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1 TODO

- (a) Measure theory notes
- (b) measure theory practice
- (c) Fredholm operators
- (d) Theorem on why you cant multiply distributions
- (e) compactness in weak topology (Banach-Agalou) and do fall 2011
- (f) Sobolev spaces
- (g) Schwartz spaces
- (h) Fourier analysis review main theorems

2 Important Notes to Self

2.1 Common Mistakes

- (a) Reflexive seperable Banach spaces need not admit a Schauder basis!! However, separable Hilbert spaces always admit an othogonal Schauder basis and thus are all isomorphic to ℓ^2 .

3 Measure Theory Definitions and Theorems

Definition 3.0.1. A measure space (X, \mathcal{F}, μ) is called σ -finite if there exists a countable cover $\{A_i\}$ of X by measurable sets $A_i \in \mathcal{F}$ with $\mu(A_i) < \infty$.

3.1 Integration

Remark. Fix a measure space $(\Omega, \mathcal{F}, \mu)$. A function on $X \in \mathcal{F}$ means a map $f : X \rightarrow \hat{\mathbb{R}}$ where $\hat{\mathbb{R}} = [-\infty, \infty]$ is the extended real numbers. We say that f is measurable if it is measurable with respect to the Borel σ -algebra on $\hat{\mathbb{R}}$.

Remark. For functions $f : \mathbb{R} \rightarrow \mathbb{R}$ the term “measurable” is a priori ambiguous because we have not specified the measure space on the domain (although our conventions do prescribe the Borel σ -algebra on the codomain). We take the standard convention that measurable means with respect to the Borel σ -algebra i.e. $f : (\mathbb{R}, \mathcal{B}) \rightarrow (\mathbb{R}, \mathcal{B})$ is measurable. Then we say that f is Lebesgue measurable if $f : (\mathbb{R}, \mathcal{L}) \rightarrow (\mathbb{R}, \mathcal{B})$ is measurable where \mathcal{L} is the σ -algebra of Lebesgue measurable sets. Notice that measurable implies Lebesgue measurable. However, neither implies that the function $f : (\mathbb{R}, \mathcal{L}) \rightarrow (\mathbb{R}, \mathcal{L})$ is measurable, a condition which has the distasteful property of not holding for all continuous [functions](#) (of course continuous functions are (Borel) measurable and Lebesgue measurable).

Definition 3.1.1. A measurable function $s : X \rightarrow \hat{\mathbb{R}}$ is *simple* if it takes on finitely many values. Clearly, any simple function can be written as,

$$s = \sum_{i=1}^n c_i \chi_{A_i}$$

for $c_i \in \mathbb{R}$. For a nonnegative simple function (i.e. $c_i \geq 0$) we define,

$$\int_X s \, d\mu = I(s) = \sum_{i=1}^n c_i \mu(A_i)$$

Remark. This is well-defined even when $\mu(A_i) = \infty$ because $c_i \geq 0$. Clearly $I(s)$ is independent of the sum representation of s .

Definition 3.1.2. For a nonnegative function $f \geq 0$ we define,

$$\int_X f \, d\mu = \sup\{I(s) \mid 0 \leq s \leq f \text{ where } s \text{ is simple}\}$$

Definition 3.1.3. We say that a measurable function f is *integrable* if,

$$\int_X |f| \, d\mu < \infty$$

Notice that the integral of f may be *definable* even if f is not integrable where the integral is definable if when writing $f = f^+ - f^-$ for nonnegative $f^+ = \max\{(f, 0)\}$ and $f^- = \max\{(-f, 0)\}$ with one of f^+ or f^- integrable in which case,

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

which of course may be infinite so we still do not say that f is integrable unless both integrals are finite in which case,

$$\int_X |f| \, d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu < \infty$$

is finite so indeed f is integrable in the previous sense.

Proposition 3.1.4. The following are basic properties of the Lebesgue integral,

(a) if $f \stackrel{\text{a.e.}}{=} g$ then,

$$\int_X f \, d\mu = \int_X g \, d\mu$$

(b) if $f \leq g$ a.e. then,

$$\int_X f \, d\mu \leq \int_X g \, d\mu$$

(c) if $f \geq 0$ a.e. then,

$$f \stackrel{\text{a.e.}}{=} 0 \iff \int_X f \, d\mu = 0$$

Theorem 3.1.5. If $f : X \rightarrow \mathbb{R}$ is a nonnegative measurable function then,

$$\int_X f \, d\mu = \int_0^\infty f^*(t) \, dt$$

where $f^*(t) = \mu(\{x \in X \mid f(x) > t\})$ is monotonic and thus Riemann integrable.

Theorem 3.1.6. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is Riemann integrable then for the Lebesgue measure μ ,

$$\int_{\mathbb{R}} f \, d\mu = \int_{\mathbb{R}} f \, dx$$

Theorem 3.1.7. Let $C_c(\mathbb{R})$ be the space of compactly supported continuous functions on \mathbb{R} with the following norm,

$$\|f\| = \int_{\mathbb{R}} |f(x)| \, dx$$

which may be computed by the Riemann integral since f is continuous. This always exists because f is supported on a compact (and thus bounded set) and it must thus be bounded. Then $C_c(\mathbb{R})$ is not complete but its completion is isomorphic to $L^1(\mathbb{R})$. Then, $\int : C_c(\mathbb{R}) \rightarrow \mathbb{R}$ is a bounded operator and is defined on the dense subspace $C_c(\mathbb{R}) \subset L^1(\mathbb{R})$ so it has a unique extension $\int : L^1(\mathbb{R}) \rightarrow \mathbb{R}$ which is exactly the Lebesgue integral on the space of Lebesgue integrable function (modulo $\stackrel{\text{a.e.}}{=}$).

Theorem 3.1.8 (Fubini). Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be σ -finite measure spaces and $f : X_1 \times X_2 \rightarrow \hat{\mathbb{R}}$ measurable. Then,

$$\int_{X_1 \times X_2} |f| \, d\mu = \int_{X_1} \left(\int_{X_2} |f(x, y)| \, d\mu_2(y) \right) d\mu_1(x) = \int_{X_2} \left(\int_{X_1} |f(x, y)| \, d\mu_1(x) \right) d\mu_2(y)$$

and if this value is finite then f is by definition integrable and furthermore,

$$\int_{X_1 \times X_2} f \, d\mu = \int_{X_1} \left(\int_{X_2} f(x, y) \, d\mu_2(y) \right) d\mu_1(x) = \int_{X_2} \left(\int_{X_1} f(x, y) \, d\mu_1(x) \right) d\mu_2(y)$$

3.2 Convergence in Measure (WIP)

Definition 3.2.1. Let (X, \mathcal{F}, μ) be a measure space and f_n a sequence of measurable functions and f a measurable function. We say that,

(a) $f_n \rightarrow f$ *globally in measure* if for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

(b) $f_n \rightarrow f$ *locally in measure* if for every finite measure $A \subset X$ and $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mu(\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

Theorem 3.2.2 (Borel-Cantelli). Let $A_n \subset X$ be a sequence of measurable sets such that,

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

Then,

$$\mu\left(\limsup_{n \rightarrow \infty} A_n\right) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 0$$

Proposition 3.2.3. Let (X, \mathcal{F}, μ) be a measure space and f_n a sequence of measurable functions and f a measurable function. Suppose either,

(a) $f_n \rightarrow f$ globally in measure or

(b) (X, \mathcal{F}, μ) is σ -finite and $f_n \rightarrow f$ locally in measure

then f_n has a subsequence converging pointwise a.e. to f .

Proof. For each $n > 0$ there exists some j_n such that for any $k \geq j_n$,

$$\mu(\{x \in X \mid |f_k(x) - f(x)| \geq \frac{1}{n}\}) < \frac{1}{2^n}$$

Let,

$$E_n = \{x \in X \mid |f_{j_n}(x) - f(x)| \geq \frac{1}{n}\}$$

I claim that the subsequence f_{j_n} converges to f on $U = X \setminus \limsup_{n \rightarrow \infty} E_n$ and that $\mu\left(\limsup_{n \rightarrow \infty} E_n\right) = 0$.

Indeed, for $x \in U$ there is some m such that for $n > m$ we have $x \notin E_n$. Thus for any $\epsilon > 0$ choose $N > m$ such that $\frac{1}{N} < \epsilon$ then for $n > N$ we have $\frac{1}{n} < \epsilon$ so,

$$|f_{j_n}(x) - f(x)| < \frac{1}{n} < \epsilon$$

because $x \notin E_n$. Furthermore, by construction,

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Therefore by Borel-Cantelli,

$$\mu\left(\limsup_{n \rightarrow \infty} E_n\right) = 0$$

as required. □

Proposition 3.2.4. If $f_n \rightarrow f$ pointwise locally almost everywhere then $f_n \rightarrow f$ locally in measure.

Proof. For any $A \in \mathcal{F}$ with $\mu(A) < \infty$ we know that,

$$\mu(\{x \in A \mid \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}) = \mu\left(\bigcup_{k=1}^{\infty} \{x \in A \mid \forall N : \exists n > N : |f_n(x) - f(x)| \geq \frac{1}{k}\}\right) = 0$$

Therefore, for each $\epsilon > 0$ we have,

$$\mu(\{x \in A \mid \forall N : \exists n > N : |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

Furthermore,

$$\begin{aligned} \{x \in A \mid \forall N : \exists n > N : |f_n(x) - f(x)| \geq \epsilon\} &= \bigcap_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} \{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\} \\ &= \limsup_{n \rightarrow \infty} \{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\} \end{aligned}$$

Now because $(A, \mathcal{F}|_A, \mu|_A)$ is a finite measure space μ is continuous with respect to descending limits of sets and thus,

$$\limsup_{n \rightarrow \infty} \mu(\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}) \geq \mu(\limsup_{n \rightarrow \infty} \{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

which implies (since the sequences are nonnegative) that,

$$\lim_{n \rightarrow \infty} \mu(\{x \in A \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

□

Remark. Even given $f_n \rightarrow f$ pointwise a.e. globally we cannot in general conclude that $f_n \rightarrow f$ globally in measure unless the measure space is finite. Indeed, consider, $f_n = \chi_{[n, \infty)}$ then $f_n \rightarrow 0$ pointwise everywhere. However, f_n does not converge to 0 in measure.

Proposition 3.2.5. Let (X, \mathcal{F}, μ) be a measure space functions $f_n, f \in L^p(X)$ such that $f_n \rightarrow f$ in the L^p -norm. Then $f_n \rightarrow f$ globally in measure.

Proof. First consider $1 \leq p < \infty$. Since we have $f_n \rightarrow f$ in $L^p(X)$ we know that $\|f_n - f\|_p \rightarrow 0$ in $L^p(X)$. Let $g_n = |f_n - f|^p$. Then we know, for any $\epsilon > 0$ and $\eta > 0$ there is some N such that for $n > N$,

$$\int_X g_n d\mu < \epsilon \eta^p$$

Then,

$$\mu(\{x \in X \mid g_n(x) \geq M\}) \leq \frac{1}{M} \int_X g_n d\mu$$

taking $M = \eta^p$ we see that,

$$\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \eta\}) = \mu(\{x \in X \mid g_n(x) \geq \eta^p\}) \leq \epsilon$$

and therefore,

$$\lim_{n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \eta\}) = 0$$

proving that $f_n \rightarrow f$ in measure. Finally, if $p = \infty$ then convergence in $L^\infty(X)$ implies uniform and thus pointwise convergence and therefore convergence in measure. □

Remark. A good counter example for the converse is the sequence $f_n = n\chi_{[0, \frac{1}{n}]}$ which converges to 0 almost everywhere and globally in measure but not in L^p -norm for any $p \geq 1$.

Corollary 3.2.6. If $f_n \rightarrow f$ converges in $L^p(X)$ then there is a subsequence converging pointwise a.e. to f .

Corollary 3.2.7. If $f_n \in L^p(X) \cap L^{p'}(X)$ and $f_n \rightarrow f$ in the L^p -norm and $f_n \rightarrow f'$ in the $L^{p'}$ -norm. Then $f \stackrel{\text{a.e.}}{=} f'$ so we see that $f_n \rightarrow f$ also in the $L^{p'}$ -norm and $f_n \rightarrow f'$ also in the L^p -norm.

Proof. Since $f_n \rightarrow f$ in L^p and $f_n \rightarrow f'$ in $L^{p'}$ we know that there are subsequences $\{f_{n_j}\}$ and $\{f_{m_j}\}$ such that pointwise a.e. $f_{n_j} \rightarrow f$ and $f_{m_j} \rightarrow f'$. However, $f_{m_j} \rightarrow f$ in L^p because it is a subsequence of a L^p convergent sequence. Then let $g_j = |f_{m_j} - f|^p$ and by Fatou's lemma,

$$\int_X \liminf_{j \rightarrow \infty} g_j \, d\mu \leq \liminf_{j \rightarrow \infty} \int_X g_j \, d\mu = 0$$

because $f_{m_j} \rightarrow f$ in L^p . However,

$$\liminf_{j \rightarrow \infty} g_j = \liminf_{j \rightarrow \infty} |f_{m_j} - f|^p \stackrel{\text{a.e.}}{=} |f' - f|^p$$

Therefore,

$$\int_X |f' - f|^p \, d\mu = 0$$

which implies that $f \stackrel{\text{a.e.}}{=} f'$. □

3.3 Lebesgue-Radon-Nikodym

Definition 3.3.1. Let (Ω, Σ) be a measurable space and μ, ν be Σ -measures. We say that,

- (a) ν is *absolutely continuous* with respect to μ ($\nu \ll \mu$) if $\mu(A) = 0 \implies \nu(A) = 0$ for all $A \in \Sigma$
- (b) μ and ν are *mutually singular* ($\mu \perp \nu$) if there is a decomposition $X = A \cup B$ such that $A \cap B = \emptyset$ and for any measurable $E \subset A$ we have $\nu(E) = 0$ and for any measurable $E \subset B$ we have $\mu(E) = 0$.

Theorem 3.3.2 (Lebesgue). Let μ and ν be σ -finite measures. Then there is a unique decomposition $\nu = \nu_c + \nu_s$ where ν_c, ν_s are σ -finite measures such that $\nu_c \ll \mu$ and $\nu_s \perp \mu$.

Theorem 3.3.3 (Radon-Nikodym). Suppose that μ, ν are σ -finite measures with $\nu \ll \mu$. Then there exists a nonnegative measurable function $f : X \rightarrow [0, \infty)$ such that for any $A \in \Sigma$,

$$\nu(A) = \int_A f \, d\mu$$

Remark. The above function is uniquely determined μ -a.e. and therefore we write suggestively,

$$f = \frac{d\nu}{d\mu}$$

to suggest that f is the derivative.

Corollary 3.3.4. If $\nu \ll \mu$ and g is a ν -integrable function then,

$$\int_A g \, d\nu = \int_A g \cdot \left(\frac{d\nu}{d\mu} \right) d\mu$$

Proof. By splitting g , it suffices to consider the case that $g \geq 0$. Then applying Fubini,

$$\begin{aligned} \int_A g \, d\nu &= \int_0^\infty \nu(\{x \in A \mid g(x) > t\}) \, dt = \int_0^\infty \left(\int_{g^{-1}((-\infty, t))} \frac{d\nu}{d\mu} \, d\mu \right) dt \\ &= \int_0^\infty \int_X \chi_{g^{-1}((-\infty, t))} \cdot \left(\frac{d\nu}{d\mu} \right) d\mu \, dt = \int_X \left(\int_0^\infty \chi_{g^{-1}((-\infty, t))} \, dt \right) \frac{d\nu}{d\mu} d\mu \\ &= \int_X g \cdot \left(\frac{d\nu}{d\mu} \right) d\mu \end{aligned}$$

because,

$$\chi_{g^{-1}((-\infty, t))}(x) = \chi_{[0, g(x))}(t)$$

and $\mu_{\mathcal{L}}([0, g(x))) = g(x)$. □

3.4 Lebesgue Differentiation

Remark. Here let $X = \mathbb{R}^n$ with the Lebesgue measure or more generally a Riemannian manifold with a σ -finite Borel measure.

Theorem 3.4.1. Let $f : X \rightarrow \hat{\mathbb{R}}$ be a locally integrable function. Then, the limit,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} f \, d\mu \stackrel{\text{a.e.}}{=} f(x)$$

exists and almost everywhere equals $f(x)$. In fact,

$$\lim_{\epsilon \rightarrow 0} \left(\frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} |f(y) - f(x)| \, d\mu(y) \right) = 0$$

almost everywhere.

