Fix a prime p and all rings will be $\mathbb{F}_p$-algebras. Therefore, we have the natural transformation $F : R \to R$ so R is an R-module in a nontrivial way which we denote as $F_* R$.

Theorem 0.0.1 (Kunz)**.** If R is Noetherian, then R is regular if and only if F is flat (equivalently $F_* R$ is a flat R-module).

Remark. From now on, we only consider noetherian rings.

Remark. Even if R is Noetherian F can be nonfinite. For example $R = \mathbb{F}_p(x_1, x_2, \dots)$ is not F-finite.

Given a local ring $(R, \mathfrak{m}, k)$ we make some definitions.

Definition 0.0.2. R is F-finite if $F_* R$ is a finite R-module. In this case the maps $F^e : R \to R$ are all finite. Tehn we write,

$$\lambda_e(R) = \frac{\#\text{gens of } F^e_* R}{[k : k^{p^e}] p^{e \dim R}}$$

and

$$s_e(R) = \frac{\max\{b \mid F^e_* R \cong R^{\oplus b} \oplus M\}}{[k : k^{p^e}] p^{e \dim R}}$$

It is clear that $\lambda_e(R) \geq 1$ and $s_E(R) \leq 1$.

Proposition 0.0.3. If any $\lambda_e(R) = 1$ then all $\lambda_e(R) = 1$ and $s_e(R) = 1$ and R is regular.

Definition 0.0.4. $e_{HK}(R) = \lim_e \lambda_e(R) \geq 1$ and $s(R) = \lim_e s_e(R) \leq 1$ called the F-signature.

Proposition 0.0.5. If R is equidimensional,

$$R \text{ is regular} \iff e_{HK}(R) = 1 \iff s(R) = 1$$

Proposition 0.0.6. $s(R) > 0$ iff R is strongly F-regular.

Remark. The number of generators of $F^e_* R$ is equal to,

$$\begin{aligned} \dim_k (F^e_* R) \otimes_R k = \ell_R(F^e_* R / \mathfrak{m} F^e_* R) &= [k : k^{p^e}] \ell_{F^e_* R}(F^e_* R / \mathfrak{m} F^e_* R) \\ &= [k : k^{p^e}] \ell_R(R / \mathfrak{m}^{[p]}) \end{aligned}$$

Proposition 0.0.7. This is extendable to non-F-finite R. For $R^\wedge$ reduced. There exists an $\mathfrak{m}$-primary ideal I and an element $u \in (I : \mathfrak{m})$ meaning $u\mathfrak{m} \subset I$ such that,

$$s_e(R) = \frac{\ell_R((I, u)^{[p^e]} / I^{[p^e]}}{p^{e \dim R}}$$

Proposition 0.0.8. For e_1, e_2 there exists I, u that work for both.

Proposition 0.0.9. The limit exists for non-F-finite R.

Given R is the function,

$$\mathfrak{p} \mapsto e_{HK}(R_\mathfrak{p} \quad \text{or} \quad \mathfrak{p} \mapsto s(R_\mathfrak{p})$$

semicontinuous on $\operatorname{Spec}(R)$?

Remark. This is false if the regular locus is not open so we should at least require that the ring be J_1. Thus we will restrict to excellent rings R.

Theorem 0.0.10 (Smirnov, Rilstra)**.** True if R is f.g. over an excellent local ring.

Remark. In general, this function is NOT constructible. It can take on infinitely many values because we have taken the limit.

Theorem 0.0.11 (Shepherd-Baron)**.** λ_e and s_e define semicontinuous functions.

Theorem 0.0.12 (Polstra)**.** The convergence $\lambda_e \to e_{HK}$ and $s_e \to s$ are uniform.

Theorem 0.0.13 ('23)**.** For any excellent R, the convergence $\lambda \to e_{HK}$ and $s_E \to s$ are uniform. If R is locally equidimensional then e_{HK} defines a semicontinuous function. And s defines a semicontinuous function if R is either Gorenstein or a quotient of a regular ring.

Let A be a reduced complete equidimensional ring. Then there exists a regular local ring P (power sieres) and a finite, generically étale ring map $P \to A$ then,

$$F^e_* P \otimes_P A \to F^e_* A$$

is injective and "birational" (i.e. is isom in codim 0). Weill use this for $R^\wedge_{\mathfrak{p}}$ but only do the calculation with,

$$P \hat{\otimes}_K K \to R^\wedge_{\mathfrak{p}} \hat{\otimes}_K K$$

where P is a power series ring.

Proposition 0.0.14. Given an excellent ring R, there exists a family of C-G n with $P(\mathfrak{p}) \to R^\wedge_{\mathfrak{p}}$ with the multiplicity of the discriminant bounded[1] and other numbers.

[1]Actually, you need to take a finite flat, quasi-finite extension S of R and you only get $P(\mathfrak{p})S^\wedge_{\mathfrak{p}}$ on a subset of $\operatorname{Spec}(S)$ that covers $\operatorname{Spec}(R)$