Mathematics GU6308 Algebraic Topology
Assignment # 1

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1 Chapter 0

1.1 Problem 26

Let the pair (X, A) have the homotopy extension property. I will use the following proposition,

Proposition 1.1 (Hatcher 0.20)**.** If (X, A) satisfies the homotopy extension property and the inclusion map $A \hookrightarrow X$ is a homotopy equivalence then it is a homotopy equivalence rel A or equivalently that A is a deformation retract of X.

In particular, I will apply this proposition to the pair $(X \times I, X \times \{0\} \cup A \times I)$ to show that $X \times I$ deformation retracts onto $X \times \{0\} \cup A \times I$. By the proposition, it suffices to show that $(X \times I, X \times \{0\} \cup A \times I)$ has the homotopy extension property and that $X \times \{0\} \cup A \times I \hookrightarrow X \times I$ is a homotopy equivalence.

The homotopy extension property for $(X \times I, X \times \{0\} \cup A \times I)$ follows directly from the homotopy extension property for (X, A). Recall that the homotopy extension property is equivent to the existence of a retract $r : X \times I \to X \times \{0\} \cup A \times I$. So we need to find a retract,

$$r' : X \times I \times I \to X \times I \times \{0\} \cup (X \times \{0\} \cup A \times I) \times I = X \times (I \times \{0\} \cup \{0\} \times I) \cup A \times I \times I$$

However, we may deform the corner $(I \times \{0\} \cup \{0\} \times I)$ to a single edge $I \times \{0\}$ so the subspace becomes,

$$X \times I \times \{0\} \cup A \times I \times I$$

so it suffices to show that $(X \times I, A \times I)$ has the homotopy extension property which is clear since we can take the retract $X \times I \to X \times \{0\} \cup A \times I$ and multiply by I to get a retract $X \times I \times I \to X \times I \times \{0\} \cup A \times I \times I$.

Now we need to show that $\iota : X \times \{0\} \cup A \times I \hookrightarrow X \times I$ is a homotopy equivalence. We already have a retract $r : X \times I \to X \times \{0\} \cup A \times I$ via the homotopy extension property such that $r \circ \iota = \mathrm{id}$. Thus it suffices to show that $\iota \circ r \sim \mathrm{id}$. Consider the following homotopy $h : X \times I \times I \to X \times I$,

$$h(x, s, t) = \begin{cases} (x, s(1 - 2t)) & t \in [0, \frac{1}{2}] \\ \iota \circ r(x, s(2t - 1)) & t \in [\frac{1}{2}, 1] \end{cases}$$

which is continuous at $t = \frac{1}{2}$ since $\iota \circ r(x, 0) = \iota(x, 0) = (x, 0)$ because $(x, 0) \in X \times \{0\} \cup A \times I$ on which it is trivial since it is a retract. Thus h is a homotopy from $h(-, -, 0) = \mathrm{id}$ to $h(-, -, 1) = \iota \circ r$. Thus $\iota : X \times \{0\} \cup A \times I \hookrightarrow X \times I$ is a homotopy equivalence.

Now let (X_1, A) be a pair satisfing the homotopy extension property and $f, g : A \to X_0$ are homotopic attaching maps. Consider a homotopy $F : A \times I \to X_0$ between f and g then $X_1 \sqcup_F (A \times I)$ contains $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$. We have shown that there is a deformation retract,

$$X_1 \times I \to X_1 \times \{0\} \cup A \times I$$

This deformation retract gives a pair of deformation retracts $X_0 \sqcup_F (X_1 \times I) \to X_0 \sqcup_f X_1$ and $X_0 \sqcup_F (X_1 \times I) \to X_0 \sqcup_g X_1$ and thus a homotopy equivalence $X_0 \sqcup_f X_1 \sim X_0 \sqcup_g X_1 \text{ rel } X_0$ proving proposition 0.18 in general for a pair (X, A) with the homotopy extension property.

