Introduction to Complex Analysis and Riemann Surfaces

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July 22, 2020

Contents

1 Holomorphic Maps

Definition: A subset $\Omega \subset \mathbb{C}$ is a domain if Ω is open and connected.

Definition: A map $f : \Omega \to \mathbb{C}$ is holomorphic at $z \in \Omega$ if the limit,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The map f is holomorphic on Ω if it is holomorphic at each $z \in \Omega$.

Proposition 1.1. Let $f : \Omega \to \mathbb{C}$ be holomorphic at $z \in \Omega$. Then we may write f as a function of two real variables as, $f(x, y) = f(x + iy)$. This done,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$$

and thus,

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0$$

Proposition 1.2.

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left[\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right] \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right]$$

Therefore, if f is holomorphic then

$$\frac{\partial f}{\partial z} = f'(z) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = 0$$

Definition: Let $U \subset \mathbb{R}^m$ then denote the vectorspace of continuous functions $U \to \mathbb{C}$ by $\mathcal{C}^0(U)$ and for $n > 0$ define,

$$\mathcal{C}^n(U) = \{ f : U \to \mathbb{R}^m \mid \forall p \in U : f'_p \text{ exists and } \forall \mathbf{v} \in \mathbb{R}^n : f'(\mathbf{v}) \in \mathcal{C}^{n-1}(U) \}$$

where $f' \cdot \mathbf{v}$ is the map $p \mapsto f'_p(\mathbf{v})$. Furthermore, the space of smooth functions is,

$$\mathcal{C}^\infty(U) = \bigcap_k \mathcal{C}^k(U)$$

Theorem 1.3. Let Ω be a domain and $f : \Omega \to \mathbb{C}$. Then the following are equivalent,

1. $f : \Omega \to \mathbb{C}$ is holomorphic.
2. $f \in \mathcal{C}^1(\Omega)$ and
$$\frac{\partial f}{\partial \bar{z}} = 0$$
3. $f \in \mathcal{C}^1(\Omega)$ and for $D \subseteq \Omega$ with piecewise $\mathcal{C}^1(\Omega)$ boundary we have
$$\oint_{\partial D} f(z) \, \mathrm{d}z = 0$$

4. $\forall B_r(w) \subsetneq \Omega$ we have,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

for all $z \in B_r(w)$.

5. $\forall w \in \Omega \exists r > 0$ such that whenever $|z - w| < r$ we have,

$$f(z) = \sum_{n=0}^{\infty} a_n (x - w)^n$$

Proof. We will show that,

$$(2) \iff (3) \implies (4) \implies (5) \implies (1) \implies (2)$$

$(4) \implies (5)$ We assume that,

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

We express the function,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - w - (z - w)} = \frac{1}{\zeta - w} \frac{1}{1 - \left(\frac{z-w}{\zeta-w}\right)} = \frac{1}{\zeta - w} \sum_{n=0}^{\infty} \left(\frac{z - w}{\zeta - w}\right)^n = \sum_{n=0}^{\infty} \frac{(z - w)^n}{(\zeta - w)^{n+1}}$$

Then, formally,

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} \left(\sum_{n=0}^{\infty} \frac{(z - w)^n}{(\zeta - w)^{n+1}}\right) \mathrm{d}\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} \frac{\mathrm{d}\zeta}{(\zeta - w)^{n+1}}\right) (z - w)^n$$

However, to interchange the sum and integral we must establish uniform and absolute convergence. We know that $|\zeta - w| = r$ and $z \in B_r(w)$ so $|z - w| < r$ and thus the sum,

$$\sum_{n=0}^{\infty} \left|\frac{z - w}{\zeta - w}\right|^n$$

converges. Furthermore,

$$\left|\left(\frac{z - w}{\zeta - w}\right)^n\right| = \left|\frac{z - w}{\zeta - w}\right|^n < \left|\frac{z - w}{\zeta - w}\right| = M < 1$$

so the functions are bounded by M^n whose sum coverges and thus by the Weierstrass M-test the series converges absolutly and uniformly. Therefore, take,

$$a_n = \frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} \frac{\mathrm{d}\zeta}{(\zeta - w)^{n+1}}$$

(5) $\implies$ (1) It is clear that if,

$$f(z) = \sum_{n=0}^{\infty} a_n (x-w)^n$$

then,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (x-w)^{n-1}$$

exists.

(1) $\implies$ (2) Suppose that $\Omega = B_\delta(w)$. For each $z \in \Omega$, let ℓ_z be the segment joining w to z and define,

$$F(z) = \int_{\ell_z} f(\zeta)\, \mathrm{d}\zeta$$

Now compute the ratio,

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left[\int_{\ell_z} f(\zeta)\,\mathrm{d}\zeta \; - \int_{\ell_{z+h}} f(\zeta)\,\mathrm{d}\zeta\right]$$

(PROGRESS) Because the integral over the tringle is zero, we have,

$$\frac{1}{h}\left[\int_{\ell_z} f(\zeta)\,\mathrm{d}\zeta \; - \int_{\ell_{z+h}} f(\zeta)\,\mathrm{d}\zeta\right] = \frac{1}{h}\int_z^{z+h} f(\zeta)\,\mathrm{d}\zeta \; = \int_0^1 f(z+th)\,\mathrm{d}t \; \to f(z)$$

where we have parametrized the path z to $z+h$ by $z+th$ for $0 \le t \le 1$. Thus, $F'(z) = f(z)$ which implies that F is $\mathcal{C}^1(\Omega)$ and holomorphic so,

$$\partial f \bar{z} = 0$$

and thus satisfies (2). Therefore, by (2) $\implies$ (5) we have that F is a power seies and thus $f = F'$ is a power series so f is $\mathcal{C}^1(\Omega)$. Furthrermore, f is holomorphic which implies that

$$\frac{\partial f}{\partial \bar{z}} = 0$$

. Therefore, we have (2).

□

Theorem 1.4. For any $z_0 \in \Omega$, either $f \equiv 0$ in a neighborhood of z_0 or we can express $f = (z-z_0)^n u(z)$ for $u(z)$ holomorphic and $u(z) \neq 0$.

Proof. In a neighborhood of z_0, we can write,

$$f(z) = \sum_{n=0}^{\infty} n_n (z-z_0)^n$$

Either $c_n = 0$ for each n so $f = 0$ or $c_N \neq 0$ for some n and $c_n - 0$ for $n < N$. Therefore,

$$f(z) = \sum_{n \geq N}^{\infty} c_n (z - z_0)^n = (z - z_0)^N \left(\sum_{m=0}^{\infty} c_{N+m} (z - z_0)^m \right) = (z - z_0)^N u(z)$$

Furthermore, $u(z_0) = c_N \neq 0$ so, by continuity, there exists a neighborhood of z_0 on which $n(z) \neq 0$. □

Theorem 1.5. Let f be holomorphic on a domain Ω. If $f \equiv 0$ on some open set inside Ω then $f \equiv 0$ on all of Ω.

Proof. Define,

$$\Omega' = \{z \in \Omega \mid f \equiv 0 \text{ on an open neighborhood of } z\}$$

Clearly Ω' is open in Ω because each $z \in \Omega'$ in inside an open neighborhood of Ω on which f vanishes so is contained in an open neighborhood of Ω'.
Take $z_1 \notin \Omega'$. Thus, f does not vanish identically on every neighborhood of z so there exists a neighborhood U such that $f(z) = (z - z_1)^N u(z)$ for $u(z) \neq 0$. Then $f(z) \neq 0$ on $U \setminus \{z_1\}$. Therefore, $U \subset (\Omega')^C$ because f is nonzero on $U \setminus \{z\}$ and thus cannot be identically zero on any neighborhood of any point of U. Thus, $(\Omega')^C$ is open so Ω' is clopen. However, Ω is connected and thus $\Omega' = \Omega$. □

Example 1.6. Consider the solution to the equation $w^2 = z$. First take the open domain $U = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \geq 0\}$ and for $z = re^{i\theta}$ with $0 < \theta < 2\pi$ define $w = r^{1/2} e^{i\theta/2} = \sqrt{z}$. The function $f(z) = w$ is perfectly holomorphic on U. However, the line we choose to remove is artificial, any cut will work with a redefinition of the angular interval. We solve this problem by taking two copies of U called (I) and (II) and then constructing a surface X by gluing (I) and (II) along the cuts such that moving across the cut in $\mathbb{C}$ corresponds to changing sheets. We can define w on all of X by $w(p) = w(z) = \sqrt{z}$ if p is on sheet (I) at position z and otherwise $w(p) = -w(z) = -\sqrt{z}$ if p is on sheet (II) at position z.

Topologically, X is a sphere minus two points. We call $\hat{X}$ the compactified version of X constrcuted by adding back the two points such that $\hat{X} \cong S^2$.

2 Meromorphic Functions

Definition: A function $f : \Omega \to \mathbb{C}$ is meromorphic if, near any $z_0 \in \Omega$, it can be written as,

$$f(z) = \sum_{n \geq -N} c_n (z - z_0)^n$$

We call N the order of the pole (assuming that $c_n \neq 0$) and c_{-1} the residue at z_0.

Theorem 2.1 (Residue)**.** Let $f : \Omega \to \mathbb{C}$ be meromorphic and $D \subset \overline{D} \subset \Omega$ be a domain in Ω with piecewise smooth boundary ∂D such that no poles of g lie on ∂D. Then,

$$\oint_{\partial D} f(z)\, \mathrm{d}z = 2\pi i \sum_{p \in D} \mathrm{Res}_{f(p)}$$

Proof. We can deform the path ∂D to a sum of small circles of radius r surrounding each pole. Since f is holomorphic on the region D minus these circles the two integrals along these paths (whose difference is the integral over the boundary) are equal. Thus,

$$\begin{aligned} \oint_{\partial D} f(z)\, \mathrm{d}z - 2\pi i \sum_{p \in D} \mathrm{Res}_p f &= \sum_{p \in D} \left[\oint_{\partial B_r(p)} f(p+z)\, \mathrm{d}z - 2\pi i \mathrm{Res}_p g \right] \\ &= \sum_{p \in D} \left[\int_0^{2\pi} i \Big(f(p + re^{i\theta}) re^{i\theta} - \mathrm{Res}_p g \Big)\, \mathrm{d}\theta \right] \end{aligned}$$

However,

$$\mathrm{Res}_p f = \lim_{z \to p} (z-p) f(z) = \lim_{h \to 0} f(p+h)h$$

and thus, for each $\epsilon > 0$ we can choose some δ such that $r < \delta$ implies that,

$$\left| f(z + rr^{i\theta}) re^{i\theta} - \mathrm{Res}_p f \right| < \epsilon$$

Therefore,

$$\left| \oint_{\partial D} f(z)\, \mathrm{d}z - 2\pi i \sum_{p \in D} \mathrm{Res}_p f \right| \le \sum_{p \in D} \left[\int_0^{2\pi} \left| f(p + re^{i\theta}) re^{i\theta} - \mathrm{Res}_p g \right| \mathrm{d}\theta \right]$$

$$\le \sum_{p \in D} \int_0^{2\pi} \epsilon = 2\pi N \epsilon$$

where N is the number of poles. Since ϵ is arbitrary,

$$\oint_{\partial D} f(z)\, \mathrm{d}z = 2\pi i \sum_{p \in D} \mathrm{Res}_p f$$

□

Theorem 2.2. Let $f : \Omega \to \mathbb{C}$ be meromorphic and $D \subset \overline{D} \subset \Omega$ be a domain in Ω with piecewise $\mathcal{C}^1$ boundary ∂D such that no poles of g lie on ∂D. Then,

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)}\, \mathrm{d}z = (\#\text{ of zeros}) - (\#\text{ of poles})$$

Theorem 2.3. At each point $p \in D$ we can expand,

$$f(z) = (z-p)^N u(z)$$

where u is holomorphic and nonvanishing. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \log f(z) = \frac{\mathrm{d}}{\mathrm{d}z} \left[(z-p)^N u(z) \right] = \frac{N}{x-p} + \frac{u'(z)}{u(z)}$$

Thus when f has either a zero ($N > 0$) or a pole ($N < 0$) the logarithmic derivative has residue,

$$\mathrm{Res}_p \left(\frac{f'}{f} \right) = N$$

Therefore the result holds by the residue theorem.

Corollary 2.4. Let $f : \Omega \to \mathbb{C}$ be holomorphic take $w \in \mathbb{C}$, then the number of solutions in D to the equation $f(z) - w = 0$ is equal to,

$$\#\{z \in D \mid f(z) = w\} = \oint_{\partial D} \frac{f'(z)}{f(z) - w} \, \mathrm{d}z$$

Proof. Since $f - w$ is holomorphic on Ω is has no poles. Therefore, the only residues are from roots of $f - w$ i.e. solutions to $f(z) - w = 0$. As above, the integral of the logarithmic derivative counts the number of such poles. □

3 An Elliptic Curve

Consider the solution to the equation $w^2 = z(z-1)(z-\lambda)$ with $\lambda \neq 0, 1$. Then we can construct a Riemann surface X on which this function $w(z)$ is everywhere holomorphic such that the compactification $\hat{X}$ is a torus. However, what complex structure does this torus have?

3.1 Complex Tori

Definition: We say that $\Lambda \subset \mathbb{C}$ is a non-degenerate lattice if there exist $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$ such that,

$$\Lambda = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\}$$

Clearly, Λ is a additive subgroup of $\mathbb{C}$ so we may consider the quotient topological group $\mathbb{C}/\Lambda$ which we call a complex torus.

Lemma 3.1. A complex torus $\mathbb{C}/\Lambda$ is topologically a torus.

Proof. The space $\mathbb{C}/\Lambda$ has universal cover $\mathbb{C} \to \mathbb{C}/\Lambda$ with group of Deck Transformations $\mathbb{Z} \times \mathbb{Z}$. □

Theorem 3.2 (Open Mapping)**.** Let $f : \Omega \to \mathbb{C}$ be holomorphic and $\Omega \subset \mathbb{C}$ is open then either f is constant or $f(\Omega) \subset \mathbb{C}$ is open.

Proof. □

Theorem 3.3. If $f : \Omega \to \mathbb{C}$ is not constant then f cannot achive a maximum inside the interior of Ω.

Proof. If f achives a maximum at $z_0 \in \Omega$ then $f(z_0)$ is a boundary point of $f(\Omega^\circ)$ which is open since f is not constant. Thus, $f(z_0) \notin f(\Omega^\circ)$ so $z_0 \notin \Omega^\circ$. $\square$

Remark 1. Since $\hat{X}$ is open and compact the image of any holomorphic map must also be open and compact unless it is constant. Thus any nonconstant holomorphic function $f : \hat{X} \to \mathbb{C}$ must have an open compact image which is impossible. Thus the only holomorphic functions $f : \hat{X} \to \mathbb{C}$ must be constant.

Remark 2. Since $\hat{X}$ is compact we have seen that there do not exist global holomorphic functions on $\hat{X}$. For example, $w(z)$ is meromorphic with zeros at $z = 0, 1, \lambda$ and a triple pole at ∞. Therefore, we must consider holomorphic forms on $\hat{X}$ instead. For example, there exists a nonvanishing holomorphic 1-form on $\hat{X}$,

$$\frac{\mathrm{d}z}{w} = \begin{cases} \frac{\mathrm{d}z}{\sqrt{z(z-1)(z-\lambda)}} & \text{on (I)} \\ \frac{-\,\mathrm{d}z}{\sqrt{z(z-1)(z-\lambda)}} & \text{on (II)} \end{cases}$$

We can check this in local coordinates.

Theorem 3.4. Every holomorphic or antiholomorphic 1-form on a Riemann surface (1-dimensional complex manifold) is closed.

