Mathematics W4043 Algebraic Number Theory
Assignment # 1

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1. Take $x, y \in \mathbb{Q}$ then write $x = p^{v_p(x)} \cdot \frac{a_1}{b_1}$ and $y = p^{v_p(x)} \cdot \frac{a_2}{b_2}$ where a_1, b_1, a_2, b_2 are all relativly prime to p and $v_p : \mathbb{Q} \to \mathbb{N}$ is the p-adic valuation. First, if $x = 0$ then by definition $|x|_p = 0$. Otherwise, a_1 and b_1 are well defined and thus, $|x|_p = p^{-v_p(x)} \neq 0$ because $p \neq 0$. Now, $|xy|_p = |p^{v_p(x)+v_p(y)} \cdot \frac{a_1 a_2}{b_1 b_2}|_p$. However, because p does not divide a_1 or a_2 we have $p \not| \, a_1 a_2$ ad similarly $p \not| \, b_1 b_2$. Thus, $v_p(xy) = v_p(x) + v_p(y)$ so,
$$|xy|_p = p^{-v_p(xy)} = p^{-v_p(x)} \cdot p^{-v_p(y)} = |x|_p |y|_p$$
Finally, because $v_p(x), v_p(y) \geq \min\{v_p(x), v_p(y)\}$ we can write,
$$x + y = p^{\min\{v_p(x), v_p(y)\}} \left(p^x \cdot \frac{a_1}{b_1} + p^y \cdot \frac{a_2}{b_2} \right)$$
where x and y are nonnegative. Thus, $v_p(x+y) \geq \min\{v_p(x), v_p(y)\}$ because $\left(p^x \cdot \frac{a_1}{b_1} + p^y \cdot \frac{a_2}{b_2} \right)$ can only contain positive powers of p. Therefore,
$$|x + y|_p \leq p^{-\min\{v_p(x), v_p(y)\}} = \max\{p^{-v_p(x)}, p^{-v_p(y)}\} = \max\{|x|_p, |y|_p\}$$
where I have used the fact that $-\min\{a, b\} = \max\{-a, -b\}$. Thus, $| \bullet |_p$ is a norm.

2. (a) Given a sequence satisfying $a_i \in \mathbb{Q}$ and $\lim\limits_{i \to \infty} |a_i|_p = 0$ then the norm of the series is
$$\left| \sum_{i=0}^{\infty} a_i \right|_p = \lim_{n \to \infty} \left| \sum_{i=0}^{n} a_i \right|_p$$
Consider the mapping $f : \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = |x|_p$. The norm, is the limit of a sequence of elements pf the image of f. Therefore, by Lemma 0.1, the limit must be contained in the closure of the image. Since $\mathrm{Im}(f) = \{p^r \mid r \in \mathbb{Z}\}$ we have that the norm of any series is an element of $\overline{\mathrm{Im}(f)} = \{p^r \mid r \in \mathbb{Z}\} \cup \{0\}$ because there is finite seperation between powers of p but there are also arbitrarily small powers of p. It remains to show that this limit exists. Because $\lim\limits_{i \to \infty} |a_i|_p = 0$ we have for any $\epsilon > 0$ there exists k such that $n > k \implies |a_i|_p < \epsilon$. By the ultrametric property, fot $n, m > k$,
$$\left| \sum_{i=n}^{m} a_i \right|_p \leq \max\{|a_n|, |a_{n+1}|, \cdots, |a_m|\} < \epsilon$$

Furthermore,

$$\left|\left|\sum_{i=0}^{m} a_i\right|_p - \left|\sum_{i=0}^{n} a_i\right|_p\right| = \left|\left|\sum_{i=0}^{n-1} a_i + \sum_{i=n}^{n} a_i\right|_p - \left|\sum_{i=0}^{m} a_i\right|_p\right| \leq \left|\sum_{i=n}^{m} a_i\right|_p < \epsilon$$

thus, the sequence of norms is Cauchy so the limit in $\mathbb{R}$ exists.

