Classical Mechanics from the Symplectic Viewpoint

Benjamin Church

April 22, 2020

Contents

1 Symplectic Geometry

Definition 1.0.1. Let V be a finite k-vectorspace and $\omega \in \bigwedge^2 V^*$ a 2-form. We say that ω is *nondegenerate* if for all nonzero $v \in V$ the map $\omega(v, -) \in V^*$ is nonzero. Equivalently, ω is nondegenerate exactly when the map $V \to V^*$ defined by $v \mapsto \omega(v, -)$ is an isomorphism.

Lemma 1.0.2. If ω is a nondegenerate 2-form on V then $\dim V = 2n$ is even.

Proof. Choose a basis $e_1, \dots, e_k$ of V. Then we have a matrix $M_{ij} = \omega(e_i, e_j)$ which is antisymmetric. Then ω is nondegenerate implies that $\det M \neq 0$. However, $M^\top = -M$ so we must have,

$$\det M = \det(-M) = (-1)^{\dim V} \det M$$

Thus $\dim V = 2n$ is even. □

Definition 1.0.3. Let M be a smooth $2n$-manifold. A *symplectic form* ω on M is a closed nondegenerate 2-form. We say that the pair (M, ω) is a *symplectic manifold.* A *symplectomorphism* $f : (M, \omega_M) \to (N, \omega_N)$ is a smooth map $f : M \to N$ such that $f^* \omega_N = \omega_M$.

Remark. Consider a vector field X on M. Such a vector field defines a flow $\phi_t : M \to M$. We consider when this flow preserves the symplectic structure. This occurs when ϕ_t is a symplectomorphism i.e. when $\phi_t^* \omega = \omega$. Now, recall that, the Lie derivative is defined via,

$$\mathcal{L}_X \omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\phi_t^* \omega \right)$$

Therefore $\phi_t : M \to M$ is symplectic iff $\mathcal{L}_X \omega = 0$.

Definition 1.0.4. We say a vector field X on M is *symplectic* if $\mathcal{L}_X \omega = 0$.

Definition 1.0.5. We say a vector field X on M is *Hamiltonian* if there exists a smooth function $H : M \to \mathbb{R}$ such that $\iota_X \omega = \mathrm{d}H$.

Lemma 1.0.6. Hamiltonain vector fields are symplectic.

Proof. Let X be Hamiltonian such that $\iota_X \omega = \mathrm{d}H$. Then, we use Cartan's magic formula,

$$\mathcal{L}_X \omega = \mathrm{d}(\iota_X \omega) + \iota_X \mathrm{d}\omega$$

Applying $\iota_X \omega = \mathrm{d}H$ and using $\mathrm{d}\omega = 0$ we find,

$$\mathcal{L}_X \omega = \mathrm{d}(\mathrm{d}H) = 0$$

□

2 Symptectic Geometry

Definition 2.0.1. A *symplectic form* on M is a closed non-degenerate 2-form ω. We say that (M, ω) is a *symplectic manifold.* A *symplectomorphism* $f : (M, \omega_M) \to (N, \omega_N)$ is a smooth map $f : M \to N$ such that $f^* \omega_N = \omega_M$.

Lemma 2.0.2. Symplectic forms can only exist on even-dimensional manifolds.

Proof. Locally, a symplectic form ω is a nondegenerate anti-symmetric bilinear form $S : T_pM \times T_pM \to \mathbb{R}$. So we have $S^\top = -S$ and $\det S \neq 0$. However,

$$\det S = \det S^\top = \det(-S) = (-1)^n \det S$$

since $\det S \neq 0$ we must have $(-1)^n = 1$ i.e. n is even. $\square$

Definition 2.0.3. We say that a vector field X on (M, ω) is symplectic if $\mathcal{L}_X \omega = 0$.

Remark. We see that the condition $\mathcal{L}_X \omega = 0$ that a vector field be symplectic is equivalent to the condition that its flows $\phi_t : M \to M$ be symplectomorphisms since,

$$\mathcal{L}_X \omega = \frac{\mathrm{d}}{\mathrm{d}t}((\phi_t)^* \omega) = 0$$

Thus, symplectic vector fields are fields whose flows preserve the symplectic structure.

Definition 2.0.4. We say that a vector field X on (M, ω) is Hamiltonian if the form $\iota_X \omega \in \Omega^1(M)$ is exact i.e. there exists a function $H : M \to \mathbb{R}$ such that,

$$\mathrm{d}H = \iota_X \omega$$

Remark. Note that since ω is non-degenerate, the map $\omega : TM \to \Omega^1(M)$ via $X \mapsto \iota_X \omega$ is an isomorphism and thus we can consider $\omega^{-1} : \Omega^1(M) \to TM$. Then the above condition is that $X = \omega^{-1}(\mathrm{d}H)$.

