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Interested in studying perverse sheaves in the Euclidean and étale topologies for studying Poincare duality for singular varieties. We studied:

(a) tiangulated categories

(b) t-structures

(c) gluing t-structues $\mathcal{D}_Z \to \mathcal{D} \to \mathcal{D}_U$ with nice functors between these triangulated categories

(d) perverse sheaves over $\mathcal{C}$

We are now going to develop the theory of perverse sheaves in the étale setting for étale sheaves and étale cohomology. The main goal for the first two weeks is to define the derived categories for $\overline{\mathbb{Q}}_\ell$-sheaves and six functor formalism. Then we study the theory of weights due to Deligne (Weil II) for varieties over $\mathbb{F}_q$.

1.1 Etale Sheaves

Assume all schemes are Noetherian. Then $X_{\text{ét}}$ is the small étale site with objects: schemes étale over X and covers and jointly surjective families of étale maps.

1.1.1 Constructibility

Let Λ be a finite "coefficient" ring. We say that an étale sheaf of sets $\mathscr{F}$ on $X_{\text{ét}}$ is locally constant constructible (lcc) if it is represented by an object of $\text{FEt}(X)$ i.e. there is a finite étale map $X' \to X$ such that,

$$\mathscr{F}(U) = \text{Hom}_X(U, X')$$

We say that a sheaf $\mathscr{F}$ is Λ-modules is *constructible* if each $\mathscr{F}_x$ is a finite Λ-module and there exists a stratification,

$$X = \bigcup X_i$$

such that $X_i \subset X$ is locally closed s.t. $\mathscr{F}|_{X_I}$ is lcc. Furthermore, $\mathscr{F} \in D^b(X, \Lambda)$ is constructible if each $H^i(\mathscr{F})$ is constructible. This gives the triangulated subcategory $D_c^i(X, \Lambda)$.

Now given a map of Noetherian schemes $f : X \to Y$ we get a six functor formalism. We have $f_* : \text{Ab}(X) \to \text{Ab}(Y)$ and $Rf_* : D^+(x, \Lambda) \to D^+(Y, \Lambda)$ and $f^* : \text{Ab}(Y) \to \text{Ab}(X)$ which is exact. Then there is an adjointness,

$$\text{Hom}_X(f^*\mathscr{F}, \mathscr{G}) = \text{Hom}_Y(\mathscr{F}, f_*\mathscr{G})$$

When $\iota : Z \hookrightarrow X$ is a closed immersion and $j : U \hookrightarrow X$ is the complementary open immersion we can define the "exceptional" functors. We define,

$$j_! : \text{Ab}(Y) \to \text{Ab}(X)$$

which is exact because it preserves stalks. Then ι_* is also exact because it preserves stalks (and extends by zero on U). Furthermore,

$$\iota^! : \text{Ab}(X) \to \text{Ab}(Z)$$

is defined by,

$$\iota^! \mathscr{F} = \iota^* \ker\left(\mathscr{F} \to j_* j^* \mathscr{F}\right)$$

which is the subsheaf of sections supported on Z.

However, $Rf_!$ is not the naive derived functor of $f_!$ unfortunately. Assume that $f : X \to Y$ is separated of finite type. Then by Nagata, there is a compactification,

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

Then we define $Rf_! = D^+(X, \Lambda) \to D^+(Y, \Lambda)$ by $Rf_! = (R\bar{f}_*) \circ j_!$.

Remark. It does not work to define $Rf_! = R(f_!)$ where $f_! = \bar{f}_* \circ j_!$. For example for a curve X over a field k and $f : X \to \operatorname{Spec}(k)$. Then,

$$\Gamma_c(X, \Lambda) = \bigoplus_{x \in X} \Gamma_x(X, \mathscr{F})$$

where Γ_x is local cohomology. But this derived functor can be too large.

Then we can also define $Rf^! : D^+(Y, \Lambda) \to D^+(X, \Lambda)$ to be the right adjoint of $Rf_!$. For example if $f = \iota$ is a closed immersion then $Rf^! = R(\iota^!)$ is actually a derived functor. If f is a smooth map of relative dimension d (and $n\Lambda = 0$ for $n \in \mathcal{O}_Y^\times$) then $Rf^! = f^*(\alpha)[2d]$

Theorem 1.1.1. Let $f : X \to Y$ be finite tpye over a field k. Then the six functors presserve $D_c^+(X, \Lambda)$ and we have biduality when $n\Lambda = 0$ for $n \in k^\times$ and Λ is an injective Λ-module. Let $f : X \to \operatorname{Spec}(k)$ and set $K_X = Rf^! \Lambda$ and $DL := \operatorname{RHom}(L, K_X) \in D^b(X, \Lambda)$. Then $L \xrightarrow{\sim} DDL$ in $D_c^b(X, \Lambda)$.

