1 Sep. 20

1.1 History

(a) 19th century

  (a) $Z(f_1, \dots, f_n) \subset \mathbb{C}^n$ using analytic tools
  (b) Riemann's idea of moduli of algebraic curves (1857)

(b) 20th century

  (a) $Z(f_1, \dots, f_n) \subset \mathbb{C}^n$ using algebraic tools (commutative algebra)
  (b) replace $\mathbb{C}$ with algebraically closed field k
  (c) number theory: want $k = \mathscr{F}_p$ or $\mathbb{Q}$ not algebraically closed. Examples: Fermat's Last Theorem:

  $$u^n + v^n = 1$$

  want geometry for,

  $$\mathbb{Q}[u, v]/(u^2 + v^2 - 1)$$

  but this only has finitely many points so how is there a "geometry".

1.1.1 Question: for any field k is there a "geometry" for $k[X_1, \dots, X_n]/I$?

First attempt (Weil and Zariski 1930s - 1940s) use Galois theory with algebraic sets in $\overline{k}^n$ for ideals $I \subset k[X_1, \dots, X_n]$. This only works for perfect fields (not $\mathscr{F}_p(t)$ which we want to consider generic families of equations over $\mathbb{Z}$). Weil's Foundations of Algebraic Geometry.

1.1.2 The Weil Conjectures (1948)

For $f_1, \dots, f_r \in \mathbb{F}_p[x_1, \dots, x_n]$ then define,

$$N_m = \{x \in \mathbb{F}_{p^m}^r \mid f_1(x) = \cdots = f_r(x) = 0\}$$

Then we define a Zeta function,

$$\zeta(s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} p^{-sm}\right)$$

which sould controll the behavior of N_m as $m \to \infty$. Furthermore,

$$Z_{\mathbb{C}}(F_1, \dots, F_r) \subset \mathbb{C}^n \quad \text{and} \quad Z_p = \mathbb{Z}(F_1, \dots, F_r) \subset \overline{\mathbb{F}_p}^n$$

are closely related where algebraic topology invariants of $Z_{\mathbb{C}}$ gives formulas for counts of Z_p.

1.1.3 1950's: Chaos (Kähler, Shimura, Nagata, etc.)

Proposal for algebraic geometry over Dedekind domains but chaotic and confusing. Then Schemes resolve all of these problems to give "geometry over any commutative ring".

1.2 Affine Algebraic Sets

Let k be an algebraically closed field. Let $\mathbb{A}^n = k^n$ and define a subset $Z \subset \mathbb{A}^n$ to be *algebraic* if $Z = Z(\Sigma)$ where $\Sigma \subset k[X_1, \dots, X_n]$ is a set of polynomials. Then $Z(\Sigma) = Z(I)$ where I is the ideal generated by Σ.

Theorem 1.2.1. The algebraic sets form (the complement of) a topology on $\mathbb{A}^n$.

Remark. We call this the Zariski topology.

There is a base of open sets given by,

$$U_f = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$$

1.2.1 Examples

The Zariski topology on $\mathbb{A}^1$ has the cofinite topology. However, $\mathbb{A}^2 \neq \mathbb{A}^1 \times \mathbb{A}^1$ as a topological space.

Remark. Some weird properties of the Zariski topology,

(a) In $\mathbb{C}^n$ any nonzero open ball is Zariski dense.

(b) $Z(f) = Z(f^n)$ and $Z(I) = Z(\sqrt{I})$.

Definition 1.2.2. For any $Y \subset \mathbb{A}^n$ define,

$$I(Y) = \{f \in k[X_1, \dots, X_n] \mid \forall y \in Y : f(y) = 0\}$$

which is a radical ideal.

Proposition 1.2.3 (Nullstellensatz)**.** For a field k and $\mathfrak{m} \subset K[X_1, \dots, X_n]$ a maximal ideal. Then $K[X_1, \dots, X_n]/\mathfrak{m}$ is a finite dimensional K-vector space.

