1 Examples of stable ∞-cagtegories

1.1 dg-Categories

Review, let $\mathcal{A}$ be an abelian cagtegory with enough projectives. Then there is a category $D^-(\mathcal{A})$ the derived category of bounded-above.

$\mathbf{Ch}\left(()\,\mathcal{A}\right)$ is naturally a dg-Category meaning it is enriched in $\mathbf{Ch}\left(()\,\mathbf{Ab}\right)$. Indeed, if $A, B \in \mathbf{Ch}\left(()\,\mathcal{A}\right)$ then we define,

$$\mathrm{Hom}\left(A_\bullet, B_\bullet\right)_n = \prod_k \mathrm{Hom}\left(A_k, B_{k-n}\right)$$

with differential,

$$(\mathrm{d}f)_k = f_{k+1} \circ \mathrm{d}_B + (-1)^{n+1}\mathrm{d}_B \circ f_k$$

Then we check that,

$$H^0(\mathrm{Hom}\left(A, B\right)_\bullet) = \mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}\left(A, B\right)$$

1.2 Program

Given a dg-category, we get a simplicially enriched category by truncating and then applying Dold-Kan and then apply the homotopy-coherent nerve.

However, this requires checking many steps. We can instead go directly with the following construction.

1.3 Construction

Definition 1.3.1. A *dg-category* is a category enriched in $\mathbf{Ch}\left(()\,\mathcal{A}\right)$. This includes the requirement that,

$$\mathrm{d}(g \circ f) = \mathrm{d}g \circ f + (-1)^{\deg g} g \circ \mathrm{d}f$$

which arises from needing to preserve the monoidal structure on $\mathbf{Ch}\left(()\,\mathcal{A}\right)$ given by graded tensor $A \otimes B$.

We will define a dg-Nerve which goes directly from dg-categories to ∞-categories which does not require passing through simplically-enriched categories.

Definition 1.3.2. Let C be a dg-category. Then the *dg-Nerve* is the simplicial set $N_{\mathrm{dg}}(C)$ where $N_{\mathrm{dg}}(C)_n$ is the set of objects X_i for $i \in \{0, \dots, n\}$ and for each ordered set $I = \{i_- < i_m < \cdots < i_1 < i_+\}$ for $m \geq 0$ of elements in $\{0, \dots, n\}$ we have map,

$$f_I \in \mathrm{Map}\left(X_{i_-}, X_{i_+}\right)$$

such that,

$$\mathrm{d}f_I = \sum_{1 \leq j \leq m} (-1)^i (f_{I \setminus \{i_j\}} - f_{\{i_j < \cdots < i_+\}} \circ f_{\{i_- < \cdots < i_j\}})$$

and the maps work as follow. If $\alpha : [m] \to [n]$ is monotone. Then the induced map α^* is given by,

$$\alpha^*(\{X_i\}, \{f_I\}) = (\{X_{\alpha(i)}\}, \{g_I\})$$

where,

$$g_J = \begin{cases} f_{\alpha(J)} & \alpha|_J \text{ injective} \\ \mathrm{id}_{X_i} & J = \{j, j'\} \text{ and } \alpha(j) = j' \\ 0 & \text{else} \end{cases}$$

Proposition 1.3.3. For any dg-category C the dg-Nerve $N_{\mathrm{dg}}(C)$ is an ∞-category.

Proof. We need to show that inner horns can be filled. However, $\Lambda_i^n \to N_{\mathrm{dg}}(C)$ this is the same data as specifying $\Delta^n \to N_{\mathrm{dg}}(C)$ except we haven't specified all the maps f_I. In fact, this has specified the maps for all I except for $I = [n]$ and $I = [n] \backslash \{i\}$. Then we set $f_{[n]} = 0$ and,

$$f_{[n] \backslash \{i\}} = \sum_{0<p<n} (-1)^{p-i} f_{\{p,\dots,n\}} \circ f_{\{0,\dots,p\}} - \sum_{p<p<n, p\neq i} (-1)^{p-i} f_{[n] \backslash \{0\}}$$

□

Remark. $\mathrm{Hom}_{N_{\mathrm{dg}}(C)}(X,Y) = \mathrm{DK}(\tau_{\geq 0} \mathrm{Map}_C(X,Y))$ which is the same result we would have gotten from the program applying Dold-Kan to the simplicially-enriched category.

Remark. Under what conditions do we get a stable ∞-category from $N_{\mathrm{dg}}(C)$?

Definition 1.3.4. Let $\mathcal{A}$ be an abelian category with enough projectives, $D^+(\mathcal{A}) = N_{\mathrm{dg}}((\mathbf{Ch}\left(()\,\mathcal{A}_{\mathrm{proj}}\right)_{\geq 0})$.

Proposition 1.3.5 (Prop 1.3.2.10)**.** $D^-(\mathcal{A})$ is stable.

Definition 1.3.6. We say that $\mathcal{A}$ is a *Grothendieck abelian category* if A has a generator and small filtered colimits of monomorphisms are monomorphisms.

Example 1.3.7. The category of R-modules is Grothendieck.

Theorem 1.3.8. Let $\mathcal{A}$ be Grothendieck. Then $\mathbf{Ch}\left(()\,\mathcal{A}\right)$ has a model structure where,

(a) cofibrations $M \to N$ are those morphisms which are injective termwise

(b) equivalences $M \to N$ are quasi-isomorphisms

(c) fibrations are those satisfying the right lifting property wrt the acyclic cofibrations.

Proposition 1.3.9 (1.3.5.6)**.** (a) If M_n is injective for all n and $M_n \cong 0$ for $n \gg 0$ then $M_\bullet$ is fibrant.

(b) If $M_\bullet$ is fibrant then each M_n is injective.

