

1 Local Cohomology

Definition 1.1. A local group or local system of groups \mathcal{L} is a locally-constant sheaf of abelian groups. We write $\mathfrak{Loc}(X)$ for the category of local systems on X .

Theorem 1.2. Let X be a locally-path-connected (AND) topological space. Then there is a equivalence of categories between the category of local groups on X and the category of actions of the fundamental groupoid $\Pi(X)$ on abelian groups.

Proof. There is a functor $\mathfrak{Loc}(X) \rightarrow \mathbf{AbGrp}^{\Pi(X)}$ sending a local system to its monodromy action. For any path $\gamma : I \rightarrow X$ and a point $\gamma(t)$ there is a open connected neighborhood $\gamma(t) \in U_t$ small enough such that $\mathcal{L}|_{U_t} \cong \underline{G}|_{U_t}$ for some abelian group G . Then $\gamma^{-1}(U_t)$ cover I which is compact so we may choose finitely many U_i which cover the path and we may assume that $U_i \cap U_{i+1} \neq \emptyset$. Then since both are connected and \mathcal{L} is constant on each we get isomorphisms,

$$\begin{array}{ccc} \mathcal{L}(U_i) & & \mathcal{L}(U_{i+1}) \\ & \searrow \sim & \swarrow \sim \\ & \mathcal{L}(W) & \end{array}$$

where W is a connected component of $U_i \cap U_{i+1}$. Thus $\mathcal{L}(U_i) \xrightarrow{\sim} \mathcal{L}(U_{i+1})$. Inductively, this gives $\mathcal{L}(U_0) \xrightarrow{\sim} \mathcal{L}(U_n)$ which, since it is well-defined after shrinking the neighborhoods admits restricting to stalks, gives the monodromy map $[\gamma] : \mathcal{L}_{\gamma(0)} \rightarrow \mathcal{L}_{\gamma(1)}$. Clearly this construction respects composition. Furthermore, we can do the exact same construction for maps $I^2 \rightarrow X$ showing that the identifications everywhere commute under homotopy. Explicitly, let $h : I^2 \rightarrow X$ be a path homotopy between $\gamma_1 : I \rightarrow X$ and $\gamma_2 : I \rightarrow X$ then for each t let $h(t, -) : I \rightarrow X$ be the path homotoping the point $\gamma_1(t)$ to $\gamma_2(t)$. Then $[h(t_2, -)] \circ [\gamma_1(t_1 \mapsto t_2)] = [\gamma_2(t_1 \mapsto t_2)] \circ [h(t_1, -)]$ as maps $\mathcal{L}_{\gamma_1(t_1)} \rightarrow \mathcal{L}_{\gamma_2(t_2)}$. Since at the endpoints $h(0, -) = h(1, -)$ is the constant path then we see that $[\gamma_1] = [\gamma_2]$. Therefore, monodromy defined a functor $M_{\mathcal{L}} : \Pi(X) \rightarrow \mathbf{AbGrp}$.

Now I claim this association $\mathcal{L} \mapsto M_{\mathcal{L}}$ is functorial. Given a morphism $\eta : \mathcal{L} \rightarrow \mathcal{L}'$ of local groups we get maps $\eta_x : \mathcal{L}_x \rightarrow \mathcal{L}'_x$ which commute with restriction and thus with the monodromy construction i.e. a natural transformation between functors $M_{\mathcal{L}}$ and $M_{\mathcal{L}'}$.

Now we need to show that $\mathcal{L} \mapsto M_{\mathcal{L}}$ is fully faithful.

Finally, $M : \mathfrak{Loc}(X) \rightarrow \mathbf{AbGrp}^{\Pi(X)}$ is essentially surjective. (PROVE THIS) □

Remark. When X is connected, then groupoid $\Pi(X)$ -representations are simply group representations of $\pi_1(X, x_0)$.

Definition 1.3. Let X be a locally-path-connected. For each $n > 1$ (for $n = 1$ the representation is simply the inner automorphism representation of a groupoid) there is a groupoid representation $\pi_n(X) : \Pi(X) \rightarrow \mathbf{AbGrp}$ which generalizes the action at each point $\pi_1(X, x_0) \curvearrowright \pi_n(X, x_0)$. By the above theorem, this corresponds to a local group $\pi_n(X)$.

2 Maps of a Proper Curve are Finite

Theorem 2.1. Let C be a proper curve over k and X is separated of finite type over k . Then any nonconstant map $f : C \rightarrow X$ over k is finite.

Proof. Since $C \rightarrow \operatorname{Spec}(k)$ is proper and $X \rightarrow \operatorname{Spec}(k)$ is separated then by Tag 01W6 the map $f : C \rightarrow X$ is proper. The fibres of closed points $x \in X$ are proper closed subschemes $C_x \hookrightarrow C$ (since if $C_x = C$ then $f : C \rightarrow X$ would be the constant map at $x \in X$) and thus finite since proper closed subsets of a curve are finite. Now I claim that if the fibres $f^{-1}(x)$ are finite at closed points $x \in X$ then all fibres are finite. Assuming this, $f : C \rightarrow X$ is proper with finite fibres and thus is finite by Tag 02OG.

To show the claim consider,

$$E = \{x \in X \mid \dim C_x = 0\}$$

Since C is Noetherian, $\dim C_x = 0$ iff C_x is finite (suffices to check for affine schemes since quasi-compact and dimension zero Noetherian rings are exactly Artinian rings which have finite spectrum). Then E is locally constructible by Tag 05F9 and contains all the closed points of X . Since X is finite type over k then X is Jacobson which implies that E is dense in every closed set. Then for any point $\xi \in X$ then $Z = \overline{\{\xi\}}$ is closed and irreducible with generic point ξ and thus $E \cap Z$ is dense in Z . Then by Tag 005K we have $\xi \in E$ so $E = X$ proving that all fibres are finite. \square

Remark. The only facts about C that I used were that $C \rightarrow \operatorname{Spec}(k)$ is proper and that C is irreducible of dimension one. The second two properties are needed for the following to hold.

Lemma 2.2. If X is an irreducible Noetherian scheme of dimension one then every nontrivial closed subset of X is finite.

Proof. Since X is quasi-compact it suffices to show this property for affine schemes $X = \operatorname{Spec}(A)$ with $\dim A = 1$ and prime nilradical. Any nontrivial closed subset is of the form $V(I)$ for some proper radical ideal $I \subset A$ with $I \supsetneq \operatorname{nilrad}(A)$. Then $\operatorname{ht}(I) = 1$ since any prime above I must properly contain $\operatorname{nilrad}(A)$ and thus have height at least one but $\dim A = 1$. Then,

$$\operatorname{ht}(I) + \dim A/I \leq \dim A$$

so $\dim A/I = 0$. Since A is Noetherian so is A/I but $\dim A/I = 0$ and thus A/I is Artinian. Therefore $\operatorname{Spec}(A/I)$ is finite proving the proposition. \square

Remark. Since $C \rightarrow \operatorname{Spec}(k)$ is proper it is finite type over k and thus C is Noetherian.

3 Action on Fibres of Fibration

Theorem 3.1. Let $F \hookrightarrow E \xrightarrow{\sim} B$ be a fibration. Then there is a groupoid action $\Pi(B)$ on the space of fibres and in particular $\pi_1(B, x_0) \rightarrow \operatorname{Aut}(F)$.

Proof. Consider a path $\gamma : I \rightarrow B$ from x_1 to x_2 and then the diagram,

$$\begin{array}{ccc} F_{x_1} & \xhookrightarrow{\quad} & E \\ \downarrow & \nearrow \tilde{\gamma} & \downarrow p \\ F_{x_1} \times I & \xrightarrow{\gamma} & B \end{array}$$

By homotopy lifting we get a map $\tilde{\gamma} : F_{x_1} \times I \rightarrow E$ lifting $\gamma : F_{x_1} \times I \rightarrow B$. Then $p \circ \tilde{\gamma} = \gamma$ so $\tilde{\gamma}(-, 1) \subset F_{x_2}$ since $p \circ \tilde{\gamma}(-, 1) = \gamma(1) = x_2$. Therefore we get a map $[\gamma] : F_{x_1} \rightarrow F_{x_2}$ via $[\gamma](x) = \tilde{\gamma}(x, 1)$.

I claim that two lifts of homotopic paths are homotopic. Given two paths $\gamma_1, \gamma_2 : I \rightarrow B$ and a path homotopy $h : I^2 \rightarrow B$ and two lifts $\tilde{\gamma}_1, \tilde{\gamma}_2 : F_{x_1} \times I \rightarrow E$ we want a map $F_{x_1} \times I^2 \rightarrow E$ above $h : I^2 \rightarrow B$. This map is defined on $F_{x_1} \times (I \times \{0, 1\} \cup \{0\} \times I)$ via $\tilde{\gamma}_1$ on $F_{x_1} \times I \times \{0\}$ and $\tilde{\gamma}_2$ on $F_{x_1} \times I \times \{1\}$ any by inclusion of the fibre F_{x_1} on $F_{x_1} \times \{0\} \times I$ (constant on I) since $h_{\{0\} \times I}$ is constant since it is a path homotopy. Then by homotopy lifting, we get $\tilde{h} : F_{x_1} \times I \times I \rightarrow E$ such that $p \circ \tilde{h} = h$ and thus $\tilde{h}(-, 1, -) : F_{x_1} \times I \rightarrow F_{x_2}$ gives a homotopy from $[\gamma_1] : F_{x_1} \rightarrow F_{x_2}$ to $[\gamma_2] : F_{x_1} \rightarrow F_{x_2}$.

Therefore, we have a representation of $\Pi(B)$ on **hTop** sending $x \mapsto F_x$ and $\gamma \mapsto [\gamma]$. \square

4 Serre - Vanishing

Remark. First we prove the result for the case \mathbb{P}_R^n .

Theorem 4.1. Let $\mathbb{P}^n = \mathbb{P}_R^n$. For any coherent $\mathcal{O}_{\mathbb{P}^n}$ -module \mathcal{F} there is some $r > 0$ such that,

$$H^i(\mathbb{P}_R^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(s)) = 0$$

for all $i > 0$ and $s \geq r$.

Proof. Since this holds for $i > n$ we may apply reverse induction on i . Assume the theorem holds for $i + 1$ and let \mathcal{F} be some coherent sheaf. Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is ample, for some $\ell > 0$ the sheaf $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(\ell)$ is generated by global sections,

$$\bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(\ell)$$

and thus tensoring by $\mathcal{O}_{\mathbb{P}^n}(-\ell)$ we get a surjection,

$$\bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n}(-\ell) \twoheadrightarrow \mathcal{F}$$

which we may extend to an exact sequence,

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n}(-\ell) \longrightarrow \mathcal{F} \longrightarrow 0$$

Since $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free it is flat (exactness can be checked on stalks) so we get a short exact sequence,

$$0 \longrightarrow \mathcal{G}(d) \longrightarrow \bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n}(d - \ell) \longrightarrow \mathcal{F}(d) \longrightarrow 0$$

Applying the LES of homology we get,

$$\bigoplus_{j=1}^N H^i(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}^n}(d - \ell)) \longrightarrow H^i(\mathbb{P}_R^n, \mathcal{F}(d)) \longrightarrow H^{i+1}(\mathbb{P}_R^n, \mathcal{G}(d))$$

By the induction hypothesis, for all sufficiently large $d \geq r_{\mathcal{G}}$ the cohomology $H^{i+1}(\mathbb{P}_R^n, \mathcal{G}(d)) = 0$ vanishes and furthermore by explicit calculation, $H^i(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}^n}(d - \ell)) = 0$ for $i > 0$ and $d \geq \ell$ so take $r_{\mathcal{F}} = \max\{\ell, r_{\mathcal{G}}\}$ and then for $d \geq r_{\mathcal{F}}$ we find,

$$H^i(\mathbb{P}_R^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)) = 0$$

proving the result by induction. \square

Theorem 4.2. Let R be a noetherian ring and $X \rightarrow \operatorname{Spec}(R)$ proper. Furthermore, let \mathcal{L} be an ample line bundle on X . Then for any coherent \mathcal{O}_X -module \mathcal{F} there is some $r > 0$ such that,

$$H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = 0$$

for all $i > 0$ and $s \geq r$.

