1 TODO!!

(a) Finish symplectic geometry course

  (a) figure out if symplectic toric is the same as projective toric variety (projectivity needed to come from a polytope and also to be Kahler)
  (b) review coisotropic reduced and write some notes
  (c) hyperkahler reduction examples
  (d) are there examples of noncompact hyperkahlers?
  (e) work out the kinks in notes on hamiltonian actions

(b) review killing homotopy groups columbia lectures and write some notes

(c) figure out those damn jet bundles and connections on principal bundles

  (a) RMK: $\pi^* E$ is NOT trivial for a vector bundle let alone a fiber bundle. it does get equiped with a canonical section but for a vector bundle this is just the trivial section, only for a principal bundle does giving a section trivialize it.
  (b) role of atiyah sequence vs jet bundle sequence
  (c)

(d) spectral sequences for tor and ext in derived category (FIND MY NOTES ON THIS!)

  (a) application to universal coefficient theorem
  (b) Kunneth spectral sequence
  (c) Kunneth formula for smash product?
  (d) why are derived functors triangulated
  (e) derived functors in terms of Kan extensions (NOTES)

(e) write notes on universal morphisms

(f) G-action of X/Y induces map Descent data X/Y to G-equivariant sheaves

  (a) isomorphism when X/Y is a G-cover i.e. $X \to Y$ is a G-torsor
  (b) write down explicit G-equivariant structure on Ω_X
  (c) Galois descent derive explicit form

(g) Weil restriction

  (a) write down trivialization after going back up
  (b) Galois descent in explicit form

(h) notes on Galois actions on schemes

(i) notes on Frobenii

(j) notes on universal constructions in math with examples

(k) fix notes on Tor in category of sheaves and Tor symmetry (do I need symmetry of flat objects a priori?).

(l) Finish stable homotopy theory course.

(m) Finish vector bundles and connections notes (in AG folder)

  (a) Kahler iff $\nabla I = 0$ where ∇ is the Levi-Civita connection
  (b) Ricci tensor and the trace bullshit
  (c) Riemann-Hilbert and existence of flat frames for integrable connections

2 What I Want to Think About

(a) Flat cohomology equal etale cohomology for smooth (affine groups) apply this to that counting rational points things

(b) work out the details for the group fixing $\mathbb{C}$ inside endomorphism group. What does an integrable structure of this kind look like, how close to a complex manifold can we get? In dimension two this should be exactly a conformal (not necessarily orientable) structure.

(c) FINISH CONFORMAL NOTES!

(d) Hilbert Class Field of curves (ASK BRIAN FOR REFERENCE)

(e) Read about Fredholm index and Riemann-Roch

(f) Cohmology and inclusion-exclusion: cohomology for vectorspaces?

3 Some Questions I Have

(a) Reduction of structure group for a scheme.

  (a) what about the algebraic group $\mathrm{SL}^{\pm} = \det^{-1}(\mu)$ what does reduction of structure group give. For a manifold this is supposed to be a pseudo-volume form but obviously that's not right.
  (b) what about $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \hookrightarrow \mathrm{GL}_2$ from the action $\mathbb{G}_m \circlearrowright \mathbb{A}^1_{\mathbb{C}}$ restricted giving an action $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \circlearrowright \mathbb{A}^2_{\mathbb{R}}$. I feel like this should give an almost complex structure. What properties does it have? What about for other fields?
  (c) What is an almost complex structure on a scheme look like?

(b) Is my calculation of an "almost almost complex structure" as reduction of structure group to $\langle\sigma\rangle \ltimes \mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$. For the case $n = 1$ this should be the conformal group justfying that I think this should correspond to the non-oriented case of a complex manifold since Riemann surfaces are exactly oriented conformal manifolds.

4 TEST

5 Introduction

Let X be a minimal variety of dimension n over $\mathcal{C}$. Throughout this note, we will assume that there are k linearly independent nowhere-vanishing holomorphic 1-forms on X. In other words, there exists a short exact sequence

$$0 \to \mathcal{O}^{\oplus k} \to \Omega^1_X \to \mathscr{F}^\vee \to 0$$

of vector bundles.

Lemma 5.0.1. The dual short exact sequence

$$0 \to \mathscr{F} \to T_X \to \mathcal{O}_X^{\oplus k} \to 0$$

gives a foliation.

Q1. When is $\mathscr{F}$ algebraic?

This is very much related to a paper of Campana and Peternell, where they ask this question in the context

5.1 Foliations

Let X be an algebraic variety over $\mathbb{C}$ or a complex manifold. We will usually assume X projective and smooth so there will be no distinction between analytic and algebraic sheaves. On X we define $\mathcal{T}_X = \Omega_X^\vee$ which is a reflexive sheaf but not necessarily a vector bundle if X is singular.

Definition 5.1.1. A *(singular) foliation* on X is a subsheaf $\mathscr{F} \subset T_X$ such that,

(a) $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$

(b) $\mathscr{F}$ is *saturated* meaning $\mathcal{T}_X / \mathscr{F}$ is torsion-free.

We write $\mathcal{N}_{\mathscr{F}} = \mathcal{T}_X / \mathscr{F}$ is the *normal bundle* of the foliation and $r_{\mathscr{F}} = \operatorname{rank} \mathscr{F}$ is the *rank* of the foliation. The *canonical bundle* of $\mathscr{F}$ is $\omega_{\mathscr{F}} = \det \mathscr{F}^\vee$ and any Weil divisor $K_{\mathscr{F}}$ such that $\mathcal{O}_X(K_{\mathscr{F}}) = \omega_{\mathscr{F}}$ is called the *canonical class*.

Proposition 5.1.2. If $\mathscr{F}$ is a foliation then $\mathscr{F}$ is reflexive.

Proof. This follows from $\mathcal{T}_X / \mathscr{F}$ being torsion-free by Tag 0EBG. □

Proposition 5.1.3. The map $\ell_{\mathscr{F}} : \mathscr{F} \otimes \mathscr{F} \to \mathcal{T}_X / \mathscr{F}$ is $\mathcal{O}_X$-linear and,

$$[\mathscr{F}, \mathscr{F}] \subset \mathscr{F} \iff \ell_{\mathscr{F}} = 0$$

Proof. Indeed, we just need to compute,

$$[fX, Y] = f[X, Y] + Y(f)X$$

so if X, Y are sections of $\mathscr{F}$ then $Y(f)X$ is zero in $\mathcal{T}_X / \mathscr{F}$ proving the claim. □

We now define a bunch of notions of singular sets of a foliation.

