1 Introduction

1.1 References

(a) A sampling of vector bundle techniques, Lazarsfeld.

1.2 Divisors

Remark. Let X be a projective variety over $k = \bar{k}$.

A divisor is a formal sum,

$$D = \sum a_i D_i$$

for $a_i \in \mathbb{Z}$ and D_i is a codimension 1 subvariety. We also will allow $a_i \in \mathbb{Q}$ or $\mathbb{R}$.

Definition 1.2.1. $N^1(X)_{\mathbb{R}} = \{\mathbb{R}\text{-divisors}\}/\sim$ where,

$$D_1 \sim D_2 \iff D_1 \cdot C = D_2 \cdot C$$

for all integral curves $C \subset X$.

Definition 1.2.2 (Ample)**.** A Line bundle $\mathcal{L}$ is *ample* if one of the following equivalent conditions hold,

(a) $\mathcal{L}^{\otimes m}$ (for some $m \geq 0$) is very ample meaning $\mathcal{L}$ defines an embedding $X \hookrightarrow \mathbb{P}^N$

(b) for any coherent sheaf $\mathscr{F}$ there exists $n(\mathscr{F})$ s.t. $m \geq n(\mathscr{F})$ implies $\mathscr{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated

(c) for any coherent sheaf $\mathscr{F}$ there exists $n(\mathscr{F})$ s.t. $m \geq n(\mathscr{F})$ implies $H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for all $i > 0$

(d) (over $\mathbb{C}$) positive in the sense of admiting a positive hermitian connection.

Theorem 1.2.3 (Nakai-Moishezon)**.** On X a line bundle $\mathcal{L}$ is ample if and only if

$$(\mathcal{L}^{\dim V} \cdot V) > 0$$

for all subvarieties $V \subset X$.

Definition 1.2.4. $\mathcal{L}$ is nef (numerically effective) if,

$$(\mathcal{L} \cdot C) \geq 0$$

for all curves $C \subset X$.

Theorem 1.2.5 (Kleiman)**.** If $\mathcal{L}$ is net then for any subvariety $V \subset X$,

$$\mathcal{L}^{\dim V} \cdot V \geq 0$$

Remark. However, $\mathcal{L} \cdot C > 0$ does not imply that $\mathcal{L}$ is ample meaning it does not imply that the intersection against all subvarities is positive.

Proposition 1.2.6. (a) non-negative linear combinations of nef divisors are nef.

(b) if $f : X \to Y$ is proper and $\mathcal{L}$ on Y is nef then $f^*\mathcal{L}$ is nef.

(c) if $f : X \to Y$ is surjective and proper and $f^*\mathcal{L}$ is nef then $\mathcal{L}$ is nef.

Corollary 1.2.7. (a) Let X be projective, D is a nef $\mathbb{R}$-divisor, and H is any ample $\mathbb{R}$-divisor. Then $D + \epsilon H$ is ample for all $\epsilon > 0$.

(b) fix $\mathbb{R}$-divisors D and H, if $(D + \epsilon H)$ is ample for all small $\epsilon > 0$ then D is nef.

Proof. For (2) we have,

$$D \cdot C = \lim_{\epsilon \to 0} (D + \epsilon H) \cdot C \geq 0$$

For (1) we need to show that,

$$(D + \epsilon H)^{\dim V} \cdot V > 0$$

for any subvariety $V \subset X$. Now,

$$(D + \epsilon H)^{\dim V} = [D^{\dim V} + \cdots + (\epsilon H)^{\dim V}] \cdot V$$

Since D is nef, all the intersections are ≥ 0 and $\epsilon^{\dim V} H^{\dim V} \cdot V > 0$ because $\epsilon > 0$ and H is ample and thus we conclude. $\square$

Proposition 1.2.8. Let $f : X \to T$ be surjective, proper and $\mathcal{L}$ is a line bundle on X. Suppose for some $t_0 \in T$, that L_{t_0} is nef on X_{t_0}. Then there exists a countable union of proper subvarities $B \subset T$ such that L_t is nef on X_t for all $t \notin B$.

Definition 1.2.9. The ample cone is,

$$\mathrm{Amp}(X) = \{D \in N^1(X)_{\mathbb{R}} \mid D \text{ is ample}\} \subset N^1(X)_{\mathbb{R}}$$

and the nef cone,

$$\mathrm{Nef}(X) = \{D \in N^1(X)_{\mathbb{R}} \mid D \text{ is nef}\} \subset N^1(X)_{\mathbb{R}}$$

The corollaries tell us that $\mathrm{Amp}(X)$ is an open convex cone and $\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$.

Example 1.2.10. $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then $N^1(X)_{\mathbb{R}} = \mathbb{R}\langle F_1, F_2 \rangle$ with basis $F_i = [\pi_i^{-1}(\mathrm{pt})]$. The F_i are both nef but $F_i^2 = 0$ so they are not ample. The ample cone is the first quadrant and the nef cone is the first quadrant plus the positive axes.

Example 1.2.11. Let E be an elliptic curve general in $\mathcal{M}_1$. Let $X = E \times E$. Then,

$$N^1(X)_{\mathbb{R}} = \mathbb{R}\langle F_1, F_2, \Delta \rangle$$

Claim: any effective class on $X = E \times E$ is nef. Indeed this is because we can freely move classes by translation until they intersect properly.

Lemma 1.2.12. Let $X = E \times E$. A class $\alpha \in N^1(X)_{\mathbb{R}}$ is nef iff $\alpha^2 \geq 0$ and $\alpha \cdot h \geq 0$ for some ample h.

Proposition 1.2.13. Let X be a surface and D an integral divisor s.t. $D^2 > 0$ and $(D \cdot H) > 0$ for some ample H, then mD is effective for some $m > 0$.

Proof. Consider,

$$\chi(X, mD) = \tfrac{1}{2}(mD) \cdot (mD - K_X) + \chi(\mathcal{O}_X)$$

Now since $D^2 > 0$ we can make $\chi(X, mD) \to \infty$ as $m \to \infty$. Furthermore, $h^2(X, mD) = h^0(X, K_X - mD) = 0$ for large enough m if $D \cdot H > 0$. Otherwise, there would be an effective $D' \sim K_X - mD$ and then $D' \cdot H > 0$ since H is ample but $D' \cdot H = K_X \cdot H - mD \cdot H < 0$ for large enough m since $D \cdot H > 0$. Therefore, we must have $h^0(X, mD) \to \infty$ as $m \to \infty$. $\square$

Remark. This proves the previous lemma using that the nef cone is closed and that eny effective class is nef.

Remark. Back to the example, let $\alpha = xF_1 + yF_2 + z\Delta$ and $h = F_1 + F_2 + \Delta$. Applying the lemma gives the inequalities of the nef cone,

$$x + y + z \geq 0 \quad xy + xz + yz \geq 0$$

This is a round cone.

1.3 Schedule

(a) Castelnuovo-Mumford regularity

(b) Introduction to Brill-Noether Theory

(c) Petri's condition and Brill-Noether Theory on K3 surfaces:

$$\mu_0 : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \to H^0(C, \omega_C)$$

for a line bundle A and a curve C when is this injective?

(d) Lazarsfeld-Mukai bundles on K3 surfaces.

(e) Proof of the Brill-Noether-Petri.

(f) $\dim = 2$ case of Fujita's conjecture.

(g) Moduli of sheaves on K3s? Other topics?

2 Mumford-Castounovo Regularity

Theorem 2.0.1 (Serre Vanishing)**.** Let $X \to \mathrm{Spec}\,(A)$ be proper and $\mathcal{L}$ ample on X. Then for any $\mathscr{F} \in \mathfrak{Coh}\,(X)$ there is some $n(\mathscr{F})$ such that for all $n \geq n(\mathscr{F})$ and $i > 0$,

$$H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

Remark. Today we want to quantify how the minimal $n(\mathscr{F})$ grows.

Definition 2.0.2. Let $X = \mathbb{P}^n_k$. Let $\mathscr{F} \in \mathfrak{Coh}\,(X)$ and $m \in \mathbb{Z}$. Then $\mathscr{F}$ is *m-regular* if,

$$H^i(X, \mathscr{F}(m - i)) = 0$$

for all $i > 0$. Then *the regularity* of $\mathscr{F}$ is,

$$\mathrm{reg}(\mathscr{F}) = \inf\{m \in \mathbb{Z} \mid \mathscr{F} \text{ is } m\text{-regular}\}$$

Remark. If $\mathscr{F}$ is supported on a finite set then $\operatorname{reg}(\mathscr{F}) = -\infty$. Otherwise $\operatorname{reg}\mathscr{F}$ is a finite number.

Example 2.0.3. Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}_X(-1,-3)$. Then,

$$H^i(X, \mathcal{L}) = 0$$

for all i by Kunneth since $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no cohomology. However,

$$\dim H^i(X, \mathcal{L}(1)) = \begin{cases} 0 & i \neq 1 \\ 1 & i = 1 \end{cases}$$

because,

$$H^1(X, \mathcal{L}(1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$$

The cohomology can all vanish but can jump up after a *positive* twist. However, $\operatorname{reg}(\mathcal{L}) = 3$ so after twisting three times the higher cohomology stays zero.