Proof. Let ω be a holomorphic 1-form then, in local coordinates z we can write $\omega = f(z)\,\mathrm{d}z$. Thus,

$$\mathrm{d}\omega \;=\; \mathrm{d}z\,\wedge \partial_z f(z)\,\mathrm{d}z \;+\; \mathrm{d}\bar{z}\,\wedge \partial_{\bar{z}} f(z)\,\mathrm{d}z \;= \partial_{\bar{z}} f(z)\,\mathrm{d}\bar{z}\,\wedge\,\mathrm{d}z \;= 0$$

The first term vanishes by antisymmetry the second because f is holomorphic so $\partial_{\bar{z}} f = 0$ by the Cauchy-Riemann equations. An identical argument flipping z and $\bar{z}$ holds in the antiholomorphic case. $\square$

Theorem 3.5. Let ψ be a meromorphic form on a Riemann surface X then summing over the residues at the poles of ψ gives,

$$\sum_{p \in X} \mathrm{Res}_p\,\psi = 0$$

Remark 3. Since $\hat{X}$ is topologically a torus we have $\pi_1(\hat{X}) \cong \mathbb{Z} \times \mathbb{Z}$ so any two paths from p_0 to p differ, up to homotopy, by concatenation with γ_1 and γ_2 the generating loops. That is, if γ and γ' are paths from p_0 to p then $\gamma' \sim \gamma * \gamma_1^n * \gamma_2^m$. Define the periods,

$$\omega_1 = \oint_{\gamma_1} \omega \quad \text{and} \quad \omega_2 = \oint_{\gamma_2} \omega$$

and the lattice Λ with periods ω_1 and ω_2. This is more conveniently expressed using de Rham cohomology which is isomorphic to the fundamental group and thus also generated by two cycles A and B which we denote by,

$$H^1_{\mathrm{dR}}(\hat{X}) = \mathbb{Z} \oint_A \oplus \mathbb{Z} \oint_B$$

Then if γ' and γ are paths with the same endpoints then we can write,

$$\int_{\gamma'} = \int_\gamma + n \oint_A + m \oint_B$$

for some $n, m \in \mathbb{Z}$ meaning that for any 1-form ω we have,

$$\int_{\gamma'} \omega = \int_\gamma \omega + n \oint_A \omega + m \oint_B \omega$$

Remark 4. There exists a meromorphic form ω_{pq} on $\hat{X}$ with simple poles at exactly p and q with residue $+1$ at p and -1 at q. Furthermore, we can fix,

$$\oint_A \omega_{pq} = 0$$

since we can add $\omega_{pq} \to \omega_{pq} + c\omega$ without chaning the pole structure since ω is holomorphic. Furthermore,

$$\oint_A \Big(\omega_{pq} + c\omega\Big) = \oint \omega_{pq} + c\omega_1$$

Since ω_1 is nonzero (ω_1 and ω_2 span a non-degenerate lattice) we can fix c such that the integral is zero.

3.1.1 Riemann Bilinear Relations and Abelian Integrals

Consider the Abelian integral of a 1-form ω on $\hat{X}$,

$$f(z) = \int_{p_0}^z \omega$$

which is a function on X_{cut} which transforms as,

$$f(z+A) = f(z) + \oint_A \omega = f(z) + \omega_1 \quad \text{and} \quad f(z+B) = f(z) + \oint_B \omega = f(z) + \omega_2$$

In local coordinates $\zeta = x + iy$,

$$\omega = \omega_\zeta \, \mathrm{d}\zeta$$

and thus,

$$i\omega \wedge \bar{\omega} = |\omega_\zeta|^2 i \, \mathrm{d}\zeta \ \vee \ \mathrm{d}\bar{\zeta} \ = 2|\omega_\zeta|^2 \, \mathrm{d}x \ \wedge \ \mathrm{d}y$$

which is proportional to the Euclidean area form by a positive quantitiy. Thus,

$$0 < \int_{\hat{X}} i\omega\wedge\bar{\omega} = \int_{X_{\mathrm{cut}}} i\omega\wedge\bar{\omega} = \int_{X_{\mathrm{cut}}} i\omega\wedge\bar{\omega} = i\int_{X_{\mathrm{cut}}} \mathrm{d}f\wedge\bar{\omega} = i\int_{X_{\mathrm{cut}}} \mathrm{d}(f\bar{\omega}) = i\int_{\partial X_{\mathrm{cut}}} f\bar{\omega}$$

by Stokes' theorem. We have used the fact that,

$$\mathrm{d}(f\bar{\omega}) \;=\; \mathrm{d}f\;\wedge\bar{\omega} + f\,\mathrm{d}\bar{\omega}$$

and that $\mathrm{d}\bar{\omega}\; = 0$ since $\bar{\omega}$ is antiholomorphic and thus closed on a Riemann surface. Thus we should compute the boundary integral,

$$i\int_{\partial X_{\mathrm{cut}}} f\bar{\omega} = i\left(\oint_A f(z)\overline{\omega(z)} - \oint_A f(z+B)\overline{\omega(z+B)} - \oint_B f(z)\overline{\omega(z)} + \oint_B f(z+A)\overline{\omega(z+A)}\right)$$

Since ω is a form on $\hat{X}$ we know $\omega(z+A) = \omega(z)$ and $\omega(z+B) = \omega(z)$. Thus,

$$\begin{aligned}
i\int_{\partial X_{\mathrm{cut}}} f\bar{\omega} &= i\left(\oint_A \Big(f(z) - f(z+B)\Big)\overline{\omega(z)} + \oint_B \Big(f(z+A) - f(z)\Big)\overline{\omega(z)}\right) \\
&= i\left(\oint_A \Big(-\oint_B \omega\Big)\overline{\omega(z)} + \oint_B \Big(\oint_A \omega\Big)\overline{\omega(z)}\right) \\
&= i\left(\Big(\oint_A \omega\Big)\overline{\Big(\oint_B \omega\Big)} - \Big(\oint_B \omega\Big)\overline{\Big(\oint_A \omega\Big)}\right) \\
&= i\left(\omega_1\bar{\omega_2} - \omega_2\bar{\omega_1}\right)
\end{aligned}$$

Therefore, the imaginary part of $\omega_1\bar{\omega_2}$ is nonzero so the two periods cannot be dependent over $\mathbb{R}$ otherwise we would be able to write $\omega_1 = \lambda\omega_2$ or $\omega_2 = \lambda\omega_1$ with $\lambda \in \mathbb{R}$ and either way, $\omega_1\bar{\omega_2} \in \mathbb{R}$.

We need to establish one other formula using this trick. We need to compute the integrals of the forms ω_{pq} about the cycles A and B. Again we use the Abelian integral,

$$f(z) = \int_{p_0}^{z} \omega$$

on X_{cut}. Then we have,

$$\begin{aligned}
\oint_{\partial X_{\mathrm{cut}}} f\omega_{pq} &= \left(\oint_A \Big(f(z) - f(z+B)\Big)\omega_{pq}(z) + \oint_B \Big(f(z+A) - f(z)\Big)\omega_{pq}(z)\right) \\
&= \left(\oint_A \Big(-\oint_B \omega\Big)\omega_{pq}(z) + \oint_B \Big(\oint_A \omega\Big)\omega_{pq}(z)\right) \\
&= \left(\omega_1 \oint_B \omega_{pq} - \omega_2 \oint_A \omega_{pq}\right) \\
&= i\left(\omega_1\bar{\omega_2} - \omega_2\bar{\omega_1}\right) = \omega_1 \oint_B \omega_{pq}
\end{aligned}$$

since we have set the integral of ω_{pq} over the A cycle to zero. However, by the residue theorem,

$$\oint_{\partial X_{\mathrm{cut}}} f\omega_{pq} = 2\pi i \sum_p \mathrm{Res}_p\, f\omega_{pq} = 2\pi i\,[f(p) - f(q)]$$

Therefore,

$$\oint_B \omega_{pq} = \frac{2\pi i}{\omega_1}\,[f(p) - f(q)]$$

3.1.2 Abel's Theorem

We can now show the equivalence between $\hat{X}$ and a complex torus $\mathbb{C}/\Lambda$.

Theorem 3.6. For each $\lambda \neq 0, 1$ there exists a non-degenerate lattice $\Lambda \subset \mathbb{C}$ such that there is a biholomorphic map $f : \hat{X} \to \mathbb{C}/\Lambda$.

Proof. Considered a closed nonvanishing holomorphic 1-form ω on $\hat{X}$ and a fixed point $p_0 \in \hat{X}$. Define the map $I : \hat{X} \to \mathbb{C}/\Lambda$ via,

$$I(p) = \int_\gamma \omega$$

where γ is some path from $p_0 \to p$. This map is well-defined because if I choose a different path γ' from p_0 to p then $\gamma' \sim \gamma * \gamma_1^n * \gamma_2^m$. Suppose that two paths $\gamma \sim \delta$ are homotopic. Then we have a map $H : D^2 \to \hat{X}$ with boundary $\gamma * \delta^{-1}$. By Stokes' Theorem,

$$\int_\gamma \omega - \int_\delta \omega = \int_{\gamma * \delta^{-1}} \omega = \int_{\partial H(D^2)} \omega = \int_{H(D^2)} \mathrm{d}\omega = 0$$

because ω is closed so $\mathrm{d}\omega = 0$. Therefore,

$$\int_\gamma \omega = \int_\delta \omega$$

In particular,

$$\int_{\gamma'} \omega = \int_{\gamma * \gamma_1^n * \gamma_2^m} \omega = \int_\gamma \omega + n\int_{\gamma_1} \omega + m\int_{\gamma_2} \omega = \int_\gamma \omega + n\omega_1 + m\omega_2$$

Therefore, $I(p)$ is defined up to an element of Λ so $I : \hat{X} \to \mathbb{C}/\Lambda$ is well-defined. Furthermore, we have proved that Λ is non-degenerate via Abelian integrals. Finally, we can show that I is one-to-one assuming Abel's theorem. If p and q are distinct points and $I(p) = I(q)$ then there exists a meromorphic function f on $\hat{X}$ with a zero at p and pole at q. Thus, the meromorphic form $f\omega$ has a unique pole at q contradicting the fact that its residues must sum to zero. Finally, $I : \hat{X} \to \mathbb{C}/\Lambda$ is a nonconstant holomorphic map between compact connected Riemann surfaces and thus surjective. □

Theorem 3.7 (Abel)**.** Let $p_1, \cdots, p_M, q_1, \ldots, q_N$ be points on $\hat{X}$. Then there exists a meromorphic function f on $\hat{X}$ with zeroes at $p_1, \ldots, p_M$ and poles at $q_1, \ldots, q_N$ if and only if $M = N$ and

$$\sum_{i=1}^{M} I(p_i) = \sum_{i=1}^{N} I(q_i)$$

Proof. First assume the Abel condition. We will first construct the logarithmic derivative of f. Consider the meromorphic form,

$$\psi = \sum_{j=1}^{M} \omega_{p_j q_j} + c\omega$$

We need to show that the function,

$$f = \exp\left(\int_{p_0}^{z} \sum_{j=1}^{M} \omega_{p_j q_j} + c\omega\right)$$

is well-defined and has the required zero-pole structure. Since,

$$\frac{\mathrm{d}f}{f} = \sum_{j=1}^{M} \omega_{p_j q_j} + c\omega$$

which has poles at p_i and q_i with residue $+1$ at each p_i and -1 at each q_i. Thus f has zeros at the p_i and simple poles at the q_i. Furthermore, cut $\hat{X}$ along A and B cycles to form X_{cut}. Given $z \in X_{\text{cut}}$ define,

$$\ell_\gamma(z) = \int_\gamma \left[\sum_{j=1}^{M} \omega_{p_j q_j} + c\omega\right]$$

by picking some path γ from p_0 to z in X_{cut}. If we choose two different paths γ and γ' then,

$$\left(\int_\gamma \sum_{j=1}^{M} \omega_{p_j q_j} + c\omega\right) - \left(\int_{\gamma'} \sum_{j=1}^{M} \omega_{p_j q_j} + c\omega\right) = \left(\int_{\partial D} \sum_{j=1}^{M} \omega_{p_j q_j} + c\omega\right) = 2\pi i N$$

Where N is the difference between the number of p poles enclosed and the number of q poles enclosed by D with boundary γ composed with γ' inversed. Since N is an integer we have shown that,

$$\exp \ell_\gamma(z) = \exp\left(\ell_{\gamma'}(z) + 2\pi i N\right) = \exp \ell_{\gamma'}(z) e^{2\pi i N} = \exp \ell_{\gamma'}(z)$$

and therefore,

$$f(z) = \exp \ell_\gamma(z)$$

is independent of the choice of path. Consider,

$$
\begin{aligned}
f(z+A) &= \exp\left(\int_{p_0}^{z+A}\sum_{j=1}^{M}\omega_{p_jq_j}+c\omega\right) = f(z)\exp\left(\oint_A\sum_{j=1}^{M}\omega_{p_jq_j}+c\omega\right)\\
&= f(z)\exp\left(c\oint_A\omega\right) = f(z)\exp\left(c\omega_1\right)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
f(z+B) &= \exp\left(\int_{p_0}^{z+B}\sum_{j=1}^{M}\omega_{p_jq_j}+c\omega\right) = f(z)\exp\left(\oint_B\sum_{j=1}^{M}\omega_{p_jq_j}+c\omega\right)\\
&= f(z)\exp\left(\frac{2\pi i}{\omega_1}\sum_{j=1}^{M}\left[f(p_i)-f(p_j)\right]+c\omega_2\right)
\end{aligned}
$$

However, we know that,

$$\sum_{j=1}^{M}I(p_j) = \sum_{j=1}^{M}I(q_j)$$

which images in $\mathbb{C}/\Lambda$. Thus, we have,

$$\sum_{j=1}^{M}\left[f(p_j)-f(q_j)\right] = m_1\omega_1+m_2\omega_2$$

viewed as a map $X_{\text{cut}}\to\mathbb{C}$. Then if we take,

$$c = -\frac{2\pi i}{\omega_1}m_2$$

we have,

$$f(z+A) = f(z)\exp\left(c\omega_1\right) = f(z)\exp\left(-2\pi i m_2\right) = f(z)$$

and

$$
\begin{aligned}
f(z+B) &= f(z)\exp\left(\frac{2\pi i}{\omega_1}\sum_{j=1}^{M}\left[f(p_j)-f(q_j)\right]+c\omega_2\right)\\
&= f(z)\exp\left(2\pi i m_1+2\pi i m_2\frac{\omega_2}{\omega_2}-2\pi i m_2\frac{\omega_2}{\omega_1}\right) = f(z)\exp\left(2\pi i m_1\right) = f(z)
\end{aligned}
$$

Therefore, f is a well-defined meromorphic function on $\hat{X}$ with the correct zero-pole structure. $\square$

4 Weierstrass Function Theory

We want to consider doubly periodic meromorphic functions. That is, meromorphic functions $f : \mathbb{C}/\Lambda \to \mathbb{C}$.

Theorem 4.1. Let $\Lambda \subset \mathbb{C}$ be a Lattice. The function,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

is a well-defined meromorphic function $\wp : \mathbb{C}/\Lambda \to \mathbb{C}$ with double poles on Λ.

Proof. We can write,

$$\begin{aligned}\frac{1}{(z+\omega)^2} &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z+\omega}\right) = -\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}z}\left[\sum_{n=0}^{\infty}\left(-\frac{z}{\omega}\right)^n\right] = -\frac{1}{\omega}\sum_{n=0}^{\infty}\left[\frac{(-1)^n n z^{n-1}}{\omega^n}\right] \\ &= \frac{1}{\omega^2}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{z^n}{\omega^n}\right]\end{aligned}$$

and this series is uniformly convergent for $|z| < |\omega|$. Thus,

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\sum_{n=1}^{\infty}(-1)^n(n+1)\frac{z^n}{\omega^n}$$

and therefore,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

converges uniformly. Furthermore,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\omega \in \Lambda^\times}\left(-\frac{2}{(z+\omega)^3}\right) = -2\sum_{\omega\in\Lambda}\frac{1}{(z+\omega)^3}$$

Thus, $\wp'(z)$ is doubly periodic with periods ω_1 and ω_2. Thus, $\wp(z+\omega_1) = \wp(z) + c_1$ so $\wp(-\frac{1}{2}z_1) = \wp(\frac{1}{2}z) + c_1$ but $\wp$ is even so $c_1 = 0$. The same for ω_2. Thus, $\wp(z+\omega_1) = \wp(z)$ and $\wp(z+\omega_2) = \wp(z)$. □

Proposition 4.2. The function,

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda^\times}\left[\frac{1}{z+\omega} - \frac{1}{\omega} + \frac{z}{\omega^2}\right]$$

is meromorphic and has simple poles on Λ.

