

1 Feb 11

1.1 Line Bundles

There exists a map,

$$\Gamma(X, \mathcal{L}^{\otimes a}) \otimes \Gamma(X, \mathcal{L}^{\otimes b}) \rightarrow \Gamma(X, \mathcal{L}^{\otimes ab})$$

since we have an isomorphism $\mathcal{L}^{\otimes a} \otimes \mathcal{L}^{\otimes b} = \mathcal{L}^{\otimes ab}$. Furthermore, since \mathcal{L} is rank 1 this map is commutative since $s \times s' = s' \otimes s$ since they only differ by a section of \mathcal{O}_X . This allows us to define the following graded ring structure.

Definition 1.1.1. Let \mathcal{L} be an invertable \mathcal{O}_X -module, \mathcal{F} any \mathcal{O}_X -module and $s \in \mathcal{L}(X)$ a global section. Then we define the following graded ring.

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

and then the following module,

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which is a graded $\Gamma_*(X, \mathcal{L})$ -module. Furthermore, there is a map,

$$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \rightarrow \mathcal{F}(X_s) = \Gamma(X_s, \mathcal{F})$$

sending $\frac{t}{s^n} \mapsto t|_{X_s} \otimes (s|_{X_s})^{\otimes -n}$.

Proposition 1.1.2. Let X be a quasi-compact, quasi-separated scheme and \mathcal{F} be quasi-coherent. Then the above map is an isomorphism.

Proof. Tag OB5K. (Compare with that Hartshorne Exercise 2.16). □

Example 1.1.3. Let A be a graded ring such that A is generated by A_1 as a A_0 -algebra (e.g. $A = k[X_0, \dots, X_n]$). Let $X = \text{Proj}(A)$ and consider the graded module $M = A(n)$ which is the graded module $M_k = A_{k+n}$. Then we can construct the Serre twists,

$$\mathcal{O}_X(n) = \widetilde{M} = \widetilde{A(n)}$$

which is an invertable \mathcal{O}_X -module. Furthermore,

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$$

Remark. This will not be invertible and these maps will not be isomorphisms in general when A does not satisfy the required conditions.

Proof. We can decompose,

$$X = \bigcup_{f \in A_1} D_+(f) = \bigcup_{f \in A_1} \text{Spec}(A_{(f)})$$

via the given assumptions. We know that,

$$\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{A(n)}|_{D_+(f)} = A[f^{-1}]_n$$

However it is clear that $A[f^{-1}]_n = A[f^{-1}]_0 \cdot f^n$ so this sheaf is free of rank 1. □

Remark. For $n = 1$ any element $f \in A_1$ gives a global section $f \in \Gamma(X, \mathcal{O}_X(1))$ such that $D_+(f) = X_s$ and hence,

$$\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(1)|_{X_s}$$

Corollary 1.1.4. In the setting above, further assume that A is generated by finitely many $f \in A_1$ as an A_0 -algebra. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} if we set,

$$M = \Gamma_*(X, \mathcal{O}_X(1), \mathcal{F})$$

as a graded A -module via the map,

$$A \rightarrow \Gamma_*(X, \mathcal{O}_X(1)) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

Then we get, $\mathcal{F} = \widetilde{M}$.

Proof. Tag □

2 Feb. 13

Definition 2.0.1. Let X be a scheme and \mathcal{L} an invertible \mathcal{O}_X -module. We say \mathcal{L} is *ample* if X is quasi-compact and $\forall x \in X : \exists n > 0 : s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $x \in X_s$.

Example 2.0.2. Let $X = \text{Proj}(A)$ where A is generated by A_1 as a A_0 -algebra and $A_1 = f_1 A_0 + \dots + f_r A_0$. Then $\mathcal{O}_X(1)$ is invertible and X is covered by $D_+(f_i)$ and is quasi-compact, and $D_+(f_i) = X_{s_i}$ where $s_i \in \Gamma(X, \mathcal{O}_X(1))$ is a section corresponding to f_i .

Proposition 2.0.3. Let X be quasi-compact and quasi-separated for $\mathcal{L} \in \text{Pic}(X)$ the following are equivalent,

- (a) \mathcal{L} is ample
- (b) for all \mathcal{O}_X -modules \mathcal{F} locally of finite type there exists $n > 0$ s.t. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections.

Proof. TAG 01Q3. □

Lemma 2.0.4. \mathcal{L} is ample iff $\mathcal{L}^{\otimes n}$ is ample for any $n > 0$.

Lemma 2.0.5. If X is affine, and \mathcal{L} is invertible, and $s \in \Gamma(X, \mathcal{L})$ then X_s is affine.

Definition 2.0.6. A scheme is noetherian if it has a finite open cover by spectra of noetherian rings.

Remark. It is equivalent to require that X is quasi-compact and $\mathcal{O}_X(U)$ is noetherian for each affine open.

Lemma 2.0.7. A locally noetherian scheme is quasi-separated.

Proof. If U, V are affines then $U \cap V$ is quasi-compact since every subspace of a noetherian space is quasi-compact. □

Definition 2.0.8. Let X be a noetherian scheme. An \mathcal{O}_X -module \mathcal{F} is *coherent* if it is quasi-coherent and locally of finite type.

Remark. It is equivalent to require that locally on affine opens $\mathcal{F}|_U = \widetilde{M}$ for a finitely-generated module M .

Remark. The inclusion functors,

$$\mathcal{Coh}(\mathcal{O}_X) \subset \mathcal{QCoh}(\mathcal{O}_X) \subset \mathcal{Mod}(\mathcal{O}_X)$$

are exact and preserved under extensions i.e. given a short exact sequence,

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

if $\mathcal{F}_1, \mathcal{F}_2$ are (quasi)-coherent then \mathcal{F}_3 is also (quasi)-coherent.

Lemma 2.0.9. A scheme of finite type over a noetherian scheme is noetherian.

Proof. Since $f : X \rightarrow Y$ is finite type f is quasi-compact but Y is quasi-compact open so its preimage X is also quasi-compact. Furthermore, for any affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(B) = V \subset Y$ such that $f(U) \subset V$ we get a ring map $B \rightarrow A$ of finite type so $B[x_1, \dots, x_n] \twoheadrightarrow A$ and since B is noetherian we see that A is noetherian so X is quasi-compact and covered by $\text{Spec}(A)$ for noetherian rings A . \square

Remark. We want to prove the following theorem. Let R be a noetherian ring, X a projective (or proper) scheme over R (then X is noetherian), and \mathcal{F} a coherent sheaf on X , then,

$$H^i(X, \mathcal{F})$$

is a finite R -module for any i and $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.

3 Feb 18

Definition 3.0.1. An immersion $j : X \rightarrow Y$ is a morphism which may be factored as $X \rightarrow U \rightarrow Y$ where $X \rightarrow U$ is a closed immersion and $U \rightarrow Y$ is an open immersion.

Definition 3.0.2. Let R be a ring, and X a scheme over R . We say X is *quasi-projective over R* iff there exists a quasi-compact immersion $j : X \rightarrow \mathbb{P}_R^n$ over R .

Remark. If X is proper over R (or just universally closed) then j is automatically a closed immersion since $\mathbb{P}_R^n \rightarrow \text{Spec}(R)$ is separated and $X \rightarrow \text{Spec}(R)$ is universally closed implies that $j : X \rightarrow \mathbb{P}_R^n$ is universally closed and in particular topologically closed and thus closed as an immersion. This gives the following lemma.

Lemma 3.0.3. X is projective over R iff X is quasi-projective and proper over R .

Theorem 3.0.4. Let R be a ring and X a scheme over R . The TFAE,

- (a) X is quasi-projective over R
- (b) X is of finite type over R and X has an ample invertible module \mathcal{L}
- (c) there exists a quasi-compact open immersion $X \hookrightarrow X'$ with X' projective over R .

Lemma 3.0.5. Let $j : X \rightarrow Y$ be a quasi-compact immersion and \mathcal{L} an ample line bundle on Y . Then $j^*\mathcal{L}$ is an ample line bundle on X .

Proof. (DO THIS!!) □

Lemma 3.0.6. Let $j : X \rightarrow Y$ be a quasi-compact immersion and X' is scheme-theoretic image. Then $j : X \rightarrow X'$ is an open immersion.

Proof. Since j is qc and qs (immersions are separated) then $j_*\mathcal{O}_X$ is quasi-coherent and thus $\mathcal{I} = \ker(\mathcal{O}_Y \rightarrow \mathcal{O}_X)$ is quasi-coherent so we find $X' = V(\mathcal{I})$ (FINSIH THIS) □

Example 3.0.7. $\text{Spec}(k[[x]]) \rightarrow \text{Spec}(k[x])$ has scheme theoretic image $\text{Spec}(k[x])$ since its image contains the generic point. However, its set theoretic image is two points.

Proof. of Theorem (2) \implies (1). Choose $r \geq 0$ and $n \geq 1$ and $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.t.

$$X = \bigcup_{i=0}^r X_{s_i}$$

and X_{s_i} affine. Write $X_{s_i} = \text{Spec}(A_i)$. Now R is finite type over R so A_i is finite type over R so we may take $a_{i1}, \dots, a_{iN_i} \in A_i$ which generate A_i as an R -algebra. Choose $m \geq 1$ and $s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ such that $a_{ij} = s_{ij} \cdot s_i^{\otimes -m}|_{X_{s_i}}$. Therefore, $s_0^{\otimes m}, \dots, s_r^{\otimes m}, s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes mn})$ generate $\mathcal{L}^{\otimes mn}$ and therefore define a morphism $\varphi : X \rightarrow \mathbb{P}_R^{r+\sum N_i}$. It suffices to check that $X_{s_i} \rightarrow D_+(T_i)$ are a closed immersion. This holds because it is given by the ring map,

$$R[\frac{T_0}{T_1}, \dots, \frac{T_r}{T_i}, \frac{T_{ij}}{T_i}] \rightarrow A_i = \mathcal{O}_X(X_{s_i})$$

given by $\frac{T_{ij}}{T_i} \rightarrow a_{ij}$ which is clearly surjective so $X_{s_i} \rightarrow D_+(T_i)$ is a closed immersion. □

Remark. If we had checked that $X_{s_{ij}} \rightarrow D(T_{ij})$ we also a closed immersion with $X_{s_{ij}}$ affine then $\varphi : X \rightarrow \mathbb{P}_R^N$ would be a *closed* immersion. We checked only that it is locally a closed immersion on X

3.1 Functorial Characterization of \mathbb{P}_R^n

Consider the functor, $F : \mathfrak{Sch}_R \rightarrow \mathfrak{Set}$ via,

$$T \mapsto \{(\mathcal{L}, s_0, \dots, s_n) \mid \mathcal{L} \in \text{Pic}(T) \mathcal{O}_T^{n+1} \xrightarrow{(s_0, \dots, s_n)} \mathcal{L} \text{ i.e. } s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \text{ generate}\} / \cong$$

where $(\mathcal{L}, s_0, \dots, s_n) \cong (\mathcal{L}', s'_0, \dots, s'_n)$ if there is an isomorphism $\alpha : \mathcal{L} \rightarrow \mathcal{L}'$ with $\alpha(s_i) = s'_i$.