Example 2.0.4. $\mathscr{F} = \mathcal{O}_{\mathbb{P}^n}(a)$ is $(-a)$-regular. If $X \subset \mathbb{P}^n$ is a degree d-hypersurface then $\iota_* \mathcal{O}_X$ is $(d-1)$-regular.

Proposition 2.0.5. Let $\mathscr{F} \in \mathfrak{Coh}(X)$ be m-regular. Then for $k \geq 0$,

(a) $\mathscr{F}$ is $(m+k)$-regular

(b) $\mathscr{F}(m+k)$ is generated by global sections

(c) the natural map,

$$H^0(X, \mathscr{F}(m)) \otimes H^0(X, \mathcal{O}_X(k)) \twoheadrightarrow H^0(X, \mathscr{F}(m+k))$$

is surjective.

Proof. By flat base change, we can ssume that k is algebraically closed. Then we do induction on $n = \dim X$. For $\mathscr{F} \in \mathfrak{Coh}(X)$ the support $\operatorname{Supp}(\mathscr{F})$ is a closed subscheme so it has finitely many components and hence there exists a hyperplane mising each generic point (using that k is infinite). Therefore, we get an exact sequence,

$$0 \longrightarrow \mathscr{F}(-1) \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

with $\mathscr{G} = \iota_*(\mathscr{F}|_H)$ supported on $H \cong \mathbb{P}^{n-1}$. When $n = 0$ the statements are obvious. By the sequence, if $\mathscr{F}$ is m-regular then $\mathscr{G}$ is m-regular. By the induction hypothesis, $\mathscr{G}$ is $(m+k)$-regular. Thus for $i > 0$ and $k \geq 0$ we have $H^i(X, \mathscr{G}(m+k-i)) = 0$ so if $H^i(X, \mathscr{F}(m+k-1-i)) = 0$ then $H^i(X, \mathscr{F}(m+k-i)) = 0$ so if $\mathscr{F}$ is $(m+(k-1))$-reglar then $\mathscr{F}$ is $(m+k)$-regular so by induction $\mathscr{F}$ is $(m+k)$-regular for all $k \geq 0$ proving (a). Now, consider the diagram,

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathscr{F}(m+k-1)) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & H^0(\mathscr{F}(m+k)) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & H^0(\mathscr{G}(m+k)) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}(m+k-1) \arrow[r] & \mathscr{F}(m+k) \arrow[r] & \mathscr{G}(m+k) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathscr{F}$ is $(m+k)$-regular $H^1(X, \mathscr{F}(m+k-1)) = 0$ so the top sequence is short exact. By the induction hypothesis, for all $k \geq 0$ the map $H^0(\mathscr{G}(m+k)) \otimes \mathcal{O}_X \twoheadrightarrow \mathscr{G}(m+k)$ is surjective (on H this is the induction hypothesis and outside H this hold because $\mathscr{G}$ vanishes). By Serre, there is some $k \gg 0$ such that $\mathscr{F}(m+k)$ is globally generated and thus by downward induction we see that $\mathscr{F}(m+k)$ is globally generated for all $k \geq 0$ proving (b). Then consider the diagram,

$$\begin{tikzcd}
 & 0 \arrow[d] \\
H^0(X, \mathscr{F}(m)) \otimes H^0(X, \mathcal{O}_X(k-1)) \arrow[r] \arrow[d] & H^0(X, \mathscr{F}(m+k-1)) \arrow[d] \\
H^0(X, \mathscr{F}(m)) \otimes H^0(X, \mathcal{O}_X(k)) \arrow[r] \arrow[d, two heads] & H^0(X, \mathscr{F}(m+k)) \arrow[d, two heads] \\
H^0(\mathscr{G}(m)) \otimes H^0(H, \mathcal{O}_H(k)) \arrow[r, two heads] & H^0(H, \mathscr{G}(m+k)) \arrow[d] \\
 & 0
\end{tikzcd}$$

By induction on n the bottom map is surjective. The bottom downward maps are surjective because $\mathscr{F}$ and $\mathscr{G}$ are m-regular so $H^1(X, \mathscr{F}(m-1)) = 0$ and likewise for $\mathscr{G}$. Now we use induction on k. The case $k = 0$ is clear so we can assume that the top map is surjective and thus the middle map is also surjective completing the induction step. Therefore,

$$H^0(X, \mathscr{F}(m)) \otimes H^0(X, \mathcal{O}_X(k)) \twoheadrightarrow H^0(X, \mathscr{F}(m+k))$$

is surjective proving (c). $\square$

Proposition 2.0.6. Given an exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

Then,

(a) if $\mathscr{F}_1$ and $\mathscr{F}_3$ are m-regular then $\mathscr{F}_2$ is m-regular

(b) if $\mathscr{F}_1$ is $(m+1)$-regular and $\mathscr{F}_2$ is m-regular then $\mathscr{F}_3$ is m-regular

(c) if $\mathscr{F}_2$ is m-regular and $\mathscr{F}_3$ is $(m-1)$-regular then $\mathscr{F}_1$ is m-regular

(d) $\text{reg}(\mathscr{F}_1) \leq \max\{\text{reg}(\mathscr{F}_2), \text{reg}(\mathscr{F}_3) + 1\}$

(e) $\text{reg}(\mathscr{F}_2) \leq \max\{\text{reg}(\mathscr{F}_1), \text{reg}(\mathscr{F}_2)\}$

(f) $\text{reg}(\mathscr{F}_3) \leq \max\{\text{reg}(\mathscr{F}_1) - 1, \text{reg}(\mathscr{F}_2)\}$

Proof. Consider the long exact sequence,

$$H^i(X, \mathscr{F}_1(m-i)) \longrightarrow H^i(X, \mathscr{F}_2(m-i)) \longrightarrow H^i(X, \mathscr{F}_3(m-i)) \longrightarrow H^{i+1}(X, \mathscr{F}_1(m-i))$$

DO THIS $\square$

Proposition 2.0.7. Consider a coherent resolution,

$$0 \longrightarrow \mathscr{F}_n \xrightarrow{\mathrm{d}_n} \mathscr{F}_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \longrightarrow \cdots \longrightarrow \mathscr{F}_1 \xrightarrow{\mathrm{d}_1} \mathscr{F}_0 \xrightarrow{\mathrm{d}_0} \mathscr{F} \longrightarrow 0$$

with each $\mathscr{F}_j$ is $(m+j)$-regular. Then $\mathscr{F}$ is m-regular and $H^0(X, \mathscr{F}_0(m)) \twoheadrightarrow H^0(\mathscr{F}(m))$.

Proof. Given $H^i(X, \mathscr{F}_j(m+j-i)) = 0$ for all $i > 0$ and $j \geq 0$. We want to show that $H^i(X, \mathscr{F}(m-i)) = 0$. DO THIS $\square$

Proposition 2.0.8. A coherent sheaf $\mathscr{F} \in \mathfrak{Coh}(X)$ is m-regular iff there exists a resolution,

$$0 \longrightarrow \mathcal{O}_X(-m-(n+1))^{\oplus a_{n+1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_X(-m-1)^{\oplus a_1} \longrightarrow \mathcal{O}_X(-m)^{\oplus a_0} \longrightarrow \mathscr{F} \longrightarrow 0$$

Proposition 2.0.9. Let $\mathscr{F} \in \mathfrak{Coh}(X)$ and $\mathcal{E}$ a vector bundle. If $\mathscr{F}$ is m-regular and $\mathcal{E}$ is ℓ-regular then $\mathscr{F} \otimes \mathcal{E}$ is $(m+\ell)$-regular.

Proof. We apply the resultion property to $\mathscr{F}$ and then applying $- \otimes \mathcal{E}$ gives a resultion of $\mathscr{F} \otimes \mathcal{E}$ since $\mathcal{E}$ is flat. Then we apply the previous proposition. $\square$

Corollary 2.0.10. If $\mathcal{E}$ is a an m-regular vector bundle then,

(a) $\mathcal{E}^{\otimes r}$

(b) $\bigwedge^r \mathcal{E}$

(c) $S^r \mathcal{E}$ (for characteristic zero).

all are (rm)-regular.

Proof. Regularity of $\mathcal{E}^{\otimes r}$ is immediate. Then consider the exact sequence,

$$0 \longrightarrow I \longrightarrow \mathcal{E}^{\otimes r} \longrightarrow \textstyle\bigwedge^r \mathcal{E} \longrightarrow 0$$

which has a section (CHECK) $\square$

Definition 2.0.11. Let X be a projective variety and $\mathcal{L}$ a globally generated line bundle. Then $\mathscr{F}$ is m-regular with respect to $\mathcal{L}$ if,

$$H^i(X, \mathscr{F} \otimes \mathcal{L}^{\otimes(m-i)}) = 0$$

for all $i > 0$.

Proposition 2.0.12 (Green's Theorem)**.** Let $W \subset H^0(X, \mathcal{O}_X(d))$ be a codimension n basepoint-free linear system. Then for $k \geq c$,

$$\zeta_k : W \otimes H^0(X, \mathcal{O}_X(k)) \twoheadrightarrow H^0(X, \mathcal{O}_X(d+k))$$

is surjective.

