Math GR6262 Algebraic Geometry
Assignment # 4

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February 24, 2020

1 Problem a

Let K/k be a finitely generated (as a k-algebra) extension of fields. Consider the variety over k given by $X = \mathrm{Spec}\,(K)$. The inclusion map $k \to K$ induces a morphism of schemes $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(k)$ making X a scheme over k. We may compute the function field via $k(X) = \mathcal{O}_{X,x}/\mathfrak{m}_x = \mathcal{O}_{X,x} = K$. Furthermore, $\mathrm{Spec}\,(K)$ is a one-point space and thus, I claim, a variety over k. Since $\mathrm{Spec}\,(K)$ is a sheaf of fields (and the zero ring over the empty set) over one point it is clearly reduced and irreducible. Therefore, it suffices to show that $\mathrm{Spec}\,(K)$ is separated of finite type over k. Since we assumed that K is a finitely generated k-algebra, the map $k \to K$ is finite type and thus $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(k)$ is a finite type morphism by definition (the morphism is clearly quasi-compact since all subsets are quasi-compact). Finally, the diagonal morphism $X \xrightarrow{\Delta} X \times X$ corresponds to the morphism of k-algebras $K \to K \otimes_k K$ given by sending $x \mapsto x \otimes_k x$. We need to show that Δ is a closed immersion. Since every nonempty open subset of X is affine (there is only one) Δ is clearly affine and the corresponding map $K \otimes_k K \to K$ on the whole affine open is surjective. Furthermore, since $K \otimes_k K \to K$ is surjective, the preimage of the maximal ideal (0) (i.e. the kernel of the multiplication map) is maximal and thus closed in $\mathrm{Spec}\,(K \otimes_k K) = X \times X$. Therefore, the image of X is closed we have shown that the diagonal morphism $X \xrightarrow{\Delta} X \times X$ is a closed immersion proving that $\mathrm{Spec}\,(K)$ is indeed separated and thus a variety over k.

Now let K/k be a finitely generated (as fields) extension of fields. Then there exist $t_1, \dots, t_n \in K$ such that $k(t_1, \dots, t_n) = K$ take the domain $A = k[t_1, \dots, t_n] \subset K$ and consider the affine scheme $X = \mathrm{Spec}\,(A)$ over k. I claim that X is a variety. Since X is affine it is separated and since its coordinate ring A is a domain X is integral. Finally, the k-algebra morphism $k \to k[t_1, \dots, t_n]$ is clearly of finite type so the corresponding morphism of schemes $X \to \mathrm{Spec}\,(k)$ makes X a scheme of finite type over k. Thus X is a variety. Furthermore, the function field of X is,

$$k(X) = k(\mathrm{Spec}\,(A)) = \mathrm{Frac}\,(A) = k(t_1, \dots, t_n) = K$$

2 Problem b

Suppose that $A \subset B$ is an extension of domains such that B is a finitely generated A-algebra and such that the inclusion map $\iota : A \to B$ induce and isomorphism $\iota : \mathrm{Frac}\,(A) \xrightarrow{\sim} \mathrm{Frac}\,(B)$. Since B is finitely generated, there exists a surjective map $A[x_1, \dots, x_n] \to B$. Let e_i be the image of x_i under this surjective map. Since the map $\mathrm{Frac}\,(A) \to \mathrm{Frac}\,(B)$ is a surjection and the inclusion $B \to \mathrm{Frac}\,(B)$ is an injection (since B is a domain) we know that any element $b \in B$ can be written as a fraction $b = \frac{a}{d}$ for $a, d \in A$. Furthermore, we know that any element $d \in A$ can be generated by the elements $f_1, \dots, f_n$. So we may take $a_i d_i \in A$ such that $\frac{a_i}{d_i} = e_i$. Now take $f = d_1 \cdots d_n$. I claim that $A_f = B_f$. Clearly, $A_f \subset B_f$ so it suffices to show that any element $\frac{b}{f^k} \in B_f$ is contained in A_f. Since e_i generate B it furthermore suffices to show that $e_i \in A_f$. However, this is clear because $e_i = \frac{a_i}{b_i}$ and,

$$b_i^{-1} = \frac{\prod_{j \neq i} b_i}{f}$$

is an element of A_f. Therefore A_f contains the generators of B_f so $A_f = B_f$. Finally, $f = d_1 \cdots d_n$ is nonzero because A is a domain.

