Math 4081 HW#1, due WEDNESDAY 1/31/18 **NAME: Benjamin Church**

1. Lee 1-5 [SECOND edition] = Lee 1-7 [FIRST edition 1-7]. So either way you will have done it (since hw 1 originally read 1. Lee 1-5, 2. Lee 1-7).

ONLY DO THE CASE WHEN $n = 2$.

Let N denote the north pole $(0, ..., 0, 1) \in S^n \subset \mathbb{R}^{n+1}$ and let S denote the south pole $(0, ..., 0, -1)$. Define the **stereographic projection** $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, ..., x^{n+1}) = \frac{(x^1, ..., x^n)}{1 - x^{n+1}}$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.
(a) For any $x \in S^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13 in LEE SECOND and FIRST). Similarly, show that $\widetilde{\sigma}(x)$is the point where the line through S and x intersects the same subspace. (For this reason, z is called **stereographic projection from the south pole.**)

Solution: The following is restricted to the case $n = 2$. Consider the line through the points $(0, 0, 1), (x, y, z) \in S^2$ which can be parametrized by $\{(tx, ty, (1-t) + tz) \mid t \in \mathbb{R}\}$. When this intersects the equitorial plane, $(1 - t) + tz = 0$ so $t = \frac{1}{1-z}$. Therefore, the intersection point is at, $\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$ which equals $\sigma(x, y, z)$ with $\mathbb{R}^2$ identified as the equitorial plane. The case for the south pole is identical.

Consider the line through the points $(0, 0, -1), (x, y, z) \in S^2$ which can be parametrized by $\{(tx, ty, -(1 - t) + tz) \mid t \in \mathbb{R}\}$. When this intersects the equitorial plane, $-(1 - t) + tz = 0$ so $t = \frac{1}{1+z}$. Therefore, the intersection point is at, $\left(\frac{x}{1+z}, \frac{y}{1+z}, 0\right)$ which equals $\tilde{\sigma}(x, y, z)$ with $\mathbb{R}^2$ identified as the equitorial plane.

(b) Show that σ is bijective, and

$$\sigma^{-1}(u^1, ..., u^n) = \frac{(2u^1, ..2u^n, |u|^2 - 1)}{|u|^2 + 1}$$

Solution:

It suffices to show that the compositions $\sigma \circ \sigma^{-1} = \mathrm{id}_{\mathbb{R}^2}$ and $\sigma^{-1} \circ \sigma = \mathrm{id}_{S^2}$. This follows directly by computation, take $(x, y, z) \in S^2$ and $(u, v) \in \mathbb{R}^2$,

$$\begin{aligned}
\sigma^{-1} \circ \sigma(x, y, z) = \sigma^{-1}\left(\frac{(x, y)}{1 - z}\right) &= \frac{\left(\frac{2x}{1-z}, \frac{2y}{1-z}, \left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2 - 1\right)}{\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2 + 1} \\
&= \frac{(2x(1 - z), 2y(1 - z), x^2 + y^2 - (1 - z)^2)}{x^2 + y^2 + (1 - z)^2} \\
&= \frac{(2x(1 - z), 2y(1 - z), -2z^2 + 2z)}{2(1 - z) + (x^2 + y^2 + z^2 - 1)} = (x, y, z) = \mathrm{id}_{S^2}(x, y, z) \\
\sigma \circ \sigma^{-1}(u, v) = \sigma\left(\frac{(2u, 2v, u^2 + v^2 - 1)}{u^2 + v^2 + 1}\right) &= \frac{(2u, 2v)}{(u^2 + v^2 + 1)\left(1 - \frac{u^2+v^2-1}{u^2+v^2+1}\right)} \\
&= \frac{(2u, 2v)(u^2 + v^2 + 1)}{2(u^2 + v^2 + 1)} = (u, v) = \mathrm{id}_{\mathbb{R}^2}(u, v)
\end{aligned}$$

(c) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\}, \sigma)$ and $(S^n \setminus \{S\}, \widetilde{\sigma})$ defines a smooth structure on S^n. (The coordinates defined by σ or $\tilde{\sigma}$ are called stereographic coordinates.)

