Contents

1 Lang-Nishimura

Theorem 1.0.1 (Lang-Nishimura)**.** Let $f : X \dashrightarrow Y$ be a rational map of k-varieties with Y proper. If X has a smooth k-point then Y has a k-point.

Proof. First we prove the case that X is a curve. Shrink to the smooth locus $U \subset X$ which intersects some generic point since X has a smooth point $x \in X$ and U is open. Hence we get a rational map $U \dashrightarrow Y$ which extends to $U \to Y$ since U is a regular curve and Y is proper.

Now we reduce to the curve case. We may shrink X such that it affine and integral with $x \in X(k)$ a smooth k-point. The goal is to show that there exists a (nonproper) curve $C \to X$ mapping to X whose image intersects the locus of definition of $f : X \dashrightarrow Y$ and contains a lift $x' \in C(k)$ as a smooth k-point of C. There is an étale neighborhood $U \to X$ of x with a lift $x' \in U(k)$ with an étale map $U \to \mathbb{A}^n_k$. Let $V \subset X$ be the domain of f then pushing and pulling gives a dense open of $\mathbb{A}^n_k$. Therefore, choose a line $L \subset \mathbb{A}^n_k$ through the origin intersecting this locus. Then the preimage $L' \subset U$ is a smooth curve passing through x' and hence $L' \to X$ satisfies the hypotheses. $\square$

Example 1.0.2. The condition that $x \in X(k)$ is a *smooth point* is necessary. For example, consider,

$$X = \text{Proj}\left(\mathbb{R}[X, Y, Z]/(X^2 + Y^2)\right)$$

and let $Y = \mathbb{P}^1_{\mathbb{C}}$ be its normalization and consider the inverse of the normalization $X \dashrightarrow Y$. Now X contains a nonsmooth $\mathbb{R}$-point $[0 : 0 : 1] \in X(\mathbb{R})$ but Y does not have an $\mathbb{R}$-point.

2 $\mathbb{E}_8$ lattice

Let $X = \text{Bl}_{P_1,\ldots,P_9}(\mathbb{P}^2)$ be the blowup at 9 points sufficiently general so there is a unique cubic C through these points and it is smooth. Then there there is a genus 1 curve $\widetilde{C} \subset X$ which is the strict transform of the unique conic through the points $P_1, \ldots, P_9$. Let $E_1, \ldots, E_9$ be the exceptional divisors. Then,

$$\widetilde{C} = 3H - (E_1 + \cdots + E_9)$$

so indeed we see that $\widetilde{C}^2 = 0$. Now the claim is that the lattice,

$$\text{Pic}\,(X) = \text{NS}\,(X)$$

contains the $\mathbb{E}_8$ lattice as a subquotient. Indeed,

$$\left\langle \widetilde{C} \right\rangle^{\perp} / \left\langle \widetilde{C} \right\rangle \cong \mathbb{E}_8$$

3 Root Stacks

4 Weierstrass Points

5 MAPP

Remark. Note that the isomorphism $X \xrightarrow{\sim} (B' \times Z)/G$ is *not* compatible with any map to A. Indeed, there may not even be a map to A since $B'/G = B$ may only be isogenous to an abelian subvariety. Even if G is trivial, the isomorphism may not be compatible with f and the projection. For example, consider $X = E \times C$ where E is an elliptic curve and C is a genus 2 curve with Jacobian $E \times E'$. Mapping to the Albanese $E \times E \times E'$, our construction gives the identity $\text{id} : E \times C$. However, the map to the Al banese does not factor through the first projection $\text{pr}_1 : X \to E$.

6 Etale fundamental groups are NOT profinitely complete

I allways thought that étale fundamental groups are profinitely complete i.e. equal to their own profinite completion. This is false in general. They are always profinite but this is weaker in general. It is true that a profinite group is the limit over its finite *continuous* quotients or equivalently,

$$G = \varprojlim G/H$$

as H runs over the finite index *open* (actually every open subgroup in a compact group is automatically finite index) normal subgroups. However, this does not necessarily include every finite index subgroup.

Remark. However, if a *topological* group G is profinite then $G \to \widehat{G}^{\text{top}}$ is an isomorphism by definition where,

$$\widehat{G}^{\text{top}} = \varprojlim_{\substack{H \triangleleft G \\ H \text{ open}}} G/H$$

Example 6.0.1. $\pi_1^{\text{ét}}(\text{Spec}(\mathbb{Q})) = \text{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ is *not* profinitely complete. Indeed, see Chapter 7 of Milne's Class Field Theory.

Remark. See these answers:

(a) Silverman's incorrect definition

(b) examples of noncomplete profinite groups

Proposition 6.0.2. However, if X is a scheme of finite type over $\mathbb{C}$ then,

$$\pi_1^{\text{ét}}(X) = \pi_1\widehat{(X(\mathbb{C}))}$$

is the profinite completion of a finitely presented group and hence is profinitely complete.

Proof. Indeed, by Riemann-Existence,

$$\pi_1^{\text{ét}}(X)\text{-FinSets} \cong \text{FÉt}_X \cong \{\text{finite covering spaces of } X\} \cong \pi_1(X(\mathbb{C}))\text{-FinSets}$$

where $\pi_1^{\text{ét}}(X)$-FinSets means *continuous* finite $\pi_1^{\text{ét}}(X)$-sets. This identifies $\pi_1^{\text{ét}}(X)$ as a topological group with $\pi_1\widehat{(X(\mathbb{C}))}$ □

Lemma 6.0.3. If G is a finitely presented group then $\widehat{G} \to \widehat{\widehat{G}}$ is an isomorphism.

Proof. See here. □

Remark. Note that the above theorem is nontrivial. In fact, it is false without the finite presentation assumption. See here.

7 Grothendieck Abelian Categories

Definition 7.0.1. Let $\mathcal{A}$ be an abelian category. We say that satisfies,

(AB3) $\mathcal{A}$ has all direct sums

(AB4) $\mathcal{A}$ is AB4 and taking direct sums is exact

(AB5) $\mathcal{A}$ is AB3 and taking filtered colimits is exact

(AB6) $\mathcal{A}$ is AB3 and given a family of filtered categories $\{I_j\}_{j \in J}$ and maps $D_j : I_j \to \mathcal{A}$ we have,

$$\prod_{j \in J} \operatorname*{colim}_{I_j} D_j = \operatorname*{colim}_{(i_j) \in \prod_{j \in J} I_j} \left(\prod_{j \in J} D_j(i_j) \right)$$

We say that $\mathcal{A}$ *has a generator* if there is an object $M \in \mathcal{A}$ such that $\text{Hom}_{\mathcal{A}}(M, -)$ is faithful. We say that $\mathcal{A}$ is a *Grothendieck category* if $\mathcal{A}$ is AB5 and has a generator.

Lemma 7.0.2. We have the following implications:

$$\text{AB6} \implies \text{AB5} \implies \text{AB4} \implies \text{AB3}$$

Lemma 7.0.3. For any unital ring R, the category Mod_R satisfies AB6 and AB4* but not AB5*.

Example 7.0.4. **Ab** thus satisfies AB6 and AB4* but not AB5*. Hence $\mathbf{Ab}^{\text{op}}$ which is isomorphic to the category of compact Hausdorff topological groups by Pontriagin duality saitsifes AB6* and AB4 but not AB5.

Lemma 7.0.5. The only abelian category satisfying AB5 and AB5* is the zero category.

Lemma 7.0.6. An AB3 abelian category $\mathcal{A}$ has a generator M if and only if for every $A \in \mathcal{A}$ there is an epimorphism,

$$\bigoplus_I M \twoheadrightarrow A$$

Proof. Suppose that,

$$\bigoplus_I M \twoheadrightarrow A$$

By definition, if $f, g : A \to B$ are two maps such that the induced maps,

$$M \to A \to B$$

are pairwise equal then $f = g$. Therefore,

$$\text{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\text{Hom}_{\mathcal{A}}(M,-)} \text{Hom}\left(\text{Hom}_{\mathcal{A}}(M, A), \text{Hom}_{\mathcal{A}}(M, B)\right)$$

is injective since it is injective after evaluation at the inclusions $\{M \to A\}_I$.

Conversely, suppose that $\mathcal{A}$ has a generator. For each $A \in \mathcal{A}$ let $I = \text{Hom}_{\mathcal{A}}(M, A)$ which is a set and there is a canonical map,

$$c : \bigoplus_I M \to A$$

via evaluation. We need to show this is an epimorphism. Indeed, if $f, g : A \to B$ are two maps such that $f \circ c = g \circ c$ this means that $f_* = g_*$ and since $\text{Hom}_{\mathcal{A}}(M, -)$ is faithful we see that $f = g$ so we conclude that c is an epimorphism. ☐

Theorem 7.0.7 (1.10.1 in Tôhoku)**.** Let $\mathcal{A}$ be a Grothendieck abelian category then $\mathcal{A}$ has enough injectives.

(IS THIS CORRECT??)

Proposition 7.0.8. Let $\mathcal{C}$ be a category and $\mathcal{A}$ satisfies any of,

(a) AB3

(b) AB4

(c) AB5

(d) AB6

(e) AB3*

(f) AB4*

(g) AB5*

(h) AB6*

(i) $\mathcal{A}$ has a generator

(j) $\mathcal{A}$ is a Grothendieck abelian category

then the same is true of $\mathrm{PSh}(\mathcal{C}, \mathcal{A}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.

DO THIS!!

Theorem 7.0.9. Let $\mathcal{C}$ be a site and $\mathcal{A}$ satisfies any of,

(a) AB3

(b) AB4

(c) AB5

(d) AB6

(e) AB3*

(f) $\mathcal{A}$ has a generator

(g) $\mathcal{A}$ is a Grothendieck abelian category

then the same is true of $\mathfrak{Sh}(\mathcal{C}, \mathcal{A})$.

Remark. Note tha even for $\mathcal{A} = \mathbf{Ab}$ the sheaf category $\mathfrak{Sh}(\mathcal{C}, \mathcal{A})$ need not be AB4* because infinite products are only left exact and do not, in general, preserve epimorphisms. For example, see here.

DO THIS!!

Theorem 7.0.10. Let $\mathcal{A}$ is a Grothendieck abelian category and $\mathcal{C}$ is a site then the inclusion,

$$\mathfrak{Sh}(\mathcal{C}, \mathcal{A}) \hookrightarrow \mathrm{PSh}(\mathcal{C}, \mathcal{A})$$

has a left adjoint called "sheafificaion".

DO THIS PROOF

Corollary 7.0.11. If $\mathcal{A}$ is a Grothendieck abelian category and $\mathcal{C}$ is a site then $\mathfrak{Sh}(\mathcal{C}, \mathcal{A})$ has enough injectives.

Theorem 7.0.12 (Gabber[1])**.** Let X be a scheme. Then $\mathfrak{QCoh}(X)$ is a Grothendieck abelian category and hence has enough injectives. Furthermore, $\mathfrak{QCoh}(X) \hookrightarrow \mathbf{Mod}_{\mathcal{O}_X}$ has a right adjoint and hence is also AB3*.

IS IT TRUE THAT ALL GROTHENDIECK ABELIAN CATEGORIES HAVE ALL PRODUCTS?? WHY DOES QCoh HAVE PRODUCTS?? JUST BECAUSE OF THE COHERATOR?

Remark. Note that products in $\mathfrak{QCoh}\,(X)$ do not agree with products in $\mathbf{Mod}_{\mathcal{O}_X}$ in general. GIVE EXAMPLE They are also not exact see here

(a) CMB

(b) Leo's answer.

(c) quasi-coherent module

(d) Tohoku.

8 Griffiths Conjecture

Conjecture 8.0.1. Let X be a smooth projective complex variety. If E is an ample vector bundle on X then it admits a hermitian metric with positive bisectional curvature.

Remark. In the case that $\operatorname{rank} E = 1$ this is exactly the Kodiara embedding theorem.

Remark. This conjecture is almost false as follows:

(a) this paper proves that if X admits a *Kahler* metic with negative bisectional curvature then $\pi_1(X)$ is infinite

(b) Brotbek and Darondeau proved that a generic complete intersection of large enough codimension and degree in $\mathbb{P}^N$ has ample cotangent bundle

(c) by Lefschetz hyperplane theorem the above examples have $\pi_1 = 0$.

The reason this does not give a counterexample to Griffiths' conjecture is exactly the stipulation that the metric on X is Kahler not just some arbitrary hermitian metric on TX.

Some other references:

(a) MO Griffiths positivity

(b) Approach to the conjecture

(c) MO reference on holomorphic (bi)sectional curvature.

9 Infinite Products are not quasi-coherent

Usually the sheaf,

$$\mathscr{F} = \prod_{i \in \mathbb{N}} \mathcal{O}_X$$

is not quasi-coherent. This may be surprising since the inclusion of presheaves into sheaves admiting a left-adjoint shows that,

$$(\lim_i \mathscr{F}_i)(U) = \lim_i \mathscr{F}_i(U)$$

and therefore,

$$\mathscr{F}(U) = \prod_{i \in \mathbb{N}} \mathcal{O}_X(U)$$

However this is just because localization does not commute with products. Indeed, if $\mathscr{F}$ were quasi-coherent, over an affine Spec (A) we must have,

$$\mathscr{F} = \widetilde{A^{\times \mathbb{N}}}$$

but this does not hold as an equality of sheaves because localization and infinte products do not commute. Indeed, consider $A = k[x]$ and localization at the element $f = x$. Then there is a natural map,

$$(A^{\times n})_f \to (A_f)^{\times n}$$

but the element $(1, x^{-1}, x^{-2}, \cdots)$ is not in the image since elements on the right must have only bounded below powers of x since they can be writen as $f^{-n}s$ for $s \in A^{\times n}$.

10 Positronium Lifetimes

Let an electron with (four) momentum p_1 and positron with momentum p_2 annihilate to two photons (or vector bosons) with momenta k_1, k_2. The leading-order Feynman diagrams give,

$$i\mathcal{M} = (-ie)^2 \epsilon(k_1)^*_\mu \epsilon(k_2)^*_\nu \; \bar{v}^{s_2}(p_2) \left[\gamma^\nu \frac{i(\not{q}_1 + m)}{q_1^2 - m^2 + i\epsilon} \gamma^\mu + \gamma^\mu \frac{i(\not{q}_2 + m)}{q_2^2 - m^2 + i\epsilon} \gamma^\nu \right] u^{s_1}(p_1)$$

where $q_1 = p_1 - k_1$ and $q_2 = p_1 - k_2$ corresponding to the t-channel and u-channel respectively. First we work out some formulas. Expanding the momenta to first-order,

$$\bar{v}(p_2) \not{a} u(p_1) = \sqrt{E_1 E_2}\, \xi'^\dagger \left[-a^0 \left(\frac{\vec{p}_1}{E_1} + \frac{\vec{p}_2}{E_2} \right) \cdot \vec{\sigma} + 2\vec{a} \cdot \vec{\sigma} \right] \xi$$

Similarly,

$$\bar{v}(p_2) \not{a} \gamma^5 u(p_1) = \sqrt{E_1 E_2}\, \xi'^\dagger \left[-2a^0 + \vec{a} \cdot \left(\frac{\vec{p}_1}{E_1} + \frac{\vec{p}_2}{E_2} \right) - i\vec{\sigma} \cdot \left[\left(\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right) \times \vec{a} \right] \right] \xi$$

And finally,

$$\begin{aligned} \bar{v}(p_2) \not{a} \not{b} u(p_1) = \sqrt{E_1 E_2}\, \xi'^\dagger \Bigg[& a^\mu b_\mu \left(\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right) \cdot \vec{\sigma} - 2a^0 (\vec{b} \cdot \vec{\sigma}) + 2b^0 (\vec{a} \cdot \vec{\sigma}) - i(\vec{a} \times \vec{b}) \cdot \left(\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right) \\ & + (\vec{a} \times \vec{b}) \times \left(\frac{p_1}{E_1} + \frac{p_2}{E_2} \right) \cdot \vec{\sigma} \Bigg] \xi \end{aligned}$$

We use the identity,

$$\gamma^\mu \gamma^\nu \gamma^\rho = g^{\mu\nu} \gamma^\rho + g^{\nu\rho} \gamma^\mu - g^{\mu\rho} \gamma^\nu - i\varepsilon^{\alpha\mu\nu\rho} \gamma_\alpha \gamma^5$$

Work in the CM frame where,

$$\begin{aligned} p_1 &= (\tfrac{1}{2} E_{\text{CM}}, \vec{p}) \\ p_2 &= (\tfrac{1}{2} E_{\text{CM}}, -\vec{p}) \\ k_1 &= (\tfrac{1}{2} E_{\text{CM}}, \vec{k}) \\ k_2 &= (\tfrac{1}{2} E_{\text{CM}}, -\vec{k}) \end{aligned}$$

Therefore,

$$q_1^2 = (p_1 - k_1)^2 = m^2 + m_B^2 - 2p_1 \cdot k_1 = m^2 + m_B^2 - \tfrac{1}{2}E_{\text{CM}}^2 + 2\vec{p} \cdot \vec{k}$$

and likewise,

$$q_2^2 = (p_1 - k_2)^2 = m^2 + m_B^2 - 2p_1 \cdot k_2 = m^2 + m_B^2 - \tfrac{1}{2}E_{\text{CM}}^2 - 2\vec{p} \cdot \vec{k}$$

Also in the CM frame,

$$\bar{v}(p_2)\not{a}u(p_1) = E_{\text{CM}}\,\xi'^{\dagger}\,[\vec{a} \cdot \vec{\sigma}]\,\xi$$

Similarly,

$$\bar{v}(p_2)\not{a}\gamma^5 u(p_1) = \xi'^{\dagger}\left[-E_{\text{CM}}a^0 - 2i\vec{\sigma} \cdot [\vec{p} \times \vec{a}]\right]\xi$$

And finally,

$$\bar{v}(p_2)\not{a}\not{b}u(p_1) = \xi'^{\dagger}\left[2a^\mu b_\mu(\vec{p} \cdot \vec{\sigma}) - E_{\text{CM}}a^0(\vec{b} \cdot \vec{\sigma}) + E_{\text{CM}}b^0(\vec{a} \cdot \vec{\sigma}) - 2i(\vec{a} \times \vec{b}) \cdot \vec{p}\right]\xi$$

We need to simplify,

$$\mathcal{M} = e^2\epsilon(k_1)_\mu^*\epsilon(k_2)_\nu^*\;\bar{v}^{s_2}(p_2)\left[\frac{\gamma^\nu\not{q}_1\gamma^\mu + m\gamma^\nu\gamma^\mu}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 - 2\vec{p} \cdot \vec{k}} + \frac{\gamma^\mu\not{q}_2\gamma^\nu + m\gamma^\mu\gamma^\nu}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 + 2\vec{p} \cdot \vec{k}}\right]u^{s_1}(p_1)$$

To do this, we define two quantities,

$$\begin{aligned} A =&\epsilon(k_1)_\mu^*\epsilon(k_2)_\nu^*\;\bar{v}^{s_2}(p_2)[\gamma^\nu\not{q}_1\gamma^\mu + m\gamma^\nu\gamma^\mu]u^{s_1}(p_1) \\ B =&\epsilon(k_1)_\mu^*\epsilon(k_2)_\nu^*\;\bar{v}^{s_2}(p_2)[\gamma^\mu\not{q}_2\gamma^\nu + m\gamma^\mu\gamma^\nu]u^{s_1}(p_1) \end{aligned}$$

such that,

$$\mathcal{M} = \frac{e^2}{2m^2 - m_B^2}\left[A \cdot \left(\frac{2m^2 - m_B^2}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 - 2\vec{p} \cdot \vec{k}}\right) + B \cdot \left(\frac{2m^2 - m_B^2}{\frac{1}{2}E_{\text{CM}}^2 - m_B^2 + 2\vec{p} \cdot \vec{k}}\right)\right]$$

To first-order in $\vec{p}$ this is,

$$\mathcal{M} = \frac{e^2}{2m^2 - m_B^2}\left[(A + B) + \left[\frac{2\vec{p} \cdot \vec{k}}{2m^2 - m_B^2}\right] \cdot (A - B)\right]$$

Now we expand A and B to first-order in $\vec{p}$. Then,

$$\begin{aligned} A = E_{\text{CM}}\xi'^{\dagger}\Big[&(\epsilon_1^* \cdot q_1)(\vec{\epsilon}_2^{\,*} \cdot \vec{\sigma}) + (\epsilon_2^* \cdot q_1)(\vec{\epsilon}_1^{\,*} \cdot \vec{\sigma}) - (\epsilon_1^* \cdot \epsilon_2^*)(\vec{q}_1 \cdot \vec{\sigma}) + i\varepsilon^{\alpha\nu\rho\mu}(\epsilon_1^*)_\nu(q_1)_\rho(\epsilon_2^*)_\mu\langle\gamma_\alpha\rangle_5 \\ &+2m(\epsilon_1^* \cdot \epsilon_2^*)\frac{\vec{p} \cdot \vec{\sigma}}{E_{\text{CM}}} - m\epsilon_2^{*0}(\vec{\epsilon}_1^{\,*} \cdot \vec{\sigma}) + m\epsilon_1^{*0}(\vec{\epsilon}_2^{\,*} \cdot \vec{\sigma}) + 2im(\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*}) \cdot \frac{\vec{p}}{E_{\text{CM}}}\Big]\xi \end{aligned}$$

where we define $\langle\gamma_\alpha\rangle_5$ as the matrix M in $\bar{v}^{s_2}(p_2)\gamma_\alpha\gamma^5u^{s_1}(p_1) = E_{\text{CM}}\xi'^{\dagger}M\xi$. We need to be careful expanding the ε term. There are four terms depending on where the 0 index appears. These are (including a minus sign from index lowering),

$$-i\langle\gamma_0\rangle_5(\vec{\epsilon}_1^{\,*} \times \vec{q}_1) \cdot (\vec{\epsilon}_2^{\,*}) + i\epsilon_1^{*0}(\langle\vec{\gamma}\rangle_5 \times \vec{q}_1) \cdot \vec{\epsilon}_2^{\,*} - iq_1^0(\langle\vec{\gamma}\rangle_5 \times \vec{\epsilon}_1^{\,*}) \cdot \vec{\epsilon}_2^{\,*} + i\epsilon_2^{*0}(\langle\vec{\gamma}\rangle_5 \times \vec{\epsilon}_1^{\,*}) \cdot \vec{q}_1$$