Corollary 3.4.2. Let $A \subset X$ be measurable. Then,

$$\lim_{\epsilon \rightarrow 0} \frac{\mu(A \cap B_\epsilon(x))}{\mu(B_\epsilon(x))} = 1$$

for almost all $x \in A$.

Proof. Apply Lebesgue differentiation to the function χ_A which is locally integrable. Then, for almost every $x \in A$,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} \chi_A \, d\mu = \lim_{\epsilon \rightarrow 0} \frac{\mu(A \cap B_\epsilon(x))}{\mu(B_\epsilon(x))} \stackrel{\text{a.e.}}{=} \chi_A(x) = 1$$

□

3.5 Egorov's and Lusin's Theorems (WIP)

Theorem 3.5.1 (Egorov). Let (X, \mathcal{F}, μ) be a finite measure space and M a separable metric space $f_n : X \rightarrow M$ a sequence of measurable functions (for the Borel σ -algebra on M) such that $f_n \rightarrow f$ pointwise a.e. then for any $\epsilon > 0$ there exists a measurable subset $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \rightarrow f$ uniformly on $X \setminus E$.

Remark. Finiteness of the measure space is essential. For example, consider $X = \mathbb{R}$ with the Lebesgue measure and $M = \mathbb{R}$. Then let $f_n = \chi_{[n, n+1]}$. Pointwise $f_n \rightarrow 0$ but this convergence is not uniform on the complement of any finite measure open set (notice that every finite measure set is contained in a finite measure open set by regularity).

3.6 •

4 Convergence Theorems for Lebesgue Integrals

4.1 A Few Remarks on Measurable Limits

Lemma 4.1.1. Let $X \in \mathcal{F}$ be measurable, and $\{f_n\}$ a sequence of measurable functions on X . Then define,

- (a) $(\liminf f_n)(x) = \liminf_{n \rightarrow \infty} f_n(x)$
- (b) $(\limsup f_n)(x) = \limsup_{n \rightarrow \infty} f_n(x)$
- (c) $(\inf f_n)(x) = \inf_{n \geq 1} f_n(x)$
- (d) $(\sup f_n)(x) = \sup_{n \geq 1} f_n(x)$

which always exist as functions $X \rightarrow \hat{\mathbb{R}}$. Then $\liminf f_n, \limsup f_n, \inf f_n, \sup f_n$ are all measurable.

Proof. Rudin RCA Theorem 1.14. □

Corollary 4.1.2. If $f_n \rightarrow f$ pointwise and f_n is a sequence of measurable functions then f is measurable.

Lemma 4.1.3. Assume that $(\Omega, \mathcal{F}, \mu)$ is a complete measure space. Then if $f \stackrel{\text{a.e.}}{=} g$ and f is measurable then g is measurable.

Proof. Let $N = \{x \in X \mid f(x) \neq g(x)\}$ and we know $\mu(N) = 0$. It suffices to show that $g^{-1}(Y) \in \mathcal{F}$ for all $Y \in \mathcal{B}$. However,

$$g^{-1}(Y) = [g^{-1}(Y) \cap N] \cup [g^{-1}(Y) \cap N^C] = [g^{-1}(Y) \cap N] \cup [f^{-1}(Y) \cap N^C]$$

where the last equality holds because,

$$\begin{aligned} x \in g^{-1}(-\infty, c) \cap N^C &\iff x \in N \text{ and } g(x) \in (Y) \iff x \in N^C \text{ and } f(x) \in (Y) \\ &\iff x \in f^{-1}(Y) \cap N^C \end{aligned}$$

since $f(x) = g(x)$ for $x \in N^C$. Now because $\mu(N) = 0$ and $g^{-1}(Y) \cap N \subset N$ we have $g^{-1}(Y) \cap N \in \mathcal{F}$ because the measure space is complete. Therefore $g^{-1}(Y) \in \mathcal{F}$ so g is measurable. □

Corollary 4.1.4. Let f_n be a sequence of measurable functions on X and f a function such that $f_n \rightarrow f$ pointwise a.e. If $(\Omega, \mathcal{F}, \mu)$ is complete then f is measurable.

Proof. We know that $\liminf f_n$ exists and that $\liminf f_n = f$ almost everywhere since wherever $f_n(x) \rightarrow f(x)$ converges we have $(\liminf f_n)(x) = \lim_{n \rightarrow \infty} f_n(x) = f(x)$. We know that $\liminf f_n$ is measurable so we find that f is measurable as well because $(\Omega, \mathcal{F}, \mu)$ is complete. \square

4.2 The Main Lemma

Lemma 4.2.1 (Fatou). Let $X \in \mathcal{F}$ be measurable, and $\{f_n\}$ a sequence of measurable functions which are nonnegative almost everywhere. Define $f : X \rightarrow \mathbb{R}$ via,

$$f(x) = \liminf_{n \rightarrow \infty} f_n(x)$$

Then f is measurable and,

$$\int_X f \, d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n \, d\mu$$

Proof. The proof from first principles is annoying. \square

Corollary 4.2.2 (Reverse Fatou). Now suppose that $\{f_n\}$ is a sequence of measurable functions on X such that $|f_n| \leq g$ almost everywhere for an integrable function g on X . Then,

$$\limsup_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X \limsup_{n \rightarrow \infty} f_n \, d\mu$$

Proof. Consider $f'_n = g - f_n$. Then f'_n is measurable and a.e. nonnegative. Applying Fatou's lemma we see that,

$$f' = \liminf_{n \rightarrow \infty} f'_n = g - \limsup_{n \rightarrow \infty} f_n$$

is measurable and,

$$\int_X (g - \limsup_{n \rightarrow \infty} f_n) \, d\mu \leq \liminf_{n \rightarrow \infty} \int_X (g - f_n) \, d\mu$$

However, g is integrable so we can pull it out of the integrals,

$$\begin{aligned} \int_X (g - \limsup_{n \rightarrow \infty} f_n) \, d\mu &= \int_X g \, d\mu - \int_X \limsup_{n \rightarrow \infty} f_n \, d\mu \\ \liminf_{n \rightarrow \infty} \int_X (g - f_n) \, d\mu &= \int_X g \, d\mu - \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X \limsup_{n \rightarrow \infty} f_n \, d\mu$$

\square

4.3 The Main Theorems

Theorem 4.3.1 (Monotone Convergence). Let $\{f_n\}$ be a pointwise a.e. non-decreasing sequence of a.e. non-negative measurable functions on X , explicitly this means

$$0 \leq f_n(x) \leq f_{n+1}(x)$$

for all n almost everywhere. Then let $f = \liminf_{n \rightarrow \infty} f_n$. Then f is measurable and,

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. Notice that f_n is nondecreasing a.e. and thus $\int f_n$ is nondecreasing. Then applying Fatou's lemma, we see that f is measurable and,

$$\int_X f d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$$

Furthermore, $f_n \leq f$ a.e. so we find that,

$$\int_X f_n d\mu \leq \int_X f d\mu$$

proving the equality. □

Remark. Notice that there is no confusion about if the hypothesis means that,

$$\mu(\{x \in X \mid \forall n : 0 \leq f_n(x) \leq f_{n+1}(x)\}^C) = 0 \quad \text{or} \quad \forall n : \mu(\{x \in X \mid 0 \leq f_n(x) \leq f_{n+1}(x)\}^C) = 0$$

because these are equivalent since the first set is the countable union of the second sequence of sets the measure is countably additive.

Theorem 4.3.2 (Dominated Convergence). Let $\{f_n\}$ be a sequence of measurable functions on X dominated almost everywhere by an integrable function g meaning that $|f_n| \leq g$ almost everywhere. Suppose that $f_n \rightarrow f$ pointwise almost everywhere where f is measurable. Then f_n and f are integrable and,

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. First, because f_n is measurable and $|f_n| \leq g$ a.e. where g is integrable we see that f_n is integrable. Furthermore, almost everywhere we have,

$$|f - f_n| \leq |f| + |f_n| \leq 2g$$

Therefore, we may apply reverse Fatou to the sequence $\{|f - f_n|\}$ bounded by the integrable function $2g$ to find that,

$$\limsup_{n \rightarrow \infty} \int_X |f - f_n| d\mu \leq \int_X \limsup |f - f_n| d\mu$$

but $\limsup |f - f_n| \stackrel{\text{a.e.}}{=} 0$ because $f \rightarrow f_n$ a.e. and thus,

$$\limsup_{n \rightarrow \infty} \int_X |f - f_n| d\mu \leq 0$$

but $|f - f_n| \geq 0$ so the limit exists because the liminf is bounded below by zero as well so,

$$\lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = 0$$

and therefore,

$$\left| \int_X f_n d\mu - \int_X f d\mu \right| \leq \int_X |f_n - f| d\mu \rightarrow 0$$

meaning that,

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

□

Remark. If the measure space is complete or $f_n \rightarrow f$ everywhere then f is automatically measurable.

4.4 Scheffé's Lemma

Lemma 4.4.1 (Scheffé). Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \rightarrow f$ a.e. with f integrable. Then,

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0 \iff \lim_{n \rightarrow \infty} \int_X |f_n| d\mu = \int_X |f| d\mu$$

Proof. One direction is obvious. If,

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0$$

then because $||f_n| - |f|| \leq |f_n - f|$ we know that,

$$\left| \int_X |f_n| d\mu - \int_X |f| d\mu \right| \leq \int_X ||f_n| - |f|| d\mu \leq \int_X |f_n - f| d\mu \rightarrow 0$$

and therefore,

$$\lim_{n \rightarrow \infty} \int_X |f_n| d\mu = \int_X |f| d\mu$$

Conversely, let $g_n = |f_n| - |f_n - f|$ then,

$$|g_n| \leq ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f|$$

so because f is integrable we can apply the dominated convergence theorem to conclude that,

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X |f| d\mu$$

because $g_n \rightarrow |f|$ almost everywhere. However,

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \lim_{n \rightarrow \infty} \left(\int_X |f_n| d\mu - \int_X |f - f_n| d\mu \right)$$

and we assume that,

$$\lim_{n \rightarrow \infty} \int_X |f_n| d\mu = \int_X |f| d\mu$$

so combining the two equations we find that,

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| \, d\mu = 0$$

□

Corollary 4.4.2. Under the hypotheses of the lemma, if

$$\lim_{n \rightarrow \infty} \int |f_n| \, d\mu = \int |f| \, d\mu$$

then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$$

Proof. By the lemma, our assumption implies that,

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| \, d\mu = 0$$

and then we conclude from

$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| \leq \int_X |f_n - f| \, d\mu \rightarrow 0$$

meaning that,

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

□

Remark. However, the converse of this corollary does *not* hold in general (it does say when f_n are nonnegative and it becomes trivial). For example, let

$$f_n = \frac{1}{2\sqrt{\pi}} \left(e^{-(x-n)^2} - e^{-(x+n)^2} \right)$$

Then $f_n \rightarrow 0$ everywhere pointwise. And likewise,

$$\int_X f_n \, d\mu = 0$$

for each n so we indeed have,

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \int_X f \, d\mu = 0$$

trivially where $f = 0$. However,

$$\int_X |f_n| \, d\mu \rightarrow 1 \quad \text{where as} \quad \int_X |f| \, d\mu = 0$$

5 Functional Analysis Theorems

5.1 Banach-Steinhaus

Theorem 5.1.1 (Banach-Steinhaus). Let X be a Banach space and Y a normed space. Then for any subset $\mathcal{M} \subset \mathcal{L}(X, Y)$ we have,

$$\mathcal{M} \text{ is pointwise bounded} \iff \mathcal{M} \text{ is uniformly bounded}$$

Corollary 5.1.2. Let X, Y be normed spaces with X Banach and $T_n \in \mathcal{L}(X, Y)$ a sequence such that $\lim_{n \rightarrow \infty} T_n x$ exists for all $x \in X$. Then there exists $T \in \mathcal{L}(X, Y)$ defined by $Tx = \lim_{n \rightarrow \infty} T_n x$.

Proof. It is clear that T is linear and thus suffices to prove that T is bounded. Consider,

$$\mathcal{M} = \{T_n \mid n \in \mathbb{N}\}$$

For any $x \in X$ the sequence $T_n x$ is bounded because it has a limit. Thus \mathcal{M} is pointwise bounded and therefore uniformly bounded by Banach-Steinhaus. In particular, there exists some $C > 0$ such that $\|T_n\| \leq C$ for all n . Therefore,

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \lim_{n \rightarrow \infty} \|T_n x\| \leq \lim_{n \rightarrow \infty} \|T_n\| \leq C$$

so T is bounded. □

5.2 Hahn-Banach

Theorem 5.2.1 (Hahn-Banach). Let X be a normed space and $U \subset X$ a linear subspace. Let $\ell : U \rightarrow \mathbb{R}$ be a bounded linear functional. Then there exists a linear functional $\tilde{\ell} : X \rightarrow K$ extending ℓ that satisfies $\|\tilde{\ell}\|_{X^*} = \|\ell\|_{U^*}$.

Proof. First, we show that given $z \in X \setminus U$ it is possible to extend ℓ to $U' = U + \text{span}\{z\}$. Indeed,

$$\ell'(u + \alpha z) = \ell(u) + \alpha a$$

is well-defined since $z \notin U$ implies the decomposition is unique and is clearly linear. Thus we need to show that a can be chosen to fix the norm. Because $U \subset U'$ we only need to show that ℓ' is bounded above by $\|\ell\|_U$ (bounded below then follows immediately by linearity) since the norm of any extension is automatically larger as it is a supremum over a superset. We need to choose a such that,

$$\ell'(u + \alpha z) = \ell(u) + \alpha a \leq \|\ell\|_U \cdot \|u + \alpha z\|$$

If $\alpha = 0$ this says nothing. If $\alpha > 0$ then we require,

$$\ell(u) + a \leq \|\ell\|_U \cdot \|u + z\|$$

for all $u \in U$ by pulling out α and rescaling u . Similarly, for $\alpha < 0$ we require for all $u \in U$ that,

$$\ell(u) - a \leq \|\ell\|_U \cdot \|u - z\|$$

However,

$$\ell(u_1) + \ell(u_2) = \ell(u_1 + u_2) \leq \|\ell\|_U \cdot \|u_1 + u_2\| \leq \|\ell\|_U \cdot (\|u_1 + z\| + \|u_2 - z\|)$$

and therefore,

$$\ell(u_2) - \|\ell\|_U \cdot \|u_2 - z\| \leq \|\ell\|_U \cdot \|u_1 + z\| - \ell(u_1)$$

for all $u_1, u_2 \in U$ and therefore the supremum of the left is less than the infimum of the right so there exists an a fitting between i.e. satisfying,

$$a \leq \|\ell\|_U \cdot \|u + z\| - \ell(u) \quad \text{and} \quad a \geq \ell(u) - \|\ell\|_U \cdot \|u - z\|$$

Therefore, such an extension ℓ' exists.

Now we complete the proof via Zorn's lemma applied to pairs (U', ℓ') where $U' \subset X$ is a linear subspace and $\ell' : U' \rightarrow \mathbb{R}$ is an extension such that $\|\ell'\|_{U'} = \|\ell\|_U$. This is a poset under inclusion $U_1 \subset U_2$ with the restriction that $\ell_2|_{U_1} = \ell_1$. Therefore, chains give compatible bounded linear functionals on their union and thus give an upper bound. By Zorn's lemma there is a maximal element but by the above construction a maximal pair (U', ℓ') cannot have any $z \in X \setminus U'$ or it could be extended to $U' + \text{span}\{z\}$ so $U' = X$ proving the theorem. \square

Corollary 5.2.2. Let X be a normed space and $x \in X$ nonzero. Then there exists $\ell \in X^*$ such that $\ell(x) = \|x\|$ and $\|\ell\| = 1$.

Proof. Let $U = \text{span}\{x\}$ then define $\ell : U \rightarrow \mathbb{R}$ via $\lambda x \mapsto \lambda\|x\|$. This is clearly linear and furthermore,

$$\|\ell\| = \sup_{\|\lambda x\|=1} \|\ell(\lambda x)\| = \sup_{\|\lambda x\|=1} \|\lambda x\| = 1$$

By Hahn-Banach there is an extension $\ell' : X \rightarrow \mathbb{R}$ such that $\|\ell'\| = \|\ell\| = 1$ and $\ell'(x) = \|x\|$. \square

Corollary 5.2.3. Let X be a normed space. Then X^* separates points.

