1 Clemens Conjecture

1.1 Notation: Mori degeneration

Consider Mori's degeneration:

Definition 1.1.1. Let S be a scheme, $t \in \mathcal{O}_S$. Let $f, g \in \mathcal{O}_S[x_0, \dots, x_n]$ be homogeneous polynomials of degrees cd and d respectively such that $g^c - f$ is not identically zero mod s for any point $s \in S$. The scheme

$$Z = V(y^c - f, ty - g) \subset \mathbb{P}_S(1^{n+1}, d) := \operatorname{Proj}\left(S[x_0, \dots, x_n, y]\right)$$

where $\deg x_i = 1$ and $\deg y = d$. If $t(s) \neq 0$ then Z_s is isomorphic to the hypersurface

$$V(g(s)^c - t(s)^c f(s)) \subset \mathbb{P}^n_{\kappa(s)} := \operatorname{Proj}\left(\kappa(s)[x_0, \dots, x_n]\right)$$

by eliminating y. Moreover, if $t(s) = 0$ then the fiber Z_s is the degree c cyclic cover of the hypersuface $\{g(s) = 0\}$ ramified along $\{f(s) = 0\}$.

Remark. Moreover, the map $\mu : Z_0 \to \mathbb{P}^n$ is given by projection away from $\{x_0 = x_1 = \cdots = x_n = 0\}$ and since f is homogeneous this point is not on Z_0 so it gives a morphism. Therefore $\mu^* \mathcal{O}(1)$ agrees with $\mathcal{O}(1)|_{Z_0}$ arising from the total space. Furthermore, note that the $\mathbb{G}_m$-action is free everywhere but along the ray $[0, 0, \dots, 0, \lambda]$ and therefore, since the hypersurface is away from this ray we see that $\mathcal{O}(1)$ is locally free along Z_0.

Let $\mu : X_0 \to \mathbb{P}^3$ be the $p = 5$ cyclic cover defined by

$$w^p = f(x, y, z)$$

Then according to Kollár there is an exact sequence,

$$0 \to \mu^* \mathcal{O}_{\mathbb{P}^3}(-p) \to \mu^* \Omega_{\mathbb{P}^3} \to \Omega_{X_0} \to \mu^* \mathcal{O}_{\mathbb{P}^3}(-1) \to 0$$

We define,

$$Q_0 := \operatorname{coker}\left(\mu^* \mathcal{O}_{\mathbb{P}^3}(-p) \to \mu^* \Omega_{\mathbb{P}^3}\right) = \operatorname{im}\left(\mu^* \Omega_{\mathbb{P}^3} \to \Omega_{X_0}\right)$$

Then Kollár proves that

$$(\wedge^2 Q_0)^{\vee\vee} = \mu^* \mathcal{O}_{\mathbb{P}^3}(1)$$

Lemma 1.1.2. Let $f : S \to X$ be a morphism from a smooth birationally ruled surface to a smooth 3-fold. Suppose $\varphi : \mathcal{L} \hookrightarrow \wedge^2 \Omega_X$ is a line bundle embedded in $\wedge^2 \Omega_X$ and $\mathcal{L}$ has a nonzero section s. Let $\overline{S} = \operatorname{im} f$ then one of the following must hold:

(a) $\overline{S} \subset V(s)$

(b) $f^*(\mathcal{L} \otimes \mathcal{O}_X(\overline{S}))$ intersects non-positively with the general fiber of $S \to C$

(c) $\overline{S} \subset V(\varphi)$

Proof. Suppose (a) does not hold. Because $H^0(S, \omega_S) = 0$ since S is ruled and $f^* \mathcal{L}$ has a nonzero section because we are not in case (a), the composition is zero

$$f^* \mathcal{L} \to f^* \wedge^2 \Omega_X \to \omega_S$$

since ω_S has no sections and $f^*\mathcal{L}$ is big.

Now consider the sequence
$$0 \to \mathcal{C} \to f^*\Omega_X \to \Omega_S$$
Let $\overline{S}$ be the image of S. Then we have a sequence,

$$0 \to \mathcal{C} \to \Omega_X|_{\overline{S}} \to \Omega_{\overline{S}} \to 0$$

and the sequence is left exact because $\overline{S}$ is a prime divisor and hence is Cartier and so $\mathcal{C}$ is a line bundle. Consider the exact sequence

$$0 \to f^*\mathcal{C} \to f^*\Omega_X \twoheadrightarrow \mathscr{F} \subset \Omega_S$$

where $\Omega_S/\mathscr{F}$ has support over the exceptional locus of $S \to \overline{S}$. Then I claim there is a sequence

$$0 \to \mathscr{F} \otimes \mathcal{C} \to \wedge^2 f^*\Omega_X \to \omega_S$$

Indeed, consider the map $f^*\Omega_X \otimes \mathcal{C} \to \wedge^2 f^*\Omega_X$. I claim this surjects onto the kernel. Indeed, if $\alpha \wedge \beta \mapsto 0$ then $\alpha - \lambda\beta$ is in the kernel. Therefore, $\alpha \wedge \beta = (\alpha - \lambda\beta) \wedge \beta$ thus is in the image of the claimed map. Moreover, since $\mathcal{C} \otimes \mathcal{C}$ maps to zero we get a map $\mathscr{F} \otimes \mathcal{C} \to \wedge^2 f^*\Omega_X$. This is injective because $\mathcal{C}$ is a line bundle and $\mathscr{F}$ is torsion-free and rank 1 so we can check injectivity at the generic point.

Therefore, since $f^*\mathcal{L} \to \wedge^2 f^*\Omega_X \to \omega_S$ is zero we get that the map factors through $f^*\mathcal{L} \to \mathscr{F} \otimes \mathcal{C}$. Hence, if the map $f^*\mathcal{L} \to \wedge^2 f^*\Omega_X$ is nonzero then we get an embedding

$$f^*\mathcal{L} \hookrightarrow \Omega_S \otimes f^*\mathcal{C}$$

We need that $f^*(\mathcal{L} \otimes \mathcal{C}^\vee)$ is big since Ω_S cannot contain a big line bundle. Indeed, there is map $S \to C$ whose general fiber is $\mathbb{P}^1$. Then we know $\Omega_S|_F \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ but a big line bundle must restrict positively to the generic fiber. □

If X is singular this might be an issue unless the singularities are not so bad that forms do not extend to the resolution

Note if $S \to X$ hits a singular point of X that needs to be resolved then the modification to the normal bundle is only over exceptional loci of S I think and therefore do not interact with the general fiber of S maybe?? Unless the map contracts something to the singularity which seems very possible.

2 Chang and Ran

let $X \subset \mathbb{P}^4$ be a general quintic hypersurface. Let it be a general hyperplane section of $Y \subset \mathbb{P}^5$ another fixed quintic. Let $S \to X$ be a smooth surface of negative kodiara dimension mapping birationally onto its image in X. There are two cases:

(a) either S fills Y as we move H

(b) S extends to a divisor of Y such that S is a section.

I THINK they show (b) does not occur and when $-K_S$ is nef (a) does not occur either.

2.1 (a)

Consider the sequences,

$$0 \to T_S \to f^*T_X \to N_f \to 0$$

and

$$0 \to N_f \to N_{\tilde{f}} \to L \to 0$$

where $\tilde{f}$ is the composite

$$S \xrightarrow{f} X \hookrightarrow Y$$

and $L = f^*\mathcal{O}(1)$.

Note that the second sequence splits in any neighbrohood of a fiber of f. Let $\tau = (N_f)_{\text{tors}}$ which is supported purely in codimension 1 (because T_S has corank 1 in f^*T_X). Since S fills Y we see that $N_{\tilde{f}}$ is generated generically by global sections. Thus

$$c_1(N_{\tilde{f}}/\tau) = c_1(N_{\tilde{f}}) - c_1(\tau)$$

is nef WHY? maybe I don't know what generically globally generated means in this context?

3 Wang 2000

Let X be a non-singular complete intersection of type $(m_1, \dots, m_k)$ in a Grassmanian $G(r, n+1)$ such that $\dim X \geq 3$ and $m = m_1 + \cdots + m_k \geq n+1$, and supposet $\overline{D} \subset X$ is an irreducible and reduced divisor. Let $f : D \to \overline{D} \subset X$ be a desingularization, ℓ denote the dimension of D and $L = f^*\mathcal{O}_G(1)$. Obviously, L is big and nef. Let K_D be the canonical bundle of D. Let S and Q be the universal subbundle and universal quotient bundle on G.

Proposition 3.0.1. *X does not cotain any reduced irreducible divisor which admits a designularization having*

$$H^0(K_D \otimes f^*Q^\vee) = 0 \quad \text{and} \quad H^1(K_D - L^{\otimes m_i}) = 0$$

for any all $i = 1, \dots, k$.

3.1 Reflexive Sheaves

Let $\mathscr{F}^{\vee\vee}$ be the double dual of $\mathscr{F}$. A coherent sheaf $\mathscr{F}$ is reflexive if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism. Define the singularity set of $\mathscr{F}$ to be the locus where $\mathscr{F}$ is not free over the local ring.

It is well-known that the sigularity set of a torsion-free sheaf on D is in codimension ≥ 2. Moreover, the singularity set of a reflexitve sheaf on D is in codimension ≥ 3. It is also well-known that, in general, any reflexive rank 1 sheaf on an integral locally factoral scheme is a line bundle.

3.2 The Proof

Assume such $\overline{D}$ exists. Consider the sequence

$$0 \to Q^\vee \to \mathcal{O}_G^{n+1} \to S^\vee \to 0$$

Pull this back and tensor with f^*Q to get

$$0 \to f^*Q \otimes f^*Q^\vee \to (f^*Q)^{n+1} \to f^*T_G \to 0$$

The top cohomology

$$h^\ell(f^*Q) = h^0(K_D \otimes f^*Q^\vee) = 0$$

vanishes by assumption and hence $H^\ell(f^*T_G) = 0$. Now we pull back the normal bundle sequence of X

$$0 \to f^*T_X \to f^*T_G \to \bigoplus L^{\otimes m_i} \to 0$$

Note that we need the smoothness of X to get the above sequence. Then we have,

$$h^{\ell-1}(L^{\otimes m_i}) = h^1(K_D - L^{\otimes m_i}) = 0$$

also by assumption and hence using this and the above calculation

$$H^\ell(f^*T_X) = 0$$

Next, consider the defining sequence of the normal sheaf

$$0 \to T_D \to f^*T_X \to N_f \to 0$$

with the above three sequences we obtain

$$H^\ell(N_f) = 0$$

and

$$c_1(N_f) = K_D + (n+1-m)L$$

where

$$m = m_1 + \cdots + m_k$$

Let $N_f^{\vee\vee}$ be the double dual of N_f which is a line bundle. The image of $N_f \to N_f^{\vee\vee}$ is torsion-free. The singularity set of the image is in codimension ≥ 2 so there is an exact sequence

$$0 \to \tau \to N_f \to N_f^{\vee\vee} \to \phi \to 0$$

with $\dim \mathrm{Supp}\,(\phi) \leq 0$. Devide these into sequences

$$0 \to \tau \to N_f \to \psi \to 0$$

and

$$0 \to \psi \to N_f^{\vee\vee} \to \phi \to 0$$

Then $H^\ell(N_f) = 0$ implies that likewise

$$H^\ell(N_f^{\vee\vee}) = 0$$

because $H^\ell(\phi) = 0$ by dimension reasons. On the other hand, we have

$$c_1(N_f^{\vee\vee}) = K_D + (n+1-m)L - c_1(\tau)$$

Note that $c_1(\tau)$ is always effective. Therefore,

$$h^\ell(N_f^{\vee\vee}) = h^0(K_D - N_f^{\vee\vee}) = h^0((m-n-1)L + c_1(\tau)) > 0$$

which is a contradiction.

3.3 Main Theorem

For $r = 1$ we identify $G(1, n+1) = \mathbb{P}^n$.

Proposition 3.3.1. A nonsingular complete intersection X of type $(m_1, \ldots, m_k)$ in $\mathbb{P}^n$ for $n \geq 4$ such that

$$m = m_1 + \cdots + m_k \geq n + 1$$

does not contain a reduced irreducible divisor which admits a desingularization having $H^0(K_D - L) = 0$ and $H^1(K_D - m_i L) = 0$ for all $i = 1, \ldots, k$.

We get thiis immediately if we identify $\mathbb{P}^n$ with $G(n, n+1)$.

