1 Formal Singularities

1.1 The Hilbert Samuel Function

In this section, let A be a Noetherian semi-local ring[1] and I an ideal of definition[2].

Definition 1.1.1. For a finite A-module M we define the *Hilbert-Samuel function*,

$$\chi_I^M(n) = \ell(M/I^n M)$$

When $(A, \mathfrak{m}, \kappa)$ is a local ring we write $\chi^M := \chi_{\mathfrak{m}}^M$.

Remark. Consider the graded algebra (ring of the tangent cone),

$$\mathbf{gr}_I(A) = \bigoplus_{n \ge 0} I^n / I^{n+1}$$

and the graded $\mathbf{gr}_I(A)$-module,

$$\mathbf{gr}_I(M) = \bigoplus_{n \ge 0} I^n M / I^{n+1} M$$

Then we see that,

$$\chi_M^I(n) = \ell(M/I^n M) = \sum_{i=0}^{n-1} \ell(I^i M / I^{i+1} M) = \sum_{i=0}^{n-1} H_{\mathbf{gr}_I(M)}(i)$$

where $H_{\mathbf{gr}_I(M)}$ is the Hilbert function[3] function of the graded $\mathbf{gr}_I(A)$-module $\mathbf{gr}_I(M)$.

Proposition 1.1.2. For any finite a polynomial $P_{M,I} \in \mathbb{Q}[x]$ such that for all $n \gg 0$,

$$\chi_I^M(n) = P_{M,I}(n)$$

and $\deg P_{M,I} = \dim M := \dim(A/\mathrm{Ann}_A(M))$. Furthermore, this polynomial has the form,

$$P_{M,I}(n) = \sum_{i=0}^{d} (-1)^i e_i \cdot \binom{n+d-i}{d-i}$$

for integers $e_i \in \mathbb{Z}$.

Proof. This follows from properties of the Hilbert function of a finite module over a finitely-generated graded A/I-algebra since A/I is Artinian. Indeed, if $x_1, \dots, x_r \in I$ generate then,

$$(A/I)[x_1, \dots, x_r] \twoheadrightarrow \mathbf{gr}_I(A)$$

makes $\mathbf{gr}_I(A)$ a finite type A/I-algebra and $\mathbf{gr}_I(M) = M \otimes_A \mathbf{gr}_I(A)$ is a finite $\mathbf{gr}_I(A)$-module. $\square$

Definition 1.1.3. The *multiplicity* of M is $e(M, I) = e_0$ and the *dimension* is $d(M, I) = \deg P_{M,I}$.

[1] A is *semi-local* if $A/\mathrm{Jac}(A)$ is Artinian or equivalently A has finitely many maximal ideals.

[2] An ideal $I \subset A$ is an *ideal of definition* if $\sqrt{I} = \mathrm{Jac}(A)$ or equivalently $\mathrm{Jac}(A)^n \subset I \subset \mathrm{Jac}(A)$ for some n.

[3] For a graded algebra $S = \bigoplus_{n \ge 0} S_n$ over an Artin ring A and a graded S-module M the Hilbert function H_M is the map $n \mapsto \ell(M_n)$

Remark. Therefore, the leading term of $P_{M,I}$ is $\frac{e(M,I)}{d!}n^d$ where $d = d(M,I)$. In particular,

$$e(M,I) = d! \cdot \lim_{n\to\infty} \frac{\chi_I^M(n)}{n^d}$$

Proposition 1.1.4. Consider an exact sequence of finte A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

then,

$$P_{I,M_2} = P_{I,M_1} + P_{I,M_3} - F$$

where F is a polynomial of degree $d < \deg P_{I,M_1}$ and with positive leading coefficient.

Proof. The exact sequence,

$$0 \longrightarrow (I^n M_2 \cap M_1)/I^n M_1 \longrightarrow M_1/I^n M_1 \longrightarrow M_2/I^n M_2 \longrightarrow M_3/I^n M_3 \longrightarrow 0$$

shows that,

$$\chi_I^{M_1}(n) + \chi_I^{M_3}(n) - \chi_I^{M_2}(n) = \ell((I^n M_2 \cap M_1)/I^n M_1)$$

By the Artin-Rees lemma, $I^n M_2 \cap M_1 \subset I^{n-k} M_1$ for $n \gg 0$ and thus for $n \gg 0$,

$$\ell((I^n M_2 \cap M_1)/I^n M_1) \le \ell(I^{n-k} M_1/I^n M_1) = \chi_I^{M_1}(n) - \chi_I^{M_1}(n-k) = P_{M_1,I}(n) - P_{M_1,I}(n-k) = F(n)$$

is a polynomial of degree strictly less than $d(M_1, I)$ with positive leading coefficient. Therefore,

$$P_{I,M_1}(n) + P_{I,M_3}(n) - P_{I,M_2}(n) = \chi_I^{M_1}(n) + \chi_I^{M_3}(n) - \chi_I^{M_2}(n) \le F(n)$$

for all $n \ge 0$ and thus these are equal as polynomials. $\square$

Corollary 1.1.5. Givne an exact sequence, $d(M_2, I) = \max\{d(M_1, I), d(M_3, I)\}$ and,

