Work over $\mathbb{C}$ and let $g \geq 2$. Let $\mathcal{H}_{g,n} \subset \mathcal{M}_{g,n}$ be the space of hyperelliptic curves with n marked points. The pointed are labeled and may or may not lie in hyperelliptic pairs or at the intersections.

$$\chi_{0,c}^{S_n}(\mathcal{H}_{g,n}) = \sum_{i \geq 0} (-1)^i W_0 H_c^i(\mathcal{H}_{g,n}, \mathbb{Q}) \in K_0(\mathbb{Q}[S_n])$$

for the mixed Hodge structure weight filtration. We view this in the Grothendieck group of S_n-representations as a virtual representation or character.

The main theorem (BCK 2023+) gives a sum-of-graphs formula,

$$z_g = \sum_{n \geq 0} \mathrm{char}_n(\chi_{0,c}^{S_n}(\mathcal{H}_{g,n})) \in \hat{\Lambda}_{\mathbb{Q}} = \varprojlim \mathbb{Q}[\![x_1, \dots, x_n]\!]^{S_n}$$

in the degree-completed ring of symmetric functions where,

$$\mathrm{char}_n\left(\sum_{\lambda n} a_\lambda S^\lambda\right) = \sum_{\lambda n} a_\lambda s_\lambda \in \Lambda_n$$

where s_λ is the Shur function,

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x_1^{c_1(T)} x_2^{c_2(T)} \cdots$$

these are semistandard fillings of the Young tableux. Or equivalently,

$$\mathrm{char}_n(W) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{tr}\, \sigma | W \cdot \psi(\sigma)$$

where,

$$\psi(\sigma) = p_{a_1} \cdots p_{a_k}$$

where σ has cycle type $a_1 \geq a_2 \geq \cdots \geq a_k$ and,

$$p_i = x_1^i + x_2^i + \cdots$$

Sample theorem $g = 2$ (Faber, '08),

$$z_2 = -\frac{1}{12} P_1 + \frac{1}{2} \frac{P_1}{P_2} - \frac{1}{6} \frac{P_1^2}{P_3} - \frac{1}{12} \frac{P_1}{3} P_2^2 - \frac{1}{6} \frac{P_2 P_3}{P_6}$$

where $P_i = 1 + p_i$.

Related work:

(a) $\chi(\mathcal{H}_{g,n})$ and $\chi(M_{g,n})$ Bini '07 Bini-Harer 11, $\chi^{\mathrm{Orb}}(\mathcal{M}_{g,n})$ Harer-Zagier '86.

(b) $\chi^{S_n}(\mathcal{H}_{g,n})$ and $\chi^{S_n}(M_{g,n})$ Gorsky '09, '14.

0.1 Boundary Complexes

Let $U \subset X$ be a nc compactification of smooth varieties. Then there is a combinatorial object $\Delta(U \subset X)$ called the boundary complex. If $D = D_1 \cup \cdots [D_t$ with $D = X \backslash U$ with each D_i smooth (snc case) is a simplicial complex whose p-simplices are irreducible components of $D_{i_0} \cap \cdots \cap D_{i_p}$ for $i_0, \ldots, i_p$.

Example 0.1.1. $U = {\mathbb{G}_m}^2$. Then we can consider $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ are toric compactifications. We could also blow up $\mathbb{P}^2$ at not a torus fixed point to get a non-toric example. These give a triangle, a square, and a triangle with a tail respectively.

It turns out that the homotopy type of this complex is an invariant of U. From Deligne Hodge II,III we get,

Theorem 0.1.2. $\widetilde{H}^{k-1}(\Delta(U \subset X), \mathbb{Q}) \cong \mathbf{gr}_{(W)_0 H_c^k(U,\mathbb{Q})}$.