But $q_1^0 = 0$ and we can compute the $\langle\vec{\gamma}\rangle_5$ terms by rearranging them into the form $\langle\vec{a} \cdot \vec{\gamma}\rangle_5$ so we can use the above identities since,

$$\langle\vec{a} \cdot \vec{\gamma}\rangle_5 = \langle\not{a}\rangle_5 = \frac{\vec{p} \times \vec{a}}{E_{\text{CM}}} \cdot (2i\vec{\sigma})$$

where $a = (0, -\vec{a})$. Therefore,

$$i\varepsilon^{\alpha\nu\rho\mu}(\epsilon_1^*)_\nu(q_1)_\rho(\epsilon_2^*)_\mu \langle\gamma_\alpha\rangle_5 = i(\vec{\epsilon_1^*}\times\vec{q}_1)\cdot\vec{\epsilon_2^*} - 2\epsilon_1^{*0}\frac{\vec{p}\times(\vec{q}_1\times\vec{\epsilon_2^*})}{E_{\text{CM}}}\cdot\vec{\sigma} - 2\epsilon_2^{*0}\frac{\vec{p}\times(\vec{\epsilon_1^*}\times\vec{q}_1)}{E_{\text{CM}}}\cdot\vec{\sigma}$$

And putting everything together (and using that $q_1^0 = 0$) we get,

$$\begin{aligned} A = -E_{\text{CM}}\xi'^\dagger\Big[&(\vec{\epsilon_1^*}\cdot\vec{q}_1)(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\vec{q}_1)(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + (\epsilon_1^*\cdot\epsilon_2^*)(\vec{q}_1\cdot\vec{\sigma}) \\ &- i(\vec{\epsilon_1^*}\times\vec{q}_1)\cdot\vec{\epsilon_2^*} + 2\epsilon_1^{*0}\frac{\vec{p}\times(\vec{q}_1\times\vec{\epsilon_2^*})}{E_{\text{CM}}}\cdot\vec{\sigma} + 2\epsilon_2^{*0}\frac{\vec{p}\times(\vec{\epsilon_1^*}\times\vec{q}_1)}{E_{\text{CM}}}\cdot\vec{\sigma} \\ &-m\left(2(\epsilon_1^*\cdot\epsilon_2^*)\frac{\vec{p}\cdot\vec{\sigma}}{E_{\text{CM}}} - \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + 2i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\frac{\vec{p}}{E_{\text{CM}}}\right)\Big]\xi \end{aligned}$$

And B is identical except for swapping ϵ_1 and ϵ_2 and swapping q_1 for q_2. Now write $\hat{A}$ and $\hat{B}$ for the unitless quantities inside the spinor inner product menaing that,

$$A = -2mE_{\text{CM}}\ \xi'^\dagger\hat{A}\xi$$

and likewise for B. Therefore, since to first-order in $\vec{p}$ we have $E_{\text{CM}} = 2m$ we have,

$$\mathcal{M} = -\left(\frac{2e^2}{1-\frac{m_B^2}{2m^2}}\right)\xi'^\dagger\left[\hat{A}+\hat{B}+\left[\frac{2\vec{p}\cdot\vec{k}}{2m^2-m_B^2}\right]\cdot(\hat{A}-\hat{B})\right]\xi$$

Now we consider, using that $\vec{q}_1+\vec{q}_2 = 2\vec{p}$ and $\vec{q}_1-\vec{q}_2 = -2\vec{k}$ the quantity

$$\begin{aligned} \hat{A}+\hat{B} &= (\vec{\epsilon_1^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{p}}{m}\cdot\vec{\sigma}) \\ &\quad + i(\vec{\epsilon_1^*}\times\tfrac{\vec{k}}{m})\cdot\vec{\epsilon_2^*} - \epsilon_1^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_2^*})}{m^2}\cdot\vec{\sigma} + \epsilon_2^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_1^*})}{m^2}\cdot\vec{\sigma} \\ &\quad - (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{p}}{m}\cdot\vec{\sigma}) \\ &\quad = (\vec{\epsilon_1^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\tfrac{\vec{k}}{m} - \epsilon_1^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_2^*})}{m^2}\cdot\vec{\sigma} + \epsilon_2^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_1^*})}{m^2}\cdot\vec{\sigma} \end{aligned}$$

and likewise,

$$\begin{aligned} \hat{A}-\hat{B} &= -(\vec{\epsilon_1^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) - (\vec{\epsilon_2^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{k}}{m}\cdot\vec{\sigma}) \\ &\quad - i(\vec{\epsilon_1^*}\times\tfrac{\vec{p}}{m})\cdot\vec{\epsilon_2^*} \\ &\quad + \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma}) - i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\tfrac{\vec{p}}{m} \\ &\quad = -(\vec{\epsilon_1^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) - (\vec{\epsilon_2^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{k}}{m}\cdot\vec{\sigma}) + \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma}) \end{aligned}$$

Finally, dropping terms to higher order in $\vec{p}$ we get,

$$\begin{aligned} \mathcal{M} = -\left(\frac{2e^2}{1-\frac{m_B^2}{2m^2}}\right)\xi'^\dagger\Big[&(\vec{\epsilon_1^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{p}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) - i(\vec{\epsilon_1^*}\times\vec{\epsilon_2^*})\cdot\tfrac{\vec{k}}{m} \\ &- \epsilon_1^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_2^*})}{m^2}\cdot\vec{\sigma} + \epsilon_2^{*0}\frac{\vec{p}\times(\vec{k}\times\vec{\epsilon_1^*})}{m^2}\cdot\vec{\sigma} \\ &- \left[\frac{2\vec{p}\cdot\vec{k}}{2m^2-m_B^2}\right]\Big((\vec{\epsilon_1^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_2^*}\cdot\vec{\sigma}) + (\vec{\epsilon_2^*}\cdot\tfrac{\vec{k}}{m})(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + (\epsilon_1^*\cdot\epsilon_2^*)(\tfrac{\vec{k}}{m}\cdot\vec{\sigma}) - \epsilon_2^{*0}(\vec{\epsilon_1^*}\cdot\vec{\sigma}) + \epsilon_1^{*0}(\vec{\epsilon_2^*}\cdot\vec{\sigma})\Big)\Big]\xi \end{aligned}$$

For photon polarizations this simplifies greatly since $\vec{\epsilon} \perp \vec{k}$ and $\epsilon^0 = 0$. Therefore, the photon amplitude is,

$$\mathcal{M} = -2e^2 \xi'^\dagger \left[(\vec{\epsilon}_1^* \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon}_2^* \cdot \vec{\sigma}) + (\vec{\epsilon}_2^* \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon}_1^* \cdot \vec{\sigma}) + (\vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*)(\tfrac{\vec{p}}{m} \cdot \hat{k})(\hat{k} \cdot \vec{\sigma}) - i(\vec{\epsilon}_1^* \times \vec{\epsilon}_2^*) \cdot \hat{k} \right] \xi$$

using that $\frac{\vec{k}}{m} = \hat{k}$ to first-order in $\vec{p}$ since the photons carry away all the energy and hence $|\vec{k}| = m$. All but the last term are suppressed by a factor of $\vec{p}/m$ which for positronium will be proportional to α.

10.1 Selection rules for 2-photon annihilation

We are free to orient our spinor basis along any direction (recall that ξ' is the flipped spinor of the physical positron). We choose to orient along the z-direction which is chosen as the direction along which $\vec{k}$ points. Then,

$$\mathcal{M}_{\pm\pm} = -2e^2 \xi'^\dagger (\pm 1 + \tfrac{\vec{p}}{m} \cdot \vec{\sigma}) \xi$$
$$\mathcal{M}_{\pm\mp} = -2e^2 \xi'^\dagger (\tfrac{p_x \mp i p_y}{m})(\sigma_x \mp i \sigma_y) \xi$$

Spin-orbit coupling means that the positronium will naturally be split into eigenstates of total angular momentum. For $n = 1$ there is only an $\ell = 0$ wavefunction so there are only two states, para (singlet $s = 0$) and ortho (triplet $s = 1$) given by,

$$|1^1 S_0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes |\psi_1\rangle$$

and

$$|1^3 S_1, m = 1\rangle = |\uparrow\uparrow\rangle \otimes |\psi_1\rangle$$
$$|1^3 S_1, m = 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_1\rangle$$
$$|1^3 S_1, m = -1\rangle = |\downarrow\downarrow\rangle \otimes |\psi_1\rangle$$

where $|\psi_0\rangle$ is the $1S$ state wavefunction. For $n = 2$ we have $\ell = 0, 1$ and hence there are more states. We have the excited $2S$ (meaning $\ell = 0$) versions of para ($s = 0$) and ortho ($s = 1$) positronium which are identical but with $|\psi_0\rangle$ replaced by $|\psi_{1,0}\rangle$. More interesting are the following, the $s = 0$ states,

$$|2^1 P_1, m = 1\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,1}\rangle$$
$$|2^1 P_1, m = 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,0}\rangle$$
$$|2^1 P_1, m = 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,0}\rangle$$

the $s = 1$ state with $j = 0$,

$$|2^3 P_0\rangle = \frac{1}{\sqrt{3}} \left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,-1}\rangle + |\downarrow\downarrow\rangle \otimes |\psi_{1,1,1}\rangle - \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,0}\rangle \right)$$

the $s = 1$ states with $j = 1$,

$$
\begin{aligned}
|2^3P_1, m = 1\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,0}\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_1, m = 0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,-1}\rangle - |\downarrow\downarrow\rangle \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_1, m = -1\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle \otimes |\psi_{1,1,0}\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,-1}\rangle\right)
\end{aligned}
$$

and finally the $s = 1$ sates with $j = 2$,

$$
\begin{aligned}
|2^3P_2, m = 2\rangle &= |\uparrow\uparrow\rangle \otimes |\psi_{1,1,1}\rangle \\
|2^3P_2, m = 1\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,0}\rangle + \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_2, m = 0\rangle &= \frac{1}{\sqrt{6}}\left(|\uparrow\uparrow\rangle \otimes |\psi_{1,1,-1}\rangle + \frac{\sqrt{2}}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,0}\rangle + |\downarrow\downarrow\rangle \otimes |\psi_{1,1,1}\rangle\right) \\
|2^3P_2, m = -1\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle \otimes |\psi_{1,1,0}\rangle + \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\psi_{1,1,-1}\rangle\right) \\
|2^3P_2, m = -2\rangle &= |\downarrow\downarrow\rangle \otimes |\psi_{1,1,-1}\rangle
\end{aligned}
$$

The spin-flip rule assigns, for the positron,

$$
|\uparrow\rangle \mapsto \xi' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\downarrow\rangle \mapsto \xi' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
$$

The amplitudes can be rewritten in the form of the matrix shown times $\xi\xi'^\dagger$. Then using the spin flip we easily see that $s = 0$ state corresponds to,

$$
\xi\xi'^\dagger = \frac{1}{\sqrt{2}}\mathrm{id}
$$

and the $s = 1$ state with spin along $\hat{n}$ corresponds to,

$$
\xi\xi'^\dagger = \frac{1}{\sqrt{2}}\hat{n} \cdot \vec{\sigma}
$$

(this is consistent with Peskin as can be seen by daggering the above expression which exchanges ξ and ξ' and replaces $\hat{n}$ by $\hat{n}^*$) where $\hat{n}$ points along the direction of $m = 0$ meaning spin $+\hat{z}$ corresponds to $\hat{n} = \hat{x} + i\hat{y}$. Therefore, taking the trace of the inner matrix,

$$
\mathcal{M}(s = 0) = 2i\sqrt{2}e^2(\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*}) \cdot \hat{k}
$$

and therefore,

$$
\begin{aligned}
\mathcal{M}_{\pm\pm}(s = 0) &= \mp 2\sqrt{2}e^2 \\
\mathcal{M}_{\pm\mp}(s = 0) &= 0
\end{aligned}
$$

so any $s = 0$ state decays into an odd S-wave EPR $j = 0$ entangled state. Since the wavefunction of a P orbital vanishes at the origin, this protects against decay of the $j = 1$ and $s = 0$ states into two photons (angular momentum of two photons cannot be $j=1$). This is also due to C conservation since the 2^1P_1 state is odd under C but any two photon state is even under C. Therefore, we expect this decay to be forbidden into any C-odd vector particles with a C-invariant interaction term. Likewise,

$$\begin{aligned}
\mathcal{M}(s=1) &= -\sqrt{2}e^2 \ \mathrm{Tr}\left(\left[(\vec{\epsilon}_1^* \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon}_2^* \cdot \vec{\sigma}) + (\vec{\epsilon}_2^* \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon}_1^* \cdot \vec{\sigma}) + (\vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*)(\tfrac{\vec{p}}{m} \cdot \hat{k})(\hat{k} \cdot \vec{\sigma}) - i(\vec{\epsilon}_1^* \times \vec{\epsilon}_2^*) \cdot \hat{k}\right](\hat{n} \cdot \sigma)\right) \\
&= -2\sqrt{2}e^2\left[(\vec{\epsilon}_1^* \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon}_2^* \cdot \hat{n}) + (\vec{\epsilon}_2^* \cdot \tfrac{\vec{p}}{m})(\vec{\epsilon}_1^* \cdot \hat{n}) + (\vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*)(\tfrac{\vec{p}}{m} \cdot \hat{k})(\hat{k} \cdot \hat{n})\right]
\end{aligned}$$

and therefore, using the completness relation

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1) &= -2\sqrt{2}e^2(\tfrac{\vec{p}}{m} \cdot \hat{n}) \\
\mathcal{M}_{\pm\mp}(s=1) &= -2\sqrt{2}e^2(\tfrac{p_x}{m} \mp i\tfrac{p_y}{m})(n_x \mp i n_y)
\end{aligned}$$

Hence for the three spin orientations we get,

$$\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1, m=1) &= -2e^2(\tfrac{p_x}{m} + i\tfrac{p_y}{m}) \\
\mathcal{M}_{\pm\pm}(s=1, m=0) &= -2\sqrt{2}e^2(\tfrac{p_z}{m}) \\
\mathcal{M}_{\pm\pm}(s=1, m=-1) &= -2e^2(\tfrac{p_x}{m} - i\tfrac{p_y}{m}) \\
\mathcal{M}_{\pm\mp}(s=1, m=\pm1) &= -4e^2(\tfrac{p_x}{m} \mp i\tfrac{p_y}{m}) \\
\mathcal{M}_{\pm\mp}(s=1, m=0) &= 0 \\
\mathcal{M}_{\pm\mp}(s=1, m=\mp1) &= 0
\end{aligned}$$

This alows the $s=1$ and $j=0$ state to decay into an even S-wave EPR $j=0$ state. The first two terms do not couple since they require both $\vec{S}$ and $\vec{L}$ in the same orientation. However, the $\mathcal{M}_{\pm\pm}$ does couple to the last term and is even under exchange of RHC and LHC photons. This is necessary to preserve parity since any ^3P state has even parity. Note that S-wave $|{+}{+}\rangle$ and $|{-}{-}\rangle$ are both $j=0$ states of the photon field since they are identical particles so are even under exchange facilitated by a π-rotation. Thus either linear combination is a valid $j=0$ state with $s=0$ and $\ell=0$. Therefore we see that it is not P but C that really forbids various positronium decays.

Finally, we analyize the 2^3P_1 and 2^3P_2 states. The easiest is 2^3P_2 for which $m=\pm2$ clearly couple to $\mathcal{M}_{\pm\mp}$. The $m=\pm1$ states do not couple. Although it at first appears that the $m=0$ state couples, it does not. Indeed, the amplitude is,

$$-2\sqrt{2}e^2[-1 + (\sqrt{2})^2 - 1] = 0$$

where the minus signs arise from the Condon–Shortley phase convention which is chosen such that the raising and lowering operators act on spherical harmonics in the way expected for the derivation of the Clebsch-Gordan coefficients in use. Therefore we get D-wave (using the rotation matrices for $j=2$ we get $\cos 2\theta$ angular dependence on amplitudes) $s=2$ photon emission with $j=2$.

For 2^3P_1 we see that none of the states couple since $|\uparrow\uparrow\rangle$ only couples to two photons when paired with $|\psi_{1,1,1}\rangle$ and likewise only the $m=0$ states and $m=-1$ states couple to eachother. This decay is not forbidden by C it is forbidden by angular momentum selection since a two photon state cannot have odd spin along an axis and thus cannot have $j=1$ since L_z is zero along the direction of motion and hence J_z has an even eigenvalue. (BETTER EXPLINATION)

10.2 Positronium Decay Rate

We build a Positronium states. For $\ell = 0$ we consider,

$$|\mathrm{Ps}(\vec{k})\,{}^1S_0\rangle = \sqrt{2E_{\vec{k}}} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{4E_{\vec{p}+\frac{\vec{k}}{2}}E_{-\vec{p}+\frac{\vec{k}}{2}}}} \tilde{\psi}_0(\vec{p}) \frac{\sqrt{4E_{\vec{p}+\frac{\vec{k}}{2}}E_{-\vec{p}+\frac{\vec{k}}{2}}}}{\sqrt{2}} \sum_s a^{s\dagger}_{\vec{p}+\frac{\vec{k}}{2}} b^{s\dagger}_{-\vec{p}+\frac{\vec{k}}{2}} |\Omega\rangle$$

we need to show that this is properly normalized. Indeed,

$$\begin{aligned}
\langle \mathrm{Ps}(\vec{k}')\,{}^1S_0\,|\,\mathrm{Ps}(\vec{k})\,{}^1S_0\rangle &= 2E_{\vec{k}} \int \frac{\mathrm{d}^3\vec{p}'}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \tilde{\psi_0}^*(\vec{p}')\tilde{\psi}_0(\vec{p}) \left[\frac{1}{2}\sum_{s's} \langle\Omega| b^{s'}_{-\vec{p}'+\frac{\vec{k}'}{2}} a^{s'}_{\vec{p}'+\frac{\vec{k}'}{2}} a^{s\dagger}_{\vec{p}+\frac{\vec{k}}{2}} b^{s\dagger}_{-\vec{p}+\frac{\vec{k}}{2}}|\Omega\rangle\right] \\
&= 2E_{\vec{k}} \int \frac{\mathrm{d}^3\vec{p}'}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \tilde{\psi_0}^*(\vec{p}')\tilde{\psi}_0(\vec{p}) \\
&\quad \cdot \left[\frac{1}{2}\sum_{s's} (2\pi)^3 \delta^{(3)}(\vec{p}' + \tfrac{\vec{k}'}{2} - \vec{p} - \tfrac{\vec{k}}{2})\delta_{ss'}\delta_{ss'}(2\pi^3)\delta^{(3)}(-\vec{p} + \tfrac{\vec{k}}{2} + \vec{p}' - \tfrac{\vec{k}'}{2})\right]
\end{aligned}$$

