1 A Stalemate among Abelian Varieties over $\mathbb{F}_q$

1.1 A new construction for algebraic integers

Definition 1.1.1. A complex number $\alpha \in \mathbb{C}$ is an algebraic integer if there exists some monic integer polynomial $p \in \mathbb{Z}[x]$ such that $p(\alpha) = 0$.

Definition 1.1.2. The minimal polynomial of α is the unique polynomial of the above form with minimal degree. The roots of the minimal polynomial are called the conjugates of α.

Definition 1.1.3. Take an algebraic integer α with conjugates $\alpha_1, \ldots, \alpha_n$. Then $\deg \alpha = n$ and $\operatorname{tr}(\alpha) = \alpha_1 + \cdots + \alpha_n$. Call α totally positive if $\alpha_1, \ldots, \alpha_n > 0$.

Example 1.1.4. Consider ζ_7 a primitive root of $x^7 - 1$. Then $\deg \zeta_7 = 6$ and $\operatorname{tr}(\zeta_7) = -1$.

Example 1.1.5. Consider $\beta = \zeta_7 + \bar{\zeta_7}$ then $\operatorname{tr}\left(\zeta_7 + \bar{\zeta_7}\right) = -1$. Then we can take $\gamma = \zeta_7 + \bar{\zeta_7} + 2$ then γ is totally positive with $\deg \gamma = 3$ and $\operatorname{tr}(\gamma) = 5$.

Example 1.1.6. If you take $\beta_n = \zeta_n + \bar{\zeta_n} + 2$ these are totally positive. The distribution of these conjugates follows $(1 - (x/2 - 1)^2)^{-\frac{1}{2}}$ in the interval $[0, 4]$.

Remark. Shur - Siegal - Smith conjectured for any $\epsilon > 0$ there are only finitely many TPAIs α such that,

$$\operatorname{tr}(\alpha) < (2 - \epsilon) \deg \alpha$$

Theorem 1.1.7 (Smith)**.** There are infinitely many TPIA α so,

$$\operatorname{tr}(\alpha) < 1.809 \deg \alpha$$

Definition 1.1.8. Let Σ be a real interval of length greater than 4. Call an algebraic integer α a Σ-algebraic integer if all its conjugates lie in Σ. Given such an α we define a measure on Borel sets,

$$\mu_\alpha(Y) = \frac{\#\{\beta \in Y \mid m_\alpha(\beta) = 0\}}{\deg \alpha}$$

Definition 1.1.9. We say that a sequence of measures μ_i weak-$*$ converges $\mu_i \to \mu$,

$$\lim_{i \to \infty} \int f \mathrm{d}\mu_i = \int f \mathrm{d}\mu$$

for all continuous $f : \Sigma \to \mathbb{R}$.

Remark. This is weak-$*$ convergence in the dual space of $C(\Sigma)$. (CHECK THIS)

Definition 1.1.10. Given μ on Σ consider the conditions,

(A) there is a sequence α_i of distinct Σ-AIs such that $\mu_{\alpha_i} \to \mu$

(B) for every nonzero integer polynomial Q,

$$\int_\Sigma \log |Q| \mathrm{d}\mu \geq 0$$

Proposition 1.1.11. Condition (A) implies (B).

Proof. Given Q and α with conjugates $\alpha_1, \dots, \alpha_n$. Consider,

$$\beta = \prod_{i \leq n} Q(\alpha_i)$$

which is invariant under every Galois automorphism and thus $\beta \in \mathbb{Z}$. Furthermore,

$$\left| \prod_{i \leq n} Q(\alpha_i) \right| \geq 1$$

unless $Q(\alpha) = 0$. Therefore,

$$\int \log |Q| \mathrm{d}\mu_\alpha \geq 0$$

Then by dominated convergence,

$$\int \log |Q| \mathrm{d}\mu \geq 0$$

$\square$

Example 1.1.12. Let Q be the minimal polynomial for some other algebraic integer β. Then,

$$\int \log |z - w| \mathrm{d}\mu_\alpha \mathrm{d}\mu_\beta \geq 0$$

showing that the conjugates of α and β are spaced apart.

Theorem 1.1.13 (Smith (2021))**.** Condition (B) implies condition (A).

Remark. Previously this required an energy condition,

$$I(\mu) = \int \int \log |z - w| \mathrm{d}\mu(z) \mathrm{d}\mu(w) \geq 0$$

1.2 Abelian Varieties

We require the Weil conjectures but actually only the parts that were known to Weil.

Theorem 1.2.1 (Weil)**.** Given a "nice" curve $C/\mathbb{F}_q$ of genus g, there are AIs,

$$\lambda_1, \dots, \lambda_g$$

suh that $|\lambda| = q^{\frac{1}{2}}$ for $i \leq g$ such that,

$$\#C(\mathbb{F}_{q^n}) = q^n + 1 - (\lambda_1^n + \bar{\lambda}_1^n + \cdots + \lambda_g^n + \bar{\lambda}_g^n)$$

Theorem 1.2.2 (Weil)**.** Given $A/\mathbb{F}_q$ of dimension g, there exist,

$$\lambda_1, \dots, \lambda_g$$

with $|\lambda| = q^{\frac{1}{2}}$ and,

$$\#A(\mathbb{F}_{q^n}) = \prod_{i \leq g} (\lambda_i^n - 1)(\bar{\lambda}_i^n - 1)$$

If A is the Jacobian of C, these λ_i agree with the previous ones for C.

1.3 Honda - Tate Theory

Consider the map $A \mapsto \lambda_1 + \bar{\lambda}_1$. This gives,

$$\{\text{simple } A/\mathbb{F}_q \text{ up to isogeny}\} \to \{\text{AIs } \alpha \text{ with conjugates in } [-2\sqrt{q}, 2\sqrt{q}] \text{ up to conjugacy}\}$$

(Note: you can write any such $\lambda \in [-2\sqrt{q}, 2\sqrt{q}]$ as a sum of a Weil q number and its conjugate under $z \mapsto z + \frac{q}{z}$ Tate showed that this map is injective. Honda showed that this map is bijective.

Notice that,

$$\#A(\mathbb{F}_{q^n}) = \prod_{i \leq g} (\lambda_i^n - 1)(\bar{\lambda}_i^n - 1) = \prod_{i \leq n} (q + 1 - \lambda_i - \bar{\lambda}_i)$$

Therefore,