Remark. Unless otherwise stated, all rings are commutative and unital.

1 Definitions

Definition 1.0.1. An element $p \in A$ is prime if (p) is a prime ideal. Equivalently p is prime if whenever $p \mid xy$ either $p \mid x$ or $p \mid y$.

Definition 1.0.2. An element $r \in A$ which is nonzero and not a unit is irreducible if whenever $r = xy$ either $x \in A^\times$ or $y \in A^\times$.

2 Domains

Definition 2.0.1. A ring A is a domain if A has no zero divisors i.e. if $ab = 0$ then $a = 0$ or $b = 0$.

Proposition 2.0.2. Let A be a domain then any nonzero prime element is irreducible.

Proof. Let $p \in A$ be a prime. Now suppose that $p = xy$ for $x, y \in A$. Thus, $p \mid xy$ so (WLOG) we have $p \mid x$ so $x = pz$ and thus $p = pzy$. However, p is nonzero and A is a domain so $zy = 1$ and thus $y \in A^\times$ proving that p is irreducible. $\square$

3 Principal Ideal Domains

Definition 3.0.1. A principal ideal domain (PID) is a domain A such that every ideal is principal.

Lemma 3.0.2. If A is a PID then A is Noetherian.

Proof. Every ideal is principal and thus finitely generated. $\square$

Lemma 3.0.3. Let A be a PID and $r \in A$ irreducible then (r) is maximal and thus r is prime.

Proof. Consider an intermediate ideal $(r) \subset J \subset A$ then since A is a PID we have $J = (a)$ so $r \in (a)$ and thus $r = ac$ so either $a \in A^\times$ in which case $J = A$ or $c \in A^\times$ in which case $J = (r)$ so (r) is maximal and thus a prime ideal. $\square$

Theorem 3.0.4. Let A be a PID and not a field then $\dim A = 1$.

Proof. Any prime ideal $\mathfrak{p} \subset A$ is principal so $\mathfrak{p} = (p)$ and p is prime. Either $p = 0$ which is prime since A is a domain or p is irreducible and so we have shown (p) is maximal. So every prime ideal is zero or maximal and thus $\dim A \le 1$. If $\dim A = 0$ then (0) is maximal so A is local and any nonzero element is thus invertible so A is a field. $\square$

Theorem 3.0.5 (Kaplansky)**.** Let A be Noetherian then A is a principal ideal ring iff every maximal ideal is prime.

Theorem 3.0.6 (Cohen)**.** A ring A is Noetherian iff every prime ideal is finitely generated.

Corollary 3.0.7. A ring A is a principal ideal ring iff every prime ideal is principal.

4 Unique Factorization Domains

Definition 4.0.1. A domain A is a unique factorization domain (UFD) if every nonzero element has a unique factorization into irreducible elements.

Definition 4.0.2. A factorization ring A is a ring such that every nonzero element has a factorization into irreducible elements.

Lemma 4.0.3. If A is a Noetherian domain then it is a factorization domain.

Proof. Take $a_0 \in A$. If a is irreducible, zero, or a unit then we are done. Then we can write, $a = a_1^{(1)} a_2^{(1)}$ for $a_1, b_1 \notin A^\times$. Continuing in this manner we get,

$$(a) \subsetneq (a_1^{(1)}, a_2^{(1)}) \subsetneq (a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, a_4^{(3)}) \subsetneq \cdots$$

(CHECK THIS) This sequence is proper since if $a = bc$ and $b \in (a)$ then $a = arc$ so $rc = 1$ and thus $c \in A^\times$ contradicting our construction. However, A is Noetherian then the sequence must terminate so at some point the factorization must become irreducible. $\square$

Theorem 4.0.4. Let A be a factorization domain. Then A is a UFD iff every irreducible is prime.

Proof. If A is a UFD and p an irreducible. Let $x, y \in A$ and $p \mid xy$ then p is in the factorization of xy and thus, by uniqueness must be in the factorization of either x or y so $p \mid x$ or $p \mid y$.

Conversely, if A is a factorization domain and every irreducible is prime then given two factorizations of x each irreducible must, by primality, divide an irreducible in the other factorization so they are equal. (DO THIS BETTER) $\square$

Corollary 4.0.5. If A is a PID then A is a UFD.

Proof. If A is a PID then it is Noetherian and thus a factorization domain. Furthermore, every irreducible is prime so A is a UFD. $\square$

4.1 Height One Prime Ideals

Proposition 4.1.1. Let A be Noetherian. Then any principal prime ideal has height at most one.

Proof. Let $\mathfrak{p} = (p) \subset A$ be a principal prime ideal. Then consider the localization which is $A_{(p)}$ Noetherian and the unique maximal ideal $pA_{(p)}$ is principal. Take $N = \text{nilrad}\left(A_{(p)}\right)$ then,

$$\dim A_{(p)}/N = \dim A_{(p)} = \mathbf{ht}\left(\mathfrak{p}\right)$$

but $A_{(p)}/N$ is a Noetherian domain and the unique maximal ideal $pA_{(p)}$ is principal so $A_{(p)}/N$ is a PID and thus $\dim A_{(p)}/N \leq 1$. $\square$

Proposition 4.1.2. If A is a UFD then every prime ideal of height one is principal.

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal with $\mathbf{ht}\left(\mathfrak{p}\right) = 1$. Take any nonzero element $x \in \mathfrak{p}$ and consider its factorization into irreducibles. Since $\mathfrak{p}$ is prime some irreducible factor $p \mid x$ must be in $\mathfrak{p}$ so $(p) \subset \mathfrak{p}$. Since A is a UFD all irreducibles are prime so $(p) \subset \mathfrak{p}$ is prime. However $\mathbf{ht}\left(\mathfrak{p}\right) = 1$ and $(p) \neq (0)$ so $(p) = \mathfrak{p}$ and thus $\mathfrak{p}$ is principal. $\square$

Theorem 4.1.3. Let A be a Noetherian domain. Then A is a UFD iff every height one prime ideal is principal.

Proof. We showed one direction above. Conversely, suppose every height one prime ideal is principal. Since A is a Noetherian domain, it suffices to show that each irreducible is prime. Let r be irreducible and consider a minimal prime $\mathfrak{p} \supset (r)$. Then by Krull's Hauptidealsatz, $\mathfrak{p}$ has height one so by our assumption $\mathfrak{p} = (p)$ is principal. However, $(r) \subset (p)$ so $p \mid r$ but r is irreducible so we must have $(r) = (p) = \mathfrak{p}$ and thus r is prime. $\square$

Theorem 4.1.4 (Krull's Hauptidealsatz)**.** Let $I \subset A$ be an ideal in a Noetherian ring A with n generators then any minimal prime ideal $\mathfrak{p} \supset I$ has height at most n.

5 Simple Modules

Definition 5.0.1. A nonzero R-module is *simple* if it has no nontrivial submodules.

Proposition 5.0.2. Let R be a ring and M an R-module. Then the following are equivalent,

(a) M is simple

(b) $\ell_R(M) = 1$

(c) $M = R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.

Proof. The first two are equivalent by definition. Clearly if $\mathfrak{m} \subset R$ is maximal then $R/\mathfrak{m}$ is simple. Now suppose that M is simple and take a nonzero $x \in M$. Then $(x) = M$ by simplicity so consider $I = \ker(R \xrightarrow{x} M) = \mathrm{Ann}_A(x) = \{r \in R \mid rx = 0\}$. Since $M = Rx$ we know that $M \cong R/I$. However, by the lattice isomorphism theorem, submodules of R/I correspond to ideals above I so since M is simple we must have I maximal. $\square$

6 Artinian Modules

Definition 6.0.1. An R-module M is *noetherian/artinian* if it satisfies the ascending/descending chain condition on submodules.

Theorem 6.0.2. An R-module M has finite length iff it is both noetherian and artinian.

Proof. If M has finite length then clearly it is noetherian and artinian since chains of submodules are bounded in length. Alternatively, simple modules are noetherian and artinian so given a composition series we see that M is noetherian and artinian by repeated extension. Now, conversely, assume that M is noetherian and artinian. By the artinian property we can take a minimal nonzero submodule $M_1 \subset M$. Then M_1 is simple. Either M/M_1 is simple or we may repeat to get $M_2 \supset M_1$ and M_2/M_1 is simple. Thus we get an ascending chain $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$ with M_{i+1}/M_i simple. Since M is Noetherian, this must terminate at $M_n = M$ so we get a finite length composition series showing that M has finite length. $\square$

7 Artinian Rings

Definition 7.0.1. A ring A is *artinian* if it satisfies the descending chain condition on ideals: given a chain of ideals,

$$I_0 \supset I_1 \supset I_2 \supset \cdots$$

the chain stabilizes $I_{n+i} = I_n$.

Remark. A is artinian iff it is artinian as a module over itself.

Proposition 7.0.2. An artinian ring has finitely many maximal ideals.

Proof. Let $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \dots$ be a list of maximal ideals. Then consider the chain,

$$\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supset \cdots$$

By the artinian condition, we must have $\mathfrak{m}_1 \cdots \mathfrak{m}_n = \mathfrak{m}_1 \cdots \mathfrak{m}_n\mathfrak{m}_{n+1}$ for some n. But then by prime avoidence $\mathfrak{m}_{n+1}$ must be one of $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ since $\mathfrak{m}_{n+1} \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n$ so $\mathfrak{m}_{n+1} \supset \mathfrak{m}_i$ and $\mathfrak{m}_i$ is maximal. $\square$

Proposition 7.0.3. Let A be an artinian ring. Then every prime ideal is maximal so $\dim A = 0$.

