1 Appendix

1.1 Curves and Genera

Lemma 1.1.1. Let X be a integral scheme proper over k then $K = H^0(X, \mathcal{O}_X)$ is a finite field extension of k and for any coherent $\mathcal{O}_X$-module $\mathscr{F}$, the cohomology $H^p(X, \mathscr{F})$ is a finite-dimensional $H^0(X, \mathcal{O}_X)$-module.

Proof. Since $\mathcal{O}_X$ is coherent, and X is proper over k so $K = H^0(X, \mathcal{O}_X)$ is a finite k-module. However, since X is integral $H^0(X, \mathcal{O}_X)$ is a domain but a finite k-algebra domain is a field and we see K/k is a finite extension of fields. Furthermore, the $\mathcal{O}_X(X)$-module structure on $H^p(X, \mathscr{F})$ gives it a K-module structure. Since X is proper over k then $H^p(X, \mathscr{F})$ is a finite k-module and thus finite as a K-module. □

Remark. Unfortunately, when k is not algebraically closed then we may not have $H^0(X, \mathcal{O}_X) = k$ even for smooth projective varieties. Therefore, some caution must be taken in defining numerical invariants of the curve such as genus. However, by [**?**, Tag 0BUG], whenever X is proper geometrically integral then indeed $H^0(X, \mathcal{O}_X) = k$. Furthermore, for proper X if $H^0(X, \mathcal{O}_X) \neq k$ then X cannot be geometrically connected by [**?**, Tag 0FD1].

Definition 1.1.2. Let C be a smooth proper curve over k with $H^0(C, \mathcal{O}_C) = K$. Then we define $g(C) := \dim_K H^0(X, \Omega_{C/k})$. If C is any curve over k then there is a unique smooth proper curve S over k which is k-birational to C. Then we define $g(C) := g(S)$.

Remark. By definition, the genus of a curve is clearly a birational invariant since there is a unique smooth complete curve in every birational equivalence class of curves.

Remark. There is a slight subtlety in this definition in the case of a non-perfect base field. It it always true that we can find a proper *regular* curve C in each birational equivalence class however when k is non-perfect the curve C may not be smooth. However, under a finite purely separable extension K/k, we can ensure that C_K admits a smooth proper model. Then we define $g(C) := g(C_K)$ in the case that C_K is a curve. The only thing that can go wrong is when C is not geometrically irreducible since then C_K will not be integral.

Definition 1.1.3. The *arithmetic genus* $g_a(C)$ of a proper curve C over k with $H^0(C, \mathcal{O}_C) = K$ is,

$$g_a(C) := \dim_K H^1(X, \mathcal{O}_C)$$

By Serre duality, if C is smooth then $H^0(C, \Omega_C) = H^1(C, \mathcal{O}_X)^\vee$ meaning that $g_a(C) = g(C)$.

Remark. The arithmetic genus depends on the projective compactification and singularities meaning it will not be a birational invariant unlike the (geometric) genus.

Example 1.1.4. Let $k = \mathbb{F}_p(t)$ for an odd prime $p = 2k + 1$ and consider the curve,

$$C = \mathrm{Spec}\left(k[x, y]/(y^2 - x^p - t)\right)$$

which is regular but not smooth at $P = (y, x^p - t)$. Consider the purely inseperable extension $K = \mathbb{F}_p(t^{1/p})$. Then $C_K = \mathrm{Spec}\left(K[x, y]/(y^2 - (x - t^{1/p})^p)\right) \cong \mathrm{Spec}\left(K[x, y]/(y^2 - x^p)\right)$. Taking the normalization of C_K gives $\mathbb{A}^1_K \to C_K$ via $t \mapsto (t^p, t^2)$. This is birational since the following ring map is an isomorphism,

$$(K[x, y]/(y^2 - x^p))_x \to K[t]_t$$

sending $x \mapsto t^2$ and $y \mapsto t^p$ which has an inverse $t \mapsto y/x^k$ since $x \mapsto t^2 \mapsto y^2/x^{2k} = x$ and $y \mapsto t^p \mapsto y^p/x^{kp} = y(y^{2k}/x^{pk}) = y$ and $t \mapsto y/x^k \mapsto t^{p-2k} = t$.

Therefore, $C_K \overset{\sim}{\dashrightarrow} \mathbb{P}^1_K$ so $g(C) = g(C_K) = 0$. However, consider the projective closure,

$$\overline{C} = \mathrm{Proj}\left(k[X,Y,Z]/(Y^2Z^{p-2} - X^p - tZ^p)\right)$$

then $\overline{C} \hookrightarrow \mathbb{P}^2_k$ is a Cartier divisor (since $\mathbb{P}^2_k$ is locally factorial) so we find that $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = k$ and $\dim_k H^1(\overline{C}, \mathcal{O}_{\overline{C}}) = \frac{1}{2}(p-1)(p-2) = k(2k-1)$ since its sheaf of ideals is $\mathcal{O}_{\mathbb{P}^2_k}(-p)$. Then $p=3$ we expect this to be an elliptic curve and we do see $g_a(\overline{C}) = 1$. However, $g(\overline{C}) = 0$ and correspondingly C is not smooth due to the positive characteristic phenomenon.

Lemma 1.1.5. Suppose that $f : X \to Y$ is a finite birational morphism of n-dimensional irreducible Noetherian schemes. Then $H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(X, \mathcal{O}_X)$ is surjective.

Proof. The map f must restrict on some open subset $U \subset X$ to an isomorphism $f|_U : U \to V$. Thus, the sheaf map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ restricts on V to an isomorphism $\mathcal{O}_Y|_V \xrightarrow{\sim} (f_*\mathcal{O}_X)|_V$. We factor this map into two exact sequences,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_Y \longrightarrow \mathscr{I} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{I} \longrightarrow f_*\mathcal{O}_X \longrightarrow \mathscr{C} \longrightarrow 0$$

with $\mathscr{K} = \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{C} = \mathrm{coker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{I} = \mathrm{Im}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. Taking cohomology and using that it vanishes in degree above n we get,

$$H^{n-1}(Y, \mathscr{I}) \longrightarrow H^n(Y, \mathscr{K}) \longrightarrow H^n(Y, \mathcal{O}_Y) \longrightarrow\!\!\!\!\!\rightarrow H^n(Y, \mathscr{I}) \longrightarrow 0$$

$$H^{n-1}(Y, \mathscr{C}) \longrightarrow H^n(Y, \mathscr{I}) \longrightarrow H^n(X, \mathcal{O}_X) \longrightarrow\!\!\!\!\!\rightarrow H^n(X, \mathscr{C}) \longrightarrow 0$$

where we have used that $f : X \to Y$ is affine to conclude that $H^p(Y, f_*\mathscr{F}) = H^p(Y, \mathscr{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$. Furthermore, $\mathscr{C}|_V = 0$ so $\mathrm{Supp}_{\mathcal{O}_Y}(\mathscr{C}) \subset X \setminus V$ but $\mathscr{C}$ is coherent so the support is closed. Since V is dense open, $\mathscr{C}$ is supported in positive codimension so $H^n(Y, \mathscr{C}) = 0$ (since $H^n(S, \mathscr{C})$ vanishes due to dimension on the closed subscheme $S = \mathrm{Supp}_{\mathcal{O}_X}(\mathscr{C})$ on which $\mathscr{C}$ is supported). Thus we have,

$$H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(X, \mathcal{O}_X)$$

proving the proposition. $\square$

Corollary 1.1.6. Let S and C be proper curves over k where S is smooth which are birationally equivalent and $H^0(S, \mathcal{O}_S) \cong H^0(C, \mathcal{O}_C)$. Then the genera satisfy,

