Riemann Surfaces Midterm

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Remark 1. Whenever I have the complex coordinate z, I will denote the real coordinates by $x, y \in \mathbb{R}$ such that $z = x + iy$ and the derivatives $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$.

Problem 1

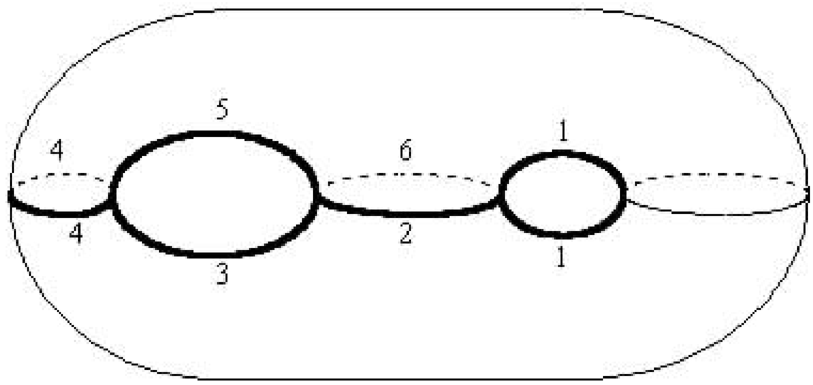
Take six pairwise distinct points $a_1, a_2, a_3, a_4, a_4, a_5, a_6 \in \mathbb{C}$ and consider the equation,

$$w^2 = \prod_{i=1}^{6} (x - a_i)$$

We wish to consider the compact Riemann surface $\hat{X}$ for this equation. I will assume, for convenience, that these points are all nonzero and not infinity although this assumption will not seriously affect my argument only remove degenerate cases.

(a)

Consider two copies of the complex plane, called sheets, labeled (I) and (II). We give each sheet three branch cuts, a_1-a_2 and a_3-a_4 and a_5-a_6. Now we compactify each sheet by adding two points at infinity ∞_{I} and ∞_{II} to close them into a sphere with three holes. When we glue these two spheres with three holes along the boundaries of the holes we get a two holed torus $\hat{X}$ which is the unique orientable compact surface of genus $g = 2$.



To make $\hat{X}$ a Riemann surface we need to choose a covering of $\hat{X}$ by holomorphic chars. For a point away from either ∞ and any branch point a_i and any cut, we simply take a disk in the complex plane (I) or (II) with standard holomorphic coordinate. For a point near the cut, we take a half-disk on (I) which continues to a half-disk on (II) across the cut. At either ∞ we take the holomorphic coordinate $z = \zeta^{-1}$. At a branch point a_i we take the holomorphic coordinate $z = a_i + \zeta^2$ where the double cover of the disk about a_i by ζ^2 is taken to be injective by sending the first covering to sheet (I) and the second covering to sheet (II). We need to show that w is well-defined and meromorphic on $\hat{X}$. Away from branch points and infinities w is a holomorphic function on $\mathbb{C}$ and thus on the holomorphic coordinate disks. Now we need to check the branch points. Near $z = a_i$ we have the holomorphic coordinate $z = \zeta^2 + a_i$ and thus,

$$w(\zeta) = \pm\sqrt{\prod_{j=1}^{6}(\zeta^2 + a_i - a_j)} = \zeta\sqrt{\prod_{j\neq i}^{6}(\zeta^2 + a_i - a_j)}$$

which is holomorphic on the ζ coordinate chart with a simple zero at $\zeta \to 0$ i.e. $z = a_i$. Next, near $z = \infty_{\mathrm{I}}$ we have the coordinate chart $z = \frac{1}{\zeta}$ and thus,

$$w(\zeta) = \pm\sqrt{\prod_{j=1}^{6}(\zeta^{-1} - a_j)} = \zeta^{-3}\sqrt{\prod_{j=1}^{6}(1 - a_j\zeta)}$$

which has a triple pole at each ∞. I should remark that $\pm$ is shorthand notation for the fact that as ζ passes the negative imaginary axis we transition from (I) to (II) meaning that w changes sign. This negative sign allows the manipulation $\pm\sqrt{\zeta^2} = \zeta$ since the sign information that would usually be lost in the choice of square root is preserved by the $\pm$ on different sheets.

(b)

We define two holomorphic forms on $\hat{X}$,

$$\omega_1 = \frac{\mathrm{d}z}{w} = \begin{cases} \frac{\mathrm{d}z}{\sqrt{\prod_{i=1}^{6}(z-a_i)}} & z \in (\mathrm{I}) \\ -\frac{\mathrm{d}z}{\sqrt{\prod_{i=1}^{6}(z-a_i)}} & z \in (\mathrm{II}) \end{cases}$$

$$\omega_2 = \frac{z\,\mathrm{d}z}{w} = \begin{cases} \frac{z\,\mathrm{d}z}{\sqrt{\prod_{i=1}^{6}(z-a_i)}} & z \in (\mathrm{I}) \\ -\frac{z\,\mathrm{d}z}{\sqrt{\prod_{i=1}^{6}(z-a_i)}} & z \in (\mathrm{II}) \end{cases}$$