Definition 5.1.4. The *singular set* of a coherent subsheaf $\mathscr{F} \subset T_X$ is,

$$\mathrm{Sing}(\mathscr{F}) = \{x \in X \mid \mathscr{F}_x \to (\mathcal{T}_X)_x \text{ is not a free direct factor}\}$$

is union of the points where the rank of $\mathscr{F}$ jumps up and where the rank of $\mathscr{F} \to T_X$ jumps down. Explicitly,

$$S_1(\mathscr{F}) = \{x \in X \mid \mathscr{F}_x \text{ is not free}\}$$

and likewise,

$$S_2(\mathscr{F}) = \{x \in X \mid \mathscr{F}(x) \to \mathcal{T}_X(x) \text{ is not injective}\}$$

Proposition 5.1.5.

Definition 5.1.6. A foliation $\mathscr{F} \subset \mathcal{T}_X$ is *regular* if $\mathrm{Sing}(\mathscr{F}) = \varnothing$ or, equivalently, if $\mathscr{F} \hookrightarrow \mathcal{T}_X$ is a sub-bundle (in the sense of locally being a free direct factor).

Remark. If $\mathcal{T}_X$ is a vector bundle then the following are equivalent,

$$\mathscr{F} \text{ is regular} \iff \mathscr{F} \subset \mathcal{T}_X \text{ is a sub-bundle} \iff \mathcal{N}_{\mathscr{F}} \text{ is a vector bundle} \iff \mathrm{Sing}(\mathscr{F}) = \varnothing$$

Definition 5.1.7. We say that $\mathscr{F}$ is 1-*Gorenstein* if $\omega_{\mathscr{F}}$ is a line bundle.

Remark. Note that $\mathscr{F}$ is automatically reflexive so $\det \mathscr{F}$ is reflexive so $\det \mathscr{F}$ is a line bundle iff $\omega_{\mathscr{F}}$ is a line bundle.

Remark. If X is regular then any $\mathscr{F}$ is automatically 1-Gorenstein because rank 1 reflexive sheaves on regular schemes are line bundles (this should follow from every Weil divisor being Cartier) it is in Hartshorne's Stable Reflexive Sheaves somewhere.

5.2 The Analytic Theory

In the analytic theory, we must (because I don't understand analytic spaces) work with smooth varities so that we get a manifold. Everywhere we work over $\mathbb{C}$. Recall the following theorem from real analysis,

Theorem 5.2.1 (Frobenius)**.** Let M be a smooth manifold and $\mathscr{F}$ a regular foliation. Then there is a collection $\{L_\alpha\}_\alpha$ of connected injectively immersed smooth manifolds $L_\alpha \to M$ (not closed) such that,

(a) M is a disjoint union of the L_α

(b) for each $p \in M$ there is a chart $U_p \subset M$ such that $U_p \cap L_\alpha$ is a countable union of slices (there are coodinates $(x^1, \dots, x^n)$ such that the components are $x^{r+1} = c_1, \dots, x^n = c_{n-r}$)

(c) for each $p \in L_\alpha$ we have $T_p L_\alpha = \mathscr{F}_p$ inside T_M.

If X is a complex manifold and $\mathscr{F} \subset T_X$ is a complex regular foliation then we can assume that the leaves L_α are immersed complex submanifolds.

Corollary 5.2.2 (Lemma 2.6, OFF)**.** Let $(X, \mathscr{F})$ be a 1-Gorenstein foliated variety. Suppose that $\mathscr{F}$ is Pfaff-regular and locally free at a point $x \in X$ then there exists an analytic open U of x a complex analytic space W, and a smooth morphism $U \to W$ of relative dimension $r_{\mathscr{F}}$ such that $\mathscr{F}|_U = \mathcal{T}_{U/W}$.

Definition 5.2.3. Let X be a smooth algebraic variety. We say an immersed manifold $\iota : L \to X$ is *algebraic* if $\iota(L) \cap Z^{\text{sm}} \subset Z$ is (analytically) open where Z is the Zariski closure of $\iota(L)$.

Remark. I choose this slightly strange condition to capture the following phenomenon. Let $\mathbb{A}^1 \to \mathbb{A}^2$ be $t \mapsto (t^2 - 1, t(t^2 - 1))$ whose image is the nodal curve X. This is an immersed complex manifold. However, the image of the open set $\overline{B_{1/2}(1)}^C$ is not open in X. We need to remove the node to get an open set.

Remark. I think the definition of algebraic leaves in Campana 2021 (that the Zariski and topological closures coincide) is wrong. For example, it predicts that the dense irrational slope foliation on an abelian surface is algebraic since each leaf is topologically and hence Zariski dense.

Lemma 5.2.4. Let $\iota : L \to X$ be an immersed submanifold such that $Z = \overline{\iota(L)}^{\text{Zar}}$ and L have the same dimension. Then L is algebraic.

Proof. Consider the map $\iota : \iota^{-1}(Z^{\text{sm}}) \to Z^{\text{sm}}$ is a local diffeomorphism of smooth manifolds since it is an immersion of manifolds of the same dimension and hence is open. ☐

Remark. Algebraicity of course implies that L is a complex (immersed) submanifold.

Lemma 5.2.5. . Let $\iota : L \to X$ be an immersed submanifold with L connected. Then if $Z = \overline{\iota(L)}^{\text{Zar}}$ has the same dimension as L then Z is irreducible.

Proof. We know $\iota : L \to Z$ is analysically open away from the singularities. However, $Z^{\text{sing}} \subset Z$ has codimension at least 2 and thus $\iota^{-1}(Z^{\text{sing}})$ also has codimension at least 2 so $L \setminus \iota^{-1}(Z^{\text{sing}})$ is connected. Thus $\iota(L \setminus \iota^{-1}(Z^{\text{sing}})) \subset Z \setminus Z^{\text{sing}}$ so it must lie in some irreducible component (the irreducible components have become disconnected by removing the singularities). Since $L \setminus \iota^{-1}(Z^{\text{sing}})$ is dense in L then $\iota : L \to Z$ is contained in some irreducible component. ☐

Remark. This is false if we don't assume that $\dim Z = \dim L$. For example, there are embedded curves $\mathbb{R} \to \mathbb{A}^3_{\mathbb{C}}$ whose closure is the union of two planes. Ineed, consider a curve which wanders in the xy-plane before following the x-axis then smoothly transitions to wandering in the xz-plane.

Proposition 5.2.6. If X is a smooth variety and $\mathscr{F}$ regular algebraic foliation on X then every algebraic leaf is an *embedded* submanifold which is the analytification of a smooth algebraic subvariety.

