Mathematics GU4051 Topology
Assignment # 11

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Problem 1.

(a). Let $X = \mathbb{R}^2 \setminus \{0\}$ and define $f : X \to S^1$ by $f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$ which is well defined because $\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = 1$. First, because the function is resricted to $(x,y) \neq 0$ so $\sqrt{x^2+y^2} \neq 0$ and thus f is continuous by analysis. Futhermore, if $(x,y) \in S^1$ then $x^2+y^2 = 1$ so $f(x,y) = (x,y)$ so $f \circ i_{S^1} = \mathrm{id}_{S^1}$. So f is a retract.

Furthmore, let $H : X \times I \to X$ by $H(x,y,t) = \left(tx + (1-t)\frac{x}{\sqrt{x^2+y^2}}, ty + (1-t)\frac{y}{\sqrt{x^2+y^2}} \right)$. Again because $(x,y) \neq 0$ this function is continuous by analysis. Now,

$$H(x,y,0) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) = i_{S^1} \circ f(x,y)$$

Furthermore, $H(x,y,1) = (x,y) = \mathrm{id}_X$. Also, if $(x,y) \in S^1$ then $x^2+y^2 = 1$ so $H(x,y,t) = (tx + (1-t)x, ty + (1-t)y) = (x,y)$. Thus, f is a deformation retract.

(b). If $f : X \to A$ is a retract then for any $x_0 \in A$ the map $f_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is a surjection. However, $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}^2, (1,0)) \cong \{e\}$ because $\mathbb{R}^2$ is convex. Therefore, there does not exist a surjection from $\pi_1(S^1, (1,0))$ to $\pi_1(\mathbb{R}^2, (1,0))$ because the former is larger then the latter. Therefore, S^1 is not a retract of $\mathbb{R}^2$.

Problem 2.

(a). Let $f : X \to A$ be a deformation retract and $a \in A$. Then, there exists a homotopy from id_X to $i \circ f$ i.e. $H : X \times I \to X$ such that $H(x,0) = x$ and $H(x,1) = i \circ f(x)$ and if $x \in A$ then $H(x,t) = x$. Therefore, for $x \in A$, $f(x) = H(x,1) = x$ so f is a retraction and thus the induced homomorphism $f_* : \pi_1(X,a) \to \pi_1(A,a)$ is a surjection. It suffices to show that f_* is also an injection. Let $\gamma : I \to X$ be a loop at a. Then consider the map $G : I \times I \to X$ given by $G(x,t) = H(\gamma(x), t)$. Now, $G = H \circ (\gamma \times \mathrm{id}_I)$ which is a composition of continuous maps and therefore continuous.

$$\begin{tikzcd} I \times I \arrow[r, "\gamma \times \mathrm{id}_I"] \arrow[rr, bend right, dashed, "G"'] & X \times I \arrow[r, "H"] & X \end{tikzcd}$$

However, $G(x,0) = H(\gamma(x),0) = \gamma(x)$ and $G(x,1) = i \circ f \circ \gamma(x)$ and $G(0,t) = H(a,t) = a$ and $G(1,t) = H(a,t) = a$ because $a \in A$. Thus, G is a path-homotopy between γ and $i \circ f \circ \gamma$. Suppose that $f_*([\gamma_1]) = f_*([\gamma_2])$ then $[f \circ \gamma_1] = [f \circ \gamma_2]$ so $f \circ \gamma_1 \sim f \circ \gamma_2$. Therefore, $i \circ f \circ \gamma_1 \sim i \circ g \circ \gamma_2$ because $i : A \to X$ is continuous. However, $\gamma_1 \sim i \circ f \circ \gamma_1$ and similarly $\gamma_2 \sim i \circ f \circ \gamma_2$ so by transitivity, $\gamma_1 \sim \gamma_2$ so $[\gamma_1] = [\gamma_2]$. Therefore, f_* is an injection.

(b). Let $T = S^1 \times S^1$ i.e. the torus embedded in $\mathbb{R}^4$ and $x_0 = (1,0) \in S^1$. Consider the projection $\pi_1 : S^1 \times S^1 \to S^1$ which is continuous by the definition of the product topology. Let $s : S^1 \to S^1 \times \{x_0\}$ be the map $s : x \mapsto (x, x_0)$. I claim that the map $f = s \circ \pi_1 : X \to S^1 \times \{x_0\} = A$ is a retraction. This is because if $p \in A$ then $p = (x, x_0)$ so $\pi_1(p) = x$ so $s(\pi_1(p)) = (x, x_0)$. Therefore, $f \circ i_A = \mathrm{id}_A$. So A is a retract of T. However,

$$\pi_1(T, x_0 \times x_0) = \pi_1(S^1 \times S^1, x_0 \times x_0) \cong \pi_1(S^1, x_0) \times \pi_1(S^1, x_0) \cong \mathbb{Z} \times \mathbb{Z}$$