We need to check that these forms are holomorphic when expressed in terms of the local holomorphic variables on coordinate charts. At the branch point a_i, in terms of the local holomorphic coordinate $z = \zeta^2 + a_i$ we express the forms,

$$\omega_1 = \frac{\mathrm{d}z}{w} = \pm\frac{\zeta\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(\zeta^2 + a_i - a_j)}} = \frac{\mathrm{d}\zeta}{\sqrt{\prod_{j\neq i}^{6}(\zeta^2 + a_i - a_j)}}$$

$$\omega_2 = \frac{z\,\mathrm{d}z}{w} = \pm\frac{(\zeta^2 + a_i)\zeta\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(\zeta^2 + a_i - a_j)}} = \frac{(\zeta^2 + a_i)\,\mathrm{d}\zeta}{\sqrt{\prod_{j\neq i}^{6}(\zeta^2 + a_i - a_j)}}$$

which are well-defined and nonzero in the limit $\zeta \to 0$. Next, at each ∞ with local holomorphic coordinate $z = \zeta^{-1}$ the forms are,

$$\omega_1 = \frac{\mathrm{d}z}{w} = -\frac{\zeta^{-2}\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(\zeta^{-1} - a_j)}} = -\frac{\zeta^{-2}\,\mathrm{d}\zeta}{\zeta^{-3}\sqrt{\prod_{j=1}^{6}(1 - a_j\zeta)}} = -\frac{\zeta\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(1 - a_j\zeta)}}$$

$$\omega_2 = \frac{z\,\mathrm{d}z}{w} = -\frac{\zeta^{-3}\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(\zeta^{-1} - a_j)}} = -\frac{\zeta^{-3}\,\mathrm{d}\zeta}{\zeta^{-3}\sqrt{\prod_{j=1}^{6}(1 - a_j\zeta)}} = -\frac{\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(1 - a_j\zeta)}}$$

Both forms are well-defined and holomorphic in the limit $\zeta \to 0$. Furthermore, we see that ω_1 has a simple zero at each ∞ and ω_2 is nonzero near ∞. Therefore, ω_1 and ω_2 cannot be linearly dependent $(1,0)$-forms since they have different pole structures and are nonzero. Therefore, neither can be a multiple of the other.

The zeros of these forms are interesting. We have shown that ω_1 has simple zeros at $z = \infty_{\mathrm{I}}$ and $z = \infty_{\mathrm{II}}$ while ω_2 is nonzero at all branch points and each ∞ but clearly has simple zeros at $z = 0_{\mathrm{I}}$ and $z = 0_{\mathrm{II}}$. We should expect this because as holomorphic $(1,0)$-forms, ω_1 and ω_2 are holomorphic sections of the canonical bundle K_X and thus (since holomorphic forms have no poles) the number of zeros of each must equal $c_1(K_X) = -\chi(\hat{X}) = 2g - 2 = 2$ where $g = 2$ is the genus of our Riemann surface.

(c)

Consider the form,

$$\omega_{a_i} = \frac{\omega_1}{z - a_i}$$

I need to check that ω_{a_i} is meromorphic with exactly a double pole at a_i. First, near each ∞, in the local holomorphic coordinate $z = \zeta^{-1}$ we have,

$$\omega_{a_i} = -\frac{\zeta^{-2}\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(\zeta^{-1} - a_j)}} \cdot \frac{1}{\zeta^{-1} - a_i} = -\frac{\zeta^{2}\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(1 - a_j\zeta)}} \cdot \frac{1}{1 - a_i\zeta}$$

Therefore ω_{a_i} has a double pole at each ∞. Next, at a branch point a_k for $k \neq i$ we write the form in terms of the local holomorphic coordinate $z = \zeta^2 + a_k$ as,

$$\omega_{a_i} = \pm\frac{\zeta\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6}(\zeta^2 + a_k - a_j)}} \cdot \frac{1}{\zeta^2 + a_k - a_i} = \frac{\mathrm{d}\zeta}{\sqrt{\prod_{j\neq k}^{6}(\zeta^2 + a_k - a_j)}} \cdot \frac{1}{\zeta^2 + a_k - a_i}$$

which is holomorphic and nonzero as $\zeta \to \infty$ since $a_k \neq a_i$. Finally, near a_i and again using the local holomorphic coordinate $z = \zeta^2 + a_i$ we find,

$$\omega_{a_i} = \pm \frac{\zeta \, \mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{6} (\zeta^2 + a_i - a_j)}} \cdot \frac{1}{\zeta^2} = \frac{\mathrm{d}\zeta}{\sqrt{\prod_{j\neq i}^{6} (\zeta^2 + a_i - a_j)}} \cdot \frac{1}{\zeta^2}$$

which has a double pole at $\zeta = 0$ i.e. at $z = a_i$.

The meromorphic $(1,0)$-form ω_{a_i} has two poles at $z = a_i$. As a meromorphic section of K_X know that,

$$(\# \text{ of zeros of } \omega_{a_i}) - (\# \text{ of poles of } \omega_{a_i}) = c_1(K_X) = 2g - 2 = 2$$

so ω_{a_i} must have four poles. We have seen this is the case because ω_{a_i} has a double pole at ∞_{I} and a double pole at ∞_{II}.

Problem 2

Let X be a compact Riemann surface and $p, q \in X$ be distinct points.