Proof. We obtain ζ by integrating $\wp$ term by term. However, $\wp$ converges uniformly to ζ converges uniformly to the integral of $\wp$ on the fundamental domain. □

Proposition 4.3. We have $\zeta(z+\omega_1) - \zeta(z) = \eta_1$ and $\zeta(z+\omega_2) - \zeta(z) = \eta_2$ where $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$.

Proof. Because $\zeta'(z+\omega_1) = -\wp(z+\omega_1) = -\wp(z) = \zeta'(z)$ we have $\zeta'(z+\omega_1) = \zeta'(z)+\eta_1$ and the same for ω_2. Furthermore, ζ has residue 1 at zero so the integral around the fundamental domain gives,

$$2\pi i = \oint_\gamma \zeta(z)\,\mathrm{d}z \; = \int_A (\zeta(z)\zeta(z+B))\,\mathrm{d}z \; + \int_B (\zeta(z+A) - \zeta(z))\,\mathrm{d}z \; = -\eta_2\omega_1 + \eta_1\omega_2$$

□

Definition: The meromorphic form $\omega_{pq}(z) = [\zeta(z-p) - \zeta(z-q)]\ \mathrm{d}z$ has simple at p and q with residue 1 at p and -1 at q. Furthermore, ω_{pq} is doubly periodic because,

$$\omega_{pq}(z+\omega_1) = [\zeta(z+\omega_1-p) - \zeta(z+\omega_1-p)]\,\mathrm{d}z = [\zeta(z-p) + \eta_1 - \zeta(z+-p) - \eta_1]\,\mathrm{d}z = \omega_{pq}(z)$$

and the same for ω_2. Thus, ω_{pq} is a meromorphic form on $\mathbb{C}/\Lambda$.

Definition: Define the holomorphic function,

$$\sigma(z) = \exp\left(\int_0^z \zeta(z')\,\mathrm{d}z'\right) = z \prod_{\omega\in\Lambda^\times} \left(1 + \frac{z}{\omega}\right) e^{-\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}$$

which has simple zeros on Λ.

Proposition 4.4. The function σ transforms as,

$$\frac{\sigma'(z+\omega_1)}{\sigma(z+\omega_1)} - \frac{\sigma'(z)}{\sigma(z)} = \zeta(z+\omega) - \zeta(z) = \eta_1$$

and therefore,

$$\sigma(z+\omega_1) = \sigma(z)e^{\eta_1 z + \mu_1}$$

with $\mu_1 = \pi i + \frac{1}{2}\eta_1\omega_1$ up to $2\pi i\mathbb{Z}$. Thus,

$$\sigma(z+\omega_1) = -\sigma(z)e^{\eta_1(z+\frac{1}{2}\omega_1)}$$

Theorem 4.5 (Abel)**.** There exists a meromorphic function $f : \mathbb{C}/\Lambda \to \mathbb{C}$ with zeros at $p_1, \dots, p_M$ and poles at $q_1, \dots, p_N$ iff $M = N$ and,

$$\sum_{j=1}^M I(p_j) = \sum_{j=1}^N I(p_i)$$

where I is the identity map on $\mathbb{C}/\Lambda$.

Proof. We can choose representatives $p_j, q_j \in \mathbb{C}$ such that,

$$\sum_{j=1}^{M} p_j = \sum_{j=1}^{N} q_j$$

as elements of $\mathbb{C}$. Now define the function,

$$f(z) = \frac{\prod_{j=1}^{M} \sigma(z - p_j)}{\prod_{i=1}^{N} \sigma(z - q_j)}$$

Now,

$$f(z + \omega_1) = \frac{\prod_{j=1}^{M} \sigma(z + \omega_1 - p_j)}{\prod_{i=1}^{N} \sigma(z + \omega_1 - q_j)} = \frac{\prod_{j=1}^{M} \sigma(z - p_j) e^{\eta_1 (z - p_j)}}{\prod_{i=1}^{N} \sigma(z - q_j) e^{\eta_1 (z - q_j)}}$$

where the factors of -1 and $e^{\frac{1}{2}\eta_1 \omega_1}$ cancel out because there are equal numbers of factors in the numerator and denominator. Therefore,

$$f(z + \omega_1) = f(z) \exp\left[\sum_{j=1}^{N} \eta_1 (p_j - q_j)\right]$$

However,

$$\sum_{j=1}^{N} (p_j - q_j) = 0$$

so $f(z + \omega_1) = f(z)$ and similarly $f(z + \omega_2) = f(z)$ so f descends as a holomorphic function of the quotient $f : \mathbb{C}/\Lambda \to \mathbb{C}$. $\square$

4.1 The Defining Differential Equation for $\wp$

We need to work out he expansion fo $\wp(z)$ near 0. We have,

$$\frac{1}{(z - \omega)^2} = \frac{1}{\omega^2} \cdot \frac{1}{(1 - \frac{z}{\omega})^2} = \frac{1}{\omega^2} \sum_{n=0}^{\infty} (m + 1) \left(\frac{z}{\omega}\right)^m$$

Therefore,

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \sum_{m=1}^{\infty} (m + 1) \frac{z^m}{\omega^{m+2}} \\
&= \frac{1}{z^2} + \sum_{m=1}^{\infty} (m + 1) z^m \left(\sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{m+2}}\right) \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k + 1) z^{2k} G_{k+1}(\Lambda)
\end{aligned}$$

where we have defined,

$$G_k(\Lambda) = \sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{2k}}$$

The odd terms vanish because if we sum an odd function over the lattice then we get zero. Explicitly,

$$\wp(z) = \frac{1}{z^2} + 3G_2(\Lambda)z^2 + 5G_3(\Lambda)z^4 + O(z^6)$$

Next,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty}(2k+1)(2k)G_{k+1}(\Lambda)z^{2k-1}$$

which we sum as,

$$\wp'(z) = -\frac{2}{z^3} + 6G_2(\Lambda)z + 20G_3(\Lambda)z^3 + O(z^5)$$

Thus, compute,

$$\begin{aligned}\wp'(z)^2 &= \left(-\frac{2}{z^3} + 6G_2(\Lambda)z + 20G_3(\Lambda)z^3 + O(z^5)\right)^2 \\ &= \frac{4}{z^6} - 24G_2(\Lambda)\frac{1}{z^2} - 80G_3(\Lambda) + O(z^2)\end{aligned}$$

Similarly, compute,

$$\begin{aligned}\wp(z)^3 &= \left(\frac{1}{z^2} + 3\Lambda_2(\Lambda)z^2 + 5G_3(\Lambda)z^4 + O(z^6)\right)^3 \\ &= \frac{1}{z^6} + 9G_2(\Lambda)\frac{1}{z^2} + 15G_3(\Lambda) + O(z^2)\end{aligned}$$

Therefore,

$$\begin{aligned}\wp'(z)^2 - 4\wp(z)^3 &= -24G_2(\Lambda)\frac{1}{z^2} - 36G_2(\Lambda)\frac{1}{z^2} - 80G_3(\Lambda) - 60G_3(\Lambda) + O(z^2) \\ &= -60G_2(\Lambda)\frac{1}{z^2} - 140G_3(\Lambda) + O(z^2)\end{aligned}$$

Which implies that,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda) = O(z^2)$$

Therefore the function,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda)$$

is holomorphic on $\mathbb{C}$ and is doubly periodic and thus constant. However, it vanishes at $z = 0$ and thus must be the zero function. Therefore, we have the differential equation,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda) = 0$$

4.1.1 Inverse to the Abel Map

Define $g_2 = 60G_2(\Lambda)$ and $g_3 = 140G_3(\Lambda)$. Then we have the fundamental differential equation,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

This implies that,

$$\int_0^z \frac{\wp'(z)\,\mathrm{d}z}{\sqrt{4\wp(z)^3 - g_2\wp(z) - g_3}} = z$$

Let $u = \wp(z)$ then we have,

$$\int_\infty^{\wp(z)} \frac{\mathrm{d}u}{\sqrt{4u^3 - g_2 u - g_3}} = z$$

Thus if we introduce the elliptic integral,

$$E(v) = \int_\infty^{v} \frac{\mathrm{d}u}{\sqrt{4u^3 - g_2 u - g_3}}$$

then we have,

$$E(\wp(z)) = z$$

5 Jacobi Function Theory

Definition: For a lattice $\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\} \subset \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ we define,

$$\theta(z|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$$

which converges because,

$$\left| e^{\pi i n^2 \tau + 2\pi i n z} \right| = e^{-\pi n^2 \mathrm{Im}(\tau)}$$

and thus the sum coverges uniformly by the M-test.

Lemma 5.1. The θ function transforms as,

$$\theta(z+1|\tau) = \theta(z|\tau) \qquad \theta(z+\tau|\tau) = e^{-\pi i\tau - 2\pi i z}\theta(z|\tau)$$

The second transformation property can be written as,

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\theta(z+\tau|\tau) = -2\pi i + \frac{\mathrm{d}}{\mathrm{d}z}\log\theta(z|\tau)$$

Proposition 5.2. The theta function has 1 zero in the fundamental domain of $\mathbb{C}/\Lambda$.

Consider the line integral,

$$\oint_C z\frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z\ = 2\pi i z_0$$

by the residue theorem. However,

$$\begin{aligned}
&\oint_{\partial X_{\mathrm{cut}}} z\frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z \\
&\quad= \int_A z\frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z\ - \int_A (z+\tau)\frac{\theta'(z+\tau|\tau)}{\theta(z+\tau|\tau)}\,\mathrm{d}z\ - \int_B z\frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z\ + \int_B (z+1)\frac{\theta'(z+1|\tau)}{\theta(z+1|\tau)}\,\mathrm{d}z \\
&\quad= \int_A z\frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z\ - \int_A (z+\tau)\left(\frac{\theta'(z|\tau)}{\theta(z|\tau)} - 2\pi i\right)\mathrm{d}z\ + \int_B \frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z \\
&\quad= 2\pi i\int_A (z+\tau)\,\mathrm{d}z\ - \int_A \tau\frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z\ + \int_B \frac{\theta'(z|\tau)}{\theta(z|\tau)}\,\mathrm{d}z \\
&\quad= 2\pi i\left(\tau+\tfrac{1}{2}\right) - \tau\left(\log\theta(1|\tau) - \log\theta(0|\tau)\right) + \left(\log\theta(\tau|\tau) - \log\theta(0|\tau)\right) \\
&\quad= 2\pi i\left(\tau+\tfrac{1}{2}\right) - \pi i\tau = \pi i(\tau+1)
\end{aligned}$$

Thus,

$$z_0 = \frac{1+\tau}{2}$$

We generalize,

$$\theta[\delta',\delta''](z|\tau) = \sum_{n\in\mathbb{Z}} \exp\left[\pi i(x+\delta')^2\tau + 2\pi i(n+\delta')(z+\delta'')\right]$$

In particular, take,

$$\begin{aligned}
\theta_1(z|\tau) = \theta[1/2,1/2](z|\tau) &= \sum_{n\in\mathbb{Z}} \exp\left[\pi i(x+\tfrac{1}{2})^2\tau + 2\pi i(n+\tfrac{1}{2})(z+\tfrac{1}{2})\right] \\
&= \exp\left[\tfrac{\pi i}{4}\tau + \pi i(z+\tfrac{1}{2})\right]\theta(z+\tfrac{1+\tau}{2}|\tau)
\end{aligned}$$

which has its zero at the origin and is odd.

Lemma 5.3. The θ_1 function transforms as,

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau) \qquad \theta_1(z+\tau|\tau) = -e^{-\pi i\tau - 2\pi i z}\theta_1(z|\tau)$$

Theorem 5.4 (Abel)**.** There exists a meromorphic function $f : \mathbb{C}/\Lambda \to \mathbb{C}$ with zeroes at $p_1, \ldots, p_M$ and poles at $q_1, \ldots, q_N$ if and only if $M = N$ and

$$\sum_{j=1}^{M} I(p_j) = \sum_{j=1}^{N} I(q_j)$$

Proof. The Abel map is given by,

$$I(p) = \int_0^p \mathrm{d}z\ = p$$

as a map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$. Assuming the Abel condition, we can choose representatives $p_1, \ldots, p_j, q_1, \ldots, q_j \in \mathbb{C}$ such that,

$$\sum_{j=1}^{M} p_j = \sum_{j=1}^{N} q_j$$

Define,

$$f(z) = \frac{\prod_{j=1}^{M} \theta_1(z - p_j|\tau)}{\prod_{j=1}^{N} \theta_1(z - q_j|\tau)}$$

Therefore,

$$f(z+1) = \frac{\prod_{j=1}^{M} \theta_1(z + 1 - p_j|\tau)}{\prod_{j=1}^{N} \theta_1(z + 1 - q_j|\tau)} = \frac{\prod_{j=1}^{M} \theta_1(z - p_j|\tau)}{\prod_{j=1}^{N} \theta_1(z - q_j|\tau)} = f(z)$$

and similarly,

$$\begin{aligned} f(z+\tau) &= \frac{\prod_{j=1}^{M} \theta_1(z + \tau - p_j|\tau)}{\prod_{j=1}^{N} \theta_1(z + \tau - q_j|\tau)} = \frac{\prod_{j=1}^{M} \theta_1(z - p_j|\tau)\left(-e^{-\pi i\tau - 2\pi i(z-p_j)}\right)}{\prod_{j=1}^{N} \theta_1(z - q_j|\tau)\left(-e^{-\pi i\tau - 2\pi i(z-q_j)}\right)} \\ &= \frac{\prod_{j=1}^{M} \theta_1(z + \tau - p_j|\tau)}{\prod_{j=1}^{N} \theta_1(z + \tau - q_j|\tau)} \exp\left[2\pi i \sum_{j=1}^{M} \Big(I(p_j) - I(q_j)\Big)\right] = f(z) \end{aligned}$$

which holds since

$$\exp\left[2\pi i \sum_{j=1}^{M} \Big(I(p_j) - I(q_j)\Big)\right] = 1$$

Thus $f : \mathbb{C} \to \mathbb{C}$ is doubly periodic on Λ so it descends to a meromorphic function $f : \mathbb{C}/\Lambda \to \mathbb{C}$. □

Proposition 5.5.

$$\wp(z) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} \log\left(\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right) + c(\tau)$$

where the constant $c(\tau)$ is given by,

$$c(\tau) = \frac{1}{3}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}$$

Theorem 5.6. The Jacobi θ functions admit product expansion,

$$\theta(z|\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})\left(1 + q^{2n-1} e^{2\pi i z}\right)\left(1 + q^{2n-1} e^{-2\pi i z}\right)$$

where $q = e^{\pi i \tau}$.