Proof. If $x_1, x_2 \in X$ are not equal then there exists $\ell \in X^*$ such that $\|\ell(x_1 - x_2)\| = \|x_1 - x_2\| \neq 0$ but $\ell(x_1 - x_2) = \ell(x_1) - \ell(x_2)$ so $\ell(x_1) \neq \ell(x_2)$. \square

Corollary 5.2.4. For each $x \in X$,

$$\|x\|_X = \sup_{\|\ell\|=1} |\ell(x)|$$

Proof. By definition,

$$\|\ell\| \geq \frac{|\ell(x)|}{\|x\|}$$

Therefore,

$$\|x\| \geq \sup_{\|\ell\|=1} |\ell(x)|$$

However, by above there exists $\ell \in X^*$ such that $\|\ell\| = 1$ and $\ell(x) = \|x\|$ and therefore,

$$\|x\| \leq \sup_{\|\ell\|=1} |\ell(x)|$$

proving the claim. \square

Corollary 5.2.5. Let $U \subset X$ be a closed linear subspace, $x \in X$ with $x \notin U$. Then there exists $\ell \in X^*$ with $\ell|_U = 0$ and $\ell(x) = \|x\|_X$.

Proof. Consider X/U which inherits a normed structure via,

$$||[x]||_{X/U} = \inf_{u \in U} ||x + u||_X$$

Then, there exists a bounded linear functional $\ell : X/U \rightarrow K$ such that $\ell([x]) = ||[x]||_{X/U}$. Take $\tilde{\ell} = \ell \circ \pi$ where $\pi : X \rightarrow X/U$ is the projection which is bounded by 1. Thus, $\tilde{\ell} \in X^*$ and $\tilde{\ell}|_U = 0$. Furthermore,

$$\tilde{\ell}(x) = \ell([x]) = ||[x]||_{X/U} \neq 0$$

because $x \notin U$ so $[x] \neq 0$ so we can scale $\tilde{\ell}$ to give the required result. \square

5.3 Open Mapping Theorem

Lemma 5.3.1. Let X be a normed space. A linear subspace $U \subset X$ is open iff $U = X$.

Proof. If U is open then $0 \in U$ there must exist some open ball $B_\epsilon(0) \subset U$ and thus for any $x \in X$ we have $x = \lambda x'$ for $x' \in B_\epsilon(0) \subset U$ so $U = X$ because it is closed under scaling. \square

Corollary 5.3.2. If $T : X \rightarrow Y$ is a linear map of topological vector spaces and Y is normed. Then if T has open image then T is surjective.

Proof. By linearity, the image $T(X) \subset Y$ is a linear subspace. Furthermore, $T(X)$ is open by assumption so by the lemma $T(X) = Y$. \square

Theorem 5.3.3 (Open Mapping). Let X, Y be Banach spaces. For $T \in \mathcal{L}(X, Y)$ then,

$$T \text{ is surjective} \iff T \text{ is open}$$

Remark. By above, the direction $T \text{ is open} \implies T \text{ is surjective}$ is trivial.

Theorem 5.3.4 (Bounded Inverse). Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. If T is bijective then $T^{-1} \in \mathcal{L}(X, Y)$.

Proof. It is obvious that T^{-1} is linear so it suffices to check that T^{-1} is bounded. Because T is bijective we have,

$$||T^{-1}|| = \sup_{y \neq 0} \frac{||T^{-1}y||}{||y||} = \sup_{x \neq 0} \frac{||x||}{||Tx||} = \left[\inf_{x \neq 0} \frac{||Tx||}{||x||} \right]^{-1}$$

Therefore, it suffices to show that T is positively bounded below in addition to being bounded above. By the Open Mapping theorem, T is open and thus a homeomorphism. Let,

$$S(X) = \{x \in X \mid ||x|| = 1\}$$

is closed so $T(S(X))$ is also closed and $0 \notin T(S(X))$ because T is injective. Thus $d(0, T(S(X))) > 0$ because $T(S(X))$ is a closed subset of a metric space. Therefore,

$$\inf_{||x||=1} ||Tx|| > 0$$

and thus T is bounded below proving the theorem. \square

Lemma 5.3.5. A linear operator $T : X \rightarrow Y$ between normed spaces is continuous iff it is bounded.

Proof. If T is bounded then,

$$\|Tx - Ty\| = \|T(x - y)\| \leq \|T\| \cdot \|x - y\|$$

so indeed T is even Lipschitz. If T is continuous then consider,

$$S(Y) = \{y \in Y \mid \|y\| = 1\}$$

is closed so $A = T^{-1}(S(Y))$ is closed. Furthermore, $0 \notin A$ because $T(0) = 0 \notin S(Y)$. Therefore, because A is closed in a metric space and $0 \notin A$ so $d(0, A) > 0$ and therefore,

$$\|T\| = \sup_{x \in A} \frac{1}{\|x\|} = \left[\inf_{x \in A} \|x\| \right]^{-1}$$

exists because $\inf_{x \in A} \|x\|$ is positive. Alternatively, $T^{-1}(B_1(0))$ is open so there is a ball $B_\delta(0) \subset T^{-1}(B_1(0))$ and thus,

$$\|Tx\| = \frac{\|x\|}{\delta} \|T(x\delta/\|x\|)\| \leq \frac{\|x\|}{\delta}$$

so T is bounded. □

Remark. We can thus give a slick proof of the bounded operator theorem. Since T is surjective it is open and thus T^{-1} is continuous and thus bounded.

Example 5.3.6. If X, Y are not both complete then the above may fail. For example, let $X = C([0, 1])$ and $Y = C^1([0, 1])$ with supremum norms and take,

$$(Tf)(t) = \int_0^t f(s) ds$$

which is linear and bounded because,

$$\|Tf\|_\infty = \sup_{t \in [0, 1]} \left| \int_0^t f(s) ds \right| \leq \|f\|_\infty$$

because f is bounded. Thus $\|T\| \leq 1$ and is injective by the fundamental theorem of calculus. Therefore it is bijective onto its image. However T is not bounded below because we can take a sequence of spikier functions with vanishing area but the same supremum norm. However, if we give Y the “correct” C^1 norm $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$ then the operator is invertible because,

$$\|Tf\|_{C^1} = \sup_{t \in [0, 1]} \left| \int_0^t f(s) ds \right| + \sup_{t \in [0, 1]} |(Tf)'(t)| \leq 2\|f\|_\infty$$

and also,

$$\|Tf\|_{C^1} \geq \|(Tf)'\|_\infty = \|f\|_\infty$$

so we see that $\|f\|_\infty \leq \|Tf\|_{C^1} \leq 2\|f\|_\infty$ so T is also bounded below and thus has closed image which is thus a Banach space (the image is the subspace of $C^1([0, 1])$ vanishing at 0).

6 Closed Image Theorem

Lemma 6.0.1. Let $T : X \rightarrow Y$ be a continuous map of Banach spaces. Then T is injective and $\text{im } T$ is closed if and only if T is bounded below.

Proof. Suppose there is $C > 0$ such that,

$$\|x\|_X \leq C\|Tx\|_Y$$

for all $x \in X$. In particular, if $Tx = 0$ then $x = 0$ so T is injective. Furthermore, if $x_n \in X$ is a sequence such that Tx_n is Cauchy (any Cauchy sequence in $\text{im } T$ is of this form) then,

$$\|x_n - x_m\| \leq C\|T(x_n - x_m)\| \leq C\|Tx_n - Tx_m\| < C\epsilon$$

for $n, m > N$. Thus, $\{x_n\}$ is Cauchy and therefore since X is Banach the sequence is convergent,

$$\lim_{n \rightarrow \infty} x_n = x$$

Furthermore, T is continuous so,

$$\lim_{n \rightarrow \infty} Tx_n = T\left(\lim_{n \rightarrow \infty} x_n\right) = Tx$$

and thus x_n is convergent and $x \in \text{im } T$ so $\text{im } T$ is closed.

Conversely, since T is injective $T : X \rightarrow \text{im } T$ is bijective and $\text{im } T \subset Y$ is closed so $\text{im } T$ is Banach. Therefore, by the bounded inverse theorem $T : X \rightarrow \text{im } T$ is invertible by a bounded operator $T^{-1} \in \mathcal{L}(\text{im } T, X)$. Thus,

$$\|x\|_X = \|T^{-1}(Tx)\|_X \leq \|T^{-1}\| \cdot \|Tx\|_Y$$

so $\|T^{-1}\|$ gives the required constant. □

Theorem 6.0.2 (Closed Graph). Let $T : X \rightarrow Y$ be a linear operator between Banach spaces. Then T is continuous if and only if its graph is closed.

6.1 Closed Range Theorem

Definition 6.1.1. Let $V \subset X$ be a subset then define $V^\perp \subset X^*$ via,

$$V^\perp = \{\ell \in X^* \mid \forall v \in V : \ell(v) = 0\} = \bigcap_{v \in V} \ker \text{ev}_v$$

Clearly this is a weak-* closed subspace and thus norm closed. Let $W \subset X^*$ be a subset then define $W^\perp \subset X$ via,

$$W^\perp = \{x \in X \mid \forall \ell \in W : \ell(x) = 0\} = \bigcap_{\ell \in W} \ker \ell$$

which is clearly a weakly closed subspace and thus norm closed.

Lemma 6.1.2. If $V \subset X$ is a subspace then $V^{\perp\perp} = \overline{V}$.

Proof. If $v \in V$ then for any $\ell \in V^\perp$ by definition $\ell(v) = 0$ so $v \in V^{\perp\perp}$ so $V \subset V^{\perp\perp}$ and thus $\overline{V} \subset V^{\perp\perp}$. Let $x \notin \overline{V}$. Since \overline{V} is a subspace, by Hahn-Banach there is some $\ell \in X$ such that $\ell|_{\overline{V}} = 0$ but $\ell(x) = 1$ so $\ell \in V^\perp$ and thus $x \notin V^{\perp\perp}$. Thus $V^{\perp\perp} = \overline{V}$. □

Remark. Recall that weakly closed and closed subspaces coincide so this is also the weak closure of V .

Lemma 6.1.3. If $W \subset X^*$ is a subspace then $W^{\perp\perp}$ is the weak-* closure of W .

Proof. If $\ell \in W$ then for any $x \in W^\perp$ by definition $\ell(x) = 0$ so $\ell \in W^{\perp\perp}$ so $W \subset W^{\perp\perp}$ and thus if Y is the weak-* closure of W then $Y \subset W^{\perp\perp}$. Now suppose that $\ell \in W^{\perp\perp}$ then $\ell|_{W^\perp} = 0$ and we need to show that $\ell \in Y$. Now ℓ descends to $\ell \in (X/W^\perp)^*$. For any finite dimensional subspace $V \subset X/W^\perp$ we can find $\ell_V \in W$ such that $\ell|_V = \ell_V$. This is because if $\eta(v) = 0$ for all $\eta \in W$ then $v \in W^\perp$ so $[v] = 0$. Thus $W \rightarrow X^* \rightarrow (X/W^\perp)^* \rightarrow V^*$ is surjective because its dual $v \mapsto (\eta \mapsto \eta(v))$ is injective and V is finite dimensional (see this answer). Therefore there is such a ℓ_V . Now we see that $\ell_V \rightarrow \ell$ pointwise i.e. in the weak-* topology as V ranges over the finite dimensional subspaces of (X/W^\perp) . Therefore $\ell \in Y$ since Y is weak-* closed and $\ell_V \in W \subset Y$. \square

Remark. This also shows that $Y \cong (X/W^\perp)^*$.

Proposition 6.1.4. Let $T : X \rightarrow Y$ be a continuous linear operator. Then,

$$(a) \ker T = (\operatorname{im} T^*)^\perp$$

$$(b) \ker T^* = (\operatorname{im} T)^\perp$$

Proof. If $Tx = 0$ then if $\ell = T^*\eta$ we have $\ell(x) = \eta(Tx) = 0$ so $\ker T \subset (\operatorname{im} T^*)^\perp$. If $x \in (\operatorname{im} T^*)^\perp$ then for any $\ell \in Y^*$ we see that $\ell(Tx) = (T^*\ell)(x) = 0$ so $Tx = 0$ by Hahn-Banach.

If $T^*\ell = 0$ then if $y = Tx$ we see that $\ell(y) = (T^*\ell)(x) = 0$ so $\ell \in (\operatorname{im} T)^\perp$. If $\ell \in (\operatorname{im} T)^\perp$ then for all $x \in X$ we have $(T^*\ell)(x) = \ell(T(x)) = 0$ so $T^*\ell = 0$ and thus $\ell \in \ker T^*$. \square

Corollary 6.1.5. We see that,

$$\overline{\operatorname{im} T} = (\ker T^*)^\perp$$

and that the weak-* closure of $\operatorname{im} T^*$ is $(\ker T)^\perp$.

Corollary 6.1.6. Let $T : X \rightarrow Y$ be a continuous linear operator. Then,

$$(a) \operatorname{im} T \text{ is dense iff } T^* \text{ is injective}$$

$$(b) \operatorname{im} T^* \text{ is weak-* dense iff } T \text{ is injective.}$$

Proof. We see that,

$$\operatorname{im} T \text{ is dense} \iff \overline{\operatorname{im} T} = Y \iff (\ker T^*)^\perp = Y \iff \ker T^* = (0) \iff T^* \text{ is injective}$$

using that $\ker T^*$ is weak-* closed and

$$\operatorname{im} T^* \text{ is weak-* dense} \iff \overline{\operatorname{im} T^*} = X^* \iff (\ker T)^\perp = X^* \iff \ker T = (0) \iff T \text{ is injective}$$

using that $\ker T$ is closed. \square

Theorem 6.1.7 (Closed Range). Let $T : X \rightarrow Y$ be a continuous map of Banach spaces and $T^* : Y^* \rightarrow X^*$ its dual. Then the following are equivalent,

$$(a) \operatorname{im} T \text{ is closed}$$

- (b) $\text{im } T^*$ is closed
- (c) $\text{im } T = (\ker T^*)^\perp = \{y \in Y \mid \forall \ell \in \ker T^* : \langle \ell, y \rangle = 0\}$
- (d) $\text{im } T^* = (\ker T)^\perp = \{\psi \in Y^* \mid \forall x \in \ker T : \langle \psi, x \rangle = 0\}$

Remark. This result can be extended to densely defined operators as long as they have closed graph. Notice that everywhere defined continuous operators between Banach spaces automatically have closed graph.

7 Examples of Banach Spaces

Example 7.0.1. Let K be a compact metric space and $C(K)$ the space of continuous functions equipped with the sup-norm $\|\bullet\|_\infty$. This is well-defined because continuous functions on a compact space are bounded and if $\|f\|_\infty = 0$ then $f = 0$ because then $|f(x)| = 0$ so $f(x) = 0$ for all x . Finally, convergence in the sup-norm is what is usually called uniform convergence which preserves continuity so $C(K)$ is complete.

Example 7.0.2. However, $C^k([0, 1])$ for $k \geq 1$ is not a Banach space because the uniform limit of differentiable functions is not differentiable.

Proposition 7.0.3. Let X, Y be normed spaces and $\mathcal{L}(X, Y)$ the space of bounded linear operators endowed with the operator norm. Then $\mathcal{L}(X, Y)$ is a normed space and is Banach if Y is Banach.

