# 1 Toric Construction of Models

In this section we discuss the results of [Tim, Models] (2018) which gives a method of explicitly constructing a regular normal crossings model of a curve and explicilty describing its special fiber using the preceeding methods characterizing curves on toric surfaces.

*Remark.* In [Tim, Models], Dokchitser uses "curve" to refer to an integral separated *geometrically connected* scheme of finite type over a field. To mitigate any confusion, we render any results quoted from his work with "curve" replaced by "geometrically connected curve" when necessary. (OKAY THIS MAY BE WRONG!!)

## 1.1 Notations and Definitions

We work in the case of a discretely definition valued field $K$ with valuation $\nu : K^\times \twoheadrightarrow \mathbb{Z}$, valuation ring $R$, uniformizer $\varpi$, and residue field $\kappa$. Given a smooth projective geometrically connected curve over $K$ our goal will be to construct a regular normal crossings model over $R$. First we need to fix some notation.

**Definition 1.1.** Given a Laruent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$ recall the Newton polygon is,

$$\Delta(f) = \mathrm{Conv}\left(\{(i,j) \in \mathbb{Z}^2 \mid a_{ij} \neq 0\}\right) \subset \mathbb{R}^2$$

We will assume througout that $\mathrm{vol}\,(\Delta) > 0$. Now we refine the Newton polygon with respect to the valuation $\nu : K^\times \twoheadrightarrow \mathbb{Z}$,

$$\Delta_\nu(f) = \mathrm{LowerConvHull}\left(\{(i,j,v(a_{ij})) \mid (i,j) \in \Delta(f) \cap \mathbb{Z}^2\}\right) \subset \mathbb{R}^2 \times \mathbb{R}$$

The projection $\pi : \Delta_\nu \to \Delta$ is a homeomorphism. Thus, for each point $P \in \Delta$ there is a unique point $(P, \nu(P)) \in \Delta_\nu$ which defines a piecewise affine function $v : \Delta \to \mathbb{R}$ extending the valuation.

The bijection $\pi : \Delta_\nu \to \Delta$ pushes the polyhedral structure on $\Delta_\nu$ onto $\Delta$. Because $\Delta_\nu$ is the lower convex hull of finitely many points in $\mathbb{R}^2 \times \mathbb{R}$ it decompses into faces of dimension $0, 1, 2$. Under the projection $\pi : \Delta_\nu \to \Delta$ the *v-vertices* $P$ of $\Delta$ are the images of the 0-faces, the *v-edges* $L$ are the images of the 1-faces, the *v-faces* $F$ are the images of the 2-faces. These define a polygonal partition of $\Delta$.

**Definition 1.2.** For each edge $L$ and face $F$ there is an associated integer $\delta_\lambda$ (with $\lambda = L$ or $F$), the *denominator*, defined as smallest positive $m$ such that $\nu(P) \in \frac{1}{m}\mathbb{Z}$ for each $P \in \lambda \cap \mathbb{Z}^2$.

*Remark.* We now consider how to restrict a polynomial with respect to the $\nu$-partition to form a Laurent polynomial supported on the faces and vertices. First, following the Notation of [Tim, Models] we define how to restrict the polynomial to some subset of a lattice.

**Definition 1.3** (Restriction)**.** Let $S \subset \mathbb{Z}^n$ be a nonempty subset of a lattice and take $\Lambda$ to be the smallest affine lattice $S \subset \Lambda \subset \mathbb{Z}^n$ containing $S$. Let $\Lambda$ have rank $r$ and choose an isomorphism $\phi : \mathbb{Z}^r \to \Lambda$. Then for a Laurent polynomial $g \in K[\mathbf{x}^{\pm 1}]$ we define the restriction $g|_S \in K[\mathbf{y}^{\pm 1}]$,

$$g|_S = \sum_{\mathbf{i} \in \phi^{-1}(S)} c_{\phi(\mathbf{i})} \mathbf{y}^{\mathbf{i}} \in K[\mathbf{y}^{\pm 1}] \quad \text{for} \quad g = \sum_{\mathbf{i} \in \mathbb{Z}^n} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$$

Note that different choices of an isomorphism $\Lambda \to \mathbb{Z}^r$ are related by an automorphism in $\mathrm{GL}_r(\mathbb{Z})$ acting on the variables $\mathbf{y}$.