(a) if $d(M_1, I) = d(M_3, I)$ then $e(M_2, I) = e(M_1, I) + e(M_3, I)$

(b) if $d(M_1, I) > d(M_3, I)$ then $e(M_2, I) = e(M_1, I)$

(c) if $d(M_1, I) < d(M_3, I)$ then $e(M_2, I) = e(M_3, I)$.

1.2 For Schemes

Let X be a Noetherian scheme and $\mathscr{F}$ a coherent sheaf on $\mathscr{F}$. Then for $x \in X$ we define the Hilbert-Samuel polynomial $P_{\mathscr{F},x} = P_{\mathscr{F}_x,\mathfrak{m}_x}$ for the module $\mathscr{F}_x$ over the local ring $\mathcal{O}_{X,x}$ with respect to the maximal ideal $\mathfrak{m}_x$. We define $e(\mathscr{F}, x) = e(\mathscr{F}_x, \mathfrak{m}_x)$ and $d(\mathscr{F}, x) = d(\mathscr{F}_x, \mathfrak{m}_x) = \dim \mathscr{F}_x$. We say the *multiplicity* of a point $x \in X$ is $m_x := e(\mathcal{O}_X, x) = e(\mathcal{O}_{X,x}, \mathfrak{m}_x)$.

1.3 Formal Germs

(GRADED RING IS AN INVARIANT AND THUS ALL HILBERT SAMUEL STUFF)

1.4 Embedding Dimension

(EMBEDDING DIMENSION ISO ON FORMAL RINGS)

2 Deformation Theory of Singularities

3 Hypersurface Singularities

3.1 Introduction

(INVARIANTS?)
(BASIC RESULTS)

Proposition 3.1.1. (MULTIPLICITY IN TERMS OF NORMAL FORM OF f!!)

3.2 Singular Hypersurfaces

Definition 3.2.1. A *hypersurface* $X \subset \mathbb{P}^{n+1}$ is a reduced subscheme of pure codimension 1.

Proposition 3.2.2. A hypersurface is a Cartier divisor and hence is defined by some,

$$F \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^{n+1}}(d)) = k[X_0, \dots, X_{n+1}]_{(d)}$$

where $d = \deg X$.

Proof. DO IT (HOW TO SHOW HEIGHT ONE IDEAL WITH NO EMBEDDED PRIMES IS PRINCIPLE?) □

Proposition 3.2.3. Let S be a hypersurface singularity. Then there exists a hypersurface $X \subset \mathbb{P}^{n+1}$ and a point $p \in X$ such that $(X, p) \cong S$ at $X \setminus \{p\}$ is smooth.

Proof. DO THIS!!! □

Proposition 3.2.4. Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface defined by F and $p \in X$ a point. Then m_p is the smallest integer e such that $F \cdot \mathcal{O}_{\mathbb{P}^{n+1}}(-d)_p \subset \mathfrak{m}_p^e$ or equivalently the smallest degree term of F in local coordinates at p.

Proof. Choosing coordinates such that p is the origin of $\mathbb{A}^{n+1} \subset \mathbb{P}^{n+1}$ we have F dehomogenize to some polynomial $f \in A = k[x_1, \dots, x_{n+1}]$. Since $\mathfrak{m}^e \subset (f)$ for $k \geq e$ (DO THIS!!!) □

3.3 The Milnor Number

(DEF)
(PROVE INVARIANCE)
(GIVE TOP INTERP)

Proposition 3.3.1. $\nu_p \geq 2\delta_p - \gamma_p + 1$

Proposition 3.3.2. Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d then every point $p \in X$ has,

$$\mu_p \leq (d-1)^{n+1}$$

with equality iff (WHAT) X is the union of d hyperplanes at p.

Proof. Up to automorphism assume $p = 0 \in \mathbb{A}^n$. Let $f \in k[x_0, \dots, x_n]$ be an equation for X on $\mathbb{A}^n$. Then clearly ∇f is a list of polynomials of degree at most $(d-1)$ and therefore,

$$\mu_p = \dim_k \widehat{\mathcal{O}_{\mathbb{P}^{n+1},0}} / (\nabla f) \leq (d-1)^{n+1}$$

(FINISH THIS) □

3.4 Plane Curve Singularities

(LOOK AT LATEX AND IPAD NOTES FOR COHOMOLOGY ARGUMENTS) (GENUS DISCREPANCY and also (NOT RELEVANT) REDUCTION DISCREPANCY IN MISC) (DEF INVARIANTS)

Proposition 3.4.1. Let X be a curve and $\nu : X^\nu \to X$ the normalization. Then $m_p = \det \nu$.

