1 Chapter 1

1.1 Exercises

1.1.1 Exercise 1.9

Exercise 1.1.1. Define the type family $\mathrm{Fin} : \mathbb{N} \to \mathcal{U}$ and dependent function $\mathrm{fmax} : \prod_{\mathrm{n}:\mathbb{N}} \mathrm{Fin}(\mathrm{succ}(\mathrm{n}))$.

We use the recursive type constructor,

$$\mathrm{Fin} = \mathrm{rec}_{\mathbb{N}}(\mathcal{U}, \mathbf{0}, \lambda \mathrm{n}.\lambda \mathrm{A}.(\mathrm{A} + \mathbf{1}))$$

This satisfies,

$$\mathrm{Fin}(0) :\equiv \mathbf{0} \quad \mathrm{Fin}(\mathrm{succ}(\mathrm{n})) :\equiv \mathrm{Fin}(\mathrm{n}) + \mathbf{1}$$

then the max function also has an inductive construction using the type family Fin,

$$\mathrm{fmax} = \mathrm{ind}_{\mathbb{N}}(\mathrm{Fin} \circ \mathrm{succ}, \star, \mathrm{c_s})$$

where,

$$c_s : \prod_{n:\mathbb{N}} \mathrm{Fin}(\mathrm{succ}(\mathrm{n})) \to \mathrm{Fin}(\mathrm{succ}(\mathrm{succ}(\mathrm{n})))$$

is the function,

$$c_s(n, q) :\equiv \mathrm{inr}(\star)$$

Which satisfies the properties,

$$\mathrm{fmax}(0) \equiv \star : \mathbf{1} \quad \mathrm{fmax}(\mathrm{succ}(\mathrm{n})) \equiv \mathrm{inr}(\star) : \mathrm{Fin}(\mathrm{succ}(\mathrm{n})) + \mathbf{1}$$

1.1.2 Exercise 1.10

Exercise 1.1.2. Show that the Ackermann function $\mathrm{ack} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ is definable using only $\mathrm{rec}_{\mathbb{N}}$ satisfying the following equations:

$$\begin{aligned} \mathrm{ack}(0, n) &\equiv \mathrm{succ}(n) \\ \mathrm{ack}(\mathrm{succ}(m), 0) &\equiv \mathrm{ack}(m, 1) \\ \mathrm{ack}(\mathrm{succ}(m), \mathrm{succ}(n)) &\equiv \mathrm{ack}(m, \mathrm{ack}(\mathrm{succ}(m), n)) \end{aligned}$$

We define the following,

$$\mathrm{ack} :\equiv \mathrm{rec}_{\mathbb{N}}(\mathbb{N} \to \mathbb{N}, \mathrm{succ}, \lambda(m : \mathbb{N}).\lambda(g : \mathbb{N} \to \mathbb{N}).\lambda(n : \mathbb{N}).c_s(n, g, m))$$

where,

$$c_s :\equiv \mathrm{rec}_{\mathbb{N}}((\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}), \lambda(g : \mathbb{N} \to \mathbb{N}).\lambda(m : \mathbb{N}).g(1), u)$$

where,

$$u :\equiv \lambda(n : \mathbb{N}).\lambda(c : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})).\lambda(g : \mathbb{N} \to \mathbb{N}).\lambda(m : \mathbb{N}).g(c(g, m))$$

Then we have the following,

$$c_s(0) \equiv \lambda(g : \mathbb{N} \to \mathbb{N}).\lambda(m : \mathbb{N}).g(1)$$

and

$$c_s(\mathrm{succ}(n)) \equiv u(n, c_s(n)) \equiv \lambda(g : \mathbb{N} \to \mathbb{N}).\lambda(m : \mathbb{N}).g(c_s(n, g, m))$$

Therefore,

$$\mathrm{ack}(0) \equiv \mathrm{succ} \quad \text{meaning } \mathrm{ack}(0, n) \equiv \mathrm{succ}(n)$$

and also,

$$\mathrm{ack}(\mathrm{succ}(m)) \equiv [\lambda(m : \mathbb{N}).\lambda(g : \mathbb{N} \to \mathbb{N}).\lambda(n : \mathbb{N}).c_s(n, g, m)](m, \mathrm{ack}(m)) \equiv \lambda(n : \mathbb{N}).c_s(n, \mathrm{ack}(m), m)$$

This means that,

$$\mathrm{ack}(\mathrm{succ}(m), n) \equiv c_s(n, \mathrm{ack}(m), m)$$

so in particular,

$$\mathrm{ack}(\mathrm{succ}(m), 0) \equiv c_s(0, \mathrm{ack}(m), m) \equiv \mathrm{ack}(m, 1)$$

and also,

$$\mathrm{ack}(\mathrm{succ}(m), \mathrm{succ}(n)) \equiv c_s(\mathrm{succ}(n), \mathrm{ack}(m), m) \equiv \mathrm{ack}(m, c_s(n, \mathrm{ack}(m), m)) \equiv \mathrm{ack}(m, \mathrm{ack}(\mathrm{succ}(m), n))$$

1.1.3 Exercise 1.11

Exercise 1.1.3. Show that for any type A we have $\neg\neg\neg A \to \neg A$.

We need to show that the type $(((A \to \mathbf{0}) \to \mathbf{0}) \to \mathbf{0}) \to (A \to \mathbf{0})$ is inhabited. Indeed consider,

$$\lambda(f : \neg\neg\neg A).\lambda(a : A).f(\lambda(g : \neg A).g(a))$$

1.1.4 Exercise 1.12

Exercise 1.1.4. Using the propositions as types interpretation, derive the following tautologies,

(a) If A then (if B then A).

(b) If A then not (not A).

(c) If (not A or not B) then not (A and B).

(a) $(\lambda(a : A).\lambda(b : B).a) : (A \to B \to A)$

(b) $(\lambda(a : A).\lambda(p : \neg A).p(a)) : (A \to (A \to \mathbf{0}) \to \mathbf{0})$

(c) $\mathrm{rec}_{\neg A + \neg B}(\neg(A \times B), [\lambda(p : \neg A).\lambda(f : A \times B).p \circ \mathrm{pr}_1(f)], [\lambda(p : \neg B).\lambda(f : A \times B).p \circ \mathrm{pr}_2(f)]) :$
$(\neg A + \neg B) \to \neg(A \times B)$

1.1.5 Exercise 1.15

Exercise 1.1.5. Show that the indiscernibility of identicals follows from path induction.

We want to show show that for any family of types $C : A \to \mathcal{U}$ there is,

$$f : \prod_{x,y:A} \prod_{p:x=_A y} C(x) \to C(y)$$

such that,

$$f(x, x, \mathrm{refl}_x) :\equiv \mathrm{id}_{C(x)}$$

We apply path induction to the family of types,

$$D(x, y, p) :\equiv C(x) \to C(y)$$

and our base case is,

$$\prod_{x:A} D(x, x, \mathrm{refl}_x) \equiv \prod_{x:A} C(x) \to C(x)$$

which is inhabited by id_C. Then we set,

$$f :\equiv \mathrm{ind}_{=_A}(D, \mathrm{id}_C)$$

which satisfies,

$$f(x, x, \mathrm{refl}_x) :\equiv \mathrm{id}_{C(x)}$$