Mathematics GU4053 Algebraic Topology
Assignment # 8

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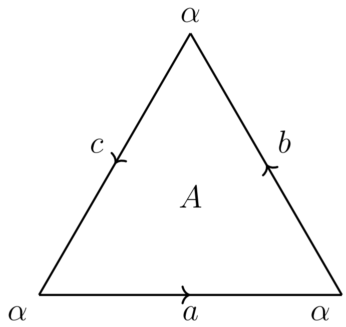
Problem 1.



Let $X = \Delta^2$ with all the vertices identified at α. Call the three "faces" of Δ^2 which are 1-simplices, a, b, and c. Finally, call the filled 2-simplex A. Now consider the chain complex,

$$0 \xrightarrow{\partial_3} \mathbb{Z}A \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial_1} \mathbb{Z}\alpha \xrightarrow{\partial_0} 0$$

The boundary maps take $\partial_2 A = a + b + c$ and $\partial_1 = 0$ because there is only one vertex so the endpoints of all 1-simplicies are the same. Now, we can calculate the homology of this complex,

$$H_0(\Delta) = \mathbb{Z}\alpha / \{0\} \cong \mathbb{Z}$$

$$H_1(\Delta) = \ker \partial_1 / \mathrm{Im}(\partial_2) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c / \langle (1,1,1) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

and finally,

$$H_2(\Delta) = \ker \partial_2 / \mathrm{Im}(\partial_3) = 0$$

Problem 2.

Let X be the Δ-complex formed by taking Δ^n and identifying all faces of the same dimension. Call α^k the single k-simplex of X. Therefore, the chain complex of X becomes,

$$0 \xrightarrow{\partial_{n+1}} \mathbb{Z}\alpha^n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_5} \mathbb{Z}\alpha^4 \xrightarrow{\partial_4} \mathbb{Z}\alpha^3 \xrightarrow{\partial_3} \mathbb{Z}\alpha^2 \xrightarrow{\partial_2} \mathbb{Z}\alpha^1 \xrightarrow{\partial_1} \mathbb{Z}\alpha^0 \xrightarrow{\partial_0} 0$$

Since a k-simplex has $k+1$ faces the boundary map acts as,

$$\partial_k \alpha^k = \sum_{i=0}^{k} (-1)^k \alpha^{k-1} = \begin{cases} \alpha^{k-1} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

Therefore,

$$\partial_k = \begin{cases} \phi_k & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

where $\phi_k : \mathbb{Z}\alpha^k \to \mathbb{Z}\alpha^{k-1}$ is the map taking the generator to the generator and thus is the identity when these groups are viewed as the abstract cyclic group $\mathbb{Z}$. Now, consider the homology of this complex for $0 < k < n$. If k is even,

$$H_k(X) = \ker \partial_k / \mathrm{Im}(\partial_{k+1}) = \{0\}/\{0\} = 0$$

Furthermore, if k is odd,

$$H_k(X) = \ker \partial_k / \mathrm{Im}(\partial_{k+1}) = \mathbb{Z}\alpha^k / \mathbb{Z}\alpha^k = 0$$

Therefore for $k < n$ we have $H_k(X) = 0$. We must now check the edge cases. For $k = 0$,

$$H_0(X) = \mathbb{Z}\alpha^0 / \{0\} \cong \mathbb{Z}$$

which reflects the fact that X is connected. Finally, for $k = n$ we have,

$$H_n(X) = \ker \partial_n / \mathrm{Im}(\partial_{n+1}) \cong \ker \partial_n \cong \begin{cases} \mathbb{Z} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Problem 3.

Suppose that A is a retract of X then there exist maps $\iota : A \hookrightarrow X$ and $r : X \to A$ such that ι is the inclusion and $r \circ \iota = \mathrm{id}_A$. Since H_n is a functor, $(r \circ \iota)_* = r_* \circ \iota_* = (\mathrm{id}_A)_* = \mathrm{id}_{H_n(A)}$. Therefore, $\iota_* : H_n(A) \to H_n(X)$ is an injection and $r_* : H_n(X) \to H_n(A)$ is a surjection.

Problem 4.