Similarly,

$$\pi_1(S^1 \times \{x_0\}, x_0 \times x_0) \cong \pi_1(S^1, x_0) \times \pi_1(\{x_0\}, x_0) \cong \mathbb{Z} \times \{e\} \cong \mathbb{Z}$$

By the folowing problem, $\mathbb{Z} \times \mathbb{Z} \not\cong \mathbb{Z}$ so $\pi_1(T, x_0 \times x_0) \not\cong \pi_1(S^1 \times \{x_0\}, x_0 \times x_0)$. Therefore, $S^1 \times \{x_0\}$ cannot be a deformation retract of T because otherwise the fundamental groups would be isomorphic.

Problem 3.

(a). Suppose that $\varphi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is a homomorphism. Then, take $\phi(1) = (a,b) \in \mathbb{Z} \times \mathbb{Z}$. Because φ is a homomorphism, $\varphi(n) = (an, bn)$. However, if $(1,0) \in \mathrm{Im}(\varphi)$ then $b = 0$ since $n \neq 0$ if $an = 1$. Similarly, if $(0,1) \in \mathrm{Im}(\varphi)$ then $a = 0$ which contradicts the claim that $an = 1$. Thus one of $(1,0)$ or $(0,1)$ is not in the image of φ so the map cannot be surjective.

(b). $\pi_1(S^3, x_0) \cong \{e\}$ and $\pi_1(S^2 \times S^1, x_0' \times y_0) \cong \pi_1(S^2, x_0') \times \pi_1(S^1, y_0) \cong \{e\} \times \mathbb{Z} \cong \mathbb{Z}$ and,

$$\pi_1(S^1 \times S^1 \times S^1, y_0 \times y_0 \times y_0) \cong \pi_1(S^1, y_0) \times \pi_1(S^1, y_0) \times \pi_1(S^1, y_0) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

No two of these groups are isomorphic. The trivial group has one element which cannot be put into bijection with either infinite group. Futhermore, if there existed an isomorphism between $\mathbb{Z}$ and $\mathbb{Z}^3$ then by composing this map with the projection down to $\mathbb{Z}^2$ we would obtain a surjective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}^2$ which we proved was impossible above. Thus, no two of these spaces are isomorphic.

Problem 4.

In an exactly analogous fashion to question 1 (a), S^{n-1} is a deformation retract of $\mathbb{R}^n \backslash \{0\}$. Therefore, $\pi(\mathbb{R}^n \backslash \{0\}, x_0) \cong \pi_1(S^{n-1}, x_0)$ so for $n > 2$ the fundamental group of $\mathbb{R}^n \backslash \{0\}$ is trivial because S^k is simply connected for $k > 1$. However, $\pi_1(\mathbb{R}^2 \backslash \{0\}, x_0) \cong \pi_1(S^1, x_0) \cong \mathbb{Z}$. Therefore, $\mathbb{R}^2 \backslash \{0\}$ is not homeomorphic to $\mathbb{R}^n \backslash \{0\}$ for $n > 2$. However, if $\mathbb{R}^2 \cong \mathbb{R}^n$ then the subspace $\mathbb{R}^2 \backslash \{0\}$ is homeomorphic to $\mathbb{R}^n \backslash \{x\}$ which is homeomorphic to $\mathbb{R}^n \backslash \{0\}$ by shifting. However the fomer is not simply connected and the latter is which is a contradiction.

Problem 5.

