1 Phonons

We have a prescription,

$$\mathcal{H}(\mu) \to Z(P, V, T)$$

Given some symmetry we can produce an effective Hamiltonian which gives the same partition function. We give the example of phonons in a 1D lattice with spacing a. Let $q_n^{(0)} = na$ and $q_n = q_n^{(0)} + u_n$ small dispacement. Then,

$$U = U_0 + \sum_j K_j \sum_n (u_n - u_{n+k})^2$$

And,

$$T = \sum_n \tfrac{1}{2} m \dot{u}_n^2$$

We replace u_n with its fourier transform $u(k)$ to give,

$$\mathcal{H} = U_0 + \tfrac{1}{2} \sum_k \left[\frac{p(k)^2}{2m} + \kappa(k) u(k) u^*(k) \right]$$

where

$$\kappa(k) = \sum_j K_j (1 - \cos k\alpha_j)$$

Then,

$$\omega(k) = \sqrt{\frac{\kappa(k)}{m}}$$

Thus in the limit $k \to 0$ we find $\kappa \propto k^2$ and $\omega(k) \propto k$ so we get a linear dispersion law with leads to $C \propto T$.

The question is if we move away from an ideal monotomic 1D lattice, do these properties change?

1.1 Phenomenological Approach

Concisder $a \ll d \ll \lambda_T$ where,

$$\lambda_T = \frac{2k}{k} = \frac{2\pi V}{T}$$

is the reduced thermal wavelength. Then w let $u(x)$ be the averaged u_n given by,

$$u(x) = \sum_n \phi_d(x - an) u_n$$

where ϕ_d is some weighting function of width d. (In class he writes ϕ_d is a step function $\frac{1}{d} \cdot 1_{[-d/2, d/2]}$ but I think it must be continuous for $u(x)$ to be continuous). Now define the density $\rho = \frac{m}{a}$ and, ignoring the internal clump energy (which is not important in the long wavelength approximation),

$$T = \rho \int \tfrac{1}{2} \dot{u}(x)^2 \mathrm{d}x$$

Now we need a form of the potential energy which we will express as,

$$U[u(x)] = \int \mathrm{d}x \, \Phi \left(u, \frac{\mathrm{d}x}{\mathrm{d}x}, \frac{\mathrm{d}^2 u}{\mathrm{d}x^{\,2}}, \cdots \right)$$

We make the following assumption,

(a). Locality: higher derivatives are vanishing in importance.

(b). translation invarianc:

$$\Phi[u(x)+b] = \Phi[u(x)]$$

(c). stability: no dependence on $\frac{\mathrm{d}u}{\mathrm{d}x}$ so we may write,

$$U[u(x)] = \int \mathrm{d}x \left\{ \frac{1}{2}K\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{1}{2}L\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^{\,2}}\right)^2 + \cdots + M\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^3 + \cdots \right\}$$

Now replacing $u(k)$ with its fourier transform gives,

$$\begin{aligned} U[u(x)] &= \int \frac{\mathrm{d}k}{2\pi}\left[\frac{1}{2}k^2 + \frac{1}{2}Lk^4 + \cdots\right]|u(k)|^2 \\ &\quad - iM\int \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{(2\pi)^2} k_1k_2(k_1+k_2)u(k_1)u(k_2)u(-k_1-k_2) \end{aligned}$$

Which, again, in the low $k \to 0$ is quadratic and thus gives linear dispersion.

Now consider dimension greater than 1. We have $u(\vec{k})$. If we restrict ourselves to Isotropic media then symmetry restricts of the form of the lowest-order terms to be,

$$\vec{u}(\vec{k})^2 \quad \text{or} \quad (\vec{k}\cdot\vec{u}(\vec{k}))^2$$

Then,

$$\mathcal{H}_{\text{eff}} = \int \frac{\mathrm{d}^dk}{(2\pi)^d}\left[\frac{1}{2}\rho\frac{\partial\vec{u}}{\partial t}\cdot\frac{\partial\vec{u}}{\partial t} + \frac{1}{2}\mu k^2\vec{u}^2 + \frac{1}{2}(\mu+\lambda)[\vec{k}\cdot\vec{u}]^2\right]$$

This gives linear dispersion but for two types of waves:

(a). Longitudinal waves have $\vec{k} \parallel \vec{u}$ and speed $c_\ell = \sqrt{\frac{2\mu+\lambda}{\rho}}$.

(b). Transverse waves have $\vec{k} \perp \vec{u}$ and speed $c_t = \sqrt{\frac{\mu}{\rho}}$.

Then the bulk energy due to photons can be expressed as,

$$E(T) = L^d \int \frac{\mathrm{d}^dk}{(2\pi)^d}\left[\frac{\hbar v_\ell k}{1-e^{\beta\hbar v_\ell}} + (d-1)\frac{\hbar v_t k}{1-e^{\beta\hbar v_t\beta}}\right] \approx A(v_\ell, v_t)T^{d+1}$$