

1 Mar. 30

Remark. Here is a reason that a -1 -category should be the initial (empty) category or the terminal category (single object and single morphism). We want the Hom spaces of a 0 -category (a set) to be -1 -categories but these are singletons or empty. Therefore, we can say that a -1 -sheaf (-2 -stack) should be a function because its a category fibered over X in -1 -categories so just a single element over open compatibly with restriction.

Given a (pre)sheaf on a topological space X we can “glue the fibers together” to get a fibered category over $\text{Open}(U)$ where the fibers are the sets $\mathcal{F}(U)$ over some open. The morphisms in this category are exactly $f|_U \rightarrow f$ where the morphism represents the restriction of the function f over $U \hookrightarrow V$. To make this a sheaf we need axioms involving the topology.

We do the same thing for stacks.

Definition 1.0.1. A 2-presheaf over a topological space X is a functor $f : \mathcal{C} \rightarrow \text{Open}(X)$ such that

- (a) pullbacks exist
- (b) every morphism in \mathcal{C} is a pullback

Remark. Some exercises:

- (a) the fibers of a 2-presheaf are groupoids.

Definition 1.0.2. Let \mathcal{C} be a 2-presheaf then \mathcal{C} is a stack if

- (a) for $a, b \in \mathcal{C}(U)$ the functor $\text{Isom}(a, b)$ is a sheaf on U
- (b) objects glue.

Definition 1.0.3. A category of geometric spaces is a category \mathcal{G} such that there is a distinguished class of “open immersions” which is

- (a) closed under composition
- (b) local in nature
- (c) preserved by pullbacks (fibered products)

Proposition 1.0.4 (Yonega). Consider the category $\text{PSh}_{\mathcal{C}}$ which is the contravariant functors from \mathcal{C} to Set . Then $X \mapsto h^X = \text{Hom}_{\mathcal{C}}(-, X)$ gives a fully faithfully embedding $\mathcal{C} \hookrightarrow \text{PSh}_{\mathcal{C}}$.

2 Geometry on PSh

What is a vectorbundle on a presheaf? If we are going to give it geometry we should know an answer to this question.

Example, the Hodge bundle. For any $S \rightarrow \mathcal{M}_g$ there is a family of curves $\pi : \mathcal{C} \rightarrow S$ and thus we get the Hodge bundle $\pi_*\Omega_{\mathcal{C}/S}$ which is a rank g vector bundle on S . These vector bundles are compatible (by cohomology and base change) with pullbacks $S' \rightarrow S \rightarrow \mathcal{M}_g$.

We call this data a vector bundle on \mathcal{M}_g .

Definition 2.0.1. A vector bundle \mathcal{E} on $\mathcal{F} \in \text{PSh}_{\mathcal{G}}$ is a vector bundle $\mathcal{E}(S)$ on each $S \in \mathcal{G}$ along with isomorphisms (DO THIS)

Exercise 2.0.2. Let \mathcal{G} be the category of open balls in \mathbb{C}^n and holomorphic maps between them. Then $\text{Man}_{\mathbb{C}} \rightarrow \text{PSh}_{\mathcal{G}}$ is a fully faithful embedding.

2.1 Fiber Products

\mathcal{G} may not have fiber products because. For example if \mathcal{G} is the category of smooth manifolds and smooth maps then fiber products of non submersions is not a smooth manifold.

However, $\text{PSh}_{\mathcal{G}}$ does have fiber products. Indeed we construct fiber products point-wise.

Exercise 2.1.1. Any fiber product in \mathcal{G} agrees with the corresponding fiber product in $\text{PSh}_{\mathcal{G}}$ (the Yoneda embedding preserves fiber products).

The Yoneda functor preserves fiber products basically by definition because

$$h^{A \times_B C}(X) = \text{Hom}_{\mathcal{G}}(X, A \times_B C) = \text{Hom}_{\mathcal{G}}(X, A) \times_{\text{Hom}_{\mathcal{G}}(X, B)} \text{Hom}_{\mathcal{G}}(X, C)$$

Definition 2.1.2. A morphism $f : F \rightarrow G$ in $\text{PSh}_{\mathcal{G}}$ is representable when for any map $S \rightarrow G$ from $S \in \mathcal{G}$ then $F \times_G S$ is representable.