Proof. 210B [Mat, Thm 5.3] $\square$

Corollary 1.2.4. If K is algebraically closed then $K \to K[X_1, \dots, X_n]/\mathfrak{m}$ is an isomorphism so we have $a_i \mapsto X_i$ and thus $X_i - a_i = 0$ in the quotient so,

$$\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$$

is the kernel of the map $K[X_1, \dots, X_n] \to K[X_1, \dots, X_n]/\mathfrak{m}$. Therefore, points in $Z(J)$ correspond to $\mathfrak{m} \supset J$. Therefore,

$$I(Z(J)) = \bigcap_{\mathfrak{m} \supset J} \mathfrak{m}$$

Theorem 1.2.5. The following hold,

(a) $I_1 \subset I_2 \implies Z(I_1) \supset Z(I_2)$

(b) $Y_1 \subset Y_2 \implies I(Y_1) \supset I(Y_2)$

(c) $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(d) $Z(I_1 \cap I_2) = Z(Y_1) \cup Z(Y_2)$

(e) $Z(I(Y)) = \overline{Y}$

(f) $I(Z(J)) = \sqrt{J}$ (Hilbert's Nullstellensatz).

Proof. The first three follow directly from definitions. Suppose that $x \notin Z(I_1)$ and $x \notin Z(I_2)$ then there is some $f_i \in I_i$ such that $f_i(x) \neq 0$ so $f_1(x) f_2(x) \neq 0$ but $f_1 f_2 \in I_1 \cap I_2$ so $x \notin Z(I_1 \cap I_2)$.

Now $Y \subset Z(I(Y))$ so $\overline{Y} \subset Z(I(Y))$. Pick $x \notin \overline{Y}$ so there is some $x \in U_f$ such that $U_f \cap \overline{Y} = \varnothing$. Therefore, $U_f \cap Y = \varnothing$ so $f|_Y = 0$ since if $x \in Y$ then $x \notin U_f$. Thus $f \in I(Y)$ so $x \notin Z(I(Y))$ proving that $Z(I(Y)) \subset \overline{Y}$.

Since $J \subset I(Z(J))$ is radical we see that $\sqrt{J} \subset I(Z(J))$. The key is to apply the Nullstellensatz. $\square$

2 Sep. 22

Proposition 2.0.1. Let K be a field and A a finitely generated K-algebra, $J \subset A$ an ideal. Then,

$$\sqrt{J} = \bigcap_{\mathfrak{m} \supset J} \mathfrak{m} = \mathrm{Jac}\,(J)$$

Proof. Replace A by $A/\sqrt{J}$ such that $J = (0)$ and $\mathrm{nilrad}\,(A) = (0)$. Choose $f \neq 0$ then f is not nilpotent so A_f is nonzero and $A_f = A[x]/(xf - 1)$ is a finitely generated K-algebra. Thus A_f has a maximal ideal $\mathfrak{m} \subset A_f$. Now under $\varphi : A \to A_f$ we see that $\varphi^{-1}(\mathfrak{m}) \subset A$ is a prime. However, $A/\varphi^{-1}(\mathfrak{m}) \hookrightarrow A_f/\mathfrak{m}$ but $A_f/\mathfrak{m}$ is a finite field extension of K so $A/\varphi^{-1}(\mathfrak{m})$ is a finite dimensional K-algebra and a domain so its a field and thus $\varphi^{-1}(\mathfrak{m})$ is maximal and $f \notin \varphi^{-1}(\mathfrak{m})$ and thus $f \notin \mathrm{Jac}\,(A)$. $\square$

Remark. Any domain D that is a finite dimensional K-algebra is a field because if $r \in D$ is nonzero then $D \xrightarrow{\times r} D$ is injective and thus surjective so $xr = 1$ for some $x \in D$ so D is a field.

Remark. Usually difficult to compute $\sqrt{J}$ given generators of J.