(a) $g_a(C) \geq g_a(S)$

(b) $g(C) = g(S)$

(c) $g(C) \leq g_a(C)$ with equality if and only if C is smooth.

Proof. Given a birational map $S \overset{\sim}{\dashrightarrow} C$ we can extend it to a birational morphism $S \to C$ since S is regular. The morphism $S \to C$ is automatically finite since it is a non-constant map of proper curves. Then the previous lemma implies that $g_a(S) \le g_a(C)$. (b). follows from the definition of $g(C)$. The third follows from the fact that $g(S) = g_a(S)$ because of Serre duality,

$$H^1(S, \mathcal{O}_S) \cong H^0(S, \Omega_{S/k})^\vee$$

using that S is smooth. Then we see that $g(C) = g(S) = g_a(S) \le g_a(C)$ proving the inequality part of (c). Finally, if C is smooth we see by Serre duality that $g(C) = g_a(C)$. Conversely, suppose that $g(C) = g_a(C)$ then $g_a(C) = g(C) = g(S) = g_a(S)$ and consider the map $f : S \to C$ which is finite birational map of integral schemes over k. In particular, f is affine so for each $y \in C$ we may choose an affine open $y \in V \subset C$ whose preimage $U = f^{-1}(V)$ is also affine. On sheaves, this gives a map of domains $\mathcal{O}_C(V) \to \mathcal{O}_S(U)$ which localizes to an isomorphism on the fraction fields. However, the localization map of a domain is injective so $\mathcal{O}_C(V) \hookrightarrow \mathcal{O}_S(U)$ is an injection. This shows that $\mathcal{O}_C \to f_* \mathcal{O}_S$ is an injection of sheaves which we extend to an exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_* \mathcal{O}_S \longrightarrow \mathscr{C} \longrightarrow 0$$

Note that $f : S \to C$ induces an isomorphism $H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$ since it is a map of fields with the same (finite) dimension over k. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, \mathscr{C}) \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, \mathscr{C}) = 0$$

I claim that $H^1(S, \mathscr{C}) = 0$. Since f is birational, $\mathscr{C}$ is supported in codimension one. Thus, the map $H^1(C, \mathcal{O}_C) \twoheadrightarrow H^1(S, \mathcal{O}_S)$ is surjective but $g_a(C) = g_a(S)$ so these vectorspaces have the same dimension so $H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S)$ is an isomorphism. Thus, from the exact sequence we have $H^0(X, \mathscr{C}) = 0$. However, $\mathrm{Supp}_{\mathcal{O}_C}(\mathscr{C})$ is a closed ($\mathscr{C}$ is coherent) dimension zero subset i.e. finitely many discrete closed points. However, a sheaf supported on a discrete set of points is zero iff it has no global sections. Therefore, $\mathscr{C} = 0$ so $\mathcal{O}_C \xrightarrow{\sim} f_* \mathcal{O}_S$. In particular $\mathcal{O}_C(V) \xrightarrow{\sim} \mathcal{O}_S(U)$ is an isomorphism which implies that the map of affine schemes $f|_U : U \to V$ is an isomorphism. Since the affine opens V cover C we see that $f : S \to C$ is an isomorphism. In particular, C is smooth. □

1.2 The Locus on Which Morphisms Agree

Lemma 1.2.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then for schemes X there is a natural bijection,

$$\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(R), X) \cong \{x \in X \text{ and local map } \mathcal{O}_{X,x} \to R\}$$

Proof. Given $\mathrm{Spec}(R) \to X$ we automatically get $\mathfrak{m} \mapsto x$ and $\mathcal{O}_{X,x} \to R_\mathfrak{m} = R$. Now, note that taking any affine open neighborhood $x \in \mathrm{Spec}(A) \subset X$ and then $A \to A_\mathfrak{p} = \mathcal{O}_{X,x}$ to give $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(A) \to X$. Clearly, this map sends $\mathfrak{m}_x \mapsto x$ and at $\mathfrak{m}_x$ has stalk map $\mathrm{id} : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ since it is the localization at $\mathfrak{p}$ of $A \to A_\mathfrak{p}$.

Thus we get an inverse as follows. Given a point $x \in X$ and a local map $\phi : \mathcal{O}_{X,x} \to R$ then take,

$$\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$$

This is inverse since $\mathfrak{m} \mapsto \mathfrak{m}_x$ (because $\mathcal{O}_{X,x} \to \mathfrak{m}_x$ is local) and $\mathfrak{m}_x \mapsto x$ and the stalk at $\mathfrak{m}$ gives $\mathcal{O}_{X,x} \xrightarrow{\mathrm{id}} \mathcal{O}_{X,x} \xrightarrow{\phi} R$.

Finally, I claim that any $f : \mathrm{Spec}(R) \to X$ factors through $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ and thus is reconstructed from $x \in X$ and $\mathcal{O}_{X,x} \to R$. Choose an affine open neighborhood $x \in \mathrm{Spec}(A) \subset X$ then consider $f^{-1}(\mathrm{Spec}(A))$ which is open in $\mathrm{Spec}(R)$ and contains the unique closed point $\mathfrak{m} \in \mathrm{Spec}(R)$ so there is some $f \in R$ s.t. $\mathfrak{m} \in D(f) \subset f^{-1}(\mathrm{Spec}(A))$ so $f \notin \mathfrak{m}$ so $f \in R^\times$ and thus $D(f) = \mathrm{Spec}(R)$. Therefore, we get a map $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ and thus $\phi : A \to R$ where $\phi^{-1}(\mathfrak{m}) = \mathfrak{p} = x$ so $A \setminus \mathfrak{p}$ is mapped inside $R^\times$ so this map factors through $A \to A_{\mathfrak{p}} \to R$ giving the desired factorization $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(A) \to X$. □

Definition 1.2.2. The locus Z on which two maps $f, g : X \to Y$ over S agree is given as the pullback,

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

with $F = (f, g)$. This is the equalizer of $f, g : X \to Y$. Furthermore $Z \to X$ is an immersion since it is the base change of $\Delta_{Y/S}$ which is an immersion.

