Mathematics GU4051 Topology
Assignment # 3

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Problem 1.

Let $(X, \mathcal{T})$ be a topological space and $f : X \to Y$ be any function. Define

$$\mathcal{S} = \{U \in \mathbf{P}(Y) \mid f^{-1}(U) \in \mathcal{T}\}$$

Since $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$ then $\emptyset, Y \in \mathcal{S}$.
Suppose that for some index set Λ, the sets $V_\lambda \in \mathcal{S}$. Then by Lemma **??**,

$$f^{-1}\left(\bigcup_{\lambda \in \Lambda} V_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda) \in \mathcal{T}$$

Because each $V_\lambda \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions. Therefore, $\bigcup\limits_{\lambda \in \Lambda} V_\lambda \in \mathcal{S}$.
Suppose that for some *finite* index set Λ, the sets $V_\lambda \in \mathcal{S}$. Then by Lemma **??**,

$$f^{-1}\left(\bigcap_{\lambda \in \Lambda} V_\lambda\right) = \bigcap_{\lambda \in \Lambda} f^{-1}(V_\lambda) \in \mathcal{T}$$

Because each $V_\lambda \in \mathcal{T}$ and $\mathcal{T}$ is closed under finite intersections. Therefore, $\bigcap\limits_{\lambda \in \Lambda} V_\lambda \in \mathcal{S}$.
Thus, $\mathcal{S}$ is a topology on Y.

Problem 2.

The basis $\mathcal{B} = \{V \times W \mid V \in \mathcal{T}_Y \text{ and } W \in \mathcal{T}_W\}$ generates the product topology $\mathcal{T}_{Y \times Z}$ on the space $Y \times Z$. Thus by Lemma **??**, the open sets in $\mathcal{T}_{Y \times Z}$ are exactly those that are unions of basis elements. Therefore,

$$U \in \mathcal{T}_{Y \times Z} \iff U = \bigcup_{\lambda \in \Lambda} V_\lambda \times W_\lambda$$

with $V_\lambda \times W_\lambda \in \mathcal{B}$ i.e. for $V_\lambda \in \mathcal{T}_Y$ and $W_\lambda \in \mathcal{T}_Z$.

Problem 3.

Let X, Y, and Z be topological spaces and $Y \times Z$ have the product topology. Suppose that $f_1 : X \to Y$ and $f_2 : X \to Z$ are continuous. Then define $F : X \to Y \times Z$ by $F : x \mapsto (f_1(x), f_2(x))$.

Take U open in $\mathcal{T}_{Y\times Z}$ so, by problem 2, $U = \bigcup_{\lambda\in\Lambda} V_\lambda \times W_\lambda$ with $V_\lambda \in \mathcal{T}_Y$ and $W_\lambda \in \mathcal{T}_Z$. Then,

$$\begin{aligned} x \in F^{-1}(U) &\iff (f_1(x), f_2(x)) \in \bigcup_{\lambda\in\Lambda} V_\lambda \times W_\lambda \iff \exists \lambda \in \Lambda : f_1(x) \in V_\lambda \text{ and } f_2(x) \in W_\lambda \\ &\iff \exists \lambda \in \Lambda : x \in f_1^{-1}(V_\lambda) \cap f_2^{-1}(W_\lambda) \iff x \in \bigcup_{\lambda\in\Lambda} f_1^{-1}(V_\lambda) \cap f_2^{-1}(W_\lambda) \end{aligned}$$

Thus,

$$F^{-1}(U) = \bigcup_{\lambda\in\Lambda} f_1^{-1}(V_\lambda) \cap f_2^{-1}(W_\lambda)$$

Now by continuity of f_1 and f_2, the sets $f_1^{-1}(V_\lambda)$ and $f_2^{-1}(W_\lambda)$ are open in X and since X is a topological space, their intersection is open. Therefore,

$$F^{-1}(U) = \bigcup_{\lambda\in\Lambda} f_1^{-1}(V_\lambda) \cap f_2^{-1}(W_\lambda) \in \mathcal{T}_X$$

because it is a union of open sets of X which shows that F is continuous.

Now let one of f_1 and f_2 be not continuous. WLOG take f_1 to be not continous. Then for some $V \in \mathcal{T}_Y$, we must have $f_1^{-1}(V) \notin \mathcal{T}_X$. Then $V \times Z \in \mathcal{T}_{Y\times Z}$ because $Z \in \mathcal{T}_Z$. Consider,

$$x \in F^{-1}(V \times Z) \iff (f_1(x), f_2(x)) \in V \times Z \iff f_1(x) \in V$$

Because for any x, $f_2(x) \in Z$. Thus, $F^{-1}(V \times Z) = f_1^{-1}(V) \notin \mathcal{T}_X$ so F cannot be continuous.

