## Mathematics GU6308 Algebraic Topology Assignment # 5

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## 1 The Gysin Sequence

For sphere bundles, there is a particularly nice exact sequence due to Gysin.

**Theorem 1.0.1** (Gysin). Let  $S^n \hookrightarrow E \xrightarrow{p} B$  be a sphere bundle which is homotopically simple. Then there is an exact sequence,

$$\cdots \xrightarrow{p_*} H_{i+1}(B) \xrightarrow{d} H_{i-n}(B) \xrightarrow{\ell} H_i(E) \xrightarrow{p_*} H_i(B) \xrightarrow{d} H_{i-n-1}(B) \xrightarrow{\cdots} \cdots$$

Furthermore, let  $C \in H^{n+1}(B; \pi_n(S^n)) = H^{n+1}(B; \mathbb{Z})$  be the primary obstruction. Then  $d(x) = x \frown C$  and  $\ell$  is the map lifting a homology class of B to its preimage in E.

*Proof.* We consider the homological Serre spectral sequence,

$$E_{p,q}^2 = H_p(B, H_q(S^n)) \implies H_{p+q}(E)$$

Note that,

$$E_{p,q}^2 = H_p(B, H_q(S^n)) = \begin{cases} H_p(B; \mathbb{Z}) & q = 0, n \\ 0 & q \neq 0, n \end{cases}$$

To choose an isomorphism  $\pi_n(S^n) \cong \mathbb{Z}$  we need an orientation of  $S^n$ . However, this is not an issue since we have assumed that the fibration is homotopically simple so there is no obstruction to choosing a consistent orientation.

Therefore, the second page of the Serre spectral sequence has two rows. The differential  $d_{p,q}^r$  has bidegree (-r, r-1) so the only relevant differentials occur at page r=n+1 giving a differential,

$$d_{p,0}^{n+1}: E_{p,0}^{n+1} \to E_{p-n-1,n}^{n+1}$$

Therefore, we can explicitly describe the  $\infty$ -page,

$$E_{p,q}^{\infty} = \begin{cases} H_p(B; \mathbb{Z}) & p < n+1, q = 0\\ \ker \left( d_{p,0}^{n+1} \right) & p \ge n+1, q = 0\\ \operatorname{coker} \left( d_{p+n+1,0}^{n+1} \right) & q = n \end{cases}$$

Since the spectral sequence converges,

$$E_{p,q}^2 \implies H_{p+q}(E)$$

there is a filtration  $F_pH_n(E)$  such that,

$$E_{p,q}^{\infty} = F_p H_{p+q}(E) / F_{p-1} H_{p+q}(E)$$

Note that  $E_{p,q}^{\infty} \neq 0$  only when q = 0, n so for fixed i = p + q we find,

$$F_i H_i(E) / F_{i-1} H_i(E) = E_{i,0}^{\infty} = \ker \left( d_{i,0}^{n+1} \right)$$

and

$$F_{i-n}H_i(E)/F_{i-n-1}H_i(E) = E_{i-n,n}^{\infty} = \operatorname{coker}\left(d_{i+1,0}^{n+1}\right)$$

and all other quotients are zero. Therefore,  $F_iH_i(E) = H_i(E)$  and  $F_{i-1}H_i(E) = \cdots = F_{i-n}H_i(E)$  and  $F_{i-n-1}H_i(E) = 0$ . This gives an exact sequence,

$$0 \longrightarrow E_{i-n,n}^{\infty} \longrightarrow H_i(E) \longrightarrow E_{i,0}^{\infty} \longrightarrow 0$$

and therefore,

$$0 \longrightarrow \operatorname{coker}\left(\operatorname{d}_{i+1,0}^{n+1}\right) \longrightarrow H_i(E) \longrightarrow \ker\left(\operatorname{d}_{i,0}^{n+1}\right) \longrightarrow 0$$

Now these are maps,

$$d_{i,0}^{n+1}: H_i(B; \mathbb{Z}) \to H_{i-n-1}(B; \mathbb{Z})$$
  
 $d_{i+1,0}^{n+1}: H_{i+1}(B; \mathbb{Z}) \to H_{i-n}(B; \mathbb{Z})$ 

Therefore, the following sequence is exact,

$$H_{i+1}(B; \mathbb{Z}) \xrightarrow{\mathrm{d}_{i+1,0}^{n+1}} H_{i-n}(B; \mathbb{Z}) \xrightarrow{\ell} H_{i}(E; \mathbb{Z}) \xrightarrow{p_*} H_{i}(B; \mathbb{Z}) \xrightarrow{\mathrm{d}_{i,0}^{n+1}} H_{i-n-1}(B; \mathbb{Z})$$