Proof. It is clear that $\mathcal{L}(X, Y)$ is a vector space. We need to check that $\|\bullet\|$ is indeed a norm on $\mathcal{L}(X, Y)$. It is obviously homogeneous and nonnegative so we need to check positivity and the triangle inequality. First, suppose that $\|T\| = 0$ then by definition $\|Tx\| = 0$ for all $x \in X$ so $Tx = 0$ meaning $T = 0$. If $T, S \in \mathcal{L}(X, Y)$ then,

$$\|X + Y\| = \sup_{\|x\|=1} \|(X + Y)x\| \leq \sup_{\|x\|=1} (\|Tx\| + \|Sx\|) \leq \|T\| + \|S\|$$

Now suppose that Y is Banach. Given a Cauchy sequence $T_n \in \mathcal{L}(X, Y)$ we know that,

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \cdot \|x\|$$

and thus for each fixed $x \in X$ the sequence $T_n x$ is Cauchy and thus converges to a limit in Y because Y is complete. It is clear that

$$Tx = \lim_{n \rightarrow \infty} T_n x$$

defines a linear operator. We need to show that it is bounded. Notice that $\|T_n\|$ is bounded by C because T_n is Cauchy and therefore,

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \lim_{n \rightarrow \infty} \|T_n x\| \\ &= \sup_{\|x\|=1} \limsup_{n \rightarrow \infty} \|T_n x\| \leq \sup_{\|x\|=1} \limsup_{n \rightarrow \infty} \|T_n\| \cdot \|x\| = \limsup_{n \rightarrow \infty} \|T_n\| \leq C \end{aligned}$$

is bounded. Thus it suffices to show that $T_n \rightarrow T$ in operator norm. Recall that,

$$\|(T_n - T_m)x\| \leq \|T_n - T_m\| \cdot \|x\|$$

for any x and T_n is Cauchy so for any $\epsilon > 0$ we can choose N large enough such that $\|T_n - T_m\| < \epsilon$ for any $n, m > N$. By the continuity of the norm on Y we find that,

$$\|(T - T_n)x\| = \lim_{m \rightarrow \infty} \|(T_m - T_n)x\| < \epsilon \|x\|$$

for $n > N$ and thus,

$$\|T - T_n\| = \sup_{\|x\|=1} \|(T - T_n)x\| < \epsilon$$

for all $n > N$ so $\|T - T_n\| \rightarrow 0$. □

Remark. A nice write up can be found [here](#).

Example 7.0.4. We will prove that for any locally compact Hausdorff space, $C_0(X)$ equipped with the uniform norm is a Banach space. However, here we consider $C_c(X)$ the space of compactly supported functions. Although $(C_c(X), \|\bullet\|_\infty)$ is clearly a normed space, it is *not* complete.

Here we provide an example. Let $X = \mathbb{R}$ and consider a smooth bump function $\chi(x)$ which satisfies $\chi(x) = 1$ if $|x| < 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Therefore, χ is compactly supported and continuous. Then, consider,

$$f_n(x) = \frac{\chi(x/n)}{1 + x^2}$$

which is clearly continuous and compactly supported since $\text{Supp}(f_n) \subset \overline{B_{2n}(0)}$. Furthermore,

$$\|f_n - f_m\|_\infty \leq \frac{2}{1 + \min(n, m)^2}$$

because for $x < \min(n, m)$ we have $f_n(x) - f_m(x) = 0$ and for $x \geq \min(n, m)$ each function is bounded by $(1 + \min(n, m)^2)^{-1}$. Therefore, f_n is a Cauchy sequence in the uniform norm and therefore converges in $C_0(X)$. Clearly $f_n \rightarrow f$ where $f(x) = \frac{1}{1+x^2}$ which is not compactly supported and since $C_c(X) \subset C_0(X)$ (and normed spaces being Hausdorff have unique limits) we see that $C_c(X)$ cannot be complete.

However, we can give $C_c(X)$ a better topology such that it is complete. Indeed,

$$C_c(X) = \lim_{\substack{K \supset X \\ \text{compact}}} C(X, K)$$

where $C(X, K)$ is the space of continuous functions supported on K . This is Banach because obviously $C(X, K) \cong C(K)$ which when given the uniform norm is Banach. Therefore, we equip $C_c(X)$ with the locally convex direct limit topology. Then the sup norm $\|\bullet\|_\infty$ is continuous on $C_c(X)$ because it is on each $C(X, K)$ but the topology is finer than the norm topology e.g. there are sequences which converge in the uniform norm which are not Cauchy in $C_c(X)$ for example the sequence f_n defined above.

8 Locally Convex Analysis

Definition 8.0.1. A subset X of a \mathbb{R} -vector space V is called *convex* if for all $x, y \in X$ and $\lambda \in [0, 1]$ the point $\lambda x + (1 - \lambda)y \in X$.

Definition 8.0.2. A topological vector space is called *locally convex* if the origin has a neighborhood basis of convex sets.

Remark. Because translation is a homeomorphism, every point of a locally convex topological vector space (LCTVS) has a convex neighborhood basis.

Definition 8.0.3. A *semi-norm* on a \mathbb{R} -vector space V is a function $p : V \rightarrow \mathbb{R}$ such that,

- (a) $p(x) \geq 0$ for all $x \in X$
- (b) $p(\lambda x) = |\lambda|p(x)$ for all $\lambda \in \mathbb{R}$ and $x \in X$
- (c) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.

Notice that a semi-norm satisfying $p(x) = 0 \implies x = 0$ is a norm.

Definition 8.0.4. Given a family of semi-norms \mathcal{P} on a vector space V , the *initial topology* $\mathcal{T}_{\mathcal{P}}$ on X is the coarsest topology for which each $p \in \mathcal{P}$ is continuous.

Proposition 8.0.5. The initial topology makes V a locally convex topological vector space.

Proof. A function $f : X \rightarrow V$ is continuous if and only if $p \circ f : X \rightarrow \mathbb{R}$ is continuous for each $p \in \mathcal{P}$. First, for $f : \mathbb{R} \times V \rightarrow V$ via $(\lambda, v) \mapsto \lambda v$ we have $p \circ f(\lambda, v) = |\lambda|p(v)$ so $p \circ f$ is the same as $m \circ (|\bullet| \times p)$ which is continuous because p is continuous and the other functions we know are continuous on \mathbb{R} . Now for $f : V \times V \rightarrow V$ we know that $p \circ f(x, y) = p(x + y) \leq p(x) + p(y)$.

Furthermore, the open sets $\{U_{\epsilon, p} = p^{-1}(B_{\epsilon}(0))\}_{\epsilon > 0, p \in \mathcal{P}}$ form a neighborhood basis of the origin and if $x, y \in U_{\epsilon, p}$ then $p(x) < \epsilon$ and $p(y) < \epsilon$ so for any $\lambda \in [0, 1]$ we have,

$$p(\lambda x + (1 - \lambda)y) \leq \lambda p(x) + (1 - \lambda)p(y) < \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon$$

so $\lambda x + (1 - \lambda)y \in U_{\epsilon, p}$ and thus $U_{\epsilon, p}$ is convex. □

9 Stone-Weierstrass Theorem

Definition 9.0.1. Let X be a locally compact Hausdorff space. Then $C_0(X, \mathbb{R})$ is the space of continuous functions that vanish at infinity meaning that for each $f \in C_0(X, \mathbb{R})$ any $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|f|_{X \setminus K} < \epsilon$.

Remark. If X is compact then $C_0(X, \mathbb{R}) = C(X, \mathbb{R})$.

Proposition 9.0.2. When $C_0(X, \mathbb{R})$ is equipped with the supremum norm it is a Banach space.

Proof. Because the greater than 1 support of f is contained in a compact set K we know that f is bounded because $f|_K$ is bounded since it is continuous on a compact space. Thus $\|f\|_{\infty}$ exists. Furthermore, it is trivial to show this is a normed space. Now suppose that $\{f_n\}$ is Cauchy sequence. Then $|f_n(x) - f_m(x)| < \|f_n - f_m\|$ so $\{f_n(x)\}$ is Cauchy for each $x \in X$. Therefore, we can define a function,

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

which exists because \mathbb{R} is complete. I claim that $f_n \rightarrow f$ in the sup norm. Indeed, for any $\epsilon > 0$ we have $\|f_n - f_m\| < \epsilon$ for $n, m > N$ with some N so,

$$|f_n(x) - f_m(x)| < \epsilon$$

Taking the limit as $m \rightarrow \infty$ we find,

$$|f_n(x) - f(x)| < \epsilon$$

and since this is a uniform bound for all $x \in X$ then $\|f_n - f\| < \epsilon$ so $f_n \rightarrow f$. Now we need to show that $f \in C_0(X, \mathbb{R})$. For continuity, it suffices to show that $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$ has an open neighborhood for all $\epsilon > 0$ and $x_0 \in X$. Notice that we can choose n large enough such that, $\|f_n - f\| < \frac{\epsilon}{3}$ and by continuity of f_n there is an open $U \subset f_n^{-1}(B_{\frac{\epsilon}{3}}(f_n(x_0)))$. We need to show that $f(U) \subset B_\epsilon(f(x_0))$. for $x \in U$,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

and therefore $U \subset f^{-1}(B_\epsilon(f(x_0)))$ and $x_0 \in U$ so we find that f is continuous. Finally, we need to show that f vanishes at infinity. For any $\epsilon > 0$ we can find some f_n such that $\|f - f_n\| < \frac{\epsilon}{2}$ and because $f_n \in C_0(X, \mathbb{R})$ also a compact $K \subset X$ such that $\|f_n|_{X \setminus K}\|_\infty < \frac{\epsilon}{2}$. Then,

$$\|f|_{X \setminus K}\| \leq \|f - f_n\| + \|f_n|_{X \setminus K}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so f vanishes at infinity. □

Remark. This proof is showing that uniform limits of continuous functions are continuous.

Theorem 9.0.3 (Stone-Weierstrass). Let X be a locally compact Hausdorff and $A \subset C_0(X, \mathbb{R})$ be a subalgebra such that,

- (a) A vanishes nowhere meaning for each $x \in X$ there is $f \in A$ with $f(x) \neq 0$
- (b) A separates points meaning for each $x, y \in X$ with $x \neq y$ there is $f \in A$ with $f(x) \neq f(y)$

then A is dense in $C_0(X, \mathbb{R})$.

Corollary 9.0.4. Polynomials with rational coefficients are dense in $C([a, b], \mathbb{R})$ and thus $C([a, b], \mathbb{R})$ is separable.

10 Duals and Weak-* Topology

10.1 Definitions

Remark. Here let K be a complete normed field.

Definition 10.1.1. Let X be a topological K -vector space. Then the *algebraic dual space* $X^\#$ is the space of linear functionals and the *continuous dual space* X' is the space of continuous (equivalently bounded if X is normed) linear functionals.

Remark. Notice that if X is a normed space then $X^* = \mathcal{L}(X, K)$ inherits the operator norm and is a Banach space with respect to this norm because K is complete.

Definition 10.1.2. Let X be a normed space. Consider the canonical map $\varphi : X \rightarrow X^{**}$ via sending $x \mapsto (\ell \mapsto \ell(x))$. First, notice that $\varphi(x)$ is a bounded operator because,

$$|\varphi(x)(\ell)| = |\ell(x)| \leq \|\ell\| \cdot \|x\|$$

and thus,

$$\|\varphi(x)\|_{X^{**}} \leq \|x\|$$

Therefore, φ is a bounded operator $\varphi \in \mathcal{L}(X, X^{**})$ and a short map $\|\varphi\| \leq 1$. We say that X is *reflexive* if φ is an isomorphism.

Proposition 10.1.3. Let X be a normed space. Then φ is an isometry and is thus injective.

Proof. For each $x \in X$, by Hahn-Banach, there exists a linear functional $\ell_x : X \rightarrow K$ such that $\ell_x(x) = \|x\|$ and $\|\ell_x\| = 1$. Therefore,

$$\|\varphi(x)\| \geq |\varphi(x)(\ell_x)| = \|x\|$$

but we already showed that $\|\varphi(x)\| \leq \|x\|$ so indeed $\|\varphi(x)\| = \|x\|$. Injectivity follows from being an isometry because if $\varphi(x) = 0$ then $\|x\| = \|\varphi(x)\| = 0$ so $x = 0$. We alternatively know that X^* separates points again by the Hahn-Banach theorem so if $x \neq y$ then there is a linear functional ℓ such that $\ell(x) \neq \ell(y)$ and thus $\varphi(x)(\ell) \neq \varphi(y)(\ell)$ so $\varphi(x) \neq \varphi(y)$. \square

Theorem 10.1.4. Let X be a normed space. If X^* is separable then X is separable.

Corollary 10.1.5. If X is a reflexive Banach space then X is separable if and only if X^* is separable.

10.2 Reflexive Spaces

Example 10.2.1. If X is finite dimensional then X is reflexive.

Example 10.2.2. On a measure space $(\Omega, \mathcal{F}, \mu)$ for $1 < p < \infty$ we have $L^p(\Omega)^* \cong L^q(\Omega)$ where $p^{-1} + q^{-1} = 1$. Therefore $L^p(\Omega)$ is reflexive. However, for $p = 1, \infty$ this does not hold.

Example 10.2.3. Indeed, here we consider the $p = \infty$ case. Consider $X = C([-1, 1], \mathbb{R})$ with the sup-norm. By Hahn-Banach, for any $\ell \in X^*$ there is some $\varphi \in X^{**}$ such that $\|\varphi\| = 1$ and $\varphi(\ell) = \|\ell\|$. If we assume that X is reflexive then φ is in the image and thus $\varphi(\ell) = \ell(x)$ for some $x \in X$. However, consider,

$$\ell(g) = \int_{-1}^0 g(t)dt - \int_0^1 g(t)dt$$

Then $\|\ell\| \leq 2$ so there exists some $f \in X$ with $\|f\| = 1$ and $\ell(f) = 2$. However, $|\ell(f)| < 2$ for all f with $\|f\| = 1$ giving a contradiction.

Theorem 10.2.4 (Riesz). Let H be a Hilbert space. Then, $\langle -, - \rangle$ defines an isomorphism $H \rightarrow \overline{H}^*$ via $x \mapsto \langle x, - \rangle$ where \overline{H}^* is the conjugate dual space or equivalently the space of continuous anti-linear functionals.

Corollary 10.2.5. Any Hilbert space is reflexive.

Proof. Since \overline{H}^* is a Hilbert space there are canonical isomorphisms by Riesz $\varphi_1 : H \rightarrow \overline{H}^*$ and $\varphi_2 : \overline{H}^* \rightarrow H^{**}$. It suffices to show that $\varphi_2 \circ \varphi_1 = \varphi$ where $\varphi : H \rightarrow H^{**}$ is the canonical map considered above. Indeed, for any $x \in H$ and $\ell \in H^*$ consider,

$$(\varphi_2 \circ \varphi_1)(x)(\ell) = \varphi_2(\langle x, - \rangle)(\ell) = \langle \langle x, - \rangle, \ell \rangle_{H^*}$$

However, by Riesz we know that $\ell = \langle y, - \rangle$ for some $y \in H$ and that φ_1 is an isometry (although anti-linear so it reverses the inner product) so,

$$(\varphi_2 \circ \varphi_1)(x)(\ell) = \langle y, x \rangle = \ell(x)$$

\square

Proposition 10.2.6. If X is a normed space and $V \subset X$ is a closed linear subspace. If X is reflexive then so is V .

Proof. We need to prove that if $\psi \in Y^{**}$ then there exists $y \in Y$ such that $\varphi_Y(y) = \psi$. Under the canonical map $Y^{**} \rightarrow X^{**}$ we see that any $\psi \in Y^{**}$ maps to the image of $\varphi_X : X \rightarrow X^{**}$ so there is some $x \in X$ such that the image ψ' of ψ under $Y^{**} \rightarrow X^{**}$ satisfies,

$$\forall \ell \in X^* : \psi'(\ell) = \psi(\ell|_Y) = \ell(x)$$

Therefore it suffices to show that $x \in Y$ since by Hahn-Banach we can extend any linear functional on Y to X such that its restriction is unchanged. Suppose not, there there exists some $m \in X^*$ such that $m|_Y = 0$ and $m(x) \neq 0$ by Hahn-Banach. Now, clearly,

$$\psi'(m) = \psi(m|_Y) = 0$$

However, $\psi'(m) = m(x) \neq 0$ giving a contradiction so $x \in Y$ and thus Y is reflexive. \square

10.3 Weak Convergence

Definition 10.3.1. Let X be a topological K -vector space. We say a sequence $x_n \in X$ *converges weakly* to $x \in X$ or $x_n \rightarrow x$ weakly if the sequence converges in the weak topology $\sigma(X, X^*)$ on X . Explicitly, $x_n \rightarrow x$ weakly if for every $\ell \in X^*$ we have,

$$\lim_{n \rightarrow \infty} \ell(x_n) = \ell(x)$$

Remark. From here on let X be a normed space.

Proposition 10.3.2. Bounded linear operators $T : X \rightarrow Y$ are weakly continuous.

Proof. If $T : X \rightarrow Y$ is bounded then it is continuous in the norm topology. Thus for any continuous linear functional $\ell \in Y^*$ we know that $T^*\ell = \ell \circ T \in X^*$ is a continuous linear functional and thus continuous in the weak topology on X . Therefore, $T : X \rightarrow Y$ is continuous in the weak topologies. \square

Proposition 10.3.3. If $x_n \in X$ is a weakly convergent sequence then $\{x_n\}$ is bounded.