*Remark.* The notational complexity of the above definition derives from making the polynomial $g|_S$ an element of a standard Laurent polynomial ring $k[\mathbf{y}^{\pm 1}]$. We can simplify the above notation using our previous abstract notation used for the toric constructions. Given a lattice $M$ and a Laurent polynomial $g \in K[M]$ and a subset $S \subset M$ we define the restriction,

$$g|_S = \sum_{m \in S} c_m \chi^m \in K[\langle S \rangle] \quad \text{where} \quad g = \sum_{m \in M} c_m \chi^m$$

where $\langle S \rangle$ is the sublattice of $M$ generated by $S$. The above definition is recovered choosing some isomorphism $\phi : \langle S \rangle \to \mathbb{Z}^r$ giving an isomorphism $K[\langle S \rangle] \cong K[\mathbf{y}]$.

**Definition 1.4** (Reduction)**.** For a Laurent polynomial $h \in K[x^{\pm 1}, y^{\pm 1}]$, there exist integers, $c, m, n \in \mathbb{Z}$ such that $\tilde{h}(x,y) = \varpi^c h(\varpi^m x, \varpi^n y)$ has coefficients in $R$ and $\tilde{h} \mod \varpi \in \kappa[x,y]$ has the same Newton polygon as $h$. Then we say that $\overline{h} = \tilde{h} \mod \varpi$ is *reduction* of $h$.

**Example 1.5.** (DO THIS!!!!!!!)

**Definition 1.6.** In paticular for $\lambda$ an edge $L$ or face $F$ we define the restriction $f|_\lambda = f|_S$ for the set, $S = \{P \in \lambda \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}$. Note, $S$ contains the vertices of $L$ or $F$.

*Remark.* Reduction gives, for each edge $L$ and face $F$, polynomials $\overline{f|_L} \in \kappa[t]$ and $\overline{f|_F} \in \kappa[x,y]$. This gives affine curves over $\kappa$ on each edge and face which we complete in a toric compactification as follows.

**Definition 1.7** (Components)**.** We define the following schemes over $\kappa$:

(a). $X_L = V(\overline{f_L}) \subset \mathbb{G}_{m,\kappa}$

(b). $X_F = V(\overline{f_F}) \subset \mathbb{G}_{m,\kappa}^2$

(c). $\overline{X}_F = X_F^\Delta$ is the completion of $X_F$ with respect to its Newton polygon $F$ i.e. the closure of the immersion $X_F \hookrightarrow \mathbb{G}_{m,\kappa}^2 \hookrightarrow \mathbb{T}_F$. By Theorem **??**, $\overline{X}_F$ is connected and, in fact, the Theorem applies for any finite extension $\kappa' \supset \kappa$ showing that $\overline{X}_F$ is geometrically connected.

**Example 1.8.** (DO EXAMPLE!!!!!!)

**Definition 1.9.** We say that $f \in k[x^{\pm 1}, y^{\pm 1}]$ is *strictly* $\Delta_\nu$*-regular* if all $X_F$ and $X_L$ are smooth over $\kappa$.

*Remark.* The condition that all $X_L$ are smooth implies that $f$ is nondegenerate with respect to its Newton polgon since it implies that $f$ restricted to each edge is smooth.

**Definition 1.10.** A Laurent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$ is $\Delta_\nu$*-regular* if each $X_F$ is smooth and for the interior edges $L$ and edges $L \subset \partial F$ with $\delta_L \neq \delta_F$ we require $X_L$ is smooth and otherwise we require $\overline{X}_F$ is *outer-regular* i.e. smooth at the points corresponding to $L$ via,

$$\overline{X}_F(\bar{\kappa}) \setminus X_K(\bar{\kappa}) \longleftrightarrow \coprod_{L \supset \partial F} X_L(\bar{\kappa})$$

*Remark.* As with toric regularity, we have defined the notion of $\Delta_\nu$-regular with respect to a given Lauret polynomial i.e. to a given affine model $C_0$ of a curve. As before, we extend this definition to an arbitrary curve in the obvious way.