(a)

First, let $U \subset X$ be a coordinate chart with a chart map $\varphi : U \to D \subset \mathbb{C}$ to a disk in the complex plane such that $p, q \in U$. Denote the holomorphic coordinate $\varphi^{-1} : D \to U$ on $U \subset X$ by z and the images of p and q in D by P an Q. Now, define,

$$g = \chi(z) \log\left(\frac{z - P}{z - Q}\right) \quad \text{and} \quad \tilde{\omega}_{pq} = \partial \left(\chi(z) \log\left(\frac{z - P}{z - Q}\right)\right) \mathrm{d}z$$

where χ is a bump function with support contained in $U = \varphi^{-1}(D)$ such that $\chi \equiv 1$ on some compact disk $B \subset D$ large enough that $P, Q \in B^\circ$. Since $\mathrm{Supp}\,(\tilde{\omega}_{pq}) \subset U$ we can extend $\tilde{\omega}_{pq}$ to a form on X by taking it zero outside U. Furthermore, since $\chi \equiv 1$ on $B \subset D$, in that smaller disk,

$$\tilde{\omega}_{pq} = \partial \left(\log\left(\frac{z - P}{z - Q}\right)\right) \mathrm{d}z \ = \left(\frac{1}{z - P} - \frac{1}{z - Q} \right) \mathrm{d}z$$

which is meromorphic with simple poles at p and q and residues $+1$ and -1 respectively. Furthermore, g is holomorphic on $B \setminus L$ where L is the line segment connecting P and Q which we can choose as the branch cut of the logarithm. The logarithm remains well-defined when traversing a loop around both P and Q because both $\log(z - P)$ and $\log(z - Q)$ pick up a factor of $2\pi i$ which cancel. Therefore, if we do not cross the branch cut L, the logarithm remains well-defined. Furthermore,

$$\bar{\partial} \tilde{\omega}_{pq} = \partial \bar{\partial} \left(\chi(z) \log\left(\frac{z - P}{z - Q}\right)\right) \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

I claim that $\eta = \bar{\partial} g$ can be extended to a smooth $(0,1)$-form on X. First, outside B we know that η is a smooth form because g is smooth and then identically zero. Furthermore, g is holomorphic on $B \setminus L$ so $\bar{\partial} g = 0$ on $B \setminus L$. Therefore, we can extend $\eta \equiv 0$ on all of B since $L \subset B^\circ$ without affecting its smoothness outside B. Thus, η is a smooth $(0,1)$-form on X. Now consider the equation,

$$\bar{\partial}\partial u = \bar{\partial}\tilde{\omega}_{pq}$$

where $\bar{\partial}\tilde{\omega}_{pq}$ is a smooth $(1,1)$-form because $\bar{\partial}\tilde{\omega}_{pq} = \partial\eta$ or because $\tilde{\omega}_{pq}$ is smooth outside B and holomorphic inside $B \setminus \{P, Q\}$ so $\bar{\partial}\tilde{\omega}_{pq} = 0$ on $B \setminus \{P, Q\}$ and can be extended to zero on all of B. Furthermore, I can write,

$$\bar{\partial}\tilde{\omega}_{pq} = \partial\bar{\partial}\left(\chi(z)\log\left(\frac{z-P}{z-Q}\right)\right)\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \;=\; \mathrm{d}\left\{\bar{\partial}\left(\chi(z)\log\left(\frac{z-P}{z-Q}\right)\right)\,\mathrm{d}\bar{z}\;\right\} \;=\; \mathrm{d}\eta$$

since both sides are zero on B and are smooth outside B. Above, I have used the identify, $\mathrm{d}(f\,\mathrm{d}\bar{z}\,) = \partial f\,\mathrm{d}z \wedge \mathrm{d}\bar{z} + \bar{\partial} f\,\mathrm{d}\bar{z} \wedge \mathrm{d}\bar{z} = \partial f\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$. Therefore,

$$\int_X \bar{\partial}\tilde{\omega}_{pq} = \int_X \mathrm{d}\eta\; = 0$$

by Stokes theorem because X has no boundary and η is a smooth $(0,1)$-form. By the PDE existence theorem proven in class, this implies the existence of a smooth solution u on X. Let $\psi = \partial u$ which is a smooth $(1,0)$-form. Then, we have shown,

$$\bar{\partial}\psi = \bar{\partial}\tilde{\omega}_{pq}$$

Define $\omega_{pq} = \tilde{\omega}_{pq} - \psi$ which has simple poles at exactly p and q since $\tilde{\omega}_{pq}$ is meromorphic inside $\varphi^{-1}(B) \subset U$ with simple poles as p and q while ψ is smooth everywhere. Furthermore,

$$\bar{\partial}\omega_{pq} = \bar{\partial}\tilde{\omega}_{pq} - \bar{\partial}\psi = 0$$

so ω_{pq} is holomorphic wherever it is smooth i.e. everywhere but its poles p and q so ω_{pq} is the desired meromorphic form.

(b)

Now suppose that p and q no not lie in the same coordinate chart of X. Let $\{U_\alpha\}$ be a covering of X by holomorphic coordinate charts. Since X is compact, we may choose this covering to include a finite number, N, charts. Let G be the graph on N vertices representing the charts with an edge between two vertices exactly when their corresponding charts intersect. For an edge E between α and β pick distinct points $P_E = P_{\alpha\beta}$ in the corresponding intersection $U_\alpha \cap U_\beta$. First, I claim that G is a connected graph. Let $C \subset G$ be a connected component. Then, let,

$$A = \bigcup_{\alpha \in C} U_\alpha \quad \text{and} \quad B = \bigcup_{\alpha \notin C} U_\alpha$$