Call the arguments of the δ-functions A, B. Then $\frac{1}{2}(A+B) = (\vec{p}' - \vec{p})$ and $A - B = \vec{k}' - \vec{k}$ and the matrix,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix}$$

has determinant -1 and therefore we can perform this change of variables on the δ-functions to get,

$$\begin{aligned}
\langle \mathrm{Ps}(\vec{k}')\,{}^1S_0\,|\,\mathrm{Ps}(\vec{k})\,{}^1S_0\rangle &= 2E_{\vec{k}} \int \frac{\mathrm{d}^3\vec{p}'}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \tilde{\psi_0}^*(\vec{p}')\tilde{\psi}_0(\vec{p}) \left[(2\pi)^6 \delta^{(3)}(\vec{p}' - \vec{p})\delta^{(3)}(\vec{k}' - \vec{k})\right] \\
&= 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k}' - \vec{k}) \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \tilde{\psi}_0^*(\vec{p})\tilde{\psi}(\vec{p}) = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k}' - \vec{k})
\end{aligned}$$

which is the desired relativistic normalization. Now we consider the P-states. Let ψ_i be a set of P-wave wavefunctions of the form $\psi_i = x^i f(|x|)$ normalized such that,

$$\int \mathrm{d}^3x\; \psi_i^*(x)\psi_j(x) = \delta_{ij}$$

Then consider the states,

$$|\mathrm{Ps}(\vec{k})\,P_\Sigma\rangle = \sqrt{2E_{\vec{k}}} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_i \frac{1}{\sqrt{4E_{p+\frac{\vec{k}}{2}}E_{-p+\frac{\vec{k}}{2}}}} \tilde{\psi}_i(p) \sqrt{4E_{p+\frac{\vec{k}}{2}}E_{-p+\frac{\vec{k}}{2}}} \sum_{s's} a^{s'\dagger}_{p+\frac{\vec{k}}{2}} \Sigma^i_{s's} b^{s\dagger}_{-p+\frac{\vec{k}}{2}} |\Omega\rangle$$

where Σ_i are a set of 2 x 2 matrices such that $\sum_i \mathrm{tr}\,\Sigma^{i\dagger}\Sigma^i = 1$. We need to check the normalization of these states,

$$\begin{aligned}
\langle \mathrm{Ps}(\vec{k}')\,P_\Sigma\,|\,\mathrm{Ps}(\vec{k})\,P_\Sigma\rangle &= 2E_{\vec{k}} \int \frac{\mathrm{d}^3p'}{(2\pi)^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{ij} \tilde{\psi}_i^*(p')\tilde{\psi}_j(p) \sum_{t'ts's} \Sigma^{j*}_{t't}\Sigma^i_{s's} \langle\Omega| b^t_{-p'+\frac{\vec{k}'}{2}} a^{t'}_{p'+\frac{\vec{k}'}{2}} a^{s'\dagger}_{p+\frac{\vec{k}}{2}} b^{s\dagger}_{-p+\frac{\vec{k}}{2}} |\Omega\rangle \\
&= 2E_{\vec{k}} \int \frac{\mathrm{d}^3p'}{(2\pi)^3} \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{ij} \tilde{\psi}_i^*(p')\tilde{\psi}_j(p) \left[\sum_{t'ts's} (\Sigma^{j\dagger})_{tt'}\Sigma^i_{s's}\delta_{ts}\delta_{t's'}\right] (2\pi)^6\delta^{(3)}(p' - p)\delta^{(3)}(\vec{k}' - \vec{k}) \\
&= 2E_{\vec{k}}(2\pi)^3\delta(\vec{k}' - \vec{k}) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{ij} \left[\tilde{\psi}_i^*(p')\tilde{\psi}_j(p)\,\mathrm{tr}\,\Sigma^{j\dagger}\Sigma^i\right] \\
&= 2E_{\vec{k}}(2\pi)^3\delta(\vec{k}' - \vec{k}) \sum_{ij} \delta_{ij}\,\mathrm{tr}\,\Sigma^{j\dagger}\Sigma^i = 2E_{\vec{k}}(2\pi)^3\delta(\vec{k}' - \vec{k})
\end{aligned}$$

using the normalization condition on Σ. Hence we get the correct relativistic normalization. Then we can consider the decay rate. For a positronium state of the form,

$$|\mathrm{Ps}\rangle = \sqrt{2M} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{iss'} C_{iss'} \tilde{\psi}_i(p) a_p^{s\dagger} b_{-p}^{s'\dagger} |\Omega\rangle$$

then we get an amplitude for two photon decay, recalling the relativisitc normalization convention for the definition of $\mathcal{M}$,

$$\mathcal{M}(\mathrm{Ps} \to 2\gamma) = \sqrt{2M} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{iss'} C_{iss'} \tilde{\psi}_i(p) \frac{1}{2m} \mathcal{M}(e^-(\vec{p}, s) + e^+(-\vec{p}, s') \to 2\gamma)$$

Now we let,

$$\mathcal{M}^{ss'}_{\alpha\beta}(\vec{p}, \vec{k}) := \mathcal{M}(e^-_s(\vec{p}) + e^+_{s'}(-\vec{p}) \to \gamma_\alpha(\vec{k}) + \gamma_\beta(-\vec{k}))$$

where $\alpha, \beta = +$ or $-$ label the photon polarizations and $a, b =\uparrow$ or $\downarrow$ are spinor indices for the polarizations of the electron and the positron (recall we use Peskin's terrible convention that the spinor indices for antiparticles are flipped with respect to the physical spin). We will need to expand this in $\vec{p}$. Then the decay rate for a fixed polarization is given by,

$$\begin{aligned}
\Gamma &= \frac{1}{2} \int \frac{1}{2M} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{4|k|^2} |\mathcal{M}(\mathrm{Ps} \to 2\gamma)|^2 (2\pi)^4 \delta^{(4)}(k + k' - p_{\mathrm{Ps}}) \\
&= \frac{2\pi}{16M} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{|k|^2} \delta(2|k| - M) |\mathcal{M}(\mathrm{Ps} \to 2\gamma)|^2 \\
&= \frac{2\pi}{16M} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{|k|^2} \delta(2|k| - M) |\mathcal{M}(\mathrm{Ps} \to 2\gamma)|^2 \\
&= \frac{1}{16M} \frac{1}{(2\pi)^2} \frac{1}{2} \int |\mathcal{M}(\mathrm{Ps} \to 2\gamma)|^2 \mathrm{d}\Omega \\
&= \frac{\pi}{(4\pi)^3} \frac{1}{2M} \int |\mathcal{M}(\mathrm{Ps} \to 2\gamma)|^2 \mathrm{d}\Omega
\end{aligned}$$

the first $\frac{1}{2}$ from the fact that the final state photons are identical particles and another $\frac{1}{2}$ comes from the δ-function. Therefore,

$$\Gamma = \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2} \int \sum_{\alpha\beta} \left| \int \frac{\mathrm{d}^3 \vec{p}}{(2\pi)^3} \sum_{iss'} C_{iss'} \tilde{\psi}_i(\vec{p}) \mathcal{M}^{ss'}_{\alpha\beta}(\vec{p}, \vec{k}) \right|^2 \mathrm{d}\Omega$$

Now we first compute the S-wave decays. Consider the $\ell = 0$ wavefunctions,

$$\psi_n(x) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-1)!}{2nn!}} e^{-\rho} L^1_{n-1}(\rho) \cdot \frac{1}{\sqrt{4\pi}}$$

where $\rho = \frac{2r}{na_0}$ and $a_0 = (\mu\alpha^2)^{-1}$ where μ is the reduced mass and L^1_{n-1} is the generalized Laguerre polynomial which are normalized so that $L^1_{n-1}(0) = 1$. Therefore,

$$\psi_n(0) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-1)!}{2n\, n!}} \cdot \frac{1}{\sqrt{4\pi}}$$

Since these have zero gradient at the origin, to first-order in $\vec{p}$ we find,

$$\Gamma_{n^1S_0} = \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2} \int 2|\psi_n(0)|^2 [8e^4] \mathrm{d}\Omega$$

the first two arises from the two final allowed polarizations and the $8e^4$ is the square of,

$$\mathcal{M}_{\pm\pm}(s=0) = \mp 2\sqrt{2}e^2$$

and using that the other polarization states have zero amplitude by conservation of angular momentum. Alternatively, we can use the ${}^1S_0$ state we constructed earlier which has $C_{0ss'} = \frac{1}{\sqrt{2}}\delta_{ss'}$ and that $\mathcal{M}^{ss'}_{\pm\pm} = \mp 2e^2\delta_{ss'}$ so therefore the internal sum gives the same result: $\mp 2\sqrt{2}e^2\psi_n(0)$. Therefore, we get,

$$\Gamma_{{}^1S_0} = \frac{1}{(4\pi)^2}\cdot\frac{\pi}{4m^2}\cdot\left(\frac{m\alpha}{n}\right)^3\frac{1}{2n^2}\cdot\frac{1}{4\pi}\cdot(16e^4) = \frac{1}{2n^5}m\alpha^5$$

In the case, $n=1$ we get,

$$\Gamma_{1^1S_0} = \tfrac{1}{2}m\alpha^5$$

and for $n=2$ we get,

$$\Gamma_{2^1S_0} = \tfrac{1}{64}m\alpha^5$$

Likewise the ${}^3S_1$ state decay to two photons is forbidden by C invariance.

Now we compute the decay of the P-states. From our expression for the Positronium state, we get a decay rate,

$$\Gamma = \frac{1}{(4\pi)^3}\cdot\frac{\pi}{4m^2}\int\sum_{\alpha\beta}\left|\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\sum_i\tilde{\psi}_i(\vec{p})\,\mathrm{tr}\left(\Sigma^\top\mathcal{M}_{\alpha\beta}(\vec{p},\vec{k})\right)\right|^2\mathrm{d}\Omega$$

Since $\psi_i(0)=0$ the zeroth-order term of $\mathcal{M}(\vec{p})$ integrates to zero. Therefore, we should write,

$$\mathcal{M}^{ss'}(\vec{p},\vec{k}) = \mathcal{M}^{ss'}(0,\vec{k}) + \vec{F}^{ss'}(\vec{k})\cdot\vec{p} + O(p^2)$$

Then to first-order,

$$\Gamma = \frac{1}{(4\pi)^3}\cdot\frac{\pi}{4m^2}\int\sum_{\alpha\beta}\left|\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\sum_i\tilde{\psi}_i(\vec{p})\left[\vec{p}\cdot\mathrm{tr}\left(\Sigma^{i\top}\vec{F}_{\alpha\beta}(\vec{k})\right)\right]\right|^2\mathrm{d}\Omega$$

Furthermore,

$$\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\tilde{\psi}_i(\vec{p})\,\vec{p} = -i\nabla\psi_i(x)\Big|_{x=0} = -i\vec{e}_i f(0)$$

because the other term $x^i\partial_j f(|x|)$ is zero at $\vec{x}=0$. Therefore,

$$\Gamma = \frac{1}{(4\pi)^3}\cdot\frac{\pi}{4m^2}|f(0)|^2\int\sum_{\alpha\beta}\left|\mathrm{tr}\left(\vec{\Sigma}^\top\cdot\vec{F}_{\alpha\beta}(\vec{k})\right)\right|^2\mathrm{d}\Omega$$

Now we set,

$$\Sigma^i = \begin{cases}\frac{1}{\sqrt{6}}\sigma^i & j=0\\ \frac{1}{2}\epsilon^{ijk}n^j\sigma^k & j=1\\ \frac{1}{\sqrt{2}}h^{ij}\sigma^j & j=2\end{cases}$$

where n is a unit vector and h^{ij} is a symmetric traceless tensor such that $\sum_{ij}h^{ij}(h^{ij})^* = 1$. Note! Peskin has a mistake here, in order for the normalization to work correctly we need $\frac{1}{\sqrt{2}}$ not $\frac{1}{\sqrt{3}}$ in the $j=2$ case. To compute $\mathrm{tr}\left(A^\top\mathcal{M}\right)$ write,

$$\mathcal{M}^{ss'} = \xi^{s'\dagger}M\xi^s$$

and therefore,

$$\mathrm{tr}\,(A^\top \mathcal{M}) = \sum_{ss'} A_{ss'}\mathcal{M}^{ss'} = \sum_{ss'} A_{ss'}\xi^{s'\dagger}M\xi^s = \mathrm{tr}\left(M\sum_{ss'} A_{ss'}\xi^s\xi^{s'\dagger}\right) = \mathrm{tr}\,(MA^\top)$$

We computed,

$$\mathrm{tr}\,(M^i_{\pm\pm}\sigma^j) = -\frac{4e^2}{m}\delta_{ij} \qquad \mathrm{tr}\,(M^i_{\pm\mp}\sigma^j) = -\frac{4e^2}{m}(\delta_{i1} \mp i\delta_{i2})(\delta_{j1} \mp i\delta_{j2})$$

Therefore, for the $j = 0$ angular momentum state,

$$\mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\pm}) = -\frac{2\sqrt{6}e^2}{m} \qquad \mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\mp}) = -\frac{4e^2}{m}\tfrac{1}{\sqrt{6}}(1-1) = 0$$

Therefore,

$$\Gamma_{2^3P_0} = \frac{1}{(4\pi)^2}\cdot\frac{\pi}{4m^2}|f(0)|^2\cdot 2\left(\frac{24e^4}{m^2}\right)$$

and

$$f(r) = \frac{1}{4\sqrt{2\pi}a_0^{3/2}}\frac{1}{a_0}e^{-r/2a_0}$$

where $a_0 = (\mu\alpha)^{-1}$ so plugging in gives,

$$\Gamma_{2^3P_0} = \frac{1}{(4\pi)^2}\cdot\frac{\pi}{4m^2}\frac{(m/2)^5\alpha^5}{32\pi}\cdot 2\left(\frac{24e^4}{m^2}\right) = \tfrac{3}{256}m\alpha^7$$

For the $j = 1$ angular momentum state,

$$\mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\pm}) = 0 \qquad \mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\mp}) = -\frac{4e^2}{2m}(\mp i\epsilon^{132}n^3 \mp i\epsilon^{231}n^3) = 0$$

and we recover the fact that the $j = 1$ does not decay into two photons. For the $j = 2$ angular momentum state,

$$\mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\pm}) = -\frac{4e^2}{\sqrt{2}m}h^{ij}\delta_{ij} = 0 \qquad \mathrm{tr}\,(\vec{\Sigma}^\top \cdot \vec{F}_{\pm\mp}) = -\frac{4e^2}{\sqrt{2}m}(h^{11} - h^{22} \mp ih^{12} \mp ih^{21})$$

We need to average over the possible polarization tensors h. However, the Peskin solutions have an error in that the standard basis of symmetric traceless tensors do not give an *orthogonal* basis of spin 2 states. Indeed, consider the matrices,

$$h_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad h_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

these correspond to states,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\hat{x}_0\rangle \otimes |\hat{x}_0\rangle - |\hat{y}_0\rangle \otimes |\hat{y}_0\rangle) \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|\hat{y}_0\rangle \otimes |\hat{y}_0\rangle - |\hat{z}_0\rangle \otimes |\hat{z}_0\rangle)$$

but the $m = 0$ states along perpendicular axes are orthogonal. Therefore $\langle \psi_1 \,|\, \psi_2 \rangle = \frac{1}{2}$. Instead we need to choose a basis of symmetric traceless matrices which is orthogonal for the physical states. A good choice are the states of definite J_z. These correspond to,

$$h_{+2} = \frac{1}{2}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h_{+1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}$$

$$h_0 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$h_{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}$$

$$h_{+2} = \frac{1}{2}\begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the calculation of $\mathcal{M}$ we put $\vec{k}$ along $\hat{z}$ so to perform the integral over $\vec{k}$ we instead need to integrate over the orientations of the spin 2 particle. If we choose coordinates with $\theta = 0$ corresponding definite spin along z then the amplitude squared is constant in the azimuthal angle ϕ. Then,

$$\begin{aligned} \frac{\mathrm{d}\Gamma_h^\pm}{\mathrm{d}\Omega} &= \frac{1}{(4\pi)^3} \cdot \frac{\pi}{4m^2} \frac{(m/2)^5 \alpha^5}{32\pi} \cdot \left(\frac{2\sqrt{2}e^2}{m}\right)^2 |h^{11}(\theta) - h^{22}(\theta) \mp 2ih^{12}(\theta)|^2 \\ &= \frac{m\alpha^7}{512} \cdot \frac{1}{4\pi} |h^{11}(\theta) - h^{22}(\theta) \mp 2ih^{12}(\theta)|^2 \\ &= \frac{m\alpha^7}{512} \cdot \frac{1}{4\pi} f_h(\theta) \end{aligned}$$

Applying the rotation matrix and then computing the amplitude and summing over the two nonzero photon polarization states,

$$\begin{aligned} f_{+2}(\theta) &= \tfrac{1}{16}(35 + 28\cos 2\theta + \cos 4\theta) \\ f_{+1}(\theta) &= \tfrac{1}{4}(5 - 4\cos 2\theta - \cos 4\theta) \\ f_0(\theta) &= \tfrac{1}{8}(9 - 12\cos 2\theta + 3\cos 4\theta) \\ f_{-1}(\theta) &= \tfrac{1}{4}(5 - 4\cos 2\theta - \cos 4\theta) \\ f_{-2}(\theta) &= \tfrac{1}{16}(35 + 28\cos 2\theta + \cos 4\theta) \end{aligned}$$

Then notice that,

$$f_{+2}(\theta) + f_{+1}(\theta) + f_0(\theta) + f_{-1}(\theta) + f_{-2}(\theta) = 8$$

so an unpolarized collection of emitts photons uniformly as they must. Therefore, averaging over polarizations and integrating we get,

$$\Gamma_{2^3P_2} = \frac{m\alpha^7}{512} \cdot \frac{8}{5} = \tfrac{1}{320} m\alpha^7$$

Morover, integrating each $f_i(\theta)$ we get the same value: $\frac{32\pi}{5}$ and hence,

$$\Gamma_h = \tfrac{1}{320} m\alpha^7$$

and therefore each state in the $j = 2$ multiplet has the same decay probability. We can see why this is true by angular momentum conservation. Since $\vec{J}$ commutes with the Hamiltonian and hence the rasing and lower operators $J_\pm$ commute with the Hamiltonian. These annihilate the electromagnetic vacuum and hence only act on the positronium state. Therefore the different m-states of the J multiplets of positronium must have the same overall decay rate and must decay to photon states which are connected by raising and lowering operators of total angular momentum.

10.3 Decay to Massive B

For a massive vector particle there are three possible physical polarizations. The $\epsilon^\pm$ are identical but the third polarization in the rest frame $\vec{\epsilon} = (0, 0, 1)$ is transformed via Lorentz bost into,

$$\epsilon^\mu = (\beta\gamma, 0, 0, \gamma)$$

For the particle with momentum $\vec{k}$ and the opposite sign on β for the other particle. From the traces, we again see that,

$$\mathcal{M}_{\pm\pm}(s = 0) = \mp \left(\frac{2\sqrt{2}e^2}{1 - \frac{m_B^2}{2m^2}} \right) \left(\frac{k}{m} \right)$$

and all other polarizations have zero amplitude. Similarly, for $s = 1$ the transverse polarizations give a similar result,

$$\mathcal{M}_{\pm\pm}(s = 1) = -\left(\frac{2\sqrt{2}e^2}{1 - \frac{m_B^2}{2m^2}} \right) \left(\tfrac{\vec{p}}{m} \cdot \hat{n} - \left(\tfrac{m_B^2}{2m^2 - m_B^2} \right) (\tfrac{\vec{p}}{m} \cdot \hat{k})(\hat{k} \cdot \hat{n}) \right)$$

$$\mathcal{M}_{\pm\mp}(s = 1) = -\left(\frac{2\sqrt{2}e^2}{1 - \frac{m_B^2}{2m^2}} \right) (\tfrac{p_x}{m} \mp i\tfrac{p_y}{m})(n_x \mp i n_y)$$

Now we need to compute the amplitudes with at least one transverse polarization.

$$\mathcal{M}_{00}(s = 1) = -\frac{2\sqrt{2}e^2}{\left(1 - \frac{m_B^2}{2m^2}\right)^2} \left(\frac{m_B}{m} \right)^2 \left(\frac{\vec{p} \cdot \hat{n}}{m} \right)$$

$$\mathcal{M}_{\pm 0}(s = 1) = -\frac{4e^2 m_B}{2m^2 - m_B^2} \left[n_z(p_x \mp i p_y) + \frac{m_B^2}{2m^2 - m_B^2}(n_x \mp i n_y) p_z \right]$$

$$\mathcal{M}_{0\pm}(s = 1) = -\frac{4e^2 m_B}{2m^2 - m_B^2} \left[n_z(p_x \pm i p_y) + \frac{m_B^2}{2m^2 - m_B^2}(n_x \pm i n_y) p_z \right]$$

This gives S-wave emission for states with $m_\ell = 0$ and $s = 1$. Notice that these go to zero as $m_B \to 0$ as they must since this is an unphysical polarization of the photon.

Let's see if any of these amplitudes break any forbidden decays into two photons. The ${}^3S_1$ states remain forbidden since the new polarizations only couple to nonzero momenta. Indeed C conservation shows this decay remains forbidden as does 2^1P_1. Indeed, the new polarizations couple to $\vec{\sigma}$ and hence give zero on $s = 0$. Thus we need only consider 2^3P_1. The new terms of interest are in $\mathcal{M}_{\pm 0}(s = 1)$ and $\mathcal{M}_{0\pm}(s = 1)$ which couple $m = 0$ to $m = \pm 1$ states. This indeed allows for decay of 2^3P_1. This shows that the massless photon imposes an additional restriction on the angular momentum selection rules.

11 Parity (and C) Violating Decay

Consider a new interaction term,

$$\mathcal{H}_{\text{int}} = \bar{\psi}(g_s + i g_p \gamma^5) \not{B} \psi$$

Therefore, the new vertex contribution in the Feynman rules is,

$$-i\Gamma^\mu = -i(g_s + i g_p \gamma^5)\gamma^\mu$$

This term is P and C violating unless $g_p = 0$ and B^μ is a vector or $g_s = 0$ and B^μ is a pseudo-vector.

11.1 $e^- + e^+ \to B$

The amplitude for this process is,

$$\mathcal{M} = -\epsilon^*_\mu \bar{v}^{s_2}(p_2)\Gamma^\mu u^{s_1}(p_1)$$

In the nonrelativistic limit in the CM frame we get,

$$\mathcal{M} = -E_{\text{CM}} \xi'^\dagger \left[g_s(\vec{\epsilon^*} \cdot \vec{\sigma}) + i g_p \epsilon^{*0} - g_p \vec{\sigma} \cdot (\tfrac{\vec{p}}{m} \times \vec{\epsilon^*}) \right] \xi$$

For $s = 0$ we get,

$$\mathcal{M}(s=0) = -E_{\text{CM}} \sqrt{2} i g_p \epsilon^{*0}$$

but in the CM frame $\epsilon^{*0} = 0$ since there is no timelike polarization. Thus $\mathcal{M}(s = 0) = 0$ as expected since it must decay to a spin 1 particle. The orbital angular momentum cannot produce a B through either interaction, interesting. For g_s this is explained by P-invariance since ${}^1P_1$ is even under P but for $g_p = 0$ we get P conservation if B is odd so g_s cannot couple to ${}^1P_1$. For g_p this is explained by C-invariance since ${}^1P_1$ is odd under P but for $g_s = 0$ we get C conservation if B is even so the term g_p cannot couple to ${}^1P_1$.