Lemma 1.2.3. Topologically, the locus on which S-morphisms $f, g : X \to Y$ agree is,

$$Z = \{x \in X \mid f(x) = g(x) \text{ and } f_x = g_x : \kappa(f(x)) \to \kappa(x)\}$$

Proof. On some S-subscheme $G \subset X$, the maps $f|_G = g|_G$ agree iff there exists $G \to Y$ such that,

$$\begin{tikzcd} G \arrow[r, dashed] \arrow[d, hook] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

commutes. In particular, for any point $x \in X$ consider $\iota : \mathrm{Spec}(\kappa(x)) \to X$ then $f \circ \iota = g \circ \iota$ iff $f(x) = g(x)$ and $f_x = g_x : \kappa(f(x)) \to \kappa(x)$. Consider a point $z \in Z$ and $\mathrm{Spec}(\kappa(z)) \to Z$, such a point is equivalent to giving a diagram,

$$\begin{tikzcd} \mathrm{Spec}(\kappa(z)) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ & X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

However, $\iota : Z \to X$ is an immersion so $\iota_x : \kappa(\iota(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism. Therefore, points $\mathrm{Spec}(\kappa(z)) \to Z$, are exactly points of X for which a lift $\mathrm{Spec}(\kappa(x)) \to Y$ exists i.e. points such that f and g agree in the required way. □

Lemma 1.2.4. If $f : X \to Y$ is an immersion then $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective for each $x \in X$ and $f_x : \kappa(f(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism.

Proof. For closed immersions, $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective by definition. Thus we get a surjection $f_x : \mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_{f(x)}$. Furthermore, topologically, $f : X \to Y$ is a homomorphism onto its image so for any open $U \subset X$ there exists an open $V \subset Y$ s.t. $U = f^{-1}(V)$ showing that,

$$(f_* \mathcal{O}_X)_{f(x)} = \varinjlim_{f(x) \in V} \mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}$$

Furthermore, for an open immersion, $f^{\flat} : f^{-1}\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism so $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. Thus the composition, $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective. Furthermore, f_x is local we get $f_x : \kappa(f(x)) \twoheadrightarrow \kappa(x)$ which is a surjection of fields and thus an isomorphism. $\square$

Lemma 1.2.5. If $Y \to S$ is separated then the locus on which $f, g : X \to Y$ over S agree is closed.

Proof. Since $X \to S$ is separated, $\Delta_{Y/S} : Y \to Y \times_S Y$ is a closed immersion. So $Z \to X$ is the base change of a closed immersion and thus a closed immersion. $\square$

Lemma 1.2.6. Let X be a reduced and Y be a separated scheme over S and $f, g : X \to Y$ be morphism over S. If $f \circ j = g \circ j$ agree on a dense subscheme $j : G \hookrightarrow X$ then $f = g$.

Proof. Consider $F = (f, g) : X \to Y \times_S Y$. Since $\Delta : Y \to Y \times_S Y$ is a closed immersion (by separateness). Then $F^{-1}(\Delta)$ is the locus on which $f = g$ which is closed because $\Delta : Y \to Y \times_S Y$ is a closed immersion. Since $f|_G = g|_G$ we get a diagram,

$$\begin{tikzcd}
G \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z \arrow[r, "\tilde{F}"] \arrow[d, hook, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\
& X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}$$

Since $\iota : Z \hookrightarrow X$ is a closed immersion with dense image, $Z \hookrightarrow X$ is surjective. By the following, $\iota : Z \to X$ is an isomorphism. Thus, $F = F \circ \iota \circ \iota^{-1} = \Delta_Y \circ \tilde{F} \circ \iota^{-1}$. By the universal property of maps $X \to Y \times_S Y$ this implies that $f = g = \tilde{F} \circ \iota^{-1}$. $\square$

Lemma 1.2.7. Let X be a scheme and consider an exact sequence of quasi-coherent $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A} \longrightarrow 0$$

and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebra. Suppose that $\mathscr{F}_x \neq 0$ for each $x \in X$. Then $\mathscr{I} \hookrightarrow \mathcal{N}$ where $\mathcal{N}$ is the sheaf of nilpotent.

Proof. Take an affine open $U = \mathrm{Spec}\,(R) \subset X$ such that $\mathcal{A}|_U = \widetilde{A}$. Then we have an surjection of rings $R \twoheadrightarrow A$ giving $R/I = A$ for $I = \ker (R \to A)$. Now, for each $\mathfrak{p} \in \mathrm{Spec}\,(R)$ we know $R_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}} \neq 0$. However, if $\mathfrak{p} \not\supset I$ then $(R/I)_{\mathfrak{p}} = A_{\mathfrak{p}} = 0$ so we must have $\mathfrak{p} \supset I$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ i.e. $I \subset \mathrm{nilrad}\,(R)$. Therefore, $\mathscr{I}|_U \hookrightarrow \mathcal{N}|_U$ for any affine open $U \subset X$ showing that $\mathscr{I}$ is comprised of nilpotents. $\square$

Corollary 1.2.8. If X is reduced and $\iota : Z \hookrightarrow X$ is a surjective closed immersion then $\iota : Z \xrightarrow{\sim} X$ is an isomorphism.

Proof. Since $\iota : Z \hookrightarrow X$ is a homeomorphism onto its image X it suffices to show that the map of sheaves $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. Since $\iota : Z \to X$ is a closed immersion $\iota^\# : \mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Z$ is a surjection and $\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras giving an exact sequence,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Furthermore,

$$\mathrm{Supp}_{\mathcal{O}_X} (\iota_* \mathcal{O}_Z) = \mathrm{Im}\,(\iota) = X$$

since $(\iota_* \mathcal{O}_Z)_x = \mathcal{O}_{Z,x}$ when $x \in \mathrm{Im}\,(\iota)$ (and zero elsewhere). by the above, $\mathscr{I} \hookrightarrow \mathscr{N} = 0$ since X is reduced to $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. ☐

Lemma 1.2.9. A rational S-map $f : X \dashrightarrow Y$ with X reduced and $Y \to S$ separated is equivalent to a morphism $f : \mathrm{Dom}\,(f) \to Y$.

Proof. For any (U, f_U) and (V, f_V) representing f there must be a dense (in X) open $W \subset U \cap V$ on which $f_U|_W = f_V|_W$ and thus $f_U|_{U \cap V} = f_V|_{U \cap V}$ since $f_U, f_V : U \cap V \to Y$ are morphisms from reduced to irreducible schemes. Now $\mathrm{Dom}\,(f)$ has an open cover (U_i, f_i) for which $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ so these morphisms glue to give $f : \mathrm{Dom}\,(f) \to Y$ ($\mathrm{Hom}_S\,(-, Y)$ is a sheaf on the Zariski site). ☐

1.3 Extending Rational Maps

Lemma 1.3.1. Regular local rings of dimension 1 exactly correspond to DVRs.