Proof. It suffices to show that each leaf L is Zariski closed. Let Z be the Zariski closure of L. Choose $p \in X$ and an open U such that $U \cap L$ is a union of slices. Since Z is closed we may shrink U so that $U \cap Z$ is connected. Since Z is irreducible, Z^{sm} is a connected embedded submanifold dense in Z. Then $L \cap Z^{\text{sm}} \cap U \subset Z^{\text{sm}} \cap U$ is open and its closure in U is a union of slices but since $Z \cap U$ is connected of dimension equal to the dimension of the slices it cannot contain more than one. Hence $L \cap U$ is a single slice and is closed. Thus taking closures $L \cap U = Z \cap U$ so $L = Z$ and hence L is smooth. ☐

Remark. Without the algebraicity assumption, the leaves of $\mathscr{F}$ do not even need to be closed. For example, the irrational slope foliation on an abelian variety.

Remark. Is it true that *every* leaf of an algebraic foliation is algebraic. I think this is true. I know how to prove this using Reeb stability if the manifold is compact.

Remark. An immersed submanifold being algebraic is a local property in the following sense.

Proposition 5.2.7. Let X be a smooth variety and $\iota : L \to X$ an immersed submanifold. Let $x \in X$ and $U \subset X$ is a Zariski open neighborhood of x then L is algebraic in X if and only if $L \cap U$ is algebraic in U.

Proof. Suppose $L \cap U$ is algebraic in U. Then $(L \cap U) \cap Z^{\text{sm}}$ is analytically open in $Z \cap U = \overline{L \cap U}^{\text{Zar}}$ where $Z = \overline{L}^{\text{Zar}}$. Then Z is irreducible so $\dim L = \dim Z$ because $\dim Z$ can be computed as $\dim Z \cap U$ so we conclude by Lemma 5.2.5. $\square$

Remark. This locality is *false* if U is an analytic open (also it is somewhat unclear what algebraicity of $L \cap U$ should mean in this case). For example, consider the graph of a bump function in the real coodinate in $\mathbb{A}^2$ (meaning $z \mapsto (z, \varphi(\Re z))$). There are analytic opens where this equals $(z, 0)$ and where this equals $(z, 1)$ so it is "algebraic" on each open. This fails because we don't have $\overline{L \cap U}^{\text{Zar}} = \overline{L}^{\text{Zar}} \cap U$ in this case (only works for Zariski opens). However, this should still be true as long as $\iota : L \to X$ is an analytic (immersed) submanifold using analytic continuation.

Definition 5.2.8. Let X be a smooth variety and $\mathscr{F} \subset \mathcal{T}_X$ a regular foliation. A closed subvariety $Z \subset X$ is a *leaf* of $\mathscr{F}$ if it is smooth and Z^{an} is a leaf of $\mathscr{F}$ as an immersed submanifold.

Remark. Being a leaf is local in the following sense.

Proposition 5.2.9. Let X be a smooth variety and $\mathscr{F} \subset \mathcal{T}_X$ a regular foliation. Let $Z \subset X$ be a closed subvariety and $x \in Z$ and $U \subset X$ an analytic open neighborhood of x. Then Z is a leaf of $\mathscr{F}$ if and only if $Z^{\text{an}} \cap U$ is a leaf of $\mathscr{F}|_U$.

Remark. If U is Zariski open then this is equivalent to $Z \cap U$ being a leaf of $\mathscr{F}|_U$ in the above sense.

Proof. This follows from analytic continuation. Any two analytic immersed submanifolds are equal if and only if they are equal on some open set. We apply this to Z and L the unique leaf through x to see that these are equal globally if and only if they are equal on U. $\square$

6 The Complex Geometric Picture

Proposition 6.0.1. There is a covariant equivalence of categories,

$$\{\mathbb{Z}\text{-Hodge Structures of type } (1,0) \oplus (0,1)\} \iff \{\text{complex Tori}\}$$

which specializes to,

$$\{\text{polarized } \mathbb{Z}\text{-Hodge Structures of type } (1,0) \oplus (0,1)\} \iff \{\text{abelian varities}\}$$

Remark. If the polarization is required to be principal then the corresponding abelian varitety is princiaplly polarized.

Corollary 6.0.2. Nonconstant morphisms $f : X \to A$ to a simple abelian variety (considered up to translation and isogeny) correspond to irreducible sub-$\mathbb{Q}$-Hodge structures of $H^1(X, \mathbb{Z})$.

Proof. Morphisms $f : X \to A$ sending a fixed base point $x_0 \in X$ to $0 \in A$ are equivalent to homomorphisms $\text{Alb}_X \to A$ which correspond to maps of Hodge structures $H^1(A, \mathbb{Z}) \to H^1(\text{Alb}_X, \mathbb{Z}) = H^1(X, \mathbb{Z})$. Considered in the isogeny category, these correspond to maps $H^1(A, \mathbb{Q}) \to H^1(X, \mathbb{Q})$ and since A is simple $H^1(A, \mathbb{Q})$ is irreducible so the map is either zero or injective. $\square$

The smoothness of the morphism $f : X \to A$ is not in question. Smoothness is equvalent to $f^*\Omega_A \to \Omega_X$ being a subbundle (of the correct dimension) i.e. if $\omega_1, \dots, \omega_g \in H^0(A, \Omega_A)$ is a basis of holomorphic 1-forms then $f^*\omega_1, \dots, f^*\omega_g \in H^0(X, \Omega_X)$ should form a partial frame (meaning they are everywhere independent).

6.1 Maps to Circles

Proposition 6.1.1. Let M be a compact smooth manifold and $\omega \in M$ a closed nonvalishing 1-form. For any $\epsilon > 0$ there exists a submersive map $f_\epsilon : M \to S^1$ and an integer n_ϵ such that,

$$||\omega - n_\epsilon^{-1} f_\epsilon^* \mathrm{d}t|| < \epsilon$$

in the L^∞ norm.

