Contents

1 I Varieties

1.1 Section 1

1.1.1 1.1 DO THIS

(a) Let Y be the plane curve $y = x^2$. Let $A(Y)$ be the affine coodinate ring

$$A(Y) = k[x, y]/(y - x^2) \cong k[x]$$

via the map $y \mapsto x^2$.

(b) Let Z be the plane curve $xy = 1$. Considre the affine coodinate ring $A(Y) = k[x, y]/(xy - 1)$. Consider a map $k[x, y]/(xy - 1) \to k[t]$ then x, y map to units but $(k[t])^\times = k^\times$ and thus the map is not surjective. Therefore there cannot be such an isomorphism.

(c) Let f be any irreducible quadratic polynomial $f \in k[x, y]$ and let W be the conic defined by f. Then write,

$$f(x, y) = a_0 + a_{1,0}x + a_{0,1}y + a_{1,1}xy + a_{2,0}x^2 + a_{0,2}y^2$$

where not all $a_{1,1}, a_{2,0}, a_{0,2}$ are zero. Let's do the characteristic not equal to two case first. When $a_{2,0} \neq 0$ we can write,

$$f(x, y) = a_{2,0}(x - ay - b)^2 + a_{0,2}(y - a'x - b')^2 + a_0'$$

1.1.2 1.2

Let $Y \subset \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Clearly, $Y \subset Z = Z(f_1, f_2, f_3)$ where $f_1 = x^2 - y$ and $f_2 = y^3 - z^2$ and $f_3 = z - x^3$. Furthermore, for any $p \in Z$ we know that $y = x^2$ and $z = x^3$ so $p = (x, x^2, x^3) \in Y$ and thus $Y = Z$. Clearly, $\dim Y = 1$ because it is is infinite and the image of $\mathbb{A}^1 \to \mathbb{A}^3$. Then,

$$I(Y) = (y - x^2, z - x^3, y^3 - z^2)$$

Now consider,

$$A(Y) = k[x, y, z]/I(Y) = k[x]$$

because $y \mapsto x^2$ and $z \mapsto x^3$.

1.1.3 1.3

Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $f_1 = x^2 - yz$ and $f_2 = xz - x$. Then $Y = Z(I)$ where $I = (x^2 - yz, xz - x)$. We need to find the minimal primes over I. Clearly $(x, y) \supset I$ and $(x, z) \supset I$ and $(z - 1, y - x^2) \supset I$. These are prime ideals and they are minimal because I has height two. Furthermore,

$$(x, y) \cap (x, z) \cap (z - 1, y - x^2) = I$$

so I has three irreducible components.

1.1.4 1.5

Let B be a k-algebra. It is clear that if $B = A(Y)$ for some affine algebraic set then $B = A(Y) = k[x_1, \dots, x_n]/I(Y)$ is finitely generated and moreover I is radical so B is reduced.

Now suppose that B is a reduced finite type k-algebra. Then there is a surjection $k[x_1, \dots, x_n] \twoheadrightarrow B$ whose kernel is some ideal I. Therefore, $B \cong k[x_1, \dots, x_n]/I$. Since B is reduced we see that I is radical and thus $I = I(Z(I))$ and therefore $B = A(Z(I))$.

1.1.5 1.7 (IN MY NOTES SOMEWHERE PRETTY OBVIOUS)

1.1.6 1.10

(a) Let $Y \subset X$ then choose a maximal chain of closed irreducibles,

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

inside Y where $n = \dim Y$. Then taking closures in X we see that,

$$\overline{Z}_0 \subsetneq \overline{Z}_1 \subsetneq \cdots \subsetneq \overline{Z}_n$$

is also a chain of closed irreducibles. Furthermore, the inclusions are strict because $\overline{Z}_i \cap Y = Z_i$ and therefore if $\overline{Z}_i = \overline{Z}_{i+1}$ then $Z_i = Z_{i+1}$ which is false. Thus, $\dim X \geq n$.

(b) Let X be a topological space covered by a family of open subsets $\{U_i\}$. By the previous part,

$$\sup \dim U_i \leq \dim X$$

Now choose a maximal chain of closed irreducibles,

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X. There is some U_i such that $Z_0 \cap U_s$ is nonempty. Then I claim that $Z_i \cap U_s$ gives such a chain. It is clear that $Z_i \cap U_s$ is closed and irreducible now if $Z_i \cap U_s = Z_{i+1} \cap U_s$ then U_s^C and Z_i cover Z_{i+1} but Z_{i+1} is irreducible so $U_s^C \cap Z_{i+1} = \varnothing$ which is impossible because $Z_0 \subset Z_{i+1}$ so this must be a chain. Thus, $\dim U_s \geq \dim X$ proving the proposition.

(c) Let $X = \mathrm{Spec}\,(\mathbb{Z}_p)$ then the point $(p) \in \mathrm{Spec}\,(\mathbb{Z}_p)$ is closed so $(0) \in \mathrm{Spec}\,(\mathbb{Z}_p)$ is open and also dense since this is an integral scheme (so all opens are dense). However, $U = \{(0)\}$ clearly has dimension zero but $\dim X = 1$ since we have a chain $(0) \subsetneq (p)$.

(d) Let Y be a closed subset of an irreducible finite-dimensional topological space X. Suppose that $\dim Y = \dim X$. If $Y \subsetneq X$ then any maximal chain in Y can be augmented to give a longer chain by adding on X (since closed sets in Y are closed in X since $Y \subset X$ is closed and irreducibility is not relative). Thus $\dim Y < \dim X$.

(e) (EXAMPLE HERE!)

1.1.7 1.11

Let $Y \subset \mathbb{A}^3$ be the curve given by (t^3, t^4, t^5). Consider the ideal,

$$I = (x^4 - y^3, x^5 - z^3, y^5 - z^4, xz - y^2, yz - x^3, x^2y - z^2) = (xz - y^2, yz - x^3, x^2y - z^2)$$

It is clear that $Y \subset Z(I)$. For any $p \in Z(I)$ we choose $t \in k$ such that $t^3 = x$ (we can do this since k is algebraically closed). Then $y^3 = x^4 = t^{12}$ so we can change t by a third root of unity such that $y = t^4$. Then $z^4 = y^5 = t^{20}$ so we can choose $z = t^5$ (WHY) and thus $Z(I) \subset Y$. Therefore $Y = Z(I)$. For dimension reasons ($\dim Y = 1$) we see that $\mathbf{ht}\,(I) = 2$. We need to show that I cannot have two generators. Then I/I^2 would have two generators as a A/I-module where $A = k[x, y, z]$. Then consider $\mathfrak{m} = (x, y, z) \subset A$ then $I/I^2 \otimes_A A/\mathfrak{m}$ would have two generators as a $A/\mathfrak{m}$-module which is a field. However,

$$M = I/I^2 \otimes_A A/\mathfrak{m} = I/\mathfrak{m}I$$

Suppose that $x^4 - y^3, x^5 - z^3, y^5 - z^4$ are dependent in M then,

$$\alpha(xz - y^2) + \beta(yz - x^3) + \gamma(x^2y - z^3) \in \mathfrak{m}I$$

However, every term in $\mathfrak{m}I$ has degree at least 3 and thus $\alpha = \beta = 0$ because they cannot cancel eachother. Furthermore, there is no z^3 in any term of an element of $\mathfrak{m}I$ and thus $\gamma = 0$. Thus $\dim M = 3$ contradicting the fact that it has two generators.

1.1.8 1.12

Consider $f = x^2(x-1)^2 + y^2 \in \mathbb{R}[x, y]$ then f is irreducible in $\mathbb{R}[x, y]$ because of unique factorization in $\mathbb{C}[x, y]$ we have,

$$f = (x(x-1) + iy)(x(x-1) - iy)$$

but neither factor is in $\mathbb{R}[x, y]$ and thus f cannot factor. Furthermore, $Z(f)$ is the union of two points $(0, 0)$ and $(1, 0)$ and thus cannot be irreducible (it's not even connected!).

2 II Schemes

3 III Cohomology

3.1 Section 2

3.1.1 2.1 DO!!

(a) Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k and $P, Q \in X$ be distinct points. Ex. II.1.19 gives an exact sequence,

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_Y \longrightarrow 0$$

where $Y = \{P, Q\}$ and $U = X \setminus Y$. Then $\mathbb{Z}_Y = \iota_P\mathbb{Z} \oplus \iota_Q\mathbb{Z}$. Taking the cohomology sequence,

$$0 \longrightarrow \Gamma(X, \mathbb{Z}_U) \longrightarrow \Gamma(X, \mathbb{Z}) \longrightarrow \Gamma(X, \mathbb{Z}_Y) \longrightarrow H^1(X, \mathbb{Z}_U)$$

However, $\Gamma(X, \mathbb{Z}) = \mathbb{Z}$ because X is connected and $\Gamma(X, \mathbb{Z}_Y) = \mathbb{Z} \oplus \mathbb{Z}$ because $P, Q \in X$. Therefore, $\Gamma(X, \mathbb{Z}) \to \Gamma(X, \mathbb{Z}_Y)$ cannot be surjective so we must have $H^1(X, \mathbb{Z}_U) \neq 0$.

(b) Let $Y \subset X = \mathbb{A}^n_k$ be the union of $n + 1$ hyperplanes in general position and let $U = X \setminus Y$.

3.1.2 2.2 DO!!

3.1.3 2.5 DO!!

3.1.4 2.6 DO!!

Let X be a noetherian topological space and let $\{\mathscr{I}_\alpha\}_{\alpha \in A}$ be a directed system of injective sheaves of abelian groups on X.

I claim that $\mathscr{I}$ is injective if and only if for every open $U \subset X$ and subsheaf $\mathscr{F} \subset \mathbb{Z}_U$ and map $f : \mathscr{F} \to \mathscr{I}$ there exists an extension to $\mathbb{Z}_U \to \mathscr{I}$. Given this property consider an injection $\mathcal{A} \hookrightarrow \mathcal{B}$ of sheaves and a map $f : \mathcal{A} \to \mathscr{I}$. Then for every local section $s \in \mathcal{B}(U)$ we take the map $\underline{\mathbb{Z}}_U \to \mathcal{B}|_U$ such that,

$$\begin{tikzcd}
\mathcal{R} \arrow[r, hook] \arrow[d] \arrow[dd, bend right] & \underline{\mathbb{Z}}_U \arrow[d] \arrow[ddl, dashed, bend left] \\
\mathcal{A}|_U \arrow[r, hook] \arrow[d] & \mathcal{B}|_U \\
\mathscr{I}|_U &
\end{tikzcd}$$

where $\mathcal{R}$ is the preimage of $\mathcal{A}$ under $\underline{Z}_U \to \mathcal{B}|_U$.

Clearly, if $\mathscr{I}$ is injective the above property holds.

Lemma 3.1.1. If X is a noetherian space then every subsheaf of $\underline{\mathbb{Z}}$ is finite type.

Proof. Let $\mathscr{F} \subset \underline{\mathbb{Z}}$ be a subsheaf. For each $x \in X$ we see that $\mathscr{F}_x \subset \mathbb{Z}$ and thus $\mathscr{F}_x = (n_x)$ for some $n_x \in \mathbb{Z}$. Thus there exists some open U_x containing x such that $n_x \in \mathscr{F}(U_x)$. Now if $y \in U_x$ then $n_x \in \mathscr{F}_y$ so $n_y \mid n_x$. Because $\mathbb{Z}$ is noetherian, there is some $x_0 \in U_x$ such that n_{x_0} is minimal and thus n_y is constant for $y \in V_x = U_{x_0} \cap U_x$. Therefore, consider $\mathbb{Z}|_V \to \mathscr{F}|_V$ by sedning $1 \mapsto n_{x_0}$ which is an isomorphism on stalks and thus is an isomorphism. Therefore, inside any open U there is a smaller (nonempty) open $V \subset U$ on which $\mathscr{F}|_V$ is finite type. Therefore, if $\mathscr{F}|_{X \setminus V}$ is finite type $\square$

3.1.5 2.7 DO!!

