Math 56: Proofs and Modern Mathematics
Homework 7 Solutions

Naomi Kraushar

December 3, 2021

Problem 1. Let $(F, +, \cdot)$ be the field of real numbers of the form $a + b\sqrt{2}$, $a, b \in \mathbb{Q}$ with the inherited $+, \cdot$. Define $P \subset F$ by $a + b\sqrt{2} \in P$ if $a - b\sqrt{2}$ is positive as an element of $\mathbb{R}$. Show that $(F, +, \cdot, P)$ is an ordered field.

Solution. We need to prove three things: the trichotomy axiom, $x, y \in P \implies x + y \in P$, and $x, y \in P \implies x+y \in P$. To make things simpler, for $x = a+b\sqrt{2}$, we define $\overline{x} = a-b\sqrt{2}$, so that $x \in P$ if and only if $\overline{x} > 0$ in $\mathbb{R}$. We have the following properties:

(i) For $x = a + b\sqrt{2} \in F$, we have $\overline{-x} = -\overline{x}$: we compute

$$\overline{-x} = \overline{-(a + b\sqrt{2})} = \overline{a - b\sqrt{2}} = -a + b\sqrt{2} = -(a - b\sqrt{2}) = -\overline{x}.$$

In particular, $\overline{0} = 0$.

(ii) For $x = a + b\sqrt{2}, y = c + d\sqrt{2} \in F$, we have $\overline{x + y} = \overline{x} + \overline{y}$: we compute

$$\begin{aligned}\overline{x + y} &= \overline{(a + b\sqrt{2}) + (c + d\sqrt{2})} = \overline{(a + c) + (b + d)\sqrt{2}} \\ &= (a + c) - (b + d)\sqrt{2} = (a - c\sqrt{2}) + (b - d\sqrt{2}) = \overline{x} + \overline{y}. \quad (1)\end{aligned}$$

(iii) For $x = a + b\sqrt{2}, y = c + d\sqrt{2} \in F$, we have $\overline{x \cdot y} = \overline{x} \cdot \overline{y}$: we compute

$$\begin{aligned}\overline{x \cdot y} &= \overline{(a + b\sqrt{2})(c + d\sqrt{2})} = \overline{(ac + 2bd) + (ad + bc)\sqrt{2}} \\ &= (ac + 2bd) - (ad + bc)\sqrt{2} = (a - b\sqrt{2})(c - d\sqrt{2}) = \overline{x} \cdot \overline{y}. \quad (2)\end{aligned}$$

The rest is just using the order axioms in $\mathbb{R}$.

Trichotomy: Let x be an arbitrary element of F. By the trichotomy axiom in $\mathbb{R}$, exactly one of the following is true: $\overline{x} = 0$, $\overline{x} < 0$ in $\mathbb{R}$, or $-\overline{x} > 0$ in $\mathbb{R}$. By property (i), this is equivalent to saying that $x = 0$, $\overline{x} < 0$ in $\mathbb{R}$, or $\overline{-x} > 0$ in $\mathbb{R}$. Hence exactly one of the following is true: $x = 0$, $x \in P$, or $-x \in P$.

$\underline{x, y \in P \implies x + y \in P}$: Suppose that $x, y \in P$. This means that $\overline{x}, \overline{y} > 0$ in $\mathbb{R}$, so by the second ordered field axiom in $\mathbb{R}$ and property (ii), we have $\overline{x+y} = \overline{x} + \overline{y} > 0$ in $\mathbb{R}$. Hence $x + y \in P$.

$\underline{x, y \in P \implies x \cdot y \in P}$: Suppose that $x, y \in P$. This means that $\overline{x}, \overline{y} > 0$ in $\mathbb{R}$, so by the second ordered field axiom in $\mathbb{R}$ and property (iii), we have $\overline{x \cdot y} = \overline{x} \cdot \overline{y} > 0$ in $\mathbb{R}$. Hence $x + y \in P$.

Problem 2. Show that in an ordered field for all $x, y \in F$, we have $|x \cdot y| = |x| \cdot |y|$.

Solution. Let P be the set of "positive" elements of F. We have four possible cases to deal with: at least one of x, y is 0, both are in P, neither are in P, or exactly one is in P.

Case 1. Suppose that at least one of x, y is 0; without loss of generality, suppose that $x = 0$. Then $|x \cdot y| = |0| = 0$, and $|x| \cdot |y| = |0| \cdot |y| = 0 \cdot |y| = 0$, so the statement is true in this case.

Case 2. Suppose that $x, y \in P$, so that $x \cdot y \in P$. Hence $|x \cdot y| = x \cdot y = |x| \cdot |y|$, so the statement is true in this case.

Case 3. Suppose that $x, y \notin P$, so that $-x, -y \in P$, and $x \cdot y \in P$. We have $|x \cdot y| = x \cdot y$, and $|x| \cdot |y| = (-x) \cdot (-y) = x \cdot y$, so the statement is true in this case.

