Physics C2801, Fall 2013 Prof. Brian Cole

Issued: **Sept. 12**	Problem Set # 2	Due: **Sept. 19**

Problem 1. Cross products in Cartesian coordinates

In lecture you were shown how obtain the cross products of the Cartesian unit vectors using cyclic permutations of $\hat{\imath} \times \hat{\jmath} = \hat{k}$. In the cyclic permutations $i \to j$, $j \to k$, $k \to i$. Cyclic permutations can also be used easily to write out the form of a cross product in Cartesian coordinates. Let

$$\begin{aligned} \vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \\ \vec{B} &= B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k} \end{aligned}$$

a. Using the above expressions for $\vec{A}$ and $\vec{B}$, evaluate the cross product $\vec{A} \times \vec{B}$. Use the *convention* $\hat{\imath} \times \hat{\jmath} = \hat{k}$ and its cyclic permutations to evaluate all nine terms in the cross product and then combine terms appropriately. You should obtain the result given on page 9 of your textbook.

b. Now evaluate the cross product starting with just the (two) $\hat{k}$ terms in $\vec{A} \times \vec{B}$ which directly follow from $\hat{\imath} \times \hat{\jmath} = \hat{k}$. From those two terms, you should be able to obtain the other four terms in the expression for the cross product by cyclically permuting the unit vectors as above and permuting $x \to y$, $y \to z$, $z \to x$. If you get into the habit of writing out the expressions for cross products in Cartesian coordinates this way rather than looking up the result in your textbook, by the end of the semester, you will be able to immediately write the full expression for $\vec{A} \times \vec{B}$ any time you need it.

c. In two dimensions, it is easy to show that $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}||\sin\theta_{AB}| = |\vec{A}||\vec{B}|\sin(|\theta_{AB}|)$ by writing the Cartesian components of vectors in terms of the corresponding polar angles (e.g. θ_A and θ_B here). Show that $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin(|\theta_{AB}|)$ with $\theta_{AB} = \theta_B - \theta_A$ taking the two-dimensional reduction of the expressions for $\vec{A}$ and $\vec{B}$ above. Note that I wrote the expression for the cross product with an absolute value on $\sin\theta_{AB}$. You don't normally see the expression written this way because implicitly the angle is taken to be positive. But in this two dimensional analysis we could end up with either sign for θ_{AB} and $\sin\theta_{AB}$. Convince yourself (you don't need to write in your solutions to the pset) that this sign applied to the direction of the cross product would be consistent with the right-hand rule. On the other hand, the magnitude of a vector is always ≥ 0 so if we treat θ_{AB} as a signed angle to keep track of the direction of the cross product, then we need to take the $|\sin\theta_{AB}|$ in writing an expression for $|\vec{A} \times \vec{B}|$.

d. The choice $\hat{\imath} \times \hat{\jmath} = \hat{k}$ corresponds to a "right-handed" coordinate system. This choice is a convention that we follow to ensure consistency when using vectors to analyze physical problems. Suppose we were to chose the opposite conventions, $\hat{\jmath} \times \hat{\imath} = \hat{k}$ which corresponds to a "left-handed" coordinate system. Show that the resulting expression for the cross product $\vec{A} \times \vec{B}$ is simply the negative of your result from part a. Convince yourself that this choice of convention results in a "left-hand" rule for the directions of cross products. **From here on out we will always use a right-handed system, but it's important that you understand this is simply a convention**.

Problem 2. Kleppner and Kolenkow problem 1.19

Problem 3. Kleppner and Kolenkow problem 1.20

Problem 4. Linear motion, analyzed with polar coordinates

A particle moves at constant velocity v in a two-dimensional plane parallel to the y axis (in the $\hat{j}$ direction) with x coordinate, $x = d$. At $t = 0$, the particle crosses the x axis. The goal of this problem will be to evaluate the time dependence of various polar coordinate quantities over the time range $-\infty < t < \infty$. We will define θ the usual way – as the angle of the position vector with respect to the x axis and with θ positive for the particle above the x axis.

a. Find expressions for $r(t)$, $\theta(t)$, and for the position vector $\vec{r}(t)$ in terms of its Cartesian components. Express your answers in terms of v, d, t and other numeric constants.

b. Find expressions for $\hat{r}(t)$ and $\hat{\theta}(t)$ in terms of their Cartesian components. Evaluate the limiting cases $t = 0, -\infty, \infty$ and show that the directions make physical sense.

c. Evaluate $\dot{r}(t)$ and $\dot{\theta}(t)$. To evaluate $\dot{\theta}$ use implicit differentiation – namely start from an expression for $\cos\theta$, $\sin\theta$, or $\tan\theta$ in terms of d, v, t and differentiate both sides. You should be able to express the result for $\dot{\theta}$ in a relatively simple form. Evaluate the same limiting cases as in part b and explain why they make physical sense.

d. Evaluate $\dot{\hat{r}}$ and $\dot{\hat{\theta}}$ and analyze the same limiting cases as above.

e. Evaluate $\dot{\vec{r}}$ from your expression in part a, and show that it can be put in the form, $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$.

f. If the particle is moving in a straight line at constant (vector) velocity, it should have zero acceleration. Evaluate all four of the terms in the expression of $\vec{a}$ in polar coordinates and show that while each of them is non-zero, the total acceleration *is* zero.

Problem 5. Kleppner and Kolenkow problem 2.9

Problem 6. Kleppner and Kolenkow problem 2.29

Problem 7. Kleppner and Kolenkow problem 2.33

Problem 8. Kleppner and Kolenkow problem 2.34

Problem 9. Rotations in two-dimensions

In lecture 3 I stated that vectors have the property that their length is invariant under rotations of the coordinate system. In this problem, we will first develop the form for the transformation of coordinates in two dimensions under a rotation. We will then write the general form for the transformation of any vector under rotations, and show explicitly that the transformations preserve the length of any vector and, more generally, leave dot products invariant.

Consider a usual two-dimensional coordinate system with the x axis horizontal. The position vector for a particle in this coordinate system is (as you well know) $\vec{r} = x\hat{i} + y\hat{j}$. Now consider how to express that same position vector in another coordinate system with x' and y' axes rotated by an angle θ_R with respect to the x and y axes of the first coordinate system. Take positive θ_R to be a counter-clockwise rotation. We will write in primed coordinates, $\vec{r} = x'\hat{i}' + y'\hat{j}'$. *Note:* observe that I didn't write $\vec{r}' = x'\hat{i}' + y'\hat{j}'$ here. $\vec{r}$ is the position vector for the particle and is unambiguous – it represents the physical position of the particle. It is the *representation* of $\vec{r}$ that depends on the choice of coordinate system so we can write

$$\vec{r} = x\hat{i} + y\hat{j} = x'\hat{i}' + y'\hat{j}' \tag{1}$$

a. Find expressions for $\hat{i}'$ and $\hat{j}'$ in terms of $\hat{i}$ and $\hat{j}$.

b. Using the results of part a and Eq. 1, find expressions for x' and y' in terms of x and y and the rotation angle θ_R. These two equations express how the coordinates transform under a rotation.

c. Explain how/why the same transformation applies to all vectors in two dimensions not just the position vector.

d. Show that the transformation in part b leaves the length of the position vector invariant.

e. Show that the dot product of any two vectors – position or other – in two dimensions is invariant under rotations.