If $M\mathcal{A}$ and $M' = M[0]$ then there is a fibrant replacement $M[0] \to Q$ where the map is a trivial cofibration this proves enough injectives.

Definition 1.3.10. Let $\mathbf{Ch}\left(()\,\mathcal{A}\right)^\circ$ be the full subcategory of fibrat objects. Let $D(A) = N_{\mathrm{dg}}(\mathbf{Ch}\left(()\,\mathcal{A}\right)^\circ)$.

Proposition 1.3.11. $D(\mathcal{A})$ is stable and is the ∞ unbounded derived category.

1.4 Spectra

Definition 1.4.1. We say an ∞-category is *pointed* if it admits a zero object.

Motivation: stable maps,

$$[X,Y]_s := \varinjlim_n [\Sigma^n X, \Sigma^n Y]$$

There is a category of topological spaces with stable maps. This gives a triangulated category with Σ acting via shift. Constructing this category "formally" we have some options,

(a) objects are (X, n) with $X \in \textbf{Top}$ and $n \in \mathbb{Z}$ and morphisms are,

$$\text{Hom}\left((X,n),(Y,m)\right) = \varinjlim_k [\Sigma^{k+n}X, \Sigma^{k+m}Y]$$

which works even for negative n, m becasuse we can choose k large enough. This "formally inverts" suspension by giving elemens like $(X, -1)$ which is the de-suspension of X

(b) construting inifinte loop spaces: a sequence E_n with maps $E_n \xrightarrow{\sim} \Omega E_{n+1}$ so these are progressive deloopings. These are also giving inverses of Σ.

Definition 1.4.2. Let $F : C \to D$ be a functor of ∞-categories then,

(a) F is *excisive* if psuhout diagrams map to pullback diagrams

(b) F is *reduced* if $F(*) = *$.

Definition 1.4.3. Let $\mathcal{C}$ admit finite limits. A *spectrum object* is a reduced excisive functor,

$$F : S_*^{\text{fin}} \to \mathcal{C}$$

The ∞-category of spectra is,

$$\text{Sp}(\mathcal{C}) = \text{Fun}\left(S_*^{\text{fin}}, \mathcal{C}\right)_{\text{exc, red}} \subset \text{Fun}\left(S_*^{\text{fin}}, \mathcal{C}\right)$$

where S_*^{fin} is the pointed category of finite spaces (full subcategory of ∞-category of pointed spaces).

Proposition 1.4.4 (1.4.2.11)**.** Let C be a pointed ∞-category with finite limits and colimits then $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence then $\mathcal{C}$ is stable.

Proposition 1.4.5. If $\mathcal{C}$ is a pointed ∞-category with finite limits and colimts then $\text{Sp}(\mathcal{C})$ is stable.

Proposition 1.4.6. Let $\mathcal{C}$ be a pointed ∞-category with finite limits then there is a tower of ∞-categories,

$$\cdots \longrightarrow C \xrightarrow{\Omega} C \xrightarrow{\Omega} C$$

and $\text{Sp}(\mathcal{C})$ is the homotopy limit.

Let $\mathcal{A}$ be an abelian group. Then the Eilenberg-Maclain spaces satisfy $K(A, n-1) \cong \Omega K(A, n)$ and hence defines a spectrum.

Proposition 1.4.7. Let $\mathcal{C}$ be an ∞-category which admits finite limits. The following are equivalent,

(a) $\mathcal{C}$ is a stable ∞-category

(b) the functor $\Omega^\infty : \text{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories

where Ω^∞ is the map sending $F \mapsto F(S^0)$ where $S^0 = * \coprod *$.

2 ∞-operads

References: Higher Algebra

2.1 Motivations

Want the symmetric monoidal category in the ∞-setting.

Recall that the definition of a symmetric monidal category is annoying:

Definition 2.1.1. A symmetric monidal category $(\mathcal{C}, \otimes, 1, \alpha, \nu, \beta)$ is,

(a) a 1-category $\mathcal{C}$

(b) a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) a natural isomorphism,

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$$

(d) a unit object 1 with a fixed isomorphism $1 \otimes 1 \xrightarrow{\sim} 1$

(e) a natural isomorphsim,

$$\beta_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

(this is the easy part, now the bad part) satisfying,

(a) $A \mapsto 1 \otimes A$ and $A \mapsto A \otimes 1$ are equivalences of categories (then from $A \otimes 1 \xrightarrow{\sim} A \otimes (1 \otimes 1) \xrightarrow{\sim} (A \otimes 1) \otimes 1$ we see that $A \xrightarrow{\sim} A \otimes 1$ by equivalence)

(b) unit should be compatible with associativity,

$$\begin{tikzcd} X \otimes (1 \otimes Y) \arrow[rr, "\alpha_{1,Y}"] \arrow[dr] & & (X \otimes 1) \otimes Y \arrow[dl] \\ & X \otimes Y & \end{tikzcd}$$

(c) some huge nasty diagram making associativity compatible for 4-fold products

2.2 Construction

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. Then we can construct a new category $C^{\otimes}$ whose objects are $\{[C_1, \ldots, C_n]\}_{C_i \in \mathcal{C}}$ with $n \geq 0$. And the morphisms,

$$[C_1, \ldots, C_n] \xrightarrow{(S,\alpha,f)} [C'_1, \ldots, C'_m]$$

for $S \subset \{1, \ldots, n\}$ and $\alpha : S \to \{1, \ldots, m\}$ and maps $\{f_j\}_{j=1}^m$ which are maps $f_j : \otimes_{i \in \alpha^{-1}(j)} C_i \to C_j$. The composition law is given by $(S, \alpha, f) \circ (S', \alpha', f')$ in the only reasonable way (WRITE DOWN).

Remark. Notice that the ordering of the object $[C_1, \ldots, C_n]$ does not matter.

