1 Math 245B Topics in algebraic geometry: Deligne-Lustzig Theory

Note: no class week of Jan 29th and zoom the week after.

The course is about $\mathbb{C}$-rep theory of finite groups of Lie type e.g. $\mathrm{GL}_3(\mathbb{F}_8)$ or $\mathrm{Sp}_8(\mathbb{F}_{27})$ or $\mathrm{SO}_5(\mathbb{F}_3)$. The goal is to construct all the (irreducible) representations.

Example 1.0.1. Consider $G = \mathrm{SL}_2(\mathbb{F}_q)$ for $p > 2$. Then $T(\mathbb{F}_q) \subset B(\mathbb{F}_q) \subset \mathrm{SL}_2(\mathbb{F}_q)$ be the torus and upper-triangular Borel. Given a character $\theta : T(\mathbb{F}_q) \to \mathbb{C}^\times$ consider the map $B \to T$ quotienting by the unipotent part then get a G-rep $\mathrm{Ind}_{B(\mathbb{F}_q)}^G(\theta)$. If θ is trivial then $\mathrm{Ind}_{B(\mathbb{F}_q)}^G(\theta) = \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_q), \mathbb{C})$ with the standard $\mathrm{SL}_2(\mathbb{F}_q)$-action. This has a subrep of the constant functions giving an exact sequence,

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ind}_{B(\mathbb{F}_q)}^G(1) \longrightarrow \mathrm{st} \longrightarrow 0$$

where st is the Steinberg. This is irreducible (exercise). Does this proceedure give all representations? No.

Example 1.0.2. If $\theta^2 \neq 1$ then $\mathrm{Ind}_{B(\mathbb{F}_q)}^G(\theta) = \mathrm{Ind}_{B(\mathbb{F}_q)}^G(\theta^{-1})$ so we get fewer representations. If $p > 2$ and $q = p^r$ then we get $\frac{q+5}{2}$ irreps of $\mathrm{SL}_2(\mathbb{F}_q)$ from this proceedure. However, there are $q+4$ conjugacy classes and thus irreps.

The other half of the reps must come from a different construction. Frobenius was able to write these down in the 1890s but we want a general proceedure for all groups of Lie type. Macdonald conjectured that these are related to characters of $T^1(\mathbb{F}_q) \subset \mathrm{SL}_{@}(\mathbb{F}_q)$ which is the nonsplit torus $\mathbb{F}_{q^2}^\times \subset \mathrm{GL}_2(\mathbb{F}_q)$ intersected with SL_2. Problem, is there is no $\mathbb{F}_q$-stable Borel containing this. Drinfeld gives us the solution. Consider the curve,

$$C = \{xy^q - yx^q = 1\} \subset \mathbb{A}^2_{\mathbb{F}_q}$$

which has commuting actions of $\mathrm{SL}_2(\mathbb{F}_q)$ are μ_{q+1} given by,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y) \mapsto (ax + by, cx + dy)$$

and

$$\zeta \cdot (x, y) \mapsto (\zeta x, \zeta y)$$

Then for $\theta : \mu_{q+1} \to \overline{\mathbb{Q}_\ell}$ (which is abstractly isomorphic to $\mathbb{C}$) then we get a representation,

$$\mathrm{SL}_2(\mathbb{F}_q) \circlearrowright H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}_\ell})[\theta]$$

where this is the part where μ_{q+1} acts by θ. These give the remaining representations.

Remark. Notice that C is a μ_{q+1}-cver of $\mathbb{P}^1_{\mathbb{F}_q} \backslash \mathbb{P}^1(\mathbb{F}_q)$.

2 Representation Theory of Finite Groups

Definition 2.0.1. Let G be a finite group and k a field. A k-representation of G is a pair (V, π) where V is a finite-dimensional k-vectorspace and $\pi : G \times V \to V$ is a k-linear action of G. A morphism of representations $f : (V, \pi) \to (V', \pi')$ is a linear map $f : V \to V'$ such that,

$$\begin{array}{ccc} G \times V & \xrightarrow{\mathrm{id} \times f} & G \times V' \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ V & \xrightarrow{\quad f \quad} & V' \end{array}$$

This category is called RepkG.

Proposition 2.0.2. RepkG is abelian and F : Rep$kG \to \mathrm{Vect}_k$ commutes with all limits and colimits. Furthermore, RepkG is monoidal and F is a monoidal functor with the usual $\otimes$ on Vect_k.

Proposition 2.0.3 (Maschke)**.** If $\#G \in k^\times$ then RepkG is semisimple.

Definition 2.0.4. Given (V, π, ρ) there is a function $\chi_V : G \to k$ via $g \mapsto \operatorname{tr} \rho(g)$ called the *character*.

Theorem 2.0.5 (Orthogonality)**.** If $\#G = k^\times$ and V, V' are G-reps then,

$$\frac{1}{\#G} \sum_{g \in G} \chi_V(g) \chi_{V'}(g^{-1}) = \dim \mathrm{Hom}_G\left(V, V'\right)$$

inside k.