Proof. By construction, these functions have the same set of poles on $\mathbb{C}$. Now let,

$$T(z|q) = \prod_{n=1}^{\infty}(1-q^{2n})\left(1+q^{2n-1}e^{2\pi iz}\right)\left(1+q^{2n-1}e^{-2\pi iz}\right)$$

Clearly, $T(z+1) = T(z)$ since $e^{x+2pii} = e^x$. Furthermore, consider,

$$\begin{aligned}
T(z+\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})\left(1+q^{2n-1}e^{2\pi iz+2\pi i\tau}\right)\left(1+q^{2n-1}e^{-2\pi iz-2\pi i\tau}\right) \\
&= \prod_{n=1}^{\infty}(1-q^{2n})\left(1+q^{2n-1+2}e^{2\pi iz}\right)\left(1+q^{2n-1-2}e^{-2\pi iz}\right) \\
&= \frac{1+q^{-1}e^{-2\pi iz}}{1+qe^{2\pi iz}}\prod_{n=1}^{\infty}(1-q^{2n})\left(1+q^{2n-1+2}e^{2\pi iz}\right)\left(1+q^{2n-1}e^{-2\pi iz}\right) \\
&= q^{-1}e^{-2\pi iz}\prod_{n=1}^{\infty}(1-q^{2n})\left(1+q^{2n-1+2}e^{2\pi iz}\right)\left(1+q^{2n-1}e^{-2\pi iz}\right) \\
&= e^{-\pi i\tau-2\pi iz}T(z)
\end{aligned}$$

Since the two functions transform in the same way their ratio,

$$f(z) = \frac{\theta(z|\tau)}{T(z)}$$

is doubly periodic. However, $f(z)$ is holomorphic since the two have the same zeros. Thus $f(z)$ is a constant $c(q)$. I claim that $c(q) = c(q^4)$ which implies that $c(q) = c(q^\ell)$ as $\ell \to \infty$. Since $|q| = |e^{\pi i\tau}| = e^{-\pi \mathrm{Im}(\tau)} < 1$. Thus $c(q) = \lim_{q\to 0} c(q)$. However, $T(z|q) \to 1$ as $q \to 0$ and $\theta(z|\tau) \to 1$ as $|q| \to 0$ thus,

$$c(q) = \lim_{q\to 0}\frac{\theta(z|\tau)}{T(z|\tau)} = 1$$

So it remains to prove this claim. Consider, $\theta(\frac{1}{2}|\tau)$ and $\theta(\frac{1}{4}|\tau)$. We know,

$$\begin{aligned}
\theta(\tfrac{1}{2}|\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2}e^{\pi i} = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} \\
T(\tfrac{1}{2}|\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2 = \prod_{n=1}^{\infty}(1-q^n)(1-q^{2n-1})
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\theta(\tfrac{1}{4}|\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2}e^{\pi in/2} = \sum_{n\in\mathbb{Z}} q^{n^2}i^n = \sum_{k=0}^{\infty} q^{(2k+1)^2}\left[i^{2k+1}+i^{-(2k+1)}\right] + \sum_{k=0}^{\infty} q^{(2k)^2}\left[i^{2k}+i^{-(2k)}\right] \\
&= \sum_{k=0}^{\infty} q^{(2k)^2}\left[(-1)^k+(-1)^k\right] = \sum_{m\in\mathbb{Z}}(-1)^m q^{4m^2}
\end{aligned}$$

and also,

$$T(\tfrac{1}{4}|\tau) = \prod_{n=1}^{\infty}\left(1-q^{2n})(1+q^{2n-1}e^{\pi i/2}\right)\left(1+q^{2n-1}e^{-\pi i/2}\right) = \prod_{n=1}^{\infty}\left(1-q^{2n})(1+iq^{2n-1}\right)\left(1-iq^{2n-1}\right)$$
$$= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2}) = \prod_{n=1}^{\infty}(1-q^{4n-2})(1-q^{4n})(1+q^{4n-2}) = \prod_{n=1}^{\infty}(1-q^{4n})(1-q^{8n-4})$$

which are exactly what we found earlier with $q \mapsto q^4$ proving the claim that $c(q) = c(q^4)$. $\square$

5.1 Modular Transformations of $\theta(z|\tau)$

Both τ and $-\tau^{-1}$ generate the same lattice so there must be a relationship between their θ functions.

Theorem 5.7 (Poisson Summation Formula)**.** Let $f(\lambda)$ be a smooth and rapidly decreasing function then,

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n z} = \sum_{n\in\mathbb{Z}} f(n+z)$$

Proof. Define,

$$u(\theta) = \sum_{n\in\mathbb{Z}} f(n+\theta)$$

so $u \in \mathcal{C}^\infty$ and periodic. Thus we can Fourier expand u as,

$$u(\theta) = \sum_{m\in\mathbb{Z}} \left\langle u, e^{-2\pi i m\theta}\right\rangle e^{2\pi i m\theta}$$

with,

$$\left\langle u, e^{-2\pi i m\theta}\right\rangle = \int_0^1 u(\theta)e^{-2\pi i m\theta}\mathrm{d}\theta = \int_0^1 \sum_{n\in\mathbb{Z}} f(n+\theta)e^{-2\pi i m\theta} = \sum_{n\in\mathbb{Z}}\int_0^1 f(n+\theta)e^{-2\pi i m\theta}\mathrm{d}\theta$$

We can commute sums and integrals because the series converges uniformly since f is rapidly decreasing. Notice that $e^{2\pi i m n} = 1$ so,

$$\begin{aligned}\left\langle u, e^{-2\pi i m\theta}\right\rangle &= \sum_{n\in\mathbb{Z}}\int_0^1 f(n+\theta)e^{-2\pi i m(n+\theta)}\,\mathrm{d}\theta \\ &= \sum_{n\in\mathbb{Z}}\int_n^{n+1} f(\eta)e^{-2\pi i m\eta}\,\mathrm{d}\eta \;= \int_{\mathbb{R}} f(\eta)e^{-2\pi i m\eta} = \hat{f}(m)\end{aligned}$$

Therefore,

$$u(\theta) = \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{2\pi i m\theta}$$

$\square$

Theorem 5.8. There is a functional equation,

$$\theta(z|-\tau^{-1}) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z^2} \theta(z\tau|\tau)$$

In particular, at $z = 0$,

$$\theta(0|-\tau^{-1}) = \sqrt{\frac{\tau}{i}} \theta(0|\tau)$$

Proof. Define $\tau = i\tau_2$ and take z to be real. Consider,

$$\begin{aligned} \sqrt{\frac{\tau}{i}} e^{\pi i \tau z^2} \theta(z\tau|\tau) &= \sqrt{\tau_2} e^{-\pi\tau_2 z^2} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 \tau_2 - 2\pi n z \tau_2} \\ &= \sqrt{\tau_2} \sum_{n\in\mathbb{Z}} e^{-\pi\tau_2 (z+n)^2} = \sum_{n\in\mathbb{Z}} e^{-\pi n^2/\tau_2 + 2\pi i n z} = \theta(z|-\tau^{-1}) \end{aligned}$$

where I have applied Poisson summation to $f(z) = \sqrt{\tau_2} e^{-\pi\tau_2 z^2}$. □

6 $\bar{\partial}$-Equations

Suppose again we want to find a meromorphic form $\omega_p(z)$ on a Riemann surface X with a douple pole at p. If X is a complex torus then we can use the Weierstrass $\wp$-function. We want to discuss this problem more generally. Let (U, z) be a chart on X with $p \in U$ and $z(p) = 0$ and $z(U) = D$ a disk in $\mathbb{C}$. We could take the form $\frac{\mathrm{d}z}{z^2}$ on D. However, we still have the problem of extending this form to X. Our first attempt is to use a bump function $\chi(z)$. Then we can take,

$$\tilde{\omega}_p(z) = -\partial_z \left(\frac{\chi(z)}{z} \right) \mathrm{d}z$$

which is well-defined on X and meromorphic inside a smaller ball $B \subset D$ on which $\chi(z) = 1$. We can correct $\tilde{\omega}_p$ to be a meromorphic form on X by adding a form ψ which cancels the derivative with respect to $\bar{z}$ which we denote by $\bar{\partial}$. Writing $\psi = \psi_z \,\mathrm{d}z$ we have,

$$\bar{\partial} \left(-\partial \left(\frac{\chi(z)}{z} \right) - \psi_z \right) = 0$$

and we restrict to smooth solutions to ψ so that the pole structure is mantained. Because $\tilde{\omega}_p$ is meromorphic inside B we know that $\bar{\partial}\tilde{\omega}_p = 0$ inside B. Furthermore, outside B we have $\tilde{\omega}_p$ is smooth. Therefore $\bar{\partial}\tilde{\omega}_p(z)$ is a smooth $(1,1)$-form which can be written as,

$$\bar{\partial}\tilde{\omega}_p(z) = -\partial\bar{\partial} \left(\frac{\chi(z)}{z} \right) \mathrm{d}\bar{z} \wedge \mathrm{d}z$$

Theorem 6.1. Consider the equation $\bar{\partial}\partial\omega = \Phi$ on a Rieman surface X where Φ is a smooth $(1,1)$-form and ω is a scalar function. The equation admits a smooth solution u if and only if,

$$\int_X \Phi = 0$$

Furthermore, in the case at hand,

$$-\int_X \partial\bar{\partial}\left(\frac{\chi(z)}{z}\right) \mathrm{d}\bar{z} \wedge \mathrm{d}z = \int_X \mathrm{d}\left(\bar{\partial}\left(\frac{\chi(z)}{z}\right) \mathrm{d}\bar{z}\right) = 0$$

since the manifold has no boundary. Therefore, there exists a solution to,

$$\bar{\partial}\partial u = \tilde{\omega}_p$$

and thus we can take $\psi = \partial u$ such that $\omega_p = \tilde{\omega}_p - \psi$ is a meromorphic form on X with a double pole at p. Proving this theorem is now going to take some work.

Definition: Define the Sobolev norm,

$$||u||^2_{(1)} = \int_X \partial u \bar{\partial} u \, \mathrm{d}z \wedge \mathrm{d}\bar{z} + \int_X |u|^2 \, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

The the Sobolev space,

$$H_{(1)}(X) = \overline{\left\{u \in \mathcal{C}^\infty(X) \mid ||u||^2_{(1)} \leq \infty\right\}}$$

is the completion of Sobolev normed functions which is a subspace of $L^2(X)$.

Definition: Define the functional $I : H_{(1)}(X) \to \mathbb{C}$ via, for $u \in H_{(1)}(X)$,

$$I(u) = \int_X \left[\tfrac{1}{2}\partial u \bar{\partial} u + u\Phi\right] \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

We will show that $I(u)$ is bounded from below. Furthermore if we take a sequence u_j such that $I(u_j) \to \inf I(u)$ then we will show that $u_\infty = \lim_{j\to 0} u_j \in H_{(1)}(X)$ and $I(u_\infty) = \inf I(u)$. Finally, we will show that if $I(u_\infty) = \inf I(u)$ then u_∞ is a generalized solution to $\bar{\partial}\partial u = \Phi$.

6.1 Proof of the Main Theorem

6.1.1 The Functional Is Bounded Below

We have that,

$$I(u) = \int_X \left[\tfrac{1}{2}\partial u \bar{\partial} u + (u - \bar{u})\Phi\right] \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

where,

$$\bar{u} = \frac{\int_X u \, \mathrm{d}^2 z}{\int_X \mathrm{d}^2 z}$$

since

$$\int_X \bar{u}\Phi = \bar{u}\int_X \Phi = 0$$

Then,

$$I(u) = \int_X \left[\tfrac{1}{2}\partial(u-\bar{u})\bar{\partial}u + \Phi\right] \geq \frac{1}{2}\int_X \partial u\bar{\partial}u - \left|\int_X (u-\bar{u})\Phi\right| = \tfrac{1}{2}||\nabla u||_{L^2} - \left|\int_X (u-\bar{u})\Phi\right|$$

Furthermore,

$$\left|\int_X (u-\bar{u})\Phi\right| \leq \frac{1}{2}||\Phi||_{L^2}||u+\bar{u}||_{L^2} \leq \epsilon||u-\bar{u}||^2_{L^2} + \frac{1}{2\epsilon}||\Phi||^2_{L^2}$$

For any $\epsilon > 0$.

Theorem 6.2 (Poincare)**.** Let X be a compact Riemannian manifold then $\exists C_X$ s.t.

$$||u-\bar{u}||_{L^2} \leq C_X||\nabla u||^2_{L^2}$$

Using this theorem,

$$\epsilon||u-\bar{u}|| \leq \epsilon C_X||\nabla u||^2_{L^2}$$

so choose ϵ small enough such that, $\epsilon C_X \leq \frac{1}{4}$. Then,

$$\begin{aligned} I(u) \geq& \tfrac{1}{2}||\nabla u||_{L^2} - \epsilon||u-\bar{u}||^2_{L^2} - \frac{1}{2\epsilon}||\Phi||^2_{L^2} \geq \tfrac{1}{2}||\nabla u||_{L^2} - \tfrac{1}{4}||\nabla u||_{L^2} - \frac{1}{2\epsilon}||\Phi||^2_{L^2} \\ =& \tfrac{1}{4}||\nabla u||_{L^2} - \frac{1}{2\epsilon}||\Phi||^2_{L^2} \end{aligned}$$

Therefore,

$$I(u) \geq -\frac{1}{2\epsilon}||\Phi||^2_{L^2}$$

so $I(u)$ is bounded below by $-C$ where,

$$C = \frac{1}{2\epsilon}||\Phi||^2_{L^2}$$

6.1.2 The Functional I Attains its Minimum

Let $\{u_j\} \in H_{(1)}(X)$ be a sequence with $I(u_j) \to \inf I(u)$. I claim that $||u_j||_{H_{(1)}(X)} \leq C$. Indeed,

$$C_1 \geq U(u_j) \geq \frac{1}{4}\int_X |\partial u_j|^2 - C \implies \int_X |\partial u_j|^2 \leq C_2$$

Furthermore, we can replace u_j by $u_j - \bar{u}_j$ without changing I so we assume that $\bar{u}_j = 0$. Then by Poincare,

$$||u_j||^2_{L^2} = ||u_j - \bar{u}_j||_{L^2} \leq C_X||\nabla u_j||^2_{L^2} \leq C_3$$

Thus each u_j is bounded since,

$$||u_j||^2_{H_{(1)}(X)} = ||\nabla u_j||_{L^2} + ||u_j||_{L^2} \leq C_2 + C_3$$

Theorem 6.3 (Banach-Alaoglu)**.** Let $\mathcal{H}$ be a seperable Hilbert space. Then any bounded sequence $\{u_j\} \subset \mathcal{H}$ with $||u_j|| \leq 1$ admits a weakly convergent subsequence in the sense that,

$$\exists u_\infty \in \mathcal{H} : \forall v \in \mathcal{H} : \lim_{k\to\infty} \langle u_k, v\rangle = \langle u_\infty, v\rangle$$

Applying the theorem we get some u_∞ such that,

$$\lim_{k\to\infty} \langle u_j, v\rangle = \langle u_\infty, v\rangle$$

for any v. Furthermore the embedding $H_{(1)}(X) \hookrightarrow L^2(X)$ is compact i.e. every bounded set is sent to a compact set. Thus u_j converges to u_∞ in $L^2(X)$. However, we do not have ∂u_j converging to ∂u_∞.

Proposition 6.4 (Lower semi-continuity of weak limits)**.** If $w_j \to w_\infty$ weakly then,

$$||w_\infty|| \leq \liminf_{j\to\infty} ||w_j||$$

Proof. Let $v = w_\infty$ then we get,

$$||w_\infty||^2 = \langle w_\infty, w_\infty\rangle = \lim_{j\to\infty} \langle w_j, w_\infty\rangle \leq \liminf_{j\to\infty} ||w_j||||w_\infty||$$

and thus,

$$||w_\infty|| \leq \liminf_{j\to\infty} ||w_j||$$

□

Using this result we find,

$$I(u_\infty) = \tfrac{1}{2}||\nabla u||^2_{L^2} + \langle \Phi, u_\infty\rangle \leq \tfrac{1}{2}\liminf_{j\to\infty}\left(||\nabla u_j||^2_{L^2} + \langle \Phi, u_j\rangle\right) = \liminf_{j\to\infty} I(u_j) = \inf I(u)$$

However, $\inf I(u) \leq I(u_\infty)$ and therefore,

$$I(u_\infty) = \inf I(u)$$

so I attains its minimum.