Theorem 3.1.1. \mathbb{P}_R^n represents this functor, $\text{Hom}_{\mathfrak{Sch}_R}(T, \mathbb{P}_R^n) = F(T)$.

Proof. Given $\varphi : T \rightarrow \mathbb{P}_R^n$ we get $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}_R^n}(1)$ and $s_i = \varphi^*(T_i)$.

Conversely, given $(\mathcal{L}, s_0, \dots, s_n)$ and $U \subset T$ and □

Theorem 3.1.2. If R is Noetherian and X is proper over R and \mathcal{L} is ample on X then,

$$X \cong \text{Proj}(\Gamma_*(X, \mathcal{L}))$$

and $\Gamma_*(X, \mathcal{L})$ is a finitely-generated graded R -algebra whose degree zero part is a finite R -module.

Remark. We will prove this using cohomology.

4 Cohomology

Theorem 4.0.1. $\mathbf{Mod}_{\mathcal{O}_X}$ is a Grothendieck abelian category so there are enough injectives.

Definition 4.0.2. Therefore, we can produce the right-derived functors $H^i(X, -)$ of the global sections functor,

$$\Gamma(X, -) : \mathbf{Mod}_{\mathcal{O}_X} \rightarrow \mathbf{Mod}_{\Gamma(X, \mathcal{O}_X)}$$

where (X, \mathcal{O}_X) is a ringed space. Since this is right-exact we find $H^0(X, -) = \Gamma(X, -)$.

Definition 4.0.3. Furthermore, given a morphism $f : X \rightarrow Y$ we can produce $R^i f_* : \mathbf{Mod}_{\mathcal{O}_X} \rightarrow \mathbf{Mod}_{\mathcal{O}_Y}$ the right-derived functors of $f_* : \mathbf{Mod}_{\mathcal{O}_X} \rightarrow \mathbf{Mod}_{\mathcal{O}_Y}$.

Remark. $\mathbf{Ab}(X) = \mathbf{Mod}_{\mathbb{Z}}$ so we may apply the theory of cohomology of \mathcal{O}_X -modules to the ringed space (X, \mathbb{Z}) to get a cohomology theory for abelian sheaves.

Lemma 4.0.4 (locality of cohomology). Given $\xi \in H^p(X, \mathcal{F})$ with $p > 0$ there exists an open covering,

$$X = \bigcup_{i \in I} U_i$$

s.t. $\xi|_{U_i} = 0$ for each $i \in I$.

Proof. □

Remark. The pullback is defined as follows,

5 Feb 20

5.1 Čech Cohomology

For any open covering \mathfrak{U} of a space X and a sheaf \mathcal{F} there is a simplicial abelian group,

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0})$$

Then $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})$ is the complex associated to the cosimplicial object.

Example 5.1.1. Given an exact sequence of sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

An obstruction to lifting a section $s \in \Gamma(X, \mathcal{H})$ is a cocycle in $\check{C}^1(\mathfrak{U}, \mathcal{F})$.

Lemma 5.1.2. Čech cohomology vanishes on injective objects in the category of presheaves.

Corollary 5.1.3. As a functor ON THE CATEGORY OF PRESHEAVES $\check{H}^i(\mathfrak{U}, -)$ are the right-derived functors of $\check{H}^0(\mathfrak{U}, -)$.

Lemma 5.1.4. Given a ringed space, (X, \mathcal{O}_X) and B is a basis of top and Cov a set of coverings s.t.

- (a) \mathfrak{U} in cov implies that its union and all finite intersections are in B
- (b) for U basis the coverings of U in Cov are cofinal

If $\mathcal{F} \in \mathcal{Mod}(\mathcal{O}_X)$ and

$$(*) \forall \mathfrak{U} \in \text{Cov} : \check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$$

Then $H^p(\mathfrak{U}, \mathcal{F}) = 0$ for any U in the basis.

6 Feb 25

Lemma 6.0.1. Let \mathfrak{U} be an open covering of X and $\mathcal{F} \in \mathcal{Mod}(\mathcal{O}_X)$ s.t. $H^p(U_{i_1} \cap \dots \cap U_{i_n}, \mathcal{F}) = 0$ for all finite intersections. Then $H^p(X, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$ for all $p \geq 0$.

Proof. See proof in Hartshorne Ex. It goes as follows,

- (a) Use an exact sequence,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

with \mathcal{I} injective.

- (b) Show for any sheaf $\check{H}^0(X, \mathcal{F}) = H^0(X, \mathcal{F})$ just by the sheaf property.
(c) By the assumptions, there is an exact sequence on check complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{I}) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{G}) \longrightarrow 0$$

- (d) this gives a long exact sequence of Cech cohomology
(e) use this exact sequence plus $\check{H}^p(\mathfrak{U}, \mathcal{I}) = 0$ for $p > 0$ (since flasque) to show that $\check{H}^p(\mathfrak{U}, \mathcal{G}) = \check{H}^{p+1}(\mathfrak{U}, \mathcal{F})$ and $\check{H}^1(\mathfrak{U}, \mathcal{F}) = \text{coker } \check{H}^0(\mathfrak{U}, \mathcal{I}) \rightarrow \check{H}^0(\mathfrak{U}, \mathcal{G})$
(f) use long exact sequence of $H^p(U_{i_0, \dots, i_n}, -)$ to show that \mathcal{G} also satisfies the hypotheses.
(g) use long exact sequence of $H^p(X, -)$ to show that the above hold for usual cohomology.
(h) then by induction we get $\check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) = \check{H}^p(\mathfrak{U}, \mathcal{G}) = H^p(X, \mathcal{G}) = H^{p+1}(X, \mathcal{F})$ and the base case holds since they are both kernels.

□

Corollary 6.0.2. Let X be a scheme whose diagonal is affine (for example a separated scheme). Let \mathfrak{U} be a covering of affine opens and \mathcal{F} a quasi-coherent \mathcal{O}_X -module. Then,

$$H^p(X, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$$

Remark. There is a Cech to cohomology spectral sequence,

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \underline{H}^q(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F})$$

Corollary 6.0.3. Let $f : X \rightarrow Y$ be a quasi-compact quasi-separated morphism of schemes. Then $R^i f_*$ sends quasi-coherent modules to quasi-coherent modules.

Lemma 6.0.4. Let $f : X \rightarrow Y$, $F \in \mathcal{Mod}(\mathcal{O}_X)$ then $R^p f_* \mathcal{F}$ is the sheaf associated to the presheaf,

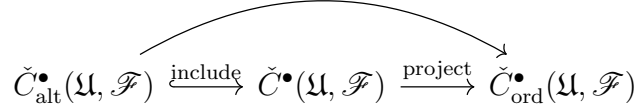
$$V \mapsto H^i(f^{-1}(V), \mathcal{F})$$

Proposition 6.0.5. We define the following modifications to the Cech complex,

$\check{C}_{\text{alt}}^\bullet$ is elements of the form $(s_{i_0 \dots i_p})$ which are antisymmetric and vanish if any two indices agree and the ordered check complex for a total order $<$ on I ,

$$\check{C}_{\text{ord}}^p = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p})$$

There are the following relations between Cech complexes,

$$\check{C}_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{include}} \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{project}} \check{C}_{\text{ord}}^\bullet(\mathcal{U}, \mathcal{F})$$


the curved arrow is an isomorphism of complexes and the horizontal arrows are homotopy equivalences.

7 Feb. 27

Proposition 7.0.1. Let R be a Noetherian ring and \mathcal{F} a coherent sheaf on \mathbb{P}_R^n . Then,

- (a) $\exists r \geq 0 : \exists m \in \mathbb{Z}$ and a surjection $\mathcal{O}_X(m)^{\oplus r} \twoheadrightarrow \mathcal{F}$
- (b) $H^i(\mathbb{P}_R^n, \mathcal{F}) = 0$ for $i \notin [0, n]$
- (c) $H^i(\mathbb{P}_R^n, \mathcal{F})$ is a finite R -module
- (d) for $i > 0$, $H^i(\mathbb{P}_R^n, \mathcal{F}(d)) = 0$ for any $d \geq d_0(\mathcal{F})$
- (e) $\bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{F}(d))$ is a finite $P = R[T_0, \dots, T_n]$ -module.

Proof. Recall that $\mathcal{O}_X(1)$ is ample so $\mathcal{F} \otimes \mathcal{O}_X(d)$ is generated by global sections for sufficiently large d and thus we get $\mathcal{O}_X^{\oplus r} \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ and thus $\mathcal{O}_X(-d)^{\oplus r} \twoheadrightarrow \mathcal{F}$.

Note that $\mathbb{P}_R^n = \bigcup_i D_+(T_i)$ which is an open cover of $n+1$ affines so by Cech cohomology, cohomology vanishes above n .

Now we apply descending induction since $H^{n+1}(\mathbb{P}_R^n, \mathcal{F}) = 0$. Now we assume (3) and (4) for degree $k+1$. For a coherent sheaf \mathcal{F} consider the exact sequence,

$$0 \longrightarrow \mathcal{G}(d) \longrightarrow \mathcal{O}_X(m+d)^{\oplus n} \longrightarrow \mathcal{F}(d) \longrightarrow 0$$

then, from the LES we get,

$$H^k(\mathbb{P}_R^n, \mathcal{O}_X(m+d)^{\oplus n}) \longrightarrow H^k(\mathbb{P}_R^n, \mathcal{F}(d)) \longrightarrow H^{k+1}(\mathbb{P}_R^n, \mathcal{G}(d))$$

For the case $d = 0$ we assume that $H^{k+1}(\mathbb{P}_R^n, \mathcal{G})$ is a finite R -module and, by computation, so is $H^k(\mathbb{P}_R^n, \mathcal{O}_X(m)^{\oplus n})$ and thus $H^k(\mathbb{P}_R^n, \mathcal{F})$ is a finite R -module. For $d \gg 0$ then we assume that $H^{k+1}(\mathbb{P}_R^n, \mathcal{G}(d)) = 0$ for sufficiently large d . Furthermore, for $k > 0$ we computed that $H^k(\mathbb{P}_R^n, \mathcal{O}_X(m)^{\oplus n}) = 0$ for $d \geq m$ and thus we see that $H^k(\mathbb{P}_R^n, \mathcal{F}(d)) = 0$ for sufficiently large d proving (3) and (4).

Finally, we also use descending induction and consider the exact sequence,

$$\bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{O}_X(m+d)^{\oplus r}) \longrightarrow \bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{F}(d)) \longrightarrow \bigoplus_{d \geq 0} H^0(\mathbb{P}_R^n, \mathcal{G}(d))$$

By computation, the first term is a submodule of a finite P -module and the last term is zero is sufficiently large degrees. Thus the middle term M has a f.g. P -submodule M' such that M/M' is finite as an R -module so M is a f.g. P -module. \square

Lemma 7.0.2. Let $f : X \rightarrow Y$ be an affine morphism of schemes. Then $H^p(X, \mathcal{F}) = H^p(Y, f_* \mathcal{F})$ for \mathcal{F} quasi-coherent.