Solution: Take $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}\tilde{\sigma} \circ \sigma^{-1}(x,y) = -\sigma(-\sigma^{-1}(x,y)) &= -\frac{(-2x,-2y)}{(x^2+y^2+1)\left(1-\frac{x^2+y^2-1}{x^2+y^2+1}\right)} \\ &= \frac{2(x,y)}{x^2+y^2+1-(x^2+y^2-1)} = (x,y)\end{aligned}$$

is a diffeomorphism. Therefore, $(S^n \setminus \{N\}, \sigma)$ and $(S^n \setminus \{S\}, \widetilde{\sigma})$ are smoothly compatible charts and thus there exists a unique maximal atlas containing these charts which defines a smooth structure on S^2.

(d) Show that this smooth structure is the same as the one defined in Example 1.31 from LEE SECOND or Example 1.20 in LEE FIRST.

Solution:

It suffices to show that σ and $\widetilde{\sigma}$ are smoothly compatible with $\phi_i^{\pm}$, because then the union of the smooth atlases is a smooth atlas which implies that the maximal atlases defined by the two sets of charts are in fact equal.
Consider $\phi_i^{\pm} \circ \sigma^{-1} : \sigma(U_i^{\pm} \cap (S^2 \setminus \{N\})) \to \phi_i^{\pm}(U_i^{\pm} \cap (S^2 \setminus \{N\}))$. Take $(x, y) \in \mathbb{R}^2$ then,

$$\phi_1^{\pm} \circ \sigma^{-1}(x,y) = \phi_1^{\pm}\left(\frac{(2x,2y,x^2+y^2-1)}{x^2+y^2+1}\right) = \frac{(2x,-z^2)}{x^2+y^2+1}$$

$$\phi_2^{\pm} \circ \sigma^{-1}(x,y) = \phi_2^{\pm}\left(\frac{(2x,2y,x^2+y^2-1)}{x^2+y^2+1}\right) = \frac{(2y,-z^2)}{x^2+y^2+1}$$

$$\phi_2^{\pm} \circ \sigma^{-1}(x,y) = \phi_2^{\pm}\left(\frac{(2x,2y,x^2+y^2-1)}{x^2+y^2+1}\right) = \frac{(2x,2y)}{x^2+y^2+1}$$

which are all diffeomorphic by basic analysis.

Similarly, for $\phi_i^{\pm} \circ \widetilde{\sigma}^{-1} : \widetilde{\sigma}(U_i^{\pm} \cap (S^n \setminus S)) \to \phi_i^{\pm}(U_i^{\pm} \cap (S^n \setminus S))$. Take $(x, y) \in \mathbb{R}^2$ then,

$$\phi_1^{\pm} \circ \tilde{\sigma}^{-1}(x,y) = \phi_1^{\pm}\left(\frac{(2x,2y,1-x^2-y^2)}{x^2+y^2+1}\right) = \frac{(2x,z^2)}{x^2+y^2+1}$$

$$\phi_2^{\pm} \circ \tilde{\sigma}^{-1}(x,y) = \phi_2^{\pm}\left(\frac{(2x,2y,1-x^2-y^2)}{x^2+y^2+1}\right) = \frac{(2y,z^2)}{x^2+y^2+1}$$

$$\phi_2^{\pm} \circ \tilde{\sigma}^{-1}(x,y) = \phi_2^{\pm}\left(\frac{(2x,2y,1-x^2-y^2)}{x^2+y^2+1}\right) = \frac{(2x,2y)}{x^2+y^2+1}$$

which are all diffeomorphic by basic analysis.

2. Lee 1.6 [SECOND EDITION] (probably more interesting than the first option)
Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from B^n to itself, which is a diffeomorphism if and only if $s = 1$.]

Solution: Let M be a nonempty smooth n-manifold for $n \geq 1$ with some smooth atlas $\mathcal{A}$ and pick a point $p \in M$. First, we will need a lemma:

Lemma 1. *For any $p \in M$ there exists a smooth atlas $\mathcal{A}'$ such that p is contained in the domain of exactly one chart in $\mathcal{A}'$ and both $\mathcal{A}$ and $\mathcal{A}'$ define the same smooth structure on M.*

Proof. Let $\mathcal{C}_p \subset \mathcal{A}$ be the set $\mathcal{C}_p = \{(U, \phi) \in \mathcal{A} \mid p \in U\}$. Because M is Hausdorff, the sigleton $\{p\}$ is closed so the sets $U \setminus \{p\}$ are open. Define $\mathcal{A}'$ by replaceing each $(U, \phi) \in \mathcal{C}_p$ with $(U \setminus \{p\}, \phi|_{U \setminus \{p\}})$ which is still a chart because $U \setminus \{p\}$ is open and the restiction of any homeomorphism is still a homeomorphism onto its image. Clearly, $\mathcal{A}'$ is a smooth atlas because the transition maps are simply restrictions of the transition maps of $\mathcal{A}$ which are smooth. To prove that $\mathcal{A}$ and $\mathcal{A}'$ define the same smooth structure, it suffices to show that their union is a smooth atlas. However, the transition maps between $\mathcal{A}$ and $\mathcal{A}'$ are also restrictions of smooth maps and are therefore smooth. $\square$

