1   On a Conjecture of Griffiths and Harris

In [4], Grifiths and Harrised proposed the following range of conjectures concerning curves contained in a general hypersurface in $\mathbb{P}^4$ of degree $d \geq 6$ (such that the canonical bundle is ample)

(a) we have d divides $\deg C$

(b) the image of the Abel-Jacobi map

$$\varphi : \mathrm{Hom}^2(X)/\mathrm{Rat}^2(X) \to J^2(X)$$

   of C is trivial

(c) the group

$$\mathrm{Hom}^2(X)/\mathrm{Alg}^2(X)$$

   is trivial

(d) the group

$$\mathrm{Alg}^2(X)/\mathrm{Rat}^2(X)$$

   is trivial

(e) if C is smooth, then C is a complete intersection with a surface $\Sigma \subset \mathbb{P}^4$.

Mark Green has explained some recent progress on (b), this note proposes to show that (e) is false, we show this by showing that the variant (e'), is also false,

$$\text{the normal bundle sequence for } C \subset X \subset \mathbb{P}^4 \text{ is split}$$

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1.1   The counterexample to (e)

We suppose $d > 2$ since the case $d = 2$ is trivial since every quadric contains a line.

   Let $X \subset \mathbb{P}^n$ for $n \geq 4$ be a smooth hypersurface of degree d, and let $C \subset X$ be a smooth curve. We suppose there exists a surface $\Sigma \subset \mathbb{P}^n$ such that C is the complete intersection of X and Σ. Since C is smooth, Σ is smooth along C, so Σ^{sing} is comprised of isolated points located outside of C (I guess because $C \subset \Sigma$ is ample). Let $\tau : \overline{\Sigma} \to \Sigma$ be the designgularization of Σ. We get a natural inclusion $C \subset \overline{\Sigma}$ and C is a member of the linear system $|\tau^* \mathcal{O}_\Sigma(d)|$ on $\overline{\Sigma}$. The class of C in $H^2(\overline{\Sigma}, \mathbb{Z})$ is divisible by d which implies

(a) d divides $C \cdot_{\overline{\Sigma}} C$

(b) d divides $K_{\overline{\Sigma}} \cdot_{\overline{\Sigma}} C$

and therefore the adjunction formula implies:

$$d \text{ divides } \deg K_C$$

1.2 •

We consider now a curve with an ordinary double point D constructed via two smooth tangent plances $P_1 \cap X = C_1$ and $P_2 \cap X = C_2$ of X meeting transversely at a point p. One such curve exists when $n \geq 4$.

We have, for $i = 1, 2$

(a) d divides $\deg C_i$

(b) d divides $\deg K_{C_i}$

from which we have

$$\deg K_D = \deg K_{C_1} + \deg K_{C_2} + 2 \equiv 2 \mod d$$

Then if S is a complete intersection surface $X \cap X_1 \cap \cdots \cap X_{n-3}$ containing D and let $D' \subset S$ be a smooth member of the linear system $|mH + D|$ on S (D' exists for m sufficiently large).

We have

$$\deg K_{D'} = (D')^2 + K_S \cdot D' = (D + mH)^2 + K_S \cdot (D + mH)$$
$$= \deg K_D + 2mH \cdot D + m^2 H^2 + mK_S \cdot H$$

the last two terms are divisible by d; according to (a) $\deg D$ is also 2 mod d hence $\deg K_{D'}$ is 2 mod d.

Since $d > 2$, D' does not satisfy condition 1.2, and provides a counterexample to (v).

1.3 Counterexample to (e)

We supppose from now on (for simplicity) that $n = 3$. We start again with the curve $D = C_1 \cup_p C_2$.

We will show the following facts

(A) the normal bundle exact sequence of $D \subset X \subset \mathbb{P}^4$ is not split

(B) if $S = X \cap X'$ is a smooth surface containing D with $\deg X' = k$ sufficiently large; let D' be a smooth curve on S in the linear system $|mH + D|$ for m sufficiently large; then the normal bundle exact sequence of $D' \subset X \subset \mathbb{P}^4$ is not split.

Lemma 1.3.1. (A) $\implies$ (b)

Proof. We denote $e_D \in H^1(C, N_{D|X}(-d))$ the class of the extension

$$0 \to N_{D|X} \to N_{D|\mathbb{P}^4} \to \mathcal{O}_D(d) \to 0$$

and for any hypersurface X' such that $X' \cap X = S$, we denote $F_D^{X'} \in H^1(D, N_{D|S}(-d))$ the class of the extension

$$0 \to N_{D|S} \to N_{D|X'} \to \mathcal{O}_D(d) \to 0$$

and the same notations for D'

We consider the exact sequence

$$0 \to N_{D|S} \to N_{D|X} \to \mathcal{O}_D(k) \to 0$$

which provides the map

$$\alpha : H^1(D, N_{D|S}(-d)) \to H^1(D, N_{D|X}(-d)) \quad \text{such that} \quad \alpha(F_D^{X'}) = e_D$$

(i) We suppose S is fixed, and let m be such that $H^1(S, \mathcal{O}_S(-D)(k - d - m)) = 0$. This implies: $H^0(S, \mathcal{O}_S(k - d)) \to H^0(D', \mathcal{O}_{D'}(k - d))$ is surjective, pour any curve D' in the liniear system $|mH + D|$ on S. We immediately deduce: if $e_{D'} - 0$, there exists a hypersurface X'' of degree k, such that $X \cap X'' = S$ and $F_{D'}^{X''} = 0$.