Proof. Any DVR R has a uniformizer $\varpi \in R$ then $\dim R = 1$ and $\mathfrak{m}/\mathfrak{m}^2 = (\varpi)/(\varpi^2) = \varpi\kappa$ which also has $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 1$ so R is regular. Conversely, if R is a regular local ring of dimension $\dim R = 1$ then, by regularity, R is a normal Noetherian domain so by $\dim R = 1$ then R is Dedekind but also local and thus is a DVR. ☐

Proposition 1.3.2. Let X be a Noetherian S-scheme and $Z \subset X$ a closed irreducible codimension 1 generically nonsingular subset (with generic point $\eta \in Z$ such that $\mathcal{O}_{X,\eta}$ is regular). Let $f : X \dashrightarrow Y$ be a rational map with Y proper over S. Then $Z \cap \mathrm{Dom}\,(f)$ is a dense open of Z.

Proof. Choose some representative (U, f_U) for $f : X \dashrightarrow Y$. Note that $\mathcal{O}_{X,\eta}$ is a regular dimension one (see Lemma 1.4.3) ring and thus a DVR. Consider the generic point $\xi \in X$ of X then, by localizing, we get an inclusion of the generic point $\mathrm{Spec}\,(\mathcal{O}_{X,\xi}) \to \mathrm{Spec}\,(\mathcal{O}_{X,\eta}) \to X$ and $\mathcal{O}_{X,\xi} = K(X) = \mathrm{Frac}\,(\mathcal{O}_{X,\eta})$. Furthermore, the inclusion of the generic point gives $\mathrm{Spec}\,(K(X)) \to U \xrightarrow{f_U} Y$ and thus we get a diagram,

$$\begin{tikzcd}
\mathrm{Spec}\,(K(X)) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\
\mathrm{Spec}\,(\mathcal{O}_{X,\eta}) \arrow[r] \arrow[ur, dashed, "\ell"] & \mathrm{Spec}\,(k)
\end{tikzcd}$$

and a lift $\mathrm{Spec}\,(\mathcal{O}_{X,\eta}) \to Y$ by the valuative criterion for properness applied to $Y \to \mathrm{Spec}\,(k)$ since $\mathcal{O}_{X,\eta}$ is a DVR. Choose an affine open $\mathrm{Spec}\,(R) \subset Y$ containing the image of $\mathrm{Spec}\,(\mathcal{O}_{X,\eta}) \to Y$ (i.e. choose a neighborhood of the image of η which automatically contains $f(\xi)$ since the map factors $\mathrm{Spec}\,(\mathcal{O}_{X,\eta}) \to \mathrm{Spec}\,\big(\mathcal{O}_{Y,f(\eta)}\big) \to \mathrm{Spec}\,(R) \to Y$) and let $\eta \in V = \mathrm{Spec}\,(A) \subset X$ be an affine open neighborhood of ξ mapping onto $\mathrm{Spec}\,(R)$. By Lemma 1.4.7, since $\mathcal{O}_{X,\eta}$ is a domain, we may shrink V so that A is a domain. Since X is irreducible $U \cap V$ is a dense open. Note that if $\eta \in U$ then $\eta \in \mathrm{Dom}\,(f)$ and thus $Z \cap \mathrm{Dom}\,(f)$ is a nonempty open of the irreducible space Z

and therefore a dense open so we are done. Otherwise, let $\mathfrak{p} \in \mathrm{Spec}\,(A)$ correspond to $\eta \in Z$ then $A_\mathfrak{p} = \mathcal{O}_{X,\eta}$ is a DVR. Take some principal affine open $D(f) \subset U \cap V$ for $f \in A$ so $f \in \mathfrak{p}$ since $\mathfrak{p} \notin D(f) \subset U \cap V$. Since $A_\mathfrak{p}$ is a DVR we may choose a uniformizer $\varpi \in \mathfrak{p}$ so the map $A \to \mathfrak{p}$ via $1 \mapsto \varpi$ is as isomorphism when localized at $\mathfrak{p}$. Since A is Noetherian both are f.g. A-modules so there must be some $s \in A \setminus \mathfrak{p}$ such that $A_s \to \mathfrak{p}_s$ is an isomorphism. Replacing A by A_s we may assume $\mathfrak{p} = (\varpi) \subset A$ is principal. Since $f \in \mathfrak{p}$ we can write $f = t\varpi^k$ for some $a \in A \setminus \mathfrak{p}$ (see Lemma 1.4.1). Then consider $\tilde{V} = \mathrm{Spec}\,(A_t)$. Since $t \notin \mathfrak{p}$ then $\eta \in \tilde{V}$ and since $f = t\varpi^k$ we have $D(f) \subset D(t) = \tilde{V}$. Now we get the following diagram,

$$\begin{tikzcd}
& & \mathrm{Spec}\,(R) \\
\mathrm{Spec}\,(A_\mathfrak{p}) \arrow[r] \arrow[urr, bend left, "\ell"] & \mathrm{Spec}\,(A_t) \arrow[ur, dashed, "f_V"] & \\
\mathrm{Spec}\,(\mathrm{Frac}\,(A)) \arrow[r] \arrow[u] & \mathrm{Spec}\,(A_f) \arrow[u] \arrow[uur, bend right, "f_U"'] &
\end{tikzcd}$$

I claim the square is a pushout in the category of affine schemes because maps $R \to A_\mathfrak{p}$ and $R \to A_f$ which agree under the inclusion to $\mathrm{Frac}\,(A)$ gives a map $R \to A_\mathfrak{p} \cap A_f \subset \mathrm{Frac}\,(A)$. However, consider,

$$x \in A_\mathfrak{p} \cap A_t \implies x = \frac{u\varpi^r}{s} = \frac{a}{f^n}$$

for $u, s, t \in A \setminus \mathfrak{p}$ and $a \in A$. Thus we get,

$$ut^n\varpi^{r+nk} = sa$$

so $a \in \mathfrak{p}^{r+nk} \setminus \mathfrak{p}^{r+nk+1}$ ($s \notin \mathfrak{p}$ which is prime) and thus $a = u'\varpi^{r+nk}$ for $u' \in A \setminus \mathfrak{p}$. Therefore,

$$x = \frac{u'\varpi^{r+nk}}{t^n\varpi^{nk}} = \frac{u'\varpi^r}{t^n} \in A_t$$

Thus, $A_\mathfrak{p} \cap A_f \subset A_f$ so we get a map $R \to A_t$. Therefore we get a map $f_{\tilde{V}} : \tilde{V} \to Y$ such that $(f|_{\tilde{V}})|_{D(f)} = (f_U)|_{D(f)}$ which implies that $\eta \in \tilde{V} \subset \mathrm{Dom}\,(f)$ so $Z \cap \mathrm{Dom}\,(f)$ is a dense open of Z. $\square$

Proposition 1.3.3. Let $C \to S$ be a proper regular Noetherian scheme with $\dim C = 1$ and $f : C \dashrightarrow Y$ a rational S-map with $Y \to S$ proper. Then f extends uniquely to a morphism $f : C \to Y$.

