Introduction to Complex Analysis and Riemann Surfaces

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Contents

1 Holomorphic Maps

Definition: A subset $\Omega \subset \mathbb{C}$ is a domain if Ω is open and connected.

Definition: A map $f : \Omega \to \mathbb{C}$ is holomorphic at $z \in \Omega$ if the limit,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The map f is holomorphic on Ω if it is holomorphic at each $z \in \Omega$.

Proposition 1.1. Let $f : \Omega \to \mathbb{C}$ be holomorphic at $z \in \Omega$. Then we may write f as a function of two real variables as, $f(x, y) = f(x + iy)$. This done,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$$

and thus,

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0$$

Proposition 1.2.

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left[\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right] \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right]$$

Therefore, if f is holomorphic then

$$\frac{\partial f}{\partial z} = f'(z) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = 0$$

Definition: Let $U \subset \mathbb{R}^m$ then denote the vectorspace of continuous functions $U \to \mathbb{C}$ by $\mathcal{C}^0(U)$ and for $n > 0$ define,

$$\mathcal{C}^n(U) = \{ f : U \to \mathbb{R}^m \mid \forall p \in U : f'_p \text{ exists and } \forall \mathbf{v} \in \mathbb{R}^n : f'(\mathbf{v}) \in \mathcal{C}^{n-1}(U) \}$$

where $f' \cdot \mathbf{v}$ is the map $p \mapsto f'_p(\mathbf{v})$. Furthermore, the space of smooth functions is,

$$\mathcal{C}^\infty(U) = \bigcap_k \mathcal{C}^k(U)$$

Theorem 1.3. Let Ω be a domain and $f : \Omega \to \mathbb{C}$. Then the following are equivalent,

1. $f : \Omega \to \mathbb{C}$ is holomorphic.
2. $f \in \mathcal{C}^1(\Omega)$ and
$$\frac{\partial f}{\partial \bar{z}} = 0$$
3. $f \in \mathcal{C}^1(\Omega)$ and for $D \subseteq \Omega$ with piecewise $\mathcal{C}^1(\Omega)$ boundary we have
$$\oint_{\partial D} f(z) \, \mathrm{d}z = 0$$

4. $\forall B_r(w) \subsetneq \Omega$ we have,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

for all $z \in B_r(w)$.

5. $\forall w \in \Omega \exists r > 0$ such that whenever $|z - w| < r$ we have,

$$f(z) = \sum_{n=0}^{\infty} a_n (x - w)^n$$

Proof. We will show that,

$$(2) \iff (3) \implies (4) \implies (5) \implies (1) \implies (2)$$

$(4) \implies (5)$ We assume that,

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

We express the function,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - w - (z - w)} = \frac{1}{\zeta - w} \frac{1}{1 - \left(\frac{z-w}{\zeta - w}\right)} = \frac{1}{\zeta - w} \sum_{n=0}^{\infty} \left(\frac{z - w}{\zeta - w}\right)^n = \sum_{n=0}^{\infty} \frac{(z - w)^n}{(\zeta - w)^{n+1}}$$

Then, formally,

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} f(\zeta) \left(\sum_{n=0}^{\infty} \frac{(z - w)^n}{(\zeta - w)^{n+1}} \right) \mathrm{d}\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} f(\zeta) \frac{\mathrm{d}\zeta}{(\zeta - w)^{n+1}} \right) (z - w)^n$$

However, to interchange the sum and integral we must establish uniform and absolute convergence. We know that $|\zeta - w| = r$ and $z \in B_r(w)$ so $|z - w| < r$ and thus the sum,

$$\sum_{n=0}^{\infty} \left| \frac{z - w}{\zeta - w} \right|^n$$

converges. Furthermore,

$$\left| \left(\frac{z - w}{\zeta - w} \right)^n \right| = \left| \frac{z - w}{\zeta - w} \right|^n < \left| \frac{z - w}{\zeta - w} \right| = M < 1$$

so the functions are bounded by M^n whose sum coverges and thus by the Weierstrass M-test the series converges absolutly and uniformly. Therefore, take,

$$a_n = \frac{1}{2\pi i} \oint\limits_{|\zeta - w| = r} \frac{f(\zeta) \, \mathrm{d}\zeta}{(\zeta - w)^{n+1}}$$

(5) $\implies$ (1) It is clear that if,

$$f(z) = \sum_{n=0}^{\infty} a_n (x-w)^n$$

then,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (x-w)^{n-1}$$

exists.