Definition 2.2.1. New category Fin_* whose objects are $\langle n \rangle := \{1, \ldots, n\} \sqcup \{*\}$ and the morphisms are $\langle n \rangle \to \langle m \rangle$ with $* \mapsto *$. This is just the category of finite sets with a disjoint basepoint added and the maps must be basepoint preserving.

Remark. Fin_* is the category of pointed finite sets and basepoint preserving maps.

Definition 2.2.2. We have special maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ which sends $i \mapsto 1$ and $k \mapsto *$ for $k \neq i$. Then we write $\langle n \rangle^{\circ} = \{1, \ldots, n\} = \langle n \rangle \backslash \{*\}$.

Remark. The data of a symmetric monoidal category is equivalent to a functor $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ given by $[C_1, \ldots, C_n] \mapsto \langle n \rangle$ satisfying some properties.

2.3 Colored Operads

Definition 2.3.1. A symmetric monoidal ∞-category should be a cocartesian fibration $p : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ satisfying $\forall n \geq 0$ the collection of $\{\rho^i\}$ induces $\rho^i_* : \mathcal{C}^{\otimes}_{\langle n \rangle} \to C^{\otimes}_{\langle 1 \rangle}$ determining $\mathcal{C}^{\otimes}_{\langle n \rangle} \cong (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^{\oplus n}$.

Remark. We will come back and motivate this definition. First we discuss operads which are like categories where we have hom spaces have arbitrary arity, they are not just 2-adic.

Definition 2.3.2. A *colored operad* $\mathcal{O}$ is the following data,

(a) a collection of objects $\mathrm{Ob}(\mathcal{O})$

(b) Hom spaces: given any finite set I and a list of objects $(\{X_i\}_{i \in I}, Y)$ we get a space $\mathrm{Mult}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ if $I = \varnothing$ we get a set $\mathrm{Mult}_{\mathcal{O}}(, Y)$ which is a unit (or something)

(c) a composition law, for any map $I \to J$ we get,

$$\prod_{j \in J} \mathrm{Mult}_{\mathcal{O}}\left(\{X_i\}_{i \in I_j}, Y_j\right) \times \mathrm{Mult}_{\mathcal{O}}(\{Y_j\}_{j \in J}, Z) \to \mathrm{Mult}_{\mathcal{O}}(\{X_i\}_{i \in J}, Z)$$

(d) $\mathrm{id}_Y \in \mathrm{Mult}_{\mathcal{O}}(\{Y\}, Y)$

(e) associativity (DO THIS)

Remark. We use the terminology Mult because in the case of a symmetric monoidal category we will get an operad with,

$$\mathrm{Mult}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) = \mathrm{Hom}\left(\bigotimes_{i \in I} X_i, Y\right)$$

Remark. From $\mathcal{O}$ we get a 1-category $\mathcal{C}$ with Objects $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{O})$ and $\mathrm{Hom}(X, Y) = \mathrm{Mult}_{\mathcal{O}}(\{X\}, Y)$.

Example 2.3.3. Given any 1-category $\mathcal{C}$ we can produce an operad $\mathcal{O}$ on the same objects with,

$$\mathrm{Mult}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) = \begin{cases} \varnothing & \#I \neq 1 \\ \mathrm{Hom}_{\mathcal{C}}(X, Y) & \# = 1 \end{cases}$$

This is left-adjoint to the "underlying category" functor.

Example 2.3.4. Given a symmetric monoidal category $(\mathcal{C}, \otimes)$ we get an operad on the same objects with,

$$\mathrm{Mult}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) = \mathrm{Hom}_{\mathcal{C}}\left(\bigotimes_{i \in I} X_i, Y\right)$$

Example 2.3.5. Given a colored operad $\mathcal{O}$, we get a new category $\mathcal{O}^{\otimes}$ whose objects are $\{X_i\}_{i \in I}$ and whose mapping sets are,

$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{X_i\}_{i \in I}, \{Y_j\}_{j \in J}) = \prod_{j \in J} \mathrm{Mult}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y_j)$$

In fact, $\mathcal{O}$ is equivalent to the data of the fibration,

$$\pi : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$$

with some requirements on π. Indeed we can recover, $\mathcal{O} = \pi^{-1}(\langle 1 \rangle)$ and we get $\mathrm{Mult}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ by considering maps in $\mathcal{O}^{\otimes}$ between the object $\{X_i\}_{i \in I} \in \pi^{-1}(\langle n \rangle)$ and $Y \in \pi^{-1} 9 \langle 1 \rangle)$.

2.4 Cocartesian morphisms

Definition 2.4.1. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor of 1-categories. Then $g : X \to Y$ in $\mathcal{C}$ is cocartesian if for all Z,

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Y, Z) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X, Z) \arrow[d] \\ \mathrm{Hom}_{\mathcal{D}}(p(Y), p(Z)) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(p(X), p(Z)) \end{tikzcd}$$

is a pullback.

Definition 2.4.2. Let $p : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then a morpism $g : X \to Y$ in $\mathcal{C}$ is cocartesian if p is an inner fibration and (WHAT)

2.5 ∞-Operads

Definition 2.5.1. $f : \langle m \rangle \to \langle n \rangle$ in Fin_* is *inertia* if for $i \in \langle n \rangle^\circ$ then $f^{-1}(i)$ consists of *exactly* one element. This means that two elements can only be mapped to the same place if their image is $*$ and also the map is surjective.