By the previous lemma, we can assume that $\mathcal{A}$ has exactly one chart containing p, namely, (U_p, ϕ_p). We will construct a new atlas $\mathcal{B}_s$ by replacing (U_p, ϕ_p) with (U_p, ϕ_s) defined as,

$$\phi_s(x) = F_s(\phi_p(x) - \phi_p(p))$$

For any $s > 0$, I claim that $\mathcal{B}_s$ is a smooth atlas. This is because for any other chart $(V, \psi) \in \mathcal{B}_s$ we know $p \notin U_p \cap V$ so the map,

$$\psi \circ \phi_s^{-1} : \phi_s(U \cap V_p) \to \psi(U \cap V_p)$$

cannot contain 0 in its domain. This is because $\phi_s(p) = 0$ but ϕ_s is an injection and $p \notin U \cap V_p$ so 0 is not in the image unde ϕ_s'. I claim that F_s is a diffeomorphism on any set not containing 0. I will show this by exhibting an inverse,

$$F_s^{-1}(x) = \frac{x}{|x|^{1-\frac{1}{s}}}$$

which is well defined $(s > 0)$ and smooth as long as $x \neq 0$. Furthermore,

$$F_s \circ F_s^{-1}(x) = \frac{|x|^{s-1}}{|x|^{(1-\frac{1}{s})(s-1)}} \frac{x}{|x|^{1-\frac{1}{s}}} = x \qquad F_s^{-1} \circ F_s(x) = \frac{|x|^{s-1}x}{|x|^{s(1-\frac{1}{s})}} = x$$

Thus, F_s is a diffeomorphism away from $x = 0$. Therefore,

$$\psi \circ \phi_s^{-1}(x) = \psi \circ \phi_p^{-1} \circ (F_s^{-1}(x) + \phi_p(p))$$

is a diffeomorphism because $x \neq 0$. Furthermore, the transition map of any two charts in $\mathcal{B}_s$ neither of which are (U_p, ϕ_s) are smooth because they are also charts of $\mathcal{A}$ which is a smooth atlas.

Finally, we need to show that $\mathcal{B}_s$ and $\mathcal{B}_{s'}$ define different smooth structures on M if $s \neq s'$.

If $\mathcal{B}_s$ and $\mathcal{B}_{s'}$ defined the same smooth structure then their union would be a smooth atlas. However,
$$\phi_s' \circ \phi_s^{-1}(x) = F_{s'}(\phi_p(\phi_p^{-1}(F_s^{-1}(x) + \phi_p(p)) - \phi_p(p)) = F_{s'}(F_s^{-1}(x))$$
which is not smooth unless $s = s'$. This holds because,
$$F_{s'} \circ F_s^{-1}(x) = \frac{|x|^{s'-1}}{|x|^{(1-\frac{1}{s})(s'-1)}} \frac{|x|^{s'-1}}{|x|^{s'-\frac{s'}{s}}} x = |x|^{\frac{s'}{s}-1} x$$
which is not differentiable at $x = 0$ unless $s = s'$. Thus, the charts of $\mathcal{B}_s$ and $\mathcal{B}_{s'}$ are not smoothly compatable. Therefore, there is a distinct smooth structure on M for each $s \in (0, \infty)$ which is an uncountable set.

3. DO ONE OF the following:

Lee 1.7 [First Edition] = Lee 1.9 [Second Edition]

(Just do the case $n = 1$.) Also check that the projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ is smooth.

