Mathematics GR6261 Commutative Algebra
Assignment # 2

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1 Problem 1

Let A be a Noetherian ring.

$(a) \implies (b)$ If A is Artinian then A has finitely many maximal ideals and every prime ideal is maximal. Thus, $\mathrm{Spec}\,(A)$ is finite and if $\mathfrak{p} \in \mathrm{Spec}\,(A)$ then $\mathfrak{p}$ is maximal so $V(\mathfrak{p}) = \{\mathfrak{p}\}$ since there are no proper ideals above it. Therefore each point is closed so $\mathrm{Spec}\,(A)$ is discrete.

$(b) \implies (c)$ Trivial.

$(c) \implies (a)$ Suppose that $\mathrm{Spec}\,(A)$ is discrete. Then for each prime $\mathfrak{p} \in \mathrm{Spec}\,(A)$ the set $\{\mathfrak{p}\}$ must be closed. Therefore, $\{\mathfrak{p}\} = V(I)$ for some ideal I so $\mathfrak{p}$ is the only prime ideal above I. However, every ideal is contained in a maximal ideal which is prime so $\mathfrak{p}$ must be maximal. Therefore A is an Noetherian ring in which every prime ideal is maximal. This implies that A is Artinian.

2 Problem 2

Let k be a field and A a k-algebra. If A is finite-dimensional then, as k-modules, $A \cong k^n$ which is Artinian because k a field and thus Artinian as a k-module. Thus A is Artinian as a A-module since every ideal is a k-submodule.

(This argument is WRONG). Conversely, if A is Artinian then it must be Noetherian as well. Take any $v_0 \in A$ and let $A_1 = k \cdot v_0$ and define a sequence of submodules inductively. Define $A_{n+1} = A_n + k \cdot v_n$ where v_n is some element of $A \setminus A_n$ which we assume is nonempty. Clearly, we have a strictly increasing infinite chain,

$$A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots$$

contradicting the fact that A is Noetherian. Thus, $A = A_n = k \cdot v_0 + \cdots + k \cdot v_n$ for some n. Therefore A is finite-dimensional. (But A_i not necessarily an A-module).

Note: the coverse here is wrong. We need A is a finite type artinian k-algebra otherwise take $A = k(t)$ clearly artinian but not finite. Here is a real proof. Because

A is Artinian it is a finite product of local Artinian rings so we may assume that A is local. Then consider the series,

$$A \supsetneq \mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots \supsetneq \mathfrak{m}^n \supsetneq \mathfrak{m}^{n+1} = 0$$

Now $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite $A/\mathfrak{m}$-module since A is noetherian so $\mathfrak{m}$ is finitely generated. Therefore, $\dim_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ is finite. Furthermore, $A/\mathfrak{m}$ is a finite type k-algebra and a field so it is a finite k-module by the Nullstellensatz. Therefore $\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ is finite. Finally, since the chain is finite we see that,

$$\dim_k A = \sum_{i=0}^{n} \dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1})$$

is finite.

3 Problem 3

Let M be an A-module and $f : M \to M$ a morphism of A-modules.

a) Suppose that f is surjective and M is Noetherian. Consider the ascending chain of submodules,

$$\ker f \subset \ker f^2 \subset \ker f^3 \subset \cdots$$

Because M is Noetherian, the chain must stabilize at some k. Suppose $f(x) = 0$. By surjectivity, there must exist $y \in M$ such that $f^k(y) = x$. Thus $f(x) = f(f^k(y)) = f^{k+1}(y) = 0$ so $y \in \ker f^{k+1} = \ker f^k$ so $x = f^k(y) = 0$. Thus, f is injective. Because f is also a surjective morphism it is an isomorphism.

b) Suppose that f is injective and M is Artinian. Consider the descending chain of submodules,

$$\mathrm{Im}(f) \supset \mathrm{Im}(f^2) \supset \mathrm{Im}(f^3) \supset \cdots$$

Because M is Artinian, this chain must stabilize at some k. Thus, for any $y \in M$ we know that $f^k(y) \in \mathrm{Im}(f^k) = \mathrm{Im}(f^{k+1})$ so $f^k(y) = f^{k+1}(x) = f^k(f(x))$ for some $x \in M$. However, f is injective and thus f^k is also injective so $y = f(x)$ since $f^k(y) = f^k(f(x))$. Therefore $y \in \mathrm{Im}(f)$ so f is surjective. Since f is also an injective morphism, f is an isomorphism.