We would expect if $\mathcal{M} = \epsilon_\mu \mathcal{M}^\mu$ then $\mathcal{M}^0 = 0$ by the Ward identity since we are in the rest frame of the produced B. However, this does not happen since the Ward identity is violated by this interaction. Does this create a problem? For the $s = 1$ states we get, using the trace tricks,

$$\mathcal{M}(s=1) = -E_{\text{CM}} \sqrt{2} \left[g_s(\vec{\epsilon^*} \cdot \hat{n}) - g_p(\tfrac{\vec{p}}{m} \times \vec{\epsilon^*}) \cdot \hat{n} \right]$$

If g_s is nonzero then any $s = 1$ state nonvanishing at the origin can decay to form a B polarized along $\hat{n}$. If $g_p = 0$ then $\mathcal{H}_{\text{int}}$ is P-invariant with B_μ odd under parity. Since the ${}^3S_1$ states are P odd the g_s coupling is alowed. Likewise P states are P even explaining why g_s does not couple to P states. Furthermore, the ${}^3S_1$ are odd under P but if $g_s = 0$ then the coupling is P invariant with B even hence the g_p can only couple to states even under P.

If g_p is nonzero there is a more complicated coupling. This coupling vanishes on S states since it is proportional to $\vec{p}$. Furthermore it vanishes on the 2^3P_2 and 2^3P_0 states because it does not couple states with parallel $\vec{p}$ and $\hat{n}$. This must be true since a single B has $j = 1$ in its rest frame. However, by inspection, it does couple to 2^3P_1 with $j = 1$ as is allowed by angular momentum conservation. This cannot be explained by P or C since all 2^3P_j are even under P and C only by $j = 1$ selection.

The tables show that only one state is allowed to decay for each of g_s and g_p and these both occur at leading order.

	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$
P	✓	✓	✗	✗	✗	✗
C	✗	✓	✓	✗	✗	✗
j	✗	✓	✓	✗	✓	✗

Table 1: $e^- + e^+ \to B$ vector (g_s) decays allowed by P, C, and J conservation.
.

	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$
P	✗	✗	✓	✓	✓	✓
C	✓	✗	✗	✓	✓	✓
j	✗	✓	✓	✗	✓	✗

Table 2: $e^- + e^+ \to B$ pseduo-vector (g_p) decays allowed by P, C, and J conservation.
.

11.2 $e^- + e^+ \to 2B$

The leading-order Feynman diagrams give,

$$i\mathcal{M} = (-i)^2 \epsilon(k_1)^*_\mu \epsilon(k_2)^*_\nu \; \bar{v}^{s_2}(p_2) \left[\Gamma^\nu \frac{i(\not{q}_1 + m)}{q_1^2 - m^2 + i\epsilon} \Gamma^\mu + \Gamma^\mu \frac{i(\not{q}_2 + m)}{q_2^2 - m^2 + i\epsilon} \Gamma^\nu \right] u^{s_1}(p_1)$$

With two particles in the final state, the configuration may contribute to overall parity so we cannot simply use P to rule out decays. Indeed, we saw that states both odd – $^1S_0$ – and even – $^3P_0$ under P decay to two photons. However, C allows us to forbid decays in the case that one coupling constant is zero so the Hamiltonian is C invariant. In either case B^μ is a vector or pseudovector i.e. has definite C so the $2B$ state has C eigenvalue $+1$. Hence the same decays are C forbidden in the vector and pseudovector cases. Since we showed that the states which are not C protected already decay to a massive (only $^3P_1$ is protected in the massless case) vector we will not get anything new in the pseudovector case. Therefore we look for C violating decays. The candidates are $^3S_1$ and $^1P_1$. Note that both have $j = 1$ so we need to work in the massive case to have a chance of seeing such decays.

For $^3S_1$ we need to consider only the nonrelativistic limit to zeroth-order in momenta. Consider,

$$
\begin{aligned}
&\bar{v}^{s_2}(p_2)\Gamma^\nu(\gamma^\alpha + m)\Gamma^\mu u^{s_1}(p_1) \\
&\quad = m\overline{\begin{pmatrix}\xi' \\ -\xi'\end{pmatrix}}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\nu \\ (g_s + ig_p)\bar{\sigma}^\nu & 0\end{pmatrix}\begin{pmatrix}m & \sigma^\alpha \\ \bar{\sigma}^\alpha & m\end{pmatrix}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\mu \\ (g_s + ig_p)\bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&\quad = m\overline{\begin{pmatrix}\xi' \\ -\xi'\end{pmatrix}}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\nu \\ (g_s + ig_p)\bar{\sigma}^\nu & 0\end{pmatrix}\begin{pmatrix}\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & m(g_s - ig_p)\sigma^\mu \\ m(g_s + ig_p)\bar{\sigma}^\mu & \bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&\quad = m\overline{\begin{pmatrix}\xi' \\ -\xi'\end{pmatrix}}\begin{pmatrix}0 & (g_s - ig_p)\sigma^\nu \\ (g_s + ig_p)\bar{\sigma}^\nu & 0\end{pmatrix}\begin{pmatrix}\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & m(g_s - ig_p)\sigma^\mu \\ m(g_s + ig_p)\bar{\sigma}^\mu & \bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&\quad = m\begin{pmatrix}\xi' \\ -\xi'\end{pmatrix}^\dagger\begin{pmatrix}0 & I \\ I & 0\end{pmatrix}\begin{pmatrix}(g_s - ig_p)\sigma^\nu m(g_s + ig_p)\bar{\sigma}^\mu & (g_s - ig_p)\sigma^\nu\bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu \\ (g_s + ig_p)\bar{\sigma}^\nu\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & (g_s + ig_p)\bar{\sigma}^\nu m(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&\quad = m\begin{pmatrix}\xi' \\ -\xi'\end{pmatrix}^\dagger\begin{pmatrix}(g_s + ig_p)\bar{\sigma}^\nu\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu & (g_s + ig_p)\bar{\sigma}^\nu m(g_s - ig_p)\sigma^\mu \\ (g_s - ig_p)\sigma^\nu m(g_s + ig_p)\bar{\sigma}^\mu & (g_s - ig_p)\sigma^\nu\bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&\quad = m\xi'^\dagger\Big[(g_s + ig_p)\bar{\sigma}^\nu\sigma^\alpha(g_s + ig_p)\bar{\sigma}^\mu + (g_s + ig_p)\bar{\sigma}^\nu m(g_s - ig_p)\sigma^\mu \\
&\qquad\quad - (g_s - ig_p)\sigma^\nu m(g_s + ig_p)\bar{\sigma}^\mu - (g_s - ig_p)\sigma^\nu\bar{\sigma}^\alpha(g_s - ig_p)\sigma^\mu\Big]\xi \\
&\quad = m\xi'^\dagger\Big[(g_s^2 - g_p^2)[\bar{\sigma}^\nu\sigma^\alpha\bar{\sigma}^\mu - \sigma^\nu\bar{\sigma}^\alpha\sigma^\mu] + 2ig_sg_p[\bar{\sigma}^\nu\sigma^\alpha\bar{\sigma}^\mu + \sigma^\nu\bar{\sigma}^\alpha\sigma^\mu] + (g_s^2 + g_p^2)m[\bar{\sigma}^\nu\sigma^\mu - \sigma^\nu\bar{\sigma}^\mu]\Big]\xi
\end{aligned}
$$

To zeroth order in momenta $q_1 = (0, -\vec{k})$ and $q_2 = (0, \vec{k})$ so $q^1 = q^2 = -k^2 = -(m^2 - m_B^2)$. Since α is spatial the second term, the P violating term, is only nonzero if exactly one of μ or ν is spatial. Therefore, we expect P violating decays into one transverse polarization and one longitudinal polarization B. Therefore,

$$
\begin{aligned}
\mathcal{M} = \left(\frac{2m}{2m^2 - m_B^2}\right)\xi'^\dagger\Big[&(g_s^2 - g_p^2)[(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma}) - (\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma})] \\
&+ 2ig_sg_p[\epsilon_2^{*0}(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma}) + (\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})\epsilon_1^{*0} - \epsilon_1^{*0}(\vec{k}\cdot\vec{\sigma})(\vec{\epsilon}_2^{\,*}\cdot\vec{\sigma}) - (\vec{\epsilon}_1^{\,*}\cdot\vec{\sigma})(\vec{k}\cdot\vec{\sigma})\epsilon_2^{*0}]\Big]\xi
\end{aligned}
$$

Now,

$$
(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{c} + i(\vec{b}\times\vec{c})\cdot\vec{\sigma}) = i\vec{a}\cdot(\vec{b}\times\vec{c}) + (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{\sigma}) - (\vec{a}\times(\vec{b}\times\vec{c}))\cdot\vec{\sigma}
$$

Antisymmetrizing over $\vec{a}$ and $\vec{c}$ gives,

$$
\begin{aligned}
(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma}) - (\vec{c}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) &= 2i\vec{a}\cdot(\vec{b}\times\vec{c}) + [(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{\sigma}) - (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{\sigma})] + (\vec{b}\times(\vec{c}\times\vec{a}))\cdot\vec{\sigma} \\
&= 2i\vec{a}\cdot(\vec{b}\times\vec{c})
\end{aligned}
$$

Therefore,

$$
\mathcal{M} = \left(\frac{2i}{1 - \frac{m_B^2}{2m^2}}\right)\xi'^\dagger\Big[(g_s^2 - g_p^2)[\tfrac{\vec{k}}{m}\cdot(\vec{\epsilon}_1^{\,*}\times\vec{\epsilon}_2^{\,*})] + 2ig_sg_p[\tfrac{\vec{k}}{m}\times(\epsilon_2^{*0}\vec{\epsilon}_1^{\,*} - \epsilon_1^{*0}\vec{\epsilon}_2^{\,*})]\cdot\vec{\sigma}\Big]\xi
$$

We get a reduction of the main term in $\mathcal{M}(s=0)$ but only transverse polarizations can be emitted from $^1S_0$ states still (WHY IS THERE SOME CONSERVATION?) Our trace tricks give,

$$
\mathcal{M}(s=1) = -\left(\frac{4\sqrt{2}g_sg_p}{1 - \frac{m_B^2}{2m^2}}\right)[\tfrac{\vec{k}}{m}\times(\epsilon_2^{*0}\vec{\epsilon}_1^{\,*} - \epsilon_1^{*0}\vec{\epsilon}_2^{\,*})]\cdot\hat{n}
$$

$$
\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1) &= 0 \\
\mathcal{M}_{\pm\mp}(s=1) &= 0 \\
\mathcal{M}_{00}(s=1) &= 0 \\
\mathcal{M}_{\pm 0}(s=1) &= \pm 8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{m^2-m_B^2}{2m^2-m_B^2}\right)(n_x \mp in_y) \\
\mathcal{M}_{0\pm}(s=1) &= \mp 8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{m^2-m_B^2}{2m^2-m_B^2}\right)(n_x \pm in_y)
\end{aligned}
$$

These alow decay of ${}^3S_1$ into P-wave (since the total probability for $B+B$ production varies as $\cos^2\theta$ away from the $s=1$ spin axis) $j=1$ with $s=1$ state of two B particles.

Finally, we need to consider the decay of ${}^1P_1$. To do this we need to expand $\mathcal{M}(s=0)$ to second-order in $\vec{p}$. We get,

$$
\begin{aligned}
\mathcal{M}_{\pm\pm}(s=1) &= \pm\frac{2\sqrt{2}(g_p^2-g_s^2)\sqrt{1-\frac{m_B^2}{m^2}}}{1-\frac{m_B^2}{2m^2}} \\
\mathcal{M}_{\pm\mp}(s=1) &= 0 \\
\mathcal{M}_{00}(s=1) &= 0 \\
\mathcal{M}_{\pm 0}(s=1) &= +8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{\sqrt{m^2-m_B^2}}{2m^2-m_B^2}\right)(p_x \mp ip_y) \\
\mathcal{M}_{0\pm}(s=1) &= -8ig_sg_p\left(\frac{m}{m_B}\right)\left(\frac{\sqrt{m^2-m_B^2}}{2m^2-m_B^2}\right)(p_x \pm ip_y)
\end{aligned}
$$

The transverse polarizations only couple to zero momentum. However, we see that the orbital angular momentum can now couple though the $\mathcal{M}_{\pm 0}$ and $\mathcal{M}_{0\pm}$ amplitudes to create spin 1 particles. This alows for the decay of ${}^1P_1$ into P-wave $j=1$ with $s=1$ state of two B-particles.

Questions: notice that these amplitudes diverge as $m_B \to 0$. Does this show that the parity violating coupling is somehow inconsistent for massless particles. It is indeed not gauge invariant so the Ward identity is violated so we may not expect the longitudal polarization to cancel in the limit. Is this a problem? Is there any reason that these extra coupling for n and for p look very similar up to signs and a factor of k?

12 The Ward Identity

Peskin shows that the Ward identity implies $Z_1 = Z_2$ where Z_1^{-1} is the vertex factor of QED and Z_2 is the electron field strength renormalization. This means that the electric charge e in terms of the bare charge e_0 is renormalized as follows,

$$
e = Z_1^{-1}Z_2\sqrt{Z_3}e_0 = \sqrt{Z_3}e_0
$$

See Peskin 10.37. Why is this cancellation necessary. I think it is to preserve gauge invariance of the renormalized Lagrangian. The bare Lagrangian,

$$
\mathcal{L} = -\tfrac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \bar{\psi}_0(i\partial\!\!\!/ - e_0A\!\!\!/_0 - m_0)\psi_0
$$

Now we renomalize, first doing field strength renormalization via rescaling,

$$\psi_0 = Z_2^{1/2}\psi \qquad A_0^\mu = Z_3^{1/2} A_\mu$$

and rearranging into counterterms to get,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi - \tfrac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\delta_2\partial\!\!\!/ - e\delta_1 A\!\!\!/ - \delta_m)\psi$$

where,

$$\delta_3 = Z_3 - 1 \quad \delta_2 = Z_2 - 1 \quad \delta_1 = Z_1 - 1 = (e_0/e)Z_2 Z_3^{1/2} - 1 \quad \delta_m = Z_2 m_0 - m$$

Now the gauge tansformation takes,

$$\psi \mapsto e^{i\alpha}\psi \quad A^\mu \mapsto A^\mu - e_0^{-1} Z_3^{-1/2}\partial^\mu \alpha$$

where the *bare* charge appears as well as the field strength renormalization since this is the gauge transformation of A_0^μ multiplied by $Z_3^{-1/2}$. But notice that the Ward identity forces $e = e_0 Z_3^{1/2}$ and therefore,

$$\psi \mapsto e^{i\alpha}\psi \quad A^\mu \mapsto A^\mu - e^{-1}\partial^\mu \alpha$$

meaning that the renormalized fields satisfy gauge invariance! Notice, there is no reason (or need) for the coincidence $Z_1 = Z_2$ in a non-gauge theory as we can simply renormalize away a shift in the coupling constant using the vertex counterterm. In QED it is actually photon field strength renormalization that renormalizes e and accounts for its running so only the photon self-energy is needed in the Callan–Symanzik equation to compute the β function. A consequence of this is that the coupling constants of A^μ to each charged species are renormalized exactly the same way (since the entire renormalization is via photon-self energy which does not depend on the particular vertex defining the coupling to the charged species) meaning there is a universal electric interaction strength for all species. Thus in the renormalized theory A^μ couples to the conserved Noether charge current as it must since it satisifes a full gauge symmetry even after renormalization.

I think Peskin's treatment is a bit backwards. First he computes the bare vertex factor Γ^μ and sees that it gives a form factor F_1 which is divergent at $q^2 = 0$ in violation of the principal that $F_1(0) = 1$ since this corresponds to e being the physical electric charge. Then he uses the LSZ reduction formula to see that we didn't include the effects of electron field-strength renormalization Z_2 whcih cancels the divergence and exactly gives $F_1(0) = 1$ via the Ward identity forced equality $Z_1 = Z_2$. However, this is still wrong! Peskin did not include the photon field-strength renormalization in this analysis. We should really have $Z_3^{1/2} Z_2 \Gamma^\mu$ as our vertex factor and this $Z_3^{1/2}$ accomodates for the fact that we're still using the bare e_0 and shifts it to e the physical charge. The difference between e_0 and e is pushed under the rug in chapter 7.

13 Coulomb Scattering in QFT (Peskin 4.4 and 5.1)

Consider the interaction term in the Hamiltonian,

$$H_I = \int \mathrm{d}^3x \, e\bar{\psi}\gamma^\mu A_\mu \psi$$

where A_μ is a source field (i.e. not quantized). Then we compute the S-matrix elements using the convention,

$$S = I + iT$$

and to leading-order,

$$\langle p's'|iT|ps\rangle = \langle p's'| - i\int \mathrm{d}t H_I|ps\rangle$$

However,

$$\psi(x)|ps\rangle = u^s(p)e^{-ipx}$$

and therefore,

$$\langle p's'|iT|ps\rangle = -ie\bar{u}^{s'}(p')\gamma^\mu u(p)\int e^{-i(p-p')x}A_\mu \mathrm{d}^4x \; = -ie\bar{u}^{s'}(p')\gamma^\mu u(p)\widetilde{A}_\mu(q)$$

where $q = p - p'$.

13.1 Time-independent potentials

Suppose that A_μ is a time-independent potential the particle is scattering off. Then from the Hamiltonian or Lagrangian formalism we expect that the energy of the particle is conserved. Indeed, the time component of the Fourier transform gives a δ-function since A_μ is constant in t. Therefore we define the scattering amplitude via the formula,

$$\langle p's'|iT|ps\rangle = (2\pi)\delta(E_f - E_i)i\mathcal{M}$$

Therefore, we get,

$$\mathcal{M} = -e\bar{u}^{s'}(p')\gamma^\mu u(p)\int e^{-i(p-p')x}A_\mu \mathrm{d}^4x \; = -ie\bar{u}^{s'}(p')\gamma^\mu u(p)\widetilde{A}_\mu(\vec{q})$$

where now $\widetilde{A}_\mu(\vec{q})$ is the spatial Fourier transform evaluated at the 3-vector $\vec{q} = \vec{p} - \vec{p}'$. There is an overall δ-function enforcing energy conservation because of time-indpendence but not momentum conservation because the potential can absorb arbitrary momentum.

13.2 Building Wavepackets

The incident wave packet $|\psi\rangle$ is built as follows,

$$|\psi\rangle_{\text{in}} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\tilde{\psi}(\vec{k})e^{i\vec{b}\cdot\vec{k}}|\vec{k}\rangle_{\text{in}}$$

with impact parameter $\vec{b}$. Then we consider asymtotic final states of pure momentum,

$$_{\text{out}}\langle\phi| = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\phi}(\vec{p})\,_{\text{out}}\langle\vec{p}|$$

Then the probability of scattering into a sector of momentum space is,

$$\begin{aligned} P(\vec{b}) &= \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\left|_{\text{out}}\langle\vec{p}\,|\,\psi\rangle_{\text{in}}\right|^2 \\ &= \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{4E_kE_{k'}}}\tilde{\psi}^*(\vec{k}')\tilde{\psi}(\vec{k})e^{i\vec{b}\cdot(\vec{k}-\vec{k}')}\left(_{\text{out}}\langle\vec{p}\,|\,\vec{k}\rangle_{\text{in}}\right)\left(_{\text{out}}\langle\vec{p}\,|\,\vec{k}'\rangle_{\text{in}}\right)^* \end{aligned}$$

Now we define,

$$\mathrm{d}\sigma = \int \mathrm{d}^2b\; P(\vec{b})$$

And hence, the integration over $\vec{b}$ gives a factor of $(2\pi)^2\delta^{(2)}(k^{\perp} - k'^{\perp})$. Furthermore, we showed that

$$({}_{\text{out}}\langle\vec{p}\,|\,\vec{k}\rangle_{\text{in}}) = i\mathcal{M}(\vec{k} \to \vec{p})(2\pi)\delta(E_k - E_p)$$

and likewise,

$$({}_{\text{out}}\langle\vec{p}\,|\,\vec{k}'\rangle_{\text{in}})^* = -i\mathcal{M}^*(\vec{k}' \to \vec{p})(2\pi)\delta^{(4)}(E_{k'} - E_p)$$

We use this second delta function and the delta function arising from integration over $\vec{b}$. Inspect,

$$\int \mathrm{d}k'_z \delta(E_{k'} - E_p) = \int \mathrm{d}k'_z\, \delta\left(\sqrt{(k'^{\perp})^2 + k'^2_z + m^2} - E_p\right) = \left|\frac{k'_z}{E_{k'}}\right|^{-1} = \frac{1}{v'}$$

Therefore,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(2E_p)^2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{v'}\tilde{\psi}^*(\vec{k}')\tilde{\psi}(\vec{k})\mathcal{M}(\vec{k} \to \vec{p})\mathcal{M}^*(\vec{k}' \to \vec{p})(2\pi)\delta(E_k - E_p)$$

where we fix $k^{\perp} = k'^{\perp}$ and $E_{k'} = E_p = E$ hence since $E_k = E_p$ we have $k'_z = \pm k_z$. If the wavefunctions are well-localized in momentum space we can ignore the $k'_z = -k_z$ solution to the δ-functions and take $\vec{k} = \vec{k}'$. Therefore, if the wavefunction is well-peaked we can more smooth functions through the integral over $\vec{k}$ to get,

$$\begin{aligned}
\mathrm{d}\sigma &= \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(2E)^2v}(2\pi)\delta(E_k - E_p)|\mathcal{M}(\vec{k} \to \vec{p})|^2\int \frac{\mathrm{d}^3k}{(2\pi)^3}\tilde{\psi}^*(\vec{k})\tilde{\psi}(\vec{k}) \\
&= \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(2E)^2}\cdot\frac{1}{v}\cdot(2\pi)\delta(E_k - E_p)\,|\mathcal{M}(\vec{k} \to \vec{p})|^2
\end{aligned}$$

Now we integrate over $\vec{p}$ to get the full cross section for scattering with this S-matrix element,

$$\begin{aligned}
\sigma &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(2E)^2v}(2\pi)\delta(E_k - E_p)|\mathcal{M}(\vec{k} \to \vec{p})|^2 \\
&= \frac{1}{4\pi^2(2E)^2v}\int \mathrm{d}|p|\mathrm{d}\Omega|\vec{p}|^2\delta(E_k - E_p)|\mathcal{M}(\vec{k} \to \vec{p})|^2 \\
&= \frac{1}{16\pi^2E^2v}\cdot\frac{E}{|k|}|k|^2\int |\mathcal{M}(\vec{k} \to \vec{p})|^2\mathrm{d}\Omega \\
&= \frac{1}{(4\pi)^2}\int |\mathcal{M}(\vec{k} \to \vec{p})|^2\mathrm{d}\Omega
\end{aligned}$$

We have constraints $|p| = |k|$ and $E_p = E_k$ but do not constrain the direction of $\vec{p}$.

