1 The Chow Group

1.1 Flat Pullback

1.2 Proper Pushforward

2 Introduction to Intersection

Let X be an integral scheme proper over $S = \mathrm{Spec}\,(R)$ of dimension 2 with R noetherian. Given integral closed subschemes $C, D \subset X$ we want to make sense of the intersection $C \frown D$. For simplicitly, suppose that C, D are prime Cartier divisors. Then we would want the intersection multiplicity at $x \in X$ to be,

$$\iota_x(C, D) = \ell_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x}/(f, g)\right)$$

where f, g are the local equations cutting out C and D i.e. there is an affine open neighborhood $x \in U = \mathrm{Spec}\,(A)$ with $C \cap U = V(f)$ and $D \cap U = V(g)$. I claim these intersection multiplicities piece together to give a meaningful cycle in $A_0(X)$.

Definition 2.0.1. Let X be a noetherian scheme and $Z_i \subset X$ its irreducible components. Then the fundamental class of X is,

$$[X] := \sum_{i=1}^{n} m_i [Z_i] \in A_*(X)$$

where the multiplicities are,

$$m_i = \ell_{\mathcal{O}_{X,\xi_i}}\left(\mathcal{O}_{X,\xi_i}\right)$$

where ξ_i is the generic point of Z_i.

Example 2.0.2. Let $X = \mathrm{Spec}\,(k[x, y]/(xy^2))$. Then the irreducible components are $Z_1 = V(x)$ and $Z_2 = V(y)$ with generic points $\xi_1 = (x)$ and $\xi_2 = (y)$. However, notice that,

$$\mathcal{O}_{X,\xi_1} = k(y)$$

is a field so $m_1 = 1$ but,

$$\mathcal{O}_{X,\xi_2} = k(x)[y]/(y^2)$$

has length 2 over itself with submodule $(0) \subset (y) \subset \mathcal{O}_{X,\xi_2}$ reflecting the doubling of the x-axis. Therefore,

$$[X] = [Z_1] + 2[Z_2]$$

Example 2.0.3. Suppose that X is noetherian dimension zero scheme. Then $X = \mathrm{Spec}\,(A)$ for an artinian ring A. Then,

$$[X] = \sum_{\mathfrak{m} \subset A} \ell_{A_{\mathfrak{m}}}\left(A_{\mathfrak{m}}\right) \cdot [V(\mathfrak{m})]$$

and we see that,

$$\deg\,[X] = \sum_{\mathfrak{m} \subset A} \ell_{A_{\mathfrak{m}}}\left(A_{\mathfrak{m}}\right) = \ell_A\left(A\right)$$

Remark. When C and D intersect properly, i.e. $\dim\,(C \cap D) = 0$, we might define the intersection class of $C, D \subset X$ as follows. Let $(C \cap D) \subset X$ be the scheme theoretic intersection,

$$
\begin{tikzcd}
C \cap D \arrow[r, hook] \arrow[d, hook] & D \arrow[d, hook] \\
C \arrow[r, hook] & X
\end{tikzcd}
$$

then define $C \frown D = \iota_*[C \cap D]$ where $\iota : C \cap D \to X$ is the inclusion. If we take a point $x \in C \cap D$ and a sufficiently small open neighborhood $U = \mathrm{Spec}\,(A)$ then notice $C \cap D \cap U = \mathrm{Spec}\,(A/(f,g))$ which is artinian so,

$$
\begin{aligned}
[C \cap D \cap U] &= \sum_{\mathfrak{m} \in V(f,g)} \ell_{A_\mathfrak{m}/(f,g)}\left(A_\mathfrak{m}/(f,g)\right) \cdot [\mathfrak{m}] = \sum_{\mathfrak{m} \in V(f,g)} \ell_{A_\mathfrak{m}}\left(A_\mathfrak{m}/(f,g)\right) \cdot [\mathfrak{m}] \\
&= \sum_{x \in C \cap D \cap U} \ell_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x}/(f,g)\right) \cdot [x]
\end{aligned}
$$

which agrees with our definition of the intersection multiplicity.