6.1.3 The Generalized Solution to the $\bar{\partial}\partial$-Equation

For any $v \in H_{(1)}(X)$ consider the function $t \mapsto I(u_\infty + tv)$ which attains its minimum at $t = 0$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} I(u_\infty + tv)\bigg|_{t=0} = 0$$

Expanding,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I(u_\infty + tv)\bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_X \left[\tfrac{1}{2}\partial(u_\infty + tv)\bar{\partial}(u_\infty + tv) + (u_\infty + tv)\Phi\right] \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \int_X \left[\tfrac{1}{2}\partial v\bar{\partial}u_\infty + \tfrac{1}{2}\partial u_\infty \bar{\partial} v + v\Phi\right] \mathrm{d}z \wedge \mathrm{d}\bar{z}\end{aligned}$$

If u_∞ is smooth then we would have,

$$\int_X v\left(\bar\partial\partial u_\infty - \Phi\right)\, \mathrm{d}z\, \wedge\, \mathrm{d}\bar z\, = 0$$

for any function $v \in H_{(1)}(X)$ which implies that,

$$\bar\partial\partial u_\infty = \Phi$$

When u_∞ is not smooth we call it a generalized solution to the equation,

$$\bar\partial\partial u_\infty = \Phi$$

meaning, by definition, that it satisfies,

$$\int_X \left[\tfrac{1}{2}\partial v\bar\partial u_\infty + \tfrac{1}{2}\partial u_\infty\bar\partial v + v\Phi\right]\, \mathrm{d}z\, \wedge\, \mathrm{d}\bar z\, = 0$$

for all $v \in H_{(1)}(X)$

6.1.4 The Generalized Solution is Smooth and Thus a Standard Solution

Definition: Let X be a n-dimensional compact Riemannian manifold. Define the Sobolev norm,

$$||u||^2_{(s)} = \sum_{|\alpha|\le S} ||\partial^\alpha u||^2_{L^2}$$

The the Sobolev space,

$$H_{(s)}(X) = \overline{\left\{u \in \mathcal{C}^\infty(X) \mid ||u||^2_{(s)} \le \infty\right\}}$$

is the completion of Sobolev normed functions which is a subspace of $L^2(X)$.

Theorem 6.5 (Sobolev Embedding)**.** For $s > \frac{n}{2} + k$ there is a compact embedding,

$$H_{(s)}(X) \hookrightarrow \mathcal{C}^k(X)$$

such that,

$$||u||_{\mathcal{C}^k(X)} \le ||u||_{(s)}$$

Remark 5. For any $f \in H_{(s)}(X)$ then there exists $f_\epsilon \in \mathcal{C}^\infty(X)$ such that $f_\epsilon \to f$ with respect to $||\cdot||_{(s)}$ and all f_ϵ are bounded with respect to this norm. Furthermore, $\partial f_\epsilon = (\partial f)_\epsilon$ then $\partial\bar\partial u = f$ in the generalized sense the $\partial\bar\partial u_\epsilon = f_\epsilon$ in the standard sense.

Proposition 6.6. Furthermore, if $u \in \mathcal{C}^\infty$ and $f \in \mathcal{C}^\infty$ and $\partial\bar\partial u = f$ then

$$||u||^2_{(1)} \le C\left(||f||^2_{(0)} + ||u||^2_{(0)}\right)$$

Proof. Suppose that $\partial\bar\partial u = f$ in the generalized sense then

$$\partial\bar\partial(\partial^\alpha u_\epsilon) = (\partial f)_\epsilon \implies ||\partial^\alpha u||^2_{(1)} \le C\left(||\partial^\alpha f||^2 + ||\bar\partial u||^2_{(0)}\right) \le C\left(C_\alpha ||f||^2_{(0)} + ||u||^2_{(0)}\right)$$

□

Theorem 6.7. $u \in H_{(1)}(X)$ if and only if $u \in L^2(X)$ and there exists $v \in L^2(X)$ such that for all $\psi \in \mathcal{C}^\infty$ that,

$$-\int_X u\partial\psi = \int_X v\psi$$

6.1.5 Rewriting the Generalized Solutions

Take $\chi \in \mathcal{C}^\infty$ and $\chi = 0$ outside of a small neighbrohood of 0 such that,

$$\int_X \chi(w)\,\mathrm{d}z\,\wedge\,\mathrm{d}\bar{z}\, = 1$$

Define $\forall \epsilon > 0$ the function $\chi_\epsilon(w) = \frac{1}{\epsilon^2}\chi\left(\frac{w}{\epsilon}\right)$ and let,

$$u_\epsilon(z) = \int_X u(w)\chi_\epsilon(z-w)\,\mathrm{d}w\,\wedge\,\mathrm{d}\bar{w}$$

which is smooth. I claim that, $||u_\epsilon||_{L^2} \leq ||u||_{L^2}$ and $||u_\epsilon - u||_{L^2} \to 0$ whenever $\epsilon \to 0$. Furthermore,

$$\begin{aligned}
\partial\bar{\partial}u_\epsilon &= \int_X u(w)\partial_z\partial_{\bar{z}}\chi_\epsilon(z-w)\,\mathrm{d}w\,\wedge\,\mathrm{d}\bar{w} \\
&= \int_X u(w)\Big[\partial_w\partial_{\bar{w}}\chi_\epsilon(z-w)\Big]\,\mathrm{d}w\,\wedge\,\mathrm{d}\bar{w}
\end{aligned}$$

However, since u is a generalized solution, for any $v \in H_{(1)}(X)$ we have,

$$\int_X \left[\tfrac{1}{2}\partial v\bar{\partial}u + \tfrac{1}{2}\partial u\bar{\partial}v + v\Phi\right]\,\mathrm{d}z\,\wedge\,\mathrm{d}\bar{z}\, = 0$$

Using the fact that $u \in H_{(1)}(X)$ if we take $v \in H_{(1)}(X)$ to be smooth then,

$$-\int_X u\partial(\bar{\partial}v) = \int_X \partial u\bar{\partial}v$$

Therefore,

$$\int_X \left[u\partial\bar{\partial}v - \Phi v\right] = 0$$

Thus,

$$\begin{aligned}
\partial\bar{\partial}u_\epsilon &= \int_X u(w)\Big[\partial_w\partial_{\bar{w}}\chi_\epsilon(z-w)\Big]\,\mathrm{d}w\,\wedge\,\mathrm{d}\bar{w} \\
&= \int_X \Phi\chi_\epsilon(z-w)\,\mathrm{d}w\,\wedge\,\mathrm{d}\bar{w}\, = \Phi_\epsilon(z)
\end{aligned}$$

Using the generalized equation.

6.1.6 A Priori Estimates

For $u, v \in \mathcal{C}^\infty$ and $\partial\bar{\partial}u = v$ then,

$$\int_X (\partial\bar{\partial}u)u = \int_X vu \implies ||\partial u||_{L^2}^2 = \left|\int_X uv\right| \leq ||u||_{L^2}\cdot||v||_{L^2} \leq \tfrac{1}{2}||u||_{L^2}^2 + \tfrac{1}{2}||v||_{L^2}^2$$

Applying this to the differential equation $\partial\bar{\partial}u_\epsilon = \Phi_\epsilon$ then $||Du_\epsilon||^2_{L^2} \leq \frac{1}{2}||\Phi_\epsilon||^2_{L^2} + \frac{1}{2}||u_\epsilon||^2_{L^2}$. Furthermore,

$$\partial\bar{\partial}D^\alpha u_\epsilon = D^\alpha \Phi_\epsilon$$

and thus,

$$||D^{\alpha+1}u_\epsilon||^2_{L^2} \leq \frac{1}{2}||D^\alpha\Phi_\epsilon||^2_{L^2} + \frac{1}{2}||D^\alpha u_\epsilon||^2 \leq ||D^\alpha\Phi||^2_{L^2} + ||u||^2_{L^2}$$

which is bounded indepenendent on ϵ. Apply the Sobolev embeddding to get for any fixed m,

$$||u_\epsilon||_{\mathcal{C}^m} \leq C$$

independent of ϵ. Then $\{u_\epsilon\}$ is equicontinuous in $\mathcal{C}^{m-1}$ and thus there exists a convergent subsequence which converges to an element of $\mathcal{C}^{m-1}$. However, the sequence $\{u_\epsilon\}$ converges to u_∞ in $H_{(1)}(X)$ and the limit is unique. Thus $u_\infty \in \mathcal{C}^{m-1}$ for each m and thus u_∞ is smooth.

7 The Moduli Space of Tori

We began by considering the Riemann surface of the form $w^2 = z(z-1)(z-\lambda)$ but we can more generally consider $w^2 = (z-a_1)(z-a_2)(z-a_3)(z-a_4)$ with distinct roots. However, we may apply a Mobius transformation,

$$z' = \frac{az+b}{cz+d}$$

to the sphere $\hat{C}$ on which the points a_1, a_2, a_3, a_4 lie which gives an equivalent complex torus. We can generically send $a_1 \mapsto 0$ and $a_2 \mapsto 1$ and $a_3 \mapsto \infty$ by choosing the appropriete Mobius tranfromation. Under this transformation we get $a_4 \mapsto \lambda$. Any permutation of a_1, a_2, a_3, a_4 results in a group acting on λ which does not change the structure of the complex torus it defines. The group orbits are generated by $\lambda \mapsto 1-\lambda$ and $\lambda \mapsto \lambda^{-1}$.

7.1 Modular Invariant

Definition: The Modular invariant j is defined as,

$$j(\lambda) = \frac{4}{27}\frac{(1-(1-\lambda)\lambda}{(1-\lambda)^2\lambda}$$

Theorem 7.1. Two complex torii are biholomorphic if and only they have the same modular invariat j.

8 Complex Manifolds

Definition: A complex n-manifold X is a second-countible Hausdorff topological space with an atlas of charts $(U_\alpha, \varphi_\alpha)$ with maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ whose domain cover X such that the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\beta \cap U_\beta)$ are holomorphic for each pair of charts.

Definition: A Riemann surface is a complex 1-manifold and thus a smooth surface with a complex structure.

Definition: A map $f : X \to Y$ between complex manifolds is holomorphic if for any point $p \in X$ there exists charts (U, φ) for X and (V, ψ) for Y such that $p \in U$ and $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic in the usual sense.

Theorem 8.1. Every holomorphic map $f : M \to N$ between connected complex manifolds with M compact is either constant or surjective in which case N is compact.

Proof. Let $f : M \to N$ be a holomorphic map with M and N compact connected complex manifold. Take a cover of charts $(U_\alpha, \varphi_\alpha)$ for M with U_α a domain and (ψ_β, V_β) for N, Then the map $\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta)$ is holomorphic on the domain $\varphi_\alpha(U_\alpha)$. Therefore, either f is constant on U_α or $\mathrm{Im}(f \circ \varphi_\alpha^{-1}) = f(U_\alpha)$ is open since $\psi_\beta \circ f(U_\alpha)$ is open. Suppose that f is nonconstant on each U_α. Then the set,

$$f(M) = f\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f(U_\alpha)$$

is open because it is a union of open sets and is compact because M is compact and f is continuous. Since N is Hausdorff, $f(M)$ is clopen and nonempty. Since N is connected we must have $f(M) = N$ which forces N to be compact since it is the image of the compact set M. Therefore, either f is surjective or f must be constant on some domain U_α which implies that f is constant on the entirety of M by connectivity and analytic continuation. □

Corollary 8.2. There are no nonconstant holomorphic maps from a compact connected complex manifold to a complex manifold with noncompact components.

Corollary 8.3. There are no nonconstant holomorphic functions on a compact connected complex manifold.

8.1 Line Bundles on Riemann Surfaces

Definition: The curvature of the bundle L with a given connection derived from the metric h is written as $F_{\bar{z}z}$ where,

$$[\nabla, \bar{\nabla}]\varphi = -F_{\bar{z}z}\varphi$$

Theorem 8.4. For any meromorphic section φ of L which is not identically zero,

$$(\#\text{ zeros of }\varphi) - (\#\text{ poles of }\varphi) = \frac{i}{2\pi}\int_X F_{\bar{z}z}\,\mathrm{d}z\,\wedge\,\mathrm{d}\bar{z}$$

Proof. Consider,

$$-\partial\bar{\partial}\log|\varphi|_h^2 = -\partial\bar{\partial}\log(\varphi\bar{\varphi}h)$$

When we are away from D the set of zeros and poles of φ we can write,

$$-\partial\bar{\partial}\log|\varphi|_h^2 = -\partial\bar{\partial}\left(\log\varphi + \overline{\log\varphi} + \log h\right) = -\partial\bar{\partial}\log h = F_{\bar{z}z}$$

because $\log\varphi$ is holomorphic and $\overline{\log\varphi}$ is anti-holomorphic on $X\setminus D$. Consider the union of disks,

$$D_\epsilon = \bigcup_{p\in D} B_\epsilon(p)$$

where we choose ϵ small enough for $B_\epsilon(p)$ to lie in the image of a single chart so we can identify $B_\epsilon(p)$ in the image of a chart with a disk on X and small enough that only one $p\in D$ lies in each disk. We can always do this because φ is a nonzero meromorphic section and thus has isolated poles and zeros. Then,

$$\int_X F_{\bar{z}z}\,\mathrm{d}z\,\wedge\,\mathrm{d}\bar{z} = \lim_{\epsilon\to 0}\int_{X\setminus D_\epsilon} F_{\bar{z}z}\,\mathrm{d}z\,\wedge\,\mathrm{d}\bar{z}$$

However,

$$\mathrm{d}(-\bar{\partial}\log|\varphi|_h^2\,\mathrm{d}\bar{z}) = -\partial\bar{\partial}\log|\varphi|_h^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z} - \bar{\partial}\bar{\partial}\log|\varphi|_h^2\,\mathrm{d}\bar{z}\wedge\mathrm{d}\bar{z} = -\partial\bar{\partial}\log|\varphi|_h^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z}$$

since $\log|\varphi|_h^2$ is a well-defined scalar function on $X\setminus D$ unlike $\log h$ whose argument transforms as a section of the nontrivial line bundle $L^{-1}\otimes\bar{L}^{-1}$. Therefore, by Stokes' theorem,

$$\begin{aligned}\int_X F_{\bar{z}z}\,\mathrm{d}z\,\wedge\,\mathrm{d}\bar{z} &= \lim_{\epsilon\to 0}\int_{X\setminus D_\epsilon}\mathrm{d}(-\bar{\partial}\log|\varphi|_h^2\,\mathrm{d}\bar{z}) = -\lim_{\epsilon\to 0}\int_{\partial(X\setminus D_\epsilon)}\bar{\partial}\log|\varphi|_h^2\,\mathrm{d}\bar{z}) \\ &= \lim_{\epsilon\to 0}\sum_{p\in D}\oint_{\partial B_\epsilon(p)}\bar{\partial}\log|\varphi|_h^2\,\mathrm{d}\bar{z}\end{aligned}$$

The minus sign is canceled by the change in orientation of the integration contours since ∂D and ∂D^C are equal but have reversed orientation. Since φ is meromorphic, near $p\in D$ we can write $\varphi = z^N u(z)$ for $u(z)\neq 0$ on $B_\epsilon(p)$. Therefore, we have,

$$|\varphi|_h^2 = |z|^{2N}|u(z)|^2 h(z)$$

which implies that,

$$\log|\varphi|_h^2 = N\log|z|^2 + \log|u(z)|^2 + \log h(z)$$

and thus,

$$\bar{\partial}\log|\varphi|_h^2 = \frac{N}{\bar{z}} + \bar{\partial}\log|u(z)|^2 + \bar{\partial}\log h(z)$$

About each $p \in D$ we can compute,

$$\lim_{\epsilon\to 0}\oint_{\partial B_\epsilon(p)} \bar{\partial}\log|\varphi|_h^2\,\mathrm{d}\bar{z} \;= \lim_{\epsilon\to 0}\oint_{|z|=\epsilon}\left[\frac{N}{\bar{z}} + \bar{\partial}\log|u(z)|^2 + \bar{\partial}\log h(z)\right]\,\mathrm{d}\bar{z}$$

Since both $\bar{\partial}\log|u(z)|^2$ and $\bar{\partial}\log h(z)$ are smooth and have no singularities on $B_\epsilon(p)$ so their integrals go to zero in the limit $\epsilon \to 0$. Therefore,

$$\lim_{\epsilon\to 0}\oint_{\partial B_\epsilon(p)} \bar{\partial}\log|\varphi|_h^2\,\mathrm{d}\bar{z} \;= \lim_{\epsilon\to 0}\oint_{|z|=\epsilon}\frac{N}{\bar{z}}\,\mathrm{d}\bar{z} \;= -\lim_{\epsilon\to 0}\overline{\oint_{|z|=\epsilon}\frac{N}{z}\,\mathrm{d}z} \;= -2\pi i N$$

Thus,

$$\int_X F_{\bar{z}z}\,\mathrm{d}z \,\wedge\, \mathrm{d}\bar{z} \;= \sum_{p\in D}\lim_{\epsilon\to 0}\oint_{\partial B_\epsilon(p)} \bar{\partial}\log|\varphi|_h^2\,\mathrm{d}z \;= -\sum_{p\in D} 2\pi i N_p$$

Which gives the theorem since N_p counts the multiplicity of each zero and the negative of the multiplicity of each pole. □

9 The Riemann-Roch Theorem

Definition: The canonical bundle, K_X of X, is the bundle of $\Lambda^{1,0}$ forms on X.