(1) $\implies$ (2) Suppose that $\Omega = B_\delta(w)$. For each $z \in \Omega$, let ℓ_z be the segment joining w to z and define,

$$F(z) = \int_{\ell_z} f(\zeta)\, \mathrm{d}\zeta$$

Now compute the ratio,

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left[\int_{\ell_z} f(\zeta)\, \mathrm{d}\zeta \; - \int_{\ell_{z+h}} f(\zeta)\, \mathrm{d}\zeta\right]$$

(PROGRESS) Because the integral over the tringle is zero, we have,

$$\frac{1}{h}\left[\int_{\ell_z} f(\zeta)\, \mathrm{d}\zeta \; - \int_{\ell_{z+h}} f(\zeta)\, \mathrm{d}\zeta\right] = \frac{1}{h}\int_{z}^{z+h} f(\zeta)\, \mathrm{d}\zeta \; = \int_0^1 f(z+th)\, \mathrm{d}t \; \to f(z)$$

where we have parametrized the path z to $z+h$ by $z+th$ for $0 \le t \le 1$. Thus, $F'(z) = f(z)$ which implies that F is $\mathcal{C}^1(\Omega)$ and holomorphic so,

$$\partial f \bar{z} = 0$$

and thus satisfies (2). Therefore, by (2) $\implies$ (5) we have that F is a power seies and thus $f = F'$ is a power series so f is $\mathcal{C}^1(\Omega)$. Furthrermore, f is holomorphic which implies that

$$\frac{\partial f}{\partial \bar{z}} = 0$$

. Therefore, we have (2).

$\square$

Theorem 1.4. For any $z_0 \in \Omega$, either $f \equiv 0$ in a neighborhood of z_0 or we can express $f = (z-z_0)^n u(z)$ for $u(z)$ holomorphic and $u(z) \neq 0$.

Proof. In a neighborhood of z_0, we can write,

$$f(z) = \sum_{n=0}^{\infty} n_n (z-z_0)^n$$

Either $c_n = 0$ for each n so $f = 0$ or $c_N \neq 0$ for some n and $c_n - 0$ for $n < N$. Therefore,

$$f(z) = \sum_{n \geq N}^{\infty} c_n (z - z_0)^n = (z - z_0)^N \left(\sum_{m=0}^{\infty} c_{N+m} (z - z_0)^m \right) = (z - z_0)^N u(z)$$

Furthermore, $u(z_0) = c_N \neq 0$ so, by continuity, there exists a neighborhood of z_0 on which $n(z) \neq 0$. $\square$

Theorem 1.5. Let f be holomorphic on a domain Ω. If $f \equiv 0$ on some open set inside Ω then $f \equiv 0$ on all of Ω.

Proof. Define,

$$\Omega' = \{z \in \Omega \mid f \equiv 0 \text{ on an open neighborhood of } z\}$$

Clearly Ω' is open in Ω because each $z \in \Omega'$ in inside an open neighborhood of Ω on which f vanishes so is contained in an open neighborhood of Ω'.
Take $z_1 \notin \Omega'$. Thus, f does not vanish identically on every neighborhood of z so there exists a neighborhood U such that $f(z) = (z - z_1)^N u(z)$ for $u(z) \neq 0$. Then $f(z) \neq 0$ on $U \setminus \{z_1\}$. Therefore, $U \subset (\Omega')^C$ because f is nonzero on $U \setminus \{z\}$ and thus cannot be identically zero on any neighborhood of any point of U. Thus, $(\Omega')^C$ is open so Ω' is clopen. However, Ω is connected and thus $\Omega' = \Omega$. $\square$

