1 Fourier-Mukai Transforms

Theorem 1.0.1 (Mukai). Let A/k be an abelian variety then there is an equivalence of categories,

$$D^b(A) \xrightarrow{\sim} D^b(A^{\vee})$$

Remark. $D^b(A)$ is the category of complexes of quasi-coherent sheaves whose cohomology sheaves are coherent and only finitely many nonzero.

1.1 Derived Categories

Let X be smooth projective over a field k. We make these assumptions because,

- (a) smooth: such that all rings are regular hence finite modules have finite projective dimension
- (b) projective: every coherent sheaf is a quotient by a vector bundle

therefore every element in $D^b(X)$ is represented by a finite complex of vector bundles.

Fact: there exists $\mathbb{R} \underline{\text{Hom}}$ and $\otimes^{\mathbb{L}}$. For a map $f: X \to Y$ (which is automatically proper because X, Y are projective varities) then for any $E \in D^b(X)$ we get $\mathbb{R} f_* E \in D^b(Y)$ and for $E \in D^b(Y)$ then $\mathbb{L} f^* E \in D^b(X)$.

Furthermore, there is a projection formula,

$$\mathbb{R}f_*(E \otimes_X^{\mathbb{L}} \mathbb{L}f^*F) \cong \mathbb{R}f_*E \otimes_Y^{\mathbb{L}} F$$

Proposition 1.1.1. If we have a Cartesian diagram,

$$X' \xrightarrow{g'} X$$

$$\downarrow^{f'} \quad \downarrow^{f}$$

$$Y' \xrightarrow{g} Y$$

and either f or g is flat then for $E \in D^b(X)$ there is a base change isomorphism,

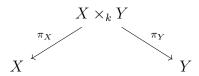
$$\mathbb{L}g^*(\mathbb{R}f_*E) \cong \mathbb{R}f'_*(\mathbb{L}g'^*E)$$

Proposition 1.1.2. If $f: X \to Y$ is smooth,

$$(\mathbb{R}f_*E)^{\vee} \cong \mathbb{R}f_*(E^{\vee} \otimes \omega_{X/Y}[\dim X - \dim Y])$$

1.2 Integral functors

Given the following situation,



And fix $\mathcal{P} \in D^b(X \times Y)$.

Definition 1.2.1. The integral functor with kernel \mathcal{P} is,

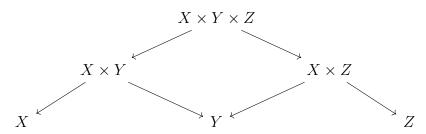
$$\Phi_{Y \to X}^{\mathcal{P}} : D^b(Y) \to D^b(X)$$

given by

$$E \mapsto \mathbb{R}\pi_{X*}(\mathcal{P} \otimes_{X \times Y}^{\mathbb{L}} \mathbb{L}\pi_Y^* E)$$

Proposition 1.2.2. The composition of integral functors is an integral functor.

Proof. To compute $\Phi_{Y\to X}^{\mathcal{P}} \circ \Phi_{Z\to Y}^{\mathcal{Q}}$ draw the diagram,



then we compute that by base change,

$$\mathbb{L}(\pi_V^{XY})^* \circ \mathbb{R}(\pi_V^{YZ})_* \cong \mathbb{R}(\pi_{XY}^{XYZ})_* \circ \mathbb{L}(\pi_{YZ}^{XYZ})^*$$

Therefore,

$$\Phi_{Y \to X}^{\mathcal{P}} \circ \Phi_{Z \to Y}^{\mathcal{Q}} = \mathbb{R}(\pi_X^{XYZ})_* [\mathcal{R} \otimes_{XYZ}^{\mathbb{L}} \mathbb{L}(\pi_Z^{XYZ})^*]$$

where

$$\mathcal{R} = (\pi_{YZ}^{XYZ})^* \mathcal{Q} \otimes_{XYZ}^{\mathbb{L}} (\pi_{XY}^{XYZ})^* \mathcal{P}$$

Proposition 1.2.3. All integral functors have both left and right adjoints which are also integral functors.

Proof.

$$\mathbb{R}\mathrm{Hom}_{X}\left(E,\mathbb{R}\pi_{X*}[\mathcal{P}\otimes^{\mathbb{L}}\mathbb{L}\pi_{Y}^{*}F\right) = \mathbb{R}\mathrm{Hom}_{X\times Y}\left(\pi_{X}^{*}E,\mathcal{P}\otimes^{\mathbb{L}}\mathbb{L}\pi_{Y}^{*}F\right)$$

$$= \mathbb{R}\mathrm{Hom}_{X\times Y}\left(\mathcal{P}^{\vee}\otimes^{\mathbb{L}}\mathbb{L}\pi_{X}^{*}E,\mathbb{L}\pi_{Y}^{*}F\right)$$

$$= \mathbb{R}\mathrm{Hom}_{Y}\left(\mathbb{R}\pi_{Y*}(\mathcal{P}^{\vee}\otimes^{\mathbb{L}}\mathbb{L}\pi_{X}^{*}E\otimes^{\mathbb{L}}\mathbb{L}\pi_{X}^{*}\omega_{X}[\dim X],F\right)$$

therefore the left adjoint is given by $\Phi_{X \to Y}^{\mathcal{P}^{\vee} \otimes^{\mathbb{L}} \mathbb{L} \pi_X^* \omega_X[\dim X]}$. Similarly the right adjoint is given by $\Phi_{X \to Y}^{\mathcal{P}^{\vee} \otimes^{\mathbb{L}} \mathbb{L} \pi_Y^* \omega_Y[\dim Y]}$.