Complex projective n-space, denoted by CP^n, is the set of all 1-dimensional complex-linear subspaces of C^{n+1}, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to CP^n$. Show that CP^n is a compact 2n-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for RP^n (done in an earlier example in Chapter 1 in both versions). We use the correspondence

$$(x^i + iy^1, ..., x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, ...x^{n+1}, y^{n+1})$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$

Solution:

We first show that $\mathbb{CP}^1$ is locally euclidean. Notation: $\mathcal{C} : \mathbb{R}^2 \to \mathbb{C}$ defined by $x + iy \mapsto (x, y)$. I will use the charts on $\mathbb{CP}^1$ defined by the charts,

$$\begin{aligned}
(\mathbb{CP}^1 \backslash \{[1,0]\}, \phi_1) \quad \text{with} \quad & \phi_1 : [x+iy, u+iv] \mapsto \mathcal{C}\left(\frac{x+iy}{u+iv}\right) \\
& \phi_1^{-1} : (x,y) \mapsto [x+iy, 1] \\
(\mathbb{CP}^1 \backslash \{[0,1]\}, \phi_2) \quad \text{with} \quad & \phi_2 : [x+iy, u+iv] \mapsto \mathcal{C}\left(\frac{u+iv}{x+iy}\right) \\
& \phi_2^{-1} : (x,y) \mapsto [1, x+iy]
\end{aligned}$$

We note that these are, in fact, inverses because. We have,

$$\phi_1^{-1} \circ \phi_1([x+iy, u+iv]) = \phi_1^{-1}\left(\frac{x+iy}{u+iv}\right) = \left[\frac{x+iy}{u+iv}, 1\right] \cong [x+iy, u+iv]$$

which is well defined because $u + iv \neq 0$ on the domain of ϕ_1. Similarly,

$$\phi_1 \circ \phi_1^{-1}(x,y) = \phi_1[x+iy, 1] = \mathcal{C}\left(\frac{x+iy}{1}\right) = (x,y)$$

Likewise,

$$\phi_2^{-1} \circ \phi_2([u+iv, x+iy]) = \phi_2^{-1}\left(\frac{u+iv}{x+iy}\right) = \left[1, \frac{u+iv}{x+iy}\right] \cong [x+iy, u+iv]$$

which is well defined because $x + iy \neq 0$ on the domain of ϕ_2. Similarly,

$$\phi_2 \circ \phi_2^{-1}(x,y) = \phi_2[1, x+iy] = \mathcal{C}\left(\frac{x+iy}{1}\right) = (x,y)$$

Consider the maps from $\mathbb{C}^2 \to \mathbb{R}^2$,

$$\phi_1 \circ \pi(x+iy, u+iv) = \phi_1([x+iy, u+iv]) = \mathcal{C}\left(\frac{x+iy}{u+iv}\right)$$

and

$$\phi_2 \circ \pi(x+iy, u+iv) = \phi_2([x+iy, u+iv]) = \mathcal{C}\left(\frac{u+iv}{x+iy}\right)$$

which are continuous because the domains are restricted to the sets where these denominators are nonzero. By the properties of quotient maps, ϕ_1 and ϕ_2 must be continuous. Similarly, the inverses are easily seen to be continuous. Therefore ϕ_1 and ϕ_2 are homeomorphisms onto their image. The Hausdorff and second countable properties are easily checked via the identification with $\mathbb{R}^4$. Therefore $\mathbb{CP}^1$ is a 2-manifold.

It remains to show that the atlas given by the charts ϕ_1 and ϕ_2 defines a smooth structure on $\mathbb{CP}^1$. Consider the transition map,

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$$

where $U_1 \cap U_2 = \mathbb{CP}^1 \backslash \{[1,0],[0,1]\}$. Now, consider, $(x,y) \in \phi_2(U_1 \cap U_2)$ then $(x,y) \neq 0$ since $\phi_2([1,0]) = (0,0) \notin \phi_2(U_1)$ and,

$$\phi_1 \circ \phi_2^{-1}(x,y) = \phi_1([1, x+iy]) = \mathcal{C}\left(\frac{1}{x+iy}\right) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

which is smooth in any region of the plane minus the origin. Similarly, the opposite transition map,

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and take $(x,y) \in \phi_1(U_1 \cap U_2)$ so $(x,y) \neq 0$ since $\phi_1([0,1]) = (0,0) \notin \phi_1(U_2)$. Then,

$$\phi_2 \circ \phi_1^{-1}(x,y) = \phi_1([x+iy, 1]) = \mathcal{C}\left(\frac{1}{x+iy}\right) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

which is likewise smooth in any region of the plane minus the origin. Therefore, the maps ϕ_1 and ϕ_2 are smoothly compatable so they define a smooth structure on $\mathbb{CP}^1$. Under this smooth stucture, we will show that $\mathbb{CP}^1$ is diffeomorphic to S^2 and thus homeomorphic because any smooth map is continuous. Therefore, $\mathbb{CP}^1$ is compact because S^2 is compact.