Proof. We use the Grothendieck spectral sequence and not that for $f : X \rightarrow Y$ affine and \mathcal{F} quasi-coherent we have $R^p f_* \mathcal{F} = 0$ for $p > 0$ since quasi-coherent higher cohomology vanishes on affine schemes. \square

Example 7.0.3. If X is a projective scheme over a Noetherian ring R . For closed immersion $X \hookrightarrow \mathbb{P}_R^n$,

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}_R^n, j_* \mathcal{F})$$

for quasi-coherent \mathcal{O}_X -modules.

Lemma 7.0.4. If $\mathcal{F} : X \rightarrow Y$ is finite and X and Y are Noetherian then f_* preserves coherent sheaves.

Proof. Since f is affine it preserves quasi-coherent modules. Since the morphism is additionally finite on rings so it changes finite modules to finite modules on the affine open level. \square

Corollary 7.0.5. For any coherent \mathcal{F} on a scheme X projective over Noetherian R then the above proposition holds with $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ where \mathcal{L} is an ample line bundle.

Remark. Let X be Noetherian over Noetherian R then let $n = \max\{\dim X_s \mid s \in \text{Spec}(R)\}$ then $H^i(X, \mathcal{F}) = 0$ for $i > n$. Warning, this is not true for quasi-projective X over a Noetherian ring. For example, consider $X = \mathbb{A}_{\mathbb{Q}}^2 \setminus \{0\} \rightarrow \mathbb{A}_{\mathbb{Q}}^2$ is quasi-projective over $R = \mathbb{Q}[x, y]$ but X does not have finitely generated cohomology.

Lemma 7.0.6. Let X be projective over a field k then X has an open cover by $\dim X + 1$ affines.

Proof. Choose $X \hookrightarrow \mathbb{P}_k^n$ show that we can find $F \in k[T_0, \dots, T_n]_d$ s.t. $\dim(X \cap V(F)) < \dim X$. Namely, choose F not vanishing at the generic points of X by graded prime avoidance. Then we can repeat to get,

$$X \cap V(F_1) \cap \dots \cap V(F_{\dim X + 1}) = \emptyset$$

and thus,

$$X = (X \cap D_+(F)) \cup \dots \cup (X \cap D_+(F_{\dim X + 1}))$$

where these factors are affine. \square

Corollary 7.0.7. $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$ for \mathcal{F} quasi-coherent on X projective over a field.

Theorem 7.0.8 (Grothendieck). If (X, \mathcal{O}_X) is a Noetherian ringed space then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$ and any \mathcal{O}_X -module \mathcal{F} .

Remark. Since we can always choose $\mathcal{O}_X = \mathbb{Z}$ in the above theorem applies to all abelian sheaves.

Lemma 7.0.9. If X is qc and qs then for \mathcal{F}_i quasi-coherent and I an arbitrary index set,

$$H^p(X, \bigoplus_{i \in I} \mathcal{F}_i) = \bigoplus_{i \in I} H^p(X, \mathcal{F}_i)$$

Remark. The above is always true in general for *finite* I since biproducts preserve exact sequences and injectives.

Proof. It is enough to show the above for Čech cohomology for finite affine open covers. Thus, it is enough to show that,

$$\left(\bigoplus_{i \in I} \mathcal{F}_i \right) (U) = \bigoplus_{i \in I} \mathcal{F}_i(U)$$

If X is affine open in X (WAIT WHAT??) □

7.1 Duality

Lemma 7.1.1. Let R be a ring, M an R -module, and X qc + sep over R . And some $n \geq 0$ such that $H^{n+1}(X, \mathcal{F}) = 0$ for all \mathcal{F} quasi-coherent. Then, the functor $F : \mathbf{Qcoh}(\mathcal{O}_X) \rightarrow \mathbf{Mod}_R$ via $\mathcal{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathcal{F}), M)$ is representable by some $\omega_{X/R, M} \in \mathbf{Qcoh}(\mathcal{O}_X)$. That is,

$$F(-) = \mathrm{Hom}_{\mathcal{O}_X}(-, \omega_{X/R, M})$$

Example 7.1.2. For $X = \mathrm{Spec}(A)$ then we have $\widetilde{N} \mapsto \mathrm{Hom}_R(N_R, M)$. Then,

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_A(N, \mathrm{Hom}_R(A, M))$$

so we would have $\omega_{A/R, M} = \mathrm{Hom}_R(\widetilde{A}, M)$.

Proof. First note that F acts on direct sums as,

$$F\left(\bigoplus_{i \in I} \mathcal{F}_i\right) = \mathrm{Hom}_R\left(H^n(X, \bigoplus_{i \in I} \mathcal{F}_i), M\right) = \mathrm{Hom}_R\left(\bigoplus_{i \in I} H^n(X, \mathcal{F}_i), M\right) = \prod_{i \in I} \mathrm{Hom}_R(H^n(X, \mathcal{F}_i), M)$$

Furthermore, F takes epis to monos since given an exact sequence,

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

then we get,

$$H^n(X, \mathcal{F}_2) \longrightarrow H^n(X, \mathcal{F}_3) \longrightarrow H^{n+1}(X, \mathcal{F}_1) = 0$$

These together shows that F takes all small colimits to products. Then if F satisfies some mild set-theoretic condition then the adjoint functor theorem gives $\omega_{X/R, M}$ as a functor on M . The ideal goes as follows. We take,

$$\omega_{X/R, M} = \mathrm{colim}_{\mathcal{C}} \mathcal{F}$$

where \mathcal{C} is a category of pairs (\mathcal{F}, α) where \mathcal{F} is a quasi-coherent sheaf and $\alpha \in F(\mathcal{F})$ and $\mathrm{Hom}_{\mathcal{C}}((\mathcal{F}, \alpha), (\mathcal{G}, \beta)) = \varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $\varphi^* \beta = \alpha$. However, this category is big so we cannot take a total colimit over it. We must resolve this set-theoretic issue.

In the case R is Noetherian and X is finite type over R then any quasi-coherent \mathcal{F} can be written as a filtered colimit,

$$\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$$

with \mathcal{F}_i coherent. This means that in the colimit defining $\omega_{X/R, M}$ we can restrict to only coherent \mathcal{F} and there is a set of isomorphism classes of coherent sheaves. □

8 Mar 3

Remark. Here X will be a Noetherian scheme.

Lemma 8.0.1. Let X be a Noetherian scheme. Any presheaf on $\mathfrak{QCo}(\mathcal{O}_X)$ which transforms colimits into limits is representable.

Lemma 8.0.2. Any quasi-coherent module \mathcal{F} on X is a filtered colimit of coherent \mathcal{O}_X -modules. (In fact \mathcal{F} is the rising union of its coherent submodules).

Corollary 8.0.3. For any $\mathcal{F} \in \mathfrak{QCo}(\mathcal{O}_X)$ there exists an exact sequence,

$$\bigoplus_{j \in J} \mathcal{G}_j \longrightarrow \bigoplus_{i \in I} \mathcal{F}_i \longrightarrow 0$$

where \mathcal{F}_i and \mathcal{G}_j are coherent.

Lemma 8.0.4. There is a set of isomorphism classes of coherent \mathcal{O}_X -modules.

Proposition 8.0.5. Let X be finite type over R Noetherian. Let n be an integer s.t. $H^{n+1}(X, \mathcal{F}) = 0$ for any $\mathcal{F} \in \mathfrak{QCo}(\mathcal{O}_X)$. Then, for any R -module M , the functor,

$$\mathcal{F} \mapsto \mathrm{Hom}_R(H^n(X, \mathcal{F}), M)$$

is representable by $\omega_{X/R, M, n} \in \mathfrak{QCo}(\mathcal{O}_X)$ i.e.

$$\mathrm{Hom}_R(H^n(X, \mathcal{F}), M) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_{X/R, M, n})$$

functorially in $\mathcal{F} \in \mathfrak{QCo}(\mathcal{O}_X)$.

Remark. For any integer p and $\mathcal{F} = \mathrm{colim} \mathcal{F}_i$ is a filtered colimit of \mathcal{O}_X -modules on a Noetherian scheme (or qcqs scheme) we have,

$$H^p(X, \mathcal{F}) = \mathrm{colim} H^p(X, \mathcal{F}_i)$$

Theorem 8.0.6. If k is a field and $n \geq 0$. Then $\omega_{\mathbb{P}_k^n/k, k, n} = \mathcal{O}_{\mathbb{P}_k^n}(-n-1)$. In particular,

$$H^n(\mathbb{P}_k^n, \mathcal{F})^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_k^n}(-n-1))$$

functorially in $\mathcal{F} \in \mathfrak{QCo}(\mathcal{O}_{\mathbb{P}_k^n})$.

Proof. It suffices to show for \mathcal{F} coherent. Pick a resolution,

$$\bigoplus_{j=1}^s \mathcal{O}_{\mathbb{P}_k^n}(e_j) \longrightarrow \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}_k^n}(d_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

Since $H^n(\mathbb{P}_k^n, -)$ is right exact (by dimension vanishing) we get,

$$\bigoplus_{j=1}^s H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(e_j)) \longrightarrow \bigoplus_{j=1}^r H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d_i)) \longrightarrow H^n(\mathbb{P}_k^n, \mathcal{F}) \longrightarrow 0$$

Then taking k -linear duals,

$$\begin{array}{ccccccc}
\bigoplus_{j=1}^s H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(e_j))^\vee & \longleftarrow & \bigoplus_{j=1}^r H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d_i))^\vee & \longleftarrow & H^n(\mathbb{P}_k^n, \mathcal{F}) & \longleftarrow & 0 \\
\parallel & & \parallel & & \vdots & & \\
\bigoplus_{j=1}^s H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-e_j - n - 1)) & \xleftarrow{t} & \bigoplus_{j=1}^r H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-d_i - n - 1)) & \longleftarrow & \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_k^n}(-n - 1)) & \longleftarrow & 0
\end{array}$$

Note that,

$$\mathcal{O}_{\mathbb{P}_k^n}(-d - n - 1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{\mathbb{P}_k^n}(d), \mathcal{O}_{\mathbb{P}_k^n}(-n - 1))$$

gives the above “transpose” map t above by functoriality in the first argument along with the fact,

$$H^0(\mathbb{P}_k^n, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}, \mathcal{G})) = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

□

9 March 5

9.1 Serre Duality for \mathbb{P}_k^n Continued.

Write $\omega = \mathcal{O}_{\mathbb{P}_k^n}(-n - 1)$ and $t : H^n(\mathbb{P}_k^n, \omega) \rightarrow k$ via the Check class,

$$\frac{1}{T_0 \cdots T_n} \mapsto 1$$

Then we know that ω represents the functor,

$$\mathcal{F} \mapsto H^n(\mathbb{P}_k^n, \mathcal{F})^\vee$$

on $\mathfrak{QCoh}(\mathcal{O}_X)$ with universal object t .