Proof. Consider $W \otimes \mathcal{O}_X$ then there is a map,

$$W \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X(d)$$

which is surjective as a map of sheaves since W is base-point free. Let $\mathcal{M}_W$ be its kernel. Then surjectivity is equivalent to $H^1(X, \mathcal{M}_W(k)) = 0$. Similarly, define,

$$0 \longrightarrow \mathcal{M}_d \longrightarrow H^0(X, \mathcal{O}_X(d)) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(d) \longrightarrow 0$$

Wind that $\mathcal{M}_d$ is 1-regular and $\bigwedge^k \mathcal{M}_d$ is k-regular. Since $\operatorname{codim}(W) = c$ we have,

$$0 \longrightarrow \mathcal{M}_W \longrightarrow \mathcal{M}_d \longrightarrow \mathcal{O}_X^{\otimes c} \longrightarrow 0$$

Using the Egan-Northcott complex we have $\mathcal{M}_W$ is $(c+1)$-regular. If $k \geq c$ then $\mathcal{M}_W$ is $(k+1)$-regular and thus,

$$H^1(X, \mathcal{M}_W(k)) = H^1(X, \mathcal{M}_W(k+1-1)) = 0$$

□

Definition 2.0.13 (Fujita)**.** Let X be a smooth projective varitety $\dim X = n$. Let D be an ample divisor. Then,

(a) $k \geq n+1$ implies that $K_X + kD$ is basepoint free

(b) $k \geq n+2$ implies that $K_X + kD$ is very ample.

Remark. This is true for curves, surfaces, and projective spaces.

Remark. h^0 can be hard to compute but χ is easier. If $H^i = 0$ for $i > 0$ then $\chi = h^0$.

Example 2.0.14. Let $X \subset \mathbb{P}^r$. What is the dimension of quadtric supersurfaces containing X. Consider $h^0(\mathscr{I}_X(2))$. We have,

$$0 \longrightarrow \mathscr{I}_X(2) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(2) \longrightarrow \mathcal{O}_X(2) \longrightarrow 0$$

Therefore, we need vanishing $H^1(\mathscr{I}_X(2)) = 0$ to compute $h^0(\mathscr{I}_X(2))$.

Example 2.0.15. Let $\ell \subset \mathbb{P}^3$ be a line. What is the dimension of degree d surfaces containing ℓ. Since ℓ is a complete intersection $\ell = H_1 \cap H_2$ for two hyperplanes. We have a Kozul resolution,

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

By the previous result, $H^0(\mathcal{O}) \twoheadrightarrow H^0(\mathcal{O}_\ell)$ is surjective and thus $H^1(\mathscr{I}_\ell(d)) = 0$. Alternatively, the resultion gives that $\mathscr{I}_\ell$ is 0-regular.

Example 2.0.16. Noether-Lefschetz: $\operatorname{Pic}(S_d) = \mathbb{Z}$ for very general hypersurface $S_d \subset \mathbb{P}^3$ of degree d. However, if $S_d \supset \ell$ then it is not very general.

Example 2.0.17. Let $H_1, \dots, H_e \subset \mathbb{P}$ be hypersurfaces with $\deg H_i = d_i$ and a complete intersection,

$$X = H_1 \cap \cdots \cap H_e$$

Then,

$$\operatorname{reg}(\mathscr{I}_X) = d_1 + \cdots + d_e - e + 1$$

3 Andres: Moduli of Vector Bundles on Curves

Let C be a curve a smooth projective curve over k. Vector bundles on C vary in continuous families.

Example 3.0.1. If C is an elliptic curve then there is a bijection between,

$$C(k) \xrightarrow{\sim} \{\text{rank 1 vector bundles of degree 1}\}$$

via the map,

$$p \mapsto \mathcal{O}_C(p) \cong \mathscr{I}_p^\vee$$

Let $\Sigma_{n,d}$ be the set of all vector bundle s on C of fixed rank n and degree d. Assume that $(n, d) = 1$. We want that $\Sigma_{n,d}$ is the k-points of some projective variety.

Let T be a varierty or a scheme, we can consider a vector bundle $\mathcal{E}$ on $T \times C$ this gives a map $T(k) \to \Sigma_{n,d}$ via $t \mapsto \mathcal{E}_{C_t} \in \Sigma_{n,d}$. Therefore, we use this as the functor of points of the desired variety.

Consider the finest topology on $\Sigma_{n,d}$ such that for all T and all $\mathcal{E}$ on $C \times T$ the induced map $T(k) \to \Sigma_{n,d}$ is continuous.

4 Brill-Noether Theory

Let C be a smooth projective curve of genus g. Then we want to consider the space of line bundles $\mathcal{L}$ on C with $V \subset H^0(C, \mathcal{L})$ of dimension $r + 1$ giving a map $C \dashrightarrow \mathbb{P}^r$ of degree d. We get a moduli space $G^r_d(C)$. We ask the following questions:

(a) when is $G^r_d(C)$ nonempty

(b) what is the dimension of $G^r_d(C)$

(c) how many components does $G^r_d(C)$ have and are they equidimensional?

Definition 4.0.1. The Brill-Noether number,

$$\rho = g - (r + 1)(g - d + r)$$

is the "expected dimension" of $G^r_d(C)$ for a general curve C.

Definition 4.0.2. There is a universal fibration $\mathscr{G}^r_d \to \mathcal{M}_g$ of the Brill-Noether moduli spaces.

Theorem 4.0.3 (Brill-Noether)**.** There is an open locus of $\mathcal{M}_g$ such that if,

(a) $\rho < 0$ then $\mathscr{G}^r_d|_U$ is empty

(b) $\rho \geq 0$ then $\mathscr{G}^r_d|_U$ has constant fiber dimension ρ and is smooth

(c) $\rho > 0$ then $\mathscr{G}^r_d$ has connected fibers (over all of $\mathcal{M}_g$).

Remark. If $\rho \geq 0$ then $G^r_d(C)$ is nonempty for all C but need not be smooth or of the correct dimension.

Example 4.0.4. Hyperelliptic curves have nontrivial $\mathfrak{g}^1_2$ but

$$\rho = g - 2(g-1) = 2 - g$$

is negative for large g .

Definition 4.0.5. Consider the space

$$W^r_d(C) = \{\mathcal{L} \mid \mathcal{L} \text{ line bundle wth } \deg \mathcal{L} = d \text{ and } \dim H^0(C, \mathcal{L}) \geq r+1\}$$

Then clearly there is a map $\beta : G^r_d(C) \to W^r_d(C)$.

4.1 Definition of Moduli Spaces

Definition 4.1.1. Let F_1, F_2 be free modules of finite rank over R and consider,

$$F_1 \xrightarrow{\varphi} F_2 \longrightarrow M \longrightarrow 0$$

Then the a^{th} fitting ideal $\mathrm{Fitt}_a(M)$ is the ideal generated by the $(\mathrm{rk} F_2 - a) \times (\mathrm{rk} F_@ - a)$ minors of the matrix representing φ. This is independent of the presentation.

Definition 4.1.2. Using the universal line bundle $\mathcal{L}$ on $C \times \mathrm{Pic}^d_C$ we define,

$$W^r_d(C) = \mathrm{Fitt}_{g-d+r-1}(R^1\nu_*\mathcal{L})$$

where $\nu : C \times \mathrm{Pic}^d_C \to \mathrm{Pic}^d_C$.

Remark. Notice that $R^1\nu_*\mathcal{L}$ has fibers $H^1(C, L)$ over the point $[L]$ for L of degree d. Choose high enough degree divisor Γ on C we get,

$$0 \longrightarrow L \longrightarrow L(\Gamma) \longrightarrow L(\Gamma)/L \longrightarrow 0$$

Then the long exact sequence gives,

$$0 \longrightarrow H^0(C, L) \longrightarrow H^0(C, L(\Gamma)) \xrightarrow{\gamma} H^0(C, L(\Gamma)/L) \longrightarrow H^1(C, L) \longrightarrow 0$$

Then by Riemann-Roch $h^0(C, L(\Gamma)) = d - g + 1 + m$ and $h^0(C, L(\Gamma)/L) = m$ where $\deg \Gamma = m$. Then we have,

$$|W^r_d(C)| = \{L \in \mathrm{Pic}^d \mid \mathrm{rank}\, \gamma \leq m - (g - d + r - 1) - 1 = m - g + d - r\}$$

which is exactly the conditions of the fitting ideal.

Remark. Naive dimension count for $W^r_d(C)$ is,

$$\dim \mathrm{Pic}^d_C - \#\{\text{minors}\} = g - (m - (m-g+d-r))(d-g+1+m-(m-g+d-r)) = g - (r+1)(g-d+r) = \rho$$

4.2 Petri's Condition

Let C be a smooth projective curve. We say that C satisfies (P) if for all $\mathcal{L} \in \mathrm{Pic}(C)$,

$$\mu_{\mathcal{L}} : H^0(C, \mathcal{L}) \otimes H^0(C, \omega_C \otimes \mathcal{L}^{\otimes -1}) \to H^0(C, \omega_C)$$

is injective.

Theorem 4.2.1 (Gieseker)**.** Petri's condition holds for a general C.