Proof. Because S^1 is a $K(\mathbb{Z}, 1)$ there are continuous maps $f_i : M \to S^1$ such that $f_i^*[S_1]$ form a basis of $H^1(X, \mathbb{Z})$. By [Prop. 17.8 of Bott, Tu, Differential Forms in Algebraic Topology] up to homotopy, we may choose the f_i smooth. By naturality of the de Rham comparision theorem,

$$\eta_i = f^* \mathrm{d}t$$

form a basis of $H^1_{\mathrm{dR}}(X)$. Thus we can write,

$$\omega = \sum_i \alpha_i \eta_i + \mathrm{d}g$$

The idea is to rationally approximate the numbers $\alpha_i \in \mathbb{R}$. Indeed, we can choose rational numbers $\frac{a_i}{n_\epsilon} \in \mathbb{Q}$ such that if let let,

$$\tilde{\omega} = \sum_i \frac{a_i}{n_\epsilon} \eta_i + \mathrm{d}g$$

then we get,

$$||\omega - \tilde{\omega}|| = \left|\left| \sum_i \left(\alpha_i - \frac{a_i}{n_\epsilon} \right) \eta_i \right|\right| < \epsilon$$

this requires choosing the rational apprxomation on the order of $\frac{\epsilon}{\mathrm{Vol}(M)}$. The let,

$$f_\epsilon = \left(\prod_i f_i^{a_i} \right) \cdot (\Pi \circ g)^{n_\epsilon}$$

where $\Pi : \mathbb{R} \to S^1$ is the universal cover. Therefore,

$$f_\epsilon^* \mathrm{d}t = \sum_i a_i f_i^* \mathrm{d}t + n_\epsilon \mathrm{d}g = \sum_i a_i \eta_i + n_\epsilon \mathrm{d}g = n_\epsilon \tilde{\omega}$$

proving the required inequality. Finally, for sufficiently small ϵ, since ω is nonvanishing we see that $\tilde{\omega}$ is also nonvanishing so f_ϵ is smooth. $\square$

Remark. Let's see what happens when we try to do this for a holomorphic 1-form $\omega \in H^0(X, \Omega_X)$. Write $\omega = \omega_1 + i\omega_2$ into its real an imaginary parts. Note that because ω is holomorphic $\mathrm{d}\omega = 0$ (this requires X compact Kahler). Therefore, we can rationally approximate, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ to get a submersive (because for small enough ϵ we can ensure that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are everywhere independent) map $f : X \to S^2$ with $f^* \mathrm{d}z = n\tilde{\omega}$ with $\tilde{\omega} = \tilde{\omega}_1 + i\tilde{\omega}_2$ but there is no reason that $\tilde{\omega}_2$ should be holomorphic. Suppose we could approximate ω by a rational form which is holomorphic. This is exactly a $\mathbb{Q}$-Hodge submodule of $H^1(X, \mathbb{Q})$ of rank 1. Therefore, we are in the buisness of showing that $H^1(X, \mathbb{Q})$ is a reducible Hodge structure.

7 Morphisms

Note: Ω_X on a normal variety is often not torsion-free.

Proposition 7.0.1. Let $f : X \to Y$ be a dominant morphism of normal varities Then the subsheaf $\mathcal{T}_f = \ker(T_X \to f * T_Y)$ is a (possibly singular) foliation.

(DOES THIS EVEN MAKE SENSE IF PULLBACK IS NOT INJECTIVE)

Proof. Consider the sequence,

$$f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_f \longrightarrow 0$$

then the left-exactness (as a functor on the opposite category) of $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ gives a sequence,

$$0 \longrightarrow \mathcal{T}_f \longrightarrow \mathcal{T}_X \longrightarrow f^*\mathcal{T}_Y$$

where we define $\mathcal{T}_f = \Omega_f^\vee$. Note that these sheaves are reflexive[1]) and that $\mathcal{T}_X/\mathcal{T}_f \hookrightarrow f^*\mathcal{T}_Y$ which is reflexive and hence torsion-free. Therefore, it suffices to show that $\mathcal{T}_f$ is closed under Lie bracket. For two local sections ξ, η of $\mathcal{T}_f$ then,

$$[\xi, \eta] \circ \mathrm{d} = (\xi \circ \mathrm{d})(\eta \circ \mathrm{d}) - (\eta \circ \mathrm{d})(\xi \circ \mathrm{d})$$

as derivations. This acts trivially on $f^{-1}\mathcal{O}_Y$ since $\xi \circ \mathrm{d}$ and $\eta \circ \mathrm{d}$ both act trivially by definition. Hence $[\xi, \eta] \circ \mathrm{d}$ factors through Ω_f menaing $[\xi, \eta]$ arises from a section of $\mathcal{T}_f$. $\square$

7.1 Leaves on Nonsmooth Varities

We have defined what it means for a leaf of a regular foliations to be algebric inside a smooth variety. Here we consider what it should mean for a non-smooth variety and a non-reglar foliation.

Definition 7.1.1. Let X be a variety and $\mathscr{F} \subset \mathcal{T}_X$ a foliation. We say that a closed subvariety $Z \subset X$ is a *leaf* of $\mathscr{F}$ if $Z \not\subset X^\circ$ and $Z \cap X^\circ$ is a leaf of $\mathscr{F}|_{X^\circ}$ where $X^\circ = X^{\mathrm{sm}} \backslash \mathrm{Sing}(\mathscr{F})$.

Proposition 7.1.2. There is a unique leaf throught any point of X°.

Proof. Obvious from the analytic statement that there is a unique leaf through any point of a smooth manifold. $\square$

Example 7.1.3. It is not necessarily true that there is a unique leaf through points of $X \setminus X^\circ$. Indeed, consider $X = \mathbb{A}^2$ and $\mathscr{F} = (x\partial_x + y\partial_y) \subset \mathcal{T}_X$. Then $\mathrm{Sing}(\mathscr{F}) = \{(0,0)\}$. There are infinitely many leaves through $(0,0) \in \mathbb{A}^2$. Indeed, any complex line through the origin is a leaf.

Definition 7.1.4. A foliation $\mathscr{F}$ on X is *algebraic* if the leaf of the regular foliation $\mathscr{F}|_{X^\circ}$ through a general point is algebraic.

Proposition 7.1.5. Let X be a variety with a foliation $\mathscr{F} \subset \mathcal{T}_X$ and $Z \subset X$ a closed subvariety of dimension $r_{\mathscr{F}}$. Then if Z is a leaf if and only if the map $\mathscr{F} \otimes \Omega_X \to \mathcal{O}_X$ restricted to Z factors as,

$$\begin{tikzcd} \mathscr{F}|_Z \otimes \Omega_X|_Z \arrow[r] \arrow[d] & \mathcal{O}_Z \\ \mathscr{F}|_Z \otimes \Omega_Z \arrow[ur, dashed] & \end{tikzcd}$$

[1] See Tag 0AY4

Proof. By definition, Z is a leaf iff for any $x \in Z^{\text{sm}} \cap X^\circ$ then $\mathscr{F}(x) = \mathcal{T}_Z(x)$ inside $\mathcal{T}_X(x)$. The above factorization is equivalent to showing that $\varphi : \mathscr{F}|_Z \otimes \mathcal{C}_Z \to \mathcal{O}_Z$ is zero. Thus L is a leaf iff on the dense open $Z^{\text{sm}} \cap X^\circ$ of Z the map φ is zero since $\mathscr{F}(x) = \mathcal{T}_Z(x) \iff \mathscr{F}(x) \subset \mathcal{C}_Z^\perp$ since $\mathscr{F}(x)$ and $\mathcal{T}_Z(x)$ have the same dimension. Finally, since $\mathcal{O}_Z$ is torsion-free, φ is zero iff it vanishes on a dense open. □

