1 Basic Definitions

Definition 1.0.1. We say a k-group G is algebraic if it is finite type over k.

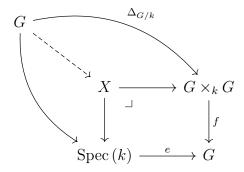
Proposition 1.0.2. A connected algebraic k-group G is geometrically connected and geometrically irreducible.

Proof. See HW 1 for proof.

Remark. Over an imperfect field k a reduced connected algebraic k-group need not be geometrically reduced (and thus not smooth). For example see W1 Exercise 4(iii).

Proposition 1.0.3. Any k-group scheme is automatically separated.

Proof. The map $e: \operatorname{Spec}(k) \to G$ is a closed immersion. Notice that $\Delta_{G/k}: G \to G \times_k G$ is the base change of e via $f = m \circ (\operatorname{id}, \iota): G \times_k G \to G$ because,



commutes giving an arrow $G \to X$. However, $X \to G \times_k G$ gives a pair of maps $f, g : X \to G$ such that $m \circ (f, \iota \circ g) = e$ then by the group axioms,

$$q = m \circ (e, q) = m \circ (m \circ (f, \iota \circ q), q) = m \circ (f, m(\iota \circ q, q)) = m \circ (f, e) = f$$

so f = g and thus $(f, g) = \Delta_{G/k} \circ f$ so the universal map factors uniquely through $\Delta_{G/k}$ and thus $G \to X$ is an isomorphism.

2 Smoothness of Algebraic Groups

Lemma 2.0.1. Let A be a local Noetherian k-algebra with residue field k. Let k'/k be a field extension $\mathfrak{m}' = \ker (A \otimes_k k' \to k')$. Assume that $A' = (A_{k'})_{\mathfrak{m}'}$ is Noetherian. Then A is regular if and only if A' is regular.

Proof. The map $A \to A'$ is a local map of local Noetherian ring. Furthermore, $k \to k'$ is flat so $A \to A_{k'}$ is flat and thus $A \to A_{k'} \to A'$ is flat because localization is flat. Thus, by the dimension formula,

$$\dim A' = \dim A + \dim A'/\mathfrak{m}A'$$

but $\mathfrak{m}A' = \mathfrak{m}'$ so dim $A'/\mathfrak{m}A' = \dim k' = 0$. Thus,

$$\dim A' = \dim A$$

Furthermore, by flatness of k'/k we see that $k' \otimes_k \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathfrak{m}'/\mathfrak{m}'^2$ canonically. Thus,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_{k'} \mathfrak{m}'/\mathfrak{m}'^2$$

Therefore,

$$\dim A' = \dim_{k'} \mathfrak{m}'/\mathfrak{m}'^2 \iff \dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$$

Remark. This fails for $A = \mathbb{Q}[[x]]$ and $k' = \mathbb{C}$ because $\mathfrak{m}' = \ker(\mathbb{Q}[[x]] \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C})$ is exactly the ideal (X). Notice that if $\alpha \in \mathbb{C}$ is transcendental then a power series like,

$$1 - \alpha x$$

is not in \mathfrak{m}' but has no inverse because its inverse,

$$1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots$$

is not in $A_{\mathbb{C}}$ because it has infinitely many \mathbb{Q} -independent coefficients. Thus A' is larger than $A_{\mathbb{C}}$ and is not Noetherian.

Remark. Consider $k = \mathbb{F}_p(t)$ and $k' = \mathbb{F}_p(t^{\frac{1}{p}})$ then let $A = \mathbb{F}_p[t]/(t^p - 1) \cong \mu_p \cong k'$. Then $A \otimes_k k' = k'[t]/(t^{\frac{1}{p}} - 1)^p$ is non-reduced but local and Noetherian. This fails because the residue field of A is k'. It is the rationality of the point \mathfrak{m} that saves the day.

Proposition 2.0.2. Let G be an algebraic k-group. Then the following are equivalent,

- (a) G is smooth
- (b) $G_{\bar{k}}$ is regular
- (c) $\mathcal{O}_{G_{\bar{k}},e_{\bar{k}}}$ is regular
- (d) $\mathcal{O}_{G,e}$ is regular

Proof. The equivalence of (a) and (b) is general. The equivalence of (b) and (c) comes from translation by $G(\bar{k})$ acting transitively on $G_{\bar{k}}$ and thus if one point is regular then every point is regular since translation is an automorphism. Equivalence of (c) and (d) follow from the lemma. \Box

Proposition 2.0.3. An algebraic k-group is smooth if and only if it is geometrically reduced.

Proof. Since $G_{\bar{k}}$ is a reduced finite type k-scheme, by generic smoothness, it has a smooth point. Therefore, every point is smooth by translation. See the proof of Lemma 3.2.3. for more details. \square

Remark. Therefore, we can add the condition that $\mathcal{O}_{G,e}$ is geometrically reduced to the above list.

Corollary 2.0.4. Over a perfect field k, an algebraic k-group is smooth if and only it it is reduced.

Proof. Over a perfect field, any reduced k-algebra is geometrically reduced by Tag 030U.