# 1 Feb 11

## 1.1 Line Bundles

There exists a map,

$$\Gamma(X, \mathcal{L}^{\otimes a}) \otimes \Gamma(X, \mathcal{L}^{\otimes b}) \to \Gamma(X, \mathcal{L}^{\otimes ab})$$

since we have an isomorphism $\mathcal{L}^{\otimes a} \otimes \mathcal{L}^{\otimes b} = \mathcal{L}^{\otimes ab}$. Furthermore, since $\mathcal{L}$ is rank 1 this map is commutative since $s \times s' = s' \otimes s$ since they only differ by a section of $\mathcal{O}_X$. This alows us to define the following graded ring structure.

**Definition** Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, $\mathscr{F}$ any $\mathcal{O}_X$-module and $s \in \mathcal{L}(X)$ a global section. Then we define the following graded ring.

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

and then the following module,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which is a graded $\Gamma_*(X.\mathcal{L})$-module. Furthermore, there is a map,

$$\Gamma_*(X, \mathcal{L}, \mathscr{F})_{(s)} \to \mathscr{F}(X_s) = \Gamma(X_s, \mathscr{F})$$

sending $\frac{t}{s^n} \mapsto t|_{X_s} \otimes (s|_{X_s})^{\otimes -n}$.

**Proposition 1.1.** Let $X$ be a quasi-compact, quasi-seperated scheme and $\mathscr{F}$ be quasi-coherent. Then the above map is an isomorphism.

*Proof.* Tag OB5K. (Compare with that Hartshorne Excercise). $\qquad\square$

**Example 1.2.** Let $A$ be a graded ring such that $A$ is generated by $A_1$ as a $A_0$-algebra (e.g. $A = k[X_0, \ldots, X_n]$). Let $X = \operatorname{Proj}(A)$ and consider the graded module $M = A(n)$ which is the graded module $M_k = A_{k+n}$. Then we can construct the Serre twists,

$$\mathcal{O}_X(n) = \widetilde{M} = \widetilde{A(n)}$$

which is an invertable $\mathcal{O}_X$-module. Furthermore,

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$$

*Remark.* This will not be invertible and these maps will not be isomorphisms in general when $A$ does not satisfy the required conditions.

*Proof.* We can decompose,

$$X = \bigcup_{f \in A_1} D_+(f) = \bigcup_{f \in A_1} \mathrm{Spec}\left(A_{(f)}\right)$$

via the given assumptions. We know that,

$$\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{A(n)}|_{D_+(f)} = \widetilde{A[f^{-1}]}_n$$

However it is clear that $A[f^{-1}]_n = A[f^{-1}]_0 \cdot f^n$ so this sheaf is free of rank 1. $\quad\square$

*Remark.* For $n = 1$ any element $f \in A_1$ gives a global section $f \in \Gamma(X, \mathcal{O}_X(1))$ such that $D_+(f) = X_s$ and hence,

$$\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(1)|_{X_s}$$

**Corollary 1.3.** In the setting above, further assume that $A$ is generated by finitely many $f \in A_1$ as an $A_0$-algebra. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ if we set,

$$M = \Gamma_*(X, \mathcal{O}_X(1), \mathscr{F})$$

as a graded $A$-module via the map,

$$A \to \Gamma_*(X, \mathcal{O}_X(1)) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

Then we get, $\mathscr{F} = \widetilde{M}$.

*Proof.* Tag $\quad\square$

# 2 Feb. 13

**Definition** Let $X$ be a scheme and $\mathcal{L}$ an invertivle $\mathcal{O}_X$-module . We say $\mathcal{L}$ is *ample* if $X$ is quasi-compact and $\forall x \in X \exists n > 0, s \; in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s$ is affine and $x \in X_s$.

**Example 2.1.** Let $X = \mathrm{Proj}\,(A)$ where $A$ is generated by $A_1$ as a $A_0$-algebra and $A_1 = f_1 A_0 + \cdots + f_r A_0$. Then $\mathcal{O}_X(1)$ is invertible and $X$ is covered by $D_+(f_i)$ and is quasi-compact, and $D_+(f_i) = X_{s_i}$ where $s_i \in \Gamma(X, \mathcal{O}_X(1))$ is a section corresponding to $f_i$.

**Proposition 2.2.** Let $X$ be quasi-compact and quasi-seperated for $\mathcal{L} \in \mathrm{Pic}(X)$ the following are equivalent,

(a). $\mathcal{L}$ is ample

(b). for all $\mathcal{O}_X$-modules $\mathscr{F}$ locally of finite type there exists $n > 0$ s.t. $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections.

*Proof.* TAG 01Q3. □

**Lemma 2.3.** $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is ample for any $n > 0$.

**Lemma 2.4.** If $X$ is affine, and $L$ is invertible, and $s \in \Gamma(X, \mathcal{L})$ then $X_s$ is affine.

**Definition** A scheme is noetherian if it has a finite open cover by spectra of noetherian rings.

*Remark.* It is equivalent to require that $X$ is quasi-compact and $\mathcal{O}_X(U)$ is noetherian.

**Lemma 2.5.** A locally noetherian scheme is quasi-seperated.

*Proof.* If $U, V$ are affines then $U \cap V$ is quasi-compact since every subspace of a noetherian space is quasi-compact. □

**Definition** Let $X$ be a neotherian scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is *coherent* if it is quasi-coherent and locally of finite type.

*Remark.* It is equivalent to require that locally on affine opens $\mathscr{F}|_U = \widetilde{M}$ for a finitely-generated module $M$.

*Remark.* The inclusion functors,

$$\mathfrak{Coh}\,(\mathcal{O}_X) \subset \mathfrak{QCoh}\,(\mathcal{O}_X) \subset \mathfrak{Mod}\,(\mathcal{O}_X)$$

are exact and preserved under extensions i.e. given a short exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

if $\mathscr{F}_1, \mathscr{F}_2$ are (quasi)-coherent then $\mathscr{F}_2$ is also (quasi)-coherent.

**Lemma 2.6.** A scheme of finite type over a noetherian scheme is noetherian.

*Proof.* Since $f : X \to Y$ is finite type $f$ is quasi-compact but $Y$ is quasi-compact open so its preimage $X$ is also quasi-compact. Furthermore, for any affine opens $\mathrm{Spec}\,(A) = U \subset X$ and $\mathrm{Spec}\,(B) = V \subset Y$ such that $f(U) \subset V$ we get a ring map $B \to A$ of finite type so $B[x_1, \ldots, x_n] \twoheadrightarrow A$ and since $B$ is noetherian we see that $A$ is noetherian so $X$ is quasi-compact and covered by $\mathrm{Spec}\,(A)$ for noetherian rings $A$. □

*Remark.* We want to prove the following theorem. Let $R$ be a noetherian ring, $X$ a projective (or proper) scheme over $\mathbb{R}$ (then $X$ is noetherian), and $\mathscr{F}$ a coherent sheaf on $X$, then,

$$H^i(X, \mathscr{F})$$

is a finite $R$-module for any $i$ and $H^i(X, \mathscr{F}) = 0$ for $i > \dim X$.