Remark. If \mathcal{G} has fiber products then every morphism between representable functors is representable.

Exercise 2.1.3. Representable morphisms are preserved by base change.

Definition 2.1.4. Given a property \mathcal{P} of morphisms in \mathcal{G} . Then we say a representable morphism $f : F \rightarrow G$ in $\text{PSh}_{\mathcal{G}}$ has property \mathcal{P} if for every $S \rightarrow G$ with $S \in \mathcal{G}$ the morphism $F \times_G S \rightarrow S$ (which is a \mathcal{G} -morphism) has property \mathcal{P} .

Remark. For this to make sense, we need \mathcal{P} to be a property preserved under base change so that $X \rightarrow Y$ has property \mathcal{P} if and only if $X_{Y'} \rightarrow Y'$ has property \mathcal{P} .

Definition 2.1.5. We can now define an open cover in $\text{PSh}_{\mathcal{G}}$. A representable morphism is open in the above sense.

3 April 4

Remark. Notice that every representable presheaf on \mathcal{G} is a sheaf when restricted to each object $X \in \mathcal{G}$.

Definition 3.0.1. A presheaf $F \in \text{PSh}_{\mathcal{G}}$ is a *sheaf* if for each $X \in \mathcal{G}$ the presheaf $X|_{\mathcal{G}}$ (restriction to the open subsets of \mathcal{G}) is a sheaf.

Remark. This will be a sheaf for the topology on \mathcal{G} induced by open embeddings.

Definition 3.0.2. Let \mathcal{G} be a category (not necessarily with fiber products). A topology on \mathcal{G} is a connection of morphisms $\mathcal{G}^{\circ} \subset \mathcal{G}$ (the “open immersions”) satisfying the following properties:

- (a) and isomorphism $f : X \rightarrow Y$ is in \mathcal{G}° (for example id_X because \mathcal{G}° is a subcategory)
- (b) openness is preserved under composition (\mathcal{G} is a subcategory)
- (c) pullbacks of morphisms in \mathcal{G}° by morphisms in \mathcal{G} exist and are in \mathcal{G}° .
- (d) the fiber product of $U_1 \rightarrow X$ and $U_2 \rightarrow X$ gives $U_1 \times_X U_2 \rightarrow X$ is open (this is implied by composition and preservation under fiber products).

Along with the data of distinguished collections of morphisms in \mathcal{G}° called covering families such that

- (a) every isomorphism $f : X \rightarrow Y$ is a covering family
- (b) given a covering on Y and a morphism $f : X \rightarrow Y$ then the base change is a cover of X
- (c) a cover of a cover is a cover meaning if $\{X_\alpha \rightarrow X\}$ is a covering family and $\{X_{\beta\alpha} \rightarrow X_\alpha\}$ are covering families then $\{X_{\beta\alpha} \rightarrow X\}$ is a covering family.

Definition 3.0.3. The category of sheaves $\mathfrak{Sh}_{\mathcal{G}} \subset \text{PSh}_{\mathcal{G}}$ is the full subcategory of objects “determined locally on covers” i.e. satisfying the usual sheaf axiom.

Exercise 3.0.4. $\mathfrak{Sh}_{\mathcal{G}}$ has all fiber products and they agree with fiber products in \mathcal{G} (when they exist) and an in $\text{PSh}_{\mathcal{G}}$ under the fully faithfully embeddings,

$$\mathcal{G} \hookrightarrow \mathfrak{Sh}_{\mathcal{G}} \hookrightarrow \text{PSh}_{\mathcal{G}}$$

Definition 3.0.5. If $X \in \mathcal{G}$ define \mathcal{G}_X the slice category of morphisms $f : Y \rightarrow X$.

Definition 3.0.6. A sheaf $F \in \mathfrak{Sh}_{\mathcal{G}}$ is *locally representable* if there is an open cover by representable sheaves. Explicitly there are representable sheaves and representable morphisms $U_i \rightarrow F$ such that for every such diagram,

$$\begin{array}{ccc} X_i & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ U_i & \longrightarrow & F \end{array}$$

for $X \in \mathcal{G}$ we have $\{X_i \rightarrow X\}$ is a covering family in \mathcal{G} .