Case 4. Suppose that exactly one of x, y is in P; without loss of generality, suppose that $x \in P, y \notin P$, so that $x \in P$, $-y \in P$. In particular, this means that $-x \cdot y = x \cdot (-y) \in P$, so $x \cdot y \notin P$. Then $|x \cdot y| = -(x \cdot y)$, and $|x| \cdot |y| = x \cdot (-y) = -(x \cdot y)$, so the statement is true in this case.

Having covered all cases, we conclude that $|x \cdot y| = |x| \cdot |y|$.

Problem 3. Show that in an ordered field if $a > b > 0$, then $b^{-1} > a^{-1} > 0$.

Solution. Method 1. Let P be the set of positive numbers in the field, and suppose that $a > b > 0$, so that $a, b, b - a \in P$. Suppose that $a^{-1} \notin P$, so that $-a^{-1} \in P$. This gives us $a(-a^{-1}) = -1 \in P$, which is false, so $a^{-1} \in P$; similarly, $b^{-1} \in P$, so $a^{-1}, b^{-1} > 0$. Finally, consider the element $b^{-1} - a^{-1}$. We have

$$b^{-1} - a^{-1} = aa^{-1}b^{-1} - a^{-1}bb^{-1} = (a - b)a^{-1}b^{-1}.$$

We know that $a - b, a^{-1}, b^{-1} \in P$, so this is in P. Hence $b^{-1} - a^{-1} \in P$, which gives us $b^{-1} > a^{-1}$, as required.

Method 2. We claim that if $x < y$ and $\lambda > 0$, then $\lambda x < \lambda y$. This is because if $x < y$, then $y - x > 0$, so $\lambda(y - x) > 0$ by the multiplicative closure of P. Multiplying out gives $\lambda y - \lambda x > 0$, i.e. $\lambda x < \lambda y$.

Now let's apply this to our situation. We have $a > b > 0$. Suppose that $b^{-1} < 0$. Multiplying this inequality by b, which is positive, gives us $1 < 0$, which is false. Hence $b^{-1} > 0$; similarly, $a^{-1} > 0$, so that $a^{-1}b^{-1}$ is positive. Now let us multiply $b > a$ by $a^{-1}b^{-1}$, which is positive: we get $a^{-1} > b^{-1}$ as required.

Problem 4 (Abbott, Exercise 1.3.5)**.** Let $A \subset \mathbb{R}$ be non-empty, bounded above, and let $cA = \{ca : a \in A\}$. If $c \geq 0$, show that $\sup(cA) = c \sup A$.

Solution. If $c = 0$, then $A = \{0\}$, so its upper bound is $0 = c \sup A$. We therefore turn to the case where $c > 0$.

Method 1. By definition, $a \leq \sup A$ for all $a \in A$, so $ca \leq c \sup A$ for all $ca \in cA$, since $c > 0$. Hence $c \sup A$ is an upper bound for cA. Now suppose that $b < c \sup A$ is also an upper bound for A. Since $b < c \sup A$, multiplying by $1/c > 0$ gives $b/c < \sup A$. Since $ca \leq b$ for all $ca \in cA$, we also have $a < b/c$ for all $a \in A$. But this means that we have found a smaller upper bound for A than $\sup A$, which is a contradiction. Hence $c \sup A$ is the smallest upper bound for cA, i.e. $c \sup A = \sup(cA)$.

Method 2. We can use Lemma 1.3.8: if $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$, then $s = \sup A$ if and only if for every $\varepsilon > 0$ there exists $a \in A$ with $a > s - \varepsilon$. By definition of the supremum, every $a \in A$ satisfies $a \leq \sup A$, so for $c \geq 0$, this implies that $ca \leq c \sup A$ for every $a \in A$. Hence $c \sup A$ is an upper bound for cA. By Lemma 1.3.8, for every $\varepsilon > 0$, there exists $a \in A$ such that $a > \sup A - \varepsilon/c$. This means that for every $\varepsilon > 0$, there exists $ca \in cA$ such that $ca > c \sup A - \varepsilon$. By Lemma 1.3.8, this means that $\sup cA = c \sup A$, as required.

Problem 5 (Abbott, Exercise 1.3.6(a,d))**.** Given sets A, B define $A + B = \{a + b : a \in A, b \in B\}$. The goal is to show that if A, B are non-empty, bounded above then $\sup(A + B) = \sup A + \sup B$.

1. Let $s = \sup A, t = \sup B$. Show that $s + t$ is an upper bound for $A + B$.
2. Use Lemma 1.3.8 to show that $\sup(A + B) = s + t$.

Solution. 1. Since $s = \sup A, t = \sup B$, we know that $a \leq s$, $b \leq t$ for all $a \in A$ and all $b \in B$. As we showed in the previous homework, this means that $a + b \leq s + t$, so that $s + t$ is an upper bound for $A + B$.

2. By Lemma 1.3.8, for every $\varepsilon > 0$, there exists $a \in A$ and $b \in B$ such that $a > s - \varepsilon/2$ and $b > t - \varepsilon/2$. Adding these as in the previous homework, we get $a + b > s + t - \varepsilon$. Hence the condition of Lemma 1.3.8 is fulfilled, and $s + t = \sup(A + B)$.