**Mathematics GU4051 Topology**
**Assignment # 8**

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# Problem 1.

Consider
$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$
and $Y = S \setminus \{(0, 1)\}$. Now, define the function $f : \mathbb{R} \to Y$ by
$$f : t \mapsto \left( \frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1} \right)$$
which is well-defined because $\frac{4t^2}{(t^2+1)^2} + \frac{(t^2-1)^2}{(t^2+1)^2} = \frac{(t^2+1)^2}{(t^2+1)^2} = 1$ and $\frac{t^2-1}{t^2+1} < 1$. Also define the map $g : Y \to \mathbb{R}$ given by,
$$g : (x, y) \mapsto \frac{x}{1 - y}$$
which is well-defined because for $y \in Y$, we have $y \neq 1$. I claim these are inverse functions. This can be checked explicitly,
$$\begin{aligned}
f \circ g(x, y) &= \left( \frac{\frac{2x}{1-y}}{\frac{4x^2}{(1-y)^2} + 1}, \frac{\frac{x^2}{(1-y)^2} - 1}{\frac{x^2}{(1-y)^2} + 1} \right) \\
&= \left( \frac{2x(1 - y)}{x^2 + (1 - y)^2}, \frac{x^2 - (1 - y)^2}{x^2 + (1 - y)^2} \right) \\
&= \left( \frac{2x(1 - y)}{x^2 + y^2 + 1 - 2y}, \frac{x^2 - 1 - y^2 + 2y}{x^2 + y^2 + 1 - 2y} \right) \\
&= \left( \frac{2x(1 - y)}{2(1 - y)}, \frac{2y(1 - y)}{2(1 - y)} \right) \\
&= (x, y)
\end{aligned}$$
in which I have used the fact that $x^2 + y^2 = 1$. Furthermore,
$$g \circ f(t) = \frac{\frac{2t}{t^2+1}}{1 - \frac{t^2-1}{t^2+1}} = \frac{2t}{(t^2 + 1) - (t^2 - 1)} = \frac{2t}{2} = t$$
Thus, $f$ and $g$ are inverse functions so both are bijections. Also, because they are rational functions with everywhere nonzero denominators on subsets of $\mathbb{R}^n$, they are continuous. Thus, $f : \mathbb{R} \to Y$ is a homeomorphism. $\mathbb{R}$ is Hausdorff and $S$ is a closed bounded subset of $\mathbb{R}^2$ so it is compact Hausdorff. $S$ is clearly bounded by 1 and is closed because it is the preimage of the closed set $\{1\}$ under the map $(x, y) \mapsto x^2 + y^2$ which is continuous. Finally, $\mathbb{R} \cong Y = S \setminus \{(0, 1)\}$ and therefore, $\hat{\mathbb{R}} \cong S$.

## Problem 2.

Suppose that $C \subset \mathbb{Q}$ contains $(a, b) \cap \mathbb{Q}$ with $a < b$. This interval must contain an irration number, i.e. $\exists r \in (a, b) \cap (R \backslash \mathbb{Q})$. Then $a < r < b$ so let $\delta = b - r$. Consider the sequence of intervals

$$I_n = (r, r + \tfrac{\delta}{n}) \subset (a, b)$$

where the last inclusion holds because $r + \frac{\delta}{n} < r + \delta = b$. Because $\frac{\delta}{n} > 0$ each interval is nonempty and must contain some rational, $\exists q_n \in I_n \cap \mathbb{Q} \subset (a, b) \cap \mathbb{Q} \subset C$. Take any point $x \neq r$ then take $\epsilon = |r - x| > 0$ so we can choose $N \in \mathbb{N}$ s.t. $N > 2\frac{\delta}{\epsilon}$. Thus, $\frac{\delta}{N} < \frac{\epsilon}{2}$ so for $n > N$ we have $q_n \in I_n \subset (r, r + \epsilon/2)$ so $|x - q_n| > |x - r| - |r - q_n| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$. Therefore, there are at most $N$ values of $n$ for which $q_n$ is within distance $\frac{\epsilon}{2}$ of $x$. Therefore, no subsequence can converge to $x$ if $x \neq r$. However, $r \notin C \subset \mathbb{Q}$ because $r$ is irrational by construction. Therefore, $\{q_n\}$ is a sequence in $C$ with no subsequence which converges in $C$. Because $C \subset \mathbb{R}$ is a metric space, sequential compactness is equivalent to compactness so $C$ cannot be compact.

Suppose that $\mathbb{Q}$ were locally compact. Then for any $x \in \mathbb{Q}$ there would exist an open set $U$ and a compact set $C$ such that $x \in U \in C$. However, $U$ is open so $\exists \delta > 0$ such that $x \in B_\delta(x) \subset U \subset C$ and $B_\delta(x) = (x - \delta, x + \delta) \cap \mathbb{Q}$ in the metric space $\mathbb{Q}$. Therefore, $(x - \delta, x + \delta) \subset C$ and $C$ is a compact subset of $\mathbb{Q}$ which is a conradiction. Therefore, $\mathbb{Q}$ is not locally compact.