Corollary 4.2.2. (a) If $\rho < 0$, for a general C, then G^r_d and W^r_d are empty

(b) if $\rho \geq 0$, for a general C, then G^r_d is smooth of diemension ρ and W^r_d is smooth away from W^{r+1}_d and has dimension ρ

(c) if $\rho \geq 1$, for a general C, then G^r_d and W^r_d are irreducible.

Proof. Consider infinitessimal deformation theory, given $(L, V) \in G^r_d(\mathbb{C})$ we consider,

$$T_{(L,V)} G^r_d = \{(L', V') \mid L' \text{ extending } L \text{ and } V' \subset H^0(L') \text{ free restricting to } V\}$$

The tangent space fits into a sequence,

$$0 \longrightarrow T_{(L,V)} \beta^{-1}(L) \to T_{(L,V)} G^r_d \xrightarrow{\beta} T_L \mathrm{Pic}^d$$

and recall that $T_L \mathrm{Pic}^d \xrightarrow{\sim} H^1(C, \mathcal{O}_C)$.

When does $\phi \in T_L \mathrm{Pic}^d$ lie in the image of $T\beta$? We can represent ϕ by a Cech 1-cocycle $\phi_{\alpha\beta} \in \mathcal{O}_C(U_{\alpha\beta})$. For a given $[L] \in H^1(C, \mathcal{O}_C^\times)$ represented by a cocycle $\{g_{\alpha\beta}\}$ then we can represent the lift with a given class by the cocycle $\{g'_{\alpha\beta} = g_{\alpha\beta}(1 + \epsilon \phi_{\alpha\beta})\}$. There needs to exist an extension (L, s) to (L', s') for $s \in W \subset H^0(L)$. For s' to be an extension of s we should have,

$$s'_\alpha = s_\alpha + \epsilon t_\alpha$$

for $t_\alpha \in \mathcal{O}_C(U_\alpha)$ and we want $s'_\beta = g'_{\alpha\beta} s'_\alpha$. This gives,

$$-\phi_{\alpha\beta} s_\alpha = t_\alpha - g_{\beta\alpha} t_\beta$$

Therefore we require that $-\phi \cdot s$ is zero in $H^1(L)$.

Thus $\phi \in \mathrm{im}\, T\beta$ is zero prcisely when $\phi \cdot W \subset H^1(L)$ is zero. Therefore,

$$\begin{aligned} \mathrm{im}\, T\beta &= \{\phi \in H^1(\mathcal{O}_C) \mid \phi \cdot W = 0\} = \{\phi \in H^1(\mathcal{O}_C) \mid \forall s : \langle \phi W, s \rangle = 0\} \\ &= \{\phi \in H^1(\mathcal{O}_C) \mid \forall s : \langle \phi, W \cdot s \rangle = 0\} \\ &= \{\phi \in H^1(\mathcal{O}_C) \mid \langle \phi, t \rangle = 0\} \end{aligned}$$

over all $t \in \mathrm{im}\,(H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C))$. Therefore,

$$\dim T_{(L,W)} G^r_d = \dim \mathrm{im}\, T\beta + (r+1)(h^0(L) - (r+1))$$

using that $\beta^{-1}(L)$ is a Grasmannian and thus,

$$T_{(L,V)} \beta^{-1}(L) = \mathrm{Hom}\left(W, H^0(L)/W\right)$$

Therefore,

$$\begin{aligned}\dim T_{(L,W)}G^r_d &= g - \dim\operatorname{im}\mu_L + (r+1)(h^0(L) - (r+1)) \\ &= g - ((r+1)h^0(\omega_C \otimes L^{-1}) - \ker\mu_L) + (r+1)(h^0(L) - (r+1)) \\ &= g + (r+1)(h^0(L) - h^0(\omega_C \otimes L^{-1}) - (r+1)) + \ker\mu_L \\ &= g + (r+1)(g - g - r) + \ker\mu_L \\ &= \rho + \ker\mu_L\end{aligned}$$

Therefore, G^r_d has tangent space of the expected imension iff μ_L is injective. We already know $\dim G^r_d \geq \rho$ from the naive dimension count. Then G^r_d is smooth at (L, W) of dimension ρ iff $\mu_L|_W$ is injective.

Then $\beta : G^r_d \to W^r_d$ is an siomrophism away from W^{r+1}_d and $W^r_d \backslash W^{r+1}_d$ is dense in W^r_d. Furthermore, μ_L is injective implies that W^r_d is smooth of dim $= \rho$ away from W^{r+1}_d. $\square$

4.3 Riemann-Roch in Geometric Terms

Let D be an effective divisor. Then,

$$r(D) = h^0(D) - 1$$

is the number of independent relations between the canonical image $\phi(D)$ meaning under the canonical embedding $\phi : C \to \mathbb{P}^{g-1}$.

Example 4.3.1. If C is hyperelliptic and D is degree d effective divisor with $r(D) = r$. Then,

$$D \sim r\mathfrak{g}^1_2 + p_1 + \cdots + p_{d-2r}$$

Example 4.3.2. If $g = 4$ and $d = 3$ and $r = 1$ then $\rho = 0$. If C is hyperelliptic then,

$$D = \mathfrak{g}^1_2 + p$$

and therefore $W^1_3 \cong C$ is 1-dimensional. If C is not hyperelliptic then under the canonical embedding $C \hookrightarrow \mathbb{P}^3$ we have $C = Q \cap S$ for a quadric Q and a cubic S surface. Then if D is degree 3 and $r(D) = 1$ then $\phi(D)$ should be colinear and hence the line is on Q. Therefore, W^1_3 is the set of linear equivalence classes of rullings on Q so $\#W^1_3 = 1$ if Q is a cone and $\#W^1_3 = 2$ if Q is smooth.

5 Oct 11. Brill Noether Theory on K3 Surfaces, Lazarsfeld-Mukai bundles

Definition 5.0.1. $X/\mathbb{C}$ is a $K3$-surface if it is a smooth, projective variety of $\dim X = 2$ such that $K_X = \Omega^2_{X/K} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Example 5.0.2. Let $X \subset \mathbb{P}^3$ be a smooth quartic then $\omega_X = \omega_{\mathbb{P}^3} \otimes \mathcal{O}_X(4) \cong \mathcal{O}_X$.

Lemma 5.0.3. Let X be a K3 surface then $\chi(X, \mathcal{O}_X) = 2$.

Proof. $\chi = h^0 - h^1 + h^2 = 2h^0 = 2$. $\square$

Proposition 5.0.4. Let $C \subset X$ be a smooth irreducible curve of genus ≥ 1 then $|C|$ has no base points and defines a morphism $\phi : X \to \mathbb{P}^g$ such that $\phi|_C : X \to \mathbb{P}^{g-1}$ is the canonical one.

Proof. The sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0$$

and use that $H^1(X, \mathcal{O}_X) = 0$ and thus,

$$H^0(X, \mathcal{O}_X(C)) \twoheadrightarrow H^0(C, \mathcal{O}_C(C)) = H^0(C, \omega_C)$$

□

Lemma 5.0.5. Let $C \subset X$ be a smooth irreducible curve with $g \geq 1$ and $\mathcal{L} = \mathcal{O}_X(C)$. Then $c_1(\mathcal{L})^2 = 2g-2$ and $h^0(X, \mathcal{L}) = g+1$. Also, if $\ell \geq 1$ then $h^0(X, \mathcal{L}^\ell) = (\ell^2/2)c_1(\mathcal{L})^2+2 = (g-1)\ell^2+2$.

Proof. Riemann-Roch gives,

$$2g-2 = C \cdot (C + K_X) = \mathcal{L}^2$$

Then Riemann-Roch for surfaces gives,

$$\chi(X, \mathcal{L}) = \tfrac{1}{2}c_1(\mathcal{L})^2 + 2 = g+1$$

also $h^2(X, \mathcal{L}) = h^0(X, \mathcal{L}^\vee) = 0$. Therefore,

$$h^0(X, \mathcal{L}) \geq g+1$$

Furthermore, $h^1(X, \mathcal{L}) = 0$ by Kodaira vanishing or something else. □

Theorem 5.0.6. Let $C \subset X$ be a smooth irreducible curve of genus $g \geq 2$. Suppose every divisor in $|C|$ is reduced and irreducible then,

(a) for all $\mathcal{L} \in \mathrm{Pic}(C)$ the number $\rho(\mathcal{L}) = g(C) - h^0(\mathcal{L})h^1(\mathcal{L}) \geq 0$

(b) Petri's condition holds for a general member $C' \in |C|$.

Remark. The assumption on the linear series is essential. For a counterexample, let $|C| = |nD|$ with $D \subset X$ a curve of genus $g \geq 2$ and $n \geq 2$. Let $\mathcal{L} = \mathcal{O}_X(D)|_D$. Claim that $\rho(\mathcal{L}) < 0$. Consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X(D-C) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0$$

5.1 Lazarsfeld-Mukai Bundle

From now on, X is a K3 surface and $C \subset X$ is a smooth irreducible curve. Recall that $V^r_d(C) \subset \mathrm{Pic}^d(C)$ is the open subset of $W^r_d(C)$ consisting of line bundles $\mathcal{L}$ such that,

(a) $h^0(\mathcal{L}) = r+1$ and $\deg \mathcal{L} = d$

(b) $\mathcal{L}$ and $\omega_C \otimes \mathcal{L}^\vee$ are globally generated.