13.3 Coulomb Potential

Consider,

$$A^\mu(x) = \left(\frac{Ze}{4\pi r}, 0\right)$$

Then we compute the Fourier transform,

$$
\begin{aligned}
\widetilde{A}_0(\vec{q}) &= \int e^{-iqx}\frac{Ze}{4\pi r}\mathrm{d}^3 r \\
&= \frac{Ze}{4\pi}\int e^{-ir|q|\cos\theta}2\pi r\,\mathrm{d}r\,\mathrm{d}\!\cos\theta \\
&= \frac{Ze}{2}\int \frac{1}{-ir|q|}\left[e^{-ir|q|}-e^{ir|q|}\right] r\,\mathrm{d}r \\
&= \frac{Ze}{|q|}\int_0^\infty \sin\left(r|q|\right)\mathrm{d}r = \frac{Ze}{|q|^2}
\end{aligned}
$$

where we must compute these integrals in the sense of distributions.

13.4 The scattering ampltitude

Now,

$$
|\mathcal{M}|^2 = \left(\frac{Ze}{|q|^2}\right)^2 e^2 |u^{s'}(p')\gamma^0 u(p)|^2
$$

We will compute the cross section in terms of the scattering angle θ between p and p'. Since $E_f = E_i$ we see that $|p| = |p'|$ and hence,

$$
|q| = 2|p|\sin\tfrac{\theta}{2}
$$

First we compute the non-relativistic limit in which,

$$
u^{s'}(p')\gamma^0 u(p) \approx 2m\xi'^{\dagger}\xi
$$

therefore the spins are unaffected in the scattering Approximating $|p'| = |p| \approx mv$ we get,

$$
|\mathcal{M}|^2 = \frac{Z^2 e^4}{(2m)^4 v^4 \sin^4\frac{\theta}{2}}(2m)^2
$$

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2}|\mathcal{M}|^2 = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\frac{\theta}{2})}
$$

Putting in dimensionful constants we get,

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2 Z^2 \hbar^2 c^2}{4(mc^2)^2(v/c)^4\sin^4\frac{\theta}{2}}
$$

Now we compute the fully relativistic case. We need to compute the spin average,

$$
\begin{aligned}
\tfrac{1}{2}\sum_{\text{spins}} |\bar{u}^{s'}(p')\gamma^0 u^s(p)|^2 &= \tfrac{1}{2}\,\mathrm{tr}\left(\gamma^0(\not{p}'+m)\gamma^0(\not{p}+m)\right) \\
&= \tfrac{1}{2}\,\mathrm{tr}\left(\gamma^0\not{p}'\gamma^0\not{p} + \gamma^0 m\gamma^0\not{p} + \gamma^0\not{p}'\gamma^0 m + m^2\right)
\end{aligned}
$$

The middle terms have an odd number of γ matrices and thus have zero trace. Furthermore,

$$
\gamma^0\not{p}'\gamma^0\not{p} = -\gamma^0\gamma^0\not{p}'\not{p} + \gamma^0\{\not{p}',\gamma^0\}\not{p} = -\not{p}'\not{p} + 2\gamma^0 p'^0\not{p}
$$

Therefore, we get, using that $\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

$$\begin{aligned}
\tfrac{1}{2}\sum_{\text{spins}} |\bar{u}^{s'}(p')\gamma^0 u^s(p)|^2 &= \tfrac{1}{2}\left[-\,\mathrm{tr}\,(\not{p}'\not{p}) + 2p'^0\,\mathrm{tr}\,(\gamma^0\not{p}) + 4m^2\right] \\
&= -2p'\cdot p + 4p'^0p^0 + 2m^2 \\
&= 2E^2 + 2\vec{p}\cdot\vec{p}' + 2m^2 \\
&= 2m^2(1+\gamma^2+\beta^2\gamma^2\cos\theta) \\
&= 2m^2\gamma^2([1-\beta^2] + 1 + \beta^2\cos\theta) \\
&= 2m^2\gamma^2(2-\beta^2(1-\cos\theta)) \\
&= 4E^2(1-\beta^2\sin^2\tfrac{\theta}{2})
\end{aligned}$$

Then we have,

$$|\mathcal{M}|^2 = \frac{Z^2e^4}{(2|p|\sin\frac{\theta}{2})^4}4E^2(1-\beta^2\sin^2\tfrac{\theta}{2}) = \frac{Z^2e^4}{4|p|^2\beta^2\sin^4\frac{\theta}{2}}(1-\beta^2\sin^2\tfrac{\theta}{2})$$

Therefore,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{(4\pi)^2} = \frac{Z^2\alpha^2}{4|p|^2\beta^2\sin^4\frac{\theta}{2}}(1-\beta^2\sin^2\tfrac{\theta}{2})$$

Putting in the dimensionful parameters we get,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2Z^2\hbar^2}{4|p|^2\beta^2\sin^4\frac{\theta}{2}}(1-\beta^2\sin^2\tfrac{\theta}{2}) = \frac{\alpha^2Z^2\hbar^2}{4m^2c^2\gamma^2\beta^4\sin^4\frac{\theta}{2}}(1-\beta^2\sin^2\tfrac{\theta}{2})$$

13.5 Helicity Structure of the Scattering Cross Section

Notice that this formula for e^--scattering off a hard Coulomb target is well-defined in the limit $m\to 0$ for fixed (relativistic) momentum (unlike the non-relativistic case). For $m\to 0$ and hence $\beta\to 1$ we get,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2Z^2\hbar^2\cos^2\frac{\theta}{2}}{4|p|^2\sin^4\frac{\theta}{2}}$$

We want to explain this additional structure in terms of relativity and the spin/helicity structure. We need to compute,

$$u^{s'}(p')\gamma^0u^s(p)$$

in the limit $m\to 0$. We can choose states of definite helicity as our basis. We choose $p=(E,0,0,E)$ then $\xi^\uparrow = \xi^+$ satisfies $(\vec{p}\cdot\vec{\sigma})\xi^+ = E\xi^+$ and likewise $\xi^- = \xi^\downarrow$ is the negative helicity eigenstate $(\vec{p}\cdot\vec{\sigma})\xi^- = -E\xi^-$. These give definite Helicity spinors,

$$u^+(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad u^-(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Now we need to rotate these spinors to get the definite helicity states for $p' = (E, E\cos\theta, 0, E\cos\theta)$. Rotation around the y-axis is generated by,

$$e^{-i\Sigma_2\theta} = \begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 & 0\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0\\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$$

and therefore,

$$u^+(p') = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad u^-(p') = \sqrt{2E}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned}
\bar{u}^+(p')\gamma^0 u^+(p) &= 2E\cos\tfrac{\theta}{2} \\
\bar{u}^+(p')\gamma^0 u^-(p) &= 0 \\
\bar{u}^-(p')\gamma^0 u^+(p) &= 0 \\
\bar{u}^-(p')\gamma^0 u^-(p) &= 2E\cos\tfrac{\theta}{2}
\end{aligned}$$

Therefore, the sum over final polarization and average over initial spins gives,

$$2E\cos\tfrac{\theta}{2}$$

which is indeed the square-root of the extra factor that appears in the scattering cross section compared to the non-relativistic cross section for Rutherford scattering. Furthermore, notice that the process is Helicity conserving and therefore not angular momentum conserving (for $\theta \neq 0$) similar to how it does not conserve momentum. The extra factor of $\cos\frac{\theta}{2}$ can be interpreted as the amplitude to connect positive (resp. negative) helicity states after rotation.

14 QFT notes

14.1 QED conserves chirality

Note that a vector interaction with a Dirac fermion like QED preserved chirality. Indeed, the interaction term,

$$e\bar{\psi}\not{A}\psi = e(\chi_L^\dagger A_\mu \bar{\sigma}^\mu \chi_L + \chi_R^\dagger A_\mu \sigma^\mu \chi_R)$$

which couples only fermions of the same chirality together. Hence for massless fermions interacting with QED all processes preserve helicity. We saw this for example with Compton scattering where the helicity is conserved in the limit $m \to 0$ and with relativistic Coulomb scattering where again helicity is conserved in the limit $m \to 0$.

14.2 Weinberg-Witten Theorem

15 Homological. numerical, and algebraic equivalence

Definition 15.0.1.

Proposition 15.0.2. Let X be a variety then Pic^0_X parametrizes algebraically trivial line bundles and hence Cartier divisors. If X is locally factorial this coincides with algebraically trivial Weil divisors.

Proposition 15.0.3. Let X be a proper normal variety. Then for Weil divisors the following are equivalent:

(a) algebraic and $\mathbb{Z}$-homological equivalence

(b) numerical and $\mathbb{Q}$-homological equivalence

SHOW THAT THESE RESULTS FAIL FOR CODIM > 1 CLASSES. EXAMPLE OF LAZARSFELD?

16 Fibral Conditions

"Fibral" criteria have two meanings:

(a) if $f : X \to Y$ is a map of good enough S-schemes then f has property $\mathcal{P}$ iff all f_s has property P for all $s \in P$

(b) if $f : X \to Y$ is good enough then f has property $\mathcal{P}$ iff all $f_y : X_y \to \mathrm{Spec}\,(\kappa(y))$ have property $\mathcal{P}$.

We will consider both types of fibral criteria in this order. We start with type (a) critera for flatness, smoothness, and being an isomorphism.

16.1 Conrad Math 248B Homework 8 Problem 2

Let S be a scheme and $f : X \to Y$ a map between flat and locally finitely presented S-schemes.

16.2 (i)

For $s \in S$, prove that if $f_s : X_s \to Y_s$ is flat at $x \in X_s$ then f is flat at x.

This is local on the source and target so we reduce to the case that X, Y, S are affine. By finite presentation we may further assume that S is finite type over $\mathbb{Z}$ by spreading out. Recall the following theorem,

Theorem 16.2.1 (Mat CRT, 22.5)**.** Let $A \to B$ be a local map of local rings and $u : M \to N$ a morphism of finite B-modules. If N is flat over A then the following are equivalent,

(a) u is injective and $N/u(M)$ is flat over A

(b) $\tilde{u} : M \otimes_A \kappa_A \to N \otimes_A \kappa_A$ is injective.

Note that f is locally finitely presented (since the diagonal of a lfp morphism is lfp and so are the composition of two and the base change). Therefore, shrinking further we can write X in affine space over Y to get $X \hookrightarrow \mathbb{A}^n_Y$. Localize so that R is a local ring. Indeed let $S = \mathrm{Spec}\,(R)$ and $Y = \mathrm{Spec}\,(A)$ and $X = \mathrm{Spec}\,(B/J)$ where $B = A[x_1, \dots, x_n]$ and $J = (f_1, \dots, f_r)$ with $x \in X$ a maximal ideal $\mathfrak{m} \subset B$ containing J and $f(\mathfrak{m}) = \mathfrak{p}$. Now we apply the theorem from Matsumura to the localization $A_\mathfrak{p} \to B_\mathfrak{m}$ and the map of modules $\tilde{u} : J_\mathfrak{m} \to B_\mathfrak{m}$. Then $\tilde{u} = (u \otimes_R (R/\mathfrak{m}_R)) \otimes_{A/m_R A} (A/\mathfrak{m}_A)$ but $A \to B$ becomes flat after applying $- \otimes_R (R/\mathfrak{m}_R)$. Note that,

$$0 \to J_\mathfrak{m} \to B_\mathfrak{m} \to B_\mathfrak{m}/J_\mathfrak{m} \to 0$$

stays exact after applying $- \otimes_R (R/\mathfrak{m}_R)$ since $B_\mathfrak{m}/J_\mathfrak{m}$ is R-flat by assumption. Also $(B_\mathfrak{m}/J_\mathfrak{m}) \otimes_R (R/\mathfrak{m}_R)$ is (A/m_RA)-flat by assumption hence,

$$0 \to J_\mathfrak{m} \otimes_R (R/\mathfrak{m}_R) \to B_\mathfrak{m} \otimes_R (R/\mathfrak{m}_R) \to (B_\mathfrak{m}/J_\mathfrak{m}) \otimes_B (R/\mathfrak{m}_R) \to 0$$

remains flat after applying $- \otimes_{A/\mathfrak{m}_RA} (A/\mathfrak{m}_A)$ and therefore $\tilde{u}$ is injective. Therefore by the theorem $(B/J)_\mathfrak{m}$ is $A_\mathfrak{p}$-flat proving the claim.

Now we prove:

Proposition 16.2.2. Let $f : X \to Y$ be a morphism of lfp flat S-schemes. If $f_{\bar{s}} : X_{\bar{s}} \to Y_{\bar{s}}$ is flat for each geometric fiber over S then f is flat.

Proof. By the above it suffices to show that f_s is flat for each $s \in S$. This follows from the flatness of $f_{\bar{s}}$ by faithfully flat descent. Indeed, let $A \to B$ be a map of k-algebras such that $A_{\bar{k}} \to B_{\bar{k}}$ is flat. Then since $B \to B_{\bar{k}}$ is faithfully flat and $A \to A_{\bar{k}} \to B_{\bar{k}}$ hence $A \to B \to B_{\bar{k}}$ is flat we see that $A \to B$ is flat. $\square$

16.3 (ii)

Prove that if f_s is smooth (resp. étale) for all $s \in S$ then f is smooth (resp. étale). Likewise for $f_{\bar{s}}$ replacing f_s.

By implication of properties, f is lfp. By part (i) we see that f is flat. Since f_s is smooth we see that the geometric fibers $X_{\bar{y}}$ of f are regular (since these are also the geometric fibers of f_s or $f_{\bar{s}}$) and hence f is smooth.

Alternatively we can use $\Omega_{X/Y}$. It suffices to show that $\Omega_{X/Y}$ is locally free of the correct rank. By checking over f_s we see that $\Omega_{X/Y}$ has the correct rank. Hence if X is reduced we would win immediately since a constant rank coherent sheaf is a vector bundle. As above we reduce to $A \to B$ a map of flat R-algebras with $B = A[x_1, \dots, x_n]/J$ then consider the sequence,

$$J/J^2 \to B^n \to \Omega_{B/A} \to 0$$

we need to show that $J/J^2 \to B^n$ is locally a split injection. Since this is true after applying $- \otimes_R \kappa_R$ and hence after applying $- \otimes_B \kappa_B = (- \otimes_R \kappa_R) \otimes_{B/\mathfrak{m}_RB} \kappa_B$ it suffices to prove the following lemma.

Lemma 16.3.1. Let $(A, \mathfrak{m}, \kappa)$ be a local ring. Let $\varphi : M \to N$ be a map from a finitely presented A-module M to a finite projective A-module N. Then the following are equivalent,

(a) φ is a split injection

(b) $\varphi \otimes_A \kappa$ is an injection.

Proof. (a) clearly implies (b). Assume (b). Since injections over κ are split, we can choose a section $N \otimes_A \kappa \to M \otimes_A \kappa$ and consider,

$$\begin{tikzcd}
 & & M \arrow[d, two heads] \\
N \arrow[r] \arrow[urr, dashed] & N \otimes_A \kappa \arrow[r] & M \otimes_A \kappa
\end{tikzcd}$$

the lift exists since N is projective. Hence we get a map $\psi : N \to M$ such that $\psi \circ \varphi : M \to M$ is an endomorphism which equals the identity over κ. Hence $\varphi \circ \varphi$ is an isomorphism by Lemma 16.6.2 so φ is a split injection. Indeed we just need to modify the map $\psi : M \to N$ to $\psi' = (\psi \circ \varphi)^{-1} \circ \psi$ and then clealy ψ' is a section since $\psi' \circ \varphi = (\psi \circ \varphi)^{-1} \circ (\psi \circ \varphi) = \mathrm{id}$. $\square$

16.4 (iii)

Prove that if f is finite type and f_s is an isomorphism for all $s \in S$ then f is quasi-finite flat with fibral-degree 1.

Isomorphisms are smooth and hence by the previous part we conclude that f is smooth. Furthermore, it is finite-type and its fibers are the fibers of some f_s hence are a single point with degree 1 so we conclude that f is quasi-finite flat with constant fibral-degree 1.

16.5 (iv)

Prove the following lemma of Deligne and Rapoport.

Proposition 16.5.1. Let $f : X \to Y$ be a quasi-finite[2] separated map of noetherian schemes that is flat with constant fibral degree. Then f is finite.

Proof. Since f is quasi-finite, to prove that f is finite it suffices to show it is proper. Therefore we must simply verify the valuative criterion of properness. Hence reduce to the case that $Y = \mathrm{Spec}\,(R)$ is a dvr (we can further assume that X has a $K = \mathrm{Frac}\,(R)$ point and find a section but it suffices to just show that $X \to \mathrm{Spec}\,(R)$ is proper since then this property holds). Since $f : X \to Y$ is quasi-finite separated it factors by ZMT as an open immersion $X \hookrightarrow \overline{X}$ and a finite map $\overline{X} \to \mathrm{Spec}\,(R)$. We need to show that f has a section. Consider the scheme-theoretic closure $Z \subset \overline{X}$ of the generic fiber. Because $X \to \mathrm{Spec}\,(R)$ is flat $X \subset Z$ and Z is R-flat and finite since it is a closed subscheme of a finite R-scheme. Hence the fibral degree of Z is constant. Furthermore, $X \subset \overline{X}$ is open so $X \subset Z$ is open but the fibral degree of X is constant by assumption so $X = Z$ $\square$

As a consequence, if f in (iii) is also separated then f is finite of degree 1 and hence an isomorphism.

Remark. Separatedness is necessary. For example, let X be $\mathbb{A}^1$ with two origins and consider $X \sqcup \mathbb{G}_m \to \mathbb{A}^1$ which is flat (since locally it the inclusion of an open) with constant fibral degree 2 but not finite.

16.6 Some Lemma

Lemma 16.6.1. Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module flat over Y. Suppose that $\mathscr{F}|_{X_y}$ is locally free of rank r at $x \in X_y$ then $\mathscr{F}$ is locally free of rank r at $x \in X$.

Proof. We reduce to a statement on local rings. Let $A \to B$ be a local homomorphism and M a finitely presented B-module flat over A such that $M/\mathfrak{m}_A M$ is free of rank r then M is free of rank r. Lifting a basis gives a map $\varphi : B^r \to M$ such that $\varphi \otimes_A \kappa_A$. Consider,

$$0 \to \ker \varphi \to B^r \to M \to \operatorname{coker} \varphi \to 0$$

then since M is a finite B-module we see that $\operatorname{coker} \varphi$ is finite. Since $(\operatorname{coker} \varphi) \otimes_A \kappa_A = 0$ then $(\operatorname{coker} \varphi) \otimes_B \kappa_B = 0$ and hence $\operatorname{coker} \varphi = 0$ by Nakayama. Therefore φ is surjective. Since M is finitely presented $\ker \varphi$ is B-finite and M is A-flat so $(\ker \varphi) \otimes_A \kappa_A = 0$ so $(\ker \varphi) \otimes_B \kappa_B = 0$ and hence $\ker \varphi = 0$ by Nakayama so φ is an isomorphism. $\square$

Remark. Of course flatness is necessary e.g. consider $k[x] \to k[x]$ and $M = k[x]/(x)$.

[2] By definition this means f is finite type. Indeed it means finite type and finite fiber degree see Tag 02NH

Remark. Consider $k[x,y]/(x^2,xy) \to k[x,y]/(x^2,xy) \to k[y]$. This is an example where $\mathscr{F}$ is a vector bundle when restricted to the fiber over any irreducible subscheme on the base but not a vector bundle.

Lemma 16.6.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Suppose that M is a finite R-module with an endomorphism $\phi : M \to M$ such that $\phi \otimes \mathrm{id} : M \otimes_R \kappa \to M \otimes_R \kappa$ is an isomorphism then ϕ is an isomorphism.