Lemma 7.1.6. Let $f : X \to Y$ be a dominant map of integral characteristic zero schemes. Let X_y be a fiber and let $Z = (X_y)_{\text{red}}$ be the reduction. Then the natural pairing $\Omega_{X/Y}^\vee \otimes \Omega_{X/Y} \to \mathcal{O}_X$ factors as,

$$\begin{tikzcd} \Omega_{X/Y}^\vee|_Z \otimes \Omega_{X_y}|_Z \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \\ \Omega_{X/Y}^\vee|_Z \otimes \Omega_Z \arrow[ur, dashed] & \end{tikzcd}$$

Proof. Let $\mathcal{N}_y$ be the conormal bundle of $Z \hookrightarrow X_y$ i.e. the sheaf of nilpotents. It suffices to show that $\mathscr{F}|_Z \otimes \mathcal{N}_y \to \mathcal{O}_Z$ is zero. This is an affine local question so reduce to an injective map of domains $\varphi : R \to A$ and localizing we may assume that R is local. Let $I = \mathfrak{m}_R A$ and $J = \sqrt{I}$. We need to show that $\text{Hom}_A\left(\Omega_{A/R}, A\right) \otimes J/J^2 \to (R/J)$ given by $(\ell, f) \mapsto \ell(\mathrm{d}f)$ is zero. This is equivalent to showing: for all $f \in J$ and $\ell \in \text{Hom}_A\left(\Omega_{A/R}, A\right)$ that $\ell(f) \in J$. Now $f^n \in I$ then I claim that for all $0 \le k \le n$,

$$f^{n-k}\ell(\mathrm{d}f)^{2k} \in I$$

The case $k = 0$ is obvious. For induction, notice that $\mathrm{d}I \subset I\Omega_{A/R}$ since $\mathrm{d}\mathfrak{m}_R = 0$ and thus,

$$\mathrm{d}(f^{n-k}\ell(\mathrm{d}f)^{2k}) = (n-k)f^{n-k-1}\mathrm{d}f\ell(\mathrm{d}f)^{2k} + (2k)f^{n-k}\ell(\mathrm{d}f)^{2k-1}\mathrm{d}\ell(\mathrm{d}f) \in I\Omega_{A/R}$$

then applying ℓ we see,

$$(n-k)f^{n-k-1}\ell(\mathrm{d}f)^{2k+1} + (2k)f^{n-k}\ell(\mathrm{d}f)^{2k-1}\ell(\mathrm{d}\ell(\mathrm{d}f)) \in I$$

multiplying by $\ell(\mathrm{d}f)$ preserves I so we get,

$$(n-k)f^{n-k-1}\ell(\mathrm{d}f)^{2(k+1)} + (2k)f^{n-k}\ell(\mathrm{d}f)^{2k}\ell(\mathrm{d}\ell(\mathrm{d}f)) \in I$$

but by the induction hypothesis the second term lies in I and hence,

$$(n-k)f^{n-k-1}\ell(\mathrm{d}f)^{2(k+1)} \in I$$

so if $n > k$ the claim is proved by induction (using characteristic zero). □

Corollary 7.1.7. Suppose $f : X \to Y$ is a map of integral schemes and there is a morphism $\mathscr{F} \to \mathcal{T}_X$ with $\mathscr{F} \to \mathcal{T}_X \to f^*\mathcal{T}_Y$ zero. Then for any irreducible component $Z \subset X_y$ (with the reduced structure) we have a factorization,

$$\begin{tikzcd} \mathscr{F}|_Z \otimes \Omega_X|_Z \arrow[r] \arrow[d, two heads] & \mathcal{O}_Z \\ \mathscr{F}|_Z \otimes \Omega_Z \arrow[ur, dashed] & \end{tikzcd}$$

Proof. The condition that $\mathscr{F} \to \mathcal{T}_X \to f^*\mathcal{T}_Y$ is zero is equivalent to $\mathscr{F} \to \mathcal{T}_X$ factoring through $\mathscr{F} \to \Omega_{X/Y}^\vee$. Then we apply the previous result since the pairing $\mathscr{F} \otimes \Omega_X \to \mathcal{O}_X$ factors though $(\Omega_{X/Y})^\vee \otimes \Omega_{X/Y} \to \mathcal{O}_X$. □

7.2 Pfaff Fields and Leaves

Definition 7.2.1. Let X be a variety. A *A Pfaff field of rank* r is a nonzero map $\eta : \Omega_X^r \to \mathcal{L}$ where $\mathcal{L} \in \mathrm{Pic}\,(X)$. The singular locus S of η is the vanishing locus of η which is defined by the ideal $\mathscr{I}_S = \mathrm{im}\,(\Omega_X^r \otimes \mathcal{L}^\vee \to \mathcal{O}_X)$.

Definition 7.2.2. A closed subscheme $Z \subset X$ is *invariant* under a Pfaff field η is,

(a) no irredcible component of Y is contained in the singular locus of η

(b) the restriction $\eta|_Y : \Omega_X^r|_Y \to \mathcal{L}|_Y$ factors as,

$$\begin{tikzcd} \Omega_X^r|_Y \arrow[r, "\eta|_Y"] \arrow[d] & \mathcal{L}|_Y \\ \Omega_Y^r \arrow[ur] & \end{tikzcd}$$

Definition 7.2.3. If $\mathscr{F}$ is a 1-Gorenstein foliation of rank r then there is an associated Pfaff field,

$$\eta_{\mathscr{F}} : \Omega_X^r \to \wedge \mathcal{T}_X^\vee \to \wedge^r \mathscr{F}^\vee = \det \mathscr{F}^\vee = \omega_{\mathscr{F}}$$

Definition 7.2.4. We say that a Pfaff field η is *regular* if $\mathrm{Sing}(\eta) = \varnothing$ and a 1-Gorenstein foliation is *Pfaff-regular* if $\mathrm{Sing}(\eta_{\mathscr{F}}) = \varnothing$.