Proof. For any point $x \notin \mathrm{Dom}\,(f)$ let $Z = \overline{\{x\}} \subset D$ for $D = C \setminus \mathrm{Dom}\,(f)$. Since $\mathrm{Dom}\,(f)$ is a dense open, by lemma 1.4.2, we have $\mathrm{codim}\,(Z, C) \geq \mathrm{codim}\,(D, C) \geq 1$ but $\dim C = 1$ so $\mathrm{codim}\,(Z, C) = 1$. Furthermore, since C is regular $\mathcal{O}_{C,x}$ is regular and thus, by the previous proposition, $Z \cap \mathrm{Dom}\,(f)$ is a dense open and in particular $x \in \mathrm{Dom}\,(f)$ meaning that $\mathrm{Dom}\,(f) = C$ so we get a morphism $C \to Y$. This is unique because C is reduced (it is regular) and Y is separated (it is proper over S) so morphisms $C \to Y$ are uniquely determined on a dense open which any representative for $f : C \dashrightarrow Y$ is defined on. $\square$

Corollary 1.3.4. Rational maps between normal proper curves are morphisms.

Corollary 1.3.5. Birational maps between normal proper curves are isomorphisms.

Proof. Let $f : C_1 \dashrightarrow C_2$ and $g : C_2 \dashrightarrow C_1$ be birational inverses of smooth proper curves. Then we know that these extend to morphisms $f : C_1 \to C_2$ and $g : C_2 \to C_1$. Furthermore, the maps $g \circ f : C_1 \to C_1$ must extend the identity on some dense open. However, since curves are separated and reduced there is a unique extension of this map so $g \circ f = \mathrm{id}_{C_1}$ and likewise $f \circ g = \mathrm{id}_{C_2}$. $\square$

Theorem 1.3.6. If k is perfect then there exists a unique normal curve in each birational equivalence class of curves.

Proof. It suffices to show existence. Given a curve X, we consider the projective closure $X \hookrightarrow \overline{X}$ which is birational and $\overline{X} \to \mathrm{Spec}\,(k)$ is proper. Then take the normalization $\overline{X}^{\nu} \to \overline{X}$ which remains proper over $\mathrm{Spec}\,(k)$ and is birational. Then $\overline{X}^{\nu}$ is regular and thus smooth over k since k is perfect and $\overline{X}^{\nu} \to X$ is birational. $\square$

1.4 Lemmas

Lemma 1.4.1. Let A be a Noetherian domain and $\mathfrak{p} = (\varpi)$ a principal prime. Then any $f \in \mathfrak{p}$ can be written as $f = t\varpi^k$ for $f \in A \setminus \mathfrak{p}$.

Proof. From Krull intersection,
$$\bigcap_{n \geq 0}^{\infty} \mathfrak{p}^n = (0)$$
so there is some n such that $f \in \mathfrak{p}^n \setminus \mathfrak{p}^{n+1}$. Thus $f = t\varpi^n$ for some $f \in A$ but if $t \in \mathfrak{p}$ then $f \in \mathfrak{p}^{n+1}$ so the result follows. $\square$

Lemma 1.4.2. Consider a closed subset $Y \subset X$ and an open $U \subset X$ with $U \cap Z \neq \varnothing$. Then $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Y \cap U, U)$.

Proof. Consider a chain of irreducible $Z_i \supsetneq Z_{i+1}$ with $Z_0 \subset Y$. I claim that $Z_i \mapsto Z_i \cap U$ and $Z_i \mapsto \overline{Z_i}$ are inverse functions giving a bijection between closed irreducible chains in X with final terms contained in Y and closed irreducible chains in U with final term contained in $Y \cap U$. Note, if $Z_i \subset Y \cap U$ then $\overline{Z_i} \subset Y$ since Y is closed in X.

First, $\overline{Z_i \cap U} \subset Z_i$ and is closed in X. Then $\overline{Z_i \cap U} \cup U^C \supset Z_i$ so because Z_i is irreducible $\overline{Z_i \cap U} = Z_i$ since by assumption $Z_i \not\subset U^C$. Conversely, if $Z_i \subset U$ is a closed irreducible subset then $\overline{Z_i}$ is closed and irreducible in X and $Z_i \subset \overline{Z_i} \cap U$ but $Z_i = C \cap U$ for closed $C \subset X$ so $Z_i \subset C$ and thus $\overline{Z_i} \subset C$ so $\overline{Z_i} \cap U \subset C \cap U = Z_i$ meaning $Z_i = \overline{Z_i} \cap U$. Thus we have shown these operations are inverse to each other.

Finally, if $Z_i \cap U - Z_{i+1} \cap U$ then $\overline{Z_i \cap U} = \overline{Z_i \cap U}$ so $Z_i = Z_{i+1}$ so the chain does not degenerate. Likewise, if $\overline{Z_i} = \overline{Z_{i+1}}$ then $\overline{Z_i} \cap U = \overline{Z_{i+1}} \cap U$ so $Z_i = Z_{i+1}$. Therefore, we get a length-preserving bijection between the chains defining $\mathrm{codim}\,(Y, X)$ and $\mathrm{codim}\,(Y \cap U, U)$. $\square$

Lemma 1.4.3. Let $Z \subset X$ be a closed irreducible subset with generic point $\eta \in Z$. Then $\mathrm{codim}\,(Z, X) = \dim \mathcal{O}_{X,\eta}$.

Proof. Take affine open neighborhood $\eta \in U = \mathrm{Spec}\,(A) \subset X$. Then for $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponding to η we get $A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$. However, $\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U)$ and $Z \cap U = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Therefore,
$$\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U) = \mathbf{ht}\,(\mathfrak{p}) = \dim A_{\mathfrak{p}} = \dim \mathcal{O}_{X,\eta}$$
$\square$

Lemma 1.4.4. Let X be a Noetherian scheme then the nonreduced locus,

$$Z = \{x \in X \mid \mathrm{nilrad}\,(\mathcal{O}_{X,x}) \neq 0\}$$

is closed.