Proof. Consider the operators $\text{ev}_{x_n} \in X^{**}$. Because x_n converges weakly, for each $\ell \in X^*$ we know that,

$$\lim_{n \rightarrow \infty} \ell(x_n) = \lim_{n \rightarrow \infty} \text{ev}_{x_n}(\ell)$$

exists and thus $\{\text{ev}_{x_n}(\ell)\}$ is bounded. Since X^* is a Banach space, we can apply Banach-Steinhaus to the sequence of pointwise bounded operators $\{\text{ev}_{x_n}\} \subset \mathcal{L}(X^*, K)$ to conclude that $\{\text{ev}_{x_n}\}$ is uniformly bounded. Therefore,

$$\|x_n\| = \sup_{\|\ell\|=1} |\ell(x_n)| = \sup_{\|\ell\|=1} |\text{ev}_{x_n}(\ell)| = \|\text{ev}_{x_n}\|$$

is uniformly bounded. \square

Proposition 10.3.4 (Lower semi-continuity of weak limits). Let $x_n \rightarrow x$ weakly. Then,

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$$

Proof. By Hahn-Banach, there exists some $\ell \in X^*$ with $|\ell(x)| = \|x\|$ and $\|\ell\| = 1$. Therefore,

$$\|x\| = |\ell(x)| = \lim_{n \rightarrow \infty} |\ell(x_n)| = \liminf_{n \rightarrow \infty} |\ell(x_n)| \leq \liminf_{n \rightarrow \infty} \|x_n\| \cdot \|\ell\| = \liminf_{n \rightarrow \infty} \|x_n\|$$

□

Lemma 10.3.5. Suppose that $C \subset X$ is convex and (norm) closed then C is weakly closed.

Proof. Let C be a closed and convex. Then for any $x_0 \notin C$ by convex Hahn-Banach there is some $\ell \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x_0) < \alpha < f(x)$ for all $x \in C$. Then $f^{-1}((-\infty, \alpha))$ is a weakly open neighborhood of x in C^C so C is weakly closed. □

Proposition 10.3.6. If $\varphi : X \rightarrow \mathbb{R}$ is convex and lower semi-continuous in the norm topology then φ is lower semi-continuous in the weak topology on X .

Proof. It suffices to show that $C_\alpha = \varphi((-\infty, \alpha])$ is weakly closed. Since φ is lower semi-continuous in the norm topology C_α is norm closed. Furthermore, because φ is convex if $a, b \in C_\alpha$ then for any $\lambda \in [0, 1]$,

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b) \leq \lambda \alpha + (1 - \lambda)\alpha = \alpha$$

Therefore, C_α is convex so it is weakly closed by above. Therefore φ is weakly lower semi-continuous. □

Remark. Since $\|\bullet\| : X \rightarrow \mathbb{R}$ is convex and norm continuous we get another proof of the weak lower semi-continuity.

Proposition 10.3.7. If X and Y are Banach spaces and $T : X \rightarrow Y$ is linear then T is bounded if and only if T is weakly continuous.

Proof. One direction is immediate. Suppose that $T : X \rightarrow Y$ is weakly continuous. To show that T is continuous it suffices to show that its graph is closed. Let $x_n \rightarrow x$ and $Tx_n \rightarrow y$ in norm. Then $x_n \rightarrow x$ weakly so $Tx_n \rightarrow Tx$ weakly and thus $Tx = y$ because $Tx_n \rightarrow y$ weakly and weak limits are unique. □

10.4 The Weak-* Topology

Remark. When X is a normed space we know that X^* is also normed (in fact it is Banach) so we get a canonical topology on X^* compatible with the vector space structure. However, in general it is not clear how to topologize X^* . One option is the weak-* topology which we now define although this will usually not agree with the norm topology when one is available.

Definition 10.4.1. The weak-* topology on X^* is the coarsest topology on which the maps $\ell \mapsto \ell(x)$ for $x \in X$ are continuous.

Proposition 10.4.2. Using weak-* topology, the canonical map $\varphi : X \rightarrow X^{**}$ is continuous.

Proof. First we must check that φ is actually well-defined. Then $\varphi(x) : \ell \mapsto \ell(x)$ is continuous by hypothesis and thus $\varphi(x) \in X^{**}$. Furthermore, we topologize X^{**} with the weak-* topology induced by the weak-* topology on X . Therefore, for any $\ell \in X^*$ we know that $e_\ell : \psi \mapsto \psi(\ell)$ is continuous. In particular, φ is continuous iff $e_\ell \circ \varphi$ is continuous for all $\ell \in X^*$. However, $e_\ell \circ \varphi : x \mapsto \ell(x)$ is continuous by the definition of the weak-* topology on X^* . □

Remark. Sometimes the weak-* topology is called the “topology of pointwise convergence” because a net f_\bullet in X^* converges to f iff $f_\bullet(x)$ converges to $f(x)$ for each $x \in X$.

Proposition 10.4.3. If X is a Banach space and $\ell_n \in X^*$ is a weak-* convergent sequence then $\{\ell_n\}$ is bounded.

Proof. For each $x \in X$, notice that,

$$\lim_{n \rightarrow \infty} \ell_n(x)$$

exists and thus $\{\ell_n\} \subset \mathcal{L}(X, K)$ is a pointwise bounded sequence of operators. Because X is Banach we can apply Banach-Steinhaus to conclude that $\{\ell_n\}$ is uniformly bounded. \square

Theorem 10.4.4 (Banach-Alaoglu). Let X be a topological vector space and $U \subset X$ a neighborhood of the origin. Then,

$$U^\circ = \{\ell \in X^* \mid \sup_{x \in U} |\ell(x)| \leq 1\}$$

is compact in the weak-* topology.

Corollary 10.4.5. Let X be a normed space and $B = \{x \in X \mid \|x\| \leq 1\}$. Then,

$$B^\circ = \{\ell \in X^* \mid \sup_{x \in U} |\ell(x)| \leq 1\} = \{\ell \in X^* \mid \|\ell\| \leq 1\} = B^*$$

is compact in the weak-* topology.

Proposition 10.4.6. Let X be a Banach space. Then $B = \{x \in X \mid \|x\| \leq 1\}$ is compact in the weak topology on X if and only if X is reflexive.

Proof. If X is reflexive then $X \rightarrow X^{**}$ is an isomorphism. Furthermore, the weak-* topology on X^{**} is equal to the weak topology on X under this isomorphism because the weak-* topology is the weakest topology for which $\psi = \text{ev}_x \mapsto \text{ev}_x(\ell) = \ell(x)$ is continuous which is exactly the weak topology on X . Furthermore, $X \rightarrow X^{**}$ is isometric so the unit balls agree. Therefore, by Banach-Alaoglu, B is compact. Conversely, suppose that B is compact. (HOW TO SHOW THIS!!!) \square

Proposition 10.4.7. If X is a separable normed space then B^* in the weak topology is metrizable.

Proof. We apply Spring 2010 part 2 Q2. Let $\{x_i\} \subset X$ be a dense countable set then $f_i = \text{ev}_{x_i}$ are continuous on X^* in the weak-* topology. Furthermore, for $\ell_1, \ell_2 \in X^*$ if $\ell_1(x_i) = \ell_2(x_i)$ for all x_i then $\ell_1 = \ell_2$ because \mathcal{C} is Hausdorff and $\{x_i\}$ is dense. Therefore f_j separate points. Furthermore, by Banach-Alaoglu, B^* is compact so we apply the problem to conclude that B^* is metrizable in the weak topology. \square

Lemma 10.4.8. Let X be an infinite dimensional normed space. In the weak topology, $\overline{S} = B$.

Proof. Let $\|x_0\| \leq 1$. Now any weakly open neighborhood contains an open of the form,

$$U = \bigcap_{i=1}^n (f_i^{-1}(B_\epsilon(0)) + x_0)$$

for some $f_i \in X^*$. Then consider the map $f : V \rightarrow K^n$ via $x \mapsto (f_1(x), \dots, f_n(x))$. Because V is not finite dimensional, f cannot be injective. Choose some $y_0 \neq 0$ with $f(y_0) = 0$. Then $f_i(ty_0) = 0$ for all t and therefore $x_0 + ty_0 \in U$. Let $g(t) = \|x_0 + ty_0\|$. Notice that $g(0) = \|x_0\| \leq 1$ and $g(t) \rightarrow \infty$ as $t \rightarrow \infty$ because,

$$g(t) \geq \|t\| \|y_0\| - \|x_0\|$$

and $\|y_0\| > 0$. Therefore, there exists some $t \in \mathbb{R}$ such that $g(t) = 1$. Therefore, $x_0 + ty_0 \in U \cap S$ so we see that x_0 is a weak limit point of S . Therefore, $B \subset \overline{S}$. However, B is weakly closed because it is convex and norm closed. Therefore, $B = \overline{S}$. \square

11 Measure Theory

Proposition 11.0.1 (Chebyshev). Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f a measurable function and $\epsilon > 0$. Then,

$$\mu(\{x \in X \mid |f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f| d\mu$$

Remark. The following establishes the continuity of the Lebesgue integral.

Proposition 11.0.2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : X \rightarrow \hat{\mathbb{R}}$ an integrable function with $X \in \mathcal{F}$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $E \in \mathcal{F}$ with $\mu(E) < \delta$,

$$\left| \int_E f d\mu \right| < \epsilon$$

Proof. Because,

$$\left| \int_E f d\mu \right| \leq \int_E |f| d\mu$$

it suffices to assume that $f \geq 0$. Now by definition,

$$\int_X f d\mu = \int_0^\infty f^*(t) dt = \lim_{s \rightarrow \infty} \int_0^s f^*(t) dt$$

where $f^*(t) = \mu(\{x \in X \mid f(x) > t\})$ exists. Because the integral is an increasing function bounded by its limit, there exists some s_0 such that,

$$\int_{s_0}^\infty f^*(t) dt < \frac{\epsilon}{2}$$

Now set $\delta = \frac{\epsilon}{2s_0}$ and suppose that $\mu(E) < \delta$. Then consider,

$$\left| \int_E f d\mu \right| = \int_0^\infty f_E^*(t) dt$$

where,

$$f_E^*(t) = \mu(\{x \in E \mid f(x) > t\}) = \mu(\{x \in X \mid f(x) > t\} \cap E) \leq \mu(E) < \delta$$

so $f_E^*(t) \leq f^*(t)$ and $f_E^*(t) < \delta$. Therefore,

$$\left| \int_E f d\mu \right| = \int_0^{s_0} f_E^*(t) dt + \int_{s_0}^\infty f_E^*(t) dt < \int_0^{s_0} \delta dt + \int_{s_0}^\infty f^*(t) dt < s_0 \delta + \frac{\epsilon}{2} = \epsilon$$

□

12 Spectral Theory

Definition 12.0.1. Let X be a complex Banach space and $T : X \rightarrow X$ a bounded operator. Define the *spectrum*,

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not invertible}\}$$

and the *resolvent*

$$\rho(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ invertible}\} = \mathbb{C} \setminus \sigma(T)$$

By the bounded inverse theorem $T - \lambda I$ is invertible iff it is bijective. Therefore, we can decompose,

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

where,

$$\sigma_p(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not injective}\}$$

$$\sigma_p(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective and } \overline{\text{Im}(T)} = X\}$$

$$\sigma_r(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective and } \overline{\text{Im}(T)} \neq X\}$$

12.1 Some Lemmas

Lemma 12.1.1. If $S : X \rightarrow Y$ and $T : Y \rightarrow Z$ are bounded operators then $T \circ S$ is bounded and,

$$\|T \circ S\| \leq \|T\| \cdot \|S\|$$

Proof. Since $\|Ty\| \leq \|T\| \cdot \|y\|$ for any $y \in Y$,

$$\|T \circ S\| = \sup_{\|x\|=1} \|T(S(x))\| \leq \sup_{\|x\|=1} \|T\| \cdot \|Sx\| = \|T\| \cdot \|S\|$$

□

Lemma 12.1.2. Let X, Y, Z be normed spaces and $T : Y \rightarrow Z$ a bounded operator. Then composition $T \circ - : \mathcal{L}(X, Y) \rightarrow \mathcal{L}(X, Z)$ is a bounded operator and thus continuous.

Proof. We have shown that for any $S \in \mathcal{L}(X, Y)$ that $T \circ S$ is bounded and,

$$\|T \circ S\| \leq \|T\| \cdot \|S\|$$

and therefore $\|T \circ -\|_{\mathcal{L}(X, Y) \rightarrow \mathcal{L}(X, Z)} \leq \|T\|_{\mathcal{L}(Y, Z)}$ so $T \circ -$ is a bounded operator. □

Lemma 12.1.3. Let X be a Banach space. Suppose that $S \in \mathcal{L}(X, X)$ has $\|S\| < 1$. Then $(I - S)$ is invertible and in fact,

$$(I - S)^{-1} = \sum_{n=0}^{\infty} S^n$$

Proof. First, we need to show that,

$$Q = \lim_{n \rightarrow \infty} Q_n \text{ where } Q_n = \sum_{i=0}^n S^i$$

exists. We know that $\|S^n\| \leq \|S\|^n = r^n$ and thus since $r = \|S\| < 1$ this is a Cauchy sequence because,

$$\|Q_{n+k} - Q_n\| = \|S^n Q_k\| \leq \|S^n\| \cdot \|Q_k\| \leq \|Q_k\| \cdot r^n$$

Furthermore, by the triangle inequality, $\|Q_k\| \leq 1 + r + \dots + r^k = \frac{1-r^{k+1}}{1-r}$ and thus,

$$\|Q_{n+k} - Q_n\| \leq r^n \cdot \frac{1-r^{k+1}}{1-r} < \frac{r^n}{1-r} \rightarrow 0$$

Therefore, the limit exists because X is a Banach space so $\mathcal{L}(X, X)$ is also Banach. Thus Q is well-defined. Now we apply the continuity of composition to see that,

$$(I - S) \circ Q = \lim_{n \rightarrow \infty} \sum_{i=0}^n (S^i - S^{i+1}) = \lim_{n \rightarrow \infty} (I - S^{n+1}) = I$$

because $\|(I - S^{n+1}) - I\| = \|S^{n+1}\| \leq r^{n+1} \rightarrow 0$. Clearly the operators commute so $Q \circ (I - S) = I$ as well. □

12.2 Basic Properties

Remark. We write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

Theorem 12.2.1. Let X be a Banach space and $T \in \mathcal{L}(X)$ a bounded operator. Then,

- (a) $\sigma(T) \subset \mathbb{C}$ is closed
- (b) for any $\lambda \in \rho(T)$ we have $B_\delta(\lambda) \subset \rho(T)$ where $\delta = \|(T - \lambda I)^{-1}\|^{-1} \leq d(\lambda, \sigma(T))$
- (c) the map $\rho(T) \rightarrow \mathcal{L}(X)$ via $\lambda \mapsto (T - \lambda I)^{-1}$ is analytic.