Problem 4.

(a). The function $\log : \mathbb{R}^+ \to \mathbb{R}$ is continuous by its integral definition (since the subspace topology on $\mathbb{R}^+$ is generated by the same metric that generates the standard topology on $\mathbb{R}$). Furthermore, log has an inverse namely exp which is also continuous because it is differentiable. Thus, log is a homeomorphism between $\mathbb{R}^+$ and $\mathbb{R}$.

(b). Let

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

Define $F : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R} \times S$ by $F : (x, y) \mapsto \left(\log\sqrt{x^2+y^2}, \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)\right)$

Now the functions $f_1 : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ and $f_2 : \mathbb{R}^2\backslash\{(0,0)\} \to S$ given by

$$f_1 : (x, y) \mapsto \log\sqrt{x^2+y^2} \text{ and } f_2 : (x, y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

are continous by ϵ, δ arguments. Then $F = (f_1, f_2)$ so by problem 3, F is continuous under the product topology on $\mathbb{R} \times S$.

Now define $G : \mathbb{R} \times S \to \mathbb{R}^2\backslash\{(0,0)\}$ by $G : (r, (x, y)) \mapsto (xe^r, ye^r)$. Thus,

$$F \circ G(r, (x, y)) = F(xe^r, ye^r) = \left(\log e^r\sqrt{x^2+y^2}, \left(\frac{xe^r}{e^r\sqrt{x^2+y^2}}, \frac{ye^r}{e^r\sqrt{x^2+y^2}}\right)\right)$$

But $(x, y) \in S$ so $x^2 + y^2 = 1$ and $e^r > 0$ thus, $F \circ G(r, (x, y)) = (r, (x, y))$.
Furthermore, for $(x, y) \neq (0, 0)$ (such that $F(x, y)$ is defined) we have,

$$
\begin{aligned}
G \circ F(x, y) &= G\left(\log \sqrt{x^2+y^2}, \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)\right) \\
&= \left(\frac{x}{\sqrt{x^2+y^2}} \exp\log\sqrt{x^2+y^2}, \frac{y}{\sqrt{x^2+y^2}} \exp\log\sqrt{x^2+y^2}\right) = (x, y)
\end{aligned}
$$

Therefore, $G \circ F = \mathrm{id}_{\mathbb{R}^2 \setminus \{(0,0)\}}$ and $F \circ G = \mathrm{id}_{\mathbb{R} \times S}$ so, in particular, F is a bijection. Since the product topology on $\mathbb{R} \times S$ is metrizable by the $\mathbb{R}^3$ Euclidean metric, we can use standard analysis facts to conculde that G extended to $\mathbb{R}^3 \to \mathbb{R}^2 \setminus \{(0,0)\}$ is continuous with respect to the Euclidean metric thus its restriction to $\mathbb{R} \times S$ is also continuous.

Problem 5.

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ define:

$$
d(\mathbf{u}, \mathbf{v}) = \begin{cases} |\mathbf{u} - \mathbf{v}| & \text{if } \mathbf{u} = t\mathbf{v} \text{ for } t \in \mathbb{R} \\ |\mathbf{u}| + |\mathbf{v}| & \text{otherwise} \end{cases}
$$

Since both $|\mathbf{u} - \mathbf{v}| \geq 0$ and $|\mathbf{u}| + |\mathbf{v}| \geq 0$ then $d(\mathbf{u}, \mathbf{v}) \geq 0$.
Since both $|\mathbf{u} - \mathbf{v}| = |\mathbf{v} - \mathbf{u}|$ and $|\mathbf{u}| + |\mathbf{v}| = |\mathbf{v}| + |\mathbf{u}|$ then $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.
Also $|\mathbf{u} - \mathbf{v}| = 0 \iff \mathbf{u} = \mathbf{v}$ and $|\mathbf{u}| + |\mathbf{v}| = 0 \iff |\mathbf{u}| = |\mathbf{v}| = 0 \iff \mathbf{u} = \mathbf{v} = 0$ then $d(\mathbf{u}, \mathbf{v}) = 0 \iff \mathbf{u} = \mathbf{v}$.