1.3 Full faithfulness

General fact: if a functor F has a left adjoint G then $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(F(X),F(Y)) = \operatorname{Hom}(GF(X),Y)$ so $GF = \operatorname{id}$ is enough to show that F is fully faithful.

Theorem 1.3.1. The functor $F := \Phi_{Y \to X}^{\mathcal{P}}$ is fully faithful if and only if

$$\mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_{Y}\left(\mathcal{O}_{y_{1}},\mathcal{O}_{y_{1}}\right)\to\mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_{Y}\left(F\mathcal{O}_{y_{1}},F\mathcal{O}_{y_{2}}\right)$$

is an isomorphism in $D^b(k)$ for all closed points $y_1, y_2 \in Y$. Equivalently, if $y_1 \neq y_2$ then $\mathbb{R}\mathrm{Hom}_Y(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0$ and if $y_1 = y_2$ then,

$$\mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_{Y}\left(\mathcal{O}_{y},\mathcal{O}_{y}\right)\stackrel{\sim}{\longrightarrow}\mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_{Y}\left(F\mathcal{O}_{y},F\mathcal{O}_{Y}\right)$$

Proof. It is clear this is necessary by considering shifts. Therefore, we need to prove sufficience. Let G be the left adjoint and write $GF = \Phi_{Y \to Y}^{\mathcal{Q}}$. The counit map $GF \to \operatorname{id}$ is realized by the map of sheaves $\mathcal{Q} \to \Delta_Y$ inside $D^b(Y \times Y)$. Let K be the fiber of this map. By assumption,

$$\mathbb{R}\mathrm{Hom}_{Y}\left(\mathcal{O}_{y_{1}},\mathcal{O}_{y_{1}}\right)\stackrel{\sim}{\longrightarrow}\mathbb{R}\mathrm{Hom}_{Y}\left(GF\mathcal{O}_{y_{1}},\mathcal{O}_{y_{2}}\right)$$

this is given by precomposition with $Q \to \Delta_Y$ meaning there is an exact triangle

$$K_{y_1} \to \mathcal{Q}_{y_1} \to \mathcal{O}_{y_1} \to K_{y_1}[+1]$$

But the assumption exactly implies that $\mathbb{R}\mathrm{Hom}_Y(K_{y_1},\mathcal{O}_{y_2})=0$. Then $K_{y_1}=0$ because the resolution of K_{y_1} must be exact at each y_2 . Therefore K=0 because it is zero on all fibers. \square

1.4 Fourier-Mukai

Let A/k be an abelian variety and A^{\vee}/k is dual Pic_A^0 . We let $\mathcal{P} \in D^b(A \times A^{\vee})$ the Poincare bundle. The claim is that:

$$\Phi_{A^{\vee} \to A}^{\mathcal{P}} : D^b(A) \to D^b(A^{\vee})$$

is an equivalence.

First we show it is fully faithful. To do this set $F = \Phi_{A^{\vee} \to A}^{\mathcal{P}}$ then

- (a) for $y_1 \neq y_2$ we need \mathbb{R} Hom $(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0$ but $F\mathcal{O}_{y_i} = \mathcal{L}_{y_i}$ is the line bundle corresponding to $y_i \in A^{\vee}$ so we need to show that \mathbb{R} Hom_A $(\mathcal{L}_{y_1}, \mathcal{L}_{y_2}) = 0$
- (b) for y = y we need to show that $\mathbb{R}\mathrm{Hom}_{A^{\vee}}(\mathcal{O}_y, \mathcal{O}_y) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_A(\mathcal{L}_y, \mathcal{L}_y)$.

For part 1 we just need to show that $H^i(\mathcal{L}_y) = 0$ for all i and $y \neq 0$ by setting $y = y_2 - y_1$.

Lemma 1.4.1 (Mumford). For any $\mathcal{L} \in A^{\vee}$ nonzero $H^{i}(\mathcal{L}) = 0$ for all i.

Proof. Consider $m: A \times A \to A$ addition. Then these satisfy the theorem of the square,

$$m^*\mathcal{L}_y \cong \pi_1^*\mathcal{L}_y \otimes \pi_2^*\mathcal{L}_y$$

Then by Kunneth:

$$H^{k}(A \times A, m^{*}\mathcal{L}_{y}) = \bigoplus_{i+j=k} H^{i}(A, \mathcal{L}_{y}) \otimes H^{j}(A, \mathcal{L}_{y})$$

Note that if $A \hookrightarrow A \times A \to A$ is the identity where $A \hookrightarrow A \times A$ is id \times 0. Therefore, the pullback map m^* is injective so,

$$\dim H^k(A, \mathcal{L}_y) \le H^k(A \times A, m^*\mathcal{L}_y) = \sum_{i+j=k} (\dim H^i(A, \mathcal{L}_y)) (\dim H^j(A, \mathcal{L}_y))$$

But we know that $H^0(A, \mathcal{L}_y) = 0$ since $y \neq 0$ so we get that all $H^i(A, \mathcal{L}_y) = 0$ by an inductive argument.

For part 2 we can use the Kozul resolution to show that,

$$\dim \operatorname{Ext}_{A^{\vee}}^{i}\left(\mathcal{O}_{y}, \mathcal{O}_{y}\right) = \begin{pmatrix} g \\ i \end{pmatrix}$$

and by Hodge theory we also know that,

$$\dim H^i(A, \mathcal{O}_A) = \begin{pmatrix} g \\ i \end{pmatrix}$$

Both sides have the same dimension.

Now we show it is essentially surjective. However, we know its left adjoint is the integral functor for $\mathcal{P}^{\vee} \otimes \pi_X^* \omega_X[\dim X] = \mathcal{P}^{\vee}[\dim X]$. Therefore, the same computation works for \mathcal{P}^{\vee} so we see that the left adjoint is also fully faithful.