Math GR6262 Algebraic Geometry
Assignment # 7

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1 Exercise 103.42.2

Let M be an A-module and $I = (f_1, \cdots, f_t)$ a finitely generated ideal. Now take $U = V(I)^C = D(f_1) \cup \cdots \cup D(f_t)$. Therefore, $D(f_i)$ gives an affine open cover of U so we may apply Cech cohomology to compute sheaf cohomology of quasi-coherent sheaves. The Cech complex $\check{C}^\bullet(\{D(f_i)\}, \widetilde{M})$ is,

$$0 \longrightarrow \prod_{i=1}^{t} \widetilde{M}(D(f_i)) \longrightarrow \prod_{i<j}^{t} \widetilde{M}(D(f_i) \cap D(f_j)) \longrightarrow \cdots \longrightarrow \widetilde{M}(D(f_1) \cap \cdots \cap D(f_t)) \longrightarrow 0$$

which is equal to,

$$0 \longrightarrow \prod_{i=1}^{t} \widetilde{M}(D(f_i)) \longrightarrow \prod_{i<j}^{t} \widetilde{M}(D(f_i f_j)) \longrightarrow \cdots \longrightarrow \widetilde{M}(D(f_1 \cdots f_t)) \longrightarrow 0$$

and therefore, using the defining property $\widetilde{M}(D(f)) = M_f$ the Cech complex becomes,

$$0 \longrightarrow \prod_{i=1}^{t} M_{f_i} \longrightarrow \prod_{i<j}^{t} M_{f_i f_j} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_t} \longrightarrow 0$$

whose cohomology gives the Cech cohomology and thus the sheaf cohomology,

$$\check{H}(\{D(f_i)\}, \widetilde{M}) \cong H(U, \widetilde{M})$$

which agree in this case because U is a separated scheme (see Tag 0BDX and Tag 01XD). To see why U is separated apply Lemma 6.0.1, using the fact that U is a subscheme of the affine scheme $\mathrm{Spec}\,(A)$ which is automatically separated.

2 Exercise 103.42.3

It will be convenient to label variables as,

$$\mathbb{A}^d_k = \mathrm{Spec}\,(k[x_0, \cdots, x_{d-1}])$$

and $n = d-1$ to line up with the definitions in projective space. Consider the projection morphism $\pi : \mathbb{A}^{n+1}_k \setminus \{(x_1, \ldots, x_n)\} \to \mathbb{P}^n_k$ and let $U = \mathbb{A}^d_k \setminus \{(x_1, \ldots, x_n)\}$ and $X = \mathbb{P}^n_k$. The schemes $D_+(X_i)$ for each variable X_i constitute an affine open cover of $\mathbb{P}^n_k$. Furthermore, $\pi^{-1}(D_+(X_i)) = D(x_i) \subset$

$k[x_1, \dots, x_d]$. Therefore, π is an affine morphism and $\mathcal{O}_U$ is a quasi-coherent $\mathcal{O}_U$-module so we have shown that,

$$H^q(\mathbb{P}^n_k, \pi_* \mathcal{O}_U) = H^q(U, \mathcal{O}_U)$$

Furthermore, denote $S = k[x_0, \cdots, x_n]$, then,

$$\pi_* \mathcal{O}_U|_{D_+(X_i)} = \mathcal{O}_U|_{D(x_i)} = \mathcal{O}_{\mathbb{A}^{n+1}_k}|_{D(x_i)} = \widetilde{S_{x_i}} = \bigoplus_{k \in \mathbb{Z}} \widetilde{(S_{x_i})_k} = \bigoplus_{k \in \mathbb{Z}} (\widetilde{S(k)_{x_i}})_0 = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_X(k)|_{D_+(X_i)}$$

Thus, because the sheaves agree on an open affine cover, we can identify,

$$\pi_* \mathcal{O}_U = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_X(k)$$