which are clearly open because they are unions of open sets and $A \cup B = X$. Furthermore, since C is a connected component, if $\alpha \in C$ and $U_\alpha \cap U_\beta \neq \varnothing$ then $\beta \in C$ which implies that if $A \cap U_\beta \neq \varnothing$ then $\beta \in C$ so $A \cap B = \varnothing$. Since X is connected, either A or B must be empty which implies that $C = G$ and thus G is connected. Suppose that $p \in U_\alpha$ and $q \in U_\beta$ then there exists a path in G from α to β given by the finite sequence of vertices $\Gamma = \{\gamma_i\}$ with $\gamma_1 = \alpha$ and $\gamma_n = \beta$. Then define the following form by summing over the edges $E = (\gamma_i, \gamma_{i+1})$ of Γ, where I denote γ_i by i for notation convenience,

$$\omega_{pq} = \sum_{i=0}^{n-1} \omega_{P_{E_n} P_{E_{n+1}}} = \sum_{i=0}^{n-1} \omega_{P_{i,i+1} P_{i+1,i+2}}$$

where I take the convention $P_{0,1} = p$ and $P_{n,n+1} = q$. The form $\omega_{P_{i,i+1} P_{i+1,i+2}}$ is defined as above since $P_{\gamma_i \gamma_{i+1}} \in U_{\gamma_i} \cap U_{\gamma_{i+1}}$ and $P_{\gamma_{i+1}\gamma_{i+2}} \in U_{\gamma_{i+1}} \cap U_{\gamma_{i+2}}$ so both points are contained in the chart $U_{\gamma_{i+1}}$. The form ω_{pq} is meromorphic because it is the sum of finitely many meromorphic forms. I claim that ω_{pq} has simple poles at exactly p and q. The possible locations for poles of ω_{pq} are each $P_{i,i+1}$. Let us require that Γ is a minimal path and therefore contains no cycles thus hitting each vertex and edge at most once. If γ_{i+1} is not the start or endpoint then,

$$\mathrm{Res}_{P_{i,i+1}} \omega_{pq} = \mathrm{Res}_{P_{i,i+1}} \left(\omega_{P_{i,i+1}, P_{i+1,i+2}}\right) + \mathrm{Res}_{P_{i,i+1}} \left(\omega_{P_{i-1,i}, P_{i,i+1}}\right) = 1 - 1 = 0$$

but all the poles of ω_{pq} are simple so ω_{pq} can only have poles at the first and last points p and q. Since there is nothing to cancel these poles in the terms $\omega_{pP_{1,2}}$ and $\omega_{P_{n-1,n} q}$ we have that ω_{pq} has simple poles at exactly p and q with residue $+1$ at p and -1 at q.

(c)

Take the same setup with $p \in U \subset X$ a coordinate chart with holomorphic coordinate z. Suppose we want a meromorphic form with a single simple pole at p. We might try,

$$\tilde{\omega}_p = \partial \left(\chi(z) \log\left(z - P\right)\right) \, \mathrm{d}z$$

where χ is a bump function with support contained in $U = \varphi^{-1}(D)$ such that $\chi \equiv 1$ on some compact disk $B \subset D$. On B we know that $\chi \equiv 1$ so we have,

$$\tilde{\omega}_p = \frac{\mathrm{d}z}{z - P}$$

which is meromorphic on B with a simple pole at P. We would want to take B large enough to contain the branch cut L of the logarithm. However, this is no longer possible because the cut now extends to infinity outside any compact disk where as before it as a segment between P and Q which could be contained in a compact disk. Nevertheless, we will strive ahead by extending $\tilde{\omega}_p$ by zero outside U such that it is a smooth $(1,0)$-form on $X \setminus \{p\}$. As before, we consider the equation,

$$\bar{\partial}\partial u = \bar{\partial}\tilde{\omega}_{pq}$$

where $\bar{\partial}\tilde{\omega}_{pq}$ is extended to be a smooth $(1,1)$-form since $\tilde{\omega}_{pq}$ is holomorphic on $B \setminus \{p\}$ and thus $\bar{\partial}\tilde{\omega}_{pq} = 0$ on $B \setminus \{P\}$ so we can extend it to zero on all of B and also $\tilde{\omega}_p$ is smooth outside B. Now consider the integral,

$$\int_X \bar{\partial}\tilde{\omega}_p = \int_X \partial\bar{\partial}\left(\chi(z)\log\left(z-P\right)\right)\, \mathrm{d}z \,\wedge\, \mathrm{d}\bar{z}$$

which is zero exactly when we have a solution to the above PDE allowing us to construct a meromorphic form with the same pole structure as $\tilde{\omega}_p$. We would want to write,

$$\bar{\partial}\tilde{\omega}_p = \partial\bar{\partial}\left(\chi(z)\log\left(z-P\right)\right)\, \mathrm{d}z \,\wedge\, \mathrm{d}\bar{z} \;=\; \mathrm{d}\left\{\bar{\partial}\left(\chi(z)\log\left(z-P\right)\right)\, \mathrm{d}\bar{z}\,\right\}$$

and finish the proof by an application of Stokes theorem to show that the integral is zero. However, this is invalid because $\bar{\partial}\left(\chi(z)\log\left(z-P\right)\right)\, \mathrm{d}\bar{z}$ is not smooth outside B since the branch cut L extends out of B. However, this does hold on $X \setminus L$ were we take L to end outside U since χ is zero outside U killing the branch cut. So we may compute the integral over X cut along L to have a boundary L_+ and L_-,