Theorem 9.1.1. For coherent modules \mathcal{F} , there is an isomorphism,

$$H^{n-i}(\mathbb{P}_k^n, \mathcal{F})^\vee = \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \omega)$$

Proof. Both sides are contravariant δ -functors in \mathcal{F} so it suffices to show that both are universal for which it suffices to show that both are coeffecable. For any coherent sheaf \mathcal{F} we can find,

$$\mathcal{O}_{\mathbb{P}_k^n}(-q) \oplus^r \twoheadrightarrow \mathcal{F}$$

and then for $i > 0$ we know,

$$H^{n-i}(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-q)) = 0 \quad \text{and} \quad \text{Ext}_{\mathcal{O}_X}^i(\mathcal{O}_{\mathbb{P}_k^n}(-q), \omega) = H^i(\mathbb{P}_k^n, \omega(q)) = H^i(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-n-1+q)) = 0$$

for sufficiently large $q \gg 0$ using our Cech calculations. □

Lemma 9.1.2. Let (X, \mathcal{O}_X) be a ringed space and \mathcal{E} a finite locally free \mathcal{O}_X -module. Then.

$$\text{Ext}_{\mathcal{O}_X}^i(\mathcal{E}, \mathcal{G}) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

Proof. Choose an injective resolution $\mathcal{G} \rightarrow \mathcal{I}^\bullet$ then,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet) = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet)) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)$$

Now I claim that $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$ is an injective resolution over $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}$. To see this, we use,

$$\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} -, \mathcal{I}^\bullet)$$

but \mathcal{I}^\bullet is injective and \mathcal{E} is flat so this is an exact functor. Taking cohomology of the first equality proves the lemma. \square

Remark. We could also just say, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, -) = \Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$ so taking their derived functors gives the same thing. However, $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} -$ is exact so taking derived functors of $\Gamma(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -) = H^i(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} -)$.

Remark. The perfect pairings,

$$\mathrm{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^i(\mathcal{F}, \omega) \times H^{n-i}(\mathbb{P}_k^n, \mathcal{F}) \rightarrow H^n(\mathbb{P}_k^n, \omega) \xrightarrow{t} k$$

factors through $H^n(\mathbb{P}_k^n, \omega)$. The first map can be realized as composition of ext classes or a cup product.

Remark. If \mathcal{F} is locally free then we have a diagram,

$$\begin{array}{ccc} H^i(\mathbb{P}_k^n, \mathcal{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \omega) \times H^{n-i}(\mathbb{P}_k^n, \mathcal{F}) & \longrightarrow & k \\ \downarrow & & \uparrow t \\ H^n(\mathbb{P}_k^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \mathcal{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \omega) & \longrightarrow & H^n(\mathbb{P}_k^n, \omega) \end{array}$$

which gives the same pairing using the unique evaluation pairing,

$$\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}_k^n}} \mathcal{F}^\vee = \mathcal{F} \otimes_{\mathbb{P}_k^n} \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_k^n}) \rightarrow \mathcal{O}_{\mathbb{P}_k^n}$$

9.2 Dualizing Sheaves in General

Definition 9.2.1. Let X be proper over k and $\dim X = n$. A *dualizing sheaf* (ω_X, t) is a pair consisting of a coherent \mathcal{O}_X -module ω_X and a map $t : H^n(X, \omega_X) \rightarrow k$ which represents the functor,

$$\mathcal{F} \mapsto H^n(X, \mathcal{F})^\vee$$

Remark. We have proven, by abstract nonsense, that such a *quasi-coherent* dualizing sheaf exists but now we want to know when such a module is actually *coherent*.

Remark. Consider the case that X is the disjoint union of a curve and a surface. Then $H^2(X, -)$ ignores cohomology on the curve since it vanishes above $H^1(X, -)$. Thus the dualizing sheaf will be zero on the curve. To fix this one looks for a dualizing complex,

$$\omega_X^\bullet \in D^b(\mathfrak{D}\mathfrak{C}\mathfrak{o}\mathfrak{h}(\mathcal{O}_X))$$

such that $H^i(X, \mathcal{F})$ is dual to $\mathrm{Ext}_{\mathcal{O}_X}^{-i}(\mathcal{F}, \omega_X^\bullet)$.

Theorem 9.2.2. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}_k^n$,

$$\iota_*\omega_X \cong \mathcal{E}xt_{\mathcal{O}_X}^c(\iota_*\mathcal{O}_X, \omega_{\mathbb{P}_k^n})$$

where $c = n - \dim X$ is the codimension.

Lemma 9.2.3. Let $\iota : X \rightarrow Y$ be a closed immersion of schemes then $\iota_* : \mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_X) \rightarrow \mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_X)$ defines an equivalence of categories onto its image which is the full subcategory of quasi-coherent \mathcal{O}_Y -modules \mathcal{F} such that $\mathcal{I} \cdot \mathcal{F} = 0$ for $\mathcal{I} = \ker(\mathcal{O}_Y \rightarrow \iota_*\mathcal{O}_X)$.

Remark. If X and Y are Noetherian schemes, then the above holds also for coherent modules.

Remark. If $f : X \rightarrow Y$ is an affine morphism, $\mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_X)$ is the category of pairs (\mathcal{F}, γ) with $\mathcal{F} \in \mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_Y)$ and $\gamma : f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}$ gives \mathcal{F} a $f_*\mathcal{O}_X$ -module structure meaning $f_*\mathcal{O}_X$ is a monoid object and $f_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}$ is an action of a monoid object.

10 Mar. 12

Lemma 10.0.1. Let A be Noetherian and M, N be finite-presentation A -modules and $X = \text{Spec}(A)$. Then,

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) = \widetilde{\text{Hom}_A(M, N)}$$

Proof. The isomorphism,

$$\text{Hom}_A(M, N)_f = \text{Hom}_{A_f}(M_f, N_f)$$

for finitely-presented modules patch together on the open sets $D(f)$ to give an isomorphism,

$$\widetilde{\text{Hom}_A(M, N)} = \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

□

Lemma 10.0.2. Let A be Noetherian and M, N be finite A -modules and $X = \text{Spec}(A)$. Then,

$$\mathcal{E}xt_{\mathcal{O}_X}^i(\widetilde{M}, \widetilde{N}) = \widetilde{\text{Ext}_A^i(M, N)}$$

Proof. This holds for $i = 0$ by the above. Then we apply dimension-shifting to prove this in general. Given a

□

Lemma 10.0.3. For $p < \dim P - \dim X$ we have,

$$\mathcal{E}xt_{\mathcal{O}_X}^p(\iota_*\mathcal{O}_X, \omega_P) = 0$$

Proof. This reduced to the algebra question, given $B = k[x_1, \dots, x_n] \twoheadrightarrow A$ then,

$$\text{Ext}_B^p(A, B) = 0$$

for $p < \dim B - \dim A$. To see this, recall we have $\iota : X \hookrightarrow P = \mathbb{P}_k^n$ then $X \cap D_+(T_i) \subset X$ and $D_+(T_i) = \text{Spec}(B)$. Then, $\omega_P|_{D_+(T_i)} = \mathcal{O}_X|_{D_+(T_i)} = \widetilde{B}$. Furthermore, $\iota : X \hookrightarrow P$ is affine (closed immersion) so $X \cap D_+(T_i) = \text{Spec}(A)$ for $A = B/I$.

Since B is Cohen-Macaulay we have vanishing for,

$$\text{depth}_I(A) \geq \dim B - \dim A$$

□

Proof.

□

Theorem 10.0.4. Every projective scheme X/k has a dualizing module ω_X and for any closed immersion $\iota : X \hookrightarrow \mathbb{P}_k^n$,

$$\iota_*\omega_X \cong \mathcal{E}xt_{\mathcal{O}_X}^c(\iota_*\mathcal{O}_X, \omega_{\mathbb{P}_k^n})$$

where $c = n - \dim X$ is the codimension.

Proposition 10.0.5. If $\iota : X \rightarrow Y$ is a closed immersion then $\iota^*\iota_*\mathcal{F} = \mathcal{F}$ for any \mathcal{O}_X -module \mathcal{F} and if $\mathcal{I} \cdot \mathcal{G} = 0$ for some \mathcal{O}_Y -module \mathcal{G} then $\mathcal{G} = \iota_*\iota^*\mathcal{G}$ where $\mathcal{I} = \ker(\mathcal{O}_Y \rightarrow \iota_*\mathcal{O}_X)$.

11 Local Property

Definition 11.0.1. A property P of ring maps is *local* if,

- (a) $P(R \rightarrow A) \implies P(R_f \rightarrow A_f)$ for all $f \in R$
- (b) $P(R_f \rightarrow A)$ for some $f \in R$ then $P(R \rightarrow A_a)$ for any $a \in A$
- (c) if $P(R \rightarrow A_{a_i})$ for $(a_1, \dots, a_r) = A$ then $P(R \rightarrow A)$.

Definition 11.0.2. We say a morphism of schemes $f : X \rightarrow Y$ is locally P for some local property P if for each $x \in X$ there is an affine open $U = \text{Spec}(A)$ with $x \in U \subset X$ and $V = \text{Spec}(R)$ with $V \subset Y$ with $f(U) \subset V$ such that $P(R \rightarrow A)$.

Lemma 11.0.3. If $f : X \rightarrow Y$ is locally P then for any affine opens $U \subset X$ and $V \subset Y$ with $f(U) \rightarrow V$ then $P(\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U))$.

Remark.

12 Smooth Maps

Definition 12.0.1. A ring map $R \rightarrow A$ is *smooth* if it is of finite presentation,

$$A \cong R[x_1, \dots, x_n]/I$$

where I is finitely generated. Then consider,

$$I/I^2 \xrightarrow{d} \bigoplus_{i=1}^n \text{Ad}x_i$$

given by,

$$f \mapsto df = \sum \frac{\partial f}{\partial x_i} dx_i$$

Then d is injective and its cokernel is a projective A -module. Since $K = \text{coker } d$ is projective and finitely generated then it is locally free so it has a rank function. We say that $R \rightarrow A$ is *smooth of relative dimension n* if K is of constant rank n .

Remark. Smoothness satisfies the following,

- (a) local

(b) preserved under composition

(c) preserved under base change

Example 12.0.2. Take $R \rightarrow R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ such that,

$$\det \left(\frac{\partial f_j}{\partial x_i} \right)_{\substack{i=1, \dots, c \\ j=1, \dots, c}}$$

maps to an invertible element of A under $R[x_1, \dots, x_n] \rightarrow A$. Then,

$$\frac{(f_1, \dots, f_c)}{(f_1, \dots, f_c)^2} \rightarrow \bigoplus_{i=1}^n A dx_i \rightarrow \text{coker } d \rightarrow 0$$

makes coker projective since the matrix for the map,

$$f_i \mapsto \frac{\partial f_i}{\partial x_j} dx_j$$

in the basis $\{f_i\}$ and $\{dx_j\}$ is invertible. Therefore, the cokernel is locally free. We think of this situation as $f = (f_1, \dots, f_c)$ defining a map $R[y_1, \dots, y_c] \rightarrow R[x_1, \dots, x_n]$ via $y_i \mapsto f_i$ thus we get a morphism $f : \mathbb{A}_R^n \rightarrow \mathbb{A}_R^c$ then $\text{Spec}(R[x_1, \dots, x_n]/(f_1, \dots, f_c))$ is the fiber above the point zero (y_1, \dots, y_c) .

