Complex Analysis and Riemann Surfaces II
Final Exam

Ben Church

October 23, 2020

1 Problem 1

Remark 1. In my notation $q > n$ is the number written as p in the problem statement. I realized too late for it to be feasible to change notation since p appears in many formulae we derived in class which I used. I apologize for any confusion this may cause.

Theorem 1.1. Let (M, ω) be a compact Kähler manifold of dimension n and let $f \in C^0(M)$ satisfy,

$$\int_M e^f \omega^n = \int_M \omega^n$$

Consider the complex Monge-Ampere equation,

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^f \omega \qquad \omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0 \qquad \sup_M \varphi = 0$$

Choose any real $q > n$ then there exists a constant $C(M, \omega, ||e^f||_q)$ only depending on M, ω, and $||e^f||_q$ such that,

$$||\varphi||_{C^0(M)} \leq C$$

Remark 2. I will use the notation $||\psi||_p = ||\psi||_{L^p(M)}$ and $|\psi||_{C^0} = ||\psi||_{C^0(M)}$.

Remark 3. The proof will require multiple steps. Let $r = \frac{q}{q-1} < \frac{n}{n-1} \leq 2$ and take any $w_0 > r$. We will split up the proof into a number of steps:

1. Use Moser iteration via the Sobolev Inequality to establish the bound,

$$||\varphi||_{C^0} = ||\varphi||_{L^\infty} \leq C_1(M, \omega, ||e^f||_q) \cdot ||\varphi||_{L^{w_0}} + 1$$

2. Use the Poincare Inequality to find, for a specific $w_0 > r$, a bound,

$$||\varphi - \varphi_{\text{avg}}||_{L^{w_0}} \leq C_2(M, \omega, ||e^f||_q) \cdot ||\varphi||_{L^1}^2$$

3. Apply Green's functions to show that,

$$||\varphi||_{L^1} \leq C_3(M, \omega, ||e^f||_q)$$

1.1 Step 1: Moser Iteration

It will be convenient in this section to define $\phi = 1 - \varphi$ such that $\phi \geq 1$ and thus $\phi^a \leq \phi^b$ when $a \leq b$.

From the defining equation, for any $p \geq 1$,

$$\int_M \phi^{p-1}(\omega_\phi^n - \omega^n) = \int_M \phi^{p-1}\left(e^f - 1\right)\omega^n$$

Recall the Holder inequality,

$$\frac{1}{r} + \frac{1}{q} = 1 \implies ||fg||_1 \leq ||f||_r ||g||_q$$

Therefore, if we choose $r > 1$ such that,

$$\frac{1}{r} + \frac{1}{q} = 1 \implies r = \frac{q}{q-1} < \frac{n}{n-1}$$

then we find,

$$\int_M \phi^{p-1}\left(e^f - 1\right)\omega^n \leq ||\phi^{p-1}(e^f - 1)||_1 \leq ||\phi^{p-1}||_r \cdot ||(e^f - 1)||_q \leq ||\phi^{p-1}||_r \cdot (||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}})$$

On the other hand,

$$\begin{aligned}
\phi^{p-1}\left(\omega_\varphi^n - \omega^n\right) &= \phi^{p-1}(\omega_\varphi - \omega) \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right) \\
&= \phi^{p-1} i\partial\bar{\partial}\varphi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right) \\
&= -\phi^{p-1} i\partial\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)
\end{aligned}$$

since, $i\partial\bar{\partial}\varphi = -i\partial\bar{\partial}\phi$. Furthermore, because ω and ω_φ are Kähler forms and thus closed,

$$\begin{aligned}
\mathrm{d}\left(\phi^{p-1} i\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)\right) &= (p-1)\phi^{p-2}\,\mathrm{d}\phi\, \wedge i\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right) \\
&\quad + \phi^{p-1} i\,\mathrm{d}(\bar{\partial}\phi)\, \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right) \\
&= (p-1)\phi^{p-2}(\partial\phi + \bar{\partial}\phi) \wedge i\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right) \\
&\quad + \phi^{p-1} i(\partial\bar{\partial}\phi + \bar{\partial}^2\phi) \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)
\end{aligned}$$