Proof. Consider the exact sequence,

$$M \xrightarrow{\phi} M \longrightarrow \operatorname{coker}\phi \longrightarrow 0$$

and apply the right-exact functor $(-) \otimes_R \kappa$ to get,

$$M \otimes_R \kappa \xrightarrow{\phi\otimes\mathrm{id}} M \otimes_R \kappa \longrightarrow (\operatorname{coker}\phi) \otimes_R \kappa \longrightarrow 0$$

But $\phi \otimes \mathrm{id}$ is an isomorphism and the sequence is exact so $(\operatorname{coker}\phi) \otimes_R \kappa = 0$ and thus $\operatorname{coker}\phi = 0$ by Nak. Therefore φ is an isomorphism by a general result on endomorphisms of finite modules. $\square$

16.7 Type (a) fibral finiteness and isomorphsism

Proposition 16.7.1 (EGA III, tome 1, Proposition 4.6.7)**.** Let S be a locally noetherian scheme. Let $f : X \to Y$ be a morphism of proper S-schemes. Let $s \in S$ and consider $f_s : X_s \to Y_s$.

(a) if f_s is a finite morphism (resp. a closed immersion) then there exists an open neighborhood $U \subset S$ of $s \in S$ such that $f|_U : X_U \to Y_U$ is finite (resp. a closed immersion).

(b) If $X \to S$ is flat then if f_s is an isomorphism then there exists an open neighborhood $U \subset X$ of $s \in S$ such that $f|_U : X_U \to Y_U$ is an isomorphism.

Corollary 16.7.2. Let S be a locally noetherian scheme. Let $f : X \to Y$ be a morphism of proper S-schemes. Let $s \in S$ and consider $f_s : X_s \to Y_s$.

(a) if f_s is finite (resp. a closed immersion) for each $s \in S$ then f is finite (resp. a closed immersion).

16.8 Type (b) Fibral Properness and Isomorphism

Proposition 16.8.1 (EGA IV.15.7.10)**.** If $f : X \to Y$ is universally submersive (e.g. flat), finite type, separated and has proper and geometrically connected fibers then f is proper.

Remark. Note! Geometrically connected implies nonempty! This is very important or else Grothendieck would be claiming that open immersions are proper!

Remark. Universally submersive is necessary e.g. consider $\mathbb{G}_m \sqcup * \to \mathbb{A}^1$.

Remark. For a local version consider this question.

Corollary 16.8.2. If $f : X \to Y$ is universally submersive (e.g. flat), finite type, separated and has fibral-degree 1. Then f is an isomorphism.

Proof. Indeed since f has geometrically connected fibers we see that f is proper but it is quasi-finite and hence finite. Therefore f is an isomorphism sinice it is a finite map of degree 1. $\square$

Remark. This also follows from the lemma of Deligne-Rapoport.

17 Flatness of Hilbert and Picard Schemes

Proposition 17.0.1. Let $f : X \to Y$ be a lfp morphism of schemes such that,

(a) the geometric fibers $X_{\bar{y}}$ are regular

(b) the fiber dimension $\dim X_y$ is constant

(c) the geometric fibers $X_{\bar{y}}$ are irreducible

(d) f is proper

(e) Y is reduced

then f is smooth.

Proof. This is local on the source and target so spreading out we reduce to the case that X and Y are affine and finite type over $\mathbb{Z}$. Then by the valuative criterion of flatness we reduce to the case that $Y = \mathrm{Spec}\,(R)$ is a dvr. Let $Z \subset X$ be the scheme-theoretic closure of the generic fiber. Since the generic fiber is irreducible Z is irreducible. Furthermore, the fiber dimension of Z is can only jump up but the fiber dimension of X is constant hence the special fibers of X and Z have the same dimension and X_s is irreducible so $X = Z$ as closed subsets and hence X is irreducible. Then we use the following lemma. $\square$

Lemma 17.0.2. Let $f : X \to Y$ be a lpf morphism of schemes such that,

(a) Y is regular (hence locally Noetherian)

(b) the fibers X_y are regular and equidimensional of constant dimension

(c) X is equidimensional

then X is regular and f is flat.

Proof. For $x \in X$ let $y = f(x)$. Consider the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ Since Y is regular, $\mathfrak{m}_y$ is generated by $\dim \mathcal{O}_{Y,y}$ elements therefore,

$$\dim_{\kappa(x)} \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y) \geq \dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 - \dim_{\kappa(x)}(\mathfrak{m}_y/\mathfrak{m}_y^2) \otimes_{\kappa(y)} \kappa(x) \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

But X_y is regular so,

$$\dim \mathcal{O}_{X_y,x} = \dim_{\kappa(x)} \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y)$$

so we need to show that,

$$\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

then this will imply that the inequalities are equalities hence that $\mathcal{O}_{X,x}$ is a regular local ring. Also it will show flatness by the miracle flatness theorem applied to $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus we apply the following. $\square$

Lemma 17.0.3. Let $f : X \to Y$ be a locally finite type morphism of locally noetherian schemes. Suppose that,

(a) X is equidimensional

(b) the fibers are equidimensional of constant dimension

(c) Y is universally catenary

then for each $x \in X$ set $y = f(x)$ the equality,

$$\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

holds.

Proof. Pulling back along $\mathrm{Spec}\,(\mathcal{O}_{Y,y}) \to Y$ and shrinking to an affine open we reduce to a finitely presented morphism $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ where A is a regular local ring and hence universally catenary. For each irreducble (reduced) component $\mathrm{Spec}\,(B_i) \subset \mathrm{Spec}\,(B)$ we have by assumption $\dim B_i = \dim B$. Since $X \to Y$ is dominant (either X is empty and the result is trivial or f is dominant since every fiber has constant hence nonnegative dimension) the map $A \to B_i$ is injective. Therefore, the dimension formula holds:

$$\dim (B_i)_{\mathfrak{m}} - \dim A = \mathrm{trdeg}_{K(A)}(K(B_i)) - \mathrm{trdeg}_{\kappa_A}(\kappa(\mathfrak{m}))$$

but by constancy and equidimensionality of the fibers we have,

$$\dim (B \otimes_A \kappa_A) = \dim (B_i \otimes_A \kappa_A) = \mathrm{trdeg}_{K(A)}(K(B_i))$$

since the fiber of each B_i is an union of irreducible components of the fiber each of which has the same dimension by assumption. Likewise we apply the dimension formula to the fiber $\mathrm{Spec}\,(B \otimes_A \kappa_A) \to \mathrm{Spec}\,(\kappa_A)$ to get that for each irreducible component $\mathrm{Spec}\,(C_j)$ we have,

$$\dim (C_j)_{\mathfrak{m}} = \mathrm{trdeg}_{\kappa_A}(K(C_j)) - \mathrm{trdeg}_{\kappa_A}(\kappa(\mathfrak{m}))$$

and since $B \otimes_A \kappa_A$ is an equidimensional finite-type κ_A-scheme,

$$\dim (B \otimes_A \kappa_A) = \dim C_j = \mathrm{trdeg}_{\kappa_A}(K(C_j))$$

Thus finally,

$$\dim (B_i)_{\mathfrak{m}} - \dim A = \dim (C_i)_{\mathfrak{m}}$$

However, B_i are the irreducible components of B so by definition $\dim B_{\mathfrak{m}} = \max_i \dim (B_i)_{\mathfrak{m}}$ and likewise $\dim (B \otimes_A \kappa_A)_{\mathfrak{m}} = \max_i \dim (C_i)_{\mathfrak{m}}$ so we conclude that,

$$\dim (B \otimes_A \kappa_A) = \dim B - \dim A$$

which is what we needed to show. $\square$

Proposition 17.0.4. Let $\mathscr{X} \to S$ be a smooth projective family of surfaces meaning the geometric fibers are smooth varities of dimension 2. Then the relative Hilbert scheme of points $\mathrm{Hilb}^n_{\mathscr{X}/S}$ is smooth over S.

Proof. Since $\mathscr{X} \to S$ is projective the Hilbert scheme exists and is projective. Furthermore, since $\mathscr{X}_{\bar{s}}$ is a smooth surface, $(\mathrm{Hilb}_{\mathscr{X}/S})_{\bar{s}} = \mathrm{Hilb}_{\mathscr{X}_{\bar{s}}}$ is smooth and irreducible of dimension $2n$. Therefore, we conclude by the previous results that $\mathrm{Hilb}_{\mathscr{X}/S} \to S$ is smooth. $\square$

Example 17.0.5. Let E be an elliptic curve and consider the nontrivial extension,

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

and let $X = \mathbb{P}_E(\mathcal{E})$. Choose the ample class $\mathcal{L} = \pi^*(\mathcal{O}_E([0])) \otimes \mathcal{O}_X(1)$ and hilbert polynomial $p(t) = t + 1$. Then I claim that,

$$\mathrm{Hilb}_X^{\mathcal{L},p} \cong \mathrm{Spec}\,(k[\epsilon])$$

The exact sequence gives a section $\sigma : E \to X$. Then $T_{[\sigma]}\mathrm{Hilb}_X = H^0(E, \mathcal{N}_{E|X})$. We'll next compute that $\mathcal{N}_{E|X} = \mathcal{O}_E$ and hence $T_{[\sigma]}\mathrm{Hilb}_X = k$ so we get a 1-dimensional tangent space. Thus it suffices to show that $\mathrm{Hilb}_X(k) = \{\sigma\}$. Consider, DO THIS!!

Example 17.0.6. Consider $X = \mathbb{P}_Y(\mathcal{E})$ for some vector bundle and consider a section $\sigma : Y \to X$ given by,

$$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

Then we need to compute,

$$0 \longrightarrow \mathcal{C}_\sigma \longrightarrow \sigma^*\Omega_X \longrightarrow \Omega_Y \longrightarrow 0$$

and there is an Euler sequence,

$$0 \longrightarrow \Omega_{X/Y}(1) \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

where the second map is the universal quotient. Therefore,

$$0 \longrightarrow (\sigma^*\Omega_{X/Y}) \otimes \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

thus we see that $\sigma^*\Omega_{X/Y} = \mathcal{E}_0 \otimes \mathcal{L}^{-1}$. Furthermore, consider the cotangent sequence,

$$0 \longrightarrow \pi^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

when we apply σ^* there is a section $\mathrm{d}\sigma : \sigma^*\Omega_X \to \Omega_Y$ and hence,

$$\mathcal{C}_\sigma = \ker \mathrm{d}\sigma = \sigma^*\Omega_{X/Y} = \mathcal{E}_0 \otimes \mathcal{L}^{-1}$$

Therefore,

$$T_{[\sigma]}\mathrm{Hilb}_X = H^0(\mathcal{N}_\sigma) = \mathrm{Hom}\,(\mathcal{E}_0, \mathcal{L}) = T_{[\sigma]}\mathrm{Hom}_\pi\,(Y, X)$$

The last equality comes from considering surjections $\mathcal{E}[\epsilon] \twoheadrightarrow \mathcal{L}'$ over $Y \times \mathrm{Spec}\,(k[\epsilon])$ up to isomorphism or equivalently flat subbundles $\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]$ fits into a sequence,

$$\mathrm{Hom}\,(\mathcal{E}_0, \mathcal{E}) \to \{\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]\} \to \ker\,(\mathrm{Def}(\mathcal{E}_0) \to \mathrm{Ob}(\mathcal{E}_0 \to \mathcal{E}))$$

which is,

$$\mathrm{Hom}\,(\mathcal{E}_0, \mathcal{E}) \to \{\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]\} = \ker\,(\mathrm{Ext}^1_{\mathcal{O}_X}\,(\mathcal{E}_0, \mathcal{E}_0) \to \mathrm{Ext}^1_{\mathcal{O}_X}\,(\mathcal{E}_0, \mathcal{E}))$$

this is the section of the long exact sequence giving,

$$\{\mathcal{E}_0' \hookrightarrow \mathcal{E}[\epsilon]\} = \mathrm{Hom}_{\mathcal{O}_X}\,(\mathcal{E}_0, \mathcal{L})$$

Example 17.0.7. Let E be an elliptic curve and consider the bundle $\mathcal{E}$ on $Y = E \times \mathbb{A}^1$ given by the extension,

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

corresponding to the extension class,

$$\xi = t \in \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) = k[t]$$

Now set $X = \mathbb{P}_Y(\mathcal{E})$ and consider the family of smooth surfaces $X \to \mathbb{P}^1$ and the ample line bundle $\mathcal{L} = \pi^*(\mathcal{O}_Y(\sigma_0)) \otimes \mathcal{O}_X(1)$ where σ_0 is the zero section of $Y \to \mathbb{A}^1$. For $t \neq 0$ we have seen that

$$\mathrm{Hilb}^{\mathcal{L},p}_{X_t} = \mathrm{Spec}\,(k[\epsilon])$$

However, $X_0 = E \times \mathbb{P}^1$ with ample $\mathcal{L}_0 = \mathcal{O}_E([0]) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathrm{Hilb}^{\mathcal{L}_0,p}_{X_0}$ contains a $\mathbb{P}^1$ parameterizing the closed subschemes $E \times \{s\} \subset E \times \mathbb{P}^1$. Therefore $\mathrm{Hilb}^{\mathcal{L},p}_{X/\mathbb{P}^1}$ is not flat over $\mathbb{P}^1$.

18 Notes

$H^1(\mathcal{O}_X)$ constant in the smooth case by Hodge theory.

Can $H^1(\mathcal{O}_X)$ jump in the normal case?

Looks like no, using vanishing cycle theory.

Questions:

(a) Hilbert schemes of points of smooth families are flat

(b) Find example of flat family where Hilbert schemes of points are not flat

(c) Example where Hilbert scheme with fixed polynomial is not flat.

19 Neutrino Cosmic Background

19.1 Relativistic Gas

Consider a relativistic gas of bosons or fermions with s species (e.g. for electrons/positrons we have $s = 4$ (two polarizations times two particle types) for neutrios we have $s = 12$ (two polarizations times six particle types) when the temperature is large compared to the heaviest neutrino mass which is basically always true). We compute,

$$Z = \left(\prod_{\{n_{\vec{k}}\}_{\vec{k}}} \prod_{\vec{k}} e^{-\beta n_{\vec{k}} E_{\vec{k}}} \right)^s = \left(\prod_{\vec{k}} \prod_{n_{\vec{k}}} e^{-\beta n_{\vec{k}} E_{\vec{k}}} \right)^s = \left[\prod_{\vec{k}} \left(\frac{1}{1 \mp e^{-\beta E_{\vec{k}}}} \right)^{\pm 1} \right]^s$$

where the energy associated to a certain mode with wavenumber $\vec{k}$ is,

$$E_{\vec{k}}^2 = \hbar^2 |k|^2 c^2 + m^2 c^4$$

For a gas in a box of large volume with respect to the thermal wavelength we compute,

$$-\log Z = \pm s \int_0^\infty \mathrm{d}k \left(\frac{V}{\pi^3} \right) \left(\frac{4\pi k^2}{8} \right) \log\left(1 \mp e^{-\beta E_{\vec{k}}}\right) = \pm \frac{sV}{2\pi^2} (\beta \hbar c)^{-3} \int_0^\infty x^2 \log\left(1 \mp e^{-\sqrt{x^2 + (\beta m c^2)^2}}\right) \mathrm{d}x$$

Then we derive the total energy from the formula,

$$U = -\frac{\partial \log Z}{\partial \beta}$$

to get,

$$\begin{aligned} U &= \pm \frac{sV}{2\pi^2\hbar^3c^3}\left[-3\beta^{-4}\int_0^\infty x^2\log\left(1\mp e^{-\sqrt{x^2+(\beta mc^2)^2}}\right)\mathrm{d}x \pm \beta^{-3}\int_0^\infty \frac{x^2(x^2+(\beta mc^2)^2)^{-\frac{1}{2}}\beta(mc^2)^2}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x\right] \\ &= \pm\frac{sV}{2\pi^2\beta^4\hbar^3c^3}\left[-\int_0^\infty (3x^2)\log\left(1\mp e^{-\sqrt{x^2+(\beta mc^2)^2}}\right)\mathrm{d}x \pm \int_0^\infty \frac{x^2(x^2+(\beta mc^2)^2)^{-\frac{1}{2}}(\beta mc^2)^2}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x\right] \end{aligned}$$

Therefore, integrating the first by parts gives,

$$\begin{aligned} U &= \frac{sV}{2\pi^2\beta^4\hbar^3c^3}\left[\int_0^\infty \frac{x^4(x^2+(\beta mc^2)^2)^{-\frac{1}{2}}}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x + \int_0^\infty \frac{x^2(x^2+(\beta mc^2)^2)^{-\frac{1}{2}}(\beta mc^2)^2}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x\right] \\ &= \frac{sV}{2\pi^2\beta^4\hbar^3c^3}\int_0^\infty \frac{x^2\sqrt{x^2+(\beta mc^2)^2}}{e^{\sqrt{x^2+(\beta mc^2)^2}}\mp 1}\mathrm{d}x \end{aligned}$$

which is much more easily derived from the expression for $-\log Z$ before the introduction of the dimensionless integration parameter x. Regardless, introduce functions,

$$u_\pm(\alpha) := \int_0^\infty \frac{x^2\sqrt{x^2+\alpha^2}}{e^{\sqrt{x^2+\alpha^2}}\mp 1}\mathrm{d}x$$

and

$$f_\pm(\alpha) := \pm\int_0^\infty x^2\log\left(1\mp e^{-\sqrt{x^2+\alpha^2}}\right)\mathrm{d}x$$

then we see,

$$U = \frac{sV(k_BT)^4}{2\pi^2\hbar^3c^3}u_\pm(\alpha)$$

and,

$$F = \frac{sV(k_BT)^4}{2\pi^2\hbar^3c^3}f_\pm(\alpha)$$

where,

$$\alpha = \frac{mc^2}{k_BT}$$

is the ratio of the Compton temperature to T. In particular,

$$\frac{F}{U} = \frac{f_\pm(\alpha)}{u_\pm(\alpha)}$$

We compute the high and low temperature limits. In the high temperature limit $\beta \to 0$ or equivalently $m \to 0$ we get,

$$\begin{aligned} u_{+,0}(\alpha) &= u_+(0) = \frac{\pi^4}{15} \\ u_{-,0}(\alpha) &= u_-(0) = \frac{7\pi^4}{120} \\ f_{+,0}(\alpha) &= f_+(0) = -\frac{\pi^4}{45} \\ f_{-,0}(\alpha) &= f_-(0) = -\frac{7\pi^4}{360} \end{aligned}$$

Therefore, for an ultra-relativistic gas we find that the ratio of free energy to total energy is a constant independent of the nature of the gas,

$$\frac{F}{U} = -\frac{1}{3}$$

This gives the familiar result, that the pressue

$$P = -\frac{\partial F}{\partial V}\bigg|_T = -\frac{F}{V} = \frac{1}{3}\frac{U}{V}$$

of a relativistic gas is $\frac{1}{3}$ the energy density. Now in the low-temperature limit, which corresponds to large α, we can approximate,

$$\begin{aligned} u_\pm(\alpha) &\approx \int_0^\infty \frac{x^2\sqrt{x^2+\alpha^2}}{e^{\sqrt{x^2+\alpha^2}}}\mathrm{d}x = \int_\alpha^\infty \frac{xu^2\mathrm{d}u}{e^u} = e^{-\alpha}\int_0^\infty \frac{\sqrt{(y+\alpha)^2-\alpha^2}\,(y+\alpha)^2\mathrm{d}y}{e^y} \\ &\approx e^{-\alpha}\int_0^\infty \frac{\frac{1}{\sqrt{2}}[y^2+2\sqrt{\alpha y}\,](y+\alpha)^2\mathrm{d}y}{e^y} \approx \sqrt{\tfrac{\pi}{2}}\,\alpha^{\frac{5}{2}}e^{-\alpha} \end{aligned}$$

Therefore as $T \to 0$ for $m > 0$ we have $\alpha \to \infty$ hence the energy content of this gas component freezes out. Furthermore,

$$\begin{aligned} f_\pm(\alpha) &\approx \pm\int_0^\infty x^2\left(\mp e^{-\sqrt{x^2+\alpha^2}}\right)\mathrm{d}x = -\int_\alpha^\infty xu\mathrm{d}ue^{-u} = e^{-\alpha}\int_0^\infty \sqrt{(y+\alpha)^2-\alpha^2}(y+\alpha)e^{-y}\mathrm{d}y \\ &\approx e^{-\alpha}\int_0^\infty \tfrac{1}{\sqrt{2}}[y+2\sqrt{y\alpha}](y+\alpha)e^{-y}\mathrm{d}y \approx \sqrt{\tfrac{\pi}{2}}\alpha^{\frac{3}{2}}e^{-\alpha} \end{aligned}$$

Therefore,

$$\frac{F}{U} \approx \alpha^{-1} = \frac{k_BT}{mc^2}$$

Now the entropy is,

$$S = \frac{U-F}{T} = k_B\frac{sV(k_BT)^3}{2\pi^2\hbar^3c^3}[u_\pm(\alpha) - f_\pm(\alpha)]$$

19.2 Neutrino Decoupling

Consider a gas of photons, electrons, and positrons. Then we get,

$$S_{\mathrm{EM}} = k_B\frac{V(k_BT)^3}{\pi^2\hbar^3c^3}\left[\frac{\pi^4}{15} + \frac{\pi^4}{45} + 2u_-(\alpha_e) - 2f_-(\alpha_e)\right]$$

Likewise, the entropy of a neutrino gas (assuming $\alpha_\nu \gg 1$ meaning $k_BT \gg m_\nu c^2$) is that of a fermionic gas with $s = 12$ so,

$$S_\nu = k_B\frac{12\pi^2V(k_BT_\nu)^3}{2\hbar^3c^3}\cdot\frac{7}{120}\cdot\frac{4}{3} = k_B\frac{7\pi^2V(k_BT_\nu)^3}{15\hbar^3c^3}$$

Therefore,

$$\frac{S_{\mathrm{EM}}}{S_\nu} = \frac{30}{7\pi^4}\left(\frac{T}{T_\nu}\right)^3\left[\frac{2\pi^4}{15} + u_-(\alpha_e) - f_-(\alpha_e)\right]$$

At electroweak unification temperature the neutrinos are thermalized with the other particles. However, as the universe cools sufficiently for the weak sector to decouple from the electromagnetic sector the entropy of each sector are independently conserved since the expansion of the universe is adiabatic. Therefore the ratio is a constant. However, at high temperature we have,

$$\frac{S_{\text{EM}}}{S_\nu} = \frac{30}{7\pi^4}\left[\frac{2\pi^4}{45} + \frac{7\pi^4}{120} + \frac{7\pi^4}{360}\right] = \frac{11}{21}$$

Therefore, at very low temperature the electrons and positrons freeze out so we get,

$$\frac{11}{21} = \frac{4}{21}\left(\frac{T}{T_\nu}\right)^3$$

and therefore,

$$T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T$$

For the CMB temperature of $T = 2.73K$ we get $T_\nu = 1.95K$. One interpretation of this lower temperature is that the neutrinos decouple before the electron and positrons freeze out of the soup but when these freeze out they dump their energy into the photon gas giving it extra heating.