$$\begin{aligned}
\int_{X_{\text{cut}}} \bar{\partial}\tilde{\omega}_p &= \int_{X_{\text{cut}}} \mathrm{d}\left\{\bar{\partial}\left(\chi(z)\log\left(z-P\right)\right)\, \mathrm{d}\bar{z}\,\right\} \\
&= \int_{L_+} (\bar{\partial}\chi(z))\log\left(z-P\right)\mathrm{d}\bar{z} \;-\int_{L_-} (\bar{\partial}\chi(z))\log\left(z-P\right)\mathrm{d}\bar{z} \\
&= 2\pi i \int_b^u (\bar{\partial}\chi(x))\log x \,\mathrm{d}x
\end{aligned}$$

where I choose the branch cut along the positive real direction in $D = \varphi(U)$, b is the distance along L that it leaves B i.e. the radius of B if we take it to be a disk in D centered at P, and u is the total length of the cut in D. Because χ is not holomorphic, this integral will not be zero.

Problem 3

Let $\tau = \tau_1 + i\tau_2$ and consider the torus $\mathbb{C}/\Lambda$ where $\Lambda = \{m + n\tau \mid m,n \in \mathbb{Z}\}$. We define the Green's function $G(z|\tau)$ on $\mathbb{C}/\Lambda$ by the formula,

$$G(z|\tau) = \log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 - \frac{2\pi}{\tau_2}\left(\mathrm{Im}(z)\right)^2$$

(a)

First, we consider the transformation properties of G. I will make use of the transformation of Jacobi θ-functions, The θ function transforms as,

$$\theta(z+1|\tau) = \theta(z|\tau) \qquad \theta(z+\tau|\tau) = e^{-\pi i\tau - 2\pi i z}\theta(z|\tau)$$

which were proved in class. First, since under the transformation $z \mapsto z+1$ the imaginary part of z is unchanged and also $\theta(z+1 \mid \tau) = \theta(z \mid \tau)$ we have,

$$G(z+1 \mid \tau) = G(z \mid \tau)$$

Furthermore, under the transformation $z \mapsto z+\tau$ we can write,

$$\begin{aligned}
G(z+\tau \mid \tau) &= \log\left|\frac{\theta_1(z+\tau|\tau)}{\theta_1'(0|\tau)}\right|^2 - \frac{2\pi}{\tau_2}\left(\mathrm{Im}(z+\tau)\right)^2 \\
&= \log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 + \log|e^{-\pi i\tau - 2\pi i z}|^2 - \frac{2\pi}{\tau_2}\left(\mathrm{Im}(z)+\tau_2\right)^2
\end{aligned}$$

Writing $z = x + iy$ we find that,

$$e^{-\pi i\tau - 2\pi i z} = e^{-\pi i\tau_1 + \pi\tau_2 - 2\pi i x + 2\pi y} = e^{\pi\tau_2 + 2\pi y} e^{-\pi i\tau_1 - 2\pi i x}$$

Since $\tau_1, \tau_2, x, y \in \mathbb{R}$ we have,

$$|e^{-\pi i\tau - 2\pi i z}| = e^{\pi\tau_2 + 2\pi y}|e^{-\pi i\tau_1 - 2\pi i x}| = e^{\pi\tau_2 + 2\pi y}$$

Which implies that,

$$\log|e^{\pi\tau_2 + 2\pi y}|^2 = 2\log e^{\pi\tau_2 + 2\pi\mathrm{Im}(z)} = 2\pi\tau_2 + 4\pi\mathrm{Im}(z)$$

Therefore,

$$\begin{aligned}
G(z+\tau \mid \tau) &= \log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 + 2\pi\tau_2 + 4\pi\mathrm{Im}(z) - \frac{2\pi}{\tau_2}\left(\mathrm{Im}(z)+\tau_2\right)^2 \\
&= \log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 + 2\pi\tau_2 + 4\pi\mathrm{Im}(z) - \frac{2\pi}{\tau_2}\left(\mathrm{Im}(z)^2 + 2\mathrm{Im}(z)\tau_2 + \tau_2^2\right) \\
&= \log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 + 2\pi\tau_2 + 4\pi\mathrm{Im}(z) - 4\pi\mathrm{Im}(z) - 2\pi\tau_2 - \frac{2\pi}{\tau_2}\left(\mathrm{Im}(z)\right)^2 \\
&= \log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 - \frac{2\pi}{\tau_2}\left(\mathrm{Im}(z)\right)^2 = G(z \mid \tau)
\end{aligned}$$

Therefore $G(z \mid \tau)$ is doubly periodic.

Furthermore, we derived the following formula in class,

$$\wp(z) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\left(\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right) + c(\tau)$$

where the constant $c(\tau)$ is given by,

$$c(\tau) = \frac{1}{3}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}$$

Therefore, we can integrate,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

twice to get,

$$\log\left(\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right) = \tfrac{1}{2}c(\tau)z^2 + c_1 z + c_2 + \log z + \sum_{\omega \in \Lambda^\times} \left[\log{(z+\omega)} - \frac{1}{2}\left(\frac{z}{\omega}\right)^2 \right]$$

Therefore,

$$\begin{aligned} \log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 &= \tfrac{1}{2}c(\tau)z^2 + \tfrac{1}{2}\overline{c(\tau)}\bar{z}^2 + c_1 z + \bar{c}_1\bar{z} + c_2 + \bar{c}_2 + \log|z|^2 \\ &+ \sum_{\omega \in \Lambda^\times} \left[\log|z+\omega|^2 - \frac{1}{2}\left(\frac{z}{\omega}\right)^2 - \frac{1}{2}\overline{\left(\frac{z}{\omega}\right)}^2 \right] \end{aligned}$$