4 Problem 4

a) Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be $\mathfrak{p}$-primary ideals. Consider the ideal,

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

By lemma 5.1, we have $\sqrt{I} = \sqrt{\mathfrak{q}_1} \cap \cdots \cap \sqrt{\mathfrak{q}_r} = \mathfrak{p} \cap \cdots \cap \mathfrak{p} = \mathfrak{p}$. Furthermore, I claim that I is primary. If $xy \in I$ then $xy \in \mathfrak{q}_i$ for each i so either $x \in \mathfrak{q}_i$ or $y \in \mathfrak{p} = \sqrt{\mathfrak{q}_i}$. If $x \in \mathfrak{q}_i$ for each i then $x \in I$ so we are done. Otherwise,

$y \in \mathfrak{p}$ for some i. However, for every i, $\mathfrak{p} = \sqrt{\mathfrak{q}_i}$ so there is some power such that $y^{n_i} \in \mathfrak{q}_i$ for each i. Take $N = n_1 + \cdots + n_r$ then $y^N \in \mathfrak{q}_i$ for all i so $y^N \in I$. Thus, I is primary and $\sqrt{I} = \mathfrak{p}$ so I is $\mathfrak{p}$-primary.

b) Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and $x \in A \setminus \mathfrak{q}$. Consider $I = (\mathfrak{q} : Ax) = \{r \in A \mid rx \in \mathfrak{q}\}$. Since $\mathfrak{q} \subset I$ we have $\sqrt{\mathfrak{q}} = \mathfrak{p} \subset \sqrt{I}$. Suppose that $r \in \sqrt{I}$ then $r^n \in I$ for some n so $r^n x \in \mathfrak{q}$ but $x \notin \mathfrak{q}$ and $\mathfrak{q}$ is primary so $r \in \sqrt{\mathfrak{q}} = \mathfrak{p}$. Furthermore, if $ab \in I$ then $abx \in \mathfrak{q}$ then $a^r \in \mathfrak{q}$ or $bx \in \mathfrak{q}$ since $\mathfrak{q}$ is primary. Thus, $a^n \in I$ or $b \in I$ and thus I is $\mathfrak{p}$-primary.

c) Suppose that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a minimal primary decomposition. Then, for any $x \in A$, by Lemma 5.2,

$$(\mathfrak{a} : x) = \bigcap_i (\mathfrak{q}_i : x)$$

Since $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary with $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ by (b) we know that $(\mathfrak{q}_i : x)$ is $\mathfrak{p}_i$-primary when $x \in A \setminus \mathfrak{q}_i$ and otherwise $(\mathfrak{q}_i : x) = A$. Then, by lemma 5.1, we know that,

$$\sqrt{(\mathfrak{a} : x)} = \bigcap_i \sqrt{(\mathfrak{q}_i : x)} = \bigcap_{i \mid x \notin \mathfrak{q}_i} \mathfrak{p}_i$$

Suppose that $\mathfrak{p} = \sqrt{(\mathfrak{a} : x)}$ is a prime ideal. By corollary 5.4, we know that $\mathfrak{p} = \mathfrak{p}_i$ for some i. Thus every prime of the form $\sqrt{(\mathfrak{a} : x)}$ is one of $\mathfrak{p}_i$, the associated primes. The decomposition is minimal so for each i there must exist some $x_i \notin \mathfrak{q}_i$ which is an element of all $\mathfrak{q}_j$ for $i \neq j$. Then, $(\mathfrak{q}_i : x_i)$ is $\mathfrak{p}_i$-primary and $(\mathfrak{q}_j : x_i) = A$ for $i \neq j$. Therefore,

$$\sqrt{(\mathfrak{a} : x_i)} = \bigcap_j \sqrt{(\mathfrak{q}_j : x_i)} = \mathfrak{p}_i$$

so each associated prime is of the form $\sqrt{(\mathfrak{a} : x)}$. We have shown that the primes associated to $\mathfrak{a}$ are exactly those primes which can be written as $\mathfrak{p} = \sqrt{(\mathfrak{a} : x)}$. Furthermore, this proves that the set of associate primes $\sqrt{\mathfrak{q}_i}$ is independent of the primary decomposition $\{\mathfrak{q}_i\}$ since "all primes of the form $\mathfrak{p} = \sqrt{(\mathfrak{a} : x)}$" is a categorization with is manifestly independent of the choice of decomposition.