Proof. Let $A = \mathcal{O}_{X,p}$ be the local ring and $\widetilde{A}$ its normalization. Consider the exact sequence of A-modules,

$$0 \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow Q \longrightarrow 0$$

However, $Q \otimes \mathrm{Frac}\,(A) = 0$ and thus $d(Q) = 0$ so we have $m_p = e(A) = e(\widetilde{A}) = \deg_p \nu$ because $\mathfrak{m}_p \widetilde{A} = (\varpi_1^{e_1} \cdots \varpi_r^{e_r})$ where $\varpi_1, \dots, \varpi_r \in \widetilde{A}$ are the uniformizers of the points $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ in the fiber over p. Thus,

$$\ell(\widetilde{A}/\mathfrak{m}_p^n \widetilde{A}) = \dim_\kappa \widetilde{A}/\mathfrak{m}_1^{ne_1} \cdots \mathfrak{m}_r^{ne_r} = \sum_{i=0}^{r} ne_i[\kappa(\mathfrak{m}_i) : \kappa] = n \left(\sum_{i=0}^{r} e_i[\kappa(\mathfrak{m}_i) : \kappa] \right) = n \deg \nu$$

□

Proposition 3.4.2. There is a relation between the curve singularity invariants,

$$\mu_p = 2\delta_p - \gamma_p + 1$$

Proof. DO THIS!!! □

3.5 Singularities of Type A_n

(An singularities and COMPUTE)

3.6 Singularities of Plane Curves of Degree d

Definition 3.6.1. A plane curve is a hypersurface $X \subset \mathbb{P}^2$.

4 Surface Singularities

(ADE TYPE)

5 Rational Singularities

6 Singularities in the Minimal Model Program

7 Resolution of Singularities

8 THAT PROBLEM, WRONG

Let $\overline{C}$ be any smooth genus 2 curve over $\mathbb{C}$ and $C = \overline{C} \setminus \{p\}$ be the affine curve obtained by removing the point $p \in \overline{C}$. I claim there is no immersion $C \to \mathbb{P}^2$.

This answers (1) (2) and (3) because if we choose $p \in \overline{C}$ to be a ramification point of the hyperelliptic cover $\overline{C} \to \mathbb{P}$ or equivalently a fixed point of the hyperelliptic involution. Then Ω_C is trivial showing that it cannot be the only immersion obstruction.

The Proof

Suppose $\iota : C \to \mathbb{P}^2$ is an immersion. Let $X = \mathbb{P}^2$ and consider the closure $f : \overline{C} \to X$. Let $D \subset X$ be the image and d the degree of D. If $f(p) \in \iota(C)$ then $D = \iota(C)$ meaning $\iota(C)$ is closed which would imply C is compact which is false. Thus $f : \overline{C} \to D$ is a homeomorphism (it is a bijective closed continuous map) and is the normalization showing that the singularity $f(p) \in D$ is unibranch.

The log-Bogomolov-Miyaoka-Yau inequality (e.g. equation (3.8) of [this paper][1]) gives an upper bound on d. From the following inequality: for any smooth surface X and divisor $D \subset X$ for each point $p \in D$ let m_p be the multiplicity γ_p the number of analytic branches, δ_p the discrepancy (change in arithmetic genus when singularity is resolved) and $\mu_p = 2\delta_p - \gamma_p + 1$ the Milnor number. The log-BMY inequality says,

$$(K_X + D)^2 \le 3(c_2(X) + (K_X + D) \cdot D) - \sum_{p \in D} \left(2 + \frac{1}{m_p}\right) \mu_p$$

For our case, $K_X = -3H$ and $D = dH$ and $c_2(X) = 3$ and $p \in D$ is the unique singular point so,

$$\left(2 + \frac{1}{m_p}\right) \mu_p \le 9 + 3d(d-3) - (d-3)^2 = d(2d-3)$$

Now the Milnor number $\mu_p = 2\delta_p - \gamma_p + 1 = 2\delta_p = (d-1)(d-2) - 2g$ where g is the geometric genus ($g = 2$ for us). Also $\mu_p \ge m_p(m_p - 1)$ so

$$\frac{\mu_p}{m_p} \ge \frac{\mu_p}{\sqrt{\mu_p + \frac{1}{4}} + \frac{1}{2}}$$

Thus,

$$\frac{\mu_p}{\sqrt{\mu_p + \frac{1}{4}} + \frac{1}{2}} \le 3d - 4 + 4$$

BUT BUT THIS DOESNT ACTUALLY GIVE A BOUND ON d SHIT. NEED $K_X \ge 0$ FOR A BOUND.

[1]: https://arxiv.org/abs/2007.01735