Proof. Since $X \rightarrow \operatorname{Spec}(R)$ is finite type and X has an ample line bundle \mathcal{L} then X must be quasi-projective over R for some immersion $\iota : X \rightarrow \mathbb{P}_R^N$ where $\mathcal{L}^{\otimes d} = \iota^* \mathcal{O}_{\mathbb{P}^N}(1)$. Since $X \rightarrow \operatorname{Spec}(R)$ is proper and $\mathbb{P}_R^N \rightarrow \operatorname{Spec}(R)$ is separated then $\iota : X \rightarrow \mathbb{P}_R^N$ is automatically proper hence a closed immersion so X is projective.

Being a closed immersion $\iota : X \rightarrow \mathbb{P}_R^N$ is affine so we may compute (the Leray spectral sequence degenerates),

$$H^i(X, \mathcal{G}) = H^i(\mathbb{P}_R^N, \iota_* \mathcal{G})$$

for any quasi-coherent sheaf on X . Therefore, considering the coherent sheaf $\mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}$ it suffices to compute,

$$H^i(\mathbb{P}_R^N, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}))$$

We will apply the projection formula noting that writing $s = nd + r$ gives,

$$\mathcal{L}^{\otimes s} = (\mathcal{L}^{\otimes d})^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r} = (\iota^* \mathcal{O}_{\mathbb{P}^N}(1))^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r} = \iota^* \mathcal{O}_{\mathbb{P}^N}(n) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}$$

Therefore, let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^N}(n)$ in the projection formula to find that,

$$\iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r} \otimes_{\mathcal{O}_X} \iota^* \mathcal{O}_{\mathbb{P}^N}(n)) = \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(n)$$

Since $\iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r})$ is coherent the previous proposition allows us to choose n large enough (taking the maximum of the n large enough to kill the cohomology of each of $r = 0, 1, \dots, d-1$) so that,

$$H^i(\mathbb{P}_R^N, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(n)) = 0$$

for any $r = 0, 1, \dots, d-1$ and $n \gg 0$. Therefore, for all sufficiently large s we have,

$$H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = H^i(\mathbb{P}_R^N, \iota^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s})) = H^i(\mathbb{P}_R^N, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(n)) = 0$$

\square

Theorem 4.3 (projection formula). Let $f : X \rightarrow Y$ be a morphism of ringed spaces \mathcal{F} a \mathcal{O}_X -module and \mathcal{E} a finite locally free \mathcal{O}_Y -module. Then,

$$R^q f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{E}) = R^q f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$$

Theorem 4.4. Let X be projective, then the functors $\text{Ext}_{\mathcal{O}_X}^i(-, \mathcal{G}) : \mathbf{Coh}(\mathcal{O}_X) \rightarrow \mathbf{Mod}_{\Gamma(X, \mathcal{O}_X)}$ for a fixed quasi-coherent \mathcal{O}_X -module \mathcal{G} are universal contravariant δ -functors.

Proof. It suffices to show that $\text{Ext}^i(-, \mathcal{G})$ are coeffaceable for all $i > 0$. Since X is projective there is an ample line bundle \mathcal{L} on X and for the coherent \mathcal{O}_X -module \mathcal{G} there is some $r > 0$ such that,

$$H^i(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = 0$$

for any $s \geq r$ and $i > 0$. Then since \mathcal{L} is ample, for any coherent \mathcal{O}_X -module \mathcal{F} for some n_0 such that for $n \geq n_0$ the coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections. Choosing $n \geq \max\{n_0, r\}$ we get a surjection,

$$\bigoplus_{j=1}^N \mathcal{O}_X \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

However, since \mathcal{L} is a line bundle we may tensor by $\mathcal{L}^{\otimes -n} = (\mathcal{L}^{\otimes n})^\vee$ to get a surjection,

$$\mathcal{H} = \bigoplus_{j=1}^N \mathcal{L}^{\otimes -n} \twoheadrightarrow \mathcal{F}$$

Furthermore, since \mathcal{L} is locally free of rank one,

$$\text{Ext}_{\mathcal{O}_X}^i(\mathcal{H}, \mathcal{G}) = \bigoplus_{j=1}^N \text{Ext}_{\mathcal{O}_X}^i((\mathcal{L}^{\otimes n})^\vee, \mathcal{G}) = \bigoplus_{j=1}^N \text{Ext}_{\mathcal{O}_X}^i(\mathcal{O}_X, \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{G}) = \bigoplus_{j=1}^N H^i(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$$

for $i > 0$ by Serre vanishing showing that $\text{Ext}_{\mathcal{O}_X}^i(-, \mathcal{G})$ is coeffaceable for all $i > 0$. \square

5 Computing Ext and Tor in the Second Argument

5.1 Ext

Definition 5.1. Let \mathcal{C} be an abelian category (possibly enriched over another category \mathcal{D}). Then if \mathcal{C} has enough injectives, $\text{Ext}_{\mathcal{C}}^i(A, -) : \mathcal{C} \rightarrow \mathcal{D}$ are the right-derived functors of $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathcal{D}$.

Lemma 5.2. $\text{Ext}_{\mathcal{C}}^i(-, M) : \mathcal{C} \rightarrow \mathcal{D}$ is a contravariant functor.

Proof. Given an injective resolution $M \rightarrow \mathcal{I}^\bullet$ and a map $A \rightarrow B$ we get a morphism of complexes $\text{Hom}_{\mathcal{C}}(B, \mathcal{I}^\bullet) \rightarrow \text{Hom}_{\mathcal{C}}(A, \mathcal{I}^\bullet)$ and thus a morphism of cohomology,

$$\text{Ext}_{\mathcal{C}}^i(B, M) \rightarrow \text{Ext}_{\mathcal{C}}^i(A, M)$$

which clearly respects composition. \square

Lemma 5.3. If P is projective then $\text{Ext}_{\mathcal{C}}^i(P, -) = 0$ for $i > 0$.

Proof. This follows immediately from the defining property that $\text{Hom}_{\mathcal{C}}(P, -)$ is exact. \square

Proposition 5.4. Given a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{C} and some $M \in \mathcal{C}$ then there is a long exact sequence,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, M) & \longrightarrow & \text{Hom}_{\mathcal{C}}(B, M) & \longrightarrow & \text{Hom}_{\mathcal{C}}(A, M) \\
& & & & & & \downarrow \\
& & \text{Ext}_{\mathcal{C}}^1(C, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^1(B, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^1(A, M) \\
& & & & & & \downarrow \\
& & \text{Ext}_{\mathcal{C}}^2(C, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^2(B, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^2(A, M) \longrightarrow \dots
\end{array}$$

Proof. Take an injective resolution $M \rightarrow \mathcal{I}^\bullet$. Then since each \mathcal{I}^n is injective the functor $\text{Hom}_{\mathcal{C}}(-, \mathcal{I}^n)$ is exact so we get an exact sequence of complexes,

$$0 \longrightarrow \text{Hom}_{\mathcal{C}}(C, \mathcal{I}^\bullet) \longrightarrow \text{Hom}_{\mathcal{C}}(B, \mathcal{I}^\bullet) \longrightarrow \text{Hom}_{\mathcal{C}}(A, \mathcal{I}^\bullet) \longrightarrow 0$$

Taking the cohomology sequence of this short exact sequence of complexes gives the desired long exact sequence. \square

Lemma 5.5. If $P_\bullet \rightarrow A$ is a projective resolution then $\text{Ext}_{\mathcal{C}}^i(A, -) = H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$.

Proof. We may use the acyclicity lemma which may be proven by the above exact sequence for $\text{Hom}_{\mathcal{C}}(-, M)$ noting that $\text{Ext}_{\mathcal{C}}^i(P_n, M) = 0$. However, a more elegant argument goes as follows. Since P_\bullet is a complex of projectives the functor $\text{Hom}_{\mathcal{C}}(P_n, -)$ is exact so for any exact sequence,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$$

we get an exact sequence of complexes,

$$0 \longrightarrow \text{Hom}_{\mathcal{C}}(P_\bullet, M) \longrightarrow \text{Hom}_{\mathcal{C}}(P_\bullet, N) \longrightarrow \text{Hom}_{\mathcal{C}}(P_\bullet, K) \longrightarrow 0$$

which gives a long exact sequence in the cohomology functors $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$ which shows that $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$ form a δ -functor. Furthermore, since \mathcal{C} has enough injectives, for any $M \in \mathcal{C}$ we can embed $M \hookrightarrow I$ into an injective I and $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, I)) = 0$ since $\text{Hom}_{\mathcal{C}}(-, I)$ is exact. Therefore, $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$ is an effaceable δ -functor and thus universal by Grothendieck. Furthermore, since $\text{Hom}_{\mathcal{C}}(-, M)$ is left-exact,

$$\begin{aligned}
H^0(\text{Hom}_{\mathcal{C}}(P_\bullet, -)) &= \ker(\text{Hom}_{\mathcal{C}}(P^0, -) \rightarrow \text{Hom}_{\mathcal{C}}(P^1, -)) = \text{Hom}_{\mathcal{C}}(\text{coker}(P^1 \rightarrow P^0), -) \\
&= \text{Hom}_{\mathcal{C}}(A, -)
\end{aligned}$$

However, $\text{Ext}_{\mathcal{C}}^i(A, -)$ are the derived functors of $\text{Hom}_{\mathcal{C}}(A, -)$ so they too form a universal δ -functor over $\text{Hom}_{\mathcal{C}}(A, -)$. Thus, since universal δ -functors with naturally isomorphic first terms are unique,

$$\text{Ext}_{\mathcal{C}}^i(A, -) = H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$$

\square

Remark. The above formalism applies exactly to any bifunctor $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ such that for any $A \in \mathcal{C}$ there are enough $F(A, -)$ -acyclics I for which $F(-, I)$ is exact and replacing ‘injective’ with this class of acyclics and ‘projective’ by any class of objects P such that $F(P, -)$ is exact. Furthermore we assume \mathcal{C} is abelian with enough injectives, \mathcal{D} is additive, and $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ is additive.

For example, in the category of \mathcal{O}_X -modules on a scheme, the bifunctor,

$$\mathcal{H}om_{\mathcal{O}_X}(-, -) : \mathcal{M}od(\mathcal{O}_X)^{\text{op}} \times \mathcal{M}od(\mathcal{O}_X) \rightarrow \mathcal{M}od(\mathcal{O}_X)$$

satisfies the following properties. First, for injective sheaves \mathcal{I} we have $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{I})$ is exact (and there are enough injectives which are obviously acyclic for $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$). Second, if \mathcal{E} is a locally-free sheaf then,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, -) = \mathcal{E}^\vee \otimes_{\mathcal{O}_X} (-)$$

and \mathcal{E}^\vee is locally free and thus flat so $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, -)$ is exact. Therefore, we see that $\mathcal{E}xt_{\mathcal{O}_X}^i(-, \mathcal{G})$ is a contravariant δ -functor, vanishing on locally free sheaves, which may be computed via cohomology of locally-free complexes. Furthermore, whenever $\mathcal{M}od(\mathcal{O}_X)$ has enough locally frees (for example whenever X has an ample line bundle) then $\mathcal{E}xt_{\mathcal{O}_X}^i(-, \mathcal{G})$ forms a universal contravariant δ -functor.

5.2 Tor

Definition 5.6. When \mathcal{C} has a right-exact cmonoid structure $- \otimes_{\mathcal{C}} -$ and \mathcal{C} has enough projectives then define $\text{Tor}_i^{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathcal{C}$ as the left-derived functors of $A \otimes_{\mathcal{C}} - : \mathcal{C} \rightarrow \mathcal{C}$.

Remark. Here it will be necessary to assume that \mathcal{C} has enough flat objects ($- \otimes_{\mathcal{C}} F$ is exact) which happens say when projectives are flat.

Lemma 5.7. $\text{Tor}_i^{\mathcal{C}}(-, M)$ is a covariant functor.

Proof. Given a map $A \rightarrow B$ and a projective resolution $P_\bullet \rightarrow M$ we get a morphism of complexes, $A \otimes_{\mathcal{C}} P_\bullet \rightarrow B \otimes_{\mathcal{C}} P_\bullet$ and thus a morphism of homology,

$$\text{Tor}_i^{\mathcal{C}}(A, M) \rightarrow \text{Tor}_i^{\mathcal{C}}(B, M)$$

□

Definition 5.8. We say an object $P \in \mathcal{C}$ is *flat* if $P \otimes_{\mathcal{C}} -$ is an exact functor.