Proof. The LHS is,

$$\frac{1}{\#G} \sum_{g \in G} \operatorname{tr}\left(g | \mathrm{Hom}\left(V, V'\right)\right)$$

and for any $w \in$ RepkG we have,

$$\frac{1}{\#G} \sum_{g \in G} \operatorname{tr}\left(g | W\right) = \dim W^G$$

□

Proposition 2.0.6. Let $\#G \in k^\times$ and $k = \bar{k}$. Then $\{\chi_V\}$ for V irreps span the space of conjugation invariant functions $G \to k$.

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Fix a finite group G and a field k s.t. $\#G \in k^\times$ and $k = \bar{k}$. If $H \subset G$ is a subgroup, then there is a functor,

$$\mathrm{Res}_H^G\left(-\right) : \mathrm{Rep}kG \to \mathrm{Rep}kH$$

which has both a left and a right adjoint given by

$$\mathrm{Ind}_H^G\left(-\right) : \mathrm{Rep}kH \to \mathrm{Rep}kG$$

which is defined by,

$$V \mapsto \{f : G \to V \mid \forall h \in H, g \in G : f(hg) = \rho_V(h) f(g)\}$$

Remark. $\dim \operatorname{Ind}_H^G(V) = [G:H]\dim V$.

Remark. A goal of Mackey theory is to understand when induced representations are irreducible.

Definition 3.0.1. We notate the induced character,

$$\chi_V^G = \chi_{\operatorname{Ind}_H^G(V)}$$

so therefore Frobenius reciprocity (the adjunction) is given by the corresponding statement for pairing characters,

$$\left\langle \chi_V^G, \chi_V^G \right\rangle_G = \left\langle \chi_V, \chi_V^G|_H \right\rangle_H$$

Recall, by character theory $\operatorname{Ind}_H^G(V)$ is absolutely irreducible iff the above pairing is 1. For $g \in G$ we write H^g for $gHg^{-1} \subset G$ and $\rho : H \to \mathrm{GL}(V)$ I write $\rho^g : gHg^{-1} \to \mathrm{GL}(V)$ with $ghg^{-1} \mapsto \rho(h)$. Note that $H \cap H^g$ only depends, up to isomorphism, on $[g] \in H\backslash G/H$.

Theorem 3.0.2.

$$\operatorname{Res}_H^G\left(\operatorname{Ind}_H^G(\rho)\right) = \bigoplus_{[g]\in H\backslash G/H} \operatorname{Ind}_{H\cap H^g}^H\left(\operatorname{Res}_{H\cap H^g}^{H^g}(\rho^g)\right)$$

Corollary 3.0.3. $\operatorname{Ind}_V^G(V)$ is irreducible iff V is irreducble and $\operatorname{Res}_{H^g\cap H}^{H^g}(\chi)$ and $\operatorname{Res}_{H^g\cap H}^{H^g}(\rho^g)$ share no common irreducible factors (other than $g = 1$).

Proof.

$$\begin{aligned}\left\langle \chi_V^G, \chi_V^G \right\rangle_G &= \left\langle \chi_V, (\chi_V^G)_H \right\rangle_H = \sum_{g\in H\backslash G/H} \left\langle \chi_V, \chi_{\operatorname{Ind}_{H\cap H^g}^H\left(\operatorname{Res}_{H\cap H^g}^{H^g}(\rho^g)\right)} \right\rangle \\ &= \sum_{g\in H\backslash G/H} \left\langle \operatorname{Res}_{H\cap H^g}^H(\chi), \operatorname{Res}_{H\cap H^g}^{H^g}(\chi^g) \right\rangle\end{aligned}$$

Each term in the sum is a positive integer so we must have exactly one of them is equal to 1. ☐

Example 3.0.4. Apply this to $G = \mathrm{SL}_2(\mathbb{F}_q)$ and $H = B(\mathbb{F}_q)$. Let,

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then,

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix}s^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

Conjugation by s preserves $T(\mathbb{F}_q)$ and axts as inversion on it. Then $B(\mathbb{F}_q) \cap sB(\mathbb{F}_q)s^{-1} = T(\mathbb{F}_q)$.

Lemma 3.0.5. $\mathrm{SL}_2(\mathbb{F}_q) = B(\mathbb{F}_q) \cup B(\mathbb{F}_q)sB(\mathbb{F}_q)$ is the Bruhat decomposition.

If we start with $\theta_1, \theta_2 : T(\mathbb{F}_q) \to \mathbb{C}^\times$ and consider them as representations of $B(\mathbb{F}_q) \to T(\mathbb{F}_q)$ then,

$$\left\langle \operatorname{Ind}_{B(\mathbb{F}_q)}^{SL_2(\mathbb{F}_q)}(\theta_1), \operatorname{Ind}_{B(\mathbb{F}_q)}^{\mathrm{SL}_2(\mathbb{F}_q)}(\theta_2) \right\rangle_G = \langle \theta_1, \theta_2 \rangle_T + \langle \theta_1, \theta_2^s \rangle_T$$

Corollary 3.0.6. If $\theta_1 = \theta_2$ we find $\operatorname{Ind}_{B(\mathbb{F}_q)}^{\mathrm{SL}_2(\mathbb{F}_q)}(\theta)$ is irred if $\theta_1 \neq \theta_1^{-1}$. If $\theta_1 \in \{\theta_2, \theta_2^{-1}\}$ then $\operatorname{Ind}_-^-(\theta_1)$ and $\operatorname{Ind}_-^-(\theta_2)$ shrea no common factors.