These five term sequences glue to form a long exact sequence as follows,

$$\cdots \xrightarrow{\ell} H_{i+1}(E; \mathbb{Z}) \xrightarrow{p_*} H_{i+1}(B; \mathbb{Z}) \xrightarrow{d_{i+1,0}^{n+1}} H_{i-n}(B; \mathbb{Z})$$

$$\parallel \qquad \qquad \parallel$$

$$H_{i+1}(B; \mathbb{Z}) \xrightarrow{d_{i+1,0}^{n+1}} H_{i-n}(B; \mathbb{Z}) \xrightarrow{\ell} H_{i}(E; \mathbb{Z}) \xrightarrow{p_*} \cdots$$

Now that we have demonstrated the existence of the Gysin sequence, we need to identify these maps. First, the morphism of fibrations,

$$E \xrightarrow{p} B$$

$$\downarrow^{p} \qquad \downarrow_{id}$$

$$B \xrightarrow{id} B$$

induces a morphism on spectral sequences which shows that the map

$$H_i(E; \mathbb{Z}) \to E_{p,0}^2 \subset E_{p,0}^2 = H_p(B; \mathbb{Z})$$

is induced by  $p: E \to B$ . Note that this is the map  $H_i(E; \mathbb{Z}) \to \ker \left( \operatorname{d}_{i,0}^{n+1} \right) \to H_i(B; \mathbb{Z})$  which appear in our long exact sequence by the properties of a morphism of spectral sequences.

We need to investigate the map  $d_{i,0}^{n+1}: H_i(B;\mathbb{Z}) \to H_{i-n-1}(B;\mathbb{Z})$ . To do this, we use the cap product structure on the homological and cohomological spectral sequences  $\frown: E_{p,q}^r \times E_r^{p',q'} \to E_{p-p',q-q'}^r$ 

which reduces to the usual cap product. Furthermore, the cap product structure is compatible with the differential. In particular, for the differential  $d_{p,0}^{n+1}: E_{p,0}^{n+1} \to E_{p-n-1,n}^{n+1}$  this is a map  $H_p(B;\mathbb{Z}) \to H_{p-n-1}(B;\mathbb{Z})$  which is cap product with some class  $e \in H^{n+1}(B;\mathbb{Z})$  i.e. the map  $e: H_p(B;\mathbb{Z}) \to H_{p-n-1}(B;\mathbb{Z})$ . Recall the definition of the primary obstruction. There is no obstruction to finding a section  $s: B^n \to E$  since  $\pi_i(F) = 0$  for i < n. Then for each (n+1)-cell  $D^n$  with an attaching map  $f: S^n \to B^n$  we have an inclusion  $h: D^{n+1} \to B$ , then pulling back the fibration gives  $h^*E \to D^{n+1}$ . We have a section  $s: \partial D^{n+1} \to h^*E$  and since  $h^*E$  is locally trivial then a map  $f: S^n \to F$ . Adding these together gives a cohomology class  $H^{n+1}(X; \pi_n(F)) = H^{n+1}(X; \mathbb{Z})$ .

Notice that if the fibration  $p: E \to B$  has a section  $s: B \to E$  then by  $p \circ s = \mathrm{id}_B$  we immediately see that  $p_* \circ s_* = \mathrm{id}$  on homology so the map  $p_*H_i(E) \to H_i(B)$  is surjective. Therefore, d = 0 and the Gysin sequence splits into short exact sequences,

$$0 \longrightarrow H_{i-n}(B) \longrightarrow H_i(E) \stackrel{p_*}{\longrightarrow} H_i(B) \longrightarrow 0$$

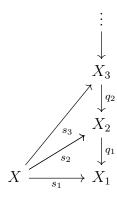
In particular, this shows that whenever  $p: E \to B$  has a section, the class e = 0. Focus on degree i = n + 1 in which we have shown there is a sequence,

$$H_1(B) \longrightarrow H_{n+1}(E) \stackrel{p_*}{\longrightarrow} H_{n+1}(B) \stackrel{\frown e}{\longrightarrow} H_0(B)$$

Choose a section  $s: B^n \to E$  on the *n*-skeleton of B. Then the map  $\smile e$  takes an n+1-cell  $D^{n+1}$  to the degree of the map  $\partial D^{n+1} \to F$  (as a map  $S^n \to S^n$ ). Thus e is the primary obstruction.  $\square$ 

## 2 Postnikov and Whitehead Towers

**Definition 2.0.1.** Let X be a path-connected space. Then a *Postnikov* tower  $\mathcal{X}$  is a diagram of spaces,



Such that,

- (a)  $s_n: X \to X_n$  induces an isomorphism  $(s_n)_*: \pi_i(X) \to \pi_i(X_n)$  for  $i \leq n$
- (b)  $\pi_i(X_n) = 0 \text{ for } i > n.$

A morphism of Postnikov towers is a morphism of diagrams.