Lemma 5.9. The following are equivalent,

- (a). P is flat
- (b). $\text{Tor}_i^{\mathcal{C}}(P, -) = 0$ for all $i > 0$
- (c). $\text{Tor}_1^{\mathcal{C}}(P, -) = 0$.

Proof. Clearly (a) \implies (b) \implies (c). Now, if $\text{Tor}_1^{\mathcal{C}}(P, -) = 0$ then for any exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we get an exact sequence,

$$\text{Tor}_1^{\mathcal{C}}(P, C) \longrightarrow P \otimes_{\mathcal{C}} A \longrightarrow P \otimes_{\mathcal{C}} B \longrightarrow C \otimes_{\mathcal{C}} P \longrightarrow 0$$

so if $\text{Tor}_1^{\mathcal{C}}(P, -) = 0$ then $P \otimes_{\mathcal{C}} -$ is exact i.e. P is flat. □

Proposition 5.10. Given a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{C} and some $M \in \mathcal{C}$ then there is a long exact sequence,

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathrm{Tor}_2^{\mathcal{C}}(A, M) & \longrightarrow & \mathrm{Tor}_2^{\mathcal{C}}(B, M) & \longrightarrow & \mathrm{Tor}_2^{\mathcal{C}}(C, M) \\
& & & & & & \downarrow \\
& & & & & & \mathrm{Tor}_1^{\mathcal{C}}(A, M) \longrightarrow \mathrm{Tor}_1^{\mathcal{C}}(B, M) \longrightarrow \mathrm{Tor}_1^{\mathcal{C}}(C, M) \\
& & & & & & \downarrow \\
& & & & & & A \otimes_{\mathcal{C}} M \longrightarrow B \otimes_{\mathcal{C}} M \longrightarrow C \otimes_{\mathcal{C}} M \longrightarrow 0
\end{array}$$

Proof. Take a flat resolution $F_{\bullet} \rightarrow M$. Then since each F^n is flat the functor $F^n \otimes_{\mathcal{C}} -$ is exact so we get an exact sequence of complexes,

$$0 \longrightarrow A \otimes_{\mathcal{C}} F_{\bullet} \longrightarrow B \otimes_{\mathcal{C}} F_{\bullet} \longrightarrow C \otimes_{\mathcal{C}} F_{\bullet} \longrightarrow 0$$

Taking the homology sequence of this short exact sequence of complexes gives the desired long exact sequence since by the acyclicity lemma we may commute $\mathrm{Tor}_i^{\mathcal{C}}(A, M)$ via a flat resolution of M . \square

Lemma 5.11. If $F_{\bullet} \rightarrow A$ is a free resolution then $\mathrm{Tor}_i^{\mathcal{C}}(A, -) = H_i(F_{\bullet} \otimes_{\mathcal{C}} -)$.

Proof. We may use the acyclicity lemma which may be proven by the above exact sequence for $\mathrm{Tor}_i^{\mathcal{C}}(-, M)$ showing that $\mathrm{Tor}_i^{\mathcal{C}}(-, M)$ forms a δ -functor and noting that $\mathrm{Tor}_i^{\mathcal{C}}(F_n, M) = 0$. However, a more elegant argument goes as follows. Since F_{\bullet} is a complex of frees the functor $F_n \otimes -$ is exact so for any exact sequence,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$$

we get an exact sequence of complexes,

$$0 \longrightarrow F_{\bullet} \otimes M \longrightarrow F_{\bullet} \otimes N \longrightarrow F_{\bullet} \otimes K \longrightarrow 0$$

which gives a long exact sequence in the homology functors $H_i(F_{\bullet} \otimes -)$ which shows that $H_i(F_{\bullet} \otimes -)$ form a (homological) δ -functor. Furthermore, since \mathcal{C} has enough frees, for any $M \in \mathcal{C}$ we have a surjection $F \twoheadrightarrow M$ for some free F and $H_i(F_{\bullet} \otimes_{\mathcal{C}} F) = 0$ since $- \otimes \mathcal{F}$ is exact (both rows and columns stay exact, it is the exactness of the columns here ensured by freeness of F which is needed for the vanishing). Therefore, $H_i(F_{\bullet} \otimes -)$ is a coeffaceable δ -functor and thus universal by Grothendieck. Furthermore, since $- \otimes_{\mathcal{C}} M$ is right-exact,

$$H_0(F_{\bullet} \otimes_{\mathcal{C}} -) = \mathrm{coker}([F_1 \otimes_{\mathcal{C}} -] \rightarrow [F_0 \otimes_{\mathcal{C}} -]) = \mathrm{coker}(F_1 \rightarrow F_0) \otimes_{\mathcal{C}} (-) = A \otimes_{\mathcal{C}} (-)$$

However, $\mathrm{Tor}_i^{\mathcal{C}}(A, -)$ are the derived functors of $A \otimes_{\mathcal{C}} -$ so they too form a universal δ -functor over $A \otimes_{\mathcal{C}} -$. Thus, since universal δ -functors with naturally isomorphic first terms are unique,

$$\mathrm{Tor}_i^{\mathcal{C}}(A, -) = H_i(F_{\bullet} \otimes_{\mathcal{C}} -)$$

\square

Proposition 5.12. Tor is symmetric: there is a natural isomorphism $\mathrm{Tor}_i^{\mathcal{C}}(A, B) = \mathrm{Tor}_i^{\mathcal{C}}(B, A)$.

Proof. Choose a flat resolution $F_{\bullet} \rightarrow A$. By the above lemma $\mathrm{Tor}_i^{\mathcal{C}}(A, B) = H_i(F_{\bullet} \otimes_{\mathcal{C}} B)$. However, by the symmetry of $- \otimes_{\mathcal{C}} -$ we have, $H_i(F_{\bullet} \otimes_{\mathcal{C}} B) = H_i(B \otimes_{\mathcal{C}} F_{\bullet})$. Furthermore, because $\mathrm{Tor}_i^{\mathcal{C}}(B, -)$ is the left-derived functor of $B \otimes_{\mathcal{C}} -$ we may compute it via acyclics (since it is a δ -functor) so $\mathrm{Tor}_i^{\mathcal{C}}(B, A) = H_i(B \otimes_{\mathcal{C}} F_{\bullet})$ and thus,

$$\mathrm{Tor}_i^{\mathcal{C}}(A, B) = H_i(F_{\bullet} \otimes_{\mathcal{C}} B) = H_i(B \otimes_{\mathcal{C}} F_{\bullet}) = \mathrm{Tor}_i^{\mathcal{C}}(B, A)$$

\square

Remark. These arguments apply to the satellites of any symmetric bifunctor $F : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$ between abelian categories such that \mathcal{C} has enough objects A for which $F(A, -)$ is exact, in particular, if $F(P, -)$ is exact for projectives (as is the tensor product).

Remark. Symmetry follows directly from the following spectral sequence argument. Let $F_\bullet^A \rightarrow A$ and $F_\bullet^B \rightarrow B$ be free resolutions. Then consider the double complex $C_{p,q} = F_p^A \otimes_{\mathcal{C}} F_q^B$. There are two spectral sequences which compute the homology of the total complex $\text{Tot}(C_{\bullet,\bullet})$. These two spectral sequences agree on their zeroth page, ${}^A E_{p,q}^0 = {}^B E_{p,q}^0 = F_p^A \otimes_{\mathcal{C}} F_q^B$. Now, the first pages are,

$$\begin{aligned} {}^A E_{p,q}^1 &= H_p(C_{\bullet,q}) = H_p(F_\bullet^A \otimes_{\mathcal{C}} F_q^B) = A \otimes_{\mathcal{C}} F_q^B & \text{in } p \text{ degree zero} \\ {}^B E_{p,q}^1 &= H_q(C_{p,\bullet}) = H_q(F_p^A \otimes_{\mathcal{C}} F_\bullet^B) = F_p^A \otimes_{\mathcal{C}} B & \text{in } q \text{ degree zero} \end{aligned}$$

where we have used the fact that $- \otimes_{\mathcal{C}} F_q^B$ and $F_p^A \otimes_{\mathcal{C}} -$ are exact (since the resolutions are free) and thus commute with taking homology. Then the second pages are,

$$\begin{aligned} {}^A E_{p,q}^2 &= H_q({}^A E_{p,\bullet}^1) = H_q(A \otimes_{\mathcal{C}} F_\bullet^B) = L^q(A \otimes_{\mathcal{C}} -)(B) & \text{in } p \text{ degree zero} \\ {}^B E_{p,q}^2 &= H_p({}^B E_{\bullet,q}^1) = H_p(F_\bullet^A \otimes_{\mathcal{C}} B) = L^p(- \otimes_{\mathcal{C}} B)(A) & \text{in } q \text{ degree zero} \end{aligned}$$

Since the second pages are supported in a single row or column both spectral sequences are converged. Therefore, we find,

$$H_n(\text{Tot}(C_{\bullet,\bullet})) = {}^A E_{0,n}^2 = {}^B E_{n,0}^2 = L^n(A \otimes_{\mathcal{C}} -)(B) = L^n(- \otimes_{\mathcal{C}} B)(A)$$

Therefore, for a bifunctor we may derive in either component to get the same satellite functors. Furthermore, when $- \otimes_{\mathcal{C}} -$ is symmetric then,

$$\begin{aligned} L^n(A \otimes_{\mathcal{C}} -)(B) &= L^n(- \otimes_{\mathcal{C}} A)(B) = L^n(B \otimes_{\mathcal{C}} -)(A) \\ L^n(- \otimes_{\mathcal{C}} B)(A) &= L^n(B \otimes_{\mathcal{C}} -)(A) = L^n(- \otimes_{\mathcal{C}} A)(B) \end{aligned}$$

so the derived functors are symmetric.

5.3 Acyclicity

Lemma 5.13. Let F be a δ -functor. Suppose there is an exact sequence,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow K \longrightarrow 0$$

where I^i are F -acyclic. Then for $i > 0$,

$$F^{n+1+i}(A) = F^i(A)$$

and $F^{n+1}(A) = \ker(F^0(I^n) \rightarrow F^0(K))$.

Proof. We prove this by induction on n . For $n = 0$, we are given a short exact sequence,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow K \longrightarrow 0$$

Taking the long exact sequence,

$$0 \longrightarrow F^0(A) \longrightarrow F^0(I^0) \longrightarrow F^0(K) \longrightarrow F^1(A) \longrightarrow F^1(I^0)$$

and

$$F^i(I^0) \longrightarrow F^i(K) \longrightarrow F^{i+1}(A) \longrightarrow F^{i+1}(I^0)$$

However, I^0 is F -acyclic so $F^i(I^0) = 0$ for $i > 0$ and thus $F^{i+1}(A) = F^i(K)$ for $i > 0$. Furthermore, for the second sequence $F^1(A) = \ker(F^0(I^0) \rightarrow F^0(K))$.

Now we assume the result holds for $n - 1$. We split the exact sequence into,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow \tilde{K} \longrightarrow 0$$

and

$$0 \longrightarrow \tilde{K} \longrightarrow I^n \longrightarrow K \longrightarrow 0$$

Applying the induction hypothesis we see that, $F^{n+i}(A) = F^i(\tilde{K})$ for $i > 0$. In particular, we will use, $F^{n+1}(A) = F^1(\tilde{K})$. Now, by the LES of the second exact sequence we find,

$$0 \longrightarrow F^0(\tilde{K}) \longrightarrow F^0(I^n) \longrightarrow F^0(K) \longrightarrow F^1(\tilde{K}) \longrightarrow F^1(I^n)$$

and

$$F^i(I^n) \longrightarrow F^i(K) \longrightarrow F^{i+1}(\tilde{K}) \longrightarrow F^{i+1}(I^n)$$

However, I^n is F -acyclic so for $i > 0$ we get,

$$F^i(K) = F^{i+1}(\tilde{K}) \quad \text{and} \quad F^1(\tilde{K}) = \text{coker}(F^0(I^n) \rightarrow F^0(K))$$

Therefore, we have $F^{n+i+1}(A) = F^{i+1}(\tilde{K}) = F^i(K)$ for $i > 0$. Furthermore,

$$F^{n+1}(A) = F^1(\tilde{K}) = \text{coker}(F^0(I^n) \rightarrow F^0(K))$$

proving the lemma. □

Theorem 5.14 (acyclicity). If F is a δ -functor and $A \rightarrow I^\bullet$ a resolution of F -acyclic objects,

$$F^n(A) = H^n(F^0(I^\bullet))$$

Proof. We may truncate the resolution by adding a cokernel K to give an exact sequence,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow K \longrightarrow 0$$

By the previous lemma, we can compute,

$$F^{n+1}(A) = \text{coker}(F^0(I^n) \rightarrow F^0(K))$$

However, by exactness, $K = \text{coker}(I^{n-1} \rightarrow I^n) = \ker(I^{n+1} \rightarrow I^{n+2})$. Furthermore, F^0 is left-exact so $F^0(K) = \ker(F(I^{n+1}) \rightarrow F(I^{n+2}))$. Therefore, for $n \geq 0$ we find,

$$F^{n+1}(A) = \text{coker}(F^0(I^n) \rightarrow F^0(K)) = \text{coker}(F^0(I^n) \rightarrow \ker(F(I^{n+1}) \rightarrow F(I^{n+2}))) = H^{n+1}(F^0(I^\bullet))$$

Furthermore, F^0 is left-exact so,

$$F^0(A) = F^0(\ker(I^0 \rightarrow I^1)) = \ker(F^0(I^0) \rightarrow F^0(I^1)) = H^0(F^0(I^\bullet))$$

□

5.4 Tor for Sheaves

Remark. Often the categories $\mathcal{M}od(\mathcal{O}_X)$, $\mathfrak{Q}\mathfrak{C}oh(\mathcal{O}_X)$, and $\mathfrak{C}oh(\mathcal{O}_X)$ do not have enough projectives. Therefore, we cannot define Tor for sheaves as a left-derived functor we need an alternative definition.