Definition 2.5.2. An ∞-operad is a functor $p : \mathcal{O}^\otimes \to N(\mathrm{Fin}_*)$ from an ∞-category $\mathcal{O}^\otimes$ where we write $\mathcal{O}^\otimes_{\langle n \rangle} = \pi^{-1}(\langle n \rangle)$ such that,

(a) if $f : \langle m \rangle \to \langle n \rangle$ is inertia then every object $C \in \mathcal{O}^\otimes_{\langle m \rangle}$ there exists a p-cocartesian morphism $\bar{f} : \mathcal{C} \to \mathcal{C}'$ lifting f

(b) for $C \in \mathcal{C}^\otimes_{\langle n \rangle}$ and $C' \in \mathcal{C}^\otimes_{\langle m \rangle}$ and $f : \langle m \rangle \to \langle n \rangle$ in Fin_* then,

$$\mathrm{Map}_f(C, C') := (\mathrm{Map}(C, C'))^\circ$$

is the connected component lying over f then,

$$\mathrm{Map}_f(C, C') \cong \prod_{1 \le i \le n}^{f} \mathrm{Map}_{\rho_i \circ f}(C, C')$$

(c) $\forall n \ge 0$ the maps $\{\rho^i_! : \mathcal{O}_{\langle n \rangle} \to \mathcal{O}_{\langle 1 \rangle}$ induces an equivaence of ∞-categories,

$$\mathcal{O}^\otimes_{\langle n \rangle} \xrightarrow{\sim} \left(\mathcal{O}^\otimes_{\langle 1 \rangle}\right)^{\oplus n}$$

Remark. We write $\mathcal{O} := \mathcal{O}^\otimes_{\langle 1 \rangle}$ which is called the underlying ∞-category of $\mathcal{O}^\otimes$.

Remark. If $\mathcal{O}$ is a colored (1-categorical) operad then $N(\mathcal{O}^\otimes) \to N(\mathrm{Fin}_*)$ is an ∞-operad.

Example 2.5.3. The trivial operad is on the trivial category,

$$\underline{\mathrm{Triv}} \subset \mathrm{Fin}_*$$

which is the full subcategory on the inertia maps. Then the inclusion map,

$$N(\underline{\mathrm{Triv}}) \to N(\mathrm{Fin}_*)$$

is an ∞-operad.

2.6 The Associative Operad

Let $\mathcal{E}_0^{\otimes}$ be the operad whose objects are $\langle n \rangle$ and whose morphism $f : \langle m \rangle \to \langle n \rangle$ such that $\#f^{-1}(i) \leq 1$ for $1 \leq i \leq n$ (weaker than inertia since it does not have to be surjective). This is the first of our associative operads.

The commutative operad is supposed to be $N(\text{Fin}_*) \to N(\text{Fin}_*)$.

Definition 2.6.1. A morphism $F : \mathcal{O}_1 \to \mathcal{O}_2$ of operads is a functor $F : \mathcal{O}_1^{\otimes} \to \mathcal{O}_2^{\otimes}$ of ∞-categories and a homotopy making the diagram,

$$\begin{tikzcd} \mathcal{O}_1^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathcal{O}_2^{\otimes} \arrow[dl] \\ & N(\text{Fin}_*) & \end{tikzcd}$$

3 Algebras and Modules

3.1 Operad Review

Perspectives on operads:

(a) Categories with "many-to-one" structure: meaning there are higher airity maps $Mult(\{X_i\}, Y)$

(b) For every operad $\mathcal{O}$ get $\mathcal{O}$-monoidal category

(c) For every operad $\mathcal{O}$ get $\mathcal{O}$-algebra object in a symmetric monoidal category

(d) for $\mathcal{O}' \to \mathcal{O}$ a map of operads get $\mathcal{O}'$-algebras in $\mathcal{O}$-monoidal categories.

Motivation: $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$ be the symmetric monoidal category of abelian groups with tensor. A commutative ring is $A \in \mathbf{Ab}$ equipped with maps,

(a) $e : \mathbb{Z} \to A$

(b) $m : A \otimes A \to A$

such that some diagrams commute,

$$\begin{tikzcd} A \otimes A \to A \arrow[r] \arrow[d] & A \otimes A \arrow[d] \\ A \otimes A \arrow[r] & A \end{tikzcd}$$

etc. We can package all these together into the following construction.

Definition 3.1.1. A *unital commutative ring* is,

(a) an object $A \in \mathbf{Ab}$

(b) for each finite set I a map $m_I : A^{\otimes I} \to A$

such that,

(a) if $\#I = 1$ then $m_I = \mathrm{id}$

(b) for $\varphi : I \to J$ then m_I is the composition,

$$A^{\otimes I} = \bigotimes_{i \in J} A^{\otimes \varphi^{-1}(j)} \xrightarrow{\bigotimes_{j \in J} \mathfrak{m}_{\varphi^{-1}(j)}} A^{\otimes J} \xrightarrow{m_J} A$$

Remark. Commutativity arrises from the swapping map $\varphi : \{1, 2\} \to \{1, 2\}$.

Definition 3.1.2. $\mathrm{ord}(I)$ is the set of linear orders on I. There is a "combine orderings" map,

$$\mathrm{comb} : \mathrm{ord}(J) \times \prod_{j \in J} \mathrm{ord}(\varphi^{-1}(j)) \to \mathrm{ord}(I)$$

given $\varphi : I \to J$.

Definition 3.1.3. A *unital associative ring* is,

(a) an object $A \in \mathbf{Ab}$

(b) for each finite set I and $o \in \mathrm{ord}(I)$ a map $m_{I,o} : A^{\otimes I} \to A$

such that,

(a) if $\#I = 1$ then $m_I = \mathrm{id}$

(b) for $\varphi : I \to J$ then $m_{I,\mathrm{comb}(o,o_j)}$ is the composition,

$$A^{\otimes I} = \bigotimes_{i \in J} A^{\otimes \varphi^{-1}(j)} \xrightarrow{\bigotimes_{j \in J} \mathfrak{m}_{\varphi^{-1}(j), o_j}} A^{\otimes J} \xrightarrow{m_{J,o}} A$$

Remark. What if we want to define a *non-unital* associative ring? In this case we just require our sets I to be nonempty. Or alternatively, let,

$$\mathrm{ord}'(I) = \begin{cases} \mathrm{ord}(I) & I \neq \varnothing \\ \varnothing & I = \varnothing \end{cases}$$

Replacing ord by ord$'$ gives the definition of a non-unital associative ring.