Example 1.6. Consider the solution to the equation $w^2 = z$. First take the open domain $U = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \geq 0\}$ and for $z = re^{i\theta}$ with $0 < \theta < 2\pi$ define $w = r^{1/2} e^{i\theta/2} = \sqrt{z}$. The function $f(z) = w$ is perfectly holomorphic on U. However, the line we choose to remove is artificial, any cut will work with a redefinition of the angular interval. We solve this problem by taking two copies of U called (I) and (II) and then constructing a surface X by gluing (I) and (II) along the cuts such that moving across the cut in $\mathbb{C}$ corresponds to changing sheets. We can define w on all of X by $w(p) = w(z) = \sqrt{z}$ if p is on sheet (I) at position z and otherwise $w(p) = -w(z) = -\sqrt{z}$ if p is on sheet (II) at position z.

Topologically, X is a sphere minus two points. We call $\hat{X}$ the compactified version of X constrcuted by adding back the two points such that $\hat{X} \cong S^2$.

2 Meromorphic Functions

Definition: A function $f : \Omega \to \mathbb{C}$ is meromorphic if, near any $z_0 \in \Omega$, it can be written as,

$$f(z) = \sum_{n \geq -N} c_n (z - z_0)^n$$

We call N the order of the pole (assuming that $c_n \neq 0$) and c_{-1} the residue at z_0.

Theorem 2.1 (Residue)**.** Let $f : \Omega \to \mathbb{C}$ be meromorphic and $D \subset \overline{D} \subset \Omega$ be a domain in Ω with piecewise smooth boundary ∂D such that no poles of g lie on ∂D. Then,

$$\oint_{\partial D} f(z)\, \mathrm{d}z = 2\pi i \sum_{p \in D} \mathrm{Res}_{f(p)}$$

Proof. We can deform the path ∂D to a sum of small circles of radius r surrounding each pole. Since f is holomorphic on the region D minus these circles the two integrals along these paths (whose difference is the integral over the boundary) are equal. Thus,

$$\begin{aligned}\oint_{\partial D} f(z)\, \mathrm{d}z - 2\pi i \sum_{p \in D} \mathrm{Res}_p\, f &= \sum_{p \in D} \left[\oint_{\partial B_r(p)} f(p+z)\, \mathrm{d}z - 2\pi i \mathrm{Res}_p\, g \right] \\ &= \sum_{p \in D} \left[\int_0^{2\pi} i \Big(f(p + re^{i\theta}) re^{i\theta} - \mathrm{Res}_p\, g \Big)\, \mathrm{d}\theta \right]\end{aligned}$$

However,

$$\mathrm{Res}_p\, f = \lim_{z \to p} (z-p) f(z) = \lim_{h \to 0} f(p+h)h$$

and thus, for each $\epsilon > 0$ we can choose some δ such that $r < \delta$ implies that,

$$\left| f(z + rr^{i\theta}) re^{i\theta} - \mathrm{Res}_p\, f \right| < \epsilon$$

Therefore,

$$\begin{aligned}\left| \oint_{\partial D} f(z)\, \mathrm{d}z - 2\pi i \sum_{p \in D} \mathrm{Res}_p\, f \right| &\le \sum_{p \in D} \left[\int_0^{2\pi} \left| f(p + re^{i\theta}) re^{i\theta} - \mathrm{Res}_p\, g \right| \mathrm{d}\theta \right] \\ &\le \sum_{p \in D} \int_0^{2\pi} \epsilon = 2\pi N \epsilon\end{aligned}$$

where N is the number of poles. Since ϵ is arbitrary,

$$\oint_{\partial D} f(z)\, \mathrm{d}z = 2\pi i \sum_{p \in D} \mathrm{Res}_p\, f$$