Theorem 3.3.2. A non-singular complete intersection X of type $(m_1, \ldots, m_k)$ in $\mathbb{P}^n$ such that $\dim X \geq 3$ and $m = m_1 + \cdots + m_k \geq n + 1$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

Proof. If $-K_D$ is nef, $-K_D + L$ and $-K_D + m_i L$ are nef and big. Therefore by Kawamata-Viehweg vanishing we obtain

$$H^0(K_D - L) = 0 \quad H^1(K_D - m_i L) = 0$$

for all i. Note that $\dim D = \dim X - 1 \geq 2$ so we may apply the vanishing results. □

4 Ideas

(a) study elliptic curves on 3-folds using Noether-Lefschetz for rank two bundles see here http://www.math.u varu/extension.pdf

(b) failure of isotriviality for lc singularities. But look at the $-K_X$ nef case https://arxiv.org/pdf/1612.0592 where there is some result.

To read:

(a) phantom on a rational surface

(b) quasi-albanese maps

(c) The work of Ein

(d) motivic invariants of birational maps

(e) https://www.youtube.com/watch?v=CLLf-oDCNR0

5 Conic Complexity

Definition 5.0.1. We say a map of schemes $f : X \to Y$ is a *fibration* (or an *algebraic fiber space* when X and Y are finite type over k) if $f_* \mathcal{O}_X = \mathcal{O}_Y$.

Definition 5.0.2. Let $f : X \to Y$ be a morphism of varities over k. We say that f is a *conic bundle* if it is proper, $f_* \mathcal{O}_X = \mathcal{O}_Y$ and the generic fiber of f is a curve of arithmetic genus zero (i.e. a conic).

Definition 5.0.3. Let X be an n-dimensional variety over a field k. The *conic complexity* $c_{\text{conic}}(X)$ of X is the minimum value of c such that there exists a normal proper birational model X_0 and a sequence of fibrations of k-varities,

$$X_0 \to X_1 \to \cdots \to X_\ell$$

so that $n - c$ of them are conic bundles.

IS IT TRUE THAT CONIC COMPLEXITY IS A GEOMETRIC NOTION?

Remark. The conic complexity is a geometic notion. Indeed, given a sequence over k we obtain a similar sequence

Here let R be an excellent DVR with fraction field $K = \mathrm{Frac}\,(R)$ and maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$.

Proposition 5.0.4. Let $\mathcal{X} \to \mathrm{Spec}\,(R)$ be a flat proper morphism with $\mathcal{X}$ integral and normal. Suppose there is a birational modification $\gamma : \mathcal{X}_{0,K} \to \mathcal{X}_K$ and a sequence of fibrations

$$\mathcal{X}_{0,K} \xrightarrow{f_{0,K}} \mathcal{X}_{1,K} \xrightarrow{f_{1,K}} \mathcal{X}_{2,K} \to \cdots \to \mathcal{X}_{\ell,K}$$

where $\mathcal{X}_{i,K}$ are normal proper K-varieties and $f_{i,K} : \mathcal{X}_{i,K} \to \mathcal{X}_{i+1,K}$ is a fibration of relative dimension r_i. Let $\delta \in \mathcal{X}_\kappa$ be the generic point of some irreduible component. Then there exists the following data:

(a) asequence of fibrations of normal integral schemes flat and proper over R,

$$\mathcal{X}_0 \xrightarrow{f_0} \mathcal{X}_1 \xrightarrow{f_1} \mathcal{X}_2 \to \cdots \to \mathcal{X}_\ell$$

(b) a sequence $\delta_i \in (\mathcal{X}_i)_\kappa$ of generic points of normal irreducible components

(c) a birational map $\epsilon : \mathcal{X}_0 \to \mathcal{X}$ such that $\epsilon : \delta_0 \mapsto \delta$

(d) a diagram,

$$\begin{tikzcd}
(\mathcal{X}_0)_K \arrow[r, "(f_0)_K"] \arrow[d, "\mu_0"] & (\mathcal{X}_1)_K \arrow[r, "(f_1)_K"] \arrow[d, "\mu_1"] & (\mathcal{X}_2)_K \arrow[r] \arrow[d, "\mu_2"] & \cdots \arrow[r] & (\mathcal{X}_\ell)_K \arrow[d, "\mu_\ell"] \\
\mathcal{X}_{0,K} \arrow[r, "f_{0,K}"] & \mathcal{X}_{1,K} \arrow[r, "f_{1,K}"] & \mathcal{X}_{2,K} \arrow[r] & \cdots \arrow[r] & \mathcal{X}_{\ell,K}
\end{tikzcd}$$

where the downward maps are birational maps

such that $f_i : \delta_i \mapsto \delta_{i+1}$ has relative dimension r_i.

THIS IS WRONG Furthermore, if $\mathcal{X} \to \mathrm{Spec}\,(R)$ admits a section through $\overline{\{\delta\}}$ then the map on closures $\overline{\{\delta_i\}} \to \overline{\{\delta_{i+1}\}}$ has geometrically connected fibers.

Note that $\overline{\{\delta_0\}} \to \overline{\{\delta\}}$ is birational since $\mathcal{X}$ is normal so $\epsilon : \mathcal{X}_0 \to \mathcal{X}$ is an isomorphism over a codimension 2 subset and hence over δ.

Remark. If κ has characteristic zero, when there is a section, the conclusion implies that $\overline{\{\delta_i\}} \to \overline{\{\delta_{i+1}\}}$ is a fibration. Otherwise, we may have to worry about inseparable maps which have nonreduced generic fiber.

Proof. This is a repeated application of the lemma of Abhyankar and Zariski [**?**, Lemma 2.22] and some form of resolution of singularities. We proceed by induction on ℓ. For $\ell = 0$ there is nothing to prove. Now suppose we have proved the lemma for $\ell - 1$. Let B be any normal integral flat proper model of $\mathcal{X}_{1,K}$ over R. By [**?**, Lemma 2.22] there is a birational modification $g : \overline{B} \to B$ such that $\mathcal{X} \dashrightarrow \overline{B}$ sends δ to a regular codimension 1 point $\delta_1' \in \overline{B}$ Then apply the inductive hypothesis to $\delta_1 \in \overline{B}$ to produce the data:

(a) a sequence of fibrations

$$\mathcal{X}_1 \xrightarrow{f_1} \mathcal{X}_2 \xrightarrow{f_2} \mathcal{X}_3 \to \cdots \to \mathcal{X}_\ell$$

(b) a sequence $\delta_i \in (\mathcal{X}_i)_\kappa$ of generic points of normal irreducible components

(c) a birational map $\epsilon_1 : \mathcal{X}_1 \to \overline{B}$ such that $\epsilon_1 : \delta_1 \mapsto \delta_1'$

(d) a diagram,

$$\begin{tikzcd}
(\mathcal{X}_1)_K \arrow[r, "(f_1)_K"] \arrow[d, "\mu_1"] & (\mathcal{X}_2)_K \arrow[r, "(f_2)_K"] \arrow[d, "\mu_2"] & \cdots \arrow[r] & (\mathcal{X}_\ell)_K \arrow[d, "\mu_\ell"] \\
\mathcal{X}_{1,K} \arrow[r, "f_{1,K}"] & \mathcal{X}_{2,K} \arrow[r, "f_{2,K}"] & \cdots \arrow[r] & \mathcal{X}_{\ell,K}
\end{tikzcd}$$

where the downward maps are birational maps

Now we produce a modification $\epsilon : \mathcal{X}_0 \to \mathcal{X}$ so that $f_0 : \mathcal{X}_0 \to \mathcal{X}_1$ is a morphism using the following standard argument. Consider the normalization of the closure of the graph of $\mu_1^{-1} \circ f_{0,K} \circ \gamma^{-1} : \mathcal{X} \dashrightarrow \mathcal{X}_1$, which we will denote by Γ. This admits a morphism

$$\Gamma \to \mathcal{X} \times_R \mathcal{X}_1$$

over R, with the projection maps π_1, π_2. By further modifying Γ we can assume $\overline{\{\delta\}}$ is normal. Indeed apply Cesnavicius's Macaulayfication CITE and Lipman's argument for resolution of codimension 2 singularities CITE. Hence we obtain $\epsilon : \mathcal{X}_0 \to \mathcal{X}$ and a morphism $f_0 : \mathcal{X}_0 \to \mathcal{X}_1$ satisfying all the required properties. Indeed, since $(f_0)_{K*}\mathcal{O}_{(\mathcal{X}_0)_K} = \mathcal{O}_{(\mathcal{X}_1)_K}$ because the same holds for $f_{0,K}$. Since $\mathcal{X}_0$ is integral so $\mathcal{O}_{\mathcal{X}_1} \hookrightarrow (f_0)_*\mathcal{O}_{\mathcal{X}_0}$ is an extension of sheaves of domains that is an isomorphism at the generic point so by normality of $\mathcal{X}_1$ it is an isomorphism. Thus f_i have geometrically connected fibers hence the same is true of $\overline{\{\delta_i\}} \to$ FUCK

This concludes the proof by induciton. ◻

Proposition 5.0.5. Let $\mathcal{X} \to \operatorname{Spec}(R)$ be a flat proper morphism with $\mathcal{X}$ a normal integral scheme. For each irreducible component $X \subset \mathcal{X}_\kappa$ of the geometric generic fiber there is an inequality:

$$c_{\text{conic}}(X) \leq c_{\text{conic}}(\mathcal{X}_{\overline{K}})$$

Proof. Cf. the proof of Proposition 3.6 of [CCJS23].

The hypothesis gives a sequence,

$$\mathcal{X}_K = \mathcal{X}_{0,K} \to \mathcal{X}_{1,K} \to \cdots \to \mathcal{X}_{\ell,K}$$

where $n - c$ of these are conic bundles.

Applying Lemma 5.0.4 with the generic point $\delta \in X \subset \mathcal{X}_\kappa$ of the irreducible component we get a sequence,

$$\mathcal{X}_0 \to \mathcal{X}_1 \to \cdots \to \mathcal{X}_\ell$$

with the data as in the lemma including a sequence of points $\delta_i \in (\mathcal{X}_i)_\kappa$. Applying (CITE) to those $\mathcal{X}_i \to \mathcal{X}_{i+1}$ for which the generic fiber is a conic bundle we see that $\overline{\delta_i} \to \overline{\delta_{i+1}}$ is also a conic bundle. Hence X has a fibration sequence with at least $n - c$ conic bundles so we conclude. ☐

Proposition 5.0.6. Let $X \to Y$ be a conic bundle between varities over k. Then the map

$$f^* : H^i_{\mathrm{nr}}(Y/k, A) \to H^i_{\mathrm{nr}}(X/k, A)$$

has 2-torsion kernel and cokernel. In particular, if 2 in invertible on A then f^* is an isomorphism.

Proof. Indeed, there is an étale open $u : U \to Y$ with dense image such that $X_U := X \times_Y U \cong U \times \mathbb{P}^1$ where $U \to Y$ has degree 2. The pull-push formula and functoriality for unramified cohomology gives:

$$\begin{tikzcd}
H^i_{\mathrm{nr}}(Y/k, A) \arrow[r, "f^*"] \arrow[d, "u^*"] & H^i_{\mathrm{nr}}(X/k, A) \arrow[d, "u'^*"] \\
H^i_{\mathrm{nr}}(U/k, A) \arrow[r, "f'^*"] \arrow[d, "u_*"] & H^i_{\mathrm{nr}}(X_U/k, A) \arrow[d, "u'_*"] \\
H^i_{\mathrm{nr}}(Y/k, A) \arrow[r, "f^*"] & H^i_{\mathrm{nr}}(X/k, A)
\end{tikzcd}$$

where the composition of the downward arrows is multiplication by 2. But $f'^* : H^i_{\mathrm{nr}}(U/k, A) \to H^i_{\mathrm{nr}}(X_U/k, A)$ is an isomorphism by (SHREIDER Lemma 4.5). Hence the multiplication by 2 map on the kernel and cokernel of f^* is zero. ☐

Definition 5.0.7. Let X/L and Y/k be varities. We say that X *degenerates to* Y if there exists a DVR R and a flat scheme $\mathcal{X} \to \mathrm{Spec}\,(R)$ such that DO THIS PROPERTLY

Theorem 5.0.8. Suppose that X/L degenerates to $Y/\bar{k}$ and $H^i_{\mathrm{nr}}(Y/\bar{k}, A) \neq 0$ where 2 is invertible on A. Then $c_{\mathrm{conic}}(X/L) > i$.

Proof. ☐

6 GAeL Talk

Joint work w/ Nathan Chen and Junyan Zhao.