Remark. Applying this construction we get:

- (a) for affine schemes and open immersions get all schemes
- (b) for varieties and open immersions get pre-varieties (no separatedness or quasi-compactness)
- (c) for open balls in \mathcal{C}^n with open holomorphic embeddings get pre-manifolds (no Hausdorffness or second countability).

4 April 6

Reminder about \mathcal{G} : maybe it doesn't have fiber products (e.g. manifolds). We require our topology is *subcanonical* meaning for any $Y \in \mathcal{G}$ the functor h^Y is a sheaf.

Our category \mathcal{G} is often a subcategory of locally ringed spaces. In most cases we can recover the sheaf of rings via maps to a ring object $\mathbb{A}^1 \in \mathcal{G}$.

Exercise 4.0.1. $\text{LRep}_{\text{AffSch}} \subset \mathbf{Sch}$.

Theorem 4.0.2. If \mathcal{G} contains all fiber products (and a terminal object) then every $M \in \text{LRep}_{\mathcal{G}}$ has $\Delta : M \rightarrow M \times M$ representable.

Remark. In the example $\mathcal{G} = \text{AffSch}$ we see that \mathbb{A}^2 with the doubled origin is not in $\text{LRep}_{\mathcal{G}}$ because its diagonal is not affine and hence not representable.

Lemma 4.0.3. If \mathcal{G} has products and fiber products. Then all maps $X \rightarrow F$ for $X \in \mathcal{G}$ and $F \in \text{PSh}_{\mathcal{G}}$ are representable if and only if $\Delta : F \rightarrow F \times F$ is representable.

Proof. First, assume that Δ is representable. Then,

$$\begin{array}{ccc} X \times_F Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & F \end{array}$$

The following diagram is also cartesian,

$$\begin{array}{ccc} X \times_F Y & \longrightarrow & X \times Y \\ \downarrow & \lrcorner & \downarrow \\ F & \longrightarrow & F \times F \end{array}$$

and $X \times Y \in \mathcal{G}$ so $X \times_F Y \in \mathcal{G}$ proving the claim. Next, suppose that $X \rightarrow F$ is always representable. For $U \in \mathcal{G}$ and $U \rightarrow F \times F$ we want to show that $U \times_{F \times F} F \in \mathcal{G}$.

$$\begin{array}{ccc} U \times_{F \times F} F & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ U \times_F U & \longrightarrow & U \times U \\ \downarrow & \lrcorner & \downarrow \\ F & \longrightarrow & F \times F \end{array}$$

We see that $U \times_F U$ is representable because $U \rightarrow F$ is representable and thus the top square is all in \mathcal{G} and hence because \mathcal{G} has fiber products we conclude. \square

Remark. Given a diagram,

$$\begin{array}{ccc} \text{Isom} & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ F & \longrightarrow & F \times F \end{array}$$

then the pullback will classify isomorphisms between the objects over U represented by F under the two maps $U \rightarrow F$.

Proof of Theorem. We need to show that $F \rightarrow F \times F$ is representable. By the lemma, it is equivalent to ask if for every diagram with $X, Y \in \mathcal{G}$ we have,

$$\begin{array}{ccc} X \times_F Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & F \end{array}$$

we want $X \times_F Y$ is representable. Choose a cover $U_i \rightarrow F$ by representable U_i . Then we pullback to get \square

4.1 Complex Analytic Spaces

4.2 Sheafification

5 April 8

Given the data $U_i \in \mathcal{G}$ along with opens immersions $U_{ij} \hookrightarrow U_i$ and $U_{ij} \rightarrow U_j$ such that $U_{ij} \times_{U_i} U_{ik} = U_{kj} \times_{U_j} U_{ik}$ MAKE THIS WORK!!!

Then we get that,

$$M^-(X) = \text{Hom}_{\mathcal{G}}(X, \coprod U_i) / \text{Hom}_{\mathcal{G}}(X, \coprod U_{ij})$$

automatically $M^-(X) \in \text{PSh}_{\mathcal{G}}^+$. This cannot be the right presheaf however, for example id_M doesn't make sense because we are not stratifying X . To do this we exactly take the sheafification. Therefore we define $M = (M^-)^+$.

