1 Lang-Nishimura

Theorem 1.0.1 (Lang-Nishimura)**.** Let $f : X \dashrightarrow Y$ be a rational map of k-varieties with Y proper. If X has a smooth k-point then Y has a point.

Proof. First we prove the case that X is a curve. Shrink to the smooth locus $U \subset X$ which intersects some generic point since X has a smooth point $x \in X$ and U is open. Hence we get a rational map $U \dashrightarrow Y$ which extends to $U \to Y$ since U is a regular curve and Y is proper.

Now we reduce to the curve case. We may shrink X such that it affine and integral with $x \in X(k)$ a smooth k-point. The goal is to show that there exists a (nonproper) curve $C \to X$ mapping to X whose image intersects the locus of definition of $f : X \dashrightarrow Y$ and contains a lift $x' \in C(k)$ as a smooth k-point of C. There is an étale neighborhood $U \to X$ of x with a lift $x' \in U(k)$ with an étale map $U \to \mathbb{A}^n_k$. Let $V \subset X$ be the domain of f then pushing and pulling gives a dense open of $\mathbb{A}^n_k$. Therefore, choose a line $L \subset \mathbb{A}^n_k$ through the origin intersecting this locus. Then the preimage $L' \subset U$ is a smooth curve passing through x' and hence $L' \to X$ satisfies the hypotheses. $\square$

Example 1.0.2. The condition that $x \in X(k)$ is a *smooth point* is necessary. For example, consider,

$$X = \text{Proj}\left(\mathbb{R}[X, Y, Z]/(X^2 + Y^2)\right)$$

and let $Y = \mathbb{P}^1_{\mathbb{C}}$ be its normalization and consider the inverse of the normalization $X \dashrightarrow Y$. Now X contains a nonsmooth $\mathbb{R}$-point $[0 : 0 : 1] \in X(\mathbb{R})$ but Y does not have an $\mathbb{R}$-point.

2 $\mathbb{E}_8$ lattice

Let $X = \text{Bl}_{P_1,\dots,P_9}(\mathbb{P}^2)$ be the blowup at 9 points sufficiently general so there is a unique cubic C through these points and it is smooth. Then there there is a genus 1 curve $\widetilde{C} \subset X$ which is the strict transform of the unique conic through the points $P_1, \dots, P_9$. Let $E_1, \dots, E_9$ be the exceptional divisors. Then,

$$\widetilde{C} = 3H - (E_1 + \cdots + E_9)$$

so indeed we see that $\widetilde{C}^2 = 0$. Now the claim is that the lattice,

$$\text{Pic}\,(X) = \text{NS}\,(X)$$

contains the $\mathbb{E}_8$ lattice as a subquotient. Indeed,

$$\left\langle \widetilde{C} \right\rangle^{\perp} / \left\langle \widetilde{C} \right\rangle \cong \mathbb{E}_8$$

3 Root Stacks

4 Weierstrass Points

5 MAPP

Remark. Note that the isomorphism $X \xrightarrow{\sim} (B' \times Z)/G$ is *not* compatible with any map to A. Indeed, there may not even be a map to A since $B'/G = B$ may only be isogenous to an abelian subvariety.

Even if G is trivial, the isomorphism may not be compatible with f and the projection. For example, consider $X = E \times C$ where E is an elliptic curve and C is a genus 2 curve with Jacobian $E \times E'$. Mapping to the Albanese $E \times E \times E'$, our construction gives the identity $\mathrm{id} : E \times C$. However, the map to the Al banese does not factor through the first projection $\mathrm{pr}_1 : X \to E$.

6 Etale fundamental groups are NOT profinitely complete

I allways thought that étale fundamental groups are profinitely complete i.e. equal to their own profinite completion. This is false in general. They are always profinite but this is weaker in general. It is true that a profinite group is the limit over its finite *continuous* quotients or equivalently,

$$G = \varprojlim G/H$$

as H runs over the finite index *open* (actually every open subgroup in a compact group is automatically finite index) normal subgroups. However, this does not necessarily include every finite index subgroup.

Remark. However, if a *topological* group G is profinite then $G \to \widehat{G}^{\text{top}}$ is an isomorphism by definition where,

$$\widehat{G}^{\text{top}} = \varprojlim_{\substack{H \triangleleft G \\ H \text{ open}}} G/H$$

Example 6.0.1. $\pi_1^{\text{ét}}(\mathrm{Spec}\,(\mathbb{Q})) = \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ is *not* profinitely complete. Indeed, see Chapter 7 of Milne's Class Field Theory.

Remark. See these answers:

(a) Silverman's incorrect definition

(b) examples of noncomplete profinite groups

Proposition 6.0.2. However, if X is a scheme of finite type over $\mathbb{C}$ then,

$$\pi_1^{\text{ét}}(X) = \widehat{\pi_1(X(\mathbb{C}))}$$

is the profinite completion of a finitely presented group and hence is profinitely complete.

Proof. Indeed, by Riemann-Existence,

$$\pi_1^{\text{ét}}(X)\text{-FinSets} \cong \text{FÉt}_X \cong \{\text{finite covering spaces of } X\} \cong \pi_1(X(\mathbb{C}))\text{-FinSets}$$

where $\pi_1^{\text{ét}}(X)$-FinSets means *continuous* finite $\pi_1^{\text{ét}}(X)$-sets. This identifies $\pi_1^{\text{ét}}(X)$ as a topological group with $\widehat{\pi_1(X(\mathbb{C}))}$ □

Lemma 6.0.3. If G is a finitely presented group then $\widehat{G} \to \widehat{\widehat{G}}$ is an isomorphism.

Proof. See here. □

Remark. Note that the above theorem is nontrivial. In fact, it is false without the finite presentation assumption. See here.

7 Grothendieck Abelian Categories

Definition 7.0.1. Let $\mathcal{A}$ be an abelian category. We say that satisfies,

(AB3) $\mathcal{A}$ has all direct sums

(AB4) $\mathcal{A}$ is AB4 and taking direct sums is exact

(AB5) $\mathcal{A}$ is AB3 and taking filtered colimits is exact

(AB6) $\mathcal{A}$ is AB3 and given a family of filtered categories $\{I_j\}_{j \in J}$ and maps $D_j : I_j \to \mathcal{A}$ we have,

$$\prod_{j \in J} \operatorname*{colim}_{I_j} D_j = \operatorname*{colim}_{(i_j) \in \prod\limits_{j \in J} I_j} \left(\prod_{j \in J} D_j(i_j) \right)$$

We say that $\mathcal{A}$ *has a generator* if there is an object $M \in \mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(M, -)$ is faithful. We say that $\mathcal{A}$ is a *Grothendieck category* if $\mathcal{A}$ is AB5 and has a generator.

Lemma 7.0.2. We have the following implications:

$$\text{AB6} \implies \text{AB5} \implies \text{AB4} \implies \text{AB3}$$

Lemma 7.0.3. For any unital ring R, the category Mod_R satisfies AB6 and AB4* but not AB5*.

Example 7.0.4. **Ab** thus satisfies AB6 and AB4* but not AB5*. Hence $\mathbf{Ab}^{\text{op}}$ which is isomorphic to the category of compact Hausdorff topological groups by Pontriagin duality saitsifes AB6* and AB4 but not AB5.

Lemma 7.0.5. The only abelian category satisfying AB5 and AB5* is the zero category.

Lemma 7.0.6. An AB3 abelian category $\mathcal{A}$ has a generator M if and only if for every $A \in \mathcal{A}$ there is an epimorphism,

$$\bigoplus_I M \twoheadrightarrow A$$

Proof. Suppose that,

$$\bigoplus_I M \twoheadrightarrow A$$

By definition, if $f, g : A \to B$ are two maps such that the induced maps,

$$M \to A \to B$$

are pairwise equal then $f = g$. Therefore,

$$\operatorname{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(M,-)} \operatorname{Hom}\left(\operatorname{Hom}_{\mathcal{A}}(M, A), \operatorname{Hom}_{\mathcal{A}}(M, B)\right)$$

is injective since it is injective after evaluation at the inclusions $\{M \to A\}_I$.

Conversely, suppose that $\mathcal{A}$ has a generator. For each $A \in \mathcal{A}$ let $I = \operatorname{Hom}_{\mathcal{A}}(M, A)$ which is a set and there is a canonical map,

$$c : \bigoplus_I M \to A$$

via evaluation. We need to show this is an epimorphism. Indeed, if $f, g : A \to B$ are two maps such that $f \circ c = g \circ c$ this means that $f_* = g_*$ and since $\operatorname{Hom}_{\mathcal{A}}(M, -)$ is faithful we see that $f = g$ so we conclude that c is an epimorphism. $\square$

Theorem 7.0.7 (1.10.1 in Tôhoku)**.** Let $\mathcal{A}$ be a Grothendieck abelian category then $\mathcal{A}$ has enough injectives.

(IS THIS CORRECT??)

Proposition 7.0.8. Let $\mathcal{C}$ be a category and $\mathcal{A}$ satisfies any of,

(a) AB3

(b) AB4

(c) AB5

(d) AB6

(e) AB3*

(f) AB4*

(g) AB5*

(h) AB6*

(i) $\mathcal{A}$ has a generator

(j) $\mathcal{A}$ is a Grothendieck abelian category

then the same is true of $\mathrm{PSh}(\mathcal{C}, \mathcal{A}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.

DO THIS!!

Theorem 7.0.9. Let $\mathcal{C}$ be a site and $\mathcal{A}$ satisfies any of,

(a) AB3

(b) AB4

(c) AB5

(d) AB6

(e) AB3*

(f) $\mathcal{A}$ has a generator

(g) $\mathcal{A}$ is a Grothendieck abelian category

then the same is true of $\mathfrak{Sh}(\mathcal{C}, \mathcal{A})$.

Remark. Note tha even for $\mathcal{A} = \mathbf{Ab}$ the sheaf category $\mathfrak{Sh}(\mathcal{C}, \mathcal{A})$ need not be AB4* because infinite products are only left exact and do not, in general, preserve epimorphisms. For example, see here.

DO THIS!!

Theorem 7.0.10. Let $\mathcal{A}$ is a Grothendieck abelian category and $\mathcal{C}$ is a site then the inclusion,

$$\mathfrak{Sh}(\mathcal{C}, \mathcal{A}) \hookrightarrow \mathrm{PSh}(\mathcal{C}, \mathcal{A})$$

has a left adjoint called "sheafificaion".

DO THIS PROOF

Corollary 7.0.11. If $\mathcal{A}$ is a Grothendieck abelian category and $\mathcal{C}$ is a site then $\mathfrak{Sh}(\mathcal{C}, \mathcal{A})$ has enough injectives.

Theorem 7.0.12 (Gabber[1]). Let X be a scheme. Then $\mathfrak{QCoh}(X)$ is a Grothendieck abelian category and hence has enough injectives. Furthermore, $\mathfrak{QCoh}(X) \hookrightarrow \mathbf{Mod}_{\mathcal{O}_X}$ has a right adjoint and hence is also AB3*.

IS IT TRUE THAT ALL GROTHENDIECK ABELIAN CATEGORIES HAVE ALL PRODUCTS?? WHY DOES QCoh HAVE PRODUCTS?? JUST BECAUSE OF THE COHERATOR?

Remark. Note that products in $\mathfrak{QCoh}(X)$ do not agree with products in $\mathbf{Mod}_{\mathcal{O}_X}$ in general. GIVE EXAMPLE They are also not exact see here

(a) CMB

(b) Leo's answer.

(c) quasi-coherent module

(d) Tohoku.

8 Griffiths Conjecture

Conjecture 8.0.1. Let X be a smooth projective complex variety. If E is an ample vector bundle on X then it admits a hermitian metric with positive bisectional curvature.

Remark. In the case that $\operatorname{rank} E = 1$ this is exactly the Kodiara embedding theorem.

Remark. This conjecture is almost false as follows:

(a) this paper proves that if X admits a *Kahler* metic with negative bisectional curvature then $\pi_1(X)$ is infinite

(b) Brotbek and Darondeau proved that a generic complete intersection of large enough codimension and degree in $\mathbb{P}^N$ has ample cotangent bundle

(c) by Lefschetz hyperplane theorem the above examples have $\pi_1 = 0$.

The reason this does not give a counterexample to Griffiths' conjecture is exactly the stipulation that the metric on X is Kahler not just some arbitrary hermitian metric on TX.

Some other references:

(a) MO Griffiths positivity

(b) Approach to the conjecture

(c) MO reference on holomorphic (bi)sectional curvature.

9 Infinite Products are not quasi-coherent

Usually the sheaf,

$$\mathscr{F} = \prod_{i \in \mathbb{N}} \mathcal{O}_X$$

is not quasi-coherent. This may be surprising since the inclusion of presheaves into sheaves admiting a left-adjoint shows that,

$$(\varprojlim_i \mathscr{F}_i)(U) = \varprojlim_i \mathscr{F}_i(U)$$

and therefore,

$$\mathscr{F}(U) = \prod_{i \in \mathbb{N}} \mathcal{O}_X(U)$$

However this is just because localization does not commute with products. Indeed, if $\mathscr{F}$ were quasi-coherent, over an affine $\mathrm{Spec}\,(A)$ we must have,

$$\mathscr{F} = \widetilde{A^{\times \mathbb{N}}}$$

but this does not hold as an equality of sheaves because localization and infinte products do not commute. Indeed, consider $A = k[x]$ and localization at the element $f = x$. Then there is a natural map,

$$(A^{\times n})_f \to (A_f)^{\times n}$$

but the element $(1, x^{-1}, x^{-2}, \cdots)$ is not in the image since elements on the right must have only bounded below powers of x since they can be writen as $f^{-n} s$ for $s \in A^{\times n}$.

10 Positronium Lifetimes

Let an electron with (four) momentum p_1 and positron with momentum p_2 annihilate to two photons (or vector bosons) with momenta k_1, k_2. The leading-order Feynman diagrams give,

$$i\mathcal{M} = (-ie)^2 \epsilon(k_1)^*_\mu \epsilon(k_2)^*_\nu \; \bar{v}^{s_2}(p_2) \left[\gamma^\nu \frac{i(\not{q}_1 + m)}{q_1^2 - m^2 + i\epsilon} \gamma^\mu + \gamma^\mu \frac{i(\not{q}_2 + m)}{q_2^2 - m^2 + i\epsilon} \gamma^\nu \right] u^{s_1}(p_1)$$

where $q_1 = p_1 - k_1$ and $q_2 = p_1 - k_2$ corresponding to the t-channel and u-channel respectively. First we work out some formulas. Expanding the momenta to first-order,

$$\bar{v}(p_2) \not{a} u(p_1) = \sqrt{E_1 E_2}\, \xi'^\dagger \left[-a^0 \left(\frac{\vec{p}_1}{E_1} + \frac{\vec{p}_2}{E_2} \right) \cdot \vec{\sigma} + 2 \vec{a} \cdot \vec{\sigma} \right] \xi$$

Similarly,

$$\bar{v}(p_2) \not{a} \gamma^5 u(p_1) = \sqrt{E_1 E_2}\, \xi'^\dagger \left[-2a^0 + \vec{a} \cdot \left(\frac{\vec{p}_1}{E_1} + \frac{\vec{p}_2}{E_2} \right) - i \vec{\sigma} \cdot \left[\left(\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right) \times \vec{a} \right] \right] \xi$$

And finally,

$$\bar{v}(p_2) \not{a} \not{b} u(p_1) = \sqrt{E_1 E_2}\, \xi'^\dagger \left[a^\mu b_\mu \left(\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right) \cdot \vec{\sigma} - 2a^0 (\vec{b} \cdot \vec{\sigma}) + 2b^0 (\vec{a} \cdot \vec{\sigma}) - i (\vec{a} \times \vec{b}) \cdot \left(\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right) \right.$$
$$\left. + (\vec{a} \times \vec{b}) \times \left(\frac{p_1}{E_1} + \frac{p_2}{E_2} \right) \cdot \vec{\sigma} \right] \xi$$

We use the identity,

$$\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho + g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu - i\varepsilon^{\alpha\mu\nu\rho}\gamma_\alpha\gamma^5$$

Work in the CM frame where,

$$\begin{aligned} p_1 &= (\tfrac{1}{2}E_{\text{CM}}, \vec{p}) \\ p_2 &= (\tfrac{1}{2}E_{\text{CM}}, -\vec{p}) \\ k_1 &= (\tfrac{1}{2}E_{\text{CM}}, \vec{k}) \\ k_2 &= (\tfrac{1}{2}E_{\text{CM}}, -\vec{k}) \end{aligned}$$

Therefore,

$$q_1^2 = (p_1 - k_1)^2 = m^2 + m_B^2 - 2p_1 \cdot k_1 = m^2 + m_B^2 - \tfrac{1}{2}E_{\text{CM}}^2 + 2\vec{p}\cdot\vec{k}$$

and likewise,

$$q_2^2 = (p_1 - k_2)^2 = m^2 + m_B^2 - 2p_1 \cdot k_2 = m^2 + m_B^2 - \tfrac{1}{2}E_{\text{CM}}^2 - 2\vec{p}\cdot\vec{k}$$

Also in the CM frame,

$$\bar{v}(p_2)\not{a}u(p_1) = E_{\text{CM}}\,\xi'^\dagger\,[\vec{a}\cdot\vec{\sigma}]\,\xi$$

Similarly,

$$\bar{v}(p_2)\not{a}\gamma^5 u(p_1) = \xi'^\dagger\left[-E_{\text{CM}}a^0 - 2i\vec{\sigma}\cdot[\vec{p}\times\vec{a}]\right]\xi$$

And finally,

$$\bar{v}(p_2)\not{a}\not{b}u(p_1) = \xi'^\dagger\left[2a^\mu b_\mu(\vec{p}\cdot\vec{\sigma}) - E_{\text{CM}}a^0(\vec{b}\cdot\vec{\sigma}) + E_{\text{CM}}b^0(\vec{a}\cdot\vec{\sigma}) - 2i(\vec{a}\times\vec{b})\cdot\vec{p}\right]\xi$$

We need to simplify,

$$\mathcal{M} = e^2\epsilon(k_1)^*_\mu\epsilon(k_2)^*_\nu\;\bar{v}^{s_2}(p_2)\left[\frac{\gamma^\nu\not{q}_1\gamma^\mu + m\gamma^\nu\gamma^\mu}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 - 2\vec{p}\cdot\vec{k}} + \frac{\gamma^\mu\not{q}_2\gamma^\nu + m\gamma^\mu\gamma^\nu}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 + 2\vec{p}\cdot\vec{k}}\right]u^{s_1}(p_1)$$

To do this, we define two quantities,

$$\begin{aligned} A =& \epsilon(k_1)^*_\mu\epsilon(k_2)^*_\nu\;\bar{v}^{s_2}(p_2)[\gamma^\nu\not{q}_1\gamma^\mu + m\gamma^\nu\gamma^\mu]u^{s_1}(p_1) \\ B =& \epsilon(k_1)^*_\mu\epsilon(k_2)^*_\nu\;\bar{v}^{s_2}(p_2)[\gamma^\mu\not{q}_2\gamma^\nu + m\gamma^\mu\gamma^\nu]u^{s_1}(p_1) \end{aligned}$$

such that,

$$\mathcal{M} = \frac{e^2}{2m^2 - m_B^2}\left[A\cdot\left(\frac{2m^2 - m_B^2}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 - 2\vec{p}\cdot\vec{k}}\right) + B\cdot\left(\frac{2m^2 - m_B^2}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 + 2\vec{p}\cdot\vec{k}}\right)\right]$$

To first-order in $\vec{p}$ this is,

$$\mathcal{M} = \frac{e^2}{2m^2 - m_B^2}\left[(A+B) + \left[\frac{2\vec{p}\cdot\vec{k}}{2m^2 - m_B^2}\right]\cdot(A-B)\right]$$

