Mathematics GU4042 Modern Algebra II
Assignment # 4

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Page 147. # 1 Let R be a Noetherian ring and $I \subset R$ be an ideal. Let $\eta : R \to R/I$ be the canonical ring homomorphism given by $\eta : r \mapsto r + I$. Trivially, η is a surjective ring homomorphism. Thus, for any ideal $J \subset R/I$ we have that $\eta^{-1}(J) \subset R$ is an ideal of R. Thus, because R is Noetherian, $\eta^{-1}(J)$ is finitely generated i.e. $\eta^{-1}(J) = (a_1, \dots, a_n)$. Therefore, by Lemma 0.1, $\eta(\eta^{-1}(J)) = (\eta(a_1), \dots, \eta(a_n))$. However, η is surjective so, by Lemma 0.2, $\eta(\eta^{-1}(J)) = J$. Therefore, $J = (\eta(a_1), \dots, \eta(a_n))$ which means that J is finitely generated. Since any ideal of R/I is finitely generated, R/I is a Noetherian ring.

Now suppose that $R[X]$ is a Noetherian ring. Consider the natural embedding of R in $R[X]$ given by the projection homomorphism: $\pi : R[X] \to R$ which acts as $\pi : a_n X^n + \cdots + a_1 X + a_0 \mapsto a_0$. Then,

$$\pi(a_n X^n + \cdots + a_1 X + a_0) = 0_R \iff a_0 = 0_R \iff a_n X^n + \cdots + a_1 X + a_0 = (a_n X^{n-1} + \cdots + a_1)X$$
$$\iff a_n X^n + \cdots + a_1 X + a_0 \in (X)$$

where the final equivalence holds by the fact that X commutes with every element of $R[X]$. Thus, $\ker \pi = (X)$. Also, π is clearly sujective because for any $r \in R$ take $r \in R[X]$ (a degree zero polynomial) so $\pi : r \mapsto r$. By the first isomorphism theorem, $R[X]/(X) \cong R$. Since $R[X]$ is Noetherian every quotient of $R[X]$ is also Noetherian. In particular, $R[X]/(X) \cong R$ is Noetherian.

Lemmas

Lemma 0.1. *Let $\phi : R \to S$ be a surjective ring homomorphism then $\phi((a_1, \dots, a_n)) = (\phi(a_1), \dots, \phi(a_n))$.*

Proof. Consider an ideal $I = (a_1, \dots, a_n) \subset R$. Then take $y \in \phi(I)$ so $\exists x \in I$ s.t. $\phi(x) = y$. By definition, $x = r_1 a_1 s_1 + \cdots + r_n a_n s_n$ with $r_1, s_1, \dots, r_n, s_n \in R$. Thus,

$$\phi(x) = \phi(r_1)\phi(a_1)\phi(s_1) + \cdots + \phi(r_n)\phi(a_n)\phi(s_n) \in (\phi(a_1), \dots, \phi(a_n))$$

Thus, $\phi(I) \subset (\phi(a_1), \dots, \phi(a_n))$. However, for each $a_i \in I$, we have $\phi(a_i) \in \phi(I)$ but because ϕ is surjective $\phi(I)$ is an ideal in S so by closure and absorption, $(a_1, \dots, a_n) \subset \phi(I)$. Therefore, $\phi(I) = (\phi(a_1), \dots, \phi(a_n))$. □

Lemma 0.2. *If $f : X \to Y$ is an sujective function and $A \subset Y$ then $f(f^{-1}(A)) = A$.*

Proof. If $y \in A$ then, by surjectivity, $\exists x \in X : f(x) = y \in A$ so $x \in f^{-1}(A)$ and thus,
$f(x) = y \in f(f^{-1}(A))$ so $A \subset f(f^{-1}(A))$. Now take $y \in f(f^{-1}(A))$ then $\exists x \in f^{-1}(A)$ s.t. $f(x) = y$ but $x \in f^{-1}(A)$ so $f(x) = y \in A$ so $f(f^{-1}(A)) \subset A$. □