Proof. Let $\mathfrak{p}$ be prime and $x \notin \mathfrak{p}$. Consider the chain,

$$(x) \supset (x^2) \supset (x^3) \supset \cdots$$

By the artinian condition $(x^n) = (x^{n+1})$ for some n so $x^n = rx^{n+1}$ for some $r \in A$. Thus,

$$x^n(rx - 1) = 0$$

However, $x^n \notin \mathfrak{p}$ so $rx - 1 \in \mathfrak{p}$ and thus $x \in A/\mathfrak{p}$ is invertible so $A/\mathfrak{p}$ is a field and thus $\mathfrak{p}$ is maximal. $\square$

Proposition 7.0.4. Let A be artinian. Then $\text{nilrad}\,(A)$ is a nilpotent ideal.

Proof. Let $I = \text{nilrad}\,(A)$. Consider the chain of ideals,

$$I \supset I^2 \supset I^3 \supset \cdots$$

By the artinian condition, $I^{n+1} = I^n$ for some n. Consider $J = \{x \in A \mid xI^n = 0\}$. If $J \neq R$ we can choose $J' \supsetneq J$ minimal (using the artinian property). Then take $y \in J'$ so by minimality $J' = J + (y)$. Suppose $J + I(y) = J'$ then, since $J \subset \text{Jac}\,(A)$ and (y) is finitely generated, by Nakayama, $J' = J + I(y) = J$ which is false so $J \subset J + I(y) \subsetneq J'$ and thus $J = J + I(y)$ by minimality so $I(y) \in J$. Therefore, $y \cdot I^{n+1} = 0$ but $I^{n+1} = I^n$ so $y \cdot I^n = 0$ and thus $y \in J$ contradicting our situation so $J = R$ and thus $I^n = 0$. $\square$

Proposition 7.0.5. Every artinian ring is a product of local artinian rings: $A_{\mathfrak{m}_i} = A/\mathfrak{m}_i^n$.

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals. Then we know that $\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r} = 0$ for some integers $n_1, \dots, n_r \in \mathbb{Z}$. Therefore, by the Chinese remainder theorem,

$$A = A/(\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r}) = \prod_{i=1}^{r} A/\mathfrak{m}_i^{n_i}$$

Furthermore, $A/\mathfrak{m}_i^{n_i}$ is local because $\mathfrak{m}_i$ is the only maximal ideal above $\mathfrak{m}_i^{n_i}$. Furthermore,

$$A_{\mathfrak{m}_i} = (A/\mathfrak{m}_i^{n_i})_{\mathfrak{m}_i} = A/\mathfrak{m}_i^{n_i}$$

since $A \setminus \mathfrak{m}_i$ is not contained in any maximal ideal of $A/\mathfrak{m}_i^{n_i}$ and thus is invertible. $\square$

Proposition 7.0.6. A ring A is artinian iff it has finite length as a module over itself.

Proof. If A has finite length as an A-module then it satisfies both the ascending and descending chain conditions on A-submodules i.e. ideals thus A is both noetherian and artinian. Conversely, let A be artinian. Since A is a finite product of local artinian rings we may reduce to the case that A is local artinian with maximal ideal $\mathfrak{m}$. Since $\text{nilrad}\,(A) = \mathfrak{m}$ then $\mathfrak{m}^n = 0$ for some n so we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

Then $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a $A/\mathfrak{m}$-module and,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

which must be finite since $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is an artinian module and thus must have finite dimension else there would be a nonterminating descending chains. Thus from the series A has finite length. $\square$

Theorem 7.0.7. A ring A is artinian iff A is noetherian and $\dim A = 0$.

Proof. If A is artinian then it has finite length over itself and thus is noetherian. Also every prime is maximal so $\dim A = 0$. Conversely, suppose that A is noetherian and $\dim A = 0$. Then $\text{Spec}\,(A)$ is a noetherian topological space which has finitely many irreducible componets so A has finitely many minimal primes which are also maximal since $\dim A = 0$. Thus A has finitely many primes all of which are maximal. Since $\dim A = 0$ we have $I = \text{Jac}\,(A) = \text{nilrad}\,(A)$ so any $f \in I$ is nilpotent so I is nilpotent because A is noetherian so I is finitely generated. Thus by the Chines remainder theorem A is a finite product of local rings so we reduce to the case that A is local with maximal ideal $\mathfrak{m}$. Then we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

but $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite $A/\mathfrak{m}$-module since A is noetherian so,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is finite and thus $\ell_A(A)$ is finite from the series showing that A is artinian. $\square$

Proposition 7.0.8. Let A be an artinian ring. Then,

$$\ell_A(A) = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i})$$

Proof. We can write, $A = A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r}$ and thus the formula immediately follows. $\square$

Proposition 7.0.9. Any finite dimensional k-algebra is artinian.

Proof. By dimensionality arguments every descending chain stabilizes. $\square$

Proposition 7.0.10. Let $A \to B$ be a local map and M an B-module of finite length. Then,

$$\ell_A(M) = \ell_B(M) \cdot [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

and in particular $\ell_A(M)$ is finite if $\kappa(\mathfrak{m}_B)$ is a finite extension of $\kappa(\mathfrak{m}_A)$.

Proof. Consider a composition series,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

Then M_i/M_{i-1} is a simple B-module so $M_i/M_{i-1} \cong B/\mathfrak{m}_B = \kappa(\mathfrak{m}_B)$ since B is local. Therefore,

$$\ell_A(M) = \sum_{i=1}^{n} \ell_A(M_i/M_{i-1}) = \sum_{i=1}^{n} \ell_A(\kappa(\mathfrak{m}_B)) = n \cdot [\kappa(B_\mathfrak{m}) : \kappa(A_\mathfrak{m})]$$

where $\ell_A(\kappa(\mathfrak{m}_B)) = \ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$ because $A \to B$ is local and,

$$\ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = \dim_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

□

Corollary 7.0.11. If A is a local artinian finite type k-algebra. Then,

$$\dim_k A = \ell_A(A) \cdot \dim_k (A/\mathfrak{m})$$

in particular A is a finite k-module.

Proof. Viewing A as a module over itself we know it has finite length since A is artinian. Furthermore, $A/\mathfrak{m}$ is a field finitely generated over k and thus a finite extension of k by the Nullstellensatz. Then applying the previous result we conclude. □

Corollary 7.0.12. Let A be an artinian finite type k-algebra. Then,

$$\dim_k A = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}) \cdot \dim_k (A/\mathfrak{m}_i)$$

Proof. Since A is artinian we can write,

$$A = \prod_{i=1}^{r} A_{\mathfrak{m}_i}$$

where $A_{\mathfrak{m}_i}$ are the local artinian factors associated to the finitely many prime ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$. The result follows from above by additivity of the dimensions. □

Remark. We can generalize this to the following proposition.

Proposition 7.0.13. Let A be local with maximal ideal $\mathfrak{m}$ and B be semi-local with maximal ideals $\mathfrak{m}_i$. Let $A \to B$ be a homomorphism of rings such that $\mathfrak{m}_i$ lie over $\mathfrak{m}$ and $[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ is finite. Let M be a finite length B-module. Then,

$$\ell_A(M) = \sum_{i=1}^{n} \ell_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \cdot [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$$

8 Weakly Associated Points

8.1 Weakly Associated Primes

Definition 8.1.1. Let A be a ring and M an A-module. Then a prime $\mathfrak{p} \subset A$ is *weakly associated* to M if $\mathfrak{p}$ is minimal over $\text{Ann}_A(m)$ for some $m \in M$. We denote these primes $\text{WAss}_A(M)$.

Lemma 8.1.2. Let M be an A module then the natural map,

$$M \to \prod_{\mathfrak{p} \in \text{WAss}_A(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Suppose that $m \in M$ maps to zero. Then $\mathfrak{p} \not\supset \text{Ann}_A(m)$ for each $\mathfrak{p} \in \text{WAss}_A(M)$ which implies $\text{Ann}_A(m) = A$ since otherwise some associated prime will be minimal over $\text{Ann}_A(m)$. Thus $m = 0$. $\square$

Lemma 8.1.3. Let M be an A-module. Then,

$$M = (0) \iff \text{WAss}_A(M) = \varnothing$$

Proof. If $M = (0)$ then this is clear. Otherwise, by the previous lemma $M \hookrightarrow (0)$ is injective so $M = (0)$. $\square$

Lemma 8.1.4. Let A be a ring and M an A-module. Then,

$$\mathfrak{p} \in \text{WAss}_A(M) \iff \mathfrak{p}A_{\mathfrak{p}} \in \text{WAss}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

Proof. Consider the exact sequence for each $m \in M$,

$$0 \longrightarrow \text{Ann}_A(m) \longrightarrow A \xrightarrow{\ m\ } M$$

Since localization is exact, we get an exact sequence,

$$0 \longrightarrow (\text{Ann}_A(m))_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}} \xrightarrow{\ m\ } M_{\mathfrak{p}}$$

Therefore, $\text{Ann}_{A_{\mathfrak{p}}}(m) = (\text{Ann}_A(m))_{\mathfrak{p}}$. If $\mathfrak{p} \supset \text{Ann}_A(m)$ is minimal then $\mathfrak{p}A_{\mathfrak{p}} \subset (\text{Ann}_A(m))_{\mathfrak{p}} = \text{Ann}_{A_{\mathfrak{p}}}(m)$ is minimal. Conversely, if $\mathfrak{p}A_{\mathfrak{p}} \supset \text{Ann}_{A_{\mathfrak{p}}}(m/s)$ is minimal then,

$$\text{Ann}_{A_{\mathfrak{p}}}(m/s) = \text{Ann}_{A_{\mathfrak{p}}}(m) = (\text{Ann}_A(m))_{\mathfrak{p}}$$

which implies that $\mathfrak{p} \supset \text{Ann}_A(m)$ is minimal because if $x \in \text{Ann}_A(m)$ and $x \notin \mathfrak{p}$ then $(\text{Ann}_A(m))_{\mathfrak{p}} = A_{\mathfrak{p}}$ and any prime $\mathfrak{q}$ such that $\mathfrak{p} \subset \mathfrak{q} \subset \text{Ann}_A(m)$ implies that $\mathfrak{q}A_{\mathfrak{p}}$ is intermediate. $\square$

Lemma 8.1.5. Let A be a ring and M an A-module. Then $\text{WAss}_A(M) \subset \text{Supp}_A(M)$ furthermore any minimal element of $\text{Supp}_A(M)$ is an element of $\text{WAss}_A(M)$.