Now we expand A and B to first-order in $\vec{p}$. Then,

$$\begin{aligned} A = E_{\text{CM}}\xi'^\dagger\Big[&(\epsilon_1^*\cdot q_1)(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma}) + (\epsilon_2^*\cdot q_1)(\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma}) - (\epsilon_1^*\cdot\epsilon_2^*)(\vec{q}_1\cdot\vec{\sigma}) + i\varepsilon^{\alpha\nu\rho\mu}(\epsilon_1^*)_\nu(q_1)_\rho(\epsilon_2^*)_\mu\,\langle\gamma_\alpha\rangle_5 \\ &+2m(\epsilon_1^*\cdot\epsilon_2^*)\frac{\vec{p}\cdot\vec{\sigma}}{E_{\text{CM}}} - m\epsilon_2^{*0}(\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma}) + m\epsilon_1^{*0}(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma}) + 2im(\vec{\epsilon}_1^{\,*}\times\vec{\epsilon}_2^{\,*})\cdot\frac{\vec{p}}{E_{\text{CM}}}\Big]\xi \end{aligned}$$

where we define $\langle\gamma_\alpha\rangle_5$ as the matrix M in $\bar{v}^{s_2}(p_2)\gamma_\alpha\gamma^5 u^{s_1}(p_1) = E_{\text{CM}}\xi'^\dagger M\xi$. We need to be careful expanding the ε term. There are four terms depending on where the 0 index appears. These are (including a minus sign from index lowering),

$$-i\,\langle\gamma_0\rangle_5\,(\vec{\epsilon_1^*}\times\vec{q}_1)\cdot(\vec{\epsilon_2^*}) + i\epsilon_1^{*0}(\langle\vec{\gamma}\rangle_5\times\vec{q}_1)\cdot\vec{\epsilon_2^*} - iq_1^0(\langle\vec{\gamma}\rangle_5\times\vec{\epsilon_1^*})\cdot\vec{\epsilon_2^*} + i\epsilon_2^{*0}(\langle\vec{\gamma}\rangle_5\times\vec{\epsilon_1^*})\cdot\vec{q}_1$$

But $q_1^0 = 0$ and we can compute the $\langle\vec{\gamma}\rangle_5$ terms by rearranging them into the form $\langle\vec{a}\cdot\vec{\gamma}\rangle_5$ so we can use the above identities since,

$$\langle\vec{a}\cdot\vec{\gamma}\rangle_5 = \langle\not{a}\rangle_5 = \frac{\vec{p}\times\vec{a}}{E_{\text{CM}}}\cdot(2i\vec{\sigma})$$

where $a = (0, -\vec{a})$. Therefore,

$$i\varepsilon^{\alpha\nu\rho\mu}(\epsilon_1^*)_\nu(q_1)_\rho(\epsilon_2^*)_\mu\,\langle\gamma_\alpha\rangle_5 = i(\vec{\epsilon_1^*}\times\vec{q}_1)\cdot\vec{\epsilon_2^*} - 2\epsilon_1^{*0}\frac{\vec{p}\times(\vec{q}_1\times\vec{\epsilon_2^*})}{E_{\text{CM}}}\cdot\vec{\sigma} - 2\epsilon_2^{*0}\frac{\vec{p}\times(\vec{\epsilon_1^*}\times\vec{q}_1)}{E_{\text{CM}}}\cdot\vec{\sigma}$$

And putting everything together (and using that $q_1^0 = 0$) we get,

$$\begin{aligned}
A = -E_{\text{CM}}\xi'^\dagger\Big[&(\vec{\epsilon_1^*}\cdot\vec{q}_1)(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\vec{q}_1)(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + (\epsilon_1^*\cdot\epsilon_2^*)(\vec{q}_1\cdot\vec{\sigma}) \\
&- i(\vec{\epsilon_1^*}\times\vec{q}_1)\cdot\vec{\epsilon_2^*} + 2\epsilon_1^{*0}\frac{\vec{p}\times(\vec{q}_1\times\vec{\epsilon_2^*})}{E_{\text{CM}}}\cdot\vec{\sigma} + 2\epsilon_2^{*0}\frac{\vec{p}\times(\vec{\epsilon_1^*}\times\vec{q}_1)}{E_{\text{CM}}}\cdot\vec{\sigma} \\
&-m\left(2(\epsilon_1^*\cdot\epsilon_2^*)\frac{\vec{p}\cdot\vec{\sigma}}{E_{\text{CM}}} - \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + 2i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\frac{\vec{p}}{E_{\text{CM}}}\right)\Big]\,\xi
\end{aligned}$$

And B is identical except for swapping ϵ_1 and ϵ_2 and swapping q_1 for q_2. Now write $\hat{A}$ and $\hat{B}$ for the unitless quantities inside the spinor inner product menaing that,

$$A = -2mE_{\text{CM}}\;\xi'^\dagger\hat{A}\xi$$

and likewise for B. Therefore, since to first-order in $\vec{p}$ we have $E_{\text{CM}} = 2m$ we have,

$$\mathcal{M} = -\left(\frac{2e^2}{1-\frac{m_B^2}{2m^2}}\right)\xi'^\dagger\left[\hat{A}+\hat{B}+\left[\frac{2\vec{p}\cdot\vec{k}}{2m^2-m_B^2}\right]\cdot(\hat{A}-\hat{B})\right]\xi$$

Now we consider, using that $\vec{q}_1+\vec{q}_2 = 2\vec{p}$ and $\vec{q}_1-\vec{q}_2 = -2\vec{k}$ the quantity

$$\begin{aligned}
\hat{A}+\hat{B} &= (\vec{\epsilon_1^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{p}}{m}\cdot\vec{\sigma}) \\
&\quad + i(\vec{\epsilon_1^*}\times\tfrac{\vec{k}}{m})\cdot\vec{\epsilon_2^*} - \epsilon_1^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_2^*})}{m^2}\cdot\vec{\sigma} + \epsilon_2^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_1^*})}{m^2}\cdot\vec{\sigma} \\
&\quad - (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{p}}{m}\cdot\vec{\sigma}) \\
&\quad = (\vec{\epsilon_1^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\tfrac{\vec{k}}{m} - \epsilon_1^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_2^*})}{m^2}\cdot\vec{\sigma} + \epsilon_2^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_1^*})}{m^2}\cdot\vec{\sigma}
\end{aligned}$$

and likewise,

$$\begin{aligned}
\hat{A}-\hat{B} &= -(\vec{\epsilon_1^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) - (\vec{\epsilon_2^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{k}}{m}\cdot\vec{\sigma}) \\
&\quad - i(\vec{\epsilon_1^*}\times\tfrac{\vec{p}}{m})\cdot\vec{\epsilon_2^*} \\
&\quad + \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma}) - i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\tfrac{\vec{p}}{m} \\
&= -(\vec{\epsilon_1^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) - (\vec{\epsilon_2^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{k}}{m}\cdot\vec{\sigma}) + \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma})
\end{aligned}$$

Finally, dropping terms to higher order in $\vec{p}$ we get,

$$\begin{aligned}\mathcal{M} = -\left(\frac{2e^2}{1-\frac{m_B^2}{2m^2}}\right)\xi'^\dagger\Bigg[&(\vec{\epsilon_1^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\tfrac{\vec{k}}{m} \\ &- \epsilon_1^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_2^*})}{m^2}\cdot\vec{\sigma} + \epsilon_2^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_1^*})}{m^2}\cdot\vec{\sigma} \\ &- \left[\frac{2\vec{p}\cdot\vec{k}}{2m^2-m_B^2}\right]\Big((\vec{\epsilon_1^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{k}}{m}\cdot\vec{\sigma}) - \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma})\Big)\Bigg]\xi\end{aligned}$$

For photon polarizations this simplifies greatly since $\vec{\epsilon}\perp\vec{k}$ and $\epsilon^0 = 0$. Therefore, the photon amplitude is,

$$\mathcal{M} = -2e^2\xi'^\dagger\left[(\vec{\epsilon_1^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + (\vec{\epsilon_1^*}\cdot\vec{\epsilon_2^*})(\tfrac{\vec{p}}{m}\cdot\hat{k})(\hat{k}\cdot\vec{\sigma}) - i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\hat{k}\right]\xi$$

using that $\frac{\vec{k}}{m} = \hat{k}$ to first-order in $\vec{p}$ since the photons carry away all the energy and hence $|\vec{k}| = m$. All but the last term are suppressed by a factor of $\vec{p}/m$ which for positronium will be proportional to α.

10.1 Selection rules for 2-photon annihilation

We are free to orient our spinor basis along any direction (recall that ξ' is the flipped spinor of the physical positron). We choose to orient along the z-direction which is chosen as the direction along which $\vec{k}$ points. Then,

$$\mathcal{M}_{\pm\pm} = -2e^2\xi'^\dagger(\pm 1 + \tfrac{\vec{p}}{m}\cdot\vec{\sigma})\xi$$
$$\mathcal{M}_{\pm\mp} = -2e^2\xi'^\dagger(\tfrac{p_x\mp ip_y}{m})(\sigma_x \mp i\sigma_y)\xi$$

Spin-orbit coupling means that the positronium will naturally be split into eigenstates of total angular momentum. For $n = 1$ there is only an $\ell = 0$ wavefunction so there are only two states, para (singlet $s = 0$) and ortho (triplet $s = 1$) given by,

$$|1^1S_0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\otimes|\psi_1\rangle$$

and

$$\begin{aligned}|1^3S_1, m=1\rangle &= |\uparrow\uparrow\rangle\otimes|\psi_1\rangle \\ |1^3S_1, m=0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\otimes|\psi_1\rangle \\ |1^3S_1, m=-1\rangle &= |\downarrow\downarrow\rangle\otimes|\psi_1\rangle\end{aligned}$$

where $|\psi_0\rangle$ is the $1S$ state wavefunction. For $n = 2$ we have $\ell = 0, 1$ and hence there are more states. We have the excited $2S$ (meaning $\ell = 0$) versions of para ($s = 0$) and ortho ($s = 1$) positronium which are identical but with $|\psi_0\rangle$ replaced by $|\psi_{1,0}\rangle$. More interesting are the following, the $s = 0$

states,

$$
\begin{aligned}
|2^1P_1, m=1\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \otimes |\psi_{1,1,1}\rangle \\
|2^1P_1, m=0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \otimes |\psi_{1,1,0}\rangle \\
|2^1P_1, m=0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \otimes |\psi_{1,1,0}\rangle
\end{aligned}
$$

the $s=1$ state with $j=0$,

$$
|2^3P_0\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,-1}\rangle + |\downarrow\downarrow\rangle \otimes |\psi_{1,1,1}\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,0}\rangle\right)
$$

the $s=1$ states with $j=1$,

$$
\begin{aligned}
|2^3P_1, m=1\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,0}\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_1, m=0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,-1}\rangle - |\downarrow\downarrow\rangle \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_1, m=-1\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle \otimes |\psi_{1,1,0}\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,-1}\rangle\right)
\end{aligned}
$$

and finally the $s=1$ sates with $j=2$,

$$
\begin{aligned}
|2^3P_2, m=2\rangle &= |\uparrow\uparrow\rangle \otimes |\psi_{1,1,1}\rangle \\
|2^3P_2, m=1\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,0}\rangle + \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_2, m=0\rangle &= \frac{1}{\sqrt{6}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,-1}\rangle + \frac{\sqrt{2}}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,0}\rangle + |\downarrow\downarrow\rangle \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_2, m=-1\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle \otimes |\psi_{1,1,0}\rangle + \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,-1}\rangle\right) \\
|2^3P_2, m=-2\rangle &= |\downarrow\downarrow\rangle \otimes |\psi_{1,1,-1}\rangle
\end{aligned}
$$

The spin-flip rule assigns, for the positron,

$$
|\uparrow\rangle \mapsto \xi' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\downarrow\rangle \mapsto \xi' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
$$

The amplitudes can be rewritten in the form of the matrix shown times $\xi\xi'^\dagger$. Then using the spin flip we easily see that $s=0$ state corresponds to,

$$
\xi\xi'^\dagger = \frac{1}{\sqrt{2}}\mathrm{id}
$$

and the $s=1$ state with spin along $\hat{n}$ corresponds to,

$$
\xi\xi'^\dagger = \frac{1}{\sqrt{2}}\hat{n}\cdot\vec{\sigma}
$$

(this is consistent with Peskin as can be seen by daggering the above expression which exchanges ξ and ξ' and replaces $\hat{n}$ by $\hat{n}^*$) where $\hat{n}$ points along the direction of $m = 0$ meaning spin $+\hat{z}$ corresponds to $\hat{n} = \hat{x} + i\hat{y}$. Therefore, taking the trace of the inner matrix,

$$\mathcal{M}(s=0) = 2i\sqrt{2}e^2(\vec{\epsilon_1^*} \times \vec{\epsilon_2^*}) \cdot \hat{k}$$

and therefore,

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s=0) &= \mp 2\sqrt{2}e^2 \\
\mathcal{M}_{\pm\mp}(s=0) &= 0
\end{aligned}$$

so any $s = 0$ state decays into an odd S-wave EPR $j = 0$ entangled state. Since the wavefunction of a P orbital vanishes at the origin, this protects against decay of the $j = 1$ and $s = 0$ states into two photons (angular momentum of two photons cannot be $j = 1$). This is also due to C conservation since the 2^1P_1 state is odd under C but any two photon state is even under C. Therefore, we expect this decay to be forbidden into any C-odd vector particles with a C-invariant interaction term. Likewise,

$$\begin{aligned}
\mathcal{M}(s=1) &= -\sqrt{2}e^2 \, \mathrm{Tr}\left(\left[(\vec{\epsilon_1^*} \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon_2^*} \cdot \vec{\sigma}) + (\vec{\epsilon_2^*} \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon_1^*} \cdot \vec{\sigma}) + (\vec{\epsilon_1^*} \cdot \vec{\epsilon_2^*})(\tfrac{\vec{p}}{m} \cdot \hat{k})(\hat{k} \cdot \vec{\sigma}) - i(\vec{\epsilon_1^*} \times \vec{\epsilon_2^*}) \cdot \hat{k}\right](\hat{n} \cdot \sigma)\right) \\
&= -2\sqrt{2}e^2\left[(\vec{\epsilon_1^*} \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon_2^*} \cdot \hat{n}) + (\vec{\epsilon_2^*} \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon_1^*} \cdot \hat{n}) + (\vec{\epsilon_1^*} \cdot \vec{\epsilon_2^*})(\tfrac{\vec{p}}{m} \cdot \hat{k})(\hat{k} \cdot \hat{n})\right]
\end{aligned}$$

and therefore, using the completness relation

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1) &= -2\sqrt{2}e^2(\tfrac{\vec{p}}{m} \cdot \hat{n}) \\
\mathcal{M}_{\pm\mp}(s=1) &= -2\sqrt{2}e^2(\tfrac{p_x}{m} \mp i\tfrac{p_y}{m})(n_x \mp i n_y)
\end{aligned}$$

Hence for the three spin orientations we get,

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1, m=1) &= -2e^2(\tfrac{p_x}{m} + i\tfrac{p_y}{m}) \\
\mathcal{M}_{\pm\pm}(s=1, m=0) &= -2\sqrt{2}e^2(\tfrac{p_z}{m}) \\
\mathcal{M}_{\pm\pm}(s=1, m=-1) &= -2e^2(\tfrac{p_x}{m} - i\tfrac{p_y}{m}) \\
\mathcal{M}_{\pm\mp}(s=1, m=\pm 1) &= -4e^2(\tfrac{p_x}{m} \mp i\tfrac{p_y}{m}) \\
\mathcal{M}_{\pm\mp}(s=1, m=0) &= 0 \\
\mathcal{M}_{\pm\mp}(s=1, m=\mp 1) &= 0
\end{aligned}$$

This alows the $s = 1$ and $j = 0$ state to decay into an even S-wave EPR $j = 0$ state. The first two terms do not couple since they require both $\vec{S}$ and $\vec{L}$ in the same orientation. However, the $\mathcal{M}_{\pm\pm}$ does couple to the last term and is even under exchange of RHC and LHC photons. This is necessary to preserve parity since any ${}^3P$ state has even parity. Note that S-wave $|{+}{+}\rangle$ and $|{-}{-}\rangle$ are both $j = 0$ states of the photon field since they are identical particles so are even under exchange facilitated by a π-rotation. Thus either linear combination is a valid $j = 0$ state with $s = 0$ and $\ell = 0$. Therefore we see that it is not P but C that really forbids various positronium decays.

Finally, we analyize the 2^3P_1 and 2^3P_2 states. The easiest is 2^3P_2 for which $m = \pm 2$ clearly couple to $\mathcal{M}_{\pm\mp}$. The $m = \pm 1$ states do not couple. Although it at first appears that the $m = 0$ state couples, it does not. Indeed, the amplitude is,

$$-2\sqrt{2}e^2[-1 + (\sqrt{2})^2 - 1] = 0$$

where the minus signs arise from the Condon–Shortley phase convention which is chosen such that the raising and lowering operators act on spherical harmonics in the way expected for the derivation of the Clebsch-Gordan coefficients in use. Therefore we get D-wave (using the rotation matrices for $j = 2$ we get $\cos 2\theta$ angular dependence on amplitudes) $s = 2$ photon emission with $j = 2$.