Definition 5.1.1. Fix $\mathcal{L} \in V^r_d(C)$. Let $\iota : C \hookrightarrow X$ be the inclusion. For each pair $(C, \mathcal{L})$ define, $\mathscr{F}_{C,\mathcal{L}}$ as the kernel of,

$$\mathrm{ev} : H^0(\mathcal{L}) \otimes_{\mathbb{C}} \mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{L}$$

Lemma 5.1.2. Let $\mathcal{E}$ be a vector bundle on X with a surjection $\varphi : \mathcal{E}|_C \twoheadrightarrow \mathcal{L}$. Then consider the exact sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{E} \longrightarrow \iota_* \mathcal{L} \longrightarrow 0$$

Then $\mathscr{F}$ is locally free.

Proof. Work locally, assume $\mathcal{L} = \mathcal{O}_X$. Then there is alocally free resolution,

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

Therefore the homological dimension of $\mathcal{L}$ is ≤ 1 and therefore the homological dimension of $\mathscr{F}$ is 0 and thus $\mathscr{F}$ is locally free. $\square$

Corollary 5.1.3. The Lazarsfeld-Mukai bundle $\mathscr{F}_{C,\mathcal{L}}$ is a vector bundle.

Proof. Consider the sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow H^0(\mathcal{L}) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \iota_* \mathcal{L} \longrightarrow 0$$

and apply the previous lemma. $\square$

Lemma 5.1.4. Let $\mathscr{F} = \mathscr{F}_{C,\mathcal{L}}$. Then,

(a) $\mathscr{F}^\vee$ is globally generated

(b) $c_1(\mathscr{F}) = -[C]$ and $c_2(\mathscr{F}) = \deg \mathscr{F} = d$

(c) $H^0(\mathscr{F}) = H^2(\mathscr{F}^\vee) = 0$ and $H^1(\mathscr{F}) = H^2(\mathscr{F}^\vee) = 0$ and,

$$h^0(\mathscr{F}^\vee) = h^0(\mathcal{L}) + h^1(\mathcal{L})$$

Proof. Consider the sequence,

$$0 \longrightarrow H^0(\mathcal{L})^\vee \otimes \mathcal{O}_X \longrightarrow \mathscr{F}^\vee \longrightarrow \iota_*(\omega_C \otimes \mathcal{L}^\vee) \longrightarrow 0$$

By assumption the third term is globally generated and $H^0(\mathscr{F}^\vee) \twoheadrightarrow H^0(\mathfrak{m}_C \otimes \mathcal{L}^\vee)$ because $H^1(X, \mathcal{O}_X) = 0$. Therefore, $\mathscr{F}^\vee$ is globally generated.

In general we have a formula,

$$c_1(\iota_* \mathcal{L}) = [C] \quad c_2(\iota_* \mathcal{L}) = [C]^2 - \iota_* c_1(\mathcal{L}) = [C]^2 - (\deg \mathcal{L})[pt]$$

Then from the exact sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow H^0(\mathcal{L}) \otimes \mathcal{O}_X \longrightarrow \iota_* \mathcal{L} \longrightarrow 0$$

We get that,

$$c_1(\mathscr{F}) = -c_1(\mathcal{L}) = -[C] \quad c_2(\mathscr{F}) = -c_1(\iota_* \mathcal{L}) c_1(\mathscr{F}) - c_2(\iota_* \mathcal{L}) = [C]^2 - [C]^2 + (\deg \mathcal{L})[pt] = (\deg \mathcal{L})[pt]$$

$\square$

Lemma 5.1.5. Let $\mathscr{F} = \mathscr{F}_{C,\mathcal{L}}$ then,

$$\chi(\mathscr{F} \otimes \mathscr{F}^\vee) =$$

6 Oct 18

6.1 Proof of (a)

Theorem 6.1.1 (Main)**.** Let $C \subset X$ be a smooth irreducible curve of genus $g \geq 2$ on the K3 surface X. Assume that every divisor in the linear series $|C|$ is reduced and irreducible. Then,

(a) for each $\mathcal{L} \in \text{Pic}(C)$ we have $\rho(\mathcal{L}) \geq 0$

(b) Petri's condition holds for a general element $C' \in |C|$.

Lemma 6.1.2. Let $\mathcal{L} \in \text{Pic}(C)$ for a smooth proper curve C with $\deg \mathcal{L} \in (0, 2g-2)$. There is a line bundle $\mathcal{L} = \mathcal{L}'(D)$ such that $\mathcal{L}'$ and $\omega_C \otimes \mathcal{L}'^{\vee}$ are globally generated and $\rho(\mathcal{L}') \leq \rho(\mathcal{L})$.

Proof. Let D_1 be the divisor of base points of $\mathcal{L}$. Then $\mathcal{L}(-D_1)$ is globally generated because $|\mathcal{L}| = |\mathcal{L}(-D_1)| + D_1$. Let D_2 be the divisor of basepoints of $K_C - c_1(\mathcal{L}) + D_1$. Then $K_C - c_1(\mathcal{L}) + D_1 - D_2$ is base-point free. I claim that $\mathcal{L}(D_2 - D_1)$ is also globally generated. If $\mathcal{L}(D_2 - D_1 - P)$ does not drop dimension then by Riemann Roch $K_C - c_1(\mathcal{L}) + D_1 + P - D_2$ must increase dimension ☐

Proof of (a). Suppose that $\rho(\mathcal{L}) < 0$ (REPLACE WITH BPF)

Let $\mathcal{E} = \mathscr{F}_{C,\mathcal{L}}^{\vee}$ which is a vector bundle since $\mathcal{L} \in V_d^r(C)$. We showed that,

$$2h^0(X, \mathscr{F} \otimes \mathscr{F}^{\vee}) \geq \chi(\mathscr{F}, \mathscr{F}) = 2 - 2\rho(\mathcal{L}) \geq 4$$

thus $\mathcal{E}$ has a nontrivial endomorphism $\varphi : \mathscr{F} \to \mathscr{F}$ meaning $\varphi \neq \lambda \text{id}$. Choose a point $x \in X$ and let λ be an eigenvalue of $\varphi(x)$. Then $\psi = \varphi - \lambda \text{id}$ is nonzero but is not of full rank at x. Thus $\det \psi \in \text{Hom}_X(\det \mathcal{E}, \det \mathcal{E}) = H^0(X, \mathcal{O}_X)$ has a zero and hence is zero. Let $\mathcal{E}_1 = \text{im}\, \psi$ and $\mathcal{E}_2 = \text{coker}\, \psi$ so there is a sequence,

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

so we have $c_1(\mathcal{E}) = c_1(\mathcal{E}_1) + c_1(\mathcal{E}_2)$ and $c_1(\mathcal{E}) = [C]$. Then if $c_1(\mathcal{E}_1)$ and $c_1(\mathcal{E}_2)$ are represented by nonzero effective divisors. We showed last time that $\mathcal{E}$ is globally generated and $H^0(X, \mathcal{E}^{\vee}) = 0$. Thus since $\mathcal{E} \twoheadrightarrow \mathcal{E}_i$ we see that $\mathcal{E}_i$ are globally generated so $c_1(\mathcal{E}_i) = [C_i]$ for some effective class C_i. (SHOW BOTH CLASSES ARE NONTRIVIAL). Hence $C \sim C_1 + C_2$ contradicting the assumption on the linear system. ☐

6.2 Mukai's Theorem

Definition 6.2.1. Let X be a proper k-scheme. A vector bundle $\mathcal{E}$ on X is *simple* if,

$$\text{Hom}_X(\mathcal{E}, \mathcal{E}) = k$$

Remark. Simple vector bundles are indecomposable. If X is geometrically irreducible then all line bundles are simple.

In this section, let X be a (smooth projective) K3 surface over $\mathbb{C}$. Therefore, all line bundles are simple.

Remark. If $\mathcal{E}$ is simple, then by Serre duality using that $\omega_X \cong \mathcal{O}_X$,

$$\text{Ext}_X^2(\mathcal{E}, \mathcal{E}) \cong \text{Ext}_X^2(\mathcal{E} \otimes \mathcal{E}^{\vee}, \omega_X) = H^0(X, \mathcal{E} \otimes \mathcal{E}^{\vee})^{\vee} = \text{Hom}_X(\mathcal{E}, \mathcal{E})^{\vee} = \mathbb{C}$$

Definition 6.2.2. Let $\mathcal{M}(X, r, c_1, c_2)$ be the moduli space of simple vector bundles on X of rank r and with Chern classes c_1 and c_2.

Remark. Because the objects of $\mathcal{M}$ are simple, the stabilzers groups are $\mathbb{G}_m$ and hence $\mathcal{M} \to M$ is a $\mathbb{G}_m$-torsor over a coarse space M.

Remark. The moduli problem has tangent-obstruction theory at a point $\mathcal{E} \in \mathcal{M}$,

$$T^i = \mathrm{Ext}^i_X(\mathcal{E}, \mathcal{E})$$

Therefore, since the fiber direction $B\mathbb{G}_m$ have trivial tangent direction we see that,

$$T_{[\mathcal{E}]}M = \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E})$$

Remark. The cup product gives a nondegenerate holomorphic 2-form on M defined by,

$$\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E}) \times \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}) = \mathbb{C}$$

Therefore, M gives an example of a holomorphic symplectic variety. When $\dim M = 2$ it turns out that M is also a K3 surface.