Proposition 2.0.4. Suppose that $C, D \subset X$ are prime Cartier divisors intersecting properly. Then,

$$
C \frown D = (\iota_C)_*[c_1(\iota_C^* \mathcal{O}_X(D))] = \iota_*[C \cap D]
$$

where $\iota_C : C \to X$ and $\iota : C \cap D \to X$ are the inclusions.

Proof. Since C is a curve, to compute c_1 of the line bundle $\mathcal{L} = \iota_C^* \mathcal{O}_X(D)$ we need a nonzero section. The effective divisor D corresponds to a section $s_D \in \Gamma(X, \mathcal{O}_X(D))$ which pulls back to $s = \iota_C^* s_D$. Since the intersection is proper, s is not identically zero and therefore,

$$
c_1(\mathcal{L}) = \sum_{p \in C} \mathrm{ord}_p\left(s/s_\mathcal{L}\right) \cdot [p]
$$

where $s_\mathcal{L}$ is a local trivializing section of $\mathcal{L}$. Choose a sufficiently small affine open $U \subset X$ with $p \in U$ trivializing $\mathcal{O}_X(D)$ then $\mathcal{O}_X(D)|_U = g^{-1}\mathcal{O}_U$ and s_D corresponds to 1. Then we can take $s_\mathcal{L} = g^{-1}$ and $s_D = 1$ which gives,

$$
\mathrm{ord}_p(s/s_\mathcal{L}) = \mathrm{ord}_p(g) = \ell_{\mathcal{O}_{C,p}}\left(\mathcal{O}_{C,p}/(g)\right) = \ell_{\mathcal{O}_{X,p}}\left(\mathcal{O}_{X,p}/(f,g)\right)
$$

Therefore,

$$
c_1(\mathcal{L}) = \sum_{p \in C} \mathrm{ord}_p\left(s/s_\mathcal{L}\right) \cdot [p] = [C \cap D]_C
$$

and pushing forward by ι_C gives the desired result. $\square$

Remark. This result proves the symmetry,

$$
(\iota_C)_*[c_1(\iota_C^* \mathcal{O}_X(D))] = (\iota_D)_*[c_1(\iota_D^* \mathcal{O}_X(C))]
$$

Remark. Notice that even when C and D do not intersect properly the quantity,

$$
C \frown D = (\iota_C)_*[c_1(\iota_C^* \mathcal{O}_X(D))]
$$

is well-defined. In particular, the self-intersection equals,

$$
C^2 := C \frown C = (\iota_C)_*[c_1(\iota_C^* \mathcal{O}_X(C))] = (\iota_C)_*[c_1(\mathcal{N}_{C/X})]
$$

where $\mathcal{N}_{C/X} = \iota_C^* \mathcal{O}_X(C) = \mathcal{O}_C \otimes \mathcal{O}_X(C) = \mathcal{O}_X/\mathscr{I} \otimes \mathscr{I}^\vee = (\mathscr{I}/\mathscr{I}^2)^\vee = \mathcal{C}_{C/X}^\vee$ is the normal bundle. In paricular,

$$
\deg C^2 = \deg \mathcal{N}_{C/X}
$$

Remark. There is a more general formula due to Serre for the intersection multiplicity. Suppose that $C, D \subset X$ are closed subschemes. Let $Z \subset C \cap D$ be an irreducible component with generic point $\xi \in Z$. Then the multiplicity of Z in $C \cap D$ is defined to be,

$$\iota(Z; C, D) := \sum_{i=0}^{\infty} (-1)^i \ell_{\mathcal{O}_{X,\xi}} \left(\mathrm{Tor}_i^{\mathcal{O}_{X,\xi}} \left(\mathcal{O}_{X,\xi}/I, \mathcal{O}_{X,\xi}/J \right) \right)$$