(b) Suppose that the series $\sum_{i=0}^{n} a_i$ and $\sum_{i=0}^{n} b_i$ are equivalent. Then,

$$\lim_{n\to\infty}\left|\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} b_i\right|_p = 0$$

Now suppose that $\left|\sum_{i=0}^{\infty} a_i\right|_p = L$. Therefore, for any $\epsilon > 0$ there exist $k_1, k_2 \in \mathbb{N}$ such that,

$$n > k_1 \implies \left|\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} b_i\right|_p < \frac{\epsilon}{2}$$

and likewise,

$$n > k_2 \implies \left|\left|\sum_{i=0}^{n} a_i\right|_p - L\right| < \frac{\epsilon}{2}$$

Now, we can write, when $n > k_1$,

$$-\frac{\epsilon}{2} < \left|\sum_{i=0}^{n} b_i\right|_p - \left|\sum_{i=0}^{n} a_i\right|_p < \frac{\epsilon}{2}$$

Therefore, for $n > \max\{k_1, k_2\}$,

$$-\epsilon < \left|\sum_{i=0}^{n} b_i\right|_p - \left|\sum_{i=0}^{n} a_i\right|_p + \left|\sum_{i=0}^{n} a_i\right|_p - L < \epsilon$$

or rearranging,

$$\left|\left|\sum_{i=0}^{n} b_i\right|_p - L\right| < \epsilon$$

Thus,

$$\left|\sum_{i=0}^{\infty} b_i\right|_p = \lim_{n\to\infty}\left|\sum_{i=0}^{n} b_i\right|_p = L = \left|\sum_{i=0}^{\infty} a_i\right|_p$$

3. (a) Take $x = \frac{a}{b} \in \mathbb{Q}$. By the fundamental theorem of arithmetic, a and b factor into products of primes. We can write $x = \frac{a}{b} = \pm p_1^{r_1} \cdots p_k^{r_k}$ with possibly negative powers where p_i runs through the primes in both a and b. Therefore, $|x|_{p_i} = p_i^{-r_i}$ and for any prime p not in the factorization, $p \not| \, x$ so $|x|_p = p^0 = 1$. Because the factoization is finite, for all but a finite number of primes, $|x|_p = 1$.

(b) Take $a \in \mathbb{Q}$ with $a \neq 0$. Write the prime factoization, $a = \pm p_1^{r_1} \cdots p_k^{r_k}$ remembering that the powers may be negative. Consider the product, which by part (a) is finite,

$$|a| \prod_p |a|_p = |a| \prod_{i=1}^{k} |a|_{p_i} = |a| \prod_{i=1}^{k} p_i^{-r_i} = \frac{|a|}{p_1^{r_1} \cdots p_r^{r_k}} = \frac{|a|}{|a|} = 1$$

4. We can define the homomorphism $i : \mathbb{Q} \to \mathbf{A} = \mathbb{R} \otimes \prod_{\mathbf{p}} \mathbb{Q}_{\mathbf{p}}$ by $i : x \to (x, (x))$ where I have set $a_{\mathbb{R}} = a \in \mathbb{Q} \subset \mathbb{R}$ and for every p, set $a_p = a \in \mathbb{Q} \subset \mathbb{Q}_p$. This function is well-defined because $x \in \mathbb{Q}$ so by the previous problem, $|x|_p = 1$ for all but a finite number of p so the sequence is in the adele group. Therefore, $i(x) \in \mathbf{A}$. This map is injective because if $i(x) = i(y)$ then $(x, (x)) = (y, (y))$ so $x = y$. Finally, this map is a homomorphism because $i(x + y) = (x + y, (x + y)) = (x, (x)) + (y, (y)) = i(x) + i(y)$ since the sum is also a rational number and p-adic addition restricts to rational addition on $\mathbb{Q}$.

   Next, take $x \in \mathbb{Z}$ then $x = \pm p_1^{r_1} \cdots p_k^{r_k}$ with positive r_i so $|x|_{p_i} = p_i^{-r_i}$ and for any p not in the factorization $|x|_p = 1$. Thus, $|x|_p \leq 1$ for every p. Suppose that $x \in \mathbb{Q}$ satisfies $|x|_p \leq 1$ for every p. We can write $x = \frac{a}{b}$ and a and b factor into products of primes so $x = \frac{a}{b} = \pm p_1^{r_1} \cdots p_k^{r_k}$ with possibly negative powers where p_i runs through the primes in both a and b. Therefore, $|x|_{p_i} = p_i^{-r_i} \leq 1$ so by hypothesis $r_i \geq 0$. Using the factorization, $x = \pm p_1^{r_1} \cdots p_k^{r_k} \in \mathbb{Z}$ because no power can be negative.

Lemmas

Lemma 0.1. *Let X be a metric space and a_n a seqeunce in $S \subset X$. If it exists, $\lim\limits_{n \to \infty} a_n \in \overline{S}$.*

Proof. Suppose that for every $\epsilon > 0$ there exists an $k \in \mathbb{N}$ such that $n > k \implies d(a_n, L) < \epsilon$. Then, $a_n \in S$ so $B_\epsilon(L) \cap S \neq \varnothing$ for every $\epsilon > 0$ which implies that $L \in \overline{S}$. $\square$