Hartshorne has computed the cohomology of the sum of twists (Hartshorne III.5, Theorem 5.1) to be,

$$H^q \left(X, \bigoplus_{n \in k} \mathcal{O}_X(k) \right) = \begin{cases} k[X_0, \cdots, X_n] & q = 0 \\ 0 & 0 < q < n \\ \frac{1}{X_0 \cdots X_n} k[X_0^{-1}, \dots, X_n^{-1}] & q = n \end{cases}$$

Reverting to our initial notation and using the isomorphism $H^q(X, \pi_* \mathcal{O}_U) = H^q(U, \mathcal{O}_U)$ we arrive at,

$$H^q(U, \mathcal{O}_U) = \begin{cases} k[x_1, \cdots, x_d] & q = 0 \\ 0 & 0 < q < n \\ \frac{1}{x_1 \cdots x_d} k[x_1^{-1}, \dots, x_d^{-1}] & q = d - 1 \end{cases}$$

3 Exercise 103.42.4

Let k be a field and $Y = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Let $\pi_i : Y \to \mathbb{P}^1_k$ be the projection maps. Now consider the invertable sheaves on Y,

$$\mathcal{O}_Y(a, b) = \pi_1^* \mathcal{O}_{\mathbb{P}^1_k}(a) \otimes_{\mathcal{O}_Y} \pi_2^* \mathcal{O}_{\mathbb{P}^1_k}(b)$$

The Künneth formula allows us to compute the cohomology of such sheaves via,

$$H^n(Y, \mathcal{O}_Y(a, b)) = \bigoplus_{p+q=n} H^p(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a)) \otimes_k H^q(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b))$$

However, we have computed the cohomology of the twists previously,

$$H^p(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a)) = \begin{cases} (k[X_0, X_1])_a & p = 0 \\ \left(\frac{1}{X_0 X_1} k[X_0^{-1}, X_1^{-1}] \right)_a & p = 1 \\ 0 & p \neq 0, 1 \end{cases}$$

Consider the case $a, b > 0$ then we have,

$$H^n(Y, \mathcal{O}_Y(a, b)) = H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a)) \otimes_k H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b)) = (k[X_0, X_1])_a \otimes (k[Y_0, Y_1])_b$$

which is exactly the ring of bigraded polynomials of bidegree (a, b). An injective $\mathcal{O}_Y$-module map $\mathcal{O}_Y \to \mathcal{O}_Y(a, b)$ is defined by $1 \mapsto F$ for some regular section $F \in H^0(Y, \mathcal{O}_Y(a, b))$ which is some (a, b)-bigraded polynomial. Then $\mathcal{O}_Y(a, b)$ and F define an effective Cartier divisor $X \subset Y$ given by the vanishing of F whose inverse ideal sheaf is $\mathcal{O}_Y(a, b)$ (see Tag 01X0) which is a locally principally closed subscheme here of codimension 1. Since $\dim_k Y = 2$ we have $\dim_k X = 1$. Furthermore, there are closed immersions,

$$X \hookrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k \hookrightarrow \mathbb{P}^3_k$$

given by the Segre embedding showing that X is a projective scheme over k. Since $\mathcal{O}_Y(a,b)$ is the inverse of the sheaf of ideals defining $X \subset Y$ there exists an exact sequence of $\mathcal{O}_Y$-modules,

$$0 \longrightarrow \mathcal{O}_Y(-a,-b) \longrightarrow \mathcal{O}_Y \longrightarrow \iota_* \mathcal{O}_X \longrightarrow 0$$

