Mathematics GU4053 Algebraic Topology
Assignment # 6

Benjamin Church

July 8, 2021

Note. My order of path concatenation follows Hatcher,

$$\gamma * \delta(x) = \begin{cases} \gamma(2x) & x \le \frac{1}{2} \\ \delta(2x-1) & x \ge \frac{1}{2} \end{cases}$$

Problem 1.

Suppose the following diagram of abelian groups commutes,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r, "c"] \arrow[d, "h"] & D \arrow[r, "d"] \arrow[d, "i"] & E \arrow[d, "j"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & D' \arrow[r, "d'"] & E' \end{tikzcd}$$

with exact rows and f, g, i, and j are isomorphims. Suppose that $h(x) = 0$ then $c' \circ h(x) = 0$. By commutativity, $i \circ c(x) = 0$ but i is an injection so $c(x) = 0$. Thus, $x \in \ker c = \mathrm{Im}\,(b)$ so there exists $y \in B$ such that $b(y) = x$ but $h(x) = 0$ so $h \circ b(y) = b' \circ g(y) = 0$ so $g(y) \in \ker b' = \mathrm{Im}\,(a')$ so there exists $z \in A'$ such that $a'(z) = g(y)$. But f is a surjection so there exists $q \in A$ such that $f(q) = z$. Then, $g \circ a(q) = a' \circ f(q) = a'(z) = g(y)$ but g is an injection so $a(q) = y$. Then $b \circ a(q) = b(y) = x$. However, the top row is exact so $\ker b = \mathrm{Im}\,(a)$ but $a(q) \in \mathrm{Im}\,(a)$ so $a(q) \in \ker b$ so $b \circ a(q) = x = 0$. Thus, h is injective.

In this proof, we never used the maps d, j, and d' so only the first four groups in the sequences are needed. Also, I only used the fact that f is a surjection, g is an injection, and i is an injection.

Problem 2.

WARNING: The following is wrong. It assumes that $\gamma_t(r) = \gamma(1-(1-r)t)$ satisfies $\gamma_t(0) = \pi(x)$ for all t which is clearly false unless γ is contained in the fiber. I should have know it was wrong because the fact that p is a fibration is not actually used.

Let $p : (E, e_0) \to (B, b_0)$ be a pointed fibration. The fiber of p is the subspace $F = p^{-1}(b_0)$. Then, define the map $\phi : F \to N_p$ by $\phi(x) = (x, e_{b_0})$ where e_{b_0} is the constant loop at b_0. This map is well-defined because $x \in F = p^{-1}(b_0)$ so $p(x) = b_0 = e_{b_0}(0)$. The projection $\pi_1 : N_p \to E$ is given by

$\pi_1(x,\gamma) = x$. Therefore, $\pi_1 \circ \phi(x) = \pi_1(x, e_{b_0}) = x$ so $\pi_1 \circ \phi = \mathrm{id}_F$. However, $\phi \circ \pi_1(x,\gamma) = \phi(x) = (x, e_{b_0})$. Define the homotopy $H : N_p \times I \to N_p$ by $H(x,\gamma,t) = (x, \gamma_t)$ where $\gamma_t(r) = \gamma(1-(1-r)t)$. Thus, $\gamma_0(r) = \gamma(1) = b_0$ and $\gamma_1(r) = \gamma(r)$. Therefore, $H(x,\gamma,0) = (x,\gamma_0) = (x, e_{b_0}) = \phi \circ \pi_1(x,\gamma)$ and $H(x,\gamma,1) = (x,\gamma_1) = (x,\gamma)$. Thus, H is a homotopy between $\phi \circ \pi_1$ and id_{N_p} so ϕ is a homotopy equivalence.

Also this doesn't work because $\pi_1(x,\gamma) = x$ is in E but not necessarily F because we only know that $p(x) = \gamma(0)$ not that $p(x) = b_0$. Here's how to actually do it.