Let $X = S^1$ be the circle with its usual topology. Write $S^1 = U_1 \cup U_2$ for a pair of arcs such that $U_{12} = U_1 \cap U_2$ is the union of two contractible spaces. Consider the Godement resolution,

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \prod_{x \in S^1} \mathbb{Z}_x \longrightarrow 0$$

3.2 Section 3

3.2.1 3.1

Let X be a noetherian scheme. If $X = \mathrm{Spec}\,(A)$ is affine then $X_{\text{red}} = \mathrm{Spec}\,(A_{\text{red}})$ is clearly affine. Conversely, suppose that $X_{\text{red}} = \mathrm{Spec}\,(A)$ is affine. There is a closed immersion $X_{\text{red}} \hookrightarrow X$ which sheaf of ideals $\mathcal{N}$ which is coherent since X is noetherian. Therefore, since $\mathcal{N}$ is the sheaf of nilpotents as an ideal $\mathscr{N}^{n+1} = 0$ for some n because locally $\mathscr{N}|_{\mathrm{Spec}(B)} = \widetilde{\mathrm{nilrad}}(\mathrm{B})$ which is finitely generated because B is Noetherian. Therefore, for any quasi-coherent sheaf $\mathscr{F}$ there is a filtration,

$$\mathscr{F} \supset \mathscr{N} \cdot \mathscr{F} \supset \mathscr{N}^2 \cdot \mathscr{F} \supset \cdots \supset \mathscr{N}^n \cdot \mathscr{F} \supset \mathscr{N}^{n+1} \cdot \mathscr{F} = 0$$

let $\mathscr{F}_i = \mathscr{N}^i \cdot \mathscr{F}$ then $\mathscr{G}_i = \mathscr{F}_i / \mathscr{F}_{i+1}$ satisfies $\mathscr{N} \cdot \mathscr{G}_i = 0$. Since $\iota : X_{\mathrm{red}} \to X$ is a closed immersion ι_* induces an equivalence of categories between quasi-coherent $\mathcal{O}_{X_{\mathrm{red}}}$-modules and quasi-coherent $\mathcal{O}_X$-modules killed by $\mathscr{N}$. Thus $\mathscr{G}_i = \iota_* \mathscr{G}_i'$ where $\mathscr{G}_i'$ is a $\mathcal{O}_{X_{\mathrm{red}}}$-module. Then $H^q(X, \mathscr{G}_i) = H^q(X, \iota_* \mathscr{G}_i') = H^q(X_{\mathrm{red}}, \mathscr{G}_i') = 0$ for $q > 0$ because $\mathscr{G}_i'$ is a quasi-coherent $\mathcal{O}_{X_{\mathrm{red}}}$-module and X_{red} is affine. Clearly $H^q(X, \mathscr{F}_{n+1}) = 0$. Now assume that $H^q(X, \mathscr{F}_{i+1}) = 0$ for $q > 0$. Using the exact sequence,

$$0 \longrightarrow \mathscr{F}_{i+1} \longrightarrow \mathscr{F}_i \longrightarrow \mathscr{G}_i \longrightarrow 0$$

we apply cohomology to find,

$$H^q(X, \mathscr{G}_i) \longrightarrow H^{q+1}(X, \mathscr{F}_{i+1}) \longrightarrow H^{q+1}(X, \mathscr{F}_i) \longrightarrow H^{q+1}(X, \mathscr{G}_i)$$

and thus $H^{q+1}(X, \mathscr{F}_{i+1}) \xrightarrow{\sim} H^{q+1}(X, \mathscr{F}_i)$ is an isomorphism for $q > 0$ and $H^1(X, \mathscr{F}_{i+1}) \twoheadrightarrow H^1(X, \mathscr{F}_i)$ is a surjection. Therefore, $H^q(X, \mathscr{F}_i) = 0$ for $q > 0$ because $H^q(X, \mathscr{F}_{i+1}) \twoheadrightarrow H^q(X, \mathscr{F}_i)$ and $H^q(X, \mathscr{F}_{i+1}) = 0$ for $q > 0$. Thus X is affine by Serre's criterion.

3.2.2 3.2

Let X be a reduced noetherian scheme. Suppose that $X = \mathrm{Spec}\,(A)$ is affine. Then the irreducible components of X are $\mathrm{Spec}\,(A/\mathfrak{p}_i)$ for the minimal primes $\mathfrak{p}_i \subset A$ which are affine.

Conversely, suppose that each irreducible component $Y \subset X$ is affine. Since X is Noetherian there are finitely many irreducible components $Y_i \subset X$. For any coherent sheaf of ideals $\mathscr{I}$ which corresponds to some closed subscheme $Z \subset X$ we want to show that $H^1(X, \mathscr{I}) = 0$. To do so, we proceed by descending induction on the number of irreducible components of X contained in the support of Z. If Z contains every component then $\mathscr{I} = (0)$ because X is reduced and thus $H^1(X, \mathscr{I}) = 0$. Now, let Y be an irreducible component not contained in Z and consider the exact sequence,

$$0 \longrightarrow \mathscr{I}_{Z \cup Y} \longrightarrow \mathscr{I}_Z \longrightarrow (\iota_Y)_* \mathscr{I}_{Z \cap Y} \longrightarrow 0$$

Because Y_1 is affine, $H^1(X, (\iota_Y)_* \mathscr{I}_{Z \cap Y}) = H^1(Y, \mathscr{I}_{Z \cap Y}) = 0$ and thus the long exact sequence gives a surjection $H^1(X, \mathscr{I}_{Z \cup Y}) \twoheadrightarrow H^1(X, \mathscr{I}_Z)$. However, $Z \cup Y$ contains more irreducible components of X than Z since $Y \not\subset Z$ so by the induction hypothesis $H^1(X, \mathscr{I}_{Z \cup Y}) = 0$. Therefore $H^1(X, \mathscr{I}_Z) = 0$ proving the result by induction. Since $H^1(X, \mathscr{I}) = 0$ for every coherent sheaf of ideals $\mathscr{I}$, we conclude that X is affine by Serre's criterion.

Here I give an alternative proof. Because X is Noetherian, there are finitely many irreducible components Z_i. We proceed by induction on the number of irreducible components so assume the theorem for r components and let X have irreducible components $Z_1, \dots, Z_{r+1}$. If there is only one irreducible component then because X is reduced, $X = Z$ and thus the statement is trivial. Now proceed by induction. Take any coherent $\mathcal{O}_X$-module $\mathscr{F}$ and consider the exact sequence,

$$0 \longrightarrow \mathscr{I}_Z \cdot \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} / \mathscr{I}_Z \mathscr{F} \longrightarrow 0$$

where $Z \subset X$ is an irreducible component. By Lemma 3.2.1, $\text{Supp}_{\mathcal{O}_X}(\mathscr{I}_Z \otimes \mathscr{F}) \subset X' = Z_1 \cup \cdots \cup Z_r$ where $Z_1, \dots, Z_r \subset X$ are the irreducible components besides Z so X' has r components and $\mathscr{I}_Z \cdot \mathscr{F}$ is the pushforward of a $\mathcal{O}_{X'}$-module $\mathscr{F}'$ (possibly with nonreduced structure). In particular, X' has the same $Z_1, \dots, Z_r$ irreducible components as X (except for Z) and thus each is affine. Likewise, $\mathscr{G} = \mathscr{F}/\mathscr{I}_Z\mathscr{F}$ is anhilated by $\mathscr{I}_Z$ and thus $\mathscr{F}/\mathscr{I}_Z\mathscr{F} = \iota_* \iota^* \mathscr{G}$. Then taking the cohomology sequence,

$$H^q(X', \mathscr{F}') \longrightarrow H^q(X, \mathscr{F}) \longrightarrow H^q(Z, \mathscr{G})$$

By assumption, Z is ample and X' has r irreducible componets all of which are affine so (perhaps after reducing X') X' by the induction hypothesis X' is affine. Since $\mathscr{F}'$ and $\mathscr{G}$ are coherent we get vanishing $H^q(X', \mathscr{F}') = 0$ and $H^q(Z, \mathscr{G}) = 0$ for all $q > 0$. Therefore, the exact sequence gives that $H^q(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $q > 0$ proving that X is affine by Serre's criterion. Thus the result holds for any number of irreducible components by induction.

Lemma 3.2.1. Let X be a reduced scheme with finitely many irreducible components $Z_1, \dots, Z_r$ corresponding to quasi-coherent sheaves of ideals $\mathscr{I}_{Z_i}$. Then,

$$X \setminus Z_i \subset \text{Supp}_{\mathcal{O}_X}(\mathscr{I}_{Z_i}) \subset \bigcup_{j \neq i} Z_j$$

Proof. If $x \notin Z$ then we know that $(\mathscr{I}_Z)_x = \mathcal{O}_{X,x}$ because $(\mathcal{O}_X/\mathscr{I}_Z)_x = 0$ proving the first inclusion.Notice that $\mathscr{I}_{Z_1} \cdots \mathscr{I}_{Z_{r+1}} \subset \mathscr{I}_X = (0)$ because X is reduced. Therefore, if $x \in X \setminus \bigcup_{j \neq i} Z_j$ then $(\mathscr{I}_{Z_j})_x = \mathcal{O}_{X,x}$ for each $j \neq i$ and thus we must have $(\mathscr{I}_{Z_i})_x = 0$ for the relation to hold proving the complement of the second inclusion. $\square$

3.2.3 3.3

Let A be a noetherian ring and $\mathfrak{a} \subset A$ an ideal. Let $X = \text{Spec}(A)$ and $Y = V(\mathfrak{a})$.

(a) We know $\Gamma_{\mathfrak{a}}(M) = \Gamma_Y(X, \widetilde{M})$ from (II.5.6) and therefore since $\widetilde{-}$ is exact and $\Gamma_Y(X, -)$ is left exact this shows that $\Gamma_{\mathfrak{a}}(-)$ is left exact. Explicitly, let $\varphi : M \to N$ be a morphism of A-modules then $m \in \ker(\varphi : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N))$ iff $\varphi(m) = 0$ and $\mathfrak{a}^n m = 0$ for some $n > 0$ iff $m \in \Gamma_{\mathfrak{a}}(\ker \varphi)$. We denote the right dertived functors of $\Gamma_{\mathfrak{a}}(-)$ by $H^i_{\mathfrak{a}}(-)$.

(b) Because $\Gamma_{\mathfrak{a}}(-) = \Gamma_Y(X, \widetilde{-})$ and $\widetilde{-}$ takes injective modules to flasque sheaves since A is noetherian and thus $H^i_{\mathfrak{a}}(-) = R^i\Gamma_{\mathfrak{a}}(-) = R^i\Gamma_Y(X, -)(\widetilde{-}) = H^i_Y(X, \widetilde{-})$ where the last equality follows from (3.6) showing that cohomology of quasi-coherent modules on noetherian schemes is computed as the derived functors of Γ_Y on the category of coherent sheaves.

Alternatively, because $\widetilde{-}$ is exact, the functors $H^q_Y(X, \widetilde{-})$ form a δ-functor on $\mathbf{Mod}_A$. Furthermore, $\mathbf{Mod}_A$ has enough injectives and $\widetilde{I}$ is flasque since A is noetherian so $H^q_Y(X, \widetilde{I}) = 0$ and thus $H^q_Y(X, \widetilde{-})$ is effacable so they form a universal δ-functor. Furthermore, since $H^0_Y(X, \widetilde{-}) = \Gamma_Y(X, \widetilde{-}) = \Gamma_{\mathfrak{a}}(-)$ we get a natural isomorphism $H^q_Y(X, \widetilde{-}) = R^q\Gamma_{\mathfrak{a}}(-) = H^q_{\mathfrak{a}}(-)$.

Alternatively, we can show this explicitly by induction and dimension shifitng. Let M be an A-module and $M \hookrightarrow I$ an embedding into an injective A-module. Then we find an exact sequence,

$$0 \longrightarrow M \longrightarrow I \longrightarrow K \longrightarrow 0$$

The long exact sequence gives,

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(I) \longrightarrow \Gamma_{\mathfrak{a}}(K) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

$$H^q_{\mathfrak{a}}(I) \longrightarrow H^q_{\mathfrak{a}}(K) \longrightarrow H^{q+1}_{\mathfrak{a}}(M) \longrightarrow H^{q+1}_{\mathfrak{a}}(I)$$

and thus $H^q_{\mathfrak{a}}(K) \xrightarrow{\sim} H^{q+1}_{\mathfrak{a}}(M)$ for $q > 0$. Furthermore, applying the exact functor $\widetilde{-}$ we get an exact sequence,

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{I} \longrightarrow \widetilde{K} \longrightarrow 0$$

which gives a long exact sequence of cohomology with supports,

$$0 \longrightarrow \Gamma_Y(X, \widetilde{M}) \longrightarrow \Gamma_Y(X, \widetilde{I}) \longrightarrow \Gamma_Y(X, \widetilde{K}) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

$$H^q_Y(X, \widetilde{I}) \longrightarrow H^q_Y(X, \widetilde{K}) \longrightarrow H^{q+1}_Y(X, \widetilde{M}) \longrightarrow H^{q+1}_Y(X, \widetilde{I})$$

using that $\widetilde{I}$ is flasque so its higher cohomology vanishes we see $H^q_Y(X, \widetilde{K}) \xrightarrow{\sim} H^{q+1}_Y(X, \widetilde{M})$ for $q > 0$. Since $\Gamma_Y(X, \widetilde{-}) = \Gamma_{\mathfrak{a}}(-)$ the cokernel sequences imply that $H^1_{\mathfrak{a}}(M) = H^1_Y(X, \widetilde{M})$ for any M proving our base case. Now we assume for induction that $H^q_{\mathfrak{a}}(-) = H^q_Y(X, \widetilde{-})$ for $q > 0$. Then we see,

$$H^{q+1}_{\mathfrak{a}}(M) = H^q_{\mathfrak{a}}(K) = H^q_Y(X, \widetilde{K}) = H^{q+1}_Y(X, \widetilde{M})$$

proving that $H^q_{\mathfrak{a}}(M) = H^q_Y(X, \widetilde{M})$ for all $q \geq 0$ and all M by induction.

(c) First consider the case $i = 0$. For any A-module M, if $m \in \Gamma_{\mathfrak{a}}(M)$ then $\mathfrak{a}^n m = 0$ for some $m > 0$ so $m \in \Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M))$ and $\Gamma_{\mathfrak{a}}(N) \subset N$ for any N meaning that $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}}(M)$. Now, note that if M has the property that $\Gamma_{\mathfrak{a}}(M) = M$ and $\varphi : M \twoheadrightarrow N$ then $\Gamma_{\mathfrak{a}}(N) = N$ because for any $x \in N$ we can lift to some $m \in M$ and $\mathfrak{a}^n m = 0$ for some $n > 0$ and thus $\mathfrak{a}^n x = \mathfrak{a}^n \varphi(m) = \varphi(\mathfrak{a}^n x) = 0$. Therefore $\Gamma_{\mathfrak{a}}(N) = N$. Now we proceed by induction and dimension shifting. Embed $M \hookrightarrow I$ into an injective A-module I giving an exact sequence,

$$0 \longrightarrow M \longrightarrow I \longrightarrow K \longrightarrow 0$$

The long exact sequence gives for any $q \geq 0$,

$$H^q_{\mathfrak{a}}(I) \longrightarrow H^q_{\mathfrak{a}}(K) \longrightarrow H^{q+1}_{\mathfrak{a}}(M) \longrightarrow H^{q+1}_{\mathfrak{a}}(I)$$

but $H^{q+1}_{\mathfrak{a}}(I) = 0$ since I is injective and thus $H^q_{\mathfrak{a}}(K) \twoheadrightarrow H^{q+1}_{\mathfrak{a}}(M)$. Therefore, if $\Gamma_{\mathfrak{a}}(H^q_{\mathfrak{a}}(K)) = H^q_{\mathfrak{a}}(K)$ for any A-module K then we see that $\Gamma_{\mathfrak{a}}(H^{q+1}_{\mathfrak{a}}(M)) = H^{q+1}_{\mathfrak{a}}(M)$ so by induction $\Gamma_{\mathfrak{a}}(H^q_{\mathfrak{a}}(M)) = H^q_{\mathfrak{a}}(M)$ for any $q \geq 0$ and any A-module M.