Definition 3.1.4. For a colored operad $\mathcal{O}^{\otimes}$ which consists of,

(a) a set of colors $\mathcal{O}$

(b) for each collection $\{X_i\}_{i \in i}$ and Y of colors there is a set,

$$\mathrm{Mult}_{\mathcal{O}}\left(\{X_i\}_{i \in I}, Y\right)$$

satisfying some associativity and unital relations.

and a symmetric monoidal category $(\mathcal{C}, \otimes)$ an <u>$\mathcal{O}$-algebra object</u> in $(\mathcal{C}, \otimes)$ is the data of,

(a) for each $X \in \mathcal{O}$ an object $A_X \in \mathcal{C}$

(b) for each $m : \{X_i\}_{i\in I} \to Y$ a morphism,

$$f_m : \bigotimes_{i\in I} A_{X_i} \to A_Y$$

satisfying,

(a) if $m = \mathrm{id}_X \in \mathrm{Mult}_{\mathcal{O}}(\{X\}, X)$ then $f_m = \mathrm{id}_{A_X}$

(b) if $\varphi : I \to J$ and $m \in \mathrm{Mult}_{\mathcal{O}}(\{Y_j\}_{j\in J}, Z)$ and $m_j \in \mathrm{Mult}_{\mathcal{O}}\left(\{X_i\}_{i\in\varphi^{-1}(j)}, Y_j\right)$ then $f_{\mathrm{comp}(m,\{m_j\})}$ equals the composition,

$$\bigotimes_{i\in I} A_{X_i} \xrightarrow{\otimes_{j\in J} f_{m_j}} \bigoplus_{j\in J} A_{Y_j} \xrightarrow{f_m} A_Z$$

Remark. An $\mathcal{O}$-algebra object in $\mathcal{C}$ is the same as a map of operads $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$.

Remark. There is a notion of maps of operads,

$$\{\text{symmetric monoidal cat}\} \hookrightarrow \{\text{operads}\}$$

An $\mathcal{O}^{\otimes}$-algebra object of $\mathcal{C}^{\otimes}$ is a map of operads $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$. Maps of operads are diagrams,

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r] \arrow[d] & \mathcal{O}^{\otimes\prime} \arrow[d] \\ N(\mathrm{Fin}_*) \arrow[r, equal] & N(\mathrm{Fin}_*) \end{tikzcd}$$

that takes cotartesian lifts of inert maps to cocartesian lifts.

3.2 Little Cubes Operad

For $n \geq 0$ construct an ∞-operad $\mathbb{E}_n^{\otimes}$. Consider $D^n \subset \mathbb{R}^n$ unit disk. There is a unique color $* \in \mathcal{O}$ and then define (writing I for $\{*\}_I$),

$$\mathrm{Mult}(I, *) = \{\text{embeddings } \coprod_{i\in I} D^n \to D^n \text{ arising from scaling and translation}\}$$

There is a composition map, for $\varphi : I \to J$,

$$\mathrm{Mult}(J, *) \times \prod_{j\in J} \mathrm{Mult}\left(\varphi^{-1}(j), *\right) \to \mathrm{Mult}(I, *)$$

which is continuous. Then we can take the topological nerve to get an ∞-operad $\mathbb{E}_n^{\otimes}$.

Remark. $\mathbb{E}_1^{\otimes}$ is discrete and $\mathbb{E}_1^{\otimes} \cong \mathrm{Assoc}^{\otimes}$ (defining unital associative algebra) which is the operad of finite sets with orderings.

Then $\mathbb{E}_0^{\otimes}$ is trivial which just imposes the existence of an object $u : \mathbb{Z} \to A$ so we get pointed objects.

Definition 3.2.1. $\mathbb{E}_{\infty}^{\otimes} = [N(\mathrm{Fin}_*) \to N(\mathrm{Fin}_*)]$ as an operad. Meaning $\mathrm{Mult}_{\mathbb{E}_{\infty}^{\otimes}}(I, *) = \{*\}$. This is because as we take the limit of n for the morphism sets of $\mathbb{E}_n^{\otimes}$ with fixed I the spaces become weakly contractible.

There are maps of operads $\mathbb{E}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \cdots \to \mathbb{E}_{\infty}^{\otimes}$. Therefore, I can always restrict an $\mathbb{E}_n^{\otimes}$-algebra to a lower-order algebra.

Remark. Recall that $\mathcal{S}$ is the ∞-category of spaces.

Theorem 3.2.2 (May)**.** Inside the monoidal category $(\mathcal{S}, \times)$. Then there is a functor,

$$\mathcal{S}_* \xrightarrow{\Omega^n} \mathrm{Alg}_{\mathbb{E}_n^{\otimes}}(\mathcal{S})$$

This induces an equivalence for $n \geq 1$,

$$\mathcal{S}_{*,\geq n} \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_n^{\otimes}}(\mathcal{S})$$

where $\mathcal{S}_{*,\geq n}$ is the ∞-category of n-connected pointed spaces and $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_n^{\otimes}}(\mathcal{S})$ is the ∞-category of grouplike $\mathbb{E}_n^{\otimes}$-algebras in spaces where we say that an algebra A is grouplike if $\pi_0(A)$ with its induced monoid structure is a group. Furthermore, there is an equivalence,

$$\mathrm{Sp}(\mathcal{S}_*)_{\geq 0} \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_{\infty}^{\otimes}}(\mathcal{S})$$

between connective spectra and $\mathbb{E}_{\infty}^{\otimes}$-algebras.

