Mathematics GU4053 Algebraic Topology
Assignment # 5

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Note. My order of path concatenation follows Hatcher,

$$\gamma * \delta(x) = \begin{cases} \gamma(2x) & x \le \frac{1}{2} \\ \delta(2x-1) & x \ge \frac{1}{2} \end{cases}$$

Problem 1.

Let $f : X \to Y$ and $g : Y \to Z$ be fibrations. Given any space A, a map $p : A \to X$ and a homotopy $h : A \times I \to Z$ consider the diagram,

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
A \arrow[u, "p"] \arrow[ur, "f \circ p"] \arrow[rr, hook] & & A \times I \arrow[u, "h"'] \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}$$

Because g is a fibration, there exists a lift $\tilde{h} : A \times I \to Y$ of h matching $f \circ p$ such that the diagram commutes. Now, rewrite the commutative diagram as,

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
A \arrow[u, "p"] \arrow[r, hook] & A \times I \arrow[ul, dashed, "\tilde{\tilde{h}}"] \arrow[u, "\tilde{h}"] \arrow[ur, "h"']
\end{tikzcd}$$

which gives a lift $\tilde{\tilde{h}}$ of $\tilde{h}$ at p because f is a fibration. Therefore, there is a map $\tilde{\tilde{h}} : A \times I \to X$ which makes the diagram commute. That is, $\tilde{\tilde{h}}(a, 0) = p(a)$ and $g \circ f \circ \tilde{\tilde{h}} = g \circ \tilde{h} = h$. Thus, $g \circ f$ is a fibration.

Likewise, let $f : X \to Y$ and $g : Y \to Z$ be cofibrations. Given any space A, a map $p : Z \to A$ and a homotopy $h : X \times I \to A$ consider the diagram,

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, hook] & & Y \arrow[r, "g"] \arrow[dl, "p\circ g"] \arrow[dd, hook] & Z \arrow[dd, hook] \arrow[dll, bend left, "p"] \\
& A & & \\
X \times I \arrow[rr, "f\times\mathrm{id}_I"] \arrow[ur, "h"] & & Y \times I \arrow[r, "g\times\mathrm{id}_I"] \arrow[ul, dashed, "\tilde{h}"] & Z \times I
\end{tikzcd}
$$

Because f is a cofibration, there exists an extension $\tilde{h} : Y \times I \to A$ of h lifting to $p \circ g$ such that the diagram commutes. Now, rewrite the commutative diagram as,

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd, hook] & Y \arrow[rr, "g"] \arrow[dd, hook] & & Z \arrow[dl, "p"] \arrow[dd, hook] \\
& & A & \\
X \times I \arrow[r, "f\times\mathrm{id}_I"] & Y \times I \arrow[rr, "g\times\mathrm{id}_I"] \arrow[ur, "\tilde{h}"] & & Z \times I \arrow[ul, dashed, "\tilde{\tilde{h}}"]
\end{tikzcd}
$$

which gives an extension $\tilde{\tilde{h}}$ of $\tilde{h}$ lifting p because f is a cofibration. Therefore, there is a map $\tilde{\tilde{h}} : Z \times I \to A$ which makes the diagram commute. Thus, $g \circ f$ is a cofibration.

Problem 2.

Let $f : X \to Y = \{*\}$ be a continuous (constant map). Given a map $g : A \to X$ and a homotopy $h : A \times I \to Y$. However, h must be a constant map so we should take $\tilde{h} : A \times I \to X$ given by $\tilde{h}(x, t) = g(x)$. Then, $f \circ \tilde{h} = h$ because both are constant maps. Also, $\tilde{h}(x, 0) = g(x)$ by construction. Thus, the following diagram commutes,

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
A \arrow[u, "p"] \arrow[r, hook] & A \times I \arrow[u, "h"'] \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}
$$

which means that f is a fibration.

Problem 3.