Since $\iota : X \to Y$ is a closed immersion and thus affine, we may identify $H^n(Y, \iota_* \mathcal{O}_X) = H^n(X, \mathcal{O}_X)$, taking the long exact sequence of cohomology, we find,

$$\begin{tikzcd}
0 \arrow[r] & H^0(Y, \mathcal{O}_Y(-a,-b)) \arrow[r] & H^0(Y, \mathcal{O}_Y) \arrow[r] & H^0(X, \mathcal{O}_X) \arrow[dll, rounded corners] \\
& H^1(Y, \mathcal{O}_Y(-a,-b)) \arrow[r] & H^1(Y, \mathcal{O}_Y) \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[dll, rounded corners] \\
& H^2(Y, \mathcal{O}_Y(-a,-b)) \arrow[r] & H^2(Y, \mathcal{O}_Y) \arrow[r] & H^2(X, \mathcal{O}_X) \arrow[r] & \cdots
\end{tikzcd}$$

Now $H^0(Y, \mathcal{O}_Y) = k$ and $H^p(Y, \mathcal{O}_Y) = 0$ for $p > 0$ and in the case $a, b > 0$ we have $H^0(Y, \mathcal{O}_Y(-a,-b)) = 0$ since there is no negative graded part of $k[X_0, X_1]$ and likewise,

$$\begin{aligned}
H^1(Y, \mathcal{O}_Y(-a,-b)) = & H^0(\mathbb{P}^1_k, \mathcal{O}_X(-a)) \otimes_k H^1(\mathbb{P}^1_k, \mathcal{O}_X(-b)) \\
& \oplus H^1(\mathbb{P}^1_k, \mathcal{O}_X(-a)) \otimes_k H^0(\mathbb{P}^1_k, \mathcal{O}_X(-b)) = 0
\end{aligned}$$

since one of the factors is zero in both cases. Plugging into the long exact sequence gives exact sequences,

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow 0$$

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y(-a,-b)) \longrightarrow 0$$

which are thus isomorphisms. Finally,

$$\begin{aligned}
\dim_k H^2(Y, \mathcal{O}_Y(-a,-b)) &= \dim_k \left(H^1(\mathbb{P}^1_k, \mathcal{O}_X(-a)) \otimes_k H^1(\mathbb{P}^1_k, \mathcal{O}_X(-b)) \right) \\
&= \dim_k \left(\frac{1}{X_0 X_1} k[X_0^{-1}, X_1^{-1}] \right)_{-a} \cdot \dim_k \left(\frac{1}{Y_0 Y_1} k[Y_0^{-1}, Y_1^{-1}] \right)_{-b} \\
&= (-a+1)(-b+1) = (a-1)(b-1)
\end{aligned}$$

and thus,

$$\begin{aligned}
& H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) = k \\
& H^1(X, \mathcal{O}_X) = H^2(Y, \mathcal{O}_Y(-a,-b)) \implies \dim_k H^1(X, \mathcal{O}_X) = (a-1)(b-1)
\end{aligned}$$

For example, take the $(11, 11)$-bigraded polynomial,

$$F = X_0^5 X_1^6 Y_0^6 Y_1^5 + X_0^6 X_1^5 Y_0^5 Y_1^6 \in H^0(X, \mathcal{O}_X(11, 11))$$

Then the curve $X = V(F) \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ defined by the vanishing of F has,

$$\begin{aligned}
H^0(X, \mathcal{O}_X) &= k \\
\dim_k H^1(X, \mathcal{O}_X) &= (11-1)(11-1) = 100
\end{aligned}$$

and X is a projective scheme over k of dimension 1.

4 Exercise 103.42.6

Let X be a locally ringed space. Notate by $\mathcal{O}_X^\times$, the sheaf of abelian groups given by $U \mapsto \mathcal{O}_X(U)^\times$. Now let $\mathcal{L}$ be an invertable sheaf on X meaning that there exists an open cover $\mathfrak{U}$ such that for each $U \in \mathfrak{U}$ we have isomorphisms $\varphi_U : \mathcal{O}_X|_U \to \mathcal{L}|_U$. Therefore, on the overlaps we have isomorphism,

$$\varphi_{ij} = \varphi_{U_i}^{-1}|_{U_i \cap U_j} \circ \varphi_{U_j}|_{U_i \cap U_j} : \mathcal{O}_X|_{U_i \cap U_j} \to \mathcal{O}_X|_{U_i \cap U_j}$$