3.3 Limits and Colimits

Theorem 3.3.1. If $\mathcal{C}$ has all small limits then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ also has all small limits and limits are computed "objectwise".

Theorem 3.3.2. If $\mathcal{C}$ has all small colimits and $X \otimes -$ preserves them then $\mathrm{Alg}_{\mathbb{E}_{\infty}^{\otimes}}(\mathcal{C})$ has small colimits.

3.4 Modules

If $A \in \mathrm{CAlg}(\mathcal{C})$ then there is a category $\mathbf{Mod}_A\mathcal{C}$.

Remark. $\mathbf{Mod}_A\mathcal{C}$ does *not* have a symmetric monoidal structure but only an operad structure.

Theorem 3.4.1. If $A \in \mathrm{CAlg}(\mathcal{C})$ then,

$$\mathrm{CAlg}(\mathbf{Mod}_A(\mathcal{C})) \cong \mathrm{CAlg}(\mathcal{C})_{A/}$$

If B is an algebra over $\mathbf{Mod}_A(\mathcal{C})$ corresponding to some
$overline B \in \mathrm{CAlg}(\mathcal{C})_{A/}$ then,

$$\mathbf{Mod}_B(\mathbf{Mod}_A(\mathcal{C})) \cong \mathbf{Mod}_{\overline{B}}(\mathcal{C})$$

Theorem 3.4.2. (a) Limits in $\mathbf{Mod}_A(\mathcal{C})$ can be computed in $\mathcal{C}$

(b) of $\mathcal{C}$ is presentable and $X \otimes -$ preserves al l small colimits then $\mathbf{Mod}_A(\mathcal{C})$ is symmetric monoidal, presentable, and $\otimes$ commutes with colimits.

4 Ring Spectra

Recalll a spectrum object in S_* os an ∞-functor,

$$F : S_*^{\text{fin}} \to S_*$$

such that,

(a) F sends homotopy pushouts to pullbacks

(b) $F(*) = *$.

Remark. For each n there is a pushout square,

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \end{tikzcd}$$

which gives a pullback square,

$$\begin{tikzcd} F(S^n) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(S^{n+1}) \end{tikzcd}$$

and thus $F(S^n) = \Omega F(S^{n+1})$. Since every object of S_*^{fin} is a finite colimit of spheres therefore the data of F is equivalent up to homotopy to the sequence $\{F(S^n)\}$ along with the data of equvalenes $F(S^n) \xrightarrow{\sim} \Omega F(S^{n+1})$. This is classically what is known as an Ω-spectrum.

Definition 4.0.1. A *spectrum* is a sequence of pointed spaces $\{X_n\}_{n\ge 0}$ along with maps $\Sigma X_n \to X_{n+1}$ (equivalently $X_n \to \Omega X_{n+1}$).

Example 4.0.2. Some spectra,

(a) $\mathcal{S} = \{S^n\}_{n\ge 0}$ is not an Ω-spectrum

(b) for $A \in \mathbf{Ab}$ we have $HA = \{K(A, n)\}_{h\ge 0}$ is an Ω-spectrum

(c) of $\{Y_n\}_{n\ge 0}$ is any sequence of spaces then define $X_0 = Y_0$ and $X_{n+1} = \Sigma X_n \vee Y_{n+1}$ and this defines a spectrum.

How do we make this into a category of specta? If X, Y are Ω-spectra then,

$$\mathrm{Hom}_{\mathrm{hSp}}(X, Y) = \left\{ f_n : X_n \to Y_n \;\middle|\; \begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ \Omega X_{n+1} \arrow[r, "\Omega f_{n+1}"] & \Omega Y_{n+1} \end{tikzcd} \right\}$$

If X, Y are *not* Ω-spectra then,

$$\mathrm{Hom}_{\mathrm{hSp}}(X, Y) = \varinjlim_{X' \subset X} \mathrm{Hom}(X', X)$$

where $X' \subset X$ is a weak homotopy equivalence where we define weak homotopy equivalence using the following notion of homotopy groups.

Definition 4.0.3. if X is spectrum $n \in \mathbb{Z}$ then define,

$$\pi_n(X) = \varinjlim_k \pi_{n+k}(X_k)$$

Example 4.0.4. $\pi_n(\mathcal{S}) = \pi_n^s(\mathcal{S}) = \pi_{2n+2}(S^{n+2}$ by Freudenthal suspension.

Example 4.0.5.

$$\pi_n(HA) = \begin{cases} A & n = 0 \\ 0 \end{cases}$$

Example 4.0.6. Homotopy groups may be supported in negative degrees. Indeed consider,

$$X_n = S^n \vee S^{n-1} \vee \cdots \vee S^1$$

then $\pi_k(X_n) \neq 0$ for all $k \in \mathbb{Z}$ using Hilton-Milnor theorem.

Proposition 4.0.7. The inclusion $N(\mathbf{Ab}) \to \mathrm{Sp}$ sending $A \mapsto HA$ is fully faithful.

Proof. Let's just check this hSp. Point is to compute $[K(A,n), K(B,n)]$. This is,

$$[K(A,n), K(B,n)] = H^n(K(A,n), B) = \mathrm{Hom}\,(H_n(A,n), \mathbb{Z}, B) = \mathrm{Hom}\,(\pi_n(K(A,n)), B) = \mathrm{Hom}\,(A, B)$$

using Hurewicz's theorem. $\square$

Definition 4.0.8. There is a natural t-structure on Sp which is $\mathrm{Sp}^{\geq 0} = \{X \mid \pi_n(X) = 0 \quad n < 0\}$ with $\mathrm{Sp}^{\leq 0}$ defined similarly.