Remark. Note that any morphism of Postnikov towers  $f: \mathcal{X} \to \mathcal{X}'$  induces a weak homotopy equivalence  $f_n: X_n \to X_n'$  because the diagram,

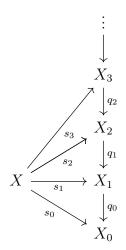
$$\pi_i(X_n) \xrightarrow{(f_n)_*} \pi_i(X'_n)$$

$$(s_n)_* \xrightarrow{(s'_n)_*} \pi_i(X)$$

commutes and either  $\pi_i(X_n) = \pi_i(X_n') = 0$  for i > n or the upward maps are isomorphism for  $i \le n$  so  $(f_n)_* : \pi_i(X_n) \xrightarrow{\sim} \pi_n(X_n')$  is an isomorphism.

**Proposition 2.0.2.** Let X be a path-connected CW complex. Then X admits a Postnikov tower  $\mathcal{X}$  of CW complexes which is unique up to homotopy equivalence.

*Proof.* Starting with the constant map  $q_0: X \to X_0 = *$  we construct a tower,



Given a map  $s_n: X \to X_n$  such that  $(s_n)_*: \pi_i(X) \to \pi_i(X_n)$  with  $i \le n$  and  $\pi_i(X_n) = 0$  for i > n. For each  $S^{n+1} \to X$  generating  $\pi_{n+1}(X)$  define  $X_{n+1}$  by attaching (n+2)-cells via the attaching maps  $S^{n+1} \to X$  to kill  $\pi_{n+1}(X)$  and for each  $S^{n+i} \to X$  generating  $\pi_{n+i}(X)$  attach an (n+i+1)-cell via the attaching maps  $S^{n+i+1} \to X$  to kill  $\pi_{n+i}(X)$ . Then we have a map  $s_{n+1}: X \hookrightarrow X_{n+1}$  satisfying the required properties. Furthermore, since  $\pi_i(X_n) = 0$  for i > n, the map  $X \to X_n$  lifts to  $X \to X_{n+1} \to X_n$  using Lemma 4.7 in Hatcher and noting that  $X_{n+1} \setminus X$  is built from cells of dimension at least n+2.

Now suppose that  $\mathcal{X}$  is a CW Postnikov tower for X and  $\mathcal{X}'$  be the tower constructed above via attaching cells to X. It suffices to show there is a morphism  $\mathcal{X} \to \mathcal{X}'$  of Postnikov towers since such a morphism is a weak homotopy equivalence on  $X_n$  and are CW complexes so any weak homotopy equivalence is automatically a homotopy equivalence. Such a morphism is given by Hatcher Proposition 4.18.

Remark. For each  $q_n: X_{n+1} \to X_n$  we can expand  $X_n \hookrightarrow X'_n$  to give a fibration  $q'_n: X'_{n+1} \to X'_n$  fitting into the diagram,

$$X_{n+1} \longleftrightarrow X'_{n+1}$$

$$\downarrow^{q_n} \qquad \qquad \downarrow^{q'_n}$$

$$X_n \longleftrightarrow X'_n$$

Therefore, we may assume that  $q_n: X_{n+1} \to X_n$  is a fibration in the definition of a Postnikov tower. This allows us to investigate the fiber  $F_{n+1} \hookrightarrow X_{n+1} \xrightarrow{q_n} X_n$  via the long exact sequence,

$$\pi_{i+1}(X_{n+1}) \xrightarrow{q_*} \pi_{i+1}(X_n) \longrightarrow \pi_i(F_{n+1}) \longrightarrow \pi_i(X_{n+1}) \xrightarrow{q_*} \pi_i(X_n)$$

