ASTR GR6001 Radiative Processes
Assignment # 5

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1 Problem 1

Consider a gas of nitrogen molecules with diameter d that collide with mean free path ℓ.

(a)

We suppose that at each collision, the mollecules come off with random velocities. The time between colisions will be distributed exponentially,

$$P(t) \propto e^{-t/c_s \ell}$$

where c_s is the sound speed. Then the time between colisions is, on average $t = \ell / c_s$. Therefore, the number of collisions is,

$$N = \frac{tc_s}{\ell}$$

Now we can calculate the RMS distance traveled,

$$\begin{aligned} \left\langle \vec{r}^2 \right\rangle &= \left\langle \left(\sum_{i=1}^{N} \Delta \vec{r}_i^2 \right) \right\rangle \\ &= \sum_{i,j} \left\langle \Delta \vec{r}_i \cdot \Delta \vec{r}_j \right\rangle \end{aligned}$$

Since we have assumed that collisions erase all correlations in velocity,

$$\left\langle \Delta \vec{r}_i \cdot \Delta \vec{r}_j \right\rangle = \ell^2 \delta_{ij}$$

and therefore,

$$\left\langle \vec{r}^2 \right\rangle = \sum_{i,j} \left\langle \Delta \vec{r}_i \cdot \Delta \vec{r}_j \right\rangle = \sum_{i=1}^{N} \ell^2 = N \ell^2$$

and therefore the RMS distance traveled is,

$$R_{\text{RMS}} = \ell \sqrt{N} = \ell \sqrt{\frac{tc_s}{\ell}} = \sqrt{\ell t c_s}$$

Since we know c_s if we can measure the distance traveled in a time t this determines ℓ. This can be measured via the rate of diffusion of particles in the gas. For example, a highly volitile odorous chemical can be released on one side of a room and the time it takes before someobdy on the other side can smell it measured. Note that this this case the sound speed c_s should be modified to be the RMS speed of the volitile particle from the Maxwell-Boltzman distribution. This corresponds simply to correcting c_s by the squareroot of the mass ratio of nitrogen molecules to the volitile chemical.

(b)

We assume that in the liquid state that the distance between molecules is approximatly d. Therefore, we can compute the number of particles as,

$$N = \frac{V_{\text{liquid}}}{\left(\frac{1}{6}\pi d^3\right)}$$

Therefore,

$$n = \frac{V_{\text{liquid}}}{V_{\text{gas}}} \cdot \frac{1}{\left(\frac{1}{6}\pi d^3\right)}$$

Now we can compute d from the mean free path. Let σ be the scattering cross section for nitrogen molecules in the gas as,

$$\ell = (\sigma n)^{-1}$$

Now we estimate,

$$\sigma \approx \pi d^2$$

Therefore,

$$d = \frac{1}{\sqrt{\pi \ell n}}$$

Now plugging in,

$$n = \frac{V_{\text{liquid}}}{V_{\text{gas}}} \cdot \frac{6(\pi \ell n)^{\frac{3}{2}}}{\pi}$$

Rearanging, we find that,

$$n^{-\frac{1}{2}} = \frac{V_{\text{liquid}}}{V_{\text{gas}}}(6\pi^{\frac{1}{2}}\ell^{\frac{3}{2}})$$

and therefore,

$$n = \left(\frac{V_{\text{gas}}}{V_{\text{liquid}}}\right)^2 \cdot \left(\frac{1}{36\pi}\right) \cdot \frac{1}{\ell^3}$$

2 Problem 2

The classical treatment of cyclotron radiation is valid for non-relativistic velocities and orbital angular momentum ℓ much greater than $\hbar$.

(a)

In the classical treatment we have an acceleration,

$$a = \frac{v_\perp^2}{r} = \frac{e v_\perp}{mc} B$$

and thus,

$$r = \frac{m c v_\perp}{eB}$$

Furthermore, the orbital angular momentum is then,

$$L = m v_\perp r = \frac{m^2 v_\perp^2 c}{eB}$$

Therefore,

$$B = \frac{m^2 v_\perp^2 c}{eL}$$

Now we must have $v_\perp \ll c$ and $L \gg \hbar$ for the classical approximation to be valid. This implies that for the classical approximation to hold we must have,

$$B \ll B_{\max} = \frac{m^2 c^3}{e\hbar} = \frac{e}{r_0 \lambda\!\!\!^{-}_e} = 4.9 \cdot 10^{13}\ \text{esu cm}^{-2}$$

(b)