We need to apply the Laplacian $\partial\bar{\partial}$ to this function. However,

$$\tfrac{1}{2}c(\tau)z^2 + c_1 z + c_2$$

is holomorphic everywhere so $\bar{\partial}$ applied to it gives zero. Furthermore, the conjugate of this holomorphic function is anti-holomorphic and thus its derivative under $\bar{\partial}$ is again anti-holomorphic so applying ∂ gives zero. Equivalently, we can use the fact that we are taking the real part of $\frac{1}{z}c(\tau)z^2 + c_1 z + c_2$ and the real and imaginary parts of any holomorphic function are harmonic by Lemma 1.1. Anyways,

$$\begin{aligned} \partial\bar{\partial}\log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 &= \partial\bar{\partial}\log|z|^2 + \partial\bar{\partial}\sum_{\omega \in \Lambda^\times} \left[\log|z+\omega|^2 - \frac{1}{2}\left(\frac{z}{\omega}\right)^2 - \frac{1}{2}\overline{\left(\frac{z}{\omega}\right)}^2 \right] \\ &= \pi\delta(z) + \sum_{\omega \in \Lambda^\times} \pi\delta(z+\omega) = \pi\sum_{\omega \in \Lambda}\delta(z+\omega) \end{aligned}$$

where I have used the identity $\partial\bar{\partial}\log|z| = \pi\delta(z)$ and the fact that,

$$\left(\frac{z}{\omega}\right)^2 + \overline{\left(\frac{z}{\omega}\right)}^2$$

is the real part of a holomorphic function and therefore harmonic. Furthermore, writing $z = x + iy$, we have, by Lemma 1.1,

$$\partial\bar{\partial} = \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

and therefore,

$$\partial\bar{\partial}(\mathrm{Im}(z))^2 = \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] y^2 = \frac{1}{2}$$

Finally, putting everything together,

$$\partial\bar{\partial}G(z \mid \tau) = \partial\bar{\partial}\log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 - \frac{2\pi}{\tau_2}\partial\bar{\partial}(\mathrm{Im}(z))^2 = \pi\sum_{\omega\in\Lambda}\delta(z+\omega) - \frac{\pi}{\tau_2}$$

Proving the result since we can identify,

$$\sum_{\omega\in\Lambda}\delta(z+\omega)$$

with the Dirac mass $\delta_{\mathbb{C}/\Lambda}(z)$ at the origin on the torus $\mathbb{C}/\Lambda$.

(b)

Let $\phi : \mathbb{C}/\Lambda \to \mathbb{C}$ be a doubly-periodic complex function. Let X be the fundamental domain of the lattice $\Lambda \subset \mathbb{C}$ and define the function,

$$u(z) = \frac{1}{2\pi}\int_X G(z-w \mid \tau)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w}$$

Since the Green's function G is doubly periodic,

$$\begin{aligned}
u(z+1) &= \frac{1}{2\pi}\int_X G(z+1-w \mid \tau)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = \frac{1}{2\pi}\int_X G(z-w \mid \tau)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = u(z) \\
u(z+\tau) &= \frac{1}{2\pi}\int_X G(z+\tau-w \mid \tau)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = \frac{1}{2\pi}\int_X G(z-w \mid \tau)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = u(z)
\end{aligned}$$

so u is doubly-periodic. Now suppose that ϕ has zero average, that is,

$$\int_X \phi(z) i \,\mathrm{d}z \wedge \mathrm{d}\bar{z} = 0$$

Then consider,

$$\begin{aligned}
\partial\bar{\partial}u(z) &= \frac{1}{2\pi}\int_X \partial\bar{\partial}G(z-w \mid \tau)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = \frac{1}{2\pi}\int_X \left[\pi\delta(z-w) - \frac{\pi}{\tau_2}\right]\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} \\
&= \frac{1}{2}\int_X \delta(z-w)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} - \frac{1}{2\tau_2}\int_X \phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = \phi(z)
\end{aligned}$$

The second integral vanishes because ϕ has zero average. The first integral works out to $\phi(z)$ because,

$$i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = i(\mathrm{d}x + i\,\mathrm{d}y) \wedge (\mathrm{d}x - i\,\mathrm{d}y) = i(i\,\mathrm{d}y \wedge \mathrm{d}x - i\,\mathrm{d}x \wedge \mathrm{d}y) = 2\,\mathrm{d}x \wedge \mathrm{d}y$$

which is twice the standard area element meaning that,

$$\frac{1}{2}\int_X \delta(z-w)\phi(w) i \,\mathrm{d}w \wedge \mathrm{d}\bar{w} = \int_X \delta(z-w)\phi(w)\,\mathrm{d}A = \phi(z)$$

by the definition of the Dirac mass on a plane. Therefore,

$$\partial\bar{\partial}u(z) = \phi(z)$$

Problem 4

Let $L \to X$ be a holomorphic line bundle over a Riemann surface X.