where I and J are the ideals defining C and D in $\mathcal{O}_{X,\xi}$ and then the intersection cycle is,

$$C \frown D = \sum_{Z \subset C \cap D} \iota(Z; C, D)$$

Notice that when C and D are prime Cartier divisors we get $I = (f)$ and $J = (g)$ and thus,

$$\mathrm{Tor}_i^{\mathcal{O}_{X,\xi}} \left(\mathcal{O}_{X,\xi}/I, \mathcal{O}_{X,\xi}/J \right) = \begin{cases} \mathcal{O}_{X,\xi}/(f,g) & i = 0 \\ \ker \left(\mathcal{O}_{X,\xi}/(g) \xrightarrow{f} \mathcal{O}_{X,\xi}/(g) \right) & i = 1 \\ 0 & i > 1 \end{cases}$$

because f is a nonzerodivisor since we assumed that C is Cartier. Furthermore, since the intersection is proper, we cannot have $f \in (g)$ and since (g) is prime (Z is a prime Cartier divisor) the map $\mathcal{O}_{X,\xi}/(g) \xrightarrow{f} \mathcal{O}_{X,\xi}/(g)$ is injective. Therefore,

$$\mathrm{Tor}_i^{\mathcal{O}_{X,\xi}} \left(\mathcal{O}_{X,\xi}/I, \mathcal{O}_{X,\xi}/J \right) = \begin{cases} \mathcal{O}_{X,\xi}/(f,g) & i = 0 \\ 0 & i > 0 \end{cases}$$

giving,

$$\iota(Z; C, D) = \ell_{\mathcal{O}_{X,\xi}} \left(\mathcal{O}_{X,\xi}/(f,g) \right)$$

which agrees with our previous formula.

2.1 Adjunction

Given a smooth subvariety $Z \subset X$ of a smooth variety X, we know

$$\omega_Z = \omega_X|_Z \otimes \bigwedge^{\mathrm{top}} \mathcal{N}_{Z|X}$$

Therefore, taking chern classes,

$$c_1(\omega_Z) = \iota^* c_1(\omega_X) + c_1(\mathcal{N}_{Z|X})$$

and thus we find,

$$K_Z = K_X|_Z + c_1(\mathcal{N}_{Z|X})$$

2.1.1 Divisors

In particular, if $Z = V(D)$ for some divisor D then if Z is smooth,

$$\mathcal{N}_{Z|X} = \iota^* \mathcal{O}_X(D)$$

and therefore,

$$\omega_Z = \iota^*(\omega_X \otimes \mathcal{O}_X(D)) = \omega_X|_Z \otimes \mathcal{O}_D(D)$$

meaning,

$$K_Z = (K_X + D)|_Z$$

In fact, even when Z is not smooth we can compute,

$$\omega_Z = \iota_* \mathscr{E}xt^1_{\mathcal{O}_X}(\iota_* \mathcal{O}_Z, \omega_X)$$

However, there is a locally-free resolution,

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

and then we get an exact sequence,

$$0 \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\iota_* \mathcal{O}_Z, \omega_X) \longrightarrow \omega_X \longrightarrow \omega_X(D) \longrightarrow \iota_* \omega_Z \longrightarrow 0$$

Therefore, $\iota_* \omega_Z$ is the cokernel of the map $\omega_X \to \omega_X(D)$ defined by $\mathrm{id}_{\omega_X} \otimes s_D$ where $s_D : \mathcal{O}_X \to \mathcal{O}_X(D)$ is the canonical section corresponding to the inclusion $\mathcal{O}_X(-D) \to \mathcal{O}_X$. Therefore,

$$\iota_* \omega_Z = \omega_X \otimes \iota_* \mathcal{O}_Z \otimes \mathcal{O}_X(D) = \omega_X(D) \otimes \iota_* \mathcal{O}_Z$$