Let's start with a general question: given an embedding

$$X \hookrightarrow \mathbb{P}^N$$

with X an n-dimensional variety. If I slice X by linear spaces Λ of dimension $n+1$ then I get a family of curves $C := X \cap \Lambda$ covering X with $\deg C = \deg X$.

the fundamental question:

Q: do there exist curves on X which are "simplier" than the linear slices.

our main result confirms a folklore conjecture that for a general complete intersection of large degree, we have $\deg C \geq \deg X$ so the linear slices are curves of minimal degree

Theorem A (Chen-C-Zhao, '24). *Let $X \subseteq \mathbb{P}^{n+r}$ be a general complete intersection variety of dimension $n \geq 1$ cut out by polynomials of degrees $d_1, \ldots, d_r \geq 2n$. Then any curve $C \subseteq X$ satisfies*

$$\deg(C) \;\geq\; (d_1 - 2n + 1) \cdots (d_r - 2n + 1).$$

Moreover, there exists $N := N(n, r)$ such that if $d_1, \ldots, d_r \geq N$, then

$$\deg(C) \;\geq\; d_1 \cdots d_r.$$

Besides intrinsic interest, our motivation is a conjecture of Bastianelli–De Poi–Ein–Lazarself–Ullery [BDELU17] on the measures of irrationality of complete intersections.

6.1 Measures of Irrationality

These are quantitative measures of "how far from being rational" a variety.

For a projective variety X of dimension n, the *degree of irrationality* and the *covering gonality* are defined as follows:

$$\operatorname{irr}(X) \;:=\; \min\Big\{\delta > 0 \mid \exists \text{ rational dominant map } X \dashrightarrow \mathbb{P}^n \text{ of degree } \delta\Big\};$$

$$\operatorname{cov.gon}(X) \;:=\; \min\Big\{c > 0 \mid \exists \text{ a curve of gonality } c \text{ through a general point } x \in X\Big\}.$$

From their descriptions, we see that the degree of irrationality is a measure of how far X is from being rational, while the covering gonality is a measure of how far X is from being uniruled. These are related by:

$$\operatorname{irr}(X) \geq \operatorname{cov.gon}(X)$$

For me, irr is the more fundamental measure. However, in practice cov. gon is much easier to study. Since we are interested in lower bounds, it suffices to bound cov. gon

BDELU prove that for a general hypersurface $X_d \subset \mathbb{P}^{n+1}$ then $\operatorname{cov.gon}(X_d)$ (and hence $\operatorname{irr}(X)$) is asymptotically $\sim d$. Their method can prove if $X_{d_1,\ldots,d_r} \subset \mathbb{P}^{n+r}$ is a general complete intersection then $\operatorname{cov.gon}(X_{d_1,\ldots,d_r}) \gtrapprox d_1 + \cdots + d_r$ an *additive* bound. They ask: are there *multiplicative bounds*

$$\operatorname{cov.gon}(X_{d_1,\ldots,d_r}) \geq C d_1 \cdots d_r$$

We prove this conjecture and give the sharpest possible constant $C = 1$.

Theorem B (Chen-C-Zhao, '24). *For any $0 < \epsilon \ll 1$, there exists an integer $N_\epsilon = N(\epsilon, n, r) > 0$ such that if $d_1, \ldots, d_r \geq N_\epsilon$, then*

$$\operatorname{cov.gon}(X_{d_1,\ldots,d_r}) \;\geq\; (1 - \epsilon) \cdot d_1 \cdots d_r.$$

It turns out that our proof Theorem B depends on Theorem A.

6.2 Idea of the Reduction of Theorem B to Theorem A

The standard technique for bounding covering gonality uses sections of the canonical bundle. If

$$\begin{tikzcd}
\mathbb{P}^1 \times S \arrow[dr] & \mathcal{C} \arrow[l, "c"'] \arrow[r] \arrow[d] & X \\
& S &
\end{tikzcd}$$

is a family of gonality $\leq c$ curves covering X and $\omega_1, \dots, \omega_r \in H^0(X, \omega_X)$ are some top-forms then we can pull these back to $\mathcal{C}$ and trace them along $\mathcal{C} \to \mathbb{P}^1 \times S$. If the sections $\omega_1, \dots, \omega_r$ separate c points then by chosing these points to be a fiber of the gonal map of the image of some general $\mathcal{C}_s$ in X we see that the trace is nonzero. Since $\mathbb{P}^1 \times S$ has no global top forms this is a contradiction.

Therefore we conclude:

$$H^0(X, \omega_X) \text{ separates } m \text{ points} \implies \operatorname{cov.gon}(X) \geq m + 1$$

Next, how do we show that $H^0(X, \omega_X)$ separates lots of points on a complete intersection $X \subset \mathbb{P}^{n+r}$. We're going to use a trick: write $X = Y \cap D$ for $D \in |dH|$ and Y is a complete intersection of one lower codimension then $K_X = (K_Y + X)|_X$ so it is enough to show the stronger property: $|K_Y + dH|$ separates m points on Y. This is the same as showing that

$$H^0(Y, \mathcal{O}_Y(K_Y + dH)) \to H^0(Y, \mathscr{I}_{p_1,\dots,p_m} \otimes \mathcal{O}_Y(K_Y + dH))$$

is surjective so it suffices to show $H^1(Y, \mathscr{I}_{p_1,\dots,p_m} \otimes \mathcal{O}_Y(K_Y + dH)) = 0$. Blowuping up the points, this is the same as showing that

$$H^1(\mathrm{Bl}_{p_1,\dots,p_m} Y, \mathcal{O}_Y(\pi^* K_Y + dH - (E_1 + \cdots + E_m))) = H^1(\mathrm{Bl}_{p_1,\dots,p_m}, \omega_{\mathrm{Bl}Y} \otimes \mathcal{O}(dH - n(E_1 + \cdots + E_m))$$

is zero. The strategy of Chen '23 is to show that $\mathcal{O}_{\mathrm{Bl}Y}(dH - n(E_1 + \cdots + E_m)$ is big and nef and use Kawamata-Vieweg vanishing. Another way of saying this is is we need to compute multi-point Seshadri constants of Y. Chen '23 shows how we can do this and get a multiplicative bound on $\operatorname{cov.gon}$ provided that we know a lower bound on the degrees of curves on Y.

However, the bound you get this way is not sharp ($C < 1$). In order to get a better bound, we replace $\mathscr{I}_{p_1,\dots,p_m}$ by a suitable multiplier ideal and use Nadel vanishing instead. This improves the method of Anghern-Siu in the limit of many points.

6.3 Proof of Theorem A

The idea is very simple, we break our complete intersection $X \rightsquigarrow X_1 \cup_Z X_2$ into two complete intersections with $\deg X_1 + \deg X_2 = \deg X$. Crutially we do this so that, Z is also a complete intersection of one higher codimension: say by degenerating one of the equations cutting out X into a product of two lower degree equations. Start with some curve $C \subset X$ and degenerate it to a curve $C' \subset X_1 \cup_Z X_2$.

By wishful thinking: suppose that when we break X the curve C' has a component on each side then by breaking off planes over and over we would immediately win. Unfortunately, this is not true. Consider the degeneration of the lines on a quadric suface to the union of two planes. *DRAW PICTURE*

However, by slightly less wishful thinking: let's assume that only one of two cases can occur:

Then we immediately get:

$$\operatorname{min.deg}(X) \geq \min\{\operatorname{min.deg}(X_1) + \operatorname{min.deg}(X_2), \operatorname{min.deg}(Z)\}$$

Because X_1, X_2, Z are also complete intersections, this alows us to do a complicated induction on degrees, codimension, and dimension simultaneously to prove the bound,

$$\operatorname{min.deg}(X) \geq (d_1 - 2n + 1) \cdots (d_r - 2n + 1)$$

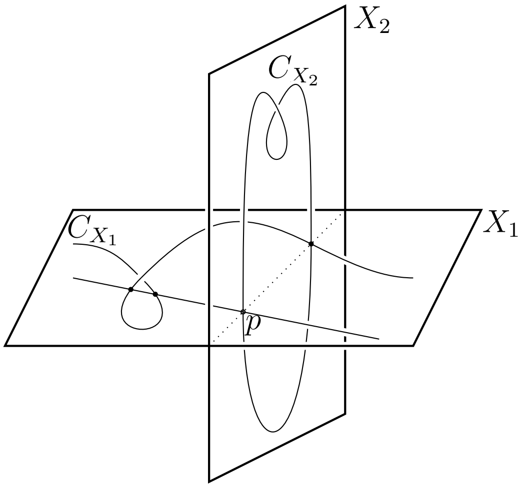

Figure 1: Case (a)

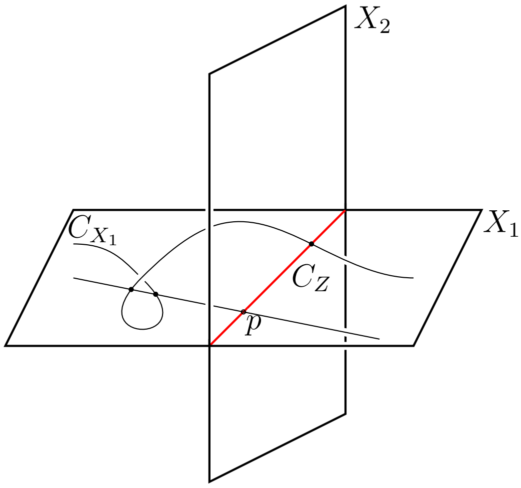

Figure 2: Case (b)

To make this work, we need to show these are the only possible degenerations. The critical input is an idea of Jun Li: if $\mathcal{X}$ is the total family of the degeneration and we have a nodal curve $C' \to \mathcal{X}_0 = X_1 \cup_Z X_2$ which deforms to the generic fiber. Suppose $p \in C'$ is a point of the curve meeting Z then p must be a node meeting two components C_1, C_2 where $C_i \subset X_i$ and they meet Z at p with the same multiplicity *provided* the following are satisfied:

(1) no component of C' meeting p is contained in Z

(2) $p \in \mathcal{X}$ is a smooth point of the *total space*.

Hence as long as C' meets Z at some point outside the singular locus of $\mathcal{X}$ then only cases (a) or (b) above are possible and hence the induction goes through. However, $\mathcal{X}$ always has singular points occuring at the intersection of the equations for X, X_1, X_2 and if we resolve these singularities then after degeneration our curve may get lost in the new exceptional components / divisors.

The trick is to instead consider the same induction for computing the minimal degree of curves on X that *cover* X. Then since these curves pass through a general point, we can always find one that hits Z outside of $\mathcal{X}^{\text{sing}}$ so the induction works. Then a trick of Reidl-Yang reduces the problem of computing $\min.\deg(X)$ to computing the degrees of curves that cover X.

7 Gael

Fujino: on the quasi-albanese map.

We construct a compactification of G as follows. Let $\mathbb{G}_m^r \hookrightarrow \mathbb{P}^r$ be the toric embedding. Then we consider

$$Z = (G \times \mathbb{P}^r)/\mathbb{G}_m^r$$

which is a $\mathbb{P}^r$-bundle over A. We claim this is the projectivization of the vector bundle associated to the $\mathbb{G}_m^r$-bundle over A (which is a product of line bundles). Note that $\mathbb{G}_m^r \hookrightarrow \mathbb{P}^r$ is the same as $\mathbb{G}_m^r \to \mathbb{A}^{r+1} \backslash \{0\}$ modulo the central $\mathbb{G}_m$. Therefore, we can see that

$$Z = (G \times \mathbb{P}^r)/\mathbb{G}_m^r = \left([(G \times \mathbb{G}_m) \times \mathbb{A}^{r+1}] \backslash \{0\} \right) / \mathbb{G}_m^{r+1} = \mathbb{P}(E \oplus \mathcal{O})$$

where $E = [(G \times \mathbb{G}_m) \times \mathbb{A}^{r+1}]/\mathbb{G}_m^{r+1}$ is the associated bundle and this is a product of line bundles.

Example 7.0.1. Let $A = E$ be an elliptic curve and $r = 1$ then there is a sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow E \longrightarrow 0$$

then $Z = \mathbb{P}(\mathcal{O} \oplus \alpha)$ where $\alpha \in \mathrm{Pic}^0_E$ and Δ is a union of the two sections.

