Math GR6262 Algebraic Geometry
Assignment # 10

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1 Tag: 0294

Consider the morphism of affine schemes,

$$\pi : X = \text{Spec}\left(\mathbb{C}[x, t, 1/(xt)]\right) \to S = \text{Spec}\left(\mathbb{C}[t]\right)$$

induced by the inclusion $\tilde{\pi} : \mathbb{C}[t] \to \mathbb{C}[x, t, 1/xt]$. First any map $\sigma : S \to X$ such that $\pi \circ \sigma = \text{id}_S$ is equivalent to a ring map, $\tilde{\sigma} : \mathbb{C}[x, t, 1/xt] \to \mathbb{C}[t]$ such that $\tilde{\sigma} \circ \tilde{\pi} = \text{id}$. However, t is not invertible in $\mathbb{C}[t]$ but xt is invertible in $\mathbb{C}[x, t, 1/xy]$ and thus would map to an invertible element of $\mathbb{C}[t]$ yet the condition $\tilde{\sigma} \circ \tilde{\pi} = \text{id}$ forces $\tilde{\sigma} : t \mapsto t$ which is a contradiction. Thus there cannot exist such a map $\sigma : S \to X$.

Now consider the open set $U \subset S$ given by the standard open $D(t) \subset \text{Spec}\left(\mathbb{C}[t]\right)$ and thus $U = \text{Spec}\left(\mathbb{C}[t, t^{-1}]\right)$. Now consider the $\mathbb{C}$-linear map $\tilde{\sigma} : \mathbb{C}[x, t, 1/xt] \to \mathbb{C}[t, t^{-1}]$ given by $x \mapsto t$ and $t \mapsto t$. Then $1/xt \mapsto 1/t^2$ is well-defined. Furthermore, $\tilde{\sigma} \circ \tilde{\pi} = \iota$ where $\iota : \mathbb{C}[t] \to \mathbb{C}[t, t^{-1}]$ is the inclusion corresponding the open embedding $U \subset S$ via,

$$\mathbb{C}[t] \to \mathbb{C}[x, t, 1/(xt)] \to \mathbb{C}[t, t^{-1}]$$

sends $t \mapsto t^{-1}$. Thus by the antiequivalence of rings and affine schemes we have a map $\sigma : U \to X$ such that $\pi \circ \sigma = \text{id}_U$ in the sense of the embedding $U \to S$.

2 Tag: 0295

Consider the morphism of affine schemes,

$$\pi : X = \text{Spec}\left(\mathbb{C}[x, t]/(x^2 + t)\right) \to S = \text{Spec}\left(\mathbb{C}[t]\right)$$

induced by the inclusion $\tilde{\pi} : \mathbb{C}[t] \to \mathbb{C}[x, t]/(x^2 + t)$. Let $U \subset S$ be an nonempty open subscheme and $\sigma : U \to X$ a section such that $\pi \circ \sigma = \iota_U$ where $\iota_U : U \to S$ is the inclusion. Because S is affine any nonepty open contains a standard open $D(f) \subset U$ for some $f \in \mathbb{C}[t]$ and then restriction we have a map $\sigma : D(f) \to X$ such that $\pi \circ \sigma = \iota_{D(f)}$ where $\iota_{D(f)} : D(f) \to S$ is the inclusion induced by the map $\tilde{\iota} : \mathbb{C}[t] \to \mathbb{C}[t]_f$. Becuse these schemse are affine, such are equivalent to a ring map $\tilde{\sigma} : \mathbb{C}[x, t]/(x^2 + t) \to \mathbb{C}[t]_f$ such that $\tilde{\sigma} \circ \tilde{\pi} = \tilde{\iota}$ which implies that $\tilde{\sigma} : t \mapsto t$. Let $\tilde{\sigma} : x \mapsto g$ then $g^2 + t = 0$ in $\mathbb{C}[t]_f$. Then write,

$$g = \frac{a}{f^n} \in \mathbb{C}[t]_f \qquad a \in \mathbb{C}[t]$$

and since $\mathbb{C}[t]_f$ is a domain we must have, $a^2 + f^{2n}t = 0$ in $\mathbb{C}[t]$. This is impossible because the degree of a^2 is even but the degree of $f^{2n}t$ is odd. Therefore there cannot exist such a section.