Proof. The subsheaf $\mathcal{N} \subset \mathcal{O}_X$ is coherent since X is Noetherian. Thus $Z = \mathrm{Supp}_{\mathcal{O}_X}(\mathcal{N})$ is closed and $\mathcal{N}_x = \mathrm{nilrad}\,(\mathcal{O}_X x)$. Locally, on $U = \mathrm{Spec}\,(A)$ we have $\mathcal{N}|_U = \widetilde{\mathrm{nilrad}\,(\mathrm{A})}$ and $\mathrm{nilrad}\,(A)$ is a f.g. A-module since A is Noetherian so,

$$\mathrm{Supp}_{\mathcal{O}_X}(\mathcal{N}) \cap U = \mathrm{Supp}_A\,(\mathrm{nilrad}\,(A)) = V(\mathrm{Ann}_A\,(\mathrm{nilrad}\,(A))$$

is closed in $\mathrm{Spec}\,(A)$. □

Lemma 1.4.5. Let X be a Noetherian scheme then X has finitely many irreducible components.

Proof. First let $X = \mathrm{Spec}\,(A)$ for a Noetherian ring A. Then the irreducible components of A correspond to minimal primes $\mathfrak{p} \in \mathrm{Spec}\,(A)$. Then $\dim A_{\mathfrak{p}} = 0$ and $A_{\mathfrak{p}}$ is Noetherian so $A_{\mathfrak{p}}$ is Artinian. $A_{\mathfrak{p}}$ must have some associated prime so $\mathrm{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}\}$. By [**?**, Tag 05BZ], then $\mathrm{Ass}_A(A) \cap \mathrm{Spec}\,(\mathbb{A}_{\mathfrak{p}}) = \mathrm{Ass}_{\mathbb{A}_{\mathfrak{p}}}(\mathbb{A}_{\mathfrak{p}}) = \{\mathfrak{p}\}$ so every minimal prime is an associated prime. However, for A Noetherian then A admits a finite composition series so there are finitely many associated primes.

Now let X be a Noetherian scheme. For any affine open $U \subset X$ we have shown that U has finitely many irreducible components. However, since X is quasi-compact there is a finite cover of affine opens and thus X must have finitely many irreducible components. □

Lemma 1.4.6. Let X be a Noetherian scheme and Y is the complement of some dense open U. Then $\mathrm{codim}\,(Y, X) \geq 1$.

Proof. It suffices to show that Y does not contain any irreducible component since then any irreducible contained in Y cannot be maximal. Since X is Noetherian, it has finitely many irreducible components Z_i. Then if $Z_j \subset Y$ for some i we would have $Z_i \cap U = \varnothing$ but then,

$$U = \bigcup_{i \neq j} Z_i$$

which is closed so $\overline{U} \subsetneq X$ contradicting our assumption that U is dense. □

Lemma 1.4.7. Let X be a Noetherian scheme and $x \in X$ such that $\mathcal{O}_{X,x}$ is a domain. Then there is an affine open neighborhood $x \in U \subset X$ with $U = \mathrm{Spec}\,(A)$ and A is a domain.

Proof. Take any affine open neighborhood $x \in U \subset X$ with $U = \mathrm{Spec}\,(A)$ and $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponding to x. Then $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is a domain. Since X is Noetherian then A is Noetherian so it has finitely many minimal primes $\mathfrak{p}_i$ (corresponding to the generic points of irreducible components of U) with $\mathfrak{p}_0 \subset \mathfrak{p}$. Since $A_{\mathfrak{p}}$ is a domain, it has a unique minimal prime and thus $\mathfrak{p}_0$ is the only minimal prime contained in $\mathfrak{p}$ (geometrically $A_{\mathfrak{p}}$ being a domain corresponds to the fact that $\mathfrak{p}$ is the generic point of a generically reduced irreducible subset which lies in only one irreducible component)

Now for any $i \neq 0$ take $f_i \in \mathfrak{p} \setminus \mathfrak{p}_0$. This is always possible else $\mathfrak{p} \subset \mathfrak{p}_0$ contradicting the minimality of $\mathfrak{p}_0$. If $f \notin \mathfrak{q}$ then $\mathfrak{q} \not\supset \mathfrak{p}_i$ for any $i \neq 0$ so $\mathfrak{q} \supset \mathfrak{p}_0$ since it must lie above some minimal prime. Thus $\text{nilrad}\,(A_f) = \mathfrak{p}_0 A_f$ is prime and $f \notin \mathfrak{p}$ since else $\mathfrak{p} \supset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ which is impossible since $\mathfrak{p} \not\supset \mathfrak{p}_i$ for any i. Now we know that $\text{nilrad}\,(A_{\mathfrak{p}}) = 0$ and A_f is Noetherian so $\text{nilrad}\,(A_{\mathfrak{p}})$ is finitely generated. Thus, there is some $g \notin \mathfrak{p}$ such that $\text{nilrad}\,(A_{fg}) = (\text{nilrad}\,(A_f))_g = 0$. Thus A_{fg} is a domain since $\text{nilrad}\,(A_{fg}) = (0)$ and is prime and $\mathfrak{p} \in A_{fg}$ because $fg \notin \mathfrak{p}$. Therefore, $x \in \text{Spec}\,(A_{fg}) \subset U$ is an affine open satisfying the requirements. $\square$

Remark. This does not imply that X is integral if $\mathcal{O}_{X,x}$ is a domain for each $x \in X$ (which is false, consider $\text{Spec}\,(k \times k)$) because it only shows there is an integral cover of X not that $\mathcal{O}_X(U)$ is a domain for each U.

Example 1.4.8. Let $X = \text{Spec}\,(k[x,y]/(xy, y^2))$. Then for the bad point $\mathfrak{p} = (x, y)$ we have $\text{nilrad}\,(\mathcal{O}_{X,\mathfrak{p}}) = (y)$. Away from the bad point, say $\mathfrak{p} = (x-1, y)$ we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left(k[x]_{(x-1)}\right)$ so $\text{nilrad}\,(\mathcal{O}_{X,\mathfrak{p}}) = (0)$. Furthermore, at the generic point $\mathfrak{p} = (y)$, we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\,(k(x))$ so $\text{nilrad}\,(\mathcal{O}_{X,\mathfrak{p}}) = (0)$.

Example 1.4.9. Consider $X = \text{Spec}\,(k[x,y,z]/(yz))$ which is the union of the x-y and x-z planes. Consider the generic point of the z-axis $\mathfrak{p} = (x, y)$ then $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left(k[x,z]_{(x)}\right)$ is a domain since the z-axis only lies in one irreducible component. However, at the generic point of the x-axis, $\mathfrak{p} = (y, z)$ we get $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left((k[x,y,z]/(yz))_{(y,z)}\right)$ has zero divisors $yz = 0$ so is not a domain since the x-axis lives in two irreducible components.