In differential geometry, such a map $f : \mathbb{C}^n \rightarrow \mathbb{C}^c$ is a submersion since the Jacobian matrix has full rank. Therefore,

$$\begin{array}{ccc} f^{-1}(\{0\}) & \hookrightarrow & \mathbb{C}^n \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & \mathbb{C}^c \end{array}$$

Then $f^{-1}(\{0\})$ is a smooth manifold by the implicit function theorem. We call this situation standard smooth.

Lemma 12.0.3. A map $R \rightarrow A$ is smooth if and only if there exist a_i s.t. $(a_1, \dots, a_r) = A$ and $R \rightarrow A_{a_i}$ is standard smooth.

Definition 12.0.4. For a standard smooth ring map, $R \rightarrow A$ we can factor,

$$\begin{array}{ccc} R & \xrightarrow{\quad} & R[x_{c+1}, \dots, x_n] \\ & \searrow & \downarrow \\ & & R[x_1, \dots, x_n]/(f_1, \dots, f_c) \end{array}$$

Then the downward map is étale.

Definition 12.0.5. A smooth morphism of schemes is a morphism which is locally smooth.

Remark. Using the previous lemma, for any smooth morphism of schemes $X \rightarrow Y$ it is locally standard smooth so we can factor,

$$\begin{array}{ccc}
X & \longleftrightarrow & U \\
\downarrow f & & \downarrow \\
Y & \longleftrightarrow & V
\end{array}
\begin{array}{c}
\searrow \text{etale} \\
\mathbb{A}_V^d \\
\swarrow \text{projection}
\end{array}$$

Definition 12.0.6. A variety X over k is smooth iff $X \rightarrow \text{Spec}(k)$ is smooth.

Definition 12.0.7. A locally noetherian scheme X is *regular* or *nonsingular* iff $\mathcal{O}_{X,x}$ is regular at each $x \in X$.

Remark. For locally Noetherian schemes it suffices to check regularity on the closed points.

Theorem 12.0.8. If $X \rightarrow \text{Spec}(k)$ is smooth then X is regular.

Theorem 12.0.9. If k is perfect then a variety is smooth iff it is regular.

Example 12.0.10. Let $k = \mathbb{F}_p(t)$ then take $\text{Spec}(k[x]/(y^2 - (x^p - t)))$ which is regular but not smooth. Consider,

$$d(y^2 - x^p + t) = 2ydy + 0$$

and thus we have,

$$(f)/(f^2) \mapsto Adx \oplus Ady$$

which is injective but the cokernel is $A \oplus A/yA$ but A/yA has torsion so cannot be projective and thus not smooth.

However, we just need to check regularity at $(y, x^p - t) = (y) \subset A$ which is a height one ideal and generated by one element so $A_{(y)}$ is regular.

13 Differentials

Remark. See Tags O8RL, O8RT.

Definition 13.0.1. For a ring map $\varphi : R \rightarrow A$ the A -module of differentials $\Omega_{A/R}$ is generated by the symbols da for $a \in A$ such that,

$$(a) \quad d(a_1 + a_2) = da_1 + da_2$$

$$(b) \quad da_1 a_2 = da_1 \cdot a_2 + a_1 \cdot da_2$$

$$(c) \quad dr = 0 \text{ for } r \in R$$

Then $d_{R/A} : R \rightarrow \Omega_{R/A}$ is the universal derivation meaning that $\Omega_{R/A}$ represents the functor $\text{Der}_R(A, -)$ i.e.

$$\text{Hom}_A(\Omega_{A/R}, M) \cong \text{Der}_R(A, M)$$

via $(f : \Omega_{A/R} \rightarrow M) \mapsto f \circ d_{A/R}$

Definition 13.0.2. Given a morphism of ringed spaces $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ then there is a universal derivation $d_{X/Y} : \mathcal{O}_X \rightarrow \Omega_{X/Y}$ s.t.

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{F}) \cong \mathrm{Der}_{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$$

Where a derivation $\varphi : \mathcal{O}_X \rightarrow \mathcal{F}$ is an abelian map such that $\varphi(fs) = f\varphi(s) + \varphi(f)s$ and under $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ we send $s \in \mathcal{O}_Y(U)$ to $\varphi(s) = 0$.

Lemma 13.0.3. For a morphism of schemes $f : X \rightarrow Y$ we have,

$$\begin{array}{ccc} X & \longleftarrow & U = \mathrm{Spec}(A) \\ \downarrow f & & \downarrow \\ Y & \longleftarrow & V = \mathrm{Spec}(R) \end{array}$$

Then we have $\Omega_{X/Y}|_U = \widetilde{\Omega_{A/R}}$.

13.1 The Diagonal

(Tag 01R1) Consider $R \rightarrow A$ then consider the map,

$$\Omega_{A/R} \xrightarrow{\sim} J/J^2$$

via $da \mapsto a \otimes 1 - 1 \otimes a$ where $J = \ker(A \otimes_R A \rightarrow A)$ via $a \otimes b \mapsto ab$. This situation generalizes to Schemes in which,

$$\Omega_{X/Y} = \Delta_{X/Y}^*(\mathcal{I})$$

where \mathcal{I} is the sheaf of ideals of $\Delta : X \rightarrow X \times_Y X$ i.e. $\mathcal{I} = \ker(\mathcal{O}_{X \times_Y X} \rightarrow \Delta_{X/Y}^* \mathcal{O}_X)$. This is the conormal sheaf of $\Delta_{X/Y} : X \rightarrow X \times_Y X$.

14 Mar. 31

14.1 Conormal Sheaf

Definition 14.1.1. Let $\iota : Z \hookrightarrow X$ be a closed immersion with $\mathcal{I} = \ker(\mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z)$. Then the conormal sheaf is $\mathcal{C}_{Z/X} = \iota^* \mathcal{I}$ and thus $\iota_* \mathcal{C}_{Z/X} = \mathcal{I}/\mathcal{I}^2$.

Remark. Affine locally $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Then we get the A/I -module $I \otimes_A A/I = I/I^2$ pushing forward to the A -module I/I^2 .

Proposition 14.1.2. Let $\iota : Z \hookrightarrow X$ is a closed immersion over S then there is an exact sequence,

$$\mathcal{C}_{Z/X} \longrightarrow \iota^* \Omega_{X/S} \longrightarrow \Omega_{Z/S} \longrightarrow 0$$

If $Z \rightarrow S$ is smooth then it is short exact.

Proof. This follows from the exact sequence, with $R \rightarrow A$ and $B = A/I$,

$$I/I^2 \xrightarrow{\alpha} \Omega_{A/R} \otimes_A B \xrightarrow{\beta} \Omega_{B/R} \longrightarrow 0$$

given by $f \mapsto df \otimes 1$ and $da \otimes a \mapsto d\bar{a}$. The second, β , is clearly surjective. Furthermore, given $B \rightarrow \Omega_{A/R} \otimes_A B / \text{Im}(\alpha)$ via $\bar{a} \mapsto da \otimes 1$ is a well-defined R -derivation and thus factors through $\Omega_{B/R}$ giving the required isomorphism. \square

Proposition 14.1.3. Given

Remark. The complex,

$$I/I^2 \rightarrow \bigoplus_{i=1}^n \text{Ad}x_i$$

for $A = R[x_1, \dots, x_n]/I$. This is a truncated version of $NL_{A/R}$, the naive cotangent complex of A/R and $\text{coker } d = \Omega_{A/R}$ and $H^{-1}(NL_{A/R}) = H^{-1}(L_{A/R})$.

Remark. In the case $A = R[x_1, \dots, x_n]/I$, we get,

$$I/I^2 \longrightarrow \Omega_{R[x_1, \dots, x_n]/R} \otimes A \longrightarrow \Omega_{A/R} \longrightarrow 0$$

and thus we see $\text{coker } d = \Omega_{A/R}$ as required above.

Definition 14.1.4. Consider the presentation $R[A] \rightarrow A$ and $J = \ker(R[A] \rightarrow A)$. Then we can define the naive cotangent complex is $NL_{A/R} := (J/J^2 \xrightarrow{d} \Omega_{R[A]/R} \otimes_{R[A]} A)$ with J/J^2 in degree -1 .

Remark. The naive cotangent complex is homotopy equivalent to the complex above for a choice of presentation therefore we see the following.

Proposition 14.1.5. A ring map $R \rightarrow A$ is smooth if the naive cotangent complex $NL_{A/R}$ is quasi-isomorphic to a projective module in degree zero. Explicitly $H^{-1}(NL_{A/R}) = 0$ and $H^0(NL_{A/R}) = \Omega_{A/R}$ is projective.

15 April 2

Theorem 15.0.1. Let $f : X \rightarrow S$ be a morphism of schemes then the FAE,

- (a) f is smooth
- (b) f is locally finite presentation, flat, and the fibers $X_s \rightarrow \text{Spec}(\kappa(s))$ are smooth.
- (c) f is locally finitely presentation and f is formally smooth.

Definition 15.0.2. A morphism $f : X \rightarrow S$ is formally smooth iff every diagram of the form,

$$\begin{array}{ccc} \text{Spec}(A/I) & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow f \\ \text{Spec}(A/I^2) & \longrightarrow & S \end{array}$$

gives a map $\text{Spec}(A/I^2) \rightarrow X$ making it commute.

15.1 Smoothness over Fields

Let k be a field and $S = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ with $\mathfrak{q} \subset S$ a prime ideal. Let $d = \dim S_{\mathfrak{q}} + \text{trdeg}_k(\kappa(\mathfrak{q}))$.

Proposition 15.1.1 (Jacobian Criterion). The following are equivalent,

- (a) S is smooth over k at \mathfrak{q}
- (b) the rank over $\kappa(\mathfrak{q})$ of the matrix,

$$\left(\frac{\partial f}{\partial x_i} \right)_{\substack{j=1, \dots, m \\ i=1, \dots, n}} \pmod{\mathfrak{p}}$$

is equal to $n - d$ (it is always at most $n - d$).

Remark. Smoothness at \mathfrak{q} implies that S is regular at \mathfrak{q} but this is not sufficient.

