1 Week 2: The Swan Conductor and the Grothendieck-Odd-Shafarevich Formula

Review of ramification: Let $\mathcal{O}_K$ be the henselian DVR of characeristic $p > 0$. Let $K = \mathrm{Frac}\,(\mathcal{O}_K)$ and $\kappa = \mathcal{O}_K/\mathfrak{m}$. We get a tower,

$$K^{\mathrm{sep}} \supset K^{\mathrm{tame}} \supset K^{\mathrm{ur}} \supset K$$

Then the Galois groups are,

$$\mathrm{Gal}\,(K^{\mathrm{ur}}/K) = \mathrm{Gal}\,(\kappa^{\mathrm{sep}}/\kappa) \quad \mathrm{Gal}\left(K^{\mathrm{tame}}/K\right) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$$

Then the inertia group is $I = \mathrm{Gal}\,(K^{\mathrm{sep}}/K^{\mathrm{ur}})$ and the wild inertia group is $P = \mathrm{Gal}\,(K^{\mathrm{sep}}/K^{\mathrm{tame}})$. From now on, we only care about ramification so assume that $K = K^{\mathrm{ur}}$.

Let L/K be finite Galois with Galois group G.

Definition 1.0.1. The ramification filtration G_i is a decreasing filtration given by,

$$G_i = \{\sigma \in G \mid \sigma(\varpi_L) - \varpi_L \in (\varpi_L)^{i+1}\}$$

Then $G_0 = I$ and G_0/G_1 is the tame inertia. Then G_1 is the wild inertia.

Remark. Let X be a geometrically integral curve over κ and $K = K(X)$. Let $j : U \hookrightarrow X$ be a nonempty open subset. let $\mathbb{F}$ be a finite field of characteristic $\ell \neq p$. Then let $\mathscr{F}$ be a $\mathbb{F}$-local system on U corresponding to a Galois representation $\rho : \mathrm{Gal}\,(K^{\mathrm{sep}}/K) \to \mathscr{F}_{\bar{\eta}}$. Then $I_x \circlearrowright \mathscr{F}_{\bar{\eta}}$ for each closed point $x \in X$. Then,

(a) if $x \in U$ then $I_x \circlearrowright \mathscr{F}_{\bar{\eta}}$ is trivial

(b) if $x \notin U$ then $I_x \circlearrowright \mathscr{F}_{\bar{\eta}}$ is interesting and we get a Swan conductor $\mathrm{Sw}_x(\mathscr{F})$.

Definition 1.0.2. The Swan conductor $\mathrm{Sw}_x(\mathscr{F})$ is defined as follows. Since $\mathscr{F}$ is a local system over a finite field $V = \mathscr{F}_{\bar{\eta}}$ is finite. Hence the action factors through a finite quotient L/K. Consider the ramification filtration G_i of $G = \mathrm{Gal}\,(L/K)$. Then,

$$\mathrm{Sw}_x(\mathscr{F}) = \sum_{i \geq 1} \frac{\dim(V/V^{G_i})}{[G_0 : G_i]}$$

which is actually a well-defined integer.

Proposition 1.0.3. The following hold about the Swan conductor,

(a) $\mathrm{Sw}_x(\mathscr{F}) = 0 \iff V$ is tamely ramified at x menaing $P_x \circlearrowright V$ trivially

(b) For $\mathscr{F}$ tamely ramified at x and some other local system $\mathscr{G}$ we have,

$$\mathrm{Sw}_x(\mathscr{F} \otimes \mathscr{G}) = (\mathrm{rank}\,\mathscr{F}) \cdot \mathrm{Sw}_x(\mathscr{G})$$

Proposition 1.0.4. Let $\mathscr{F}$ be a free lisse $\mathcal{O}_E$-local system for some finite $E/\mathbb{Q}_\ell$. Define $\mathrm{Sw}_x(\mathscr{F}) := \mathrm{Sw}_x(\mathscr{F}/\varpi_E\mathscr{F})$ where $\mathscr{F}/\varpi_E\mathscr{F}$ is a κ_E-local system.

Example 1.0.5. (a) Kummer sheaf $\mathcal{L}(\chi)$. For $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a multiplicative character. Consider the Kummer cover $\mathbb{G}_m \to \mathbb{G}_m$ via $x \mapsto x^{q-1}$ with Galois groip $\mathbb{F}_q^\times$. Define $\mathcal{L}(\chi)$ to be the local system on $\mathbb{G}_m$ corresponding to $\pi_1(\mathbb{G}_m) \to \mathbb{F}_q^\times \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$. Then $\mathbb{G}_m \subset \mathbb{P}^1$ has boundary consisting of $\{0, \infty\}$ and the two Swan conductors are,

$$\mathrm{Sw}_0(\mathcal{L}(\chi)) = \mathrm{Sw}_\infty(\mathcal{L}(\chi)) = 0$$

since the group has order coprime to p and thus has no wild ramification.