If $p > 2$ then there are $q-3$ characters θ with $\theta \neq \theta^{-1}$ and therefore $\frac{q-3}{2}$ irreps of $\mathrm{SL}_2(\mathbb{F}_q)$. Then,

$$\mathrm{Ind}_{-}^{-}(1) = 1 + \mathrm{st}$$

and for $\alpha \neq 1$ with $\alpha^2 = 1$

$$\mathrm{Ind}_{-}^{-}(\alpha) = R(\alpha)_+ + R(\alpha)_+$$

with $R(\alpha)_+$ and $R(\alpha)_-$ are nonisomorphic representations of the same dimension. Therefore we have found,

$$\frac{q-3}{2} + 4 = \frac{q+5}{2}$$

representations.

Definition 3.0.7. A representation of $\mathrm{SL}_2(\mathbb{F}_q)$ that does not contain any of the previous representation as a summand is called *cuspidal*.

Example 3.0.8. Consider $\mathrm{SL}_2(\mathbb{Z}_p) \hookrightarrow \mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{F}_p)$ and let $\mathrm{SL}_2(\mathbb{Z}_p)$ act on V via a cuspidal rep of $\mathrm{SL}_2(\mathbb{F}_p)$ then c-Ind to $\mathbb{Q}_p$ is cuspidal.

4 ℓ-adic Cohomology

Let X be a smooth projective $\mathbb{F}_q$-variety. Then can define,

$$\zeta_X(T) = \exp\left(\sum_{n\geq 1} \#X(\mathbb{F}_{q^n})\frac{T^n}{n}\right) \in \mathbb{Q}[\![T]\!]$$

Example 4.0.1. $X = \mathrm{Spec}\,(\mathbb{F}_q)$ then,

$$\zeta_X(T) = \frac{1}{1-T}$$

If $X = \mathbb{P}^1_{\mathbb{F}_q}$ then,

$$\zeta_X(T) = \frac{1}{(1-T)(1-qT)}$$

If $X = E$ is an elliptic curve over $\mathbb{F}_q$ then,

$$\zeta_X(T) = \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)}$$

Conjecture 4.0.2 (Weil)**.** ζ_X is a rational function.

Proof. Weil's idea: we are counting fixed points of Frob_q^r on $X_{\overline{\mathbb{F}}_q}$. Now, if M is a compact oriented manifold and $\psi : M \to M$ continuous with isolated fixed points then,

$$\#\mathrm{fix}(\psi) = \sum_i (-1)^i \,\mathrm{tr}\,(\psi_* | H^i_{\mathrm{sing}}(M, \mathbb{R}))$$

This implies that the exponential generating function for $\#\mathrm{fix}(\psi^n)$ is a rational function. ☐

Is there an "algebraic definition" of singular cohomology for X smooth projective over $\mathbb{C}$. Then $H^0_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) = \pi_1(X(\mathbb{C}))^{\text{ab}}$ but $\mathbb{C}^\times$ has a $\mathbb{Z}$-cover $\exp : \mathbb{C} \to \mathbb{C}^\times$ which is not algebraic. However, Riemann existence proves that all *finite* covering spaces *are* algebraic. Therefore, $H^1_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$ has an algebraic definnition.

Serre gives a simple argument that shows there cannot exist a cohomology theory for smooth projective $\mathbb{F}_q$-varities which is valued in $\mathbb{Q}$-vectorspaces such that $H^1(E, \mathbb{Q})$ is a two-dimensional $\mathbb{Q}$-vectorspace. This is because $\text{End}\,(E)$ is a quaternion algebra and this cannot act on $\mathbb{Q}^2$ in the necessary way.

So we could hope to define a cohomology theory with values in $\mathbb{Z}/\ell^n\mathbb{Z}$ for $\ell \neq p$ this gives a theorey with values in $\varprojlim \mathbb{Z}/\ell^n\mathbb{Z} = \mathbb{Z}_\ell$ and thus in $\mathbb{Z}_\ell[\ell^{-1}] = \mathbb{Q}_\ell$.