For 2^3P_1 we see that none of the states couple since $|\uparrow\uparrow\rangle$ only couples to two photons when paired with $|\psi_{1,1,1}\rangle$ and likewise only the $m = 0$ states and $m = -1$ states couple to eachother. This decay is not forbidden by C it is forbidden by angular momentum selection since a two photon state cannot have odd spin along an axis and thus cannot have $j = 1$ since L_z is zero along the direction of motion and hence J_z has an even eigenvalue. (BETTER EXPLINATION)

10.2 Positronium Decay Rate

We build a Positronium states. For $\ell = 0$ we consider,

$$|\mathrm{Ps}(\vec{k})\,{}^1S_0\rangle = \sqrt{2E_{\vec{k}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{4E_{\vec{p}+\frac{\vec{k}}{2}}E_{-\vec{p}+\frac{\vec{k}}{2}}}}\tilde{\psi}_0(\vec{p})\frac{\sqrt{4E_{\vec{p}+\frac{\vec{k}}{2}}E_{-\vec{p}+\frac{\vec{k}}{2}}}}{\sqrt{2}}\sum_s a^{s\dagger}_{\vec{p}+\frac{\vec{k}}{2}}b^{s\dagger}_{-\vec{p}+\frac{\vec{k}}{2}}|\Omega\rangle$$

we need to show that this is properly normalized. Indeed,

$$\begin{aligned}
\langle \mathrm{Ps}(\vec{k}')\,{}^1S_0\,|\,\mathrm{Ps}(\vec{k})\,{}^1S_0\rangle &= 2E_{\vec{k}}\int \frac{\mathrm{d}^3\vec{p}'}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\tilde{\psi_0}^*(\vec{p}')\tilde{\psi}_0(\vec{p})\left[\frac{1}{2}\sum_{s's}\langle\Omega|b^{s'}_{-\vec{p}'+\frac{\vec{k}'}{2}}a^{s'}_{\vec{p}'+\frac{\vec{k}'}{2}}a^{s\dagger}_{\vec{p}+\frac{\vec{k}}{2}}b^{s\dagger}_{-\vec{p}+\frac{\vec{k}}{2}}|\Omega\rangle\right] \\
&= 2E_{\vec{k}}\int \frac{\mathrm{d}^3\vec{p}'}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\tilde{\psi_0}^*(\vec{p}')\tilde{\psi}_0(\vec{p}) \\
&\quad\cdot\left[\frac{1}{2}\sum_{s's}(2\pi)^3\delta^{(3)}(\vec{p}' + \tfrac{\vec{k}'}{2} - \vec{p} - \tfrac{\vec{k}}{2})\delta_{ss'}\delta_{ss'}(2\pi^3)\delta^{(3)}(-\vec{p} + \tfrac{\vec{k}}{2} + \vec{p}' - \tfrac{\vec{k}'}{2})\right]
\end{aligned}$$

Call the arguments of the δ-functions A, B. Then $\frac{1}{2}(A + B) = (\vec{p}' - \vec{p})$ and $A - B = \vec{k}' - \vec{k}$ and the matrix,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix}$$

has determinant -1 and therefore we can perform this change of variables on the δ-functions to get,

$$\begin{aligned}
\langle \mathrm{Ps}(\vec{k}')\,{}^1S_0\,|\,\mathrm{Ps}(\vec{k})\,{}^1S_0\rangle &= 2E_{\vec{k}}\int \frac{\mathrm{d}^3\vec{p}'}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\tilde{\psi_0}^*(\vec{p}')\tilde{\psi}_0(\vec{p})\left[(2\pi)^6\delta^{(3)}(\vec{p}' - \vec{p})\delta^{(3)}(\vec{k}' - \vec{k})\right] \\
&= 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k}' - \vec{k})\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\tilde{\psi}_0^*(\vec{p})\tilde{\psi}(\vec{p}) = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k}' - \vec{k})
\end{aligned}$$

which is the desired relativistic normalization. Now we consider the P-states. Let ψ_i be a set of P-wave wavefunctions of the form $\psi_i = x^i f(|x|)$ normalized such that,

$$\int \mathrm{d}^3x\; \psi_i^*(x)\psi_j(x) = \delta_{ij}$$

Then consider the states,

$$|\mathrm{Ps}(\vec{k})\, P_\Sigma\rangle = \sqrt{2E_{\vec{k}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_i \frac{1}{\sqrt{4E_{p+\frac{\vec{k}}{2}}E_{-p+\frac{\vec{k}}{2}}}}\tilde{\psi}_i(p)\sqrt{4E_{p+\frac{\vec{k}}{2}}E_{-p+\frac{\vec{k}}{2}}}\sum_{s's} a^{s'\dagger}_{p+\frac{\vec{k}}{2}}\Sigma^i_{s's}b^{s\dagger}_{-p+\frac{\vec{k}}{2}}|\Omega\rangle$$

where Σ_i are a set of 2 x 2 matrices such that $\sum_i \mathrm{tr}\,\Sigma^{i\dagger}\Sigma^i = 1$. We need to check the normalization of these states,

$$\begin{aligned}
\langle \mathrm{Ps}(\vec{k}')\, P_\Sigma \,|\, \mathrm{Ps}(\vec{k})\, P_\Sigma\rangle &= 2E_{\vec{k}}\int \frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{ij}\tilde{\psi}^*_i(p')\tilde{\psi}_j(p)\sum_{t'ts's}\Sigma^{j*}_{t't}\Sigma^i_{s's}\langle\Omega|b^t_{-p'+\frac{\vec{k}'}{2}}a^{t'}_{p'+\frac{\vec{k}'}{2}}a^{s'\dagger}_{p+\frac{\vec{k}}{2}}b^{s\dagger}_{-p+\frac{\vec{k}}{2}}|\Omega\rangle \\
&= 2E_{\vec{k}}\int \frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{ij}\tilde{\psi}^*_i(p')\tilde{\psi}_j(p)\left[\sum_{t'ts's}(\Sigma^{j\dagger})_{tt'}\Sigma^i_{s's}\delta_{ts}\delta_{t's'}\right](2\pi)^6\delta^{(3)}(p'-p)\delta^{(3)}(\vec{k}'-\vec{k}) \\
&= 2E_{\vec{k}}(2\pi)^3\delta(\vec{k}'-\vec{k})\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{ij}\left[\tilde{\psi}^*_i(p')\tilde{\psi}_j(p)\,\mathrm{tr}\,\Sigma^{j\dagger}\Sigma^i\right] \\
&= 2E_{\vec{k}}(2\pi)^3\delta(\vec{k}'-\vec{k})\sum_{ij}\delta_{ij}\,\mathrm{tr}\,\Sigma^{j\dagger}\Sigma^i = 2E_{\vec{k}}(2\pi)^3\delta(\vec{k}'-\vec{k})
\end{aligned}$$

using the normalization condition on Σ. Hence we get the correct relativistic normalization. Then we can consider the decay rate. For a positronium state of the form,

$$|\mathrm{Ps}\rangle = \sqrt{2M}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{iss'}C_{iss'}\tilde{\psi}_i(p)a^{s\dagger}_p b^{s'\dagger}_{-p}|\Omega\rangle$$

then we get an amplitude for two photon decay, recalling the relativisitc normalization convention for the definition of $\mathcal{M}$,

$$\mathcal{M}(\mathrm{Ps}\to 2\gamma) = \sqrt{2M}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{iss'}C_{iss'}\tilde{\psi}_i(p)\frac{1}{2m}\mathcal{M}(e^-(\vec{p},s)+e^+(-\vec{p},s')\to 2\gamma)$$

Now we let,

$$\mathcal{M}^{ss'}_{\alpha\beta}(\vec{p},\vec{k}) := \mathcal{M}(e^-_s(\vec{p})+e^+_{s'}(-\vec{p})\to\gamma_\alpha(\vec{k})+\gamma_\beta(-\vec{k}))$$

where $\alpha,\beta = +$ or $-$ label the photon polarizations and $a,b =\uparrow$ or $\downarrow$ are spinor indices for the polarizations of the electron and the positron (recall we use Peskin's terrible convention that the spinor indices for antiparticles are flipped with respect to the physical spin). We will need to expand this in $\vec{p}$. Then the decay rate for a fixed polarization is given by,

$$\begin{aligned}
\Gamma &= \frac{1}{2}\int\frac{1}{2M}\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{4|k|^2}|\mathcal{M}(\mathrm{Ps}\to 2\gamma)|^2(2\pi)^4\delta^{(4)}(k+k'-p_{\mathrm{Ps}}) \\
&= \frac{2\pi}{16M}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{|k|^2}\delta(2|k|-M)|\mathcal{M}(\mathrm{Ps}\to 2\gamma)|^2 \\
&= \frac{2\pi}{16M}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{|k|^2}\delta(2|k|-M)|\mathcal{M}(\mathrm{Ps}\to 2\gamma)|^2 \\
&= \frac{1}{16M}\frac{1}{(2\pi)^2}\frac{1}{2}\int|\mathcal{M}(\mathrm{Ps}\to 2\gamma)|^2\mathrm{d}\Omega \\
&= \frac{\pi}{(4\pi)^3}\frac{1}{2M}\int|\mathcal{M}(\mathrm{Ps}\to 2\gamma)|^2\mathrm{d}\Omega
\end{aligned}$$

the first $\frac{1}{2}$ from the fact that the final state photons are identical particles and another $\frac{1}{2}$ comes from the δ-function. Therefore,

$$\Gamma = \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2} \int \sum_{\alpha\beta} \left| \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \sum_{iss'} C_{iss'} \tilde{\psi}_i(\vec{p}) \mathcal{M}^{ss'}_{\alpha\beta}(\vec{p},\vec{k}) \right|^2 \mathrm{d}\Omega$$

Now we first compute the S-wave decays. Consider the $\ell = 0$ wavefunctions,

$$\psi_n(x) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-1)!}{2nn!}} e^{-\rho} L^1_{n-1}(\rho) \cdot \frac{1}{\sqrt{4\pi}}$$

where $\rho = \frac{2r}{na_0}$ and $a_0 = (\mu\alpha^2)^{-1}$ where μ is the reduced mass and L^1_{n-1} is the generalized Laguerre polynomial which are normalized so that $L^1_{n-1}(0) = 1$. Therefore,

$$\psi_n(0) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-1)!}{2n\,n!}} \cdot \frac{1}{\sqrt{4\pi}}$$

Since these have zero gradient at the origin, to first-order in $\vec{p}$ we find,

$$\Gamma_{n^1S_0} = \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2} \int 2|\psi_n(0)|^2 [8e^4] \mathrm{d}\Omega$$

the first two arises from the two final allowed polarizations and the $8e^4$ is the square of,

$$\mathcal{M}_{\pm\pm}(s=0) = \mp 2\sqrt{2}e^2$$

and using that the other polarization states have zero amplitude by conservation of angular momentum. Alternatively, we can use the ${}^1S_0$ state we constructed earlier which has $C_{0ss'} = \frac{1}{\sqrt{2}}\delta_{ss'}$ and that $\mathcal{M}^{ss'}_{\pm\pm} = \mp 2e^2 \delta_{ss'}$ so therefore the internal sum gives the same result: $\mp 2\sqrt{2}e^2\psi_n(0)$. Therefore, we get,

$$\Gamma_{^1S_0} = \frac{1}{(4\pi)^2} \cdot \frac{\pi}{4m^2} \cdot \left(\frac{m\alpha}{n}\right)^3 \frac{1}{2n^2} \cdot \frac{1}{4\pi} \cdot (16e^4) = \frac{1}{2n^5} m\alpha^5$$

In the case, $n = 1$ we get,

$$\Gamma_{1^1S_0} = \tfrac{1}{2} m\alpha^5$$

and for $n = 2$ we get,

$$\Gamma_{2^1S_0} = \tfrac{1}{64} m\alpha^5$$

Likewise the ${}^3S_1$ state decay to two photons is forbidden by C invariance.

Now we compute the decay of the P-states. From our expression for the Positronium state, we get a decay rate,

$$\Gamma = \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2} \int \sum_{\alpha\beta} \left| \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \sum_i \tilde{\psi}_i(\vec{p})\, \mathrm{tr}\,(\Sigma^\top \mathcal{M}_{\alpha\beta}(\vec{p},\vec{k})) \right|^2 \mathrm{d}\Omega$$

Since $\psi_i(0) = 0$ the zeroth-order term of $\mathcal{M}(\vec{p})$ integrates to zero. Therefore, we should write,

$$\mathcal{M}^{ss'}(\vec{p},\vec{k}) = \mathcal{M}^{ss'}(0,\vec{k}) + \vec{F}^{ss'}(\vec{k}) \cdot \vec{p} + O(p^2)$$

Then to first-order,

$$\Gamma = \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2} \int \sum_{\alpha\beta} \left| \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \sum_i \tilde{\psi}_i(\vec{p})\, [\vec{p} \cdot \mathrm{tr}\,(\Sigma^{i\top} \vec{F}_{\alpha\beta}(\vec{k}))] \right|^2 \mathrm{d}\Omega$$

Furthermore,

$$\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\tilde{\psi}_i(\vec{p})\,\vec{p} = -i\nabla\psi_i(x)\bigg|_{x=0} = -i\vec{e}_i f(0)$$

because the other term $x^i\partial_j f(|x|)$ is zero at $\vec{x}=0$. Therefore,

$$\Gamma = \frac{1}{(4\pi)^3}\cdot\frac{\pi}{4m^2}|f(0)|^2\int\sum_{\alpha\beta}\left|\mathrm{tr}\,(\vec{\Sigma}^\top\cdot\vec{F}_{\alpha\beta}(\vec{k}))\right|^2\mathrm{d}\Omega$$

Now we set,

$$\Sigma^i = \begin{cases} \frac{1}{\sqrt{6}}\sigma^i & j=0 \\ \frac{1}{2}\epsilon^{ijk}n^j\sigma^k & j=1 \\ \frac{1}{\sqrt{2}}h^{ij}\sigma^j & j=2 \end{cases}$$

where n is a unit vector and h^{ij} is a symmetric traceless tensor such that $\sum_{ij}h^{ij}(h^{ij})^* = 1$. Note! Peskin has a mistake here, in order for the normalization to work correctly we need $\frac{1}{\sqrt{2}}$ not $\frac{1}{\sqrt{3}}$ in the $j=2$ case. To compute $\mathrm{tr}\,(A^\top\mathcal{M})$ write,

$$\mathcal{M}^{ss'} = \xi^{s'\dagger}M\xi^s$$

and therefore,

$$\mathrm{tr}\,(A^\top\mathcal{M}) = \sum_{ss'}A_{ss'}\mathcal{M}^{ss'} = \sum_{ss'}A_{ss'}\xi^{s'\dagger}M\xi^s = \mathrm{tr}\left(M\sum_{ss'}A_{ss'}\xi^s\xi^{s'\dagger}\right) = \mathrm{tr}\,(MA^\top)$$

We computed,

$$\mathrm{tr}\,(M^i_{\pm\pm}\sigma^j) = -\frac{4e^2}{m}\delta_{ij} \qquad \mathrm{tr}\,(M^i_{\pm\mp}\sigma^j) = -\frac{4e^2}{m}(\delta_{i1}\mp i\delta_{i2})(\delta_{j1}\mp i\delta_{j2})$$

Therefore, for the $j=0$ angular momentum state,

$$\mathrm{tr}\,(\vec{\Sigma}^\top\cdot\vec{F}_{\pm\pm}) = -\frac{2\sqrt{6}e^2}{m} \qquad \mathrm{tr}\,(\vec{\Sigma}^\top\cdot\vec{F}_{\pm\mp}) = -\frac{4e^2}{m}\frac{1}{\sqrt{6}}(1-1) = 0$$

Therefore,

$$\Gamma_{2^3P_0} = \frac{1}{(4\pi)^2}\cdot\frac{\pi}{4m^2}|f(0)|^2\cdot 2\left(\frac{24e^4}{m^2}\right)$$

and

$$f(r) = \frac{1}{4\sqrt{2\pi}a_0^{3/2}}\frac{1}{a_0}e^{-r/2a_0}$$

where $a_0 = (\mu\alpha)^{-1}$ so plugging in gives,

$$\Gamma_{2^3P_0} = \frac{1}{(4\pi)^2}\cdot\frac{\pi}{4m^2}\frac{(m/2)^5\alpha^5}{32\pi}\cdot 2\left(\frac{24e^4}{m^2}\right) = \frac{3}{256}m\alpha^7$$

For the $j=1$ angular momentum state,

$$\mathrm{tr}\,(\vec{\Sigma}^\top\cdot\vec{F}_{\pm\pm}) = 0 \qquad \mathrm{tr}\,(\vec{\Sigma}^\top\cdot\vec{F}_{\pm\mp}) = -\frac{4e^2}{2m}(\mp i\epsilon^{132}n^3 \mp i\epsilon^{231}n^3) = 0$$

and we recover the fact that the $j = 1$ does not decay into two photons. For the $j = 2$ angular momentum state,

$$\mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\pm}) = -\frac{4e^2}{\sqrt{2}m} h^{ij}\delta_{ij} = 0 \qquad \mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\mp}) = -\frac{4e^2}{\sqrt{2}m}(h^{11} - h^{22} \mp ih^{12} \mp ih^{21})$$

We need to average over the possible polarization tensors h. However, the Peskin solutions have an error in that the standard basis of symmetric traceless tensors do not give an *orthogonal* basis of spin 2 states. Indeed, consider the matrices,

$$h_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad h_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

these correspond to states,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\hat{x}_0\rangle \otimes |\hat{x}_0\rangle - |\hat{y}_0\rangle \otimes |\hat{y}_0\rangle) \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|\hat{y}_0\rangle \otimes |\hat{y}_0\rangle - |\hat{z}_0\rangle \otimes |\hat{z}_0\rangle)$$

but the $m = 0$ states along perpendicular axes are orthogonal. Therefore $\langle \psi_1 \,|\, \psi_2 \rangle = \frac{1}{2}$. Instead we need to choose a basis of symmetric traceless matrices which is orthogonal for the physical states. A good choice are the states of definite J_z. These correspond to,

$$h_{+2} = \frac{1}{2}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h_{+1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}$$

$$h_0 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$h_{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}$$

$$h_{+2} = \frac{1}{2}\begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the calculation of $\mathcal{M}$ we put $\vec{k}$ along $\hat{z}$ so to perform the integral over $\vec{k}$ we instead need to integrate over the orientations of the spin 2 particle. If we choose coordinates with $\theta = 0$ corresponding definite spin along z then the amplitude squared is constant in the azimuthal angle ϕ. Then,

$$\begin{aligned}
\frac{\mathrm{d}\Gamma_h^\pm}{\mathrm{d}\Omega} &= \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2}\frac{(m/2)^5\alpha^5}{32\pi} \cdot \left(\frac{2\sqrt{2}e^2}{m}\right)^2 |h^{11}(\theta) - h^{22}(\theta) \mp 2ih^{12}(\theta)|^2 \\
&= \frac{m\alpha^7}{512} \cdot \frac{1}{4\pi}|h^{11}(\theta) - h^{22}(\theta) \mp 2ih^{12}(\theta)|^2 \\
&= \frac{m\alpha^7}{512} \cdot \frac{1}{4\pi} f_h(\theta)
\end{aligned}$$

Applying the rotation matrix and then computing the amplitude and summing over the two nonzero photon polarization states,

$$\begin{aligned}
f_{+2}(\theta) &= \tfrac{1}{16}(35 + 28\cos 2\theta + \cos 4\theta) \\
f_{+1}(\theta) &= \tfrac{1}{4}(5 - 4\cos 2\theta - \cos 4\theta) \\
f_{0}(\theta) &= \tfrac{1}{8}(9 - 12\cos 2\theta + 3\cos 4\theta) \\
f_{-1}(\theta) &= \tfrac{1}{4}(5 - 4\cos 2\theta - \cos 4\theta) \\
f_{-2}(\theta) &= \tfrac{1}{16}(35 + 28\cos 2\theta + \cos 4\theta)
\end{aligned}$$

Then notice that,

$$f_{+2}(\theta) + f_{+1}(\theta) + f_0(\theta) + f_{-1}(\theta) + f_{-2}(\theta) = 8$$

so an unpolarized collection of emitts photons uniformly as they must. Therefore, averaging over polarizations and integrating we get,

$$\Gamma_{2^3P_2} = \frac{m\alpha^7}{512} \cdot \frac{8}{5} = \tfrac{1}{320} m\alpha^7$$

Morover, integrating each $f_i(\theta)$ we get the same value: $\frac{32\pi}{5}$ and hence,

$$\Gamma_h = \tfrac{1}{320} m\alpha^7$$

and therefore each state in the $j = 2$ multiplet has the same decay probability. We can see why this is true by angular momentum conservation. Since $\vec{J}$ commutes with the Hamiltonian and hence the rasing and lower operators $J_\pm$ commute with the Hamiltonian. These annihilate the electromagnetic vacuum and hence only act on the positronium state. Therefore the different m-states of the J multiplets of positronium must have the same overall decay rate and must decay to photon states which are connected by raising and lowering operators of total angular momentum.

10.3 Decay to Massive B

For a massive vector particle there are three possible physical polarizations. The $\epsilon^\pm$ are identical but the third polarization in the rest frame $\vec{\epsilon} = (0, 0, 1)$ is transformed via Lorentz bost into,

$$\epsilon^\mu = (\beta\gamma, 0, 0, \gamma)$$

For the particle with momentum $\vec{k}$ and the opposite sign on β for the other particle. From the traces, we again see that,

$$\mathcal{M}_{\pm\pm}(s = 0) = \mp \left(\frac{2\sqrt{2}e^2}{1 - \frac{m_B^2}{2m^2}} \right) \left(\frac{k}{m} \right)$$

and all other polarizations have zero amplitude. Similarly, for $s = 1$ the transverse polarizations give a similar result,

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s = 1) &= -\left(\frac{2\sqrt{2}e^2}{1 - \frac{m_B^2}{2m^2}} \right) \left(\tfrac{\vec{p}}{m} \cdot \hat{n} - \left(\tfrac{m_B^2}{2m^2 - m_B^2} \right) (\tfrac{\vec{p}}{m} \cdot \hat{k})(\hat{k} \cdot \hat{n}) \right) \\
\mathcal{M}_{\pm\mp}(s = 1) &= -\left(\frac{2\sqrt{2}e^2}{1 - \frac{m_B^2}{2m^2}} \right) (\tfrac{p_x}{m} \mp i\tfrac{p_y}{m})(n_x \mp i n_y)
\end{aligned}$$

Now we need to compute the amplitudes with at least one transverse polarization.

$$\mathcal{M}_{00}(s=1) = -\frac{2\sqrt{2}e^2}{\left(1-\frac{m_B^2}{2m^2}\right)^2}\left(\frac{m_B}{m}\right)^2\left(\frac{\vec{p}\cdot\hat{n}}{m}\right)$$
$$\mathcal{M}_{\pm 0}(s=1) = -\frac{4e^2 m_B}{2m^2-m_B^2}\left[n_z(p_x \mp i p_y) + \frac{m_B^2}{2m^2-m_B^2}(n_x \mp i n_y)p_z\right]$$
$$\mathcal{M}_{0\pm}(s=1) = -\frac{4e^2 m_B}{2m^2-m_B^2}\left[n_z(p_x \pm i p_y) + \frac{m_B^2}{2m^2-m_B^2}(n_x \pm i n_y)p_z\right]$$

This gives S-wave emission for states with $m_\ell = 0$ and $s = 1$. Notice that these go to zero as $m_B \to 0$ as they must since this is an unphysical polarization of the photon.