Lemma 7.2.5. Let X be a smooth variety and $\mathscr{F}$ a rank r foliation. Let $Z \subset X$ be an irreducible subvariety of dimension $r_{\mathscr{F}}$ such that $Z \not\subset \mathrm{Sing}(\mathscr{F})$. Then,

$$Z \backslash \mathrm{Sing}(\mathscr{F}) \text{ is a leaf of } \mathscr{F}|_{X\backslash \mathrm{Sing}(\mathscr{F})} \iff Z \text{ is invariant under } \eta_{\mathscr{F}}$$

Proof. Let $x \in Y \backslash \mathrm{Sing}(\mathscr{F})$ be a smooth point of Y and $v_1, \dots, v_r$ a local frame of $\mathscr{F}$. Then the map,

$$\eta|_U : \Omega_X^r|_U \to \omega_{\mathscr{F}}|_U$$

is given by,

$$\alpha \mapsto \alpha(v_1, \dots, v_r)\omega$$

where $\omega \in H^0(U, \omega_{\mathscr{F}})$ is a generator such that $\omega(v_1, \dots, v_r) = 1$. Since $\omega_{\mathscr{F}}$ is torsion-free, to show that Y is invariant under $\eta_{\mathscr{F}}$ it suffices to show that it kills the kernel of $\Omega_U^r|_Y \to \Omega_Y^r$ at the dense set of smooth points of Y. Therefore, Y is invariant under η if and only if any $f \in H^0(U, \mathcal{O}_U)$ vanishing on $Y \cap U$ and any local $(r-1)$-form β has $(\mathrm{d}f \wedge \beta)(v_1, \dots, v_r) = 0$ (this generates the kernel of $\Omega_X^r|_Y \to \Omega_Y^r$) which happens exactly if $\mathrm{d}f(v_i) = 0$ for each i meaning Y is tangent to $\mathscr{F}$ at x. Since the smooth points of Y are dense and Y is irreducible this means that Y is the closure of a leaf of $\mathscr{F}|_{X\backslash \mathrm{Sing}(\mathscr{F})}$ iff it is invariant under η. ☐

Lemma 7.2.6. Let $f : X \to Y$ be a dominant map of integral characteristic zero schemes. Let X_y be a fiber and let $Z = (X_y)_{\mathrm{red}}$ be the reduction. Let $\eta : \Omega_X^r \to \mathcal{L}$ be a Pfaff field which factors as $\Omega_X^r \to \Omega_{X/Y}^r \to \mathcal{L}$. Then the restriction of η factors as,

$$\begin{tikzcd} \Omega_{X_y}^r|_Z \arrow[r] \arrow[d, two heads] & \mathcal{L}|_Z \\ \Omega_Z^r \arrow[ur, dashed] & \end{tikzcd}$$

Proof. This is an affine local question so reduce to an injective map of domains $\varphi : R \to A$ and localizing we may assume that R is local and $\mathcal{L}$ is free. Let $I = \mathfrak{m}_R A$ and $J = \sqrt{I}$. We need to show that for all $f \in J$ and forms $\beta \in \Omega^{r-1}_{A/R}$ then, $\eta(\mathrm{d}f \wedge \beta) \in J$ since the elements $\mathrm{d}f \wedge \beta$ generate the kernel. The map $\Omega_{A/R} \to \Omega^r_{A/R} \to A$ defined by $x \mapsto \eta(x \wedge \beta)$ is linear. We proved in the similar lemma previously that any linear map $\ell : \Omega_{A/R} \to A$ satisfies $\ell(\mathrm{d}f) \in J$. $\square$

7.3 Relations between notions of singularities

Lemma 7.3.1. Let X be a smooth variety, $\mathscr{F}$ a foliation of rank r. Then,

$$S(\eta_{\mathscr{F}}) \subset S_1(\mathscr{F}) \cup S_2(\mathscr{F}) = \mathrm{Sing}(\mathscr{F})$$

and,

$$\mathrm{Sing}(\eta_{\mathscr{F}}) \backslash S_1(\mathscr{F}) = S_2(\mathscr{F})$$

Proof. If $\mathscr{F}_x \to (\mathcal{T}_X)_x$ is a direct factor then $\mathscr{F}(x) \to \mathcal{T}_X(x)$ is clearly injective (the exact sequence is split) so $S_1(\mathscr{F}) \cup S_2(\mathscr{F}) \subset \mathrm{Sing}(\mathscr{F})$. Moreover, since X is smooth $(\mathcal{T}_X)_x$ is free so if $\mathscr{F}_x$ is also free then I claim if $\mathscr{F}(x) \to \mathcal{T}_X(x)$ is injective then it is a direct factor. Consider $\mathcal{T}_X/\mathscr{F}$. By upper-semi-continuity, the locus where this has rank $n-r$ is open. By the injectivity, x in this locus and hence $\mathcal{T}_X/\mathscr{F}$ is constant rank on a neighborhood of x. Since X is reducedm $\mathcal{T}_X/\mathscr{F}$ is locally free at x hence $(\mathcal{T}_X/\mathscr{F})_x$ is free so the sequence splits proving the claim.

Now we need to consider singularities of the Pfaff field. Let $x \in S_1(\mathscr{F})^c$ then choose an open U on which $\mathscr{F}$ and hence $\omega_{\mathscr{F}}$ is locally free. Therefore,

$$x \in \mathrm{Sing}(\eta_{\mathscr{F}}) \iff x \in x \in S_2(\mathscr{F})$$

since the map $\mathscr{F}(x) \to \mathcal{T}_X(x)$ is injective iff the map $\Omega^r_X(X) \to \det \mathscr{F}^\vee(x)$ is nonzero. $\square$

7.4 Pullbacks of Foliations (WIP)

Definition 7.4.1. Let $U \subset X$ be open. We say U is *big* if $\mathrm{codim}\left(U^C\right) X \geq 2$.

Definition 7.4.2. Let $f : X \to Y$ be a dominant separable map of varieties and $\mathscr{F} \subset \mathcal{T}_Y$ a foliation. Then we can form a foliation $f^\# \mathscr{F}$ by taking the pullback,

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f^*\mathscr{F} \arrow[d] \\ \mathcal{T}_X \arrow[r] & f^*\mathcal{T}_Y \end{tikzcd}$$

and then taking the saturation of the image, $f^\#\mathscr{F} = (\mathrm{im}\,(\mathcal{P} \to \mathcal{T}_X))^{\mathrm{sat}}$.

Proposition 7.4.3. If $\mathscr{F} \subset \mathcal{T}_X$ is closed under lie bracket then $\mathscr{F}^{\mathrm{sat}}$ is a foliaiton.