Theorem 4.0.3 (Grothendieck-Deligne-Artin)**.** Yes this is possible. There is a functor

$$H^i_{\text{ét}}(-, \mathbb{Q}_\ell) : \{\text{sm proj varities over } {}^{\text{op}}\overline{\mathbb{F}}_p\} \to \{\text{fin dim } \mathbb{Q}_\ell\text{-vector spaces}\}$$

such that,

(a) $H^i_{\text{ét}}(X, \mathbb{Q}_\ell) = 0$ unless $0 \le i \le 2 \dim X$

(b) $H^0_{\text{ét}}(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[\pi_0(X)]$

(c) If X lift to $\widetilde{X}$ over $\mathbb{C}$ then,
$$H^i_{\text{sing}}(\widetilde{X}(\mathbb{C}), \mathbb{Q}_\ell) = H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$$

(d) $H^i_{\text{ét}}(X, \mathbb{Q}_\ell) = H^{2d-i}(X, \mathbb{Q}_\ell)^\vee$ if X is equidimensional of dimension d

(e) if $\psi : X \to X$ has isolated fixed points then,
$$\#\text{fix}(\psi) = \sum_i (-1)^i \,\text{tr}\,(\psi_* | H^i_{\text{ét}}(X, \mathbb{Q}_\ell))$$

(f) if X is over $\mathbb{F}_q$ then,
$$\#X(\mathbb{F}_{q^n}) = \sum_i (-1)^i \,\text{tr}\,(\text{Frob}_q^n | H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$$

Theorem 4.0.4. There are also functors,

$$H^i_c(-, \mathbb{Q}_\ell) : \{\text{varities over } {}^{\text{op}}\overline{\mathbb{F}}_p \text{ with proper maps}\} \to \{\text{fin dim } \mathbb{Q}_\ell\text{-vector spaces}\}$$

such that,

(a) $H^i_c(X, \mathbb{Q}_\ell) = H^i X, \mathbb{Q}_\ell)$ if X is proper / projective

(b) $H^i_c(X, \mathbb{Q}_\ell) = 0$ unless $0 \le i \le 2 \dim X$

(c) If X is smooth and affine then $H^i_c(X, \mathbb{Q}_\ell) = 0$ for $0 \le i \le \dim X$

(d) If $Z \subset X$ is closed then is the a LES,

$$\cdots \longrightarrow H^i_c(U, \mathbb{Q}_\ell) \longrightarrow H^i_c(X, \mathbb{Q}_\ell) \longrightarrow H^i_c(Z, \mathbb{Q}_\ell) \longrightarrow H^{i+1}_c(U, \mathbb{Q}_\ell) \longrightarrow \cdots$$

(e) if $\psi : X \to X$ has isolated fixed points then,

$$\#\mathrm{fix}(\psi) = \sum_i (-1)^i \operatorname{tr}\left(\psi_* | H^i_c(X, \mathbb{Q}_\ell)\right)$$

(f) if X is over $\mathbb{F}_q$ then,

$$\#X(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q^n | H^i_c(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\right)$$

Let C be the Drinfeld curve over $\mathbb{F}_q$ equipped with actions of $\mathrm{SL}_2(\mathbb{F}_q)$ and μ_{q+1}. Let θ be a character of μ_{q+1} with values in $\mathbb{Q}_\ell$.

Definition 4.0.5 (Deligne-Lustzig induction)**.** Let $[\theta]$ denote $\mathrm{Hom}_{\mu_{p+1}}(\theta, -)$ then let,

$$R(\theta) = H^0_c(C_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)[\theta] - H^1_c(C_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)[\theta] + H^2_c(C_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)[\theta]$$

in the grothendieck group of representations.

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Recall the Drinfeld curve C (for fixed $q = p^r$) given by,

$$\{XY^q - YX^q = 1\} \subset \mathbb{A}^2_{\mathbb{F}_q}$$

This has an action of $\mathrm{SL}_2(\mathbb{F}_q)$ given by,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y) = (ax + by, cx + dy)$$

and by μ_{q+1} given by,

$$\zeta \cdot (x, y) = (\zeta x, \zeta y)$$

Observation: $C(\mathbb{F}_q) = \varnothing$. For some character,

$$\theta : \mu_{q+1} \to \overline{\mathbb{Q}}_\ell^\times$$

we define the virtual representation,

$$R'(\theta) = H^2_c(C_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\theta] - H^1_c(C_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\theta]$$

Here for $W \in \mathrm{Rep}\mu_{q+1}$ we write,

$$W[\theta] = \{w \in W \mid \zeta \cdot w = \theta(\zeta) \cdot w\}$$

We start by computing,

$$R'(1) = H^i_c(C_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)^{\mu_{q+1}} = H^i_c(C_{\overline{\mathbb{F}}_q}/\mu_{q+1}, \overline{\mathbb{Q}}_\ell)$$

Lemma 5.0.1. The map $C \to \mathbb{P}^1_{\mathbb{F}_q} \backslash \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_q)$ is a quotient map by the μ_{q+1}-action.

Proof. Since $[\zeta \cdot X, \zeta \cdot Y] = [X, Y]$ the map is μ_{q+1}-invariant.

The action is clearly free since $(0,0)$ is not on the curve.