(b) Artin-Schrier sheaf $\mathcal{L}(\psi)$ for a nontrivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. We have the Artin-Schreier cover $\mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^q - x$ with Galois group $\mathbb{F}_q$. Define $\mathcal{L}(\psi)$ to be the local system associated to $\pi_1(\mathbb{A}^1) \to \mathbb{F}_q^\times \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times$. Then for $\mathbb{A}^1 \subset \mathbb{P}^1$ and we have,

$$\mathrm{Sw}_\infty(\mathcal{L}(\psi)) = 1$$

To see this, consider the behavior at infinity. We have the equation $y^q - y = x$ let $y = u^{-1}$ and $x = v^{-1}$ so,

$$v = \frac{u^q}{1 - u^{q-1}}$$

and the automorphisms act via $y \mapsto y + a$ so

$$u \mapsto \frac{u}{1 + au} = u - au^2 + a^2u^3 + \cdots$$

which visibly lies in G_1 and not G_2 (for $a \neq 0$) so the entire Galois group is wild inertia of level 1 (besides the trivial element of course). Therefore,

$$\mathrm{Sw}_\infty(\mathcal{L}(\psi)) = \sum_{i \ge 1} \frac{\dim\left(V/V^{G_i}\right)}{[G : G_i]} = \frac{\dim\left(V/V^{G_1}\right)}{[G : G_1]} = \dim V = 1$$

1.1 The Trace Formula

Theorem 1.1.1 (Grothendieck-Ogg-Shafarevich)**.** Let $\mathscr{F}$ be a $\overline{\mathbb{Q}}_\ell$-local system on $U \subset X$. Then,

$$\chi_c(U, \mathscr{F}) = \chi_c(U, \overline{\mathbb{Q}}_\ell) \cdot (\mathrm{rank}\, \mathscr{F}) - \sum_{x \in X \setminus U} \mathrm{Sw}_x(\mathscr{F})$$

Remark. Also we know that $\chi(U, \mathscr{F}) = \chi_c(U, \mathscr{F})$.

1.2 Applications

Definition 1.2.1. Let $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ and $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. Define the *Gauss sum*,

$$G(\chi, \psi) = \sum_{a \in \mathbb{F}_q^\times} \chi(a)\psi(a)$$

Remark. Deligne noticed that,

$$G(\chi, \psi) = \sum_{a \in \mathbb{G}_m(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_a \mid \mathcal{L}(\chi)_a \otimes \mathcal{L}(\psi)_a)$$

By the Grothendieck-Lefschetz fixed-point formula,

$$G(\chi, \psi) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}\left(\mathrm{Frob} \mid H^i_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi))\right)$$

Notice that,

$$H^0_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = 0 \quad H^2_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = H^0(\mathbb{G}_m, \mathcal{L}(\chi^{-1}) \otimes \mathcal{L}(\psi^{-1}))^\vee = 0$$

since there are no global sections for nontrivial characters. Then we apply the GOS formula,

$$\chi_c(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = -\mathrm{Sw}_0(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) - \mathrm{Sw}_\infty(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi))$$

but both are tamely ramified at 0 and $\mathcal{L}(\chi)$ is tamely ramified at infinity and thus,

$$\chi_c(\mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = -1$$

and thus,

$$\dim H^1_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi)) = 1$$

Therefore, we see that,

$$G(\chi, \psi) = -\operatorname{tr}\left(\mathrm{Frob} \mid H^1_c(\mathbb{G}_m, \mathcal{L}(\chi) \otimes \mathcal{L}(\psi))\right)$$

and is a 1-dimensional space so there is a single eigenvalue. By Weil II we see that this eigenvalue has absolute value $q^{\frac{1}{2}}$ and thus,

$$|G(\chi, \psi)| = q^{\frac{1}{2}}$$

Remark.

$$|G(\chi, \psi)|^2 = \sum_{a,b} \chi(a)\overline{\chi}(b)\psi(a)\overline{\psi}(b) = \sum_{a,b} \chi(a-b)\psi(a)\overline{\psi}(b)$$

1.3 Kloosterman Sums

Fix $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. For $n \geq 1$ and $a \in \mathbb{F}_q$ define the Kloosterman sum,

$$K_{n,a} = \sum_{x_1 \cdots x_n = a} \psi(x_1 + \cdots + x_n)$$

Trivial bound,

$$|K_{n,a}| \leq q^{n-1}$$

Deligne gives,

$$|K_{n,a}| \leq n q^{\frac{n-1}{2}}$$

Write the Kloosterman sums as sums of traces of Frobenius. Let,

$$V_a^{n-1} = \{x_1 \cdots x_n = a\} \subset \mathbb{A}^n$$

which is smooth for $a \neq 0$. Consider the maps $\sigma : \mathbb{A}^n \to \mathbb{A}$ and $\pi : \mathbb{A}^n \to \mathbb{A}$ taking the sum and product respectively. We use the sheaves $\mathscr{F} = \iota^* \sigma^* \mathcal{L}(\psi)$ where $\iota : V_a^{n-1} \subset \mathbb{A}^n$ is the inclusion. Then $\mathrm{Frob}_x \circlearrowright \mathscr{F}_{\bar{x}}$ via $\psi(x_1 + \cdots + x_n)$. Therefore, by the Grothendieck trace formula,

$$K_{n,a} = \sum_{i=0}^{2n-2} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_x \mid H^i_c(V_a^{n-1}, \mathscr{F})\right)$$

Then Deligne showed the following.

Theorem 1.3.1 (Deligne). (a) $H_c^i(V_a^{n-1}, \mathscr{F}) = 0$ for $i \neq n-1$

(b) $\dim H_c^i(V_a^{n-1}, \mathscr{F}) = n$.

Corollary 1.3.2. Then by Weil II we see that $|K_{n,a}| \leq nq^{\frac{n-1}{2}}$.

Theorem 1.3.3 (Deligne). (a) the Kloosterman sheaf $\mathrm{Kl}_n := R^{n-1}\pi_!\mathscr{F})$ satisfies $\mathrm{Kl}_n|_{\mathbb{G}_m}$ is lisse of rank n

(b) direct image of Kl_n on $\mathbb{P}^1$ has stalk 0 at ∞

(c) $\dim (\mathrm{Kl}_n)_0 = 1$

(d) $\mathrm{Sw}_0(\mathrm{Kl}_n|_{\mathbb{G}_m}) = 0$

(e) $\mathrm{Sw}_0(\mathrm{Kl}_n|_{\mathbb{G}_m}) = 1$.