d) Clearly, if $\mathfrak{p} = (\mathfrak{a} : x)$ is prime then $\mathfrak{p} = \sqrt{\mathfrak{p}} = \sqrt{(\mathfrak{a} : x)}$ so $\mathfrak{p}$ is an $\mathfrak{a}$-associate. However, I believe the converse is false. All $\mathfrak{a}$-associates are primes of the form $\mathfrak{p} = \sqrt{(\mathfrak{a} : x)}$ but not necessarily of the form $(\mathfrak{a} : x)$.

e) Let S be a multiplicative subset and $\iota : A \to S^{-1}A$ the localization map. Suppose that $\mathfrak{a}$ has minimal primary decomposition,

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

Now,

$$\iota^{-1}(S^{-1}\mathfrak{a}) = \iota^{-1}\left(S^{-1}\mathfrak{q}_1 \cap \cdots \cap S^{-1}\mathfrak{q}_r\right) = \iota^{-1}(S^{-1}\mathfrak{q}_1) \cap \cdots \cap \iota^{-1}(S^{-1}\mathfrak{q}_r)$$

Therefore we need to investigate $\iota^{-1}(S^{-1}\mathfrak{q}_i)$ for each i. We have,

$$x \in \iota^{-1}(S^{-1}\mathfrak{q}) \iff (x, 1) \in S^{-1}\mathfrak{q} \iff \exists s \in S : sx \in \mathfrak{q} \iff \exists s \in S : x \in (\mathfrak{q} : s)$$

If $S \cap \mathfrak{q} \neq \varnothing$ then take $s \in S \cap \mathfrak{q}$ and $(\mathfrak{q} : s) = A$ so $\iota^{-1}(S^{-1}\mathfrak{q}) = A$. However, if $S \cap \mathfrak{q} = \varnothing$ then for any $s \in S$ we have $s \notin \mathfrak{q}$ but if $s \in \mathfrak{p} = \sqrt{\mathfrak{q}}$ then $s^n \in \mathfrak{q}$. However, S is multiplicative so $s^n \in S$ and thus $s \notin \mathfrak{q}$ since S and $\mathfrak{q}$ have trivial intersection. Therefore, $s \notin \mathfrak{p}$. Applying Lemma 5.5, $(\mathfrak{q} : s) = \mathfrak{q}$ so we have,

$$x \in \iota^{-1}(S^{-1}\mathfrak{q}) \iff \exists s \in S : x \in (\mathfrak{q} : s) \iff x \in \mathfrak{q}$$

Therefore, $\iota^{-1}(S^{-1}\mathfrak{q}) = \mathfrak{q}$ if $S \cap \mathfrak{q} = \varnothing$ and $\iota^{-1}(S^{-1}\mathfrak{q}) = A$ otherwise. Finally,

$$\iota^{-1}(S^{-1}\mathfrak{a}) = \bigcap_{i | S \cap \mathfrak{q}_i = \varnothing} \mathfrak{q}_i$$

since when $S \cap \mathfrak{q}_i \neq \varnothing$ we have $\iota^{-1}(S^{-1}\mathfrak{q}_i) = A$ which does not contribute to the intersection.

Now let Σ be a set of primes associated to $\mathfrak{a}$ which are isolated in $\mathrm{Spec}\,(A)$. I take this to mean that Σ is separated from every $\mathfrak{a}$-associated prime it does not contain. In particular, if $\mathfrak{p}$ is $\mathfrak{a}$-associated and $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ intersects Σ then $\mathfrak{p} \in \Sigma$. That is, if $\mathfrak{p}$ is an $\mathfrak{a}$-associate prime such that there exists $\mathfrak{p}' \in \Sigma$ with $\mathfrak{p} \subset \mathfrak{p}'$ then $\mathfrak{p} \in \Sigma$. Now take,

$$S = A \setminus \bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}$$

Clearly, $1 \in S$ and if $x, y \in S$ then $x, y \notin \mathfrak{p}$ for any $\mathfrak{p} \in \Sigma$ so $xy \notin \mathfrak{p}$ for each $\mathfrak{p}$ and thus $xy \in S$. Therefore, S is multiplicative. Take,

$$\mathfrak{a}_\Sigma = \iota^{-1}(S^{-1}\mathfrak{a}) = \bigcap_{i | S \cap \mathfrak{q}_i = \varnothing} \mathfrak{q}_i$$