Definition 5.15. Let X be a scheme such that $\mathfrak{C}oh(\mathcal{O}_X)$ has enough locally-frees (e.g. X has an ample line bundle). Given a coherent sheaf \mathcal{F} and a resolution $\mathcal{E}_\bullet \rightarrow \mathcal{F}$ by locally free coherent sheaves, we define,

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, -) = H_i(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} -)$$

Proposition 5.16. $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is a universal homological δ -functor.

Proof. First, given an exact sequence of coherent sheaves,

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3 \longrightarrow 0$$

we get an exact sequence of complexes,

$$0 \longrightarrow \mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{G}_1 \longrightarrow \mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{G}_2 \longrightarrow \mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{G}_3 \longrightarrow 0$$

since \mathcal{E}^n is locally-free and thus flat. Taking homology gives a long exact sequence of $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, -)$ sheaves making it a homological δ -functor. It suffices to show that $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is coexactable. Since there are enough locally-free sheaves for any coherent \mathcal{G} we can find a locally-free and a surjection $\mathcal{E}' \twoheadrightarrow \mathcal{G}$. Then, since $- \otimes_{\mathcal{O}_X} \mathcal{E}'$ is exact then,

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}) = H_i(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{E}) = 0$$

where $\mathcal{E}_\bullet \rightarrow \mathcal{F}$ is a locally-free resolution. Therefore, $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is coexactable. \square

Remark. Since $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is universal it will agree with any other reasonable definition (any definition which is a universal δ -functor) because there is a unique universal δ -functor over,

$$\mathcal{T}or_0^{\mathcal{O}_X}(\mathcal{F}, -) = H_0(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} -) = \text{coker}(\mathcal{E}^1 \rightarrow \mathcal{E}^0) \otimes_{\mathcal{O}_X} - = \mathcal{F} \otimes_{\mathcal{O}_X} -$$

where the second equality follows from right-exactness of $- \otimes_{\mathcal{O}_X} \mathcal{G}$.

Remark. Since $- \otimes_{\mathcal{O}_X} - : \mathcal{M}od(\mathcal{O}_X) \times \mathcal{M}od(\mathcal{O}_X) \rightarrow \mathcal{M}od(\mathcal{O}_X)$ is a symmetric bifunctor with enough locally-frees which are flat. Then since $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, -)$ are the left-satellite functors of $\mathcal{F} \otimes_{\mathcal{O}_X} -$ we can apply the acyclicity lemma to show that we map compute sheaf Tor from a locally free resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{G}$,

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H_i(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_\bullet)$$

which shows the symmetry of $\mathcal{T}or_i^{\mathcal{O}_X}(-, -)$.

6 Depth of Field

First we calculate the size of the circle of confusion. Let the lense have aperature D and focal length f . The image distance is given by,

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f}$$

then,

$$i = \frac{fo}{o - f}$$

Therefore, we can compute the change in image distance as o changes,

$$\frac{di}{do} = \frac{f}{o - f} - \frac{fo}{(o - f)^2} = -\frac{f^2}{(o - f)^2}$$

For a depth of Δo we have a spread of image depths,

$$\Delta i \approx \frac{f^2 \Delta o}{(o - f)^2}$$

Then the width of the circle of confusion is given by,

$$\frac{C}{D} = \frac{\Delta i}{f + \Delta i} \approx \frac{\Delta i}{f}$$

Therefore,

$$C = \frac{fD}{(o - f)^2} \Delta o$$

For a fixed allowable circle of confusion C_{\max} for the desired resolution, we find the depth of field,

$$\text{DOF} = 2 \frac{C}{D} \cdot \frac{(o - f)^2}{f} = \frac{2(o - f)^2 NC}{f^2}$$

where $N = f/D$ is the focal ratio.

6.1 Hyperfocal Distance

At some focal distance H , all objects beyond H are in focus. This occurs when,

$$\frac{i - f}{f} = \frac{C}{D}$$

and

$$i = \frac{fH}{H - f}$$

Then,

$$\frac{H}{H - f} - 1 = \frac{f}{H - f} = \frac{C}{D}$$

Therefore,

$$H = \frac{f(D + C)}{C} = \frac{f^2}{CN} + f$$

Alternatively, if we focus at infinity and ask beyond which everything is in focus then,

$$\frac{i - f}{i} = \frac{C}{D}$$

and

$$i = \frac{fH}{H-f}$$

Then,

$$1 - \frac{H-f}{H} = \frac{f}{H} = \frac{C}{D}$$

Therefore,

$$H = \frac{fD}{C} = \frac{f^2}{NC}$$

7 Morphisms from Proper to Affine Schemes

Let $X \rightarrow \operatorname{Spec}(R)$ be proper and $\operatorname{Spec}(A) \rightarrow \operatorname{Spec}(R)$ be affine. Now,

$$\operatorname{Hom}_R(X, \operatorname{Spec}(A)) = \operatorname{Hom}_R(A, \Gamma(X, \mathcal{O}_X))$$

The map $X \rightarrow \operatorname{Spec}(A)$ is given as follows, consider $\varphi_x : A \rightarrow \Gamma(X, \mathcal{O}_X) \rightarrow \mathcal{O}_{X,x}$ then $x \mapsto \varphi_x^{-1}(\mathfrak{m}_x)$. Therefore, all maps $X \rightarrow \operatorname{Spec}(A)$ are constant if $x \mapsto \operatorname{res}_x^{-1}(\mathfrak{m}_x)$ is a fixed ideal independent of x .

8 Irreducible Polynomials over \mathbb{Z}

Consider the map $\operatorname{Spec}(\mathbb{Z}[X]) \rightarrow \operatorname{Spec}(\mathbb{Z})$. The fibres are, over the generic point (0), we have $\operatorname{Spec}(\mathbb{Q}[X]) \rightarrow \operatorname{Spec}(\mathbb{Q})$ which corresponds to ideals of the form $(f(X))$ for f an irreducible polynomial $f \in \mathbb{Q}[X]$. The fibres over (p) are $\operatorname{Spec}(\mathbb{F}_p[X]) \rightarrow \operatorname{Spec}(\mathbb{F}_p)$ whose primes are of the form $(f(X))$ for f an irreducible polynomial $f \in \mathbb{F}_p[X]$. Therefore we get an explicit description of $\operatorname{Spec}(\mathbb{F}[X])$, we have the primes, $(f(X))$ for irreducible $f \in \mathbb{Q}[X]$ (for which we may clear denominators to get $f \in \mathbb{Z}[X]$) and $(p, f(X))$ for irreducible $f \in \mathbb{F}_p[X]$ (choosing some representative in $\mathbb{Z}[X]$) and finally of course (0) and (p) are prime (corresponding to the generic points of the fibres).

Suppose $f \in \mathbb{Z}[X]$ were irreducible then any prime (strictly) above (f) must be of the form $(p, f(X))$ otherwise f would be a nontrivial product. Then we have $\dim \mathbb{Z}[X]/(f) = 1$ furthermore, (COMPLETE THIS ARGUMENT ...)

9 Normalization

Example 9.1. Consider $X = \operatorname{Spec}(A)$ with $A = k[x, y]/(y^2 - x^2(x+1))$. Then consider,

$$A \rightarrow k[t] \quad x \mapsto t^2 - 1 \quad y \mapsto t(t^2 - 1)$$

Then $y^2 = t^2(t^2 - 1)$ and $x^2(x - 1) = t^2(t^2 - 1)^2$ so this map is well-defined. This gives a dominant map,

$$\mathbb{A}_k^1 \rightarrow \operatorname{Spec}(A)$$

Furthermore, I claim that,

$$\operatorname{Frac}(A) \hookrightarrow k(t)$$

is an isomorphism, clearly $\text{Frac}(A) \rightarrow k(t)$ is injective. The inverse map is $t \mapsto y/x$ then $y/x \mapsto t \mapsto y/x$ and $t \mapsto y/x \mapsto t$. Furthermore, $x \mapsto (t^2 - 1) \mapsto (y^2/x^2 - 1) = x$ and $y \mapsto t(t^2 - 1) \mapsto y/x(y^2/x^2 - 1) = y$. Thus the map $\mathbb{A}^1 \rightarrow \text{Spec}(A)$ is a dominant birational morphism. Furthermore,

$$\overline{A} = k[y/x] = k[t] \subset \text{Frac}(A)$$

because $t = y/x$ satisfies the monic $t^2 - x - 1$ so $\mathbb{A}^1 \rightarrow \text{Spec}(A)$ is the normalization.

Example 9.2. Consider the cusp $X = \text{Spec}(A)$ with $A = k[x, y] = (y^2 - x^3)$. Then consider,

$$A \rightarrow k[t] \quad x \mapsto t^2 \quad y \mapsto t^3$$

Then $y^2 \mapsto t^6$ and $x^2 \mapsto t^6$ so this is well-defined. This gives a dominant map,

$$\mathbb{A}_k^1 \rightarrow \text{Spec}(A)$$

Furthermore, I claim that,

$$\text{Frac}(A) \hookrightarrow k(t)$$

is an isomorphism. Send $t \mapsto y/x$ then $t \mapsto y/x \mapsto t$ and $y/x \mapsto t \mapsto y/x$. Then $y \mapsto t^3 \mapsto y^3/x^3 = y$ and $x \mapsto t^2 \mapsto y^2/x^2 = x$. Therefore, $\mathbb{A}_k^1 \rightarrow \text{Spec}(A)$ is a dominant birational morphism. Furthermore,

$$\overline{A} = k[y/x] = k[t] \subset k(t) = \text{Frac}(A)$$

because $t = y/x$ satisfies the monic $t^2 - x$.

Example 9.3. Consider the tachnode $X = \text{Spec}(A)$ with $A = k[x, y]/(x^2 - y^4)$. Then consider,

$$A \rightarrow k[t, s]/(s^2 - 1) \quad x \mapsto t \quad y \mapsto t^2 s$$

Then $x^4 \mapsto t^4$ and $y^2 \mapsto t^4$ so this is well-defined. this gives a dominant map,

$$\text{Spec}(k[t, s]/(s^2 - 1)) = \mathbb{A}_k^1 \coprod \mathbb{A}_k^1 \rightarrow \text{Spec}(A)$$

On the irreducible components $\mathfrak{p}_+ = (y - x^2)$ and $\mathfrak{p}_- = (y + x^2)$ of $\text{Spec}(A)$ we have,

$$\mathcal{O}_{X, \mathfrak{p}_+} = \text{Frac}(k[x, y]/(y - x^2)) \quad \mathcal{O}_{X, \mathfrak{p}_-} = \text{Frac}(k[x, y]/(y + x^2))$$

and thus the map $\text{Spec}(k[t, s]/(s^2 - 1)) \rightarrow \text{Spec}(A)$ gives an isomorphism on each component and $\text{Spec}(k[t, s]/(s^2 - 1))$ is normal.