1.2 Problem 29

Let X be a CW complex obtained from a subcomplex A by attaching an n-cell D^n via an attaching map $g : S^{n-1} \to A$. Consider a homotopy $f_t : A \to Y$ and a map $f_0' : D^n \to Y$ such that $f_0 \circ g$ and $f_0'|_{S^{n-1}}$ agree. Using the projection idea to form a deformation retract $D^n \times I \to D^n \times \{0\} \cup S^{n-1} \times I$ we get the following explicit homotopy extension to $f' : D^n \times I \to Y$ via,

$$f_t'(x) = \begin{cases} f_{t'(x)} \circ g\left(\frac{x}{|x|}\right) & 2|x| > (2-t) \\ f_0'\left(\frac{2x}{2-t}\right) & 2|x| \leq (2-t) \end{cases}$$

where we have defined,

$$t'(x) = \frac{2|x| - (2-t)}{|x|}$$

Note that when $2|x| = (2-t)$ then $y = \frac{2x}{2-t} = \frac{x}{|x|} \in S^{n-1}$ so $f_0'(y) = f_0 \circ g(y)$ and $t' = 0$ and thus the maps glue to give a continuous homotopy extension.

2 Chapter 4.1

2.1 Problem 3

Let X be an H-space i.e. a pointed space (X, x_0) with a multiplication map $\mu : X \times X \to X$ which is unital in the homotopy category meaning $\mu \circ (\mathrm{id} \times e) \sim \mu \circ (e \times \mathrm{id}) \sim \mathrm{id}$ where $e : * \to X$ is the inclusion of the identity at the basepoint x_0.

Now consider the induced group map $\mu_* : \pi_n(X \times X) \to \pi_n(X)$ which gives,

$$\begin{tikzcd} \pi_n(X \times X) \arrow[r, "\mu_*"] \arrow[d, "\sim"] & \pi_n(X) \\ \pi_n(X) \oplus \pi_n(X) \arrow[ur, "\mu_*"] & \end{tikzcd}$$

Since μ_* is a group homomorphism, we have (denoting multiplication / addition in π_n as $*$),

$$\mu_*((\gamma_1, \gamma_2) * (\delta_1, \delta_2)) = \mu_*(\gamma_1, \gamma_2) * \mu_*(\delta_1, \delta_2)$$

However, by definition of the product group,

$$(\gamma_1, \gamma_2) * (\delta_1, \delta_2) = (\gamma_1 * \delta_1, \gamma_2 * \delta_2)$$

which gives,

$$\mu_*(\gamma_1 * \delta_1, \gamma_2 * \delta_2) = \mu_*(\gamma_1, \gamma_2) * \mu_*(\delta_1.\delta_2)$$

To clarify the notation, I will write $\gamma_1 \circ \gamma_2 := \mu_*(\gamma_1.\gamma_2)$ which denotes the map sending a pair of classes of maps to the class of their "product" under μ i.e. $(\gamma_1 \circ \gamma_2)(t) = \gamma_1(t) \circ \gamma_2(t) = \mu(\gamma_1(t), \gamma_2(t))$. Using this notation, we have,

$$(\gamma_1 * \delta_1) \circ (\gamma_2 * \delta_2) = (\gamma_1 \circ \gamma_2) * (\delta_2 \circ \delta_2)$$

Furthermore, both operations are unital, letting e denote the class of the constant map at x_0 then $\gamma * e = \gamma * e = \gamma$ (here these represent classes of paths) and we know $\mu_* \circ (\mathrm{id} \times e) = \mu_* \circ (e \times \mathrm{id}) = \mathrm{id}$ so $\gamma \circ e = e \circ \gamma = \gamma$ (since $* \to X$ gives the map including the class of the constant map into $\pi_n(X)$).

The Eckmann-Hilton argument (proved in Lemma 4) shows that any pair of unital operations which satisfy the above intertwining relation must agree and further must be commutative and associative. Therefore, $\circ = *$ so we find that $\gamma_1 * \gamma_2 = \mu_*(\gamma_1, \gamma_2)$ and we also get for free that $\pi_1(X)$ is abelian for any H-space X.

2.2 Problem 10

Consider the quasi-circle X which contains an arc and S the graph of $\sin(1/x)$ on $(0, \pi]$ completed with the interval $L = [-1, 1]$ at zero. Thus the map $\mathbb{R} \to X$ sarting at 0 going around the arc and tracing the $\sin(1/x)$ curve hits everything except for $L = [-1, 1]$. Call the image of this map $C \subset X$ then $X = C \cup L$.