How to define $D_c^b(X, \overline{\mathbb{Q}}_\ell)$? Let k be a finite field or a separably closed field $\ell \in k^\times$. Let X be seperated of finite type over k. Fact 1 for $E/\mathbb{Q}_\ell$ finite extension then $\mathcal{O}_E$ have triangulated cat $D_c^b(X, \mathcal{O}_E)$ and standard nontrivial t-structure. For $r \geq 1$ let $\mathcal{O}_r = \mathcal{O}_E/\lambda^r$ where λ is a uniformizer. Then $\mathcal{O}_r$ is constructible.

Let $D_{ctf}^b(X, \mathcal{O}_r) \subset D_c^b(X, \mathcal{O}_r)$ be defined by objects isomorphic to bounded complexes of flat $\mathcal{O}_r$-module that give $D_c^b(X, \mathcal{O}_E)$ is defined as $K = (K_r)_r$ where $K_r \in D_{ctf}^b(X, \mathcal{O}_r)$ and $K_{r+1} \otimes^{\mathbb{L}} \mathcal{O}_r \cong K_r$. Moreover,

$$\operatorname{Hom}_{D_c^b(X, \mathcal{O}_E)}(K, L) = \varprojlim_{V} \operatorname{Hom}_{D_c^b(X, \mathcal{O}_r)}(K_r, L_r)$$

Furthermore, $D_c^b(X, \mathcal{O}_E)$ is a triangulated category. Furthermore, there is a standard t-structure on $D_c^b(X, \mathcal{O}_E)$. The problem is that $\tau_{\leq n}$ and $\tau_{\geq n}$ dont preserve $D_{cft}^b(X, \mathcal{O}_r)$ but Deligne defined them in such a way to preserve everything you want.

Remark. There are some serious problems with trying to first define an abelian category of $\mathbb{Z}_\ell$-sheaves and then take derived categories. For example, if $X = \operatorname{Spec}(k)$ then sheaves of $\mathbb{Z}/\ell^n$-modules is the same as $\mathbb{Z}/\ell^n[\operatorname{Gal}(\bar{k}/k)]$-modules but the limit of these categories gives the category of continuous $\operatorname{Gal}(\bar{k}/k)$-representations over $\mathbb{Z}_\ell$ which is not an abelian category. Look at Bhatt-Scholtze and condensed mathematics.

1.2 Theory of Weights

Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$ and let $X = X_0 \otimes \overline{\mathbb{F}_q}$. Then $F : X \to X$ is the geometric Frobenius i.e. $F = \mathrm{id} \otimes \mathrm{Spec}\,(F)$ where $F : \mathbb{F}_q \to \mathbb{F}_q$ is the inverse of $x \mapsto x^p$. Let $|X_0|$ denote the closed points and $x \in |X_0|$ we have $\#\kappa(x) = N(x) = q^{d(x)}$. For X_0 geometrically connected over $\mathbb{F}_q$ we have an exact sequence,

$$1 \longrightarrow \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X_0, \bar{x}) \longrightarrow \mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \longrightarrow 1$$

and $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) = \hat{\mathbb{Z}}$ generated by geometric Frobenius. Therefore, we have an exact squence,

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(X, \bar{x}) \arrow[r] & \pi_1^{\text{ét}}(X_0, \bar{x}) \arrow[r] & \mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(X, \bar{x}) \arrow[r] \arrow[u, equal] & W(X_0, \bar{x}) \arrow[r] \arrow[u, hook] & \mathbb{Z} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

Definition 1.2.1. A Weil sheaf on X_0 is a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}$ on X with an isomorphism $\varphi : F^* \mathscr{F} \to \mathscr{F}$.

Remark. Let $g : X \to X_0$ be the canonical map. Then $g \circ F = g$ so there is a natural isomorphism $\eta : F^* \circ g^* \to g^*$ and thus for any $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}_0$ on X_0 we have an natural isomorphism $\eta : F^* g^* \mathscr{F}_0 \xrightarrow{\sim} g^* \mathscr{F}_0$ and thus $\mathscr{F} := g^* \mathscr{F}_0$ and $\eta : F^* \mathscr{F} \to \mathscr{F}$ gives a Weil sheaf.

Exercise 1.2.2. For $X_0 = \mathrm{Spec}\,(\mathbb{F}_q)$ we have,

$$\{\text{Weil Sheaves}\} \iff \{\text{continuous reps of } W(X_0) \cong \mathbb{Z} \text{ on finite dimensional } \overline{\mathbb{Q}}_\ell \text{ vector spaces}\}$$

furthermore, the subcategory of $\overline{\mathbb{Q}}_\ell$-sheaves correspond to those representations that extend to $\hat{Z}$.