The claim is that $U_i \rightarrow M$ is open and make M be locally representable.

Remark. Suppose that $\Delta : M \rightarrow M \times M$ is representable. Then for any cover $U_i \rightarrow M$ is

6 April —

7 April 13

Question: given $\pi : U \rightarrow X$ in \mathcal{G} does F satisfy the sheaf condition for π meaning is,

$$F(X) \longrightarrow F(U) \rightrightarrows F(U \times_X U)$$

Theorem 7.0.1. If there is $s : X \rightarrow U$ such that $\pi \circ s = \text{id}$ then any presheaf F satisfies the sheaf condition for $\pi : U \rightarrow X$.

Proof. We get maps $\sigma_i : U \rightarrow U \times_X U$ given by $(\sigma \circ \pi, \text{id})$ and $(\text{id}, \sigma \circ \pi)$. Now we get $s^* \pi^* = \text{id}$ so we see that π^* is injective giving the first part. For gluing, given $s \in F(U)$ such that $\pi_1^* s = \pi_2^* s$. Then take $t = \sigma^* s$ and we need to show that $\pi^* t = s$. Now $\pi_1 \circ \sigma_2 = \sigma \circ \pi$ and thus,

$$\pi^* \sigma^* s = \sigma_2^* \pi_1^* s = \sigma_2^* \pi_2^* s = s$$

because $\pi_1^* s = \pi_2^* s$ and $\pi_2 \circ \sigma_2 = \text{id}$. \square

Corollary 7.0.2. Suppose we have a diagram,

$$\begin{array}{ccc} V & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X \end{array}$$

If F satisfies the sheaf condition for $V \rightarrow X$ then it satisfies the sheaf condition for $U \rightarrow X$.

Proof. Consider,

$$\begin{array}{ccc} V \times_X U & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ V & \longrightarrow & X \end{array}$$

We know automatically that F satisfies the sheaf condition for $V \times_X U \rightarrow V$ because there is a section $V \rightarrow V \times_X U$. By assumption F is a sheaf for $V \rightarrow X$. \square

8 April 20

8.1 Quasi-Coherent Sheaves on Algebraic Spaces

For every $\text{Spec}(A) \rightarrow X$ where X is an algebraic space, I want an A -module M such that for $\text{Spec}(A') \rightarrow \text{Spec}(A)$ we have M' is the pullback of M .

Proposition 8.1.1. If $\pi : X \rightarrow Y$ is a quasi-compact and quasi-separated morphism of algebraic spaces and \mathcal{F} is quasi-coherent then $\pi_* \mathcal{F}$ is quasi-coherent.

8.2 Čech Cohomology

For the Zariski topology (and also other cohomology) on X an algebraic space with quasi-compact and affine diagonal then Čech cohomology works for quasi-coherent sheaves.

9 Example

Consider \mathcal{M}_3^a the moduli space of genus 3 curves with no nontrivial automorphisms over \mathbb{C} . Let $\mathcal{G} = \mathbf{Sch}_{\mathbb{C}}$ then $\mathcal{M}_3^a \in \mathbf{PSh}_{\mathcal{G}}$ is the functor,

$S \mapsto \{\pi : C \rightarrow S \text{ relative dim 1 smooth geometrically integral fibers of genus 3 with no automorphisms}\}$

Consider $\pi_* \Omega_{C/S}$ is a rank 3 vector bundle (by cohomology and base change) and thus we get a closed embedding,

$$C \hookrightarrow \mathbb{P}_S(\pi_* \Omega_{C/S})$$

over S . These are all embedded as plane quartic curves. Quartic divisors are parametrized by \mathbb{P}^{14} there is an open $U \subset \mathbb{P}^{14}$ where the associated plane quadric is a smooth irreducible curve. Then,

$$\mathcal{M}_3^a = U/\text{PGL}_3$$

To see this, consider,

$$\begin{array}{ccc}
B_U & \longrightarrow & U \\
\downarrow & \lrcorner & \downarrow \\
B & \longrightarrow & \mathcal{M}_3^a
\end{array}$$