Remark. If we want to preserve a product structure, we could choose Z instead associated to $\mathbb{G}_m^r \hookrightarrow (\mathbb{P}^1)^r$. Indeed, this is the same as taking $E = L_1 \oplus \cdots \oplus L_r$ and then letting

$$Z = \mathbb{P}(L_1 \oplus \mathcal{O}) \times_A \cdots \times_A \mathbb{P}(L_r \oplus \mathcal{O})$$

Let G be a semi-abelian group. Then there is a compactification $G \subset Z$ with a map $Z \to A$ making it a projective bundle of rank the rank of the torus. Furthermore

$$\Omega_Z(\log Z \backslash G) = \mathcal{O}_Z^{\oplus \dim G}$$

so

$$\bar{q}(G) = \dim G \qquad \bar{p}_m(G) = 1$$

If V is smooth quasi-projective then there exists a pair

$$(\mathrm{Alb}(V), \mathrm{alb}(V))$$

which is universal for maps to semi-abelian varities and $\dim \mathrm{Alb}(V) = \bar{q}(V)$. This is construced via compactifying $V \hookrightarrow X$ with boundary D and taking

$$\mathrm{Alb}(V) = H^0(X, \Omega_X(\log D))^\vee / H_1(V, \mathbb{Z})_{\text{tors-free}}$$

Furthermore, any semi-abelian variety is of the form

$$G \cong \mathbb{C}^{\dim G}/\pi_1(G)$$

and $\pi_1(G)$ is free-abelian of rank $2 \dim A + r$ where A is the abelian part and r is the rank of the torus. Note that there is a map $V \hookrightarrow X$ which induces $\mathrm{Alb}(V) \to \mathrm{Alb}(X)$ this is the abelian part corresponding to

$$H^0(X, \Omega_X(\log D))^\vee \twoheadrightarrow H^0(X, \Omega_X)^\vee$$

Proposition 7.0.2. Let $H \subset G$ be a smooth subvariety and G a semi-abelian variety. Then the following are equivalent:

(a) H is a translate of a semiableian subvarity

(b) $\bar{\kappa}(H) = 1$ (meaning $\bar{p}_m(H) = 1$ for all $m \gg 0$)

(c) $H^0(\overline{V}, \Omega^i(\log \overline{V} \backslash V)) = \binom{\dim H}{i}$

(d) $\bar{p}_m(H) = 1$ for all $m \geq 0$

7.1 Right Birationality For Open Varities

Definition 7.1.1. A map if weakly proper birational if it is the composition of proper birational maps, codimension ≥ 2 opens, and their rational inverses.

Two varities are WPB-equivalent iff there is a WPB-map between them. WPB-equivalence implies they have the same log-invariants.

Remark. WPB-equivalence is not saturated. Not sure what an example is.

Definition 7.1.2. $f : U \to V$ is WWPB if f is a compoisition of things in the saturation and their inverses. Indeed, let

$$\mathcal{W} = \{f : U \to V \mid \exists g : V \to W \text{ or } h : W \to V \text{ such that } g \circ f \text{ or } f \circ h \text{ is WPB}\}$$

Then WWPB is the localization of the inverses of maps in $\mathcal{W}$. We can continue to get WWWPB etc.

W^∞PB maps preserve log invariants.

Proposition 7.1.3. If $f : V \dashrightarrow U$ is WWPB between affine varities then f is an isomorphism.

Theorem 7.1.4. If V is a smooth quasi-projective surface with $\bar{q}(V) = 2$ and either

(a) $q(\overline{V}) > 0$ and $\bar{p}_1(V) = \bar{p}_2(V) = 1$

(b) $q(\overline{V}) = 0$ and $\bar{p}_1(V) = \bar{p}_2(V) = \bar{p}_3(V) = 1$

then V is WWPB equivalent to a semi-abelian variety.

Corollary 7.1.5. If V is affine and $\bar{p}_1(V) = \bar{p}_3(V) = 1$ and $\bar{q}(V) = 2$ then $V \cong \mathbb{G}_m^2$.

7.2 June 19

A *pair* (X, D) is a projective normal variety X with a $\mathbb{Q}$-divisor D which we write

$$D = \sum b_i D_i$$

We require that $K_X + D$ is $\mathbb{Q}$-Cartier. Notation: $\pi : (Y, B_Y) \to (X, D)$ is a resolution such that $D_Y - \pi_*^{-1} D$ is actually a divisor with smooth (not just SNC!) support. Then we write

$$K_Y + D_Y = \pi^*(K_X + D) + \sum_i a_i E_i$$

and these a_i are the discrepancies. If we choose such a resolution, we don't need to take into account the coefficients of D_Y nor do we need to check on all resolutions like plt etc. For klt we still need the assumption that the coefficients of D are < 1.

Definition 7.2.1. Let X be a variety. Then X is *demi-normal* if X is S_2 and X has at worst SNC singularities in codimension 1.

On a demi-normal variety, there is a well-defined K_X because we can do it at all codimension 1 points.

If X is demi-normal and K_X is $\mathbb{Q}$-Cartier then

$$\nu : (X^\nu, \Delta^\nu) \to X$$

we can pullback and define

$$K_{X^\nu} + \Delta^\nu = \nu^* K_X$$

Definition 7.2.2. A variety X is slc (semi-log canonical) if X is demi-normal, $\mathbb{Q}$-Gorenstein, and (X^ν, Δ^ν) is log canonical.

8 Log Blowups

Blowups have two problems:

(a) don't commute with base change

(b) functor of points is nasy

Log blowups solve both problems.

Recall: $I \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals then $\mathrm{Bl}_I X$ is the terminal objectin the category of $t : T \to X$ such that $t^{-1}I \cdot \mathcal{O}_T$ is invertible.

The log blowup: let $X = (X, P_X, \alpha)$ be a log-scheme. Recall $\overline{P}_X = P_X / P_X^\times$ is a finitely generated monoid. Let I be a sheaf of ideals in $\overline{P}_X$. Then $\mathrm{Bl}_I X$ is the terminal object in the category of log schemes $t : T \to X$ such that $t^{-1}I \cdot \overline{P}_T$ is locally generated by one element.

Proposition 8.0.1. The log blowup:

(a) commutes with (strict) base change

(b) it represents the functor

$$(T \to X) \mapsto \begin{cases} * & t^{-1}I \cdot \overline{P}_T \text{ is locally generated by one element} \\ \varnothing & \text{else} \end{cases}$$

Note, the second property means that it is a log-monomorphism.

If X_σ is affine toric, $S_\sigma = \sigma^\vee \cap M$ and $X_\sigma = \mathrm{Spec}\,(R[S_\sigma])$. If $I \subset S_\sigma$ is an ideal, generates an ideal in $\overline{P}_{X_\sigma}$. Then the log blowup is

$$\mathrm{Bl}_I X_\sigma = \mathrm{Bl}_I X_\sigma = X_{\sigma_I}$$

is just the ordinary (toric) blowup.

Recall that strict morphism $f : X \to Y$ of log schemes is one such that $P_X = f^* P_Y$.

8.1 Non-toric example

Let $X = (\operatorname{Spec}(k), \mathbb{N}^2 \oplus k^\times) \to \mathbb{A}^2$ which is a strict morphism. Then let $I = ((1,0),(0,1)) \subset \mathbb{N}^2$ then the blowup of $\mathbb{A}_2$ at 0 pulls back to the blowup at I. Furthermore, the log structure on the pullback $\mathbb{P}^1$ has two distinguished points $0, \infty$ the log structure is $\mathbb{N}$ generically and $\mathbb{N}^2$ at the two distinguished points.

8.2 Moduli of log-curves

Fix positive integers $2g - 2 + n > 0$.

Definition 8.2.1. A *stable n-marked curve of genus g* is a tuple $(\pi : C \to S, p_1, \dots, p_n, p_i : S \to C)$ such that

(a) π is proper, flat, finitely presented

(b) the p_i are disjoint passing through the smooth locus of π

(c) the geometric fibers of π are connectded nodal curves of genus g

(d) the automorphism groups of the fibers $(C_t, p_1, \dots, p_n)$ is finite, equivalently the log-canonical bundle $\omega_\pi(p_1 + \cdots + p_n)$ is ample.

Then $\overline{\mathcal{M}}_{g,n}$ represents the functor sending T to families of stable n-marked curves over T.

(a) $\mathcal{M}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}$ is an open immersion with

(b) $\Delta := \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ is a normal crossings divisor and $\overline{\mathcal{M}}_{g,n}$ is smooth

(c) $\overline{\mathcal{M}}_{g,n}$ is proper

(d) $\dim \overline{\mathcal{M}}_{g,n} = 3d - 3 + n$.

Proof. Of properness. Consider $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ this is a proper map because it is the universal curve and $\overline{\mathcal{M}}_{g,n}$ is proper so we win by induction (unless $g = 0, 1$ in which case we didn't talk about $\overline{\mathcal{M}}_{g,0}$). □

9 Generic Vanishing

$a : X \to A$ inducing a surjection $\mathrm{Alb}_X \to A$. Then consider,

$$V^i(a, \mathscr{F}) := \{\alpha \in \mathrm{Pic}^0_A \mid h^i(X, \mathscr{F} \otimes a^*\alpha) \neq 0\}$$

We say that $\mathscr{F}$ satisfies generic vanishing (is GV) if

$$\operatorname{codim}\left(V^i(a, \mathscr{F})\right) \geq i$$

Theorem 9.0.1 (Hacon)**.** If $\operatorname{codim}(V^i(A, \mathscr{F})) > i$ for all i then $\mathscr{F} = 0$.

This motivates the following definition

Definition 9.0.2. $\mathscr{F}$ is GV_k if

$$\operatorname{codim}\left(V^i(a,\mathscr{F})\right) \geq i+k$$

for all $i > 0$.

Hacon's theorem says we must make no condition for $i = 0$.

Proposition 9.0.3. If $\mathscr{F}$ is GV sheaf and $n = \dim X$ then

$$V^0(X,\mathscr{F}) \supset V^1(X,\mathscr{F}) \supset \cdots \supset V^n(X,\mathscr{F})$$

and moreover, if $W \subset V^i(X,\mathscr{F})$ is a component of codimension $k > i$ then it is also a component of $V^k(X,\mathscr{F})$.

Theorem 9.0.4 (Ein-Lazarsfeld, '97)**.** If X is smooth projective and $p_1(X) = p_2(X) = 1$ then alb_X is surjective.

Key ingredient: Green-Lazarsfeld (Simpson) generic vanishing theorem.

Theorem 9.0.5. Let X be smooth projective and $a : X \to A$ is a map inducing a surjection $\mathrm{Alb}_X \to A$. Let k be the fiber dimension of a then for all $i \geq k$

$$\operatorname{codim}_{A^\vee} V^i(X,\omega_X) \geq i-k$$

and $\mathrm{R}^i a_*\omega_X$ are GV and the components of $V^i(X,\omega_X)$ and of $V^i(A,a_*\omega_X)$ are torsion-translates of subtori.

Proof of Ein-Lazarsfeld '97. Apply the above theorem to $\mathrm{alb}_X : X \to \mathrm{Alb}_X$. First $\{0\}$ is an isolated point of $V^0(X,\omega_X)$. We show this by contradiction. If it was positivie dimension then by Simpson, there exists $B \subset \mathrm{Pic}^0_X$ a sub abelian variety of dimension > 0 such that $B \subset V^0(X,\omega_X)$. Consider the multiplication map

$$H^0(X,\omega_X \otimes \alpha) \otimes H^0(X,\omega_X \otimes \alpha^{-1}) \to H^0(X,\omega_X^{\otimes 2})$$

Now for $\alpha \in B$ both terms on the left hand side are nonzero. By assumption $H^0(X,\omega_X^{\otimes 2})$ is one dimensional. This shows that the unique effective divisor in $|2K_X|$ has infinitely many connected components since we can split it for any α. This is a contradiciton.

Now let $q := h^0(X,\Omega_X) = \dim \mathrm{Alb}_X$. Then,

$$V^0(X,\omega_X) = V^0(\mathrm{Alb}_X, \mathrm{alb}_*\omega_X)$$

because pushforward preserves sections and the projection formula. Therefore, $\{0\}$ is an isolated point of $V^0(\mathrm{Alb}_X,\mathrm{alb}_*\omega_X)$. We know $\mathrm{alb}_*\omega_X$ is GV so we can use propagation to say it is a component of $V^q(\mathrm{Alb}_X,\mathrm{alb}_*\omega_X)$. This means $h^q(\mathrm{Alb}_X,\mathrm{alb}_*\omega_X) \geq 1$ but if $\mathrm{alb}_*\omega_X$ has support in dimension $< q$ this would be zero. Therefore alb_X is surjective. ☐

9.1 Popa-Schnell

If (X, D) with D a smooth boundary. Let $a : X \to A$ be as before then $a * \omega_X(D)$ is GV. Shibata then proves the results on components: $V^k(X, \omega_X(D))$ are torsion-translates of sub tori.

Proposition 9.1.1. If V is smooth quasi-projective with $\bar{p}_1(V) = \bar{p}_2(V) = 1$ then $\mathrm{alb}_{\overline{V}} : \overline{V} \to \mathrm{Alb}_{\overline{V}}$ is surjective.