Then take any $\mathbf{w} \in \mathbb{R}^2$. First, suppose that $\mathbf{u} = t\mathbf{v}$ for $t \in \mathbb{R}$ so $d(\mathbf{u}, \mathbf{v}) = |\mathbf{u} - \mathbf{v}|$. Then by the triangle inequality for the Euclidean norm,

$$
|\mathbf{u} - \mathbf{v}| = |\mathbf{u} - \mathbf{w} + \mathbf{w} - \mathbf{v}| \leq |\mathbf{u} - \mathbf{w}| + |\mathbf{w} - \mathbf{v}| \leq (|\mathbf{u}| + |\mathbf{w}|) + (|\mathbf{w}| + |\mathbf{v}|)
$$

Therefore $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ because $|\mathbf{u} - \mathbf{w}| \leq d(\mathbf{u}, \mathbf{w})$.

Otherwise, it cannot be that $\mathbf{u} = t\mathbf{w}$ and $\mathbf{w} = t'\mathbf{v}$ else $\mathbf{u} = t \cdot t'\mathbf{v}$.
If $\mathbf{w}$ is not a multiple of either $\mathbf{u}$ or $\mathbf{v}$ then,

$$
d(\mathbf{u}, \mathbf{v}) = |\mathbf{u}| + |\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{w}| + |\mathbf{w}| + |\mathbf{v}| = d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})
$$

If $\mathbf{w} = t\mathbf{u}$ then using Lemma **??**,

$$
d(\mathbf{u}, \mathbf{v}) = |\mathbf{u}| + |\mathbf{v}| = |\mathbf{u}| - |\mathbf{w}| + |\mathbf{w}| + |\mathbf{v}| \leq |\mathbf{u} - \mathbf{w}| + |\mathbf{w}| + |\mathbf{v}| = d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})
$$

If $\mathbf{w} = t\mathbf{v}$ then using Lemma **??**,

$$
d(\mathbf{u}, \mathbf{v}) = |\mathbf{u}| + |\mathbf{v}| = |\mathbf{u}| + |\mathbf{w}| + |\mathbf{v}| - |\mathbf{w}| \leq |\mathbf{u}| + |\mathbf{w}| + |\mathbf{v} - \mathbf{w}| = d(\mathbf{u}, \mathbf{w}) + d(\mathbf{v}, \mathbf{w})
$$

Therefore, for all vectors, $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ so d is a metric.

However, d does not generate the standard topology on $\mathbb{R}^2$. Consider

$$B_{\frac{1}{2}}((1,0))^{\text{Rail}} = \{(x,0) \mid x \in \left(\tfrac{1}{2}, \tfrac{3}{2}\right)\}$$

This equality holds because if $\mathbf{v} \neq (x,0) = x \cdot (1,0)$ then $d(\mathbf{v},(1,0)) = |\mathbf{v}| + |(1,0)| \geq 1$.

Now, suppose $\exists \delta \in \mathbb{R}^+ : B_\delta((1,0))^{\text{Std.}} \subset B_\delta((1,0))^{\text{Rail}}$ then $(1,\delta) \in B_\delta((1,0))^{\text{Std.}} \subset B_\delta((1,0))^{\text{Rail}}$ which is a contradiction. Thus, $B_{\frac{1}{2}}((1,0))^{\text{Rail}}$ is not an open set of the standard topology but it is by definition open in the topology generated by this new metric.

Problem 6.

(a). Let $X = \{a,b\}$ and $\mathcal{T} = \{\{a\},\{a,b\},\emptyset\}$. Suppose a metric d generates $\mathcal{T}$. Then let $\delta = d(a,b)$ then $b \in B_\delta(b)$ but $a \notin B_\delta(b)$ because $d(a,b) \not< \delta = d(a,b)$. Thus $B_\delta(a) \notin \mathcal{T}$. Thus, $\mathcal{T}$ cannot be generated by the metric d.

(b). Let X be a finite set and d be a metric on X. Consider $x \in X$ and define

$$\delta_x = \min_{y \in X \setminus \{x\}} d(x,y)$$

which exists and is positive because each $d(x,y) > 0$. Then $x \in B_{\delta_x}(x)$ but for any other $y \in X$ s.t. $x \neq y$, we have $y \notin B_{\delta_x}(x)$ because

$$\delta_x = \min_{y \in X \setminus \{x\}} d(x,y) < d(x,y)$$

So $B_{\delta_x}(x) = \{x\}$ is open in the topology generated by d. For any $S \subset X$, $S = \bigcup_{x \in S} \{x\}$ is open because each $\{x\}$ is open. Thus, any metric on X generates the discrete topology.