10 A Very Weird Scheme

For finite products we have,

$$\text{Spec}(A \times B) = \text{Spec}(A) \coprod \text{Spec}(B)$$

where we take the coproduct in the category of schemes. In particular, the primes of $A \times B$ are simply $\mathfrak{p}_1 \times B$ or $A \times \mathfrak{p}_2$ for primes $\mathfrak{p}_1 \subset A$ and $\mathfrak{p}_2 \subset B$. However, for infinite product this fails. Consider,

$$X = \text{Spec}\left(\prod_{i=0}^{\infty} k\right) \quad R = \prod_{i=0}^{\infty} k$$

where k field. The prime ideals of this ring are not just the kernels of the projections $R \rightarrow k$ which are maximal ideals. To see this, consider the ideal I of functions $\mathbb{N} \rightarrow k$ which have finite support. Clearly $I \rightarrow R \rightarrow k$ is surjective for each projection so I is not contained in any of the described primes. It turns out that prime ideals of R correspond to ultrafilters \mathcal{F} of \mathbb{N} where $\mathfrak{p}(\mathcal{F})$ for some ultrafilter is the following,

$$\mathfrak{p}(\mathcal{F}) = \{(a_i) \mid \{i \mid a_i = 0\} \in \mathcal{F}\}$$

Therefore, the principal ultrafilter \mathcal{F}_i above $\{i\}$ gives exactly $\mathfrak{p}(\mathcal{F}_i) = \ker \pi_i$ but there are many more nonprincipal ultrafilters.

11 Coproducts in the Category of Schemes

Proposition 11.1. The forgetful functor $F : \mathbf{Sch} \rightarrow \mathbf{Top}$ preserves colimits.

Remark. Let $\mathrm{Hom}_{\mathbf{Top}}(F(X), S) = \mathrm{Hom}_{\mathbf{Sch}}(X, T(S))$

12 NOTE LOOK UP THE PROOF FOR PROJ -; LOCALLY FREE

13 Ravi Excercises

Remark. Maps $\mathrm{Spec}(k) \rightarrow \mathbb{P}_k^n$ are equivalent to giving a line bundle \mathcal{L} on $\mathrm{Spec}(k)$ i.e. a one-dimensional k -vectorspace $V \cong k$ and $n + 1$ sections $s_i \in V$ not all zero. We call this point $[s_0, \dots, s_n] \in \mathbb{P}_k^n$ up to isomorphism $\varphi : V \cong V'$ and $\varphi(s_i) = s'_i$. This is simply global scalling by k^\times . Furthermore, for any extension K/k we can describe $\mathbb{P}_k^n(K)$ similarly but with $s_i \in K$.

Definition 13.1. Projection from a rational point $\mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$ given a projection point $p \in \mathbb{P}_k^n$. We define this as follows: by an automorphism of \mathbb{P}_k^n let $p = [1 : 0 : \dots : 0]$. Take the dense open $U = D(X_0) \setminus \{0\} = \mathrm{Spec}(x_1, \dots, x_n) \setminus \{(0)\}$. Then consider the map $U \rightarrow \mathbb{P}_k^{n-1}$ via $\mathcal{L} = \mathcal{O}_U$ and $s_i = x_i$. These global sections generate because we have removed the point at which they all vanish. This rational map $\mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$ has domain $\mathrm{Dom}(f) = \mathbb{P}_k^n \setminus \{p\}$.

13.1 6.5 F

Consider the conic $C = V(X^2 + Y^2 = Z^2) \subset \mathbb{P}_k^2$. Consider the map $\mathbb{P}_k^1 \rightarrow \mathbb{P}_k^2$ defined by the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2)$ and the sections $X_0^2 - X_1^2, 2X_0X_1, X_0^2 + X_1^2$. The image is exactly $C = V(X^2 + Y^2 = Z^2)$ and thus $C \cong \mathbb{P}_k^1$. However, if characteristic of $k = 2$ then these sections are $X_0^2 + X_1^2, 0, X_0^2 + X_1^2$ which does not define a map since these may all vanish simultaneously. In fact, $V(X^2 + Y^2 = Z^2)$ is not smooth in characteristic two since $X^2 + Y^2 = (X + Y)^2$ so we get $X + Y = \pm Z$ the union of two lines in \mathbb{P}_k^2 .

We can also describe an isomorphism as follows. First, lets do a change of coordinates $X \mapsto \frac{1}{2}(X + Z)$ and $Z \mapsto \frac{1}{2}(X - Z)$ then $C = V(XZ + Y^2)$. Take the point $p = [1 : 0 : 0]$ use the projection $\mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^1$ away from p . On the affine $D(X)$ this is the map $U = \mathrm{Spec}(k[y, z]/(z + y^2)) \setminus \{0\} \rightarrow \mathbb{P}_k^1$ via $(y, z) \mapsto [y : z]$. Now $U = \mathrm{Spec}(k[y, y^{-1}]) = \mathbb{G}_m^k$ and the map is $\mathbb{G}_m^k \rightarrow \mathbb{P}_k^1$ via $t \mapsto [t, t^2]$. This is a rational map $C \dashrightarrow \mathbb{P}_k^1$ of smooth projective curves so it extends to $C \rightarrow \mathbb{P}_k^1$ which is inverse to the previous map.

13.2 6.5 G

Consider $C = \text{Spec}(k[x, y]/(y^2 - x^3 - x^2))$. Then we construct a rational map $C \dashrightarrow \mathbb{A}_k^1$ via projecting from $p = (0, 0)$. Explicitly, consider $U = D(x)$ and consider, $f : U \rightarrow \mathbb{A}_k^1$ via $t \mapsto y/x$. Inversely we define $g : \mathbb{A}_k^1 \rightarrow C$ generated by the ring map $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$. Note that we have seen this is the normalization $\mathbb{A}_k^1 \rightarrow C$ of C . Then $g \circ f : U \rightarrow C$ is $x \mapsto y^2/x^2 - 1 = x$ and $y \mapsto y/x(y^2/x^2 - 1) = y$. Furthermore, $f \circ g : \mathbb{G}_m^k \rightarrow \mathbb{A}_k^1$ is $t \mapsto y/x \mapsto t$. Therefore, these are inverse rational maps showing that $C \xrightarrow{\sim} \mathbb{A}_k^1$ is birational. However we cannot extend this rational map to p since $\mathcal{O}_{C,p} = \text{Spec}((k[x, y]/(y^2 - x^2))_{(x,y)})$ is not a domain and thus not regular.

This gives a formula for the rational points of C by $\mathbb{A}_L^1 \dashrightarrow C_L$. Via $t \mapsto (t^2 - 1, t(t^2 - 1))$ which hit every L -rational point on C . Thus,

$$C(L) = \{(t^2 - 1, t(t^2 - 1)) \mid t \in L\}$$

We see that C is a rational curve i.e. $C \xrightarrow{\sim} \mathbb{P}_k^1$.

13.3 6.5 H

Consider the quadric surface,

$$Q = V(X^2 + Y^2 - Z^2 - W^2) \subset \mathbb{P}_k^3$$

First, we do a change of variables,

$$X \mapsto \frac{1}{2}(X + Z) \quad Z \mapsto \frac{1}{2}(X - Z) \quad Y \mapsto \frac{1}{2}(Y + W) \quad W \mapsto \frac{1}{2}(Y - W)$$

which gives,

$$Q = V(XZ + YW) \subset \mathbb{P}_k^3$$

Now we project from the point $p = [1 : 0 : 0 : 0]$ on $U = D(X) \setminus \{p\}$ this gives the map,

$$f : \text{Spec}(k[y, z, w]/(z + yw)) \setminus \{0\} \rightarrow \mathbb{P}_k^2$$

via sections y, z, w . We describe an inverse $\mathbb{P}_k^2 \dashrightarrow Q$ as follows, consider $\mathbb{P}_k^2 = \text{Proj}(k[T_0, T_1, T_2])$ then on $D(T_0 T_2)$ take $\text{Spec}(k[t_0, t_1]) \rightarrow \text{Spec}(k[y, z, w]/(z + yw))$ via $y \mapsto -t_1$ and $z \mapsto -t_1^2/t_0$ and $w \mapsto -t_1/t_0$ which is the map $(t_0, t_1) \mapsto (-t_1, -t_1^2/t_0, -t_1/t_0)$. This gives,

$$g : D(T_0 T_2) \rightarrow D(XW)$$

and thus $\mathbb{P}_k^2 \dashrightarrow Q$. Furthermore, $g \circ f : D(XW) \rightarrow U$ is,

$$(y, z, w) \mapsto [y : z : w] = [y/w : z/w : 1] \mapsto (-z/w, -z^2/wy, -w/y) = (y, z, w)$$

restriction of the identity since $z + wy = 0$. Furthermore, $f \circ g : D(T_0 T_1 T_2) \rightarrow D(T_0 T_1 T_2)$ is,

$$(t_0, t_1) \mapsto (-t_1, t_1^2/t_0, -t_1/t_0) \mapsto [-t_1 : -t_1^2/t_0 : -t_1/t_0] = [-t_0 t_1 : -t_1^2 : -t_1] = [t_0 : t_1 : 1] = (t_0, t_1)$$

Thus we have $\mathbb{P}_k^2 \xrightarrow{\sim} Q$ via $(t_0, t_1) \mapsto (-t_1, -t_1^2/t_0, -t_1/t_0)$ on $D(T_0 T_1 T_2) \cong D(XZW)$ and thus, clearing denominators and sending $t_1 \mapsto -t_1$, we get,

$$Q(L) = \{[t_0 : t_1 t_0 : -t_1^2 : t_1] \mid t_0, t_1 \in L^\times\} \cup \{[0 : t_0 : t_1 : 0] \mid t_1, t_2 \in L^\times\} \cup \{[0 : t_0 : 0 : t_1] \mid t_1, t_2 \in L^\times\}$$

13.4 6.5 I

Consider the rational map $c : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ given by $[x : y : z] \mapsto [1/x : 1/y : 1/z]$ on $D(xyz)$. Since \mathbb{P}_k^2 is smooth, we can extend over smooth codimension one irreducibles i.e. $V(x)$ and $V(y)$ and $V(z)$ such that c is defined on a dense open of each. In particular, on $D(yz)$ we have $[x : y : z] \mapsto [1 : x/y : x/z]$ is equivalent to c restricted to $D(xyz)$ and likewise on $D(xy)$ and $D(xz)$. Thus,

$$\text{Dom}(f) \supset D(xy) \cup D(yz) \cup D(xz) = \mathbb{P}_k^2 \setminus \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$$

The remaining closed set is codimension two so we generically will not be able to extend over it. Indeed, if we try $[x : y : z] \mapsto [y : x : xy/z]$ on $D(z)$ then at $[0 : 0 : 1]$ this is not defined so it does not work.

13.5 6.5 J

Show that there are no dominant rational maps $\mathbb{P}_k^1 \rightarrow F_k^n$ where $F_k^n = \text{Proj}(k[X, Y, Z]/(X^n + Y^n - Z^n))$ is the Fermat curve for $n > 2$.

14 Which Hypersurfaces are Isomorphic to Projective Space?

First, what is a hypersurface.

Definition 14.1. A hypersurface $H \subset \mathbb{P}_k^n$ is a codimension one integral closed subscheme i.e. a prime divisor on \mathbb{P}_k^n .

Theorem 14.2. Every hypersurface $H \subset \mathbb{P}_k^n$ is of the form $V(F)$ for some $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$.

Proof. Since H is a prime divisor and \mathbb{P}_k^n is locally factorial (in particular regular) then H is Cartier so its associated sheaf of ideals $\mathcal{I} \cong \mathcal{O}_{\mathbb{P}_k^n}(-d)$ is invertible. Then the inclusion map $\mathcal{O}_{\mathbb{P}_k^n}(-d) \hookrightarrow \mathcal{O}_{\mathbb{P}_k^n}$ is given by some regular section $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$ and thus $H = V(F)$. \square

Remark. In the case $n = 1$ hypersurfaces are exactly points and since $\mathbb{P}_L^0 = \text{Spec}(L)$ then for any finite extension L/k we can easily find $\text{Spec}(L) \rightarrow \mathbb{P}_k^1$ so hypersurfaces of \mathbb{P}_k^1 are exactly of the form \mathbb{P}_L^0 . We wonder how this generalizes to $n > 1$. Furthermore, note that we will use the fact that H is effective Cartier and argue, from the exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}_k^n} \longrightarrow \iota_* \mathcal{O}_H \longrightarrow 0$$

and the associated LES,

$$\begin{array}{ccccc} H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}) & \longrightarrow & H^0(H, \mathcal{O}_H) & \longrightarrow & H^1(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-d)) \\ \parallel & & & & \downarrow \\ k & & & & 0 \end{array}$$

to argue that for $n > 1$ we get a surjection $k \rightarrow H^0(H, \mathcal{O}_H)$ showing that we cannot have extensions of k . Note that this argument does not hold for $n = 1$ since $H^1(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}(-d)) \neq 0$ and we can, in fact, have extensions of the base field.