Exercise 1.2.3. In the rank 1 case let X_0 be a normal geometrically connected / $\mathbb{F}_q$ then,

$$\mathrm{Im}\left(\pi_1(X, \bar{x}) \to W(X_0, \bar{x})^{\text{ét}}\right)$$

is a finite group times a pro-p group.

1.3 Weights

Fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathcal{C}$.

Definition 1.3.1. Let $\mathscr{F}_0$ be a Weil sheaf on X_0 and $\beta \in \mathbb{R}$ then,

(a) $\mathscr{F}_0$ is (pointwise) ι-pure of weight β if $\forall x \in |X_0|$ and α is an eigenvalue of $F_x \circlearrowright (\mathscr{F}_0)_{\bar{x}}$ then $|\iota\alpha| = q^{\frac{\beta d(x)}{2}}$.

(b) $\mathscr{F}_0$ is ι-mixed if there is a filtration,

$$0 = \mathscr{F}_0^{(0)} \subset \cdots \subset \mathscr{F}_0^{(r)} = \mathscr{F}_0$$

by Weil subsheaves s.t. $\mathscr{F}_0^{(i+1)} / \mathscr{F}_0^{(i)}$ is ι-pure of some weight

(c) $\mathscr{F}_0$ is pure of weight β / mixed if it is ι-pure of weight β / ι-mixed for any ι.

Example 1.3.2. The sheaf $\mathscr{F}_0 = \mathbb{Q}(1)$ is pure of weight -2 since $F\zeta_{\ell^n} = \zeta_{\ell^n}^{q^{-1}}$. If X_0 is normal geometrically connected then any rank 1 smooth Weil sheaf on X_0 is ι-pure.

Theorem 1.3.3 (Deligne)**.** For $f_0 : X_0 \to Y_0$ and $\mathscr{F}_0$ on X a ι-mixed of largest weight β then,

$$R^k(f_0)_*\mathscr{F}_0$$

is ι-mixed with weights $\le \beta + k$.

Example 1.3.4. For $f_0 : X_0 \to \mathrm{Spec}\,(\mathbb{F}_q)$ smooth and proper of dimension d and $\mathscr{F}_0$ a smooth sheaf on X_0 that is ι-pure of weight β then by Poincare duality,

$$F \circlearrowright H^k(X_{\text{ét}}, \mathscr{F}) \cong H^{2d-k}(X_{\text{ét}}, \mathscr{F}^\vee(d))^\vee$$

By Deligne's theorem, the left hand side has weights $\le \beta + k$. Furthermore, the sheaf $\mathscr{F}^\vee(d)$ is ι-pure of weight $-\beta - 2d$ and thus by Deligne's theorem the cohomology group has weights $\le -\beta - k$ and thus dualizing the weights are $\ge \beta + k$. Therefore, the weights are all $\beta + k$ because both sides are isomorphic..

Theorem 1.3.5 (semi-continuity)**.** Let $j_0 : U_0 \to X_0$ be a dense open immersion with $S_0 = X_0 \setminus U_0$ with the reduced scheme structure. Let $\mathscr{F}_0$ be a smooth Weil sheaf on X_0. Assume that there is $\beta \in \mathbb{R}$ such that $\forall x \in |U_0|$ and any eigenvalue α of $f_x \circlearrowright \mathscr{F}_{\bar{x}}$ then $|\iota\alpha| \le q^{\frac{\beta d(x)}{2}}$ then $\forall s \in |S_0|$ and any eigenvalue α of $F_s \circlearrowright \mathscr{F}_{\bar{s}}$ then $|\iota\alpha| \le q^{\frac{\beta d(s)}{2}}$.

Proof. We can always take a chain of curves connecting x and s so we reduce to the case $\dim X_0 = 1$ with X_0 geometrically irreducible and affine. Recall,

$$L(X_0, \mathscr{F}_0, t) = \prod_{x \in |X_0|} \det{(1 - F_x t^{d(x)} | \mathscr{F}_{\bar{x}})}^{-1} = \frac{\det{(1 - Ft | H^1_c(X, \mathscr{F}))}}{\det{(1 - Ft | H^0_c)} \det{1 - Ft | H^2_c)}}$$