Let $p : E \to B$ be a continuous map and let $\pi : E^I \to N_p$ be the map $\pi(\gamma) = (\gamma(0), p \circ \gamma)$. Suppose there exists a section $s : N_p \to E^I$ such that $\pi \circ s = \mathrm{id}_{N_p}$. Now, for any topological space Z with a map $f : Z \to E$ and homotopy $h : Z \times I \to B$ such that the following diagram commutes,

$$
\begin{tikzcd}
Z \arrow[drr, bend left, "f"] \arrow[dr, dashed, "m"] \arrow[ddr, bend right, "\hat{h}"'] & & E^I \arrow[d, "ev_0"] & \\
& N_p \arrow[ur, "s"] \arrow[r] \arrow[d] & E \arrow[d, "p"] & Z \arrow[l, "f"] \arrow[d, hook] \arrow[ul, dashed, "u"'] \\
& B^I \arrow[r, "ev_0"] & B & Z \times I \arrow[l, "h"] \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}
$$

Since we have a homotopy $h : Z \times I \to B$ by the adjunction relation there is a map $\hat{h} : Z \to B^I$. Because N_p is the pullback of $B^I \to B \leftarrow Z \times I$ and there are maps f and $\hat{h}$ from Z which make the square commute, there exists a unique map $m = (f, \hat{h}) : Z \to N_p$ which makes the diagram commute. Then, define $u = s \circ m$. Finally, $u : Z \to E^I$ corresponds by the adjunction relation to a map $\hat{h} : Z \times I \to E$. We must check that this homotopy makes the rightmost square commute. Because $\pi \circ s = \mathrm{id}_{N_f}$ we know that $\pi \circ s(x, \gamma) = (s(x, \gamma)(0), p \circ s(x, \gamma)) = (x, \gamma)$. Therefore, $s(x, \gamma)(0) = x$ and $p \circ s(x, \gamma) = \gamma$. Now,

$$\tilde{h}(z, 0) = (s \circ m)(z)(0) = s(m(z))(0) = s(f(z), \hat{h}(z))(0) = f(z)$$

Likewise,

$$p \circ \tilde{h}(z, t) = [p \circ s \circ m(z)](t) = [p \circ s(f(z), \hat{h}(z))](t) = \hat{h}(z)(t) = h(z, t)$$

Therefore, the left square commutes so $p : E \to B$ is a fibration.

Conversely, suppose that $p : E \to B$ is a fibration. We have maps $\pi_1 : N_p \to E$ such that $\pi_1(x, \gamma) = x$ and $\pi_2 : N_p \to B^I$ such that $\pi_2(x, \gamma) = \gamma$. By the adjunction relation, there is a map $h : N_p \times I \to B$ where $h(x, \gamma, t) = \pi_2(x, \gamma)(t) = \gamma(t)$. However, by the definition of N_p we know that $p(x) = \gamma(0) = h(x, \gamma, 0)$ Therefore, the following diagram commutes,

$$
\begin{tikzcd}
E^I \arrow[r, "ev_0"] & E \arrow[r, "p"] & B \\
& N_p \arrow[ul, dashed, "\hat{\tilde{h}}"] \arrow[u, "\pi_1"] \arrow[r, hook] & N_p \times I \arrow[ul, dashed, "\hat{h}"] \arrow[u, "h"']
\end{tikzcd}
$$

Because p is a fibration, there exists a map $\tilde{h} : N_p \times I \to E$ which makes the diagram commute. By the adjuction relation, this gives a map $\hat{\tilde{h}} : N_p \to E^I$. I claim that this is the section we are searching for. Consider,

$$\pi \circ \hat{\tilde{h}}(x, \gamma) = (\hat{\tilde{h}}(x, \gamma)(0), p \circ \hat{\tilde{h}}(x, \gamma)) = (\tilde{h}(x, \gamma, 0), p \circ \hat{\tilde{h}}(x, \gamma))$$

However, the consider the loop,

$$p \circ \hat{\tilde{h}}(x, \gamma)(t) = p \circ \tilde{h}(x, \gamma, t) = h(x, \gamma, t) = \gamma(t)$$