Proof. Since $\mathfrak{p} \supset \text{Ann}_A(m)$ we know $M_{\mathfrak{p}} \neq 0$ since m is nonzero in $M_{\mathfrak{p}}$. Furthermore, suppose that $\mathfrak{p} \in \text{Supp}_A(M)$ is minimal. Then $\text{Supp}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}\}$ and $M_{\mathfrak{p}} \neq 0$ so $\text{WAss}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \{A_{\mathfrak{p}}\}$ and thus $\mathfrak{p} \in \text{WAss}_A(M)$. $\square$

Lemma 8.1.6. Let A be a ring and M an A-module and $S \subset A$ a multiplicative subset. Then.

(a) $\mathrm{WAss}_A\left(S^{-1}M\right) = \mathrm{WAss}_{S^{-1}A}\left(S^{-1}M\right)$

(b) $\mathrm{WAss}_A\left(M\right) \cap \mathrm{Spec}\left(S^{-1}A\right) = \mathrm{WAss}_A\left(S^{-1}M\right)$.

Proof. We have,

$$\mathfrak{p} \in \mathrm{WAss}_A\left(S^{-1}M\right) \iff \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{WAss}_{A_{\mathfrak{p}}}\left(S^{-1}M_{\mathfrak{p}}\right)$$

For $\mathfrak{p} \in \mathrm{Spec}\left(S^{-1}A\right)$ (i.e. $S \subset A \setminus \mathfrak{p}$) we have $S^{-1}M_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $(S^{-1}A)_{\mathfrak{p}} = A_{\mathfrak{p}}$ so both equalities hold. Otherwise, $\mathfrak{p}A_{\mathfrak{p}}$ containes an element of S so $\mathfrak{p}A_{\mathfrak{p}}$ has some nonzero divisor on $S^{-1}M_{\mathfrak{p}}$ and thus $\mathfrak{p} \notin \mathrm{WAss}_A\left(S^{-1}M\right)$. $\square$

Proposition 8.1.7. Let A be a ring M an A-module then $\mathfrak{p} \in \mathrm{Supp}_A\left(M\right)$ if and only if there exists $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \in \mathrm{WAss}_A\left(M\right)$. Therefore,

$$\bigcap_{\mathfrak{p} \in \mathrm{Supp}_A(M)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{WAss}_A(M)} \mathfrak{p}$$

Proof. Take $\mathfrak{p} \in \mathrm{Supp}_A\left(M\right)$ so $M_{\mathfrak{p}} \neq 0$ and then $\mathrm{Ass}_A\left(M_{\mathfrak{p}}\right) \neq \varnothing$. Using the previous lemma, there exists $\mathfrak{q} \in \mathrm{Ass}_A\left(M_{\mathfrak{p}}\right) = \mathrm{Ass}_A\left(M\right) \cap \{\mathfrak{q} \mid \mathfrak{q} \subset \mathfrak{p}\}$. Furthermore, the support is an upward set (if $\mathfrak{q} \subset \mathfrak{p}$ and $M_{\mathfrak{q}} \neq 0$ then $M_{\mathfrak{p}} \neq 0$ since $M_{\mathfrak{p}} \to M_{\mathfrak{q}}$ is localization). Thus, if we have $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \in \mathrm{Ass}_A\left(M\right) \subset \mathrm{Supp}_A\left(M\right)$ then $\mathfrak{p} \in \mathrm{Supp}_A\left(M\right)$. $\square$

Lemma 8.1.8. Let $M \hookrightarrow N$ be an injection of A-modules. Then $\mathrm{WAss}_A\left(M\right) \subset \mathrm{WAss}_A\left(N\right)$.

Proof. This follows because the set of annihilators of elements of M is a subset of the set of annihilators of elements of N. $\square$

Lemma 8.1.9. Consider an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

then,

$$\mathrm{WAss}_A\left(M_2\right) \subset \mathrm{WAss}_A\left(M_1\right) \cup \mathrm{WAss}_A\left(M_3\right)$$

Proof. Let $\mathfrak{p} \in \mathrm{WAss}_A\left(M_2\right)$ and $\mathfrak{p} \notin \mathrm{WAss}_A\left(M_1\right)$. Using the previous lemma it suffices to consider the case that A is local with maximal ideal $\mathfrak{p}$ (since we may localize the exact sequnence at $\mathfrak{p}$). Then $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A\left(m\right)$ for some $m \in M_2$ not in the image of $M_1 \to M_2$ (else $\mathfrak{p} \in \mathrm{WAss}_A\left(M_1\right)$). Therefore $\bar{m} \in M_3$ is nonzero and $\mathrm{Ann}_A\left(\bar{m}\right) \supset \mathrm{Ann}_A\left(m\right)$ but $\mathrm{Ann}_A\left(\bar{m}\right)$ is proper since $\bar{m}$ is nonzero and thus contained in $\mathfrak{p}$. Since $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A\left(m\right)$ it must also be minimal over $\mathrm{Ann}_A\left(\bar{m}\right)$ and thus we conclude that $\mathfrak{p} \in \mathrm{WAss}_A\left(M_3\right)$. $\square$

Lemma 8.1.10. Let A be a ring and M and A-module. Then,

$$\bigcup_{\mathfrak{p} \in \mathrm{WAss}_A(M)} = \{\text{zero divisors on } M\}$$

Proof. Let $m \in M$ have zero divisors then there is exists a minimal prime (by Zorn's Lemma) above $\mathrm{Ann}_A\left(m\right)$ which must be associated. Conversely, if $f \in \mathfrak{p} \in \mathrm{WAss}_A\left(M\right)$ then $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A\left(m\right)$ for some $m \in M$. Then $R = (A/\mathrm{Ann}_A\left(m\right))_{\mathfrak{p}}$ has a unique minimal prime $\mathfrak{p}$ so $\mathfrak{p} = \mathrm{nilrad}\left(R\right)$ and thus $gf^n \in \mathrm{Ann}_A\left(m\right)$ for some least $n > 0$ and $g \notin \mathfrak{p}$. Thus $gf^n m = 0$ so $f(gf^{n-1}m) = 0$ but $gf^{n-1}m \neq 0$ because n is minimal so f is a zero divisor. $\square$

Proposition 8.1.11. Let $(A, \mathfrak{m})$ be a local ring then $\mathfrak{m} \in \mathrm{WAss}_A\left(A\right)$ iff $\mathfrak{m} = \{\text{zero divisors}\}$.

Proof. Immediate from the above since zero divisors are not units and thus contained in $\mathfrak{m}$. $\square$

Corollary 8.1.12. Given a prime $\mathfrak{p} \in \text{Spec}(A)$ and an A-module M we have,

$$\mathfrak{p} \in \text{WAss}_A(A) \iff \mathfrak{p}A_{\mathfrak{p}} = \{\text{zero divisors of } A_{\mathfrak{p}}\}$$

Proposition 8.1.13. Let A be reduced then $\text{WAss}_A(A)$ are exactly the minimal primes of A.

Proof. The minimal primes are in $\text{WAss}_A(A)$ by Lemma 8.1.5. Because $\mathfrak{p} \in \text{WAss}_A(A) \iff \mathfrak{p}A_{\mathfrak{p}} \in \text{WAss}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ is suffices to consider the case of a reduced local ring $(R, \mathfrak{m})$ and $\mathfrak{m} \in \text{WAss}_R(R)$. Then $\mathfrak{m}$ is minimal over $\text{Ann}_R(x)$ for some $x \in \mathfrak{m}$ so $\mathfrak{m} = \sqrt{\text{Ann}_R(x)}$. Thus $x^n \in \text{Ann}_R(x)$ so $x^{n+1} = x \cdot x^n = 0$ so $x = 0$ because R is reduced a contradiction unless $\mathfrak{m} = 0$ so R is a field so $\mathfrak{m}$ is minimal showing that $\mathfrak{p}A_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ and thus $\mathfrak{p} \subset A$ are minimal primes and that $A_{\mathfrak{p}}$ is a field. $\square$

Lemma 8.1.14. Let A be a ring and $\mathfrak{p} \subset A$ a prime then $\text{WAss}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

Proof. For nonzero $a \in A/\mathfrak{p}$ (i.e. $a \notin \mathfrak{p}$) the set $\text{Ann}_A(a) = \{r \in A \mid ra \in \mathfrak{p}\} = \mathfrak{p}$ since $\mathfrak{p}$ is prime and therefore therefore $\mathfrak{p}$ is the unique minimal prime over an annihilator. $\square$

Proposition 8.1.15. Let A be a ring and M a Noetherian A-module. Then,

(a) there exists a finite filtration,

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

such that each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \text{Supp}_A(M)$

(b) for any such filtration, $\text{WAss}_A(M) \subset \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$

(c) $\text{WAss}_A(M)$ is finite.

