Math 56: Proofs and Modern Mathematics
Homework 5 Solutions

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Problem 1 (Axler 3.D.7)**.** Suppose V and W are finite dimensional and $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}$$

1. Show that E is a subspace of $\mathcal{L}(V, W)$.
2. Suppose $v \neq 0$. What is $\dim E$?

Solution. 1. We need to show that E contains 0 and is closed under addition and scalar multiplication. The zero element in $\mathcal{L}(V, W)$ is the map that sends everything to 0 in W, which clearly sends v to 0 and so is in E. If T_1 and T_2 are in E, so that $T_1 v = T_2 v = 0$, then $(T_1 + T_2)(v) = T_1 v + T_2 v = 0 + 0 = 0$, so $T_1 + T_2 \in E$. Finally, if $T \in E$ and λ is a scalar, then $(\lambda T)(v) = \lambda(Tv) = \lambda(0) = 0$, so $\lambda T \in E$. Hence E is a subspace of $\mathcal{L}(V, W)$.

2. There are a few different ways we can prove this. I'll give you a few that I thought of or saw in your solutions.

Method 1. Define the following map:

$$\begin{aligned} F : \mathcal{L}(V, W) &\to W \\ F(T) &= Tv. \end{aligned}$$

This is a linear map, since $(T + T')(v) = Tv + T'v$, and $(\lambda T)(v) = \lambda(Tv)$. By definition, $\operatorname{null} F = \{T \in \mathcal{L}(V, W) : Tv = 0\} = E$. Since $v \neq 0$, we can extend v to a basis $v = v_1, v_2, \ldots, v_n$ of V, where $n = \dim V$. This means that for any $w \in W$, we can find $T \in \mathcal{L}(V, W)$ such that $Tv = w$: just let $Tv = w$ and $Tv_i = 0$ for $i \geq 2$, and extend linearly to a map on V. Hence F is surjective, so $\dim \operatorname{range} F = \dim W$. By the rank-nullity theorem, we have

$$\begin{aligned} \dim \operatorname{null} F + \dim \operatorname{range} F &= \dim \mathcal{L}(V, W) \\ \implies \dim E = \dim \operatorname{null} F &= \dim \mathcal{L}(V, W) - \dim \operatorname{range} F \\ &= \dim \mathcal{L}(V, W) - \dim W \\ &= \dim V \dim W - \dim W \\ &= (\dim V - 1) \dim W. \end{aligned}$$

Method 2. Since $v \neq 0$, we can extend v to a basis $v = v_1, v_2, \ldots, v_n$ of V, where $n = \dim V$. Let $V' = \operatorname{span}(v_2, \ldots, v_n)$, so that $V = \operatorname{span}(v) \oplus V'$. Define the following map.

$$\begin{aligned} F : E &\to \mathcal{L}(V', W) \\ F(T)(v') &= T(v'), \end{aligned}$$

where $v' \in V' \subset V$ is any element in V'. What F does is take a map in E, which means it's a map $V \to W$, and restrict it to $V' \subset V$. Since $(T + T')v' = Tv' + T'v'$, and $(\lambda T)v' = \lambda(Tv')$, the map F is linear. Since $V = \operatorname{span}(v) \oplus V'$, we can write every element in V uniquely as $av + v'$ for some scalar a and some $v' \in V'$. Using this, we define an inverse for F given by

$$\begin{aligned} F^{-1} : \mathcal{L}(V', W) &\to E \\ F^{-1}(T)(av + v') &= T(v'). \end{aligned}$$

Note that for any $T \in \mathcal{L}(V', W)$, we have $F^{-1}(T)(v) = F^{-1}(T)(v + 0) = 0$, so that this is indeed a map in E. In the same way as above, F^{-1} is linear. For a map $T \in E$, and an element $av + v' \in V$, we have $FF^{-1}(T)(av + v') = Tv' = T(av + v')$, since T is linear and $Tv = 0$, and for a map $T \in \mathcal{L}(V, W)$ and an element $v' \in V'$ we have $F^{-1}F(T)v' = Tv'$. Hence F^{-1} is indeed an inverse for F, so F is an isomorphism, and therefore

$$\dim E = \dim \mathcal{L}(V', W) = \dim V \dim W = (\dim V - 1) \dim W.$$

Method 3. We can prove this explicitly by finding a basis for E. Since $v \neq 0$, we can extend v to a basis $v = v_1, v_2, \ldots, v_n$ of V, where $n = \dim V$. Let $w_1, \ldots, w_m$ be a basis for W. Define the linear maps T_{ij}, for $i = 2, \ldots, n$ and $j = 1, \ldots, m$ by

$$\begin{aligned} T_{ij}(v_i) &= w_j \\ T_{kj}(v_k) &= 0 \qquad \text{(for } k \neq i.\text{)} \end{aligned}$$

Note that all of these map $v_1 = v$ to 0, so they are all in E. We claim that the set of T_{ij} is a basis for E. We need to prove that this set is linearly independent and spans E.