$\square$

Theorem 2.2. Let $f : \Omega \to \mathbb{C}$ be meromorphic and $D \subset \overline{D} \subset \Omega$ be a domain in Ω with piecewise $\mathcal{C}^1$ boundary ∂D such that no poles of g lie on ∂D. Then,

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)}\, \mathrm{d}z = (\#\text{ of zeros}) - (\#\text{ of poles})$$

Theorem 2.3. At each point $p \in D$ we can expand,

$$f(z) = (z-p)^N u(z)$$

where u is holomorphic and nonvanishing. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \log f(z) = \frac{\mathrm{d}}{\mathrm{d}z} \left[(z-p)^N u(z) \right] = \frac{N}{x-p} + \frac{u'(z)}{u(z)}$$

Thus when f has either a zero ($N > 0$) or a pole ($N < 0$) the logarithmic derivative has residue,

$$\operatorname{Res}_p \left(\frac{f'}{f} \right) = N$$

Therefore the result holds by the residue theorem.

Corollary 2.4. Let $f : \Omega \to \mathbb{C}$ be holomorphic take $w \in \mathbb{C}$, then the number of solutions in D to the equation $f(z) - w = 0$ is equal to,

$$\#\{z \in D \mid f(z) = w\} = \oint_{\partial D} \frac{f'(z)}{f(z) - w} \, \mathrm{d}z$$

Proof. Since $f - w$ is holomorphic on Ω is has no poles. Therefore, the only residues are from roots of $f - w$ i.e. solutions to $f(z) - w = 0$. As above, the integral of the logarithmic derivative counts the number of such poles. $\square$

3 Taylor's Theorem

Theorem 3.1. A function $f : U \to \mathbb{C}$ of a real variable defined on open $U \subset \mathbb{R}$ is the restriction of some holomorphic function $f : \Omega \to \mathbb{C}$ on a domain Ω containing $U \subset \mathbb{R}$ iff f is (real) analytic.

Proof. $\square$

Theorem 3.2 (Cauchy)**.** Let $f : \Omega \to \mathbb{C}$ be holomorphic, for any subset $D \subset \Omega$ homeomorphic to a disc with $\mathcal{C}^1(I)$ boundary and $w \in D^\circ$ we have,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint_{\partial D} \frac{f(z)}{(z-w)^{n+1}} \, \mathrm{d}z$$

Corollary 3.3 (Cauchy Estimate)**.** Let $f : \Omega \to \mathbb{C}$ be holomorphic and $w \in \Omega$. Let $r > 0$ be such that $B_r(w) \subset \Omega$ then,

$$\frac{|f^{(n)}(w)|}{n!} \le \frac{\sup\{|f(z)| \mid z \in \partial B_r(w)\}}{r^n}$$

Proof. Via the Cauchy derivative formula,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint_{\partial B_r(w)} \frac{f(z)}{(z-w)^{n+1}} \, \mathrm{d}z$$

Taking the norm of both sides,

$$\begin{aligned}\frac{|f^{(n)}(w)|}{n!} &= \frac{1}{2\pi}\left|\oint_{\partial B_r(w)} \frac{f(z)}{(z-w)^{n+1}}\,\mathrm{d}z\right| \\ &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(z)}{(re^{i\theta})^{n+1}} ire^{i\theta}\,\mathrm{d}t\right| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z)|}{r^n}\,\mathrm{d}t \le \frac{\sup\{|f(z)| \mid z \in \partial B_r(w)\}}{r^n}\end{aligned}$$

□

Corollary 3.4. Let $f : \Omega \to \mathbb{C}$ be holomorphic and $w \in \Omega$. If $B_r(w) \subset \Omega$ then the talor series about w has radius of convergence at least r.