Then B_U is exactly $\text{Isom}(\mathbb{P}_B(\pi_*\Omega_{C/S}), \mathbb{P}_B^3)$ which is a Zariski PGL_3 -torsor OR MAYBE where $C^{\text{univ}} \rightarrow U$ is the universal curve over U . Maybe it's actually correct to take the PGL_3 -torsor D over B of sections of $\mathbb{P}_S(\pi_*\Omega_{C/S})$ and tak

Let's check this. The T -points $T \rightarrow B_U$ are exactly given by the following data (a, b, γ) where $a : T \rightarrow B$ and $b : T \rightarrow U$ and an isomorphism $\gamma : C_T \xrightarrow{\sim} C_T^{\text{univ}}$ which is the data of a map

$$T \rightarrow \text{Isom}_{B \times U}(C_{B \times U}, C_{B \times U}^{\text{univ}})$$

However, since $C^{\text{text}} \rightarrow U$ is universal the maps $b : T \rightarrow U$ are exactly classified by isomorphism classes $[C_T]$ as closed subschemes since because these curves are canonically embedded, as $C \hookrightarrow \mathbb{P}_B(\pi_*\Omega_{X/S})$ and $C^{\text{univ}} \hookrightarrow \mathbb{P}_U^3$ the isomorphism $C_T \xrightarrow{\sim} C_T^{\text{univ}}$ induces an isomorphism $\mathbb{P}(\pi_*\Omega_{X/S}) \xrightarrow{\sim} \mathbb{P}^3$. For a fixed such isomorphism there is a unique map $T \rightarrow U$ defined by the image of C_T in \mathbb{P}^3 therefore,

$$B_U = \text{Isom}_{B \times U}(C_{B \times U}, C_{B \times U}^{\text{univ}}) = \text{Isom}_B(\mathbb{P}_B(\pi_*\Omega_{X/S}), \mathbb{P}_B^3)$$

10 April 25

Definition 10.0.1. A *sieve* over X is a sub-presheaf of h_X .

Example 10.0.2. $S_{U \rightarrow X}$ for any U

Definition 10.0.3. Given τ , a sieve is called a *covering sieve* if it contains some cover.

Example 10.0.4. If \mathcal{C} is the category of opens of a topological space X then a sieve on X is a collection of open in X stable under subsets.

Theorem 10.0.5. A presheaf \mathcal{F} satisfies,

$$\mathcal{F} \text{ is a sheaf} \iff \text{Hom}(S, \mathcal{F}) = \text{Hom}(h_X, \mathcal{F}) \text{ for all covering sieves } S \subset h_X$$

Remark. This shows that the category of sheaves recovers the covering sieves because we can take the sieves to be those that satisfy,

$$\text{Hom}(S, \mathcal{F}) = \text{Hom}(h_X, \mathcal{F})$$

for all sheaves \mathcal{F} .

10.1 Example

We want to show that \mathcal{M}_g^a is an algebraic space. First we need to show it is a sheaf in the étale (or smooth) topology. In fact, this will work in the fpqc topology.

Theorem 10.1.1. The data of (X, \mathcal{L}) where \mathcal{L} is ample descends because we can descend the algebra,

$$\bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$$

which determines X via embedding into projective space.

11 April 27

Suppose $G \curvearrowright X$ where G is a geometric group. We want to define X/G as an algebraic space.

Example 11.0.1. We want,

- (a) $X \rightarrow Y$ is a G -bundle then $X/G = Y$
- (b) $\mathbb{Z}/2 \curvearrowright \mathbb{A}^1$ via $x \mapsto -x$ is not a G -bundle and we want $\mathbb{A}^1/(\mathbb{Z}/2)$ to be the GIT quotient.

What is the definition of X/G as an algebraic space? There should be a G -invariant map $X \rightarrow X/G$ such that any G -invariant map $X \rightarrow Y$ factors uniquely through $X \rightarrow X/G \rightarrow Y$,

$$\begin{array}{ccc} X & \longrightarrow & X/G \\ & \searrow & \downarrow \text{dashed} \\ & & Y \end{array}$$

We call this the categorical quotient. There are some problems with this definition,

- (a) it might not exist
- (b) $X \rightarrow X/G$ might not be a G -bundle e.g. $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ via $x \mapsto x^2$ is the quotient $\mathbb{A}^1/(\mathbb{Z}/2)$ but this is not a $\mathbb{Z}/2$ -bundle (ramified over the origin).