3.2.4 3.4 DO!!

Let A be a noetherian ring, $\mathfrak{a} \subset A$ an ideal, and M an A-module.

(a) If M has an M-regular sequence $x_1 \in \mathfrak{a}$ of length 1 meaning $M \xrightarrow{x_1} M$ is injective and $M/x_1 M \neq 0$. Suppose that $m \in \Gamma_{\mathfrak{a}}(M)$ then $\mathfrak{a}^n m = 0$ so in particular $x_1^n m = 0$ but $M \xrightarrow{x_1} M$ is injective and so $M \xrightarrow{x_1^n} M$ is also injective showing that $m = 0$ so $\Gamma_{\mathfrak{a}}(M) = 0$.

Now let M be finitely generated and assume that there does not exist a M-regular sequence in $\mathfrak{a}$ then $\mathfrak{a}$ is contained in the set of zero divisors on M which is the union of the finitely many associated primes of M since M is finitely generated. By prime avoidance, $\mathfrak{a}$ is contained in some associated prime $\mathfrak{p} = \mathrm{Ann}_A(m)$ meaning that $\mathfrak{a} m = 0$ so $m \in \Gamma_{\mathfrak{a}}(M)$ is nonzero and thus $\Gamma_{\mathfrak{a}}(M) \neq 0$.

(b) Let M be finitely generated. We want to show that for any A-module M and $n \geq 0$ the following are equivalent,

(a) there exists a M-regular sequence in $\mathfrak{a}$ of length n

(b) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$

We have shown this for $n = 1$. Now assume the equivalence for n. First, suppose there is a length n regular sequence $x_1, \dots, x_{n+1} \in \mathfrak{a}$ then,

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1 M \longrightarrow 0$$

and $M/x_1 M$ has a regular sequence in $\mathfrak{a}$ of length n. Applying the long exact sequence,

$$H^i_{\mathfrak{a}}(M/x_1 M) \longrightarrow H^{i+1}_{\mathfrak{a}}(M) \xrightarrow{x_1} H^{i+1}_{\mathfrak{a}}(M) \longrightarrow H^{i+1}_{\mathfrak{a}}(M/x_1 M)$$

By the induction hypothesis $H^i_{\mathfrak{a}}(M/x_1 M) = 0$ for $i < n$ so the map $H^{i+1}_{\mathfrak{a}}(M) \xrightarrow{x_1} H^{i+1}_{\mathfrak{a}}(M)$ is injective. However, $\Gamma_{\mathfrak{a}}(H^{i+1}_{\mathfrak{a}}(M)) = H^{i+1}_{\mathfrak{a}}(M)$ so for any $m \in H^{i+1}_{\mathfrak{a}}(M)$ there is a $k > 0$ such that $\mathfrak{a}^k m = 0$ and thus $x_1^k \cdot m = 0$ so $m = 0$ by injectivity. Therefore $H^i_{\mathfrak{a}}(M) = 0$ for any $i < n + 1$ proving the second condition by induction.

Now suppose that $H^i_{\mathfrak{a}}(M) = 0$ for $i < n + 1$. Since $\Gamma_{\mathfrak{a}}(M) = 0$ we know there exists an M-regular element $x_1 \in \mathfrak{a}$ such that the sequence,

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1 M \longrightarrow 0$$

is exact. Applying the long exact sequence we get,

$$H^i_{\mathfrak{a}}(M) \xrightarrow{x_1} H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/x_1 M) \longrightarrow H^{i+1}_{\mathfrak{a}}(M)$$

By the hypothesis we see $H^i_{\mathfrak{a}}(M) = 0$ and $H^{i+1}_{\mathfrak{a}}(M) = 0$ for $i < n$ meaning that $H^i_{\mathfrak{a}}(M/x_1 M) = 0$ for $i < n$ so by the induction hypothesis $M/x_1 M$ has a regular sequence $x_2, \dots, x_{n+1} \in \mathfrak{a}$ of length n. Therefore, $x_1, \dots, x_n$ is an M-regular sequence in $\mathfrak{a}$ of length $n + 1$.

Therefore we can define $\mathrm{depth}_{\mathfrak{a}}(M) = \min\{n \in \mathbb{Z} \mid H^n_{\mathfrak{a}}(M) \neq 0\}$. Then every M-regular sequence in $\mathfrak{a}$ may be extended to a maximal sequence and all such maximal sequences have length n.

3.2.5 3.5 CHECK!!

Let X be a noetherian scheme and $x \in X$ a closed point. We want to show the following are equivalent:

(a) $\text{depth}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \geq 2$

(b) if U is any open neighborhood of x then $\Gamma(U, \mathcal{O}_X) \to \Gamma(U \setminus \{x\}, \mathcal{O}_X)$ is an isomorphism.

Let $Y = \{x\} \subset U$ is closed and let $U^\times = U \setminus Y$ the punctured neighborhood. Applying the excision sequence (III.2.3 (e)) for cohomology with supports,

$$0 \longrightarrow H^0_Y(U, \mathcal{O}_U) \longrightarrow H^0(U, \mathcal{O}_U) \longrightarrow H^0(U^\times, \mathcal{O}_{U^\times}) \longrightarrow H^1_Y(U, \mathcal{O}_U)$$

so we need to show that $H^i_Y(U, \mathcal{O}_U) = 0$ for $i = 0, 1$ in order to show that $H^0(U, \mathcal{O}_U) \xrightarrow{\sim} H^0(U^\times, \mathcal{O}_U)$ is an isomorphism. Let $V = \text{Spec}\,(A)$ be an affine open neighborhood of $x = \mathfrak{p} \in \text{Spec}\,(A)$ then $Y = V(\mathfrak{p})$. Applying excision for cohomology with supports (III.2.3 (f)),

$$H^i_Y(U, \mathcal{O}_U) \cong H^i_Y(V, \mathcal{O}_V) = \varinjlim_{x \in V} H^i_Y(V, \mathcal{O}_V) = \varinjlim_{f \in A \setminus \mathfrak{p}} H^i_{\mathfrak{p}}(A_f) = H^i_{\mathfrak{p}}(A_{\mathfrak{p}}) = H^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$$

Therefore, if $\text{depth}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \geq 2$ then $H^i_Y(U, \mathcal{O}_U) = H^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) = 0$ for $i < 2$ proving the required statement.

Conversely suppose that $\Gamma(U, \mathcal{O}_X) \to \Gamma(U \setminus \{x\}, \mathcal{O}_X)$ is an isomorphism for any open neighborhood. In paricular, choose $U = \text{Spec}\,(A)$ to be an affine open neighborhood of $x = \mathfrak{p} \in \text{Spec}\,(A)$. Applying the excision sequence (III.2.3 (e)) for cohomology with supports,

$$0 \longrightarrow H^0_Y(U, \mathcal{O}_U) \longrightarrow H^0(U, \mathcal{O}_U) \longrightarrow H^0(U^\times, \mathcal{O}_{U^\times}) \longrightarrow H^1_Y(U, \mathcal{O}_U) \longrightarrow H^1(U, \mathcal{O}_U)$$

but $H^0(U, \mathcal{O}_U) \to H^0(U^\times, \mathcal{O}_{U^\times})$ is an isomorphism and U is affine so $H^1(U, \mathcal{O}_U) = 0$ and thus $H^i_Y(U, \mathcal{O}_U) = 0$ for $i = 0, 1$. Applying excision for cohomology with supports (III.2.3 (f)),

$$H^i_Y(U, \mathcal{O}_U) \cong \varinjlim_{x \in V} H^i_Y(V, \mathcal{O}_V) = \varinjlim_{f \in A \setminus \mathfrak{p}} H^i_{\mathfrak{p}}(A_f) = H^i_{\mathfrak{p}}(A_{\mathfrak{p}}) = H^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$$

Therefore, $H^i_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) = H^i_Y(U, \mathcal{O}_U) = 0$ for $i < 2$ proving that $\text{depth}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) = \text{depth}_{\mathfrak{p}}(A_{\mathfrak{p}}) \geq 2$

3.2.6 3.6 CHECK!!

Let X be a noetherian scheme and choose a finite cover $U_i = \text{Spec}\,(A_i)$ of noetherian affine opens.

(a) Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathscr{F}|_{U_i} = \widetilde{M_i}$ for some A_i-module M_i. Embed $M_i \hookrightarrow I_i$ where I_i is an injective A_i-module. Let $j_i : U_i \hookrightarrow X$ be the open inclusion and define,

$$\mathscr{G} = \bigoplus_{i=1}^n (j_i)_*(\widetilde{I_i})$$

The natural map $\mathscr{F} \to \mathscr{G}$ is injective because for any $x \in X$ there is some i such that $x \in U_i$ and $\mathscr{F}_x \to \mathscr{G}_x$ is $(M_i)_x \hookrightarrow (I_i)_x$ in the i-component which is injective. Since X is Noetherian j is quasi-compact and quasi-separated (U is retrocompact) so $f_*(\widetilde{I_i})$ is quasi-coherent and the finite sum of quasi-coherent modules is quasi-coherent so $\mathscr{G}$ is quasi-coherent.

Furthermore, $(j_i)_*$ is right adjoint to $(j_i)^* = (j_i)^{-1}$ which is exact because j_i is an open immersion. Therefore, j_i preserves injective quasi-coherent modules. However, since I_i is injective and there is an equivalence of categories between A_i-modules and quasi-coherent $\mathcal{O}_{U_i}$-modules we see that $\widetilde{I_i}$ is injective in the category of quasi-coherent $\mathcal{O}_{U_i}$-modules. Therefore, $f_*(\widetilde{I_i})$ is injective in the category of quasi-coherent $\mathcal{O}_X$-modules. Furthermore, the direct sum of injectives is injective so $\mathscr{G}$ is injective in $\mathfrak{QCoh}(X)$ proving that $\mathfrak{QCoh}(X)$ has enough injectives. (CHECK!!)

Furthermore, let $\mathcal{M} \hookrightarrow \mathcal{N}$ be an injection of quasi-coherent $\mathcal{O}_X$-modules and suppose there is a map $\mathcal{M} \to \mathscr{G}$. Then locally $\mathcal{M}|_{U_i} = \widetilde{M_i}$ and $\mathcal{N}|_{U_i} = \widetilde{N_i}$ and there is an injection $M_i \hookrightarrow N_i$

(b) Let $\mathscr{I} \in \mathfrak{QCoh}(X)$ be injective and $U \subset X$ an open where $j : U \to X$ is the inclusion which is quasi-compact and quasi-separated since X is noetherian. Let $\mathcal{M}, \mathcal{N} \in \mathfrak{QCoh}(U)$ be quasi-coherent $\mathcal{O}_U$-modules with an injection $\mathcal{M} \hookrightarrow \mathcal{N}$ and given a map $\mathcal{M} \to \mathscr{I}|_U$. Then $\iota_* \mathcal{M} \hookrightarrow \iota_* \mathcal{N}$ is injective and both are quasi-coherent $\mathcal{O}_X$-modules (since U is retrocompact). By quotienting $\mathcal{M} \subset \mathcal{N}$ by the kernel of $\mathcal{M} \to \mathscr{I}|_U$ we can reduce to the case that $\mathcal{M} \to \mathscr{I}|_U$ is injective. Now view $\mathcal{M} \subset \mathscr{I}|_U$ as a submodule. Then by (II.5.15) there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{M}' \subset \mathscr{I}$ such that $\mathcal{M}|_U = \mathcal{M}$ and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}'$ such that $\mathcal{M}' \subset \mathcal{N}'$ and $\mathcal{N}'|_U = \mathcal{N}$. Thus we have a diagram,

$$\begin{tikzcd} \mathcal{M}' \arrow[r, hook] \arrow[rd, hook] & \mathcal{N}' \arrow[d, dashed] \\ & \mathscr{I} \end{tikzcd}$$

restricting to U we get a diagram,

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[rd, hook] & \mathcal{N} \arrow[d, dashed] \\ & \mathscr{I}|_U \end{tikzcd}$$

and therefore $\mathscr{I}|_U$ is injective. In particular, $\mathscr{I}|_{U_i} = \widetilde{I_i}$ where I_i is a quasi-coherent A_i-module since $\mathscr{I}|_{U_i}$ is an injective quasi-coherent $\mathcal{O}_{U_i}$-module and the category of quasi-coherent $\mathcal{O}_{U_i}$-modules is equivalent to the category of A_i-modules. By (3.4) $\widetilde{I_i}$ is flasque.