Lemma 2.0.5. Hamiltonian vector fields are symplectic.

Proof. Let X be Hamiltonian. Then consider,

$$\mathcal{L}_X \omega = \iota_X \mathrm{d}\omega + \mathrm{d}\iota_X \omega$$

Since ω is a symplectic form $\mathrm{d}\omega = 0$ and since X is Hamiltonainm $\iota_X \omega$ is exact and thus closed so $\mathrm{d}\iota_X \omega = 0$. Therefore,

$$\mathcal{L}_X \omega = 0$$

so X is symplectic. $\square$

Lemma 2.0.6. Symplectic and Hamiltonian vector fields form Lie subalgebras.

Proof. We know that,

$$\mathcal{L}_{[X,Y]} \omega = \mathcal{L}_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega$$

so if X, Y are symplectic then so is $[X, Y]$. Furthermore,

$$\iota_{[X,Y]} \omega = \mathcal{L}_X \iota_Y \omega - \iota_Y \mathcal{L}_X \omega$$

However, $\mathcal{L}_X \omega = 0$ since Hamiltonian fields are symplectic. Furthermore, by Cartan's formula,

$$\iota_{[X,Y]} \omega = \mathcal{L}_X \iota_Y \omega = \iota_X(\mathrm{d}\iota_Y \omega) + \mathrm{d}(\iota_X \iota_Y \omega)$$

However, since $\iota_Y \omega$ is exact it is closed and thus,

$$\iota_{[X,Y]} \omega = \mathrm{d}(\iota_X \iota_Y \omega) = \mathrm{d}(\omega(Y, X))$$

which is exact so $[X, Y]$ is Hamiltonian. $\square$

Remark. We have $\mathcal{L}_X \mathrm{d}\omega = \mathrm{d}(\mathcal{L}_X \omega)$ because d is a natural transformation in the sense that $f^*\mathrm{d} = \mathrm{d}f^*$ for any smooth map and, in particular, for the flow of X.

Definition 2.0.7. Let $f, g : M \to \mathbb{R}$ be functions and let $X_f = \omega^{-1}(\mathrm{d}f)$ and $X_g = \omega^{-1}(\mathrm{d}g)$ be the associated Hamiltonian vector fields. Then we define the *Poisson bracket* via,

$$\{f, g\} = \omega(X_f, X_g)$$

Remark. From the definitions of X_f and X_g,

$$\begin{aligned} \{f, g\} &= \omega(X_f, X_g) = \mathrm{d}f(X_g) = X_g(f) = \mathcal{L}_{X_g} f \\ &= -\omega(X_g, X_f) = -\mathrm{d}g(X_f) = -X_f(g) = -\mathcal{L}_{X_f} g \end{aligned}$$

So $\{f, g\}$ represents the flow of f along the vector field generated by g.

Lemma 2.0.8. $[X_f, X_g] = -X_{\{f,g\}}$

Proof. We have shown that if X and Y are Hamiltonian then,

$$\iota_{[X,Y]}\omega = \mathrm{d}(\omega(Y, X))$$

Therefore,

$$X_{\omega(Y,X)} = \omega^{-1}(\mathrm{d}(\omega(Y, X))) = [X, Y]$$

Now applying this to X_f and X_g we find,

$$[X_f, X_g] = \omega^{-1}(\mathrm{d}(\omega(X_g, X_f)) = -\omega^{-1}(\mathrm{d}\{f, g\}) = -X_{\{f,g\}}$$

□

Proposition 2.0.9. The Poisson bracket on smooth functions forms a Lie algebra.

Proof. Clearly the Poisson bracket is bilinear. Furthermore, it is antisymmetric because,

$$\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\}$$

The Jacobi identity is equivalent to the fact that $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ via $\xi \mapsto [\xi, -]$ is a Lie algebra homomorphism.

In the current case, $\mathrm{ad}_f(g) = \{f, g\} = -X_f(g)$ so $\mathrm{ad}_f = -X_f$ as a derivation. Then we know that,

$$[\mathrm{ad}_f, \mathrm{ad}_g] = [-X_f, -X_g] = -X_{\{f,g\}} = \mathrm{ad}_{\{f,g\}}$$

since the commutator of vector fields is their comutator as differential operators. □

Proposition 2.0.10. The map $f \mapsto -X_f = -\omega^{-1}(\mathrm{d}f)$ is a homomorphism of Lie algebras from smooth functions to Hamiltonian vector fields.

Proof. Immediate from $-X_{\{f,g\}} = [X_f, X_g] = [-X_f, -X_g]$. □