Then, $H^0_c(X, \mathscr{F}) = 0$ because X is affine and $\mathscr{F}_0$ is smooth so there are no global sections with compact support. Furthermore,

$$H^2_c(X, \mathscr{F}) \cong H^0(X, \mathscr{F}^\vee(1))^\vee$$

by Poincare duality. Fix $x \in |U_0|$ and because $\mathscr{F}_0$ is lisse then it corresponds to some representation V of $W(X_0, \bar{x})$ on $\mathscr{F}_{\bar{x}}$. Then, $H^0(X, \mathscr{F}) = V^{\pi_1(X)}$ and likewise,

$$H^2_c(X, \mathscr{F}) = H^0(X, \mathscr{F}^\vee)^\vee(-1) = V_{\pi_1(X)}(-1)$$

because the dual of invariants are coinvariants. Take α an eigenvalue of F acting on $V_{\pi_1(X)}$ then $\alpha^{d(x)}$ is an eigenvalue of $F^{d(x)} = F_x$ acting on $V_{\pi_1(X)}$ which is a quotient of $\mathscr{F}_{\bar{x}}$ and thus $q|\iota\alpha| \le q^{\frac{\beta d(x)}{2}+1}$ and thus $q\alpha$ is an eigenvalue of $F \circlearrowright H^2_c$ iff $(q\alpha)^{-1}$ is a zero of $\det{(1 - Ft|H^2_c)}$. Therefore, the possible poles of $\iota L(X_0, \mathscr{F}_0, t)$ is $\iota(q\alpha)^{-1}$ for α as above. Therefore $\iota L(X_0, \mathscr{F}_0, f)$ has no poles for $|t| < q^{-\frac{\beta}{2}-1}$. However,

$$L(X_0, \mathscr{F}_0, t) = L(U_0, \mathscr{F}_0|_{U_0}, t) \cdot \prod_{s \in S_0} \det{(1 - F_s t | \mathscr{F}_{\bar{s}})}^{-1}$$

I claim that $\iota L(U_0, \mathscr{F}_0|_{U_0}, t)$ converges and has no zeros for $|t| < q^{-\frac{\beta}{2}-1}$. Then,

$$\iota\left(\frac{L'(U_0)}{L(U_0)}\right) = \iota\left(\sum_{n=1}^{\infty}\left(\sum_{\substack{x\in|U_0|\\ d(x)|n}} d(x)\iota\mathrm{Tr}\left(F_x^{\frac{n}{d(x)}}\right)\right)t^{n-1}\right)$$

Then,

$$|\iota\mathrm{Tr}\left(F_x^{\frac{n}{d(x)}}\right)| \le r\left(q^{\frac{\beta d(x)}{2}}\right)^{\frac{n}{d(x)}} = rq^{\frac{\beta n}{2}}$$

Therefore,

$$\sum_{\substack{x\in|U_0|\\ d(x)|n}} d(x) = \#U_0(\mathbb{F}_{q^n}) \le Cq^n$$

and therefore the logarithmic derivative is dominated by,

$$\sum_{n=1}^{\infty} rCq^{n\left(\frac{\beta}{2}+1\right)}t^{n-1}$$

which converges absolutely for $|t| < q^{-\frac{\beta}{2}-1}$ and thus $|\iota\alpha| \le q^{\beta+1}$.

Now we apply the same argument to $\mathscr{G}_0 = \mathscr{F}_0^{\otimes k}$ so α^k is an eigenvalue of $F_s \circlearrowright \mathscr{G}_{\bar{s}}$ and thus $|\iota\alpha^k| \le q^{\frac{k\beta}{2}+1}$ which implies that,

$$|\iota\alpha| \le q^{\frac{\beta}{2}+\frac{1}{k}}$$

and thus taking $k \to \infty$ this goes to $q^{\frac{\beta}{2}}$ so $|\iota\alpha| \le q^{\frac{\beta}{2}}$. ☐

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Theorem 2.0.1. Let $f_* : X_0 \to Y_0$ be separated of finite type over $\mathbb{F}_q$ and $\mathscr{F}_0$ a Weil sheaf on X_0. If $\mathscr{F}_0$ is ι-mixed of largest weight β then for all k the sheaf $R^k(f_0)_!\mathscr{F}_0$ is ι-mixed of weight at most $\beta + k$.

Remark. Today we will prove this for $X_0 \to \mathrm{Spec}\,(\mathbb{F}_q)$ where X_0 is a smooth curve, $\mathscr{F}_0$ is smooth and ι-pure of weight β. Recall that purity means that for each $x \in X_0$ and α an eigenvalue of $F_x \circlearrowright \mathscr{F}_{\bar{x}}$ then $|\iota\alpha| = q^{\frac{d(x)\beta}{2}}$

Theorem 2.0.2 (Semicontinuity)**.** Let $j_0 : U_0 \hookrightarrow X_0$ be a dense open immersion and let S_0 be the complement. Let $\mathscr{F}_0$ be a smooth Weil sheaf thus corresponding to some $\pi_1(X_0)$-representation V. For $\beta \in \mathbb{R}$ consider the condition,

$$(*)_x : \forall\alpha \text{ eigenvalue of } F_x \circlearrowright \mathscr{F}_{\bar{x}} : |\iota\alpha| \le q^{\frac{d(x)\beta}{2}}$$

If $(*)_x$ holds for $x \in |U_0|$ then $(*)_x$ holds on the boundary.