20 MIT OCW 8.06 Darwin Term

20.1 HM 2.4

Recall the Feynman-Hellman lemma that if H_λ is a continuous familiy of Hamiltonians with a continuous family of eigenstates $|\psi_\lambda\rangle$ which energy E_λ then,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E_\lambda = \langle\psi_\lambda|\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda}|\psi_\lambda\rangle$$

We apply this to the Hydrogen effective Hamiltonian,

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} - \frac{e^2}{r}$$

The hydrogen atom energies are,

$$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2} \qquad a_0 = \frac{\hbar^2}{me^2}$$

In solving the radial equation one sets $n = N + \ell + 1$ where N is the degree of the radial polynomial.

(a) Let $\lambda = e^2$ be the parameter then we get $\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda} = -\frac{1}{r}$ and therefore,

$$\left\langle\frac{1}{\lambda}\right\rangle = -\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \frac{me^2}{\hbar^2 n^2} = \frac{1}{a_0 n^2}$$

(b) For the parameter $\lambda = \ell$ where in the radial equation we can consider ℓ as a continuous parameter we get,

$$\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda} = \frac{\hbar^2}{2m}\frac{2\ell+1}{r^2}$$

Therefore,

$$\left\langle \frac{1}{r^2} \right\rangle = \frac{2m}{\hbar^2(2\ell+1)} \frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda}$$

where we fix N because it corresponds to the number of nodes in the radial equation and hence is an adiabatic invariant and we vart n according to $n = N + \ell + 1$. Therefore we get,

$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \frac{me^4}{\hbar^2} \frac{1}{n^3}$$

and hence,

$$\left\langle \frac{1}{r^2} \right\rangle = \frac{2m}{\hbar^2(2\ell+1)} \frac{me^4}{\hbar^2} \frac{1}{n^3} = \left(\frac{me^2}{\hbar^2} \right)^2 \frac{2}{2\ell+1} \frac{1}{n^3} = \frac{1}{a_0^2} \frac{2}{2\ell+1} \frac{1}{n^3}$$

20.2 HW 3.2

Consider the radial equation,

$$u'' = \frac{2m}{\hbar^2} \left[V_{\text{eff}}(r) - E \right] u$$

multiply this by u' and integrate to get,

$$\int_0^\infty u'' u' \mathrm{d}r = \frac{2m}{\hbar^2} \int_0^\infty \left[V_{\text{eff}}(r) - E \right] u u' \mathrm{d}r$$

Now $u''u'$ is the derivative of $\frac{1}{2}(u')^2$ and uu' is the derivative of $\frac{1}{2}u^2$ so integrating the RHS by parts we get,

$$\tfrac{1}{2}(u')^2 \Big|_0^\infty = \frac{2m}{\hbar^2} \left[\left[V_{\text{eff}}(r) - E \right] \tfrac{1}{2}u^2 \Big|_0^\infty - \int_0^\infty V'_{\text{eff}} \tfrac{1}{2} u^2 \, \mathrm{d}r \right]$$

However, the boundary conditions for bound states in the radial equation are $u(0) = u(\infty) = 0$ and likewise since $u = r\psi$ we get $u'(0) = \psi(0)$ and $u'(\infty) = 0$ and therefore,

$$\psi(0)^2 = \frac{2m}{\hbar^2} \int_0^\infty V'_{\text{eff}} \, u^2 \mathrm{d}r = \frac{2m}{\hbar^2} \int_0^\infty \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} \psi(r)^2 \, r^2 \mathrm{d}r = \frac{2m}{4\pi\hbar^2} \int \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} \psi(r)^2 \, \mathrm{d}^3 r$$

Now if we consider a state with $\ell = 0$ we get $V_{\text{eff}} = V$ and $\psi(\vec{r}) = \psi(r)$ and we can choose ψ real so we conclude,

$$|\psi(0)|^2 = \frac{2m}{4\pi\hbar^2} \left\langle \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} \right\rangle$$

20.3 The Darwin Term

Therefore using the previous two exercises,

$$|\psi_{n,0}(0)|^2 = \frac{2me^2}{4\pi\hbar^2} \left\langle \frac{1}{r^2} \right\rangle = \frac{1}{2\pi} \cdot \frac{1}{a_0^3} \cdot \frac{2}{n^3} = \frac{1}{\pi a_0^3 n^3}$$

Therefore, for the Darwin term,

$$\delta H_{\text{Darwin}} = \frac{\hbar^2}{8m^2c^2} \nabla^2 V = \frac{\pi\hbar^2 e^2}{2m^2c^2} \delta^{(3)}(\vec{r})$$

we only get an energy shift for $\ell = 0$ which in first-order perturbation theory is,

$$\Delta E_{n,0} = \langle \delta H_{\text{Darwin}} \rangle = \nabla^2 V = \frac{\pi\hbar^2 e^2}{2m^2c^2} |\psi_{n,0}(0)|^2 = \frac{\hbar^2 e^2}{m^2c^2 a_0^3} \cdot \frac{1}{n^3} = \alpha^4 mc^2 \cdot \frac{1}{2n^3}$$

21 Semisimple Algebras

21.1 Jacobson Radical

Here a ring is always unital and associative but not necessarily commutative.

Definition 21.1.1. The *Jacobson radical* $J(R)$ is defined as,

$$J_l(R) = \bigcap_{\substack{M \text{ simple} \\ \text{left } R\text{-mod}}} \mathrm{Ann}_R(M)$$

$$J_r(R) = \bigcap_{\substack{M \text{ simple} \\ \text{right } R\text{-mod}}} \mathrm{Ann}_R(M)$$

Note that these are two-sided ideals because $\mathrm{Ann}_R(M)$ is always a two-sided ideal. Indeed if M is a left R-module then if $xM = 0$ then clearly $rxM = 0$ but al $xrM = x(rM) = xM = 0$. Similarly for M a right R-module.

We use the terminology "the" Jacobson radical because of the following theorem we will now prove.

Theorem 21.1.2. For any unital ring R,

$$J_l(R) = J_r(R)$$

as two-sided ideals.

Remark. This is false for nonunital rings. GIVE EXAMPLE

Remark. Due to the symmetry one might ask if we can define the Jacobson radical more symmetrically as the intersection of the annihlators of simple bimodules. (CAN YOU DO THIS??)

We now will prove this result as follows.

Proposition 21.1.3. The following hold:

$$J_l(R) = \bigcap_{\substack{\mathfrak{m} \text{ maximal} \\ \text{left ideal}}} \mathfrak{m} = \{x \in R \mid \forall r \in R : 1 + rx \text{ is left invertible}\}$$

and likewise

$$J_r(R) = \bigcap_{\substack{\mathfrak{m} \text{ maximal} \\ \text{right ideal}}} \mathfrak{m} = \{x \in R \mid \forall r \in R : 1 + xr \text{ is right invertible}\}$$

Proof. The argument for the two statements is identical so we will only do it for the left case. Since if $\mathfrak{m}$ is a maximal left ideal then $R/\mathfrak{m}$ is a simple left R-module it is clear that $J_l(R) \subset \mathfrak{m}$ since if $x(R/\mathfrak{m}) = 0$ then $x = x \cdot 1 \in \mathfrak{m}$.

Conversely suppose that $x \in \mathfrak{m}$ for each maximal left ideal $\mathfrak{m}$. Let M be a simple left R-module and $m \in M$ nonzero. Then $Rm \subset M$ is a nonzero submodule so $Rm = M$ and hence ${}_RR \to M$

sending $r \mapsto rm$ is surjective and hence $M \cong R/\mathrm{Ann}_R(m)$ and $\mathrm{Ann}_R(m)$ is a maximal left[3] ideal (maximal otherwise there would be proper submodules of $R/\mathrm{Ann}_R(m)$). Now it is clear that,

$$\mathrm{Ann}_R(M) = \bigcap_{m \in M} \mathrm{Ann}_R(m)$$

which is an intersection of maximal left ideals and hence $x \in \mathrm{Ann}_R(M)$ so $x \in J_l(R)$.

Now we show the second equality. Suppose that $x \in \mathfrak{m}$ for each maximal left ideal. Then if $1 + rx$ does not have a left inverse then $R(1 + rx)$ is a proper left ideal and hence contained in a maximal left ideal $\mathfrak{m}$ but then $x \in \mathfrak{m}$ so $rx \in \mathfrak{m}$ so $1 \in \mathfrak{m}$ giving a contradiction. Conversely, if $(1 + rx)$ does have a left inverse for all $r \in R$ suppose that $x \notin \mathfrak{m}$ then there exists r such that $1 + rx \in \mathfrak{m}$ so $1 \in R(1 + rx) \subset \mathfrak{m}$ giving a contradiction. To see the existence of such r consider the left ideal $\mathfrak{m} + Rx$ which is strictly larget than $\mathfrak{m}$ since $x \notin \mathfrak{m}$. Hence by maximality $\mathfrak{m} + Rx = R$ so it contains 1 and hence there exists $r \in R$ so that $1 + rx \in \mathfrak{m}$. $\square$

Lemma 21.1.4. The following conditions on $x \in R$ are equivalent,

(a) $\forall r \in R : 1 + rx$ has a left inverse

(b) $\forall r \in R : 1 + rx$ has a two-sided inverse

and similarly,

(a) $\forall s \in R : 1 + xs$ has a two-sided inverse

(b) $\forall s \in R : 1 + xs$ has a right inverse

Proof. We just need to show (a) $\implies$ (b) since other implications are similar or trivial. Suppose (a) then there exists s such that $s(1 + rx) = 1$ and hence,

$$s = 1 - srx$$

so applying (a) with r replaced by $-sr$ we see that s has a left inverse s' which means that s is invertible (since it also has a right inverse $1 + rx$) and hence $1 + rx$ is also invertible. $\square$

Theorem 21.1.5. For any unital ring R,

$$J_l(R) = \{x \in R \mid 1 + RxR \subset R^\times\} = J_r(R)$$

Proof. Indeed, we showed that if $x \in J_l(R)$ then $\forall r \in R : 1 + rx$ has a two-sided inverse by the above lemma. But $J_l(R)$ is a two-sided ideal so this means that $xs \in J_l(R)$ hence $\forall r, s \in R : 1 + rxs$ is a unit. The reverse inclusion is clear. Likewise, if $x \in J_r(R)$ then $\forall s \in R : 1 + xs$ has a two-sided inverse but $J_r(R)$ is a two-sided ideal so this means that $rx\text{ß}nJ_r(R)$ hence $\forall r, s \in R : 1 + rxs$ is a unit. Thus we conclude. $\square$

From now on we write $J(R) = J_l(R) = J_r(R)$.

[3]Unlike $\mathrm{Ann}_R(M)$ the ideal $\mathrm{Ann}_R(m)$ is *not* two-sided. The point is that for any $r \in R$ we have $rM = M$ so if $xM = 0$ then $xrM = x(rM) = xM = 0$ but the same does not work for $x \in \mathrm{Ann}_R(m)$ since rm might not be annihilated by x. This shows that even if m generates M we may have $\mathrm{Ann}_R(m) \supsetneq \mathrm{Ann}_R(M)$ in the noncommutative case because we cannot take $xm = 0$ and use it to conclude that $xrm = 0$. Indeed, if $M = R/\mathfrak{m}$ then $\mathrm{Ann}_R(\bar{1}) = \mathfrak{m}$ but $\mathrm{Ann}_R(M) = \{x \in R \mid xR \subset \mathfrak{m}\}$ the largest two-sided ideal contained in $\mathfrak{m}$, which is, in general, smaller than $\mathfrak{m}$ since $\mathfrak{m}$ is only a left ideal.

21.2 Nilpotence

Proposition 21.2.1. Let I be a left (resp. right) ideal consisting of nilpotent elements then $I \subset J(R)$.

Proof. Since I consists of nilpotent elements for each $x \in I$ we have rx is nilpotent for each r. Hence $1 + rx$ is a unit so $x \in J(R)$. □

Definition 21.2.2. We say that a module M is

(a) *Noetherian* if every ascending chain of submodules stabilizes

(b) *Artinian* if every descending chain of submodules stabilizes

We say that R is

(a) *left (resp. right) Noetherian* if ${}_R R$ (resp. R_R) is Noetherian as a left (resp. right) R-module

(b) *left (resp. right) Artinian* if ${}_R R$ (resp. R_R) is Artinian as a left (resp. right) R-module

R is left (resp. right)

Remark. Note that there exist left artinian rings that are not right artinian (see Lam, a First Course in Noncommutative Rings, p.22).

Proposition 21.2.3. Let R be left (resp. Artinian) noetherian. Then $J(R)$ is nilpotent.

Proof. Let $J = J(R)$. Consider the descending chain of left ideals,

$$J \supset J^2 \supset J^2 \supset \cdots$$

This must stabilize so we have $J^m = J^n$ for $m \geq n$ and some fixed n. Let $m = 2n$ and $I = J^n$ so $I^2 = I$. If $I \neq 0$ then there exists a left ideal K such that $IK \neq 0$ (e.g. $K = R$) since R is left Artinian there is a minimal such K by Zorns' lemma. If $y \in K$ then $Iy \subset$ FINISH □

Proposition 21.2.4. Let R be left (resp. right) Artinian then R is left (resp. right) Noetherian.

Proof. DO THIS PROOF!! □

21.3 Semisimple Rings

(SHOW AUTOMATIC FINITNESS CONDITIONS NOETHERIAN AUTOMATICALLY!!)
(GIVE EXAMPLE WHY INFINITE PRODUCT OF FIELDS NOT SEMISIMPLE)

Definition 21.3.1. A module M is *semisimple* if one of the following equivalent properties holds,

(a) M is a direct sum of simple modules

(b) M is the sum of its irreducible submodules

(c) every submodule of M is a direct summand

Definition 21.3.2. A ring R is left (resp. right) semisimple if ${}_R R$ (resp. R_R) is a semisimple left (resp. right) R-module.

Proposition 21.3.3. If R is left (resp. right) semisimple if and only if the category of left (resp. right) R-modules is semisimple in the sense that all exact sequences split.

Proof. See [Rotman, An Introduction to Homological Algebra, Prop. 4.5] for details. If R is left semisimple then ${}_RR$ hence hence all free left R-modules are semisimple. However, every left R-module is a quotient of a free module and hence semisimple because submodules of semisimple module are direct summands so quotients are also direct summands. Then every module is both injective and projective since all injections and surjections split since the modules are all semisimple. $\square$

Lemma 21.3.4. Let M be an R-module such that the intersection of all maximal submodules is zero. If M is Artinian then M is semisimple.

Proof. Since M is Artinian the poset of finite intersections of maximal submodules satisfies Zorn's lemma and hence has a minimal element which must equal the intersection of al maximal submodules which is zero. Hence there is a collection $\{N_i\}$ of maximal submodules such that $\bigcap_i N_i = (0)$. Therefore, the map,

$$M \to \bigoplus_i M/N_i$$

is injective but M/N_i is simple and hence M is a submodule of a semisimple module and hence is semisimple. $\square$

Proposition 21.3.5. The following are equivalent,

(a) R is left semisimple

(b) R is right semisimple

(c) $J(R) = 0$ and R is left Artinian

(d) $J(R) = 0$ and R is right Artinian.

Proof. Recall that $J(R)$ is the intersection of all maximal left ideals and also all maximal right ideals. Hence if $J(R) = 0$ then both ${}_RR$ and R_R satisfy that the intersection of their maximal submodules is (0). If $J(R) = 0$ and R is left (resp. right) Artinian then by the above lemma ${}_RR$ (resp R_R) is semisimple so we conclude that R is left (resp. right) semisimple. Hence it suffices to show that if R is left (resp. right) semisimple then $J(R) = (0)$ and R is both left and right Artinian.

If R is left semisimple then ${}_RR$ is a direct sum of simple modules. However, ${}_RR$ is trivially finitely generated but an infinite direct product cannot be finitely generated so it is a finite direct sum of simple modules. Hence R is left Artinian. $\square$

(HOW DO I SHOW IT IS RIGHT ARTINIAN!???)

Definition 21.3.6. A ring R is *simple* if it has no nontrivial two-sided ideals.

Remark. Note that R being simple is much weaker than ${}_RR$ being a simple left R-module. For example, $R = M_n(k)$ is simple but is certainally not simple as a left module since it has nontrivial left ideals (e.g. matrices with some columns zero). Indeed, we have the following result.

Proposition 21.3.7. The following are equivalent,

(a) R is a division ring

(b) ${}_R R$ is a simple left R-module

(c) R_R is a simple right R-module

Proof. Indeed, these are equivalent to R having no nontrivial left (resp. right) ideals. Thus if $x \in R$ is nonzero then $Rx = R$ so there is $yx = 1$ so every nonzero element has a left inverse. However, $y \in R$ then also has a left inverse hence every nonzero element is invertible. The same argument holds if R_R is a simple right R-module. $\square$

Proposition 21.3.8. Let R be semisimple. Then R is left and right Artinian.

21.4 Artin-Wedderburn Theorem

Lemma 21.4.1 (Shur)**.** Let M_1, M_2 be simple R-modules. Then any nonzero endomorphism $\varphi : M_1 \to M_2$ is an isomorphism. Hence $\mathrm{End}_R(M)$ is a division ring.

Proof. Indeed $\ker \varphi \subset M_1$ and $\operatorname{im} \varphi \subset M_2$ are submodules. Since M_1 and M_2 are simple these are either (0) or M_1 respectively M_2. If $\ker \varphi = M_1$ then $\varphi = 0$. If $\operatorname{im} \varphi = (0)$ then $\varphi = 0$ hence φ is a bijection and hence invertible. $\square$

Theorem 21.4.2 (Artin-Wedderburn)**.** Let R be a semisimple ring. Then, R is isomorphic to a finite direct product of matrix rings:

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where D_i are division rings. Moreover the division rings D_i and the integers $r, n_1, \ldots, n_r$ are a complete set of invariants of R.

Proof. Since R is semisimple, R is semisimple as a right R-module. Therefore we can decompose,

$$R = \bigoplus_{i=1}^{r} I_i^{\oplus n_i}$$

where I_i are the minimal nonzero right ideals (i.e. the simple submodules of R_R). This sum is finite since R is right Artinian. Since I_i is simple we see that $D_i = \mathrm{End}_R(I_i)$ is a division ring by Shur's lemma. Therefore,

$$R \cong \mathrm{End}_R(R_R) \cong \mathrm{End}_R\left(I_1^{\oplus n_i}\right) \times \cdots \times \mathrm{End}_R\left(I_r^{\oplus n_r}\right)$$

since the I_i are distinct simple modules there are no nonzero maps between them by Shur's lemma. Therefore,

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

$\square$

22 TEST

$\mathscr{P}$

23 Affine Torsors

$\mathbb{G}_a^n$-torsors in the Zariski, étale, fppf topologies all coincide. This is because they are classified by $H^1(X, \mathbb{G}_a^n)$ but $\mathbb{G}_a^n = \mathcal{O}_X^{\oplus n}$ as a sheaf of rings on these sites which is coherent. Therefore,

$$H^1_{\text{Zar}}(X, \mathbb{G}_a^n) = H^1_{\text{ét}}(X, \mathbb{G}_a^n) = H^1_{\text{fppf}}(X, \mathbb{G}_a^n)$$

Now consider affine bundles, meaning the transition maps are allowed to be any affine linear transformation of $\mathbb{A}^n$ not just translations. Let E_n be the affine algebraic group of affine linear transformations meaning there is an exact sequence,

$$0 \longrightarrow \mathbb{G}_a^n \longrightarrow E_n \longrightarrow \mathrm{GL}_n \longrightarrow 0$$

Then affine bundles are classified by $H^1(X, E_n)$. For any topology coarser than fppf (should work for any topology such that descent is effective for covers) we get comparison maps,

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathbb{G}_a^n) \arrow[r] \arrow[d, equal] & H^0(X, E_n) \arrow[r] \arrow[d] & H^0(X, \mathrm{GL}_n) \arrow[r] \arrow[d, equal] & H^1(X, \mathbb{G}_a^n) \arrow[r] \arrow[d, equal] & H^1(X, E_n) \arrow[r] \arrow[d] & H^1(X, \mathrm{GL}_n) \arrow[r] \arrow[d, equal] & H^2(X, \mathbb{G}_a^n) \arrow[d, equal] \\
0 \arrow[r] & H^0_\tau(X, \mathbb{G}_a^n) \arrow[r] & H^0_\tau(X, E_n) \arrow[r] & H^0_\tau(X, \mathrm{GL}_n) \arrow[r] & H^1_\tau(X, \mathbb{G}_a^n) \arrow[r] & H^1_\tau(X, E_n) \arrow[r] & H^1_\tau(X, \mathrm{GL}_n) \arrow[r] & H^2_\tau(X, \mathbb{G}_a^n)
\end{tikzcd}$$

where the maps $H^i(X, \mathrm{GL}_n) \to H^i_\tau(X, \mathrm{GL}_n)$ are isomorphism for $i = 0, 1$ by Hilbert 90 and the sheaf property. The maps $H^i(X, \mathbb{G}_a^n) \to H^i_\tau(X, \mathbb{G}_a^n)$ are isomorphisms for all i since the sheaf is quasi-coherent. Hence, the maps $H^i(X, E_n) \to H^i_\tau(X, E_n)$ are also isomorphisms by a diagram chase. Hence étale or fppf $\mathbb{A}^n$-bundles are Zariski-locally trivial.