Definition: Let L be a line-bundle over X then,

$$H^0(X,L) = \{\varphi \in \Gamma(X,L) \mid \bar{\partial}\varphi = 0\}$$

is the space of holomorphic sections.

Definition: Let L be a line-bundle over X with a metric h. Then the first Chern class is defined to be,

$$c_1(L) = \int_X F_{\bar{z}z}\,\mathrm{d}z \,\wedge\, \mathrm{d}\bar{z}$$

Lemma 9.1. Let L be a line-bundle over X then $c_1(L^n) = nc_1(L)$ for any integer n.

Theorem 9.2 (Riemann-Roch)**.**

$$\dim H^0(X,L) - \dim H^0(X, L^{-1}\otimes K_X) = c_1(L) + \tfrac{1}{2}c_1(K_X^{-1})$$

Theorem 9.3. Let $\chi(X)$ be the Euler characteristic of X i.e. $\chi(X) = 2-2g$ where g is the genus of X. Then,

$$c_1(K_X^{-1}) = \chi(X) = 2-2g$$

Corollary 9.4. $\dim H^0(X, K_X) = g$ where g is the genus of X.

Proof. Apply Riemann-Roch to $L = K_X$,

$$\dim H^0(X, K_X) - \dim H^0(X, K_X^{-1} \otimes K_X) = c_1(K_X) + \tfrac{1}{2}c_1(K_X^{-1})$$

However, $K_X^{-1} \otimes K_X$ is the trivial bundle and X is compact so the trivial bundle has no nonconstant holomorphic sections. Thus, $\dim H^0(X, K_X^{-1} \otimes K_X) = 1$. Furhtermore, $c_1(K_X) = -c_1(K_X^{-1})$. Therefore,

$$\dim H^0(X, K_X) - 1 = -\tfrac{1}{2}c_1(K_X^{-1}) = -1 + g$$

□

9.1 Point Bundles and Construction on Meromorphic Forms

Let $P \in X$ be some point and fix a holomorphic coordinate chart U containing P with local coordinate z. For $k \in \mathbb{Z}$ define the bundle $[kP]$ by its single transition function $t_{0,\infty} : U \setminus \{P\} \to \mathbb{C}$ between the holomorphic charts U and $U_\infty = X \setminus \{P\}$.[1] We define this transition function via, $t_{0,\infty}(z) = z^k$. Furthermore, we may define the section $1_{kP} \in \Gamma(X, [kP])$ by $1_{kP}|_{U_\infty} = 1$ and $1_{kP}|_U(z) = z^k$. This is indeed a section of $[kP]$ because $t_{0,\infty} 1_{kP}|_{U_\infty} = 1_{kP}|_{U_\infty}$ since $1_{kP}|_{U_\infty}$ is the constant value 1 and, on $U \cap U_\infty$ we have $t_{0,\infty}(z) = 1_{kP}|_U(z) = z^k$.

Clearly, the section 1_{kP} has a unique pole at P with order k. Therefore, applying the theorem relating poles and zeros of meromorphic sections to the first Chern class of the corresponding bundle, we find,

$$c_1([kP]) = (\#\text{ zeros of } \varphi) - (\#\text{ poles of } \varphi) = k$$

Let $P, Q \in X$ be two distinct points. Consider the bundle $L = [-P] \otimes [-Q]$. Then we have shown that,

$$c_1(L) = c_1([-P]) + c_1([-Q]) = -2$$

and therefore every meromorphic section of L must have exactly two more poles than zeros which implies that it must have at least one pole and thus cannot be holomorphic. Therefore, $\dim H^0(X, L) = 0$. Applying the Riemann-Roch theorem,

$$\dim H^0(X, L) - \dim H^0(X, L^{-1} \otimes K_X) = c_1(L) + \tfrac{1}{2}c_1(K_X^{-1})$$

which implies that,

$$\dim H^0(X, [P] \otimes [Q] \otimes K_X) = 2 - \tfrac{1}{2}c_1(K_X^{-1})$$

Furthermore, taking $L = K_X$ in the Riemann-Roch theorem we find that,

$$\dim H^0(X, K_X) - \dim H^0(X, K_X^{-1} \otimes K_X) = c_1(K_X) + \tfrac{1}{2}c_1(K_X^{-1})$$

[1] If U_∞ is not a holomorphic chart but rather is covered by such charts then we may take transition functions equal to 1 between them.

However, $K_X^{-1} \otimes K_X$ is the trivial bundle whose holomorphic sections are simply holomorphic functions on X which must be constant since X is compact. Thus, $\dim H^0(X, K_X^{-1} \otimes K_X) = 1$. Furthermore, $c_1(K_X^{-1}) = -c_1(K_X)$ so we have,

$$\dim H^0(X, K_X) = 1 - \tfrac{1}{2}c_1(K_X)$$

Therefore,

$$\dim H^0(X, [P] \otimes [Q] \otimes K_X) = \dim H^0(X, K_X) + 1 = 2 - \tfrac{1}{2}c_1(K_X) = g + 1$$

where g turns out to be the genus of X. Take a basis of independent holomorphic sections of X, $\psi_, \dots, \psi_g$. Then $\psi_1 1_P 1_Q, \dots, \psi_g 1_P 1_Q$ are independnet holomorphic sections of the bundle $[P] \otimes [Q] \otimes K_X$. Since the dimension of the space of all such holomorphic sections has dimension $g + 1$ there exists an indpendnet holomorphic section $\Phi \in \Gamma(X, [P] \otimes [Q] \otimes K_X)$. Now, $\varphi = \Phi 1_P^{-1} 1_Q^{-1}$ is a meromorphic section of K_X since we are dividing out the dependence on the transition functions of $[P]$ and $[Q]$. I claim that φ is exactly the meromorphic 1-form with simple poles at P and Q we are looking for. First, since $\varphi 1_P 1_Q = \Phi$ is a holomorphic section, φ can only have poles at P and Q, the zeros of $1_P 1_Q$, and the poles at P and Q must be, at most, first-order. Thus, if either pole of φ exists it must be simple. Furthermore, φ cannot have a single simple pole since the sum of the residues of the poles of a meromorphic form on a complex Riemann surface must be zero but the residues at each simple pole must be nonzero. Thus, φ either has no poles or has simple poles exactly at P and Q. However, if φ had no poles it would be a holomorphic section of K_X implying that φ is a linear combination,

$$\varphi = \alpha_1 \psi_1 + \cdots + \alpha_g \psi_g$$

which implies that,

$$\Phi = \varphi 1_P 1_Q = \alpha \psi_1 1_P 1_Q + \cdots \alpha_g \psi 1_P 1_Q$$

contradicting the independence of Φ. Therefore, φ is a meromorphic section of K_X i.e. a meromorphic 1-form on X with simple poles at exactly P and Q.

9.2 Index Theorems

Let $L \to X$ be a holomorphic line bundle on X a compact Riemann surface. We have an operator $\bar{\partial} : \Gamma(X, L) \to \Gamma(X, L \otimes \bar{K}_X)$. We want to introduce the adjoint map which requires an inner product on these spaces of sections. A metric h on L and a metric $g_{\bar{z}z}$ on K_X^{-1} gives an L^2 metric on the spaces of sections via

$$\varphi \in \Gamma(X, L) \implies ||\varphi||^2 = \int_X \bar{\varphi} \varphi h$$
$$\varphi \in \Gamma(X, L \otimes \bar{K}_X) \implies ||\varphi||^2 = \int_X \bar{\varphi} \varphi h g_{\bar{z}z}$$

Then we define the adjoint via,

$$\left\langle \bar{\partial}\varphi, \psi \right\rangle_{L\otimes\bar{K}} = \left\langle \varphi, \bar{\partial}^\dagger \psi \right\rangle_L$$

Then we have maps,

$$\begin{aligned}
\Delta_+ &= \bar{\partial}^\dagger \bar{\partial} : \Gamma(X, L) \to \Gamma(X, L) \\
\Delta_- &= \bar{\partial}\bar{\partial}^\dagger : \Gamma(X, L \otimes \bar{K}_X) \to \Gamma(X, L \otimes \bar{K}_X)
\end{aligned}$$

I claim that if $\lambda \neq 0$ then λ is an eigenvalue of $\Delta_+ \iff \lambda$ is an eigenvalue of Δ_-. Suppose that $\Delta_+ \varphi = \lambda\varphi$ then $\bar{\partial}\bar{\partial}^\dagger\bar{\partial}\varphi = \lambda\bar{\partial}\varphi$. Thus, $\Delta_-\bar{\partial}\varphi = \lambda\bar{\partial}\varphi$ so $\bar{\partial}\varphi$ is an eigenvector of Δ_- with eigenvalue λ since $\bar{\partial}\varphi \neq 0$ otherwise $\Delta_+\varphi = 0$ implying $\lambda = 0$. The other direction is identical.

It remains to compare the zero eigenvalues of Δ_+ and Δ_-. We are interested in $\ker \Delta_+$ and $\ker \Delta_-$ because both operators constructed from $\bar{\partial}$ vanish exactly on holomorphic sections. Note that,

$$\dim\ker\Delta_+ - \dim\ker\Delta_- = \mathrm{Tr}\left(e^{-t\Delta_+}\right) - \mathrm{Tr}\left(e^{-t\Delta_-}\right)$$

Furthermore, the operator $e^{-t\Delta_+}$ satisfies the heat eqation $(\partial_t + \Delta_+)e^{-t\Delta_+} = 0$ with initial value $e^{-t\Delta_+}|_{t=0} = I$.

9.2.1 The Spaces of Holomorphic Sections

If $\varphi \in \ker\Delta_+$ then $\bar{\partial}^\dagger\partial\varphi = 0$ which implies that,

$$||\bar{\partial}\varphi||^2 = \left\langle \bar{\partial}\varphi, \bar{\partial}\varphi \right\rangle = \left\langle \varphi, \bar{\partial}^\dagger\bar{\partial}\varphi \right\rangle = 0$$

which implies that $\bar{\partial}\varphi = 0$. Since $H^0(X, L)$ is the space of holomorphic sections which is exactly a smooth section satisying the Cauchy-Riemann equation $\bar{\partial}\varphi = 0$, we have,

$$\varphi \in \ker\Delta_+ \iff \bar{\partial}\varphi = 0 \iff \varphi \in H^0(X, L)$$

implying that $\ker\Delta_+ = \ker\bar{\partial} = H^0(X, L)$.

To calculate the kernel of Δ_- we need to derive an expression for $\bar{\partial}^\dagger$. We require that, for $\varphi \in \mathcal{C}^\infty(X, L)$ and $\varphi \in \mathcal{C}^\infty(X, L \otimes \bar{K}_X)$ we have[2],

$$\left\langle \bar{\partial}\varphi, \psi \right\rangle_{L\otimes\bar{K}_X} = \left\langle \varphi, \bar{\partial}^\dagger\psi \right\rangle_L$$

Therefore,

$$\int_X h(\bar{\partial}\varphi)\bar{\psi} = \int_X \varphi\overline{(\bar{\partial}^\dagger\psi)}hg_{\bar{z}z}$$

[2]To define the true adjoint on a Hilbert space we must also impose the defining adjoint on the limits of smooth functions with may no longer be smooth. What we have defined is called the "formal adjoint."

However, $g_{\bar{z}z}$ is a metric on $\bar{K}_X^{-1}$ and therefore a postive section of $K_X \otimes \bar{K}_X$ which is a $1,1$-form on X. We can integrate the first expression by parts,

$$\begin{aligned}\int_X h(\bar{\partial}\varphi)\bar{\psi} &= -\int_X \varphi\bar{\partial}\left(h\bar{\psi}\right) \\ &= -\int_X \varphi\overline{\partial\left(h\psi\right)} = -\int_X \varphi h\overline{h^{-1}\partial(h\psi)}g^{\bar{z}z}g_{\bar{z}z} = -\int_X \varphi\overline{h^{-1}\partial(h\psi)}g^{\bar{z}z}hg_{\bar{z}z}\end{aligned}$$

Therefore,

$$\int_X \varphi\overline{(\partial^\dagger\psi)}hg_{\bar{z}z} = -\int_X \varphi\overline{h^{-1}\bar{\partial}(h\psi)}g^{\bar{z}z}hg_{\bar{z}z}$$

Since this must hold for all possible φ and ψ we must have,

$$\bar{\partial}^\dagger\psi = -g^{\bar{z}z}\left(h^{-1}\partial(h\psi)\right) = -g^{\bar{z}z}\nabla\psi$$

As before, we have $\ker\Delta_- = \ker\bar{\partial}^\dagger$ because if $\Delta_-\varphi = 0$ then,

$$\left\langle\bar{\partial}^\dagger\varphi, \bar{\partial}^\dagger\varphi\right\rangle = \left\langle\bar{\partial}\bar{\partial}^\dagger\varphi, \varphi\right\rangle = 0$$

which implies that $\bar{\partial}^\dagger\varphi = 0$. However,

$$\bar{\partial}^\dagger\psi = 0 \iff \partial(h\psi) = 0$$

since $g^{\bar{z}z}h^{-1}$ is nonvanishing. h is a nonvanishing section of $L^{-1}\otimes\bar{L}^{-1}$ so,

$$\psi \in \Gamma(X, L\otimes\bar{K}_X) \iff \bar{\Psi} = h\psi \in \Gamma(X, \bar{L}^{-1}\otimes\bar{K}_X) \iff \Psi = h\bar{\psi} \in \Gamma(X, L^{-1}\otimes K_X)$$

Therefore, $\psi \in \ker\bar{\partial}^\dagger \iff \Psi \in H^0(X, L^{-1}\otimes K_X)$ since,

$$\bar{\partial}^\dagger\psi = 0 \iff \partial(h\psi) = 0 \iff \bar{\partial}(h\bar{\psi}) = 0 \iff \bar{\partial}\Psi = 0$$

Thus, $\dim\ker\bar{\partial}^\dagger = \dim H^0(X, L^{-1}\otimes K_X)$ since there is a correspondence between their elements. We have shown,

$$\begin{aligned}\dim\ker\Delta_+ &= \dim\ker\bar{\partial} = \dim H^0(X, L) \\ \dim\ker\Delta_- &= \dim\ker\bar{\partial}^\dagger = \dim H^0(X, L^{-1}\otimes K_X)\end{aligned}$$

9.2.2 Exponential Laplacian Operators

Theorem 9.5. The space $L^2(X, L)$ admits an orthonormal basis $\{\varphi_j\}$ of eigenfunctions of Δ_+ with the following properties,

1. φ_j is smooth and $\Delta_+\varphi_j = \lambda_j\varphi_j$ for some $\lambda_j \geq 0$.
2. Each λ_j has finite multiplicity.
3. The set $\{\lambda_j\}$ has no acumulation point except ∞ and $\lambda_j \to 0$ at a rational rate i.e. $\lambda_j \geq Cj^p$ for some $p \in \mathbb{Q}^+$.

Similary, $L^2(X, L^{-1} \otimes K_X)$ admits an orthonormal basis $\{\psi_i\}$ of eigenfunctions of Δ_-.