Let $p : E \to B$ be a based fibration with fiber $F = p^{-1}(b_0)$. Define $\phi : F \to N_p$ by $\phi(x) = (x, e_{b_0})$ where e_{b_0} is the constant loop at b_0 (this is well defined because $e_{b_0}(0) = b_0 = p(x)$ and $e_{b_0}(1) = b_0$). Define a homotopy $g : N_p \times I \to B$ sending $(x,\gamma,t) \mapsto \gamma(t)$. Then $g_0(x,\gamma) = \gamma(0) = p(x)$ so setting $\tilde{g}_0(x,\gamma) = x$ we can apply the homotopy lifting property to the fibration $p : E \to B$,

$$\begin{tikzcd}
N_p \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d] & E \arrow[d, "p"] \\
N_p \times I \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] & B
\end{tikzcd}$$

gives a homotopy $\tilde{g} : N_p \times I \to E$ satisfying $p \circ \tilde{g}(x,\gamma,t) = g(x,\gamma,t) = \gamma(t)$. Thus we may define,

$$h : N_p \times I \to N_p \quad \text{via} \quad h(x,\gamma,t) = (\tilde{g}(x,\gamma,t), \gamma|_{[t,1]})$$

This is well-defined because $p \circ \tilde{g}(x,\gamma,t) = \gamma(t) = \gamma|_{[t,1]}(0)$ so $h(x,\gamma,t) \in N_p$. Furthermore,

$$h_0(x,\gamma) = (\tilde{g}_0(x,\gamma),\gamma) = (x,\gamma) \implies h_0 = \mathrm{id}_{N_p}$$

Notice that $p \circ \tilde{g}_1 = g_1$ sends $(x,\gamma) \mapsto \gamma(1) = b_0$ giving a map $\tilde{g}_1 : N_p \to F$ Furthermore, $h_1(x,\gamma) = (\tilde{g}_1(x,\gamma), e_{b_0}) = \phi \circ \tilde{g}_1(x,\gamma)$. So we see that h gives a homotopy between id_{N_p} and $\phi \circ \tilde{g}_1$. Finally, $\tilde{g}_1 \circ \phi(x) = \tilde{g}_1(x, e_{b_0})$ so consider $\tilde{g}(x, e_{b_0}, t)$ which satisfies $p \circ \tilde{g}(x, e_{b_0}, t) = g(x, e_{b_0}, t) = b_0$ so $\tilde{g}(x, e_{b_0}, t) \in F$. Therefore $\tilde{g}(-, e_{b_0}, -)$ is a homotopy $F \times I \to F$ from $\tilde{g}_0(-, e_{b_0}) = \mathrm{id}_F$ to $\tilde{g}_1(-, e_{b_0}) = \tilde{g}_1 \circ \phi$. Therefore, $\phi : F \to N_p$ is a homotopy equivalence.

Problem 3.

Let $f : X \to Y$ be a map of pointed spaces. Consider the projection $\pi_1 : N_f \to X$ given by $\pi_1(x,\gamma) = x$. Take any space Z and maps $g : Z \to N_f$ and $h : Z \times I \to X$ such that the following diagram commutes,

$$\begin{tikzcd}
Z \arrow[r, "\tilde{g}_0"] \arrow[d, hook, "\iota"] & N_f \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & PY \arrow[d, "\mathrm{ev}_0"] \\
Z \times I \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

The outside rectangle is a lifting diagram for $\mathrm{ev}_0 : PY \to Y$. I claim that ev_0 is a fibration. It is the fibrant replacement of $* \to Y$ i.e. $PY = E_{* \to Y}$. Consider a diagram,

$$\begin{tikzcd}
Z \arrow[r, "\tilde{h}_0"] \arrow[d, hook, "\iota"'] & PY \arrow[d, "\mathrm{ev}_0"] \\
Z \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & Y
\end{tikzcd}$$

Let $\gamma_x = \tilde{h}_0(x)$ and note that $\gamma_x(1) = y_0$ and $\gamma_x(0) = \mathrm{ev}_0 \circ \tilde{h}(x) = h(x, 0)$. Then $h(x, -)$ is a path starting at $\gamma_x(0)$. Thus we can define $\tilde{h} : Z \times I \to PY$ via,

$$\tilde{h}(x, t) = \gamma_x * (-h(x, -)|_{[0,t]})$$

Notice that $\tilde{h}(x, t)(1) = \gamma_x(1) = y_0$ so this is a well-defined function $\tilde{h} : Z \times I \to PY$. Finally, $\mathrm{ev}_0 \circ \tilde{h}(x, t) = h(x, t)$ so $\mathrm{ev}_0 \circ \tilde{h} = h$ so this is a lift proving that ev_0 is a fibration. See Hatcher 4.64 for more details.