Write $X := \overline{V}$ for some compactification. Recall there is a diagram,

$$\begin{tikzcd} & & V \arrow[r, hook] \arrow[d] & X \arrow[d] & \\ 0 \arrow[r] & \mathbb{G}_m^r \arrow[r] & \mathrm{Alb}_V \arrow[r] & \mathrm{Alb}_X \arrow[r] & 0 \end{tikzcd}$$

If $\bar{q}(V) = q(X) = \dim X$ and $\bar{p}_1(V) = \bar{p}_2(V) = 1$ then alb_X is surjective and generically finite then $p_1(X) \neq 0$. But

$$0 \neq p_1(X) \leq \bar{p}_1(V) = h^0(X, \omega_X(D)) = 1$$

and therefore $p_1(X) = 1$ and $p_2(X) = 1$ by the same argument. Since $q(X) = \dim X$ then by Cheng-Hacon we know alb_X is birational. Therefore, $V \to \mathrm{Alb}_V$ is birational. To show WWPB it suffices to show that every component of D is contracted. This we don't know how to do in arbitrary dimension but we do know how in dimension 2.

10 Lecture 4

Theorem 10.0.1. Let S be a smooth quasi-projective surface with $\bar{q}(S) = 2$ and either

(a) $q(\overline{V}) > 0$ and $\bar{p}_1(V) = \bar{p}_2(V) = 1$ or

(b) $q(\overline{V}) = 0$ and $\bar{p}_1(V) = \bar{p}_2(V) = \bar{p}_3(V)$

then S is WWPB equivalent to a quasi-abelian variety.

Corollary 10.0.2. If S is affine and $\bar{q}(S) = 2$ and $\bar{p}_1(S) = \bar{p}_2(S) = \bar{p}_3(S) = 1$ then $S = \mathbb{G}_m^2$.

Example 10.0.3. Let E be the elliptic curve with $j(E) = 12^3$. This has complex multiplication $\omega : E \to E$ order 3. Consider, a resolution

$$X \to (E \times E)/\langle \omega \times -\omega \rangle$$

then X is an elliptic K3 surface. The fibration has three fibers of type IV^*. Let S be the complement of the singular fibers. Let F_i be the fibers with reduced structure,

$$\bar{p}_m(S) = \dim H^0(X, \omega_X(F_1 + F_2 + F_3)^{\otimes m}) = h^0(X, \mathcal{O}_X(F_1 + F_2 + F_3)^{\otimes m}) = \begin{cases} 1 & m \leq 2 \\ > 1 & m \geq 3 \end{cases}$$

and $\bar{q}(S) = 2$ so we really need the condition on $\bar{p}_3(S)$. The albanese of S is $\mathbb{G}_m^2$ and the map factors through $S \to \mathbb{P}^1 \backslash \{0, 1, \infty\}$.

11 Ideas

(a) Roitman to take quotient by gonality fibers.

12 PCMI Notes

Define the moduli spaces.

Proposition 12.0.1. $\mathcal{M}(X, r, \mathcal{L}, T_i) \to M_B(X, r, \mathcal{L}, T_i)$ is a $\mathbb{G}_m$-gerbe.

TODO

Theorem 12.0.2. If there is one irreducible topological rank r complex local system $\mathbb{L}_{\mathbb{C}}$ with determiniant $\mathcal{L}$ and monodromies in T_i at infinity, then there is a non-empty open subscheme $S^\circ \subset S$ such that for any two closed points $s, s' \in |S|$ of residual characteristic $p \neq p'$ the following hold:

(a) for any prime $\ell \neq p$ there is one arithmetic local system $\mathbb{L}_{\ell,\bar{s}}$ on $X_{\bar{s}}$

(b) which has determinant $\mathcal{L}$, with quasi-unipotent monodromies $T_{i,\ell,\bar{s}}$ at ininifty such that $T_i^{ss} = T_{i,\ell,\bar{s}}^{ss}$

(c) which is irreducible over $\overline{\mathbb{Q}}_\ell$

(d) for $\ell = p$ there is one arithmetic local system $\mathbb{L}_{p,\bar{s}'}$ on $X_{\bar{s}'}$ with (s), (3) where ℓ is replaced by p

(e) for any prime ℓ, the topological pullback $(\mathrm{sp}^{\top}_{\mathbb{C},\bar{s}})^* \mathbb{L}_{\ell,\bar{s}}$ have properties (2) and (3) as topological local systems.

Proof. □

12.1 Grothendieck Specialization for Fundamental Groups

Let $X_S \to S$ be a smooth morphism, where S is any scheme. Consider two field valued points $\eta : \mathrm{Spec}\,(F) \to S$ and $s : \mathrm{Spec}\,(k) \to S$ with the property that $\mathrm{im}\, s$ lies in the Zariski closure of $\mathrm{im}\, \eta$. Therefore, there is an irreducible subscheme $Z \subset S$ such that $s \in Z$ is a point and $\mathcal{O}(Z) \to F$ is injective. Let $\widehat{Z}$ be the completion along s and $F \hookrightarrow \widehat{F}$ a field extension such that $\widehat{F}$ contains $\mathcal{O}(\widehat{Z})$.

If X is not proper, we assume there exists a relative compactification: $X_S \hookrightarrow \overline{X}_S$ such that,

(a) $\overline{X}_S \to S$ is smooth proper,

(b) $\widehat{X}_X \backslash X_S \to S$ is a relative normal crossings divisor with smooth components.

We call this a *good* compactification.

Therefore, we have a diagram

$$\mathrm{Spec}\left(\widehat{F}\right) \to \hat{Z} \leftarrow s$$

together with the scheme over it

$$\begin{tikzcd}
X_{\widehat{Z}} \arrow[r] \arrow[d] & X_{\widehat{Z}} & X_s \arrow[l] \arrow[d] \\
\mathrm{Spec}\left(\widehat{F}\right) \arrow[r] & \widehat{Z} & s \arrow[l]
\end{tikzcd}$$

We denote by $\overline{\widehat{F}} \supset \widehat{F}$ and $\overline{k} \supset k$ algebraic closures, the latter defining a morphism $\bar{s} \to s$. Then, upon choosing an S-point $x_S : S \to X_S$, one defines a specialization homomorphism

$$\mathrm{sp}_{\widetilde{F},s} : \pi_1^t(X_{\widehat{F}}, x_{\widehat{F}}) \to \pi_1^t(X_s, x_s)$$

which is the composite of the functoriality homomorphism

$$\pi_1^t(X_{\widehat{F}}, x_{\widehat{F}}) \to \pi_1^t(X_{\widehat{Z}}, x_s)$$

and the inverse of the base change isomorphism

$$\pi_1^t(X_s, x_s) \xrightarrow{\sim} \pi_1^t(X_{\widehat{Z}}, x_s)$$

Finally, one has the functoriality homomorphism

$$\pi_1^t(X_{\widehat{F}}, x_{\widehat{F}}) \to \pi_1^t(X_F, x_F)$$

which is an isomorphism in restriction to the geometric fundamental groups

$$\pi_1^t(X_{\overline{\widehat{F}}}, x_{\overline{\widehat{F}}}) \xrightarrow{\sim} \pi_1^t(X_{\overline{F}}, x_{\overline{F}})$$

Taken together, this defines the specialization homomorphism

$$\mathrm{sp}_{F,s} : \pi_1^t(X_F, x_F) \to \pi_1^t(X_s, x_s)$$

which, when retricted to the geometric fundamental groups, defines the specialization homomorphism

$$\mathrm{sp}_{\overline{F},\bar{s}} : \pi_1^t(X_{\overline{F}}, x_{\overline{F}}) \to \pi_1^t(X_{\bar{s}}, x_{\bar{s}})$$

Theorem 12.1.1. $\mathrm{sp}_{F,s}$ and $\mathrm{sp}_{\overline{F},\bar{s}}$ are surjective, and $\mathrm{sp}_{\overline{F},\bar{s}}$ induces an isomorphism on the pro-p'-completion.

12.2 Open Questions

13 Some references

(a) Konno invariant this proves that the Konno invariant (fiber gemetric genus of a rational pencil) is approximately the number of canonical sections.

(b) Some birational invariants proves that for K3 the fibering genus and irrationality are asymtotically equal

(c) Holomorphic 2-forms shows that there are lots of examples with two forms with no zeros so they cannot be classified probably.

(d) They show that genus $g \leq 14$ K3 surfaces have irr at most 4 using derived categories and bridgeland stability.

(e) example of a family of CY3s such that the $\mathbb{Q}$-hodge structures are all the same but the integral hodge

(f) Fano Manifolds with some positivity they show some things are covered by rational surfaces

(g) Lazarsfeld prove that Seshadri constantss of smooth surfaces are at least one

(h) Voisin proves that fibering genus of hyperkahler is bounded in terms of the betti number if the Hodge

Some facts about Abelian varities:

(a) Polarization is open in the space of maps

(b) we can lift abelian varities but not always with the polarization

Zero cycles:

(a) talk on mumford's theorem

(b) master's thesis on zero cycles

(c) Interesting survey: Cycles, derived categories, and rationality

(d) zero cycles on surfaces in any characteristic

(e) proves some Chow groups are not representable using Hodge theory

To understand for writing the appendix:

(a) Nisnevich motive of a stack

(b) Blakers-Massey need to find a reference in $\mathbb{A}^1$-homotopy theory

(c) here is Roy's master list

(d) https://arxiv.org/pdf/1902.08857

What we need to understand four-fold contractions:

(a) contractions always have rational singularities

(b) Towards Kotschick conjecture Schireieder

(c) The classification paper of Hao and Schreieder

(d) flips for 3-folds and 4-folds not sure what this is for

(e) extremal rays in higher dimension gives the general fibers of the contractions

(f) small contractions of four-dimensional algebaic manifolds (Kawmata) shows that small contractions must contract a locus with positive normal bundle we need this

(g) contractions of smooth varities this gives the $(3, 2)$ case when not constant fiber dimension

(h) gives the $(3, 1)$ case

(i) Mori's MMP for 3-folds

(j) if the picard number is large then no small contractions

14 Quotient Singularities

Lemma 14.0.1. Let $R \to R'$ be a finite étale map of excellent DVRs with fraction field extension K'/K. Let L be the Galois closure of K' over K and S be the integral closure of R' inside L. Then

(a) S is also the integral closure of R inside L

(b) $R' \to S$ and hence $R \to S$ are finite étale.

Proof. Since L/K' is finite, (USE EXCELLENCE HERE) $R' \to S$ is finite so $R \to S$ is finite hence integral. If $x \in L$ is integral over R then it is integral over R' so by definition $x \in S$. We need to show that $R' \to S$ is unramified. Choose an algebraic closure $K \hookrightarrow \overline{K}$ then L is the compositum of all embeddings $\sigma : K' \hookrightarrow \overline{K}$ over K. Therefore, it suffices to prove the following claim. ☐

Lemma 14.0.2. Let L_1, L_2 be field extension of K inside a fixed algebraic closure $\overline{K}$. Let $R \subset K$ be an excellent DVR such that $K = \mathrm{Frac}\,(R)$. Then if L_1/K and L_2/K are unramified in the sense that their integral closures are unramified over R then so is their compositum L_1L_2/K.

Lemma 14.0.3. We put $L_1, L_2 \subset E$ into some Galois extension E/K. I claim there is a maximal intermediate field E

Proof. ☐

CRAP what I actually showed is if you have a finite *quasi-étale* map $U \twoheadrightarrow X$ from a smooth scheme then X has finite quotient singularities. But what if this is ramified in codimension 1. Oops. I guess this was always obvious.

Question: if $U \to X$ is a quasi-etale cover by a smooth scheme then does X have finite quotient singularities?

15 classification and 1-forms

15.1 The main technical result

Theorem 15.1.1. Let $g : X \to S$ be a flat projective morphism whose fibers satisfy

(a) X_s are klt varities (in particular normal irreducible)

(b) $K_{X_s} \sim_{\mathbb{Q}} 0$

and there is a surjective morphism $g : X \to A$ where A is an abelian variety. Then there is an isogeny $\pi : B \to A$ with kernel G such that in the diagram

$$\begin{tikzcd}
F \times B \arrow[dr, dashed, "\sigma"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_A B \arrow[r] \arrow[d] & B \arrow[d, "\pi"] \\
& X \arrow[r] & A
\end{tikzcd}$$

$\sigma : F \times B \xrightarrow{\sim} X \times_A B$ is an isomorphism and hence $X \cong F \times^G B$ where $F = f^{-1}(0_A)$.

15.2 Types of Contractions

15.3 Properties of 1-forms

Definition 15.3.1. Let X be a smooth projective variety (or more generally a Kähler manifold) and $\omega \in H^0(X, \omega_X)$ a global 1-form. We say that

(NWV) ω is *nowhere vanishing* if $\omega_x \in \Omega_{X,x}$ is nonzero for all $x \in X$

(S) ω satisfies *Schreieder's condition* if for every finite étale cover $\tau : X' \to X$ the complex $(H^\bullet(X', \mathbb{C}), \wedge \tau^*\omega)$ is exact.