Theorem 14.3. Let $H \subset \mathbb{P}_k^n$ be a degree d hypersurface i.e. $H = V(F)$ for $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$ and $n > 1$. Then $H \cong \mathbb{P}_L^{n-1}$ for some L/k exactly when $L = k$ and either $d = 1$ or $n = 2$ and $d = 2$.

Proof. Suppose that $H \cong \mathbb{P}_L^{n-1}$ and consider the inclusion $\iota : H \hookrightarrow \mathbb{P}_k^n$ and let $X = \mathbb{P}_k^n$. Then for the ample sheaf $\mathcal{L} = \iota^* \mathcal{O}_X(1)$ we have $\mathcal{L} \in \text{Pic}(X) \cong \text{Pic}(\mathbb{P}_L^{n-1})$ so \mathcal{L} correspond to $\mathcal{O}_{\mathbb{P}_k^{n-1}}(k)$ for some $k \in \mathbb{Z}$. Therefore, we must have,

$$H^p(H, \mathcal{L}^{\otimes \ell}) = H^p(\mathbb{P}_k^{n-1}, \mathcal{O}_{\mathbb{P}_k^{n-1}}(k\ell))$$

In particular,

$$\dim_k H^p(H, \mathcal{L}^{\otimes \ell}) = (\dim_k L) \cdot \begin{cases} \binom{k\ell+n-1}{n-1} & p = 0 \\ 0 & p \neq 0, n-1 \\ \binom{-k\ell-1}{n-1} & p = n-1 \end{cases}$$

Furthermore, since ι is a closed immersion (and thus affine) we have,

$$H^p(H, \mathcal{L}^{\otimes \ell}) = H^p(X, \iota_* \mathcal{L}^{\otimes \ell}) = H^p(X, \iota_* \mathcal{O}_H \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell))$$

using the projection formula. Then, there is an exact sequence of sheaves,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \iota_* \mathcal{O}_H \longrightarrow 0 \\ & & \parallel & & & & \\ & & \mathcal{O}_X(-d) & & & & \end{array}$$

Twisting by $\mathcal{O}_X(\ell)$ gives,

$$0 \longrightarrow \mathcal{O}_X(\ell-d) \longrightarrow \mathcal{O}_X(\ell) \longrightarrow \iota_* \mathcal{O}_H \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell) \longrightarrow 0$$

Now denote $\mathcal{F} = \iota_* \mathcal{O}_H$ and $\mathcal{F}(\ell) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell)$ which is the sheaf whose cohomology we wish to compute. Taking the LES of cohomology we get,

$$0 \longrightarrow H^0(X, \mathcal{O}_X(\ell-d)) \longrightarrow H^0(X, \mathcal{O}_X(\ell)) \longrightarrow H^0(H, \mathcal{L}^{\otimes \ell}) \longrightarrow H^1(X, \mathcal{O}_X(\ell-d)) = 0$$

since $n > 1$. First, for $\ell = 0$ the first sequence gives $H^0(X, \mathcal{O}_X) \twoheadrightarrow H^0(H, \mathcal{O}_H)$ and thus $k \twoheadrightarrow L$ which is a k -morphism so $L = k$ since it is an extension. Furthermore, from the above short exact sequence, we see that,

$$h^0(H, \mathcal{L}^{\otimes \ell}) = h^0(X, \mathcal{O}_X(\ell)) - h^0(X, \mathcal{O}_X(\ell-d)) = \binom{\ell+n}{n} - \binom{\ell-d+n}{n}$$

In particular, for $d > 1$ and $\ell = 1$ we have,

$$h^0(H, \mathcal{L}) = h^0(X, \mathcal{O}_X(1)) = n+1$$

This must equal (since $L = k$),

$$h^0(H, \mathcal{L}) = \binom{k+n-1}{n-1} = \binom{k+n-1}{k} = r(k)$$

which is zero for $k < 0$ and monotonically increasing for $k > 0$. Note that $r(0) = 1$ and $r(1) = n$ and $r(2) = \frac{1}{2}(n+1)n$. Since $r(1) < r(2) < r(3)$ and $r(1) = n$ then either $r(2) = n+1$ or $r(k) \neq n+1$ for all k . However, $\frac{1}{2}n(n+1) = n+1$ exactly when $n = 2$ for $n > 0$ forcing the case $n = 2$ when

$d > 1$. In particular for the case $n = 2$ and $d = 2$ we get a plane conic which we know is isomorphic to \mathbb{P}_k^1 . Also, we need to consider the case $d = 1$ in which H is a hyperplane and it is easy to see that $H \cong \mathbb{P}_k^{n-1}$ via the map $\mathbb{P}_k^{n-1} \hookrightarrow \mathbb{P}_k^n$ defined by $\mathcal{O}_{\mathbb{P}_k^{n-1}}(1)$ and the n sections perpendicular to $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1))$ which has image H proving the claim.

Note further that we get,

$$H^{n-1}(X, \mathcal{O}_X(\ell)) \longrightarrow H^{n-1}(H, \mathcal{L}^{\otimes \ell}) \longrightarrow H^n(X, \mathcal{O}_X(\ell - d)) \longrightarrow H^n(X, \mathcal{O}_X(\ell)) \longrightarrow H^n(H, \mathcal{O}_H)$$

and otherwise $H^p(X, \mathcal{O}_X(\ell)) = H^{p+1}(X, \mathcal{O}_X(\ell - d))$ so $H^p(H, \mathcal{O}_H) = 0$ for $p \neq 0, n - 1$. Since $\dim H = n - 1$ we have $H^n(H, \mathcal{O}_H) = 0$ and also $H^{n-1}(X, \mathcal{O}_X(\ell)) = 0$ so we find,

$$0 \longrightarrow H^{n-1}(H, \mathcal{L}^{\otimes \ell}) \longrightarrow H^n(X, \mathcal{O}_X(\ell - d)) \longrightarrow H^n(X, \mathcal{O}_X(\ell)) \longrightarrow 0$$

so we have,

$$h^{n-1}(H, \mathcal{L}^{\otimes \ell}) = h^{n-1}(X, \mathcal{O}_X(\ell - d)) - h^{n-1}(X, \mathcal{O}_X(\ell)) = \binom{d - \ell - 1}{n} - \binom{-\ell - 1}{n}$$

which does have the correct degree in $(-\ell)$ i.e. $n - 1$ to be $h^{n-1}(\mathbb{P}_k^{n-1}, \mathcal{O}_{\mathbb{P}_k^{n-1}}(k\ell))$. \square

15 Random Comalg Facts

Lemma 15.1. Let (p_1) and (p_2) be incommensurable prime ideals. Then $(p_1) \cap (p_2) = (p_1 p_2)$.

Proof. Clearly $(p_1 p_2) \subset (p_1) \cap (p_2)$ so it suffices to show that if $a = p_1 x = p_2 y$ then $a \in (p_1 p_2)$. Since $a \in (p_1)$ and $p_2 \notin (p_1)$ we get $y \in (p_1)$ and likewise $x \in (p_2)$ showing that $a \in (p_1 p_2)$. \square

16 Open Questions

- (a). Coproducts in the Category of Schemes vs Affine Schemes why are they different but agree with LRS coproducts in the first case which agree with Top coproducts since the Forget : LRS \rightarrow Top has a right-adjoint (Raymond chat).
- (b). Which Hypersurfaces are Rational? GOOD QUESTION. I think all quadric hypersurfaces are rational even though only the conic $X^2 + Y^2 - Z^2$ is on the nose isomorphic to \mathbb{P}_k^1 . Can we prove this? Projection from a point?
- (c). Example of an affine curve which does not embed in \mathbb{A}_k^2
- (d). Does unirational imply finite domination by rational variety in general?

17 To Do on Thesis

- (a). Example of non-arithmetic curve with no Δ_ν -regular equation, try the think with weakly Δ -nondegenerate by never Δ -nondegenerate.
- (b). Is the elliptic curve example I gave toric?

- (c). find example which is toric: use the
- (d). Explicit example of curve not on toric surface?
- (e). Explicit example of curve not on a Hirzburch surface?
- (f). Example of curve which is toric but never weakly Δ -nondegenerate?
- (g).

18 When is a Sheaf a Pushforward

THE FOLLOWING IS NOT QUITE CORRECT BUT APPROXIMATELY

Lemma 18.1. Let $\iota : Z \hookrightarrow X$ be a closed embedding and $U = X \setminus Z$. Then if \mathcal{F} is a sheaf of \mathcal{O}_X -modules then $\mathcal{F} = \iota_* \iota^{-1} \mathcal{F}$ if and only if $\mathcal{F}|_U = 0$. Furthermore, $\mathcal{F} = \iota_* \iota^* \mathcal{F}$ if and only if $\mathcal{I} \cdot \mathcal{F} = 0$ where \mathcal{I} is the ideal sheaf of $Z \hookrightarrow X$. Furthermore, if Z is reduced then these notions agree.

Proof. □

Remark. Given simply topological maps, a sheaf \mathcal{F} is a pushforward of some sheaf on a closed subset exactly when it is zero on the complement. However, if we ask for this sheaf to be the pushforward of a sheaf of \mathcal{O}_Z -modules then we need the stronger $\mathcal{I} \cdot \mathcal{F} = 0$.

19 Cayley-Hamilton

Theorem 19.1. Let $A \in M_n(R)$ be a square matrix over a ring R and $p_A(\lambda) = \det(\lambda I - A)$ be its characteristic polynomial. Then $p_A(A) = 0$.

Proof. First, I argue in the case that $R = k$ is a field. Matrices $A \in M_n(k)$ correspond to closed points of $X = \mathbb{A}_k^{n^2} = \text{Spec}(k[a_{ij}])$. Now the fundamental observation is that $p_A(A)$ is a matrix of polynomials in a_{ij} and thus gives a morphism $p : X \rightarrow X$ via the ring map $k[a_{ij}] \rightarrow k[a_{ij}]$ sending a_{ij} to the i, j entry of the matrix $p_A(A)$ with $A = (a_{ij})$.

Now, if p_A is separable (i.e. has distinct roots over \bar{k}) then A is diagonalizable over \bar{k} (eigenvectors with distinct eigenvalues are independent). Then $A = BDB^{-1}$ with D diagonal (these matrices defined over \bar{k}) and it is clear that $p_A(BDB^{-1}) = Bp_A(D)B^{-1} = 0$ since $p_A(\lambda) = 0$ for each eigenvalue. Furthermore, this case occurs exactly when the discriminant $\Delta(p_A) \neq 0$ which is a polynomial in a_{ij} so $\Delta : X \rightarrow \mathbb{A}_k^1$ gives a global function. We have shown that for any closed point $A \in D(\Delta)$, i.e. some matrix over \bar{k} with $\Delta(p_A) = 0$, that $p_A(A) = 0$ so the map $p : X \rightarrow X$ vanishes on the closed points of $D(\Delta)$ which is dense since it is open and nontrivial (any diagonal matrix over \bar{k} with nonrepeated entries satisfies this, I guess I used \bar{k} is infinite here) in an irreducible variety X . Thus $p : X \rightarrow X$ is the zero map since it vanishes on a dense set (using that X is a variety). In particular p is the zero polynomial in a_{ij} .

Now, for an arbitrary ring R take a matrix $A \in M_n(R)$ then $p(a_{ij}) = p_A(A)$ is an integer coefficient polynomial in a_{ij} (meaning the coefficients are in the image $\mathbb{Z} \rightarrow R$). However, for each prime $\mathfrak{p} \in \text{Spec}(R)$, the above argument shows that $\overline{p_A(A)} \in \kappa(\mathfrak{p})$ is zero since it is the characteristic

polynomial applied to the matrix $\bar{A} \in M_n(\kappa(\mathfrak{p}))$ over the field $\kappa(\mathfrak{p})$. Thus $p_A(A) \in \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$ so $p_A(A) \in \text{nilrad}(R)$ for any A thus the coefficients are in $\text{nilrad}(R)$ (we can see this because reducing p in $\kappa(\mathfrak{p})$ gives the zero polynomial). However, the coefficients are in the image of $\mathbb{Z} \rightarrow R$ then $\text{nilrad}(R) \cap \text{Im}(\mathbb{Z}) = \text{nilrad}(\mathbb{Z}/(n))$ where $n = \ker(\mathbb{Z} \rightarrow R)$ (DAMN DOESNT WORK) \square

20 Quasi-Compactness and Noetherian Spaces

Definition 20.1. A topological space X is Noetherian if every descending chain of closed sets stabilizes.

