Mathematics GR6261 Commutative Algebra
Assignment # 1

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1 Problem 1

(a)

Let A be a ring and M an A-module. We know that $M_{\mathfrak{p}} = 0$ if and only if for each $m \in M$ then $(m, s) \sim (0, 1)$ iff there exists $u \in S = A \setminus \mathfrak{p}$ such that $u \cdot m = 0$. Therefore if there exists $u \in \text{Ann}_A(M) \cap S$ i.e. if $\text{Ann}_A(M) \not\subset \mathfrak{p}$ then $u \cdot m = 0$ for all m so $M_{\mathfrak{p}} = 0$. Suppose that $\mathfrak{p} \in \text{Supp}_A(A)$ then we know that $M_{\mathfrak{p}} \neq 0$ and thus $\text{Ann}_A(M) \subset \mathfrak{p}$ which means that $\mathfrak{p} \in V(\text{Ann}_A(M))$. Therefore,

$$\text{Supp}_A(A) \subset V(\text{Ann}_A(M))$$

Now, suppose that M is of finite type so that we may write $M = Am_1 + \cdots + Am_r$ for constants $m_i \in M$. If $M_{\mathfrak{p}}$ then we know that for each $m \in M$, in particular for m_i there exist elements $u_i \in S$ such that $u_i \cdot m_i = 0$ for each i. Take $u = u_1 \dots u_r$. Clearly, for any $m \in M$ we can write, $m = a_1 \cdot m_1 + \cdots + a_i \cdot m_i$ and thus

$$u \cdot m = (u_2 \cdots u_r a_1) \cdot (u_1 \cdot m_1) + \cdots + (u_1 \cdots u_{r-1} \cdot a_r) \cdot (u_r \cdot m_r) = 0$$

so $u \in \text{Ann}_A(M)$ but $u \in S$ so $\text{Ann}_A(M) \not\subset \mathfrak{p}$. Therefore, if $\mathfrak{p} \notin \text{Supp}_A(A)$ then $\mathfrak{p} \notin V(\text{Ann}_A(M))$ which implies that,

$$\text{Supp}_A(A) = V(\text{Ann}_A(M))$$

(b)

Assume that A is Noetherian and M is of finite type. By the above, since M has finite type,

$$\text{Supp}_A(A) = V(\text{Ann}_A(M))$$

Now suppose that $\text{Supp}_A(A) \subset V(I)$ for some ideal $I \subset A$ then we have that $V(\text{Ann}_A(M)) \subset V(I)$ and thus if $\text{Ann}_A(M) \subset \mathfrak{p}$ then we know that $I \subset \mathfrak{p}$. This implies that $I \subset \sqrt{\text{Ann}_A(M)}$ because it is contained in every prime ideal above $\text{Ann}_A(M)$ and thus also their intersection. Since A is Noetherian, the ideal I is finitely generated so by Lemma 5.1 we know that there exists a power N such that $I^N \subset \text{Ann}_A(M)$ and thus $I^N \cdot M = 0$.

2 Problem 2

(a)

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $\mathfrak{a}$ is finitely generated and both $A/\mathfrak{a}$ and $A/\mathfrak{b}$ are Noetherian. Since $\mathfrak{a}$ is finitely generated we may write it as $\mathfrak{a} = Aa_1 + \cdots + Aa_n$ for constants $a_i \in \mathfrak{a}$. Consider the map $A^n \to \mathfrak{a}/\mathfrak{a}\mathfrak{b}$ given by $(c_i) \mapsto \left[\sum_{i=1}^n a_i c_i\right]_{\mathfrak{a}\mathfrak{b}}$. Clearly, this map is surjective because $\mathfrak{a}$ is generated by the set $\{a_i\}$ over A. Furthermore, the ideal $\mathfrak{b}^n \subset A^n$ is sent to (0) because if $(b_i) \subset \mathfrak{b}^n$ then $\sum_{i=1}^n a_i b_i \in \mathfrak{a}\mathfrak{b}$. Therefore, this map factors through the quotient as,