Proof. Since $M \neq (0)$ there is some $\mathfrak{p} \in \text{WAss}_A(M)$ so we have an injection $A/\mathfrak{p} \to M$ let $M_1 \subset M$ be the image of this map so $M_1/M_0 \cong A/\mathfrak{p}_1$. Now take M/M_1 and $\mathfrak{p}_2 \in \text{WAss}_A(M/M_1)$ then we have an injection $A/\mathfrak{p}_2 \to M/M_1$ so take $\bar{M}_2$ to be the image inside M/M_1 and M_2 its preimage in M. Then $M_2/M_1 \cong A/\mathfrak{p}_2$ and continuing by induction we construct a sequence,

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$$

with $M_i/M_{i-1} = A/\mathfrak{p}_i$ and

$$\mathfrak{p}_i \in \text{WAss}_A(M/M_{i-1}) \subset \text{Supp}_A(M/M_{i-1}) \subset \text{Supp}_A(M)$$

However, M is Noetherian so this sequence must stabilize but it is striclty increasing when $M_i \subset M$ is proper. Thus, $M_n = M$ for some n.

For any such filtration we get exact sequences,

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow A/\mathfrak{p}_{i+1} \longrightarrow 0$$

Assume for induction that $\text{Ass}_A(M_i) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_i\}$ then, by Lemma 8.1.9,

$$\text{WAss}_A(M_{i+1}) \subset \text{WAss}_A(M_i) \cup \text{WAss}_A(A/\mathfrak{p}_{i+1}) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_{i+1}\}$$

proving (b) by induction. (c) follows directly from (a) and (b). $\square$

8.2 Associated Primes

Definition 8.2.1. Let A be a ring and M an A-module. We say that $\mathfrak{p} \subset A$ is an *associated prime* of M if $\mathfrak{p} = \mathrm{Ann}_A(m)$ for some $m \in M$. We write $\mathrm{Ass}_A(M)$ for the set of associated primes of M.

Remark. Note $\mathfrak{p} = \mathrm{Ann}_A(m) \iff A/\mathfrak{p} \hookrightarrow M$ via $a \mapsto a \cdot m$.

Remark. Clearly $\mathrm{Ass}_A(M) \subset \mathrm{WAss}_A(M)$. We will see equality holds when A is Noetherian.

Lemma 8.2.2. Given an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

we have,

$$\mathrm{Ass}_A(M_2) \subset \mathrm{Ass}_A(M_1) \cup \mathrm{Ass}_A(M_3)$$

Proof. If $\mathfrak{p} \in \mathrm{Ass}_A(M)$ then we have an embedding

$$A/\mathfrak{p} \hookrightarrow M_2$$

which is injective and $\iota(A/\mathfrak{p}) \cap N_1 = (0)$ then we get an injective map $A/\mathfrak{p} \to M_3$ so $\mathfrak{p} \in \mathrm{Ass}_A(M_3)$. If $\iota(A/\mathfrak{p}) \cap M_1 \neq (0)$ then take nonzero $n \in \iota(A/\mathfrak{p}) \cap M_1$. Then $\mathrm{Ann}_A(n) = \mathrm{Ann}_A(\iota(x))$ for $x \in A/\mathfrak{p}$ nonzero. However, if $a \cdot \iota(x) = 0$ then $\iota(a \cdot x) = 0$ but ι is injective so $a \cdot x = 0$ and thus $\mathrm{Ann}_A(\iota(x)) = \mathrm{Ann}_A(x) = \mathfrak{p}$ because if $a \cdot x \in \mathfrak{p}$ for $x \notin \mathfrak{p}$ then $a \in \mathfrak{p}$. ☐

Lemma 8.2.3. Let $S_{M,\mathfrak{p}} = \{\mathrm{Ann}_A(m) \subset \mathfrak{p} \mid m \in M \setminus \{0\}\}$ then any maximal element in $S_{M,\mathfrak{p}}$ is a prime ideal.

Proof. Let $\mathfrak{q} \in S_{M,\mathfrak{p}}$ be maximal with $\mathfrak{q} = \mathrm{Ann}_A(m)$ for $m \neq 0$. Suppose $ab \in \mathfrak{q}$ and $a, b \notin \mathfrak{q}$. Then $\mathfrak{q} \subsetneq \mathrm{Ann}_A(am)$ since $b \in \mathrm{Ann}_A(am) \setminus \mathrm{Ann}_A(m)$ so by maximality $\mathrm{Ann}_A(am) \not\subset \mathfrak{p}$. Choose $s \in \mathrm{Ann}_A(am) \setminus \mathfrak{p}$. Then $a \in \mathrm{Ann}_A(sm)$ so $\mathrm{Ann}_A(m) \subsetneq \mathrm{Ann}_A(sm)$ and thus by maximality we can choose $t \in \mathrm{Ann}_A(sm) \setminus \mathfrak{p}$ so $st \in \mathrm{Ann}_A(m) \subset \mathfrak{p}$ but $s, t \notin \mathfrak{p}$ contradicting the primality of $\mathfrak{p}$. Thus $\mathfrak{q}$ is prime. ☐

Proposition 8.2.4. Let A be Noetherian and M be an A-module. Then,

$$\mathrm{Ass}_A(M) = \mathrm{WAss}_A(M)$$

In particular, $\mathrm{Ass}_A(M) \neq \varnothing$ and all other properties of $\mathrm{WAss}_A(M)$ apply to $\mathrm{Ass}_A(M)$.

Proof. $\mathrm{Ass}_A(M) \subset \mathrm{WAss}_A(M)$ is obvious. If $\mathfrak{p} \in \mathrm{WAss}_A(M)$ then $\mathfrak{p} \supset \mathrm{Ann}_A(m)$ for some $m \in M$ and thus m is nonzero in $M_{\mathfrak{p}}$ so $\mathfrak{p} \in \mathrm{Supp}_A(M)$. Let A be Noetherian then ascending chains in $S_{M,\mathfrak{p}}$ stabilize and thus by Zorn's Lemma every annhilator $\mathrm{Ann}_A(m) \subset \mathfrak{p}$ is contained in some maximal $\mathrm{Ann}_A(m') \subset \mathfrak{p}$. Thus, if $\mathfrak{p} \in \mathrm{WAss}_A(M)$ then $\mathfrak{p}$ is a minimal prime over some $\mathrm{Ann}_A(m)$ so $\mathfrak{p} = \mathrm{Ann}_A(m')$ since $\mathrm{Ann}_A(m')$ is prime and $\mathrm{Ann}_A(m) \subset \mathrm{Ann}_A(m') \subset \mathfrak{p}$. ☐

Lemma 8.2.5. Let A be a ring and M an A-module and $S \subset A$ a multiplicative subset. Then.

(a) $\mathrm{Ass}_A(S^{-1}M) = \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$

(b) $\mathrm{Ass}_A(M) \cap \mathrm{Spec}(S^{-1}A) \subset \mathrm{Ass}_A(S^{-1}M)$ with equality when A is Noetherian.

Proof. Tag 05BZ. ☐

Proposition 8.2.6. Let A be a Noetherian ring and M a finite A-module. Then,

(a) there exists a finite filtration,

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

such that each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_A(M)$

(b) for any such filtration, $\mathrm{Ass}_A(M) \subset \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$

(c) $\mathrm{Ass}_A(M)$ is finite.

Proof. M is a Noetherian module so this applies directly from Prop. 8.2.6. □

8.3 Primary Decomposition

Remark. In this section we let A be a Noetherian ring.

Definition 8.3.1. An A-module M is called coprimary if $\mathrm{Ass}_A(M) = \{\mathfrak{p}\}$ and if $N \subset M$ we say that N is $\mathfrak{p}$-primary if M/N is coprimary with $\mathrm{Ass}_A(M/N) = \{\mathfrak{p}\}$.

Lemma 8.3.2. M is coprimary iff any zero divisor of M is locally nilpotent i.e. if $a \cdot m = 0$ for some $m \in M \setminus \{0\}$ then $\forall m' \in M : a^n \cdot m' = 0$ for some n.

Proof. Assume that M is coprimary, $\mathrm{Ass}_A(M) = \{\mathfrak{p}\}$. If $x \in M$ is nonzero then Ax is a nonzero submodule of M so $\mathrm{Ass}_A(Ax) = \{\mathfrak{p}\}$ since it is nonempty. Therefore, $\mathfrak{p}$ is a minimal element in $\mathrm{Supp}_A(Ax) = V(\mathrm{Ann}_A(x))$ because $Ax \cong A/\mathrm{Ann}_A(x)$. Thus, $\sqrt{\mathrm{Ann}_A(x)} = \mathfrak{p}$. If a is a zero divisor of M then $a \in \mathfrak{p}$ so $a^n \in \mathrm{Ann}_A(x)$ so a is locally nilpotent. Converely, assume that the set of zero divisors equals the set of locally nilpotent elements. Take $\mathfrak{p}$ to be the ideal of all locally nilpotents. Take $\mathfrak{q} \in \mathrm{Ass}_A(M)$ then $\mathfrak{q} = \mathrm{Ann}_A(x)$ for some x. If $a \in \mathfrak{p}$ then $a^n \cdot x = 0$ for some n implies that $a^n \in \mathfrak{q}$ so $a \in \mathfrak{q}$. so $\mathfrak{p} \subset \mathfrak{q}$. Furthermore,

$$\bigcup_{\mathfrak{q} \in \mathrm{Ass}_A(M)} \mathfrak{q} = \{\text{zero divisors}\} = \mathfrak{p}$$

so for any $\mathfrak{q} \in \mathrm{Ass}_A(M)$ we have $\mathfrak{q} \subset \mathfrak{p}$. Thus, $\mathfrak{p} = \mathfrak{q}$ so $\mathrm{Ass}_A(M)$ constains a unique prime. □

Corollary 8.3.3. If $I \subset A$ is an ideal then $\mathrm{Ass}_A(A/I) = \{\mathfrak{p}\}$ if and only if I is a primary ideal and in that case $\sqrt{I} = \mathfrak{p}$.

