

# Mathematics GU4042 Modern Algebra II

## Assignment # 2

Benjamin Church

December 21, 2017

### Page 115.

1. Let  $\phi : R \rightarrow S$  be a ring homomorphism and let  $A \subset R$  be a subring. For  $x, y \in \phi(A)$  we have  $\exists a, b \in A$  s.t.  $\phi(a) = x$  and  $\phi(b) = y$ .

Now,  $a + b, ab, 1_R \in A$  so  $\phi(a + b) = \phi(a) + \phi(b) = x + y \in \phi(A)$  and  $\phi(ab) = \phi(a)\phi(b) = xy \in \phi(A)$ . Also  $\phi(1_R) = 1_S \in \phi(A)$ . Finally,  $-x = -\phi(a) = \phi(-a) \in \phi(A)$  because  $a \in A \implies -a \in A$ . Therefore,  $\phi(A)$  is a subring of  $S$ .

2. Let  $\phi : R \rightarrow S$  be a ring homomorphism and let  $B \subset S$  be a subring. Let  $x, y \in \phi^{-1}(B)$  then  $\phi(x), \phi(y) \in B$  thus,  $\phi(x) + \phi(y) \in B$  so  $\phi(x + y) \in B$  equivalently,  $x + y \in \phi^{-1}(B)$ . Also,  $\phi(x)\phi(y) \in B$  so  $\phi(xy) \in B$ . Therefore,  $xy \in \phi^{-1}(B)$ . Also  $\phi(1_R) = 1_S \in B$  so  $1_S \in \phi^{-1}(B)$ . Finally,  $\phi(-x) = -\phi(x) \in B$  so  $-x \in \phi^{-1}(B)$ . Therefore,  $\phi^{-1}(B)$  is a subring of  $R$ .
3. Let  $\phi : R \rightarrow S$  be a surjective ring homomorphism and let  $I \subset R$  be an ideal. By problem 1, since  $I$  is an additive subgroup of  $R$  then  $\phi(I)$  is an additive subgroup of  $S$ . Take  $x \in \phi(I)$  and  $s \in S$  thus  $\exists a \in I$  s.t.  $x = \phi(a)$ . Since  $\phi$  is surjective,  $\exists r \in R$  s.t.  $s = \phi(r)$ . Now  $sx = \phi(r)\phi(a) = \phi(ra) \in \phi(I)$  because  $ra \in I$  by absorption. Similarly,  $xs = \phi(a)\phi(r) = \phi(ar) \in \phi(I)$  because  $ar \in I$  by absorption. Therefore,  $\forall x \in \phi(I), \forall s \in S : xs, sx \in \phi(I)$  so  $\phi(I)$  is an ideal of  $S$ .
4. Let  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  be the unique ring homomorphism from  $\mathbb{Z}$ .  $\phi(1) = (1, 1)$  so  $\phi(n) = ([n]_2, [n]_2)$ . Now consider the ideal  $(3) \subset \mathbb{Z}$ . However,  $\phi((3)) = \{(0, 0), (1, 1)\}$  is not an ideal because  $(0, 1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $(1, 1) \in \phi((3))$  but  $(0, 1) \cdot (1, 1) = (0, 1) \notin \phi((3))$ .
5. Let  $\phi : R \rightarrow S$  be a ring homomorphism and let  $J \subset S$  be an ideal. By problem 2, since  $J$  is an additive subgroup of  $S$  then  $\phi^{-1}(J)$  is an additive subgroup of  $R$ . Take  $x \in \phi^{-1}(J)$  then  $\phi(x) \in J$  and take  $r \in R$ . Now  $\phi(rx) = \phi(r)\phi(x) \in J$  and  $\phi(xr) = \phi(x)\phi(r) \in J$  by the absorption property of  $J$ . Thus,  $rx, xr \in \phi^{-1}(J)$  so  $\phi^{-1}(J)$  is an ideal of  $R$ .
6. Let  $F = \{a + ib \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}$ . Now  $F$  is a subring of  $\mathbb{C}$  if it is closed under addition and multiplication and contains inverses and identities.

Let  $x, y \in F$  then  $x = a_x + ib_x$  and  $y = a_y + ib_y$  thus  $x + y = (a_x + a_y) + i(b_x + b_y) \in F$  because  $a_x + a_y, b_x + b_y \in \mathbb{Q}$ . Also,  $xy = (a_x a_y - b_x b_y) + i(a_x b_y + b_x a_y) \in F$  because  $a_x a_y - b_x b_y \in \mathbb{Q}$  and  $a_x b_y + b_x a_y \in \mathbb{Q}$ .

Also,  $1 \in F$  since  $1 = 1+i0$  and  $-x = -a-ib \in F$  because  $1, 0, -a, -b \in \mathbb{Q}$ . Since  $x+(-x) = 0$  and  $-x+x = 0$ , inverses hold. Furthermore,  $F$  is a field because for any  $x \in F \setminus \{0\}$ ,  $x = a+ib$  then take  $y = \frac{1}{a^2+b^2}(a-ib)$ . Now,  $xy = yx = (a+ib)(a-ib)\frac{1}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$ . This exists because  $a^2+b^2 = 0$  only when  $a = b = 0$  i.e.  $x = 0$  which is the case we excluded.

Let  $\phi : \mathbb{Z}[i] \rightarrow F$  be the ring homomorphism given by  $\phi(x) = x$  which is trivially injective. Now any  $x \in F$  can be written as  $x = \frac{p_1}{q_1} + i\frac{p_2}{q_2} = \frac{p_1q_2}{q_1q_2} + i\frac{p_2q_1}{q_1q_2}$  with  $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ . Then,  $x = \phi(p_1q_2 + ip_2q_1)\phi(q_1q_2)^{-1}$  which is defined because  $q_1 \neq 0$  and  $q_2 \neq 0$  and since  $\mathbb{Z}$  is a domain,  $q_1q_2 \neq 0$ . Also,  $\phi$  is injective so  $\phi(q_1q_2) \neq 0$ . Therefore, by the universal mapping property of the field of fractions,  $F \cong Q_{\mathbb{Z}[i]}$ .