Note that (NWV) $\implies$ (S) by CITE

15.4 Conjecture: Hao

As an application of these ideas, we also verify the $\dim X = 4$ case of a conjecture of Hao [**?**, Conj. 1.5].

Theorem C. *Let $f : X \to A$ be a morphism from a smooth projective 4-fold X to a simple abelian variety A. The following are equivalent:*

(a) there exists a holomorphic one-form $\omega \in H^0(A, \Omega^1_A)$ such that $f^*\omega$ is nowhere vanishing;

(b) $f : X \to A$ is smooth.

Proof. Note that the cases $\dim A \leq 1$ are trivial so we may assume that $\dim A \geq 2$. We first run MMP to X. □

15.5 Conjecture: Me

Assuming the abundance conjecture, we are also able to prove a conjectue of Chen, Hao, and the author CITE HERE.

Theorem D. *Let X be a smooth projective good minimal model. Then if g is the maximal number of PLI 1-forms on X then there exists a smooth morphism $X \to A$ over an abelian variety of dimension $\dim A = g$.*

Proof. We apply the main construction CITE to conclude that $X \cong (Y \times B)/G$ where B is isogenous to the image of the generic fiber of the Iitaka fibration inside Alb_X. Also by the main construction $\dim B \geq g$ and $G \circlearrowright B$ freely. Therefore, we have a smooth morphism $X \to B/G$. If $\dim B = \dim B/G > g$ then there is a frame of PLI forms on X pulled back from B/G contradicting the maximality of g hence $g = \dim B$ so we conclude. □

15.6 Conjecture: Hao + Schreieder Conjecture 1.8

Assuming the abundance conjecture and termination of flops we can also prove a conjecture of Hao and Schreieder [**?**, HS21(1)]

Theorem E. *Let $X \to A$ be a smooth morphism from a smooth projective variety X to an abelian variety A. If $\kappa(X) \geq 0$ and assuming X admits a good minimal model then there is a birational model*

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & A & \end{tikzcd}$$

with $X' \to A$ an isotrivial smooth projective morphism (i.e. an analytic fiber bundle).

Proof. Let $X^{\min}$ be a good minimal model. We apply the main construction CITE to $X^{\min} \to A$ to conclude that $X^{\min} \cong (Y \times B)/G$ where $B \to A$ is an isogeny. Also by the main construction $\dim B \geq g$ and $G \circlearrowright B$ freely. Let $Y' \to Y$ be a G-equivariant resolution of singularities. Then $X' = (Y' \times B)/G$ gives the requisite model. $\square$

15.7 Conjecture: Hao + Schreieder Conjecture 1.7

16 Upper Triangular Automorphisms

We call a map $\varphi : X \times Y \to X \times Y$ *upper triangular* if there is a diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\varphi"] \arrow[d] & X \times Y \arrow[d] \\ Y \arrow[r, "\sigma"] & Y \end{tikzcd}$$

Lemma 16.0.1. Let X, Y be normal varieties and $\varphi : X \times Y \to X \times Y$ an upper triangular automorphism. Then $\sigma : Y \to Y$ is also an isomorphism.

Proof. By the diagram, it is clear that σ is surjective. Consider the fiber $\sigma^{-1}(y)$ over y then we get an isomorphism,

$$\varphi : X \times \sigma^{-1}(y) \to X \times \{y\}$$

and the only way for these to be isomorphic is $\sigma^{-1}(y)$ to be a reduced point so we win. $\square$

Example 16.0.2. There is an upper triangular bijection $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ with a non-upper triangular inverse,

$$\varphi(x, y) = \begin{cases} (2x, y/2) & y \text{ even} \\ (2x+1, (y-1)/2) & y \text{ odd} \end{cases}$$

Indeed, this is clearly upper triangular and its inverse

$$\varphi^{-1}(x, y) = \begin{cases} (x/2, 2y) & x \text{ even} \\ ((x-1)/2, 2y+1) & x \text{ odd} \end{cases}$$

is *lower* triangular.

17 Meng-Popa Conjecture C

Assuming MMP and adbundance we also prove Conjecture C of [?]

Theorem F. *Let $f : X \to A$ be an algebraic fiber space, with X a smooth projective variety, A an abelian variety, and general fiber F with $\kappa(F) \geq 0$. Assume that F admits a good minimal model. If f is smooth away from a closed set of codimension at least 2 in A, then there exists an isogeny $A' \to A$ such that*

$$X \times_A A' \sim F \times A'$$

i.e. X becomes birational to a product after an étale base change.

Proof. By Lai CITE X admits a good minimal model $X^{\min}$. Then it suffices to show that $X^{\min} \to A \times X^{\mathrm{can}}$ is surjective by MAIN RESULT. But this is imlied by the fact that $X_{\mathrm{can},f} \times_A A' \cong F_{\mathrm{can}} \times A'$ and $X_{\mathrm{can},f} \to X_{\mathrm{can}}$ is surjective. □

18 Hao's Conjecture Redux

Lemma 18.0.1. Suppose that $f : X \to Y$ is a flat map between smooth varieties. Let $y \in Y$ be a point such that X_y is singular (i.e. f is singular at some point over y) then there exists a reduced divisor $Z \subset Y$ containing y and an open set $V \subset Z$ such that $\omega|_V$ is nonvanishing for any 1-form $\omega \in H^0(Y, \Omega_Y)$ with $f^*\omega$ nonvanishing.

Proof. We are going to reduce to the case that f is quasi-finite flat. Let $\dim Y = m$ and $\dim X = n + r$. Let $x \in X_y$ be a singular point and choose a regular sequence $s_1, \ldots, s_m \in \mathfrak{m}_y$ by flatness $f^\# s_1, \ldots, f^\# s_m \in \mathfrak{m}_x$ is a regular sequence. Since X is regular, it can be extended to a regular sequence

$$f^\# s_1, \ldots, f^\# s_m, g_1, \ldots, g_r \in \mathfrak{m}_x$$

such that $\bar{g}_1, \ldots, \bar{g}_r \in \mathfrak{m}_x / \mathfrak{m}_x^2$ are independent. Indeed, $\bar{g}_1$ is not a zero-divisor in $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/(f^\# s_1, \ldots, f^\# s_m, g_1, \ldots, g_r)$ as long as it is not contained in any associated prime. As long as $\mathfrak{m}_x$ is not an associated prime (which it is not because $\mathcal{O}_{X,x}$ is Cohen-Macaulay) then by prime avoidance

$$\mathfrak{m}_x \not\subset \mathfrak{m}_x^2 \cup \bigcup_{\mathfrak{p} \in \mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X_y,x})} \mathfrak{p}$$

so there exists an element g_1 and we repeat to build the requisite sequence.

Set $I = (g_1, \ldots, g_r)$ and note that $\mathcal{O}_{X,x}/I$ is regular so I is prime. Let $X' \subset X$ be the closure of the point corresponding to I. Then $x \in X'$ and X' is smooth at x. Furthermore, $f' : X' \to Y$ has the same tangent rank at x as f. By construction, X'_y is finite so f' is quasi-finite at x' and flat by Tag 00MF. Let $D \subset X'$ be the singular locus of f'. By Zariski-Nagata purity, D is pure of codimension 1 in an open set $U \subset X'$ containing x. Shrinking U, we can ensure $U \to Y$ is quasi-finite flat. Since $X' \to Y$ is quasi-finite, $Z = \overline{f(D \cap V)}$ is a reduced divisor. Let V be the image of the étale locus of $D \cap V \to Z$ which is an open set.

NO FAIL, NO REASON FORMS POINT ALONG Z IF f IS FULL RANK AT THE POINT OOPS. □

What I can prove is that we can assume that f is flat and smooth away from codimension 2 and is a $\mathbb{Z}$-homology bundle for the purposes of Hao's conjecture.

Note also we can assume the fibers are generically reduced and lci hence reduced.

19 Questions to Ponder

(a) for $\dim A = 2$ we just need to show there are no isolated points of $\Delta(f)$. By the Purity result, it would suffice to show that if Z is a positive dimensional component of the singular locus mapped to a point $a \in A$ then its tangent map must surject onto $\mathbb{P}(T_a A)$. How do I show this?

(b)

20 Some Interesting Papers

(a) Ando On the Normal bundle of an Exceptional curve in a higher dimensional algebraic manifold classifies those curves that can be contracted to get an algebraic space in higher dimensions.

(b) Vector Bundles and Adjunction the authors classify ample vector bundles E on X of rank $\dim X - 1$ such that $K_X + \det E$ is not nef. This is used in the classification of fibers of contractions.

(c) Casagrande: Fano 4-folds with a small contraction shows that small contractions only occur for small picard number and gives results on quasi-elementary and special contractions.

(d) Casagrande: Fano 4-folds with rational fibrations shows that small contractions only occur for small picard number and gives results on quasi-elementary and special contractions.

(e) Casagrande: On the birational geometry of Fano 4-folds classifies Fano 4-folds with respect to contraction types

(f) Fano-Mori elementary contractions with reducible general fiber they determine conditions to ensure that if $\varphi : X \to Y$ is a divisorial contraction then $\operatorname{Exc}\varphi \to \varphi(\operatorname{Exc}\varphi)$ has irreducible general fiber.

(g) NODAL CURVES ON SURFACES OF GENERAL TYPE shows that certain Severi varities are smooth of the correct dimension.

21 Some Good Mathoverflow references

(a) Local system on an open with finite monodromy is trivialized by a finite ramified covered of the whol

(b) canonical sheaf on singular variety

(c) canonical bundle of a normal variety look at Sandor's answer

(d) simple abelian varities have effective = nef = ample also discuss when "every effective divisor is ample" implies picard rank 1

(e) singular locus of the discriminant

(f) classifying genus 3 curves

(g) smooth blowups along singular varieties and likewise this question

(h) minimal models are actually minimal

(i) cohomology and base change without noetherian assumptions

(j) Discriminant of a Conic Bundle

(k) Bertini in positive characterstic for basepoint free linear series

22 To Read About maps to Abelian Varieties and Derived Categories

(a) Popa-Schnell Derived Invariance shows that the Picard varities of derived equivalent varities are isogenous.

(b) Leiblich-Olsson prove something about derived equivalence implies the fibers of Iitaka are derived equ

(c) Derived invariance of the Albanese relative canonical ring shows moreover that the relative canonical model over the albanese is somewhat preserved by Derived equivalence.

(d) Kawamata K-equivalence nice survey of what we know for general type varities.

(e) Kawamata derived categories survey

(f) rationality and derived categories

23 To Read About Conic Bundles and del Pezzo Fibrations

(a)

24 Exercises on Coherent Modules

24.0.1 13.6.B

Let A be coherent as an A-module we need to show that if M is finitely-presented then it is coherent. Indeed, let $A^n \to M$ be a map and K the kernel. We need to show that K is finite. Choose a presentation

$$A^r \to A^s \to M \to 0$$

24.0.2 13.8.A

Coherent implies finitely presented implies finitely generated. This is by definition: if M is coherent it is, finitely generated by definition and so taking $A^n \twoheadrightarrow M$ the kernel is finite by coherence hence M is finitely presented.

24.0.3 13.8.B

0 is coherent since for any map $A^n \to 0$ its kernel is A^n and hence finite.

24.1 C-I

For the next exercises, let

$$0 \to M \to N \to P \to 0$$

be an exact sequence of A-modules.

24.1.1 13.8.C

Let N be finitely generated. Then there is a surjection $A^n \twoheadrightarrow N \twoheadrightarrow P$ so P is finitely generated.

24.1.2 13.8.D

Suppose M, P are finitely generated. Lift a generating set of P to N to get a map $A^n \to N$ whose composition with $N \to P$ is surjective. Therefore its image plus M is all of N but M is finite so we win by taking $A^{n+m} \to N$ given by a generating set for M and the lifts.

24.1.3 13.8.E

Let N, P be finitely generated. Then M may not be. Indeed, this holds for any P finite but not finitely preseted. Consider

$$0 \to I \to A \to A/I \to 0$$

for an ideal and $A = k[x_1, x_2, \dots]$ and $I = (x_1, x_2, \dots)$ then I is not finite but the other two are by definition.

24.1.4 13.8.F

Let N be coherent and M finitely generated. We want to show M is coherent. In fact, we just need that M is a fintely generated submodule of N since for any map $A^n \to M$ the kernel is the same as the kernel of $A^n \to M \to N$ since $M \hookrightarrow N$ is injective and this kernel is finitely generated because N is coherent.