Let $f : C \to D$ be an isomorphism in the category of $\mathbf{Ch}(\mathbf{Ab})$. Therefore there is a map $g : D \to C$ such that $f \circ g = \mathrm{id}_D$ and $g \circ f = \mathrm{id}_C$. Therefore, $(g \circ g)_n = g_n \circ f_n = (\mathrm{id}_C)_n = \mathrm{id}_{C_n}$. Similarly, $(f \circ g)_n = f_n \circ g_n = (\mathrm{id}_D)_n = \mathrm{id}_{D_n}$. Therefore each $f_n : C_n \to D_n$ is an isomorphism of groups.

Conversely, suppose that we have a sequence of isomorphisms of groups $f_n : C_n \to D_n$. Therefore, we have maps $g_n : C_n \to D_n$ such that $g_n \circ f_n = \mathrm{id}_{C_n}$ and $f_n \circ g_n = \mathrm{id}_{D_n}$. Therefore we have the commutative diagram,

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+3}"] & C_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d, "f_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r, "\partial_{n-2}"] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r, "\partial_{n+3}"] & D_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d, "g_{n+2}"] & D_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "g_{n+1}"] & D_n \arrow[r, "\partial_n"] \arrow[d, "g_n"] & D_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "g_{n-1}"] & D_{n-2} \arrow[r, "\partial_{n-2}"] \arrow[d, "g_{n-2}"] & \cdots \\
\cdots \arrow[r, "\partial_{n+3}"] & C_{n+2} \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] & C_{n-2} \arrow[r, "\partial_{n-2}"] & \cdots
\end{tikzcd}$$

The squares commute because,

$$\begin{aligned}(g_n \circ f_n) \circ \partial_{n+1} &= g_n \circ (f_n \circ \partial_{n+1}) = g_n \circ (\partial_{n+1} \circ f_{n+1}) \\ &= (g_n \circ \partial_{n+1}) \circ f_{n+1} = \partial_{n+1} \circ (g_{n+1} \circ f_{n+1})\end{aligned}$$

However, $g_n \circ f_n = \mathrm{id}_{C_n}$ so $g \circ f = \mathrm{id}_C$. An identical argument shows that $f \circ g = \mathrm{id}_D$. Therefore, f is an isomorphism in the category $\mathbf{Ch}(\mathbf{Ab})$.

Problem 5.

Suppose we have a complex A given by,

$$\cdots \xrightarrow{\partial_7} A_6 \xrightarrow{\partial_6} A_5 \xrightarrow{\partial_5} A_4 \xrightarrow{\partial_4} A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0$$

such that each boundary map is zero i.e. $\partial_i = 0$. Therefore, the homology of this complex is,

$$H_i(A) = \ker \partial_i / \mathrm{Im}(\partial_{i+1}) = A_i/\{e\} \cong A_i$$

because the kernel of the zero map is the entire domain and the image of the zero map is trivial.

Problem 6.

Consider two maps $f, g : A \to B$ in the category $\mathbf{Ch}(\mathbf{Ab})$. Then we can take the sum $f + g$ to be given by the component morphisms $(f+g)_n = f_n + g_n$ of abelian groups. Now, consider the action of this map on the homology groups, $f_* : H_n(A) \to H_n(B)$ takes,

$$\begin{aligned}(f+g)_*(\alpha \mathrm{Im}(\partial^A_{i+1})) &= (f+g)_n(\alpha)\mathrm{Im}(\partial^B_{i+1}) = (f_n + g_n)(\alpha)\mathrm{Im}(\partial^B_{i+1}) = f_n(\alpha)\mathrm{Im}(\partial^B_{i+1}) + g_n(\alpha)\mathrm{Im}(\partial^B_{i+1}) \\ &= f_*(\alpha \mathrm{Im}(\partial^A_{i+1})) + g_*(\alpha \mathrm{Im}(\partial^A_{i+1}))\end{aligned}$$

Theefore $(f+g)_* = f_* + g_*$. In other notation, $H_n(f+g) = H_n(f) + H_n(g)$.

Problem 7.