Let's see if any of these amplitudes break any forbidden decays into two photons. The ${}^3S_1$ states remain forbidden since the new polarizations only couple to nonzero momenta. Indeed C conservation shows this decay remains forbidden as does 2^1P_1. Indeed, the new polarizations couple to $\vec{\sigma}$ and hence give zero on $s = 0$. Thus we need only consider 2^3P_1. The new terms of interest are in $\mathcal{M}_{\pm 0}(s=1)$ and $\mathcal{M}_{0\pm}(s=1)$ which couple $m = 0$ to $m = \pm 1$ states. This indeed allows for decay of 2^3P_1. This shows that the massless photon imposes an additional restriction on the angular momentum selection rules.

11 Parity (and C) Violating Decay

Consider a new interaction term,

$$\mathcal{H}_{\text{int}} = \bar{\psi}(g_s + i g_p \gamma^5)\not{B}\psi$$

Therefore, the new vertex contribution in the Feynman rules is,

$$-i\Gamma^\mu = -i(g_s + i g_p \gamma^5)\gamma^\mu$$

This term is P and C violating unless $g_p = 0$ and B^μ is a vector or $g_s = 0$ and B^μ is a pseudo-vector.

11.1 $e^- + e^+ \to B$

The amplitude for this process is,

$$\mathcal{M} = -\epsilon^*_\mu \bar{v}^{s_2}(p_2)\Gamma^\mu u^{s_1}(p_1)$$

In the nonrelativistic limit in the CM frame we get,

$$\mathcal{M} = -E_{\text{CM}}\xi'^\dagger\left[g_s(\vec{\epsilon^*}\cdot\vec{\sigma}) + i g_p \epsilon^{*0} - g_p \vec{\sigma}\cdot\left(\tfrac{\vec{p}}{m}\times\vec{\epsilon^*}\right)\right]\xi$$

For $s = 0$ we get,

$$\mathcal{M}(s=0) = -E_{\text{CM}}\sqrt{2} i g_p \epsilon^{*0}$$

but in the CM frame $\epsilon^{*0} = 0$ since there is no timelike polarization. Thus $\mathcal{M}(s=0) = 0$ as expected since it must decay to a spin 1 particle. The orbital angular momentum cannot produce a B through either interaction, interesting. For g_s this is explained by P-invariance since ${}^1P_1$ is even

	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$
P	✓	✓	✗	✗	✗	✗
C	✗	✓	✓	✗	✗	✗
j	✗	✓	✓	✗	✓	✗

Table 1: $e^- + e^+ \to B$ vector (g_s) decays allowed by P, C, and J conservation.

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	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$
P	✗	✗	✓	✓	✓	✓
C	✓	✗	✗	✓	✓	✓
j	✗	✓	✓	✗	✓	✗

Table 2: $e^- + e^+ \to B$ pseduo-vector (g_p) decays allowed by P, C, and J conservation.

.

under P but for $g_p = 0$ we get P conservation if B is odd so g_s cannot couple to $^1P_1$. For g_p this is explained by C-invariance since $^1P_1$ is odd under P but for $g_s = 0$ we get C conservation if B is even so the term g_p cannot couple to $^1P_1$.

We would expect if $\mathcal{M} = \epsilon_\mu \mathcal{M}^\mu$ then $\mathcal{M}^0 = 0$ by the Ward identity since we are in the rest frame of the produced B. However, this does not happen since the Ward identity is violated by this interaction. Does this create a problem? For the $s = 1$ states we get, using the trace tricks,

$$\mathcal{M}(s=1) = -E_{\text{CM}}\sqrt{2}\left[g_s(\vec{\epsilon^*}\cdot\hat{n}) - g_p(\tfrac{\vec{p}}{m}\times\vec{\epsilon^*})\cdot\hat{n}\right]$$

If g_s is nonzero then any $s = 1$ state nonvanishing at the origin can decay to form a B polarized along $\hat{n}$. If $g_p = 0$ then $\mathcal{H}_{\text{int}}$ is P-invariant with B_μ odd under parity. Since the $^3S_1$ states are P odd the g_s coupling is alowed. Likewise P states are P even explaining why g_s does not couple to P states. Furthermore, the $^3S_1$ are odd under P but if $g_s = 0$ then the coupling is P invariant with B even hence the g_p can only couple to states even under P.

If g_p is nonzero there is a more complicated coupling. This coupling vanishes on S states since it is proportional to $\vec{p}$. Furthermore it vanishes on the 2^3P_2 and 2^3P_0 states because it does not couple states with parallel $\vec{p}$ and $\hat{n}$. This must be true since a single B has $j = 1$ in its rest frame. However, by inspection, it does couple to 2^3P_1 with $j = 1$ as is allowed by angular momentum conservation. This cannot be explained by P or C since all 2^3P_j are even under P and C only by $j = 1$ selection.

The tables show that only one state is allowed to decay for each of g_s and g_p and these both occur at leading order.

11.2 $e^- + e^+ \to 2B$

The leading-order Feynman diagrams give,

$$i\mathcal{M} = (-i)^2 \epsilon(k_1)^*_\mu \epsilon(k_2)^*_\nu\ \bar{v}^{s_2}(p_2)\left[\Gamma^\nu \frac{i(\not{q}_1 + m)}{q_1^2 - m^2 + i\epsilon}\Gamma^\mu + \Gamma^\mu \frac{i(\not{q}_2 + m)}{q_2^2 - m^2 + i\epsilon}\Gamma^\nu\right] u^{s_1}(p_1)$$

With two particles in the final state, the configuration may contribute to overall parity so we cannot simply use P to rule out decays. Indeed, we saw that states both odd – ${}^1S_0$ – and even – ${}^3P_0$ under P decay to two photons. However, C allows us to forbid decays in the case that one coupling constant is zero so the Hamiltonian is C invariant. In either case B^μ is a vector or pseudovector i.e. has definite C so the $2B$ state has C eigenvalue $+1$. Hence the same decays are C forbidden in the vector and pseudovector cases. Since we showed that the states which are not C protected already decay to a massive (only ${}^3P_1$ is protected in the massless case) vector we will not get anything new in the pseudovector case. Therefore we look for C violating decays. The candidates are ${}^3S_1$ and ${}^1P_1$. Note that both have $j = 1$ so we need to work in the massive case to have a chance of seeing such decays.

For ${}^3S_1$ we need to consider only the nonrelativistic limit to zeroth-order in momenta. Consider,

$$
\begin{aligned}
&\bar{v}^{s_2}(p_2)\Gamma^\nu(\gamma^\alpha + m)\Gamma^\mu u^{s_1}(p_1)\\
&\quad= m\overline{\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\nu\\ (g_s + ig_p)\bar{\sigma}^\nu & 0\end{pmatrix}\begin{pmatrix}m & \sigma^\alpha\\ \bar{\sigma}^\alpha & m\end{pmatrix}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\mu\\ (g_s + ig_p)\bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&\quad= m\overline{\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\nu\\ (g_s + ig_p)\bar{\sigma}^\nu & 0\end{pmatrix}\begin{pmatrix}\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & m(g_s - ig_p)\sigma^\mu\\ m(g_s + ig_p)\bar{\sigma}^\mu & \bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&\quad= m\overline{\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\nu\\ (g_s + ig_p)\bar{\sigma}^\nu & 0\end{pmatrix}\begin{pmatrix}\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & m(g_s - ig_p)\sigma^\mu\\ m(g_s + ig_p)\bar{\sigma}^\mu & \bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&\quad= m\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}^\dagger\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}(g_s - ig_p)\sigma^\nu m(g_s + ig_p)\bar{\sigma}^\mu & (g_s - ig_p)\sigma^\nu\bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\\ (g_s + ig_p)\bar{\sigma}^\nu\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & (g_s + ig_p)\bar{\sigma}^\nu m(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&\quad= m\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}^\dagger\begin{pmatrix}(g_s + ig_p)\bar{\sigma}^\nu\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & (g_s + ig_p)\bar{\sigma}^\nu m(g_s - ig_p)\sigma^\mu\\ (g_s - ig_p)\sigma^\nu m(g_s + ig_p)\bar{\sigma}^\mu & (g_s - ig_p)\sigma^\nu\bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\\
&\quad= m\xi'^\dagger\Big[(g_s + ig_p)\bar{\sigma}^\nu\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu + (g_s + ig_p)\bar{\sigma}^\nu m(g_s - ig_p)\sigma^\mu\\
&\qquad\qquad - (g_s - ig_p)\sigma^\nu m(g_s + ig_p)\bar{\sigma}^\mu - (g_s - ig_p)\sigma^\nu\bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\Big]\xi\\
&\quad= m\xi'^\dagger\Big[(g_s^2 - g_p^2)[\bar{\sigma}^\nu\sigma^\alpha\bar{\sigma}^\mu - \sigma^\nu\bar{\sigma}^\alpha\sigma^\mu] + 2ig_sg_p[\bar{\sigma}^\nu\sigma^\alpha\bar{\sigma}^\mu + \sigma^\nu\bar{\sigma}^\alpha\sigma^\mu] + (g_s^2 + g_p^2)m[\bar{\sigma}^\nu\sigma^\mu - \sigma^\nu\bar{\sigma}^\mu]\Big]\xi
\end{aligned}
$$

To zeroth order in momenta $q_1 = (0, -\vec{k})$ and $q_2 = (0, \vec{k})$ so $q^1 = q^2 = -k^2 = -(m^2 - m_B^2)$. Since α is spatial the second term, the P violating term, is only nonzero if exactly one of μ or ν is spatial. Therefore, we expect P violating decays into one transverse polarization and one longitudinal polarization B. Therefore,

$$
\begin{aligned}
\mathcal{M} = \left(\frac{2m}{2m^2 - m_B^2}\right)\xi'^\dagger\Big[&(g_s^2 - g_p^2)[(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma}) - (\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma})]\\
&+ 2ig_sg_p[\epsilon_2^{*0}(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma}) + (\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})\epsilon_1^{*0} - \epsilon_1^{*0}(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma}) - (\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})\epsilon_2^{*0}]\Big]\xi
\end{aligned}
$$

Now,

$$
(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{c} + i(\vec{b}\times\vec{c})\cdot\vec{\sigma}) = i\vec{a}\cdot(\vec{b}\times\vec{c}) + (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{\sigma}) - (\vec{a}\times(\vec{b}\times\vec{c}))\cdot\vec{\sigma}
$$

Antisymmetrizing over $\vec{a}$ and $\vec{c}$ gives,

$$
\begin{aligned}
(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma}) - (\vec{c}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) &= 2i\vec{a}\cdot(\vec{b}\times\vec{c}) + [(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{\sigma}) - (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{\sigma})] + (\vec{b}\times(\vec{c}\times\vec{a}))\cdot\vec{\sigma}\\
&= 2i\vec{a}\cdot(\vec{b}\times\vec{c})
\end{aligned}
$$

Therefore,

$$\mathcal{M} = \left(\frac{2i}{1-\frac{m_B^2}{2m^2}}\right)\xi'^{\dagger}\Big[(g_s^2-g_p^2)[\tfrac{\vec{k}}{m}\cdot(\vec{\epsilon}_1^{\,*}\times\vec{\epsilon}_2^{\,*})] + 2ig_sg_p[\tfrac{\vec{k}}{m}\times(\epsilon_2^{*0}\vec{\epsilon}_1^{\,*}-\epsilon_1^{*0}\vec{\epsilon}_2^{\,*})]\cdot\vec{\sigma}\Big]\xi$$

We get a reduction of the main term in $\mathcal{M}(s=0)$ but only transverse polarizations can be emitted from ${}^1S_0$ states still (WHY IS THERE SOME CONSERVATION?) Our trace tricks give,

$$\mathcal{M}(s=1) = -\left(\frac{4\sqrt{2}g_sg_p}{1-\frac{m_B^2}{2m^2}}\right)[\tfrac{\vec{k}}{m}\times(\epsilon_2^{*0}\vec{\epsilon}_1^{\,*}-\epsilon_1^{*0}\vec{\epsilon}_2^{\,*})]\cdot\hat{n}$$

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1) &= 0\\
\mathcal{M}_{\pm\mp}(s=1) &= 0\\
\mathcal{M}_{00}(s=1) &= 0\\
\mathcal{M}_{\pm 0}(s=1) &= \pm 8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{m^2-m_B^2}{2m^2-m_B^2}\right)(n_x\mp in_y)\\
\mathcal{M}_{0\pm}(s=1) &= \mp 8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{m^2-m_B^2}{2m^2-m_B^2}\right)(n_x\pm in_y)
\end{aligned}$$

These alow decay of ${}^3S_1$ into P-wave (since the total probability for $B+B$ production varies as $\cos^2\theta$ away from the $s=1$ spin axis) $j=1$ with $s=1$ state of two B particles.

Finally, we need to consider the decay of ${}^1P_1$. To do this we need to expand $\mathcal{M}(s=0)$ to second-order in $\vec{p}$. We get,

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1) &= \pm\frac{2\sqrt{2}(g_p^2-g_s^2)\sqrt{1-\frac{m_B^2}{m^2}}}{1-\frac{m_B^2}{2m^2}}\\
\mathcal{M}_{\pm\mp}(s=1) &= 0\\
\mathcal{M}_{00}(s=1) &= 0\\
\mathcal{M}_{\pm 0}(s=1) &= +8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{\sqrt{m^2-m_B^2}}{2m^2-m_B^2}\right)(p_x\mp ip_y)\\
\mathcal{M}_{0\pm}(s=1) &= -8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{\sqrt{m^2-m_B^2}}{2m^2-m_B^2}\right)(p_x\pm ip_y)
\end{aligned}$$

The transverse polarizations only couple to zero momentum. However, we see that the orbital angular momentum can now couple though the $\mathcal{M}_{\pm 0}$ and $\mathcal{M}_{0\pm}$ amplitudes to create spin 1 particles. This alows for the decay of ${}^1P_1$ into P-wave $j=1$ with $s=1$ state of two B-particles.

Questions: notice that these amplitudes diverge as $m_B\to 0$. Does this show that the parity violating coupling is somehow inconsistent for massless particles. It is indeed not gauge invariant so the Ward identity is violated so we may not expect the longitudal polarization to cancel in the limit. Is this a problem? Is there any reason that these extra coupling for n and for p look very similar up to signs and a factor of k?

12 Weights

Let (G, T) be a pair of a Lie group (or algberaic group) and a maximal torus. Then the character lattice $X(T)$ is the abelian group of characters. Often we work inside $X(T)_{\mathbb{R}}$.

Definition 12.0.1. The action $T \circlearrowright G$ by conjugation induces an action $T \circlearrowright \mathfrak{g}$ and hence a weight decomposition,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\Phi \subset X(T)$ is the set of *roots* i.e. nonzero characters of T such that,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall t \in T : t \cdot X = \alpha(t) X\}$$

is nonempty.

This is a special case of the general theory of weights.

Definition 12.0.2. A *weight* of a Lie algebra over k is a k-linear map $\lambda \in \mathfrak{g}^*$ such that $\lambda([x, y]) = 0$.

Remark. A weight is just a morphism of Lie algebras to the 1-dimensional abelian Lie algebra. Indeed, for any algebra a character should be an algebra map to the corresponding "trivial" 1-dimensional such algebra. For example, for associative algebras this recovers a multiplicative character.

Remark. Clearly any weight of $\mathfrak{g}$ factors through $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ which is abelian. Therefore, it is most natural to consider weights only for abelian Lie algebras. This is why we pass to a Cartan subalgebra (corresponding to a maximal torus).

For now fix a pair $(\mathfrak{g}, \mathfrak{h})$ of a Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Definition 12.0.3. Let $\rho : \mathfrak{g} \to \mathrm{End}\,(V)$ be a representation of a Lie algebra $\mathfrak{g}$ on a vectorspace V. Let λ be a weight of $\mathfrak{h}$. Then the *weight space* of V with weight λ is the subspace,

$$V_\lambda := \{v \in V \mid \forall H \in \mathfrak{h} : \rho(H) \cdot v = \lambda(H) \cdot v\}$$

A *weight* of ρ is a nonzero weight λ of $\mathfrak{h}$ such that V_λ is nonempty.

For example, if we consdier the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,(\mathfrak{g})$ then the weights of ad are exactly the roots under the identification of $X(T)_k = \mathfrak{h}^*$ if $(\mathfrak{g}, \mathfrak{h})$ is the Lie algebra of (G, T). Indeed, it means there is a root vector $X \in \mathfrak{g}$ such that,

$$\forall H \in \mathfrak{h} : [H, X] = \mathrm{ad}_H \cdot X = \alpha(H) X$$

For other representations of $\mathfrak{g}$ we can think of weights as living in the same space as the root lattice.

Remark. Moreover the roots act on the weights of any representation as follows. For $X \in \mathfrak{g}_\alpha$ meaning a root vector corresponding to the root $\alpha \in \Phi$ and $v \in V_\lambda$ meaning a weight vector corresponding to the weight λ then consider, for any $H \in \mathfrak{h}$,

$$\rho(H)(\rho(X) \cdot v) = \rho(X)(\rho(H) \cdot v) + [\rho(H), \rho(X)] \cdot v = \lambda(H)(\rho(X) \cdot v) + \alpha(H)\rho(X) \cdot v = (\lambda + \alpha)(H)\, \rho(X) \cdot v$$

and therefore $\rho(X) \cdot v$ is either zero or a weight vector of weight $\lambda + \alpha$.

Lemma 12.0.4. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

Proof. Choose $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. For any $H \in \mathfrak{h}$,

$$[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]] = \alpha(H)[X, Y] + \beta(H)[X, Y]$$

and therefore we see that $[X, Y]$ is a root vector for $\alpha + \beta$. ☐

12.1 The Weyl Group Action

Roots and weights are computed with respect to a fixed Cartan $\mathfrak{h}$ or maximal torus T. However, the answer should not really depend on the choice. Indeed it does not because any two Cartans or maximal tori are conjugate (at least over $\mathbb{C}$) so we can define an abstract isomorphism of roots systems between the two induced by this conjugation. However this suggests extra symmetry since there are conjugation actions that fix the maximal torus. Let,

$$W = N_G(T)/T$$

be the Weyl group. Then we claim that W acts on the roots and weights. Indeed for $w \in W$ and $\alpha \in \Phi$ consider $w \cdot \alpha$ meaning the character $(w \cdot \alpha)(t) = \alpha(w^{-1}tw)$ which is well-defined since T is abelian. However, if $X \in \mathfrak{g}_\alpha$ is a root vector then for any $t \in T$ we have,

$$t \cdot (w \cdot X) = w(w^{-1}tw) \cdot X = w \cdot \alpha(w^{-1}tw)X = (w \cdot \alpha)(t) w \cdot X$$

Thus $w : \mathfrak{g}_\alpha \xrightarrow{\sim} \mathfrak{g}_{w \cdot \alpha}$ gives an isomorphism permuting the roots and the root spaces. Similarly, if λ is a weight of ρ and $v \in V_\lambda$ is a weight vector then for any $H \in \mathfrak{h}$,

$$\rho(H) \cdot (\rho(w) \cdot v) = \rho(w)\rho(w^{-1}Hw) \cdot v = \rho(w)\lambda(w^{-1}Hw)v = (w \cdot \lambda)(H)\, \rho(w) \cdot v$$

and thus $w : V_\lambda \xrightarrow{\sim} V_{w \cdot \lambda}$. Therefore when drawing the root system we should respect the symmetry under the Weyl group. The best way to do this is to choose a Weyl-invariant inner product on $X(T)_{\mathbb{R}}$.

