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Lemma 3.0.1. Let $I, J \subset R$ be ideals. Then,

$$V(IJ) = V(I \cap J) = V(I) \cup V(J)$$

Proof. If $I \subset \mathfrak{p}$ then $\mathfrak{p} \supset I \cap J \subset IJ$ so it is clear that,

$$V(I) \cup V(J) \subset V(I \cap J) \subset V(IJ)$$

Thus suppose that $\mathfrak{p} \supset IJ$ but $\mathfrak{p} \notin V(I) \cup V(J)$. Then there is $x \in I$ and $y \in J$ such that $x, y \notin \mathfrak{p}$ so that $\mathfrak{p} \not\supset I$ and $\mathfrak{p} \not\supset J$. Then $xy \in IJ \subset \mathfrak{p}$ so $xy \in \mathfrak{p}$ contradicting the primality of $\mathfrak{p}$ and proving the claim. □

Proposition 3.0.2. Let R be a comutative ring and $I, J \subset R$ are ideals. If any of the following are true,

(a) $I + J = R$

(b) $\text{nilrad}\,(R/IJ) = (0)$

then $I \cap J = IJ$.

Proof. If $I + J = R$ then for any $r \in I \cap J$ consider $1 = x + y$ with $x \in I$ and $y \in J$ and $r = rx + ry \in IJ$ so $I \cap J \subset IJ \subset I \cap J$ proving equality.

Now suppose that $\text{nilrad}\,(R/IJ) = (0)$. Consider the ideal $(I \cap J)/IJ \subset R/IJ$. I claim that it is contained in the nilradical. Indeed, for any prime $\mathfrak{p}$ of R/IJ, that is a prime of R above IJ because $V(IJ) = V(I \cap J)$ and thus $(I \cap J)/IJ \subset \text{nilrad}\,(R/IJ)$ so $I \cap J = IJ$. □