Proof. Consider,

$$T_{N,w}(z) = \sum_{n=0}^{N} \frac{f^{(n)}(w)}{n!}(z-w)^n$$

For $z \in B_r(w)$ consider,

$$\sum_{n=0}^{N} \left|\frac{f^{(n)}(w)}{n!}(z-w)^n\right| = \sum_{n=0}^{N} \frac{|f^{(n)}(w)|r^n}{n!}\left(\frac{|z-w|}{r}\right)^n \le \sum_{n=0}^{N} M_r x^n$$

where $M_r = \sup\{|f(z)| \mid z \in \partial B_r(w)\}$ and $x = |z-w|/r < 1$ since $z \in B_r(w)$. Then,

$$\sum_{n=0}^{N} M_r x^n = M_r \sum_{n=0}^{N} x^n$$

converges for $N \to \infty$ so $\lim\limits_{N\to\infty} T_{N,w}(z)$ converges absolutly on $B_r(w)$. □

Theorem 3.5. An entire function $f : \mathbb{C} \to \mathbb{C}$ is globally a power series. In particular, f is everywhere equal to its taylor series about any point.

Proof. Since for any $r > 0$ and $w \in \mathbb{C}$ we have $B_r(w) \subset \mathbb{C}$ the above argument shows that the radius of convergence of $T_w(z)$ is infinite. □

Remark 1. This is completely false for everywhere real analytic functions. For example,

$$f(x) = \frac{1}{1+x^2}$$

is analytic but has finite radius of covergence about each point. This is because its extension to the complex plane has poles at $x = \pm i$ and thus is not entire.

Theorem 3.6. Let $f : \Omega \to \mathbb{C}$ be holomorphic and take $w \in \Omega$ and $r > 0$ s.t. $B_r(w) \subset \Omega$. Then, for all $z \in B_r(w)$,

$$T_w(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n = f(z)$$

Furthermore, the error term,

$$R_{N,w}(z) = f(z) - T_{N,w}(z) =$$

Proof. We know that the sum,

$$T_w(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n$$

is absolutly and uniformly convergent. By the Cauchy integral formula,

$$\frac{f^{(n)}(w)}{n!} = \frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{(\zeta - w)^{n+1}} \, \mathrm{d}\zeta$$

and thus,

$$\begin{aligned}
T_w(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{(\zeta - w)^{n+1}} \, \mathrm{d}\zeta \right) (z-w)^n \\
&= \frac{1}{2\pi i} \left(\oint_{\partial B_r(w)} f(\zeta) \sum_{n=0}^{\infty} \left[\frac{(z-w)^n}{(\zeta - w)^{n+1}} \right] \mathrm{d}\zeta \right) \\
&= \frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{(\zeta - w) - (z-w)} \, \mathrm{d}\zeta \; = \frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \; = f(z)
\end{aligned}$$

where I may interchange the integrals and sums by uniform convergence. Furthermore, I have used the series,

$$\frac{(z-w)^n}{(\zeta - w)^{n+1}} = \frac{1}{\zeta - w} \frac{(z-w)^n}{(\zeta - w)^n} = \frac{1}{\zeta - w} \cdot \frac{1}{1 - \frac{z-w}{\zeta - w}} = \frac{1}{(\zeta - w) - (z-w)}$$

which converges because $\zeta \in \partial B_r(w)$ so $|\zeta - w| = r$ and $z \in B_r(w)$ so $|z - w| < r$ and thus,