24.1.5 13.8.G

Let N, P be coherent. We want to show that M is coherent. First, M is finite because N is finite and P is finitely presented. However, since N, P are moreover coherent we just need $M = \ker (N \to P)$ not that $N \to P$ is surjective. Indeed, for any surjection $A^n \to N$ the kernel of $A^n \to N \to P$ surjects onto M and this is finite since P is coherent.

Now we need to consder a map $A^n \to M$ and prove that it has finite kernel. This follows from before since M is a finitely generated submodule of a coherent module.

24.1.6 13.8.H

Let M be finitely generated and N coherent. We want to show that P is coherent. Since N is finitely generated, it is clear that P is also. Consider a map $A^n \to P$ we need to show it has finite kernel. We can lift these generators and add a generating set for M to get a diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^m & \xrightarrow{\psi} & A^{(n+m)} & \longrightarrow & A^n & \longrightarrow & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow \tilde{\varphi} & & \downarrow \varphi & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0
\end{array}$$

the snake lemma gives an exact sequence

$$0 \to \ker \psi \to \ker \tilde{\varphi} \to \ker \varphi \to 0 \to \operatorname{coker} \tilde{\varphi} \to \operatorname{coker} \varphi \to 0$$

but N is coherent so $\ker \tilde{\varphi}$ is finite and hence $\ker \varphi$ is also finite via the surjection $\ker \varphi \twoheadrightarrow \ker \varphi$.

24.1.7 13.8.I

Let M, P be coherent. We want to show N is coherent. We already showed N is finite so it suffices to consider a map $A^n \to N$ and consider the diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & A^n & \longrightarrow & A^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \tilde{\varphi} & & \downarrow \varphi & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0
\end{array}$$

the snake lemma gives an exact sequence

$$0 \to \ker \tilde{\varphi} \to \ker \varphi \to M$$

Then $\ker \varphi$ is finite since P is coherent. Morover M is coherent so the kernel of the map $\ker \varphi \to M$ is also finite proving the claim.

24.1.8 13.8.J

The direct sum of coherent modules is coherent using the exact sequence

$$0 \to M \to M \oplus N \to N \to 0$$

and the previous exercise.

24.1.9 13.8.K

Let M be finitely generated, N coherent and $\phi : M \to N$ any map. We want to show that $\operatorname{im} \phi$ is coherent. It is finite by definition so we use that a finite submodule of a coherent module is coherent.

24.1.10 13.8.L

Let $\phi : M \to N$ be a map of coherent modules. We want to show that $\ker \varphi$ and $\operatorname{coker} \varphi$ are coherent. Indeed, for the kernel we use that $\ker \varphi$ is finite since it is the kernel of a map from a finite module to a coherent module and that a finite submodule of a coherent module is coherent. For $\operatorname{coker} \varphi$ we use that $\operatorname{im} \phi$ is coherent by the previous exercise and the exact sequence

$$0 \to \operatorname{im} \phi \to N \to \operatorname{coker} \varphi \to 0$$

then gives that $\operatorname{coker} \varphi$ is coherent by H.

24.1.11 13.8.M

Let M, N be coherent submodules of a coherent module P. Consider the map

$$M \oplus N \to P$$

which has image $M + N$ and kernel $M \cap N$. Hence both are coherent.

24.1.12 13.8.N

Let A be coherent. Then A^n is coherent hence any finitely presented module is the cokernel of a map of coherent modules and hence coherent.

24.1.13 13.8.O

Let M be finitely presented and N coherent. Consider $A^m \to A^n \to M \to 0$ and consider

$$0 \to \mathrm{Hom}_A(M, N) \to N^n \to N^n$$

hence $\mathrm{Hom}_A(M, N)$ is a kernel of maps of coherent modules thus coherent.

24.1.14 13.8.P

Let M be finitely presented and N coherent. Consider $A^m \to A^n \to M \to 0$ and consider

$$N^m \to N^n \to M \otimes_A N \to 0$$

hence $M \otimes_A N$ is a cokernel of maps of coherent modules thus coherent.

24.1.15 13.8.Q

If $f \in A$ and M is finite (resp. finitely presented, coherent) then we need to show M_f is the same. This is clear for finite and finitely presented because $(-)_f$ is exact. Furthermore, let M be coherent then M_f is finite. Let $A_f^n \to M_f$ be a map. Clearing denominators, there is a map $A^n \to M$ whose localization has the same image. Since M is coherent this submodule is finitely presented so $A_f^n \to M_f$ has finite kernel.

24.1.16 13.8.R

Let $(f_1, \ldots, f_n) = A$. Let M_{f_i} be finite for all i. We want to show that M is finite. Let $m_{ij} \in M_{f_i}$ be a generating set for each i. We can assume these arise from $m_{ij} \in M$. This gives a map $\bigoplus_{i,j} \to M$ such that its localization at each f_i is surjective. Let Q be the cokernel then $Q_{f_i} = 0$ for all f_i. Hence for each $q \in Q$ there is n so that $f_i^{r_i} q = 0$ and thus $1 \in (f_1, \ldots, f_n)$ implies that 1 can be written as a linear combination in $f_1^{r_1}, \ldots, f_n^{r_n}$ hence $q = 1 \cdot q = 0$.

24.1.17 13.8.S

Let $(f_1, \ldots, f_n) = A$. Let M_{f_i} be coherent for all i. We want to show that M is coherrent.

We know M is finite by above. Let $\phi : A^n \to M$ be a map. Then $(\ker \phi)_{f_i} = \ker \phi_{f_i}$ is finite since M_{f_i} is coherent and hence $\ker \phi$ is finite so M is coherent.

The same works to show that if M_{f_i} is finitely presented then M is finitely presented.

25 Sheaves Stuff

Question: does there exists an fpqc torsor for a reasonable group not representable by an algebraic space?

Lemma 25.0.1. Descent holds along a τ-cover for sheaves in the τ-topology. Explicitly, let $\mathcal{C}_\tau$ be a site and consider the natural map

$$\mathrm{Shv}_S(\mathcal{C}_\tau) \to \mathrm{DD}_{S'/S}(\mathrm{Shv}_{S'}(\mathcal{C}_\tau))$$

is an equivaence of categories.

Remark. Note that $\mathrm{Shv}_S(\mathcal{C}_\tau)$, the slice category of sheaces on $\mathcal{C}_\tau$ over the representable h^S (in presheaves if h^S is not a τ-sheaf), is equivalent to $\mathrm{Shv}(\mathcal{C}_{\tau/S})$ the sheaves on the slice category over S. Indeed, the map $\varphi : F \to S$ gives a map $F(T) \to \mathrm{Hom}\,(T, S)$ so it lives over the slice category already. Conversely, given a sheaf G on the slice category we define F via

$$T \mapsto \{(\alpha, \beta) \mid \alpha : T \to S \text{ and } \beta \in G(\alpha : T \to S)\}$$

Proof. This is just unwinding definitions. For full faithfulness, we need to show that

$$\mathrm{Hom}_S\,(F, G) \to \mathrm{Hom}_{S'}\,(F_{S'}, G_{S'}) \rightrightarrows \mathrm{Hom}_{S' \times_S S'}\,(F_{S' \times_S S'}, G_{S' \times_S S'})$$

is an equalizer. This is exactly the sheaf condition for $\mathrm{Hom}\,(F, G)$. Indeed, let's prove it. Let $\varphi, \psi : F \to G$ be S-morphisms that become equal upon pulling back to S'. For any $T \to S$ consider the cover $T_{S'} \to T$ then $\varphi_{T_{S'}} = \psi_{T_{S'}}$ so by local uniqueness: $\varphi_T = \psi_T$. Now suppose that $\varphi' : F_{S'} \to G_{S'}$ is equalized. Let φ be defined as follows: $\varphi_T(x) \in G(T)$ is obtained by gluing $\varphi_{T_{S'}}(x|_{T_{S'}})$ along $T_{S'} \to T$ which exists because of the overlap condition on φ_T.

Now we prove essential surjectivity. Let (G, α) be a descent datum. We produce a sheaf F as follows. Base changing along $T \to S$ we can replace S by any T so it suffices to produce $F(S)$. Define $F(S)$ as the limit (equalizer) of the diagram

$$\begin{tikzcd}
 & & G(\pi_1 : S' \times_S S' \to S') \arrow[dd, "\alpha"] \\
F(S) \arrow[r] & G(S') \arrow[ur] \arrow[dr] & \\
 & & G(\pi_2 : S' \times_S S' \to S')
\end{tikzcd}$$

□

26 Symbols of Differential Operators and Pseduodifferential Operators

Every citation is to [1] Raymond Wells "Differential Analysis on Complex Manifolds"

26.1 Structures on Manifolds

Let K be a complete valued field (either $\mathbb{R}$ or $\mathbb{C}$ we will care about). For $D \subset K^n$ an open subset we have the following:

(a) $K = \mathbb{R}$

  (a) $\mathcal{E}(D)$ are the C^∞-functions on D

  (b) $\mathcal{A}(D)$ are the *real-analytic* functions on D

(b) $K = \mathbb{C}$

  (a) $\mathcal{O}(D)$ is the complex-valued *holomorphic functions* on D

In general, let $\mathcal{S}$ be a subsheaf of the sheaf of continuous functions on K^n in the standard topology.

Definition 26.1.1 (1, Definition 1.1)**.** An $\mathcal{S}$-*structure* $\mathcal{S}_M$ on a K-manifold M is a subsheaf of the sheaf $\mathcal{C}_M$ of K-valued continuous functions on M such that for any chart (U, φ) of M the natural isomorphism

$$\varphi^\# : \mathcal{C}_{\varphi(U)} \xrightarrow{\sim} \varphi_* \mathcal{C}_M$$

identifies $\varphi_* \mathcal{S}_M$ with $\mathcal{S}_{\varphi(U)}$ defined via the open $\varphi(U) \subset K^n$.

For our three classes of functions we have defined for

(a) $\mathcal{S} = \mathcal{E}$ a *differentiable* (or C^∞) manifold and the function in $\mathcal{E}_M$ are called C^∞-functions on (open subsets of) M

(b) $\mathcal{S} = \mathcal{A}$ a *real-analytic* (or C^ω) manifold and the functions in $\mathcal{A}_M$ are called *real-analytic functions* on (open subsets) of M

(c) $\mathcal{S} = \mathcal{O}$ a *complex-analytic* (or *holomorphic* or simply *complex*) *manifold*, and the functions in $\mathcal{O}_M$ are called *holomorphic* (or *complex-analytic functions*) on (open subsets) of M.

Definition 26.1.2. An $\mathcal{S}$-*morphism* $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is a continuous map $F : M \to N$ suhc that

$$f \in \mathcal{S}_N(N) \implies f \circ F \in \mathcal{S}_M(F^{-1}(U))$$

equivalently the morphism of sheaves $F^\# : \mathcal{C}_N \to F_* \mathcal{C}_M$ induces a morphism of sheaves between the subsheaves $\mathcal{S}_N$ and $F_* \mathcal{S}_M$.

Definition 26.1.3. Let X be an $\mathcal{S}$-manifold. An $\mathcal{S}$-structure on a topological K-vector bundle $\pi : E \to X$ is a $\mathcal{S}$-manifold structure on E such that $\pi : E \to X$ becomes an $\mathcal{S}$-morphism and there exists local trivializations by $\mathcal{S}$-isomorphisms.

26.2 Sobolev Spaces

Recall that there is a sobolev norm for compactly supported differentiable functions $f : \mathbb{R}^n \to \mathbb{C}^m$ defined by

$$||f||^2_{s,\mathbb{R}^n} = \int |\hat{f}(y)|^2 (1 + |y|^2)^s \mathrm{d}y$$

where $\hat{f}$ is the Fourier transform

$$\hat{f}(y) = (2\pi)^{-n} \int e^{-i\langle x,y\rangle} f(x)\mathrm{d}x$$

In $\mathbb{R}^n$ this norm is equivalent to the norm

$$\left[\sum_{|\alpha|\leq s} \int_{\mathbb{R}^n} |D^\alpha f|^2 \mathrm{d}x\right]^{1/2}$$

Then we let $W(\mathbb{R}^n, \mathbb{C}^m)$ to be the completion of $\mathcal{E}(\mathbb{R}^n, \mathbb{C}^m)$ with respect to either norm. Note that $||\bullet||_s$ is defined for all $s \in \mathbb{R}$ but we shall deal only with integral values in our applications.