The frequency of the cyclotron radiation is,

$$\omega = \frac{v_\perp}{r} = \frac{eB}{mc}$$

Now let,

$$B = \frac{1}{10} B_{\max} = \frac{1}{10} \cdot \frac{m^2 c^3}{e\hbar}$$

and thus,

$$\omega = \frac{1}{10} \cdot \frac{mc^2}{\hbar} = \frac{1}{10} \cdot \frac{c}{\lambda\!\!\!^{-}_e} = 1.2 \cdot 10^{19}\ \text{Hz}$$

3 Problem 3

We know that, the spontaneous emission rate is given by radiation damping,

$$A_{21} = 3\Gamma = \frac{8\pi^2 e^2}{mc\lambda_0^2} f_{21} = \frac{8\pi^2 r_0 c}{\lambda_0^2} f_{21} \qquad g_1 f_{12} = g_2 f_{21}$$

given,

$$r_0 = 2.82 \cdot 10^{-13}\ \text{cm}$$

so we have,

$$\Gamma = 2.23\ \text{GHz} \left(\frac{\lambda_0}{1000\ \text{Å}} \right)^{-2} f_{21}$$

In the case of Lyα transition $n = 2 \mapsto 1$ we have $\lambda_0 = 1215.7$ Å and $g_2 = 6$ and $g_1 = 2$ since there are three oribital angular momentum states at level $n = 2$. Thus,

$$A_{21} = 6.26 \cdot 10^8\ \text{s}^{-1}$$

Furthermore,

$$\begin{aligned} B_{21} &= \frac{c^2}{2h\nu_0^3} A_{21} = \frac{\lambda_0^3}{2hc} A_{21} \\ &= \frac{4\pi^2 r_0}{h} \lambda_0 f_{21} = (2.33 \cdot 10^9\ \text{cm}^2\ \text{s Hz sr erg}^{-1}\ \text{s}^{-1}) \left(\frac{\lambda_0}{1000\text{Å}} \right) \end{aligned}$$

Therefore, in our case,

$$B_{21} = 2.83 \cdot 10^9\ \text{cm}^2\ \text{s Hz sr erg}^{-1}\ \text{s}^{-1}$$

and finally,

$$B_{12} = B_{21}\frac{g_2}{g_1} = 3B_{21} = (7.00 \cdot 10^9 \text{ cm}^2 \text{ s Hz sr erg}^{-1} \text{ s}^{-1}) \left(\frac{\lambda_0}{1000\text{Å}}\right)$$

so in the case of Lyα we have,

$$B_{12} = 8.49 \cdot 10^8 \text{ cm}^2 \text{ s Hz sr erg}^{-1} \text{ s}^{-1}$$

Consider a Hdrogenic atom of atomic number Z. For the $n = 2 \mapsto 1$ transition we have an analogous line. The oscillator stengths f_{12} and f_{21} and the multiplicities g_1 and g_2 are determined by the geometry and quantum numbers which are unchanged. The only change in the above expressions is the energy difference and thus the wavelength λ_0. Now since the energy levels are,

$$E_n = -\frac{me^4 Z^2}{2\hbar^2} \cdot \frac{1}{n^2}$$

and thus we have,

$$\lambda_0 = \frac{hc}{E_2 - E_1} \propto Z^{-2}$$

Therefore, we have,

$$\begin{aligned} A_{21} &= (6.26 \cdot 10^8 \text{ s}^{-1})\, Z^4 \\ B_{21} &= (2.83 \cdot 10^9 \text{ cm}^2 \text{ s Hz sr erg}^{-1} \text{ s}^{-1})\, Z^{-2} \\ B_{12} &= (8.49 \cdot 10^8 \text{ cm}^2 \text{ s Hz sr erg}^{-1} \text{ s}^{-1})\, Z^{-2} \end{aligned}$$

4 Problem 4

The cross section is given by,

$$\sigma(\nu) = \frac{\pi e^2}{mc}\, f_{12}\, \phi(\nu) = \pi r_0 c\, f_{12}\, \phi(\nu)$$

where $\phi(\nu)$ is the line width computed as a convolution of the Lorentizan natural line profile with the Gaussian dopler brodening,

$$\begin{aligned} \phi_{\text{nat}}(\nu) &= \frac{\frac{\Gamma}{4\pi^2}}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} \\ \phi_D(\nu) &= \frac{1}{\Delta\nu\sqrt{\pi}} e^{-\left(\frac{\nu-\nu_0}{\Delta\nu}\right)^2} \end{aligned}$$

where,

$$\Gamma = \frac{8\pi^2 e^2}{3mc\lambda_0^2}\, f_{21} = \frac{8\pi^2 r_0 c}{3\lambda_0^2}\, f_{21} \qquad \Delta\nu = \frac{b\nu_0}{c} = \sqrt{\frac{2kT}{m_H}}\left(\frac{\nu_0}{c}\right) \qquad g_1 f_{12} = g_2 f_{21}$$