$$\left| \frac{z-w}{\zeta - w} \right| < 1$$

Now we may compute the error term as follows,

$$\begin{aligned}
R_{w,N}(z) = f(z) - T_{w,N}(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n - \sum_{n=0}^{N} \frac{f^{(n)}(w)}{n!}(z-w)^n \\
&= \sum_{n=N+1}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n = \sum_{n=N+1}^{\infty} \left(\frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{(\zeta - w)^{n+1}} \, \mathrm{d}\zeta \right) (z-w)^n \\
&= \frac{1}{2\pi i} \left(\oint_{\partial B_r(w)} f(\zeta) \sum_{n=N+1}^{\infty} \left[\frac{(z-w)^n}{(\zeta - w)^{n+1}} \right] \mathrm{d}\zeta \right) \\
&= \frac{(z-w)^{N+1}}{2\pi i} \left(\oint_{\partial B_r(w)} \frac{f(\zeta)}{(\zeta - w)^{N+1}(\zeta - z)} \, \mathrm{d}\zeta \right)
\end{aligned}$$

where,

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \left[\frac{(z-w)^n}{(\zeta - w)^{n+1}} \right] &= \frac{(z-w)^{N+1}}{(\zeta - w)^{N+2}} \sum_{n=0}^{\infty} \left[\frac{z-w}{\zeta - w} \right]^n \\
&= \frac{(z-w)^{N+1}}{(\zeta - w)^{N+2}} \cdot \frac{1}{1 - \frac{z-w}{\zeta - w}} = \frac{(z-w)^{N+1}}{(\zeta - w)^{N+1}(\zeta - z)}
\end{aligned}$$

$\square$

Lemma 3.7. Let $f : \mathbb{R} \to \mathbb{R}$ be $(n+1)$-differentiable on $[a, b]$ and $f^{(k)}(a) = 0$ for each $n \leq k$. Then there exists some $\xi \in [a, b]$ such that,

$$f(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$

Proof. Suppose that f is $(n+1)$-differentiable on $[a, b]$ and $f^{(n)}(a) = 0$. Consider the function,

$$g(x) = f(x) - \frac{f(b)}{(b-a)^{n+1}}(x-a)^{n+1}$$

Then g satisfies the same condition that $g^{(k)}(a) = 0$ for $k \leq n$ and g is $(n+1)$-differentiable on $[a, b]$ but also $g(b) = 0$. Now, for each $k \leq n+1$ I claim that there exists $\xi_k \in [a, b]$ such that $g^{(k)}(\xi_k) = 0$. For $k = 0$, by the mean value theorem, there exists $\xi_0 \in [a, b]$ such that,

$$g'(\xi_0) = \frac{g(b) - g(a)}{b-a} = 0$$

Now we proceed by induction. Suppose we have $\xi_k \in [a, b]$ such that $g^{(k)}(\xi_k) = 0$. Then for $k \leq n$ we also know that $g^{(k)}(a) = 0$. Then, since $g^{(k)}$ is differentiable for $k \leq n$, by the mean value theorem, there exists $\xi_{k+1} \in [a, \xi_k] \subset [a, b]$ such that,

$$g^{(k+1)}(\xi_{k+1}) = \frac{g^{(k)}(\xi_k) - g^{(k)}(b)}{b-a} = 0$$

Proving the claim by induction. Finally,

$$g^{(n+1)}(\xi_{n+1}) = f^{(n+1)}(\xi_{n+1}) - f(b)\frac{(n+1)!}{(b-a)^{n+1}} = 0$$

which implies that,

$$f(b) = \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}(b-a)^{n+1}$$

$\square$

Theorem 3.8 (Lagrange Error Form)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be $(n+1)$-differentiable on $[a, b]$. Then the remainder term,

$$R_{a,n}(b) = f(b) - T_{a,n}(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$

for some $\xi \in (a, b)$.

Proof. Consider the function,

$$R_{a,n}(x) = f(x) - T_{a,n}(x)$$

which is $(n+1)$-differentiable on $[a,b]$ and satisfies $R_{a,n}^{(k)}(a) = 0$ for each $k \leq n$ so by the lemma, there exists $\xi \in [a,b]$ such that,

$$R_{a,n}(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$

Because $R_{a,n}^{(n+1)}(b) = f^{(n+1)}(b)$ since the Taylor partial sum has order n. $\square$