Finally, consider the projection map $\pi : \mathbb{C}^2 \backslash \{0\} \to \mathbb{CP}^1$. The chart, $\psi : \mathbb{C}^2 \backslash \{0\} \to \mathbb{R}^4$ given by,

$$\psi : (x+iy, u+iv) \to (x,y,u,v)$$

is obviously smoothly compatable with itself and therefore defines a smooth structure on $\mathbb{C}^2 \backslash \{0\}$. Therefore, the coordinate representation of the projection,

$$\phi_1 \circ \pi \circ \psi^{-1}(x,y,u,v) = \phi_1([x+iy, u+iv]) = \mathcal{C}\left(\frac{x+iy}{u+iv}\right)$$

which is smooth on a domain in which $(u,v) \neq 0$. For a point such that $u = v = 0$, the image of the projection is not within the domain of ϕ_1 so we must use the other chart,

$$\phi_2 \circ \pi \circ \psi^{-1}(x,y,u,v) = \phi_1([x+iy, u+iv]) = \mathcal{C}\left(\frac{u+iv}{x+iy}\right)$$

which is smooth on a domain in which $(x,y) \neq 0$. Therefore, one of these coordinate representations is well defined and smooth whenever $(x,y,u,v) \neq 0$ which always holds on $\mathbb{C}^2 \backslash \{0\}$. Thus, π is a smooth map.

4. Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.

Solution:

Notation: $\mathcal{C} : \mathbb{R}^2 \to \mathbb{C}$ defined by $x + iy \mapsto (x, y)$.

I will use the atlas on $\mathbb{CP}^1$ defined by the charts,

$$\begin{aligned}
(\mathbb{CP}^1\backslash\{[1,0]\}, \phi_1) \quad &\text{with} \quad \phi_1 : [x+iy, u+iv] \mapsto \mathcal{C}\left(\frac{x+iy}{u+iv}\right) \\
&\phantom{\text{with}} \quad \phi_1^{-1} : (x, y) \mapsto [x+iy, 1] \\
(\mathbb{CP}^1\backslash\{[0,1]\}, \phi_2) \quad &\text{with} \quad \phi_2 : [x+iy, u+iv] \mapsto \mathcal{C}\left(\frac{u+iv}{x+iy}\right) \\
&\phantom{\text{with}} \quad \phi_2^{-1} : (x, y) \mapsto [1, x+iy]
\end{aligned}$$

and the atlas on S^1 given by the steriographic projections defined above,

$$\begin{aligned}
(S^2\backslash\{(0,0,1)\}, \sigma) \quad &\text{with} \quad \sigma : (x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \\
&\phantom{\text{with}} \quad \sigma^{-1} : (x,y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right) \\
(S^2\backslash\{(0,0,-1)\}, \tilde{\sigma}) \quad &\text{with} \quad \tilde{\sigma} : (x,y,z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right) \\
&\phantom{\text{with}} \quad \sigma^{-1} : (x,y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{x^2+y^2+1}\right)
\end{aligned}$$

Now, I define the map $F : S^2 \to \mathbb{CP}^1$ by,

$$F : (x,y,z) \mapsto \begin{cases} \left[\frac{x+iy}{1-z}, 1\right] & z \neq 1 \\ [1, 0] & z = 1 \end{cases}$$

It remains to check that F is bijective, smooth, and has smooth inverse. First, we show that F is a bijection by exhibiting an inverse function,

$$F^{-1} : [x+iy, u+iv] \mapsto \begin{cases} \left(\frac{2\alpha}{\alpha^2+\beta^2+1}, \frac{2\beta}{\alpha^2+\beta^2+1}, \frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1}\right) & u+iv \neq 0 \\ (0,0,1) & u+iv = 0 \end{cases}$$

where $\alpha + i\beta = \frac{x+iy}{u+iv}$ which is well defined in $\mathbb{CP}^1$ because if $x + iy$ and $u + iv$ are scalled by the same nonzero complex number then their ratio $\alpha + i\beta$ remains constant. The following calculation shows that F is a bijection,

$$\begin{aligned}
F \circ F^{-1}([x+iy, u+iv]) &= \begin{cases} F\left(\frac{2\alpha}{\alpha^2+\beta^2+1}, \frac{2\beta}{\alpha^2+\beta^2+1}, \frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1}\right) & u+iv \neq 0 \\ F(0,0,1) & u+iv = 0 \end{cases} \\
&= \begin{cases} \left[\frac{2\alpha+2i\beta}{\alpha^2+\beta^1+1-(\alpha^2+\beta^2+1)}, 1\right] & u+iv \neq 0 \\ [1,0] & u+iv = 0 \end{cases} \\
&= \begin{cases} [\alpha+i\beta, 1] & u+iv \neq 0 \\ [1,0] & u+iv = 0 \end{cases}
\end{aligned}$$