Theorem 6.2.3 (Mukai)**.** The moduli space $M(X, r, c_1, c_2)$ is smooth.

Proof. By descent along the flat map $\mathcal{M} \to M$ it suffices to show that $\mathcal{M}$ is smooth. Alternatively we can develop directly tangent-obstruction theory for M. Either way, it suffices to show that obstruction classes $\mathrm{ob}(E) \in \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E})$ vanish. Let $\mathcal{E} \in \mathcal{M}$ be a closed point (corresponding to a simple vector bundle $\mathcal{E}$ on X) and a small extension of Artin local k-algebras $A \subset B$,

$$\begin{tikzcd}
\mathrm{Def}_{\mathcal{M}}(B) \arrow[r] \arrow[d, "\det"] & \mathrm{Def}_{\mathcal{M}}(A) \arrow[r, "\mathrm{ob}"] \arrow[d, "\det"] & \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}) \arrow[d, "\mathrm{tr}"] \\
\mathrm{Def}_{\mathrm{Pic}}(B) \arrow[r] & \mathrm{Def}_{\mathrm{Pic}}(A) \arrow[r, "\mathrm{ob}"] & \mathrm{Ext}^2_X(\mathcal{O}_X, \mathcal{O}_X)
\end{tikzcd}$$

but Pic_X is smooth so we see that $\mathrm{tr} \circ \mathrm{ob} = 0$. However, using Serre duality,

$$\begin{tikzcd}
\mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}) \arrow[r, "\sim"] \arrow[d, "\mathrm{tr}"] & H^0(X, \mathcal{E} \otimes \mathcal{E}^\vee) \arrow[r, "\sim"] \arrow[d, "\mathrm{tr}"] & \mathrm{Hom}_X(\mathcal{E}, \mathcal{E}) \arrow[d, "\mathrm{tr}"] \\
\mathrm{Ext}^2_X(\mathcal{O}_X, \mathcal{O}_X) \arrow[r, "\sim"] & H^0(X, \mathcal{O}_X) \arrow[r, equal] & H^0(X, \mathcal{O}_X)
\end{tikzcd}$$

but since $\mathcal{E}$ is simple the map $\mathrm{tr} : \mathrm{Hom}_X(\mathcal{E}, \mathcal{E}) \to H^0(X, \mathcal{O}_X)$ is an isomorphism. Thus $\mathrm{tr} \circ \mathrm{ob} = 0$ implies that $\mathrm{ob} = 0$. □

6.3 Proof of (b)

Sketch of Proof of (b). Recall that for $\mathcal{L} \in V^r_d(C')$ we know that the tangent space of $V^r_d(C')$ and $G^r_d(C')$ are isomorphic and hence injectivity of $\mu_{\mathcal{L}}$ is equivalent to $V^r_d(C')$ being smooth of the expected dimension. Consider the variety,

$$\mathcal{V}^r_d = \{(C', \mathcal{L}) \mid C' \in |C| \text{ smooth curve and } \mathcal{L} \in V^r_d(C')\}$$

and denote,

$$\pi^r_d : \mathcal{V}^r_d \to |C|$$

the natural map. By generic smoothness, to show that $V^r_d(C')$ is smooth (and hence $\mu_{\mathcal{L}}$ is injectve) for a generic C' is suffices to show that $\mathcal{V}^r_d$ is smooth.

Consider the fibration,

$$\pi : \mathscr{G} \to M = M(X, r+1, [C], d)$$

where $\mathscr{G}$ is the space of pairs $(\mathcal{E}, V)$ for a simple vector bundle $\mathcal{E}$ of rank $r+1$ of X with $c_1(\mathcal{E}) = [C]$ and $c_2(\mathcal{E}) = d$ and $V \subset H^0(X, \mathcal{E})$ of dimension $r+1$. By Mukai's theorem M is smooth and hence $\mathscr{G}$ is smooth since it is a Grassmannian bundle so we can compute the tangent space at the point $\mathcal{E} = \mathscr{F}^\vee_{C,\mathcal{L}}$ to get,

$$\begin{aligned}
\dim \mathscr{G} &= \dim M + (r+1)(\dim H^0(X, \mathcal{E}) - r - 1) \\
&= \operatorname{Ext}^1_X(\mathcal{E}, \mathcal{E}) + (r+1)(\dim H^0(X, \mathcal{E}) - r - 1) \\
&= 2\rho(r, d, g) + (r+1)(g - d + r) = g + \rho(r, d, g)
\end{aligned}$$

using a lemma we proved last time. Thus it suffices to show that $\mathcal{V}^r_d$ has an open embedding in $\mathscr{G}$.

Let $U \subset \mathscr{G}$ denote the open set consisting of pairs (E, V) such that,

(a) E is globally generated and $H^1(X, E) = H^2(X, E) = 0$

(b) the natural map $\mathrm{ev} : V \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{E}$ drops rank on a smooth curve C_V and $\operatorname{coker} \mathrm{ev}$ is a line bundle on C_V.

Then we have exact sequences,

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow V^\vee \otimes \mathcal{O}_X \longrightarrow \mathcal{L}_V \longrightarrow 0$$

$$0 \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \omega_C \otimes \mathcal{L}_V^\vee \longrightarrow 0$$

(WHY) SHOW THE EQUAIVALENCE ☐

7 Oct 25

7.1 Setup

X is a smooth projective surface over $\mathbb{C}$ and L a line bundle on X. Then we have the following two facts,

adjunction if $C \subset X$ is an effective curve then,

$$p_a(C) - 1 = \tfrac{1}{2} C \cdot (C + K_X)$$

Hodge index if D, H are divisors on X with $H^2 \geq 0$ and $D \cdot H = 0$ then $D^2 \leq 0$ and $D^2 = 0$ iff $D \sim 0$.

Remark. If C is integral then $p_a(C) \geq 0$.

7.2 Linear Systems

Let $V \subset H^0(X, L)$ be a linear system. Then the base locus is,

$$\mathrm{Bs}(V) = \{p \in X \mid \forall s \in V : s(p) = 0 \in L(p)\}$$

Note we use the notation $L(p) = L_p / \mathfrak{m}_p L_p$. Consider the map,

$$\Phi_V : X \backslash \mathrm{Bs}(V) \to \mathbb{P}(V)$$

Note that $p \in \mathrm{Bs}(V)$ iff $H^0(X, L) \to H^0(Z, \mathcal{L}|_Z)$ is zero.

Proposition 7.2.1. $\mathrm{Bs}(V) = \varnothing$ then $\Phi_V : X \to \mathbb{P}(V)$ is a closed embedding iff,

(a) V separates points meaning $\forall p, q \in X$ with $p \neq q$ there is $s \in V$ with $s(p) = 0$ and $s(q) \neq 0$ or vice versa

(b) V separates tangent directions,
$$\{s \in V \mid s_p \in \mathfrak{m}_p L_p\}$$
generates $\mathfrak{m}_p L_p / \mathfrak{m}_p^2 L_p$ as a vector space.

Remark. We can reformulate the conditions as follows,

(a) $p, q \in X$ with $p \neq q$ let $Z = \{p, q\}$ redced thne,
$$H^0(X, L) \to H^0(Z, L|_Z)$$
is surjective

(b) $p \in X$ and $t \in \mathfrak{m}_p / \mathfrak{m}_p^2$ and Z is cut out by $\mathfrak{m}_p^2 + (t)$ locally then,
$$H^0(X, L) \to H^0(Z, L|_Z)$$
is surjective.

Theorem 7.2.2 (Reider)**.** Let L be a nef line bundle,

(a) let $(L \cdot L) \geq 5$. Let p be a base point of $|K_X + L|$. Then there is n effective divisor $D \subset X$ wuth $p \in D$ such that either,

   (a) $(L \cdot D) = 0$ and $D^2 = -1$
   (b) $(L \cdot D) = 1$ and $D^2 = 0$

(b) $(L \cdot L) \geq 10$. Let $p \in X$ and $q \in X$ with $p \neq q$ which are not separated by $|K_X + L|$ or $q \in \mathfrak{m}_p / \mathfrak{m}_p^2$ and p, q not separated by $|K_X + L|$. Then there is an effective divisor $D \subset X$ with $Z_{p,q} \subset D$ such that one of the three conditions holds,

   (a) $(L \cdot D) = 0$ and $(D \cdot D) \in \{-1, -2\}$
   (b) $(L \cdot D) = 1$ and $(D \cdot D) \in \{0, -1\}$
   (c) $(L \cdot D) = 2$ and $(D \cdot D) = 0$.

Example 7.2.3. Let $X = \mathbb{P}^2$ and $L = \mathcal{O}_X(2)$ then $(L \cdot L) = 4$ and $K_X = \mathcal{O}_X(-3)$ then $K_X + L = \mathcal{O}_X(-1)$ which has every point as a base point. Let $D \subset X$ and $D \in |kH|$ then $D^2 = k^2$ but $L \cdot D = 2k$ so these cannot satisfy the conclusion of the theorem. This shows that $(L \cdot L) \geq 5$ is strict in the theorem.