Corollary 2.0.3. With the same notation as above,

(a) if $\mathscr{F}_0|_{U_0}$ is ι-pure of weight β then $\mathscr{F}_0$ is ι-pure of weight β

(b) assume X_0 is normal and geometrically irreducible and $\mathscr{F}_0$ is irreducible (as a representation). If $\mathscr{F}_0|_{U_0}$ is ι-mixed then $\mathscr{F}_0$ is ι-pure.

Proof. For (a) we apply semi-continuity to $\mathscr{F}_0$ and $\mathscr{F}_0^\vee$. Now for (b) set $\mathscr{G}_0 = \mathscr{F}_0|_{U_0}$. Because $\mathscr{G}_0$ is ι-mixed, there is a filtration, $\mathscr{G}_0^{(i)}$ such that the graded pieces are ι-pure. Since each term in the filtration is constructible, by shrinking U_0 we can assume that each term is smooth. Since X_0 is normal $W(U_0) \twoheadrightarrow W(X_0)$ so we see that $\mathscr{G}_0 = \mathscr{F}_0|_{U_0}$ is irreducible so there are no subrepresentations thus the filtration is trivial so $\mathscr{G}_0$ must be ι-pure. □

2.0.1 Two More Results that use L-function Techniques

Definition 2.0.4. We say that $\mathscr{F}_0$ a Weil sheaf on X_0 is *ι-real* if for each $x \in X_0$ then,

$$\iota \det\left(1 - F_x t^{d(x)} | \mathscr{F}_{\bar{x}}\right) \in \mathbb{R}[t]$$

Example 2.0.5. If $\mathscr{F}_0$ is smooth and ι-pure of weight β then $\mathscr{F}_0 \oplus \overline{\mathscr{F}_0}$ is ι-real where $\overline{\mathscr{F}_0} = \mathscr{F}_0^\vee \otimes \mathcal{L}_0(\beta)$ where $\mathcal{L}_0(\beta)$ corresponds to the representation $W(X_0) \to W(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ via $F \mapsto q^\beta$.

Remark. Suppose that $\mathscr{F}_0$ is ι-real then,

$$\iota \det\left(1 - F_x t^{d(x)} | (\mathscr{F}^{\otimes 2k})_{\bar{x}}\right) \in \mathbb{R}_{\geq 0}[t]$$

Definition 2.0.6. Let X_0 be geometrically irreducible and normal. Recall that any rank 1 smooth Weil sheaf is ι-pure. For $\mathscr{F}_0$ smooth Weil sheaf on X_0 let $\mathscr{F}_0$ be an irreducible constituent. Then, the det ι-weight of $\mathscr{G}_0$ is defined to be $\beta/\mathrm{rk}(\mathscr{G}_0)$ where β is the ι-weight of $\det \mathscr{G}_0$ the top exterior power.

Remark. If $\mathscr{F}_0$ has det ι-weight with multiplicity $m(\beta)$ then the det ι-weights of,

$$\bigwedge^k \mathscr{F}_0$$

are,

$$\sum n_\beta \beta \text{ where } \sum n_\beta = k \text{ and } 0 \leq n_\beta \leq m(\beta)$$

This takes some work. We need to study the $G_0 = \overline{\rho(\pi_1(X))} \subset \mathrm{GL}(\mathbb{Q}_\ell)$ where $\mathscr{F}_0^{SS}$ corresponds to a representation $\rho : W(X_0) \to \mathrm{GL}(\mathbb{Q}_\ell)$.

Remark. If X_0 is a smooth curve, $H^2_c(X, \mathscr{F}) = H^0(X, \mathscr{F}^\vee)^\vee(-1) = (\mathscr{F}_{\bar{x}})_{\pi_1(X,\bar{x})}(-1)$. Therefore, $\mathscr{F}_0 \twoheadrightarrow \mathscr{G}_0$ where this is the largest quotient constant on X this corresponds to the representations,

$$\begin{array}{c} W(X, \bar{x}) \circlearrowright \mathscr{F}_{\bar{x}} \\ \downarrow \\ W(\mathbb{F}_q) \circlearrowright (\mathscr{F}_{\bar{x}})_{\pi_1(X)} \end{array}$$

Therefore, we conclude that a det ι-weight of $\mathscr{G}_0$ is β iff an eigvenvalue of $F \circlearrowright H^2_c(\mathscr{F})$ has ι-weight $\beta + 2$.