Consider a map $f : S^n \to X$. I claim that any such map has image contained in $X_a = X \setminus \{(x, \sin(1/x)) \mid 0 < x < a\}$ for some $a > 0$ which is contractible and thus f is nullhomotopic implying that $\pi_n(X) = 0$. To show this suppose that S^n hits points $(x, \sin(1/x))$ for arbitrarily small x. By connectivity of S^n then f must hit $\{(x, \sin(1/x) \mid 0 < x \leq 1\}$. Choose a point $y \in L$ and an open neighborhood V of y which is not connected (this will hold as long as it does not wrap around the arc) such that $X \setminus X_a \subset V$. Then consider the connected components U_i of $f^{-1}(V)$ which are open since S^n is locally path-connected. Now $\overline{U}_i$ is also path-connected so $f(\overline{U_i})$ must be contained in a path component of V and thus either hits L or is contained entirely in S. However, in the latter case, since $\overline{U}_i$ is compact then $f(\overline{U}_i) \subset \{(x, \sin(1/x) \mid [a_i, b_i]\}$ for some closed interval $[a_i, b_i]$ with $a_i, b_i > 0$ since it must be compact and thus closed so $f(\overline{U}_i)$ must hit any limit points so it cannot get arbitrarily close to L since L is not in the image. Finally, let $W = f^{-1}(X^\circ_{a/2})$ then since $X^\circ_{a/2}$ and V cover X we get an open cover U_i and W of S^n such that $f(U_i) \subset X_{a_i}$ and $f(W) \subset X_{a/2}$. Since S^n is compact we cana take this cover to be finite. Then taking $b = \min\{a_i, a/2\}$ which is positive for finitely many a_i we find that $f(U_i) \subset X_{a_i} \subset X_b$ and $f(W) \subset X_{a/2} \subset X_b$. Since these opens form a cover of S^n we find that $f(S^n) \subset X_b$ which is contractible and thus $f : S^n \to X$ is nullhomotopic so $\pi_n(X) = 0$ for $n > 0$.

However, I claim that X is not contractible showing, by Whitehead's theorem that X is not homotopy equivalent to a CW complex. If X we contractible then the map $f : X \to S^1$ given by projecting down the sine curve and contracting L would be nullhomotopic. However, such a nullhomotopy would imply the existence of a lift $\tilde{f} : X \to \mathbb{R}$ over $p : \mathbb{R} \to S^1$ by the homotopy lifting property (since constant maps trivially have lifts). However, exercise 7 of §1.3 asks us to show this is impossible. $f : C \to S^1$ is the identity loop minus the base point and thus must lift to the injective increasing path $(0, 2\pi) \subset \mathbb{R}$ (up to choice of basepoint). However, X is compact (closed and bounded in $\mathbb{R}^2$) so its image under $\tilde{f}$ must be compact and thus contain $[0, 2\pi] \subset \mathbb{R}$ so $\tilde{f}(L) \supset \{0, 2\pi\}$. However, $f(L) = x_0$ is the basepoint of S^1 so $\tilde{f}(L) \subset p^{-1}(x_0)$ which is discrete and L is connected so $\tilde{f}$ must be a single point contradicting the fact that $\tilde{f}(L) \supset \{0, 2\pi\}$ showing that no lift,

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

can exist. Thus X is not contractible.