Some possible answers,

- (a) define X/G as the categorical quotient
- (b) define X/G as the categorical quotient by only when $X \rightarrow X/G$ is a G -bundle
- (c) defined it as the presheaf (or the sheafification)

$$Y \mapsto X(Y)/G(Y)$$

- (d) define it as the presheaf sending Y to equivariant maps,

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array}$$

where $P \rightarrow Y$ is a principal G -bundle. We would have to take this up to isomorphism and this will be bad if there are nontrivial automorphisms of the map $P \rightarrow X$.

- (e) Consider the presheaf h^X/h^G and take the sheafification to get X/G . If $G \curvearrowright X$ freely then h^X/h^G is a separated presheaf.

Example 11.0.2. If $Y = *$ and P is the trivial $\mathbb{Z}/2$ -bundle over Y then the two maps $P \rightarrow \mathbb{A}^1$ whose image is ± 1 have no automorphisms but the map $P \rightarrow \mathbb{A}^1$ whose image is 0 does have an automorphism because the action is not free.

Definition 11.0.3. The action $G \curvearrowright X$ is free if it is free on T -points $G(T) \curvearrowright X(T)$.

Example 11.0.4. An elliptic curve is $E = \mathbb{C}/\Lambda$ analytically and indeed $h^E = (h^{\mathbb{C}}/h^{\Lambda})^{++}$. Algebraically we have $\Lambda \curvearrowright \text{Spec}(\mathbb{C}[t]) = \mathbb{A}_{\mathbb{C}}^1$ where Λ is a discrete group viewed as a scheme. And we can define $\mathbb{A}_{\mathbb{C}}^1/\Lambda$ which is an algebraic space but not isomorphic to an elliptic curve! However $(\mathbb{A}_{\mathbb{C}}^1/\Lambda)^{\text{an}} \cong \mathbb{C}/\Lambda$ so $(\mathbb{A}_{\mathbb{C}}^1/\Lambda)^{\text{an}} \cong E^{\text{an}}$ but the isomorphism (the Weierstrass \wp -function) is not algebraic.

12 May 6

Question: how many times do you need to plus h^U/h^R to get a stack?

12.1 How do we tell if a stack is DM

Proposition 12.1.1. If \mathcal{M} is DM then $\mathcal{I}_{\mathcal{M}} \rightarrow \mathcal{M}$ is unramified.

Proof. Consider,

$$\begin{array}{ccc} R & \longrightarrow & U \times U \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{M} \times \mathcal{M} \end{array}$$

where the downward maps are étale. However, $R \rightarrow U$ is étale and hence unramified so $R \rightarrow U \times U$ is unramified. \square

Proposition 12.1.2. If \mathcal{M} is an algebraic stack and $\mathcal{I}_{\mathcal{M}} \rightarrow \mathcal{M}$ is unramified then \mathcal{M} is DM.

Proof. Given a smooth cover $U \rightarrow \mathcal{M}$. For each point $p \in U$ where the relative dimension $n > 0$ we want to slice to find an étale neighborhood. \square

13 May. 9

Let \mathcal{M} be an algebraic stack (meaning a locally representable stack in the smooth topology on schemes).

Definition 13.0.1. The inertia stack,

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{M}} & \longrightarrow & \mathcal{M} \\ \downarrow \Delta & & \downarrow \Delta \\ \mathcal{M} & \xrightarrow{\Delta} & \mathcal{M} \times \mathcal{M} \end{array}$$

is defined by the above fiber product.

Proposition 13.0.2. Let \mathcal{M} be an algebraic stack, then \mathcal{M} is as an algebraic space if and only if $\mathcal{I}_{\mathcal{M}} \rightarrow \mathcal{M}$ is an isomorphism.

Proposition 13.0.3. The following are equivalent,

- (a) \mathcal{M} is DM
- (b) Δ is unramified
- (c) $\Omega_{\mathcal{M}/\mathcal{M} \times \mathcal{M}} = 0$
- (d) $\Omega_{\mathcal{I}_{\mathcal{M}}/\mathcal{M}} = 0$.