Definition: The operator $e^{-t\Delta_+}$ is defined by its action on an arbitrary section $\varphi \in \Gamma(X, L)$ which may be written in the basis as,

$$\varphi = \sum_j c_j \varphi_j$$

which converges in the sense,

$$\left|\left|\varphi - \sum_{j=1}^{n} c_j \varphi_j\right|\right| \to 0$$

Now we define,

$$e^{-t\Delta_+}\varphi = \sum_j e^{-t\lambda_j} c_j \varphi_j$$

This converges in the Hilbert space since if the sequence $\{d_j = e^{-t\lambda_j} c_j\}$ is square summable if we take $t \geq 0$ since,

$$\sum_j |d_j|^2 = \sum_j e^{-2t\lambda_j} |c_j|^2 \leq \sum_j |c_j|^2 = ||\varphi||$$

since $\lambda_i \geq 0$ and thus $t\lambda_j \geq 0$.

The eigenfunctions of $e^{-t\Delta_+}$ are clearly the eigenfunctions $\{\varphi_j\}$ which have eigenvalues $e^{-t\lambda_j}$ and therefore, for any positive $t > 0$, we have,

$$\mathrm{Tr}\left(e^{-t\Delta_+}\right) = \sum_j e^{-t\lambda_j}$$

which implies that,

$$\mathrm{Tr}\left(e^{-t\Delta_+}\right) - \mathrm{Tr}\left(e^{-t\Delta_-}\right) = \dim\ker\Delta_+ - \dim\ker\Delta_- = \dim H^0(X, L) - \dim H^0(X, L^{-1} \otimes K_X)$$

9.2.3 Introducing the Heat Kernels

We can express any section $\varphi \in L^2(X, L)$ by,

$$\varphi = \sum_\lambda c_\lambda \varphi_\lambda$$

where $c_\lambda = \langle \varphi, \varphi_\lambda \rangle$. Therefore, we can write,

$$\begin{aligned}(e^{-t\Delta_+})\varphi(z) &= \sum_\lambda e^{-t\lambda} c_\lambda \varphi_\lambda(z) = \sum_\lambda e^{-t\lambda} \left(\int_X \varphi(w) \overline{\varphi_\lambda(w)} h(w) g_{\bar{w}w} \right) \varphi_\lambda(z) \\ &= \int_X \left\{ \sum_\lambda e^{-t\lambda} \varphi_\lambda(z) \overline{\varphi_\lambda(w)} h(w) g_{\bar{w}w} \right\} \varphi(w)\end{aligned}$$

Therefore we can write,

$$(e^{-t\Delta_+}\varphi)(z) = \int_X K_t^+(z,w)\varphi(w) \quad \text{with} \quad K_t^+(z,w) = \sum_\lambda e^{-t\lambda}\varphi_\lambda(z)\overline{\varphi_\lambda(w)}h(w)g_{\bar{w}w}$$

This section $K_t^+(z,w)$ is the heat kernel. We have $K_t^+ \in \Gamma(X, L_z \otimes L_w^{-1} \otimes \bar{K}_w \otimes K_w)$. Then we can write,

$$\mathrm{Tr}\left(e^{-t\Delta_+}\right) = \int K_t^+(z,z) = \sum_\lambda e^{-t\lambda}\int_X \varphi_\lambda(z)\overline{\varphi_\lambda(z)}h(z)g_{\bar{w}w} = \sum_\lambda e^{-t\lambda}\langle\varphi_\lambda,\varphi_\lambda\rangle = \sum_\lambda e^{-t\lambda}$$

We notice that the operator $e^{-t\Delta_+}$ satisfies the heat equation,

$$(\partial_t + \Delta_+)e^{-t\Delta_+} = 0$$

We will now investigate the theory of the heat equation. To solve the heat equation in local coordinates we need to ues Fourier transforms on the complex plane.

9.2.4 Complex Fouier Transforms

Consider a function $u : \mathbb{C} \to \mathbb{C}$ which is smooth and rapidly decreasing. Then we define the Fourier transform,

$$\hat{u}(\zeta) = \int_{\mathbb{C}} e^{-\pi i(z\bar{\zeta}+\zeta\bar{z})}u(z)\,\mathrm{d}^2 z$$

Then we have the Fourier inversion formula,

$$\hat{\hat{u}}(z) = u(-z)$$

or equivalently,

$$u(z) = \int_{\mathbb{C}} e^{i\pi(z\bar{\zeta}+\zeta\bar{z})}\hat{u}(\zeta)\,\mathrm{d}^2\zeta$$

Furthermore,

$$\widehat{\partial u}(\zeta) = \int_{\mathbb{C}} e^{-\pi i(z\bar{\zeta}+\zeta\bar{z})}\partial u(z)\,\mathrm{d}^2 z \; = -\int_{\mathbb{C}} \partial e^{-\pi i(z\bar{\zeta}+\zeta\bar{z})}u(z)\,\mathrm{d}^2 z \; = -\pi i\bar{\zeta}\hat{u}(\zeta)$$

9.2.5 Solving the Heat Equation

In local coordinates,

$$\bar{\partial}^\dagger\psi = -g^{z\bar{z}}\nabla_z\psi = -g^{z\bar{z}}h^{-1}\partial(h\psi) = -g^{z\bar{z}}(\partial\psi + (\partial\log h)\psi) = -g^{z\bar{z}}\left(\partial\psi + \Gamma_z^h\psi\right)$$

where $\Gamma_z^h = \partial\log h$. Then we have,

$$\Delta_+\varphi = \bar{\partial}^\dagger(\partial\varphi) = -g^{z\bar{z}}\left(\partial\bar{\partial}\varphi + \Gamma_z^h\bar{\partial}\varphi\right) = -g^{z\bar{z}}\partial\bar{\partial}\varphi - g^{z\bar{z}}\Gamma_z^h\bar{\partial}\varphi$$

We look for a solution $f(t) = e^{-t\Delta_+}u$ and $f(0) = u$ of the form,

$$f(t,z) = \int_{\mathbb{C}} e^{\pi i(z\bar{\zeta}+\zeta\bar{z})} a(t,z,\zeta)\hat{u}(\zeta)\,\mathrm{d}^2\zeta$$

and $a(0,z,\zeta) = 1$ such that $f(0,z) = u(z)$ by Fourier inversion. Fist compute,

$$\begin{aligned}
\Delta_+ \left(e^{\pi i(z\bar{\zeta}+\zeta\bar{z})}a\right) &= \left\{-g^{z\bar{z}}\partial\bar{\partial} - g^{z\bar{z}}\Gamma_z^h\bar{\partial}\right\}\left(e^{\pi i(z\bar{\zeta}+\bar{z}\zeta)}a\right) \\
&= -g^{z\bar{z}}\partial\left(e^{\pi i(z\bar{\zeta}+\bar{z}\zeta)}(\pi i\zeta a + \bar{\partial}a)\right) - g^{z\bar{z}}\Gamma_z^h e^{\pi i(z\bar{\zeta}+\bar{z}\zeta)}\left(\pi i\zeta a + \bar{\partial}a\right) \\
&= e^{\pi i(z\bar{\zeta}+\zeta\bar{z})}\left\{-g^{z\bar{z}}\left[(\pi i\bar{\zeta})(\pi i\zeta a + \bar{\partial}a) + (\pi i\zeta\partial a + \partial\bar{\partial}a\right] - g^{z\bar{z}}\Gamma_z^h\left(\pi i\zeta + \bar{\partial}a\right)\right\}
\end{aligned}$$

Define,

$$\kappa(a) = -g^{z\bar{z}}\left[(\pi i\bar{\zeta})(\pi i\zeta a + \bar{\partial}a) + (\pi i\zeta\partial a + \partial\bar{\partial}a)\right] - g^{z\bar{z}}\Gamma_z^h\left(\pi i\zeta + \bar{\partial}a\right)$$

then we have,

$$\Delta_+\left(e^{\pi i(z\bar{\zeta}+\zeta\bar{z})}a\right) = e^{\pi i(z\bar{\zeta}+\zeta\bar{z})}\kappa(a)$$

We compute,

$$(\partial_t + \Delta_+)f = \int_{\mathbb{C}} e^{\pi i(z\bar{\zeta}+\bar{z}\zeta)}\left\{\partial_t a + \kappa\right\}\hat{u}(\zeta)\,\mathrm{d}^2\zeta$$

We would like to solve $\partial_t a + \kappa(a) = 0$. At this point, we look for a series solution of the form,

$$a(t,z,\zeta) = e^{-\pi^2 g^{z\bar{z}}\zeta\bar{\zeta}t}\sum_k b_k(t,z,\zeta)$$

where b_k is a polynomial in $t,\zeta,\bar{\zeta}$ with terms $t^p\zeta^\alpha\bar{\zeta}^\beta$ such that $k = 2p - \alpha - \beta$ and coefficients which are smooth functions of z and $\bar{z}$. This implies that,

$$\partial_t a(t,z,\zeta) + g^{z\bar{z}}\pi^2\zeta\bar{\zeta}a = e^{-\pi^2 g^{z\bar{z}}\zeta\bar{\zeta}t}\sum_k \partial_t b_k(t,z,\zeta)$$

Therefore, the equation $\partial_t a + \kappa(a) = 0$ reduces to,

$$\sum_{k=0}^{\infty}\partial_t b_k + \sum_{k=0}^{\infty} c_1(b_k) + \sum_{k=0}^{\infty} d_0(b_k) = 0$$

where,

$$\begin{aligned}
c_1(b) &= -g^{z\bar{z}}\pi i\left(\bar{z}\bar{\partial}b + \zeta\partial b + \zeta\Gamma_z b - \pi^2 t\bar{\zeta}\zeta\left(\bar{\zeta}\partial g^{z\bar{z}} + \zeta\bar{\partial}g^{z\bar{z}}\right)b\right) \\
d_0(b) &= g^{z\bar{z}}\left(-\Gamma_z\partial b + \Gamma\pi^2 t\zeta\bar{\zeta}\partial g^{z\bar{z}}b - \partial\bar{\partial}b + \pi^2 i\zeta\bar{\zeta}\partial\bar{\partial}g^{z\bar{z}}b\right. \\
&\quad \left.+\pi^2 t\zeta\bar{\zeta}\partial g^{z\bar{z}}\partial b + \pi^2 t\zeta\bar{\zeta}\partial g^{z\bar{z}}\partial b - (\pi^2 t\zeta\bar{\zeta})^2\Gamma_z|\partial g^{z\bar{z}}|^2 b\right)
\end{aligned}$$

∂_t decreases the weight by 2 and c_1 decreases the weight by 1 while d_0 does not change the weight. Therefore, we should write,

$$\sum_{k=0}^{\infty}\partial_t b_k + \sum_{k=1}^{\infty} c_1(b_{k-1}) + \sum_{k=2}^{\infty} d_0(b_{k-2}) = 0$$

We require that this equation hold at each order,

$$\begin{aligned}
&\partial_t b_0 = 0 \\
&\partial_t b_1 + c_1(b_0) = 0 \\
&\partial_t b_2 + c_1(b_1) + d_0(b_0) = 0 \\
&\quad\vdots \\
&\partial_t b_k + c_1(b_{k-1}) + d_0(b_{k-2}) = 0
\end{aligned}$$

Thus we can take,

$$\begin{aligned}
b_0 &= 1 \\
b_1 &= -\int_0^1 c_1(b_0)\,\mathrm{d}t \\
b_2 &= -\int_0^1 [c_1(b_1) + d_0(b_0)]\,\mathrm{d}t \\
&\vdots \\
b_k &= -\int_0^1 [c_1(b_{k-1}) + d_0(b_{k-2})]\,\mathrm{d}t
\end{aligned}$$

which gives an inductive solution in formal power series. To get a convergent series we truncate,

$$a_N(t,z,\zeta) = e^{-\pi^2 t g^{z\bar{z}}\zeta\bar{\zeta}} \sum_{k\leq N} b_K(t,z,\zeta)$$

Then,

$$\partial_t a_N + \kappa(a_N) = e^{-\pi^2 t g^{z\bar{z}}\zeta\bar{\zeta}} E_N(t,z,\zeta)$$

where E_N has Weight $N+2$. Because we defined a such that,

$$f(t,z) = \int_{\mathbb{C}} e^{i\pi(z\bar{\zeta}+\zeta\bar{z})} a(t,z,\zeta)\hat{\phi}(\zeta)\mathrm{d}^2\zeta = \int_{\mathbb{C}} \left(\int_{\mathbb{C}} e^{i\pi[z(\bar{\zeta}-\bar{w})+\bar{z}(\zeta-w)]} a(t,z,\zeta)\,\mathrm{d}^2\zeta \right) \varphi(w)\mathrm{d}^2 w$$

Therefore, we have found the kernel,

$$K_t^+(t,z,\zeta) = \int_{\mathbb{C}} e^{i\pi[z(\bar{\zeta}-\bar{w})+\bar{z}(\zeta-w)]} a(t,z,\zeta)\,\mathrm{d}^2\zeta$$

Therefore,

$$K_t^+(z,z) = \int_{\mathbb{C}} a(t,z,\zeta)\,\mathrm{d}^2\zeta \; = \int_{\mathbb{C}} e^{-\pi^2 t g^{z\bar{z}}\zeta\bar{\zeta}} \sum_{k=0}^{N} b_k(t,z,\zeta)\,\mathrm{d}^2\zeta$$

Since b_k for $k \geq 3$ depends on t^p for $p \geq 2$ then in the limit $t \to 0$ these terms die under the integral even under the change of variables $\eta = t^{1/2}\zeta$ which makes the numerator t invariant. Therefore, we need only consider $b_0 + b_1 + b_2$ if we will take the limit $t \to 0$. We notice that only terms propostional to 1 or $\zeta\bar{\zeta}$ can contribute since the integral is rotationally invariant. Furthermore, only covariant terms may remain since $K_t^+(z,z)$ is the section of a line bundle. Furthermore, since b_1 is first-order in ζ, by symmetry it integrates to zero. Dropping such terms we find,

$$b_2^+(t,z,\zeta) = \pi^2 \left(\tfrac{1}{2}t^2|\zeta|^2 F - \tfrac{1}{2}t^2|\zeta|^2 R + \tfrac{1}{3}\pi^3 t^3|\zeta|^4 R\right)$$

where $F = -\partial\bar{\partial}\log h$ and $R = -\partial\bar{\partial}\log g^{z\bar{z}}$ are the curvatures of the bundles L and K_X^{-1}. Therefore,

$$K_t^+(z,z) = \int_{\mathbb{C}} e^{-\pi^2 t g^{z\bar{z}}\zeta\bar{\zeta}}(b_0 + b_2)\,\mathrm{d}^2\zeta \;= \frac{\kappa_1}{\pi^2 t} + \frac{\kappa_2}{\pi^2}F(z) + \frac{\kappa_3}{\pi^2}R(z)$$

where

$$\begin{aligned}
\kappa_1 &= \int e^{-\zeta\bar{\zeta}}\,\mathrm{d}^2\zeta \\
\kappa_2 &= \int e^{-\zeta\bar{\zeta}}\tfrac{1}{2}|\zeta|^2\,\mathrm{d}^2\zeta \\
\kappa_3 &= \int e^{-\zeta\bar{\zeta}}\left(-\frac{1}{2}|\zeta|^2 + \tfrac{1}{3}|\zeta|^4\right)\,\mathrm{d}^2\zeta
\end{aligned}$$

9.2.6 Computing the Heat Kernels

10 Function Theory on General Riemann Surfaces

Let X be a compact Riemann surface of genus g. By Riemann-Roch,

$$\dim H^0(X, K_X) = g$$

We first fix a basis of homology cycles i.e. representatives for the homology classes of X. There are $2g$ such homology cycles. Once we choose representatives, we form the surface with boundary X_{cut} by cutting along the cycles. Call these cycles A_I and B_I. Since we have g independent holomorphic forms we may fix a basis ω_I such that,

$$\oint_{A_J} \omega_I = \delta_{IJ}$$

this fixes the period matrix,

$$\Omega_{IJ} = \oint_{B_J} \omega_I$$

The period matrix is symmetric, $\Omega_{IJ} = \Omega_{JI}$ and $\mathrm{Im}(\Omega) > 0$. We construct the Abel map, for $p_0 \in X$ we construct the map $I : X \to \mathbb{C}^g/\Lambda$ by,

$$p \mapsto \left(\int_{p_0}^p \omega_1, \ldots, \int_{p_0}^p \omega_g\right)$$

where $\Lambda = \{m_I + \Omega_{IL} n_L \mid m_I, n_L \in \mathbb{Z}\}$. Therefore, we identify X with a g-dimensional torus $\mathbb{C}^g/\Lambda$. To obtain meromorphic functions on X we cnonsider θ-functions of $\mathbb{C}^g$ and restrict them to $I(X)$. We fix $\zeta \in \mathbb{C}^g$ and then define our trial function on X by,

$$f(p) = \theta\left(\zeta + \int_{p_0}^{p} \omega \,\middle|\, \Omega\right)$$

where we define the g-dimensional θ-function as,

$$\theta(\zeta|\Omega) = \sum_{m \in \mathbb{Z}^g} e^{\pi i \langle m, \Omega m\rangle + 2\pi i \langle m, \zeta\rangle}$$

with $\zeta \in \mathbb{C}^g$.