which, as $\mathcal{O}_X|_{U_i \cap U_j}$-module maps are determined uniquely by $e_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$ which is a unit because the map it defines is an isomorphism. Thus, $e = (e_{ij})_{ij}$ is an element of the first Cech complex group, $\check{C}^1(\mathfrak{U}, \mathcal{O}_X^\times)$. Consider the Cech complex,

$$0 \longrightarrow \prod_{i_0} \mathcal{O}_X^\times(U_{i_0}) \longrightarrow \prod_{i_0 < i_1} \mathcal{O}_X^\times(U_{i_0} \cap U_{i_1}) \longrightarrow \prod_{i_0 < i_1 < i_2} \mathcal{O}_X^\times(U_{i_0} \cap U_{i_1} \cap U_{i_2})$$

Furthermore, on triple overlaps,

$$\begin{aligned} \varphi_{ij}|_{ijk} \circ \varphi_{jk}|_{ijk} &= \varphi_{U_i}^{-1}|_{U_{ijk}} \circ \varphi_{U_j}|_{U_{ijk}} \circ \varphi_{U_j}^{-1}|_{U_{ijk}} \circ \varphi_{U_k}|_{U_{ijk}} \\ &= \varphi_{U_i}^{-1}|_{U_{ijk}} \circ \varphi_{U_k}|_{U_i \cap U_j \cap U_k} = \varphi_{ik}|_{ijk} \end{aligned}$$

which clearly implies that $e_{ij}|_{U_{ijk}} \cdot e_{jk}|_{U_{ijk}} = e_{ik}|_{U_{ijk}}$. However, the Cech differential map $\mathrm{d} : \check{C}^1(\mathfrak{U}, \mathcal{O}_X^\times) \to \check{C}^2(\mathfrak{U}, \mathcal{O}_X^\times)$ acts via,

$$(\mathrm{d}\alpha)_{ijk} = \alpha_{jk}|_{U_{ijk}} \cdot \alpha_{ik}^{-1}|_{U_{ijk}} \cdot \alpha_{ij}|_{U_{ijk}}$$

Therefore, by the overlap identity,

$$(\mathrm{d}e)_{ijk} = e_{jk}|_{U_{ijk}} \cdot e_{ik}|_{U_{ijk}}^{-1} \cdot e_{ij}|_{U_{ijk}} = 1$$

Thus e is in the kernel of the Cech differential $\mathrm{d} : \check{C}^1(\mathfrak{U}, \mathcal{O}_X^\times) \to \check{C}^2(\mathfrak{U}, \mathcal{O}_X^\times)$ and thus e represents a Cech cohomology class $[e] \in \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$. Furthermore, if $\tilde{\varphi}_{U_i} : \mathcal{O}_X|_{U_i} \to \mathcal{L}|_{U_i}$ is another choice of locally trivializing isomorphisms then denote $\tilde{e}_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ for the element determining the isomorphisms,

$$\tilde{\varphi}_{ij} = \tilde{\varphi}_{U_i}^{-1}|_{U_{ij}} \circ \tilde{\varphi}_{U_j}|_{U_{ij}} : \mathcal{O}_X|_{U_i \cap U_j} \to \mathcal{O}_X|_{U_i \cap U_j}$$

Then we may consider the isomorphisms $t_i = \tilde{\varphi}_{U_i}^{-1} \circ \varphi_{U_i} : \mathcal{O}_X|_{U_i} \to \mathcal{O}_X|_{U_i}$ which are defined by an element $f_i \in \mathcal{O}_X^\times(U_i)$. Then we find that,

$$\begin{aligned} \tilde{\varphi}_{ij} = \tilde{\varphi}_{U_i}^{-1}|_{U_{ij}} \circ \tilde{\varphi}_{U_j}|_{U_{ij}} &= \tilde{\varphi}_{U_i}^{-1}|_{U_{ij}} \circ \varphi_{U_i}|_{U_{ijk}} \circ \varphi_{U_i}^{-1}|_{U_{ijk}} \circ \varphi_{U_j}|_{U_{ijk}} \circ \varphi_{U_j}|_{U_{ijk}}^{-1} \circ \tilde{\varphi}_{U_j}|_{U_{ij}} \\ &= t_i|_{U_{ij}} \circ \varphi_{ij} \circ t_j^{-1}|_{U_{ij}} \end{aligned}$$