To show that $\mathscr{I}$ is flasque, it suffices to show that $\mathrm{res} : \mathscr{I}(X) \to \mathscr{I}(U)$ is surjective. Consider the filtration,

$$\tilde{U}_i = U \cup \bigcup_{j=1}^{i} U_i$$

with $\tilde{U}_0 = U$ and $\tilde{U}_n = X$. Take a section $s_0 \in \mathscr{I}(U) = \mathscr{I}(\tilde{U}_0)$. For induction, let $s_i \in \mathscr{I}(\tilde{U}_i)$ be a section over $\tilde{U}_i$ such that $s_i|_U = s_0$. Since $\mathscr{I}|_{U_{i+1}} = \widetilde{I}_{i+1}$ is flasque,

$$\mathrm{res} : \mathscr{I}(U_{i+1}) \to \mathscr{I}(\tilde{U}_i \cap U_{i+1})$$

is surjective and thus we can lift to $s'_i \in \mathscr{I}(U_{i+1})$ such that $s'_i|_{\tilde{U}_i \cap U_{i+1}} = s_i|_{\tilde{U} \cap U_{i+1}}$ therefore we can glue to get a section $s_{i+1} \in \mathscr{I}(\tilde{U}_{i+1})$ such that $s_{i+1}|_{\tilde{U}_i} = s_i$ and $s_{i+1}|_{U_{i+1}} = s'_i$ and $s_{i+1}|_U = s_i|_U = s_0$. Thus, by induction, we get a section $s_n \in \mathscr{I}(X)$ such that $s|_U = s_0$ so $\mathscr{I}$ is flasque.

(c) Let $\iota : \mathfrak{QCoh}(X) \hookrightarrow \mathfrak{Sh}(X)$ be the inclusion of categories from quasi-coherent $\mathcal{O}_X$-modules to abelian sheaves on X. Then there is a diagram of functors,

$$\begin{tikzcd} \mathfrak{QCoh}(X) \arrow[rr, "\iota"] \arrow[dr, "\Gamma'"'] & & \mathfrak{Sh}(X) \arrow[dl, "\Gamma"] \\ & \mathbf{Ab} & \end{tikzcd}$$

Then since ι takes injectives to flasques which are Γ-acyclic, there is a Grothendieck spectral sequence $E_2^{p,q} = R^p\Gamma \circ R^q\iota \implies R^{p+q}\Gamma'$ but $R^p\Gamma = H^p(X, -)$ and $R^0\iota = \iota$ and $R^q\iota = 0$ for $q > 0$ because ι is exact. Therefore, $H^p(X, -) = R^p\Gamma'(X, -)$.

Alternatively, we compute the derived functors of Γ' on $\mathfrak{QCoh}(X)$ applied to $\mathscr{F}$ by taking an injective resolution in $\mathfrak{QCoh}(X)$,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^2 \longrightarrow \cdots$$

then applying ι gives a flasque resolution of $\iota(\mathscr{F})$ in $\mathfrak{Sh}(X)$ because ι is exact. Therefore,

$$H^p(X, \iota(\mathscr{F})) = H^p(\Gamma(X, \iota(\mathscr{I}^\bullet))) = H^p(\Gamma(X, \mathscr{I}^\bullet))$$

so we can compute abelian sheaf cohomology of $\iota(\mathscr{F})$ (i.e. of $\mathscr{F}$ viewed in $\mathfrak{Sh}(X)$) via taking injective resolutions in $\mathfrak{QCoh}(X)$.

3.2.7 3.7 DO!!

Let A be a noetherian ring, $X = \mathrm{Spec}\,(A)$, $\mathfrak{a} \subset A$ an ideal, and let $U \subset X$ be the open $X \setminus V(\mathfrak{a})$.

(a) Let M be an A-module. Because A is Noetherian, $\mathfrak{a} = (f_1, \dots, f_r)$ is finitely generated. Consider the map $\varphi_n : \mathrm{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \widetilde{M})$ sending $\psi : \mathfrak{a}^n \to M$ to the section $s \in \Gamma(U, \widetilde{M})$ such that $s|_{D(f_i)} = \psi_{f_i}(1)$ where $\psi_{f_i} : \mathfrak{a}^n_{f_i} \to M_{f_i}$ maps $1 = f_i^n / f_i^n$ to $\psi_{f_1}(1)$. Suppose that $\varphi_n(\phi) = 0$. Then $\psi_{f_i} = 0$ for each i

(b) Let I be an injective A-module. Then for any open $U \subset X$ the complement $X \setminus U$ is closed and thus $X \setminus U = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$. Then consider,

$$\Gamma(U, \widetilde{I}) = \varinjlim_n \mathrm{Hom}_A(\mathfrak{a}^n, I)$$

and the map $\Gamma(X, \widetilde{I}) \to \Gamma(U, \widetilde{I})$ is given by,

$$I \to \varinjlim_n \mathrm{Hom}_A(\mathfrak{a}^n, I)$$

defined by $\mathrm{Hom}_A(A, I) \to \mathrm{Hom}_A(\mathfrak{a}^n, I)$ from $\mathfrak{a}^n \hookrightarrow A$. However, since I is injective the map $I = \mathrm{Hom}_A(A, I) \to \mathrm{Hom}_A(\mathfrak{a}^n, I)$ is surjective meaning that $\Gamma(X, \widetilde{I}) \to \Gamma(U, \widetilde{I})$ is surjective so $\widetilde{I}$ is flasque.

3.2.8 3.8

Let $A = k[x_0, x_1, x_2, \dots]$ with relations $x_0^n x_n = 0$ for each n. Now let I be an injective A-module and $A \hookrightarrow I$ an injective map. Consider the map $I \to I_{x_0}$. If we assume this is surjective then $\frac{1}{x_0}$ must have a preimage $m \in I$. Therefore, $m = \frac{1}{x_0}$ so there exists some n such that $x_0^n(x_0 m - 1) = 0$ in I. Then $x_{n+1}x_0^n(x_0 m - 1) = 0$ but $x_{n+1}x_0^{n+1} = 0$ and therefore $x_{n+1}x_0^n = 0$ in I contracting the fact that $A \hookrightarrow I$ is injective. Therefore $I \to I_{x_0}$ cannot be surjective.

3.3 4

3.3.1 4.8

Let X be a noetherian separated scheme. Define the cohomological dimension $\mathrm{cd}\,(X)$ of X as the minimal integer n such that $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves $\mathscr{F}$ and all $i > n$.

(a) To show we can replace quasi-coherent with coherent in the definition, it suffices to show that fixing i if $H^i(X, \mathscr{F}) = 0$ for all coherent sheaves $\mathscr{F}$ then $H^i(X, \mathscr{G}) = 0$ for all quasi-coherent sheaves $\mathscr{G}$. However, by (Ex. II.5.15(e)) we can write any quasi-coherent sheaf $\mathscr{G}$ as a direct limit over coherent subsheaves,

$$\mathscr{G} = \varinjlim \mathscr{F}_\alpha$$

and then by III.2.9 we have,

$$H^q(X, \mathscr{G}) = H^q(X, \varinjlim \mathscr{F}_\alpha) = \varinjlim H^q(X, \mathscr{F}_\alpha) = 0$$

(b) Let X be quasi-projective over a field k so there is an ample line bundle $\mathcal{L}$ on X. Clearly for any finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we know $H^i(X, \mathcal{E}) = 0$ for all $i > \mathrm{cd}\,(X)$. Therefore, it suffices to assume $H^i(X, \mathcal{E})$ for all finite locally free $\mathcal{E}$ and all $i > n$ and conclude that $n \geq \mathrm{cd}\,(X)$. We need to show that for each coherent sheaf $\mathscr{F}$ that $H^i(X, \mathscr{F}) = 0$ for $i > n$. We proceed by descending induction on i. For $i > \mathrm{cd}\,(X)$ this is obvious. Now assume for i and use the ampleness of $\mathcal{L}$ to choose a surjection from a finite locally free module $\mathcal{E}$ which is a sum of twists of $\mathcal{L}$. Extending to an exact sequence,

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathcal{E} \longrightarrow \mathscr{F} \longrightarrow 0$$

Therefore, we get a long exact sequence,

$$H^i(X, \mathcal{E}) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow H^{i+1}(X, \mathscr{G}) \longrightarrow H^{i+1}(X, \mathcal{E})$$

For $i > n$ we have $H^i(X, \mathcal{E}) = H^{i+1}(X, \mathcal{E}) = 0$ and thus $H^i(X, \mathscr{F}) \xrightarrow{\sim} H^{i+1}(X, \mathscr{G})$ and by the induction hypothesis $H^{i+1}(X, \mathscr{G}) = 0$ so $H^i(X, \mathscr{F}) = 0$ and thus by induction $n \geq \mathrm{cd}\,(X)$.

(c) Suppose that X has a covering by $r + 1$ affine open subsets $\mathrm{U}(=)\{U_i\}$. On a Noetherian separated scheme, Cech cohomology on affine covers computes derived cohomology for quasi-coherent sheaves and thus,

$$H^i(X, \mathscr{F}) = \check{H}^i(\mathrm{U}(,)\mathscr{F}) = H^i(\check{C}^\bullet(\mathrm{U}(,)\mathscr{F}))$$

However, for $i > r$ we have $\check{C}^i(\mathrm{U}(,)\mathscr{F}) = 0$ because there are only $r + 1$ values for the $i + 1$ indices and repetition is not allowed. Therefore, for $i > r$ we find $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves and thus $\mathrm{cd}\,(X) \leq r$.

(d) Let X be quasi-projective over dimension r over a field k. We need to show that X has a cover by $\dim X + 1$ affine open subsets. Given this, by (c) we immediately see that $\text{cd}\,(X) \le \dim X$.

Now we prove the claim by induction on $r = \dim X$. We can take the projective closure of X under an immersion $j : X \to \mathbb{P}^n$ to reduce to the case that X is projective. This suffices because an affine open cover of $\overline{X}$ intersects to an affine open cover of X because $\overline{X}$ is separated. First, projective schemes of dimension 0 are affine since they are a finite discrete set of (possibly nonreduced) points and thus lie in the complement of a suitable hyperplane not passing through the finitely many points. Given a projective scheme $X \subset \mathbb{P}^n_k$ of dimension $r+1$ take a general hyperplane section $X \cap H \subset \mathbb{P}^{n-1}_k$ such that $\dim X \cap H = r$. Then by induction, $X \cap H$ can be covered by $r+1$ affine opens $U_0, \dots, U_r$ which are the complements of hyperplane sections in H. Thus, these extend to opens $U'_0, \dots, U'_r$ of X which are the complements of hyperplane sections in $\mathbb{P}^n_k$ because we can always choose a hyperplane intersecting H at a given hyperplane of H. These cover $X \cap H$ and $U_{r+1} = X \cap (\mathbb{P}^n \setminus H)$ is affine because $X \hookrightarrow \mathbb{P}^n_k$ is affine and the complement of a hyperplane is affine. Thus $U'_0, \dots, U'_r, U_{r+1}$ is an affine open cover of X proving the claim by induction.

(e) Suppose that Y is the set-theoretic intersection of hypersurfaces $H_1, \dots, H_r$ of codimension r in $X = \mathbb{P}^n_k$. Then $U_i = X \setminus H_i$ are affine opens and because $Y = H_1 \cap \cdots \cap H_r$ set-theoretically we have $U_1 \cup \cdots \cup U_r = X \setminus Y$. Therefore, pulling back to $X \setminus Y$ the open cover $U_1, \dots, U_r$ is affine (because X is separated) and therefore $\text{cd}\,(X \setminus Y) \le r - 1$.

Notice this argument works in the more general situation that X is a quasi-projective scheme, $Y \subset X$ is a set-theoretic complete intersection $D_1 \cap \cdots \cap D_r$ for ample divisors $D_i \subset X$ then $\text{cd}\,(X \setminus Y) \le r - 1$. This is because $U_i = X \setminus D_i$ is an affine open and,

$$U_1 \cup \cdots U_r = X \setminus (D_1 \cap \cdots D_r) = X \setminus Y$$

since $Y = D_1 \cap \cdots \cap D_r$ set-theoretically. Then $U_1, \dots, U_r$ forms an affine open cover of $X \setminus Y$ showing that $\text{cd}\,(X \setminus Y) \le r - 1$.

Remark. For a projective scheme X the complement of an ample divisor D is always affine. This is because we can find an embedding $X \hookrightarrow \mathbb{P}^n$ such that $D = X \cap H$ set-theoretically and thus $X \setminus D = X \cap (\mathbb{P}^n \setminus H)$ is ample since $X \hookrightarrow \mathbb{P}^n$ is affine. However, if X is merely quasi-projective this may not be true because $j : X \hookrightarrow \mathbb{P}^n$ may not be affine so the pullback $X \cap (\mathbb{P}^n \setminus H)$ need not be affine. This happens when the inclusion $j : X \hookrightarrow \overline{X}$ into the projective closure is not an affine map. For example, let $X = \mathbb{A}^2 \setminus \{(0,0)\}$. Then $\mathcal{O}_X$ is ample but the divisor $V(1+x) = \mathbb{A}^2 \setminus \{x = 1 \text{ or } (x,y) = (0,0)\}$ is not affine. This is because $j : X \to \overline{X} = \mathbb{P}^2$ is not affine.

3.3.2 4.9

Let $X = \text{Spec}\,(k[x_1, x_2, x_3, x_4])$ be affine four-space over k. Let $Y = Y_1 \cup Y_2$ where $Y_1 = V(x_1, x_2)$ and $Y_2 = V(x_3, x_4)$. If we suppose that Y is a set theoretic complete intersection of dimension 2 in X then $\text{cd}\,(X \setminus Y) \le 1$ by the extended version of Ex. III.4.8(e). Let $U = X \setminus Y$. To reach a contradiction we will show that $H^2(U, \mathcal{O}_U) \neq 0$.