(ii) We consider the exact sequence

$$0 \to \mathcal{O}_S(-d) \to \mathcal{O}_S(D')(-d) \to N_{D'|S}(-d) \to 0$$

This gives a map

$$\delta_{D'} : H^1(D', N_{D'|S}(-d)) \to H^2(S, \mathcal{O}_S(-d))$$

we also have the natural map given by cup-product:

$$\beta : H^2(S, \mathcal{O}_S(-d)) \to \mathrm{Hom}\left(H^0(S, \mathcal{O}_S(d)), H^2(S, \mathcal{O}_S)\right)$$

we verify easily that β is injective as soon as $K_S \geq 0$.

It is then well known that the image $\beta \circ \delta_{D'}(F_{D'}^{X''}) \in \mathrm{Hom}\left(H^0(\mathcal{O}_S(d)), H^2(\mathcal{O}_S)\right)$ is identified as the composite

$$H^0(\mathcal{O}_S(d)) \to H^1(T_S) \xrightarrow{\smile \lambda_{D'}} H^2(\mathcal{O}_S)$$

this map is the cup product with the class $\lambda_{D'} \in H^1(\Omega_S)$ corresponding to D, and the map $H^0(\mathcal{O}_S(d)) \to H^1(T_S)$ is provided by the exact sequence

$$0 \to T_S \to T_{X''|S} \to \mathcal{O}_S(d) \to 0$$

It is also easy to verify that $\smile \lambda_{D'}$ only depends on the "class in primitive cohomology" of D' i.e. if $\smile \lambda_{D'} = \smile \lambda_{D''}$ if $\lambda_{D'} = \lambda_{D''} + n\lambda_H$ for $n \in \mathbb{Z}$. We thus deduce $\smile \lambda_{D'} = \smile \lambda_D$.

(iii) We choose some k (and S) such that we have $H^1(\mathcal{O}_S(D)(-d)) = 0$ (it is easily seen that this condition is satisfied for k big enough).

(iv) We suppose towards a contradiciton that $e_{D'} = 0$, or D' is a smooth and chosen as in (i): there exists some X'' such that $F_{D'}^{X''} = 0$. We then deduce $\beta \circ \delta_{D'}(F_{D'}^{X''}) = 0$, and applying (ii) that $\smile \lambda_{D'} = 0$. Still according to (ii), it comes from $\smile \lambda_D$, where $\beta \circ \delta_D(F_D^{X''}) = 0$. But the choice of k, made in (iii), implies that δ_D is injective. Since β is also injective, we have deduced $F_D^{X''} = 0$, an immediately $e_D = 0$, which contradicts (A).

$\square$

1.3.1 Proof of (A)

The normal exact sequence of $D \subset X \subset \mathbb{P}^4$ write as

$$0 \to N_{D|X} \to N_{D|\mathbb{P}^4} \to \mathcal{O}_D(d) \to 0$$

it is clear that it is enough to prove that

$$h^0(N_{D|\mathbb{P}^4}(-d)) = 0$$

We consider the following exact sequences (E_1) and (E_2):

$$0 \to N_{D|\mathbb{P}^4}(-d) \to N_{D|\mathbb{P}^4}(-d)|_{C_1} \oplus N_{D|\mathbb{P}^4}(-d)|_{C_2} \to N_{D|\mathbb{P}^4}(-d)|_p \to 0$$

$$0 \to N_{C_1|\mathbb{P}^4}(-d) \to N_{D|\mathbb{P}^4}(-d)|_{C_1} \to \mathcal{O}_p \to 0$$

We have $h^0(N_{C_i|\mathbb{P}^4}(-d)) = 1$.