Let $X = Y = S^1$ and take the universal cover $\tilde{X} = \mathbb{R}$ with the standard covering map $g : \tilde{X} \to Y$ given by $g(r) = e^{2\pi i r}$. Also, take $p : Y \to X$ be the covering map given by $p(z) = z^n$. Now, the continuous map p induces an injective homomorphism $p_* : \pi_1(Y) \to \pi_1(X)$. Now, $\pi_1(Y) \cong \mathbb{Z}$ so the entire homomorphism is determined by the image of the generator. The path $\gamma : I \to Y$ given by $\gamma(t) = e^{2\pi i t}$ generates the entire group because the path $\tilde{\gamma} : I \to \mathbb{R}$ given by $\tilde{\gamma}(t) = t$ is a lift of γ to the universal cover since $g \circ \tilde{\gamma}(t) = e^{2\pi i t} = \gamma(t)$. Therefore, γ corresponds to the deck transformation taking 0 to 1 which generates the group of integer shifts. Thus, γ generates $\pi_1(Y)$. Futhermore, $p_*([\gamma]) = [p \circ \gamma]$ where $p \circ \gamma(t) = (e^{2\pi i t})^n = e^{2\pi n i t} = \gamma^n(t)$ since γ^n corresponds to a shift by n in the group of deck transformations of $\tilde{X}$ over X. Therefore, the generator of $\pi_1(Y)$ is mapped to the n^{th} power of the generator of $\pi_1(Y)$ and thus, $p_*(\pi_1(Y)) = (\pi_1(X))^n \cong n\mathbb{Z}$. However, $n\mathbb{Z} \triangleleft \mathbb{Z}$ so $p_*(\pi_1(Y)) \triangleleft \pi_1(X)$ and by a theorem from class, this implies that $p : Y \to X$ is a Galois cover. Furthermore, when $p : Y \to X$ is a Galois cover, the group of deck transformations is given by,

$$D_{Y \to X} \cong \pi_1(X)/p_*(\pi_1(Y)) \cong \mathbb{Z}/n\mathbb{Z}$$

Problem 6.

Take the map $p : Y \to X$ which is a 3-fold cover. Let $f \in D_{Y \to X}$ be a deck transformation. Consider the point $-1 \in S^1 \subset X$ whose preimage is $\pi^{-1}(-1) = \{r_1, r_2, r_3\}$ where $r_1 = (-1, a)$ and $r_2 = (i, b)$ and $r_3 = (-i, b)$. Let $\gamma : I \to S$ be the loop at -1 given by $\gamma(t) = -e^{2\pi i t}$ which goes arround the left circle S^1 once counterclockwise. This path lifts uniquely at r_1 to the path $\tilde{\gamma}_1(t) = (-e^{2\pi i t}, a)$. In particular, the lifted path is a loop. However, the lift of γ at r_2 and r_3 are paths $\tilde{\gamma}_2(t) = (ie^{\pi i t}, a)$ and $\tilde{\gamma}_3(t) = (-ie^{\pi i t}, a)$ respectivly which are not loops but instead go arround half circles in Y. Suppose that $f(r_1) \neq r_1$ then, since $p \circ f = p$, we have $f(r_1) \in p^{-1}(x_0)$ so WLOG assume that $f(r_1) = r_2$. Then, $f \circ \tilde{\gamma}_1$ is a path in Y such that $f \circ \tilde{\gamma}(0) = f(r_1) = r_2$ and $p \circ (f \circ \tilde{\gamma}_1) = (p \circ f) \circ \tilde{\gamma}_1 = p \circ \tilde{\gamma}_1 = \gamma$. Thus, $f \circ \tilde{\gamma}_1 = \tilde{\gamma}_2$, the unique lift of γ at r_2. However, $\tilde{\gamma}_1(0) = \tilde{\gamma}_1(1)$ and therefore $f \circ \tilde{\gamma}_1(0) = f \circ \tilde{\gamma}_1(1)$ so $\tilde{\gamma}_2(0) = \tilde{\gamma}_2(1)$ which contradicts the fact that the unique lift at r_2 is not a closed loop. The same argument applies if $f(r_1) = r_3$ were assumed. Therefore, $f(r_1) = r_1$ so by Lemma **??**, $f = \text{id}$ because the deck transformations act freely. Therefore, $D_{Y \to X} \cong \{\text{id}\}$, the group of deck transformations is trivial.

Lemmas

Note: I assume that all spaces are path-connected

Lemma 0.1. Let $p : Y \to X$ be a covering map, if $f \in D_{Y \to X}$ is such that $f(y_0) = y_0$ for some $y_0 \in p^{-1}(x_0)$ then $f = \text{id}$. Equivalently, the action of $D_{Y \to X}$ on a fiber $p^{-1}(x_0)$ is free.

Proof. Suppose that $f(y_0) = y_0$ and take any $y \in Y$. Take a path γ from y_0 to y and consider its image under p. The path $p \circ \gamma$ takes x_0 to $p(y)$. By the path lifting lemma, there exists a unique lift of $p \circ \gamma$ at y_0. However, $\gamma(0) = y_0$ so γ is clearly the unique lift. However, $f \circ \gamma(0) = f(y_0) = y_0$ and $p \circ (f \circ \gamma) = (p \circ f) \circ \gamma = p \circ \gamma$ because f is a deck transformation. Therefore, $f \circ \gamma = \gamma$ by uniqueness. In particular, $f \circ \gamma(1) = f(y) = \gamma(1) = y$. Therefore, $f = \text{id}$. $\square$