Proof. Consider $I \subset A$ and A/I is coprimary then take $x, y \in A$ such that $y \notin I$ and $\bar{x} \cdot \bar{y} = 0$ in A/I. Then $\bar{x}$ is a zero divisor of A/I so it is locally nilpotent by the above. Thus, $\bar{x}^n \cdot 1 = 0$ for some n so $x^n \in I$ so $x \in \sqrt{I}$ and thus I is primary. Furthermore,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Supp}_A(A/I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(A/I)} \mathfrak{p} = \mathfrak{p}$$

since $\mathrm{Ass}_A(M)$ is the set of minimal primes of $\mathrm{Supp}_A(M)$ and $\mathrm{Ass}_A(A/I) = \mathfrak{p}$. □

Definition 8.3.4. Let M be an A-module and $N \subset M$. We say N has a primary decomposition if,

$$N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

where each Q_i is primary. Moreover, we say that this decomposition is irredundant if

(a) if $i \neq j$ then $\mathrm{Ass}_A(M/Q_i) \neq \mathrm{Ass}_A(M/Q_j)$

(b) we cannot remove any Q_j from the intersection.

Lemma 8.3.5. Let M be an A-module then,

(a) If $Q_1, Q_2 \subset M$ are $\mathfrak{p}$-primary then $Q_1 \cap Q_2$ is $\mathfrak{p}$-primary.

(b) If $N = Q_1 \cap \cdots \cap Q_n$ is a irredundant primary decomposition and for each i, Q_i is $\mathfrak{p}_i$-primary then,
$$\mathrm{Ass}_A(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

Proof. Consider the injection,

$$0 \longrightarrow M/Q_1 \cap Q_2 \hookrightarrow M/Q_1 \oplus M/Q_2$$

which implies that,

$$\mathrm{Ass}_A(M/Q_1 \cap Q_2) \subset \mathrm{Ass}_A(M/Q_1 \oplus M/Q_2) = \mathrm{Ass}_A(M/Q_1) \cup \mathrm{Ass}_A(M/Q_2) = \{\mathfrak{p}\}$$

proving the first. For the second, consider the injection,

$$M/N \hookrightarrow M/Q_1 \oplus \cdots \oplus M/Q_n$$

which implies that,

$$\mathrm{Ass}_A(M/N) \subset \mathrm{Ass}_A(M/Q_1) \cup \cdots \cup \mathrm{Ass}_A(M/Q_n) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

We need to show that $\mathfrak{p}_i \in \mathrm{Ass}_A(M/N)$ for each i. We have the exact sequence,

$$0 \longrightarrow N \longrightarrow Q_2 \cap \cdots \cap Q_n \longrightarrow M/Q_1$$

and therefore,

$$(Q_2 \cap \cdots \cap Q_n)/N \hookrightarrow M/Q_1$$

which implies that,
$$\mathrm{Ass}_A((Q_2 \cap \cdots \cap Q_n)/N) \subset \mathrm{Ass}_A(M/Q_1) = \{\mathfrak{p}_1\}$$

so since it is nonempy we have,

$$\{\mathfrak{p}_1\} = \mathrm{Ass}_A((Q_2 \cap \cdots \cap Q_n)/N) \subset \mathrm{Ass}_A(M/N)$$

where the inclusion holds via the exact sequence,

$$0 \longrightarrow N \longrightarrow Q_2 \cap \cdots \cap Q_n \longrightarrow M/N$$

The same argument holds for each i. ☐

Theorem 8.3.6. Let M be Noetherian. For each $\mathfrak{p} \in \mathrm{Ass}_A(M)$, there exist $Q_{\mathfrak{p}} \subset M$ which are $\mathfrak{p}$-primary such that,
$$\bigcap_{\mathfrak{p} \subset \mathrm{Ass}_A(M)} Q_{\mathfrak{p}} = 0$$

Proof. Fix $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and consider the set $S_\mathfrak{p} = \{Q \subset M \mid \mathfrak{p} \notin \mathrm{Ass}_A(Q)\} \neq \varnothing$ since the zero module is contained in this set. Since M is Noetherian ascending chains stabilize so by Zorn's lemma there exists a maximal element $Q_\mathfrak{p} \in S_\mathfrak{p}$. We know,

$$\mathrm{Ann}_A(M/Q_\mathfrak{p}) \neq \varnothing$$

since we have $M/Q_\mathfrak{p} \neq (0)$. Otherwise, $M = Q_\mathfrak{p}$ which implies $\mathfrak{p} \in \mathrm{Ass}_A(Q_\mathfrak{p})$ but $Q_\mathfrak{p} \in S_\mathfrak{p}$. Let $\mathfrak{p}' \in \mathrm{Ass}_A(M/Q_\mathfrak{p})$ and suppose that $\mathfrak{p}' \neq \mathfrak{p}$ then we have,

$$A/\mathfrak{p}' \hookrightarrow M/Q_\mathfrak{p}$$

The image of this embedding is a submodule, $Q_\mathfrak{p} \subsetneq Q' \subset M$ such that $Q'/Q_\mathfrak{p} \cong A/\mathfrak{p}'$ implying that,

$$\mathrm{Ass}_A(Q'/Q_\mathfrak{p}) = \{\mathfrak{p}'\}$$

Thus we have an exact sequence,

$$0 \longrightarrow Q_\mathfrak{p} \longrightarrow Q' \longrightarrow A/\mathfrak{p} \longrightarrow 0$$

which implies that $\mathrm{Ass}_A(Q') \subset \mathrm{Ass}_A(Q_\mathfrak{p}) \cup \mathrm{Ass}_A(A/\mathfrak{p}') = \mathrm{Ass}_A(Q_\mathfrak{p}) \cup \{\mathfrak{p}'\}$. However, this contradicts the fact that $Q_\mathfrak{p}$ is maximal in $S_\mathfrak{p}$ since $Q' \in S_\mathfrak{p}$ as long as $\mathfrak{p}' \neq \mathfrak{p}$. Therefore, $\mathfrak{p}' = \mathfrak{p}$ so $\mathrm{Ass}_A(A/Q_\mathfrak{p}) = \{\mathfrak{p}\}$. Now consider,

$$\mathrm{Ass}_A\left(\bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(M)} Q_\mathfrak{p}\right) \subset \bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(M)} \mathrm{Ass}_A(Q_\mathfrak{p}) = \varnothing$$

because for any $\mathfrak{p}$ we know $\mathfrak{p} \notin \mathrm{Ass}_A(Q_\mathfrak{p})$. Therefore,

$$\bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(M)} Q_\mathfrak{p} = (0)$$

since it has no associated primes. $\square$

Corollary 8.3.7. If M is a finite A-module then any submodule has a primary decomposition.

Proof. Let $N \subset M$ be a submodule. Apply the theorem to $\bar{M} = M/N$ which has finite type so $\mathrm{Ass}_A(M/N)$ is finite. Write, $\mathrm{Ass}_A(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Therefore, there exist primary ideals Q_i such that,

$$Q_{\mathfrak{p}_1} \cap \cdots \cap Q_{\mathfrak{p}_r} = (0)$$

in M/N. Take Q_i to be the preimage of $Q_{\mathfrak{p}_i}$. Thus,

$$Q_1 \cap \cdots \cap Q_r = N$$

and

$$M/Q_i \cong \bar{M}/Q_{\mathfrak{p}_i} \implies \mathrm{Ass}_A(M/Q_i) = \{\mathfrak{p}_i\}$$

$\square$

8.4 Weakly Associated Points

Definition 8.4.1. Let X be a scheme and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then we define,

(a) $x \in X$ is *weakly associated* to $\mathscr{F}$ if $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is weakly associated to $\mathscr{F}_x$

(b) $\text{WAss}_{\mathcal{O}_X}(\mathscr{F})$ is the set of weakly associated points of $\mathscr{F}$

(c) the (weakly) associated points of X are $\text{WAss}_{\mathcal{O}_X}(\mathcal{O}_X)$.

Proposition 8.4.2. Let $X = \text{Spec}(A)$ and $\mathscr{F} = \widetilde{M}$ be a quasi-coherent $\mathcal{O}_X$-module then we have,

$$\text{WAss}_{\mathcal{O}_X}(\mathscr{F}) = \text{WAss}_A(M)$$

Proof. Immediate consequence of Lemma 8.1.4. □

Proposition 8.4.3. Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then,

$$\mathscr{F} = 0 \iff \text{WAss}_{\mathcal{O}_X}(\mathscr{F}) = 0$$

Proof. Choose an affine open cover $U_i = \text{Spec}(A_i)$ such that $\mathscr{F}|_{U_i} = \widetilde{M_i}$. Then $\text{WAss}_A(M_i) = \text{WAss}_{\mathcal{O}_X}(\mathscr{F}) \cap U_i = \varnothing$ so $M_i = 0$ and thus $\mathscr{F} = 0$. □

Proposition 8.4.4. Let X be a scheme and $\mathscr{F} \to \mathscr{G}$ a morphism of quasi-coherent $\mathcal{O}_X$-modules. If $\mathscr{F}_x \to \mathscr{G}_x$ is injective for each $x \in \text{WAss}_{\mathcal{O}_X}(\mathscr{F})$ then $\mathscr{F} \to \mathscr{G}$ is injective.