## Problem 3.

### (a)

In this problem, we will use the fact that a continuous $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition,

$$\lim_{x \to \infty} f(x) = \pm\infty \qquad \lim_{x \to -\infty} f(x) = \pm\infty$$

if and only if $\forall M \in \mathbb{R} : \exists c \in \mathbb{R} : |x| > c \implies |f(x)| > M$. First, suppose that $f : \mathbb{R} \to \mathbb{R}$ is proper. Given $M \in \mathbb{R}$, consider the set $[-M, M] \subset \mathbb{R}$ which is compact because it is closed and bounded. Then, because $f$ is proper, the set $f^{-1}([-M, M])$ is compact. In partiuclar, it is bounded by $c$. Thus, if $x \in f^{-1}([-M, M])$ then $|x| \leq c$. Therefore, if $|x| > c$ then $x \notin f^{-1}([-M, M])$ so $f(x) \notin [-M, M]$ and therefore, $|f(x)| > M$ so the function, which is continuous by assumption, satisfies the above limit condition.

Conversely, let $f$ be a continuous function satisfying the above limit properties. Let $C \subset \mathbb{R}$ be compact. Then by Heine-Borel, $C$ is closed and bounded. Since $C$ is closed and $f$ is continuous then $f^{-1}(C)$ is closed. Take a bound $M$ for $C$. By the limit property, $\exists c \in \mathbb{R} : |x| > c \implies |f(x)| > M$ thus,

$$x \in f^{-1}(C) \implies f(x) \in C \implies |f(x)| \leq M \implies |x| \leq c$$

Therefore, $f^{-1}(C)$ is closed and bounded so by Heine-Borrel it is compact. Therefore, $f$ is proper.

### (b)

Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a nonconstant polynomial with $a_n \neq 0$. Then,

$$\lim_{x \to \pm\infty} \frac{f(x)}{a_n x^n} = \lim_{x \to \pm\infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{a_n x^n} = \lim_{x \to \pm\infty} \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n}\right) = 1$$

Therefore, $f(x)$ and $a_n x^n$ have the same asymptotics. In particular,

$$\lim_{x \to \infty} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \pm\infty$$

because these conditions hold for $a_n x^n$. From analysis, $f(x)$ is continuous because each term is contiuous. Thus, $f(x)$ is a proper map.

## Problem 4.

Let $f : X \to Y$ be a proper map and let $X$ and $Y$ be Hausdorff spaces. Define the map $\hat{f} : \hat{X} \to \hat{Y}$ by,

$$\hat{f}(x) = \begin{cases} f(x) & x \in X \\ \infty & x = \infty \end{cases}$$

Let $C \subset \hat{Y}$ be a closed set. Then, either $\infty \notin C$ and $C$ is compact or $\infty \in C$ and $C \cap Y$ is closed in $Y$. In the first case, $C$ is compact so becaues $f$ is proper and $\infty$ does not map into $C$, $\hat{f}^{-1}(C) = f^{-1}(C)$ is a compact set not containing $\infty$ and thus is closed in $\hat{X}$. In the second case, $C \cap Y$ is closed in $Y$ and $\infty \in C$ so $\hat{f}^{-1}(C) = f^{-1}(C \cap Y) \cup \{\infty\}$. By continuity, $f^{-1}(C \cap Y)$ is closed in $X$ so $f^{-1}(C \cap Y) \cup \{\infty\}$ is closed in $\hat{X}$.

Conversely, suppose the function $\hat{f} : \hat{X} \to \hat{Y}$ given by,

$$\hat{f}(x) = \begin{cases} f(x) & x \in X \\ \infty & x = \infty \end{cases}$$

is continuous. Then, take a closed set $C \subset Y$ and consider the set $D = C \cup \{\infty\} \subset \hat{Y}$. Because $C = D \cap Y$ is closed in $Y$ and $\infty \in D$ then $D$ is closed in $\hat{Y}$. Therefore, by continuity, $\hat{f}^{-1}(D) = f^{-1}(C) \cup \{\infty\}$ is closed in $\hat{X}$. Becuase the inverse image contains $\infty$, $\hat{f}^{-1}(D) \cap X = f^{-1}(C)$ must be closed in $X$. Therefore, $f : X \to Y$ is continuous. Likewise, take a compact set $C \subset Y$ then $C$ is closed in $\hat{Y}$ so because $\infty \notin C$ and by continuity, $\hat{f}^{-1}(C) = f^{-1}(C)$ is closed in $\hat{X}$. However, $\infty \notin f^{-1}(C)$ so the set must be compact in $X$ to be closed in $\hat{X}$. Therefore, $f^{-1}(C)$ is compact so $f$ is a proper map.