Claim that the map is surjective. Indeed, given $[1 : T] \in \mathbb{P}^1(\overline{\mathbb{F}}_q) \backslash \mathbb{P}^1(\mathbb{F}_q)$. We want to find some $\lambda \in \overline{\mathbb{F}}_q^\times$ such that $[\lambda : \lambda T]$ is on the curve:

$$\lambda^{q+1}(T^q - T) = 1$$

which solvable since $T^q \neq T$ and $\overline{\mathbb{F}}_q^\times$ has all $(q+1)$-roots.

If $(\lambda, \lambda T)$ and $(\lambda', \lambda' T)$ are two different solutions then $\lambda = \zeta \lambda'$ for $\zeta \in \mu_{q+1}$ which is true because the solutions are exactly the $(q+1)$-roots of $(T^q - T)^{-1}$.

Therefore, $C(\overline{\mathbb{F}}_q)/\mu_{q+1} = \mathbb{P}^1(\overline{\mathbb{F}}_q) \backslash \mathbb{P}^1(\mathbb{F}_q)$. In fact, this is an isomorphism of schemes. $\square$

Now we compute! Let $U = \mathbb{P}^1_{\mathbb{F}_q} \backslash \mathbb{P}^1(\mathbb{F}_q)$. Take the long-exact sequence,

$$\begin{array}{ccccccccccc} 0 \longrightarrow & H^0_c(U_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) & \longrightarrow & H^0(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell) & \longrightarrow & H^0(Z_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) & \longrightarrow & H^1_c(U_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) & \longrightarrow & H^1(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell) & \longrightarrow 0 \\ & & & \| & & \| & & & & \| & \\ & & & \overline{\mathbb{Q}}_\ell & & 1 \oplus \mathrm{st} & & & & 0 & \end{array}$$

and furthermore $H^2_c(U_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) = H^2(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell(-1)$. The map $H^0(\mathbb{P}^1) \to H^0(Z)$ is injective so we see that,

$$H^0_c(U_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) = 0 \quad \text{and} \quad H^1_c(U_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) = \mathrm{st}$$

Therefore,

$$R'(1) = \mathrm{st} - 1$$

Because there are no μ_{q+1}-fixed points, the trace formula tells us that,

$$\mathrm{tr}\,(\zeta | H^2_c(C)) - \mathrm{tr}\,(\zeta | H^1_c(C)) = 0$$

This characterizes the regular representation of μ_{q+1}. So the character of the virtual representation, $H^1_c(C) - H^2_c(C)$ is a multiple of the regular representation of μ_{q+1}.

If we then apply $[\theta]$ for $\theta \neq 1$ we get an actual representation since $H^2_c(C)$ is trivial as an $\mathrm{SL}_2(\mathbb{F}_q)$-representation. The degree of $H^1_c(C)[\theta]$ us then the same as the degree of $H^1_c(C)[1] - H^2_c(C)[1] = \mathrm{st} - 1$ which has dimension $q - 1$. This argument works because this virtual character is the same as the regular representation and thus contains every irrep with equal degree.

Theorem 5.0.2. If $\theta \neq 1$ then $H^1_c(C_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\theta]$ is cuspidal.

Proof. Consider,

$$U = \left\langle \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(\mathbb{F}_q)$$

Then,

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} T \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} B \to \mathrm{Rep} \overline{\mathbb{Q}}_\ell \mathrm{SL}(\mathbb{F}_q)$$

where the first map is given by quotienting by U and the second by induction. To show that our given representation is orthogonal to the image, it suffices to show it restricted to B is orthogonal to $\mathrm{Rep}\overline{\mathbb{Q}}_\ell T$. Therefore, it suffices to show that,

$$(H^1_c(C)[\theta])_U = (H^1_c(C)[\theta])^U = 0$$

So we need to understand $H^1_c(C/U, \overline{\mathbb{Q}}_\ell)$ with the action on μ_{q+1}. What is the quotient by U. Notice that,

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot (x, y) = (x + by, y)$$

so we expect that $C \to \mathbb{G}_m$ sending $(x, y) \mapsto y$ is the quotient map with fiber $\mathbb{F}_q$. □

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Since the actions of SL_2 and μ_{q+1} commute we see that,

$$(H^1_c(C)[\theta])_U = (H^1_c(C)[\theta])^U = (H^1_c(C))^U[\theta] = H^1_c(C/U)[\theta]$$

Lemma 6.0.1. The map $f : C \to \mathbb{A}^1 \backslash \{0\}$ via $(x, y) \mapsto y$ induces the quotient by U.