Linearly independent: Suppose that we have a set of scalars a_{ij}, $i = 2, \ldots, n$, $j = 1, \ldots, m$ such that

$$a_{21}T_{21} + \cdots + a_{2m}T_{2m} + \cdots + a_{n1}T_{n1} + \cdots + a_{nm}T_{nm} = 0.$$

Consider the vector $v_2 \in V$. Since $a_{21}T_{21} + \cdots + a_{2m}T_{2m} + \cdots + a_{n1}T_{n1} + \cdots + a_{nm}T_{nm} = 0$, we have

$$a_{21}T_{21}v_2 + \cdots + a_{2m}T_{2m}v_2 + \cdots + a_{n1}T_{n1}v_2 + \cdots + a_{nm}T_{nm}v_2 = 0.$$

Terms on the left-hand side of the form $a_{ij}T_{ij}v_2$ where $i \neq 2$ are 0, and $T_{2j}v_2 = w_j$, so this simplifies to

$$a_{21}w_1 + \cdots + a_{2m}v_m = 0.$$

By linear independence of $w_1, \ldots, w_m$, this means that $a_{21}, \ldots, a_{2m} = 0$. Applying the same method for each v_i gives $a_{ij} = 0$ for $j = 1, \ldots, m$. Hence $a_{ij} = 0$ for all $i = 2, \ldots, n$ and $j = 1, \ldots, m$, so the set of these T_{ij} is linearly independent as required.

Spans E: Let T be an arbitrary map in E, so that $Tv = 0$. For $i = 2, \ldots, n$, we have

$$Tv_i = a_{i1}w_j + \cdots + a_{im}w_m$$

for some scalars $a_{i1}, \ldots, a_{im}$. Since T is completely determined by the images of the basis elements $v = v_1, v_2, \ldots, v_n$, we can conclude that

$$T = a_{21}T_{21} + \cdots + a_{2m}T_{2m} + \cdots + a_{n1}T_{n1} + \cdots + a_{nm}T_{nm},$$

since applying the maps on both sides to a basis vector gives us the same output, namely 0 for v, and $a_{i1}w_j + \cdots + a_{im}w_m$ for v_i, where $i = 2, \ldots, n$. Hence the set of T_{ij} spans E as required.

We have a basis for E, namely, the set of T_{ij}, where $i = 2, \ldots, n$ and $j = 1, \ldots, m$. This gives $\dim E = (n-1)m = (\dim V - 1)\dim W$.

Method 4. Since $v \neq 0$, we can extend it to a basis $v = v_1, \ldots, v_n$ of V. Let $w_1, \ldots, w_n$ be a basis for W. For each linear map $T \in \mathcal{L}(V, W)$, we can define its matrix $M(T)$ with respect to these bases; this gives us an isomorphism from $\mathcal{L}(V, W)$ to $\mathbb{F}^{m,n}$, which has dimension M. By definition, T is in E if and only if the first column of $M(T)$ is 0, so E corresponds to the subspace of $\mathbb{F}^{m,n}$ where the first column is all 0-s, hence $\dim E = m(n-1) = \dim W(\dim V - 1)$.

Problem 2 (Axler 3.D.14)**.** Suppose $v_1, v_2, \ldots, v_n$ is a basis of V. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by $Tv = M(v)$ is an isomorphism of V onto $\mathbb{F}^{n,1}$. Here $M(v)$ is the matrix of v with respect to the basis $v_1, \ldots, v_n$.

Solution. Recall that for $v = a_1v_1 + \cdots + a_nv_n$, the matrix of v with respect to the basis $v_1, \ldots, v_n$ is given by

$$\mathcal{M}(v) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

This defines a map $T : V \to \mathbb{F}^{n,1}$ given by $Tv = \mathcal{M}(v)$. We want to show that this map is an isomorphism, i.e., that it is linear and invertible.