Lemma 20.2. Subspaces of Noetherian subspaces are Noetherian.

Proof. Let $S \subset X$ with X noetherian. Then the closed sets of S are exactly $S \cap Z$ for $Z \subset X$ closed. Thus descending chains of closed sets in S stabilize. \square

Definition 20.3. A space is quasi-compact if every open cover has a finite subcover.

Lemma 20.4. Noetherian spaces are quasi-compact.

Proof. Let U_α be an open cover of X which is Noetherian. Then consider the poset A under inclusion of finite unions of the U_α all of which are open sets of X . Since X is Noetherian any ascending chain of opens must stabilize so any chain in A has a maximum. Then by Zorn's lemma A has a maximal element which must be X since the U_α form a cover. Therefore there exists a finite subcover. \square

Corollary 20.5. Every subset of a noetherian topological space is quasi-compact.

Definition 20.6. A continuous map $f : X \rightarrow Y$ is quasi-compact if for each quasi-compact open $U \subset Y$ then $f^{-1}(U)$ is quasi-compact open.

20.1 The Case for Schemes

Lemma 20.7. Affine schemes are quasi-compact.

Proof. Let U_i be an open cover of $\text{Spec}(A_i)$. Since $D(f)$ for $f \in A$ forms a basis of the topology on $\text{Spec}(A_i)$ we can shrink to the case $U_i = D(f_i)$. Then.

$$X = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D(\{f_i \mid i \in I\})$$

And thus the ideal $I = (\{f_i \mid i \in I\})$ is not contained in any maximal ideal so $I = (1)$. Therefore, there are f_1, \dots, f_n such that $a_1 f_1 + \dots + a_n f_n = 1$ and thus $(f_1, \dots, f_n) = (1)$ which implies that,

$$X = D((f_1, \dots, f_n)) = \bigcup_{i=1}^n D(f_i)$$

so X is quasi-compact. \square

Definition 20.8. A scheme X is *locally Noetherian* if for every affine open U the ring $\mathcal{O}_X(U)$ is Noetherian. X is *Noetherian* if it is quasi-compact and locally-Noetherian.

Lemma 20.9. If $(f_1, \dots, f_n) = A$ and A_{f_i} is Noetherian then A is Noetherian.

Proof. For any ideal $I \subset A$ we know $I_{f_i} \subset A_{f_i}$ is finitely generated. Clearing denominators and collecting the finite union of these finite generators gives a map $A^N \rightarrow I$ which is surjective when localized $A_{f_i}^N \twoheadrightarrow I_{f_i}$. Consider the A -module $K = \text{coker}(A^N \rightarrow I)$ then for any $x \in K$ we have $f_i^{n_i} \cdot x = 0$ for each i but $f_i^{n_i}$ generate the unit ideal (since $D(f_i^{n_i}) = D(f_i)$ which cover $\text{Spec}(A)$) so $x = 0$ to $A^N \twoheadrightarrow I$ so I is finitely generated showing that A is Noetherian. \square

Lemma 20.10. If X has an open affine cover $U_i = \text{Spec}(A_i)$ with A_i noetherian then X is locally noetherian. Moreover, if the cover can be made finite then X is noetherian.

Proof. Let $V = \text{Spec}(B) \subset X$ be an affine open, Then $V \cap U_i \subset V$ is open so it may be covered by principal opens $D(f_{ij}) \subset V \cap U_i$ for $f_{ij} \in B$. Since V is quasi-compact we may find a finite subcover. We need to show that $B_{f_{ij}}$ is Noetherian then since $D(f_{ij})$ cover V we use the lemma to conclude that B is Noetherian. However, $D(f_{ij}) \subset V \cap U_i$ can be covered by principal opens (of $U_i = \text{Spec}(A_i)$) $W_{ijk} \subset D(f_{ij}) \subset U_i = \text{Spec}(A_i)$ and each $(A_i)_{f_{ijk}}$ is Noetherian since A_i is, so using the same lemma we find that $B_{f_{ij}}$ is Noetherian.

Now suppose the cover is finite and let V_j be any open cover of X . We need to show X is quasi-compact so we must show that V_i has a finite subcover. Consider $U_i \cap V_j$ which is open in the affine $U_i = \text{Spec}(A_i)$ so it may be covered by principal opens $D(f_{ijk}) \subset U_i \cap V_j$. Now,

$$U_i = \bigcup_{j,k} D(f_{ijk})$$

but U_i is affine and thus quasi-compact so we may find an finite subcover which only uses finitely many V_i but the cover U_i of X is also finite so only finitely many V_i are needed to cover X . \square

Corollary 20.11. $X = \text{Spec}(A)$ is Noetherian iff A is a Noetherian ring.

Proof. If X is Noetherian then $\mathcal{O}_X(X) = A$ is a Noetherian ring (X is affine and thus quasi-compact). Conversely $\text{Spec}(A)$ is a finite Noetherian affine cover so X is Noetherian. \square

Remark. It is not the case that for a Noetherian scheme we must have $\mathcal{O}_X(X)$ a noetherian ring even for varieties. See <http://sma.epfl.ch/ojangure/nichtnoethersch.pdf>.

Lemma 20.12. If A is Noetherian then $\text{Spec}(A)$ is a Noetherian topological space.

Proof. Every descending chain of subsets is of the form $V(I_1) \supsetneq V(I_2) \supsetneq V(I_3) \supsetneq \dots$ but the ideals,

$$\sqrt{I_1} \subsetneq \sqrt{I_2} \subsetneq \sqrt{I_3} \subsetneq \dots$$

stabilize since A is Noetherian and thus so does the chain of closed subsets. \square

Lemma 20.13. If X is a Noetherian scheme then its underlying topological space is Noetherian.

Proof. Choose a finite covering $U_i = \text{Spec}(A_i)$ by Noetherian rings. Then for any descending chain of closed subsets $Z_1 \supsetneq Z_2 \supsetneq Z_3 \supsetneq \dots$ we know $Z \cap U_i$ stabilizes at n_i since $\text{Spec}(A_i)$ is a Noetherian space. Thus, Z stabilizes at $\max n_i$ which exists since the cover is finite. \square

Remark. The converses of the above are false and so is X Noetherian. Let R be a non-Noetherian valuation ring. Then $\text{Spec}(R)$ has two points and thus is Noetherian as a topological space but not as a scheme since R is not a Noetherian ring.

Lemma 20.14. If X is locally Noetherian then any immersion $\iota : Z \hookrightarrow X$ is quasi-compact.

Proof. Closed immersions are affine and thus quasi-compact so it suffices okay to show that open immersions are quasi-compact. Let $j : U \rightarrow X$ be an open immersion. It suffices to check that $j^{-1}(U_i)$ is quasi-compact on an affine open cover $U_i = \text{Spec}(A_i)$ with A_i Noetherian. But $j : j^{-1}(U_i) \rightarrow U_i \cap U$ is a homeomorphism and $\text{Spec}(A_i)$ is a Noetherian topological space so every subset is quasi-compact and, in particular, $U_i \cap U$ is quasi-compact so $j^{-1}(U_i)$ is also. \square

Remark. When X is Noetherian then it is a Noetherian space so any inclusion map $\iota : Z \hookrightarrow X$ for any subset $Z \subset X$ is quasi-compact since every subset is quasi-compact. In particular, every subset of X is retrocompact.

20.2 Quasi-Compact Morphisms

Lemma 20.15. A morphism $f : X \rightarrow Y$ is quasi-compact iff Y has a cover by affine opens V_i such that $f^{-1}(V_i)$ is quasi-compact.

Proof. Clearly if f is quasi-compact then any affine open cover V_i of Y satisfies $f^{-1}(V_i)$ is quasi-compact since V_i is a quasi-compact open by virtue of being affine open.

Now assume that such a cover exists. Let $U \subset Y$ be a quasi-compact open. Then U is covered by finitely many V_1, \dots, V_n . Then $U \cap V_i$ is open in V_i which is affine so it is covered by standard opens W_{ij} . Since U is quasi-compact then we can choose finitely many W_{ij} . Now $f^{-1}(V_i)$ is quasi-compact by assumption so it has a finite cover by affine opens,

$$f^{-1}(V_i) = \bigcup_{j=1}^N \tilde{V}_{ij}$$

Then $f : \tilde{V}_{ik} \rightarrow V_i$ is a morphism of affine schemes so $f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$ is a principal affine. Therefore,

$$f^{-1}(U) = \bigcup_{i=1}^n f^{-1}(V_i \cap U) = \bigcup_{i,j} f^{-1}(W_{ij}) = \bigcup_{i,j,k} f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$$

is a finite union of principal affines each of which is quasi-compact so $f^{-1}(U)$ is quasi-compact. \square

Proposition 20.16. X is quasi-compact iff any morphism $X \rightarrow T$ for some affine scheme T is quasi-compact.

Proof. If X is quasi-compact then $f : X \rightarrow T$ is quasi-compact since T is an affine open cover of itself and $f^{-1}(T)$ is quasi-compact. Conversely, if $f : X \rightarrow T$ is quasi-compact with T affine then T is quasi-compact open in T so $X = f^{-1}(T)$ is quasi-compact. \square

Lemma 20.17. The base change of a quasi-compact morphism is quasi-compact.

Proof. (DO THIS) \square

21 Affine Morphisms

Definition 21.1. A morphism $f : X \rightarrow Y$ is *affine* if the preimage of every affine open is affine.

Lemma 21.2. Every morphism of affine schemes is affine and thus quasi-compact.

Proof. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ and $f : X \rightarrow Y$ be a morphism of affine schemes given by a ring map $\varphi : B \rightarrow A$. Then, any affine open $\operatorname{Spec}(C) = V \subset Y$ can be covered by principal opens $D(f_i)$ for $f_i \in B$. Note that under $\psi : B \rightarrow C$ we see that $D(f_i) = D(\psi(f_i))$ since $D(f_i) \subset \operatorname{Spec}(C)$. Since $D(\psi(f_i))$ cover $\operatorname{Spec}(C)$ then $\psi(f_i) \in C$ generate the unit ideal. Then we have $f^{-1}(D(f_i)) = D(\varphi(f_i))$ which is affine and $\varphi(f_i)$ generate the unit ideal of $\Gamma(f^{-1}(V), \mathcal{O}_X)$ so f^{-1} is affine. \square

Remark. An alternative proof goes as follows. Consider the pullback diagram,

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

then open immersions are stable under base change so $f^{-1}(U) = U \times_Y X = \operatorname{Spec}(C \otimes_B A)$ if affine.

Remark. In fact, by Tag 01S8, a morphism $f : X \rightarrow S$ is affine iff X is relatively affine over S meaning $X = \mathbf{Spec}_S(\mathcal{A})$ for some quasi-coherent \mathcal{O}_S -algebra \mathcal{A} .

Lemma 21.3. Let $f : X \rightarrow Y$ be a morphism and W_i an affine open cover of Y such that $f^{-1}(W_i)$ is affine. Then f is affine.

Proof. Let $\operatorname{Spec}(A) = V \subset Y$ be affine open. Then $V_i = V \cap W_i$ is open in the affine open $V = \operatorname{Spec}(A)$ so it can be covered by principal opens $D(f_{ij}) \subset V \cap W_i$ for $f_{ij} \in A$. Since $f : f^{-1}(W_i) \rightarrow W_i$ is a morphism of affine schemes, the preimage of the affine open $D(f_{ij}) \subset V \cap W_i$ is affine $f^{-1}(D(f_{ij}))$ (note that $D(f_{ij}) \subset V \cap W_i$ is not necessarily a principal affine open of W_i). But since $D(f_{ij})$ cover $\operatorname{Spec}(A)$ the $f_{ij} \in A$ generate the unit ideal and thus $f^\#(f_{ij}) \in \Gamma(f^{-1}(V), \mathcal{O}_X)$ generate the unit ideal and $(f^{-1}(V))_{f_{ij}} = f^{-1}(D(f_{ij}))$ is affine so $f^{-1}(V)$ is affine. \square

Lemma 21.4. The base change of an affine morphism is affine.