However, $\bar{\partial}\phi \wedge \bar{\partial}\phi = 0$ and $\bar{\partial}^2\phi = 0$. Therefore,

$$\begin{aligned}
\mathrm{d}\left(\phi^{p-1} i\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)\right) &= (p-1)\phi^{p-2}\partial\phi \wedge i\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right) \\
&\quad + \phi^{p-1} i\partial\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)
\end{aligned}$$

Then applying Stokes theorem on the closed manifold M we find,

$$-\int_M \phi^{p-1} i\partial\bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right) = \int_M (p-1)\phi^{p-2} i\partial\phi \wedge \bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)$$

and therefore,

$$\int_M \phi^{p-1}(\omega_\varphi^n - \omega^n) = \int_M (p-1)\phi^{p-2} i\partial\phi \wedge \bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)$$

However, note that,

$$\frac{1}{n}|\nabla\phi|^2_\omega\, \omega^n = i\partial\phi \wedge \bar{\partial}\phi \wedge \omega^{n-1}$$

and that $\omega > 0$ and $\omega_\varphi > 0$ which implies that,

$$\begin{aligned}\frac{p-1}{n}\int_M \phi^{p-2}|\nabla\phi|^2_\omega\, \omega^n &= (p-1)\int_M \phi^{p-2} i\partial\phi \wedge \bar{\partial}\phi \wedge \omega^{n-1} \\ &\leq \int_M (p-1)\phi^{p-2} i\partial\phi \wedge \bar{\partial}\phi \wedge \left(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}\right)\end{aligned}$$

Finally, note that,

$$\phi^{p-2}|\nabla\phi|^2_\omega = |\phi^{\frac{p}{2}-1}\nabla\phi|^2_\omega = \left(\frac{2}{p}\right)^2 |\nabla\phi^{\frac{p}{2}}|^2_\omega$$

Therefore, we have calculated,

$$\int_M |\nabla\phi^{\frac{p}{2}}|^2_\omega\, \omega^n \leq \frac{np^2}{4(p-1)}\int_M \phi^{p-1}(\omega_\varphi^n - \omega^n) \leq \frac{np^2}{4(p-1)}||\phi^{p-1}||_r \cdot (||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}})$$

Recall the Sobolev inequality for (M, ω) which states that for any positive $\eta \in C^1(M)$ there exists C depending only on M and ω such that,

$$\left(\int_M \eta^{\frac{n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \leq C\left(\int_M \left(|\nabla\eta|^2 + \eta^2\right)\omega^n\right)$$

We apply this inequality in the case $\eta = \phi^{\frac{p}{2}}$. Then we find that,

$$\left(\int_M \phi^{p\cdot\frac{2n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \leq C\left(\int_M \left(|\nabla\phi^{\frac{p}{2}}|^2 + \phi^p\right)\omega^n\right)$$

Plugging in our previous result,

$$\left(\int_M \phi^{p\cdot\frac{n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \leq C\left[\frac{np^2}{4(p-1)}\cdot(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}})\left(\int_M \phi^{(p-1)r}\,\omega^n\right)^{\frac{1}{r}} + \int_M \phi^p\,\omega^n\right]$$

Define $\chi = \frac{n}{n-1}$ and let $w = pr$. Since $p > 1$ is arbitrary and $1 < r < \frac{n}{n-1} = \chi$ then $w \geq 2$ is arbitrary. Now let $\xi = \chi/r > 1$ and using the fact that $\phi \geq 1$ so we may increase the powers as we wish in this inequality, we find,

$$\left(\int_M \phi^{w\cdot\xi}\omega^n\right)^{\frac{1}{\chi}} \leq C\left[\frac{np^2}{4(p-1)}\cdot(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}})\left(\int_M \phi^w\,\omega^n\right)^{\frac{1}{r}} + \int_M \phi^p\,\omega^n\right]$$

However, by the Holder inequality,

$$||\phi^p||_{L^1} \leq ||\phi^p||_r ||1||_q = ||\phi^p||_r \cdot \mathrm{Vol}\,(M)^{\frac{1}{q}}$$