12.2 Killing Form

Definition 12.2.1. The *Killing form* of $\mathfrak{g}$ is the map,

$$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

defined by $K(X, Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$. This is a bilinear form.

Definition 12.2.2. An involution $\theta : \mathfrak{g} \to \mathfrak{g}$ of a real Lie algebra is called a *Cartan* involution if $B_\theta(X, Y) := -K(X, \theta Y)$ is positive-definite.

Proposition 12.2.3. If $\mathfrak{g}$ is a semisimple real Lie algebra then the following are equivalent,

(a) $\mathrm{id} : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution

(b) K is negative definite

(c) $\mathfrak{g}$ is the Lie algebra of a compact semisimple Lie group.

Corollary 12.2.4. If $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{R}}$ then complex conjugation on $\mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$ if and only if $\mathfrak{g}_0$ is the Lie algebra of a compact Lie group.

Theorem 12.2.5 (Cartan)**.** $\mathfrak{g}$ is semisimple if and only if K is nondegenerate.

Corollary 12.2.6. If G is compact and $Z(\mathfrak{g}) = 0$ then $\mathfrak{g}$ is semisimple.

Given a Cartan involution θ we get a Cartan pair $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ which are the $+1$ and -1 eigenspaces. Since θ is a Lie algebra homomorphism we see that,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

so $\mathfrak{k}$ is a Lie subalgebra while any subalgebra of $\mathfrak{p}$ is abelian. The data of $(\mathfrak{k}, \mathfrak{p})$ determines θ.

In particular K is negative definite on $\mathfrak{k}$ and positive definitie on $\mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an othogonal decomposition with respect to K.

In the case that $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ then a Cartan subalgebra is given by $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ where $\mathfrak{t}$ is the Lie algbra of a maximal torus T of the compact group corresponding to $\mathfrak{g}_0$. Then there is a natural identification $X(T)_{\mathbb{R}} = \mathfrak{t}^*$. We choose to instead make the identification $X(T)_{\mathbb{R}} = i\mathfrak{t}^*$ and the Killing form is positive-definite on $i\mathfrak{t}$ hence defines an isomorphism $i\mathfrak{t} \cong i\mathfrak{t}^*$ and therefore gives an inner produce on $X(T)_{\mathbb{R}}$ which is Weyl invariant. This is where the metric on the root lattice arises from.

In the theory of algebraic groups this metric is not defined so simply since there is not a good notion of compact real form. Instead we consider for each root $a \in \Phi(G, T)$ a dual coroot $a^\vee \in \Phi^\vee(G, T)$ which is the unique cocharacter $a^\vee : \mathbb{G}_m \to S_a$ where $S_a := T \cap D(Z_G(T_a))$ such that $a \circ a^\vee$ is $z \mapsto z^2$ as a map $\mathbb{G}_m \to \mathbb{G}_m$. This associatation between roots and coroots replaces the metric provided by the Killing form as a way to take inner products between roots.

12.3 SU(2)

If we choose $T = U(1) \subset \mathrm{SU}(2)$ as the diagonal torus,

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}$$

Then the weight space decomposition is familar from physics. Indeed, $\mathfrak{h}$ is 1-dimensional generated by the matrix,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is the Pauli matrix σ_z. In terms of angular momentum $J_z = \frac{1}{2}\sigma_z$ which is why there will be some factors of 2 floating around compared to the physics literature. Indeed, usually physcists parametrize this digonal maximal torus in terms of rotations around the z-axis under the standard action on $\mathbb{R}^3$ via conjugation on traceless Hermitian of determinant 1 (equivalently on unit imaginary quaterions) as,

$$T = \left\{ \begin{pmatrix} e^{\frac{i\theta_z}{2}} & 0 \\ 0 & e^{-\frac{i\theta_z}{2}} \end{pmatrix} \right\}$$

Anyway the weight spaces are exactly the eigenspaces of $\rho(H)$ and since this is an action of $U(1)$ the eigenvalues must be integers (hence the eigenvalues of J_z are half-integers). This is exactly what we do when we describe the states of J multiplets in terms of their J_z eigenvalue m. The simulatenous diagonalization of J and J_z is nothing other than the weight decomposition.

Now consider the adjoint action $\mathrm{SU}(2) \to \mathrm{GL}(\mathfrak{su}(2)_{\mathbb{C}})$. We get,

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} a & be^{2i\theta} \\ ce^{-2i\theta} & d \end{pmatrix}$$

therefore the roots are ± 2 with root vectors,

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

these are exactly the rasing and lowering operators. Note that X, Y live in $\mathfrak{su}(2)_{\mathcal{C}}$ not in $\mathfrak{su}(2)$ corresponding to the fact that in physics we define them as $L_x + iL_y$ and $L_x - iL_y$ which require a factor of i. The root vectors act additivley on the weights of any representation. This shows that the raising and lower operators shift the weight up and down by 2 (in the physicists notation by 1 i.e. a whole integer rather than by a half-integer while the weights may be half-integers).

12.4 SU(3)

The maximal torus $T \subset G = \mathrm{SU}(3)$ is given by matrices,

$$\begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{i\theta_3} \end{pmatrix}$$

with determinant 1. Therefore we take the Lie algebras $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is the Lie algebra of 3 x 3 anti-Hermtian matrices and $\mathfrak{h}$ is the imaginary diagonal matrices.

There are three standard characters that produce $e^{i\theta_i}$ called L_i. It is easier to work with the complexified Lie group $G_{\mathcal{C}} = \mathrm{SL}_3$ in which case we get the (complex) maximal torus,

$$\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}$$

where $t_1 t_2 t_3 = 1$. Then L_i sends this matrix to t_i so we see that $L_1 + L_2 + L_3 = 0$. Therefore, these define three points in the two-dimensional space $X(T)_{\mathbb{R}}$ with barycenter at the origin. Thus it is conventional to represent their span as a triangular lattice. These are the fundamental weights, the weights of the fundamental representation. Now we compute the roots. Now,

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} = \begin{pmatrix} a & t_1 t_2^{-1} b & t_1 t_3^{-1} c \\ t_2 t_1^{-1} d & e & t_2 t_3^{-1} f \\ t_3 t_1^{-1} g & t_3 t_2^{-1} h & i \end{pmatrix}$$

Therefore, the roots and corresponding root vectors are,

$$L_1 - L_2 \qquad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_2 - L_3 \qquad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_1 - L_3 \qquad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_2 - L_1 \qquad E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_2 - L_3 \qquad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_3 - L_2 \qquad E_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_3 - L_1 \qquad E_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where E_{ij} has a single 1 in the (i, j) entry and zeros elsewhere.

13 Appearence of the adjoint representation for mesons

Let $\mathfrak{g}$ be a Lie algebra and V an faithful $\mathfrak{g}$-representation meaning that the map,

$$\mathfrak{g} \to \mathrm{End}\,(V)$$

is injective. This gives a map of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$. Then the adjoint representation lives as a subrepresentation of $V^* \otimes V$. Indeed, we simply take,

$$\mathfrak{g} \hookrightarrow \mathrm{End}\,(V) \cong V^* \otimes V$$

Indeed, the first is actually a map of $\mathfrak{g}$-representations (or $U\mathfrak{g}$-modules) where $\mathfrak{g}$ is given the adjoint representation. Indeed, we need to check that,

$$\rho(\mathrm{ad}_X(Y)) = X \cdot \rho(Y)$$

but by definition of the action of $\mathfrak{g}$ on $\mathrm{End}\,(V)$ we have,

$$X \cdot \varphi = [\rho(X), \varphi]$$

and thus we see that,

$$\rho(\mathrm{ad}_X(Y)) = \rho([X, Y]) = [\rho(X), \rho(Y)] = X \cdot \rho(Y)$$

as expected.

Furthermore, if V is equipped with a $\mathfrak{g}$-compatible Hermitian inner product (meaning $\langle X \cdot v, w \rangle + \langle v, X \cdot w \rangle = 0$ so – if it exists – the associated G-representation is unitary) then it induces an isomorphism of $\mathfrak{g}$-representations $\overline{V} \to V^*$. Therefore, we also get a copy of $\mathrm{adj} \subset \overline{V} \otimes V$. This is what physicists mean when they say the following. Let λ_i be a collection of matrices with the specified commutation relations and e_i the basis vectors of the space on which λ_i acts. Let $\overline{e}_i$ be the complex conjugate basis. Then we get a singlet representation by choosing $\overline{e}_\alpha^* e_\alpha$ and we get an adjoint multiplet by choosing $s_j = \overline{e}_\alpha \lambda_j e_\alpha$. The first is looking at $\mathrm{id} \in \mathrm{End}(V)$ which is invariant under $\mathfrak{g}$. The second is looking at the image of $\mathfrak{g} \hookrightarrow \mathrm{End}(V)$ where each $X \in \mathfrak{g}$ is represented in terms of $\overline{V} \otimes V$ via $\overline{e}_j \otimes X_{ij} e_i$ where X_{ij} is the matrix representation in the basis e_i.

14 The Ward Identity

Peskin shows that the Ward identity implies $Z_1 = Z_2$ where Z_1^{-1} is the vertex factor of QED and Z_2 is the electron field strength renormalization. This means that the electric charge e in terms of the bare charge e_0 is renormalized as follows,

$$e = Z_1^{-1} Z_2 \sqrt{Z_3} e_0 = \sqrt{Z_3} e_0$$

See Peskin 10.37. Why is this cancellation necessary. I think it is to preserve gauge invariance of the renormalized Lagrangian. The bare Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (i\partial\!\!\!/ - e_0 A\!\!\!/_0 - m_0) \psi_0$$

Now we renomalize, first doing field strength renormalization via rescaling,

$$\psi_0 = Z_2^{1/2} \psi \qquad A_0^\mu = Z_3^{1/2} A_\mu$$

and rearranging into counterterms to get,

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi - \tfrac{1}{4}\delta_3 F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\delta_2 \partial\!\!\!/ - e\delta_1 A\!\!\!/ - \delta_m)\psi$$

where,

$$\delta_3 = Z_3 - 1 \quad \delta_2 = Z_2 - 1 \quad \delta_1 = Z_1 - 1 = (e_0/e) Z_2 Z_3^{1/2} - 1 \quad \delta_m = Z_2 m_0 - m$$

Now the gauge tansformation takes,

$$\psi \mapsto e^{i\alpha}\psi \quad A^\mu \mapsto A^\mu - e_0^{-1} Z_3^{-1/2} \partial^\mu \alpha$$

where the *bare* charge appears as well as the field strength renormalization since this is the gauge transformation of A_0^μ multiplied by $Z_3^{-1/2}$. But notice that the Ward identity forces $e = e_0 Z_3^{1/2}$ and therefore,

$$\psi \mapsto e^{i\alpha}\psi \quad A^\mu \mapsto A^\mu - e^{-1} \partial^\mu \alpha$$

meaning that the renormalized fields satisfy gauge invariance! Notice, there is no reason (or need) for the coincidence $Z_1 = Z_2$ in a non-gauge theory as we can simply renormalize away a shift in the coupling constant using the vertex counterterm. In QED it is actually photon field strength renormalization that renormalizes e and accounts for its running so only the photon self-energy is needed in the Callan–Symanzik equation to compute the β function. A consequence of this is that the coupling constants of A^μ to each charged species are renormalized exactly the same way (since the entire renormalization is via photon-self energy which does not depend on the particular vertex defining the coupling to the charged species) meaning there is a universal electric interaction strength for all species. Thus in the renormalized theory A^μ couples to the conserved Noether charge current as it must since it satisifes a full gauge symmetry even after renormalization.

I think Peskin's treatment is a bit backwards. First he computes the bare vertex factor Γ^μ and sees that it gives a form factor F_1 which is divergent at $q^2 = 0$ in violation of the principal that $F_1(0) = 1$ since this corresponds to e being the physical electric charge. Then he uses the LSZ reduction formula to see that we didn't include the effects of electron field-strength renormalization Z_2 whcih cancels the divergence and exactly gives $F_1(0) = 1$ via the Ward identity forced equality $Z_1 = Z_2$. However, this is still wrong! Peskin did not include the photon field-strength renormalization in this analysis. We should really have $Z_3^{1/2} Z_2 \Gamma^\mu$ as our vertex factor and this $Z_3^{1/2}$ accomodates for the fact that we're still using the bare e_0 and shifts it to e the physical charge. The difference between e_0 and e is pushed under the rug in chapter 7.

15 Coulomb Scattering in QFT (Peskin 4.4 and 5.1)

Consider the interaction term in the Hamiltonian,

$$H_I = \int \mathrm{d}^3 x \, e\bar{\psi}\gamma^\mu A_\mu \psi$$

where A_μ is a source field (i.e. not quantized). Then we compute the S-matrix elements using the convention,

$$S = I + iT$$

and to leading-order,

$$\langle p's'|iT|ps\rangle = \langle p's'| -i \int \mathrm{d}t H_I |ps\rangle$$

However,

$$\psi(x)|ps\rangle = u^s(p)e^{-ipx}$$

and therefore,

$$\langle p's'|iT|ps\rangle = -ie\bar{u}^{s'}(p')\gamma^\mu u(p) \int e^{-i(p-p')x} A_\mu \mathrm{d}^4 x \;= -ie\bar{u}^{s'}(p')\gamma^\mu u(p)\widetilde{A}_\mu(q)$$

where $q = p - p'$.

15.1 Time-independent potentials

Suppose that A_μ is a time-independent potential the particle is scattering off. Then from the Hamiltonian or Lagrangian formalism we expect that the energy of the particle is conserved. Indeed, the time component of the Fourier transform gives a δ-function since A_μ is constant in t. Therefore we define the scattering amplitude via the formula,

$$\langle p's'|iT|ps\rangle = (2\pi)\delta(E_f - E_i)i\mathcal{M}$$

Therefore, we get,

$$\mathcal{M} = -e\bar{u}^{s'}(p')\gamma^\mu u(p) \int e^{-i(p-p')x} A_\mu \mathrm{d}^4 x \ = -ie\bar{u}^{s'}(p')\gamma^\mu u(p)\widetilde{A}_\mu(\vec{q})$$

where now $\widetilde{A}_\mu(\vec{q})$ is the spatial Fourier transform evaluated at the 3-vector $\vec{q} = \vec{p} - \vec{p}\,'$. There is an overall δ-function enforcing energy conservation because of time-indpendence but not momentum conservation because the potential can absorb arbitrary momentum.

15.2 Building Wavepackets

The incident wave packet $|\psi\rangle$ is built as follows,

$$|\psi\rangle_{\text{in}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \tilde{\psi}(\vec{k}) e^{i\vec{b}\cdot\vec{k}} |\vec{k}\rangle_{\text{in}}$$

with impact parameter $\vec{b}$. Then we consider asymtotic final states of pure momentum,

$${}_{\text{out}}\langle\phi| = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{\phi}(\vec{p})\, {}_{\text{out}}\langle\vec{p}|$$

Then the probability of scattering into a sector of momentum space is,

$$\begin{aligned}
P(\vec{b}) &= \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} \left| {}_{\text{out}}\langle\vec{p}\,|\,\psi\rangle_{\text{in}} \right|^2 \\
&= \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{4E_k E_{k'}}} \tilde{\psi}^*(\vec{k}')\tilde{\psi}(\vec{k}) e^{i\vec{b}\cdot(\vec{k}-\vec{k}')} ({}_{\text{out}}\langle\vec{p}\,|\,\vec{k}\rangle_{\text{in}})({}_{\text{out}}\langle\vec{p}\,|\,\vec{k}'\rangle_{\text{in}})^*
\end{aligned}$$

Now we define,

$$\mathrm{d}\sigma = \int \mathrm{d}^2 b\, P(\vec{b})$$

And hence, the integration over $\vec{b}$ gives a factor of $(2\pi)^2\delta^{(2)}(k^\perp - k'^\perp)$. Furthermore, we showed that

$$({}_{\text{out}}\langle\vec{p}\,|\,\vec{k}\rangle_{\text{in}}) = i\mathcal{M}(\vec{k} \to \vec{p})(2\pi)\delta(E_k - E_p)$$

and likewise,

$$({}_{\text{out}}\langle\vec{p}\,|\,\vec{k}'\rangle_{\text{in}})^* = -i\mathcal{M}^*(\vec{k}' \to \vec{p})(2\pi)\delta^{(4)}(E_{k'} - E_p)$$

We use this second delta function and the delta function arising from integration over $\vec{b}$. Inspect,

$$\int \mathrm{d}k'_z \delta(E_{k'} - E_p) = \int \mathrm{d}k'_z\, \delta\left(\sqrt{(k'^\perp)^2 + k'^2_z + m^2} - E_p\right) = \left|\frac{k'_z}{E_{k'}}\right|^{-1} = \frac{1}{v'}$$

Therefore,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{(2E_p)^2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{v'} \tilde{\psi}^*(\vec{k}')\tilde{\psi}(\vec{k})\mathcal{M}(\vec{k} \to \vec{p})\mathcal{M}^*(\vec{k}' \to \vec{p})(2\pi)\delta(E_k - E_p)$$

where we fix $k^\perp = k'^\perp$ and $E_{k'} = E_p = E$ hence since $E_k = E_p$ we have $k'_z = \pm k_z$. If the wavefunctions are well-localized in momentum space we can ignore the $k'_z = -k_z$ solution to the

δ-functions and take $\vec{k} = \vec{k}'$. Therefore, if the wavefunction is well-peaked we can more smooth functions through the integral over $\vec{k}$ to get,

$$\begin{aligned}
d\sigma &= \frac{d^3p}{(2\pi)^3} \frac{1}{(2E)^2 v} (2\pi)\delta(E_k - E_p)|\mathcal{M}(\vec{k} \to \vec{p})|^2 \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}^*(\vec{k})\tilde{\psi}(\vec{k}) \\
&= \frac{d^3p}{(2\pi)^3} \frac{1}{(2E)^2} \cdot \frac{1}{v} \cdot (2\pi)\delta(E_k - E_p)\, |\mathcal{M}(\vec{k} \to \vec{p})|^2
\end{aligned}$$

Now we integrate over $\vec{p}$ to get the full cross section for scattering with this S-matrix element,

$$\begin{aligned}
\sigma &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{(2E)^2 v} (2\pi)\delta(E_k - E_p)|\mathcal{M}(\vec{k} \to \vec{p})|^2 \\
&= \frac{1}{4\pi^2 (2E)^2 v} \int d|p| d\Omega |\vec{p}|^2 \delta(E_k - E_p)|\mathcal{M}(\vec{k} \to \vec{p})|^2 \\
&= \frac{1}{16\pi^2 E^2 v} \cdot \frac{E}{|k|} |k|^2 \int |\mathcal{M}(\vec{k} \to \vec{p})|^2 d\Omega \\
&= \frac{1}{(4\pi)^2} \int |\mathcal{M}(\vec{k} \to \vec{p})|^2 d\Omega
\end{aligned}$$

We have constraints $|p| = |k|$ and $E_p = E_k$ but do not constrain the direction of $\vec{p}$.

15.3 Coulomb Potential

Consider,

$$A^\mu(x) = \left(\frac{Ze}{4\pi r}, 0 \right)$$

Then we compute the Fourier transform,

$$\begin{aligned}
\widetilde{A}_0(\vec{q}) &= \int e^{-iqx} \frac{Ze}{4\pi r} d^3r \\
&= \frac{Ze}{4\pi} \int e^{-ir|q|\cos\theta} 2\pi r \, dr \, d\cos\theta \\
&= \frac{Ze}{2} \int \frac{1}{-ir|q|} \left[e^{-ir|q|} - e^{ir|q|} \right] r \, dr \\
&= \frac{Ze}{|q|} \int_0^\infty \sin(r|q|) dr = \frac{Ze}{|q|^2}
\end{aligned}$$

where we must compute these integrals in the sense of distributions.