Proof. Clearly (b) implies (a) since $\sigma(T) = \rho(T)^C$. Choose $\lambda_0 \in \rho(T)$ and set $C = \|(T - \lambda_0 I)^{-1}\|$ which exists by the definition of $\rho(T)$. Let $\delta = C^{-1}$ and consider $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \delta$. Then compute,

$$(T - \lambda I) = (T - \lambda_0 I) - (\lambda - \lambda_0)I = (I - \lambda_0 I) \circ [I - (\lambda - \lambda_0) \cdot (T - \lambda_0 I)^{-1}]$$

Now let $S = (\lambda - \lambda_0) \cdot (T - \lambda_0 I)^{-1}$ and notice that,

$$\|S\| = |\lambda - \lambda_0| \cdot \|(T - \lambda_0 I)^{-1}\| < \delta \cdot C = 1$$

Therefore, $I - S$ is invertible so $(T - \lambda I) = (I - \lambda_0 I) \circ (I - S)$ is the composition of invertible bounded operators and thus is invertible itself (as a bounded operator meaning its inverse is also in $\mathcal{L}(X)$). Therefore, $\lambda \in \rho(T)$ showing that $B_\delta(\lambda_0) \subset \rho(T)$ so $\rho(T)$ is open proving (b). Furthermore, we know the explicit inverse of $I - S$ is,

$$(I - S)^{-1} = \sum_{n=0}^{\infty} S^n$$

Therefore, by the continuity of the composition,

$$(T - \lambda I)^{-1} = (I - S)^{-1} \circ (T - \lambda_0 I)^{-1} = \sum_{n=0}^{\infty} S^n \circ (T - \lambda_0 I)^{-1} = \sum_{n=0}^{\infty} (T - \lambda_0 I)^{-(n+1)} \cdot (\lambda - \lambda_0)^n$$

whenever $|\lambda - \lambda_0| < \delta$. Therefore, the map $\lambda \mapsto (T - \lambda I)^{-1}$ is locally computed by an operator-valued power series in $(\lambda - \lambda_0)$ and thus is analytic. Finally, if $\lambda \in \sigma(T)$ then $\lambda \notin \rho(T)$ so $|\lambda - \lambda_0| \geq \delta$ and thus,

$$d(\lambda_0, \sigma(T)) = \inf_{\lambda \in \sigma(T)} |\lambda - \lambda_0| \geq \delta = C^{-1} = \|(T - \lambda_0 I)^{-1}\|^{-1}$$

□

12.3 Spectral Radius

Definition 12.3.1. For an operator $T : X \rightarrow X$ define the *spectral radius*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

Proposition 12.3.2. When $|\lambda| > \|T\|$ the following series converges,

$$(T - \lambda I)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda}\right)^n$$

Proof. Let $S = \frac{T}{\lambda}$. Since $\|T\| < |\lambda|$ then $\|S\| < 1$ so by the previous lemma the sum exists and,

$$\sum_{n=0}^{\infty} \left(\frac{T}{\lambda}\right)^n = (I - S)^{-1} = -\lambda(T - \lambda I)^{-1}$$

□

Corollary 12.3.3. For any $|\lambda| > \|T\|$ we see that $\lambda \in \rho(T)$ and thus $r(T) \leq \|T\|$.

Proposition 12.3.4. We have the following formula,

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$$

Proof. DO THIS!!!!

□

Remark. Notice that $\|T^n\| \leq \|T\|^n$ and therefore we again see that $r(T) \leq \|T\|$.

Remark. We always know that $1 = \|\text{id}\| \leq \|T\| \cdot \|T^{-1}\|$. However, the following tells us exactly when this inequality is strict.

Proposition 12.3.5. An invertible operator with $\|T^{-1}\| = \|T\|^{-1}$ has $\sigma(T) \subset S^1 \cdot \|T\|$.

Proof. Assume that $\|T^{-1}\|^{-1} = \|T\|$. We know that $r(T) \leq \|T\|$ so it suffices to show that if $\lambda \in \sigma(T)$ then $|\lambda| \geq \|T\|$. Indeed, since by assumption $0 \in \rho(T)$ we see that $B_\delta(0) \subset \rho(T)$ where $\delta = \|T^{-1}\|^{-1} = \|T\|$ and thus if $\lambda \in \sigma(T)$ then $\lambda \notin B_\delta(0)$ so $|\lambda| > \|T\|$ proving the result. □

Remark. In general, if T is invertible then $\sigma(T)$ is contained in the annulus,

$$\sigma(T) \subset \{\lambda \in \mathbb{C} \mid \|T^{-1}\|^{-1} \leq |\lambda| \leq \|T\|\}$$

12.4 Some More Properties of the Spectrum

Proposition 12.4.1. Let $T : X \rightarrow X$ be a bounded operator on a Banach space. Then,

- (a) $\sigma(T) \subset \mathbb{C}$ is compact
- (b) if $X \neq \{0\}$ then $\sigma(T)$ is nonempty

Proof. We know that $\sigma(T) \subset \mathbb{C}$ is closed and also

$$\sup_{\lambda \in \sigma(T)} |\lambda| = r(T) \leq \|T\|$$

so $\sigma(T)$ is bounded and thus compact. Now suppose that $\sigma(T)$ is empty. For any $\ell \in \mathcal{L}(X)^*$ we see that the function $g : \mathbb{C} \rightarrow \mathbb{C}$ given by $\lambda \mapsto \ell((T - \lambda I)^{-1})$ is entire because $\lambda \mapsto (T - \lambda I)^{-1}$ is analytic everywhere and ℓ is continuous and linear so,

$$g(\lambda) = \ell((T - \lambda I)^{-1}) = \ell\left(\sum_{n=0}^{\infty} (T - \lambda_0 I)^{-(n+1)} \cdot (\lambda - \lambda_0)^n\right) = \sum_{n=0}^{\infty} \ell((T - \lambda_0 I)^{-(n+1)}) \cdot (\lambda - \lambda_0)^n$$

for $|\lambda - \lambda_0| < \delta$ and thus g is holomorphic about λ_0 for each $\lambda_0 \in \rho(T) = \mathbb{C}$. Since g is continuous on $\overline{B_{2\|T\|}(0)}$ it is bounded. Furthermore, for $|\lambda| > \|T\|$ we know that the following series is convergent,

$$(T - \lambda I)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda}\right)^n$$

and therefore,

$$g(\lambda) = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{\ell(T^n)}{\lambda^n}$$

is convergent but $|\ell(T^n)| \leq \|\ell\| \cdot \|T^n\| \leq \|\ell\| \cdot \|T\|^n$ and therefore for $|\lambda| > 2\|T\|$,

$$|g(\lambda)| \leq -\frac{1}{|\lambda|} \sum_{n=0}^{\infty} \|\ell\| \left(\frac{\|T\|}{|\lambda|} \right)^n = \frac{\|\ell\|}{|\lambda| - \|T\|} < \frac{\|\ell\|}{\|T\|}$$

is bounded. Therefore g is entire and bounded and thus must be constant. Furthermore, the above calculation shows that $g(\lambda) \rightarrow 0$ as $|\lambda| \rightarrow \infty$ so $g = 0$ and thus $(T - \lambda I)^{-1} = 0$ which is impossible unless $X = \{0\}$ in which case the $I = 0$ is invertible. Thus if $X \neq \{0\}$ then $\sigma(T)$ is nonempty. \square

13 Compact Operators on Normed Spaces

13.1 Definition and Basic Properties

Remark. First we recall the following generalization of Heine-Borel to any complete metric space.

Proposition 13.1.1. Let X be a complete metric space. Then $S \subset X$ is compact if and only if S is closed and totally bounded meaning that for any $\epsilon > 0$ there exists a finite open cover of S by balls of radius ϵ .

Definition 13.1.2. A linear operator $T : X \rightarrow Y$ between normed spaces is compact if for any bounded $B \subset X$ then $\overline{T(B)} \subset Y$ is compact.

Example 13.1.3. If Y is finite dimensional then every closed and bounded subset is compact. Therefore, bounded linear operators with finite dimensional codomains are compact.

Proposition 13.1.4. Compact operators are bounded and thus continuous.

Proof. Consider the bounded subset $B = \{x \in X \mid \|x\| \leq 1\}$. Then, $\overline{T(B)}$ is compact and in particular $T(B) \subset \overline{T(B)}$ is bounded. Therefore,

$$\sup_{\|x\| \leq 1} \|Tx\| = \sup_{y \in T(B)} \|y\| < \infty$$

\square

Proposition 13.1.5. Let $T_n : X \rightarrow Y$ be a sequence of compact linear operators with Y Banach. Suppose that $T_n \rightarrow T$ in the norm topology on $\mathcal{L}(X, Y)$. Then T is compact.

Proof. It suffices to show that for any bounded $B \subset X$ that $T(B)$ is totally bounded. Let B be bounded from zero by M . For each $\epsilon > 0$ we can find an n such that $\|T - T_n\| < \frac{\epsilon}{2M}$ and a finite cover $\{B_{\frac{\epsilon}{2}}(y_i)\}_{i \in I}$ of $T_n(B)$. For $x \in B$ we know that,

$$\|(T - T_n)x\| \leq \|T - T_n\| \cdot \|x\| < \frac{\epsilon}{2}$$

But $T_n x \in T_n(B)$ so there is some y_i such that $T_n x \in B_{\frac{\epsilon}{2}}(y_i)$ thus $Tx \in B_{\epsilon}(y_i)$ because,

$$\|y - Tx\| \leq \|y - T_n x\| + \|T_n x - Tx\| < \epsilon$$

Therefore, $\{B_{\epsilon}(y_i)\}$ provides a finite cover of $T(B)$ by ϵ -balls so $T(B)$ is precompact. \square

Remark. We did not assume that X is Banach in the above proposition. However, if Y is

Corollary 13.1.6. Let Y be a Banach space. The space of compact operators $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y)$ is closed in the norm topology on $\mathcal{L}(X, Y)$.

Proof. Since $\mathcal{L}(X, Y)$ is a normed space it is sequential. We have shown that, when Y is Banach, limits of compact operators are compact and thus $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y)$ is sequentially closed and thus closed. \square

Corollary 13.1.7. Let X be a normed space and Y a Banach space and $T_n \in \mathcal{L}(X, Y)$ a sequence of finite rank operators converging to $T \in \mathcal{L}(X, Y)$. Then T is compact.

Proof. Finite rank bounded operators are compact and therefore $T_n \rightarrow T$ implies that T is compact. \square

Proposition 13.1.8. Let $T : X \rightarrow Y$ be compact. If $x_n \rightarrow x$ weakly then $Tx_n \rightarrow Tx$ in norm.

Proof. By replacing x_n by $x_n - x$ we can assume that $x_n \rightarrow 0$ weakly. Because bounded operators are continuous in the weak topology so $Tx_n \rightarrow 0$ weakly. Assume that Tx_n does not converge in norm. Then there is some $\epsilon > 0$ such that $\|Tx_n\| \geq \epsilon$ infinitely often. Passing to a subsequence, we may assume that $\|Tx_n\| \geq \epsilon$ for all n and $x_n \rightarrow 0$ weakly.

Furthermore, since X is Banach, weakly convergent sequences are bounded so for some bounded set B we have $\{x_n\} \subset B$ and thus $\{Tx_n\} \subset \overline{T(B)}$ with $\overline{T(B)}$ compact. Because Y is a metric space, $\overline{T(B)}$ is sequentially compact and thus $\{Tx_n\}$ has a convergent subsequence. Therefore, $Tx_{n_j} \rightarrow y$ in norm and,

$$\|y\| = \lim_{j \rightarrow \infty} \|Tx_{n_j}\| \geq \epsilon$$

so $y \neq 0$. However, then $Tx_{n_j} \rightarrow y$ weakly contradicting the fact that $Tx_{n_j} \rightarrow 0$ weakly because weak limits are unique and any subsequence of a convergent subsequence is also convergent to the same limit. \square

Corollary 13.1.9. Let X be a separable Hilbert space and $T : X \rightarrow Y$ a compact operator. Choose any orthonormal basis $\{e_i\}$ of X . Then $Te_i \rightarrow 0$ in norm.

Proof. By above, it suffices to show that $e_i \rightarrow 0$ weakly. For any $x \in X$,

$$\|x\|^2 = \sum_{i=1}^{\infty} |\langle e_i, x \rangle|^2 < \infty \implies \lim_{i \rightarrow \infty} \langle e_i, x \rangle = 0$$

proving that $e_i \rightarrow 0$ weakly. \square

13.2 Examples and Ascoli's Theorem

Theorem 13.2.1 (Ascoli). Let X be a compact Hausdorff space. Then a subset $B \subset C(X)$ is precompact if and only if it is (pointwise) equicontinuous and pointwise bounded meaning that,

- (a) for all $\epsilon > 0$ there is a neighborhood U_x of each $x \in X$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in B$ and $y \in U_x$
- (b) $\sup_{f \in B} |f(x)| < \infty$ for each $x \in X$.

Proposition 13.2.2. Let X, Y be metric spaces with Y compact and X equipped with a Borel measure. Let $K : X \times Y \rightarrow \mathbb{R}$ be a function with $K(-, y)$ integrable for each $y \in Y$ and such that $\forall \epsilon > 0$ there exists $g \in L^1(X)$ with $\|g\|_1 < \epsilon$ and $\delta > 0$ such that $|K(-, y_1) - K(-, y_2)| < |g|$ almost everywhere whenever $|y_1 - y_2| < \delta$. Then the linear operator $T : C_0(X) \rightarrow C(Y)$ defined by,

$$(Tf)(y) = \int_X K(x, y) f(x) d\mu_X$$

is compact.

Proof. First we need to show that Tf is well-defined. For each $y \in Y$ the function $K(x, y)f(x)$ is integrable because,

$$|(Tf)(y)| = \left| \int_X K(x, y) f(x) d\mu_X \right| \leq \int_X |K(x, y) f(x)| d\mu_X \leq \left(\int_X |K(x, y)| d\mu_X \right) \|f\|_\infty$$

and $K(-, y)$ is integrable. Furthermore, this shows that,

$$\|Tf\|_\infty \leq \sup_{y \in Y} \left(\int_X |K(x, y)| d\mu_X \right) \cdot \|f\|_\infty$$

However, I claim that the integral is continuous. Indeed, for any $\epsilon > 0$ there exists $\delta > 0$ and $g \in L^1(X)$ so that whenever $|y_1 - y_2| < \delta$ we have,

$$\left| \int_X |K(x, y_1)| d\mu_X - \int_X |K(x, y_2)| d\mu_X \right| \leq \int_X |K(x, y_1) - K(x, y_2)| d\mu_X \leq \int_X |g| d\mu_X < \epsilon$$

Thus the integral is absolutely continuous and Y is compact so the supremum exists,

$$\|T\| \leq \sup \int_X |K(x, y)| d\mu_X$$

Furthermore, if $|y_1 - y_2| < \delta$ then,

$$|(Tf)(y_1) - (Tf)(y_2)| \leq \int_X |K(x, y_1) - K(x, y_2)| \cdot |f(x)| d\mu_X \leq \left(\int_X |g| d\mu_X \right) \|f\|_\infty \leq \|f\|_\infty \epsilon$$

so we see that Tf is uniformly continuous. Let $B \subset C_0(X)$ be bounded so suppose it is bounded in norm by a constant M . We have seen that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|(Tf)(y_1) - (Tf)(y_2)| < \frac{\epsilon}{M} \cdot \|f\|_\infty$ whenever $|y_1 - y_2| < \delta$. Therefore, for all $f \in B$,

$$|(Tf)(y_1) - (Tf)(y_2)| < \frac{\epsilon}{M} \|f\|_\infty \leq \epsilon$$

because $\|f\|_\infty \leq M$. Thus $T(B)$ is equicontinuous and uniformly bounded because T is bounded and B is bounded so thus pointwise bounded. Indeed,

$$|(Tf)(y)| \leq \left(\int_X |K(x, y)| d\mu_X \right) \|f\|_\infty \leq \left(\int_X |K(x, y)| d\mu_X \right) M$$

is a bound over all $f \in B$. Therefore by Ascoli's theorem, $T(B)$ is precompact so T is a compact operator. \square

Remark. If X is compact and μ_X is a finite Borel measure then the conditions on K follow immediately from K being continuous. Integrability follows from,

$$\int_X |K(-, y)| d\mu_X \leq \int_X \|K(-, y)\|_\infty d\mu_X = \mu(X) \cdot \|K(-, y)\|_\infty < \infty$$

Furthermore, because X is compact, $X \times Y$ is compact so K is automatically uniformly continuous. Therefore, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|y_1 - y_2| < \delta$ implies that

$$|K(x, y_1) - K(x, y_2)| < \frac{\epsilon}{2\mu(X)}$$

(here I'm actually using something a bit weaker than uniform continuity on $X \times Y$ I'm using that $K(x, -)$ is uniformly continuous on Y *uniformly* in $x \in X$ meaning the δ can be chosen independently of $x \in X$). Furthermore, the constant function $\frac{\epsilon}{\mu(X)}$ is integrable and $\int_X \frac{\epsilon}{\mu(X)} d\mu_X < \epsilon$ by definition since $\mu(X)$ is finite.

13.3 Spectrum of Compact Operators

Proposition 13.3.1. Let $T : X \rightarrow X$ be a compact operator on a normed space. If $\dim X = \infty$ then $0 \in \sigma(T)$.