**Definition 1.11.** A curve $C$ over $K$ is (strictly) $\Delta_\nu$-regular if $C$ is birationally equivalent to some affine $U_f \subset \mathbb{G}^2_{m,K}$ for some (strictly) $\Delta_\nu$-regular Laruent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$.

*Remark.* We need one more notion in order to describe the model of $C$ which is a combinatorial connectivity between two adjacent faces $F_1, F_2$ sharing an edge $L$.

**Definition 1.12** (Slopes)**.** Edges are either *inner/interior* meaning they form the boundary between two $\nu$-faces $F_1$ and $F_2$ or *outer/exterior* on the boundary of $\Delta$. For an edge $L$ there is a unique affine function $L^*_{(F_1)} : \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ with $L^*_{(F_1)}|_L = 0$ and $L^*_{(F_1)}|_{F_1} \geq 0$. Then the edge has two corresponding integers called the *slopes* Defined as follows. Choose $P_0, P_1 \in \mathbb{Z}^2$ with $L^*_{(F_1)}(P_0) = 0$ and $L^*_{(F_1)}(P_1) = 1$. Then,

$$s_1^L = \delta_L(\nu_1(P_1) - \nu_1(P_0)) \qquad s_2^L = \begin{cases} \delta_L(\nu_2(P_1) - \nu_2(P_0)) & L \text{ inner} \\ \lfloor s_1^L - 1 \rfloor & L \text{ outer} \end{cases}$$

where $\nu_i : \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ is the unique affine function which agrees with $\nu$ on $F_i$ (recall that the faces are defined such that $\nu$ is affine when restricted to each face. Given the slopes, we may consider a sequence of rational numbers $\frac{m_i}{d_i} \in \mathbb{Q}$ such that,

$$s_1^L = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \frac{m_2}{d_2} > \cdots > \frac{m_r}{d_r} > \frac{m_{r+1}}{d_{r+1}} \quad \text{and} \quad \begin{vmatrix} m_i & m_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = m_i d_{i+1} - m_{i+1} d_i = 1$$

Then $r(L)$, the minimal length of this sequence, and the denominators $d_i$ of this minimal sequence, are important combinatorial parameter of the edge $L$. It turns out such a minimal sequence is unique.

*Remark.* The existence of such a sequence needs some consideration. Take all rational numbers in $[s_1^L, s_2^L] \cap \mathbb{Q}$ with denominators bounded by the largest denominator of $s_1^L$ and $s_2^L$. This is a shifted Farey series. We define the Farey series $F^n$ to be the ordered sequence of rational numbers with denominator less than or equal to $n$ put in lowest terms. Then, if $\frac{a}{b} < \frac{c}{d}$ are consecutive terms in the Farey series then $\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}$. Therefore,

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc = 1$$

[Tim, Remark 3.15] [HW, Ch. III, Thm. 28, Thm. 29] Furthermore, if the sequence in $[s_1^L, s_2^L] \cap \mathbb{Q}$ with bounded denominators contains consecutive terms,

$$\frac{a}{b} > \frac{a+c}{b+d} > \frac{b}{d}$$

then we must have,

$$a(b+d) - b(a+c) = ab + ad - ab - bc = ad - bc = 1$$

meaning that $\frac{a}{b} > \frac{b}{d}$ have the required adjacency property and thus $\frac{a+c}{b+d}$ may be removed from the sequence. We will reinterpret this as a blowdown of regular normal crossings models after we state the main theorems describing the structure of such models from the above combinatorial data.

## 1.2 Main Theorems

We describe the properties of the model $\mathcal{C}_\Delta$ over $R$ associated to some polygonal parition of $\Delta$ defined by a Laurent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$.

