Mathematics GU4042 Modern Algebra II
Assignment # 2

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(a). Let $\phi : R \to S$ be a ring homomorphism and let $A \subset R$ be a subring. For $x, y \in \phi(A)$ we have $\exists a, b \in A$ s.t. $\phi(a) = x$ and $\phi(b) = y$.

Now, $a + b, ab, 1_R \in A$ so $\phi(a + b) = \phi(a) + \phi(b) = x + y \in \phi(A)$ and $\phi(ab) = \phi(a)\phi(b) = xy \in \phi(A)$. Also $\phi(1_R) = 1_S \in \phi(A)$. Finally, $-x = -\phi(a) = \phi(-a) \in \phi(A)$ because $a \in A \implies -a \in A$. Therefore, $\phi(A)$ is a subring of S.

(b). Let $\phi : R \to S$ be a ring homomorphism and let $B \subset S$ be a subring. Let $x, y \in \phi^{-1}(B)$ then $\phi(x), \phi(y) \in B$ thus, $\phi(x) + \phi(y) \in B$ so $\phi(x + y) \in B$ equivalently, $x + y \in \phi^{-1}(B)$. Also, $\phi(x)\phi(y) \in B$ so $\phi(xy) \in B$. Therefore, $xy \in \phi^{-1}(B)$. Also $\phi(1_R) = 1_S \in B$ so $1_S \in \phi^{-1}(B)$. Finally, $\phi(-x) = -\phi(x) \in B$ so $-x \in \phi^{-1}(B)$. Therefore, $\phi^{-1}(B)$ is a subring of R.

(c). Let $\phi : R \to S$ be a sujective ring homomorphism and let $I \subset R$ be an ideal. By problem 1, since I is an additive subgroup of R then $\phi(I)$ is an additive subgroup of S. Take $x \in \phi(I)$ and $s \in S$ thus $\exists a \in I$ s.t. $x = \phi(a)$. Since ϕ is surjective, $\exists r \in R$ s.t. $s = \phi(r)$. Now $sx = \phi(r)\phi(a) = \phi(ra) \in \phi(I)$ because $ra \in I$ by absorption. Similarly, $xs = \phi(a)\phi(r) = \phi(ar) \in \phi(I)$ because $ar \in I$ by absorption. Therefore, $\forall x \in \phi(I), \forall s \in S : xs, sx \in \phi(I)$ so $\phi(I)$ is an ideal of S.

(d). Let $\phi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be the unique ring homomorphism from $\mathbb{Z}$. $\phi(1) = (1, 1)$ so $\phi(n) = ([n]_2, [n]_2)$. Now consider the ideal $(3) \subset \mathbb{Z}$. However, $\phi((3)) = \{(0, 0), (1, 1)\}$ is not an ideal because $(0, 1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $(1, 1) \in \phi((3))$ but $(0, 1) \cdot (1, 1) = (0, 1) \notin \phi((3))$.

(e). Let $\phi : R \to S$ be a ring homomorphism and let $J \subset S$ be an ideal. By problem 2, since J is an additive subgroup of S then $\phi^{-1}(J)$ is an additive subgroup of R. Take $x \in \phi^{-1}(J)$ then $\phi(x) \in J$ and take $r \in R$. Now $\phi(rx) = \phi(r)\phi(x) \in J$ and $\phi(xr) = \phi(x)\phi(r) \in J$ by the absorption property of J. Thus, $rx, xr \in \phi^{-1}(J)$ so $\phi^{-1}(J)$ is an ideal of R.

(f). Let $F = \{a + ib \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}$. Now F is a subring of $\mathbb{C}$ if it is closed under addition and multiplication and contains inverses and identities.

Let $x, y \in F$ then $x = a_x + ib_x$ and $y = a_y + ib_y$ thus $x + y = (a_x + a_y) + i(b_x + b_y) \in F$ because $a_x + a_y, b_x + b_y \in \mathbb{Q}$. Also, $xy = (a_x a_y - b_x b_y) + i(a_x b_y + b_x a_y) \in F$ because $a_x a_y - b_x b_y \in \mathbb{Q}$ and $a_x b_y + b_x a_y \in \mathbb{Q}$.

Also, $1 \in F$ since $1 = 1 + i0$ and $-x = -a - ib \in F$ because $1, 0, -a, -b \in \mathbb{Q}$. Since $x + (-x) = 0$ and $-x + x = 0$, inverses hold. Furthermore, F is a field because for any $x \in F \backslash \{0\}$, $x = a + ib$ then take $y = \frac{1}{a^2+b^2}(a - ib)$. Now, $xy = yx = (a + ib)(a - ib)\frac{1}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$. This exists because $a^2 + b^2 = 0$ only when $a = b = 0$ i.e. $x = 0$ which is the case we excluded.

Let $\phi : \mathbb{Z}[i] \to F$ be the ring homomorphism given by $\phi(x) = x$ which is trivially injective. Now any $x \in F$ can be writen as $x = \frac{p_1}{q_1} + i\frac{p_2}{q_2} = \frac{p_1 q_2}{q_1 q_2} + i\frac{p_2 q_1}{q_1 q_2}$ with $p_1, p_2, q_1, q_2 \in \mathbb{Z}$. Then, $x = \phi(p_1 q_2 + i p_2 q_1)\phi(q_1 q_2)^{-1}$ which is defined because $q_1 \neq 0$ and $q_2 \neq 0$ and since $\mathbb{Z}$ is a domain, $q_1 q_2 \neq 0$. Also, ϕ is injective so $\phi(q_1 q_2) \neq 0$. Therefore, by the universal mapping property of the field of fractions, $F \cong Q_{Z[i]}$.