Proposition 4.0.9. $(\mathrm{Sp}^{\geq 0}, \mathrm{Sp}^{\leq 0})$ is a t-structure and its heart $\mathrm{Sp}^{\heartsuit} = N(\mathbf{Ab})$ meaning $X \in \mathrm{Sp}^{\heartsuit}$ satisfies $X \cong H\pi_0(X)$.

Proof. This is just because its heart is objects with trivial higher homotopy groups (setlike). $\square$

4.1 Smash Product

For $X, Y \in \mathbf{Top}_*$ then $X \wedge Y = (X \times Y)/(X \vee Y)$.

Example 4.1.1. $S^0 \wedge X \cong X$ and $S^1 \wedge X \cong \Sigma X$ (by definition). Then $S^m \wedge S^n \cong S^{m+n}$ because $(\mathbb{R}^m)_\infty \wedge (\mathbb{R}^n)_\infty \cong (\mathbb{R}^{m+n})_\infty$.

Example 4.1.2. There is a weird sign,

$$\begin{tikzcd} S^1 \wedge S^1 \arrow[r, "\text{flip}"] \arrow[d] & S^1 \wedge S^1 \arrow[d] \\ S^2 \arrow[r, "-1"] & S^2 \end{tikzcd}$$

because it is orientation reversing.

Proposition 4.1.3. Some properties,

(a)

$$\mathrm{Hom}_{\mathbf{Top}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}_*}\Big(X, \mathrm{Hom}_{\mathbf{Top}_*}(Y, Z)\Big)$$

(b) $X \wedge Y \cong Y \wedge X$

(c) $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.

Theorem 4.1.4 (HA, 4.8.2.19)**.** There exists a symmetric monoidal structure $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ with unit $\mathcal{S}$ which commutes with small colimits in both variables. If $\mathcal{C}$ is a symmetric monoidal ∞-cat, stable and presentable and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits in each variable then there exists a unique up to homotopy symmetric monoidal functor $F : \mathrm{Sp}^{\otimes} \to \mathcal{C}^{\otimes}$ such that the underlying $\mathrm{Sp} \to \mathcal{C}$ preserves small colimts.

Remark. Note that $HA \otimes HB \ncong H(A \otimes B)$ this acts more like a derived tensor product.

Example 4.1.5. $\mathcal{S} \wedge \mathcal{S} \to \mathcal{S}$ is an isomorphism and gives the trivial ring spectrum.

If R is an $\mathbb{E}_1$-ring then,

$$\pi_* R = \bigoplus_{n \in \mathbb{Z}} \pi_* R$$

is a graded ring. Then $\pi_n R = [\mathcal{S}[n], R]$ with $\mathcal{S}[n]_k = \mathcal{S}_{n+k}$ and we have,

$$[\mathcal{S}[n], R] \times [\mathcal{S}[m], R] \to [\mathcal{S}[n] \otimes \mathcal{S}[m], R \otimes R] \to [\mathcal{S}[n+m], R] = \pi_{n+m}(R)$$

a multiplication map.

Remark. Note that if R is an $\mathbb{E}_\infty$-ring then $\pi_* R$ is graded commutative from the fact that $\mathcal{S}[n] \otimes \mathcal{S}[m]$ is antisymmetric.

Notions of left (right) module. All certain algebra objects in Sp. The left-module operad $\underline{\mathrm{LM}}$ has two colors A, M and,

$$\mathrm{Mult}_{\mathrm{LM}}\left(\{X_i\}_{i \in I}, A\right) = \begin{cases} \mathrm{ord}(I) & X_i = A \text{ for all } i \\ \varnothing & \text{else} \end{cases}$$

and likewise,

$$\mathrm{Mult}_{\mathrm{LM}}\left(\{X_i\}_{i \in I}, M\right) = \begin{cases} \mathrm{ord}(I) & M \text{ is largest if all but exactly one are } M \\ \varnothing & \text{else} \end{cases}$$

A left module is an LM-algebra object of Sp,

$$\begin{tikzcd} {}_R \arrow[r] \arrow[d] & {} \arrow[d] \\ \{R\} \arrow[r, hook] & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \end{tikzcd}$$

Proposition 4.1.6. We have,

(a) ${}_R$ is stable

(b) natural t-structure (connective and anti-connected)

(c) if $\pi_n R = 0$ for $n \neq 0$ then ${}_R \cong \mathcal{D}(\mathbf{Mod}_{\pi_0(R)})$ preserving t-structures.

Let R be a discrete commutative ring then $\mathrm{Alg}^{\mathrm{dg}}(R)$ has a model structure with,

(a) weak equivalences are quasi-isomorphisms

(b) fibrations are levelwise surjective

Proposition 4.1.7. $N(\mathrm{Alg}^{\mathrm{dg}}(R)^c)[W^{-1}] \cong \mathrm{Alg}_{\mathbb{E}_1}(R)$ where c means the category of cofibrant objects. If $\mathbb{Q} \subset R$ then,

$$N(\mathrm{CAlg}(R)^c)[W^{-1}] \cong \mathrm{Alg}_{\mathbb{E}_\infty}(R)$$

with graded commutative on the left.

5 Feb. 23

Definition 5.0.1. Let R be connecteive then $P \in_R$ is *projective* if $\mathrm{Hom}\,(P,-) :_R^{\to} \mathcal{S}$ preserve gemoetric realization.

Remark. (a) in classical setting geometric realization is coequalizer so this recoveres the usual definition of preserving colimits

(b) $_R$ of all not necessarily connective objects has no nonzero projective objects.

Proposition 5.0.2. The following are equivalent,

(a) P is projective

(b) for all $Q \in_R$ and $i > 0$ we have $\mathrm{Ext}^i_R\,(P,Q) := \pi_0(\mathrm{Hom}\,(P,Q[i])) = 0$

(c) Given fiber sequence,

$$N' \to N \to N''$$

the map $\mathrm{Ext}^0_R\,(P,N) \to \mathrm{Ext}^0_R\,(P,N'')$ is surjective.