Proof. Otherwise, T would have a bounded inverse $T^{-1} : X \rightarrow X$ but then $TT^{-1} = I$ would be compact. Furthermore, on a normed space, I is compact iff the unit ball is precompact iff $\dim X < \infty$ giving a contradiction. \square

Proposition 13.3.2. Let $T : X \rightarrow X$ be a compact operator on a normed space. For any $\lambda \neq 0$,

$$\dim \ker(T - \lambda I) < \infty$$

Proof. This kernel is the λ -eigenspace, $V_\lambda = \ker(T - \lambda I)$. Notice that $T|_{V_\lambda} = \lambda I_{V_\lambda}$ by definition and therefore $I_{V_\lambda} = \lambda^{-1}T|_{V_\lambda}$ is compact. However, on a normed space, the identity is compact iff the unit ball is precompact iff $\dim V_\lambda < \infty$. \square

Lemma 13.3.3. Let $T : X \rightarrow X$ be a compact operator on a Banach space and $\lambda \neq 0$. If $(T - \lambda I)$ is surjective then $\ker(T - \lambda I) = \{0\}$.

Proof. Let $T_\lambda = (T - \lambda I)$. Define $V_\lambda^{(i)} = \ker T_\lambda^{i+1}$. Since $\lambda \neq 0$, by the previous lemma $V_\lambda^{(1)} = V_\lambda = \ker T_\lambda$ is finite dimensional. Then clearly $V_\lambda^{(i)} \subset V_\lambda^{(i+1)}$. Furthermore, $T_\lambda : V_\lambda^{(i+1)} \rightarrow V_\lambda^{(i)}$ is surjective (because T_λ is surjective) and

$$\ker T_\lambda|_{V_\lambda^{(i+1)}} = \ker T_\lambda = V_\lambda^{(1)}$$

because $\ker T_\lambda \subset V_\lambda^{(i+1)}$. Thus by rank-nullity,

$$\dim V_\lambda^{(i+1)} = \dim V_\lambda^{(i)} + \dim V_\lambda^{(1)}$$

Therefore, $\dim V_\lambda^{(i)} = i \cdot \dim V_\lambda$. Assume that $\dim V_\lambda \neq 0$ then this is an increasing sequence of closed finite-dimensional subspaces.

Choose $x_i \in V_\lambda^{(i)}$ such that $\|x_i\| = 2$ and $\|x_i - u\| \geq 1$ for all $u \in V_\lambda^{(i-1)}$ by taking complements. Then consider,

$$\|Tx_{i+1} - Tx_i\| = \|T_\lambda x_{i+1} + \lambda x_{i+1} - T_\lambda x_i - \lambda x_i\|$$

However, $T_\lambda x_{i+1} \in V_\lambda^{(i)}$ and $x_i, T_\lambda x_i \in V_\lambda^{(i)}$ so therefore,

$$\|Tx_{i+1} - Tx_i\| = |\lambda| \cdot \|x_{i+1} - \lambda^{-1}(T_\lambda x_i + \lambda x_i - T_\lambda x_{i+1})\| \geq |\lambda|$$

Since $|\lambda| > 0$, no subsequence of $\{Tx_i\}$ can converge but $\{x_i\}$ is bounded contradicting the compactness of T . Therefore, $V_\lambda = \{0\}$. \square

Lemma 13.3.4. Let $T : X \rightarrow X$ be a compact operator on a Banach space and $\lambda \neq 0$. Then $\text{im}(T - \lambda I)$ is closed.

Proof. Since $V_\lambda = \ker(T - \lambda I)$ is finite dimensional it has a closed complement $X = V_\lambda \oplus E_\lambda$. Thus it suffices to show the theorem for $T|_{E_\lambda}$ since $\text{im}(T - \lambda I) = \text{im}(T - \lambda I)|_{E_\lambda}$ because it is zero on V_λ . Thus we may assume that $V_\lambda = \{0\}$.

Let $T_\lambda = T - \lambda I$. Suppose that $\text{im} T_\lambda$ is not closed. Then T_λ is not bounded below so there exists a sequence x_i with $\|x_i\| = 1$ and $T_\lambda x_i \rightarrow 0$. Therefore, by compactness, there exists a convergent subsequence of $\{Tx_i\}$ say $\{Tx_j\}$ then $Tx_j \rightarrow y$. Therefore,

$$(T - \lambda I)y = \lim_{j \rightarrow \infty} (T - \lambda I)x_j = 0$$

because a subsequence of a convergent sequence converges to the same limit. Therefore,

$$\lim_{j \rightarrow \infty} Tx_j = \lambda \lim_{j \rightarrow \infty} x_j$$

and since $Tx_j \rightarrow y$ we see that $x_j \rightarrow \lambda^{-1}y$. Thus, $\|y\| = |\lambda| \cdot \lim_{j \rightarrow \infty} \|x_j\| = |\lambda| > 0$ and,

$$Ty = T \left(\lim_{j \rightarrow \infty} Tx_j \right) = T \lim_{j \rightarrow \infty} \lambda x_j = \lambda \lim_{j \rightarrow \infty} Tx_j = \lambda y$$

Since $V_\lambda = \{0\}$ we see that $y = 0$ contradicting $\|y\| > 0$. Therefore, $\text{im} T_\lambda$ is closed. \square

Theorem 13.3.5. Let $T : X \rightarrow X$ be a compact operator on an infinite dimensional Banach space. Then the following hold,

- (a) $0 \in \sigma(T)$
- (b) the eigenspaces V_λ for $\lambda \neq 0$ are finite dimensional and $\text{im}(T - \lambda I)$ is closed
- (c) $\sigma(T) \setminus \{0\}$ consists only of eigenvalues
- (d) $\sigma(T) \setminus \{0\}$ is countable so it consists of a sequence of eigenvalues $\{\lambda_i\}$
- (e) for any $r > 0$ there are finitely many $\lambda \in \sigma(T)$ with $|\lambda| \geq r$ so $\lim_{i \rightarrow \infty} \lambda_i = 0$.

Proof. We have already shown (a) and (b). To show (c) we need to prove that if $\lambda \neq 0$ and $V_\lambda = \{0\}$ then $T_\lambda = (T - \lambda I)$ is invertible meaning that $\lambda \notin \sigma(T)$. Indeed, we know that $\text{im } T_\lambda$ is closed so $\text{im } T_\lambda = (\ker T_\lambda^*)^\perp$. However, T^* is compact so $T_\lambda^* = T^* - \lambda I$ and we may apply our previous results. In particular, $\text{im } T_\lambda^*$ is closed so $\text{im } T_\lambda^* = (\ker T_\lambda)^\perp = X^*$ because $V_\lambda = \ker T_\lambda = \{0\}$. By a lemma above, $\ker T_\lambda^* = \{0\}$ because T_λ^* is surjective. Therefore, $\text{im } T_\lambda = X$ and T_λ is injective so by the bounded inverse theorem T_λ is invertible.

Notice that (e) implies (d) because,

$$\sigma(T) \setminus \{0\} = \bigcup_{n=1}^{\infty} \{\lambda \in \sigma(T) \mid |\lambda| \geq \frac{1}{n}\}$$

is a countable increasing union of finite sets and thus is countable. Then we can enumerate $\sigma(T) \setminus \{0\}$ via a sequence $\{\lambda_i\}$ and it follows that $\lambda_i \rightarrow 0$ because all but finitely many have $|\lambda_i| < \epsilon$ for any $\epsilon > 0$. Therefore it suffices to prove the main statement of (e).

Suppose there is an infinite sequence $\lambda_i \in \sigma(T) \setminus \{0\}$ with $|\lambda_i| \geq r$. Then we know $V_i = V_{\lambda_i} \neq \{0\}$ by (c). Let $E_n = \text{span}\{V_1, \dots, V_n\}$ is closed since it is finite dimensional. Since eigenspaces are disjoint these eigenspaces are nonempty $E_n \subsetneq E_{n+1}$ so there exists a sequence $x_n \in E_n$ with $\|x_n\| = 2$ and $\|x_n - u\| \geq 1$ for all $u \in E_{n-1}$. Furthermore, if $n > m$ then,

$$\|Tx_n - Tx_m\| = \|\lambda_n x_n - \lambda_m x_m\| \geq |\lambda_n| \geq r$$

because $\lambda_n^{-1} \lambda_m x_m \in E_m \subset E_{n-1}$. Therefore, no subsequence of $\{Tx_n\}$ can possibly be Cauchy contradicting the compactness of T . Thus, there can only be finitely many such eigenvalues. \square

Corollary 13.3.6 (Fredholm Alternative). Let $T : X \rightarrow X$ be a compact operator on a Banach space and $\lambda \neq 0$. Then either $Tu - \lambda u = v$ has a unique solution for each $v \in X$ or there are finitely many linearly independent solutions to $Tu - \lambda u = 0$ and then $Tu - \lambda u = v$ has a solution if and only if,

$$v \in (\ker T^*)^\perp = \{v \in X \mid \forall \ell \in X^* : \ell \circ T = 0 \implies \ell(v) = 0\}$$

in which case the space of all such solutions is $u + u_0$ where $Tu_0 - \lambda u_0 = 0$ which is a finite dimensional affine space.

Proof. Since $\lambda \neq 0$ we see that either λ is an eigenvalue or $(T - \lambda I)$ is invertible. In the latter case $Tu - \lambda u = v$ has a unique solution for all v . In the former case the space $V_\lambda = \ker(T - \lambda I)$ of solutions to $Tu - \lambda u = 0$ is finite dimensional. Furthermore, $\text{im}(T - \lambda I)$ is closed so we know that,

$$\text{im}(T - \lambda I) = (\ker T^*)^\perp = \{v \in X \mid \forall \ell \in X^* : \ell \circ T = 0 \implies \ell(v) = 0\}$$

by the closed range theorem and therefore $Tu - \lambda u = v$ has a solution if and only if $v \in (\ker T^*)^\perp$. Finally, if $u, u' \in X$ are such that $T_\lambda u = T_\lambda u' = v$ then $T_\lambda(u' - u) = 0$ so $u' - u \in \ker T_\lambda = V_\lambda$ and thus $u' = u + u_0$ where $u_0 \in V_\lambda$ and thus is a solution to $Tu - \lambda u = 0$. Therefore, $T_\lambda^{-1}(v) = u_p + V_\lambda$ for any particular solution u_p with $T_\lambda u_p = v$. \square

13.4 The Approximation Property

Definition 13.4.1. Let X, Y be normed spaces. We say that $\mathcal{L}(X, Y)$ has the approximation property if the finite rank operators are dense in the space of compact operators $\mathcal{K}(X, Y)$ with the norm topology. Because $\mathcal{L}(X, Y)$ is a metric space it is sequential and thus this is equivalent to the property that for any compact operator $T \in \mathcal{K}(X, Y)$ there exists a sequence of finite rank operators $T_n \in \mathcal{L}(X, Y)$ such that $T_n \rightarrow T$ in norm.

Definition 13.4.2. We say that X has the approximation property if $\mathcal{L}(X)$ does.

Remark. There are Banach spaces that do not have the approximation property but they are difficult to write down.

Proposition 13.4.3. If Y is a Hilbert space then $\mathcal{L}(X, Y)$ has the approximation property.

Proof. Let $T : X \rightarrow Y$ be compact. Then for any $\epsilon > 0$ there exists a finite cover of $T(B)$ by ϵ -balls $\{B_\epsilon(y_i)\}$. Let $K = \text{span}\{y_1, \dots, y_n\}$ be the span which is finite and thus closed. Therefore, there is an orthogonal projection operator $P_\epsilon : Y \rightarrow K$. Consider $T_\epsilon = P_\epsilon \circ T$ which has image inside $K \subset Y$ and thus finite rank. Now, for any $x \in B$ there is some y_i with $\|Tx - y_i\| < \epsilon$ and thus,

$$\|Tx - T_\epsilon x\| \leq \|Tx - y_i\| + \|T_\epsilon x - y_i\| < \epsilon + \epsilon$$

because $T_\epsilon x = P_\epsilon Tx$ and $y_i \in K$ so $\|P_\epsilon Tx - y_i\| \leq \|Tx - y_i\| < \epsilon$. Therefore,

$$\|T - T_\epsilon\| = \sup_{\|x\| \leq 1} \|Tx - T_\epsilon x\| \leq 2\epsilon$$

meaning that $T_\epsilon \rightarrow T$ as $\epsilon \rightarrow 0$. If we want an actual sequence, let $T_n = T_{\frac{1}{n}}$ and then,

$$\|T - T_n\| \leq \frac{2}{n}$$

and therefore $T_n \rightarrow T$ in norm. □

Remark. Notice where the proof fails for Y any Banach space. In a general Banach space, it is true that K is complemented (because it is finite dimensional) so there still exists a continuous projection $P_\epsilon : Y \rightarrow K$. And still, because $y_i \in K$,

$$\|P_\epsilon Tx - y_i\| = \|P_\epsilon Tx - P_\epsilon y_i\| = \|P_\epsilon(Tx - y_i)\| \leq \|P_\epsilon\| \cdot \|Tx - y_i\| < \epsilon \|P_\epsilon\|$$

because P_ϵ is bounded. However, we cannot conclude that this goes to zero as $\epsilon \rightarrow 0$ because the bound $\|P_\epsilon\|$ may not be well-controlled as $\epsilon \rightarrow 0$ unlike the Hilbert space case where we can ensure that $\|P_\epsilon\| = 1$ uniformly by taking orthogonal projections. Therefore, we can replace Hilbert space by any Banach space which has the property that finite dimensional subspaces have norm-1 projection operators.

Proposition 13.4.4. Suppose that Y has a net of operators $P_\alpha \in \mathcal{L}(Y, Y)$ such that $P_\alpha \rightarrow I$ strongly and $\{P_\alpha\} \subset \mathcal{L}(Y, Y)$ is pointwise bounded.

14 Spectral Theory of Self-Adjoint Operators (WORK IN PROGRESSS!!!)

Proposition 14.0.1. Let $T : H \rightarrow H$ be self-adjoint and bounded. Then $\sigma(T) \subset \mathbb{R}$.

Proof. Let λ not be real and consider the imaginary part,

$$\begin{aligned} \Im(\langle (T - \lambda I)x, x \rangle) &= \frac{1}{2} \left(\langle (T - \lambda I)x, x \rangle - \overline{\langle (T - \lambda I)x, x \rangle} \right) \\ &= \frac{1}{2} (\langle (T - \lambda I)x, x \rangle - \langle x, (T - \lambda I)x \rangle) \\ &= \frac{1}{2} (\langle Tx, x \rangle - \bar{\lambda} \langle x, x \rangle - \langle x, Tx \rangle + \lambda \langle x, x \rangle) = \Im(\lambda) \|x\|^2 \end{aligned}$$

because $\langle Tx, x \rangle = \langle x, Tx \rangle$ since $T = T^*$. Therefore,

$$\Im(\lambda) \|x\|^2 = \Im(\langle (T - \lambda I)x, x \rangle) \leq |\langle (T - \lambda I)x, x \rangle| \leq \|(T - \lambda I)x\| \cdot \|x\|$$

Therefore,

$$\|x\| \leq \Im(\lambda)^{-1} \|(T - \lambda I)x\|$$

proving that $T - \lambda I$ is injective and has closed image. Furthermore, $H = \ker T^* \oplus \overline{\text{im}(T - \lambda I)}$ but $\ker T^* = \ker T = 0$ because $T = T^*$ and thus $T - \lambda I$ has dense image and thus is surjective because we say that its image is closed. Thus by the bounded inverse theorem $T - \lambda I$ is invertible so $\lambda \in \rho(T)$ proving the claim. \square

Corollary 14.0.2. Let $T : H \rightarrow H$ be a compact self-adjoint operator then there is a countable list of eigenvalues $\lambda_i \in \mathbb{R}$ such that $\lambda_i \rightarrow 0$ and $\sigma(T) = \{\lambda_i\} \cup \{0\}$.

Theorem 14.0.3. Let H be a separable Hilbert space and $T : H \rightarrow H$ a compact self-adjoint operator. Then there exists an orthonormal basis of H consisting of eigenvectors of T . Explicitly there is an isomorphism,

$$H \cong V_0 \oplus \overline{\bigoplus_{\lambda \in \sigma(T) \setminus \{0\}} V_\lambda}$$

where V_0 admits an orthonormal basis (by separability) and each V_λ admits an orthonormal basis because $\dim V_\lambda < \infty$.

15 Some Facts about L^p Space

Proposition 15.0.1 (Minkowski). $L^p(\Omega)$ is a Banach space. In particular,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proposition 15.0.2. If $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Then $L^p(\Omega) \subset L^1(\Omega)$ for all $p \geq 1$ and the inclusion is bounded.

