1 The Yoneda Embedding

Lemma 1.0.1. Let $\eta : \text{Hom}(A, -) \to \text{Hom}(B, -)$ be a natural transformation. Then η is uniquely determined by $\eta_A(\text{id}_A)$ via $\eta_X(f) = f \circ \eta_A(\text{id}_A)$ for any $f \in \text{Hom}(A, X)$.

Proof. Let $f : A \to X$ be some map. Consider the naturality diagram,

$$\begin{array}{ccc} \text{Hom}(A, A) & \xrightarrow{f_*} & \text{Hom}(A, X) \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_X} \\ \text{Hom}(B, A) & \xrightarrow{f_*} & \text{Hom}(B, X) \end{array}$$

Consider the element $\text{id}_A \in \text{Hom}(A, A)$ which, under the upper path, maps to $\eta_X(f_*(\text{id}_A)) = \eta_X(f \circ \text{id}_A) = \eta_X(f)$ and, under the lower path, $f_*(\eta_A(\text{id}_A)) = f \circ \eta_A(\text{id}_A)$. Therefore,

$$\eta_X(f) = f \circ \eta_A(\text{id}_A)$$

□

Corollary 1.0.2. Natural transformations $\eta : \text{Hom}(A, -) \to \text{Hom}(B, -)$ are in one-to-one correspondence with functions $\text{Hom}(B, A)$. We say f^* is the natural transformation $f^*_X(g) = g \circ f$ for any $g \in \text{Hom}(A, X)$.

Theorem 1.0.3. Let $\mathcal{C}$ be any category. The functor $Y : \mathcal{C}^{\text{op}} \to \mathbf{Set}^{\mathcal{C}}$ sending $A \mapsto h^A$ where $h^A = \text{Hom}(A, -)$ and $f \mapsto f^*$ described above is fully faithful.

Proof. Clearly $(\text{id}_A)^* = \text{id}_{h^A}$ since $(\text{id}_A)^*(f) = f \circ \text{id}_A = f$ and for $f \in \text{Hom}(B, A)$ and $g \in \text{Hom}(C, B)$ then $(f \circ g)^* = g^* \circ f^*$ since for any $q \in \text{Hom}(A, X)$ we send,

$$(f \circ g)^*(q) = q \circ (f \circ g) = (q \circ f) \circ g = g^*(f^*(q))$$

The above corollary proves that Y is fully faithful. □

Lemma 1.0.4. Let $F : \mathcal{C} \to \mathcal{D}$ be fully faithful then $X \cong Y \iff F(X) \cong F(Y)$.

Proof. If $F(X) \cong F(Y)$ then there are morphisms $f \in \text{Hom}(F(X), F(Y))$ and $g \in \text{Hom}(F(Y), F(X))$ which are inverses. However, since F is full there exist morphisms $\tilde{f} : \text{Hom}(X, Y)$ and $g \in \text{Hom}(Y, X)$ such that $F(\tilde{f}) = f$ and $F(\tilde{g}) = g$. Then,

$$F(\tilde{f} \circ \tilde{g}) = F(\tilde{f}) \circ F(\tilde{g}) = f \circ g = \text{id}_{F(Y)} \quad \text{and} \quad F(\tilde{g} \circ \tilde{f}) = F(\tilde{g}) \circ F(\tilde{f}) = g \circ f = \text{id}_{F(X)}$$

However, since F is faithful then,

$$\tilde{f} \circ \tilde{g} = \text{id}_Y \quad \text{and} \quad \tilde{g} \circ \tilde{f} = \text{id}_X$$

proving that $X \cong Y$. □

Definition 1.0.5. We say a functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if $F \cong h^A$ for some $A \in \mathcal{C}$.