Furthermore,

$$||\phi^p||_r = \left(\int_M \phi^w \, \omega^n \right)^{\frac{1}{r}}$$

which implies that,

$$\left(\int_M \phi^{w \cdot \xi} \omega^n \right)^{\frac{1}{\chi}} \leq C \left[\frac{np^2}{4(p-1)} \cdot (||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}}) + \mathrm{Vol}\,(M)^{\frac{1}{q}} \right] \cdot \left(\int_M \phi^w \, \omega^n \right)^{\frac{1}{r}}$$

Now define,

$$K(M, \omega, ||e^f||_q) = Cn(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}})$$

such that,

$$C \left[\frac{np^2}{4(p-1)} \cdot (||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}}) + \mathrm{Vol}\,(M)^{\frac{1}{q}} \right] \leq K \left[\frac{p^2}{4(p-1)} + 1 \right]$$

Thus, exponentiating by $1/p$ we find,

$$\left(\int_M \phi^{w \cdot \xi} \omega^n \right)^{\frac{1}{w \cdot \xi}} \leq K^{\frac{1}{p}} \left[\frac{p^2}{4(p-1)} + 1 \right]^{\frac{1}{p}} \left(\int_M \phi^w \, \omega^n \right)^{\frac{1}{w}}$$

and therefore,

$$||\phi||_{w\xi} \leq K^{\frac{1}{p}} \left[\frac{p^2}{4(p-1)} + 1 \right]^{\frac{1}{p}} ||\phi||_w$$

for any $w \geq 2$ where C does not depend on w. We may apply this inequality inductively, on a sequence $w_k = w_0 \xi^k$ and thus $p_k = w_0 / r \xi^k$ where $w_0 > r$ so that $p_k > 1$. Then, at each step we have,

$$||\phi||_{w_{k+1}} \leq K^{\frac{1}{p_k}} \left[\frac{p_k^2}{4(p_k-1)} + 1 \right]^{\frac{1}{p_k}} ||\phi||_{w_k}$$

and thus we find,

$$||\phi||_{w_{k+1}} \leq \prod_{j=0}^{k} \left(K^{\frac{1}{p_k}} \left[\frac{p_k^2}{4(p_k-1)} + 1 \right]^{\frac{1}{p_k}} \right) ||\phi||_{w_k}$$

Recall that $\xi > 1$ and $p_0 = w_0/r > 1$ since $r < \frac{n}{n-1} \leq 2$. First,

$$\prod_{j=0}^{k} K^{\frac{1}{p_k}} = K^{\sum\limits_{j=0}^{k} p_j^{-1}}$$

but the series is geometric,

$$\sum_{j=0}^{k}\frac{1}{p_k}=\frac{r}{w_0}\sum_{j=0}^{k}\frac{1}{\xi^j}\leq\frac{r}{w_0}\frac{1}{1-\xi}$$

and thus converges in the limit $k\to\infty$ since $\xi>1$. Furthermore, since $\xi>1$ there exists some N such that for $j\geq N$ we have $p_j>2$ and thus,

$$\frac{p_k^2}{4(p_k-1)}<p_k$$

which implies that when $k>N$,

$$\begin{aligned}\prod_{j=0}^{k}\left[\frac{p_k^2}{4(p_k-1)}+1\right]^{\frac{1}{p_k}}&=\prod_{j=0}^{N}\left[\frac{p_k^2}{4(p_k-1)}+1\right]^{\frac{1}{p_k}}\cdot\prod_{j=N}^{k}\left[\frac{p_k^2}{4(p_k-1)}+1\right]^{\frac{1}{p_k}}\\&\leq\prod_{j=0}^{N}\left[\frac{p_k^2}{4(p_k-1)}+1\right]^{\frac{1}{p_k}}\cdot\prod_{j=N}^{k}p_k^{p_k}\\&=\prod_{j=0}^{N}\left[\frac{p_k^2}{4(p_k-1)}+1\right]^{\frac{1}{p_k}}\cdot\prod_{j=N}^{k}\left(\frac{w_0}{r}\right)^{\frac{1}{p_k}}\cdot\xi^{\frac{j}{p_k}}\\&=\prod_{j=0}^{N}\left[\frac{p_k^2}{4(p_k-1)}+1\right]^{\frac{1}{p_k}}\cdot\left(\frac{w_0}{r}\right)^{\sum_{j=N}^{k}\frac{1}{p_k}}\cdot\xi^{\sum_{j=N}^{k}\frac{j}{p_k}}\end{aligned}$$