This shows that the elements must satisfy, $\tilde{e}_{ij} \cdot e_{ij}^{-1} = f_i|_{U_{ij}} \cdot f_j^{-1}|_{U_{ij}}$. Furthermore, the Cech differential map $\mathrm{d} : \check{C}^0(\mathfrak{U}, \mathcal{O}_X^\times) \to \check{C}^1(\mathfrak{U}, \mathcal{O}_X^\times)$ acts via,

$$(\mathrm{d}\alpha)_{ij} = \alpha_i|_{U_{ij}} \cdot \alpha_j^{-1}|_{U_{ij}}$$

Therefore, let $f = (f_i)_i$ then $\mathrm{d}f = \tilde{e} \cdot e^{-1}$ which implies that $[\tilde{e}] = [e]$ in $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$ so the cohomology class $[e]$ associated to the invertable sheaf $\mathcal{L}$ is well-defined. The map $\mathcal{L} \mapsto [e]$ is well-defined for sheaves which are locally trivialized on $\mathfrak{U}$. Therefore we get a well-defined map,

$$\mathrm{Pic}\,(X) \to \check{H}^1(X, \mathcal{O}_X^\times) = \varinjlim_{\mathfrak{U}} \check{H}(\mathfrak{U}, \mathcal{O}_X^\times)$$

via decomposing,

$$\text{Pic}\,(X) = \bigcup_{\mathfrak{U}} \text{Pic}\,(\mathfrak{U}, X) \quad \text{where} \quad \text{Pic}\,(\mathfrak{U}, X) = \{\mathcal{L} \in \text{Pic}\,(X) \mid \forall U \in \mathfrak{U} : \mathcal{L}|_U \cong \mathcal{O}_U\}$$

and mapping,

$$\text{Pic}\,(\mathfrak{U}, X) \to \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \to \varinjlim_{\mathfrak{U}} \check{H}(\mathfrak{U}, \mathcal{O}_X^\times) = \check{H}^1(X, \mathcal{O}_X^\times)$$

using the constructed map. This map is an homomorphism because given invertable sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ and isomorphisms $\varphi^r_{U_i} : \mathcal{O}_X|_{U_i} \to \mathcal{L}_r$ corresponding to cohomology classes $[e^r]$ then there is a natural map,

$$\varphi^1_{U_i} \otimes \varphi^2_{U_i} : \mathcal{O}_X|_{U_i} \to \mathcal{L}_1|_{U_i} \otimes_{\mathcal{O}_X|_{U_i}} \mathcal{L}_2|_{U_i}$$

which therefore gives overlap maps,

$$\varphi^{\otimes}_{ij} = ((\varphi^1_{U_i})^{-1} \circ \varphi^1_{U_j}) \otimes ((\varphi^2_{U_i})^{-1} \circ \varphi^2_{U_j}) = \varphi^1_{ij} \otimes \varphi^2_{ij}$$

and thus, $\varphi^{\otimes}_{ij}(1) = e^1_{ij} \otimes e^2_{ij} \mapsto e^1_{ij} e^2_{ij}$ under the natural identification,

$$\mathcal{O}_X(U_{ij}) \otimes_{\mathcal{O}_X(U_{ij})} \mathcal{O}_X(U_{ij}) \to \mathcal{O}_X(U_{ij})$$

Therefore, the invertable sheaf $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ maps to the cohomology class $[e^1 e^2] = [e^1][e^2]$ so this map is a homomorphism.