Proof. It suffices to check that $\mathscr{F}^{\mathrm{sat}}$ is also closed under the Lie bracket. Ineed,

$$\mathscr{F}^{\mathrm{sat}} \otimes \mathscr{F}^{\mathrm{sat}} \to \mathcal{T}_X/\mathscr{F}^{\mathrm{sat}} = (\mathcal{T}_X/\mathscr{F})_{\text{tors-free}}$$

but there is a dense open U on which $\mathscr{F}|_U = \mathscr{F}^{\mathrm{sat}}|_U$ and on U this map is zero so because the target is torsion-free the map must be zero (since the image of the map is torsion). $\square$

WHAT IS RELATIONSHIP BETWEEN PULLBACK FOLIATION AND PFAFF FIELD

7.5 Existence of a Morphism

Now consider the following situation. Let X be a normal projective vareity and $\mathscr{F}$ an algebraically integrable foliation on X. Then let $W \subset \mathrm{Chow}(X)$ be some component of the closure of the set of $\mathbb{C}$-points corresponding to the algebraic leaves of $\mathscr{F}$ which dominates X. Then let $U \subset X \times W$ be the universal cycle giving a diagram,

$$\begin{array}{ccc} U & \xrightarrow{e} & X \\ \big\downarrow{\scriptstyle \pi} & & \\ W & & \end{array}$$

Notice that such a component exists because by assumption a general point of X is contained in an algebraic leaf of $\mathscr{F}$ so some component of the closure must have a universal cycle dominating X.

Remark. Notice that Pfaff fields and foliations don't usually pullback well (the intution is that sections of Ω pullback nicely not dual sections). However, in this case there is the following trick using the product structure.

Definition 7.5.1. Given a 1-Gorenstein foliation, $\mathscr{F} \subset \mathcal{T}_X$ and Pfaff field $\eta_{\mathscr{F}} : \Omega_X^r \to \omega_{\mathscr{F}}$ define,

$$\eta_{W \times X} : \Omega^r_{W \times X} = \wedge^r(\pi_1^* \Omega_W \oplus \pi_2^* \Omega_X) \to \wedge^r(\pi_2^* \Omega_X) = \pi_2^* \Omega_X^r \xrightarrow{\pi_2^* \eta_{\mathscr{F}}} \pi_2^* \omega_{\mathscr{F}}$$

and likewise we can define a foliation,

$$\mathscr{F}_{W \times X} = \pi_2^* \mathscr{F} \subset \pi_2^* \mathcal{T}_X \subset \mathcal{T}_{W \times X}$$

Indeed, $\eta_{W \times X} = \eta_{\mathscr{F}_{W \times X}}$ since,

$$\Omega^r_{W \times X} \to \det \mathscr{F}^\vee_{W \times X}$$

factors through $\Omega^r_{W \times X} \to \pi_2^* \Omega^r_X$.

Lemma 7.5.2. The Pfaff field $\eta_{W \times X}|_U : \Omega^r_{W \times X}|_U \to e^* \omega_{\mathscr{F}}$ factors through $\Omega^r_{W \times X}|_U \twoheadrightarrow \Omega^r_{U/W}$ giving a Pfaff field $\eta_U : \Omega^r_{U/W} \to e^* \omega_{\mathscr{F}}$. Similarly, the map $e^* \mathscr{F} \to \mathcal{T}_{W \times X}|_U$ factors as,

$$e^* \mathscr{F} \to \Omega^\vee_{U/W} \to \mathcal{T}_{W \times X}|_U$$

Proof. Let $K = \ker(\Omega^r_{W \times X}|_U \to \Omega^r_{U/W})$ then we know that $K \to \Omega^r_{W \times X}|_U \to e^* \omega_{\mathscr{F}}$ vanishes on the dense set of fibers which are leaves (with the reduced structure) by Lemma **??**. Since $e^* \omega_{\mathscr{F}}$ is torsion-free this map is zero.

Similarly, consider the map $e^* \mathscr{F} \to \mathcal{T}_{W \times X}|_U \to \mathcal{N}_{U|W \times X}$. As before, this map vanishes on the set of fibers which are leaves (with the reduced structure). Since $\mathcal{N}_{U|W \times X}$ is reflexive and hence torsion-free we see that the map is zero so we get a factorization $e^* \mathscr{F} \to \mathcal{T}_U \to \mathcal{T}_{W \times X}|_U$. Morover, consider $e^* \mathscr{F} \to \mathcal{T}_U \to f^* \mathcal{T}_W$ then again this is zero on the dense set of leaves. However, f is not flat so $f^* \mathcal{T}_W$ may fail to be reflexive (MAYBE THIS IS WHERE THE PFAFF FIELD IS MORE USEFUL). Assuming $f^* \mathcal{T}_W$ is torsion-free then we conclude that the map $e^* \mathscr{F} \to \mathcal{T}_U \to f^* \mathcal{T}_W$ is zero and hence factors as,

$$e^* \mathscr{F} \to \Omega^\vee_{U/W} \to \mathcal{T}_U$$

□

Remark. In particular, if $\mathscr{F}$ is a foliation then every irreducible component of f is a leaf of $\mathscr{F}$.

Lemma 7.5.3. Each fiber of $\pi : U \to W$ is connected and pure dimension $r_{\mathscr{F}}$. If no irreducible component of $\pi^{-1}(w)$ is contained in $\mathrm{Sing}(\mathscr{F})$ then $\pi^{-1}(w)$ is irreducible, has smooth reduction, and $e(\pi^{-1}(w))_{\mathrm{red}}$ is a leaf[2].

Proof. First, note that every cycle in a connected component of $\mathrm{Chow}(X)$ whose general fiber is geometrically connected and equidimensional is connected and equidimensional. To see this, note that a flat limit of a proper geometrically connected equidimensional scheme is (geometrically) connected and equidimensional so this follows from [Kollar, RCAV, Cor 3.16]. Therefore, it suffices to show that each irreducible component Z of $\pi^{-1}(w)$ is a leaf. Indeed, leafs are disjoint so this implies that $e(\pi^{-1}(w))$ with the reduced structure is a leaf. Using that $\eta_U : \Omega^r_{U/W} \to e^*\omega_{\mathscr{F}}$ factors the map $e^*\Omega^r_X \to e^*\omega_{\mathscr{F}}$ Lemma 7.2.6 proves that Z is invariant under η. Therefore, Lemma 7.2.5 proves that Z is a leaf. □

Remark. We could also probably prove this using the similar results for the whole $\mathscr{F}$ not the associated Pfaff field. But its tricky (see above comments) so this I guess is why to use the Pfaff field.

Corollary 7.5.4. If $\mathscr{F}$ is a regular foliation then every fiber of $\pi : U \to W$ is a leaf possibly with some nonreduced structure.

Proposition 7.5.5. If X is normal and $\mathscr{F}$ is regular then $e : U \to X$ is an isomorphism.