Proposition 5.0.3. TFAE:

(a) P is projective

(b) there exists a free module R-module M with P a retract of M meaning $P \to M \to P$ with $P \to P$ an equivalence.

Proof. First show (a) $\implies$ (b). There exists $R^{\oplus I} \to P$ such that $\pi_0(R^{\oplus n}) \twoheadrightarrow \pi_0(P)$ is a surjection. Consider the fiber sequence,

$$N \to R^{\oplus I} \to P$$

then N is connective by the surjection of π_0. Consider $\mathrm{Hom}_R\,(P,-)$ applied to the fiber sequence, by surjection on $\mathrm{Ext}^0_R\,(P,-)$ we get $P \to R^{\oplus I}$ satisfying the required properties.

For (b) $\implies$ (a) we have projectivity is preserved by retract. For,

$$P \to S \to P$$

then $\mathrm{Ext}^i_R\,(P,Q)$ is a retract of $\mathrm{Ext}^i_R\,(S,Q)$ then STP free module are projective. ☐

Remark. What are the examples of ring spectra:

(a) $K(A)$ for A a discrete ring

(b) $\mathcal{S}$

(c) simplicial commutative rings.

Remark. In the followign definition we don't need any connectivity assumptions.

Definition 5.0.4. M is flat over R if,

(a) $\pi_0(M)$ is a flat $\pi_0(N)$-module

(b) $\pi_m(R) \otimes_{\pi_0(R)} \pi_0(M) \xrightarrow{\sim} \pi_n(M)$ for all n.

Remark. Flatness is closed under coproduct, retract, filtered colimits. If R is connective then projective implies flat.

Proposition 5.0.5. Let N, R be connective. The following are equivalent,

(a) N is flat

(b) if M is a discrete right R-module then $M \otimes_R N$ is discrete.

Proof. Spectral sequence,

$$\mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q = \pi_{p+q}(M \otimes_R N)$$

Then (a) $\implies$ (b) because LHS $= 0$ if $p \neq 0$ and thus,

$$\pi_p(M \otimes_R N) = (\pi_* M \otimes_{\pi_* R} \pi_* N)_p = \cong (\pi_* M \otimes_{\pi_0(R)} \pi_0(N))_p$$

For (b) $\implies$ (a) we use $- \otimes_R N :_R \to \mathcal{S}$ and that $\overset{\heartsuit}{R} = \mathbf{Mod}_{\pi_0(R)}$. Then (b) says that we restrict to,

$$- \otimes_R N : \mathbf{Mod}_{\pi_0(R) \to}$$

which in particular is a map to . However, $- \otimes_R N$ is exact (preserves fiber sequences) and hence is exact on the heart. Then $- \otimes_R N = - \otimes_{\pi_0(R)} \pi_0(N)$ (using the spectral sequence and the fact that $M \otimes_R N$ is discrete for M discrete) and thus is exact meaning $\pi_0(N)$ is flat over $\pi_0(R)$. The rest uses the spectral sequence. $\square$

Lemma 5.0.6. A map $f : M \to N$ of flat R-modules is an equivalence iff $\pi_0(f) : \pi_0(M) \to \pi_0(N)$ is an isomorphism.

Proposition 5.0.7. Let R be connective and M/R is flat then,

$$M \text{ is projective } \iff \pi_0(M) \text{ is projective over } \pi_0(R)$$

Proof. We show a weaker version. If $\pi_0(M)$ is free over $\pi_0(R)$ then can find a free module $R^{\oplus n} \to M$ which is an isomorphism on π_0. Then apply lemma to conclude. $\square$

5.1 Localization

Let R be an $\mathbb{E}_\infty$-ring then $\pi_*(R)$ is graded-commutative. Consider $S \subset \pi_*(R)$ set of homog. elements closed under multiplication and containing 1. Then,

(a) M is S-nilp if all $x \in \pi_m(M)$ are killed by some $s \in S$

(b) M is S-local if for each $s \in S$ the map $\pi_* M \xrightarrow{s} \pi_* M$ is an isomorphism (not necessarily a graded map)

This produces two subcategories ${}_R^{\text{S-nilp}}$ and ${}_R^{\text{S-Loc}}$. We want some adjoints which will be localizations.

(a) ${}_R^{\text{S-nilp}}$ is stable ∞-category, closed under small colimits generated under colimits by $R/Rs[n]$ for $s \in S$. If $s \in \pi_d(R)$ then this arises from,

$$R[d] \xrightarrow{s} R \to R/Rs$$

(b) $M \in_R^{\text{S-loc}} \iff$ for all $s \in S$ and $n \in \mathbb{Z}$ then $\operatorname{Hom}(R/R_s[n], N)$ is contactible iff $\forall M \in^{\text{S-nilp}}$ we have $\operatorname{Hom}_R(M, N)$ contractible.

(1) gives a right adjoint $G :_R \to_R^{\text{S-nilp}}$ to the inclusion. Then we take a fiber sequence,

$$G(M) \to M \to S^{-1}M$$

defining $S^{-1}M$ and,

$$\operatorname{Hom}(N, G(M)) \xrightarrow{\sim} \operatorname{Hom}(N, M) \to \operatorname{Hom}\left(N, S^{-1}M\right)$$

so the last term is contractible proving that $S^{-1}M \in \mathbf{Mod}_{R^{\text{S-loc}}}$.

Remark. The functor $S^{-1}(-)$ gives a left adjoint to the inclusion.

Proposition 5.1.1. $({}_R^{\text{S-nilp}}, {}_R^{\text{S-loc}})$ gives a t-structure on ${}_R$ with trivial heart.

Remark. $\pi_*(S^{-1}M) = S^{-1}\pi_*(M)$.