However, if $u + iv \neq 0$ then $[x + iy, u + iv] \sim [\alpha + i\beta, 1]$ and otherwise $[x + iy, 1] \sim [1, 0]$. Therefore, $F \circ F^{-1} = \mathrm{id}_{\mathbb{CP}^1}$. Likewise,

$$
\begin{aligned}
F^{-1} \circ F(x, y, z) &= \begin{cases} F^{-1}\left(\left[\frac{x+iy}{1-z}, 1\right]\right) & z \neq 1 \\ F^{-1}([1, 0]) & z = 1 \end{cases} \\
&= \begin{cases} \left(\frac{2x(1-z)}{x^2+y^2+(1-z)^2}, \frac{2y(1-z)}{x^2+y^2+(1-z)^2}, \frac{x^2+y^2-(1-z)^2}{x^2+y^2+(1-z)^2}\right) & z \neq 1 \\ (0, 0, 1) & z = 1 \end{cases} \\
&= \begin{cases} (x, y, z) & z \neq 1 \\ (0, 0, 1) & z = 1 \end{cases}
\end{aligned}
$$

where the last line follows because $x^2 + y^2 + z^2 = 1$. Therefore $F^{-1} \circ F = \mathrm{id}_{S^2}$. Now, we must check that F and F^{-1} are smooth. The charts $\sigma, \tilde{\sigma}$ cover S^2 so we need to show that some choice of chart on $\mathbb{CP}^1$ makes the coordinate representation smooth. Consider,

$$
\phi_1 \circ F \circ \sigma^{-1}(x, y) = \phi_1 \circ F\left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right) = \phi_1([x + iy, 1]) = (x, y)
$$

This map is a diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. Because $z = \frac{x^2+y^2-1}{x^2+y^2+1} \neq 1$ we have that the domain of σ is mapped to within the domain of ϕ_1. Similarly,

$$
\begin{aligned}
\phi_2 \circ F \circ \tilde{\sigma}^{-1}(x, y) &= \phi_2 \circ F\left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{1 - x^2 - y^2}{x^2 + y^2 + 1}\right) \\
&= \begin{cases} \phi_2\left(\left[\frac{2x+2iy}{x^2+y^2+1-(1-x^2-y^2)}, 1\right]\right) & (x, y) \neq 0 \\ \phi_2([1, 0]) & x = y = 0 \end{cases} \\
&= \begin{cases} \phi_2\left(\left[\frac{x+iy}{x^2+y^2}, 1\right]\right) & (x, y) \neq 0 \\ \phi_2([1, 0]) & x = y = 0 \end{cases} \\
&= \begin{cases} (x, -y) & (x, y) \neq 0 \\ (0, 0) & x = y = 0 \end{cases} \\
&= (x, -y)
\end{aligned}
$$

This map is a diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. Because $F(x, y, z) = [0, 1]$ only when $(x, y, z) = (0, 0, -1)$ which is not in the domain of $\tilde{\sigma}$. Therefore, F maps the domain of $\tilde{\sigma}$ to inside the domain of ϕ_2. Because every point is in one of S^2 is contained in one of these domains, F is a smooth map. In fact, the coordinate representations of F are diffeomorphisms and F is a bijection so immediately, the coordinate maps, $(\phi_1 \circ F \circ \sigma^{-1})^{-1} = \sigma \circ F^{-1} \circ \phi_1^{-1}$ and $(\phi_2 \circ F \circ \tilde{\sigma}^{-1})^{-1} = \tilde{\sigma} \circ F^{-1} \circ \phi_2^{-1}$ are also smooth (since they are the inverses of diffeomorphisms). Thus, F is a diffeomorphism so $\mathbb{CP}^1 \cong S^2$.

5. Consider spherical coordinates on $\mathbb{R}^3$ (not including the line $x = y = 0$) ρ, ϕ, θ defined in terms of the Euclidean coordinates x, y, z by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

(a) Express $\partial/\partial\rho$, $\partial/\partial\phi$, and $\partial/\partial\theta$ as linear combinations of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$. (The coefficients in these linear combinations will be functions on $\mathbb{R}^3 \setminus (x = y = 0)$.)

