0.1 Examples

(a) {stone spaces} $\iff$ {boolean algebras}

(b) {affine schemes} $\iff$ {comm rings}$^{\text{op}}$.

Have the notion of a dualizing object in both cases. For example $S = \text{Spec}\,(A)$ then,

$$\text{Hom}_{\text{Aff}}\left(\text{Spec}\,(A)\,, \mathbb{A}^1_{\mathbb{Z}}\right) \cong \text{Hom}_{\text{CRing}}\,(\mathbb{Z}[x], A) \cong A$$

The dualizing object for Stone spaces and Boolean algebras is $\mathbb{B} = \{0,1\}_{\text{disc}}$ because maps to these give clopen subsets.

There are adjunctions, **Top** $\to$ **Locales** and **Top** $\to$ **Topoi** but these are not equvalence of categories.

Let X be a topological space. Then $\mathcal{O}(X)$ is the lattice of open subsets with union and intersection.

Definition 0.1.1. A poset with all colimits and finite limits is called a frame with distributative property (c.f. distributive lattice).

Definition 0.1.2. Locales $=$ Frame$^{\text{op}}$. Then there is a functor $\widetilde{F} : \text{Top} \to \text{Loc}$ given by $X \mapsto \mathcal{O}(X)$ and $f : X \to Y$ is sent to $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ (goes the opposite way in Loc.

Remark. This is NOT faithful since $X = \{0,1\}_{\text{indiscrete}}$ then $\widetilde{F} = U$ treats X and $*$ the same.

Remark. To to recover $\mathcal{O}(X)$ from a top space X. Let $S = \{0,1\}$ with $\{1\}$ open but $\{0\}$ not this is the Serpinski space. Then $\text{Hom}\,(X, S) = \mathcal{O}(X)$.

Remark. The category Frame $=$ Loc$^{\text{op}}$ is naturally "algebraic" meaning the forgetful functor Frame $\to$ Set has a left adjoint.

However, Top$^{\text{op}}$ is not algebraic in any real sense.

This functor $u : \text{Top} \to \text{loc}$ is fully faithful in many cases for example on sober spaces.

Remark. Locales are "pointless" topology

Remark. Let $X \in \text{Top}$ then we get a category,

$$\text{Sh}(X) = \{\text{sheaves on } X\}$$

We can define what a sheaf is on any local and Sh factors through Top $\to$ Locales.

Definition 0.1.3. A *topos* is a category equivalent to $\text{Sh}(\mathcal{C})$ where $\mathcal{C}$ is not necessarily localic (it is just some category with a grothendieck topology).

Remark. An abelian group A is a surjection $\mathbb{Z}^N \twoheadrightarrow A$.

Definition 0.1.4. A *logos* ξ is a category that can be presented as a left-exact locaization of a presheaf category:

$$f : \text{Pr}(\mathcal{C}) \to \xi$$

meaning f admits a fully-faitful right-adjoint and f preserves finite limits where $\mathcal{C}$ is small.

Remark. The category of abelian groups is locally presentable but not a logos.

0.2 Map of Logoi

Impose conditions so that they "look like" continuous map of topological spaces. Let $f : X \to Y$ be a map of topological spaces then I get included maps between their categories of sheaves $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ and $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ where f^* is left-adjoint to f_* and f^* preserves finite limits.

Definition 0.2.1. A map of logoi $f^* : \xi \to \eta$ is cocontinuous and preserves finite limits.

Remark. Then there is a right adjoint by the adjoint functor theorem for locally presentable categories.

Example 0.2.2. If X is a topological space. Then $\mathrm{Sh}(X)$ is a logos with

$$F : \mathrm{Pr}(\mathcal{O}(X)) \to \mathrm{Sh}(X)$$

sheafification.

The same thing also works for locales. Then covers are $\{U_i \to U\}$ such that $\coprod U_i = U$.

Then the category $D = \mathrm{Nat}(\mathrm{Fin}, \mathrm{Set})$ is a logos. This is $\mathrm{Pr}(\mathrm{Fin}^{\mathrm{op}})$ is the left exact cocompletion of the trivial category. That is, if ξ is a logis, then an object of ξ is exacty a map,

$$D \to \xi$$

of logoi. This is because an object of a category is a map $\{*\} \to \mathcal{C}$ then we take the completion so that $\{*\}$ becomes a logos. This is completely analogous to how we get elements of a ring A via,

$$\mathrm{Hom}\left(\mathbb{Z}[X], A\right)$$