Remark. If S is equidimensional of dimension d then the nonsmooth locus of $\text{Spec}(S)$ is the vanishing locus of all $(n - d) \times (n - d)$ minors of the matrix,

$$\left(\frac{\partial f}{\partial x_i} \right)_{\substack{j=1, \dots, m \\ i=1, \dots, n}}$$

Example 15.1.2. Let $Z = V(F_1, \dots, F_c) \subset \mathbb{P}_k^n = X$ be a complete intersection of type (d_1, \dots, d_c) . We saw that,

$$\omega_Z \cong \mathcal{E}xt_{\mathcal{O}_X}^i(\mathcal{O}_Z, \omega_X) = \omega_X(d_1 + \dots + d_c)|_Z$$

Theorem 15.1.3. If $Z \subset X = \mathbb{P}_k^n$ is a smooth projective scheme equidimensional of $\dim Z = n - c$ then,

$$\omega_Z = \mathcal{E}xt_{\mathcal{O}_X}^c(\mathcal{O}_Z, \omega_X) \cong \omega_X|_Z \otimes \bigwedge^c \mathcal{C}_{Z/X}^\vee$$

Proof. See HAR [III.7.11 + II.8.17] The exact sequence,

$$0 \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \Omega_{X/k}|_Z \longrightarrow \Omega_{Z/k} \longrightarrow 0$$

implies that $\mathcal{C}_{Z/X}$ is a vector bundles of rank c . Then Z is locally a complete intersection so we can conclude using our previous argument. \square

Lemma 15.1.4. Consider an exact sequence of locally free sheaves,

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow 0$$

then there is a canonical isomorphism of line bundles,

$$\bigwedge^{\text{top}} \mathcal{F}_1 \otimes_{\mathcal{O}_X} \bigwedge^{\text{top}} \mathcal{F}_3 = \bigwedge^{\text{top}} \mathcal{F}_2$$

Proposition 15.1.5. For $X = \mathbb{P}_k^n$ we know,

$$\omega_X \cong \mathcal{O}_X(-n-1) \cong \bigwedge^n \Omega_{X/k}$$

Proof. There is a short exact sequence of locally free sheaves,

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \bigoplus_{i=0}^n \mathcal{O}_X(-1) dT_i \longrightarrow \mathcal{O}_X \longrightarrow 0$$

on $D_+(T_i)$ given by,

$$d\left(\frac{T_i}{T_j}\right) = \frac{dT_i}{T_j} - \frac{T_i dT_j}{T_j^2}$$

and,

$$\frac{dT_i}{T_j} \mapsto \frac{T_i}{T_j}$$

Then, taking top exterior powers gives.

$$\bigwedge^{n+1} \left(\bigoplus_{i=0}^n \mathcal{O}_X(-1) \right) = \mathcal{O}_X(-n-1)$$

□

Lemma 15.1.6 (Adjunction). Let $Z \subset X$ be a smooth closed subscheme. Then,

$$\bigwedge^{\dim Z} \Omega_{Z/k} \cong \bigwedge^{\dim X} \Omega_{X/k}|_Z \otimes \bigwedge^c \mathcal{C}_{Z/X}^\vee$$

Proof. This is exactly the top exterior power of the sequence,

$$0 \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \Omega_{X/k}|_Z \longrightarrow \Omega_{Z/k} \longrightarrow 0$$

□

Remark. This implies that $K_Z = K_X|_Z + c_1(N_{Z/X})$.

Theorem 15.1.7. Let Z be a smooth projective variety over k . Then $\omega_Z \cong \bigwedge^{\dim Z} \Omega_{Z/k}$.

Proof. Choose an embedding $Z \hookrightarrow \mathbb{P}_k^n$ then we use adjunction for dualizing sheaves and for canonical sheaves which are the same to conclude. □

16 Varieties

Definition 16.0.1. A *variety* is an integral separated scheme with $X \rightarrow \operatorname{Spec}(k)$ finite type.

Remark. Problems: products of varieties are not varieties. E.g. $k = \mathbb{Q}$ and $X = \operatorname{Spec}(\mathbb{Q}(i))$ then $X \times_{\mathbb{Q}} X = \operatorname{Spec}(\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i))$ is not integral.

The problem is that varieties are not geometrically integral.

Definition 16.0.2. A scheme X over k is geometrically integral if $X_{\bar{k}} = X \times_k \bar{k}$ is integral.

Remark. There are not “enough” rational points of a general variety. For example, given,

$$X = V(x^2 + y^2 + z^2) \subset \mathbb{P}_{\mathbb{Q}}^3$$

then $X(\mathbb{Q}) = \emptyset$. We say $X(k) = \operatorname{Hom}_k(\operatorname{Spec}(k), X)$ is the set of k -rational points.

Remark. Alternative definition: we require that varieties are geometrically integral. Then products of varieties are varieties.

Definition 16.0.3. A *curve* is a variety of dimension one.

Lemma 16.0.4. A curve is regular iff it is normal.

Proof. A Noetherian local ring of dimension one is regular iff it is normal iff it is a DVR. \square

Lemma 16.0.5. A curve is either affine or projective.

16.1 Rational Maps

Definition 16.1.1. Let X, Y be varieties over k . A rational map $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, f) with $U \subset X$ is a dense open and $f : U \rightarrow Y$ is a morphism on U such that $(U, f_U) \sim (V, f_V)$ iff there is $W \subset U \cap V$ with $f_U|_W = f_V|_W$. Since X, Y are reduced and separated this implies that $f_U|_{U \cap V} = f_V|_{U \cap V}$.

Remark. Rational maps cannot be composed e.g. const map to a point then projection away from that point is undefined. To fix this we ask that the rational maps be dominant.

Definition 16.1.2. A rational map $f : X \dashrightarrow Y$ is dominant if for any representative (f, U) then $f(U) \subset Y$ is dense.

Theorem 16.1.3. The category varieties over k with rational maps is anti-equivalent to the category of finitely generated extensions of k . Sending $(f : X \dashrightarrow Y) \mapsto f_\eta : K(Y) \rightarrow K(X)$.

Definition 16.1.4. The function field $K(X)$ of an integral scheme is $\mathcal{O}_{X, \xi}$ where $\xi \in X$ is the generic point. For any affine open $\text{Spec}(A) \subset X$ then A is a domain and $(0) \subset A$ is the generic point of $\text{Spec}(A)$ and also of X . Then, $\mathcal{O}_{X, \xi} = \text{Frac}(A)$. If X is finite type over k then A is a finitely-generated k -algebra so $K(X) = \text{Frac}(A)$ is a finitely-generated field extension of k .

Lemma 16.1.5. Let X, Y be varieties over k . Any k -map $\varphi : K(Y) \rightarrow K(X)$ induces a rational map $X \dashrightarrow Y$ acting by φ on the generic points.

Proof. Let $U = \text{Spec}(A) \subset X$ be an open and $V = \text{Spec}(B) \subset Y$ an open. Then $B = k[b_1, \dots, b_n]$ we can write $\varphi(b_i) = \frac{a_i}{a'_i}$ for $a_i, a'_i \in A$ with $a'_i \neq 0$. Then replace U by $D(a'_1, \dots, a'_n) \subset \text{Spec}(A) \subset X$. Then we get,

$$\begin{array}{ccc} K(X) & \xleftarrow{\varphi} & K(Y) \\ \uparrow & & \uparrow \\ A' & \xleftarrow{\quad} & B \end{array}$$

This is dominant since $\phi^{-1}(0) = (0)$ because ϕ is injective. \square

Definition 16.1.6. Varieties X, Y over k are *birational* iff they are isomorphic in the rational category iff $K(X) = K(Y)$ as k -extensions.

17 Curves

Theorem 17.0.1. There is an anti-equivalence of categories between the category of normal projective curves over k and the category of transcendence degree one field extensions of k .

Proposition 17.0.2. The functor $C \mapsto K(C)$ is essentially surjective.

Proof. Given a transcendence degree one field K/k we know $K = k(X)$ for some affine $X \subset \mathbb{A}_k^n$ then take the closure $\overline{X} \subset \mathbb{P}_k^n$. Since $X \subset \overline{X}$ is a dense open then $K(\overline{X}) = K(X) = K$. Then we normalize to get $\overline{X}^\nu \rightarrow \overline{X}$ which is birational so get $K(\overline{X}^\nu) = K(X)$. \square

Lemma 17.0.3. Let $X \dashrightarrow Y$ be a rational map from a normal curve to a projective variety. Then it extends to $X \rightarrow Y$.

Proof. We have $X \dashrightarrow Y \subset \mathbb{P}_k^n$ since Y is closed it is enough to extend $X \rightarrow \mathbb{P}_k^n$. After replacing \mathbb{P}_k^n by a smaller projective space we may assume the map is $[f_0 : \cdots : f_n]$ for not all zero $f_i \in k(X)$. For a closed point $x \in X$ the ring $\mathcal{O}_{X,x}$ is a DVR with uniformizer ϖ since X is a normal curve. Then $K(X) = \text{Frac}(\mathcal{O}_{X,x})$ so we can write $f_i = u_i \varpi^{n_i}$ for $u_i \in \mathcal{O}_{X,x}^\times$ and $n_i \in \mathbb{Z}$. Let $N = \min n_i$. Now,

$$[f_0 : \cdots : f_n] = [\varpi^N f_0 : \cdots : \varpi^N f_n]$$

so take $g_i = \varpi^N f_i \in \mathcal{O}_{X,x}$ and there is some $g_i \in \mathcal{O}_{X,x}^\times$ so there exists $U \subset X$ open such that $g_i \in \mathcal{O}_X(U)$ and $g_j \in \mathcal{O}_X(U)^\times$. Therefore, can extend,

$$[g_0 : \cdots : g_n] : U \rightarrow D_+(T_i) \subset \mathbb{P}_k^n$$

\square

Proposition 17.0.4. The functor $C \mapsto K(C)$ is fully faithful.

Proof.

\square

Proposition 17.0.5. If Y is projective then its normalization Y^ν is projective.

18 April 16

Definition 18.0.1. Let $T \subset X$ be a proper closed subscheme of X . Then,

$$[T]_{\dim X - 1} = \sum_{Z \subset T} m_{Z,T} [Z]$$

where Z is a prime divisor with generic point $\xi \in Z$ and,

$$m_{Z,T} = \ell_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{T,\xi'})$$

Definition 18.0.2. Let $f : X \rightarrow Y$ be a morphism of schemes. Then we get a morphism $f^* : \text{Div}(Y) \rightarrow \text{Div}(X)$ via,

$$[Z] \mapsto [f^{-1}(Z)]_{\dim X - 1}$$

where $f^{-1}(Z) = X \times_Y Z$.

Let $f : X \rightarrow Y$ be a flat morphism of varieties. We get a map $f^* : \text{Div}(Y) \rightarrow \text{Div}(X)$ and $f^{-1}(Z)$ has pure codimension one. Furthermore, we get the following.

Proposition 18.0.3. $f^*\text{div}_Y(g) = \text{div}_X(g \circ f)$

Therefore, we get $f^* : \text{Cl}(Y) \rightarrow \text{Cl}(X)$. Furthermore, there is a diagram,

$$\begin{array}{ccc} \text{Pic}(Y) & \xrightarrow{f^*} & \text{Pic}(X) \\ \downarrow & & \downarrow \\ \text{Cl}(Y) & \longrightarrow & \text{Cl}(X) \end{array}$$

Example 18.0.4. For nonsingular curves, any nonconstant map $f : X \rightarrow Y$ is flat.

18.1 Ramification

Let $f : X \rightarrow Y$ be a nonconstant map of nonsingular curves. Then for $x \in X$ we have $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ which is a map of DVRs. Then we define the ramification index e_x via $\varpi_{f(x)} \mapsto \varpi_x^{e_x}$.