Then,

$$\phi = \phi_{\text{nat}} * \phi_D$$

First, for $T = 0$ we have $\Delta\nu \to 0$ and thus $\phi = \phi_{\text{nat}}$. In particular,

$$\phi(\nu_0) = \phi_{\text{nat}}(\nu_0) = \frac{4}{\Gamma}$$

Therefore,

$$\sigma(\nu_0) = \frac{4\pi r_0 c}{\Gamma} f_{12} = \frac{3\lambda_0^2}{2\pi} \cdot \frac{f_{12}}{f_{21}} = \frac{3\lambda_0^2}{2\pi} \cdot \frac{g_2}{g_1}$$

In our case, we have $\lambda_0 = 1215.7$ Å. For the $n = 2 \mapsto n = 1$ transition, there are 4 states at $n = 2$ and 2 at $n = 1$ so $g_2/g_1 = 3$. Therefore,

$$\sigma(\nu_0) = 2.12 \cdot 10^{-10}\ \text{cm}^2$$

For $T > 0$ we need to compute the convolution,

$$\phi = \phi_{\text{nat}} * \phi_D$$

and thus,

$$\begin{aligned}\phi(\nu_0) &= \frac{4\pi}{\pi\Gamma\Delta\nu\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{\nu}{\Delta\nu}\right)^2}}{1 + \left(\frac{4\pi\nu}{\Gamma}\right)^2}\, \mathrm{d}\nu = \frac{\sqrt{\pi}}{\Delta\nu} e^{\left(\frac{\Gamma}{4\pi\Delta\nu}\right)^2} \left(1 - \operatorname{erf}\left[\frac{\Gamma}{4\pi\Delta\nu}\right]\right) \\ &= \frac{1}{\Delta\nu\sqrt{\pi}} \left(1 - \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma}{4\pi\Delta\nu}\right] + \left[\frac{\Gamma}{4\pi\Delta\nu}\right]^2 + O\left(\left[\frac{\Gamma}{4\pi\Delta\nu}\right]^3\right)\right)\end{aligned}$$

Now we know that,

$$r_0 = 2.82 \cdot 10^{-13}\ \text{cm}$$

Therefore,

$$\Gamma = 2.23\ \text{GHz} \left(\frac{\lambda_0}{1000\ \text{Å}}\right)^{-2} f_{21} \qquad \Delta\nu = 12.8\ \text{GHz} \left(\frac{T}{100\ \text{K}}\right)^{\frac{1}{2}} \cdot \left(\frac{\lambda_0}{1000\ \text{Å}}\right)^{-1}$$

and so we have,

$$y = \frac{\Gamma}{4\pi\Delta\nu} = 1.39 \cdot 10^{-3} \cdot \left(\frac{T}{100\ \text{K}}\right)^{-\frac{1}{2}} \cdot \left(\frac{\lambda_0}{1000\ \text{Å}}\right)^{-1} f_{21}$$

Now,

$$\sigma(\nu_0) = \frac{\sqrt{\pi} r_0 c\, f_{12}}{\Delta\nu} \phi(\nu_0) = \frac{\sqrt{\pi} r_0 c\, f_{12}}{\Delta\nu} \left(1 - \frac{2}{\sqrt{\pi}}\sqrt{\pi}\, y + y^2 + O(y^3)\right)$$

Plugging in,

$$\sigma(\nu_0) = 1.17 \cdot 10^{-12}\ \text{cm}^2 \left(\frac{T}{100\ \text{K}}\right)^{-\frac{1}{2}} \cdot \left(\frac{\lambda_0}{1000\ \text{Å}}\right) f_{12} \left(1 - \frac{2}{\sqrt{\pi}}\sqrt{\pi}\, y + y^2 + O(y^3)\right)$$

In our case,

$$\begin{aligned}\lambda_0 &= 1215.7\ \text{Å} \\ T &= 1000\ \text{K} \\ f_{12} &= 0.4216 \\ f_{21} &= 0.1388\end{aligned}$$

Therefore,

$$y = 4.96 \cdot 10^{-4}$$

and thus,

$$\sigma(\nu_0) = 1.86 \cdot 10^{-13}\,\mathrm{cm}^2$$