Theorem 13.0.4. Let $\pi : X \rightarrow Y$ be proper flat, whose geometric fibers are reduced and connected with Y locally noetherian. Let \mathcal{L} be a line bundle on \mathcal{L} . Then the presheaf sending $Z \rightarrow Y$ to data (\mathcal{M}_Z, φ) where $\varphi : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}|_Z$ is an isomorphism of line bundles on X_Z for the diagram,

$$\begin{array}{ccc}
X_Z & \longrightarrow & X \\
\downarrow \pi_Z & & \downarrow \pi \\
Z & \longrightarrow & Y
\end{array}$$

This is represented by a locally closed subscheme of Y . If the fibers of π are integral then a closed subscheme.

Remark. If we assume π is finitely presented then we can drop noetherian assumptions because it is the pullback of a noetherian case and therefore the theorem holds (showing it can be pulled back from a *flat* noetherian case is the tricky part).

Remark. The above theorem works for a schematic morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of algebraic stacks because it holds after all base changes to schemes.

Proposition 13.0.5. For $g \geq 2$ the stack \mathcal{M}_g sending B to families of smooth genus g curves (flat schematic smooth map of relative dimension 1 with integral curves of genus g fibers) is algebraic. It is a stack in the fpqcK topology.

Proof. Any family of curves $\mathcal{C} \rightarrow B$ has $\omega_{\mathcal{C}/B}$ relatively ample and hence we get an embedding into projective space locally. Using cohomology and base change,

$$\mathcal{C} \hookrightarrow \mathbb{P}(\pi_* \omega_{\mathcal{C}/B}^{\otimes 3})$$

Therefore, consider the locus of canonically embedded curves $H \hookrightarrow \text{Hilb}_{\mathbb{P}^n}$ and then $H \rightarrow \mathcal{M}_g$ is a PGL_n -torsor. \square

Proposition 13.0.6. The stack of dimension n smooth projective varieties X where $\det \Omega_X = \omega_X$ is ample is an algebraic stack for the fpqcK topology.

Proof. Let $X \rightarrow B$ be such a family. Then there is some $\omega_{X_0}^{\otimes N}$ which is very ample with vanishing higher cohomology. Then we consider the open substack of \mathcal{M} where $h^{>0}(X, \omega_X^{\otimes N}) = 0$ and h^0 is constant which is open by semicontinuity. Then the very ample locus is open (for flat maps the closed embedding locus is open). Then we get $\mathcal{M}_{X_0} \subset \mathcal{M}$ open and $H \rightarrow \mathcal{M}_{x_0}$ is a PGL_n -torsor. \square

14 May 11

Proposition 14.0.1. Suppose that \mathcal{M} is an algebraic stack. Then the following are equivalent,

- (a) $\Omega_{\mathcal{J}/\mathcal{M}} = 0$
- (b) $\Omega_{\Delta} = 0$
- (c) \mathcal{M} has a representable étale cover by a scheme so is DM.

Proof. By pullback (b) implies (a) and $\mathcal{J} \rightarrow \mathcal{M}$ has a section so (a) implies (b) by pullback as well. We showed previously that if \mathcal{M} is DM then $\Omega_{\Delta} = 0$ since Δ admits an étale cover by an unramified morphism. Therefore, we just need to show that if Ω_{Δ} then \mathcal{M} is DM.

Start with a smooth cover $U \rightarrow \mathcal{M}$ by a scheme U . We need to slice U to make it étale. We can shrink and take disjoint union so we may take $U = \text{Spec}(A)$. Now consider,

$$\begin{array}{ccc} R & \xrightarrow{\pi_1} & U \\ \downarrow \pi_2 & & \downarrow \\ U & \longrightarrow & \mathcal{M} \end{array}$$

To slice a smooth morphism we just need that its restriction to the fiber cuts down the dimension of the differentials by one. Consider the sequence for $R \rightarrow U \times U \rightarrow \mathcal{M}$,

$$\pi_1^* \Omega_{U/\mathcal{M}} \oplus \pi_2^* \Omega_{U/\mathcal{M}} \longrightarrow \Omega_{R/\mathcal{M}} \longrightarrow \Omega_{R/(U \times U)} \longrightarrow 0$$