However, $S \cap \mathfrak{q}_i = \varnothing$ exactly when,

$$\mathfrak{q}_i \subset \bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p} = U$$

Clearly, if $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i} \in \Sigma$ then $\mathfrak{q}_i \subset \mathfrak{p}_i \subset U$ so $S \cap \mathfrak{q}_i = \varnothing$. Conversely, if $\mathfrak{q}_i \subset U$ then $\mathfrak{q}_i \subset \mathfrak{p}'$ for some $\mathfrak{p}' \in \Sigma$. Thus, $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i} \subset \mathfrak{p}'$ so, since Σ is isolated, $\mathfrak{p}_i \in \Sigma$. Therefore,

$$\mathfrak{a}_\Sigma = \iota^{-1}(S^{-1}\mathfrak{a}) = \bigcap_{i | \mathfrak{p}_i \in \Sigma} \mathfrak{q}_i$$

However, $\iota^{-1}(S^{-1}\mathfrak{a})$ is independent of the primary decomposition and thus,

$$\mathfrak{a}_\Sigma = \bigcap_{i | \mathfrak{p}_i \in \Sigma} \mathfrak{q}_i$$

is also independent of the decomposition.

5 Lemmas

Lemma 5.1. Let $I, J \subset A$ be ideals. Then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

Proof. Note that $I, J \supset I \cap J$ so $\sqrt{I}$ and $\sqrt{J}$ lie above $\sqrt{I \cap J}$ so $\sqrt{I} \cap \sqrt{J} \supset \sqrt{I \cap J}$. Furthermore, if $x \in \sqrt{I} \cap \sqrt{J}$ then $x^n \in I$ and $x^m \in J$ for some m and n. Then, $x^{m+n} \in I \cap J$ so $x \in \sqrt{I \cap J}$. Thus, $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. $\square$

Lemma 5.2. For ideals $I_i, J \subset A$,

$$\left(\bigcap_i I_i : J\right) = \bigcap_i (I_i : J)$$

Proof. We know that,

$$\left(\bigcap_i I_i : J\right) = \{x \in A \mid xJ \subset \bigcap_i I_i\}$$

However, xJ is a subset of $\cap I_i$ iff $xJ \subset I_i$ for each i. Thus,

$$\left(\bigcap_i I_i : J\right) = \bigcap_i (I_i : J)$$

$\square$

Lemma 5.3. If $\mathfrak{p} \supset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ are all prime ideals then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.

Proof. Suppose not. Then there exists $x_i \in \mathfrak{p}_i$ such that $x_i \notin \mathfrak{p}$ for each i. Therefore, we have elements not in $\mathfrak{p}$ such that the product $x_1 \cdots x_r \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{p}$ is in $\mathfrak{p}$ which contradicts the primality of $\mathfrak{p}$. $\square$

Corollary 5.4. If $\mathfrak{p} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ are all prime ideals then $\mathfrak{p} = \mathfrak{p}_i$ for some i.

Proof. We know that $\mathfrak{p} \supset \mathfrak{p}_i$ for some i but $\mathfrak{p} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r \subset \mathfrak{p}_i$ so $\mathfrak{p} = \mathfrak{p}_i$. $\square$

Lemma 5.5. Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and $s \notin \mathfrak{p}$ then $(\mathfrak{q} : s) = \mathfrak{q}$.

Proof. If $x \in (\mathfrak{q} : s)$ then $xs \in \mathfrak{q}$ thus either $x \in \mathfrak{q}$ or $s^n \in \mathfrak{q}$ so $s \in \mathfrak{p} = \sqrt{\mathfrak{q}}$ which is false. Thus $x \in \mathfrak{q}$. Conversely, if $x \in \mathfrak{q}$ then $xs \in \mathfrak{q}$ so $(\mathfrak{q} : s) = \mathfrak{q}$. $\square$