I claim that this map is, in fact, an isomorphism. Let $\mathcal{L}$ be an invertable sheaf represented by the cohomology class $[e] = [1]$ then we know that $e_{ij} = t_i|_{U_{ij}} \cdot t_j^{-1}|_{U_{ij}}$ for some set of invertable sections t_i. Therefore, modify the isomorphism $\varphi_{U_i} : \mathcal{O}_X|_{U_i} \to \mathcal{L}|_{U_i}$ which gave rise to this cohomology representative via $\tilde{\varphi}_{U_i} = t_i \varphi_{U_i}$ which are still isomorphism because $t_i \in \mathcal{O}_X(U_i)^\times$ is invertable. Therefore,

$$\tilde{\varphi}^{-1}_{U_i}|_{U_{ij}} \circ \tilde{\varphi}_{U_j}|_{U_{ij}} = (t_i|^{-1}_{U_{ij}} \cdot t_j|_{U_{ij}}) \varphi^{-1}_{U_i}|_{U_{ij}} \circ \varphi_{U_j}|_{U_{ij}} = \text{id}_{\mathcal{O}_X(U_{ij})}$$

this map takes $1 \mapsto (t_i|^{-1}_{U_{ij}} \cdot t_j|_{U_{ij}}) e_{ij} = 1$ so as a morphism of $\mathcal{O}_X|_{U_{ij}}$-modules is the identity map. Thus $\tilde{\varphi}_{U_i}|_{U_{ij}} = \tilde{\varphi}_{U_j}|_{U_{ij}}$, so the isomorphisms $\tilde{\varphi}_{U_i} \in \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_X|_{U_i}, \mathcal{L}|_{U_i})$ glue since they agree on this open cover to a global isomorphism $\tilde{\varphi} : \mathcal{O}_X \to \mathcal{L}$ so $\mathcal{L}$ is a trivial invertable sheaf. Thus $\text{Pic}\,(X) \to \check{H}^1(X, \mathcal{O}_X)$ is injective. It remains to prove that it is surjective. Given any cohomology class $[e] \in \check{H}^1(X, \mathcal{O}_X^\times)$ we may construct an invertable sheaf as follows. Define $\mathcal{L}$ via,

$$\mathcal{L}(V) = \{f_i \in \mathcal{O}_X(U_i \cap V) \mid f_i|_{U_{ij} \cap V} \cdot e_{ij}|_{U_{ij} \cap V} = f_j|_{U_{ij} \cap V}\}$$

It is clear that this is an invertable sheaf if e_{ij} satisfies the transition property given by its Cech differential vanishing and that $\mathcal{L} \mapsto [e]$.

Finally, we use the general fact that $H^1(X, \mathscr{F}) = \check{H}^1(X, \mathscr{F})$ to conclude that,

$$\text{Pic}\,(X) \cong H^1(X, \mathcal{O}_X^\times)$$

5 Exercise 103.42.7

On a previous homework assignment we showed that the affine variety,

$$X = \text{Spec}\left(k[x, y]/(y^2 - f(x))\right)$$

where k is a field and $f(x) = (x - t_1) \cdots (x - t_n)$ for $n \geq 3$ and odd admits nontrivial invertable sheaves so that $\text{Pic}\,(X)$ is nontrivial. Thus $H^1(X, \mathcal{O}_X^\times) = \text{Pic}\,(X)$ which implies that $H^1(X, -)$ is not the zero functor even though X is affine.

6 Lemmas

Lemma 6.0.1. Let X be a separated scheme and $Z \to X$ an injection then Z is separated.

Proof. Consider the map,

$$Z \hookrightarrow X \longrightarrow \operatorname{Spec}(\mathbb{Z})$$

The second map is separated by definition. The first map is separated because it is an injection (see Tag 0DVA). Since the composition of separated maps is separated, then Z is a separated scheme. $\square$