which is a bounded series in the limit $k\to\infty$ because,

$$\sum_{j=N}^{\infty}\frac{1}{p_k}=\frac{r}{w_0}\sum_{j=N}^{\infty}\frac{1}{\xi^j}<\infty\quad\text{and}\quad\sum_{j=N}^{\infty}\frac{j}{p_k}=\frac{r}{w_0}\sum_{j=N}^{\infty}\frac{j}{\xi^j}<\infty$$

are both bounded in the limit $k\to\infty$. Therefore, there exits a uniform constant $C_1(M,\omega,q,||e^f||_q)$ (not depending on k) which depends only on M, ω, $||e^f||_q$ and $r=\frac{q}{q-1}$ (so thus on the value of $q>n$) such that,

$$||\phi||_{w_{k+1}}\leq C_1||\phi||_{w_0}$$

for all sufficiently large k. However as $k\to\infty$ we have $w_{k+1}\to\infty$ since $\xi>1$ so,

$$||\phi||_{L^\infty}=\lim_{w\to\infty}||\phi||_w=\lim_{k\to\infty}||\phi||_{w_k}\leq C_1||\phi||_{L_{w_0}}$$

At last,

$$||\varphi||_{L^\infty}=||1-\phi||_{L^\infty}\leq||\phi||_{L^\infty}+1\leq C_1||\phi||_{L_{w_0}}+1$$

which proves the claim.

1.2 Step 2: Poincare Inequality

Recall that we have derived the inequality,

$$\int_M |\nabla \phi^{\frac{p}{2}}|^2_\omega \, \omega^n \le \frac{np^2}{4(p-1)} \int_M \phi^{p-1}(e^f - 1)\omega^n \le \frac{np^2}{4(p-1)} ||\phi^{p-1}(e^f - 1)||_1$$

Via the Holder inequality,

$$||\phi^{p-1}(e^f - 1)||_1 \le ||\phi^{p-1}||_r ||e^f - 1||_q \le ||\phi^{p-1}||_r \left(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}} \right)$$

since we have defined,

$$r = \frac{q}{q-1} \quad \text{such that} \quad \frac{1}{r} + \frac{1}{q} = 1$$

Recall the Poincare inequality (specialized for $p = 2$),

$$||u - u_{\mathrm{avg}}||_2 \le ||\nabla u||_2$$

and apply it to the case $u = \phi^{\frac{p}{2}}$. Then we have,

$$||\phi^{\frac{p}{2}} - (\phi^{\frac{p}{2}})_{\mathrm{avg}}||_2^2 \le ||\nabla \phi^{\frac{p}{2}}||_2^2 \le \frac{np^2}{4(p-1)} ||\phi^{p-1}||_r \left(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}} \right)$$

However, $||\phi^{p-1}||_r = ||\phi||_{r(p-1)}^{p-1}$ so take the $p-1$ root of both sides to get,

$$||\phi^{\frac{p}{2}} - (\phi^{\frac{p}{2}})_{\mathrm{avg}}||_2^{\frac{2}{p-1}} \le \left(\frac{np^2}{4(p-1)} \right)^{\frac{1}{p-1}} \left(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}} \right)^{\frac{1}{p-1}} ||\phi||_{r(p-1)}$$

Chose $p_0 = 1 + \frac{1}{r}$ such that $r(p_0 - 1) = 1$ and choose $w_0 = p_0 \chi$. Now we verify that,

$$w_0 = p_0 \frac{n}{n-1} = \frac{n}{n-1} + \frac{n}{n-1}\frac{1}{r} > \frac{n}{n-1} + 1 > 2 > r$$

because $r < \frac{n}{n-1} < 2$. This implies that our choice for w_0 is a valid one for the previously derived inequality. Plugging in,

$$||\phi^{\frac{p_0}{2}} - (\phi^{\frac{p_0}{2}})_{\mathrm{avg}}||_2^{2r} \le \left(\frac{np_0^2 r}{4} \right)^r \left(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}} \right)^r ||\phi||_1$$