Definition 2.0.2. Say that $f \in k[x_1, \ldots, x_n]$ is *radical* if $f \in k$ and no repeated irreducible factors. The hypersurface $Z(f)$ for non-constant f is radical.

Definition 2.0.3. A topological space Y is *irreducible* if $Y \neq \varnothing$ and $Y \neq Y_1 \cup Y_2$ for closed $Y_1, Y_2 \subsetneq Y$. Otherwise, Y is *reducible*.

Remark. If Y is irreducible then every nonempty open $U \subset Y$ is dense. This is because $Y = (Y \setminus U) \cup \overline{U}$ but if U is nonempty then $Y \setminus U$ is a proper subset so $\overline{U} = Y$.

Definition 2.0.4. A topological space Y is *noetherian* if it satisfies the DCC for closed sets meaning if,

$$Z_1 \supset Z_2 \supset Z_3 \supset \cdots$$

is a descending chain then it stabilizes menaing $Z_n = Z_{n+1}$ for all sufficiently large n.

Example 2.0.5. $\mathbb{A}^n$ is noetherian because closed sets correspond to ideals and $k[x_1, \ldots, x_n]$ is noetherian.

Proposition 2.0.6. Let $Z \subset \mathbb{A}^n$ be an algebraic set. Then Z is irreducible if and only if $I(Z)$ is prime.

Proof. Irreducibles are nonempty and prime ideals I are proper subsets. Thus consider the case that $I(Z) \neq (1)$ equivalently that Z is nonempty. We see that,

$$Z = Z_1 \cup Z_2 \iff I(Z) = I(Z_1) \cap I(Z_2)$$

and $Z_i \subsetneq Z$ iff $I(Z_1) \supsetneq I(Z)$. Therefore, irreducibility of Z is equivalent to the condition that if $I(Z) = I_1 \cap I_2$ with I_1 and I_2 radical then $I_1 = I(Z)$ or $I_2 = I(Z)$ which is equivalent to in $A = k[x_1, \dots, x_n]/I(Z)$ the property that if $(0) = J_1 \cap J_2$ then either $J_1 = (0)$ or $J_2 = (0)$. Therefore, we reduce to showing the following: if A is a nonzero reduced ring, then A is a domain iff $J_1 \cap J_2 = (0)$ for radical ideals J_1, J_2 then either $J_1 = (0)$ or $J_2 = (0)$.

If A is a domain then $J_1 J_2 \subset J_1 \cap J_2 = (0)$ so if $a_i \in J_i$ are nonzero then $a_1 a_2 \in J_1 J_2$ so $a_1 a_2 = 0$ contradicting the fact that A is a domain. Now suppose that A has this property. Choose $f, g \in A$ such that $fg = 0$ then let $Q = \sqrt{(f)} \cap \sqrt{(g)}$. If $a \in Q$ then $a^n = pf$ and $a^m = qg$ so $a^{n+m} = pqfg = 0$ and thus $a \in \text{nilrad}\,(A)$ but A is reduced so $Q = (0)$ and thus either $f = 0$ or $g = 0$ by the assumption. □

Corollary 2.0.7. If f is radical then $Z(f)$ is irreducible iff f is irreducible.

Proof. Both are equivalent to (f) being prime. □

Theorem 2.0.8. Every noetherian topological space is a finite union of irreducible closed sets,

$$Y = Y_1 \cup \cdots \cup Y_r$$

which is unique if we require the irredudency,

$$Y_i \not\subset \bigcup_{j \neq i} Y_i$$

Furthermore, in the irredudent case, the Y_i are exactly the maximal irreducible subsets (i.e. irreducible components).

Corollary 2.0.9. Every algebraic set Z is a finite union of irreducible closed subsets.

Definition 2.0.10. An *affine variety* is a irreducible algebraic set.

3 Dimension and Regular Functions

Lemma 3.0.1. If $Y \subset X$ is irreducible in the subspace topology and $Y \subset Z_1 \cup Z_2$ for closed $Z_j \subset X$ then $Z \subset Z_1$ or $Y \subset Z_2$.