Proof. The map is U invariant. Now the action is $(x, y) \mapsto (x + by, y)$. Surjectivity, given $y \neq 0$ there is always a root of,

$$x^q y - y^q x - 1 = 0$$

in $\overline{\mathbb{F}}_q$. Given two solutions we need to show they are related by the action. Given two solutions x_1, x_2 we want to find b such that $x_2 = x_1 + b$. Let $b = y^{-1}(x_2 - x_1) \in \overline{\mathbb{F}}_q$ this is the unique choice of b. Thus we need to show that $b \in \mathbb{F}_q$. This is equivalent to showing that $b^q = b$. Indeed,

$$b^q = y^{-q}(x_2^q - x_1^q) = y^{-(q+1)} y (x_2^q - x_1^q) = y^{-(q+1)} y^q (x_2 - x_1) = b$$

using the defining equations. Now some algebraic geometry facts will tell us $C/U \to \mathbb{A}^1 \backslash \{0\}$ is an isomorphism. □

Now,

$$H^i_c(\mathbb{A}^n) = \begin{pmatrix} 1 & i = 2n \\ 0 & i \neq 2n \end{pmatrix}$$

Furthermore, the μ_{q+1} action on $\mathbb{A}^n$ is trivial on cohomology. Since $\{0\} \hookrightarrow \mathbb{A}^1$ is μ_{q+1}-equivariant so by the LES all of the cohomology $H^i_c(\mathbb{A}^1 \backslash \{0\})$ has the trivial μ_{q+1}-reepresentation.

Corollary 6.0.2. For $\theta \neq 1$ then $H^1_c(C)[\theta]$ is cuspidal.

Remark. Assume $p > 2$ then there are $q + 4$ irreps of $\mathrm{SL}_2(\mathbb{F}_q)$ we have already found $\frac{q+5}{2}$ of them, missing $\frac{q+3}{2}$ of them. However, there are q nontrivial θ. Notice that,

$$\frac{q+3}{2} = \frac{q-1}{2} + 2$$

We claim that θ and θ^{-1} give the same irrep and θ of order two gives two irreps.

Remark. The map Frob : $C \to C$ is $\mathrm{SL}_2(\mathbb{F}_q)$-equivariant but not μ_{q+1}-equivariant since $(\zeta \cdot x, \zeta \cdot y) \mapsto (\zeta^q \cdot x^q, \zeta^q \cdots y^q) = (\zeta^{-1}x^q, \zeta^{-1}y^q)$ so F indues an μ_{q+1}-invariant map Frob : $C \to C'$ where C' is given the inverse μ_{q+1}-representation. Thus F induces and $\mathrm{SL}_2(\mathbb{F}_q)$-equivariant isomorhism,

$$H^1_c(C) \to H^1_c(C)$$

which takes $H^1_c(C)[\theta] \xrightarrow{\sim} H^1_c(C)[\theta^{-1}]$. Next, Mackey formula.

Theorem 6.0.3 (Geometric Mackey formula)**.** Let θ_1, θ_2 be nontrivial then,

$$\left\langle H^1_c(C)[\theta_1], H^1_c(C)[\theta_2] \right\rangle_{\mathrm{SL}_2} = \langle \theta_1, \theta_2 \rangle_{\mu_{q+1}} + \left\langle \theta_1, \theta_2^{-1} \right\rangle_{\mu_{q+1}}$$

Proposition 6.0.4. We have isomorphisms as $\mathrm{SL}_2(\mathbb{F}_q)$-representations,

$$H^1_c(C)[\theta_1] \cong H^1_c(C)[\theta_1^{-1}] \cong (H^1_c(C)[\theta_1])^\vee$$

Define $R'(\theta) = H^1_c(C)[\theta] - H^2_c(C)[\theta]$. Then we have, using duality,

$$\langle R'(\theta_1), R'(\theta_2) \rangle = \langle 1, R'(\theta_1) \otimes R'(\theta_2) \rangle = \dim(H^1_c(C)[\theta_1] \otimes H^1_c(C)[\theta_2])^{\mathrm{SL}_2(\mathbb{F}_q)}$$

Now we write,

$$H^*_c(X) := \sum_{i=0}^{2\dim X} (-1)^i H^i_c(X)$$

for the virtual representation. This behaves well with respect to Kunneth. Then consider,

$$H^*_c(C \times C)[\theta_1 \times \theta_2]^{\mathrm{SL}_2(\mathbb{F}_q)}$$

We want to compute,

$$H^*_c(C \times C/\mathrm{SL}_2(\mathbb{F}_q))$$

as a virtual $\mu_{q+1} \times \mu_{q+1}$-represetation.

Let $Z = C \times C \subset \mathbb{A}^4$. Then we decompose $Z = Z_0 \cup Z_{\neq 0}$ where Z_0 is cut out by $xw - yz = 0$.

Lemma 6.0.5. Z_0 is $\mu_{q+1} \times \mu_{q+1} \times \mathrm{SL}_2(\mathbb{F}_q)$-stable.

Proof. This is clear for the $\mu_{q+1} \times \mu_{q+1}$. Then, we can compute,

$$xw - yz \mapsto (ax + by)(cz + dw) - (cx + dy)(az + bw) = xw - yz$$

□

7 Jan 25

Deligne-Lustzig induction for SL_2 part IV.