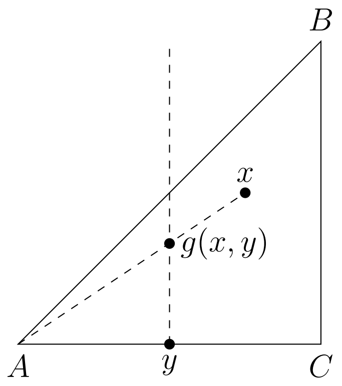
Also $\tilde{h} \circ \iota = \pi_1$ so $\tilde{h}(x, \gamma, 0) = \pi_1(x, \gamma) = x$. Therefore,

$$\pi \circ \hat{\tilde{h}}(x, \gamma) = (\tilde{h}(x, \gamma, 0), p \circ \hat{\tilde{h}}(x, \gamma)) = (x, \gamma)$$

Thus, $\pi \circ s = \mathrm{id}_{N_p}$ so s is a section.

Problem 4.

Let $p : \Delta^2 \to \Delta^1$ be the projection of a 2-simplex onto a side which is a 1-simplex. We will consider a 2-simplex as a right triangle for simplicity. We will geometrically define the a map $g : \Delta^2 \times \Delta^1 \to \Delta^2$ such that $p \circ g(x, y) = y$ and $g(x, (p(x)) = x$.



Clearly, if $y = p(x)$ then x lies on the perpendicular bisector of AC at y so $g(x, p(x)) = x$. Furthermore, $p(g(x, y)) = y$ because $g(x, y)$ lies directly above y on the perpendicular bisector of AC at y. Given such a map, define the section $s : N_p \to (\Delta^2)^I$ given by $s(x, \gamma) = g(x, \gamma(-))$. Then,

$$\pi \circ s(x, \gamma) = \pi(g(x, \gamma(-)) = (g(x, \gamma(0)), p \circ g(x, \gamma(-)))$$

but $\gamma(0) = p(x)$ because $(x, \gamma) \in N_p$ and $g(x, p(x)) = x$ and $p \circ g(x, \gamma(-)) = \gamma$. Therefore,

$$\pi \circ s(x, \gamma) = (g(x, \gamma(0)), p \circ g(x, \gamma(-))) = (x, \gamma)$$

so $\pi \circ s = \mathrm{id}_{N_p}$. Therefore, s is a section of the map $\pi : E^I \to N_p$ so by the above problem $p : \Delta^2 \to \Delta^1$ is a fibration. However, p cannot be a fiber bundle because the inverse image of the left endpoint is a line segment while the inverse image of the right endpoint is a sinlge point. Since the fibers are not homeomorphic the map cannot be a fiber bundle.

Problem 5.

Let (X, x_0) and (Y, y_0) be pointed spaces. Consider the map $\mathrm{Hom}\,(\Sigma X, Y) \to \mathrm{Hom}\,(X, \Omega Y)$ which send maps,

$$((x \wedge t) \mapsto f(x \wedge t)) \mapsto (x \mapsto (t \mapsto f(x \wedge t)))$$