Now we prove that π_1 is a fibration by showing that the (strict) pullback of a fibration is a fibration. Indeed, returning to the original diagram, we get maps $\pi_2 \circ \tilde{g}_0 : Z \to PY$ and $f \circ g : Z \times I \to Y$ such that the outer rectangle commutes. By the homotopy lifting property of the fibration $\mathrm{ev}_0 : PY \to Y$ there is a lift $\tilde{g}' : Z \times I \to PY$. However, by the universal property of the pullback we get a map $\tilde{g} : Z \times I \to N_f$ from the pair $g : Z \times I \to X$ and $\tilde{g}' : Z \times I \to PY$ making the square commute. Now $\pi_1 \circ \tilde{g} = g$ and I claim that $\tilde{g} \circ \iota = \tilde{g}_0$. Indeed, $\pi_1 \circ \tilde{g} \circ \iota = g \circ \iota = \pi_1 \circ \tilde{g}_0$ and $\pi_2 \circ \tilde{g} \circ \iota = \tilde{g}' \circ \iota = \pi_2 \circ \tilde{g}_0$ so by the universal property of the pullback $\tilde{g} \circ \iota = \tilde{g}_0$. Therefore we get a lift in the leftmost square proving that $\pi_1 : N_f \to X$ is a fibration.

Let $\pi = \pi_1 : N_f \to X$ be the fibration considered above and take, $\phi : F \to N_\pi$, the natural inclusion on the fiber $F = \pi^{-1}(x_0)$ which is given by $\phi(x_0, \gamma) = (x_0, \gamma, e_{x_0})$ for $(x_0, \gamma) \in \pi^{-1}(x_0)$. Since $(x_0, \gamma) \in N_f$ we have $\gamma(0) = f(x_0) = y_0$ and $\gamma(1) = y_0$. Therefore, γ is a loop so $F \cong \Omega Y$ via $(x_0, \gamma, e_{x_0}) \mapsto \gamma$. Thus, ϕ can be viewed as a map $\phi : \Omega Y \to N_\pi$. However, as proven in problem (2), $\phi : F \to N_\pi$ is a homotopy equivalence when π is a fibration. Therefore, $\phi : \Omega Y \to N_\pi$ is a homotopy equivalence.

Problem 4.

Consider the covering map $p : S^n \to \mathbb{RP}^n$ given by the quotient map on antipodal points. We know from covering space theory that for $m \geq 2$, the map $p_* : \pi_m(S^n) \to \pi_m(\mathbb{RP}^n)$ is an isomorphism. However, since we have some fancy new long exact sequences it seems a shame not to use them!

The covering map $p : S^n \to \mathbb{RP}^n$ is a fibration with fiber S^0. This fibration induces the long exact sequence,

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_4(S^0) \arrow[r] & \pi_4(S^n) \arrow[r] & \pi_4(\mathbb{RP}^n) \arrow[r] & \pi_3(S^0) \arrow[r] & \pi_3(S^n) \arrow[r] & \pi_3(\mathbb{RP}^n) \arrow[dllll, rounded corners] \\
& \pi_2(S^0) \arrow[r] & \pi_2(S^n) \arrow[r] & \pi_2(\mathbb{RP}^n) \arrow[r] & \pi_1(S^0) \arrow[r] & \pi_1(S^n) \arrow[r] & \pi_1(\mathbb{RP}^n)
\end{tikzcd}$$

However, $\pi_m(S^0) = 0$ for any $m > 0$ because S^0 is a disjoint union of points. Therefore, for each $m \geq 2$, we can pick out the exact sequence,

$$0 \longrightarrow \pi_m(S^n) \xrightarrow{f} \pi_m(\mathbb{RP}^m) \longrightarrow 0$$

Because this sequence is exact, $\ker f = \mathrm{Im}\,(0) = 0$ and $\mathrm{Im}\,(f) = \ker 0 = \pi_m(\mathbb{RP}^m)$ so f is an isomorphism. Therefore, $\pi_m(S^n) \cong \pi_m(\mathbb{RP}^n)$ for $m \geq 2$.

Problem 5.