1.5 Reflexive Sheaves (WIP)

Definition 1.5.1. Recall the dual of a $\mathcal{O}_X$ module $\mathscr{F}$ is the sheaf $\mathscr{F}^\vee = \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. We say that a coherent $\mathcal{O}_X$-module $\mathscr{F}$ is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism.

Lemma 1.5.2. Let X be an integral locally Noetherian scheme and $\mathscr{F}, \mathscr{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathscr{G}$ is reflexive then $\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is reflexive.

Proof. See [**?**, Tag 0AY4]. $\square$

In particular, since $\mathcal{O}_X$ is clearly reflexive, this lemma shows that for any coherent $\mathcal{O}_X$-module then $\mathscr{F}^\vee$ is a reflexive coherent sheaf. We say the map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ gives the reflexive hull $\mathscr{F}^{\vee\vee}$ of $\mathscr{F}$.

Definition 1.5.3. Let $\mathcal{R}$ be the full subcategory $\mathfrak{Coh}\,(\mathcal{O}_X)$ of coherent reflexive $\mathcal{O}_X$-modules. $\mathcal{R}$ is an additive category and in fact has all kernels and cokernels defined by taking reflexive hulls of the sheaf kernel and cokernel. Furthermore, $\mathcal{R}$ inherits a monoidal structure from the tensor product defined using the reflexive hull as follows,

$$\mathscr{F} \otimes_{\mathcal{R}} \mathscr{G} = (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$$

Finally, we define $\text{RPic}\,(X)$ to be group of constant rank one reflexives induced by the monoidal structure on $\mathcal{R}$. Explicitly, $\text{RPic}\,(X)$ is the group of isomorphism classes of constant rank one reflexive coherent $\mathcal{O}_X$-modules with multiplication $(\mathscr{F}, \mathscr{G}) \mapsto (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$ and inverse $\mathscr{F} \mapsto \mathscr{F}^\vee$.

The importance of reflexive sheaves derives from their correspondence to Weil divisors. Here we let X be a normal integral separated Noetherian scheme.

Proposition 1.5.4. If D is a Weil divisor then $\mathcal{O}_X(D)$ is reflexive of constant rank one.

Proof. (CITE OR DO). □

Theorem 1.5.5. Let X be a normal integral separated Noetherian scheme. There is an isomorphism of groups $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{RPic}(X)$ defined by $D \mapsto \mathcal{O}_X(D)$.

Proof. (DO OR CITE) □

We summarize the important results as follows.

Theorem 1.5.6. Let X be a Noetherian normal integral scheme. Then for any Weil divisors D, E,

(a) $\mathcal{O}_X(D+E) = (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee}$

(b) $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\vee}$

(c) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(E)) = \mathcal{O}_X(E-D)$

(d) if E is Cartier then $\mathcal{O}_X(D+E) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$

Proof. (DO OR CITE) □

Finally, we have a result which controls when the dualizing sheaf can be expressed in terms of a divisor.

Proposition 1.5.7. Let X be a projective variety over k. Then,

(a) if X is normal then its dualizing sheaf ω_X is reflexive of rank 1 and thus X admits a canonical divisor K_X s.t. $\omega_X = \mathcal{O}_X(K_X)$

(b) if X is Gorenstein then ω_X is an invertible module so K_X is Cartier.

Proof. (FIND CITATION OR DO). □

2 Pic Functors

QUESTION 2.0.1. In this section I did everything for $k = \mathbb{C}$ since this is what is given in Harris-Mumford. Can we extend this to other fields?

QUESTION 2.0.2. I assumed that the toric surface S is smooth here for two reasons. (1) to ensure that $\mathbf{Pic}_{X/k}$ exists and (2) such that $C \hookrightarrow S$ is Cartier. Can we drop this assumption?

Here, we investigate the behavior of very general curves with respect to embeddings onto smooth toric surfaces. Our result is that for sufficiently large genus, very general curves cannot embed in any smooth toric surface. Intuitively, a very general curve is a curve the coefficients of whose defining equations do not satisfy any algebraic relations over $\mathbb{Q}$. Specifically, we define a very general curve as follows.

Definition 2.0.3. We say a smooth proper curve C over $\mathbb{C}$ with genus g is *very general* if its class $[P] \in \mathcal{M}_g$ in the moduli space of smooth proper curves of genus g does not lie in any proper subvariety of $\mathcal{M}_g$ defined over $\mathbb{Q}$.

To prove the required result, we will make use of the following theorem of Harris and Mumford which restricts the birationality classes of surfaces on which nontrivial families of very general curves lie.

Theorem 2.0.4 (Harris-Mumford)**.** Let C be a generic curve of genus $g \geq 23$ and S an algebraic surface containing C such that C moves in a nontrivial linear system on S meaning that $\dim H^0(S, \mathcal{O}_S(C)) > 1$. Then S is a ruled surface birational to $C \times \mathbb{P}^1$.

Proof. See the introduction of [**?**]. □

Beyond this, we need a short foray into the theory of Picard schemes. Grothendieck introduced the notion of Picard schemes in two 1962 Bourbaki talks [**?**] which generalizes the Picard group of X to a group scheme representing a Picard functor over X. First, we need a relative notion of the Picard group.

Definition 2.0.5. Let $f : X \to S$ be a morphism of schemes. Then we define the relative Picard group,

$$\mathrm{Pic}\,(X/S) = H^0(S, R^1 f_* \mathcal{O}_X^\times)$$

In particular, if $S = \mathrm{Spec}\,(k)$ then $\mathrm{Pic}\,(X/S) = H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}\,(X)$.

Lemma 2.0.6. Let $f : X \to S$ be a morphism with $f^\# : \mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ then the sequence,

$$0 \longrightarrow \mathrm{Pic}\,(S) \longrightarrow \mathrm{Pic}\,(X) \longrightarrow \mathrm{Pic}\,(X/S) \longrightarrow H^2(S, f_*\mathcal{O}_X^\times) \longrightarrow H^2(X, \mathcal{O}_X^\times)$$

from the low-degree terms of the Leray spectral sequence is exact. When f admits a section $S \to X$ i.e. an S-point then $\mathrm{Pic}\,(X) \twoheadrightarrow \mathrm{Pic}\,(X/S)$ is surjective so,

$$\mathrm{Pic}\,(X/S) \cong \frac{\mathrm{Pic}\,(X)}{\mathrm{Pic}\,(S)}$$

Proof. The Leray spectral sequence gives an exact sequence of low degree terms,

$$\begin{tikzcd}
0 \arrow[r] & H^1(S, f_*\mathcal{O}_X^\times) \arrow[r] \arrow[d, equal] & H^1(X, \mathcal{O}_X^\times) \arrow[r] \arrow[d, equal] & H^0(S, R^1 f_*\mathcal{O}_X^\times) \arrow[r] \arrow[d, equal] & H^2(S, f_*\mathcal{O}_X^\times) \arrow[r] \arrow[d, equal] & H^2(X, \mathcal{O}_X^\times) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Pic}\,(S) \arrow[r] & \mathrm{Pic}\,(X) \arrow[r] & \mathrm{Pic}\,(X/S) \arrow[r] & H^2(S, \mathcal{O}_S^\times) \arrow[r] & H^2(X, \mathcal{O}_X^\times)
\end{tikzcd}$$