2 Additive Categories

Definition 2.0.1. A category $\mathcal{C}$ is pre-additive if its hom sets have the structure of an abelian group and composition of maps distributes over addition. Explicitly, for $X, Y, Z \in \mathcal{C}$, there exits a binary operation,

$$+ : \mathrm{Hom}\,(X, Y) \times \mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(X, Y)$$

such that $(\mathrm{Hom}\,(X, Y), +)$ is an abelian group and, for $f, g : X \to Y$ and $h, k : Y \to Z$ we have $h \circ (f + g) = h \circ f + h \circ g$ and $(h + k) \circ f = h \circ f + k \circ f$. This is equvalent to the requirement that hom is a functor,

$$\mathrm{Hom}\,(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$$

Lemma 2.0.2. In a pre-additive cateogory, there exists an identity element $0 \in \mathrm{Hom}\,(X, Y)$ such that $0 + f = f + 0 = f$ for $f \in \mathrm{Hom}\,(X, Y)$ and $f \circ 0 = 0$ for $f \in \mathrm{Hom}\,(Y, Z)$ and $0 \circ f = 0$ for $f \in \mathrm{Hom}\,(Z, X)$.

Proof. The hom sets are abelian groups by definiton and thus must have unique identiy elements satisfying $f + 0 = 0 + f = f$ for all $f \in \mathrm{Hom}\,(X, Y)$. Furthermore, for $f \in \mathrm{Hom}\,(Y, Z)$ we have $f \circ 0 = f \circ (0 + 0) = f \circ 0 + f \circ 0$ and thus $f \circ 0 = 0_{XZ}$. Furthermore for $f \in \mathrm{Hom}\,(Z, X)$ we know that $0 \circ f = (0 + 0) \circ f = 0 \circ f + 0 \circ f$ so $0 \circ f = 0_{ZY}$. $\square$

Definition 2.0.3. A biproduct of an indexed set $\{X_i\}_I$ is an object $X = \bigoplus_I X_i$ along with projection maps $\pi_i : X \to X_i$ and inclusion maps $\iota_i : X_i \to X$ such that $(X, \{\pi_i\}_I)$ is the product of $\{X_i\}_I$ and $(X, \{\iota_i\}_I)$ is the coproduct of $\{X_i\}_I$.

Proposition 2.0.4. Let $\mathcal{C}$ be a pre-additive category. Every finite product and finite coproduct is a biproduct. In particular, finite products and coproducts are equal.

Proof. Let $X \times Y$ be the product of X and Y. Consider the diagram,

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "\iota_X"] & & X \\
& X \times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"] & \\
Y \arrow[ur, "\iota_Y"] \arrow[rr, "\mathrm{id}_Y"] & & Y
\end{tikzcd}$$

where the maps $\iota_X : X \to X \times Y$ and $\iota_Y : Y \to X \times Y$ are defined via the universal property of the product applied to $(\mathrm{id}_X, 0)$ and $(0, \mathrm{id}_Y)$ respectivly where $0 \in \mathrm{Hom}\,(X, Y)$ is the identiy element of the abelian group. The universal property gives,

$$\pi_X \circ \iota_X = \mathrm{id}_X \quad \pi_Y \circ \iota_X = 0$$
$$\pi_X \circ \iota_Y = 0 \quad \pi_Y \circ \iota_Y = \mathrm{id}_Y$$

so the diagram commutes. We need to show that $X \times Y$ is universal with respect to the maps ι_X and ι_Y. Suppose we have maps $f_X : Z \to X$ and $f_Y : Z \to Y$ then define $\tilde{f} = f_X \circ \pi_X + f_Y \circ \pi_Y$.

$$\begin{tikzcd}
& X \arrow[rr, "\mathrm{id}_X"] \arrow[dl, "f_X"'] \arrow[dr, "\iota_X"] & & X \\
Z & & X \times Y \arrow[ll, dashed, "\tilde{f}"'] \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"] & \\
& Y \arrow[ul, "f_Y"] \arrow[ur, "\iota_Y"] \arrow[rr, "\mathrm{id}_Y"] & & Y
\end{tikzcd}$$

This map satisfies the required universal property because,

$$\tilde{f} \circ \iota_X = (f_X \circ \pi_X + f_Y \circ \pi_Y) \circ \iota_X = f_X \circ \pi_X \circ \iota_X + f_Y \circ \pi_Y \circ \iota_X = f_X + 0 = f_X$$

and likewse,

$$\tilde{f} \circ \iota_Y = (f_X \circ \pi_X + f_Y \circ \pi_Y) \circ \iota_Y = f_X \circ \pi_X \circ \iota_Y + f_Y \circ \pi_Y \circ \iota_Y = 0 + f_Y = f_Y$$