Furthermore,

$$(\phi^{\frac{p_0}{2}})_{\mathrm{avg}} = \frac{1}{\mathrm{Vol}\,(M)} \int_M \phi^{\frac{p_0}{2}} \, \omega^n \le \frac{1}{\mathrm{Vol}\,(M)} \int_M \phi \, \omega^n = \frac{1}{\mathrm{Vol}\,(M)} ||\phi||_1$$

because $p_0 = 1 + \frac{1}{r} < 2$ since $r > 1$ and $\phi \ge 1$. Now,

$$||\phi^{\frac{p_0}{2}}||_2 \le ||\phi^{\frac{p_0}{2}} - (\phi^{\frac{p_0}{2}})_{\mathrm{avg}}||_2 + ||(\phi^{\frac{p_0}{2}})_{\mathrm{avg}}||_2 \le ||\phi^{\frac{p_0}{2}} - (\phi^{\frac{p_0}{2}})_{\mathrm{avg}}||_2 + \frac{1}{\sqrt{\mathrm{Vol}\,(M)}} ||\phi||_1$$

Combining this with the earlier inequality we find,

$$||\phi^{\frac{p_0}{2}}||_2 \leq \left(\frac{np_0^2 r}{4}\right)^{\frac{1}{2}} \left(||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}}\right)^{\frac{1}{2}} ||\phi||_1^{\frac{1}{2r}} + \frac{1}{\sqrt{\mathrm{Vol}\,(M)}}||\phi||_1 \leq K||\phi||_1$$

since $||\phi||_1 > 1$ so $||\phi||_1 \geq ||\phi||_1^{\frac{1}{2r}}$ where $K(M, \omega, q, ||e^f||_q)$ is a constant. Next,

$$||\phi^{\frac{p_0}{2}}||_2^2 = ||\phi||_{p_0}^{p_0} \geq ||\phi||_{p_0}$$

since $p_0 > 1$ and $\phi \geq 1$. Which implies that,

$$||\phi||_{p_0} \leq K^2 ||\phi||_1^2$$

Finally, recall the inequality we derived for any $p > 1$,

$$\left(\int_M \phi^{p \cdot \frac{n}{n-1}} \omega^n\right)^{\frac{n-1}{n}} \leq C\left[\frac{np^2}{4(p-1)} \cdot (||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}}) \left(\int_M \phi^{(p-1)r}\,\omega^n\right)^{\frac{1}{r}} + \int_M \phi^p\,\omega^n\right]$$

If we specialize to the case $p = p_0 = 1 + \frac{1}{r}$ and $\chi = \frac{n}{n-1}$ and $w_0 = p_0 \chi$ we find,

$$\left(\int_M \phi^{w_0} \omega^n\right)^{\frac{1}{\chi}} \leq C\left[\frac{np_0^2 r}{4} \cdot (||e^f||_q + \mathrm{Vol}\,(M)^{\frac{1}{q}}) \left(\int_M \phi\,\omega^n\right)^{\frac{1}{r}} + \int_M \phi^{p_0}\,\omega^n\right] \leq K' \left(\int_M \phi^{p_0}\,\omega^n\right)$$

where $K'(M, \omega, q, ||e^f||_q)$ is a constant. The last line follows because $p_0 > 1$ and $1/r < 1$ and $1 \leq \phi$ so,

$$\left(\int_M \phi\,\omega^n\right)^{\frac{1}{r}} \leq \left(\int_M \phi^{p_0}\,\omega^n\right)^{\frac{1}{r}} \leq \left(\int_M \phi^{p_0}\,\omega^n\right)$$

Taking both sides to the $1/p_0$ power, we find that,

$$||\phi||_{w_0} \leq (K')^{\frac{1}{p_0}} ||\phi||_{p_0}$$

Combining this with the previous result we have,

$$||\phi||_{w_0} \leq K'' ||\phi||_1^2$$

where the constant K'' only depends on M, ω, q and $||e^f||_q$ (note that $p_0 = 1 + \frac{1}{r} = 1 + \frac{q-1}{q}$ depends only on q) which proves the claim.

