1 Picard Scheme

Theorem 1.0.1. Let X be a proper k-scheme. Then $\mathrm{Pic}\left(\right) X/k$ is represented by a lft k-scheme.

Remark. However, this does not hold for proper flat families in general even for curves.

From here on let $f : X \to S$ be a flat, locally finitely presented, proper morphism where $S = \mathrm{Spec}\,(R)$ is a DVR.

Theorem 1.0.2 (8.3.2)**.** $_{X/S}$ is represented by an algebraic space if and only if f is cohomologically flat in degree 0.

Remark. In fact, the above holds when S is any reduced scheme.

This is a problem since we want to study non-cohomologically flat situations. We fix this in the next section.

1.1 Rigidified Picard Scheme

Proposition 1.1.1 (8.1.6)**.** f admits a rigidifying subscheme meaning a closed subscheme $Y \subset X$ which is flat, locally finitely presented, proper and such that for any $T \to S$ the map,

$$\Gamma(X_T, \mathcal{O}_{X_T}^\times) \to \Gamma(Y_T, \mathcal{O}_{Y_T}^\times)$$

is injective.

Definition 1.1.2. Let $Y \hookrightarrow X$ be a rigidifying subscheme. Then we define the rigidified Picard functor,

$$\mathrm{Pic}_{X/S|Y} : (T \to S) \mapsto \{(\mathcal{L}, \varphi) \mid \mathcal{L} \in \mathrm{Pic}\,(X_T) \text{ and } \varphi : \mathcal{L}|_Y \xrightarrow{\sim} \mathcal{O}_Y\}/\cong$$

The condition of being a rigidifying subscheme shows exactly that there are no nontrivial automorphism of $(\mathcal{L}, \varphi)$.

FGA shows that the functor,

$$(T \to S) \mapsto (f_T)_* \mathcal{O}_{X_T}$$

is representable by a linear scheme V_X over X. This is a vector bundle over X iff f is cohomologically flat in degree 0. Furthermore, the subsheaf of units,

$$(T \to S) \mapsto (f_T)_* \mathcal{O}_{X_T}^\times$$

is represented by an open subscheme,

$$V_X^\times \subset V_X$$

Now V_X is a ring scheme and $V_X^\times$ is a group scheme.

Proposition 1.1.3. Let $Y \hookrightarrow X$ be a rigidifier. There is an exact sequence of fppf sheaves of abelian groups,

$$0 \longrightarrow V_X^\times \longrightarrow V_Y^\times \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow \mathrm{Pic}_{X/S|Y} \longrightarrow 0$$

where the last map forgets the rigidification. It is surjective in the fppf topology because by definition any class in $\mathrm{Pic}\left(\right) X/S$ is fppf locally represented by a line bundle.

Theorem 1.1.4. Let $Y \hookrightarrow X$ be a rigidifier. Then