15.4 The scattering ampltitude

Now,

$$|\mathcal{M}|^2 = \left(\frac{Ze}{|q|^2} \right)^2 e^2 |u^{s'}(p')\gamma^0 u(p)|^2$$

We will compute the cross section in terms of the scattering angle θ between p and p'. Since $E_f = E_i$ we see that $|p| = |p'|$ and hence,

$$|q| = 2|p| \sin \tfrac{\theta}{2}$$

First we compute the non-relativistic limit in which,

$$u^{s'}(p')\gamma^0 u(p) \approx 2m\xi'^\dagger \xi$$

therefore the spins are unaffected in the scattering Approximating $|p'| = |p| \approx mv$ we get,

$$|\mathcal{M}|^2 = \frac{Z^2 e^4}{(2m)^4 v^4 \sin^4 \frac{\theta}{2}}(2m)^2$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2}|\mathcal{M}|^2 = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\frac{\theta}{2})}$$

Putting in dimensionful constants we get,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2 Z^2 \hbar^2 c^2}{4(mc^2)^2 (v/c)^4 \sin^4 \frac{\theta}{2}}$$

Now we compute the fully relativistic case. We need to compute the spin average,

$$\begin{aligned}
\tfrac{1}{2}\sum_{\text{spins}} |\bar{u}^{s'}(p')\gamma^0 u^s(p)|^2 &= \tfrac{1}{2}\operatorname{tr}\left(\gamma^0(\not{p}' + m)\gamma^0(\not{p} + m)\right) \\
&= \tfrac{1}{2}\operatorname{tr}\left(\gamma^0\not{p}'\gamma^0\not{p} + \gamma^0 m\gamma^0\not{p} + \gamma^0\not{p}'\gamma^0 m + m^2\right)
\end{aligned}$$

The middle terms have an odd number of γ matrices and thus have zero trace. Furthermore,

$$\gamma^0\not{p}'\gamma^0\not{p} = -\gamma^0\gamma^0\not{p}'\not{p} + \gamma^0\{\not{p}', \gamma^0\}\not{p} = -\not{p}'\not{p} + 2\gamma^0 p'^0 \not{p}$$

Therefore, we get, using that $\operatorname{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

$$\begin{aligned}
\tfrac{1}{2}\sum_{\text{spins}} |\bar{u}^{s'}(p')\gamma^0 u^s(p)|^2 &= \tfrac{1}{2}\left[-\operatorname{tr}(\not{p}'\not{p}) + 2p'^0 \operatorname{tr}(\gamma^0\not{p}) + 4m^2\right] \\
&= -2p'\cdot p + 4p'^0 p^0 + 2m^2 \\
&= 2E^2 + 2\vec{p}\cdot\vec{p}' + 2m^2 \\
&= 2m^2(1 + \gamma^2 + \beta^2\gamma^2\cos\theta) \\
&= 2m^2\gamma^2([1-\beta^2] + 1 + \beta^2\cos\theta) \\
&= 2m^2\gamma^2(2 - \beta^2(1-\cos\theta)) \\
&= 4E^2(1 - \beta^2\sin^2\tfrac{\theta}{2})
\end{aligned}$$

Then we have,

$$|\mathcal{M}|^2 = \frac{Z^2 e^4}{(2|p|\sin\frac{\theta}{2})^4} 4E^2(1 - \beta^2\sin^2\tfrac{\theta}{2}) = \frac{Z^2 e^4}{4|p|^2\beta^2\sin^4\frac{\theta}{2}}(1 - \beta^2\sin^2\tfrac{\theta}{2})$$

Therefore,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{(4\pi)^2} = \frac{Z^2\alpha^2}{4|p|^2\beta^2\sin^4\frac{\theta}{2}}(1 - \beta^2\sin^2\tfrac{\theta}{2})$$

Putting in the dimensionful parameters we get,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2 Z^2\hbar^2}{4|p|^2\beta^2\sin^4\frac{\theta}{2}}(1 - \beta^2\sin^2\tfrac{\theta}{2}) = \frac{\alpha^2 Z^2\hbar^2}{4m^2c^2\gamma^2\beta^4\sin^4\frac{\theta}{2}}(1 - \beta^2\sin^2\tfrac{\theta}{2})$$

15.5 Helicity Structure of the Scattering Cross Section

Notice that this formula for e^--scattering off a hard Coulomb target is well-defined in the limit $m \to 0$ for fixed (relativistic) momentum (unlike the non-relativistic case). For $m \to 0$ and hence $\beta \to 1$ we get,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2 Z^2 \hbar^2 \cos^2 \frac{\theta}{2}}{4|p|^2 \sin^4 \frac{\theta}{2}}$$

We want to explain this additional structure in terms of relativity and the spin/helicity structure. We need to compute,

$$u^{s'}(p')\gamma^0 u^s(p)$$

in the limit $m \to 0$. We can choose states of definite helicity as our basis. We choose $p = (E, 0, 0, E)$ then $\xi^\uparrow = \xi^+$ satisfies $(\vec{p} \cdot \vec{\sigma})\xi^+ = E\xi^+$ and likewise $\xi^- = \xi^\downarrow$ is the negative helicity eigenstate $(\vec{p} \cdot \vec{\sigma})\xi^- = -E\xi^-$. These give definite Helicity spinors,

$$u^+(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u^-(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Now we need to rotate these spinors to get the definite helicity states for $p' = (E, E\cos\theta, 0, E\cos\theta)$. Rotation around the y-axis is generated by,

$$e^{-i\Sigma_2 \theta} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

and therefore,

$$u^+(p') = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad u^-(p') = \sqrt{2E}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned}
\bar{u}^+(p')\gamma^0 u^+(p) &= 2E\cos\tfrac{\theta}{2} \\
\bar{u}^+(p')\gamma^0 u^-(p) &= 0 \\
\bar{u}^-(p')\gamma^0 u^+(p) &= 0 \\
\bar{u}^-(p')\gamma^0 u^-(p) &= 2E\cos\tfrac{\theta}{2}
\end{aligned}$$

Therefore, the sum over final polarization and average over initial spins gives,

$$2E\cos\frac{\theta}{2}$$

which is indeed the square-root of the extra factor that appears in the scattering cross section compared to the non-relativistic cross section for Rutherford scattering. Furthermore, notice that the process is Helicity conserving and therefore not angular momentum conserving (for $\theta \neq 0$) similar to how it does not conserve momentum. The extra factor of $\cos\frac{\theta}{2}$ can be interpreted as the amplitude to connect positive (resp. negative) helicity states after rotation.

16 QFT notes

16.1 QED conserves chirality

Note that a vector interaction with a Dirac fermion like QED preserved chirality. Indeed, the interaction term,

$$e\bar{\psi}\not{A}\psi = e(\chi_L^\dagger A_\mu \bar{\sigma}^\mu \chi_L + \chi_R^\dagger A_\mu \sigma^\mu \chi_R)$$

which couples only fermions of the same chirality together. Hence for massless fermions interacting with QED all processes preserve helicity. We saw this for example with Compton scattering where the helicity is conserved in the limit $m \to 0$ and with relativistic Coulomb scattering where again helicity is conserved in the limit $m \to 0$.

16.2 Weinberg-Witten Theorem

17 Homological. numerical, and algebraic equivalence

Definition 17.0.1.

Proposition 17.0.2. Let X be a variety then Pic^0_X parametrizes algebraically trivial line bundles and hence Cartier divisors. If X is locally factorial this coincides with algebraically trivial Weil divisors.

Proposition 17.0.3. Let X be a proper normal variety. Then for Weil divisors the following are equivalent:

(a) algebraic and $\mathbb{Z}$-homological equivalence

(b) numerical and $\mathbb{Q}$-homological equivalence

SHOW THAT THESE RESULTS FAIL FOR CODIM > 1 CLASSES. EXAMPLE OF LAZARSFELD?

18 Fibral Conditions

"Fibral" criteria have two meanings:

(a) if $f : X \to Y$ is a map of good enough S-schemes then f has property $\mathcal{P}$ iff all f_s has property P for all $s \in P$

(b) if $f : X \to Y$ is good enoguh then f has property $\mathcal{P}$ iff all $f_y : X_y \to \text{Spec}\,(\kappa(y))$ have property $\mathcal{P}$.

We will consider both types of fibral criteria in this order. We start with type (a) critera for flatness, smoothness, and being an isomorphism.

18.1 Conrad Math 248B Homework 8 Problem 2

Let S be a scheme and $f : X \to Y$ a map between flat and locally finitely presented S-schemes.

18.2 (i)

For $s \in S$, prove that if $f_s : X_s \to Y_s$ is flat at $x \in X_s$ then f is flat at x.

This is local on the source and target so we reduce to the case that X, Y, S are affine. By finite presentation we may further assume that S is finite type over $\mathbb{Z}$ by spreading out. Recall the following theorem,

Theorem 18.2.1 (Mat CRT, 22.5)**.** Let $A \to B$ be a local map of local rings and $u : M \to N$ a morphism of finite B-modules. If N is flat over A then the following are equivalent,

(a) u is injective and $N/u(M)$ is flat over A

(b) $\tilde{u} : M \otimes_A \kappa_A \to N \otimes_A \kappa_A$ is injective.

Note that f is locally finitely presented (since the diagonal of a lfp morphism is lfp and so are the composition of two and the base change). Therefore, shrinking further we can write X in affine space over Y to get $X \hookrightarrow \mathbb{A}^n_Y$. Localize so that R is a local ring. Indeed let $S = \mathrm{Spec}\,(R)$ and $Y = \mathrm{Spec}\,(A)$ and $X = \mathrm{Spec}\,(B/J)$ where $B = A[x_1, \ldots, x_n]$ and $J = (f_1, \ldots, f_r)$ with $x \in X$ a maximal ideal $\mathfrak{m} \subset B$ containing J and $f(\mathfrak{m}) = \mathfrak{p}$. Now we apply the theorem from Matsumura to the localization $A_{\mathfrak{p}} \to B_{\mathfrak{m}}$ and the map of modules $\tilde{u} : J_{\mathfrak{m}} \to B_{\mathfrak{m}}$. Then $\tilde{u} = (u \otimes_R (R/\mathfrak{m}_R)) \otimes_{A/m_R A} (A/\mathfrak{m}_A)$ but $A \to B$ becomes flat after applying $- \otimes_R (R/\mathfrak{m}_R)$. Note that,

$$0 \to J_{\mathfrak{m}} \to B_{\mathfrak{m}} \to B_{\mathfrak{m}}/J_{\mathfrak{m}} \to 0$$

stays exact after applying $- \otimes_R (R/\mathfrak{m}_R)$ since $B_{\mathfrak{m}}/J_{\mathfrak{m}}$ is R-flat by assumption. Also $(B_{\mathfrak{m}}/J_{\mathfrak{m}}) \otimes_R (R/\mathfrak{m}_R)$ is $(A/m_R A)$-flat by assumption hence,

$$0 \to J_{\mathfrak{m}} \otimes_R (R/\mathfrak{m}_R) \to B_{\mathfrak{m}} \otimes_R (R/\mathfrak{m}_R) \to (B_{\mathfrak{m}}/J_{\mathfrak{m}}) \otimes_B (R/\mathfrak{m}_R) \to 0$$

remains flat after applying $- \otimes_{A/\mathfrak{m}_R A} (A/\mathfrak{m}_A)$ and therefore $\tilde{u}$ is injective. Therefore by the theorem $(B/J)_{\mathfrak{m}}$ is $A_{\mathfrak{p}}$-flat proving the claim.

Now we prove:

Proposition 18.2.2. Let $f : X \to Y$ be a morphism of lfp flat S-schemes. If $f_{\bar{s}} : X_{\bar{s}} \to Y_{\bar{s}}$ is flat for each geometric fiber over S then f is flat.

Proof. By the above it suffices to show that f_s is flat for each $s \in S$. This follows from the flatness of $f_{\bar{s}}$ by faithfully flat descent. Indeed, let $A \to B$ be a map of k-algebras such that $A_{\bar{k}} \to B_{\bar{k}}$ is flat. Then since $B \to B_{\bar{k}}$ is faithfully flat and $A \to A_{\bar{k}} \to B_{\bar{k}}$ hence $A \to B \to B_{\bar{k}}$ is flat we see that $A \to B$ is flat. ☐

18.3 (ii)

Prove that if f_s is smooth (resp. étale) for all $s \in S$ then f is smooth (resp. étale). Likewise for $f_{\bar{s}}$ replacing f_s.

By implication of properties, f is lfp. By part (i) we see that f is flat. Since f_s is smooth we see that the geometric fibers $X_{\bar{y}}$ of f are regular (since these are also the geometric fibers of f_s or $f_{\bar{s}}$) and hence f is smooth.

Alternatively we can use $\Omega_{X/Y}$. It suffices to show that $\Omega_{X/Y}$ is locally free of the correct rank. By checking over f_s we see that $\Omega_{X/Y}$ has the correct rank. Hence if X is reduced we would win immediately since a constant rank coherent sheaf is a vector bundle. As above we reduce to $A \to B$ a map of flat R-algebras with $B = A[x_1, \dots, x_n]/J$ then consider the sequence,

$$J/J^2 \to B^n \to \Omega_{B/A} \to 0$$

we need to show that $J/J^2 \to B^n$ is locally a split injection. Since this is true after applying $-\otimes_R \kappa_R$ and hence after applying $- \otimes_B \kappa_B = (- \otimes_R \kappa_R) \otimes_{B/\mathfrak{m}_R B} \kappa_B$ it suffices to prove the following lemma.

Lemma 18.3.1. Let $(A, \mathfrak{m}, \kappa)$ be a local ring. Let $\varphi : M \to N$ be a map from a finitely presented A-module M to a finite projective A-module N. Then the following are equivalent,

(a) φ is a split injection

(b) $\varphi \otimes_A \kappa$ is an injection.

Proof. (a) clearly implies (b). Assume (b). Since injections over κ are split, we can choose a section $N \otimes_A \kappa \to M \otimes_A \kappa$ and consider,

$$\begin{tikzcd} & & M \arrow[d, two heads] \\ N \arrow[r] \arrow[urr, dashed] & N \otimes_A \kappa \arrow[r] & M \otimes_A \kappa \end{tikzcd}$$

the lift exists since N is projective. Hence we get a map $\psi : N \to M$ such that $\psi \circ \varphi : M \to M$ is an endomorphism which equals the identity over κ. Hence $\varphi \circ \varphi$ is an isomorphism by Lemma 18.6.2 so φ is a split injection. Indeed we just need to modify the map $\psi : M \to N$ to $\psi' = (\psi \circ \varphi)^{-1} \circ \psi$ and then clealy ψ' is a section since $\psi' \circ \varphi = (\psi \circ \varphi)^{-1} \circ (\psi \circ \varphi) = \text{id}$. ☐

18.4 (iii)

Prove that if f is finite type and f_s is an isomorphism for all $s \in S$ then f is quasi-finite flat with fibral-degree 1.

Isomorphisms are smooth and hence by the previous part we conclude that f is smooth. Furthermore, it is finite-type and its fibers are the fibers of some f_s hence are a single point with degree 1 so we conclude that f is quasi-finite flat with constant fibral-degree 1.

18.5 (iv)

Prove the following lemma of Deligne and Rapoport.

Proposition 18.5.1. Let $f : X \to Y$ be a quasi-finite separated map of noetherian schemes that is flat with constant fibral degree. Then f is finite.

Proof. Since f is quasi-finite, to prove that f is finite it suffices to show it is proper. Therefore we must simply verify the valuative criterion of properness. Hence reduce to the case that $Y = \text{Spec}\,(R)$ is a dvr (we can further assume that X has a $K = \text{Frac}\,(R)$ point and find a section but it suffices to just show that $X \to \text{Spec}\,(R)$ is proper since then this property holds). Since $f : X \to Y$ is quasi-finite separated it factors by ZMT as an open immersion $X \hookrightarrow \overline{X}$ and a finite map $\overline{X} \to \text{Spec}\,(R)$. We need to show that f has a section. Consider the scheme-theoretic closure $Z \subset \overline{X}$ of the generic fiber. Because $X \to \mathrm{Spec}\,(R)$ is flat $X \subset Z$ and Z is R-flat and finite since it is a closed subscheme of a finite R-scheme. Hence the fibral degree of Z is constant. Furthermore, $X \subset \overline{X}$ is open so $X \subset Z$ is open but the fibral degree of X is constant by assumption so $X = Z$ $\square$

As a consequence, if f in (iii) is also separated then f is finite of degree 1 and hence an isomorphism.

Remark. Separatedness is necessary. For example, EXAMPLE!!

18.6 Some Lemma

Lemma 18.6.1. Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module flat over Y. Suppose that $\mathscr{F}|_{X_y}$ is locally free of rank r at $x \in X_y$ then $\mathscr{F}$ is locally free of rank r at $x \in X$.

Proof. We reduce to a statement on local rings. Let $A \to B$ be a local homomorphism and M a finitely presented B-module flat over A such that $M/\mathfrak{m}_A M$ is free of rank r then M is free of rank r. Lifting a basis gives a map $\varphi : B^r \to M$ such that $\varphi \otimes_A \kappa_A$. Consider,

$$0 \to \ker \varphi \to B^r \to M \to \mathrm{coker}\,\varphi \to 0$$

then since M is a finite B-module we see that $\mathrm{coker}\,\varphi$ is finite. Since $(\mathrm{coker}\,\varphi) \otimes_A \kappa_A = 0$ then $(\mathrm{coker}\,\varphi) \otimes_B \kappa_B = 0$ and hence $\mathrm{coker}\,\varphi = 0$ by Nakayama. Therefore φ is surjective. Since M is finitely presented $\ker \varphi$ is B-finite and M is A-flat so $(\ker \varphi) \otimes_A \kappa_A = 0$ so $(\ker \varphi) \otimes_B \kappa_B = 0$ and hence $\ker \varphi = 0$ by Nakayama so φ is an isomorphism. $\square$

Remark. Of course flatness is necessary e.g. consider $k[x] \to k[x]$ and $M = k[x]/(x)$.

Remark. Consider $k[x, y]/(x^2, xy) \to k[x, y]/(x^2, xy) \to k[y]$. This is an example where $\mathscr{F}$ is a vector bundle when restricted to the fiber over any irreducible subscheme on the base but not a vector bundle.

Lemma 18.6.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Suppose that M is a finite R-module with an endomorphism $\phi : M \to M$ such that $\phi \otimes \mathrm{id} : M \otimes_R \kappa \to M \otimes_R \kappa$ is an isomorphism then ϕ is an isomorphism.

Proof. Consider the exact sequence,

$$M \xrightarrow{\phi} M \longrightarrow \mathrm{coker}\,\phi \longrightarrow 0$$

and apply the right-exact functor $(-) \otimes_R \kappa$ to get,

$$M \otimes_R \kappa \xrightarrow{\phi \otimes \mathrm{id}} M \otimes_R \kappa \longrightarrow (\mathrm{coker}\,\phi) \otimes_R \kappa \longrightarrow 0$$

But $\phi \otimes \mathrm{id}$ is an isomorphism and the sequence is exact so $(\mathrm{coker}\,\phi) \otimes_R \kappa = 0$ and thus $\mathrm{coker}\,\phi = 0$ by Nak. Therefore φ is an isomorphism by a general result on endomorphisms of finite modules. $\square$

18.7 Type (a) fibral finiteness and isomorphsism

Proposition 18.7.1 (EGA III, tome 1, Proposition 4.6.7)**.** Let S be a locally noetherian scheme. Let $f : X \to Y$ be a morphism of proper S-schemes. Let $s \in S$ and consider $f_s : X_s \to Y_s$.

(a) if f_s is a finite morphism (resp. a closed immersion) then there exists an open neighborhood $U \subset S$ of $s \in S$ such that $f|_U : X_U \to Y_U$ is finite (resp. a closed immersion).

(b) If $X \to S$ is flat then if f_s is an isomorphism then there exists an open neighborhood $U \subset X$ of $s \in S$ such that $f|_U : X_U \to Y_U$ is an isomorphism.