3 Problem c

Let X and Y be varieties over k such that $k(X) \cong k(Y)$ as k-algebras i.e. the varieties X and Y are birational over k. We may restrict our attention to fixed affine opens of X and Y or equivalently to the case that $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ are affine. Since X any Y are varieties over the field k, the rings A and B are finitely generated k-algebra domains. Since we may compute the function field on any affine open, $k(X) \cong \mathrm{Frac}\,(A)$ and $k(Y) \cong \mathrm{Frac}\,(B)$ so we have $\mathrm{Frac}\,(A) \cong \mathrm{Frac}\,(B)$. Now consider,

$$\begin{tikzcd} A \arrow[d, hook] & B \arrow[d, hook] \\ \mathrm{Frac}\,(A) \arrow[r, "\sim"] & \mathrm{Frac}\,(B) \end{tikzcd}$$

so A injects into $\mathrm{Frac}\,(B)$. Let C be the subring of $\mathrm{Frac}\,(B)$ generated by the image of A and B which is also a finitely generated k-algebra by Lemma **??** and a domain since $C \subset \mathrm{Frac}\,(B)$. The inclusion $A \subset C$ makes C a finitely-generated A'-algebra (since both are finitely generated k-algebras) and since $B \subset C \subset \mathrm{Frac}\,(B)$,

$$\mathrm{Frac}\,(A) \cong \mathrm{Frac}\,(B) = \mathrm{Frac}\,(C)$$

Therefore, by the previous problem, there exists $g \in B$ such that $B_g \cong C_g$. Since C is a domain the map $C \to C_g$ is injective. These maps give an inclusion,

$$A \hookrightarrow C \hookrightarrow C_g \xrightarrow{\sim} B_g$$

Furthermore $\mathrm{Frac}\,(A) \cong \mathrm{Frac}\,(B_g)$ and $B_g = B[g^{-1}]$ is a finitely-generated A-algebra domain so there exists $f \in A$ such that $A_f \cong (B_g)_f = B_{f'g}$ where $f \in A$ has image in $B_g \subset \mathrm{Frac}\,(B)$ and thus has denominator a power of g which we multiply out to get f'. This isomorphism gives us an isomorphism of affine opens,

$$\begin{tikzcd} \mathrm{Spec}\,(A) & \mathrm{Spec}\,(B) \\ \mathrm{Spec}\,(A_f) \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{Spec}\,(B_{f'g}) \arrow[u, hook] \end{tikzcd}$$

Proving the proposition. Note that all ring maps produces are actually maps of k-algebras and thus the induced morphisms of schemes are indeed morphisms of schemes over $\mathrm{Spec}\,(k)$ as required for the induced isomorphism of affine opens to be an isomorphism as varieties over k.

4 Problem d

Remark. I read this problem incorrectly and thought it was maps $X \to \mathbb{A}^1_k$ rather than $\mathbb{A}^1_k \to X$ so I kept both solutions but present the relevant one first.

Let k be an algebraically complete field. Take the affine surface,

$$X = \mathrm{Spec}\,(k[x, y, z]/(xyz - 1))$$

and let $A = k[x, y, z]/(xyz - 1)$ such that $X = \mathrm{Spec}\,(A)$. The polynomial $xyz - 1$ is irreducible so $(xyz-1)$ is prime and has height 1 (because $xyz-1$ is minimal over zero. Therefore, since $k[x, y, z]$ is a f.g. k-algebra domain,

$$\dim A = \dim k[x, y, z] - \mathbf{ht}\,((xyz - 1)) = 2$$

Therefore, $\mathrm{Spec}\,(A)$ is an affine variety of dimension two and thus a surface. Consider the maps,

$$\mathrm{Hom}_{\mathbf{Sch}(k)}\left(\mathbb{A}^1_k, \mathrm{Spec}\,(A)\right) = \mathrm{Hom}_{k-\mathrm{alg}}\,(k[t], A)$$