Proof. Since e is proper and X is normal it suffices to show that e is injective. Since on each fiber of $\pi : U \to W$ the map e is a closed immersion, it suffices to show that no two fibers intersect in X. We have shown that every fiber is a leaf with possibly nonreduced structure. Since distinct leaves are disjoint, we need to show that the same leaf cannot appear more than once as a fiber. Indeed, let Z_1, Z_2 be two fibers such that $(Z_1)_{\mathrm{red}} = (Z_2)_{\mathrm{red}} = Z$ where Z is an irreducible closed subscheme (the underlying leaf). Then as cycles $[Z_1] = m_1[Z]$ and $[Z_2] = m_2[Z]$ for some integers $m_1, m_2 \in \mathbb{Z}$. However, these are cycles over a connected component of the Chow scheme so the degree of the cycle with respect to some ample class H on X is constant [Kollar RCAV, Prop. 3.12]. Since $H^{n-r} \cdot Z > 0$ then $m_1 = m_2$ and hence $[Z_1] = [Z_2]$ as cycles. Therefore, these correspond to the same $\mathbb{C}$-point of the Chow scheme and hence these are the fibers over the same $\mathbb{C}$ point. □

Theorem 7.5.6. Let X be a normal scheme and $\mathscr{F} \subset T_X$ be a regular algebraic foliation of rank r. Then there exists a morphism $f : X \to W$ of relative dimension r such that $\mathscr{F} \subset \mathcal{T}_f$ and this is an isomorphism away from codimension 2.

Proof. (NEED THE FOLLOWING TO BE TRUE) Want to say $\mathscr{F} \to \mathcal{T}_X \to f^*\mathcal{T}_W$ is zero. (NEED $f^*\mathcal{T}_X$ TORSION-FREE). Then for every smooth fiber this is an isomorphism. (WHAT ABOUT SINGULAR FIBERS) □

(IS f FLAT???? CAN NONREDUCED FIBERS REALLY HAPPEN?)

Example 7.5.7. The conclusion that $\mathscr{F} = \mathcal{T}_f$ *only on a dense open* cannot be strengthended. Indeed,

Proposition 7.5.8. Suppose that a foliation $\mathscr{F} \subset T_X$ is algebric. Then,

(a) there is a Zariski open $U \subset X$ and a morphism $f : U \to S$ such that $T_f = \mathscr{F}|_U$.

(b) if $\mathscr{F}$ is a vector bundle then f is smooth

(c) if $\mathscr{F}$ is furthermore a sub-bundle then S is smooth. (WAIT BUT IF X IS SMOOTH AND f IS SMOOTH IT IMPLIES THAT S IS SMOOTH).

[2] Recall that e is a closed immersion when restricted to any fiber since $U \subset W \times X$ is a closed immersion

7.6 The Case $\kappa = -\infty$

Suppose we assume that $\mu_{\alpha,\min}(\mathscr{F}^\vee) > 0$ (in the notation of Campana) then the Foliation is algebraic.

7.7 An Idea

Consider the canonical fibration $f : X \to S$. If X has a nonvanishing 1-form ω then Popa-Schnell shows that the map $X \to \mathrm{Alb}_X$ cannot fully contract the fiber F of $X \to S$. If we can show that $\dim \mathrm{Alb}_F < \dim \mathrm{Alb}_X$ then this means that Alb_X is reducible.

8 Reflexivity

Proposition 8.0.1. Let $\mathscr{F}$ be a coherent sheaf on an integral scheme X. Then $\mathscr{F}^*$ is reflexive.

Proof. Let $\psi : \mathrm{id} \to (-)^{**}$ be the double dual natural transformation. There are maps,

$$\mathscr{F}^* \xrightarrow{\psi_{\mathscr{F}^*}} \mathscr{F}^{***} \xrightarrow{\psi^*_{\mathscr{F}}} \mathscr{F}^*$$

which acts as follows, for $\varphi \in \mathscr{F}^*$ and $\ell \in \mathscr{F}^{**}$,

$$\varphi \mapsto (\ell \mapsto \ell(\varphi)) \mapsto [(\ell \mapsto \ell(\varphi)) \circ \psi_{\mathscr{F}}]$$

the result of which is the function,

$$x \mapsto^{\psi_{\mathscr{F}}} (\varphi' \mapsto \varphi'(x)) \mapsto \varphi(x)$$

which is just φ so indeed this is the identity. Thus it suffices to show that the second map is injective. However, $\mathscr{F}^{***}$ is torsion-free and hence it suffices to check this at the generic point where it becomes finite-dimensional linear algebra. $\square$

Consider a separable map $f : X \to C$ where X is a smooth proper (integral) surface over k and C a smooth proper (integral) curve over k. Then there is a sequence,

$$0 \longrightarrow f^*\Omega_C \longrightarrow \Omega_X \longrightarrow \Omega_{X/C} \longrightarrow 0$$

which is injective on the left by generic smoothness and the fact that Ω_C and Ω_X are vector bundles. Then we get an exact sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{T}_X \longrightarrow f^*\mathcal{T}_C \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_X}\big(\Omega_{X/C}, \mathcal{O}_X\big) \longrightarrow 0$$

again using that Ω_X is a vector bundle where we set $\mathscr{F} = \Omega_{X/C}^\vee$. Since $\mathscr{F}$ is reflexive and X is a regular surface we see that $\mathscr{F}$ is a vector bundle. By generic smoothness, $\mathscr{F}$ has rank 1. I claim that $\mathscr{F}$ is closed under Lie bracket.

Indeed, this a local question so we reduce to $R \to A \to B$ are ring maps then $\mathrm{Hom}_B\left(\Omega_{B/A}, B\right) \to \mathrm{Hom}_B\left(\Omega_{A/R}, B\right)$ is closed under Lie bracket. Indeed, for $X, Y \in \mathrm{Hom}_B\left(\Omega_{B/A}, B\right)$ we need to show that $(X \circ \mathrm{d})(Y \circ \mathrm{d}) - (Y \circ \mathrm{d})(X \circ \mathrm{d})$ is a A-derivation not just an R-derivation. This is basically obvious because $X \circ \mathrm{d}$ and $Y \circ \mathrm{d}$ kill A.

Now we get an exact sequence,

$$0 \longrightarrow \mathcal{T}_X/\mathscr{F} \longrightarrow f^*\mathcal{T}_C \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_X}\big(\Omega_{X/C}, \mathcal{O}_X\big) \longrightarrow 0$$

Since $f^*\mathcal{T}_C$ is torsion-free we see that so is $\mathcal{T}_X/\mathscr{F}$ so $\mathscr{F}$ is automatically saturated.