow{\sim} \{\text{2-azumaya algebras}\}/\text{morita equivalence}$$

2 Detour into dgLand

Definition 2.0.1. A dg-algera is a differential graded algebra and a dg-algebra is a graded module over this dg-algebra.

Remark. We want a modoidal cateogory in which $\mathrm{Br}\,(X)$ represents the units.

Example 2.0.2. $H^1(X, \mathbb{G}_m) = \mathfrak{Coh}\,(X)^\times$

Proposition 2.0.3. $\mathrm{Br}\,(X) = (\mathcal{O}_X\text{-de-CAT}/\sim_{\mathrm{mor}})^\times$.

Definition 2.0.4. An A-dg-cat is a cateogry enriched in A-dg-modules.

Theorem 2.0.5. There exists a well-defined cateory C consisting of $\mathcal{O}_X$-dg-cat up to morita equivalence and a map,

$$\varphi : H^2(X, \mathbb{G}_m) \hookrightarrow C$$

whose image is precise the invertables of C under the $\otimes$-structure. We define,

$$\varphi(A) = *_A$$

the category with one object and A of morphisms. Then $\mathrm{Br}\,(X) \circlearrowright C$ by automorphisms.

Definition 2.0.6. A 2-azumaya algebra $\mathcal{A}$ over X is a 2-stack of dg-categories such that,

(a) $\mathcal{A}|_U \cong C|_U$ for a cover $U \to X$

(b) $\mathrm{Hom}\,(\mathcal{A}, \mathcal{A}) \cong C$ in C.

Proposition 2.0.7. $H^3(X, \mathbb{G}_m) \cong \{\text{2-azumaya algebra}\}$

Proof. $\mathcal{A}$ is given by a class in $\mathrm{Br}\,(U)$ over each double intersection. Define the cover to go from,

$$H^1(X, \mathrm{Br}\,(X)) \to H^3(X, \mathbb{G}_m)$$

$\square$