(a)

A metric h on L is a strictly positive section of $L^{-1} \otimes \bar{L}^{-1}$. It makes sense to call such a section positive because, if $\{t_{\alpha\beta}\}$ are a defining set of transition functions for the line bundle L, then h transforms as,

$$h_\alpha = t_{\alpha\beta}^{-1} \bar{t_{\alpha\beta}}^{-1} h_\beta = |t_{\alpha\beta}|^{-2} h_\beta$$

but $|t_{\alpha\beta}|^{-2}$ is a positive real so the transition functions preserve positivity of components of the section h. Such a metric allows the definition of the Chern unitary connection which acts on a holomorphic section via, $\nabla\varphi = h^{-1}\partial(h\varphi)$ and $\overline{\nabla}\varphi = \bar{\partial}\varphi$. This connection defines a curvature via the commutator,

$$[\nabla, \overline{\nabla}]\varphi = -F_{z\bar{z}}\varphi$$

where $F_{z\bar{z}} = -\partial\bar{\partial}\log h$. Then the first Chern class of L is defined by,

$$c_1(L) = \frac{i}{2\pi}\int_X F_{z,\bar{z}}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

(b)

Let L and L' be line bundles over X. Take h and h' to be metrics on L and L' respectively. I claim that $h \cdot h'$ is a metric on $L \otimes L'$. Suppose that X_μ is a covering of X my holomorphic charts. Then L is given by transition functions $t_{\alpha\beta}$ and L' by $t'_{\alpha\beta}$ between these charts. Consider the way that $h \cdot h'$ transforms from chart β to chart α on the intersection $X_\alpha \cap X_\beta$. We have,

$$(h \cdot h')_\alpha = h_\alpha \cdot h'_\alpha = |t_{\alpha\beta}|^{-2} h_\beta |t'_{\alpha\beta}|^{-2} h'_\beta = |t_{\alpha\beta} t'_{\alpha\beta}|^{-2} h_\beta \cdot h'_\beta = |t_{\alpha\beta} t'_{\alpha\beta}|^{-2} (h \cdot h')_\beta$$

Therefore, $h \cdot h'$ is a section of the bundle $(L \otimes L') \otimes (\bar{L} \otimes \bar{L}')^{-1}$. Furthermore, since h and h' are positive, their product $h \cdot h'$ is also positive since its components are the product of positive numbers. Therefore, $h \cdot h'$ is a metric on $L \otimes L'$. Thus, the curvature of the product bundle takes the form,

$$F^{L\otimes L'}_{z\bar{z}} = -\partial\bar{\partial}\log h^{L\otimes L'} = -\partial\bar{\partial}\log(h \cdot h') = -\partial\bar{\partial}\log h - \partial\bar{\partial}\log h' = F^L_{z\bar{z}} + F^{L'}_{z\bar{z}}$$

Then, the Chern class of $L \otimes L'$ becomes,

$$\begin{aligned} c_1(L \otimes L') &= \frac{i}{2\pi}\int_X F^{L\otimes L'}_{z\bar{z}}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \frac{i}{2\pi}\int_X F^L_{z\bar{z}}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} + \frac{i}{2\pi}\int_X F^{L'}_{z\bar{z}}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} = c_1(L) + c_1(L') \end{aligned}$$

(c)

Let L_0 be the trivial bundle on X defined by having all transition functions equal the constant function 1. Apply the main theorem on the relationship between the Chern class of a bundle and the zero-pole structure of its meromorphic sections.

Theorem 0.1. *Let L be a line bundle over X and $\varphi \in \Gamma(X, L)$ a meromorphic section of L which is not identically zero. Then,*

$$(\#\text{ zeros of }\varphi) - (\#\text{ poles of }\varphi) = \frac{i}{2\pi}\int_X F_{\bar{z}z}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}$$

Clearly, the constant function 1 is a holomorphic section of L_0 with no zeros or poles. Therefore, by the above theorem, $c_1(L_0) = 0$. We can also compute this value explicitly by considering the metric $h = 1$ on L_0 with is a section of $L_0^{-1} \otimes \bar{L_0}^{-1}$ since the transition functions of this bundle, being the products of inverses and conjugates of the transition functions of L_0, are also the constant function 1. Furthermore, $h = 1$ is clearly a strictly positive choice for the metric. Then we have,

$$F_{z\bar{z}} = -\partial\bar{\partial}\log h = 0$$

so clearly,

$$c_1(L_0) = \frac{i}{2\pi}\int_X F_{z\bar{z}}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} = 0$$

Problem 5

Let X be a compact Riemann surface and let $L \to X$ be a holomorphic line bundle on X

(a)

Define $H^0(X, L)$ to be the vectorspace of holomorphic sections of L and let K_X be the canonical bundle of holomorphic 1-forms on X i.e. $\Lambda^{1,0}$. Then the Riemann-Roch theorem states,

$$\dim H^0(X, L) - \dim H^0(X, L^{-1}\otimes K_X) = c_1(L) + \tfrac{1}{2}c_1(K_X^{-1})$$

where c_1 denotes the first Chern class.

(b)

Let $P \in X$ be some point and fix a holomorphic coordinate chart U containing P with local coordinate z. For $k \in \mathbb{Z}$ define the bundle $[kP]$ by its single transition function $t_{0,\infty} : U \setminus \{P\} \to \mathbb{C}$ between the holomorphic charts U and $U_\infty = X \setminus \{P\}$.[1]

[1] If U_∞ is not a holomorphic chart but rather is covered by such charts then we may take transition functions equal to 1 between them.