Lastly, we must show that $\tilde{f}$ is unique. Suppose there exits a map $\tilde{f} : X \times Y \to Z$ such that $\tilde{f} \circ \iota_X = f_X$ and $\tilde{f} \circ \iota_Y = f_Y$. Consider the map $I : X \times Y \to X \times Y$ given by,

$$I = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$$

Therefore,

$$\pi_X \circ I = \pi_X \circ \iota_X \circ \pi_X + \pi_X \circ \iota_Y \circ \pi_Y = \pi_X + 0 = \pi_X$$

and furthermore,

$$\pi_Y \circ I = \pi_Y \circ \iota_X \circ \pi_X + \pi_Y \circ \iota_Y \circ \pi_Y = 0 + \pi_Y = \pi_Y$$

However, by the universal property of the product, there exists a unique map, namely $\mathrm{id}_{X \times Y}$, satisfying these properties. Thus, $I = \mathrm{id}_{X \times Y}$. Thus,

$$\tilde{f} = \tilde{f} \circ \mathrm{id}_{X \times Y} = \tilde{f} \circ I = \tilde{f} \circ \iota_X \circ \pi_X + \tilde{f} \circ \iota_Y \circ \pi_Y = f_X \circ \pi_X + f_Y \circ \pi_Y$$

so the map we constructed earlier is unique.

Similarly, let $X \coprod Y$ be the coproduct of X and Y. A similar argument will hold reversing all arrows. □

Remark. Additionally, we see that a terminal object T (empty product) is also initial (empty coproduct) because $\mathrm{Hom}\,(T, X)$ must have a zero element $0 : T \to X$ and for any map $f : T \to X$ we know that $f \circ 0_{TT} = 0_{TX}$ but $0_{TT} = \mathrm{id}_T$ is the unique map $T \to T$ so $f = 0_{TX}$ and thus T is also initial.

Definition 2.0.5. A category is additive if it is pre-additive, has a zero object, and has all finite biproducts. The preceding dicussion implies that it is enough to check that either all finite products or all finite coproducts exit.

Proposition 2.0.6. In an additive category, the zero map is the indentity obeject of the **Ab**-enriched hom-sets.

Proof. □

Definition 2.0.7. A functor $T : \mathcal{C} \to \mathcal{D}$ is *additive* if it preserves finite biproducts.

Proposition 2.0.8. A functor $T : \mathcal{C} \to \mathcal{D}$ is additive iff the map on enriched hom-sets,

$$T_{X,Y} : \mathrm{Hom}_{\mathcal{C}}\,(X, Y) \to \mathrm{Hom}_{\mathcal{D}}\,(T(X), T(Y))$$

is a homomorphism in the category of abelian groups.

Proof. A biproduct $X \oplus Y$ with its projections and inclusions is completely characterized by the property $\mathrm{id}_{X \oplus Y} = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$. Thus T preserves the biproduct structure iff it preserves addition i.e. iff,

$$\mathrm{id}_{T(X \oplus Y)} = T(\mathrm{id}_{X \oplus Y}) = T(\iota_X \circ \pi_X + \iota_Y \circ \pi_Y) = T(\iota_X) \circ T(\pi_X) + T(\iota_Y) \circ T(\pi_Y)$$

□

DEF-COMPLEX PROP ADD-FUNC PRESERVE COMPLEXES

3 Normality

Definition 3.0.1. A morphism $f : X \to Y$ is called,

(a) split on the left (admits a left inverse) if there exists a map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$

(b) split of the right (admits a right inverse) if there exists a map $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$

(c) an isomorphism if there exists a map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Lemma 3.0.2. The following hold in any category:

(a) a morphism split on the left is monic

(b) a morphism split of the right is epic

(c) a morphism is split both on the left and on the right if and only if it is an isomorphism.

Proof. First, if $f : X \to Y$ admits a left inverse $g : Y \to X$ then for any $a, b : Z \to X$ such that $f \circ a = f \circ b$ we have $g \circ f \circ a = g \circ f \circ b$ and thus $a = b$ since $g \circ f = \mathrm{id}_X$. Dually, if $f : X \to Y$ admits a right inverse $g : Y \to X$ then for any $a, b : Y \to Z$ such that $a \circ f = b \circ f$ we have $a \circ f \circ g = b \circ f \circ g$ and thus $a = b$ since $f \circ g = \mathrm{id}_Y$.