**Theorem 1.13** (DK, Thm. 3.13)**.** Let $C$ be a smooth projective $\Delta_\nu$-regular curve birational to $U_f$ for a $\Delta_\nu$-regular Laurent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$. Then $\mathcal{C}_\Delta/R$ is a regular normal crossing model of $C$ and the special fiber $\mathcal{C}_\kappa$ geometrically decomposes into components as follows:

(a). each $\nu$-face $F$ of $\Delta$ gives a smooth complete curve $\overline{X}_F \times_\kappa \kappa^{\mathrm{sep}}$ (CORRECT?) over $\kappa^{\mathrm{sep}}$ with multiplicity $\delta_F$ and genus $g_F = |\{P \in F^\circ \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}|$

(b). each $\nu$-edge $L$ with sequence $\frac{m_i}{d_i} \in \mathbb{Q}$ $(0 \le i \le r+1)$ gives $|X_L(\kappa^{\mathrm{sep}})|$ chains of length $r$ of closed subschemes intersecting transversally each isomorphic to $\mathbb{P}^1_{\kappa^{\mathrm{sep}}}$ with multiplicities in $\mathcal{C}_\kappa$ given by $\delta_L d_1, \dots, \delta_L d_r$.

Furthermore, the $\mathrm{Gal}\,(\kappa^{\mathrm{sep}}/\kappa)$-action on $\mathcal{C}_\kappa \times_\kappa \kappa^{\mathrm{sep}}$ is given by acting on each component $X_F \times_\kappa \kappa^{\mathrm{sep}}$ and permuting the $\mathbb{P}^1_{\kappa^{\mathrm{sep}}}$ chains via the natural action of $\mathrm{Gal}\,(\kappa^{\mathrm{sep}}/\kappa)$ on $|X_L(\kappa^{\mathrm{sep}})|$.

**QUESTION 1.14.** In the paper, the decomposition is certianally geometric but he says the scheme is $\overline{X}_F$ rather than base changing over the algebraic closure. How should I write it? Also, is it the case that $\overline{X}_F \times_\kappa \kappa^{\mathrm{sep}}$ is smooth over $\kappa^{\mathrm{sep}}$? This is just the invariance of smoothness under base change right?

*Remark.* The genus of $\overline{X}_F$ is exactly the number of lattice points interior to the Newton polygon defining $X_F \subset \mathbb{G}^2_{m,\kappa}$ by Baker's theorem. Recall this Newton polygon is the the restriction of $f$ to the set $S = \{P \in F \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}$ so the lattice generated by $S$ only has lattice points where at points of $\mathbb{Z}^2$ where $\nu(P) \in \mathbb{Z}$ explaining the genus formula above.

*Remark.* (ASK JOHAN ABOUT THIS???) What about $f = \varpi x^2 + y^2 + 1$ which has $\nu(i,j) = \frac{1}{2} i$ and thus the single interior point $P$ has $\nu(P) = \frac{1}{2}$. However, $g = \varpi^2 x^2 + y^2 + 1$ has $\nu(P) = 1$.

(GIVE THIS AS EXAMPLE)

**Theorem 1.15** (DK, Thm. 3.14)**.** Let $f \in K[x^{\pm 1}, y^{\pm 1}]$ be any Laurent polynomial defining a 1-dimensional scheme $C_0 = U_f \subset \mathbb{G}^2_{m,K}$. Then $\mathcal{C}_\Delta/R$ is a proper flat model of the toric completion $C = C_0^\Delta$ with respect to the Newton polygon $\Delta = \Delta(f)$. The special fiber $\mathcal{C}_\kappa$ is a union of closed subschemes $\overline{X}_F$ indexed by $\nu$-faces $F$ and chains $X_L \times_\kappa \Gamma_L$ where $\Gamma_L$ is a union of $\mathbb{P}^1_\kappa$ intersecting transversally as follows:

(a). for each $\nu$-edge $F$: the scheme $\overline{X}_F$ has multiplicity $\delta_F$ and, via Theorem **??** are geometrically connected and have arithmetic genus $g_F = |\{P \in F^\circ \cap \mathbb{Z}^2 \mid \nu(P) \in \mathbb{Z}\}|$.

(b). for each $\nu$-edge $L$ choose a sequence $\frac{m_i}{d_i} \in \mathbb{Q}$ $(0 \le i \le r+1)$ then let $\Gamma_L = \Gamma_L^1 \cup \cdots \cup \Gamma_L^r$ with each $\Gamma_L^i$ isomorphic to $\mathbb{P}^1_k$ embedded with multiplicity $\delta_L d_i$ and meeting transversally where we identify $0 \in \Gamma_L^i$ with $\infty \in \Gamma_L^{i+1}$. If $r = 0$ then let $\Gamma_L = \mathrm{Spec}\,(k)$.