Since $R \rightarrow U \times U$ is unramified we see that $\Omega_{R/U \times U} = 0$ and thus any function $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ for a point $\mathfrak{m} \subset A$ then it has a nonzero differential using the sequence (with structure map $\pi_2 : R \rightarrow U$),

$$\pi_2^* \Omega_{U/\mathcal{M}} \longrightarrow \Omega_{R/\mathcal{M}} \longrightarrow \Omega_{R/U} \longrightarrow 0$$

So we see from these sequences that $\pi_1^* \Omega_{U/\mathcal{M}} \twoheadrightarrow \Omega_{R/U}$ therefore for any covector in $(\Omega_{R/U})_p$ arises locally from pullback of some form $\Omega_{U/\mathcal{M}}$ which locally df for $f \in \mathfrak{m}$ on $\text{Spec}(A) \subset U$. \square

14.1 Stack of Algebraic Curves

If $g \geq 2$ then $H^0(C, \mathcal{T}_C) = 0$ and therefore there are no infinitesimal automorphisms. This proves that $\Omega_{\mathcal{M}_g/\mathcal{M}_g} = 0$ so \mathcal{M}_g is DM. Furthermore, because the stabilizers are closed subschemes of PGL_n we see that \mathcal{M}_g has affine diagonal (in particular quasi-compact). However, it is unramified so we see that every genus g curve has finitely many automorphisms.

Consider genus g curves with distinct smooth marked points $p_1, \dots, p_n \in C$ such that,

$$\mathcal{O}_C(p_1 + \dots + p_n)$$

is very ample with $h^1 = 0$. We claim this is an Artin stack by exactly the same argument: embed into \mathbb{P}^n in families.

15 May 13

Remark. Amazing theorem is that we don't need smooth covers, flat is enough.

Theorem 15.0.1 (04S6). Let F be an fppf sheaf and $f : U \rightarrow F$ a representable (by algebraic spaces) morphism which is surjective flat and locally finitely presented. Then F is an algebraic space.

Theorem 15.0.2 (06DC). Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of stacks in the fppf topology. Suppose that \mathcal{X} is an algebraic stack, f is representable by algebraic stacks which is surjective locally of finite presentation and flat. Then \mathcal{Y} is an algebraic space.

Return to $\mathcal{M}_{g,n}^{vp}$ is the moduli stack of families $f : \mathcal{C} \rightarrow B$ with f flat finitely presented with n sections whose geometric fibers are 1-dimensional schemes where the sections are distinct smooth points $p_1, \dots, p_n \in \mathcal{C}_{\bar{s}}$ and $\mathcal{C}_{\bar{s}}$ has arithmetic genus $g = 1 - \chi(C, \mathcal{O}_C)$ and,

$$\mathcal{O}(p_1 + \dots + p_n)$$

is very ample with vanishing h^1 .

Remark. The genus can be weird for example $g(\mathbb{P}^1 \sqcup \mathbb{P}^1) = -1$. But this really is the right notion because χ is locally constant in flat families.

Proposition 15.0.3. Why is $B \hookrightarrow \mathcal{C}$ via the sections σ_i effective Cartier divisors. We need to show $\mathcal{I}_{B/\mathcal{C}}$ is invertible. There is a sequence,

$$0 \longrightarrow \mathcal{I}_{B/\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{O}_B \longrightarrow 0$$

We construct this as follows. Let $U \subset \text{Hilb}(\mathbb{P}^N)$ be the locus where the intersection with a fixed hyperplane H is degree n . Then $V = Z \cap H \rightarrow U$ is flat of degree n where Z is the universal family over U . Then we take our parameter space $V \times_U \cdots \times_U V \setminus \Delta$ where Δ is the locus where the points are equal. This gives canonical sections of V pulled back to here thus labeling the points. Then we mod out by the PGL_n action fixing H to get our stack.

Then $U \rightarrow \mathcal{M}_{g,n}$ where $U \subset \mathcal{M}_{g,N}$ is the smooth geometric fibers locus where $N = n + d$ where d is large enough so that $N > 2g + 2$.

Remark. Consider M_g the stack of nodal genus g curves with connected fibers.