Therefore, we need to consider all k-algebra morphisms $A \to k[t]$ which is equivalent to a k-algebra map $k[x, y, z] \to k[t]$ such that the ideal $(xyz - 1)$ maps to zero. Such a map takes,

$$\begin{aligned} x &\mapsto f \\ y &\mapsto g \\ z &\mapsto h \end{aligned}$$

for polynomials $f, g, h \in k[t]$. However, we must have $xyz \mapsto 1$ so $fgh = 1$ which implies that $\deg(fgh) = 1$ and thus $f, g, h \in k^\times$ are units. Any prime $\mathfrak{p} \subset k[t]$ cannot contain any units but must contain zero, $\mathfrak{p} \cap k = (0)$. Thus, under any map $\phi : A \to k[t]$ which necessarily maps A inside $k \subset k[t]$, we have $\phi^{-1}(\mathfrak{p}) = \phi^{-1}(0) = \ker\phi = (x-f, y-g, z-h)$ such that $fgh = 1$ and $f, g, h \in k^\times$. This is a fixed closed point of $X = \mathrm{Spec}\,(A)$. Therefore, each point of $\mathbb{A}^1_k$ (i.e. primes of $k[t]$) maps to some given fixed closed point of X so all morphisms of k-schemes $\mathbb{A}^1_k \to X$ is constant.

For the opposite problem, take $\mathbb{P}^2_k = \mathrm{Proj}\,(k[x_0, x_1, x_2])$. The scheme $\mathbb{P}^2_k$ is a surface over k (see Lemma **??**.) Furthermore $\mathbb{A}^1_k = \mathrm{Spec}\,([t])$ is an affine scheme over k and thus we have the natural equivalence,

$$\mathrm{Hom}_{\mathbf{Sch}(k)}\left(\mathbb{P}^2_k, \mathbb{A}^1_k\right) \cong \mathrm{Hom}_{k-\mathbf{alg}}\left(k[t], \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k})\right)$$

Therefore, we need only consider the k-algebra maps $k[t] \to \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k})$. By Lemma **??**, $\Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \cong k$. Therefore, we need to consider all k-algebra maps $k[t] \to k$. However, because these maps must preserve the k-algebra structure such a map is uniquely determined by the image of t. Let $\mathrm{ev}_x : k[t] \to k$ be the unique k-algebra map sending $t \mapsto x$ for $x \in k$. Now by the correspondence, these maps induce all morphisms of schemes $\mathbb{P}^2_k \to \mathbb{A}^1_k$ over k. Consider the point $p \in \mathbb{P}^2_k$ then the preimage of the map $\Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \to \mathcal{O}_{\mathbb{P}^2_k, p}$ must take the unique maximal ideal $\mathfrak{m}_p$ to the unique prime (also maximal) ideal (0) of $\Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \cong k$. Therefore, $\tilde{\mathrm{ev}}_x(p) = \mathrm{ev}_x^{-1}(\mathfrak{m}_p) = \mathrm{ev}_x^{-1}(0) = (t - x) \in \mathbb{A}^1_k$. Therefore, the map $\tilde{\mathrm{ev}}_x : \mathbb{P}^2_k \to \mathbb{A}^1_k$ is the constant map sending $\mathbb{P}^2_k$ to the closed point corresponding to $x \in k$. Therefore any choice of map $\mathbb{P}^2_k \to \mathbb{A}^1_k$ of schemes over k is constant.

5 Problem e

Let k be a field. We need to construct a surjective map $\mathbb{A}^1_k \to \mathbb{P}^1_k$. We will first consider the problem under base change $k \to \bar{k}$ to the algebraic closure. In terms of concrete classical varieties we can easily produce a morphism $\mathbb{A}^1_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ via $s \mapsto [s : s^2 - 1]$. I claim that this map is surjective on the closed points which take the form $(x - s) \in \mathbb{A}^1_{\bar{k}}$ and $(xs - yt) \in \mathbb{P}^1_{\bar{k}}$ respectively. In the affine patch $\mathbb{A}^1_{\bar{k}} \subset \mathbb{P}^1_{\bar{k}}$ given by $[t : 1]$ we can find s such that $[s : s^2 - 1] = [t : 1]$ since $s = (s^2 - 1)t$ has a solution in s for each t over an algebraically closed field. The "point at infinity" $[1 : 0]$ is the image of $s = \pm 1$. Since a morphism of varieties is determined on its closed points (and is a surjection when it surjects on closed points) we have constructed a surjection $\mathbb{A}^1_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. Furthermore, we consider the base change,

$$\begin{tikzcd}
\mathbb{A}^1_k \arrow[r, dashed, two heads] & \mathbb{P}^1_k \\
\mathbb{A}^1_{\bar{k}} \arrow[u, two heads] \arrow[r, two heads] & \mathbb{P}^1_{\bar{k}} \arrow[u, two heads]
\end{tikzcd}$$

We need to show that this morphism descends to a map $\mathbb{A}^1_k \to \mathbb{P}^1_k$ making this square commute. However, the Galois action $\mathrm{Gal}\left(\bar{k}/k\right)$ on the schemes $\mathbb{A}^1_{\bar{k}}$ and $\mathbb{P}^1_{\bar{k}}$ commute with the constructed map since it is given by polynomials in the ground field. Therefore, pulling back an element of $\mathbb{A}^1_k$ to $\mathbb{A}^1_{\bar{k}}$ and applying the mapping to $\mathbb{P}^1_k$ is well-defined since all such pullbacks are permuted by the Galois action and are thus mapped to conjugates under $\mathbb{A}^1_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. Finally, the descended map is also surjective because the square commutes and all other maps are surjective.