Theorem 2.0.7. X_0 geometrically irreducible smooth curve, $\mathscr{F}_0$ smooth on X_0 and $\mathscr{F}_0$ is ι-real then each irreducible factor of $\mathscr{F}_0$ is ι-pure.

Proof. We may assume that X_0 is affine. Let β be the largest det ι-weight of $\mathscr{F}_0$. Then $2k\beta$ should be the largest det ι-weight on $\mathscr{F}_0^{\otimes 2k}$. Now consider the L-function,

$$\frac{\iota \det\left(1 - Ft|H^1_c(\mathscr{F}^{\otimes 2k})\right)}{\iota(1 - Ft|H^2_C(\mathscr{F}^{\otimes 2k}))} = L(X_0, \mathscr{F}_0^{\otimes 2k}, t) = \prod_{x \in X} \frac{1}{\iota \det\left(1 - F_x t^{d(x)}|\mathscr{F}_{\bar{x}}^{\otimes 2k}\right)}$$

By the second remark this converges for $|t| < q^{-\frac{2k\beta+2}{2}}$ and thus each factor converges in this region and by remark 1 the local factor on the right are positive real polynomials. Thus if α is an eigenvalue of $F_x \circlearrowright \mathscr{F}_{\bar{x}}$ then $|\iota\alpha|^{-2k} \geq q^{-\frac{2k\beta+2}{2}}$ and thus taking $k \to \infty$ we see that,

$$|\iota\alpha| \leq q^{\frac{\beta d(x)}{2}}$$

$\square$

Example 2.0.8. Let $\mathscr{F}_0 = \mathscr{G}_0 \oplus \mathscr{G}_0'$ with ranks 3 and 2 and det ι-weights $\beta > \beta'$ respectively. Let the eigenvalues of $F_{\bar{x}}$ be $\alpha_1, \alpha_2, \alpha_3$ and α_1', α_2' respectively. The first inequality says that $|\iota\alpha_i| \leq q^{\frac{\beta d(x)}{2}}$ but $|\iota\alpha_1\alpha_2\alpha_3| = q^{\frac{3\beta d(x)}{2}}$ which implies that $|\iota\alpha_i| = q^{\frac{\beta d(x)}{2}}$.

Corollary 2.0.9. Let $\mathscr{F}_0$ be ι-real on X_0 which is finite type over $\mathbb{F}_q$ then,

(a) $\mathscr{F}_0$ is ι-mixed

(b) if X_0 is geometrically irreducicble and normal and $\mathscr{F}_0$ si smooth, then rach irreducible component of $\mathscr{F}_0$ is ι-pure.

Definition 2.0.10. Let X_0 be a smooth curve, $\mathscr{F}_0$ a Weil-sheaf on X_0. Define $f_n^{\mathscr{F}_0} : X_0(\mathbb{F}_{q^n}) \to \mathcal{C}$ by,

$$f_n^{\mathscr{F}_0}(x) = \iota \mathrm{tr}\left(F_x^{\frac{n}{d(x)}}|\mathscr{F}_{\bar{x}}\right)$$

There is also a norm,

$$(f_n, g_n)_n = \sum_{x \in X_0(\mathbb{F}_{q^n})} f_n(x)\overline{g_n(x)} \quad \text{and} \quad ||f_n||_n^2 = (f_n, f_n)_n$$

Proposition 2.0.11. If $\mathscr{F}_0$ is ι-mixed and $H^0_c(X, \mathscr{F}) = 0$ then,

$$\sup\left\{\rho \mid \limsup_n \frac{||f_n^{\mathscr{F}_0}||_n^2}{q^{n(1+\rho)}} > 0\right\} = \sup_{x \in |X_0|} \sup_{\substack{\alpha \text{e.v. of} \\ F_x \circlearrowright \mathscr{F}_{\bar{x}}}} \frac{\log|\iota\alpha|^2}{\log q^{d(x)}}$$

Proof. We will not prove this. However, if $\mathscr{F}_0$ is smooth and ι-pure then we study $\log \iota L(X_0, \mathscr{F}_0 \otimes \overline{\mathscr{F}_0}; t)$. $\square$

2.1 Fourier Transform

Fix $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ a nontrivial additive character. Thenm $\mathcal{L}_0(\psi)$ a smooth sheaf of $\mathbb{A}_0^1$ corresponding to $W(\mathbb{A}_0^1, \bar{0}) \to \mathbb{F}_q \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times$ defined by the Artin-Schier extension. Then for $x \in \mathbb{A}_0^1$ then $\psi_x : k(x) \to \overline{\mathbb{Q}}_\ell^\times$ by $\psi_x(y) = \psi(\mathrm{tr}\,(xy))$.