Problem 7.

For $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$ define:

$$d'(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{n} |u_i - v_i|$$

Then each $|u_i - v_i| \geq 0$ so we get $d'(\mathbf{u},\mathbf{v}) \geq 0$. Also each $|u_i - v_i| = |v_i - u_i|$ so $d'(\mathbf{u},\mathbf{v}) = d'(\mathbf{v},\mathbf{u})$. Also, $d'(\mathbf{u},\mathbf{v}) = 0 \iff \forall i \in \{1,\ldots,n\} : |u_i - v_i| = 0 \iff u_i = v_i \iff \mathbf{u} = \mathbf{v}$. Now,

$$d'(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{n} |u_i - v_i| = \sum_{i=1}^{n} |u_i - w_i + w_i - v_i| \leq \sum_{i=1}^{n} |u_i - w_i| + \sum_{i=1}^{n} |w_i - v_i| = d'(\mathbf{u},\mathbf{w}) + d'(\mathbf{w},\mathbf{v})$$

by the triangle inequality for the absolute value function. Thus, d' is a metric.

It remains to be shown that this metric generates the standard topology on $\mathbb{R}^n$. Using the notation $B_\delta(\mathbf{x})' = \{\mathbf{y} \in \mathbb{R}^n \mid d'(\mathbf{x},\mathbf{y}) < \delta\}$, I claim that $B_{\frac{\delta}{n}}(\mathbf{x}) \subset B_\delta(\mathbf{x})' \subset B_\delta(\mathbf{x})$ because:

$$\mathbf{y} \in B_{\frac{\delta}{n}}(\mathbf{x}) \implies d(\mathbf{x},\mathbf{y}) = |\mathbf{x} - \mathbf{y}| < \delta/n \implies |x_i - y_i| \leq |\mathbf{x} - \mathbf{y}| < \delta/n$$
$$\implies d'(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| < \delta \implies \mathbf{y} \in B_\delta(\mathbf{x})$$

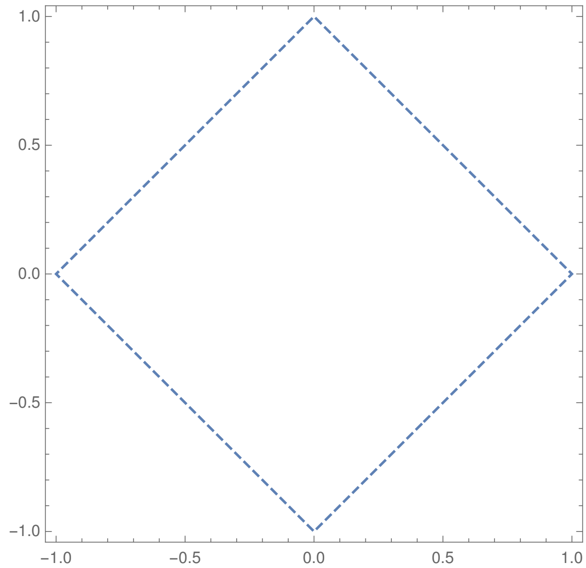

Figure 1: An open "ball" in $\mathbb{R}^2$ under the metric d' with radius 1 centered at $(0,0)$

Furthermore, because

$$d'(\mathbf{x},\mathbf{y})^2 = \left(\sum_{i=1}^{n} |x_i - y_i|\right)^2 = \sum_{i=1}^{n} |x_i - y_i|^2 + \sum_{i \neq j} |x_i - y_i||x_j - y_j| \geq \sum_{i=1}^{n} |x_i - y_i|^2$$

we have

$$\mathbf{y} \in B_\delta(\mathbf{x})' \implies d'(\mathbf{x},\mathbf{y}) < \delta \implies d(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \leq \sum_{i=1}^{n} |x_i - y_i| < \delta \implies \mathbf{x} \in B_\delta(\mathbf{x})$$

Suppose that $U \in \mathcal{T}_d$ then $\forall \mathbf{x} \in U : \exists \delta > 0 : x \in B_\delta(\mathbf{x}) \subset U$ thus $\mathbf{x} \in B_\delta(\mathbf{x})' \subset B_\delta(\mathbf{x}) \subset U$ so $\exists \delta > 0 : \mathbf{x} \in B_\delta(\mathbf{x})' \subset U$ thus $U \in \mathcal{T}_{d'}$.