2.3 Problem 14

Let $f : X \to Y$ be a homotopy equivalence between CW complexes with no $(n+1)$-cells and let $g : Y \to X$ be a homotopy inverse. By cellular approximation we may take these maps to be cellular $f : X^n \to Y^n$ and $g : X^n \to Y^n$. Furthermore, we have homotopies $h_1 : X \times I \to X$ and $h_2 : Y \times I \to Y$ taking $g \circ f$ and $f \circ g$ to the identities. Again by cellular approximation we may take these maps to be cellular (note that on $X \times \{0, 1\} \subset X \times I$ this map is cellular since it is $g \circ f$ and id then we can take our homotopy rel $X \times \{0, 1\}$ meaning that our new cellular homotopy still takes $g \circ f$ to id). Thus we get a homotopy $h_1 : (X \times I)^{n+1} \to X^{n+1}$. But since X has no $(n+1)$-cells we have $X^{n+1} = X^n$ and $(X \times I)^{n+1} = X^n \times I$ so we have a homotopy $h_1 : X^n \times I \to X^n$ from $g \circ f$ to id. We similarly get a cellular homotopy $h_2 : Y^n \times I \to Y^n$ from $f \circ g$ to id. Thus $X^n \sim Y^n$.

2.4 Problem 20

Suppose that X is a finite connected CW complex of dimension $\dim X = n$ and $\pi_i(Y)$ is finite for $i \le n$. First note that X is path-connected so its image must land in a path component $Y_i \in \pi_0(Y)$. Thus,

$$[X, Y] = \bigcup_{Y_i \in \pi_0(Y)} [X, Y_i]$$

Since $\pi_0(Y)$ is finite by assumption, it suffices to show that each $[X, Y_i]$ is finite so we may reduce to the case that Y is path-connected. Further, when Y is path-connected $\langle X, Y \rangle = [X, Y]/\pi_1(Y, y_0)$ but $\pi_1(Y, y_0)$ is finite so $[X, Y]$ is finite iff $\langle X, Y \rangle$ is finite so it suffices to show the pointed version. We will induct on the number of (positive dimensional) cells of X. If X has a single (positive dimensional) then $X \cong S^n$ in which case $\langle X, Y \rangle = \pi_n(Y)$ which we assume is finite. Now suppose that X is constructed by attaching a k-cell D^k for $k \le n$ to a subcomplex A via an attaching map $a : S^{k-1} \to A$. Then $X = A \sqcup_a D^k$ and we get maps $S^{k-1} \to A \to X$. The inclusion induces a map,

$$[X, Y] \xrightarrow{\Phi} [A, Y]$$

Consider the kernel $\ker \Phi$, if a map $f : X \to Y$ restricts to $f|_A : A \to Y$ which is homotopic tvia $(f|_A)_t : A \times I \to Y$ then by the homotopy extension property of the CW pair (X, A) we may extend this to a homotopy $f_t : X \times I \to Y$ such that $f_1|_A$ is constant i.e. $f : X \to Y$ is homotopic to a pair (f', c) with $c : A \to Y$ constant and $f' : D^k \to Y$ restricted such that $f'|_{S^{k-1}} = c \circ a$ a constant. Therefore, f' defines a class in $\pi_k(Y)$ since as we homotope it we must fix its value on $\partial D^k = S^{k-1}$ to be exactly the point $c(A)$. Thus, $\ker \Phi = [S_k, Y]$. A similar argument shows that the fibres of

$[X, Y] \to [A, Y]$ correspond exactly to classes $[S^k, Y]$. Since $\pi_k(Y)$ is finite by assumption then the fibre $[S^k, Y]$ is also finite (using Y path-connected) and thus the map $[X, Y] \to [A, Y]$ has finite fibres. Furthermore, by induction we assume that $[A, Y]$ is finite which implies that $[X, Y]$ is finite as well since it maps onto a finite set with finite fibres proving by induction that $[X, Y]$ is finite for any finite connected CW complex X with $\dim X = n$.

3 Chapter 4.2

3.1 Problem 2

First note that the action of paths $[\gamma]$ on π_n is natural in the following sense: given a map $f : X \to Y$ we get a commutative diagram,

$$\begin{tikzcd} \pi_n(X, x_0) \arrow[r, "\gamma"] \arrow[d, "f_*"] & \pi_n(X, x_1) \arrow[d, "f_*"] \\ \pi_n(Y, f(x_0)) \arrow[r, "f_*(\gamma)"] & \pi_n(Y, f(x_1)) \end{tikzcd}$$