Solution:

For any function $f(x, y, z)$ on the set $\mathbb{R}^3 \setminus \{0\}$ we find that,

$$\begin{aligned}
\frac{\partial f(x,y,z)}{\partial\rho} &= \frac{\partial x}{\partial\rho}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial\rho}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial\rho}\frac{\partial f}{\partial z} \\
&= \sin\phi\cos\theta\frac{\partial f}{\partial x} + \sin\phi\sin\theta\frac{\partial f}{\partial y} + \cos\phi\frac{\partial f}{\partial z} \\
\frac{\partial f(x,y,z)}{\partial\phi} &= \frac{\partial x}{\partial\phi}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial f}{\partial z} \\
&= \rho\cos\phi\cos\theta\frac{\partial f}{\partial x} + \rho\cos\phi\sin\theta\frac{\partial f}{\partial y} + 0\cdot\frac{\partial f}{\partial z} \\
\frac{\partial f(x,y,z)}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial f}{\partial z} \\
&= (-\rho\sin\phi\sin\theta)\frac{\partial f}{\partial x} + \rho\sin\phi\cos\theta\frac{\partial f}{\partial y} + \rho\sin\phi\cos\theta(-\sin\theta)\frac{\partial f}{\partial z}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial\rho} &= \sin\phi\cos\theta\frac{\partial f}{\partial x} + \sin\phi\sin\theta\frac{\partial f}{\partial y} + \cos\phi\frac{\partial f}{\partial z} \\
\frac{\partial}{\partial\phi} &= \rho\cos\phi\cos\theta\frac{\partial}{\partial x} + \rho\cos\phi\sin\theta\frac{\partial}{\partial y} \\
\frac{\partial}{\partial\theta} &= (-\rho\sin\phi\sin\theta)\frac{\partial}{\partial x} + \rho\sin\phi\cos\theta\frac{\partial}{\partial y} + \rho\sin\phi\cos\theta(-\sin\theta)\frac{\partial}{\partial z}
\end{aligned}$$

(b) Express $d\rho$, $d\phi$, and $d\theta$ as linear combinations of dx, dy, and dz.

Solution:

Note that $\rho^2 = x^2 + y^2 + z^2$ so,

$$2\rho\mathrm{d}\rho = 2x\mathrm{d}x + 2y\mathrm{d}y + 2z\mathrm{d}z \implies \mathrm{d}\rho = \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{\mathrm{d}\rho} = \frac{x\mathrm{d}x + y\mathrm{d}y + x\mathrm{d}z}{\sqrt{x^2+y^2+z^2}}$$

Similarly, $z = \rho\cos\phi$ and $\sin\phi = \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}$. Therefore,

$$\begin{aligned}
\mathrm{d}z &= -\mathrm{d}\phi\rho\sin\phi + \mathrm{d}\rho\cos\phi \\
\mathrm{d}\phi &= \frac{\frac{z}{\rho} - \mathrm{d}\rho}{\sqrt{x^2+y^2}} = \frac{z\frac{x\mathrm{d}x+y\mathrm{d}y+z\mathrm{d}z}{x^2+y^2+z^2} - \mathrm{d}z}{\sqrt{x^2+y^2}} \\
&= \frac{zx\mathrm{d}x + zy\mathrm{d}y - (x^2+y^2)\mathrm{d}z}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}
\end{aligned}$$

Finally, using $\tan\theta = \frac{y}{x}$ and differentiating both sides,

$$\begin{aligned}
\mathrm{d}y &= x\sec^2\theta d\theta + \mathrm{d}x\tan\theta \\
d\theta &= \frac{\mathrm{d}y - \mathrm{d}x\tan\theta}{x\sec^2\theta} = \frac{\sin^2\theta}{x}\left(\mathrm{d}y - \mathrm{d}x\tan\theta\right) = \frac{x}{x^2+y^2}\left(\mathrm{d}y - \mathrm{d}x\frac{y}{x}\right) \\
&= \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}
\end{aligned}$$

6. Let V and W be finite dimensional vector spaces and let $A : V \to W$ be a linear map. Show that the dual map $A^* : W^* \to V^*$ is given in coordinates as follows. Let $\{e_i\}$ and $\{f_j\}$ be bases for V and W, and let $\{e^i\}$ and $\{f^j\}$ be the corresponding dual bases for V^* and W^*. If $Ae_i = A_i^j f_j$ then $A^* f^j = A_i^j e^i$.