1.3 Step 3: Greens Functions

In this section we will show that,

$$||\varphi||_{L^1} \leq C_3(M, \omega)$$

By Green's formula $\forall x \in M$ we have,

$$\varphi(x) = \frac{1}{\mathrm{Vol}\,(M)} \int_M \varphi\,\omega^n - \frac{1}{\mathrm{Vol}\,(M)} \int_M G(x,y)\Delta\varphi(y)\omega^n(y)$$

Where,

$$\mathrm{Vol}\,(M) = \int_M \omega^n$$

and $G(x,y)$ is the Green's function of Δ_ω and $\Delta_\omega G(x,y) = \delta_x(y)$. We know that G is bounded below by $-C_\omega$. Taking the trace of $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0$ gives,

$$\mathrm{tr}_\omega\,\omega_\varphi = n + \Delta_\omega\varphi > 0$$

and therefore, since $G(x,y) + C_\omega \geq 0$,

$$\frac{1}{\mathrm{Vol}\,(M)} \int_M (G(x,y) + C_\omega)\,\Delta_\omega\varphi(y)\omega^n \geq -\frac{n}{\mathrm{Vol}\,(M)} \int_M (G(x,y) + C_\omega)\,\omega^n$$

Since φ is continuous and M is compact, it must achieve its maximum at some $x_m \in M$. At that point $\varphi(x_m) = \sup_M \varphi = 0$ and thus,

$$\int_M \varphi\,\omega^n = \int_M G(x,y)\Delta_\omega\varphi(y)\omega^n(y) = \frac{1}{\mathrm{Vol}\,(M)} \int_M (G(x,y) + C_\omega)\Delta_\omega\varphi(y)\omega^n(y)$$

where we may add in a constant to the Green's function because $\int_M \Delta_\omega\varphi\omega^n = 0$ since M has no boundary and thus, by Stokes theorem,

$$\int_M i\partial\bar{\partial}\phi \wedge \omega^{n-1} = \int_M i\,\mathrm{d}\left(\bar{\partial}\phi \wedge \omega^{n-1}\right) = 0$$

the equality holds because ω is closed and $\bar{\partial}\phi \wedge \bar{\partial}\phi = 0$. Thus its trace $\Delta_\omega\varphi\,\omega^n$ also integrates to zero.

Now,

$$\int_M \varphi\,\omega^n \geq -n \int_M (G(x,y) + C_\omega)\,\omega^n$$

Since $\sup_M \varphi = 0$ we have $\varphi \leq 0$ meaning that,

$$||\varphi||_{L^1} = \int_M |\varphi|\,\omega^n = -\int_M \varphi\,\omega^n \leq n \int_M (G(x,y) + C_\omega)\,\omega^n$$

which proves the claim.

1.4 The Full Theorem

Recall that the Poincare inequality gives the estimate,

$$||\phi||_{w_0} \leq C_2||\phi||_1^2$$

where $w_0 = p_0\chi = (1+\frac{1}{r})\frac{n}{n-1} > r$. Since $w_0 > r$, we may apply the result obtained via Moser iteration,

$$||\varphi||_{C^0} \leq C_1||\varphi||_{w_0} + 1 \leq C_1||\phi||_{w_0} + C_1\mathrm{Vol}\,(M)^{\frac{1}{w_0}} + 1$$

to find that,

$$||\varphi||_{C^0} \leq C_1C_2||\phi||_1^2 + C_1\mathrm{Vol}\,(M)^{\frac{1}{w_0}} + 1 = C_1C_2||\varphi||_1^2 + C_4$$

where C_4 is a constant depending on C_1, C_2, M, and ω. Finally, the Green's function argument gives,

$$||\varphi||_1 \leq C_3$$

and thus,

$$||\varphi||_{C^0} \leq C_1C_2C_3^2 + C_4$$

which proves the theorem since each constant only depends on M, ω, q, and $||e^f||_q$.