We might say that this forms a functor of groupoid actions. Restricting to the case of a covering map $p : \tilde{X} \to X$ and paths γ between points on the same fibre $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ we get,

$$\begin{tikzcd} \pi_n(\tilde{X}, \tilde{x}_0) \arrow[r, "\gamma"] \arrow[d, "p_*"] & \pi_n(\tilde{X}, \tilde{x}_1) \arrow[d, "p_*"] \\ \pi_n(X, x_0) \arrow[r, "p_*(\gamma)"] & \pi_n(X, x_0) \end{tikzcd}$$

However, the two vertical isomorphisms are not clearly commensurable since they have non-canonically isomorphic source groups. However, if the vertical maps on π_1 have the same image (which always happens for the universal cover since $\pi_1(\tilde{X}) = 0$) i.e. $p_*\pi_1(\tilde{X}, \tilde{x}_0) = p_*\pi_1(\tilde{X}, \tilde{x}_1)$ there there is a deck transformation $f : \tilde{X} \to \tilde{X}$ which takes $\tilde{x}_1$ to $\tilde{x}_0$. Furthermore, since by definition, $p \circ f = p$ we may extend the diagram to,

$$\begin{tikzcd} \pi_n(\tilde{X}, \tilde{x}_0) \arrow[r, "\gamma"] \arrow[d, "p_*"] & \pi_n(\tilde{X}, \tilde{x}_1) \arrow[r, "f_*"] \arrow[d, "p_*"] & \pi_n(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi_n(X, x_0) \arrow[r, "p_*(\gamma)"] & \pi_n(X, x_0) \arrow[r, "\mathrm{id}"] & \pi_n(X, x_0) \end{tikzcd}$$

Then the leftmost and rightmost maps are, for $n \geq 2$, the fixed isomorphism $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ and the bottom map is exactly the action of the loop $p_*(\gamma) \in \pi_1(X, x_0)$ on $\pi_n(X, x_0)$ so using the isomorphism we can describe this action as the composition $f_* \circ \gamma$ acting on $\pi_n(\tilde{X}, \tilde{x}_0)$. Finally, by the path-lifting property, $\pi_1(X, x_0)$ is surjected onto by loops of the form $p_*(\gamma)$ for paths with endpoints in the fibre $p^{-1}(x_0)$.

Now consider the covering map $p : S^n \to \mathbb{RP}^n$ and choose a basepoint $x_0 \in \mathbb{RP}^n$ in the image of the north and south poles $N, S \in S^n$. There are two homotopy classes of paths with endpoints in the fibre $p^{-1}(x_0) = \{N, S\}$, constant loops which clearly act trivially, and paths which go from one pole to the other. Let γ be a path from N to S in S^n and let $f : S^n \to S^n$ be the corresponding deck transformation which swaps these poles, namely the antipodal map $f = -\mathrm{id}$. Thus we only need to compute $f_* \circ [\gamma] : \pi_n(S^n, N) \to \pi_n(S^n, N)$. The group $\pi_n(S^n, N)$ is generated by the class $g = [\mathrm{id} : S^n \to S^n]$ so it suffices to check where this generator ends up. First, $[\gamma](g)$ corresponds to the map $R_\pi : S^n \to S^n$ which rotates the n-sphere by a half-turn as to swap N and S this corresponds to inverting a pair of axes one through the poles and one perpendicularly. Then the map $f_*([\gamma](g)) = f_* \circ R_\pi$ applies another inversion to each of the $n+1$ coordinates such that $f_* \circ R_\pi : S^n \to S^n$ is based at N. Furthermore, since inverting any coordinate of the class g sends it to its inverse $-g$ then we see that $f_* \circ R_\pi = (-1)^{n+3} g$. Therefore, when n is odd the action of $[p \circ \gamma]$ on $\pi_n(\mathbb{RP}^n)$ is trivial and when n is even it is the negation map $\mathbb{Z} \to \mathbb{Z}$. Since $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ generated by this loop $[p \circ \gamma]$ we see that $\pi_1(\mathbb{RP}^n) \circlearrowright \pi_n(\mathbb{RP}^n)$ trivially when n is odd and via negation when n is even.