   Solution:

   Suppose that $Ae_i = A_i^j f_j$ then, $A^* f^j$ is a linear functional on V such that,

$$(A^* f^j)(e_k) = f^j(Ae_k) = f^j(A_i^r f_r) = A_i^r f^j(f_r) = A_i^r \delta_r^j = A_i^j$$

   However, $A^* f^j$ can be expressed in the dual basis, $A^* f^j = C_i^j e^i$ and $C_i^j e^i(e_k) = C_i^j \delta_k^i = C_k^j$. Thus, $A^* f^j = A_i^j e^i$.

7. Let V be a finite dimensional vector space and let $\langle \cdot, \cdot \rangle$ be an inner product on V. The inner product determines an isomorphism $\phi : V \to V^*$.

(a) Show that the isomorphism ϕ is given in coordinates as follows. Let $\{e_i\}$ be a basis for V, let $\{e^i\}$ be the dual basis, and write $g_{ij} = \langle e_i, e_j \rangle$. Then $\phi(e_i) = g_{ij}e^j$.

Solution:

Let the isomorphism $\phi : V \to V^*$ be given by $\phi(v) \mapsto \langle v, \cdot \rangle$. Then,

$$\phi(e_i)(e_k) = \langle e_i, e_k \rangle = g_{ik}$$

however, $\phi(e_i) \in V^*$ so $\phi(e_i)$ can be expressed in terms of the dual basis $\phi(e_i) = C_{ij}e^j$ and $C_{ij}e^j(e_k) = C_{ij}\delta^j_k = C_{ik}$ so $C_{ik} = g_{ik}$. Therefore, $\phi(e_i) = g_{ij}e^j$.

(b) The inner product, together with the isomorphism ϕ, define an inner product on V^*. Write this in coordinates as $g^{ij} = \langle e^i, e^j \rangle$. Show that the matrix (g^{ij}) is the inverse of the matrix (g_{ij}).

Solution:

Given the inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ and the isomorphism $\phi : V \to V^*$, we can define an inner product on the dual space by, $\langle u, w \rangle = \langle \phi^{-1}(u), \phi^{-1}(w) \rangle$ for $u, w \in V_*$. Now, define the upper components by $g^{ij} = \langle e^i, e^j \rangle$. Consider,

$$\phi^{-1}(e^i) = C^{ij}e_j \implies \phi(C^{ij}e_j) = C^{ij}\phi(e_j) = C^{ij}g_{jk}e^k \implies C^{ij}g_{jk} = \delta^i_k$$

Thus $C = g^{-1}$ but g is symmetric because,

$$g_{ij} = \langle e_i, e_j \rangle = \langle e_j, e_i \rangle = g_j i$$

so C is also symmetric and $g_{ij}C^{jk} = \delta^j_i$. Now, define $g^{ij} = \langle e^i, e^j \rangle$ then,

$$\begin{aligned}
g_{ij}g^{jk} &= g_{ij}\left\langle e^j, e^k \right\rangle = \left\langle g_{ij}e^j, e^k \right\rangle = \left\langle \phi(e_i), e^k \right\rangle = \left\langle e_i, \phi^{-1}(e^k) \right\rangle = \left\langle e_i, C^{kl}e_l \right\rangle \\
&= \langle e_i, e_l \rangle C^{kl} = g_{il}C^{kl} = \delta^i_k \\
g^{ij}g_{jk} &= \left\langle e^i, e^j \right\rangle g_{jk} = \left\langle e^i, g_{jk}e^j \right\rangle = \left\langle e^i, \phi(e_k) \right\rangle = \left\langle \phi^{-1}(e_i), e_k \right\rangle = \left\langle C^{il}e_l, e_k \right\rangle = \\
&= C^{il}\langle e_l, e_k \rangle = C^{il}g_{lk} = \delta^i_k
\end{aligned}$$

Therefore, g^{ij} is the inverse matrix of g_{ij}. In particular, $C^{ij} = g^{ij}$.

8. How difficult was this assignment? How many hours did you spend on it?

   I would not say this assignment was exactly difficult. It was time consuming and at times tedious but the ideas were not too difficult. Rather, I got bogged down in computations and notation. I spent about 7 - 8 hours on it.