Linear: Suppose that v, v' are two elements in V. Using the given basis, we can write $v = a_1v_1 + \cdots + a_nv_n$ and $v' = b_1v_1 + \cdots + b_nv_n$. Then

$$\begin{aligned}
T(v+v') &= T\left((a_1v_1 + \cdots + a_nv_n) + (b_1v_1 + \cdots + b_nv_n)\right) \\
&\qquad \text{(writing } v, v' \text{ in terms of the given basis)} \\
&= T\left((a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n\right) \\
&\qquad \text{(distributivity and commutative of addition in } V) \\
&= \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} && \text{(definition of } T) \\
&= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} && \text{(addition in } \mathbb{F}^{n,1}) \\
&= Tv + Tv' && \text{(definition of } T.)
\end{aligned}$$

Hence T is additive.

Next, suppose that v is an element of V, with $v = a_1v_1 + \cdots + a_nv_n$, and λ is a scalar. Then

$$\begin{aligned}
T(\lambda v) &= T\left(\lambda(a_1v_1 + \cdots + a_nv_n)\right) \\
&= T(\lambda a_1v_1 + \cdots + \lambda a_nv_n) && \text{(distributivity in } V) \\
&= \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix} && \text{(definition of } T) \\
&= \lambda \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} && \text{(scalar multiplication in } \mathbb{F}^{n,1}) \\
&= \lambda Tv && \text{(definition of } T.)
\end{aligned}$$

Hence T is also homogeneous, and so is a linear map.

We now want to prove that T is invertible. There are two similar ways we can approach this.

Method 1. We can define an inverse for T. Define the map

$$S : \mathbb{F}^{n,1} \to V$$
$$S\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1v_1 + \cdots + a_nv_n$$

By definition, we have $STv = v$ and $TS\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)$. Hence S is an inverse for T, so T is invertible and therefore an isomorphism.

Method 2. We can show that T is both injective and surjective.

Injective: Suppose that $Tv = 0$. We can write $v = a_1v_1 + \cdots + a_nv_n$, so that $Tv = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. By definition, if this is the zero matrix in $\mathbb{F}^{n,1}$, then we must have $a_1 = \cdots = a_n = 0$, so that $v = 0$. Hence T contains only the zero vector, so T is injective.

Surjective: Consider an arbitrary matrix $M = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ in $\mathbb{F}^{n,1}$. Let $v = a_1v_1 + \cdots + a_nv_n$; by definition, this means that $Tv = M$. Hence T is surjective.

Note: We can use the rank-nullity theorem, along with the fact that $\dim V = \dim \mathbb{F}^{n,1} = n$, to show that T is invertible using only one of the above properties.

Problem 3. Compute the matrix of the linear map $T : V \to W$ with respect to the bases $\mathcal{B}$ of V and $\mathcal{C}$ of W for the following choices: $V = \mathcal{P}_2(R)$, $W = \mathcal{P} + 3(R)$, $\mathcal{B} = (1, x, x^2)$, $\mathcal{C} = (1, x, x^2, x^3)$,

$$(Tp)(x) = \int_0^x p(t)dt + p'(x).$$

Solution. We compute the image of each basis element in $\mathcal{B}$ under T. For $p(x) = x^n$, we have

$$\int_0^x p(t)dt = \int_0^x t^n dt = \left[\frac{1}{n+1}t^{n+1}\right]_0^x = \frac{1}{n+1}x^{n+1},$$

and $p'(x) = nx^{n-1}$. This gives us

$$\begin{aligned} T1(x) &= x \\ Tx(x) &= \frac{1}{2}x^2 + 1 \\ Tx^2(x) &= \frac{1}{3}x^3 + 2x. \end{aligned}$$

Hence the matrix of T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Problem 4. Let v_1, v_2 be the basis $\mathcal{B}$ of V and define a new basis $\mathcal{C}$ of V by $w_1 = v_1$, $w_2 = v_1 + v_2$. Let $T \in \mathcal{L}(V, V)$ be defined by $Tw_1 = 2w_1$, $Tw_2 = -3w_2$. What is $M(T)$ in the basis $\mathcal{C}$? What is $M(T)$ in the basis $\mathcal{B}$?

Solution. By definition, the matrix for T with respect to $\mathcal{C}$ is just

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

To compute $M(T)$ in the basis $\mathcal{B}$, we're going to need to compute Tv_1 and Tv_2 in terms of v_1 and v_2. To do this, we note that $v_1 = w_1$ and $v_2 = w_2 - v_1 = w_2 - w_1$. So we have

$$Tv_1 = Tw_1 = 2w_1 = 2v_1$$
$$Tv_2 = T(w_2 - w_1) = Tw_2 - Tw1 = -3w_2 - 2w_1 = -3(v_1 + v_2) - 2v_1 = -5v_1 - 3v_2$$

Hence $M(T)$ with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 2 & -5 \\ 0 & -3 \end{bmatrix}.$$