Proof. Let $f \in L^p(\Omega)$ we need to show that $\|f\|_1 < \infty$. Apply Hölder's inequality to $g = \frac{1}{\mu(\Omega)}$ then,

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q = \|f\|_p$$

However, $\|fg\|_1 = \frac{1}{\mu(\Omega)} \|f\|_1$ and thus $\|f\|_1 \leq \|f\|_p \cdot \mu(\Omega)$ so the inclusion is bounded by $\mu(\Omega)$. \square

Corollary 15.0.3. If $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Then $L^p(\Omega) \subset L^q(\Omega)$ for all $p \geq q$ and the inclusion is bounded.

Proof. For $f \in L^p(\Omega)$ then $g = |f|^{\frac{p}{s}} \in L^s(\Omega)$ for any $s \in [1, p]$ and thus by above,

$$\|g\|_1 = \|f\|_{\frac{p}{s}}^{\frac{p}{s}} \leq \|g\|_s \cdot \mu(\Omega) = \|f\|_p^{\frac{p}{s}} \cdot \mu(\Omega)$$

Therefore,

$$\|f\|_{\frac{p}{s}}^{\frac{p}{s}} \leq \|f\|_p^{\frac{p}{s}} \cdot \mu(\Omega)^{\frac{1}{s}}$$

Because $q \in [1, p]$ we see that we can set $q = \frac{p}{s}$ for some $s \in [1, p]$ and therefore,

$$\|f\|_q \leq \|f\|_p \cdot \mu(\Omega)^{\frac{1}{p}}$$

showing that $f \in L^1(\Omega)$ and that the inclusion is bounded by $\mu(\Omega)^{\frac{1}{p}}$. \square

Proposition 15.0.4 (Extended Hölder). If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $f \in L^a(\Omega)$ and $g \in L^b(\Omega)$ then,

$$\|fg\|_c \leq \|f\|_a \cdot \|g\|_b$$

Proof. Consider $\tilde{f} = |f|^c$ and $\tilde{g} = |g|^c$ in the usual Hölder inequality with $\frac{c}{a} + \frac{c}{b} = 1$. Then,

$$\|\tilde{f}\tilde{g}\|_1 \leq \|\tilde{f}\|_{\frac{a}{c}} \cdot \|\tilde{g}\|_{\frac{b}{c}}$$

Therefore,

$$\|fg\|_c^c \leq \|f\|_a^c \cdot \|g\|_b^c$$

and thus,

$$\|fg\|_c \leq \|f\|_a \cdot \|g\|_b$$

□

Remark. We can get a better bound for $L^p(\Omega) \subset L^q(\Omega)$ as follows. Let $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Then consider, extended Hölder applied to f and $g = 1$,

$$\|f\|_q = \|fg\|_q \leq \|f\|_p \cdot \|g\|_r = \|f\|_p \cdot \mu(\Omega)^{\frac{1}{r}}$$

Therefore,

$$\|f\|_q \leq \|f\|_p \cdot \mu(\Omega)^{\frac{1}{r}}$$

so the inclusion is bounded by $\mu(\Omega)^{\frac{1}{q} - \frac{1}{p}}$. In particular, take $p = \infty$ then,

$$\|f\|_q \leq \|f\|_\infty \cdot \mu(\Omega)^{\frac{1}{q}}$$

16 Fourier Series

Proposition 16.0.1. Let $\phi \in C(S^1)$ have a continuous extension $f \in C(\overline{\Omega_\delta})$ such that $f|_{\Omega_\delta}$ is holomorphic and $f|_{S^1} = \phi$ where,

$$\Omega_\delta = \{z \in \mathbb{C} \mid 1 - \delta < |z| < 1\}$$

Then the Fourier coefficients,

$$(\mathcal{F}\phi)_n = \int_0^1 e^{-2\pi i n \theta} \phi(\theta) d\theta$$

decay exponentially for $n < 0$ meaning $|(\mathcal{F}\phi)_n| \leq Cr^{|n|}$ for some $0 < r < 1$.

Proof. Consider the loop $\gamma_{s,n}$ defined by $\gamma_{s,n}(t) = se^{2\pi i n t}$. Then, the integrals,

$$a_{n,s} = \frac{1}{2\pi i} \oint_{\gamma_s} z^{-(n+1)} f(z) dz = \int_0^1 s^{-n} e^{-2\pi i n \theta} f(\gamma_s(\theta)) d\theta$$

for $s \in (1 - \delta, 1)$ do not depend on s . Furthermore, because f is continuous on $\overline{\Omega_\delta}$ I claim that,

$$(\mathcal{F}\phi)_n = \lim_{s \rightarrow 1} a_{n,s}$$

Indeed, consider,

$$|(\mathcal{F}\phi)_n - a_{n,s}| \leq \int_0^1 |s^{-n} f(\gamma_s(\theta)) - \phi(\theta)| dt \leq \int_0^1 [1 - s^{-n}] \cdot |f(\gamma_s(\theta)) - \phi(\theta)| dt$$

However, because $\overline{\Omega_\delta}$ is compact we know that f is uniformly continuous so for each $\epsilon > 0$ there is a $\delta' > 0$ such that when $|z - z'| < \delta'$ that $|f(z) - f(z')| < \frac{\epsilon}{2}$. In particular,

$$|f(\gamma_s(\theta)) - \phi(\theta)| = |f(se^{2\pi it}) - f(e^{2\pi it})| < \frac{\epsilon}{2}$$

when $|se^{2\pi it} - e^{2\pi it}| = (1 - s) < \delta'$. Therefore, choose δ'' such that $(1 - \delta'')^{-n} - 1 < \frac{\epsilon}{2M}$ where $M = \sup_{z \in \overline{\Omega_\delta}} |f(z)|$ which exists because f is continuous on a compact set. Thus if $|1 - s| < \min(\delta', \delta'')$,

$$|(\mathcal{F}\phi)_n - a_{n,s}| \leq \int_0^1 [|1 - s^{-n}| |f(\gamma_s(\theta))| + |f(\gamma_s(\theta)) - \phi(\theta)|] dt < \frac{\epsilon}{2M} \cdot M + \frac{\epsilon}{2} = \epsilon$$

meaning that,

$$\lim_{s \rightarrow 1} a_{n,s} = (\mathcal{F}\phi)_n$$

Then, by the integral theorem, $a_{n,s}$ is actually constant in s . Therefore,

$$(\mathcal{F}\phi)_n = a_{n,\delta'} = s^{-n} \int_0^1 e^{-2\pi in\theta} f(\gamma_{\delta'}(\theta)) d\theta \leq s^{-n} M$$

Therefore, for $n < 0$ we have,

$$|(\mathcal{F}\phi)_n| \leq M s^{|n|}$$

□

Remark. If we replace holomorphic by anti-holomorphic or swap the annulus to the outside of S^1 then we get exponential decay of the Fourier coefficients for $n > 0$. Thus, we get the following corollary.

Corollary 16.0.2. Let $\phi \in C(S^1)$ have a holomorphic extension on some annulus containing S^1 ,

$$\Omega_\delta = \{z \in \mathbb{C} \mid 1 - \delta < |z| < 1 + \delta\}$$

Then the Fourier coefficients,

$$(\mathcal{F}\phi)_n = \int_0^1 e^{-2\pi in\theta} \phi(\theta) d\theta$$

decay exponentially meaning $|(\mathcal{F}\phi)_n| \leq Cr^{|n|}$ for some $0 < r < 1$.

Proposition 16.0.3. Let $\phi \in C^k(S^1)$. Then, there exists a constant C such that for all n ,

$$|(\mathcal{F}\phi)_n| \leq C(1 + |n|)^{-k}$$

Proof. The case of $n = 0$ is trivial so let $n \neq 0$. Applying integration by parts,

$$(\mathcal{F}\phi)_n = \int_0^1 e^{-2\pi int} \phi(t) dt = (2\pi in)^{-k} \int_0^1 e^{-2\pi it} \phi^{(k)}(t) dt$$

Therefore,

$$|(\mathcal{F}\phi)_n| \leq \frac{1}{|n|^k} \cdot \left| \frac{1}{2\pi} \int_0^1 e^{-2\pi it} \phi^{(k)}(t) dt \right| \leq \frac{1}{|n|^k} \left(\frac{1}{2\pi} \int_0^1 |\phi^{(k)}(t)| dt \right) \leq \frac{1}{(1 + |n|)^k} \left(\frac{2^k}{2\pi} \|\phi^{(k)}\|_1 \right)$$

□

Example 16.0.4. Lacunary series give a good test case. For example,

$$f(z) = \sum_{n=1}^{\infty} n^{-2} z^{n^2}$$

Because,

$$|f(z)| \leq \sum_{n=1}^{\infty} n^{-2} |z|^{n^2}$$

converges when $|z| \leq 1$ we see that f is holomorphic on the open disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and continuous on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Continuity on the circle follows from the M -test which shows that,

$$f(e^{2\pi it}) = \sum_{n=1}^{\infty} n^{-2} e^{in^2 t}$$

converges uniformly and absolutely since the series of supremums,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

converges. Therefore, $f(t)$ is a uniform limit of continuous functions and thus is itself continuous. However, we can see that f cannot extend holomorphically to any annulus containing S^1 for the following reason. Let $\phi(t) = f(e^{2\pi it})$ which is continuous. Then it is immediate from the definition and Fourier inversion that,

$$a_n = (\mathcal{F}\phi)_n = \begin{cases} n^{-1} & n > 0 \text{ and is a square} \\ 0 & \text{else} \end{cases}$$

Therefore, although $a_n \in \ell^1$ we see that a_n does not decay exponentially in fact does not decay faster than n^{-1} meaning that ϕ cannot be $C^k(S^1)$ for $k > 1$. In fact, it is clear that $\psi \notin C^1(S^1)$ therefore there cannot be a holomorphic extension containing S^1 else ϕ would be smooth. However, of course we see that the $n < 0$ coefficients are all zero and thus do (trivially) decay exponentially.

Proposition 16.0.5. A function $\phi : S^1 \rightarrow \mathbb{C}$ is real analytic if and only if there exists some annulus $\Omega_\delta = \{z \in \mathbb{C} \mid 1 - \delta < |z| < 1 + \delta\}$ and a holomorphic function f on Ω_δ such that $f|_{S^1} = \phi$.

Proof. If there is such a holomorphic extension f then obviously ϕ is real analytic. Conversely, if ϕ is real analytic then for each point $z \in S^1$ there exists an open $U_z \subset \mathbb{C}$ and a holomorphic f_{U_z} extension of ϕ on U_z . Explicitly, we can use the disk $B_{r_z}(z)$ where r_z is the radius of convergence (in \mathbb{C} not along the arc) of the Taylor series at z . Therefore, it suffices to check that if $U_z \cap U_{z'} \neq \emptyset$ then $f_{U_z}|_{U_z \cap U_{z'}} = f_{U_{z'}}|_{U_z \cap U_{z'}}$. This holds because both are holomorphic and their difference vanishes on $S^1 \cap U_z \cap U_{z'}$ which has limit points so their difference is identically zero on the connected set $U_z \cap U_{z'}$ (by shrinking the cover we can ensure that the overlaps are connected). Since S^1 is compact, there is a finite subcover by sets U_z so there is a holomorphic function glued on,

$$U = \bigcup_{i=1}^n U_i$$

and then S^1 can be uniformly thickened while still inside U giving the required annulus. \square

Remark. A holomorphic function has no maximal definition set in general. We would like to

17 Operator Topologies

Proposition 17.0.1. Let X and Y be normed spaces and $T_n \in \mathcal{L}(X, Y)$ a sequence of bounded operators with $\|T_n\| \leq M$ and $T \in \mathcal{L}(X, Y)$ a bounded operator. Suppose that there is a dense set $D \subset X$ such that $\forall x \in D : T_n x \rightarrow T x$ converges then $T_n \rightarrow T$ converges in the strong topology.

Proof. We need to show that for any $x \in X : T_n x \rightarrow T x$ converges. Choose a sequence $x_k \in D$ such that $x_k \rightarrow x$. Then consider,

$$\|T_n x - T_n x_k\| \leq \|T_n\| \cdot \|x - x_k\| \leq M \|x - x_k\|$$

Therefore,

$$\|T_n x - T x\| \leq \|T_n x - T_n x_k\| + \|T_n x_k - T x_k\| + \|T x_k - T x\| \leq M \|x - x_k\| + \|T_n x_k - T x_k\| + \|T\| \cdot \|x_k - x\|$$

However, we know that, for each k ,

$$\lim_{n \rightarrow \infty} \|T_n x_k - T x_k\| = 0$$

by assumption and thus,

$$\limsup_{n \rightarrow \infty} \|T_n x - T x\| \leq (M + \|T\|) \|x - x_k\|$$

Since $x_k \rightarrow x$ taking $k \rightarrow \infty$ we see that,

$$\limsup_{n \rightarrow \infty} \|T_n x - T x\| = 0$$

and therefore $T_n x \rightarrow T x$ for each $x \in X$. □

Remark. Notice that in the proof we do not get a uniform bound on $\|T_n - T\|$ tending to zero and therefore T_n may not converge to T in the norm topology only in the strong operator (pointwise convergence) topology.

Remark. The crux of the proof is,

$$\lim_{n \rightarrow \infty} T_n x = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} T_n x_k = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} T_n x_k = \lim_{k \rightarrow \infty} T x_k = T x$$

and justifying the swapping of the limits. This is justified because,

$$\lim_{k \rightarrow \infty} T_n x_k = T_n x$$

uniformly in n because $\|T_n x - T_n x_k\| \leq M \|x - x_k\|$. See the following proposition.

Proposition 17.0.2. Let X be a metric space and $a_{n,m} \in X$ a double sequence. Suppose that,

$$\lim_{m \rightarrow \infty} a_{n,m} = a_n$$

converges uniformly in n and that for all m ,

$$\lim_{n \rightarrow \infty} a_{n,m} = a_m$$

exists and that,

$$\lim_{m \rightarrow \infty} a_m = a$$

exists. Then,

$$a = \lim_{m \rightarrow \infty} a_m = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{n,m} = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{n,m} = \lim_{n \rightarrow \infty} a_n$$

Proof. For any $\epsilon > 0$ there is some M such that for any $m > M$ and all n ,

$$d(a_n, a_{n,m}) < \epsilon$$

Now,

$$d(a_n, a) \leq d(a_n, a_{n,m}) + d(a_{n,m}, a_m) + d(a_m, a) < \epsilon + d(a_{n,m}, a_m) + d(a_m, a)$$

Then,

$$\limsup_{n \rightarrow \infty} d(a_n, a) < \limsup_{n \rightarrow \infty} d(a_{n,m}, a_m) + \epsilon + d(a_m, a) = \epsilon + d(a_m, a)$$

However, $a_m \rightarrow a$ and therefore,

$$\limsup_{n \rightarrow \infty} d(a_n, a) \leq \epsilon + \limsup_{m \rightarrow \infty} d(a_m, a) = \epsilon$$

Since ϵ is arbitrary we see that,

$$\limsup_{n \rightarrow \infty} d(a_n, a) = 0$$

and therefore,

$$\lim_{n \rightarrow \infty} a_n = a = \lim_{m \rightarrow \infty} a_m$$

proving the proposition. □

18 Proving Sets Are Meager

Definition 18.0.1. A subset $B \subset V$ of a K -vector space is absorbent if for all $v \in V$ there exists a $r > 0$ such that if $|t| > r$ then $v \in tB$.

Lemma 18.0.2. Let X be an infinite dimensional normed space and $V \subset X$ a linear subspace and $B_V \subset V$ an absorbent subset. If B_V is precompact then V is meager in X .

Proof. We see that,

$$V = \bigcup_{n=1}^{\infty} nB_V$$

because for each $v \in V$ there is some r such that for $|t| > r$ we have $v \in tB_V$ and thus for $n > r$ we have $v \in nB_V$. Therefore it suffices to prove that nB_V is nowhere dense for each n . Because multiplication by n is a homeomorphism it suffices to show that B_V is nowhere dense. Indeed, $\overline{B_V}$ is compact. So if $x \in \overline{B_V}^\circ$ then there is a ball $B_\epsilon(x) \subset \overline{B_V}^\circ$ and thus $\overline{B_\epsilon(x)} \subset \overline{B_V}$ so $\overline{B_\epsilon(x)}$ is a closed subset of a compact set and thus compact. However, since X is infinite dimensional, by Riesz lemma, the closed unit ball, which is homeomorphic to $\overline{B_\epsilon(x)}$, is not compact. Thus, $\overline{B_V}^\circ = \emptyset$ and therefore B_V is nowhere dense proving that V is of first category. □