7.3 Fujita's Conjecture

Definition 7.3.1 (Fujita 1985)**.** Let X be a compact complex manifold of dimension n and L an ample line bundle.

(a) $m \geq n+1 \implies K_X \otimes L^{\otimes m}$ is base point free

(b) $m \geq n+2 \implies K_X \otimes L^{\otimes m}$ is very ample.

Proof in the $n=2$ case. (a) We know X is projective use Nkai-Moishecon. Let $(L \cdot L) \geq 1$ then mL is nef if $m \geq 3$ then $(mL \cdot mL) \geq 3^2 \geq 5$. Then is p is a base point of $|K_X + K|$ then there is an effective divisor $D \subset X$ with $p \in D$ such that $(mL \cdot D) = 0$ and $D^2 = 1$ or $(mL \cdot D) = 1$ which is not possible since $m > 1$ and hence we have $(L \cdot D) = 0$ and $D^2 = -1$. We write,

$$D = D_1 + \cdots + D_r$$

But L is ample so $(D_i \cdot L) > 0$ and D must have some component since $p \in D$ and thus $(D \cdot L) > 0$ giving a contradiction.

(b) Use the same sort of argument with the second part of Reider's theorem.

□

7.4 Pluricanonical Mappings

Let X be a surface of general type. Consider the pluricanonical maps,

$$\Phi_m : X \dashrightarrow \mathbb{P}(H^0(mK_X))$$

defined by the complete linear system $|mK_X|$.

Proposition 7.4.1. If X is minimal then K_X is nef and $K_X^2 \geq 1$.

Definition 7.4.2. A (-2)-curve on X is a smooth rational curve $C \subset X$ with $C^2 = -2$.

Proposition 7.4.3. If X is minimal then X has finitely many -2-curves. In fact, it is at most $\rho(X) - 1$.

Theorem 7.4.4 (Bombieri)**.** Let X be a minimal surface of general type. Let,

$$F = \bigcup C \subset X$$

be the union of the -2-curves.

(a) if $m \geq 4$ or $m \geq 3$ and $K_X^2 \geq 2$ then Φ_m is a morphism

(b) if $m \geq 5$ or $m \geq 4$ and $K_X^2 \geq 2$ or $m \geq 3$ and $K_X^2 \geq 3$ then Φ_m is an embedding on $X \backslash F$.

Proof. Let $L = (m-1)K_X$ is nef then $L \cdot L \geq 5$. Apply Reider's theorem. Let p be a base point of $|K_X + L| = |mK|$. Then there is an effective divisor $D \subset X$ with $p \in D$ such that $(L \cdot D) = 0$ and $D^2 = 1$ since $L \cdot D = 1$ is impossible. Then,

$$-1 = D^2 = D \cdot (D + K_X) = 2p_a(D) - 2$$

which is a contradiction. □

7.5 Bogomolov's Theorem

Theorem 7.5.1 (Bogomolov)**.** Let E be a vector bundle of rank e on a surface X. If $c_1(E)^2 > \frac{2e}{e-1}c_2(E)$ then E is H-unstable with respect to every ample class H.

Remark. $c_2(E) \in H^4(X, \mathbb{Z})$ so we view $c_2(E)$ as an integer under the canonical isomorphism $H^4(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ using that X is oriented (as a complex manifold).

7.5.1 Stability for Curves

Let C be a smooth projective irreducible curve. Let E be a vector bundle on C.

Definition 7.5.2. The slope,

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E}$$

where $\deg E = \deg \det E$.

Example 7.5.3. Let $C = \mathbb{P}^1$ then $\mu(\mathcal{O}_C) = 0$ and $\mu(\mathcal{O}_C(1)) = 1$ and,

$$\mu(\mathcal{O}_C(a) \oplus \mathcal{O}_C(b)) = \frac{a+b}{2}$$

Definition 7.5.4. Let $F \subset E$ be a coherent subsheaf. Then F is locally free of constant rank almost everywhere. Then $c_1(F) := \det F^{\vee\vee}$ is a line bundle

Definition 7.5.5. The slope of a torsion-free sheaf F is,

$$\mu(F) = \frac{\deg F}{\operatorname{rank} F}$$

Definition 7.5.6. E is *stable* if, for every $F \subset E$ with,

$$0 < \operatorname{rank} F < \operatorname{rank} E$$

we have $\mu(F) < F(E)$ and *semistable* if $\mu(F) \leq \mu(E)$.

Remark. It is trivial that line bundles are stable.

Example 7.5.7. Let $C = \mathbb{P}^1$ then $\mathcal{O} \oplus \mathcal{O}(1)$ is unstable because $\mathcal{O}(1) \subset \mathcal{O} \oplus \mathcal{O}(1)$,

$$\mu(\mathcal{O}(1)) > \mu(\mathcal{O} \oplus \mathcal{O}(1)) = \tfrac{1}{2}$$

Theorem 7.5.8. Let E be a vector bundle on C and L a line bundle on C,

(a) E is (semi)-stable iff $E \otimes L$ is (semi)-stable

(b) E is semistable, $\deg E < 0$ implies $H^0(C, E) = 0$

(c) if E is semi-stable then $\operatorname{Sym}_n(E)$ is semi-stable for $n \geq 1$.

Remark. The last statement is not true for stable instead of semi-stable or in positive characteristic.

7.5.2 Stability For Surfaces

Let H be an ample divisor on a surface X.

Definition 7.5.9. The H-slope is defined,

$$\mu_H(E) := \frac{c_1(E) \cdot H}{\operatorname{rank} E}$$

Definition 7.5.10. For $F \subset E$ we have $c_1(F) = \det F^{\vee\vee}$ is a reflexive sheaf of rank 1 and hence is a line bundle on a surface. Then we can set,

$$\mu_H(F) = \frac{c_1(F) \cdot H}{\operatorname{rank} F}$$

Definition 7.5.11. We say E is H-stable if for every $F \subset E$ with,

$$0 < \operatorname{rank} F < \operatorname{rank} E$$

if $\mu_H(F) < \mu_H(E)$ and semi-stable if $\mu_H(F) \leq \mu_H(E)$.

8 Nov. 1 Bogomolov's Theorem

Let $(X, \mathcal{L})$ be a polarized surface over $\mathbb{C}$.

Definition 8.0.1. A sheaf $\mathscr{F}$ on X is called torsion-free if for all $U \subset X$ open, the group $\mathscr{F}(U)$ is torsion-free module over $\mathcal{O}_X(U)$.

Remark. Recall that $\mathscr{F}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. There is a natural morphism $\mathscr{F} \to \mathscr{F}^{\vee\vee}$.

Proposition 8.0.2. $\mathscr{F}$ is torsion-free iff $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective.

Definition 8.0.3. We say that $\mathscr{F}$ is reflexive if $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorpism.

Proposition 8.0.4. Any reflexive sheaf on a regular $\dim X \leq 2$ scheme is locally free.

Example 8.0.5. Let $p \in |X|$ be a closed point and $\mathscr{I}_p \hookrightarrow \mathcal{O}_X$ the sheaf of ideals. Then $\mathscr{I}_p$ is torsion-free but not locally-free.

Definition 8.0.6. The rank of a torsion-free sheaf $\mathscr{F}$ is defined to be,

$$\operatorname{rank} \mathscr{F} = \ell(\mathscr{F}_{\text{ét}})$$

where $\eta \in X$ is the generic point.

Definition 8.0.7. A sheaf $\mathscr{F}$ is called *μ-semistable* (with respect to $\mathcal{L}$) if $\mathscr{F}$ is torsion-free for all nontrivial proper subsheaves $\mathcal{E} \subset \mathscr{F}$ we have,

$$\frac{c_1(\mathcal{E}) \cdot c_1(\mathcal{L})}{\operatorname{rank} \mathcal{E}} \leq \frac{c_1(\mathcal{L}) \cdot c_1(\mathscr{F})}{\operatorname{rank} \mathscr{F}}$$

Remark. For the definition of μ-stable you need nontrivial proper subsheaves with strictly smaller rank. To see why this is necessary, consider,

$$0 \longrightarrow \mathscr{I}_p \longrightarrow \mathcal{O}_X \longrightarrow k_p \longrightarrow 0$$

Then we get $c_1(\mathscr{I}_p) = 0$ and hence we don't get a strict inequality,

$$\frac{c_1(\mathscr{I}_p) \cdot c_1(\mathcal{L})}{\operatorname{rank} \mathscr{I}_p} \leq \frac{c_1(\mathcal{O}_X) \cdot c_1(\mathcal{L})}{\operatorname{rank} \mathcal{O}_X}$$

Example 8.0.8. (a) if $\mathscr{F}$ has rank 1 then $\mathscr{F}$ is μ-semistable for all polarizatios

(b) if $\mathscr{F}$ is μ-semistable and $\mathscr{H} \in \operatorname{Pic} X$ then $F \otimes \mathscr{H}$ is μ-semistable

(c) if $\mathscr{F}$ is μ-semistable then $\mathscr{F}^{\vee\vee}$ is μ-semistable.

Definition 8.0.9. Let $\mathscr{F}$ be torsion-free of rank r, then $\Delta(\mathscr{F}) = 2rc_2 - (r-1)c_1^2$.