(c). The subscheme $X_L \times \{0\} \subset X_L \times \Gamma_L^1$ is identified with $\overline{X}_{F_1} \setminus X_{F_1}$ for the $\nu$-face $F_1$ boardering $L$ and, when $L$ is inner, likewise $X_L \times \{\infty\} \subset X_L \times \Gamma_L^r$ is identified with $\overline{X}_{F_2} \setminus X_{F_2}$ for the other $\nu$-face $F_2$ boardering $L$. These intersections are transversal and, in fact, as a scheme the intersection is $V(\overline{f_L}^{\delta_L}) \subset X_L \subset \overline{X}_F$.

Furthermore, the model $\mathcal{C}_\Delta$ is geometricall regular ar, (a). the smooth locus of $X_F$, for each $\nu$-face $L$

(b). the smooth locus of $X_L \times \Gamma_L$, for each $\nu$-edge $L$

(c). the smooth points of $\overline{X}_F \setminus X_F$ corresponding to $L$ when $L \subset \partial F$ is an outer edge with $\delta_L = \delta_F$ and $r = 0$.

Furthermore, if $C_0$ is $\Delta_\nu$-regular then $C = C_0^\Delta$ is smooth and thus the unique smooth proper curve birational to $C_0$ and $\mathcal{C}_\Delta/R$ is a regular normal corossings model of $C$.

# 2 Relationships Between Toric Notions of Regularity

We have discussed a number of regularity conditions on curves originating from their compatiblity in some sense with a certain set of toric embeddings. The utility of these conditions is the ability to verify them from the equation defining some affine model of the curve in $\mathbb{G}_{m,k}^2$. Although these notions are clearly related, we here show that they are, indeed, inequivalent. In this situation, we have a discrete valued field $K$ with valuation ring $R$ and residue field $\kappa$. On the special fiber, we will distinguish between the arithmetic ($\kappa$ non-algebraically closed) and geometric ($\kappa$ arithmetically closed) situations. The main result of this section is as follows.

**Proposition 2.1.** Let $C$ be a smooth curve over $K$. Then we have the following implications,

$$\begin{array}{ccc} C \text{ is strict } \Delta_\nu\text{-regular} & \Longrightarrow & C \text{ is nondegenerate} \\ \Downarrow & & \Downarrow \\ C \text{ is } \Delta_\nu\text{-regular} & \Longrightarrow & C \text{ is weakly nondegenerate} \end{array}$$

Furthermore, no implication is reversible.

The fact that $\Delta$-nondegeneracy implies weak $\Delta$-nondegeneracy is simply an application of Baker's theorem (recall that this notion was created, by design, as a weaker form of $\Delta$-nondegeneracy, hence the name). Likewise, strict $\Delta_\nu$-regularity implying $\Delta_\nu$-regularity is also a consequence of Baker's theorem since the outer-regular condition introduced in the definition of $\Delta_\nu$-regularity is satisfied when each $X_F$ is smooth and $X_L$ is smooth since these imply that $\overline{X}_F$ is smooth as well via Baker's theorem.

That $\Delta_\nu$-regularity implies weak nondegeneracy follows from the main theorem

# 3 Stuff

It does not suffice to take *any* affine open as the following example shows, we must indeed take a sufficiently small open so the notion of birationality here is actually necessary.

**Example 3.1.** There exists a smooth affine curve $C$ over $k$ with no immersion $C \hookrightarrow \mathbb{P}_k^2$ and, in particular, no immersion $C \hookrightarrow \mathbb{G}_{m,k}^2$. Thus, there are smooth affine curves which are not affine plane curves.

Take an algebraically closed field $k$. Show the following, (a). if $C \hookrightarrow \mathbb{P}^2_K$ is an immersion then there is a plane curve $\bar{C} \subset \mathbb{A}^2_k$ (closed immersion) an an open immersion $C \hookrightarrow \bar{C}$.