Finally, an isomorphism is clearly split on the left and right. To prove the converse, it suffices to show that the left and right inverses agree. Indeed if f has a left inverse $g_L : B \to A$ and right inverse $g_R : B \to A$ such that $g_L \circ f = \mathrm{id}_A$ and $f \circ g_R = \mathrm{id}_B$ then,

$$g_L = g_L \circ \mathrm{id}_B = g_L \circ (f \circ g_R) = (g_L \circ f) \circ g_R = \mathrm{id}_A \circ g_R = g_R$$

□

Remark. Due to the previous result, we alternatively call a morphisms split on the left a "split mono" and a morphism split on the right a "split epi".

Remark. Split monos and epis are important because every functor preserves them unlike usual monos and epis.

Lemma 3.0.3. Every equalizer is a monomorphism. Dually, every coequalizer is an epimorphism.

Proof. Let $f, g : X \to Y$ be morphisms and $e : E \to X$ be the equalizer. Given two maps $a, b : E \to X$ such that $q = e \circ a = e \circ b$ clearly we have $f \circ q = g \circ q$ because $f \circ e = g \circ e$. Therefore, q factors *uniquely* through E meaning that $a = b$. □

Corollary 3.0.4. Every kernel is a monomorphism. Dually, every cokernel is a epimorphism.

Definition 3.0.5. Let $\mathcal{A}$ be a category with zero maps.

(a) a monomorphism is *normal* if it is the kernel of some morphism

(b) an epimorphism is *conormal* if it is the cokernel of some morphism

(c) $\mathcal{C}$ is *normal* if every monomorphism is the kernel of some morphism

(d) $\mathcal{C}$ is *conormal* if every epimorphism is the cokernel of some morphism

(e) $\mathcal{C}$ is *binormal* if it is normal and conormal.

Proposition 3.0.6. Let $\mathcal{A}$ be a category with zero maps and $f : A \to B$ a morphism. Then,

(a) if f is a monomorphism and also a conormal epimorphism then f splits uniquely on the left

(b) if f is an epimorphism and also a normal monomorphism then f splits uniquely on the right.

(c) if f is a normal mono and also a conormal epi then f is an isomorphism.

(HERE I THINK ITS ALREADY AN ISOMORPHISM)

Proof. If f is a conormal epi, it is the cokernel of $a : K \to A$. Thus, $f \circ a = f \circ 0 = 0$ but f is monic so $a = 0$. Therefore, we have a diagram,

$$\begin{tikzcd} K \arrow[r, "0"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[dl, dashed, "g"] \\ & A & \end{tikzcd}$$

where $\mathrm{id}_A \circ a = 0$ because $a = 0$ and thus id_A factors through the cokernel $f : A \to B$ as $g \circ f$ for a unique $g : B \to A$.

The second is exactly the dual statement and thus is an application of the first in $\mathcal{A}^{\mathrm{op}}$. The third follows directly by applying both the previous statements. However, we will spell it out for clarity.

Let $f : A \to B$ be a normal mono and a conormal epi meaning f must be a kernel and a cokernel of some maps,

$$K \xrightarrow{\ a\ } A \xrightarrow{\ f\ } B \xrightarrow{\ b\ } C$$

where $f : A \to B$ is the cokernel of $a : K \to A$ and the kernel of $b : B \to C$. Then $f \circ a = 0 = f \circ 0$ so, since f is monic, $a = 0$. Furthermore, $b \circ f = 0 = 0 \circ f$ so, since f is an epic, $b = 0$. Therefore, consider the diagram,

$$\begin{tikzcd} K \arrow[r, "0"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "0"] \arrow[dl, dashed] & C \\ & A & B \arrow[u, "\mathrm{id}_B"'] \arrow[ul, dashed] & \end{tikzcd}$$

Where $\mathrm{id}_A \circ a = 0$ and $b \circ \mathrm{id}_B = 0$ (since $a = 0$ and $b = 0$) which implies that id_A factors through the cokernel $f : A \to B$ and id_B lifts over the kernel $f : A \to B$. Thus f has a left inverse $g_L : B \to A$ and right inverse $g_R : B \to A$ such that $g_L \circ f = \mathrm{id}_A$ and $f \circ g_R = \mathrm{id}_B$. Thus f is is both left and right split and thus is an isomorphism. Alternatively, the splittings are unique but notice that $g_R \circ f = (g_L \circ f) \circ g_R \circ f = g_L \circ f = \mathrm{id}_A$ so $g_R = g_L$ by uniqueness of the factorization. $\square$

Corollary 3.0.7. Let $\mathcal{A}$ be a binormal category. Then any morphism in $\mathcal{A}$ which is both monic and epic is an isomorphism.