## Problem 5.

Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ continuous with $X_1, X_2$ nonempty and $Y_1, Y_2$ Hausdorff. Suppose that $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is proper. Take compact $C_1 \subset Y_1$ and $C_2 \subset Y_2$. Now, $(f_1 \times f_2)^{-1}(C_1 \times C_2) = f_1^{-1}(C_1) \times f_2^{-1}(C_2)$ is compact because $f_1 \times f_2$ is proper. Now, by Lemma 0.1, this implies that $f_1^{-1}(C_1)$ and $f_2^{-1}(C_2)$ are compact and therefore, $f_1$ and $f_2$ are proper.

Conversely, let $f_1$ and $f_2$ be proper. Let $C \subset Y_1 \times Y_2$ be compact. The maps $\pi_1 : Y_1 \times Y_2 \to Y_1$ and $\pi_2 : Y_1 \times Y_2 \to Y_2$ are continuous so $\pi_1(C)$ and $\pi_2(C)$ are compact. Therefore, because $f_1$ and $f_2$ are proper, $f_1^{-1}(\pi_1(C))$ and $f_2^{-1}(\pi_2(C))$ are compact and therefore, $f_1^{-1}(\pi_1(C)) \times f_2^{-1}(\pi_2(C))$ is compact. Then, because $Y_1$ and $Y_2$ is Hausdorff, $Y_1 \times Y_2$ is Hausdorff so $C$ is closed. Thus, $(f_1 \times f_2)^{-1}(C)$ is closed.

Now, if $(x, y) \in (f_1 \times f_2)^{-1}(C)$ then $(f_1(x), f_2(y)) \in C$ so $f_1(x) \in \pi_1(C)$ and $f_2(y) \in \pi_2(C)$ so $x \in f_1^{-1}(\pi_1(C))$ and $y \in f_2^{-1}(\pi_2(C))$ so finally, $(x, y) \in f_1^{-1}(\pi_1(C)) \times f_2^{-1}(\pi_2(C))$. Therefore,

$$(f_1 \times f_2)^{-1}(C) \subset f_1^{-1}(\pi_1(C)) \times f_2^{-1}(\pi_2(C))$$

However, the former is closed and the latter is compact so $(f_1 \times f_2)^{-1}(C)$ is compact. Thus, $f_1 \times f_2$ is proper.

# Problem 6.

Let $f : X \to Y$ and $g : Y \to Z$ be continuous and $g \circ f$, proper and let $Y$ be Hausdorff. Let $C \subset Y$ be compact. By continuity, $g(C)$ is compact and since $g \circ f$ is proper, $(g \circ f)^{-1}(g(C)) = f^{-1}(g^{-1}(g(C)))$ is compact. However, $C \subset g^{-1}(g(C))$ and $C$ is compact in a Hausdorff space so $C$ is closed. Thus, $f^{-1}(C)$ is closed and $f^{-1}(C) \subset f^{-1}(g^{-1}(g(C)))$ which is compact. Therefore, $f^{-1}(C)$ is closed in a compact set and thus compact so $f$ is a proper map.

# Problem 7.

Let $f : X \to Y$ and $g : Y \to Z$ be continuous and $g \circ f$, proper and let $f$ be surjective. Let $C \subset Z$ be compact. Since $g \circ f$ is proper, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ is compact. Thus, $f(f^{-1}(g^{-1}(C)))$ is compact because $f$ is continuous. However, since $f$ is surjective, by Lemma 0.2, $f(f^{-1}(g^{-1}(C))) = g^{-1}(C)$ is compact. Therefore, $g$ is proper.

# Lemmas

**Lemma 0.1.** *If $X$ and $Y$ are nonempty and $X \times Y$ is compact then $X$ and $Y$ are compact.*

*Proof.* Let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of $X$. Then, $\{U_\lambda \times Y \mid \lambda \in \Lambda\}$ is an open cover of $X \times Y$ so there exists a finite subcover indexed by $\Lambda'$. Take any $x \in X$ and some $y \in Y$ (which exists because $Y \neq \varnothing$) then because $\Lambda'$ indexes a finite cover, $\exists \lambda \in \Lambda' : (x, y) \in U_\lambda \times Y$ so $x \in U_\lambda$ thus, $\{U_\lambda \mid \lambda \in \Lambda'\}$ is a finite subcover of $X$ so $X$ is compact. The argument for $Y$ is identical. $\square$

**Lemma 0.2.** *If $f : X \to Y$ is surjective, then for any $A \subset Y$ we have $f(f^{-1}(A)) = A$.*

*Proof.* If $a \in A$ then by surjectivity, $\exists a \in X$ s.t. $f(x) = a$ so $x \in f^{-1}(A)$ thus $f(a) = x \in f(f^{-1}(A))$ so $A \subset f(f^{-1}(A))$. If $a \in f(f^{-1}(A))$ then $\exists x \in f^{-1}(A)$ s.t. $f(x) = a$ but $f(x) \in A$ so $a \in A$ thus, $f(f^{-1}(A)) \subset A$ so $f(f^{-1}(A)) = A$. $\square$