$$\begin{tikzcd}
A^n \arrow[rr, two heads] \arrow[dr] & & \mathfrak{a}/\mathfrak{a}\mathfrak{b} \\
& (A/\mathfrak{b})^n \arrow[ur, two heads] &
\end{tikzcd}$$

where the map $(A/\mathfrak{b})^n \to \mathfrak{a}/\mathfrak{a}\mathfrak{b}$ is still surjective. Furthermore, $A/\mathfrak{b}$ is Noetherian so, by Lemma 5.2, $(A/\mathfrak{b})^n$ is also Noetherian and then surjectivity forces the image $\mathfrak{a}/\mathfrak{a}\mathfrak{b}$ to be Noetherian as well. Next, consider the exact sequence given by the third isomorphism theorem,

$$0 \longrightarrow \mathfrak{a}/\mathfrak{a}\mathfrak{b} \longrightarrow A/\mathfrak{a}\mathfrak{b} \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

Because $\mathfrak{a}/\mathfrak{a}\mathfrak{b}$ and $A/\mathfrak{a}$ are Noetherian we must have that $A/\mathfrak{a}\mathfrak{b}$ is Noetherian as well.

(b)

Suppose that A is not Noetherian and consider the poset Σ of all ideals of A which are not finitely generated ordered by inclusion. Since A is not Noetherian Σ is nonempty. Let $\mathcal{I}$ be a chain inside Σ then consider the ideal,

$$U = \bigcup_{I \in \mathcal{I}} I$$

which is, in fact, an ideal because $\mathcal{I}$ is totally ordered so all pairs of elements in U lie in a common ideal in $\mathcal{I}$ (c.f. problem 3). Suppose that U is finitely generated then there must exist some $x_1, \ldots, x_n \in U$ such that $U = Ax_1 + \cdots + Ax_n$. However, each generator must lie in some element of the union, $x_i \in I_i$. Since $\mathcal{I}$ is totally ordered and $\{I_i\}$ is a finite set of ideals, there must exists a maximum I_m of the I_i. Then, $x_i \in I_m$ for all i so $U = Ax_1 + \cdots + Ax_n \subset I_m$ which implies that $U = I_m$ so I_m is finitely generated which is a contradiction. Therefore, $U \in \Sigma$ so every chain has a maximum. Thus, by Zorn's lemma, Σ has maximal elements.

Let $\mathfrak{m} \in \Sigma$ be a maximal element. Consider $x, y \notin \mathfrak{m}$. Then $\mathfrak{m} + (x)$ and $\mathfrak{m} + (y)$ lie above $\mathfrak{m}$ so by maximality neither $\mathfrak{m} + (x)$ nor $\mathfrak{m} + (y)$ can be in Σ and thus are finitely generated. Every ideal of $A/\mathfrak{m}$ is the image of an ideal in A above $\mathfrak{m}$ which must be finitely generated because $\mathfrak{m}$ is maximal in Σ. However, the image of a finitely generated ideal is finitely generated and thus every ideal of $A/\mathfrak{m}$ is finitely generated so $A/\mathfrak{m}$ is Noetherian. Furthermore we have surjective maps, $A/\mathfrak{m} \to A/(\mathfrak{m} + (x))$ and $A/\mathfrak{m} \to A/(\mathfrak{m} + (y))$ so both quotients are Noetherian as well. Let $K = (\mathfrak{m} + (x))(\mathfrak{m} + (y))$ which is finitely generated because both $\mathfrak{m} + (x)$ and $\mathfrak{m} + (y)$ are. By the previous problem, since both are finitely generated, A/K is Noetherian as well. However, $K = \mathfrak{m}^2 + \mathfrak{m}x + \mathfrak{m}y + (xy)$. If $xy \in \mathfrak{m}$ then $K \subset \mathfrak{m}$. Then by Lemma 5.3, since A/K is Noetherian, K is finitely generated, and $K \subset \mathfrak{m}$ we would have that $\mathfrak{m}$ is finitely generated contradincting $\mathfrak{m} \in \Sigma$. Therefore, $xy \notin \mathfrak{m}$ so $\mathfrak{m}$ is a prime ideal (since $\mathfrak{m} \neq A$ since A is obviously finitely generated). Therefore, if A is not Noetherian then there must exist a prime ideal of A is not finitely generated. Equivalently, if every prime ideal of A is finitely generated then A is Noetherian.