Proposition 3.0.8. Let $\mathcal{A}$ be a category with zero maps and $f : A \to B$ a morphism. Then,

(a) if f is a normal monomorphism then $f = \ker \operatorname{coker} f$

(b) if f is a conormal epimorphism then $f = \operatorname{coker} \ker f$

Proof. If f is a normal monomorphism then $f = \ker g$ for some $g : B \to C$. Consider the diagram,

$$\begin{tikzcd}
 & Z \arrow[dl, dashed] \arrow[d, "t"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "k"] & C \\
 & K \arrow[ur, dashed, "\tilde{g}"'] &
\end{tikzcd}$$

Because $f = \ker g$ we know that $g \circ f = 0$ and thus g factors through the cokernel of $f : A \to B$ which is K. Then if $k \circ t = 0$ we see that $g \circ t = \tilde{g} \circ k \circ t = 0$ meaning that t factors uniquely through $f : A \to B$ because $f = \ker g$ showing that $f = \ker k$. The second statement is exactly dual. $\square$

4 Images and Coimages (DOOO!!!! COMPARE WITH MAGIC SQUARE)

Definition 4.0.1. The image $\operatorname{im} f$ of a morphism $f : X \to Y$ is the smallest subobject of Y such that f factors through $\operatorname{im} f \to Y$. Explicitly, this is a factorization $f = m \circ e$ with m monic such that for any other factorization $f = m' \circ e'$ with m monic as in the diagram,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, "e'"'] & & Y \\
 & \operatorname{im} f \arrow[ur, hook, "m"'] \arrow[d, dashed, "v"'] & \\
 & Z \arrow[uur, hook, "m'"'] &
\end{tikzcd}$$

there is a unique arrow $v : \operatorname{im} f \to Z$ making the diagram commute.

Definition 4.0.2. The coimage $\operatorname{coim} f$ of a morphism $f : X \to Y$ is the image of f in the opposite category or equivalently the largest quotient of X such that f factors through $X \to \operatorname{coim} f$. Explicitly, this is a factorization $f = m \circ e$ with e epic such that for any other factorization $f = m' \circ e'$ with e epic as in the diagram,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "e"] \arrow[ddr, two heads, "e'"'] & & Y \\
 & \operatorname{coim} f \arrow[ur, "m"'] & \\
 & Z \arrow[u, dashed, "v"'] \arrow[uur, "m'"'] &
\end{tikzcd}$$

there is a unique arrow $v : Z \to \operatorname{coim} f$ making the diagram commute.

5 Abelian Categories (DO!!!)

DEFINITON OF AB-CAT
DEF OF IM AND COIM
IM = COIM

Definition 5.0.1. We say that a sequence,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a complex if $g \circ f = 0$ giving a monomorphism $\mathrm{Im}\,(f) \to \ker g$. We say the sequence is *exact* if this morphism is also epic i.e. an isomorphism by the above lemma.

Proposition 5.0.2. $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$
is exact iff $(X \xrightarrow{f} Y) = \ker g$ and,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact iff $(Y \xrightarrow{g} Y) = \mathrm{coker}\, f$.

Proof. DO THIS PROOF □

Definition 5.0.3. ABELIAN FUNCTOR

Definition 5.0.4. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then we say that,

(a) F is *left-exact* if F preserves kernels

(b) F is *right-exact* if F preserves cokernels

(c) F is *exact* if F preserves exact sequences

Proposition 5.0.5. F is exact iff F is left and right-exact.

Proof. DO THIS!! □

Proposition 5.0.6. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{B}$ be an adjoint pair of additive functors between abelian categories. Then F is right-exact and G is left-exact.

Proof. Left-adjoints preserve colimits and right-adjionts preserve limits. □

6 Homology