3 Tag: 0299

Consider the scheme

$$X = \mathrm{Spec}\left(\mathbb{Z}[x, \frac{1}{x(x-1)(2x-1)}\right) = \mathrm{Spec}\left(\frac{\mathbb{Z}[x,y]}{\big(x(x-1)(2x-1)y-1\big)}\right)$$

Choose the closed subscheme $Z \subset X$ by taking the ideal,

$$I = \big(x^2 - 3x + 1, y^2 - 18y + 1, x(x-1)(2x-1)y - 1\big) \supset \big(x(x-1)(2x-1)y - 1\big)$$

Then take,

$$Z = \mathrm{Spec}\,(\mathbb{Z}[x,y]/I) \subset X$$

It is clear that $\mathbb{Z}[x,y]/I$ is finite as a $\mathbb{Z}$-module because, x and y both satisfy monic (single variable) polynomials i.e. their solutions are algebraic integers α, β making,

$$\mathbb{Z}[x,y]/(x^2 - 3x + 1, y^2 - 18y + 1) = \mathbb{Z}[\alpha, \beta]$$

a $\mathbb{Z}$-module of finite type. I claim that I is not the unit ideal because there exist simultaneous solutions to the polynomials in $\overline{\mathbb{Q}}$. In particular, let,

$$\alpha = \tfrac{1}{2}(3 - \sqrt{5}) \qquad \beta = 9 + 4\sqrt{5}$$

which solve all three polynomials. Then since $1 \notin I$ we have maximal ideals $(p, I) \in V(I) = \mathrm{Spec}\,(\mathbb{Z}[x,y]/I) = Z$ for each prime $p \in \mathbb{Z}$ which map down to $(p) \subset \mathbb{Z}$ under the map $Z \to \mathrm{Spec}\,(\mathbb{Z})$. Therefore, the map $Z \to \mathrm{Spec}\,(\mathbb{Z})$ is surjective.

4 Tag: 02EO

A famous theorem of Ernst S. Selmer says that the cubic,

$$3x^3 + 4y^3 + 5z^3 = 0$$

has nontrivial solutions over $\mathbb{F}_p$ for each prime $p \in \mathbb{Z}$ (in fact also for prime powers) but has non nontrivial rational solutions. We will use this fact to construct an example.

Consider the $\mathbb{Z}$-algebra,

$$R = \mathbb{Z}[x, y, z, x', y', z']/(3x^3 + 4y^3 + 5z^3, xx' + yy' + zz' - 1)$$

First, I claim that there is no ring map $R \to \mathbb{Q}$ which is equivalent to a $\mathbb{Z}$-algebra map because we send $1 \mapsto 1$. This is equivalent $\mathbb{Z}$-algebra map $\mathbb{Z}[x, y, z, x', y', z'] \to \mathbb{Q}$ which sends the polynomials $3x^3 + 4y^3 + 5z^3$ and $xx' + yy' + zz' - 1$ to zero and thus is determined by the image of the variables x, y, z, x', y', z'. However, there are no nontrivial rational solutions to $3x^3 + 4y^3 + 5z^3 = 0$ which implies that $x \mapsto 0$ and $y \mapsto 0$ and $z \mapsto 0$ under such a map. Thus $xx' + yy' + zz' - 1 \mapsto -1$ and thus no ring map $\mathbb{Z}[x, y, z, x', y', z'] \to \mathbb{Q}$ can factor through the quotient.

Now for each $p \in \mathbb{Z}$ prime there exists $(a_x, a_y, a_z) \in \mathbb{F}_p^3$ not all zero such that $(x, y, z) \mapsto (a_x, b_y, c_z)$ sends $3x^3 + 4y^3 + z^3$ to zero. Choose one $v \in \{x, y, z\}$ such that $v \mapsto a_v$ with $a_v \neq 0$. Then send

$v' \mapsto a_v^{-1}$. For all other variables $w \in \{x, y, z\}$ not equal to v send $w' \mapsto 0$. Thus the polynomial, $xx' + yy' + zz' - 1 \mapsto 0$ because all products are zero except for exactly one pair which multiplies to 1. This gives a ring map $\mathbb{Z}[x, y, z, x', y', z'] \to \mathbb{F}$ sending the ideal $I = (3x^3 + 4y^3 + 5z^3, xx' + yy' + zz' - 1)$ to zero and thus a map $R \to \mathbb{F}_p$. However, any ring map $R \to \mathbb{F}_p$ is surjective because 1 generates $\mathbb{F}_p$ as an abelian group. Therefore the surjective map $R \to \mathbb{F}_p$ factors through the quotient by the kernel $\mathfrak{m} \subset R$ to give an isomorphism $R/\mathfrak{m} \cong \mathbb{F}_p$. Since $\mathbb{F}_p$ is a field, this implies that the kernel $\mathfrak{m}$ is a maximal ideal completing the proof.