Let $f : C \to D$ be a morphism in $\mathbf{Ch}(\mathbf{Ab})$. Define the kernel and image of f as the complexes $\ker f$ and $\mathrm{Im}(f)$ given by,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\partial_{n+3}"] & \ker f_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d] & \ker f_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d] & \ker f_n \arrow[r, "\partial_n"] \arrow[d] & \ker f_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & \ker f_{n-2} \arrow[r, "\partial_{n-2}"] \arrow[d] & \cdots \\
\cdots \arrow[r, "\partial_{n+3}"] & C_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d, "f_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r, "\partial_{n-2}"] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r, "\partial_{n+3}"] & \mathrm{Im}(f_{n+2}) \arrow[r, "\partial_{n+2}"] \arrow[d] & \mathrm{Im}(f_{n+1}) \arrow[r, "\partial_{n+1}"] \arrow[d] & \mathrm{Im}(f_n) \arrow[r, "\partial_n"] \arrow[d] & \mathrm{Im}(f_{n-1}) \arrow[r, "\partial_{n-1}"] \arrow[d] & \mathrm{Im}(f_{n-2}) \arrow[r, "\partial_{n-2}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where the boundary maps are restricted to the groups $\ker f_n$ and $\mathrm{Im}(f_n)$. These maps are well-defined because if $f_{n+1}(x) = 0$ then $\partial_{n+1} \circ f_{n+1}(x) = f_n \circ \partial_n(x) = 0$. Therefore $\partial_{n+1} \ker f_{n+1} \subset \ker f_n$. Furthermore, if $y \in \mathrm{Im}(f_{n+1})$ then $y = f_{n+1}(x)$ we have $\partial_{n+1}(y) = \partial_{n+1} \circ f_{n+1}(x) = f_n \circ \partial_{n+1}(x)$ so $\partial_{n+1}(y) \in \mathrm{Im}(f_n)$ so $\partial_{n+1}\mathrm{Im}(f_{n+1}) \subset \mathrm{Im}(f_n)$. Furthermore, the property $\partial_n \circ \partial_{n+1} = 0$ holds for the restricted maps so each row is a complex.

From the fundamental theorem of group homomorphisms, we can form a short exact sequence in each column and an isomorphism $\phi_n : C_n/\ker f_n \to \mathrm{Im}(f_n)$. Therefore there is a morphisms of complexes $\phi : C/\ker f \to \mathrm{Im}(f)$ given by,

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial_{n+3}} & C_{n+2}/\ker f_{n+2} & \xrightarrow{\partial_{n+2}} & C_{n+1}/\ker f_{n+1} & \xrightarrow{\partial_{n+1}} & C_n/\ker f_n & \xrightarrow{\partial_n} & C_{n-1}/\ker f_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2}/\ker f_{n-2} & \xrightarrow{\partial_{n-2}} & \cdots \\
 & & \downarrow{\scriptstyle \phi_{n+2}} & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} & & \\
\cdots & \xrightarrow{\partial_{n+3}} & \mathrm{Im}(f_{n+2}) & \xrightarrow{\partial_{n+2}} & \mathrm{Im}(f_{n+1}) & \xrightarrow{\partial_{n+1}} & \mathrm{Im}(f_n) & \xrightarrow{\partial_n} & \mathrm{Im}(f_{n-1}) & \xrightarrow{\partial_{n-1}} & \mathrm{Im}(f_{n-2}) & \xrightarrow{\partial_{n-2}} & \cdots
\end{array}$$

where the boundary map decends to the quotient because $\partial_{n+1} \ker f_{n+1} \subset \ker f_n$. Because,

$$\begin{aligned}
\partial_{n+1} \circ \phi_{n+1}(x \ker f_{n+1}) &= \partial_{n+1}(f_{n+1}(x) \ker f_{n+1}) = \partial_{n+1}(f_{n+1}(x)) \ker f_n \\
&= f_n(\partial_{n+1}(x)) \ker f_n = f_n \circ \partial_{n+1}(x \ker f_{n+1})
\end{aligned}$$

the squares commute so ϕ is a morphism in $\mathbf{Ch}(\mathbf{Ab})$. However, each ϕ_n is an isomorphism so by problem 4, $\phi : C/\ker f \to \mathrm{Im}(f)$ is an isomorphism of complexes.