Proof. Consider the sequence,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}$$

Since $\mathscr{F}_x \to \mathscr{G}_x$ is an injection $\mathscr{K}_x = 0$ for each $x \in \text{WAss}_{\mathcal{O}_X}(\mathscr{F})$. Furthermore, $\text{WAss}_{\mathcal{O}_X}(\mathscr{K}) \subset \text{WAss}_{\mathcal{O}_X}(\mathscr{F})$ and thus $\text{WAss}_{\mathcal{O}_X}(\mathscr{K}) = \varnothing$ so $\mathscr{K} = 0$. □

8.5 Associated Points: the Noetherian Case

Remark. By analogy, we might define an *associated point* of $\mathscr{F}$ on X to be a point $x \in X$ such that $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is an associated prime of $\mathscr{F}_x$. However, this definition is problematic because, in general, associated primes do not play nicely with localization. In particular $\mathfrak{p} \in \text{Ass}_A(M) \implies \mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ but the converse may not hold. Therefore, we may have a scheme X and a quasi-coherent sheaf $\mathscr{F}$ such that on an affine open $U = \text{Spec}(A)$ with $\mathscr{F}|_U = \widetilde{M}$ we have $\mathfrak{p} \in \text{Ass}_A(M)$ but $\mathfrak{p} = x \in X$ is not as associated point of $\mathscr{F}$ on X. To recify this pathology, we only consider associated points on locally noetherian schemes in which case there is no difference between weakly associated points and associated points.

Definition 8.5.1. Let X be a locally noetherian scheme and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. We say $x \in X$ is an *associated point* of $\mathscr{F}$ if x is a *weakly associated point*. Likewise we write,

$$\text{Ass}_{\mathcal{O}_X}(\mathscr{F}) = \text{WAss}_{\mathcal{O}_X}(\mathscr{F})$$

Remark. Notice this definition is purely notational. In the locally noetherian case we simply will write $\text{Ass}_{\mathcal{O}_X}(\mathscr{F})$ for $\text{WAss}_{\mathcal{O}_X}(\mathscr{F})$ as a reminder that these sets behave as expected for associated points in the case of Noetherian rings.

Proposition 8.5.2. Let X be noetherian and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module. Then $\text{Ass}_{\mathcal{O}_X}(\mathscr{F})$ is finite.

Proof. Since X is quasi-compact we may choose a finite open cover $U_i = \text{Spec}(A_i)$ with A_i Noetherian on which $\mathscr{F}|_{U_i} = \widetilde{M_i}$ for finite A_i-modules. Then $\text{Ass}_{\mathcal{O}_X}(\mathscr{F}) \cap U = \text{Ass}_{A_i}(M_i)$ each of which is finite since M_i is a Noetherian module. □

9 Depth

9.1 Definitions

Definition 9.1.1. Let A be a ring $I \subset A$ an ideal and M a finite A-module. Then $x_1, \dots, x_r \in I$ are an *M-regular sequence in I* if

(a) x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$ for each $i \in \{1, \dots, r\}$

(b) $M/(x_1, \dots, x_r)M$ is nonzero.

We say that $\operatorname{depth}_I(M)$ is the supremum of the lengths of M-regular sequence in I unless $IM = M$ in which case $\operatorname{depth}_I(M) = \infty$.

Remark. If $IM \subsetneq M$ then $\operatorname{depth}_I(M) = 0$ iff $I \subset \{\text{zero divisors on } M\}$.

Remark. If $(A, \mathfrak{m})$ is a local ring then we define $\operatorname{depth}(M) := \operatorname{depth}_{\mathfrak{m}}(M)$.

9.2 The Cohomological Criterion

Lemma 9.2.1. Let A be a Noetherian ring, $I \subset R$ an ideal, and M a finite A-module with $IM \neq M$. Then the following are equivalent,

(a) $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < n$ and all finite A-modules N with $\operatorname{Supp}_A(N) \subset V(I)$

(b) $\operatorname{Ext}^i_A(A/I, M) = 0$ for all $i < n$

(c) there exists a finite A-module N with $\operatorname{Supp}_A(N) = V(I)$ and $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < n$

(d) there exists an M-regular sequence $x_1, \dots, x_n \in I$ of length n

and therefore $\operatorname{depth}_I(M) = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}^i_A(A/I, M) \neq 0\}$.

Proof. Clearly (a) $\implies$ (b) $\implies$ (c). Now we show that (c) $\implies$ (d).

Finally, we need to show that (d) $\implies$ (a). (DOOOOOOOOOOOOOOOOOOOOOOO!! OR SPLIT UP THIS PROOF!!) ☐

Remark. From here on, let A be a Noetherian ring and $I \subset A$ an ideal and M a finite A-module with $IM \neq M$.

Lemma 9.2.2. Consider an exact sequence of finite A-modules such that $IM_i \neq M_i$,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

Then the following hold,

(a) $\operatorname{depth}_I(M_2) \geq \min\{\operatorname{depth}_I(M_1), \operatorname{depth}_I(M_3)\}$

(b) $\operatorname{depth}_I(M_1) \geq \min\{\operatorname{depth}_I(M_2), \operatorname{depth}_I(M_3) + 1\}$

(c) $\operatorname{depth}_I(M_3) \geq \min\{\operatorname{depth}_I(M_1) - 1, \operatorname{depth}_I(M_2)\}$

Proof. Apply the functor $\operatorname{Hom}_A(A/I, -)$ to give the long exact sequence,

$$\operatorname{Ext}^i_A(A/I, M_1) \longrightarrow \operatorname{Ext}^i_A(A/I, M_2) \longrightarrow \operatorname{Ext}^i_A(A/I, M_3) \longrightarrow \operatorname{Ext}^{i+1}_A(A/I, M_1)$$

If $i < n = \min\{\operatorname{depth}_I(M_1), \operatorname{depth}_I(M_3)\}$ then $\operatorname{Ext}^i_A(A/I, M_2) = 0$ applying the cohomological criterion and the exact sequence so $\operatorname{depth}_I(M_3) \geq n$. The other parts follow similarly. $\square$

Lemma 9.2.3. Let x be a nonzerodivisor on M then $\operatorname{depth}_I(M/xM) = \operatorname{depth}_I(M) - 1$.

Proof. Applying the previous Lemma to the exact sequence,

$$0 \longrightarrow M \xrightarrow{\times x} M \longrightarrow M/xM \longrightarrow 0$$

gives $\operatorname{depth}_I(M/xM) \geq \operatorname{depth}_I(M) - 1$. However, for any M/xM-regular sequence $x_1, \dots, x_n \in I$ we get a M-regular sequence $x, x_1, \dots, x_n \in I$ and thus $\operatorname{depth}_I(M) \geq \operatorname{depth}_I(M/xM) + 1$. $\square$

Corollary 9.2.4. Any M-regular sequence $x_1, \dots, x_r \in I$ can be extended to a regular sequence of length $\operatorname{depth}_I(M)$ and thus all maximal regular sequences have the same length.

Proof. Given an M-regular sequence $x_1, \dots, x_r \in I$ we apply the previous Lemma to show that,

$$\operatorname{depth}_I(M/(x_1, \dots, x_r)M) = \operatorname{depth}_I(M) - r$$

and thus there exists a regular sequence $x_{r+1}, \dots, x_d \in I$ for $M/(x_1, \dots, x_r)M$ meaning that $x_1, \dots, x_r, \cdots, x_d \in$ gives a M-regular sequence of length $\operatorname{depth}_I(M)$ extending $x_1, \dots, x_r$. $\square$

9.3 Vanishing Criteria on Ext

(GRADE AND (Ischebeck))

9.4 Locality of Depth

Proposition 9.4.1. Let A be a noetherian ring, $I \subset A$ an ideal, and M a finite A-module. Then,

$$\operatorname{depth}_I(M) = \inf\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I)\}$$

Proof. DOOOOOOOOOOO!!!! $\square$

9.5 Additional Lemmas

Proposition 9.5.1. Let A be Noetherian ring, $I \subset A$ an ideal, and M a finite A-module. Then there exists an exact sequence of finite A-modules,

$$0 \longrightarrow K \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_i are finite free A-modules and $r = \operatorname{depth}(A) - \operatorname{depth}(M)$. Furthermore, given any such sequence, $\operatorname{depth}(K) = \operatorname{depth}(A)$.

Proof. There always exists a surjection $F_0 \twoheadrightarrow M$ from a finite free module F_0 because M is finite. Extending to an exact sequence,

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

gives $\operatorname{depth}_I(K) \geq \min\{\operatorname{depth}_I(A), \operatorname{depth}_I(M) + 1\}$ because F_0 is free so clearly $\operatorname{depth}_I(F_0) = \operatorname{depth}_I(A)$ by the cohomological criterion. Thus either $\operatorname{depth}_I(K) \geq \operatorname{depth}_I(A)$ already or $\operatorname{depth}_I(K) \geq \operatorname{depth}_I(M) + 1$. Therefore, repeating this process r times we see that $\operatorname{depth}_I(K_r) \geq \operatorname{depth}_I(M)$ $\square$

9.6 Cohen-Macaulay Rings

(IS THIS CORRECT AS STATED!!)

Proposition 9.6.1. Let A be a ring, $I \subset A$ an ideal, and M a finite A-module. Then,

$$\operatorname{depth}_I(M) \leq \min_{\mathfrak{p} \in \operatorname{WAss}_A(M)} \dim A/\mathfrak{p} \leq \dim \operatorname{Supp}_A(M)$$

Definition 9.6.2. Let A be a Noetherian local ring. A finite A-module M is *Cohen-Macaulay* if,

$$\operatorname{depth}(M) = \dim \operatorname{Supp}_A(M)$$

We say that A is Cohen-Macaulay if it is Cohen-Macaulay as an A-module i.e. if $\operatorname{depth}(A) = \dim A$.

Lemma 9.6.3. If A is a Cohen-Macaualy Noetherian local ring then for any prime $\mathfrak{p} \in \operatorname{Spec}(A)$ the local ring $A_{\mathfrak{p}}$ is Cohen-Macaulay.

Proof. Tag 0AAG ☐

Remark. This Lemma allows for the following definition.

Definition 9.6.4. A ring A is Cohen-Macaulay if A is Noetherian and $A_{\mathfrak{p}}$ is Cohen-Macaualy for each $\mathfrak{p} \in \operatorname{Spec}(A)$.

(UNIVERSALLY CATENARY ETC..)
(FIX THIS STATEMENT!!)

