1 Elementary Situation

Fix r points in general possition $x_1, \dots, x_r \in \mathbb{P}^2$. How many curves of degree d pass through these poitns? Equivalently, compute $n = h^0(\mathbb{P}^2, \mathcal{O}(d)(-x_1 - \cdots - x_r))$. We expect,

$$n = \max\{\tfrac{1}{2}(d+1)(d+2) - r, 0\}$$

Proof. By induction it suffices to show that X_r is not a base point of $|\mathcal{O}(d)(-x_1 - \cdots - x_{r-1})$. Indeed we can pick $x_r \notin C$. $\square$

What about,

$$H^0(\mathbb{P}^2, \mathcal{O}(d)(-2x_1 - \cdots - 2x_r))$$

i.e. linear series of degree d curves singular at x_i? We expect,

$$n = \max\{\tfrac{1}{2}(d+1)(d+2) - 3r, 0\}$$

There are only two sets of exceptions: $d = 2, r = 2$ and $d = 4$, $r = 5$ in which case we expect $\dim = 0$ but $\dim = 1$ since there is a family of singular conics.

Lemma 1.1. On $\mathbb{P}^n$, suppoose singular general points at $x_1, \dots, x_r$ fail to impose independent conditions on $\mathcal{O}(d)$ then there exists a curve containting $x_1, \dots, x_r$ such that any elment of $|\mathcal{O}(d)|$ singular on the x_i is singular on C.

Proof. Consider the embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ defined by $|\mathcal{O}(d)|$. Then sections of $\mathcal{O}(d)$ correspond to hyperplanes of $\mathbb{P}^N$ and H_d is singular at x_i iff H contains $T_{x_i}\mathbb{P}^n$. Then general x_i fail to impose independent conditions iff $\langle T_{x_1}\mathbb{P}^n, \dots, T_{x_r}\mathbb{P}^n \rangle$ fails to have the maximum dimension. Consider the $(r-1)$-secant variety of $\mathbb{P}^n \subset \mathbb{P}^N$ defined as follows,

$$Z_{r-1} = \{(x_1, \dots, x_r, z) \mid z \in \langle x_1, \dots, x_r \rangle\} \subset (\mathbb{P}^n)^r \times \mathbb{P}^N$$

Then consider the projection $\pi : Z_{r-1} \to \mathbb{P}^N$. We define $\sigma_{r-1}(\mathbb{P}^n) = \overline{\pi(Z_{r-1})}$. Then $\sigma_{r-1}(\mathbb{P}^n)$ fials to have the expected dimension iff π has positive dimensional fibres. Now take C in a fibre to give the needed curve. $\square$

Remark. The following is conjectured. Let X be a smooth projective variety of $\dim X = n$ and $\mathcal{L}$ a (very) ample line bundle. Then for all but finitely many (r, d) for all general $x_1, \dots, x_r \in X$,

$$h^0(X, \mathcal{L}^{\otimes d}(-2x_1 - \cdots - 2x_r)) = \max\{h^0(\mathcal{L}^{\otimes d}) - (n+1)r, 0\}$$