3 Problem 3

Let A be a ring and Σ the set of ideals of A containing only zero-divisors. The set Σ is naturally a poset under set inclusion. Let $\mathcal{I}$ be a chain in Σ. Then consider the ideal,

$$U = \bigcup_{I \in \mathcal{I}} I$$

This is an ideal because if $x, y \in U$ then $x \in I$ and $y \in I'$ for $I, I' \in \mathcal{I}$ but $\mathcal{I}$ is totally ordered so either $I \subset I'$ or $I' \subset I$ and thus the larger contains both x and y and therefore the sum and the multiples are in U. Also each $x \in U$ is contained in some ideal $I \in \Sigma$ so x is a zero-divisor and thus $U \in \Sigma$. However, $\forall I \in \mathcal{I} : I \subset U$. Since every chain has a maximum, by Zorn's Lemma there exist maximal elements of Σ above every ideal $I \in \Sigma$. Suppose $\mathfrak{m} \in \Sigma$ is maximal and $xy \in \mathfrak{m}$. Since xy is a zero-divisor we must have $axy = 0$ for some $a \neq 0$ so either $ax = 0$ or y is a zero-divisor. Thus either x or y is a zero-divisor. Without loss of generality, suppose that x is a zero-divisor then so is ax for any $a \in A$ so $(x) \in \Sigma$. Since the sum of zero-divisors is a zero divisor, $\mathfrak{m} + (x) \in \Sigma$ but $\mathfrak{m} \subset \mathfrak{m} + (x)$ contradicting maximality unless $\mathfrak{m} + (x) = \mathfrak{m}$ and thus $x \in \mathfrak{m}$ so $\mathfrak{m}$ is prime ($\mathfrak{m} \neq A$ since $1 \in A$ is not a zero-divisor but $\mathfrak{m} \in \Sigma$).

Now, if x is a zero-divisor then so is ax for any $a \in A$ so $(x) \in \Sigma$. By Zorn's Lemma, there exists a maximal element $\mathfrak{m}_x \in \Sigma$ above (x) so $x \in \mathfrak{m}_x$ and we know that $\mathfrak{m}_x$ is prime. Therefore, each zero-divisor is contained in a prime ideal containing only zero-divisors. Let $Z \subset A$ be the zero divisors of A. Then,

$$Z = \bigcup_{x \in Z} \mathfrak{m}_x$$

because if $x \in Z$ then $x \in \mathfrak{m}_x$ and thus in the union and each ideal of the union is contained in Z so the enite union is as well. Therefore, the zero-divisors of A are a union of prime ideals of A.

4 Problem 4

Let $N = \sqrt{(0)}$ the nilradical of A which is the ideal of all nilpotent elements and the intersection of all prime ideals. Suppose that N is prime. Leet $I \subset A$ be any ideal then

$$N \in V(I) \iff I \subset N \iff \forall \mathfrak{p} \in \mathrm{Spec}\,(A) : I \subset \mathfrak{p} \iff V(I) = \mathrm{Spec}\,(A)$$

Suppose that $U \subset \mathrm{Spec}\,(A)$ is open and nonempty. Then U^C is closed and proper so $U^C = V(I) \neq \mathrm{Spec}\,(A)$ for some I which implies that $N \notin V(I)$ so $N \in U$. However, any closed set containing U must be of the form $V(J)$ for some ideal J. Since $N \in U \subset V(J)$ we know that $V(J) = \mathrm{Spec}\,(A)$. Thus, $\overline{U} = \mathrm{Spec}\,(A)$ so U is dense. Therefore, $\mathrm{Spec}\,(A)$ is irreducible.

Conversely, suppose that N is not prime. Then there exist elements $x, y \notin N$ such that $xy \in N$. Now, $V(xy) = V(x) \cup V(y)$ where $V(x)$ and $V(y)$ are closed proper sets because $x, y \notin N$ so (x) and (y) both cannot be contained in every prime ideal else they would be elements of the intersection N. However, $xy \in N$ so $(xy) \subset \mathfrak{p}$ for each prime ideal so $V(xy) = \mathrm{Spec}\,(A)$. Thus we have written $\mathrm{Spec}\,(A) = V(x) \cup V(y)$ as the union of closed proper subsets so $\mathrm{Spec}\,(A)$ is not irreducible.