Conversely, if $U \in \mathcal{T}_{d'}$ then $\forall \mathbf{x} \in U : \exists \delta > 0 : \mathbf{x} \in B_\delta(\mathbf{x})' \subset U$ thus $\mathbf{x} \in B_{\frac{\delta}{n}}(\mathbf{x}) \subset B_{\mathbf{x}}(\delta)' \subset U$ so $\exists \tilde{\delta} = \delta/n > 0 : \mathbf{x} \in B_{\tilde{\delta}}(\mathbf{x}) \subset U$ thus $U \in \mathcal{T}_d$. Therefore, $\mathcal{T}_d = \mathcal{T}_{d'}$.

Lemmas

Lemma 0.1. For any index set Λ, $f^{-1}\left(\bigcup_{\lambda\in\Lambda} V_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{-1}(V_\lambda)$ and $f^{-1}\left(\bigcap_{\lambda\in\Lambda} V_\lambda\right) = \bigcap_{\lambda\in\Lambda} f^{-1}(V_\lambda)$

Proof.

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{\lambda\in\Lambda} V_\lambda\right) &\iff f(x) \in \bigcup_{\lambda\in\Lambda} V_\lambda \iff \exists \lambda \in \Lambda : f(x) \in V_\lambda \\ &\iff \exists \lambda \in \Lambda : x \in f^{-1}(V_\lambda) \iff x \in \bigcup_{\lambda\in\Lambda} f^{-1}(V_\lambda) \end{aligned}$$

Thus, $f^{-1}\left(\bigcup\limits_{\lambda\in\Lambda} V_\lambda\right) = \bigcup\limits_{\lambda\in\Lambda} f^{-1}\left(V_\lambda\right)$. Also,

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{\lambda\in\Lambda} V_\lambda\right) &\iff f(x) \in \bigcap_{\lambda\in\Lambda} V_\lambda \iff \exists\lambda \in \Lambda : f(x) \in V_\lambda \\ &\iff \exists\lambda \in \Lambda : x \in f^{-1}\left(V_\lambda\right) \iff x \in \bigcup_{\lambda\in\Lambda} f^{-1}\left(V_\lambda\right) \end{aligned}$$

Thus, $f^{-1}\left(\bigcup\limits_{\lambda\in\Lambda} V_\lambda\right) = \bigcup\limits_{\lambda\in\Lambda} f^{-1}\left(V_\lambda\right)$. □

Lemma 0.2. Let the basis $\mathcal{B}$ generate a topology $\mathcal{T}$ then $U \in \mathcal{T} \iff U = \bigcup\limits_{\lambda\in\Lambda} B_\lambda$ with $B_\lambda \in \mathcal{B}$

Proof. If $U \in \mathcal{T}$ then $\forall x \in U : \exists V_x \in \mathcal{B} : x \in B_x \subset U$. Then

$$\bigcup_{x\in U}\{x\} \subset \bigcup_{x\in U} B_x \subset U$$

However, $\bigcup\limits_{x\in U}\{x\} = U$ so

$$U = \bigcup_{x\in U} B_x$$

Conversely, each $B_\lambda \in \mathcal{B}$ is open and thus

$$U = \bigcup_{\lambda\in\Lambda} B_\lambda$$

is also open because it is the union of open sets. □

Lemma 0.3. $\big||\mathbf{u}| - |\mathbf{v}|\big| \leq |\mathbf{u} - \mathbf{v}|$

Proof. By the triangle inequality,

$$|\mathbf{u}| \leq |\mathbf{u} - \mathbf{v}| + |\mathbf{v}| \text{ so } |\mathbf{u}| - |\mathbf{v}| \leq |\mathbf{u} - \mathbf{v}|$$

Similarly,

$$|\mathbf{v}| \leq |\mathbf{v} - \mathbf{u}| + |\mathbf{u}| \text{ so } |\mathbf{v}| - |\mathbf{u}| \leq |\mathbf{v} - \mathbf{u}|$$

Thus,

$$-|\mathbf{u} - \mathbf{v}| \leq |\mathbf{u}| - |\mathbf{v}| \leq |\mathbf{u} - \mathbf{v}|$$

So,

$$\big||\mathbf{u}| - |\mathbf{v}|\big| \leq |\mathbf{u} - \mathbf{v}|$$

□