(b). for any closed curve $\bar{C} \subset \mathbb{A}^2_k$ we have $\Omega_{\bar{C}/k} = \mathcal{O}_C$

(c). thus, since $C \hookrightarrow \bar{C}$ is étale we have $\Omega_{C/k} = \mathcal{O}_C$ so it suffices to construct a smooth affine curve with nontrivial canonical bundle $\Omega_{C/k}$.

(d). Choose a curve $C$ with genus $g(C) \geq 2$ then $\deg \Omega_{C/k} \geq 2$ and choose a point $P \in C$ such that $K_X \not\sim (2g-2)[P]$ for any $k \in \mathbb{Z}$.

(e). Show that $U = C \setminus \{P\}$ is affine,

(f). Then $U \hookrightarrow C$ is étale so $\Omega_{U/k} = f^* \Omega_{C/k}$ so $K_C \sim [P]$.

(g). Show that this is nontrivial using the exact sequence,

$$\mathbb{Z} \longrightarrow \mathrm{Cl}\,(C) \longrightarrow \mathrm{Cl}\,(U) \longrightarrow 0$$

the first map sending $1 \mapsto [P]$ so we need to show that $K_X \not\sim (2g-2)[P]$ for any $k$.

**Lemma 3.2.** Let $C \hookrightarrow \mathbb{A}^2_k$ be a smooth curve embedded in the affine plane. Then the canoncial bundle $\Omega_{C/k} \cong \mathcal{O}_C$ is trivial.

*Proof.* Note that $C = \mathrm{Spec}\,(R)$ with $R = k[x,y]/I$ where $I = \ker\,(k[x,y] \to \Gamma(C, \mathcal{O}_C))$. Furthermore, $I = (f)$ since $\dim C = 1$ thus $\mathbf{ht}\,(I) = 1$ but $C$ is irreducible and thus $I$ is prime and since $k[x,y]$ is a UFD $I = (f)$ since each height one prime is principal. Furthermore, $C$ is smooth so $(f, f_x, f_y) = k[x,y]$ where $f_x, f_y$ are the partial derivatives of $f$ with respect to $x$ and $y$. Therefore, we can choose $g, h \in k[x,y]$ such that $g f_x + h f_y = 1$ in $R$. Now, consider the $R$-module of differentials, $\Omega_{R/k} = (R\mathrm{d}x \oplus R\mathrm{d}y)/(f_x \mathrm{d}x + f_y \mathrm{d}y)$.

Consider the map $\phi : R \to \Omega_{R/l}$ sending $1 \mapsto h\mathrm{d}x - g\mathrm{d}y$. Note that,

$$\mathrm{d}x = g f_x \mathrm{d}x + h f_y \mathrm{d}x = h f_y \mathrm{d}x - g f_y \mathrm{d}y \implies f_y \mapsto \mathrm{d}x$$

$$dy = g f_x \mathrm{d}y + h f_y \mathrm{d}y = g f_x \mathrm{d}y - h f_x \mathrm{d}x \implies -f_x \mapsto \mathrm{d}y$$

so $\phi$ is surjective. Furthermore, suppose that $\phi(a) = 0$ then $\phi(f_x a) = \phi(f_y a) = 0$ so in $R\mathrm{d}x \oplus R\mathrm{d}y$ we have $a\mathrm{d}x, a\mathrm{d}y \in (f_x \mathrm{d}x + f_y \mathrm{d}y)$ meaning $a\mathrm{d}x = c_1(f_x \mathrm{d}x + f_y \mathrm{d}y)$ and $a\mathrm{d}y = c_2(f_x \mathrm{d}x + f_y \mathrm{d}y)$ giving $c_1 f_y = 0$ and $c_2 f_x = 0$ and $c_1 f_x = c_2 f_y = a$ since $R\mathrm{d}x \oplus R\mathrm{d}y$ is free. But then

$$a = g f_x a + h f_y a = g f_x c_2 f_y + h f_y c_1 f_x = 0$$

since $c_2 f_x = c_1 f_y = 0$ so $\phi$ is injective. Thus $\Omega_{R/k} \cong R$ and sheafifying gives, $\Omega_{C/k} \cong \mathcal{O}_C$. ☐