because ω_X is locally free. By the projection formula,

$$\iota_* \iota^* \omega_X(D) = \iota_*(\mathcal{O}_Z \otimes \iota^* \omega_X(D)) = \iota_* \mathcal{O}_Z \otimes \omega_X(D)$$

and therefore,

$$\omega_Z = \iota^* \omega_X(D)$$

2.2 Adjunction for Surfaces

Let X be a smooth surface which is a complete intersection in $P = \mathbb{P}^{c+2}$. Then $X \subset \mathbb{P}^{c+2}$ is cut out by r equations $f_1, \dots, f_r$ of degrees $d_1, \dots, d_c$. Because $\dim X = 2$ these form a regular sequence meaning the Kozul complex is exact,

$$0 \longrightarrow \mathcal{O}_P(-\sum_{i=1}^{c} d_i) \longrightarrow \bigoplus_{i=1}^{c} \mathcal{O}_P(-\sum_{j \neq i} d_j) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{c} \mathcal{O}_P(-d_i) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_X \longrightarrow 0$$

which gives a locally free resolution of $\mathcal{O}_X$. Therefore, we can compute,

$$\iota_* \omega_X = \mathscr{E}xt^c_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P)$$

using this resolution via,

$$\iota_* \omega_X = H^c(\mathscr{H}om_{\mathcal{O}_P}(K_\bullet, \omega_P)) = \mathrm{coker} \left(\bigoplus_{i=1}^{c} \omega_P(\sum_{j \neq i} d_j) \xrightarrow{f_1, \dots, f_c} \omega_P(\sum_{i=1}^{c} d_i) \right) = \omega_P(d_1 + \cdots + d_c) \otimes_{\mathcal{O}_P} \mathcal{O}_X$$

Therefore, we find that,

$$\omega_X = \omega_P(d_1 + \cdots + d_c)|_X = \mathcal{O}_X(d - c - 3)$$

where $d = d_1 + \cdots + d_c$ is the total degree. Therefore, when X is smooth we find,

$$K_X = (d - c - 3)H$$

where $H = \iota^* c_1(\mathcal{O}_P(1))$ is the hyperplane class of the embedding $X \hookrightarrow P$.

Now consider a divisor $C \subset X$. By the adjunction formula,

$$\omega_C = \omega_X(C)|_C$$

Therefore, by Riemann-Roch for singular curves,

$$2p_a - 2 = \deg \omega_C = \deg \omega_X(C)|_C$$

However, we have shown,

$$D \frown (K_X + C) = c_1(\omega_X(C)|_C)$$

and therefore,

$$D \cdot (K_X + C) = \deg [D \frown (K_X + C)] = \deg \omega_X(C)|_C$$

so we find that,

$$2p_a - 2 = C \cdot (K_X + C)$$

However,

$$C \cdot K_X = \deg \omega_X|_C = \deg \mathcal{O}_P(d - c - 3)|_C = (d - c - 3) \deg \mathcal{O}_P(1)|_C$$

we define $\deg C = \deg \mathcal{O}_P(1)|_C$ which implies that,

$$C \cdot K_X = (d - c - 3) \deg C$$

Alternatively, we can use adjunction,

$$C \cdot K_X = C \cdot (d - c - 3)H = (d - c - 3)C \cdot H$$

and we write $\deg C = C \cdot H$ for the intersection of C with a generic hyperplane. Therefore,

$$2p_a - 2 = C^2 + (d - c - 3) \deg C$$

In particular, consider the case of (-1)-curves i.e. rational curves with $C^2 = -1$. Then we find,

$$\deg C = \frac{1}{3 + c - d}$$

Therefore, we can only have (-1)-curves when $d = c + 2$ i.e. when $\omega_X = \mathcal{O}_X(-1)$. In this case, $\deg C = 1$ meaning that the (-1)-curves are lines in P. Furthermore, lines in P have $\deg C = 1$ and $p_a = 0$ meaning that for a line $L \subset X$ it has self-intersection,

$$L^2 = c + 1 - d$$

3 General Intersection Theory

4 Chern Classes