3.2 Problem 12

Let $f : X \to Y$ be a map of connected CW complexes such that $f_* : \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$ is an isomorphism. Consider universal covers $p_X : \tilde{X} \to X$ and $p_Y : \tilde{Y} \to Y$ which we may give CW complex structures. Since $f \circ p_X : \tilde{X} \to Y$ is a map from a simply-connected (connected and locally-path-connected) space we get a lift,

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "\tilde{f}"] \arrow[d, "p_X"'] & \tilde{Y} \arrow[d, "p_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We further assume that $\tilde{f}_* : H_i(\tilde{X}, \mathbb{Z}) \xrightarrow{\sim} H_i(\tilde{Y}, \mathbb{Z})$ is an isomorphism. Since $\tilde{X}$ and $\tilde{Y}$ are simply-connected CW complexes we may apply the homological version of Whitehead's theorem to conclude that $\tilde{f}$ is a homotopy equivalence. Now consider the diagram,

$$\begin{tikzcd} \pi_n(\tilde{X}) \arrow[r, dashed, "\tilde{f}_*"] \arrow[d, "(p_X)_*"'] & \pi_n(\tilde{Y}) \arrow[d, "(p_Y)_*"] \\ \pi_n(X) \arrow[r, "f_*"'] & \pi_n(Y) \end{tikzcd}$$

but $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a homotopy equivelnce so $\tilde{f}_* : \pi_n(\tilde{X}) \xrightarrow{\sim} \pi_n(\tilde{Y})$ is an isomorphism. Furthermore since p_X and p_Y are covering maps, for $n \geq 2$ they induce isomorphism $(p_X)_* : \pi_n(\tilde{X}) \xrightarrow{\sim} \pi_n(X)$ and $(p_Y)_* : \pi_n(\tilde{Y}) \xrightarrow{\sim} \pi_n(Y)$. Thus by commutativity of the diagram $f_* : \pi_n(X) \xrightarrow{\sim} \pi_n(Y)$ is an isomorphism for $n \geq 2$ and since we already know this map is an isomorphism for $n = 1$ then we may apply the standard Whitehead's theorem to show that $f : X \to Y$ is a homotopy equivalence.

3.3 Problem 14

Let X be a CW complex of dimension $\dim X = n$ and Y a subcomplex of X which is homotopy equivalent to S^n. Then consider the natruality square of the Hurewicz map,

$$\begin{tikzcd} \pi_n(Y) \arrow[r] \arrow[d, "h_n"] & \pi_n(X) \arrow[d, "h_n"] \\ H_n(Y) \arrow[r] & H_n(X) \end{tikzcd}$$

However, since $Y \sim S^n$ we know that Y is $(n-1)$-connected so $h_n : \pi_n(Y) \xrightarrow{\sim} H_n(Y)$ is an isomorphism. Furthermore, the homological LES associated to the pair (X, Y) gives,

$$H_{n+1}(X,Y) \longrightarrow H_n(Y) \longrightarrow H_n(X) \longrightarrow H_n(X,Y)$$

But since (X, Y) is a CW pair (and thus good) we have $H_{n+1}(X, Y) = \tilde{H}_{n+1}(X/Y) = 0$ since $\dim X/Y = n$. Therefore the map $H_n(Y) \hookrightarrow H_n(X)$ is an injection. Returning to the Hurewicz diagram, the left and bottom maps are injective which implies that the top map $\pi_n(Y) \to \pi_n(X)$ must be injective as well.

3.4 Problem 15

Let M be a closed simply-connected 3-manifold. First I argue that M must be orientable since M is simply-connected via Lemma 4. By Hurewicz theorem for 1-connected spaces we find that $H_1(M, \mathbb{Z}) = 0$ and $h_2 : \pi_2(M) \xrightarrow{\sim} H_2(M, \mathbb{Z})$ is an isomorprhism. Furthermore, by Poincare duality, there is a fundamental class $[M] \in H_d(M, \mathbb{Z})$ and the map $H^i(M, \mathbb{Z}) \xrightarrow{\sim} H_{d-i}(M, \mathbb{Z})$ defined by the cap product $\alpha \mapsto [M] \frown \alpha$ is an isomorphism. Furthermore, the universal coefficent theorem for cohomology gives an exact sequence,