Consider the cohomology with supports sequence,

$$H^2(X, \mathcal{O}_X) \longrightarrow H^2(U, \mathcal{O}_U) \longrightarrow H^3_Y(X, \mathcal{O}_X) \longrightarrow H^3(X, \mathcal{O}_X)$$

Since $H^q(X, \mathcal{O}_X) = 0$ for $q > 0$ there is an isomorphism $H^2(U, \mathcal{O}_U) \xrightarrow{\sim} H^3_Y(X, \mathcal{O}_X)$ so it suffices to show that $H^3_Y(X, \mathcal{O}_X) \neq 0$. Furthermore, by Mayer-Vietoris for cohomology with supports,

$$H^3_{Y_1}(X, \mathcal{O}_X) \oplus H^3_{Y_2}(X, \mathcal{O}_X) \longrightarrow H^3_Y(X, \mathcal{O}_X) \longrightarrow H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \longrightarrow H^4_{Y_1}(X, \mathcal{O}_X) \oplus H^4_{Y_2}(X, \mathcal{O}_X)$$

Furthermore, consider the cohomology with supports sequences,

$$H^q_{Y_i}(X, \mathcal{O}_X) \longrightarrow H^q(X, \mathcal{O}_X) \longrightarrow H^q(X \setminus Y_i, \mathcal{O}_X) \longrightarrow H^{q+1}_{Y_i}(X, \mathcal{O}_X) \longrightarrow H^{q+1}(X, \mathcal{O}_X)$$

But $H^q(X, \mathcal{O}_X) = 0$ for $q > 0$ and $H^q(X \setminus Y_i, \mathcal{O}_X) = 0$ for $q > 1$ because Y_i is the complete intersection of $V(x_1) \cap V(x_2)$ (or $V(x_3) \cap V(x_4)$) so $\mathrm{cd}\,(X \setminus Y_i) \leq 1$. Therefore, $H^q_{Y_i}(X, \mathcal{O}_X) = 0$ for $q > 2$. Thus, returning to the Mayer-Vietoris sequence,

$$0 \longrightarrow H^3_Y(X, \mathcal{O}_X) \longrightarrow H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \longrightarrow 0$$

gives an isomorphism $H^3_Y(X, \mathcal{O}_X) \xrightarrow{\sim} H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X)$ so it suffices to show that $H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \neq 0$. Applying the cohomology with supports in $P = Y_1 \cap Y_2$ sequence,

$$H^3(X, \mathcal{O}_X) \longrightarrow H^3(X \setminus P, \mathcal{O}_X) \longrightarrow H^4_P(X, \mathcal{O}_X) \longrightarrow H^4(X, \mathcal{O}_X)$$

using that $H^q(X, \mathcal{O}_X) = 0$ for $q > 0$ we get an isomorphism $H^3(X \setminus P, \mathcal{O}_X) \xrightarrow{\sim} H^4_P(X, \mathcal{O}_X)$ so, in total we have,

$$H^2(U, \mathcal{O}_U) \xrightarrow{\sim} H^3_Y(X, \mathcal{O}_X) \xrightarrow{\sim} H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \xrightarrow{\sim} H^3(X \setminus P, \mathcal{O}_X)$$

and it suffices to show that $H^3(X \setminus P, \mathcal{O}_X) \neq 0$.

Now we take the cover $U_i = D(x_i)$ of $X \setminus P$ and consider the Cech complex begining in degree 3,

$$\bigoplus_{i=1}^{4} k[x_1, x_2, x_3, x_4]_{x_1 \cdots \hat{x_i} \cdots x_4} \longrightarrow k[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}] \longrightarrow 0$$

where the map is the alternating sum. Notice that $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ cannot be in the image if all $i_j < 0$ because each term in the image comes from a ring with not every x_i inverted. Therefore this is not surjective so $H^3(X \setminus P, \mathcal{O}_X) \neq 0$ proving that $H^2(U, \mathcal{O}_U) \neq 0$ so Y cannot be a set-theoretic complete intersection.

3.3.3 4.10

3.4 5

3.4.1 5.2

(a) Let X be a projective scheme over k and $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k. Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. We will prove that $P(n) = \chi(\mathscr{F}(n))$ is a rational polynomial by induction on $\dim \operatorname{Supp}_{\mathcal{O}_X}(\mathscr{F})$. First, notice that under the embedding $\iota : X \hookrightarrow \mathbb{P}^r_k$ associated to $\mathcal{O}_X(1)$ we have,

$$\begin{aligned} H^q(X, \mathscr{F}(n)) &= H^q(X, \mathscr{F} \otimes \mathcal{O}_X(n)) = H^q(X, \mathscr{F} \otimes \iota^* \mathcal{O}_{\mathbb{P}}(n)) = H^q(\mathbb{P}^r_k, \iota_*(\mathscr{F} \otimes \iota^* \mathcal{O}_{\mathbb{P}}(n)) \\ &= H^q(\mathbb{P}^r_k, \iota_* \mathscr{F} \otimes \mathcal{O}_{\mathbb{P}}(n)) = H^q(\mathbb{P}^r_k, (\iota_* \mathscr{F})(n)) \end{aligned}$$

using the projection formula and thus $\chi(X, \mathscr{F}(n)) = \chi(\mathbb{P}^r_k, \iota_* \mathscr{F}(n))$ and $\iota_* \mathscr{F}$ is a coherent sheaf on $\mathbb{P}^r_k$ with the same support (under the embedding $\iota : X \hookrightarrow \mathbb{P}^r_k$). Thus we reduce to the case of coherent sheaves on $X = \mathbb{P}^r_k$.

Consider the base case $\dim \operatorname{Supp}_{\mathcal{O}_X}(\mathscr{F}) = 0$ then the support is a discrete set of points and thus $\mathscr{F}(n) \cong \mathscr{F}$ so $\chi(\mathscr{F}(n))$ is a constant integrer and thus $P_{\mathscr{F}} \in \mathbb{Q}[z]$.

Now proceed by induction. We want to choose a section $\ell \in \Gamma(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}}(1))$ such that $\mathscr{F}(-1) \xrightarrow{\cdot \ell} \mathscr{F}$ is injective. To check that $\mathscr{F}(-1) \to \mathscr{F}$ is injective it suffices to on the stalks at the associated points $x \in \operatorname{Ass}_{\mathcal{O}_X}(\mathscr{F})$ of which there are finitely many (since $\mathscr{F}$ is coherent and $\mathbb{P}^r_k$ is Noetherian). Thus we may choose such an $\ell \in \Gamma(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}}(1))$ by ensuring that $\ell_x \notin \mathfrak{m}_x$ for $x \in \operatorname{Ass}_{\mathcal{O}_X}(\mathscr{F})$ then $\mathscr{F}_x \to \mathscr{F}_x$ via multiplication by ℓ_x is an isomorphism because $\mathcal{O}_{X,x}$ is local and $\mathscr{F}_x \to \mathscr{F}_x$ becomes an isomorphism after tensoring by $\kappa(x)$ since the image $\ell(x) \in \kappa(x)$ is nonzero. Therefore, we get an exact sequence,

$$0 \longrightarrow \mathscr{F}(-1) \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \otimes \mathcal{O}_H \longrightarrow 0$$

where $H = V(\ell)$ is a hyperplane and $\operatorname{coker}(\mathscr{F}(-1) \to \mathscr{F}) = \mathscr{F} \otimes \mathcal{O}_H$ via right exactness of $\mathscr{F} \otimes -$. Notice, if we only ensured that ℓ not vanish at the generic points of the componetns of $\operatorname{Supp}_{\mathcal{O}_X}(\mathscr{F})$ then $\mathscr{F}(-1) \to \mathscr{F}$ would have a nonzero kernel but one with strictly smaller dimensional support. Indeed, let $\mathscr{G} = \mathscr{F} \otimes \mathcal{O}_H$, then from the previous calculation, we see that $\mathscr{G}_x = 0$ for $x \in \operatorname{Ass}_{\mathcal{O}_X}(\mathscr{F})$ and $\operatorname{Supp}_{\mathcal{O}_X}(\mathscr{G}) \subset \operatorname{Supp}_{\mathcal{O}_X}(\mathscr{F})$ so we must have,

$$\dim \operatorname{Supp}_{\mathcal{O}_X}(\mathscr{G}) \le \operatorname{Supp}_{\mathcal{O}_X}(\mathscr{F}) - 1$$

In fact, we have equality because $s|_Z$ is a regular section of $\mathcal{O}_Z(1)$ where $Z = \operatorname{Supp}_{\mathcal{O}_X}(\mathscr{F})$ and thus $Z \cap H \subset Z$ is Cartier so the equality follows from Krull. Anyway, from the exact sequence twisted by $\mathcal{O}_{\mathbb{P}}(n)$,

$$\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{G}(n))$$

However, by the induction hypothesis $P_{\mathscr{G}}(n) = \chi(\mathscr{G}(n))$ for a polynomial $P_{\mathscr{G}} \in \mathbb{Q}[z]$ and therefore since $P_{\mathscr{F}}(n) - P_{\mathscr{F}}(n-1) = P_{\mathscr{G}}(n)$ is a polynomial it implies that $P_{\mathscr{F}} \in \mathbb{Q}[z]$ proving the claim by induction.

(b) Let $S = k[x_0, \dots, x_r]$. Recall that for a graded S-module M we define the Hilbert function $\varphi_M(n) = \dim_k M_n$ and the Hilbert polynomial $P_M \in \mathbb{Q}[z]$ is the unique polynomial agreeing with φ_M for $n \gg 1$. Now let $M = \Gamma_*(\mathscr{F})$ so $M_n = H^0(\mathbb{P}^r_k, \mathscr{F}(n))$. For $n \gg 0$ we know that $\chi(\mathscr{F}(n)) = H^0(\mathbb{P}^r_k, \mathscr{F}(n))$ by vanishing of cohomology. Therefore $P_{\mathscr{F}}(n) = \varphi_M(n)$ for $n \gg 0$ and $P_{\mathscr{F}} \in \mathbb{Q}[z]$ proving that $P_{\mathscr{F}} = P_M$ by uniqueness.

3.4.2 5.3

Let X be a projective scheme of dimension r over a field k. The *arithmetic genus* of X is defined by,

$$p_a(X) = (-1)^r \left(\chi(\mathcal{O}_X) - 1\right)$$

Note that being projective is equivalent to being quasi-projective and proper so χ is defined for any coherent $\mathcal{O}_X$-module so, in particular, for $\mathcal{O}_X$ itself.

(a) Let X be a projective integral scheme over an algebraically closed field k. By Lemma **??** the scheme X is proper over k so by Lemma **??**, $\mathcal{O}_X(X) = H^0(X, \mathcal{O}_X)$ is a finite and thus algebraic extension of k. Since k is algebraically closed, $\mathcal{O}_X(X) = k$ and thus

$$\dim_k H^0(X, \mathcal{O}_X) = 1$$

Therefore,

$$\begin{aligned} p_a(X) &= (-1)^{r+1} + (-1)^r \sum_{i=0}^{r} (-1)^i \dim_k H^i(X, \mathcal{O}_X) \\ &= (-1)^{r+1} + (-1)^r + (-1)^r \sum_{i=1}^{r} (-1)^i \dim_k H^i(X, \mathcal{O}_X) \\ &= \sum_{i=1}^{r} (-1)^{i+r} \dim_k H^i(X, \mathcal{O}_X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X) \end{aligned}$$

In particular, when X is a projective curve,

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$$

(b) In section I, we defined $p_a(Y) := (-1)^r(P_Y(0) - 1)$ where P_Y is the Hilbert polynomial of the embedding $\iota : Y \hookrightarrow \mathbb{P}^N_k$. However, in the previous exercise we showed that $P_Y(n)$ agrees with $\chi(\mathcal{O}_Y(n))$ where $\mathcal{O}_Y(n) = \iota^* \mathcal{O}_{\mathbb{P}^N_k}(n)$ and therefore $P_Y(0) = \chi(\mathcal{O}_Y)$ so the two definitions agree.

(c) We want to show that p_a is a birational invariant for nonsingular projective curves over an algebraically closed field k. This is simply because each birational class of curves has a single nonsingular projective model (MAYBE GIVE A BETTER PROOF?).

In particular, a degree 3 plane curve has $p_a(X) = 1$ and thus cannot be birational to $\mathbb{P}^1$.

3.4.3 5.4

Let X be a projective scheme over a field k and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Consider the map,

$$P : K(X) \to \mathbb{Q}[z]$$

sending the class of the coherent sheaf $\mathscr{F}$ to its Hilbert polynomial: $[\mathscr{F}] \mapsto P_{\mathscr{F}}$ where $P_{\mathscr{F}}(n) := \chi(\mathscr{F}(n))$ is the Hilbert polynomial. This is well-defined because given an exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

of coherent sheaves, then $[\mathscr{F}_2] = [\mathscr{F}_1] + [\mathscr{F}_3]$ but we also know $P_{F_2} = P_{\mathscr{F}_1} + P_{\mathscr{F}_2}$ and therefore $P([\mathscr{F}_2]) = P([\mathscr{F}_1] + [\mathscr{F}_3])$. Furthermore, this map is unique for the condition that $P([\mathscr{F}]) = P_{\mathscr{F}}$ since $K(X)$ is generated by these classes.

Now let $X = \mathbb{P}^r_k$ and let $L_i \subset \mathbb{P}^r_k$ be a linear space of dimension i for each $i = 0, 1, \dots, r$. Then notice,

$$\chi(\mathcal{O}_{L_i}(n)) = \binom{n+i}{i} = \tfrac{1}{i!}(n+i)(n+i-1)\cdots(n+1)$$

We want to show that,

(a) $K(X)$ is free abelian generated by $[\mathcal{O}_{L_i}]$ for $i = 0, 1, \dots, r$

(b) the map $P : K(X) \to \mathbb{Q}[z]$ is injective.

First notice that (a) $\implies$ (b) because the polynomials P_{L_i} are $\mathbb{Q}$-linearly independent. To show this, suppose that,

$$\sum_{i=0}^{r} a_i P_{L_i} = 0$$

Since the leading order term n^r only appears in P_{L_r} so we must have $a_r = 0$ and thus,

$$\sum_{i=0}^{r-1} a_i P_{L_i} = 0$$

reducing to the $r-1$ case proving the linear independence by induction.