Proposition 10.1. The θ-function transforms as,

$$\theta(\zeta + n \mid \Omega) = \theta(\zeta \mid \Omega)$$

and

$$\theta(\zeta + \Omega n \mid \Omega) = e^{-\pi i \langle n, \Omega n\rangle - 2\pi i \langle n, \zeta\rangle} \theta(\zeta \mid \Omega)$$

Now we need to work out the transformation properties of our induced function f.

Proposition 10.2. The function f on X_{cut} transforms as,

$$f(z + A_K) = \theta\left(\zeta_I + \int_{p_0}^{z} \omega_I + \int_{A_K} \omega_I \,\middle|\, \Omega\right) = \theta\left(\zeta_I + \int_{p_0}^{z} \omega_I + \delta_{IK} \,\middle|\, \Omega\right)$$

Using the first transformation property of θ,

$$f(z + A_K) = \theta\left(\zeta_I + \int_{p_0}^{z} \omega_I \,\middle|\, \Omega\right) = f(z)$$

Furthermore,

$$f(z + B_K) = \theta\left(\zeta_I + \int_{p_0}^{z} \omega_I + \int_{B_K} \omega_I \,\middle|\, \Omega\right) = \theta\left(\zeta_I + \int_{p_0}^{z} \omega_I + \Omega_{IL}\delta_{LK} \,\middle|\, \Omega\right)$$

Using the second transformation property of θ with $n = \delta_{IK}$, we find,

$$f(z + B_K) = e^{-\pi i \Omega_{KK} - 2\pi i \left(\zeta_K + \int_{p_0}^{z} \omega_K\right)} f(z)$$

Proposition 10.3. Either $f(z) \equiv 0$ on X or f has exactly g zeros $p_1, \ldots, p_g$ satisfying $I(p_1) + \cdots + I(p_g) = \Delta - \zeta$ for some fixed vector Δ.

Proof. The number of zeros of f is computed via,

$$\oint_{\partial X_{\mathrm{cut}}} \frac{\mathrm{d}f\,(z)}{f(z)} = \sum_{L=1}^{g} \left[\oint_{A_L} \left(-\frac{\mathrm{d}f\,(z + B_L)}{f(z + B_L)} + \frac{\mathrm{d}f\,(z)}{f(z)}\right) + \oint_{B_L} \left(\frac{\mathrm{d}f\,(z + A_L)}{f(z + A_L)} - \frac{\mathrm{d}f\,(z)}{f(z)}\right)\right]$$

Due to the first transformation property, the second term vaishes. We may rewrite the second transformation property as,

$$\log f(z+B_K) = -\pi i \Omega_{KK} - 2\pi i \left(\zeta_K + \int_{p_0}^{z} \omega_K \right) + \log f(z)$$

Therefore,

$$\mathrm{d}f(z+B_L)\, f(z+B_L) = -2\pi i \omega_L + \frac{\mathrm{d}f\,(z)}{f(z)}$$

This implies that,

$$\oint_{\partial X_{\text{cut}}} \frac{\mathrm{d}f\,(z)}{f(z)} = 2\pi i \sum_{L=1}^{g} \oint_{A_L} \omega_L = 2\pi i g$$

Therefore, f has exactly g zeros. To complete the prood we reintroducte the Abelian integrals. □

10.1 Abelian Integrals

Define the Abelian integral on X_{cut},

$$g_I(z) = \int_{p_0}^{z} \omega_I$$

which is well-defined and transforms as,

$$g_I(z+A_K) = \int_{p_0}^{z} \omega_I + \oint_{A_K} \omega_I = g(z) + \delta_{IK}$$

and

$$g_I(z+B_K) = \int_{p_0}^{z} \omega_I + \oint_{B_K} \omega_I = g(z) + \Omega_{IK}$$

Let C be a contour contianing all the poles of f. Using the Residue Theorem,

$$\oint_C g_I(z) \frac{\mathrm{d}f}{f} = 2\pi i \sum_{j=1}^{g} g_I(p_j) = 2\pi i \left[I(p_1) + \cdots + I(p_g) \right]_I$$

However, we can compute this integral by extending the curve to the boundary of X_{cut} and computing,

$$\begin{aligned}\oint_{\partial X_{\text{cut}}} g_I(z) \frac{\mathrm{d}f\,(z)}{f(z)} = \sum_{L=1}^{g} \Bigg[\oint_{A_L} \left(-g_I(z+B_L) \frac{\mathrm{d}f\,(z+B_L)}{f(z+B_L)} + g_I(z) \frac{\mathrm{d}f\,(z)}{f(z)} \right) \\ + \oint_{B_L} \left(g_I(z+A_L) \frac{\mathrm{d}f\,(z+A_L)}{f(z+A_L)} - g_I(z+A_L) \frac{\mathrm{d}f\,(z)}{f(z)} \right) \Bigg]\end{aligned}$$

The first term becomes,

$$\oint_{A_L}\left(-g_I(z+B_L)\frac{\mathrm{d}f\,(z+B_L)}{f(z+B_L)}+g_I(z)\frac{\mathrm{d}f\,(z)}{f(z)}\right)$$
$$=\oint_{A_L}\left(-[g_I(z)+\Omega_{IL}]\left(\frac{\mathrm{d}f\,(z)}{f(z)}-2\pi i\omega_L(z)\right)+g_I(z)\frac{\mathrm{d}f\,(z)}{f(z)}\right)$$
$$=\oint_{A_L}\left(2\pi i\omega_L(z)g_I(z)+2\pi i\omega_L(z)\Omega_{IL}-\Omega_{IL}\frac{df(z)}{f(z)}\right)$$
$$=2\pi i\oint_{A_L}\omega_L(z)g_I(z)+2\pi i\Omega_{IL}-\Omega_{IL}(2\pi i n_L)$$

where I have,

$$n_L=\frac{1}{2\pi i}\oint_L\frac{\mathrm{d}f\,(z)}{f(z)}$$

which is an integer because it is the change of the log going about a closed loop in X. The second term becomes,

10.2 The Enhanced Abel Map

Definition: The space $\mathrm{Sym}^g(X)$ is the quotient space $\mathrm{Sym}^g(X)=X^g/\sim$ where $\sim$ removes ordering by $(x_1,\dots,x_g)\sim(x_{\sigma(1)},\dots,x_{\sigma(g)})$ for any permutation $\sigma\in S_g$. Then $\mathrm{Sym}^g(X)$ is a g-manifold.

Theorem 10.4 (Jacobi Inversion)**.** The Abel map $I:\mathrm{Sym}^g(X)\to\mathbb{C}^g/\Lambda$ given by $I(p_1,\dots,p_g)=I(p_1)+\cdots+I(p_g)$ is onto. Furthermore, for a given ζ any point in the g-tuple satisfying $I(p_1)+\cdots+I(p_g)=\Delta-\zeta$ satisfies,

$$\theta\left(\zeta+\int_{p_0}^{p_j}\omega\,\middle|\,\Omega\right)=0$$

for each $j=1,\dots,g$. Furthermore, let Z be the variety of ζ for which $f\equiv 0$ identically. Then for $\zeta\notin Z$ the solution $(p_1,\dots,p_g)$ of $I(p_1)+\cdots+I(p_g)=\Delta-\zeta$ is unique.

Proof. Given any $v\in\mathbb{C}^g/\Lambda$ take $\zeta=\Delta-v$. If $\zeta\notin Z$ then the zeros of $f(p)$ satisfy $I(p_1)+\cdots+I(p_g)=\Delta-\zeta=v$ so $I(p_1,\dots,p_g)=v$. However, $I(\mathrm{Sym}^g(X))$ contains the complement of Z which is open and $I(\mathrm{Sym}^g(X))$ is a closed g-dimensional subvariety. Thus, $I(\mathrm{Sym}^g(X))=\mathbb{C}^g/\Lambda$.

Consider the subvariety,

$$W=\left\{(p_1,\dots,p_g,\zeta)\in\mathrm{Sym}^g(X)\times I(X)\,\middle|\,I(p_1)+\cdots+I(p_g)=\Delta-\zeta\text{ and }\theta\left(\zeta+\int_{p_0}^{p}\omega\,\middle|\,\Omega\right)=0\right\}$$

If $\zeta\notin Z$ then there exists $(p_1,\dots,p_g,\zeta)\in W$ so $\dim W\geq g$ since Z has codimension greater than 1. Furthermore, if $(p_1,\dots,p_g,\zeta)\in W$ then the projection $W\to\mathrm{Sym}^g(X)$ by $(p_1,\dots,p_g,\zeta)\mapsto(p_1,\dots,p_g)$ is injective since,

$$\zeta=\Delta-[I(p_1)+\cdots+I(p_g)]$$

Therefore $\dim W \le \dim \mathrm{Sym}^g(X) = g$ implying that $\dim W = g$ so the projection is surjective since its image is a closed g-dimensional subvariety. Thus, $W \cong \mathrm{Sym}^g(X)$. Thus, sending $(p_1, \dots, p_g) \mapsto (p_1, \dots, p_g, \zeta)$ with the only possible ζ must lie in W so we have,

$$\theta\left(\zeta + \int_{p_0}^{p} \omega \,\middle|\, \Omega\right) = 0$$

This proves the uniqueness as well because there are exactly g zeros of $f(p)$ and these are exactly the soltions to,

$$I(p_1) + \cdots + I(p_g) = \Delta - \zeta$$

□

Theorem 10.5. Let $e \in \mathbb{C}^g$ satisfy $\theta(e|\Omega) = 0$ and $f(p)$ defined by θ is not identically zero on X. Define,

$$E_e(x,y) = \theta\left(e + \int_{x}^{y} \omega \,\middle|\, \Omega\right)$$

Then there exists points $z_1, \dots, z_{g-1}$ and $w_1, \dots, w_{g-1}$ such that,

$$E_e(x,y) = 0 \iff x = y \text{ or } x = z_i \text{ or } y = w_i$$

Proof. Fix $x \in X$ such that $E_e(x,y)$ is not identically zero as a function of y. We can write,

$$E_e(x,y) = \theta\left(e + \int_{x}^{p_0} \omega + \int_{p_0}^{y} \omega \,\middle|\, \Omega\right)$$

By the lemma, there exist $y_1, \cdots, y_g$ with $E_e(x, y_i) = 0$ and,

$$I(y_1) + \cdots I(y_g) = \Delta - \left(e + \int_{x}^{p_0} \omega\right)$$

Since $y = x$ is a root of $E_e(x,y)$ then we may set $y_g = x$. Furthermore,

$$I(x) = \int_{p_0}^{x} \omega$$

and thus,

$$I(y_1) + \cdots + I(y_{g-1}) = \Delta - e$$

since the extra terms cancel. Therefore, $y_1, \cdots y_{g-1}$ are independent of x. So these become the fixed points $w_1, \dots, w_{g-1}$. The same argument holds for the first coordinate. □

Definition: We can generalize the g-dimensional θ-function to,

$$\theta[\delta'\delta''](z|\Omega) = \sum_{n \in \mathbb{Z}} e^{\pi i \langle n+\delta',\, \Omega(n+\delta')\rangle + 2\pi i \langle z+\delta',\, z+\delta''\rangle}$$

Definition: Define the function,

$$E(x,y) = \frac{\theta[\delta]\left(\int_x^y \omega \mid \Omega\right)}{h_\delta(x)h_\delta(y)}$$

where $h_\delta(x)$ has zeros at precisely $z_1, \ldots, z_{g-1}$ and $h_\delta(y)$ has zeros at precisely $w_1, \ldots, w_{g-1}$ which are actually the same because θ is even and thus the numerator is symmetric in x and y. However, if h has no poles then h cannot be a function it must be a section of a line bundle S_δ with $c_1(S_\delta) = g - 1$.

Theorem 10.6. Let $\Theta = \{\zeta \mid \theta(\zeta|\Omega) = 0\}$ then $\Theta = \Delta - I(\mathrm{Sym}^{g-1}(X))$.

Proof. Assume that $\zeta \in \Delta - I(\mathrm{Sym}^{g-1}(X))$ then $\zeta = \Delta - [I(p_1) + \cdots + I(p_{g-1})]$. Pick a point p_g then we have,

$$\eta = \zeta - I(p_g) = \Delta - [I(p_1) + \cdots + I(p_g)]$$

Therefore, the map,

$$f(z) = \theta\left(\eta + \int_{p_0}^{z} \omega \,\middle|\, \Omega\right)$$

has zeros at exactly $p_1, \cdots, p_g$. Take $z = p_g$ then,

$$\theta(\zeta|\Omega) = \theta(\eta + I(p_g)|\Omega) = \theta\left(\eta + \int_{p_0}^{p_g} \omega \,\middle|\, \Omega\right) = 0$$

Thus $\zeta \in \Theta$. Then equality holds by dimension counting. $\square$

Definition: Let S be a holomorphic line bundle. S is called a spin bundle if $S^2 = S \otimes S = K_X$.

10.2.1 Key Facts about Spin Bundles on Riemann Surfaces

Let $I(S) = I(p_1 + \cdots + p_N - q_1 - \cdots - q_N)$ where p_i, q_j are the poles and zeros of meromorphic sections of S.

1. Given a bundle S consider $\Theta + I(S) - \Delta$. Then S is a spin bundle iff $\Theta + I(S) - \Delta$ is symmetric in the sense that $\zeta \in V \iff -\zeta \in V$.

2. A translate of Θ by $\delta' + \Omega\delta''$ is symmetric $\iff \delta', \delta'' \in (\frac{1}{2}\mathbb{Z})^g$

3. Θ is symmetric so Θ corresponds to a spin bundle S_0 with $I(S_0) = \Delta$

4. To each $\delta = (\delta', \delta'')$ we can consider $\theta(\delta' + \Omega\delta'' + \zeta|\Omega)$ or $\theta[\delta', \delta''](\zeta|\Omega)$. We find,

   $$\theta[\delta'\delta''](-\zeta|\Omega) = (-1)^{\delta' \cdot \delta''}\theta[\delta'\delta''](\zeta|\Omega)$$

   We say S is an even/odd spin bundle if $\theta[\delta]\zeta|\Omega)$ is even/odd.

5. There are 2^{2g} spin bundles.

10.3 Classifying Line Bundles

Definition: $\mathrm{Pic}_k = \{L \text{ line bundle over } X \mid c_1(L) = k\}$

10.3.1 Line Bundles of Chern Class Zero

Consider Pic_0. Let L have $c_1(L) = 0$. Then for any metric h on L we have,

$$c_1(L) = \frac{i}{2\pi} \int_X F_h \, \mathrm{d}z \, \wedge \, \mathrm{d}\bar{z}$$

where $F_h = -\partial\bar{\partial} \log h$. Now consider a new metric $\tilde{h} = he^{-u}$ for a scalar holomorphic function u. Then,

$$F_{\tilde{h}} = -\partial\bar{\partial} \log \tilde{h} = -\partial\bar{\partial} \log h + \partial\bar{\partial} u$$

We want to solve the equation,

$$\partial\bar{\partial} u + F_h = 0 \iff \partial\bar{\partial} u = -F_h$$

which we have shown is solvalbe because,

$$\int_X (-F_h) \, \mathrm{d}z \, \wedge \, \mathrm{d}\bar{z} \, = 2\pi i \, c_1(L) = 0$$