We have the Drinkfeld curve C and need to prove the Mackey formula for $H^1_c(C)[\theta]$ for $\theta : \mu_{p+1} \to \overline{\mathbb{Q}}_\ell^\times$. Geometrically, this mseans understanding,

$$H^*_c(C \times C/\mathrm{SL}_2(\mathbb{F}_q))$$

as a virtual representation of $\mu_{q+1} \times \mu_{q+1}$. We broke up,

$$C \times C = Z_0 \cup Z_{\neq 0}$$

into $\mathrm{SL}_2(\mathbb{F}_Q) \times \mu_{q+1}$ stable parts. We showed last time that,

$$Z_{\neq 0}/G \sim \{U^{q+1} - ab = 1\} \subset \mathbb{A}^1 \backslash \{0\} \times \mathbb{A}^2$$

with action,

$$(\zeta_1, \zeta_2) \cdot (u, a, b) = (\zeta_1 \zeta_2 U, \zeta_1 \zeta_2^{-1} a, \zeta_1^{-1} \zeta_2 b)$$

Question is, how to compute $H_c^*(V)$ as a virtual representation. This is equivalent to computing traces,

$$\mathrm{Tr}\left((\zeta_1, \zeta_2) \mid H_c^*(V)\right)$$

Use the $\mathbb{G}_m$-action $\lambda \cdot (U, a, b) = (U, \lambda^{-1} a, \lambda b)$ and compare to $\mathrm{Tr}\left(- \mid H_c^*(V^{\mathbb{G}_m})\right)$.

Proposition 7.0.1. Since V is affine $\exists t \in \mathbb{G}_m(\overline{\mathbb{F}}_q)$ such that $V^{\mathbb{G}_m} = V^t$.

Proposition 7.0.2. If γ is a finite-order automorphism of a variety, $\gamma = su$ with u having p-power order and s prime-to-p-order and $su = us$. Then,

$$\mathrm{Tr}\left(\gamma \mid H_c^*(V)\right) = \mathrm{Tr}\left(u \mid H_c^*(V^s)\right)$$

Lemma 7.0.3. Suppose that $\Gamma \times \mathbb{G}_m$ acts on an affine variety V. Then,

$$\mathrm{Tr}\left(\gamma \mid H_c^*(V)\right) = \mathrm{Tr}\left(\gamma \mid H_c^*(V^{\mathbb{G}_m})\right)$$

Proof. Choose $t \in \mathbb{G}_m(\overline{\mathbb{F}}_q)$ such that $V^t = V^{\mathbb{G}_m}$. Then for each $\gamma \in \Gamma$ we have,

$$\mathrm{Tr}\left(\gamma \mid H_c^*(V^{\mathbb{G}_m})\right) = \mathrm{Tr}\left(\gamma \mid H_c^*(V^t)\right)$$

Then write $\gamma = su$ as before. We see that,

$$\mathrm{Tr}\left(\gamma \mid H_c^*(V^{\mathbb{G}_m})\right) = \mathrm{Tr}\left(u \mid (V^t)^s\right) = \mathrm{Tr}\left(u \mid (V^s)^t\right)$$

Then let $g = ut$ and t has prime-to-p-order so we get,

$$\mathrm{Tr}\left(u \mid (V^s)^t\right) = \mathrm{Tr}\left(g \mid V^s\right)$$

However, $\mathbb{G}_m$ acts trivially on cohomology since $\mathbb{G}_m$ is connected. Therefore,

$$\mathrm{Tr}\left(g \mid V^s\right) = \mathrm{Tr}\left(u \mid V^s\right) = \mathrm{Tr}\left(us \mid V\right) = \mathrm{Tr}\left(\gamma \mid V\right)$$

$\square$

Now we apply this to $V \subset \mathbb{A}^1 \backslash \{0\} \times \mathbb{A}^2$ with $\mathbb{G}_m$-action is $\lambda \cdot (U, a, b) = (U, \lambda^{-1} a, \lambda b)$. Then $V^{\mathbb{G}_m} = \mu_{q+1} \times \{0\} \times \{0\}$ with $\mu_{q+1} \times \mu_{q+1}$ acting via,

$$(\zeta_1, \zeta_2) \cdot \zeta = \zeta_1 \zeta_2 \zeta$$

Now consider,

$$Z_0 \subset C \times C \subset \mathbb{A}^4$$

cut out by the equations,

$$\begin{aligned} xy^q - yx^q &= 1 \\ zt^q - tz^q &= 1 \\ xt - yz &= 0 \end{aligned}$$

Lemma 7.0.4. The map $\mu_{q+1} \times C \to Z_0$ given by,

$$(\zeta, x, y) \mapsto (x, y, \zeta x, \zeta y)$$

is a $\mathrm{SL}_2(\mathbb{F}_q)$-equivariant isomorphism.