Proposition 9.6.5. Let R be a regular local ring and M a finite A-module. Then any exact sequence of finite A-modules

9.7 Dimension

Proposition 9.7.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. Then,

$$\dim A/(f) \geq \dim A - 1$$

with equality iff f is a nonzero divisor.

Proof. https://math.stackexchange.com/questions/2085779/the-dimension-modulo-a-principal-ideal-in-a-noetherian-local-ring ☐

9.8 Properties

Proposition 9.8.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$ a nonzero divisor. Then A is Cohen-Macaulay iff $A/(f)$ is Cohen-Macaulay.

Proof. We have $\operatorname{depth}(A/(f)) = \operatorname{depth}(A) - 1$ and $\dim A/(f) = \dim A - 1$. ☐

10 Projective and Global Dimension

10.1 Projective Dimension

Definition 10.1.1. Let M be an A-module. Then the projective dimension $\mathrm{pd}_A(M)$ is the minimal length r of a projective resolution of M,

$$0 \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and $\mathrm{pd}_A(M) = \infty$ if there does not exist a finite-length projective resolution of M.

Lemma 10.1.2 (Schanuel's lemma)**.** Let A be a ring and M an A-module. Let,

$$0 \longrightarrow K \xrightarrow{c_1} P_1 \xrightarrow{p_1} M \longrightarrow 0 \qquad\qquad 0 \longrightarrow L \xrightarrow{c_2} P_2 \xrightarrow{p_2} M \longrightarrow 0$$

be two short exact sequences of A-module where P_i are projective. Then there exists an isomorphism of short exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "(c_1\ \mathrm{id})"] \arrow[d] & P_1 \oplus P_2 \arrow[r, "(p_1\ 0)"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "(\mathrm{id}\ c_2)"'] & P_1 \oplus P_2 \arrow[r, "(p_2\ 0)"'] & M \arrow[r] & 0
\end{tikzcd}$$

Proof. Using projectivity of P_1 and P_2 we get maps $a : P_1 \to P_2$ and $P_2 \to P_1$ over M meaning that $p_2 \circ a = p_1$ and $p_1 \circ b = p_2$. Therefore, we get a diagram,

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "(c_1\ \mathrm{id})"] & P_1 \oplus P_2 \arrow[r, "(p_1\ 0)"] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u, dashed] \arrow[d, dashed] & P_1 \oplus P_2 \arrow[r, "(p_1\ p_2)"] \arrow[u, "t"'] \arrow[d, "s"] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "(\mathrm{id}\ c_2)"'] & P_1 \oplus P_2 \arrow[r, "(p_2\ 0)"'] & M \arrow[r] & 0
\end{tikzcd}$$

where $t(x, y) = (x + b(y), y)$ and $s(x, y) = (x, y + a(x))$ such that,

$$(p_1, 0) \circ t = p_1 \circ (\mathrm{id} + b) = p_1 + p_2 \quad \text{and} \quad (0, p_2) \circ s = p_2 \circ (\mathrm{id} + a) = p_1 + p_2$$

so the diagram commutes inducing maps $N \to K \oplus P_2$ and $N \to P_1 \oplus L$ where $N = \ker(P_1 \oplus P_2 \to M)$. It is clear that t and s are isomorphisms and thus the induced maps are also isomorphisms proving the claim. □

Lemma 10.1.3. Let A be a ring and M an A-module with finite projective dimension. Then for any projective resolution,

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

the module $\ker(P_k \to P_{k-1})$ is projective for $k \geq \mathrm{pd}_A(M) - 1$.

Proof. We proceed by induction on $\mathrm{pd}_A(M)$. For the case $\mathrm{pd}_A(M) = 0$ then M is projective so the exact sequence,

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

splits so $P_0 = K \oplus M$ proving that K is also projective giving the case $k = 0$. Replacing M by $K = \ker(P_0 \to M)$ we prove $\ker(P_k \to P_{k-1})$ is projective for all k.

Now for induction suppose $\mathrm{pd}_A(M) = d + 1$ and let,

$$0 \longrightarrow \tilde{P}_{d+1} \longrightarrow \cdots \longrightarrow \tilde{P}_1 \longrightarrow \tilde{P}_0 \longrightarrow M \longrightarrow 0$$

be a minimal length projective resolution. By Schanuel's lemma,

$$\tilde{P}_0 \oplus \ker(P_0 \to M) \cong P_0 \oplus \ker(\tilde{P}_0 \to M)$$

If $\mathrm{pd}_A(M) = 1$ and $k = 0$ then $\ker(\tilde{P}_0 \to M)$ is projective meaning that $\ker(P_0 \to M)$ is projective as well. Otherwise let $k > 0$ and consider the projective resolutions,

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow \ker(P_0 \to M) \longrightarrow 0$$

$$0 \longrightarrow \tilde{P}_{d+1} \longrightarrow \cdots \longrightarrow \tilde{P}_1 \longrightarrow \ker(\tilde{P}_0 \to M) \longrightarrow 0$$

We cannot directly apply induction because these are not resolutions of the same module. However, applying $- \oplus \tilde{P}_0$ to the first sequence and $- \oplus P_0$ to the second we get projective resolutions of $M' = \tilde{P}_0 \oplus \ker(P_0 \to M) \cong P_0 \oplus \ker(\tilde{P}_0 \to M)$

$$\cdots \longrightarrow P_3 \oplus \tilde{P}_0 \longrightarrow P_2 \oplus \tilde{P}_0 \longrightarrow P_1 \oplus \tilde{P}_0 \longrightarrow M' \longrightarrow 0$$

$$0 \longrightarrow \tilde{P}_{d+1} \oplus P_0 \longrightarrow \cdots \longrightarrow \tilde{P}_1 \oplus P_0 \longrightarrow M' \longrightarrow 0$$

because direct sum is exact and preserves projectives. From the second sequence $\mathrm{pd}_A(M') \le d$ so we may apply induction and find that $\ker(P_k \oplus \tilde{P}_0 \to P_{k-1} \oplus \tilde{P}_0) = \ker(P_{k+1} \to P_k) \oplus \tilde{P}_0$ is projective for $k \ge d - 1$ and thus $\ker(P_k \to P_{k-1})$ is projective for $k \ge d$ completing the proof. $\square$

Lemma 10.1.4. Let A be a Noetherian ring and M a finite A-module. Then the following are equivalent,

(a) $\mathrm{pd}_A(M) \le d$

(b) there exists a resolution of M by finite modules F_i and P_d,

$$0 \longrightarrow P_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the F_i are finite free and P_d is finite projective.

Proof. Clearly the second implies the first since F_i are projective. Given $\mathrm{pd}_A(M) \le d$ we know $d - 1 \ge \mathrm{pd}_A(M) - 1$. Since A is Noetherian and M is finite we can build a finite free resolution,

$$0 \longrightarrow P_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

by taking a generating set for M and the kernel $\ker(F_k \to F_{k-1})$ is again a finite A-module by the Noetherian property. Then let $P_d = \ker(F_{d-1} \to F_{d-2})$. Since the F_k are projective, by the previous lemma P_d is projective and finite as a submodule of a finite module. $\square$

Lemma 10.1.5. Let A be a Noetherian local ring and M a finite A-module. Then the following are equivalent,

(a) $\text{pd}_A(M) \le d$

(b) there exists a resolution of M by finite free modules F_i,

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Proof. This follows from above noting that finite projective A-modules are free because A is local. □

Proposition 10.1.6. Let A be a ring and M an A-module. Then the following are equivalent,

(a) $\text{pd}_A(M) \le n$

(b) $\text{Ext}^i_A(N, M) = 0$ for all A-modules A and all $i \ge n+1$

(c) $\text{Ext}^{n+1}_A(N, M) = 0$ for all A-modules.

Proof. (DO THIS!!!) □

Lemma 10.1.7. Consider an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

(a) if $\text{pd}_A(M_2) \le n$ then $\text{pd}_A(M_1) \le n$ and $\text{pd}_A(M_3) \le n+1$

(b) if $\text{pd}_A(M_1) \le n$ and $\text{pd}_A(M_3) \le n$ then $\text{pd}_A(M) \le n$

(c) if $\text{pd}_A(M_1) \le n$ and $\text{pd}_A(M) \le n+1$ then $\text{pd}_A(M_3) \le n+1$.

Proof. Combine the long exact sequence of Ext groups and the previous result. □

10.2 Global Dimension

Definition 10.2.1. Let A be a ring. The global dimension $\text{gldim}(A)$ is the supremum of $\text{pd}_A(M)$ over all A-modules M.

Theorem 10.2.2. Let A be a ring. The following are equivalent,

(a) $\text{gldim}(A) \le n$

(b) $\text{pd}_A(M) \le n$ for all A-modules M

(c) $\text{pd}_A(M) \le n$ for all finite A-modules M

(d) $\text{pd}_A(M) \le n$ for all cyclic A-modules M.