Proof. Then $Y = (Y \cap Z_1) \cap (Y \cap Z_2)$. □

Remark. This is why for an irredundant decomoposition,

$$X = Z_1 \cup \cdots \cup Z_n$$

into its irreducible components then every irreducible $Y \subset X$ lies inside some Z_i. Therefore, the Z_i are indeed maximal irreducible subsets.

Definition 3.0.2. The (combinatorial) *dimension* of a topological space X is,

$$\dim(X) = \sup\{n \geq 0 \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X \text{ of irreducible closed } Z_j \subset X\}$$

Furthermore we set $\dim(\varnothing) = -\infty$.

Remark. We may have $\dim(X) = \infty$.

Definition 3.0.3. For a commutative ring A,

$$\dim A = \sup\{n \geq 0 \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset A \text{ for prime } \mathfrak{p}_j \subset A\}$$

and we set $\dim(0) = -\infty$.

Definition 3.0.4. For $Y \subset \mathbb{A}^n$ an algebraic set, we define the coordinate ring $k[Y] := k[x_1, \dots, x_n]/I(Y)$. Notice this depends on the embedding into affine space not necessarily the intrinsic structure of Y.

Remark. We see that there are inclusion reversing equivalences,

$$\{\text{Radical ideals of } k[Y]\} \iff \{\text{closed subsets } Z \subset Y\}$$

and likewise,

$$\{\text{Prime ideals of } k[Y]\} \iff \{\text{irreducible closed subsets } Z \subset Y\}$$

Therefore,

$$\dim Y = \dim k[Y]$$

Remark. For irreducible closed $Z \subset X$ where X is an affine algebraic set, does there exist a maximal length chain,

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d \subset X$$

with some $Z_j = Z$? If X is reducible then the answer is no because we can have irreducible components of different dimensions. However, for irreducble algebraic sets the answer is yes.

Theorem 3.0.5. Let B be a domain finitely generated over a field k. Then,

(a) $\dim B = \operatorname{trdeg}_k(\operatorname{Frac}(B))$ which is, in particular, finite

(b) For any prime $\mathfrak{p} \subset B$,

$$\dim B = \dim B/\mathfrak{p} + \dim B_{\mathfrak{p}}$$

Remark. We interpret the second part of this theorem as follows. The primes of $B/\mathfrak{p}$ are exactly the primes containing $\mathfrak{p}$ and thus we consider maximal chains,

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

and the primes of $B_{\mathfrak{p}}$ are exactly the primes contained in $\mathfrak{p}$ and thus we consider maximal chains,

$$\mathfrak{p} = \mathfrak{q}_m \supsetneq \cdots \subsetneq \mathfrak{q}_0$$

and thus splicing them together, by the theorem, gives a maximal length chain in B containing $\mathfrak{p}$.

Proof. For (a) [Mat. Thm. 5.6] for (b) [Mat, Ex. 5.1] (the solution is in the back of the book and uses part (a) and induction for non-algebraically closed fields even if you only care about the case of algebrcially closed fields). ☐

Theorem 3.0.6 (Krull)**.** For all local noetherian rings, $\dim A < \infty$.

Proof. See [Mat, Thm. 13.5] and [AM, Cor. 11.11]. ☐

Corollary 3.0.7. $\dim \mathbb{A}^n = \dim k[x_1, \dots, x_n] = \mathrm{trdeg}_k(k(x_1, \dots, x_n)) = n$

Corollary 3.0.8. Let $Z \subset \mathbb{Z}^n$ be irreducble and closed. Then,

(a) For nonempty open $U \subset Z$ (so $\overline{U} = Z$ because Z is irreducible) then $\dim U = \dim Z$

(b) If $Z = Z(f)$ for irreducible f then $\dim Z = n - 1$

(c) For each $x \in Z$ we have $\dim k[Z]_{\mathfrak{m}_z} = \dim Z$.