Now we prove (a) and (b) by induction on r. The caes $r = 0$ is trivial because the Grothendieck group of finite k-modules is clearly free abelian on one generator $[k]$. Now for $X = \mathbb{P}^{r+1}_k$ consider a hyperplane $H \subset X$ so $H \cong \mathbb{P}^r_k$ and we may take $L_r = H$. In fact, we may take a flag on linear spaces,

$$L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_r = H \subsetneq L_{r+1} = X$$

so that $\mathcal{O}_{L_i}$ have support contained in H. Let $U = X \setminus H \cong \mathbb{A}^{r+1}_k$. Now by Exercise (II.6.10c) there is an exact sequence,

$$K(H) \longrightarrow K(X) \longrightarrow K(U) \longrightarrow 0$$

Where the map $K(H) \to K(X)$ sends $[\mathscr{F}] \mapsto [\iota_* \mathscr{F}]$. Notice that $P_{\iota_* \mathscr{F}}(n) = \chi(X, \iota_* \mathscr{F}(n)) = \chi(H, \mathscr{F}(n)) = P_{\mathscr{F}}(n)$ because $\iota^* \mathcal{O}_X(1) = \mathcal{O}_H(1)$ and $H^q(X, \iota_* \mathscr{F}) = H^q(H, \mathscr{F})$ and using the projection formula. Therefore, there is a commutative diagram,

$$\begin{tikzcd} K(H) \arrow[r, "\iota_*"] \arrow[dr, "P"'] & K(X) \arrow[d, "P"] \\ & \mathbb{Q}[z] \end{tikzcd}$$

However, by the induction hypothesis, $P : K(H) \to \mathbb{Q}(z)$ is injective and therefore $K(H) \to K(X)$ is injective. Furthermore, $K(U) \cong \mathbb{Z} \cdot [\mathcal{O}_U]$ because $U \cong \mathbb{A}^{r+1}_k$ and thus every finite module has a finite free resolution by Hilbert's theorem on syzygies[1] and thus $K(U)$ is generated by $[\mathcal{O}_U]$. Since $\mathbb{Z}$ is projective, the sequence splits giving,

$$K(X) = K(H) \oplus K(U)$$

Furthermore, because we assumed the linear spaces L_i form a flag inside H for $i \le r$ we see that $K(H)$ is a free abelian group generated by $[\mathcal{O}_{L_i}]$ for $i = 0, 1, \dots, r$ by the induction hypothesis. Additionally, the coherent sheaves $\mathcal{O}_{L_i}$ have support inside H and thus map to zero under $K(X) \to K(U)$ whereas $[\mathcal{O}_{L_{r+1}}] = [\mathcal{O}_X] \mapsto [\mathcal{O}_U]$ which is the generator and therefore we can choose a section $K(U) \to K(X)$ via $[\mathcal{O}_U] \to [\mathcal{O}_X]$. Thus, from the splitting $K(X) = K(H) \oplus K(U)$ we see that $K(X)$ is a free $\mathbb{Z}$-module generated by $[\mathcal{O}_{L_i}]$ for $i = 0, 1, \dots, r, r+1$ proving (a) and thus also (b) for $r+1$ and thus for all r by induction.

3.4.4 5.5

Let $X = \mathbb{P}^r_k$ and $Y \subset X$ be a closed subscheme of dimension $q \ge 1$ which is a complete intersection. We want to prove the following,

(a) for all $n \in \mathbb{Z}$ the natural map,

$$H^0(X, \mathcal{O}_X(n)) \twoheadrightarrow H^0(Y, \mathcal{O}_Y(n))$$

is surjective

(b) Y is connected

(c) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$

(d) $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$

First (a) $\implies$ (b) because $H^0(X, \mathcal{O}_X) \twoheadrightarrow H^0(Y, \mathcal{O}_Y)$ is thus one dimensional so Y is connected. Furthermore, (a) and (c) $\implies$ (d) because $\dim_k H^0(Y, \mathcal{O}_Y) = 1$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < q$ and therefore,

$$p_a(Y) = (-1)^q(\chi(\mathcal{O}_Y) - 1) = \sum_{i=1}^{q} (-1)^{q-i} \dim_k H^i(Y, \mathcal{O}_Y) = \dim_k H^q(Y, \mathcal{O}_Y)$$

Thus it suffices to prove (a) and (c).

We proceed by descending induction on q. For $q = r$ we consider the case $Y = X$ for which (a) is obvious and we know $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < r$. Now assume (a) and (c) for dimension $q+1$. Let Y be a complete intersection of dimension q then Y is the intersection of a hypersurface of degree d and a complete intersection W of dimension $q+1$. Therefore, $Y \subset W$ is a closed subscheme cut out by a section of $\mathcal{O}_W(d)$ so there is an exact sequence,

$$0 \longrightarrow \mathcal{O}_W(n-d) \longrightarrow \mathcal{O}_W(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0$$

Therefore, we get an exact sequence,

[1] The fact that U is regular and affine is not enough as this only shows there is a finite locally free resolution but we need aditionally that on affine space finite projective modules are free.

$$H^0(W, \mathcal{O}_W(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)) \longrightarrow H^1(W, \mathcal{O}_W(n-d))$$

However, by assumption (c) of the induction hypothesis $H^1(W, \mathcal{O}_W(n-d)) = 0$ because $1 < q+1$ so $H^0(W, \mathcal{O}_W(n)) \twoheadrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective. By assumption (a), the map $H^0(X, \mathcal{O}_X(n)) \twoheadrightarrow H^0(W, \mathcal{O}_W(n))$ is surjective and therefore,

$$H^0(X, \mathcal{O}_X(n)) \twoheadrightarrow H^0(W, \mathcal{O}_W(n)) \twoheadrightarrow H^0(Y, \mathcal{O}_Y(n))$$

is surjective. Furthermore, the long exact sequence contains,

$$H^i(W, \mathcal{O}_W(n)) \longrightarrow H^i(Y, \mathcal{O}_Y(n)) \longrightarrow H^{i+1}(W, \mathcal{O}_W(n-d))$$

By assumption (c), when $i > 0$ and $i+1 < q+1$ we know that $H^i(W, \mathcal{O}_W(n)) = H^{i+1}(W, \mathcal{O}_W(n)) = 0$ and therefore $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$. This proves (a) and (c) by induction for all complete intersections of dimension $q \geq 1$.

3.4.5 5.6 DO!!

Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbb{P}^3_k$ over a field k. Since $\text{Pic}\,(Q) = \mathbb{Z} \oplus \mathbb{Z}$ so effective Cartier divisors correspond to nonzero sections of $\mathcal{O}_Q(a, b)$ so divisors on Q are bigraded in degree (a, b).

(a)

(b)

(c)

(d)

3.4.6 5.7 DO!!

Let X, Y, Z be proper schemes over a noetherian ring A and $\mathcal{L}$ and invertible sheaf.

(a) If $\mathcal{L}$ is ample on X and $\iota Z \hookrightarrow X$ is a closed embedding then consider $\iota^* \mathcal{L}$. For any coherent $\mathcal{O}_Z$-module $\mathscr{F}$ consider $\mathscr{F} \otimes \iota^* \mathcal{L}^{\otimes n}$. We know that,

$$H^0(Z, \mathscr{F} \otimes \iota^* \mathcal{L}^{\otimes n}) = H^0(X, \iota_*(\mathscr{F} \otimes \iota^* L^{\otimes n}))$$

but by the projection formula,

$$\iota_*(\mathscr{F} \otimes \iota^* L^{\otimes n}) = \iota_* \mathscr{F} \otimes \mathcal{L}^{\otimes n}$$

which is generated by global sections for $n \gg 0$ because $\iota_* \mathscr{F}$ is coherent and $\mathcal{L}$ is ample. Therefore, we get a surjection,

$$\bigoplus_{i \in I} \mathcal{O}_X \twoheadrightarrow \iota_* \mathscr{F} \otimes \mathcal{L}^{\otimes n}$$

and pulling back gives a surjection,

$$\bigoplus_{i \in I} \mathcal{O}_Z \twoheadrightarrow \mathscr{F} \otimes \iota^* \mathcal{L}^{\otimes n}$$

so $\mathscr{F} \otimes \iota^* \mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$ and thus $\iota^* \mathcal{L}$ is ample.

(b) If $\mathcal{L}$ is ample on X then $\mathcal{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red} by (a) using the closed immersion $X_{\text{red}} \hookrightarrow X$. Conversely suppose that $\mathcal{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red}. To show that $\mathcal{L}$ is ample, it suffices to show that for each coherent sheaf $\mathscr{F}$ there exists a constant $n_{\mathscr{F}}$ such that for all $n \ge n_{\mathscr{F}}$ and $q > 0$ that $H^q(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$. Consider the filtration,

$$\mathscr{F} \supset \mathcal{N} \cdot \mathscr{F} \supset \mathcal{N}^2 \cdot \mathscr{F} \supset \cdots \supset \mathcal{N}^n \cdot \mathscr{F} \supset \mathcal{N}^{n+1} \cdot \mathscr{F} = 0$$

let $\mathscr{F}_i = \mathcal{N}^i \cdot \mathscr{F}$ then $\mathscr{G}_i = \mathscr{F}_i / \mathscr{F}_{i+1}$ satisfies $\mathcal{N} \cdot \mathscr{G}_i = 0$. Since $\iota : X_{\text{red}} \to X$ is a closed immersion ι_* induces an equivalence of categories between quasi-coherent $\mathcal{O}_{X_{\text{red}}}$-modules and quasi-coherent $\mathcal{O}_X$-modules killed by $\mathcal{N}$. Thus $\mathscr{G}_i = \iota_* \mathscr{G}'_i$ where $\mathscr{G}'_i$ is a $\mathcal{O}_{X_{\text{red}}}$-module. The twisted exact sequence,

$$0 \longrightarrow \mathscr{F}_{i+1} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathscr{F}_i \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathscr{G}_i \otimes \mathcal{L}^{\otimes n} \longrightarrow 0$$

gives an exact sequence,

$$H^q(X, \mathscr{F}_{i+1} \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^q(X, \mathscr{F}_i \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^q(X, \mathscr{G}_i \otimes \mathcal{L}^{\otimes n})$$

Using the projection formula, $\mathscr{G}_i \otimes \mathcal{L}^{\otimes n} = \iota_* \mathscr{G}'_i \otimes \mathcal{L}^n = \iota_*(\mathscr{G}'_i \otimes (\iota^* \mathcal{L})^{\otimes n})$ and thus,

$$H^q(X, \mathscr{G}_i \otimes \mathcal{L}^{\otimes n}) = H^q(X_{\text{red}}, \mathscr{G}'_i \otimes (\mathcal{L} \otimes \mathcal{O}_{X_{\text{red}}})^{\otimes n})$$

which vanishes for $q > 0$ and $n \ge n_{\mathscr{G}'_i}$. Because $\mathscr{F}_{n+1} = 0$ vanishing holds for $i = n+1$. Thus we proceed by descending induction by assuming that $H^q(X, \mathscr{F}_{i+1} \otimes \mathcal{L}^{\otimes n}) = 0$ for $q > 0$ and $n \ge n_{i+1}$. Then if $n \ge n_i = \max\{(n_i, n_{\mathscr{G}_i})\}$ and $q > 0$ we see that $H^q(X, \mathscr{F}_i \otimes \mathcal{L}^{\otimes n})$ from the exact sequence. Thus, by induction, vanishing holds for $\mathscr{F} = \mathscr{F}_0$ and $n \ge n_0$ meaning that $\mathcal{L}$ is ample on X.

(c) If $\mathcal{L}$ is ample on X then any irreducible component $Z \hookrightarrow X$ is included via a closed immersion and thus $\mathcal{L}|_Z$ is ample on Z.

Conversely, suppose that X is reduced and $\mathcal{L}|_Z$ is ample for each irreducible component $Z \subset X$. Because X is Noetherian, there are finitely many irreducible components Z_i. We proceed by induction on the number of irreducible components so assume the theorem for r components and let X have irreducible components $Z_1, \dots, Z_{r+1}$. If there is only one irreducible component then because X is reduced $X = Z$ and thus the statement is trivial. Now proceed by induction. Take any coherent $\mathcal{O}_X$-module $\mathscr{F}$ and consider the exact sequence,

$$0 \longrightarrow \mathscr{I}_Z \cdot \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} / \mathscr{I}_Z \mathscr{F} \longrightarrow 0$$

where $Z \subset X$ is an irreducible component. By Lemma 3.2.1,

$$\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{I}_Z \otimes \mathscr{F}) \subset X' = Z_1 \cup \cdots \cup Z_r$$

where $Z_1, \dots, Z_r \subset X$ are the irreducible components besides Z so X' has r components and $\mathscr{I}_Z \cdot \mathscr{F}$ is the pushforward of a $\mathcal{O}_{X'}$-module $\mathscr{F}'$ (possibly with nonreduced structure but ampleness is preserved under reduction). Likewise, $\mathscr{G} = \mathscr{F} / \mathscr{I}_Z \mathscr{F}$ is anhilated by $\mathscr{I}_Z$ and thus $\mathscr{F} / \mathscr{I}_Z \mathscr{F} = \iota_* \iota^* \mathscr{G}$. Twisting by $\mathcal{L}^{\otimes n}$ and applying the projection formula gives an exact sequence,

$$0 \longrightarrow j_*(\mathscr{F}' \otimes \mathcal{L}^{\otimes n}|_{X'}) \longrightarrow \mathscr{F} \otimes \mathcal{L}^{\otimes n} \longrightarrow \iota_*(\mathscr{G} \otimes \mathcal{L}^{\otimes n}|_Z) \longrightarrow 0$$

Then taking the cohomology sequence,

$$H^q(X', \mathscr{F}' \otimes \mathcal{L}|_{X'}^{\otimes n}) \longrightarrow H^q(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^q(Z, \mathscr{G} \otimes \mathcal{L}|_Z^{\otimes n})$$

By assumption, $\mathcal{L}|_Z$ is ample and $\mathcal{L}|_{X'}$ is ample when restricted to the r irreducible components of X' so (perhaps after reducing X') by the induction hypothesis $\mathcal{L}|_{X'}$ is ample. Since $\mathscr{F}'$ and $\mathscr{G}$ are coherent there exist integers n_0' and n_Z such that for all $q > 0$,

$$n \ge n_0' \implies H^q(X', \mathscr{F}' \otimes \mathcal{L}|_{X'}^{\otimes n}) = 0 \quad \text{and} \quad n \ge n_Z \implies H^q(Z, \mathscr{G} \otimes \mathcal{L}|_Z^{\otimes n}) = 0$$

Therefore, for $n \ge n_0 = \max\{n_0', n_Z\}$ and $q > 0$ the exact sequence gives that $H^q(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$ proving that $\mathcal{L}$ is ample on X. Thus the result holds for any number of irreducible components by induction.