It suffices to show

(i) $H^0(N_{D|\mathbb{P}^4}(-d)|_{C_1}) \cong H^0(N_{C_i|\mathbb{P}^4}(-d))$

(a) $H^0(N_{C_1|\mathbb{P}^4}(-d)) \oplus H^0(N_{C_2|\mathbb{P}^4}(-d)) \hookrightarrow H^0(N_{D|\mathbb{P}^4}(-d)_p)$

(i) By Riemann-Roch an duality, $H^0(N_{D|\mathbb{P}^4}(-d)|_{C_i}) = H^0(N_{C_i|\mathbb{P}^4}(-d))$ if and only if the inclusion

$$H^0(N_{D|\mathbb{P}^4}^\vee(-d) \otimes K_{C_i}) \hookrightarrow H^0(N_{C_i|\mathbb{P}^4}^\vee(d) \otimes K_{C_i})$$

is strict, or for $d \geq 3$, the sheaf $N_{C_i|\mathbb{P}^4}^\vee(d) \otimes K_{C_i}$ is generated by global sections. The conslusion is then immediate, using the dual of E_2.

(ii) the section of $H^0(N_{C_i|\mathbb{P}^4}(-d))$ comes from the canonical section of $N_{C_i|P_i}(-d)$ (where P_i are the planes defining C_i) for $i = 1, 2$. The assertion results immediately from the fact that the tangent spaces of P_1 and P_2 are transverse at the point p, and the local description of $N_{D|\mathbb{P}^4}$.

Remark. It is natural to think that a curve of type $C_1 \cup_p C_2$ furnishes counterexamples to (e) and (e'), indeed we consider the reducible surface $P = P_1 \cup_p P_2$, the union of planes P_1 and P_2 meeting transversally at the point p. Then its scheme-theoretic intersection with X is not the reducible curve $C_1 \cup_p C_2$, but has an embedded point, so that D is not a global complete intersection $P \cap X$.

Conserning the other points of the conjecture of Griffiths and Harris, we can make the (perhaps obvious) remark:

Lemma 1.3.2. (ii) $\implies$ (i)

Proof. We suppose that the general hypersurface X contains a curve of degree m, and that its image under the Abel-Jacobi map φ_X is not zero.

There exists an irreducible variety W equipped with a proper map $p : W \to \mathscr{X}$ where $\mathscr{X} = \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}(d))$, such that the fiber over X parametrizes curves of degree m inside X; two such curves are homologous and for general X, we hace $\forall C, C' \in p^{-1}(X) : \varphi_X(C - C') = 0$; in fact, this remains true for all smooth X. In effect, if H denotes a plane section of X, we have, for general X, $\varphi_X(dC - mH) = 0$ for all $C \in p^{-1}(X)$. By irreducibility of W, this remains true for all $X \in \mathscr{X}$. Then $\varphi_X(C - D')$ is a torsion point, constant over the connected components of $p^{-1}(X) \times p^{-1}(X)$. But the normality of $\mathscr{X}$ and irreducibility of W imply that if $W \to W_1 \to X$ is the Stein factorization of p, each irreducible component of the product $W_1 \times_{\mathscr{X}} W_1$ dominates $\mathscr{X}$. This leads easily to the result.

We fix a line Δ in $\mathbb{P}^4$ and we write $\mathscr{X}_\Delta$ for the family of hypersufraces of degree d containig Δ. We notation $W_\Delta := p^{-1}(\mathscr{X}_\Delta)$; we have then a normal function ν_Δ defined as follows on $\mathscr{X}_\Delta$: let X be a smooth point of $\mathscr{X}_\Delta$ and let $C \in p^{-1}(X)$; then $\deg(m\Delta - C) = 0$ and we can set $\nu_\Delta(X) = \varphi_X(m\Delta - C)$.

But it is known that the group of normal functions on $\mathscr{X}_\Delta$ is cyclic generated by the normal unction ν_Δ^H defined by: $\nu_\Delta^H(X) = \varphi_X(d\Delta - H)$ (it suffices to generalize the argument of [3], section 3). We then deduce that there exist a k such that for all smooth X in $\mathscr{X}_\Delta$ we have $\Phi_X(m\Delta - C) = k\varphi_X(d\Delta - H)$ for $C \in p^{-1}(X)$.

As Δ is deformation equivalent to Δ' we have in fact $k = k'$; on $\mathscr{X}_\Delta \cap \mathscr{X}_{\Delta'}$, so $(m - kd)\Phi_X(\Delta - \Delta') = 0$. But by Griffiths [3], if X is general in $\mathscr{X}_\Delta \cap \mathscr{X}_{\Delta'}$ then $\Phi_X(\Delta - \Delta') \in J(X)$ is not a torsion point. Then $m - kd = 0$ which proves (i). $\qquad \square$

1.4   Conclusion

In paragraph 1 we cleared the necessary condition 1.2 for the curve C to be a complete intersection $X \cap \Sigma$ of X with a surface $\Sigma \subset \mathbb{P}^4$. If d divides the degree of C, this condition is automatically saisfied when C is sub-canical (i.e. $K_C = \mathcal{O}_C(m)$ for some m). Likewise, it seems difficult to construct by an analogous process to the one described in paragraphs 1 and 2, a sub-canonical curve for which the normal bundle sequence is not eact. It does therefore not exclude that (e) or (e') might hold for sub-canonical curves.