A section $s : S \to X$ meaning $f \circ s = \mathrm{id}_S$ gives a left-inverse $s^* : H^p(X, \mathcal{O}_X^\times) \to H^p(S, \mathcal{O}_S^\times)$ to the map $f^* : H^p(S, \mathcal{O}_S^\times) \to H^p(X, \mathcal{O}_X^\times)$. In particular, the final map of the exact sequence is injective giving the required short exact sequence. □

Definition 2.0.7. Let X be a scheme over S. Then for any S-scheme $T \to S$ there is a map $\mathrm{Pic}\,(T) \to \mathrm{Pic}\,(X \times_S T)$ induced by the projection. Therefore, we may define the Picard presheaf on the big étale site,

$$\mathrm{Pic}_{X/S} : (\mathbf{Sch}_S)^{\mathrm{op}}_{\text{ét}} \to \mathbf{Ab} \qquad T \mapsto \mathrm{Pic}\,(X \times_S T/T)$$

and $\mathrm{Pic}^{\text{ét}}_{X/S}$ the associated sheaf for the étale topology. If it exists, the Picard scheme $\mathbf{Pic}_{X/S}$ is the unique scheme representing this sheaf,

$$\mathrm{Hom}_S\left(-, \mathbf{Pic}_{X/S}\right) = \mathrm{Pic}^{\text{ét}}_{X/S}$$

Remark. In particular,

$$\mathbf{Pic}_{X/S}(S) = \mathrm{Hom}_S\left(S, \mathbf{Pic}_{X/S}\right) = \mathrm{Pic}\,(X \times_S S/S) = \mathrm{Pic}\,(X/S)$$

so for $S = \mathrm{Spec}\,(k)$ the k-points of $\mathbf{Pic}_{X/S}$ are exactly $\mathrm{Pic}\,(X)$.

In his Bourbaki talk, Grothendieck gave conditions for the Picard scheme to exist and relations between the geometry of $\mathbf{Pic}_{X/S}$ and cohomological invariants of line bundles.

Theorem 2.0.8 (FGA V. Thm. 3.1)**.** Let $f : X \to S$ be a morphism of locally Noetherian schemes which is

(a) projective

(b) flat

(c) fiberwise geometrically integral.

Then a separated finite type over S scheme $\mathbf{Pic}_{X/S}$ exists representing the functor $\mathrm{Pic}^{\text{ét}}_{X/S}$.

In particular, when $f : X \to \mathrm{Spec}\,(k)$ is a projective geometrically integral variety over k then the Picard scheme $\mathbf{Pic}_{X/k}$ exists. The topology of the Picard scheme is related to a powerful equivalence relation on line bundles known as algebraic equivalence which is the algebraic version of topological homotopy equivalences of bundles.

Definition 2.0.9. Let X be a scheme over S. We say that $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}\,(X)$ are *algebraically equivalent* $\mathcal{L}_1 \sim \mathcal{L}_2$ if there is a *connected* scheme T over S, closed points $t_1, t_2 \in T$, and a line bundle $\mathcal{L} \in \mathrm{Pic}\,(X \times_S T)$ such that $\mathcal{L}|_{X \times \{t_1\}} \cong \mathcal{L}_1$ and $\mathcal{L}|_{X \times \{t_2\}} \cong \mathcal{L}_2$.

Proposition 2.0.10. Let $\mathbf{Pic}^0_{X/k} \hookrightarrow \mathbf{Pic}_{X/k}$ be the connected component of the identity. Then $\mathbf{Pic}^0_{X/k}(k)$ is exactly the group of line bundles $\mathcal{L} \in \mathrm{Pic}^0(X)$ algebraically equivalent to zero.

Proof. Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}\,(X)$ be algebraically equivalent. Then the line bundle $\mathcal{L} \in \mathrm{Pic}\,(X \times_k T)$ defines (up to an element of $\mathrm{Pic}\,(T)$) a morphism $T \to \mathbf{Pic}_{X/k}$ then $\mathcal{L}_1 \cong \mathcal{L}|_{X \times \{t_1\}}$ and $\mathcal{L}_2 \cong \mathcal{L}|_{X \times \{t_2\}}$ are the pullback under the inclusions $\mathrm{Spec}\,(\kappa(t_i)) \hookrightarrow T$ i.e. $\mathcal{L}_i$ correspond to the points $\mathrm{Spec}\,(\kappa(t_i)) \to \mathbf{Pic}_{X/k}$. However, T is connected so its image under $T \to \mathbf{Pic}_{X/k}$ is connected as wells so the points $\mathrm{Spec}\,(\kappa(t_i)) \to \mathbf{Pic}_{X/k}$ corresponding to $\mathcal{L}_i$ under the identification $\mathbf{Pic}_{X/k}(k) = \mathrm{Pic}\,(X)$ lie in the same connected component. ☐

Remark. From the above discussion, we can define the Neron-Severi group $\mathrm{NS}\,(S)$,

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}\,(X) \longrightarrow \mathrm{NS}\,(S) \longrightarrow 0$$

Therefore, the Neron-Severi group is the group of line bundles modulo algebraic equivalence or equivalently the group of connected components of $\mathbf{Pic}_{X/k}$.

Theorem 2.0.11. Let $X \to \mathrm{Spec}\,(k)$ and assume that $\mathrm{Pic}\,(X/k)$ exists representing $\mathrm{Pic}^{\text{ét}}_{X/l}$. Then the Zariski tangent space at the trivial bundle has a canonical identification,

$$T_0\mathbf{Pic}_{X/k} = H^1(X, \mathcal{O}_X)$$

thus $\dim \mathbf{Pic}_{X/k} \leq \dim_k H^1(X, \mathcal{O}_X)$ with equality exactly when $\mathbf{Pic}_{X/k}$ is smooth at 0. Since $\mathbf{Pic}_{X/k}$ is a group scheme, in this case $\mathbf{Pic}_{X/k}$ is everywhere smooth of dimension $\dim_k H^1(X, \mathcal{O}_X)$.

Proof. See [**?**, Thm. 5.11] and [**?**, Cor. 5.13]. ☐

Remark. For a smooth proper curve C over k of genus g. Then $T_0\mathbf{Pic}_{X/k} = H^1(X, \mathcal{O}_X)$ has dimension g so $\mathbf{Pic}^0_{X/k}$ is an abelian variety of dimension g which is the Jacobian variety of C.