We define this transition function via, $t_{0,\infty}(z) = z^k$. Furthermore, we may define the section $1_{kP} \in \Gamma(X, [kP])$ by $1_{kP}|_{U_\infty} = 1$ and $1_{kP}|_U(z) = z^k$. This is indeed a section of $[kP]$ because $t_{0,\infty} 1_{kP}|_{U_\infty} = 1_{kP}|_U$ since $1_{kP}|_{U_\infty}$ is the constant value 1 and, on $U \cap U_\infty$ we have $t_{0,\infty}(z) = 1_{kP}|_U(z) = z^k$.

Clearly, the section 1_{kP} has a unique pole at P with order k. Therefore, applying the theorem relating poles and zeros of meromorphic sections to the first Chern class of the corresponding bundle, we find,

$$c_1([kP]) = (\#\text{ zeros of } \varphi) - (\#\text{ poles of } \varphi) = k$$

(c)

Let $P \in X$ be two distinct points. Consider the bundle $L = [-2P]$. Then we have shown that,

$$c_1(L) = -2$$

and therefore every meromorphic section of L must have exactly two more poles than zeros which implies that it must have at least one pole and thus cannot be holomorphic. Therefore, $\dim H^0(X, L) = 0$. Applying the Riemann-Roch theorem,

$$\dim H^0(X, L) - \dim H^0(X, L^{-1} \otimes K_X) = c_1(L) + \tfrac{1}{2} c_1(K_X^{-1})$$

which implies that,

$$\dim H^0(X, [2P] \otimes K_X) = 2 - \tfrac{1}{2} c_1(K_X^{-1})$$

Furthermore, taking $L = K_X$ in the Riemann-Roch theorem we find that,

$$\dim H^0(X, K_X) - \dim H^0(X, K_X^{-1} \otimes K_X) = c_1(K_X) + \tfrac{1}{2} c_1(K_X^{-1})$$

However, $K_X^{-1} \otimes K_X$ is the trivial bundle whose holomorphic sections are simply holomorphic functions on X which must be constant since X is compact. Thus, $\dim H^0(X, K_X^{-1} \otimes K_X) = 1$. Furthermore, $c_1(K_X^{-1}) = -c_1(K_X)$ so we have,

$$\dim H^0(X, K_X) = 1 - \tfrac{1}{2} c_1(K_X)$$

Therefore,

$$\dim H^0(X, [2P] \otimes K_X) = \dim H^0(X, K_X) + 1 = 2 - \tfrac{1}{2} c_1(K_X) = g + 1$$

where g turns out to be the genus of X. Take a basis of independent holomorphic sections of X, $\psi_, \ldots, \psi_g$. Then $\psi_1 1_{2P} \ldots, \psi_g 1_{2P}$ are independent holomorphic sections of the bundle $[2P] \otimes K_X$. Since the dimension of the space of all such holomorphic sections has dimension $g + 1$ there exists an independent holomorphic section $\Phi \in \Gamma(X, [2P] \otimes K_X)$. Now, $\varphi = \Phi 1_{2P}^{-1}$ is a meromorphic section of K_X since we are dividing out the dependence on the transition functions of $[2P]$. I claim that φ is exactly the meromorphic 1-form with a double pole at P we are looking for. First, since $\varphi 1_{2P} = \Phi$ is a holomorphic section, φ can only have a pole at P at at most of order 2. Furthermore, φ cannot have a single simple pole since the sum of the residues of the poles of a meromorphic form on a complex Riemann surface must be zero but the residue at the simple pole must be nonzero. Thus, φ either has no poles or has a double poles exactly at P. However, if φ had no poles it would be a holomorphic section of K_X implying that φ is a linear combination,

$$\varphi = \alpha_1 \psi_1 + \cdots + \alpha_g \psi_g$$

which implies that,

$$\Phi = \varphi 1_{2P} = \alpha \psi_1 1_{2P} + \cdots \alpha_g \psi 1_{2P}$$

contradicting the independence of Φ. Therefore, φ is a meromorphic section of K_X i.e. a meromorphic 1-form on X with exactly one double pole at P.

1 Lemmas

Lemma 1.1. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then both f and its conjugate or the equivalently the real and imaginary parts of f are harmonic with respect to the Laplacian,

$$\partial \bar{\partial} = \frac{1}{4} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right]$$

where $z = x + iy$ and $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$.

Proof. First, we know that

$$\partial = \frac{1}{2} \left[\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right] \quad \text{and} \quad \bar{\partial} = \frac{1}{2} \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right]$$

and therefore,

$$\begin{aligned} \partial \bar{\partial} &= \frac{1}{4} \left[\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right] \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right] = \frac{1}{4} \left[\frac{\partial^2}{\partial x^2} - i \frac{\partial}{\partial y} \frac{\partial}{\partial x} + i \frac{\partial}{\partial x} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} \right] \\ &= \frac{1}{4} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \end{aligned}$$

Furthermore, since f is holomorphic, $\bar{\partial} f = 0$ so $\partial \bar{\partial} f = 0$. Furthermore, $\bar{f}$ is anti-holomorphic since $\partial \bar{f} = \overline{\bar{\partial} f} = 0$. Therefore, $\partial \bar{\partial} \bar{f} = \bar{\partial} \partial \bar{f} = 0$. This implies that any linear combination of f and $\bar{f}$ are zeros of the Laplacian operator $\partial \bar{\partial}$ and, in particular, the real part $\frac{1}{2}(f + \bar{f})$ and imaginary part $\frac{1}{2i}(f - \bar{f})$ are harmonic. $\square$