Let E be a Hermitian vector bundle on X. Let $\mathcal{E}_k(X, E)$ be the C^k-sections of E. Let $\mathcal{D}(X, E) \subset \mathcal{E}(X, E)$ be the compactly supported section. Choosing a strictly positive smooth measure μ on X (e.g. arising from a metric). Then we define an inner product on $\mathcal{E}(X, E)$ via

$$\langle \xi, \eta \rangle = \int_X \langle \xi(x), \eta(x) \rangle_E \,\mathrm{d}\mu$$

where $\langle -, - \rangle_E$ is the Hermitian metric on E. Now we define a Sobolev norm $||\bullet||_s$ on $\mathcal{E}(X, E)$. To do this, we choose a partition of unity $\{\rho_\alpha\}$ subordinate to a finite cover by charts $\{(U_\alpha, \varphi_\alpha)\}$ over which E has a trivialization

$$\begin{tikzcd}
E|_{U_\alpha} \arrow[r, "\tilde{\varphi}_\alpha"] \arrow[d] & \widetilde{U}_\alpha \times \mathbb{C}^m \arrow[d] \\
U_\alpha \arrow[r, "\varphi_\alpha"] & \widetilde{U}_\alpha
\end{tikzcd}$$

where $\varphi_\alpha : U_\alpha \to \widetilde{U}_\alpha \subset \mathbb{R}^n$ is a diffeomorphism.

Finally, we define, for $\xi \in \mathcal{E}(X, E)$

$$||\xi||_{s,E} = \sum_\alpha ||\tilde{\varphi}_\alpha^* \rho_\alpha \xi||_{s,\mathbb{R}^n}$$

We let $W^s(X, E)$ be the completion of $\mathcal{E}(X, E)$ with respect to $||\bullet||_s$. The norm $||\bullet||_s$ defined on $\mathcal{E}(X, E)$ depends on the choice of paritions of unity, the local trivialization, and the Hermitian and metric structure. However, the topology on $W^s(X, E)$ is independent of these choices.

26.3 Differential Operators

Definition 26.3.1 (1, p. 113)**.** Let E, F be differentiable $\mathbb{C}$-vector bundles over a differentiable manifold X. Let

$$L : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$$

be a $\mathbb{C}$-linear map. We say that L is a *differntial operator* if for any choice of local coordinates and local trivializations there exists a linear partial differential operator $\widetilde{L}$ such that the dagram

$$
\begin{array}{ccc}
\mathcal{E}(U)^p & \xrightarrow{\widetilde{L}} & \mathcal{E}(U)^q \\
\| & & \| \\
\mathcal{E}(U, U \times \mathbb{C}^p) & \longrightarrow & \mathcal{E}(U, U \times \mathbb{C}^q) \\
\cup\!\!\uparrow & & \cup\!\!\uparrow \\
\mathcal{E}(X, E)|_U & \xrightarrow{L} & \mathcal{E}(X, F)|_U
\end{array}
$$

commutes. That is, for $(f_1, \ldots, f_p) \in \mathcal{E}(U)^p$ we have

$$\widetilde{L}(\underline{f})_i = \sum_{\substack{i=1 \\ |\alpha| \leq k}} p a_\alpha^{ij} D^\alpha f_j$$

A differential operator is said to be of *order k* if $\widetilde{L}$ can be taken to be of the above form.

Definition 26.3.2. Suppose X is a compact C^∞-manifold. We define $\mathrm{OP}_k(E, F)$ as the the vector space of $\mathbb{C}$-linear mappings

$$T : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$$

such that there is a continuous extension of T

$$\overline{T}_s : W^s(X, W) \to W^{s-k}(X, F)$$

for all s. These are the *operators of order k* mapping E to F.

Proposition 26.3.3 (1, Prop. 2.1)**.** Let $L \in \mathrm{OP}_k(E, F)$. Then L^* exists and moreover $L^* \in \mathrm{OP}_k(F, E)$ and the extension

$$(\overline{L}^*)_s : W^s(X, F) \to W^{s-k}(X, E)$$

is given by the adjoint map

$$(\overline{L}_{k-s})^* : W^s(X, F) \to W^{s-k}(X, E)$$

Proposition 26.3.4 (1, Prop. 2.2)**.** $\mathrm{Diff}_k(E, F) \subset \mathrm{OP}_k(E, F)$

Proof. This is a local calculation and we use

$$\widehat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi)$$

$\square$

26.4 Symbol

We review how [1] defines the symbol. Let $U \subset T^*X$ be the complement of the zero section and $\pi : U \to X$ the projection. For $k \in \mathbb{Z}$ we define

$$\mathrm{Smbl}_k(E, F) = \{\sigma \in \mathrm{Hom}_U(\pi^* E, \pi^* F) \mid \forall \rho > 0, (x, v) \in U : \sigma(x, \rho v) = \rho^k \sigma(x, v)\}$$

Then we define a linear map

$$\sigma_k : \mathrm{Diff}_k(E, F) \to \mathrm{Smbl}_k(E, F)$$

where $\sigma_k(L)$ is called the *k-symbol* of the differential operator L. For $(x,v) \in U$ we define a linear map

$$\sigma_k(L)(x,v) : E_x \to F_x$$

as follows: let $e \in E_x$ be given and choose $g \in \mathcal{E}(X)$ and $f \in \mathcal{E}(X,E)$ such that $\mathrm{d}g_x = v$ and $f(x) = e$ then we define

$$\sigma_k(L)(x,v)e = L\left(\frac{i^k}{k!}(g - g(x))^k f\right)(x) \in F_x$$

Proposition 26.4.1. The symbol map σ_k gives rise to an exact sequence

$$0 \to \mathrm{Diff}_{k-1}(E,F) \to \mathrm{Diff}_k(E,F) \xrightarrow{\sigma_k} \mathrm{Smbl}_k(E,F)$$

26.5 Algebraic Symbols

In the algebraic category, a map $\varphi : \pi^* F \to \pi^* G$ extends to a map over all of T^*X as long as $\dim X \geq 2$ since the zero section has codimension $\dim X$. Therefore, the data of φ is equivalent to the data of

$$\varphi : F \to \pi_* \pi^* G = \begin{cases} G \otimes \bigoplus_{n \geq 0} \mathrm{Sym}^n(T_X) & \dim X \geq 2 \\ G \otimes \bigoplus_{n \in \mathbb{Z}} T_X^{\otimes n} & \dim X = 1 \end{cases}$$

and the k-symbols are those maps that are homogeneous of degree k i.e.

$$\mathrm{Smbl}_k(E,F) = \mathrm{Hom}\left(E, F \otimes \mathrm{Sym}^k(T_X)\right)$$

We can describe the symbol map as follows. The jet bundle or bundle of principal parts (or total symbols) satisfies

$$\mathrm{Diff}_k(E,F) = \mathrm{Hom}\left(J^k(E), F\right)$$

and there is an exact sequence

$$0 \to E \otimes \mathrm{Sym}^k(\Omega_X) \to J^k(E) \to J^{k-1}(E) \to 0$$

and therefore applying $\mathscr{H}om(-,F)$ we get and exact sequence of sheaves

$$0 \to \mathrm{Diff}_{k-1}(E,F) \to \mathrm{Diff}_k(E,F) \xrightarrow{\sigma_k} \mathrm{Hom}\left(E, F \otimes \mathrm{Sym}^k(T_X)\right) \to 0$$

the last map is identified with the symbol map.

26.6 Pseudodifferential Operators

Definition 26.6.1 (1, Def. 3.7)**.** A linear map $L : \mathcal{D}(X,E) \to \mathcal{E}(X,E)$ is a *pseuduodifferential operator* on X if for any coordinate chart (U,φ) trivializing E and F and any open $U' \subset U$ with compact closure there is a $r \times p$ matrix $p^{ij} \in S_0^m(U)$ so that the induced

$$L_U : \mathcal{D}(U')^p \to \mathcal{E}(U)^r$$

via extending by zero to U and applying L is a matrix of canonical pseudodifferential operators d

27 Talk Harvard-MIT

27.1 Pretalk

27.1.1 Stable Maps

27.1.2 Covering Gonality and Separating Points

27.1.3 Multiplier Ideals

27.2 Introduction

The main question: given a projective variety X what is the geometry of the curves on X. More precisely suppose $X \subset \mathbb{P}^N$ has a fixed embedding in projective space. We would like to ask:

(a) what possible values for the numerical invariants of curves on X can appear e.g.

  (a) degree (computed against $\mathcal{O}_X(1)$)

  (b) genus

  (c) gonality

(b) we also have a natural source of curves on X arsing from the embedding: taking a linear space Λ of dimension $N - \dim X - 1$ we get linear slice curves $C_\Lambda := X \cap \Lambda$ that cover X. We would like to know, how close are the curves covering X to linear (or higher degree) slices?

When $X \subset \mathbb{P}^{n+r}$ is a general complete intersection cut out by homogeneous polynomials of degrees $d_1, \dots, d_r$, we write X is CI of type $(d_1, \dots, d_r)$ the following result gives a first step towards these questions:

Theorem G (Chen-C-Zhao, '24)**.** *Let $X \subseteq \mathbb{P}^{n+r}$ be a general complete intersection variety of dimension $n \geq 1$ cut out by polynomials of degrees $d_1, \dots, d_r \geq 2n$. Then any curve $C \subseteq X$ satisfies*

$$\deg(C) \;\geq\; (d_1 - 2n + 1) \cdots (d_r - 2n + 1).$$

Moreover, there exists $N := N(n, r)$ such that if $d_1, \dots, d_r \geq N$, then

$$\deg(C) \;\geq\; d_1 \cdots d_r.$$

Besides intrinsic interest, our motivation is a conjecture of Bastianelli–De Poi–Ein–Lazarself–Ullery [BDELU17] on the measures of irrationality of complete intersections.

27.3 Measures of Irrationality

These are quantitative measures of "how far from being rational" a variety.

For a projective variety X of dimension n, the *degree of irrationality* and the *covering gonality* are defined as follows:

$$\operatorname{irr}(X) \;:=\; \min\Big\{\delta > 0 \mid \exists \text{ rational dominant map } X \dashrightarrow \mathbb{P}^n \text{ of degree } \delta\Big\};$$

$$\operatorname{cov.gon}(X) \;:=\; \min\Big\{c > 0 \mid \exists \text{ a curve of gonality } c \text{ through a general point } x \in X\Big\}.$$

From their descriptions, we see that the degree of irrationality is a measure of how far X is from being rational, while the covering gonality is a measure of how far X is from being uniruled. These are related by:

$$\operatorname{irr}(X) \geq \operatorname{cov.gon}(X)$$

For me, irr is the more fundamental measure. However, in practice cov. gon is much easier to study. Since we are interested in lower bounds, it suffices to bound cov. gon

BDELU prove that for a general hypersurface $X_d \subset \mathbb{P}^{n+1}$ then $\operatorname{cov.gon}(X_d)$ (and hence $\operatorname{irr}(X)$) is asymptotically $\sim d$. Their method can prove if $X_{d_1,\dots,d_r} \subset \mathbb{P}^{n+r}$ is a general complete intersection then $\operatorname{cov.gon}(X_{d_1,\dots,d_r}) \gtrapprox d_1 + \cdots + d_r$ an *additive* bound. They ask: are there *multiplicative bounds*

$$\operatorname{cov.gon}(X_{d_1,\dots,d_r}) \geq C d_1 \cdots d_r$$

We prove this conjecture and give the sharpest possible constant $C = 1$.

Theorem H (Chen-C-Zhao, '24)**.** *For any $0 < \epsilon \ll 1$, there exists an integer $N_\epsilon = N(\epsilon, n, r) > 0$ such that if $d_1, \dots, d_r \geq N_\epsilon$, then*

$$\operatorname{cov.gon}(X_{d_1,\dots,d_r}) \;\geq\; (1-\epsilon) \cdot d_1 \cdots d_r.$$

It turns out that our proof Theorem B depends on Theorem A.

28 Ample Line bundles on Abelian Varities

The following is used implicitly in PS14.

Proposition 28.0.1. Let A be an abelian variety over a field of characteristic $p > \dim X$ and $\mathcal{L}$ an ample line bundle. Then $H^0(A, \mathcal{L}) \neq 0$.

Proof. Note that $\mathrm{Td}_A = 1$ so by Grothendieck-Riemann-Roch

$$\chi(\mathcal{L}) = \int_A \operatorname{char}(\mathcal{L}) = \frac{1}{n!} c_1(\mathcal{L})^n > 0$$

Note this also shows that $\deg_L(A)$ is divisible by $n!$ (hence also the degree of any embedding $A \hookrightarrow \mathbb{P}^N$). Now since A lifts over $W_2(k)$ (see Mumford's book) and $p > \dim A$ Deligne-Illusie applies to get Kodaira vanishing (note we only need to lift A not $\mathcal{L}$ to apply Deligne-Illusie) so

$$H^{>0}(A, \mathcal{L}) = H^{>0}(A, \mathcal{L} \otimes \omega_A) = 0$$

and hence

$$H^0(X, \mathcal{L}) = \frac{1}{n!} c_1(\mathcal{L})^n > 0$$

□