Therefore, $\mathrm{Spec}\,(A)$ is irreducible if and only if the nilradical is prime.

5 Problem 5

Lemata

Lemma 5.1. Let A be a ring with a finitely generated $I \subset A$ and any ideal $J \subset A$ such that $I \subset \sqrt{J}$. Then there exists an integer N such that $I^N \subset J$.

Proof. I is finitely generated so let $I = (x_1, \dots, x_\ell)$. We know that $I \subset \sqrt{J}$ so for each $1 \leq i \leq \ell$ there exists a positive integer n_i such that $x_i^{n_i} \in J$. Take $N = n_1 + \cdots + n_\ell$. The ideal I^N is generated by monomials of the form $x_1^{r_1} \cdots x_\ell^{r_\ell}$ such that $r_1 + \cdots + r_\ell = N = n_1 + \cdots + n_\ell$. However, because these are all positive integers there must exist at least one r_i such that $r_i \geq n_i$ and thus $x_i^{r_i} = x_i^{n_i} \cdot x_i^{r_i - n_i} \in J$ and thus, by ideal absorption, the entire monomial $x_1^{r_1} \cdots x_\ell^{r_\ell} \in J$. Therefore $I^N \subset J$ because each generator is contained in J. $\square$

Lemma 5.2. Let A be a Noetherian ring then A^n is Noetherian also.

Proof. Proceede by induction on n. The case $n = 1$ is trivial. Assume true for $n - 1$ then consider the exact sequence,

$$0 \longrightarrow A \longrightarrow A^n \longrightarrow A^{n-1} \longrightarrow 0$$

since both A and A^{n-1} are Noetherian by assumption we get that A^n is Noetherian. Thus, the result holds by induction. $\square$

Lemma 5.3. Let $K \subset A$ be a finitely generated ideal such that A/K is Noetherian then any ideal of A above K is finitely generated.

Proof. Let $I \supset K$ be an ideal and $\bar{I}$ its image under $\pi : A \to A/K$. Since A/K is Noetherian we know that $\bar{I}$ is finitely generated. Write $\bar{I} = (A/K)x_1 + \cdots + (A/K)x_n$ for $x_n \in I$ (i.e. I choose some lifts to A of the generators). Let $J = Ax_1 + \cdots + Ax_n$. I claim that $I = J + K$. Clearly, $J \subset I$ and $K \subset I$ so $J + K \subset I$. Furthermore, if $x \in I$ then $\bar{x} \in \bar{I}$ so we can write,

$$\bar{x} = \bar{a}_1 x_1 + \cdots + \bar{a}_n x_n$$

and therefore, $\pi(a_1 x_1 + \cdots + a_n x_n - x) = 0$ which implies that,

$$a_1 x_1 + \cdots + a_n x_n - x \in \ker \pi = K$$

proving the claim. However, both J and K are finitely generated and thus I is as well. $\square$

Lemma 5.4. A topological space X is irreducible if and only if it cannot be written as the union of proper closed sets.

Proof. First, note that $Z_1 \cup Z_2 = X \iff (X \setminus Z_1) \subset Z_2$ because both express the condition that if $x \notin Z_1$ then $x \in Z_2$.

Let $Z_1, Z_2 \subset X$ be proper closed sets. If $Z_1 \cup Z_2 = X$ then $(X \setminus Z_1) \subset Z_2$ which is proper so $(X \setminus Z_1)$ is not dense and thus X is not irreducible. Suppose that for every choice of $Z_1, Z_2 \subset X$ that $Z_1 \cup Z_2 \neq X$. For an nonempty open set U take $Z_1 = X \setminus U$ which is proper and closed. Then if $U \subset Z_2$ for closed Z_2 we know that $Z_1 \cup Z_2 = X$ so Z_2 must be non proper and thus $\overline{U} = X$ so X is irreducible. $\square$