Proof. (DO THIS) \square

Lemma 21.5. Affine morphisms are quasi-compact.

Proof. If $f : X \rightarrow Y$ is affine then any affine open cover V_i of Y gives $f^{-1}(V_i)$ is affine and thus quasi-compact so f is quasi-compact. \square

22 Separatedness

Definition 22.1. A morphism $f : X \rightarrow Y$ with diagonal $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is,

- (a). *separated* if the diagonal $\Delta_{X/Y}$ is a closed immersion
- (b). *affine-separated* if the diagonal $\Delta_{X/Y}$ is affine
- (c). *quasi-separated* if the diagonal $\Delta_{X/Y}$ is quasi-compact

Lemma 22.2. Any morphism of affine schemes is separated. Furthermore, affine morphisms are separated.

Proof. For a map $\text{Spec}(A) \rightarrow \text{Spec}(B)$ the diagonal is $\text{Spec}(A) \rightarrow \text{Spec}(A \otimes_B A)$ given by $A \otimes_B A \rightarrow A$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ which is surjective so the diagonal is a closed immersion. The second fact is Tag 01S7. \square

Lemma 22.3. The composition of (quasi/affine)-separated morphisms are (quasi/affine)-separated.

Proof. (DO THIS) \square

Lemma 22.4. For any morphism $f : X \rightarrow Y$ the diagonal $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is an immersion.

Proof. Let V_i be an affine cover of Y then choose an affine open cover U_{ij} of X with $f(U_{ij}) \subset V_i$. Then the diagonal of the affine map $U_{ij} \rightarrow V_j$ is $U_{ij} \rightarrow U_{ij} \times_{V_i} U_{ij}$ which is a closed immersion since it corresponds to $A_{ij} \otimes_{B_i} A_{ij} \rightarrow A_{ij}$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ is surjective. Therefore $f : X \rightarrow Y$ is locally on X a closed immersion and thus an immersion. \square

Remark. Therefore, to show that $f : X \rightarrow Y$ is separated, it suffices to show that the diagonal is closed (here equivalently meaning that the map or its image is closed).

Lemma 22.5. If X is Noetherian then every morphism $f : X \rightarrow S$ is quasi-compact and quasi-separated.

Proof. Every subset of X is quasi-compact since X is (topologically) Noetherian. Then apply the first part to the diagonal $\Delta_{X/S} : X \rightarrow X \times_S X$ which is then quasi-compact and thus $f : X \rightarrow S$ is quasi-separated. \square

Lemma 22.6. Let $f : X \rightarrow S$ be affine-separated/quasi-separated with $S = \text{Spec}(A)$ affine. Then for any two affine opens $U, V \subset X$ the intersection $U \cap V$ is affine/quasi-compact.

Proof. Consider the pullback diagram,

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \times_S V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

where $U \cap V = \Delta_{X/S}(U \times_S V)$ using the basechange of an open immersion is an open immersion. Then since S is affine, $U \times_S V$ is affine and thus quasi-compact open of $X \times_S X$. Then if f is affine-separated then $\Delta_{X/S}$ is affine so $U \cap V = \Delta_{X/S}(U \times_S V)$ is affine. If f is quasi-separated then $\Delta_{X/S}$ is quasi-compact so $U \cap V = \Delta_{X/S}(U \times_S V)$ is quasi-compact. \square

Remark. In the separated case, we see that $U \cap V$ is affine and $\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$ is surjective.

Remark. Tag 01KO gives a generalization of this lemma. For the separated case see Tag 01KP.

Lemma 22.7. Let $f : X \rightarrow Y$ be quasi-compact and quasi-separated and \mathcal{F} be a quasi-coherent \mathcal{O}_X -module then $f_* \mathcal{F}$ is a quasi-coherent \mathcal{O}_Y -module.

Proof. Since this is local on Y we can restrict to the case that Y is affine. Then $X = f^{-1}(Y)$ is quasi-compact (when Y is not affine $f^{-1}(V)$ will be quasi-compact) so take a finite affine open cover U_i and since $f : X \rightarrow Y$ is quasi-separated over an affine then by the above lemma $U_i \cap U_j$ is quasi-compact so it has a finite affine open cover U_{ijk} . Then, by the sheaf property, there is an exact sequence of sheaves on Y

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i f_*(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} f_*(\mathcal{F}|_{U_{ijk}})$$

which works because these are finite sums. However, $f : U_{ijk} \rightarrow Y$ is a morphism of affine schemes and since \mathcal{F} is quasi-coherent we have $\mathcal{F}|_{U_{ijk}} = \widehat{M}_{ijk}$ so $f_*(\mathcal{F}|_{U_{ijk}}) = \widehat{M}_{ijk}$ as an $\mathcal{O}_Y(Y)$ -module. Thus, $f_*\mathcal{F}$ is a kernel of quasi-coherent \mathcal{O}_Y -modules and thus is quasi-coherent. \square

Remark. If X is Noetherian then $f : X \rightarrow Y$ is automatically quasi-compact and quasi-separated so there is no issue in the above lemma.

23 Sets Cut Out By Some Function

Theorem 23.1. Every closed subset $E \subset \mathbb{R}^n$ is the vanishing of some smooth function.

Proof. Since \mathbb{R}^n is a metric space, it is hereditarily paracompact so the complement $E^C \subset \mathbb{R}^n$ is paracompact. Since \mathbb{R}^n is separable, E^C is covered by countably many balls $B_{r_i}(a_i)$ for $a_i \in E^C$ since it is open so, by paracompactness, we may shrink the radii such that this cover is locally finite. Choose a smooth bump function,

$$g : [0, \infty) \rightarrow [0, \infty)$$

such that $g([0, 1)) > 0$ and $g([1, \infty)) = 0$ e.g.

$$g(x) = \begin{cases} \exp\left(-\frac{1}{1-x}\right) & x < 1 \\ 0 & x \geq 1 \end{cases}$$

Then consider,

$$f(x) = \sum_{x \in X} g(|x - a_i|/r_i)$$

Since $g(|x - a_i|/r_i) = 0$ for $x \notin B_{r_i}(a_i)$ and the cover is locally finite, this is a finite sum so f is well-defined and smooth. Furthermore,

$$f(x) = 0 \iff x \notin \forall i \in I : x \notin B_{r_i}(a_i) \iff x \notin E^C \iff x \in E$$

\square

Remark. This easily generalizes to show that any closed subset $Z \subset X$ of a smooth manifold is cut out by closed sets.

Our next question is what does the vanishing of analytic or holomorphic functions look like. We have one result.

Proposition 23.2. A nontrivial vanishing set of analytic functions in \mathbb{R}^n (or holomorphic functions in \mathbb{C}^n) has positive codimension. Explicitly, it does not contain any nonempty open.

Proof. This is clear because analytic and holomorphic functions which vanish on a nonempty open vanish everywhere. \square

24 Cousins Problems

Here we let X be a complex manifold and \mathcal{O}_X be its sheaf of holomorphic functions and \mathcal{K}_X be its sheaf of meromorphic functions. The Cousin problems are the following questions given a cover U_i and a meromorphic function $f_i \in \Gamma(U_i, \mathcal{K}_X)$ on U_i .

Definition 24.1. The Cousin problems ask the following.

- (a). (First or additive Cousin Problem) if $(f_i - f_j)|_{U_i \cap U_j}$ is holomorphic for each pair i, j then does there exist a global meromorphic function $f \in \Gamma(X, \mathcal{K}_X)$ such that $f|_{U_i} - f_i$ is holomorphic?
- (b). (Second or multiplicative Cousin Problem) if $(f_i/f_j)|_{U_i \cap U_j}$ is non-vanishing holomorphic for each pair i, j then does there exist a global meromorphic function $f \in \Gamma(X, \mathcal{K}_X)$ such that $f|_{U_i}/f_i$ is holomorphic and non-vanishing?

Notice that set of pairs $\{(U_i, f_i)\}$ in the first Cousin problem defines a global section of the sheaf $\mathcal{K}_X/\mathcal{O}_X$ exactly because $(f_i - f_j)|_{U_i \cap U_j} \in \mathcal{O}_X(U_i \cap U_j)$ is holomorphic. Likewise, the set of pairs $\{(U_i, f_i)\}$ in the second Cousin problem defined a global section of the sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ exactly because $(f_i/f_j)|_{U_i \cap U_j} \in \mathcal{O}_X^\times(U_i \cap U_j)$ is holomorphic and nonvanishing. Therefore, we can restate the Cousin problems as follows.

Definition 24.2. The Cousin problems ask the following.

- (a). (First Cousin Problem) is the map $H^0(X, \mathcal{K}_X) \rightarrow H^0(X, \mathcal{K}_X/\mathcal{O}_X)$ surjective?
- (b). (Second Cousin Problem) is the map $H^0(X, \mathcal{K}_X^\times) \rightarrow H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ surjective?

Now we can solve these problems using the following two exact sequences of sheaves,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{K}_X/\mathcal{O}_X \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{K}_X^\times \longrightarrow \mathcal{K}_X^\times/\mathcal{O}_X^\times \longrightarrow 0$$

and we can relate the sheaf cohomology needed in the two problems via the exponential exact sequence,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

Theorem 24.3. The first cousin problem is solvable when $H^1(X, \mathcal{O}_X) = 0$.

Proof. The first exact sequence gives a cohomology exact sequence,

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{K}_X) \longrightarrow H^0(X, \mathcal{K}_X/\mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{K}_X)$$

Clearly, if $H^1(X, \mathcal{O}_X) = 0$ then, by exactness, $H^0(X, \mathcal{K}_X) \rightarrow H^0(X, \mathcal{K}_X/\mathcal{O}_X)$ is surjective. \square

Remark. By Cartan's theorem B, we know $H^1(X, \mathcal{O}_X) = 0$ for any Stein manifold. So the first Cousin problem is always solvable for Stein manifolds.

Theorem 24.4. The second cousin problem is solvable when $H^1(X, \mathcal{O}_X^\times) = 0$ or when $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ and $H^2(X; \mathbb{Z}) = 0$.

Proof. The second exact sequence gives a cohomology exact sequence,

$$0 \longrightarrow H^0(X, \mathcal{O}_X^\times) \longrightarrow H^0(X, \mathcal{K}_X^\times) \longrightarrow H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{K}_X^\times)$$

Clearly, if $H^1(X, \mathcal{O}_X^\times) = 0$ then, by exactness, $H^0(X, \mathcal{K}_X^\times) \rightarrow H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$ is surjective. Now consider the cohomology of the exponential sequence,

$$H^1(X; \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X; \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X)$$

Then if $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathcal{O}_X) = 0$ we get an isomorphism (the first Chern class) $H^1(X, \mathcal{O}_X^\times) = H^2(X; \mathbb{Z})$ so if $H^2(X; \mathbb{Z}) = 0$ then $H^1(X, \mathcal{O}_X^\times) = 0$ giving the surjection. \square

Remark. For Stein manifolds we always have $H^p(X, \mathcal{O}_X) = 0$ for $p > 0$ by Cartan's theorem B. Therefore, the second cousin problem is solvable for Stein manifolds when $H^2(X; \mathbb{Z}) = 0$.

25 The Topology of Schemes

Here I want to ask what the topology of schemes “looks like” from the perspective of algebraic topology. The importance of the analytification functor $X \mapsto X^{\text{an}}$ is that it allows us to compute the “correct” topological invariants to complex varieties. However, what happens if we try to compute algebraic topology on the Zariski topology?

Lemma 25.1. Suppose X is a topological space with a dense point $\xi \in X$. Then X is contractible.

Proof. Consider the homotopy $h : X \times I \rightarrow X$ defined by,

$$h(x, t) = \begin{cases} x & t = 0 \\ \xi & t > 0 \end{cases}$$

This is continuous because no nontrivial closed set $Z \subset X$ contains ξ so $h^{-1}(Z) = Z \times \{0\}$ which is closed. Furthermore $h^{-1}(X) = X \times I$ so h is continuous. \square

Remark. In particular, we see that every irreducible scheme is contractible.

However, there are example of varieties which have nontrivial homotopy type.

Example 25.2. <https://math.stackexchange.com/questions/2701914/connected-non-contractible-schemes>