(d) First, let $f : X \to Y$ be a finite morphism and $\mathcal{L}$ ample on Y. Then I claim that $f^*\mathcal{L}$ is ample on X. Let $\mathscr{F}$ be any coherent $\mathcal{O}_X$-module then by the projection formula $f_*(\mathscr{F} \otimes f^*\mathcal{L}^{\otimes n}) = f_*\mathscr{F} \otimes \mathcal{L}^{\otimes n}$. Furthermore, f is affine so f_* preserves cohomology showing that,

$$H^q(X, \mathscr{F} \otimes f^*\mathcal{L}^{\otimes n}) = H^q(Y, f_*(\mathscr{F} \otimes f^*\mathcal{L}^{\otimes n})) = H^q(Y, f_*\mathscr{F} \otimes \mathcal{L}^{\otimes n})$$

Because $\mathscr{F}$ is coherent and $f : X \to Y$ is proper then $f_*\mathscr{F}$ is coherent so there exists an integer $n_{f_*\mathscr{F}}$ such that for all $n \ge n_{f_*\mathscr{F}}$ and $q > 0$ we have,

$$H^q(X, \mathscr{F} \otimes f^*\mathcal{L}^{\otimes n}) = H^q(Y, f_*\mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

and therefore $f^*\mathcal{L}$ is ample on X.

Now suppose that $f : X \to Y$ is finite and surjective and $f^*\mathcal{L}$ is ample. We now will show that $\mathcal{L}$ is ample by by Noetherian induction on Y. By (b) and (c) $\mathcal{L}$ is ample iff $\mathcal{L}|_{Y_{\text{red}}}$ is ample iff $\mathcal{L}|_Z$ is ample for each irreducible component $Z \subset Y_{\text{red}}$. Let $\mathcal{P}$ be the property of closed subsets $Z \subset Y$ that $\mathcal{L}|_Z$ is ample. Then if Y has $\mathcal{P}$ meaning $\mathcal{L}|_{Y_{\text{red}}}$ is ample then $\mathcal{L}$ is ample proving the claim. Thus, towards Noetherian induction, it suffices to show that if $Z \subset Y$ is a closed subset such that every proper closed subset $C \subsetneq Z$ has $\mathcal{P}$ then Z has $\mathcal{P}$. Notice if Z is reducible this is automatic because $\mathcal{L}|_Z$ is ample iff $\mathcal{L}|_Z$ restricted to irreducible component is ample by (c) thus we need only consider the case that Z is irreducible.

Base changing by $Z \hookrightarrow Y$ we get a finite surjective map $X_Z \to Z$ where $X_Z \hookrightarrow X$ is a closed immersion so $(f^*\mathcal{L})|_{X_Z}$ is ample. Since $X_Z \to Z$ is surjective, some $\xi \in X_Z$ must hit the generic point $\eta \in Z$. Give $W = \overline{\{\xi\}}$ the reduced subscheme structure then composing with the closed immersion $W \hookrightarrow X_Z$ gives a finite map $f' : W \to Z$ which is dominant because $\xi \mapsto \eta$ and thus surjective since $f' : W \to Z$ is closed. Since $(f')^*\mathcal{L} = (f^*\mathcal{L})|_W$ is ample using the closed immerison $W \hookrightarrow X$ and both W and Z are integral we have reduced to the integral case.

We will show that $\mathcal{L}|_Z$ is ample by using Serre's criterion. For any coherent $\mathcal{O}_Z$-module $\mathscr{F}$, by Ex. III.4.2(b) there is a coherent $\mathcal{O}_W$-module $\mathscr{G}$ and a morphism $\beta : f_*\mathscr{G} \to \mathscr{F}^{\oplus r}$ which is an isomorphism at the generic point $\eta \in Z$. Extend to an exact sequence,

$$0 \longrightarrow \ker\beta \longrightarrow f_*\mathscr{G} \xrightarrow{\beta} \mathscr{F}^{\oplus r} \longrightarrow \operatorname{coker}\beta \longrightarrow 0$$

Taking the stalk at η gives an exact sequence,

$$0 \longrightarrow (\ker\beta)_\eta \longrightarrow (f_*\mathscr{G})_\eta \xrightarrow{\beta} \mathscr{F}_\eta^{\oplus r} \longrightarrow (\operatorname{coker}\beta)_\eta \longrightarrow 0$$

but β is an isomorphism at η so $(\ker\beta)_\eta = (\operatorname{coker}\beta)_\eta = 0$ and thus their supports are proper closed subsets C_1 and C_2 of Z. In particular, $\ker\beta$ and $\operatorname{coker}\beta$ are extensions of coherent sheaves on C_1 and C_2 (with possibly nonreduced structure) but by the induction hypothesis $\mathcal{L}|_{(C_i)_{\text{red}}}$ is ample and thus $\mathcal{L}|_{C_i}$ is ample. Since $\ker\beta$ and $\operatorname{coker}\beta$ are coherent there exists n_0' such that for $n \geq n_0'$ and $q > 0$,

$$H^q(X, \ker\beta \otimes \mathcal{L}^{\otimes n}) = H^q(X, \iota_*\iota^* \ker\beta \otimes \mathcal{L}|_{C_1}^{\otimes n}) = H^q(C_1, \iota^* \ker\beta \otimes \mathcal{L}|_{C_1}^{\otimes n}) = 0$$

and likewise $H^q(X, \operatorname{coker}\beta \otimes \mathcal{L}^{\otimes n}) = 0$. Now split the exact sequence into short exact sequences,

$$0 \longrightarrow \ker\beta \longrightarrow f_*\mathscr{G} \longrightarrow \mathscr{C} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{C} \longrightarrow \mathscr{F}^{\oplus r} \longrightarrow \operatorname{coker}\beta \longrightarrow 0$$

and consider the long exact sequences after twisting,

$$H^q(Z, \ker\beta \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^q(Z, f_*\mathscr{G} \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^q(Z, \mathscr{C} \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^{q+1}(Z, \ker\beta \otimes \mathcal{L}^{\otimes n})$$

$$H^q(Z, \mathscr{C} \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^q(Z, \mathscr{F} \otimes \mathcal{L}^{\otimes n})^{\oplus r} \longrightarrow H^q(Z, \operatorname{coker}\beta \otimes \mathcal{L}^{\otimes n}) \longrightarrow H^{q+1}(Z, \mathscr{C} \otimes \mathcal{L}^{\otimes n})$$

giving $H^q(Z, f_*\mathscr{G} \otimes \mathcal{L}^{\otimes n}) \xrightarrow{\sim} H^q(Z, \mathscr{C} \otimes \mathcal{L}^{\otimes n})$ and $H^q(Z, \mathscr{C} \otimes \mathcal{L}^{\otimes n}) \twoheadrightarrow H^q(Z, \mathscr{F} \otimes \mathcal{L}^{\otimes n})^{\oplus r}$ for $q > 0$ and $n \geq n_0'$ by the vanishing of cohomology for $\ker\beta$ and $\operatorname{coker}\beta$. Furthermore, using that f is affine and the projection formula,

$$H^q(Z, f_*\mathscr{G} \otimes \mathcal{L}^{\otimes n}) = H^q(Z, f_*(\mathscr{G} \otimes f^*\mathcal{L}^{\otimes n})) = H^q(W, \mathscr{G} \otimes f^*\mathcal{L}^{\otimes n})$$

By assumption, $f^*\mathcal{L}$ is ample so because $\mathscr{G}$ is coherent there exists an integer n_1 such that for $n \geq n_1$ and $q > 0$ we have $H^q(Z, \mathscr{G} \otimes f^*\mathcal{L}^{\otimes n}) = 0$. Thus, the exact sequence shows that $H^q(Z, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for $q > 0$ and $n \geq n_0 = \max\{n_0', n_1\}$ proving that $\mathcal{L}$ is affine by Serre's criterion and thus showing that Z satisfies $\mathcal{P}$.

3.4.7 5.8 DO!!

We prove that one-dimensional proper schemes X over an algebraically closed field k are projective.

(a) Let X be irreducible and nonsingular. Then X is a nonsingular complete curve over k and thus projective by II.6.7.

(b) Let X be integral and $\nu : \tilde{X} \to X$ be its normalization.

(c)

(d)

3.4.8 5.9 DO!!

3.4.9 5.10

Let X be a projective scheme over a noetherian ring A. First, notice that if $\mathscr{F} \twoheadrightarrow \mathscr{G}$ is a surjection of coherent sheaves then we may extend to an exact sequence,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

Twisting by $\mathcal{O}_X(n)$ and taking the long exact sequence gives,

$$0 \longrightarrow \Gamma(X, \mathscr{K}(n)) \longrightarrow \Gamma(X, \mathscr{F}(n)) \longrightarrow \Gamma(X, \mathscr{G}(n)) \longrightarrow H^1(X, \mathscr{K}(n))$$

Since $\mathscr{K}$ is coherent, there exists a $n_{\mathscr{K}}$ such that for all $n \geq n_{\mathscr{K}}$ we have $H^1(X, \mathscr{K}(n)) = 0$ and thus $\Gamma(X, \mathscr{F}(n)) \twoheadrightarrow \Gamma(X, \mathscr{G}(n))$ is surjective.

Now, we will prove the proposition by induction on r. The cases $r = 0, 1, 2$ are trivial. Now suppose the result holds for r and let

$$\mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \cdots \longrightarrow \mathscr{F}_r \longrightarrow \mathscr{F}_{r+1}$$

be an exact sequence of coherent sheaves on X. Then we can split this into sequences,

$$\mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \cdots \longrightarrow \mathscr{F}_{r-1} \longrightarrow \mathscr{K}_r \longrightarrow 0$$

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F}_r \longrightarrow \mathscr{C} \longrightarrow 0$$

for subsheaves $\mathscr{K} \subset \mathscr{F}_r$ and $\mathscr{C} \subset \mathscr{F}_{r+1}$. By the induction hypothesis there is an integer n_1 such that for all $n \geq n_1$ we have,

$$\Gamma(X, \mathscr{F}_1(n)) \longrightarrow \Gamma(X, \mathscr{F}_2(n)) \longrightarrow \cdots \longrightarrow \Gamma(X, \mathscr{F}_{r-1}(n)) \longrightarrow \Gamma(X, \mathscr{K}(n))$$

and from the long exact sequence of the twist of the second short exact sequence,

$$0 \longrightarrow \Gamma(X, \mathscr{K}(n)) \longrightarrow \Gamma(X, \mathscr{F}_r(n)) \longrightarrow \Gamma(X, \mathscr{C}(n)) \longrightarrow H^1(X, \mathscr{K}(n))$$

and because $\mathscr{K}$ is coherent for $n \geq n_2$ we have $H^1(X, \mathscr{K}(n)) = 0$ and thus the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{K}(n)) \longrightarrow \Gamma(X, \mathscr{F}_r(n)) \longrightarrow \Gamma(X, \mathscr{C}(n)) \longrightarrow 0$$

is exact. Furthermore, for $n \geq n_3$ we know that $\Gamma(X, \mathscr{F}_{r-1}(n)) \twoheadrightarrow \Gamma(X, \mathscr{K}(n))$ is surjecitve. Lastly, $\Gamma(X, \mathscr{C}(n)) \hookrightarrow \Gamma(X, \mathscr{F}_{r+1}(n))$ is injective because Γ is right exact. Thus, for $n \geq n_0 = \max(n_1, n_2, n_3)$, we can patch these together to get a long exact sequence

$$\Gamma(X, \mathscr{F}_1(n)) \longrightarrow \Gamma(X, \mathscr{F}_2(n)) \longrightarrow \cdots \longrightarrow \Gamma(X, \mathscr{F}_r(n)) \longrightarrow \Gamma(X, \mathscr{F}_{r+1}(n))$$

proving the claim by induction.