6 Lemmata

Lemma 6.1. Let A be a ring. Then, $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) \cong A$

Proof. $\mathbb{P}^n_A$ is covered by affine opens $D_+(x_i) \cong \mathrm{Spec}\left((A[x_0, \dots, x_n]_{x_i})_0\right)$ where we must take the degree zero part. Therefore, by the sheaf property, the sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A) \longrightarrow \prod_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n_A}(D_+(x_i)) \longrightarrow \prod_{i,j} \mathcal{O}_{\mathbb{P}^n_A}(D_+(x_i) \cap D_+(x_j))$$

is exact. Thus $\mathcal{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A)$ is the kernel of the second map. Consider an arbitrary element $z = \left(\frac{s_1}{x_1^{r_1}}, \dots, \frac{s_n}{x_n^{r_n}}\right)$ where s_i is homogeneous of degree r_i. Suppose z is in the kernel then, in the i,j-entry, z maps to,

$$\frac{s_i}{x_i^{r_i}} - \frac{s_j}{x_j^{r_j}} = 0$$

which implies that $x_j^{r_j} s_i = x_i^{r_i} s_j$. However, since $x_i \neq x_j$ are irreducible we must have $x_i^{r_i} \mid s_i$ and $x_j^{r_j} \mid s_j$ i.e. $s_i = u_i x_i^{r_i}$ and $s_j = u_j x_j^{r_j}$. However both fractions are supposed to have degree zero so their quotient u_i and u_j must have degree zero meaning that $u_i, u_j \in A$. Furthermore,

$$\frac{s_i}{x_i^{r_i}} - \frac{s_j}{x_j^{r_j}} = 0 \implies u_i = u_j$$

Therefore $z = (u, \dots, u)$ so the kernel is isomorphic to A. $\square$

Lemma 6.2. Let k be an algebraically closed field then the scheme $\mathbb{P}^n_k = \mathrm{Proj}\,(k[x_0, \dots, x_n])$ is a variety over k of dimension n.

Proof. Since $k[x_0, \dots, x_n]$ is a finitely generated k-algebra and $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = k$ then the correspondence

$$\mathrm{Hom}_{\mathbf{Sch}}\,(\mathbb{P}^n_k, \mathrm{Spec}\,(k)) = \mathrm{Hom}_{\mathbf{Ring}}\left(k, \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})\right) = \mathrm{Hom}_{\mathbf{Ring}}\,(k, k)$$

gives a canonical morphism $\mathbb{P}^n_k \to \mathrm{Spec}\,(k)$ (corresponding to id_k) of finite type (since each affine open corresponds to a finitely generated k-algebra). Furthermore the affine open cover $D_+(x_i) = \mathrm{Spec}\left(k[x_0, \dots, x_n]_{(x_i)}\right)$ are integral domains so $\mathbb{P}^n_k$ is integral. Finally, $\mathbb{P}^n_k = \mathrm{Proj}\,(k[x_0, \dots, x_n])$ is generically separated for any graded ring. $\square$

Lemma 6.3. Let $A, B \subset D$ be finitely generated k-algebras. Then the subring $C \subset D$ generated by A and B is a finitely generated k-algebra.

Proof. Since A and B are finitely generated k-algebras there are surjective maps $k[x_1, \dots, x_m] \to A$ and $k[x_1, \dots, x_m] \to B$. Then consider the maps,

$$k[x_1, \dots, x_{n+m}] \xrightarrow{\sim} k[x_1, \dots, x_m] \otimes_k k[x_1 \dots, x_n] \twoheadrightarrow A \otimes_k B \twoheadrightarrow C$$

each of which surjects. The last map $A \otimes_k B \to C$ is given by multiplication $a \otimes_k b \mapsto ab$ which clearly surjects onto C since it is generated by all products of A and B. $\square$