Since $x_0 \wedge t_0 = x_0 \sim t$ we must have $f(x_0 \wedge t) = y_0$ be constant. Thus, the function f is taken to $x \mapsto (t \mapsto f(x \wedge t))$ which has the property that $x_0 \mapsto (t \mapsto f(x_0 \wedge t) = f(x_0 \wedge t_0) = y - 9$. Thus, $x_0 \mapsto e_0$ the basepoint of ΩY. Thus, this correspondence sends based maps to based maps. Furthermore, given a based map in $\mathrm{Hom}\,(X, \Omega Y)$ written as $x \mapsto (t \mapsto f(x, t))$ for $f \in \mathrm{Hom}\,(X \times S^1, Y)$, we know that $x_0 \mapsto e_0$ the constant map at y_0. Thus, $f(x_0, t) = y_0$. Furthermore, x must map to a based map $S^1 \to Y$ so $f(x, t_0) = y_0$. Thus, f is constant on $X \vee S^1$ so f as a function on $X \times S^1$ decends to the quotient $X \wedge S^1 = X \times S^1 / X \vee S^1$. Thus, f is in the image of the correspondence so the mapping is surjective. Furthermore, the correspondence is one-to-one because if $f(x, t) = g(x, t)$ then clearly $f(x \wedge t) = g(x \wedge t)$ because these are simply restriction maps. Thus, $\mathrm{Hom}\,(\Sigma X, Y) \cong \mathrm{Hom}\,(X, \Omega Y)$. It suffices to show that this correspondence descends to homotopy classes. This is clear because any homotopy $H : (X \wedge S^1) \times I \to Y$ is in correspondence to a homotopy $H : X \times I \to \Omega Y$ by exactly the same correspondence outlied above taken on each time slice. Thus, a homotopy between maps in one Hom space is equivalent to a homotopy between the corresponding maps in the other. Therefore,

$$\langle \Sigma X, Y \rangle = \langle X, \Omega Y \rangle$$

Problem 6.

Let X be a CW complex equal to an increasing union of subcomplexes,

$$X_1 \subset X_2 \subset X_3 \subset X_4 \subset \cdots \qquad \text{and} \qquad X = \bigcup_{i=1}^{\infty} X_i$$

such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. I claim that, picking a basepoint $\{x_0\}$, the complex $X = \varinjlim X_i$ is the direct limit (colimit) of the system,

$$\{x_0\} \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow X_4 \hookrightarrow \cdots$$

The projection functor $\pi : \mathbf{Top} \to \mathbf{hTop}$ maps topological spaces to themselves and maps maps to homotopy classes is cocontinuous so,

$$\pi(X) = \pi(\varinjlim X_i) \cong \varinjlim \pi(X_i)$$

However, each inclusion map ι is nullhomotopic. Therefore, in **hTop** each of these maps is a constant map so the one point space satisfies the universal property of the limit. In particular, let $j_1 : X_i \to X = *$ be the universal cones. Take any other space, A with maps $f_i : X_i \to A$ such that $f_i = f_{i+1} \circ \iota_{i+1}$. However, ι_{i+1} is a constant map so f_i must be constant. Moveover, f_i and f_j must have the same image because $f_i(x) = f_{i+1}(\iota_{i+1}(x)$ and both are contant. Therefore, this entire system factors though $X = *$ via a map from $* \to \{f_i(x)\}$. Thus, the one point space is the direct limit of this system in **hTop**. However, the original directed system in **Top** must project via π down to this system in **hTop**. Since colimits are unique, X is isomorphic to $*$ in hTop which implies that X is homotopy equivalent to $*$ i.e. X is contractable.

Problem 7.

Consider the pointed space (X, x_0) and the suspension $\Sigma X \cong X \times I/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)$. Consider the homotopy, $X \times I \to \Sigma X$ defined by $h(x,t) = (x, t/2)$. Then, $h(x,0) = (x,0)$ but $(x_1, 0) \sim (x_2, 0)$ so $h(x,0)$ is constant. Also, $h(x,1) = (x, 1/2) = \iota(x)$. Therefore, $\iota : X \hookrightarrow \Sigma X$ is nullhomotopic.

Problem 8.

Let X be a pointed CW complex. The infinite suspension of X is given by,

$$\Sigma^\infty(X) = \bigcup_{i=1}^{\infty} \Sigma^i(X)$$

Choosing a basepoint x_0, by problem 7, the inclusion maps in the following chain are nullhomotopic,

$$\{x_0\} \hookrightarrow X \hookrightarrow \Sigma X \hookrightarrow \Sigma^2 X \hookrightarrow \Sigma^3 X \hookrightarrow \Sigma^4 X \hookrightarrow \Sigma^5 X \hookrightarrow \Sigma^6 X \hookrightarrow \cdots$$

which, by problem 6, implies that the total complex $\Sigma^\infty(X)$ is contractable.