For i > n-1 we have  $\pi_{i+1}(X_n) = 0$  and for i > n+1 we have  $\pi_i(X_{n+1}) = 0$ . Therefore, for i > n+1 we have  $\pi_i(F_{n+1}) = 0$ . Furthermore, the diagram,

$$\pi_i(X_{n+1}) \xrightarrow{(q_{n+1})_*} \pi_i(X_n)$$

$$(s_{n+1})_* \qquad (s'_n)_*$$

$$\pi_i(X)$$

Therefore, for  $i \leq n$  both upward maps are isomorphisms so  $(q_{n+1})_*: \pi_i(X_{n+1}) \to \pi_i(X_n)$  is an isomorphism. Therefore,  $\pi_i(F_{n+1}) = 0$  from the long exact sequence when i < n. Thus, we only need to consider the cases i = n, n + 1. For i = n we get,

$$0 \longrightarrow \pi_n(F_{n+1}) \longrightarrow \pi_n(X_{n+1}) \stackrel{\sim}{\longrightarrow} \pi_n(X_n)$$

and thus  $\pi_n(F_{n+1}) = 0$ . For i = n+1 we get,

$$0 \longrightarrow \pi_{n+1}(F_{n+1}) \longrightarrow \pi_{n+1}(X_{n+1}) \xrightarrow{q_*} 0$$

and thus  $\pi_{n+1}(F_{n+1}) = \pi_{n+1}(X_{n+1}) = \pi_n(X)$ . Thus, we find that  $F_{n+1} = K(\pi_n(X), n)$ .

**Definition 2.0.3.** Let X be a connected CW complex and  $\mathcal{X}$  its Postnikov tower. Then we define a completion,

$$\hat{X} = \varprojlim_n X_n$$

## 2.1 Whitehead Towers

**Definition 2.1.1.** Let X be a connected space. Then a Whitehead tower  $\mathcal{X}$  is a sequence of spaces,

where  $X^n$  is n-connected and the morphism  $q^n: X^n \to X^{n-1}$  induces an isomorphism,

$$(q^n)_* : \pi_i(X^n) \xrightarrow{\sim} \pi_i(X^{n-1})$$

for all  $i \geq n+1$ . A morphism  $f: \mathcal{X} \to \mathcal{X}'$  of Whitehead towers is a morphism  $f^n: X^n \to X'^n$  of sequences such that  $f^0: X^0 \to X^0$  is the identity.

Remark. Given any morphism  $f: \mathcal{X} \to \mathcal{X}'$  of Whitehead towers, we have a commuting square,

$$\pi_i(X^{n+1}) \xrightarrow{q_*^{n+1}} \pi_i(X^n)$$

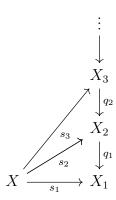
$$f_*^{n+1} \downarrow \qquad \qquad \downarrow f_*^n$$

$$\pi_i(X'^{n+1}) \xrightarrow{q_*'^{n+1}} \pi_i(X'^n)$$

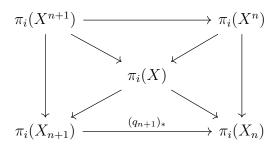
For  $i \geq n+2$  the maps  $q_*^{n+1}$  are isomorphism. For  $i \leq n+1$  we have  $\pi_i(X^{n+1}) = \pi_i(X'^{n+1}) = 0$  and thus if  $f_*^n$  is an isomorphism for all i then  $f_*^{n+1}$  is also an isomorphism for each i. Furthermore, since  $f^0 = \operatorname{id}$ , by induction we see that  $f_*^n : \pi_i(X^n) \to \pi_i(X^n)$  is an isomorphism for each i. Thus any morphism  $f: \mathcal{X} \to \mathcal{X}'$  on Whitehead towers gives a weak homotopy equivalence  $f^n: X^n \to X'^n$  on each  $X^n$ .

**Theorem 2.1.2.** Let X be a connected CW complex. Then X admits a Whitehead tower which is unique up to homotopy equivalence.

*Proof.* We may take a Postnikov tower of fibrations for X,



Consider the map  $s_n: X \to X_n$ . Then we define the Whitehead tower  $X^n$  to be the homotopy fiber of  $s_n: X \to X_n$ . Therefore, we get a fibration  $X^n \hookrightarrow N_{s_n} \to X_n$  where  $N_{s_n}$  is homotopy equivalent to X. Furthermore,  $\pi_i(N_{s_n}) \xrightarrow{\sim} \pi_i(X_n)$  is an isomorphism for  $i \leq n$  and  $\pi_{n+1}(X_n) = 0$  so by the long exact sequence of a fibration, we find that  $\pi_i(X^n) = 0$  for  $i \leq n$ . Furthermore, for i > n we know  $\pi_i(X_n) = 0$  and thus  $\pi_i(X^n) \to \pi_i(X)$  is an isomorphism for i > n. Using the functoriality of homotopy fibers, we get a diagram,



For i > n+1 the upper triangle are isomorphism proving that the spaces  $X^n$  form a Whitehead tower. Using a similar argument we see that Whitehead towers and Postnikov towers are dual and existence and uniqueness of Whitehead towers thus carries over from our discussion of Postnikov towers.