Definition 2.1.1. Let $T\psi : D_c^b(\mathbb{A}_0^1, \overline{\mathbb{Q}}_\ell) \to D_c^b(\mathbb{A}_0^1, \overline{\mathbb{Q}}_\ell)$ be defined by,

$$T_\psi(K_0) = R(\pi_0^1)_!(\pi_0^{2*}(k_0) \otimes m_0^*(\mathcal{L}_0(\psi)))[1]$$

where $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the multiplication map ($\mathbb{A}^1$ is a ring object).

Remark. By proper base change, for all $x \in \mathbb{A}_0^1(\overline{\mathbb{F}}_q)$ we have,

$$T_\psi(K_x)_{\bar{x}} = R\Gamma_c(\mathbb{A}^1, K \otimes \mathcal{L}(\psi_x))[1]$$

Example 2.1.2. For $K - 0 \in D_C^b(\mathbb{A}_0^1, \overline{\mathbb{Q}}_\ell)$ define $f_n^{K_0}(t) := \sum(-1)^i f_n^{H^i(K_0)}$ then,

$$\mathrm{tr}\left(F_x | T_\psi(K_0)_{\bar{x}}\right) = -\mathrm{tr}\left(F^{d(x)} | R\Gamma_c(\mathbb{A}^1, K \otimes \mathcal{L}(\psi_x))\right)$$

By Grothendieck-Lefschetz trace formula,

$$= - \sum_{y \in \mathbb{A}_0^1(\mathbb{F}_q^{d(x)})} \mathrm{Tr}\left(f_y | K_y)\right) \psi_x(-y)$$

Therefore,

$$f_n^{T_\psi(K_0)}(t) = - \sum_{y \in \mathbb{F}_{q^n}} f_n^{K_0}(y) \psi_t(-y)$$

and thus,

$$||f_n^{T_\psi(K_0)}||_n^2 = \sum_{t \in \mathbb{F}_{q^n}} \sum_{y,z \in \mathbb{F}_{q^n}} f_n^{K_0}(y) \overline{f_n^{K_0}(Z)} \psi_t(-y) \overline{\psi_t(-z)} = q^n ||f_n^{K_0}||_n^2$$

which uses a version of Placherel's formula.

Theorem 2.1.3 (Fourier inversion)**.** $T_{\psi^{-1}} \circ T_\psi(K_0) = K_0(-1)$ for any K_0.

Proof. First,

$$T_{\psi^{-1}} \circ T_\psi(K_0) = R\pi_!^1\left(\pi^{2*} R\pi_!^1[\pi^{2*} K_0 \otimes m^* \mathcal{L}_0(\psi)] \otimes m^* \mathcal{L}_0(\psi^{-1})\right)[2]$$

By the projection formula, ☐

Theorem 2.1.4. Let X_0 be geometrically irreducible smooth curve, $j_0 : X_0 \hookrightarrow \overline{X_0}$ and $\mathscr{F}_0$ smooth ι-pure of weight β. Then,

(a) ι-weights of $F \circlearrowright H^i(\overline{X}, j_* \mathscr{F})$ are $\leq \beta + i$

(b) ι-weights of $F \circlearrowright H_c^i(X, \mathscr{F})$ are $\leq \beta + i$.

Remark. by Poincare duality there is a perfect pairing,

$$H^i(\overline{X}, j_* \mathscr{F}) \times H^{2-i}(\overline{X}, j_* \mathscr{F}^\vee)(1)$$

are dual. Then (1) implies that $H^i(\overline{X}, j_* \mathscr{F})$ is ι-pure of weight $\beta + i$. Furthermore (1) $\iff$ (2) because of the exact sequence,

$$0 \longrightarrow j_! \mathscr{F} \longrightarrow j_* \mathscr{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathbb{Q}$ supported on the boundary. Then $\mathcal{Q}_s$ has weights less than β by semi-continuity.

Proof. Proof of (1) in the case $i = 1$. We may shrink X_0 + some minor arguments $\overline{X}_0 \to \mathbb{P}^1$ we reduce to the case the following claim over a point: consider $U_0 \subset \mathbb{A}_0^1 \subset \mathbb{P}_0^1$ and $\mathscr{G}_0$ geometrically irreducible smooth ι-pure sheaf of weight β on U_0 (unramified at ∞?) then we need ι-weights of $F \circlearrowright H_c^1(U, \mathscr{G})$ are at most $\beta + 1$. ☐

3 Oct. 5

3.1 Review of t-Structures

Definition 3.1.1. A t-structure on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ such that,

(a) letting $\mathcal{D}^{\le n} := \mathcal{D}^{\le 0}[-n]$ and $\mathcal{D}^{\ge n} := \mathcal{D}^{\ge 0}[-n]$ then $\mathcal{D}^{\le n} \subset \mathcal{D}^{\le 1}$ and $\mathcal{D}^{\ge 1} \subset \mathcal{D}^{\ge 0}$.