Proof. Given $(x, y) \in C$ we want to show there are at most $q+1$ options for (z, t) s.t $(x, y, z, t) \in Z_0(\overline{\mathbb{F}}_q)$. Write,

$$t = \frac{yz}{x}$$

and then z to satisfy,

$$z^{q+1}\left(\frac{y}{x}\right)^{q-1} z^{q+1}\left(\frac{y}{x}\right) = 1$$

which has $q+1$ roots. Therefore φ is a bijection on $\overline{\mathbb{F}}_q$-points. We can easily verify smoothness and then conclude. □

Corollary 7.0.5. $Z_0/G \cong \mu_{q+1} \times \mathbb{A}^1$ with $\mu_{q+1} \times \mu_{q+1}$ acting via,

$$(\zeta_1, \zeta_2) \cdot (\zeta, z) = (\zeta_1^{-1}\zeta_2\zeta, \zeta_1^2 z)$$

Theorem 7.0.6 (Mackey)**.** Let θ_1, θ_2 be nontrivial characters of μ_{q+1}. Then the pairing,

$$\left\langle H_c^1(C)[\theta_1], H_c^1(C)[\theta_2]\right\rangle_{\mathrm{SL}_2(\mathbb{F}_q)} = \langle \theta_1, \theta_2\rangle_{\mu_{q+1}} + \left\langle \theta_1, \theta_2^{-1}\right\rangle_{\mu_{q+1}}$$

Proof. As discussed before,

$$\langle -, - \rangle = \dim H_c^*(C \times C)^{\mathrm{SL}_2(\mathbb{F}_q)}[\theta_1 \times \theta_2]$$

We can break this up into,

$$\dim H_c^*(Z_0)^{\mathrm{SL}_2(\mathbb{F}_q)}[\theta_1 \times \theta_2] + \dim H_c^*(Z_{\neq 0})^{\mathrm{SL}_2(\mathbb{F}_q)}[\theta_1 \times \theta_2]$$

which equals,

$$= \dim H_c^*(Z_0/\mathrm{SL}_2(\mathbb{F}_q))[\theta_1 \times \theta_2] + \dim H_c^*(Z_{\neq 0}/\mathrm{SL}_2(\mathbb{F}_q))[\theta_1 \times \theta_2]$$

which is by our computations,

$$\mathrm{Ind}_{\mu_{q+1}^{(1)}}^{\mu_{q+1}\times\mu_{q+1}}(1)\,[\theta_1 \times \theta_2] + \dim \mathrm{Ind}_{\mu_{q+1}^{(2)}}^{\mu_{q+1}\times\mu_{q+1}}(1)\,[\theta_1 \times \theta_2]$$

where the first is embedded by the diagonal and the second by the anti-diagonal. By Frobenius reciprocity,

$$= \langle 1, \theta_1 \otimes \theta_2\rangle_{\mu_{q+1}} + \left\langle 1, \theta_1 \otimes \theta_2^{-1}\right\rangle_{\mu_{q+1}}$$

□

Corollary 7.0.7. $H_c^1(C)[\theta]$ is an irrep of $\dim = q-1$ if $\theta^2 \neq 1$. Then,

$$-H_c^1(C)[\theta_0] = (C)_+ + (C)_-$$

is a sum of two irreps. By a counting arugment this has produced all the irreps for $p > 2$.

Remark. We can also reinterpret parabolic induction in terms of Deligne-Lustzig induction. Indeed,

$$H_c^0\left(\frac{\mathrm{SL}_2(\mathbb{F}_q)}{U(\mathbb{F}_q)}\right)[\alpha]$$

gives the parabolic induction so we consider $\frac{\mathrm{SL}_2(\mathbb{F}_q)}{U(\mathbb{F}_q)}$ a 0-dimensional variety with $\mathrm{SL}_2(\mathbb{F}_q) \times \mu_{q-1}$-formula.

8 Jan 27

$\mathrm{GL}_3(\mathbb{F}_q)$ does act on F but since the action extends to GL_3 nothing interesting happens on cohomoloy. Therefore we need a different construction.

8.1 A general theory of "relative position"

Either we choose the condition,

$$(L_1 \subsetneq L_2, L_1^* \subsetneq L_2^*)$$

We want a relative position map,

$$F(\overline{\mathbb{F}}_q) \times F(\overline{\mathbb{F}}_q) \to Q$$

For $\mathbb{P}^1$ we have,

$$\mathbb{P}^1(\overline{\mathbb{F}}_q) \times \mathbb{P}^1(\overline{\mathbb{F}}_q) \to \{0, 1\}$$

where this measures just if two lines are equal. However, this cannot be made an algebraic map because there are connectivity issues.

A better way is to use the Bruhat decomposition: there are 2 left B-orbits on G/B. In the above case, there are 2 left B-orbits on $\mathbb{P}^1$. There are also 2 left G-orbits on $\mathbb{P}^1 \times \mathbb{P}^1$ or $G \backslash C^G/G \times G/B = B \backslash G/B$.

In general, for GL_3 want to look at $\mathrm{GL}_3 \backslash F \times F/\mathrm{GL}_3$.

Exercise 8.1.1. Let S_n be the symmetric group and conflate $\sigma \in S_n$ with the corresponding permutation matrix in GL_n. Let B be the upper triangular Borel. Then,

$$\mathrm{GL}_n = \bigsqcup_{\sigma \in S_n} B\sigma B$$

Geometrically, this translates to,

$$\mathrm{GL}_n/B \times \mathrm{GL}_n/B = \bigsqcup_{\sigma \in S_n} O(s)$$

where $O(s)$ is the GL_n-orbit of $(1, \sigma)$.