Remark. $\dim k[Z]_{\mathfrak{m}_z}$ corresponds to chains of irreducibles beginning at $Z_0 = \{z\}$.

Proof. (c) is immediate. Now we do (a). We see that U is irreducible because $\overline{U} = Z$ since $Y \subset U$ is closed then $\overline{Y} \cap U = Y$. Suppose that,

$$Y_0 \subsetneq \cdots \subsetneq Y_n \subset U$$

is a chain of closed irreducible subsets of U then,

$$\overline{Y}_0 \subsetneq \cdots \subsetneq \overline{Y}_n \subset Z$$

is a chain of closed irreducible subsets of Z since $\overline{Y}_i \cap U = Y_i$ so the containments are proper. Therefore, $\dim U \leq \dim Z$. Now, by the previous theorem, we can choose a maximal chain such that $Z_0 = \{z\}$ with $z \in U$ (the point can be chosen arbitrarily) and get,

$$Z_0 \subsetneq \cdots \subsetneq Z_n = Z$$

so take $Y_j = Z_j \cap U$ which is clearly closed in U and irreducible. Since $z \in U \cap Z_j$ we see that Y_j is nonempty but open in Z_j and thus $\overline{Y}_j = Z_j$ and thus the containments must be proper since $Z_j \subsetneq Z_{j+1}$.

Finally to show (b) we apply the dimension formula,

$$\dim Z(f) + \mathbf{ht}\left((f)\right) = n$$

so it suffices to prove that $(0) \subsetneq (f)$ is a minimal nonzero prime. However, for any nonzer $\mathfrak{p} \subset (f)$ take an irreducible element $g \in \mathfrak{p}$ (factor any element and by primality its irreducible factors are inside $\mathfrak{p}$) and thus $(g) \subset \mathfrak{p} \subset (f)$ so $g = fr$ but g is irreducible and f is not a unit so r is a unit and thus $(g) = (f)$ so $\mathfrak{p} = (f)$. ☐

Proposition 3.0.9. Let A be a Noetherian domain and suppose that $f \in A$ is nonzero and (f) is prime. Then (f) is a minimal nonzero prime.

Proof. then take $x \in \mathfrak{p}$ so $x = fr$ but $f \notin \mathfrak{p}$ so $r \in \mathfrak{p}$ and thus $f\mathfrak{p} = \mathfrak{p}$. Thus if $\mathfrak{p}$ is finitely generated (it is because we are in a Noetherian ring) then there is $r \in (f)$ such that $(r-1)\mathfrak{p} = 0$ by Nakayama but in a domain this implies $\mathfrak{p} = 0$ because $r - 1 \neq 0$. ☐

Remark. The closed sets $Z \subset \mathbb{A}^n$ whose irreduclbe components are all of dimension $n-1$ are *exactly* $Z = Z(f)$ fo nonconstant $f \in k[x_1, \dots, x_n]$ (look at irreduclbe components $Z_j = Z(f_j)$).

3.0.1 "Nice" functions on algebraic sets

We have $k[Z] = k[x_1, \dots, x_n]/I(Z) \hookrightarrow \mathrm{Func}(Z, k)$ by sending $g \mapsto (z \mapsto g(z))$ because these functions by definition do not care about polynomials that vanish on Z. Consider $U = Z_f \subset Z$ then we get $\alpha_f : k[Z]_f \to \mathrm{Func}(Z_f, k)$ because f is nonvanishing on U and thus f^{-1} makes sense as a function.

Definition 3.0.10. For any open $U \subset Z$ nonempty we define,

$$\mathcal{O}_Z(U) = \{\varphi : U \to k \mid \forall u \in U : \exists u \in V \subset U : \varphi|_V = \tfrac{g}{h} \text{ for } g, h \in k[Z] \text{ and } h|_V \text{ nonvanishing } \}$$

Proposition 3.0.11. The map $\alpha_f : k[Z]_f \xrightarrow{\sim} \mathcal{O}_Z(Y)$ is an isomorphism.