Theorem 8.0.10 (Bogomolov)**.** If $\mathscr{F}$ is μ-semistable on $(X, \mathcal{L})$ then $\Delta(\mathscr{F}) \geq 0$.

Remark. Since $\Delta(\mathscr{F})$ is independent of the polarization, Bogomolov's theorem gives an obstruction to be μ-semistable with respect to *any* polarization.

Proposition 8.0.11. Recall,

$$\operatorname{char} \mathscr{F} = \operatorname{rank} \mathscr{F} + c_1(\mathscr{F}) + \tfrac{1}{2}(c_1^2 - 2c_2)$$

Then we can compute with $r = \operatorname{rank} \mathscr{F}$,

$$\log\left(\frac{\operatorname{char} \mathscr{F}}{r}\right) = \log\left(1 + \square\right) = \left[\frac{c_1}{r} + \frac{c_1^2 - 2c_2}{2r}\right] - \frac{c_1^2}{2r} = \frac{c_1}{r} + \frac{1}{2r^2}\left((r-1)c_1^2 - 2rc_2\right)$$

Therefore,

$$\log\left(\frac{\operatorname{char} \mathscr{F}}{r}\right) = \frac{c_1}{r} + \frac{1}{2r^2}\Delta(\mathscr{F})$$

Because log sends multiplication to addition, we have,

$$\frac{\Delta(\mathscr{F} \otimes \mathscr{G})}{(\operatorname{rank} \mathscr{F})^2(\operatorname{rank} \mathscr{G})^2} = \frac{\Delta(\mathscr{F})}{(\operatorname{rank} \mathscr{F})^2} + \frac{\Delta(\mathscr{G})}{(\operatorname{rank} \mathscr{G})^2}$$

Proposition 8.0.12. (a) if $\mathscr{F}$ is a line bundle then $\Delta(\mathscr{F}) = 0$

(b) if $\mathscr{F}$ is locally free then $\Delta(\mathscr{F}) = \Delta(\mathscr{F}^\vee)$

(c) for $\mathscr{H} \in \operatorname{Pic} X$ we have,

$$\frac{\Delta(\mathscr{F} \otimes \mathscr{H})}{(\operatorname{rank} \mathscr{F})^2 1^2} = \frac{\Delta(\mathscr{F})}{(\operatorname{rank} \mathscr{F})^2} + 0 \implies \Delta(\mathscr{F} \otimes \mathscr{H}) = \Delta(\mathscr{F})$$

(d) if $\mathscr{F}$ is locally free, then $\Delta(\operatorname{End}(\mathscr{F})) = 2(\operatorname{rank} \mathscr{F})^2 \Delta(\mathscr{F})$

(e) if $\mathscr{F}$ is torsion free, then,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^{\vee\vee} \longrightarrow \mathcal{Q} \longrightarrow 0$$

and then,

$$\Delta(\mathscr{F}^{\vee\vee}) = 2rc_2(\mathscr{F}^{\vee\vee}) - (c-1)c_1(\mathscr{F}^{\vee\vee}) = 2r(c_2(\mathscr{F}) + \ell(\mathcal{Q})) - (r-1)c_1(\mathscr{F})^2 = \Delta(\mathscr{F}) + 2r\ell(\mathcal{Q})$$

Remark. By the last property, since $\ell(\mathcal{Q}) \ge 0$ we see that if $\Delta(\mathscr{F}^{\vee\vee}) \le 0$ then $\Delta(\mathscr{F}) \le 0$. Therefore, it suffices to prove the theorem for reflexive and hence locally free $\mathscr{F}$,

Proof. Proof reductions,

(a) can assume $\mathscr{F} \cong \mathscr{F}^{\vee\vee}$ by above remark

(b) $\Delta(\text{End}\,(\mathscr{F})) = 2r^2\Delta(\mathscr{F})$ so can assume that $\det \mathscr{F} = \mathcal{O}_X$.

We need to show that $c_2(\mathscr{F}) \le 0$ for $\mathscr{F}$ such that,

(a) $\mathscr{F}$ is a vector bundle

(b) $\det \mathscr{F} \cong \mathcal{O}_X$

(c) $\mathscr{F}$ is μ-semistable for some $\mathcal{L}$.

Consider,

$$\mathscr{F}_n = \text{Sym}_{nr}(\mathscr{F})$$

We use the following lemmas. ◻

Lemma 8.0.13. (a) $\det \mathscr{F}_n \cong \mathcal{O}_X$

(b) there is a formula,

$$\chi(X, \mathscr{F}_n) = -\frac{\Delta(\mathscr{F})n^{r+1}r^r}{2(r+1)!} + O(n^r)$$

Proof. Represent $\mathscr{F}$ as $[\xi] \in H^1(X, \text{SL}_n)$ because $\det \mathscr{F} \cong \mathcal{O}_X$. Then $\text{SL}_r \circlearrowright \text{Sym}_{nr}(\mathbb{C}^r)$ gives an action $\text{SL}_r \circlearrowright \det \text{Sym}_{nr}(\mathbb{C}^r)$ which is a character of SL_r and hence is trival. Therefore, the map $H^1(X, \text{SL}_r) \to H^1(X, \mathbb{G}_m)$ given by taking determinants is trivial.

Consider, $\pi : \mathbb{P}_X(\mathscr{F}) \to X$. Look at $\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1)$. Then,

$$\chi(\mathbb{P}(\mathscr{F}), \mathcal{O}_{\mathbb{P}(\mathscr{F})}(nr)) = \chi(X, R\pi_* \mathcal{O}_{\mathbb{P}(\mathscr{F})}(nr)) = \chi(X, \text{Sym}_{nr}(\mathscr{F})) = \chi(X, \mathscr{F}_n)$$

Now, by Riemann-Roch we have,

$$\chi(\mathbb{P}(\mathscr{F}), \mathcal{O}_{\mathbb{P}(\mathscr{F})}(nr)) = \frac{(nr)^{r+1}c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r+1}}{(r+1)!} + O(n^r)$$

But by the projective bundle formula (or the Grothendieck definition of Chern classes),

$$c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^r - \pi^* c_1(\mathscr{F}) c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r-1} + \pi^* c_2(\mathscr{F}) c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r-2} = 0$$

We assumed that $c_1(\mathscr{F}) = 0$. Therefore,

$$c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r+1} = -\pi^* c_2(\mathscr{F}) c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r-1}$$

Now we have,

$$\deg\left(c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r+1}\right) = -\deg \pi_*(\pi^* c_2(\mathscr{F}) c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r-1}) = -\deg\left(c_2(\mathscr{F}) \cdot \pi_*[c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1)^{r-1})]\right)$$

Now $\pi_*[c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1))^{r-1}] = [X]$ since on each fiber since this is H^{r-1} on $\mathbb{P}^{r-1}$ where H is the hyperplane class. Since $c_1(\mathscr{F}) = 0$ we have $\Delta(\mathscr{F}) = 2rc_2(\mathscr{F})$ and thus,

$$\chi(X, \mathscr{F}_n) = -\frac{\Delta(\mathscr{F}) n^{r+1} r^r}{2(r+1)!} + O(n^r)$$

$\square$

Remark. We explicitly complete the GRR calculaiton. Let $\widetilde{X} = \mathbb{P}_X(\mathscr{F})$. By GRR,

$$\chi(\widetilde{X}, \mathscr{G}) = \deg\left(\mathrm{char}(\mathscr{G}) \cdot \mathrm{td}_{\widetilde{X}}\right)$$

and because $R\pi_* \mathcal{O}_{\widetilde{X}}(nr) = \mathrm{Sym}_{nr}(\mathscr{F})\,[0]$ we have that,

$$\chi(X, \mathrm{Sym}_{nr}(\mathscr{F})) = \chi(X, R\pi_* \mathcal{O}_{\widetilde{X}}(nr)) = \chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(nr)) = \deg\left(\mathrm{char}(\mathcal{O}_{\widetilde{X}}(nr)) \cdot \mathrm{td}_{\widetilde{X}}\right)$$

Now let $\xi = c_1(\mathcal{O}_{\widetilde{X}}(1))$ then we see,

$$\chi(X, \mathrm{Sym}_{nr}(\mathscr{F})) = \deg\left(e^{nr\xi} \cdot \mathrm{td}_{\widetilde{X}}\right)$$

Let $d = \dim \widetilde{X} = r - 1 + \dim X = r + 1$. Then the leading term as a polynomial in n gives,

$$\chi(X, \mathrm{Sym}_{nr}(\mathscr{F})) = \frac{(nr)^d \xi^d}{d!} \cdot 1 + O(n^{d-1})$$

because the first term of $\mathrm{td}_{\widetilde{X}}$ is 1. This gives,

Proof. Now we complete the proof. From the lemma, to show that $\Delta(\mathscr{F}) \le 0$ it suffices to show that $\chi(X, \mathscr{F}_n) \le Cn^r$ as $n \to \infty$. This will follow if we show that $H^0(X, \mathscr{F}_n) \le C_1 n^r$ as $n \to \infty$ and $H^2(X, \mathscr{F}_n) \le C_2 n^r$ as $n \to \infty$. $\square$