Then we find,

$$f^*[y] = \sum_{x \in f^{-1}(y)} e_x[x]$$

To see this, consider $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$. Now, $y \in Y$ is the closed subscheme cut out by a sheaf of ideals $M_y \subset \mathcal{O}_Y$ such that $(M_y)_y = \mathfrak{m}_y \subset \mathcal{O}_{Y,y}$. Now, $f^{-1}(y) \subset X$ is the closed subscheme of X cut out by $f^*M_y \subset \mathcal{O}_X = f^*\mathcal{O}_Y$. If $\varpi \in \mathcal{O}_{Y,y}$ is uniformizer. Then $M_y \subset \mathcal{O}_Y$ is locally generated by π near y . Therefore, f^*M_y is generated by pullback of ϖ_y . But under $\mathcal{O}_{Y,Y} \rightarrow \mathcal{O}_{X,x}$ we get $\varpi_y \mapsto \varpi_x^{e_x}$. Therefore,

$$\ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\varpi_x^e \mathcal{O}_{X,x}) = e_x$$

18.2 Finite Pushforward

let $f : X \rightarrow Y$ be finite and dominant morphism of varieties. Let, $f_* : \text{Div}(X) \rightarrow \text{Div}(Y)$ via,

$$[Z] \mapsto \deg(Z/f(Z))[f(Z)]$$

Since $\dim X = \dim Y$ and f finite we see that $f(Z) \subset Y$ is a prime divisor (it is closed since f is proper). So we define,

$$\deg(Z/f(Z)) = [K(Z) : K(f(Z))]$$

Definition 18.2.1. Let $f : X \dashrightarrow Y$ be a dominant rational map of varieties with $\dim X = \dim Y$. Then we define the *degree*,

$$\deg(f) = [K(X) : K(Y)]$$

which is finite since they are finitely-generated field extensions of k with the same transcendence degree.

Proposition 18.2.2. Pushforward is functorial with respect to finite dominant morphisms. Also, $f_*\text{div}_X(g) = \text{div}_Y(N_{X/Y}(g))$ where $N_{X/Y} : K(X)^\times \rightarrow K(Y)^\times$ is the norm of the finite extension $k(X)/K(Y)$. Therefore, we get an induced map,

$$f_* : \text{Cl}(X) \rightarrow \text{Cl}(Y)$$

Example 18.2.3. If $f : X \rightarrow Y$ is a finite morphism of nonsingular curves over k . Let $x \in X$ be a closed point and $y = f(x) \in Y$. Then,

$$f_*[x] = [\kappa(x) : \kappa(y)][y]$$

If k is algebraically closed then $f_*[x] = [y]$.

When $f : X \rightarrow Y$ is a finite flat map. Then there is a multiplicative norm map,

$$N_f : f_*\mathcal{O}_X \rightarrow \mathcal{O}_Y$$

of degree $d = \deg(X/Y)$ which induces a norm map,

$$N_f : \text{Pic}(X) \rightarrow \text{Pic}(Y)$$

as follows. Consider $N_f : f_*\mathcal{O}_X^\times \rightarrow \mathcal{O}_Y^\times$ which is a homomorphism. This induces,

$$N_f : H^1(X, \mathcal{O}_X^\times) \rightarrow H^1(Y, \mathcal{O}_Y^\times)$$

(See OBCX). There is a commutative diagram,

$$\begin{array}{ccc} \text{Pic}(X) & \xrightarrow{N_f} & \text{Pic}(Y) \\ \downarrow & & \downarrow \\ \text{Cl}(X) & \xrightarrow{f_*} & \text{Cl}(Y) \end{array}$$

Remark. We can show that,

$$N_f(\mathcal{L}) = \bigwedge^d (f_*\mathcal{L}) \otimes \bigwedge^d (f_*\mathcal{O}_X)^{\otimes -1}$$

Alternatively,

Lemma 18.2.4. In the case of curves. For $D \in \text{Div}(X)$ we have,

$$\deg(D) = \deg(f_*D)$$

Proof. Suppose that $D = [x]$ and $y = f(x) \in Y$. Then $f_*[x] = [\kappa(x) : \kappa(y)][y]$ then,

$$\deg(f_*[x]) = [\kappa(x) : \kappa(y)][\kappa(y) : k] = [\kappa(x) : k] = \deg[x]$$

Then the lemma holds by additivity. □

Corollary 18.2.5. Let X be a projective curve then $\deg(\text{div}_X(f)) = 0$ for any $f \in K(X)^\times$.

Proof. Suppose k is infinite. Choose $X \hookrightarrow \mathbb{P}_k^N$. Then project away from X to get,

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}_k^N \\ & \searrow & \downarrow \\ & & \mathbb{P}_k^{N-1} \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ & & \mathbb{P}_k^1 \end{array}$$

Thus we get $\varphi : X \rightarrow \mathbb{P}_k^1$ which is nonconstant and thus finite flat. So any projective curve is a finite cover of \mathbb{P}_k^1 . Then we use,

$$f_*(\operatorname{div}_X(f)) = \operatorname{div}_{\mathbb{P}_k^1}(N_\varphi(f)) = 0$$

since we have proven the fact for \mathbb{P}_k^1 . □

Corollary 18.2.6. For a projective curve we get the following diagram,

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ \operatorname{Pic}(X) & \longrightarrow & \operatorname{Cl}(X) & \xrightarrow{\deg} & \mathbb{Z} \\ & & \uparrow & \nearrow \deg & \\ & & \operatorname{Div}(X) & & \\ & & \uparrow & \nearrow & \\ & & K(X)^\times & & \end{array}$$

which defines degree maps $\deg : \operatorname{Cl}(X) \rightarrow \mathbb{Z}$ and $\deg : \operatorname{Pic}(X) \rightarrow \mathbb{Z}$.

Theorem 18.2.7 (Riemann-Roch). For X projective curve and $\mathcal{L} \in \operatorname{Pic}(X)$ we have,

$$\chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X) = \deg \mathcal{L}$$

where,

$$\chi(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L})$$

Remark. We will use the following notation,

$$h^p(X, \mathcal{F}) = \dim_k H^p(X, \mathcal{F})$$

19 April 21

Lemma 19.0.1. Let X be a normal curve. Let,

$$D = D' + \operatorname{div}_X(f)$$

Then multiplication by f defines an isomorphism,

$$\mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D')$$

Proof. If $U \subset X$ is open then,

$$\mathcal{O}_X(D)(U) = \{g \in K(X) \mid \operatorname{div}_U(g) + D \geq 0\}$$

and,

$$\mathcal{O}_X(D')(U) = \{g \in K(X) \mid \operatorname{div}_U(g) + D' \geq 0\}$$

But $\operatorname{div}_U(g) + D = \operatorname{div}_U(fg) + D'$ giving the result. □

Lemma 19.0.2. Let X be a projective curve and $f \in K(X)^\times$. then,

$$\deg(\operatorname{div}_X(f)) = 0$$

Proof. Write $\operatorname{div}_X(f) = D - D'$ where D, D' are effective. Then,

$$\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$$

Therefore,

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D'))$$

We write,

$$D = \sum n_x [x]$$

Then we have an exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \bigoplus_{x \in X} \mathcal{O}_{X,x} / \mathfrak{m}_x^{n_x} \longrightarrow 0$$

Therefore,

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + \sum \dim_k(\mathcal{O}_{X,x} / \mathfrak{m}_x^{n_x})$$

However, since $\mathcal{O}_{X,x}$ is a DVR,

$$\sum \dim_k(\mathcal{O}_{X,x} / \mathfrak{m}_x^{n_x}) = \sum n_x \dim_k \kappa(x) = \deg D$$

Therefore,

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + \deg D$$

However, the same holds for D' . Thus, since $\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D'))$ we get $\deg D = \deg D'$ proving the proposition. \square

Theorem 19.0.3 (Riemann-Roch). Let X be a projective curve. Then for any $\mathcal{L} \in \operatorname{Pic}(X)$,

$$\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}_X) + \deg \mathcal{L}$$

Definition 19.0.4. Let X be a projective curve over k with $H^0(X, \mathcal{O}_X) = k$. The *genus* of X is,

$$g = \dim_k H^1(X, \mathcal{O}_X)$$

Remark. Curves are always Cohen-Macaulay so we can apply Serre duality to show that,

$$g = \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X) = \dim_k H^0(X, \Omega_X)$$

where the last equality holds when X is smooth. Furthermore, we get $\dim_k H^1(X, \omega_X) = 1$.

Corollary 19.0.5. If ω_X is invertible and $H^0(X, \mathcal{O}_X) = k$. Then,

$$\deg(\omega_X) = 2g - 2$$

If X is smooth then $\deg(\Omega_X^1) = 2g - 2$.

Proof. We know $\chi(X, \omega_X) = g - 1$ by Serre duality. But by Riemann-Roch,

$$\deg(\omega_X) = \chi(X, \omega_X) + (g - 1) = 2g - 2$$

\square

Remark. If ω_X is invertible we may write $c_1(\omega_X) = K_X$ the canonical divisor. Then we can rewrite Riemann Roch as,

$$\chi(X, \mathcal{L}) = \deg(c_1(\mathcal{L}) - \frac{1}{2}K_X)$$

Remark. We know,

$$\chi(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^1(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}) - \dim_k H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee)$$

Therefore, we can write Riemann-Roch as,

$$h^0(X, \mathcal{L}) - h^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee) = \deg \mathcal{L} + 1 - g$$

Proposition 19.0.6. Let X be projective curve and $\mathcal{L} \in \text{Pic}(X)$. Then,

- (a) $\deg \mathcal{L} < 0 \implies H^0(X, \mathcal{L}) = 0$
- (b) $\deg \mathcal{L}$ and $H^0(X, \mathcal{L}) \neq 0 \implies \mathcal{L} \cong \mathcal{O}_X$

If ω_X is invertible, then,

- (a) $\deg \mathcal{L} > 2g - 2 \implies H^1(X, \mathcal{L}) = 0$
- (b) $\deg \mathcal{L} = 2g - 2$ and $H^1(X, \mathcal{L}) \neq 0 \implies \mathcal{L} \cong \omega_X$.

Example 19.0.7. Let k be a field and X a smooth projective curve over k with $H^0(X, \mathcal{O}_X) = k$ and $H^1(X, \mathcal{O}_X) = 0$ i.e. $g = 0$.

Suppose that X has a k -rational point $p \in X$. Then consider $\mathcal{L} = \mathcal{O}_X(p)$ so $\deg \mathcal{L} = 1$. Therefore, we have $H^1(X, \mathcal{L}) = 0$ so by Riemann-Roch,

$$\dim_k H^0(X, \mathcal{L}) = 2$$

Therefore, \mathcal{L} has two independent sections and thus defines a rational map $X \dashrightarrow \mathbb{P}^1$ of degree one. Since both are smooth and projective this extends to a morphism $X \rightarrow \mathbb{P}^1$. However, we can also use the following. Suppose that s_0, s_1 both vanish at q . Then consider $\mathcal{O}_X(p - q)$ which has sections s_0 and s_1 but still we have $H^1(X, \mathcal{O}_X(p - q)) = 0$ but by Riemann-Roch,

$$\dim_k H^0(X, \mathcal{O}_X(p - q)) = 1$$

a contradiction so s_0 and s_1 cannot all vanish. Then $X \rightarrow \mathbb{P}_k^1$ is degree one so it is an isomorphism.

Lemma 19.0.8. Let $f : X \rightarrow Y$ be a nonconstant morphism of normal projective curves. Then,

$$\deg(f^*D) = \deg f \cdot \deg D$$

Remark. Let X be smooth projective of genus zero. Then consider $\mathcal{L} = \omega_X^\vee$. Then $\deg \mathcal{L} = 2$ so X is a conic plane curve by Riemann Roch.