Proof. Tag 065T. □

Lemma 10.2.3. Let A be a ring, M an A-module, and $S \subset A$ a multiplicative subset then,

(a) $\text{pd}_{S^{-1}A}(S^{-1}M) \le \text{pd}_A(M)$

(b) $\mathrm{gldim}\,(S^{-1}A) \le \mathrm{gldim}\,(A)$

Proof. The functor $S^{-1}(-) : \mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}$ is exact and preserves projectives because it is left-adjoint to restriction which is also exact. Therefore, if M has a projective A-resolution of length n then $S^{-1}M$ has a projective $S^{-1}A$-resolution of length at most n so $\mathrm{pd}_{S^{-1}A}\,(S^{-1}M) \le \mathrm{pd}_A\,(M)$. Notice that for any $S^{-1}A$-module M, we have $M = S^{-1}M_A$ viewing M_A as an A-module under the restriction function. Thus, applying the first part

$$\begin{aligned}\mathrm{gldim}\,\left(S^{-1}A\right) &= \sup\{\mathrm{pd}_{S^{-1}A}\,(M) \mid M \in \mathbf{Mod}_{S^{-1}A}\} \le \sup\{\mathrm{pd}_A\,(M_A) \mid M \in \mathbf{Mod}_{S^{-1}A}\} \\ &\le \sup\{\mathrm{pd}_A\,(M) \mid M \in \mathbf{Mod}_A\} = \mathrm{gldim}\,(A)\end{aligned}$$

□

Proposition 10.2.4. Let R be a Noetherian ring. Then,

$$\mathrm{gldim}\,(R) = \sup\{\mathrm{gldim}\,(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}\,(R)\} = \sup\{\mathrm{gldim}\,(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \mathrm{mSpec}\,(R)\}$$

Proof. DOO!!!!!!!!! □

10.3 Auslander-Buchsbaum

(MOST GENERAL VERSION!!)

10.4 Regular Rings

Remark. Throughout let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring.

Lemma 10.4.1. We always have,

$$\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \ge \dim R$$

Proof. By Nakayma, $n = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ is the minimal number of generators of $\mathfrak{m}$. Then by Krull's ideal theorem, $\dim R = \mathbf{ht}\,(\mathfrak{m}) \le n$. □

Corollary 10.4.2. When R is a Noetherian local ring, $\dim R$ is finite.

Proof. $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ is finite because $\mathfrak{m}$ is finitely generated since R is Noetherian. □

Definition 10.4.3. We say that R is a *regular local ring* if $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Proposition 10.4.4. Let R be a regular local ring. Then $\mathrm{gldim}\,(R) \le \dim R$.

Proof. DO!!!! □

Proposition 10.4.5. Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring then $\mathrm{pd}_R\,(\kappa) \ge \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

Proof. Tag 00OA. □

Proposition 10.4.6. If $\mathrm{pd}_R\,(\kappa) < \infty$ then $\dim R \ge \mathrm{pd}_R\,(\kappa)$.

Proof. Tag 00OB. □

Proposition 10.4.7. Let R be a Noetherian local ring. If $\mathrm{pd}_R\,(\kappa) < \infty$ then R is a regular local ring.

Proof. The above propositions give $\dim R \geq \mathrm{pd}_R(\kappa) \geq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ but $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$. $\square$

Proposition 10.4.8. Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring. Then $\mathrm{gldim}(R) < \infty$ if and only if R is a regular local ring in which case $\mathrm{gldim}(R) = \dim R$.

Proof. If R is regular local then $\mathrm{gldim}(R) \leq \dim R$. Conversely, if $\mathrm{gldim}(R)$ is finite then $\mathrm{pd}_R(\kappa) < \infty$ so R is reglar local. In this case, $\mathrm{pd}_R(\kappa) = \dim R$ and $\mathrm{gldim}(R) \leq \dim R$ so $\mathrm{gldim}(R) = \dim R$. $\square$

Lemma 10.4.9. If R is reglar local then $R_\mathfrak{p}$ is regular local for each prime $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. If R is regular local then $\mathrm{gldim}(R) < \infty$ and thus $\mathrm{gldim}(R_\mathfrak{p}) \leq \mathrm{gldim}(R) < \infty$. Since $R_\mathfrak{p}$ is local and noetherian, $R_\mathfrak{p}$ is regular local as well. $\square$

Definition 10.4.10. A noetherian ring R is *regular* if $R_\mathfrak{p}$ is regular local for each $\mathfrak{p} \in \mathrm{Spec}(R)$.

Remark. The preceeding Lemma says that a regular local ring is regular.

Remark. It suffices to check regularity at $R_\mathfrak{m}$ for maximal ideals $\mathfrak{m} \in \mathrm{mSpec}(R)$ since $R_\mathfrak{p}$ is a localization of some $R_\mathfrak{m}$ and we have shown that localization preserves being regular local.

Proposition 10.4.11. Let R be a Noetherian ring. The following are equivalent for each $n \in \mathbb{N}$,

(a) $\mathrm{gldim}(R) \leq n$

(b) for each $\mathfrak{m} \in \mathrm{mSpec}(R)$ the ring $R_\mathfrak{m}$ is regular and $\dim R_\mathfrak{m} \leq n$

(c) for each $\mathfrak{p} \in \mathrm{mSpec}(R)$ the ring $R_\mathfrak{p}$ is regular and $\dim R_\mathfrak{p} \leq n$.

Therefore, if $\mathrm{gldim}(R) < \infty$ then R is regular and if R is regular then

$$\mathrm{gldim}(R) = \sup\{\dim R_\mathfrak{m} \mid \mathfrak{m} \in \mathrm{mSpec}(R)\} = \sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$$

Proof. This follows from,

$$\mathrm{gldim}(R) = \sup\{\mathrm{gldim}(R_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R)\}$$

and the fact that $\mathrm{gldim}(R_\mathfrak{m}) < \infty$ is equivalent to regularity of $R_\mathfrak{m}$ in which case $\mathrm{gldim}(R_\mathfrak{m}) = \dim R_\mathfrak{m}$. $\square$

Remark. Notice that even when R is regular $\mathrm{gldim}(R)$ may be infinite simply because the dimensions of $R_\mathfrak{m}$ for $\mathfrak{m} \in \mathrm{mSpec}(R)$ may be unbounded even when R is Noetherian. In this case, $\dim R = \infty$ so if $\dim R$ is finite then $\mathrm{gldim}(R)$ is finite iff R is regular.

11 Pseudomorphisms

Lemma 11.0.1. Let $f : X \to Y$ be a morphism of schemes such that for each weakly associated point $y \in Y$ there exists a point $x \in X$ such that $f(x) = y$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. Then the map on sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective.

Proof. To show that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective, it suffices to show that $\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y$ is injective on each weakly associated point $y \in Y$. Furthermore, we know there exists $x \in X$ with $f(x) = y$ and the composition $\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y \to \mathcal{O}_{X,x}$ is injective and thus $\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y$ is injective. $\square$

Remark. In particular, if $f : X \to Y$ is a dominant map of integral schemes then $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective.

Example 11.0.2. Consider the map $\text{Spec}\,(k[x]) \to \text{Spec}\,(k[x,y]/(xy, y^2))$. Although this map hits the generic point (y), it does not hit the embedded associated point (x, y^2) at the origin and thus $k[x,y]/(xy,y^2) \to k[x]$ is not injective since $y \mapsto 0$.

Definition 11.0.3. We say an immersion $\iota : Y \hookrightarrow X$ is *scheme theoretically dense* if the scheme theoretic image is X.

Lemma 11.0.4. An open immersion $\iota : U \to X$ is scheme theoretically dense iff U contained all weakly associated points of X.

Proof. □

When can we ensure that the coker of $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is supported in codimension one.

11.1 Annhiliators

Remark. Here we let X be a scheme. Warning: many of these results do not hold for arbitrary locally ringed spaces. In particular, the kernel of quasi-coherent sheaves need not be quasi-coherent on an arbitrary locally ringed space. However, this holds locally on schemes because kernels and cokerns of sheaves associated to modules are associated to modules.

Definition 11.1.1. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then we define the sheaf of annihilators:

$$\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F}) = \ker\left(\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})\right)$$

Lemma 11.1.2. Let $\mathscr{F}, \mathscr{G}$ be quasi-coherent $\mathcal{O}_X$-modules with $\mathscr{F}$ finitely presented. Then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is quasi-coherent.

Proof. Locally on $U \subset X$ we have a presentation,

$$\bigoplus_{i=1}^m \mathcal{O}_U \longrightarrow \bigoplus_{j=1}^n \mathcal{O}_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Applying the functor $\mathscr{H}om_{\mathcal{O}_U}(-, \mathscr{G})$ gives,

$$0 \longrightarrow \mathscr{H}om_{\mathcal{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U) \longrightarrow \bigoplus_{j=1}^n \mathscr{G}|_U \longrightarrow \bigoplus_{i=1}^m \mathscr{G}|_U$$

Since $\mathscr{G}$ is quasi-coherent and finite sums and kernels of quasi-coherent sheaves are quasi-coherent we see that $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is locally quasi-coherent and thus quasi-coherent. □

Lemma 11.1.3. If $\mathscr{F}$ is finitely presented then $\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ is quasi-coherent.

Proof. From the previous lemma, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})$ is quasi-coherent. Therefore, the kernel,

$$\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F}) = \ker\left(\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})\right)$$

is quasi-coherent. □

Proposition 11.1.4. Let $\mathscr{F}$ be finitely presented. Then $\text{Supp}_{\mathcal{O}_X}(\mathscr{F})$ is closed and the quasi-coherent sheaf of ideals $\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ gives a scheme structure on $\text{Supp}_{\mathcal{O}_X}(\mathscr{F})$. Furthermore, $\mathscr{F}$ is naturally a $\mathcal{O}_X / \mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ - module.

Lemma 11.1.5. Let $f : X \to Y$ be a morphism of schemes. Assume that $\mathcal{O}_Y$ and $f_* \mathcal{O}_X$ are coherent on Y. Furthermore, for each generic point of an irreducible component $\xi \in Y$ assume that there exists some $x \in X$ with $f(x) = \xi$ and $\mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,x}$ surjective. Then $\mathscr{C} = \text{coker}\left(\mathcal{O}_Y \to f_* \mathcal{O}_X\right)$ has $Z = \text{Supp}_{\mathcal{O}_Y}(\mathscr{C})$ in positive codimension.

12 Singularities of Curves

Definition 12.0.1. NORMALIZATION

Proposition 12.0.2. Normalization of a curve exists and is regular.

(CAN WE GET $H^0(O_X)$ is the same?)