4 Appendix

4.1 A Intersection Theory

4.1.1 6.7

Let X be a nonsingular projective 3-fold with Chern classes c_1, c_2, c_3. Then we apply Grothendieck-Riemann-Roch,

$$\mathrm{ch}(f_!\mathcal{E}) = f_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_X))$$

to the morphism $f : X \to \mathrm{Spec}\,(k)$. To give,

$$\chi(\mathcal{E}) = \deg\left(\mathrm{ch}(\mathcal{L}) \cdot \mathrm{td}(\mathcal{T}_X)\right)_n$$

where pushing forward onto a point selects the dimension zero (i.e. codimension 3) part and takes degrees. Thus it suffices to compute the Todd class,

$$\mathrm{td}(\mathcal{T}_X) = 1 + \tfrac{1}{2}c_1(\mathcal{T}_X) + \tfrac{1}{12}(c_1(\mathcal{T}_X)^2 + c_2(\mathcal{T}_X)) + \tfrac{1}{24}c_1(\mathcal{T}_X)c_2(\mathcal{T}_X)$$

and by definition $c_i(\mathcal{T}_X) = c_i$. For a line bundle $\mathcal{L}$ with $c(\mathcal{L}) = 1 + D \in A^*(X)$ for some divisor D we have,

$$\mathrm{ch}(\mathcal{L}) = 1 + D + \tfrac{1}{2}D \cdot D + \tfrac{1}{6}D \cdot D \cdot D$$

and thus we find,

$$\begin{aligned}(\mathrm{ch}(\mathcal{L}) \cdot \mathrm{td}(\mathcal{T}_X))_n &= \tfrac{1}{24}c_1c_2 + D \cdot \tfrac{1}{12}(c_1^2 + c_2) + \tfrac{1}{2}D^2 \cdot \tfrac{1}{2}c_1 + \tfrac{1}{6}D^3 \\ &= \tfrac{1}{12}D \cdot (D + c_1) \cdot (2D + c_1) + \tfrac{1}{12}D \cdot c_2 + \tfrac{1}{24}c_1c_2\end{aligned}$$

For $D = 0$ we find, $\chi(\mathcal{O}_X) = \frac{1}{24}c_1c_2$ and therefore $p_a(X) = 1 - \chi(\mathcal{O}_X) = 1 - \frac{1}{24}c_1c_2$. Furthermore, $c_1 = -K_X$ and therefore,

$$\chi(\mathcal{L}) = \tfrac{1}{12}D \cdot (D - K_X) \cdot (2D - K_X) + \tfrac{1}{12}D \cdot c_2 + 1 - p_a$$

4.1.2 6.8

Let $\mathcal{E}$ be a locally free sheaf of rank 2 on $X = \mathbb{P}^3$. Hirzburch Riemann-Roch shows that,

$$\chi(\mathcal{E}) = \deg\left(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_X)\right)_n$$

First notice,

$$\mathrm{ch}(\mathcal{E}) = 2 + c_1(\mathcal{E}) + \tfrac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \tfrac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}))$$

Then we compute,

$$(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_X))_n = \tfrac{2}{24}c_1c_2 + c_1(\mathcal{E}) \cdot \tfrac{1}{12}(c_1^2 + c_2) + \tfrac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) \cdot \tfrac{1}{2}c_1 + \tfrac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}))$$

From the Euler sequence on $X = \mathbb{P}^n_k$,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus n+1} \longrightarrow \mathcal{T}_X \longrightarrow 0$$

we see that $c(\mathcal{T}_X) = (1 + c_1(\mathcal{O}_X))^{n+1} = (1+H)^{n+1}$ where $H \in A^1(X)$ is the hyperplane class. For the case $n = 3$,

$$c(\mathcal{T}_X) = 1 + 4H + 6H^2 + 4H^3$$

Therefore,

$$(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_X))_n = 2H^3 + \tfrac{11}{6} c_1(\mathcal{E}) \cdot H^2 + (c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) \cdot H + \tfrac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}))$$

Now, because $A(X) = \mathbb{Z}[H]/(H^4)$ we must have $c_i(\mathcal{E}) = d_i H$ for integers d_i. Thus,

$$(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_X))_n = [2 + \tfrac{11}{6} d_1 + (d_1^2 - 2d_2) + \tfrac{1}{6}(d_1^3 - 3d_1 d_2)] H^3$$

Therefore, since $\int_X H^3 = \deg H^3 = 1$ we find,

$$\chi(\mathcal{E}) = 2 + \tfrac{1}{6}(d_1^3 + 11 d_1) + d_1^2 - 2d_2 - \tfrac{1}{2} d_1 d_2$$

Notice that $n^3 \equiv n \mod 6$ and thus $d_1^3 + 11d_1 \equiv d_1^3 - d_1 \equiv 0 \mod 6$ so $\frac{1}{6}(d_1^3 + 11d_1)$ is an integer. Furthermore, $2 + d_1^2 - 2d_2$ is obviously an integer. Since $\chi(\mathcal{E})$ is an integer this implies that $d_1 d_2$ is divisible by 2 that is $d_1 d_2 \equiv 0 \mod 2$.

4.1.3 6.9 DO!!

Let $\iota : X \hookrightarrow \mathbb{P}^4_k$ be a smooth surface of degree d. Consider the normal sequence,

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \iota^* \mathcal{T}_{\mathbb{P}^4} \longrightarrow \mathcal{N}_{X/\mathbb{P}^4} \longrightarrow 0$$

Applying Chern classes we find that,

$$c(\mathcal{T}_X) \cdot c(\mathcal{N}_{X/\mathbb{P}^4}) = c(\iota^* \mathcal{T}_{\mathbb{P}}^4)$$

From the Euler sequence,

$$c(\mathcal{T}_{\mathbb{P}}^4) = (1+H)^5$$

where H is the hyperplane class. Therefore in $A^*(X)$,

$$c(\iota^* \mathcal{T}_{\mathbb{P}^4}) = \iota^* c(\mathcal{T}_{\mathbb{P}^4}) = (1 + \iota^* H)^5 = 1 + 5\iota^* H + 10(\iota^* H)^2$$

However, $(\iota^* H)^2 = \iota^* H^2$ is the class of d points on X. Now expand,

$$(1 + c_1 + c_2)(1 + c_1(\mathcal{N}) + c_2(\mathcal{N})) = 1 + (c_1 + c_1(\mathcal{N})) + (c_1 c_1(\mathcal{N}) + c_2 + c_2(\mathcal{N}))$$

Therefore, matching terms,

$$c_1 + c_1(\mathcal{N}) = 5\iota^* H$$
$$c_1 c_1(\mathcal{N}) + c_2 + c_2(\mathcal{N}) = 10(\iota^* H)^2$$

and plugging in gives,

$$c_2(\mathcal{N}) + c_1 \cdot (5\iota^* H - c_1) + c_2 = 10(\iota^* H)^2$$

Therefore,

$$c_2(\mathcal{N}) = 10(\iota^* H)^2 + c_1^2 - c_2 - 5c_1 \cdot \iota^* H$$

Finally, taking degrees, and using $K_X = -c_1$ and $c_2 = -K_X^2 + 12(p_a(X) + 1)$ we find,

$$\deg(c_2(\mathcal{N}) = 10d + 2K_X^2 - 12(p_a(X) + 1) + 5K_X \cdot \iota^* H$$

Finally, $X = dH^2$ in $A^*(\mathbb{P}^4_k)$ so we know that $\deg X \cdot X = d^2$ and furthermore we have $\iota_* c_2(\mathcal{N}) = X \cdot X$ and thus $\deg c_2(\mathcal{N}) = d^2$ giving a relation,

$$10d - d^2 + 2K_X^2 - 12(p_a(X) + 1) + 5K_X \cdot \iota^* H = 0$$

(a)

(b) Let $X \subset \mathbb{P}^4_k$ be a K3 surface. Then by definition $K_X = 0$ and $h^1(X, \mathcal{O}_X) = 0$ so, using Serre duality $h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X)$ since $\omega_X = \mathcal{O}_X$, we find $p_a(X) = 1$. Therefore,

$$10d - d^2 = 24$$

meaning that $d^2 - 10d + 24 = (d-4)(d-6) = 0$ and thus $d = 4$ or $d = 6$.

(c) Let $X \subset \mathbb{P}^4_k$ be an abelian surface. Then $K_X = 0$ and $c_1 = c_2 = 0$ so $p_a = -1$. Therefore,

$$10d - d^2 = 0$$

which implies that $d = 10$.

(d)

4.1.4 6.10

Suppose that X is an abelian 3-fold with an embedding $\iota : X \hookrightarrow \mathbb{P}^5$. Then consider the normal sequence,

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \iota^* \mathcal{T}_{\mathbb{P}^5} \longrightarrow \mathcal{N}_{X/\mathbb{P}^5} \longrightarrow 0$$

Therefore,

$$c(\mathcal{T}_X) c(\mathcal{N}_{X/\mathbb{P}^5}) = c(\iota^* \mathcal{T}_{\mathbb{P}^5})$$

However, $\mathcal{T}_X$ is trivial so $c(\mathcal{T}_X) = 1$ and therefore,

$$c(\mathcal{N}_{X/\mathbb{P}^5}) = \iota^* c(\mathcal{T}_{\mathbb{P}^5})$$

From the Euler sequence,

$$c(\mathcal{T}_{\mathbb{P}^5}) = (1 + H)^6$$

In particular we find,

$$c_3(\mathcal{N}_{X/\mathbb{P}^5}) = \binom{6}{3} \iota^* H^3 = 20 \iota^* H^3$$

which is nonzero because $\iota_* c_3(\mathcal{N}_{X/\mathbb{P}^5}) = 20 \iota_* \iota^* H^3 = 20 X \cdot H^3 = 20 d H^5$ where d is the degree of X in $\mathbb{P}^5$ and thus $\deg c_3(\mathcal{N}_{X/\mathbb{P}^5}) = 20d$. However, $\mathcal{N}_{X/\mathbb{P}^5}$ is a vector bundle of rank $\operatorname{codim}(X, \mathbb{P}^5) = 2$ and must have $c_3(\mathcal{N}_{X/\mathbb{P}^5}) = 0$ leading to a contradiction. Thus $\mathcal{T}_X$ cannot be trivial so X cannot be an abelian surface.

4.2 B Transcendental Methods

4.2.1 6.1

Consider the open unit disk $D^\circ \subset \mathbb{C}$. Let X be a scheme of finite type over $\mathbb{C}$ such that $X_h \cong D^\circ$. Thus we must have $\dim X = 1$ and $\pi_1^{\text{ét}}(X) = 0$. Therefore, because curves of positive genus always admit étale covers, we must have $X \cong \mathbb{A}^1$ or $X \cong \mathbb{P}^1$ (open subschemes of $\mathbb{A}^1$ involve removing finitely many points and thus are not simply connected). Clearly $\mathbb{P}^1$ cannot work because it is compact. Therefore we must have $X \cong \mathbb{A}^1$ in which case $X_h \cong \mathbb{C}$. However, I claim that D° is not biholomorphic to $\mathbb{C}$. To see this, notice that D° is biholomorphic to $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ via the map,

$$z \mapsto i \cdot \frac{1+z}{1-z}$$

Furthermore, there are no nonconstant maps $f : \mathbb{C} \to \mathfrak{h}$ because then $\exp(if) : \mathbb{C} \to \mathbb{C}$ is bounded because $|e^{if}| = e^{-\mathrm{Im}(f)} \le 1$ and therefore constant by Liouville's theorem. Thus we cannot have a biholomorphic map $f : D^\circ \to \mathbb{C}$ showing that no such X exists.

4.2.2 6.2

Let $z_1, z_2, \dots \in \mathbb{C}$ be an infinite sequence with $|z_n| \to \infty$ as $n \to \infty$. Let $\mathscr{I} \subset \mathcal{O}_{\mathbb{C}}$ be the sheaf of ideals of holomorphic functions vanishing at all z_n. First we need to show that $\mathscr{I}$ is nonzero. Using the hypothesis that $|z_n| \to \infty$, the Weierstrass factorization theorem (or equivalently the solvability of the second cousins problem on a complex manifold with $\mathrm{Pic}(X) = 0$ using that the points z_i are isolated and thus taking $f_i = z - z_i$ on a small disk about z_i) implies that there exists an entire function f with a simple pole at each z_i. Thus $f \in \Gamma(\mathbb{C}, \mathscr{I})$ so $\mathscr{I} \neq 0$. In particular, $V(\mathscr{I}) = \{z_i \mid i \in \mathbb{N}\}$ is an infinite set and $V(\mathscr{I}) \neq \mathbb{C}$.

Now let $X = \mathbb{A}^1_{\mathbb{C}}$. Coherent sheaves of ideals $\mathscr{J} \subset \mathcal{O}_X$ correspond to Zariski closed subsets $Z \subset \mathbb{A}^1_{\mathbb{C}}$ which are finite (unless $\mathscr{J} = 0$) and thus $\mathscr{J}_h$ cannot correspond to $\mathscr{I}$ as sheaves of ideals because $\mathscr{I}$ cuts out an infinite subset. Explicilty, $\mathscr{J} = \widetilde{(p)}$ for some $p \in \mathbb{C}[z]$ because $\mathbb{C}[z]$ is a PID and f has finitely many roots. Then $\mathscr{J}_h = (p) \cdot \mathcal{O}_{\mathbb{C}}$ which cannot equal $\mathscr{I}$ because $p \in \mathscr{J}_h$ viewed as a holomorphic function which has finitely many roots but every section of $\mathscr{I}$ (of which at least one exits) vanishes at all z_i of which there are infinitely many.

However, any section $s \in \Gamma(X, \mathscr{I})$ is an entire function vanishing at the z_i and thus $\frac{s}{f}$ is entire. Therefore $\mathscr{I} = (f) \cdot \mathcal{O}_{\mathbb{C}}$ which implies that $\mathscr{I} \cong \mathcal{O}_{\mathbb{C}} = (\mathcal{O}_X)_h$ as coherent sheaves.

Remark. To apply sovability of the second cousins problem, we need that the set of points $\{z_i\}$ is discrete. Here we show that $\{z_i\}$ being discrete is the same as $|z_n| \to \infty$. First, if $|z_n| \to \infty$ is it clear that $\{z_i\}$ is discrete since all but finitely many have $|z_i| > M$ for each M so $\{z_i\} \cap D_M$ is finite and thus discrete because $\mathbb{C}$ is Hausdorff. Conversely, if $\{z_i\}$ is discrete, then for each compact $\overline{D_M}$ we have $\{z_i\} \cap \overline{D_M}$ is compact and discrete and thus finite. Therefore $|z_i| > M$ for all but finitely many z_i for each $M > 0$ meaning that there is some n_M such that $n \geq n_M \implies |z_n| > M$ implying that $|z_n| \to \infty$.

4.2.3 6.3 DO!!

4.2.4 6.4 DO!!

4.2.5 6.5 DO!!

4.2.6 6.6 DO!!

4.3 C Weil Conjectures