For $m, n \in \mathbb{Z}_{>1} \cup \{\infty\}$ let $X = \mathbb{RP}^m \times S^n$ and $Y = \mathbb{RP}^n \times S^m$. Using the previous problem, for $i \geq 2$,

$$\pi_i(X) = \pi_i(\mathbb{RP}^m) \times \pi_i(S^n) \cong \pi_i(S^m) \times \pi_i(S^n) \cong \pi_i(S^n) \times \pi_i(S^m) \cong \pi_i(\mathbb{RP})^n \times \pi_i(S)^m \cong \pi_i(\mathbb{RP}^n \times S^m) = \pi_i(Y)$$

For $i = 0$ this statement is trivial because both spaces are connected. For $i = 1$ we must check the formula explicitly,

$$\pi_1(\mathbb{RP}^m \times S^n) \cong (\mathbb{Z}/2\mathbb{Z}) \times 1 \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \pi_1(\mathbb{RP}^n \times S^m) \cong (\mathbb{Z}/2\mathbb{Z}) \times 1 \cong \mathbb{Z}$$

so $\pi_1(\mathbb{RP}^m \times S^n) \cong \pi_1(\mathbb{RP}^n \times S^m)$. I have used the formula $\pi_1(S^n) = 1$ for $n > 1$ and $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 1$ because S^n is a double cover of $\mathbb{RP}^n$ which is the universal cover.

An alternative proof of this fact using covering spaces goes as follows. Because the product of covering maps is a covering map, the product of simply connected spaces is simply connected, and th universal cover is unique up to isomorphism, we know that $\tilde{X} = S^m \times S^n$ and $\tilde{Y} = S^n \times S^m$ because S^n is simply connected and the universal cover of $\mathbb{RP}^m$ is S^m. Therefore, $\tilde{X} \cong \tilde{Y}$. However, for $n \geq 2$ the covering map $p : \tilde{X} \to X$ induces an isomorphism, $p_* : \pi_i(\tilde{X}) \to \pi_i(X)$. Therefore,

$$\pi_i(X) \cong \pi_i(\tilde{X}) \cong \pi_i(\tilde{Y}) \cong \pi_i(Y)$$

Problem 6.

Consider the long exact sequence of abelian groups such that every third map ι_n is injective,

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{\iota_n} B_n \longrightarrow C_n \xrightarrow{f_n} A_{n-1} \xrightarrow{\iota_{n-1}} B_{n_1} \longrightarrow \cdots$$

Since ι_n is injective, $\ker \iota_n = 0 = \mathrm{Im}\,(f_{n+1})$ so f_{n+1} is the zero map. Likewise, ι_{n-1} is injective and the sequence is exact so $\ker \iota_{n-1} = \mathrm{Im}\,(f_n) = 0$ so f_n is the zero map. Therefore, the sequence,

$$0 \longrightarrow A_n \xrightarrow{\iota_n} B_n \longrightarrow C_n \longrightarrow 0$$

is short exact.

Problem 7.

Suppose that the sequence of abelian groups,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[l, bend left, "g"] \arrow[r, "h"] & C \arrow[r] & 0 \end{tikzcd}$$

is short exact and the map $g : B \to A$ satisfies $g \circ f = \mathrm{id}_A$. For define the homomorphism $F : B \to A \oplus C$ by $F(x) = (g(x), h(x))$. Because the kernel of the last zero map is C, the map h is surjective. Also, g is a left inverse so g is surjective. Thus, F is surjective. Furthermore, suppose that $(g(x), h(x)) = 0$ then $h(x) = 0$ so $x \in \ker h = \mathrm{Im}\,(f)$ so there exists $y \in B$ such that $f(y) = x$ but $g \circ f(y) = y$ so $g(x) = y = 0$. Thus, $y = 0$ so $f(y) = x = 0$ so F is injective. Therefore, F is an isomorphism. Thus, $B \cong A \oplus C$.

Problem 8.

Let (X, A) be a pointed pair. We showed in class that the following sequence induced by the inclusion $\iota : A \to X$,

$$\cdots \longrightarrow \pi_2(X, A) \longrightarrow \pi_1(A) \xrightarrow{\iota_*} \pi_1(X) \longrightarrow \pi_1(X, A) \longrightarrow \pi_0(A) \xrightarrow{\iota_*} \pi_0(X)$$

is long exact. Suppose that there exists a retraction $r : X \to A$. Then we know, $r \circ \iota = \mathrm{id}_A$. Therefore, $r_* \circ \iota_* = \mathrm{id}_{\pi_n(A)}$. Therefore, ι_* is an injection. Applying the result of problem 6 to this long exact sequence, we have the following short exact sequence for each n,

$$0 \longrightarrow \pi_n(A) \xrightarrow{\iota_*} \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow 0$$

However, $r_* : \pi_n(X) \to \pi_n(A)$ is a left inverse of ι_* so by problem 7 this short exact sequence splits. Therefore, $\pi_n(X) \cong \pi_n(A) \oplus \pi_n(X, A)$.