1 Introduction

The λ-calculus (here we first introduce pure untyped λ-calculus) is a formal system designed to capture of the notions of "function" and "composition".

Remark. A λ-expression will be a finite string made from the symbols λ, ., and an infinite list of variable symbols $x, y, z, \cdots$ or $a, b, c, \cdots$ or $x_1, x_2, x_3, \ldots$ whatever you want to call them. We think of $\lambda x.M$ as the function that takes in x and returns M where M is an expression possibly involving x.

Definition 1.0.1. A *well-formed* λ-expression is defined recursively via,

(a) any variable is a λ-expression

(b) if M is a λ-expression then $\lambda x.M$ is a λ-expression

(c) if M and N are λ-expressions then (MN) is a λ-expression.

Remark. We have already said that we should interpret $\lambda x.M$ as a function. Then (M, N) is an "application" of the function M to N. We think of $((\lambda x.M)N)$ "evaluating" to $M[x := N]$ which is how this system captures the essence of functions and computation as evaluation though substitution. Although intutively we think of λ-expressions as functions, that actually take an input and produce a well-defined output, this is actually difficult to define because we will have to decide when a computation is "finished" and in fact computations may not halt complicating our desire to call these things functions. A major goal will be to somehow interpret these objects as honest-to-god functions. For now, we take a different perspective not that λ-expressions are "machines" but rather they are formal strings in a formal system. To create a formal system we need "rules of inference" which are conventionally called conversions and reductions.

Definition 1.0.2. REDUCTIONS AND COVERESIONS

EXAMPLES
NORMAL EXPRESSIONS
CHURCH-ROSSER
UNIQUENESS OF NORMAL FORM

Example 1.0.3. A λ-expression need not admit a normal form. For example let,

$$\omega = \lambda x.(xx)$$

this is the function that applies its input to itself. What happens if we apply ω to itsef then we get,

$$\Omega = (\omega\omega)$$

and it is easy to see that,

$$\Omega \triangleright_{\beta,1} \Omega$$

and there are no other possible β-reductions. Therefore Ω does not have a β-normal form. We can think of it as corresponding to a computation that does not halt.

2 Arithmetic and Logic in the λ-Calculus

CHURCH ENCODING AND SIMPLE FUNCTIONS

3 "Consistency" of the λ-Calculus

EARLY FORM INCONSISTENT

Look at section here https://plato.stanford.edu/entries/lambda-calculus/
and in Barendregt's book.
THIS FORM IS CONSISTENT IN THE SENSE OF CANNOT DERIVE EVERYTHING
CURY'S PARADOX

4 Recursion and Fixed Points

5 λ-Models

6 CORRECT NOTION OF LATTICES

7 Scott's Model D_∞

DEFINITION

SHOW THAT THE TWO FIXED POINT NOTIONS AGREE