20 April 23

20.1 Plane Curves

Definition 20.1.1. A *plane curve* is a curve with a closed embedding $X \hookrightarrow \mathbb{P}_k^2$.

Remark. \mathbb{P}_k^2 is smooth so $X \subset \mathbb{P}_k^2$ is a prime divisor and actually an effective Cartier divisor. For $F \in k[T_0, T_1, T_2]$ of degree d , actually X is an effective Cartier divisor and $\mathcal{O}_{\mathbb{P}_k^2}(X) \cong \mathcal{O}_{\mathbb{P}_k^2}(d)$.

Definition 20.1.2. Let X be a scheme. An *effective Cartier divisor* is a closed subscheme $D \subset X$ which is locally cut out by one nonzerodivisor. Meaning for each $x \in X$ we have $x \in U = \text{Spec}(A)$ and $D \cap U = \text{Spec}(A/(f))$ for f a nonzero divisor. In particular, $\mathcal{O}_X(D)$ is invertible.

Then $\mathcal{O}_X(D) = \mathcal{I}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X)$ comes with a canonical section $1 : \mathcal{I} \hookrightarrow \mathcal{O}_X$ whose divisor of zeros is D .

Remark. For a plane curve $X \subset \mathbb{P}_k^2$ there is $\mathcal{I}_X = \mathcal{O}_{\mathbb{P}_k^2}(-d)$. We get SES,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Then from the long exact sequence we find,

$$H^0(X, \mathcal{O}_X) = k \quad g = \dim_k H^1(X, \mathcal{O}_X) = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$$

Remark. For $X \subset \mathbb{P}^N$ a variety, the Weil divisor class of $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$ is often denoted H and then people write $\mathcal{O}_X(H) = \mathcal{O}_X(1)$.

20.2 Genus Zero

Proposition 20.2.1. Let X be a smooth projective curve with $H^0(X, \mathcal{O}_X) = k$ and $g = 0$. Then X is a plane conic.

Proof. We choose $\mathcal{L} = \omega_X^{\otimes -2}$. By Riemann-Roch and trivial vanishing $\dim_k H^0(X, \mathcal{L}) = 3$. Pick a basis $s_0, s_1, s_2 \in H^0(X, \mathcal{L})$. Take D_0 , the vanishing of s_0 . There is an exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_{D_0} \longrightarrow 0$$

Therefore, s_1, s_2 generate $\mathcal{L}|_{D_0}$ because if s_1, s_2 both vanish on D_0 then they must be dependent. Therefore, s_0, s_1, s_2 generate \mathcal{L} over X . Then we get a morphism $\phi : X \rightarrow \mathbb{P}_k^2$. I claim, ϕ is a closed immersion and the image is a conic.

We may check this after base change to \bar{k} . Then $X \cong \mathbb{P}_k^1$ and $\mathcal{L} = \mathcal{O}_X(2)$ and can check this directly. \square

Lemma 20.2.2. Let $f : X \rightarrow Y$ be a morphism of varieties over k . Then $f : X \rightarrow Y$ has property P iff $f_{\bar{k}} : X_{\bar{k}} \rightarrow Y_{\bar{k}}$ has property P for the following properties,

- (a) closed immersion
- (b) open immersion
- (c) flat
- (d) smooth

Furthermore, for coherent \mathcal{F} and $i \geq 0$,

$$H^i(X, \mathcal{F}) \otimes_k \bar{k} = H^i(X_{\bar{k}}, \mathcal{F}_{\bar{k}})$$

20.3 Separating Points and Tangent Spaces

Let $k = \bar{k}$ and X projective variety over k . Choose $\mathcal{L} \in \text{Pic}(X)$ and $V \subset H^0(X, \mathcal{L})$ a k -vectorspace. If V generates \mathcal{L} i.e. it defines a surjective morphism $\mathcal{O}_X^V \rightarrow \mathcal{L}$. This gives a morphism,

$$\phi_V : X \rightarrow \mathbb{P}(V) = \text{Proj}(\text{Sym}_k(V)) \cong \mathbb{P}_k^{\dim V - 1}$$

Definition 20.3.1. We say that V *separates points* if $\forall x, y \in X : \exists s \in V : s(x) = 0$ and $s(y) \neq 0$

Now consider the cotangent space,

$$T_x^* = \mathfrak{m}_x / \mathfrak{m}_x^2 = \Omega_{X/k, x}^1 \otimes_{\mathcal{O}_{X, x}} \kappa(x)$$

Furthermore, the tangent space is its dual,

$$T_x X = \text{Hom}_{\kappa(x)}(\mathfrak{m}_x / \mathfrak{m}_x^2, \kappa(x)) = \text{Hom}_{\mathcal{O}_{X, x}}(\Omega_{X/k}^1, \kappa(x)) = \text{Der}_k(\mathcal{O}_{X, x}, \kappa(x))$$

We say that V *separates tangent vectors* if $\forall \vartheta \in T_x : \exists s : s = 0$ and one of the following equivalent conditions,

- (a) $\vartheta(s) \neq 0$
- (b) $s_x \in \mathcal{L}_x / I_{\vartheta} \mathcal{L}_x$ is nonzero
- (c) for any choice isomorphism $\psi : \mathcal{L}_X \xrightarrow{\sim} \mathcal{O}_{X, x}$ we have $\vartheta(\psi(s_x)) \neq 0$
- (d) $\vartheta^*(s)$ is not zero in $\Gamma(\text{Spec}(k[\epsilon]), \vartheta^* \mathcal{L})$

Remark. Ways to think about tangent vectors,

- (a) a derivation $\vartheta : \mathcal{O}_{X, x} \rightarrow k$
- (b) an Ideal $\mathfrak{m}_x^2 \subset I \subset \mathfrak{m}_x$ given by $I / \mathfrak{m}_x^2 = \ker \vartheta$
- (c) a closed subscheme defined by the ideal I
- (d) a morphism $\text{Spec}(k[\epsilon]) \rightarrow X$

Lemma 20.3.2. If V separates points and tangent vectors then ϕ_V is a closed immersion. (Requires X projective and $k = \bar{k}$).

20.4 Linear Systems

Let X be a smooth projective curve over $k = \bar{k}$.

Definition 20.4.1. A *linear series* \mathfrak{g}_d^r of degree d and dimension r is a pair (\mathcal{L}, V) as above with $\deg \mathcal{L} = d$ and $\dim_k(V) = r + 1$.

Example 20.4.2. $(\mathcal{O}_{\mathbb{P}}(1), H^0(\mathcal{O}_{\mathbb{P}}(1)))$ is a \mathfrak{g}_1^1 .

Proposition 20.4.3. If $g_X > 0$ then there is no \mathfrak{g}_1^1 .

Remark. For a linear system, (\mathcal{L}, V) we introduce the following notation,

- (a) $V(-x) = \{s \in V \mid s(x) = 0\}$

$$(b) \ V(-x - y) = \{s \in V \mid s(x) = s(y) = 0\}$$

$$(c) \ v(-2x) = \{s \in V \mid s_x \in \mathfrak{m}_x^2\}$$

Proposition 20.4.4. The linear system (\mathcal{L}, V) gives a closed immersion $X \hookrightarrow \mathbb{P}(V)$ if,

$$(a) \ \forall x \in X \text{ closed } \dim_k V(-x) = \dim V - 1$$

$$(b) \ \forall x, y \in X \text{ closed (allow } x = y) \text{ we have,}$$

$$\dim_k V(-x - y) = \dim V - 2$$

21 Riemann-Hurevich

Let $k = \bar{k}$ and $f : X \rightarrow Y$ be a nonconstant morphism of smooth curves.

There are two maps,

$$f^* \Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

and also a trace map,

$$f_* \omega_X \longrightarrow \omega_Y$$

This is defined by,

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_* \omega_X, \omega_Y) = H^1(Y, f_* \omega_X)^\vee$$

and because f is finite and thus affine,

$$H^1(Y, f_* \omega_X)^\vee = H^1(X, \omega_X)^\vee$$

so we take the standard trace $t : H^1(X, \omega_X) \rightarrow k$.

Example 21.0.1. Let k have characteristic p . Look at,

$$\begin{array}{ccc} \mathbb{P}_k^1 & \longrightarrow & \mathbb{P}_k^1 \\ \uparrow & & \uparrow \\ \mathrm{Spec}(k[x]) & \longrightarrow & \mathrm{Spec}(k[x]) \end{array}$$

sending $x \mapsto x^p$. Then the map $f^* \Omega_Y \rightarrow \Omega_X$ is the zero map.

Lemma 21.0.2. Let $f : X \rightarrow Y$ be a morphism of smooth projective curves over k (any field). The following are equivalent,

$$(a) \ f^* \Omega_{Y/k} \rightarrow \Omega_{X/k}^1 \text{ nonzero}$$

$$(b) \ \Omega_{X/Y} \text{ supported at finitely many points}$$

$$(c) \ \text{there exists an open } U \subset X \text{ s.t. } f|_U : U \rightarrow Y \text{ is étale}$$

$$(d) \ k(X)/k(Y) \text{ is separable}$$

We assume the lemma holds. Then,

$$0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

which is injective because $f^*\Omega_Y^1 \rightarrow \Omega_X$ is a nonzero map of line bundles.

We know,

$$\begin{aligned} \deg \Omega_X &= 2g_X - 2 \\ \deg \Omega_Y &= 2g_Y - 2 \\ \deg f^*\Omega_Y &: \deg f(2g_Y - 2) \\ \deg \Omega_{X/Y} &\geq 0 \end{aligned}$$

So we find,

$$2g_X - 2 \geq \deg f(2g_Y - 2)$$

Lemma 21.0.3. Let $x \in X$ be a closed point. Let $s \in \mathcal{O}_{X,x}$ be a uniformizer. Then $ds \in \Omega_{X/k}$ is a local generator.

Recall that e_x is the ramification index of $f : X \rightarrow Y$ at x defined by $\mathfrak{m}_y \cdot \mathcal{O}_{X,x} = \mathfrak{m}_x^{e_x}$. Choose uniformizers $t \in \mathcal{O}_{Y,y}$ and $u \in \mathcal{O}_{X,x}$ then $t \mapsto u \cdot s^e$ with $u \in \mathcal{O}_{X,x}^\times$. Then,

$$dt \mapsto d(us^e) = dus^e + eus^{e-1}ds = \left(s^e \frac{du}{ds} + eus^{e-1} \right) ds$$

Therefore,

$$(\Omega_{X/Y})_x \cong \mathcal{O}_{X,x} / \left(eus^{e-1} + \frac{du}{ds}s^e \right) \mathcal{O}_{X,x}$$

This has length $e - 1$ unless $e = 0$ in k which case it is larger than e . Putting everything together,

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \sum_{x \in X} \begin{cases} e_x - 1 & e_x \in k^\times \\ > e_x - 1 & e_x = 0 \end{cases}$$

In particular, in characteristic zero,

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \sum_{x \in X} (e_x - 1)$$

and furthermore,

$$K_X = f^*K_Y + \sum_{x \in X} (e_x - 1)[x]$$

in $\text{Cl}(X) = \text{Pic}(X)$.