(b) $\mathrm{Hom}_{\mathcal{D}}\left(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 1}\right) = 0$

(c) for every $X \in \mathcal{D}$ there is a distringulished triangle,

$$X^{\le 0} \longrightarrow X \longrightarrow X_{\ge 1} \longrightarrow X^{\le 0}[1]$$

where $X_{\le 0} \in \mathcal{D}^{\le 0}$ and $X^{\ge 1} \in \mathcal{D}^{\ge 1}$.

Proposition 3.1.2. Axioms (a) and (c) imply the existence of truncation functors $\tau_{\le n} : \mathcal{D} \to \mathcal{D}^{\le n}$ right adjoint to $\mathcal{D}^{\le n} \subset \mathcal{D}$ and $\tau_{\ge n} : \mathcal{D} \to \mathcal{D}^{\ge n}$ left adjoint to $\mathcal{D}^{\ge n} \subset \mathcal{D}$ and thus the exact triangles in (c) are actually functorial.

Definition 3.1.3. The *heart* of a t-structure on $\mathcal{D}$ is $\mathcal{D}^{=}\mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$.

Theorem 3.1.4. The category $\mathcal{D}$ is an abelian category and a sequence $A \to E \to B$ in $\mathcal{D}$ is exact iff there exists an exact triangle in $\mathcal{D}$ $A \to E \to B \to A[1]$ (where only the connecting map is extra data).

Theorem 3.1.5. For each n, there is an additive functor $H^n : \mathcal{D} \to \mathcal{D}$ given by $H^n(X) = \tau_{\ge n}\tau_{\le n} X = \tau_{\le n}\tau_{\ge n} X$. We rtie,

$$H^n(X) \in \mathcal{D}^{\le n} \cap D^{\ge n} \cong \mathcal{D}$$

where the last isomorphism is given by shifting. Furthermore, H^n is cohomological meaning it sends short exact sequences to long exact sequences.

Definition 3.1.6. Given trianguled categories $\mathcal{D}_1, \mathcal{D}_2$ with t-structures we say that a triangulated functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ is

$$\textit{t-left exact} \text{ if } F(\mathcal{D}_1^{\ge 0}) \subset \mathcal{D}_2^{\ge 0}$$
$$\textit{t-right exact} \text{ if } F(\mathscr{F}_1^{\le 0}) \subset \mathcal{D}_2^{\le 0}$$

4 Oct. 19

Recall $B_0 \in D_{\text{mixed}}(X_0)$ then,

$$\omega(B_0) := \max_{\nu} \left(\omega(\mathcal{H}^{\nu}(B_0)) - \nu\right)$$

And $D_{\le w}(X_0)$ if

Then B_0 is pure of weight w if $B_0 \in D_{\le w}(X_0) \cap D_{\ge w}(X_0)$.

Lemma 4.0.1 (Semi-continuity of weights)**.** Let $j : U_0 \hookrightarrow X_0$ be an open dense, $\iota : Y_0 \hookrightarrow X_0$ the closed complement. Let $\overline{B}_0 \in X_0$ be τ-mixed such that $j^*(\overline{B}_0) = B_0$ and assume that $H_p^0(\iota^*(\overline{B}_0)) = 0$ then

$$w(\overline{B}_0) = w(B_0)$$

Proof. The inequality $w(B_0) \leq w(\overline{B}_0)$ is obvious so we just need to show the other direction. We apply the Fourier transform,

$$\begin{tikzcd}
& \mathbb{A}_0^r \times \mathbb{A}_0^r \times \mathbb{A}_0^s \arrow[r, "m_{12}"] \arrow[dl, "p_{13}"'] \arrow[d, "p_{23}"] & \mathbb{A}_0^1 \\
\mathbb{A}_0^r \times \mathbb{A}_0^s & \mathbb{A}_0^r \times \mathbb{A}_0^s &
\end{tikzcd}$$

For $B \in D_c^b(\mathbb{A}_0^r \times \mathbb{A}_0^s)$ then,

$$T_\psi(B) = R(p_{13})_!(p_{23}^* B \otimes^{\mathbb{L}} m_{12}^* \mathcal{L}_0(\psi))$$

$\square$