Corollary 18.7.2. Let S be a locally noetherian scheme. Let $f : X \to Y$ be a morphism of proper S-schemes. Let $s \in S$ and consider $f_s : X_s \to Y_s$.

(a) if f_s is finite (resp. a closed immersion) for each $s \in S$ then f is finite (resp. a closed immersion).

18.8 Type (b) Fibral Properness and Isomorphism

Proposition 18.8.1 (EGA IV.15.7.10)**.** If $f : X \to Y$ is universally submersive (e.g. flat), finite type, separated and has proper and geometrically connected fibers then f is proper.

Remark. Note! Geometrically connected implies nonempty! This is very important or else Grothendieck would be claiming that open immersions are proper!

Remark. Universally submersive is necessary e.g. consider $\mathbb{G}_m \sqcup * \to \mathbb{A}^1$.

Remark. For a local version consider this question.

Corollary 18.8.2. If $f : X \to Y$ is universally submersive (e.g. flat), finite type, separated and has fibral-degree 1. Then f is an isomorphism.

Proof. Indeed since f has geometrically connected fibers we see that f is proper but it is quasi-finite and hence finite. Therefore f is an isomorphism sinice it is a finite map of degree 1. □

Remark. This also follows from the lemma of Deligne-Rapoport.

19 Flatness of Hilbert and Picard Schemes

Proposition 19.0.1. Let $f : X \to Y$ be a lfp morphism of schemes such that,

(a) the geometric fibers $X_{\bar{y}}$ are regular

(b) the fiber dimension $\dim X_y$ is constant

(c) the geometric fibers $X_{\bar{y}}$ are irreducible

(d) f is proper

(e) Y is reduced

then f is smooth.

Proof. This is local on the source and target so spreading out we reduce to the case that X and Y are affine and finite type over $\mathbb{Z}$. Then by the valuative criterion of flatness we reduce to the case that $Y = \mathrm{Spec}\,(R)$ is a dvr. Let $Z \subset X$ be the scheme-theoretic closure of the generic fiber. Since the generic fiber is irreducible Z is irreducible. Furthermore, the fiber dimension of Z is can only jump up but the fiber dimension of X is constant hence the special fibers of X and Z have the same dimension and X_s is irreducible so $X = Z$ as closed subsets and hence X is irreducible. Then we use the following lemma. □

Lemma 19.0.2. Let $f : X \to Y$ be a lpf morphism of schemes such that,

(a) Y is regular (hence locally Noetherian)

(b) the fibers X_y are regular and equidimensional of constant dimension

(c) X is equidimensional

then X is regular and f is flat.

Proof. For $x \in X$ let $y = f(x)$. Consider the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ Since Y is regular, $\mathfrak{m}_y$ is generated by $\dim \mathcal{O}_{Y,y}$ elements therefore,

$$\dim_{\kappa(x)} \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y) \geq \dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 - \dim_{\kappa(x)}(\mathfrak{m}_y/\mathfrak{m}_y^2) \otimes_{\kappa(y)} \kappa(x) \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

But X_y is regular so,

$$\dim \mathcal{O}_{X_y,x} = \dim_{\kappa(x)} \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y)$$

so we need to show that,

$$\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

then this will imply that the inequalities are equalities hence that $\mathcal{O}_{X,x}$ is a regular local ring. Also it will show flatness by the miracle flatness theorem applied to $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus we apply the following. □

Lemma 19.0.3. Let $f : X \to Y$ be a locally finite type morphism of locally noetherian schemes. Suppose that,

(a) X is equidimensional

(b) the fibers are equidimensional of constant dimension

(c) Y is universally catenary

then for each $x \in X$ set $y = f(x)$ the equality,

$$\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

holds.

Proof. Pulling back along $\mathrm{Spec}\,(\mathcal{O}_{Y,y}) \to Y$ and shrinking to an affine open we reduce to a finitely presented morphism $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ where A is a regular local ring and hence universally catenary. For each irreducble (reduced) component $\mathrm{Spec}\,(B_i) \subset \mathrm{Spec}\,(B)$ we have by assumption $\dim B_i = \dim B$. Since $X \to Y$ is dominant (either X is empty and the result is trivial or f is dominant since every fiber has constant hence nonnegative dimension) the map $A \to B_i$ is injective. Therefore, the dimension formula holds:

$$\dim (B_i)_{\mathfrak{m}} - \dim A = \mathrm{trdeg}_{K(A)}(K(B_i)) - \mathrm{trdeg}_{\kappa_A}(\kappa(\mathfrak{m}))$$

but by constancy and equidimensionality of the fibers we have,

$$\dim (B \otimes_A \kappa_A) = \dim (B_i \otimes_A \kappa_A) = \mathrm{trdeg}_{K(A)}(K(B_i))$$

since the fiber of each B_i is an union of irreducible components of the fiber each of which has the same dimension by assumption. Likewise we apply the dimension formula to the fiber $\mathrm{Spec}\,(B \otimes_A \kappa_A) \to \mathrm{Spec}\,(\kappa_A)$ to get that for each irreducible component $\mathrm{Spec}\,(C_j)$ we have,

$$\dim (C_j)_{\mathfrak{m}} = \mathrm{trdeg}_{\kappa_A}(K(C_j)) - \mathrm{trdeg}_{\kappa_A}(\kappa(\mathfrak{m}))$$

and since $B \otimes_A \kappa_A$ is an equidimensional finite-type κ_A-scheme,

$$\dim (B \otimes_A \kappa_A) = \dim C_j = \mathrm{trdeg}_{\kappa_A}(K(C_j))$$

Thus finally,

$$\dim (B_i)_{\mathfrak{m}} - \dim A = \dim (C_i)_{\mathfrak{m}}$$

However, B_i are the irreducible components of B so by definition $\dim B_{\mathfrak{m}} = \max_i \dim (B_i)_{\mathfrak{m}}$ and likewise $\dim (B \otimes_A \kappa_A)_{\mathfrak{m}} = \max_i \dim (C_i)_{\mathfrak{m}}$ so we conclude that,

$$\dim (B \otimes_A \kappa_A) = \dim B - \dim A$$

which is what we needed to show. ☐

Proposition 19.0.4. Let $\mathscr{X} \to S$ be a smooth projective family of surfaces meaning the geometric fibers are smooth varities of dimension 2. Then the relative Hilbert scheme of points $\mathrm{Hilb}^n_{\mathscr{X}/S}$ is smooth over S.

Proof. Since $\mathscr{X} \to S$ is projective the Hilbert scheme exists and is projective. Furthermore, since $\mathscr{X}_{\bar{s}}$ is a smooth surface, $(\mathrm{Hilb}_{\mathscr{X}/S})_{\bar{s}} = \mathrm{Hilb}_{\mathscr{X}_{\bar{s}}}$ is smooth and irreducible of dimension $2n$. Therefore, we conclude by the previous results that $\mathrm{Hilb}_{\mathscr{X}/S} \to S$ is smooth. ☐

Example 19.0.5. Let E be an elliptic curve and consider the nontrivial extension,

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

and let $X = \mathbb{P}_E(\mathcal{E})$. Choose the ample class $\mathcal{L} = \pi^*(\mathcal{O}_E([0])) \otimes \mathcal{O}_X(1)$ and hilbert polynomial $p(t) = t + 1$. Then I claim that,

$$\mathrm{Hilb}^{\mathcal{L},p}_X \cong \mathrm{Spec}\,(k[\epsilon])$$

The exact sequence gives a section $\sigma : E \to X$. Then $T_{[\sigma]}\mathrm{Hilb}_X = H^0(E, \mathcal{N}_{E|X})$. We'll next compute that $\mathcal{N}_{E|X} = \mathcal{O}_E$ and hence $T_{[\sigma]}\mathrm{Hilb}_X = k$ so we get a 1-dimensional tangent space. Thus it suffices to show that $\mathrm{Hilb}_X(k) = \{\sigma\}$. Consider, DO THIS!!

Example 19.0.6. Consider $X = \mathbb{P}_Y(\mathcal{E})$ for some vector bundle and consider a section $\sigma : Y \to X$ given by,

$$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

Then we need to compute,

$$0 \longrightarrow \mathcal{C}_\sigma \longrightarrow \sigma^* \Omega_X \longrightarrow \Omega_Y \longrightarrow 0$$

and there is an Euler sequence,

$$0 \longrightarrow \Omega_{X/Y}(1) \longrightarrow \pi^* \mathcal{E} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

where the second map is the universal quotient. Therefore,

$$0 \longrightarrow (\sigma^* \Omega_{X/Y}) \otimes \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

thus we see that $\sigma^* \Omega_{X/Y} = \mathcal{E}_0 \otimes \mathcal{L}^{-1}$. Furthermore, consider the cotangent sequence,

$$0 \longrightarrow \pi^* \Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

when we apply σ^* there is a section $\mathrm{d}\sigma : \sigma^* \Omega_X \to \Omega_Y$ and hence,

$$\mathcal{C}_\sigma = \ker \mathrm{d}\sigma = \sigma^* \Omega_{X/Y} = \mathcal{E}_0 \otimes \mathcal{L}^{-1}$$

Therefore,

$$T_{[\sigma]} \mathrm{Hilb}_X = H^0(\mathcal{N}_\sigma) = \mathrm{Hom}\,(\mathcal{E}_0, \mathcal{L}) = T_{[\sigma]} \mathrm{Hom}_\pi\,(Y, X)$$

The last equality comes from considering surjections $\mathcal{E}[\epsilon] \twoheadrightarrow \mathcal{L}'$ over $Y \times \mathrm{Spec}\,(k[\epsilon])$ up to isomorphism or equivalently flat subbundles $\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]$ fits into a sequence,

$$\mathrm{Hom}\,(\mathcal{E}_0, \mathcal{E}) \to \{\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]\} \to \ker\left(\mathrm{Def}(\mathcal{E}_0) \to \mathrm{Ob}(\mathcal{E}_0 \to \mathcal{E})\right)$$

which is,

$$\mathrm{Hom}\,(\mathcal{E}_0, \mathcal{E}) \to \{\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]\} = \ker\left(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E}_0) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E})\right)$$

this is the section of the long exact sequence giving,

$$\{\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]\} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{L})$$

Example 19.0.7. Let E be an elliptic curve and consider the bundle $\mathcal{E}$ on $Y = E \times \mathbb{A}^1$ given by the extension,

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

corresponding to the extension class,

$$\xi = t \in \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) = k[t]$$

Now set $X = \mathbb{P}_Y(\mathcal{E})$ and consider the family of smooth surfaces $X \to \mathbb{P}^1$ and the ample line bundle $\mathcal{L} = \pi^*(\mathcal{O}_Y(\sigma_0)) \otimes \mathcal{O}_X(1)$ where σ_0 is the zero section of $Y \to \mathbb{A}^1$. For $t \neq 0$ we have seen that

$$\mathrm{Hilb}^{\mathcal{L},p}_{X_t} = \mathrm{Spec}\,(k[\epsilon])$$

However, $X_0 = E \times \mathbb{P}^1$ with ample $\mathcal{L}_0 = \mathcal{O}_E([0]) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathrm{Hilb}^{\mathcal{L}_0,p}_{X_0}$ contains a $\mathbb{P}^1$ parameterizing the closed subschemes $E \times \{s\} \subset E \times \mathbb{P}^1$. Therefore $\mathrm{Hilb}^{\mathcal{L},p}_{X/\mathbb{P}^1}$ is not flat over $\mathbb{P}^1$.

20 Notes

$H^1(\mathcal{O}_X)$ constant in the smooth case by Hodge theory.

Can $H^1(\mathcal{O}_X)$ jump in the normal case?

Looks like no, using vanishing cycle theory.

Questions:

(a) Hilbert schemes of points of smooth families are flat

(b) Find example of flat family where Hilbert schemes of points are not flat

(c) Example where Hilbert scheme with fixed polynomial is not flat.

21 Neutrino Cosmic Background

21.1 Relativistic Gas

Consider a relativistic gas of bosons or fermions with s species (e.g. for electrons/positrons we have $s = 4$ (two polarizations times two particle types) for neutrios we have $s = 12$ (two polarizations times six particle types) when the temperature is large compared to the heaviest neutrino mass which is basically always true). We compute,

$$Z = \left(\prod_{\{n_{\vec{k}}\}_{\vec{k}}} \prod_{\vec{k}} e^{-\beta n_{\vec{k}} E_{\vec{k}}} \right)^s = \left(\prod_{\vec{k}} \prod_{n_{\vec{k}}} e^{-\beta n_{\vec{k}} E_{\vec{k}}} \right)^s = \left[\prod_{\vec{k}} \left(\frac{1}{1 \mp e^{-\beta E_{\vec{k}}}} \right)^{\pm 1} \right]^s$$

where the energy associated to a certain mode with wavenumber $\vec{k}$ is,

$$E_{\vec{k}}^2 = \hbar^2 |k|^2 c^2 + m^2 c^4$$

For a gas in a box of large volume with respect to the thermal wavelength we compute,

$$-\log Z = \pm s \int_0^\infty \mathrm{d}k \left(\frac{V}{\pi^3} \right) \left(\frac{4\pi k^2}{8} \right) \log\left(1 \mp e^{-\beta E_{\vec{k}}}\right) = \pm \frac{sV}{2\pi^2} (\beta \hbar c)^{-3} \int_0^\infty x^2 \log\left(1 \mp e^{-\sqrt{x^2 + (\beta m c^2)^2}}\right) \mathrm{d}x$$

Then we derive the total energy from the formula,

$$U = -\frac{\partial \log Z}{\partial \beta}$$

to get,

$$\begin{aligned} U &= \pm \frac{sV}{2\pi^2 \hbar^3 c^3} \left[-3\beta^{-4} \int_0^\infty x^2 \log\left(1 \mp e^{-\sqrt{x^2 + (\beta m c^2)^2}}\right) \mathrm{d}x \pm \beta^{-3} \int_0^\infty \frac{x^2 (x^2 + (\beta m c^2)^2)^{-\frac{1}{2}} \beta (m c^2)^2}{e^{\sqrt{x^2 + (\beta m c^2)^2}} \mp 1} \mathrm{d}x \right] \\ &= \pm \frac{sV}{2\pi^2 \beta^4 \hbar^3 c^3} \left[-\int_0^\infty (3x^2) \log\left(1 \mp e^{-\sqrt{x^2 + (\beta m c^2)^2}}\right) \mathrm{d}x \pm \int_0^\infty \frac{x^2 (x^2 + (\beta m c^2)^2)^{-\frac{1}{2}} (\beta m c^2)^2}{e^{\sqrt{x^2 + (\beta m c^2)^2}} \mp 1} \mathrm{d}x \right] \end{aligned}$$

Therefore, integrating the first by parts gives,

$$\begin{aligned}
U &= \frac{sV}{2\pi^2\beta^4\hbar^3c^3}\left[\int_0^\infty \frac{x^4(x^2+(\beta mc^2)^2)^{-\frac{1}{2}}}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x + \int_0^\infty \frac{x^2(x^2+(\beta mc^2)^2)^{-\frac{1}{2}}(\beta mc^2)^2}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x\right] \\
&= \frac{sV}{2\pi^2\beta^4\hbar^3c^3}\int_0^\infty \frac{x^2\sqrt{x^2+(\beta mc^2)^2}}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x
\end{aligned}$$

which is much more easily derived from the expression for $-\log Z$ before the introduction of the dimensionless integration parameter x. Regardless, introduce functions,

$$u_\pm(\alpha) := \int_0^\infty \frac{x^2\sqrt{x^2+\alpha^2}}{e^{\sqrt{x^2+\alpha^2}}\mp 1}\mathrm{d}x$$

and

$$f_\pm(\alpha) := \pm\int_0^\infty x^2\log\left(1\mp e^{-\sqrt{x^2+\alpha^2}}\right)\mathrm{d}x$$

then we see,

$$U = \frac{sV(k_BT)^4}{2\pi^2\hbar^3c^3}u_\pm(\alpha)$$

and,

$$F = \frac{sV(k_BT)^4}{2\pi^2\hbar^3c^3}f_\pm(\alpha)$$

where,

$$\alpha = \frac{mc^2}{k_BT}$$

is the ratio of the Compton temperature to T. In particular,

$$\frac{F}{U} = \frac{f_\pm(\alpha)}{u_\pm(\alpha)}$$

We compute the high and low temperature limits. In the high temperature limit $\beta \to 0$ or equivalently $m \to 0$ we get,

$$\begin{aligned}
u_{+,0}(\alpha) &= u_+(0) = \frac{\pi^4}{15} \\
u_{-,0}(\alpha) &= u_-(0) = \frac{7\pi^4}{120} \\
f_{+,0}(\alpha) &= f_+(0) = -\frac{\pi^4}{45} \\
f_{-,0}(\alpha) &= f_-(0) = -\frac{7\pi^4}{360}
\end{aligned}$$

Therefore, for an ultra-relativistic gas we find that the ratio of free energy to total energy is a constant independent of the nature of the gas,

$$\frac{F}{U} = -\frac{1}{3}$$

This gives the familiar result, that the pressue

$$P = -\left.\frac{\partial F}{\partial V}\right|_T = -\frac{F}{V} = \frac{1}{3}\frac{U}{V}$$

of a relativistic gas is $\frac{1}{3}$ the energy density. Now in the low-temperature limit, which corresponds to large α, we can approximate,

$$\begin{aligned}u_{\pm}(\alpha) &\approx \int_0^\infty \frac{x^2\sqrt{x^2+\alpha^2}}{e^{\sqrt{x^2+\alpha^2}}}\mathrm{d}x = \int_\alpha^\infty \frac{xu^2\mathrm{d}u}{e^u} = e^{-\alpha}\int_0^\infty \frac{\sqrt{(y+\alpha)^2-\alpha^2}\,(y+\alpha)^2\mathrm{d}y}{e^y} \\ &\approx e^{-\alpha}\int_0^\infty \frac{\frac{1}{\sqrt{2}}[y^2+2\sqrt{\alpha y}\,](y+\alpha)^2\mathrm{d}y}{e^y} \approx \sqrt{\tfrac{\pi}{2}}\,\alpha^{\frac{5}{2}}e^{-\alpha}\end{aligned}$$

Therefore as $T \to 0$ for $m > 0$ we have $\alpha \to \infty$ hence the energy content of this gas component freezes out. Furthermore,

$$\begin{aligned}f_{\pm}(\alpha) &\approx \pm\int_0^\infty x^2\left(\mp e^{-\sqrt{x^2+\alpha^2}}\right)\mathrm{d}x = -\int_\alpha^\infty xu\mathrm{d}ue^{-u} = e^{-\alpha}\int_0^\infty \sqrt{(y+\alpha)^2-\alpha^2}(y+\alpha)e^{-y}\mathrm{d}y \\ &\approx e^{-\alpha}\int_0^\infty \tfrac{1}{\sqrt{2}}[y+2\sqrt{y\alpha}](y+\alpha)e^{-y}\mathrm{d}y \approx \sqrt{\tfrac{\pi}{2}}\alpha^{\frac{3}{2}}e^{-\alpha}\end{aligned}$$

Therefore,

$$\frac{F}{U} \approx \alpha^{-1} = \frac{k_BT}{mc^2}$$

Now the entropy is,

$$S = \frac{U-F}{T} = k_B\frac{sV(k_BT)^3}{2\pi^2\hbar^3c^3}[u_{\pm}(\alpha) - f_{\pm}(\alpha)]$$

21.2 Neutrino Decoupling

Consider a gas of photons, electrons, and positrons. Then we get,

$$S_{\text{EM}} = k_B\frac{V(k_BT)^3}{\pi^2\hbar^3c^3}\left[\frac{\pi^4}{15} + \frac{\pi^4}{45} + 2u_-(\alpha_e) - 2f_-(\alpha_e)\right]$$

Likewise, the entropy of a neutrino gas (assuming $\alpha_\nu \gg 1$ meaning $k_BT \gg m_\nu c^2$) is that of a fermionic gas with $s = 12$ so,

$$S_\nu = k_B\frac{12\pi^2V(k_BT_\nu)^3}{2\hbar^3c^3}\cdot\frac{7}{120}\cdot\frac{4}{3} = k_B\frac{7\pi^2V(k_BT_\nu)^3}{15\hbar^3c^3}$$

Therefore,

$$\frac{S_{\text{EM}}}{S_\nu} = \frac{30}{7\pi^4}\left(\frac{T}{T_\nu}\right)^3\left[\frac{2\pi^4}{15} + u_-(\alpha_e) - f_-(\alpha_e)\right]$$

At electroweak unification temperature the neutrinos are thermalized with the other particles. However, as the universe cools sufficiently for the weak sector to decouple from the electromagnetic sector the entropy of each sector are independently conserved since the expansion of the universe is adiabatic. Therefore the ratio is a constant. However, at high temperature we have,

$$\frac{S_{\text{EM}}}{S_\nu} = \frac{30}{7\pi^4}\left[\frac{2\pi^4}{45} + \frac{7\pi^4}{120} + \frac{7\pi^4}{360}\right] = \frac{11}{21}$$

Therefore, at very low temperature the electrons and positrons freeze out so we get,

$$\frac{11}{21} = \frac{4}{21}\left(\frac{T}{T_\nu}\right)^3$$

and therefore,

$$T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T$$

For the CMB temperature of $T = 2.73K$ we get $T_\nu = 1.95K$. One interpretation of this lower temperature is that the neutrinos decouple before the electron and positrons freeze out of the soup but when these freeze out they dump their energy into the photon gas giving it extra heating.