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\left(H_{i-1}(M, \mathbb{Z}), \mathbb{Z}\right) \longrightarrow H^i(X, \mathbb{Z}) \longrightarrow \mathrm{Hom}\left(\mathbb{Z}, H_i(X, \mathbb{Z})\right)\mathbb{Z} \longrightarrow 0$$

Thus, since $H_1(X, \mathbb{Z}) = 0$ and $H_0(X, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$ because $\mathbb{Z}$ is projective as a $\mathbb{Z}$-module the exact sequence gives $H^1(X, \mathbb{Z}) = 0$. Then applying Poincare duality shows that $H_2(M, \mathbb{Z}) = 0$ so by Hurewicz $\pi_2(M) = 0$. Therefore, M is 2-connected so we get a Hurewicz isomorphism $h_3 : \pi_3(M) \to H_3(M, \mathbb{Z}) = [M]\mathbb{Z}$.

Recall that the Hurewicz map $h_n : \pi_n(X) \to H_n(X, \mathbb{Z})$ is defined via $[f] \mapsto f_*([S^n])$. Since $h_3 : \pi_3(M) \to H_3(M, \mathbb{Z})$ is an isomorphism there is some $f : S^3 \to M$ such that $h_3([f]) = f_*([S^n]) = [M]$. Since $H_3(S^3, \mathbb{Z}) = [S^n]\mathbb{Z}$ then $f_* : H_3(S^3, \mathbb{Z}) \to H_3(M, \mathbb{Z})$ takes the generator to the generator and thus is an isomorphism. Furthermore, for $0 < i < 3$ we know that $H_i(S^3, \mathbb{Z}) = H_i(M, \mathbb{Z}) = 0$ so $f_* : H_i(S^3, \mathbb{Z}) \xrightarrow{\sim} H_i(M, \mathbb{Z})$ is an isomorphism for all i (since both vanish above $\dim S^3 = \dim M = 3$). Now we use the fact that every manifold has the homotopy type of a CW complex to replace M by some homotopy equivalent CW complex X thus $f_* : H_i(S^3, \mathbb{Z}) \xrightarrow{\sim} H_i(X, \mathbb{Z})$ is an isomorphism of simply-connected CW complexes and thus $f : S^3 \to X$ is a homotopy equivalence by the homological version of Whitehead's theorem. Therefore, $M \sim X \sim S^3$.

4 Lemmas

Lemma 4.1 (Eckmann-Hilton)**.** Suppose that $*$ and $\circ$ are untial operations satisfing the following intertwining relation,

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

Then $* = \circ$ and both are commutative and associative.

Proof. First we need to show that the units agree i.e. $1_* = 1_\circ$. We simply plug into the relation,

$$(1_* * 1_\circ) \circ (1_\circ * 1_*) = (1_* \circ 1_\circ) * (1_\circ \circ 1_*)$$

The left-hand side is $1_\circ \circ 1_\circ = 1_\circ$ and the right-hand side is $1_* * 1_* = 1_*$ so indeed $1_\circ = 1_*$.

Now apply the relation to get,

$$(a * 1) \circ (1 * b) = (a \circ 1) * (1 \circ b)$$

but since the units agree this is simply $a \circ b = a * b$ so the operations agree. For commutativity, note that,

$$(1 \circ a) \circ (b \circ 1) = (1 \circ b) \circ (a \circ 1)$$

so applying the unit properties $a \circ b = b \circ a$. Finally, for associativity, note that,

$$(a \circ b) \circ (1 \circ c) = (a \circ 1) \circ (b \circ c)$$

gives,

$$(a \circ b) \circ c = a \circ (b \circ c)$$

□

Lemma 4.2. Let M be a simply-connected manifold then M is orientable.

Proof. If M were not orientable then it would have a connected orientation double cover $p : \tilde{M} \to M$ but since $\pi_1(M) = 0$, by the classiciation of covering spaces, M has no nontrivial (in paricular two-sheeted) connected covering spaces (in the case that M is orientable the orientation cover degenerates to the trivial disconnected two-sheeted cover). In particular, if M is not orientable then $\pi_1(M)$ must have an index two subgroup corresponding to orientation preserving loops. □