22 MIT OCW 8.06 Darwin Term

22.1 HM 2.4

Recall the Feynman-Hellman lemma that if H_λ is a continuous familiy of Hamiltonians with a continuous family of eigenstates $|\psi_\lambda\rangle$ which energy E_λ then,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E_\lambda = \langle\psi_\lambda|\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda}|\psi_\lambda\rangle$$

We apply this to the Hydrogen effective Hamiltonian,

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} - \frac{e^2}{r}$$

The hydrogen atom energies are,

$$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2} \qquad a_0 = \frac{\hbar^2}{me^2}$$

In solving the radial equation one sets $n = N + \ell + 1$ where N is the degree of the radial polynomial.

(a) Let $\lambda = e^2$ be the parameter then we get $\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda} = -\frac{1}{r}$ and therefore,

$$\left\langle\frac{1}{\lambda}\right\rangle = -\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \frac{me^2}{\hbar^2 n^2} = \frac{1}{a_0 n^2}$$

(b) For the parameter $\lambda = \ell$ where in the radial equation we can consider ℓ as a continuous parameter we get,

$$\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda} = \frac{\hbar^2}{2m}\frac{2\ell+1}{r^2}$$

Therefore,

$$\left\langle\frac{1}{r^2}\right\rangle = \frac{2m}{\hbar^2(2\ell+1)}\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda}$$

where we fix N because it corresponds to the number of nodes in the radial equation and hence is an adiabatic invariant and we vart n according to $n = N + \ell + 1$. Therefore we get,

$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \frac{me^4}{\hbar^2}\frac{1}{n^3}$$

and hence,

$$\left\langle\frac{1}{r^2}\right\rangle = \frac{2m}{\hbar^2(2\ell+1)}\frac{me^4}{\hbar^2}\frac{1}{n^3} = \left(\frac{me^2}{\hbar^2}\right)^2\frac{2}{2\ell+1}\frac{1}{n^3} = \frac{1}{a_0^2}\frac{2}{2\ell+1}\frac{1}{n^3}$$

22.2 HW 3.2

Consider the radial equation,

$$u'' = \frac{2m}{\hbar^2}\left[V_{\text{eff}}(r) - E\right]u$$

multiply this by u' and integrate to get,

$$\int_0^\infty u''u'\mathrm{d}r = \frac{2m}{\hbar^2}\int_0^\infty \left[V_{\text{eff}}(r) - E\right]uu'\mathrm{d}r$$

Now $u''u'$ is the derivative of $\frac{1}{2}(u')^2$ and uu' is the derivative of $\frac{1}{2}u^2$ so integrating the RHS by parts we get,

$$\tfrac{1}{2}(u')^2\Big|_0^\infty = \frac{2m}{\hbar^2}\left[\left[V_{\text{eff}}(r) - E\right]\,\tfrac{1}{2}u^2\Big|_0^\infty - \int_0^\infty V'_{\text{eff}}\,\tfrac{1}{2}u^2\,\mathrm{d}r\right]$$

However, the boundary conditions for bound states in the radial equation are $u(0) = u(\infty) = 0$ and likewise since $u = r\psi$ we get $u'(0) = \psi(0)$ and $u'(\infty) = 0$ and therefore,

$$\psi(0)^2 = \frac{2m}{\hbar^2}\int_0^\infty V'_{\text{eff}}\,u^2\mathrm{d}r = \frac{2m}{\hbar^2}\int_0^\infty \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}\psi(r)^2\,r^2\mathrm{d}r = \frac{2m}{4\pi\hbar^2}\int \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}\psi(r)^2\,\mathrm{d}^3r$$

Now if we consider a state with $\ell = 0$ we get $V_{\text{eff}} = V$ and $\psi(\vec{r}) = \psi(r)$ and we can choose ψ real so we conclude,

$$|\psi(0)|^2 = \frac{2m}{4\pi\hbar^2}\left\langle \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}\right\rangle$$

22.3 The Darwin Term

Therefore using the previous two exercises,

$$|\psi_{n,0}(0)|^2 = \frac{2me^2}{4\pi\hbar^2}\left\langle \frac{1}{r^2}\right\rangle = \frac{1}{2\pi}\cdot\frac{1}{a_0^3}\cdot\frac{2}{n^3} = \frac{1}{\pi a_0^3 n^3}$$

Therefore, for the Darwin term,

$$\delta H_{\text{Darwin}} = \frac{\hbar^2}{8m^2c^2}\nabla^2 V = \frac{\pi\hbar^2 e^2}{2m^2c^2}\delta^{(3)}(\vec{r})$$

we only get an energy shift for $\ell = 0$ which in first-order perturbation theory is,

$$\Delta E_{n,0} = \langle \delta H_{\text{Darwin}}\rangle = \nabla^2 V = \frac{\pi\hbar^2 e^2}{2m^2c^2}|\psi_{n,0}(0)|^2 = \frac{\hbar^2 e^2}{m^2c^2a_0^3}\cdot\frac{1}{n^3} = \alpha^4 mc^2\cdot\frac{1}{2n^3}$$

23 Semisimple Algebras

23.1 Jacobson Radical

Here a ring is always unital and associative but not necessarily commutative.

Definition 23.1.1. The *Jacobson radical* $J(R)$ is defined as,

$$J_l(R) = \bigcap_{\substack{M \text{ simple} \\ \text{left } R\text{-mod}}} \mathrm{Ann}_R(M)$$

$$J_r(R) = \bigcap_{\substack{M \text{ simple} \\ \text{right } R\text{-mod}}} \mathrm{Ann}_R(M)$$

Note that these are two-sided ideals because $\mathrm{Ann}_R(M)$ is always a two-sided ideal. Indeed if M is a left R-module then if $xM = 0$ then clearly $rxM = 0$ but al $xrM = x(rM) = xM = 0$. Similarly for M a right R-module.

We use the terminology "the" Jacobson radical because of the following theorem we will now prove.

Theorem 23.1.2. For any unital ring R,

$$J_l(R) = J_r(R)$$

as two-sided ideals.

Remark. This is false for nonunital rings. GIVE EXAMPLE

Remark. Due to the symmetry one might ask if we can define the Jacobson radical more symmetrically as the intersection of the annihlators of simple bimodules. (CAN YOU DO THIS??)

We now will prove this result as follows.

Proposition 23.1.3. The following hold:

$$J_l(R) = \bigcap_{\substack{\mathfrak{m}\text{ maximal}\\ \text{left ideal}}} \mathfrak{m} = \{x \in R \mid \forall r \in R : 1 + rx \text{ is left invertible}\}$$

and likewise

$$J_r(R) = \bigcap_{\substack{\mathfrak{m}\text{ maximal}\\ \text{right ideal}}} \mathfrak{m} = \{x \in R \mid \forall r \in R : 1 + xr \text{ is right invertible}\}$$

Proof. The argument for the two statements is identical so we will only do it for the left case. Since if $\mathfrak{m}$ is a maximal left ideal then $R/\mathfrak{m}$ is a simple left R-module it is clear that $J_l(R) \subset \mathfrak{m}$ since if $x(R/\mathfrak{m}) = 0$ then $x = x \cdot 1 \in \mathfrak{m}$.

Conversely suppose that $x \in \mathfrak{m}$ for each maximal left ideal $\mathfrak{m}$. Let M be a simple left R-module and $m \in M$ nonzero. Then $Rm \subset M$ is a nonzero submodule so $Rm = M$ and hence ${}_RR \to M$ sending $r \mapsto rm$ is surjective and hence $M \cong R/\mathrm{Ann}_R(m)$ and $\mathrm{Ann}_R(m)$ is a maximal left[2] ideal (maximal otherwise there would be proper submodules of $R/\mathrm{Ann}_R(m)$). Now it is clear that,

$$\mathrm{Ann}_R(M) = \bigcap_{m \in M} \mathrm{Ann}_R(m)$$

which is an intersection of maximal left ideals and hence $x \in \mathrm{Ann}_R(M)$ so $x \in J_l(R)$.

Now we show the second equality. Suppose that $x \in \mathfrak{m}$ for each maximal left ideal. Then if $1 + rx$ does not have a left inverse then $R(1 + rx)$ is a proper left ideal and hence contained in a maximal left ideal $\mathfrak{m}$ but then $x \in \mathfrak{m}$ so $rx \in \mathfrak{m}$ so $1 \in \mathfrak{m}$ giving a contradiction. Conversely, if $(1 + rx)$ does have a left inverse for all $r \in R$ suppose that $x \notin \mathfrak{m}$ then there exists r such that $1 + rx \in \mathfrak{m}$ so $1 \in R(1 + rx) \subset \mathfrak{m}$ giving a contradiction. To see the existence of such r consider the left ideal $\mathfrak{m} + Rx$ which is strictly larget than $\mathfrak{m}$ since $x \notin \mathfrak{m}$. Hence by maximality $\mathfrak{m} + Rx = R$ so it contains 1 and hence there exists $r \in R$ so that $1 + rx \in \mathfrak{m}$. ☐

[2] Unlike $\mathrm{Ann}_R(M)$ the ideal $\mathrm{Ann}_R(m)$ is *not* two-sided. The point is that for any $r \in R$ we have $rM = M$ so if $xM = 0$ then $xrM = x(rM) = xM = 0$ but the same does not work for $x \in \mathrm{Ann}_R(m)$ since rm might not be annihilated by x. This shows that even if m generates M we may have $\mathrm{Ann}_R(m) \supsetneq \mathrm{Ann}_R(M)$ in the noncommutative case because we cannot take $xm = 0$ and use it to conclude that $xrm = 0$. Indeed, if $M = R/\mathfrak{m}$ then $\mathrm{Ann}_R(\bar{1}) = \mathfrak{m}$ but $\mathrm{Ann}_R(M) = \{x \in R \mid xR \subset \mathfrak{m}\}$ the largest two-sided ideal contained in $\mathfrak{m}$, which is, in general, smaller than $\mathfrak{m}$ since $\mathfrak{m}$ is only a left ideal.

Lemma 23.1.4. The following conditions on $x \in R$ are equivalent,

(a) $\forall r \in R : 1 + rx$ has a left inverse

(b) $\forall r \in R : 1 + rx$ has a two-sided inverse

and similarly,

(a) $\forall s \in R : 1 + xs$ has a two-sided inverse

(b) $\forall s \in R : 1 + xs$ has a right inverse

Proof. We just need to show (a) $\implies$ (b) since other implications are similar or trivial. Suppose (a) then there exists s such that $s(1 + rx) = 1$ and hence,

$$s = 1 - srx$$

so applying (a) with r replaced by $-sr$ we see that s has a left inverse s' which means that s is invertible (since it also has a right inverse $1 + rx$) and hence $1 + rx$ is also invertible. $\square$

Theorem 23.1.5. For any unital ring R,

$$J_l(R) = \{x \in R \mid 1 + RxR \subset R^\times\} = J_r(R)$$

Proof. Indeed, we showed that if $x \in J_l(R)$ then $\forall r \in R : 1 + rx$ has a two-sided inverse by the above lemma. But $J_l(R)$ is a two-sided ideal so this means that $xs \in J_l(R)$ hence $\forall r, s \in R : 1 + rxs$ is a unit. The reverse inclusion is clear. Likewise, if $x \in J_r(R)$ then $\forall s \in R : 1 + xs$ has a two-sided inverse but $J_r(R)$ is a two-sided ideal so this means that $rx ßn J_r(R)$ hence $\forall r, s \in R : 1 + rxs$ is a unit. Thus we conclude. $\square$

From now on we write $J(R) = J_l(R) = J_r(R)$.

23.2 Nilpotence

Proposition 23.2.1. Let I be a left (resp. right) ideal consisting of nilpotent elements then $I \subset J(R)$.

Proof. Since I consists of nilpotent elements for each $x \in I$ we have rx is nilpotent for each r. Hence $1 + rx$ is a unit so $x \in J(R)$. $\square$

Definition 23.2.2. We say that a module M is

(a) *Noetherian* if every ascending chain of submodules stabilizes

(b) *Artinian* if every descending chain of submodules stabilizes

We say that R is

(a) *left (resp. right) Noetherian* if ${}_RR$ (resp. R_R) is Noetherian as a left (resp. right) R-module

(b) *left (resp. right) Artinian* if ${}_RR$ (resp. R_R) is Artinian as a left (resp. right) R-module

R is left (resp. right)

Remark. Note that there exist left artinian rings that are not right artinian (see Lam, a First Course in Noncommutative Rings, p.22).

Proposition 23.2.3. Let R be left (resp. Artinian) noetherian. Then $J(R)$ is nilpotent.

Proof. Let $J = J(R)$. Consider the descending chain of left ideals,

$$J \supset J^2 \supset J^2 \supset \cdots$$

This must stabilize so we have $J^m = J^n$ for $m \geq n$ and some fixed n. Let $m = 2n$ and $I = J^n$ so $I^2 = I$. If $I \neq 0$ then there exists a left ideal K such that $IK \neq 0$ (e.g. $K = R$) since R is left Artinian there is a minimal such K by Zorns' lemma. If $y \in K$ then $Iy \subset$ FINISH □

Proposition 23.2.4. Let R be left (resp. right) Artinian then R is left (resp. right) Noetherian.

Proof. DO THIS PROOF!! □

23.3 Semisimple Rings

(SHOW AUTOMATIC FINITNESS CONDITIONS NOETHERIAN AUTOMATICALLY!!)
(GIVE EXAMPLE WHY INFINITE PRODUCT OF FIELDS NOT SEMISIMPLE)

Definition 23.3.1. A module M is *semisimple* if one of the following equivalent properties holds,

(a) M is a direct sum of simple modules

(b) M is the sum of its irreducible submodules

(c) every submodule of M is a direct summand

Definition 23.3.2. A ring R is left (resp. right) semisimple if ${}_R R$ (resp. R_R) is a semisimple left (resp. right) R-module.

Proposition 23.3.3. If R is left (resp. right) semisimple if and only if the category of left (resp. right) R-modules is semisimple in the sense that all exact sequences split.

Proof. See [Rotman, An Introduction to Homological Algebra, Prop. 4.5] for details. If R is left semisimple then ${}_R R$ hence hence all free left R-modules are semisimple. However, every left R-module is a quotient of a free module and hence semisimple because submodules of semisimple module are direct summands so quotients are also direct summands. Then every module is both injective and projective since all injections and surjections split since the modules are all semisimple. □

Lemma 23.3.4. Let M be an R-module such that the intersection of all maximal submodules is zero. If M is Artinian then M is semisimple.

Proof. Since M is Artinian the poset of finite intersections of maximal submodules satisfies Zorn's lemma and hence has a minimal element which must equal the intersection of al maximal submodules which is zero. Hence there is a collection $\{N_i\}$ of maximal submodules such that $\bigcap_i N_i = (0)$. Therefore, the map,

$$M \to \bigoplus_i M/N_i$$

is injective but M/N_i is simple and hence M is a submodule of a semisimple module and hence is semisimple. □

Proposition 23.3.5. The following are equivalent,

(a) R is left semisimple

(b) R is right semisimple

(c) $J(R) = 0$ and R is left Artinian

(d) $J(R) = 0$ and R is right Artinian.

Proof. Recall that $J(R)$ is the intersection of all maximal left ideals and also all maximal right ideals. Hence if $J(R) = 0$ then both ${}_RR$ and R_R satisfy that the intersection of their maximal submodules is (0). If $J(R) = 0$ and R is left (resp. right) Artinian then by the above lemma ${}_RR$ (resp R_R) is semisimple so we conclude that R is left (resp. right) semisimple. Hence it suffices to show that if R is left (resp. right) semisimple then $J(R) = (0)$ and R is both left and right Artinian.

If R is left semisimple then ${}_RR$ is a direct sum of simple modules. However, ${}_RR$ is trivially finitely generated but an infinite direct product cannot be finitely generated so it is a finite direct sum of simple modules. Hence R is left Artinian. ☐

(HOW DO I SHOW IT IS RIGHT ARTINIAN!???)

Definition 23.3.6. A ring R is *simple* if it has no nontrivial two-sided ideals.

Remark. Note that R being simple is much weaker than ${}_RR$ being a simple left R-module. For example, $R = M_n(k)$ is simple but is certainally not simple as a left module since it has nontrivial left ideals (e.g. matrices with some columns zero). Indeed, we have the following result.

Proposition 23.3.7. The following are equivalent,

(a) R is a division ring

(b) ${}_RR$ is a simple left R-module

(c) R_R is a simple right R-module

Proof. Indeed, these are equivalent to R having no nontrivial left (resp. right) ideals. Thus if $x \in R$ is nonzero then $Rx = R$ so there is $yx = 1$ so every nonzero element has a left inverse. However, $y \in R$ then also has a left inverse hence every nonzero element is invertible. The same argument holds if R_R is a simple right R-module. ☐

Proposition 23.3.8. Let R be semisimple. Then R is left and right Artinian.

23.4 Artin-Wedderburn Theorem

Lemma 23.4.1 (Shur)**.** Let M_1, M_2 be simple R-modules. Then any nonzero endomorphism $\varphi : M_1 \to M_2$ is an isomorphism. Hence $\mathrm{End}_R(M)$ is a division ring.

Proof. Indeed $\ker \varphi \subset M_1$ and $\operatorname{im} \varphi \subset M_2$ are submodules. Since M_1 and M_2 are simple these are either (0) or M_1 respectively M_2. If $\ker \varphi = M_1$ then $\varphi = 0$. If $\operatorname{im} \varphi = (0)$ then $\varphi = 0$ hence φ is a bijection and hence invertible. ☐

Theorem 23.4.2 (Artin-Wedderburn)**.** Let R be a semisimple ring. Then, R is isomorphic to a finite direct product of matrix rings:

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where D_i are division rings. Moreover the division rings D_i and the integers $r, n_1, \ldots, n_r$ are a complete set of invariants of R.

Proof. Since R is semisimple, R is semisimple as a right R-module. Therefore we can decompose,

$$R = \bigoplus_{i=1}^{r} I_i^{\oplus n_i}$$

where I_i are the minimal nonzero right ideals (i.e. the simple submodules of R_R). This sum is finite since R is right Artinian. Since I_i is simple we see that $D_i = \mathrm{End}_R(I_i)$ is a division ring by Shur's lemma. Therefore,

$$R \cong \mathrm{End}_R(R_R) \cong \mathrm{End}_R\left(I_1^{\oplus n_i}\right) \times \cdots \times \mathrm{End}_R\left(I_r^{\oplus n_r}\right)$$

since the I_i are distinct simple modules there are no nonzero maps between them by Shur's lemma. Therefore,

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

□

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$\mathscr{P}$