24 DGLA

Definition 24.0.1. A *Differential graded Lie algebra* (DGLA) is a $\mathbb{Z}$-graded vector space $L = \bigoplus_{i \in \mathbb{Z}} L^i$ with a bilinear map $[\bullet, \bullet] : L \times L \to L$ and a linear map $\mathrm{d} : L \to \mathcal{L}$ satisfying the conditions on homogeneous elements,

(a) $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{|a| \cdot |b|}[b, a] = 0$

(b) $[a, [b, c]] = [[a, b], c] + (-1)^{|a| \cdot |b|}[b, [a, c]]$

(c) $\mathrm{d}(L^i) \subset L^{i+1}$ and $\mathrm{d}^2 = 0$ and $\mathrm{d}[a, b] = [\mathrm{d}a, b] + (-1)^p [a, \mathrm{d}b]$ so d is derivation with respect to the bracket.

Remark. Note that L^0 and L^{even} are Lie algebras.

Definition 24.0.2. A linear map $f : L \to L$ is a *derivation of degree n* if $f(L^i) \subset L^{i+n}$ such that,

$$f([a, b]) = [f(a), b] + (-1)^{n|a|}[a, f(b)]$$

Remark. Note that if $a \in L^i$ then $\mathrm{ad}_a : L \to L$ given by $[a, -]$ is a derivation of degree i and d is a derivation of degree 1.

Definition 24.0.3. Denote,

$$Z^i(L) = \ker (\mathrm{d} : L^i \to L^{i+1})$$

and

$$B^i(L) = \mathrm{im} (\mathrm{d} : L^{i-1} \to L^i)$$

and

$$H^i(L) = Z^i(L)/B^i(L)$$

The Mauer-Cartan equation of a DGLA L is,

$$\mathrm{d}a + \tfrac{1}{2}[a, a] = 0$$

Definition 24.0.4. A morphism $f : L_1 \to L_2$ of DGLAs is a graded linear map commuting with the bracket and differential. We say that f is a *quasi-isomorphism* if $f_* : H^i(L_1) \to H^i(L_2)$ are isomorphisms.

Definition 24.0.5. A DGLA L is *Formal* if it is quasi-isomorphic to $H^\bullet(L)$.

Proposition 24.0.6. Let $D : L \to L$ be a derivation, then $\ker D$ is a graded Lie subalgebra.

Proof. We need to show that if $a, b \in \ker D$ then, $[a, b] \in \ker D$ since as a subalgebra it automatically satifies the other properties. Indeed,

$$D([a, b]) = [Da, b] + (-1)^{|a|}[a, Db] = 0$$

so $[a, b] \in \ker D$. □

Proposition 24.0.7. Let L be a DGLA and $a \in L^i$ then,

(a) if i is even $[a, a] = 0$

(b) if i is odd $[a, [a, b]] = \frac{1}{2}[[a, a], b]$ for $b \in L$ and $[[a, a], a] = 0$.

Proof. Indeed,

$$[a, a] + (-1)^{i^2}[a, a] = 0$$

but i is even so we see that $[a, a] = 0$. Now we use the Jacobi identity,

$$[a, [a, b]] = [[a, a], b] + (-1)^{i^2}[a, [a, b]]$$

but i is odd so we see that,

$$[a, [a, b]] = \tfrac{1}{2}[[a, a], b]$$

Furthermore, for $a = b$ then,

$$[a, [a, a]] + (-1)^{i^2}[[a, a], a] = 0$$

hence these are minus eachother so from the above we conclude,

$$[[a, a], a] = 0$$

□

Example 24.0.8. Let M be a complex manifold and E a holomorphic vector bundle on M. Let $\mathcal{A}^{p,q}(E)$ be the sheaf of C^∞-differential forms valued in E of type (p, q). Then consider,

$$L = \bigoplus_q \Gamma(M, \mathcal{A}^{0,q}(\mathrm{End}\,(E)))[-q]$$

with differential $\bar{\partial}_E$ and the natural bracket defined by,

$$[\varphi \otimes \omega, \varphi' \otimes \omega'] = [\varphi, \varphi'] \otimes (\omega \wedge \omega')$$

This is a DGLA. Indeed, it is clear that $[\bullet, \bullet]$ is graded skewsymmetric since there is an extra sign in $[\varphi, \varphi']$ additionally to $\wedge$ which is graded commutative. Then the Jacobi identity follows from the Jacobi identity for $\mathrm{End}\,(E)$ and associativity of $\wedge$. The derivation property of d follows from the fact that,

$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + \omega(-1)^q \wedge \mathrm{d}\omega'$$

25 Deformation Complexes

Definition 25.0.1. Let $\mathscr{X}$ be a DM stack. Let $L^\bullet \in D(\mathcal{O}_{\mathscr{X}_{\text{ét}}})$. We say that,

(a) $L^\bullet$ is *admissible* if,

  (a) $h^i(L^\bullet) = 0$ for all $i > 0$

  (b) $h^i(L^\bullet)$ is coherent for $i = 0, -1$.

(b) $L^\bullet$ is *perfect* (with amplitude contained in $[a, b]$) if locally it quasi-isomorphic to a complex,

$$0 \to \mathcal{E}^a \to \cdots \to \mathcal{E}^b \to 0$$

where each $\mathcal{E}^i$ is a vector bundle living in degree i.

Definition 25.0.2. Let $E^\bullet \in D(\mathcal{O}_{\mathscr{X}_{\text{ét}}})$ be admissible. Then a homomorphism $\psi : E^\bullet \to \mathbb{L}_{\mathscr{X}}$ is an *obstruction theory* if,

(a) $h^0(\phi)$ is an isomorphism

(b) $h^{-1}(\phi)$ is surjective.

Remark. Given a lifting problem,

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, hook] & \mathscr{X} \\ \overline{T} & \end{tikzcd}$$

with ideal sheaf $\mathcal{J}$, we use the map ϕ to pullback the obstruction class $\omega(g) \in \mathrm{Ext}^1\,(g^*\mathbb{L}_{\mathscr{X}}, \mathcal{J})$ to an obstruction,

$$\mathrm{ob}_E(g) := \phi^*\omega(g) \in \mathrm{Ext}^1\,(g^*E^\bullet, \mathcal{J})$$

Definition 25.0.3. An obstruction theory $E^\bullet \to \mathbb{L}_{\mathscr{X}}$ is *perfect* if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$.

Theorem 25.0.4. The following are equivalence,

(a) $\phi : E^\bullet \to \mathbb{L}_{\mathscr{X}}$ is an obstruction problem

(b) for any lifting problem $(T, \overline{T}, g)$ the obstruction $\mathrm{ob}_{E^\bullet}(g) \in \mathrm{Ext}^1\,(g^*E^\bullet, \mathcal{J})$ vanishes if and only if there is a solution to the lifting problem and if $\mathrm{ob}_{E^\bullet}(g) = 0$ then the extensions form a torsor under $\mathrm{Ext}^0\,(g^*E^\bullet, \mathcal{J}) = \mathrm{Hom}\,(g^*h^0(E^\bullet), \mathcal{J})$.

25.1 Examples

Example 25.1.1. If $\mathscr{X}$ is smooth then $\mathbb{L}_{\mathscr{X}}$ is a perfect deformation theory and there are no obstructions to lifting.

Example 25.1.2. Let $C, X \to S$ be proper S-schemes. Consider the Hom scheme $\underline{\mathrm{Hom}}_S\left(C, X\right)$ constructed from the Hilbert scheme. Recall that when X is smooth, there is a classical tangent-obstruction theory,

$$T^i = \mathrm{Ext}^i_{\mathcal{O}_C}\left(f^*\Omega_{X/S}, \mathcal{O}_C\right)$$

to deforming a map $f : C \to X$ over S.

Let $H = \underline{\mathrm{Hom}}_S\left(C, X\right)$ be the Hom scheme and consider the diagram,

$$\begin{tikzcd} H \times C \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_1"] & X \\ C & \end{tikzcd}$$

Therefore we get maps,

$$\mathrm{ev}^*\mathbb{L}_X \to \mathbb{L}_{H\times C} = \pi_1^*\mathbb{L}_H \oplus \pi_2^*\mathbb{L}_C \to \pi_1^*\mathbb{L}_H$$

using the functoriality of the cotangent complex. We want to push this forward to H but the adjunction goes the wrong way. To fix this, we use Grothendieck duality. Assume that C admits a dualizing complex $\omega^\bullet_{C/S}$. Then applying $- \otimes^{\mathbb{L}} \omega^\bullet_{C/S}$ we get,

$$(\mathrm{ev}^*\mathbb{L}_X) \otimes^{\mathbb{L}} \omega^\bullet_{C/S} \to (\pi_1^*\mathbb{L}_H) \otimes^{\mathbb{L}}_{C/S} \xrightarrow{\sim} \pi_1^!\mathbb{L}_H$$

Now we can use the correct adjunction to get,

$$\mathbf{R}\pi_*([\mathrm{ev}^*\mathbb{L}_X] \otimes^{\mathbb{L}} \omega^\bullet_{C/S}) \to \mathbb{L}_H$$

Therefore,

$$E^\bullet = \mathbf{R}\pi_*([\mathrm{ev}^*\mathbb{L}_X] \otimes^{\mathbb{L}} \omega^\bullet_{C/S})$$

is our candidate for an obstruction theory.

Remark. This makes sense because we should be computing $\mathrm{Ext}^1_{\mathcal{O}_C}\left(f^*\Omega_X, \mathcal{O}_C\right) = H^1(C, f^*\mathcal{T}_X)$ on the fibers in the smooth case where as $\mathbf{R}\pi_*\mathbb{L}_X$ would be more like computing $H^1(f^*\Omega_X)$ on the fibers. In either case we will take $\mathrm{RHom}\left(-, \mathcal{O}_C\right)$ on the base but this does not dualize $f^*\Omega_X$ to $f^*\mathcal{T}_X$ *inside* the $\mathbf{R}\pi_*$.

Theorem 25.1.3. Let $S = \mathrm{Spec}\left(k\right)$. If C is gorenstein then $E^\bullet \to \mathbb{L}_H$ is an obstruction theory. If C is a curve and X is smooth the obstruction theory is perfect.

Remark. IS GORENSTEIN REALLY NEEDED??

Proof. DO THIS!!! ☐

Lemma 25.1.4 (Existence of the Mumford Complex)**.** Let $\pi : X \to S$ be a projective map

25.2 Moduli Stack of PRojective Varieties

Let $\mathcal{M}$ and $\mathscr{X}$ be DM stacks and $p : \mathcal{M} \to \mathscr{X}$ be a flat relatively Gorenstein projective morphism (has constant relative dimension and that the relative dualizing complex $\omega^\bullet_{\mathcal{M}/\mathscr{X}}$ is a line bundle $\omega_{\mathcal{M}/\mathscr{X}}$.

If $G^\bullet \in D^+(\mathcal{O}_\mathscr{X})$ when $p^! G^\bullet = p^* G^\bullet \otimes^{\mathbb{L}} \omega_{\mathcal{M}/\mathscr{X}}$ and thus for any complex $F^\bullet \in D^-(\mathcal{O}_\mathcal{M})$ there are natural isomorphisms,

$$\mathrm{Ext}^k_{\mathcal{O}_\mathcal{M}}(F^\bullet, p^* G^\bullet) \to \mathrm{Ext}^k_{\mathcal{O}_\mathcal{M}}\left(F^\bullet \otimes^{\mathbb{L}}, p^! G^\bullet\right) \to \mathrm{Ext}^k_{\mathcal{O}_\mathscr{X}}\left(\mathbf{R}p_*(F^\bullet \otimes^{\mathbb{L}} \omega_{\mathcal{M}/\mathscr{X}}), G^\bullet\right)$$

The connecting map on cotangent complexes,

$$\mathbb{L}_{\mathcal{M}/\mathscr{X}} \to p^* \mathbb{L}_X[1]$$

which we call the Kodaira-Spencer map since it reproduces it in the classical setting then induces a map,

$$E^\bullet = \mathbf{R}p_*(\mathbb{L}_{\mathcal{M}/\mathscr{X}} \otimes^{\mathbb{L}} \omega^\bullet_{\mathcal{M}/\mathscr{X}})[-1] \to \mathbb{L}^\bullet_\mathscr{X}$$

Proof. In the above situation. If $p : \mathcal{M} \to \mathscr{X}$ is *universal* then $E^\bullet \to \mathbb{L}^\bullet_X$ is an obstruction theory for X. ☐

Proof. DO THIS!! ☐

Corollary 25.2.1. If p is smooth of relative dimension ≤ 2 then $E^\bullet$ is a perfect obstruction theory.

25.3 Cones

25.4 Construction of Virtual Fundamental Classes

Proposition 25.4.1. If $E^\bullet$ is a perfect obstruction theory with $h^0(E^\bullet)$ locally free and $h^1(E^\bullet) = 0$ then X is smooth, the virtual dimension of $[X, E^\bullet]$ is $\dim X$ and $[X, E^\bullet] = [X]$.

Proposition 25.4.2. Let X be smooth and $E^\bullet$ a perfect obstruction theory for X. If $h^0(E^\bullet)$ is locally free then the virtual fundamental class is,

$$[X, E^\bullet] = c_r(h^1(E^{\bullet\vee})) \cdot [X]$$

where $r = \operatorname{rank} h^1(E^{\bullet\vee})$.

Proof. If $F^\bullet \to E^\bullet$ is a global resolution of $E^\bullet$, then $C(F^\bullet) = \mathrm{im}\,(F_0 \to F_1)$. ☐

26 pseudo-Torsors

Definition 26.0.1. Let $G \to X$ be an X-group scheme. Then a *pseudo G-torsor* $T \to X$ is an X-scheme together with an X-action $\rho : G \times_X T \to T$ such that the natural map of T-schemes,

$$G \times_X T \xrightarrow{(\rho, \pi_2)} T \times_X T$$

is an isomorphism. A morphism $f : T \to T'$ of pseudo-torsors is a G-equivariant morphism of X-schemes.

Remark. If $T \to X$ is a pseudo G-torsor then for any $X' \to X$ we have $T \times_X X' \to X'$ is a $G \times_X X'$-pseudo-torsor.

Indeed, the map,

$$G_{X'} \times_{X'} T_{X'} \to T_{X'} \times_{X'} T_{X'}$$

is just the base change of $G \times_X T \to T \times_X T$ along $X' \to X$ over the unique projection to X.

Remark. This is a map of T-schemes for the second projection structure. Note that it is G-equivariant for the actions $g \cdot (g', t) = (gg', t)$ and $g \cdot (t, t') = (g \cdot t, t')$ which is an action over T. Note that,

$$T \times_X G \xrightarrow{(\pi_1, \rho)} T \times_X T$$

is likewise an isomorphism of T-schemes using the first projection structure and is equivariant for the actions $g \cdot (t, g') = (t, gg')$ and $g \cdot (t, t') = (t, g \cdot t')$.

We can also consider,

$$G \times_X T \xrightarrow{(\rho, \pi_2)} T \times_X T$$

as an isomorphism of X-schemes. Now over X these both have $G \times_X G$-actions (the second G makes $G \times_X T \to T$ equivariant but it is not an action over T since $G \times_X T \to T$ is a map of T-schemes only if G acts trivially). However, it is not an isomorphism of $G \times_X G$-pseudo-torsors since it does not intertwine both actions. The action on the section factor of the RHS corresponds to the anti-diagonal action $g \cdot (g', t) = (g' g^{-1}, g \cdot t)$ on the LHS not the action on the second factor. Indeed, as $G \times_X G$-torsors over X, in general, $G \times_X T$ and $T \times_X T$ are not isomorphic. For exmaple, if T is the frame bundle of a vector bundle E such an isomorphism would imply that $E \oplus E \cong \mathcal{O}_X^{\oplus \operatorname{rank} E} \oplus E$.

What is confusing is that $T \times_X T$ is isomorphic to $G \times_X T$ for either G-action *individually* but not simultaneously (IS THIS FALSE!!!!). The other isomorphism is given by,

$$G \times_X T \xrightarrow{(\rho, \pi_2)} T \times_X T$$

then $g \cdot (g', t) = (g', g \cdot t)$ is compatible with $g \cdot (t, t') = (t, g \cdot t')$

Proposition 26.0.2. If a pseudo-torsor $\pi : T \to X$ has a section if and only if it is a trivial torsor i.e. there is a G-equivariant X-isomorphism,

$$G \xrightarrow{\sim} T$$

Proof. Given an isomorphism $G \xrightarrow{\sim} T$ we immediately transfer the zero section. Thus if $\sigma : X \to T$ is a section then consider the map $\varphi : G \to T$,

$$G \xrightarrow{\mathrm{id} \times \sigma} G \times_X T \xrightarrow{\rho} T$$

which is clearly G-equivariant. We claim this is an isomorphism. Indeed, consider the diagram,

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\mathrm{id} \times \sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\mathrm{id} \times \sigma"] \\ G \times_X T \arrow[r] & T \times_X T \end{tikzcd}$$

which is Cartesian because if $(f, g) : S \to G \times_X T$ and $h : S \to T$ agree as maps $G \to T \times_X T$ then $g = \sigma \circ s$ where $s : S \to X$ is the structure map and $h = \rho \circ (f \times g) = (f \times \sigma \circ s)$ hence this data is the same as $f : S \to G$ and we compute $g = \sigma \circ s = \sigma \circ \pi \circ f$ and $h = \varphi \circ f$. Hence by base change φ is an isomorphism. $\square$

Proposition 26.0.3. A scheme $T \to X$ equipped with a G-action over X is a pseudo G-torsor if and only if the sheaf h^T is a pseudo h^G-torsor in the sense that for any test scheme $S \to X$ then $T(S)$ is either empty or a $G(S)$-torsor.

Proof. First suppose that T is a pseudo G-torsor and that $T(S)$ is nonempty. Choose a map $S \to T$ then pulling back to S the G_S-torsor $T_S \to S$ has a section and hence is isomorphic to $G_S \to S$. Therefore, $T(S) = T_S(S) = G_S(S) = G(S)$.

Conversely, suppose that h^T is a pseudo h^G-torsor. The action $h^G \circlearrowright h^T$ induces $G \circlearrowright T$ over X. To show that $G \times_X T \to T \times_X T$ is an isomorphism it suffices to show that the associated map of sheaves is an isomrphism. However, working in the category of X-schemes, the map of sets,

$$G(S) \times T(S) \to T(S) \times T(S)$$

is a bijection if and only if $T(S)$ is a $G(S)$-torsor or is empty. Indeed, if $T(S)$ is nonempty then considering the fiber over $T \times \{t_0\}$ this says that $G(S) \to T(S)$ via $g \mapsto g \cdot t_0$ is bijective. $\square$

Example 26.0.4. It is clear from the definitions that $T = \varnothing$ is a pseduo G-torsor. For any pseudo G-torsor $T \to U$ and $U \hookrightarrow X$ an open embedding we also see that $T \to X$ is a pseduo G-torsor. Clearly we should require some sort of surjectivity or we're not get the right notion

However, the worst example is probably the following:

Example 26.0.5. Let $G = *$ and $X = \mathrm{Spec}\,(R)$ with $(R, \mathfrak{m}, \kappa)$ a dvr with $K = \mathrm{Spec}\,(R)$. Then $\mathrm{Spec}\,(K) \sqcup \mathrm{Spec}\,(\kappa) \to \mathrm{Spec}\,(R)$ is a G-torsor. Indeed for $G = \{*\}$ we are just considering maps $T \to X$ such that $\Delta_{T/X} : T \to T \times_X T$ is an isomorphism i.e. monomorphisms of schemes. Given such an example we can base change it by any group to get a nonflat but surjective G-torsor. Now monomorphisms of affine schemes are the same as monomorphisms in the category of affine schemes (since maps into an affine scheme are controlled at the level of global sections) hence correspond to epimorphisms of rings. The map $R \to K \times \kappa$ is an epimorphism of rings because if $\varphi, \psi : K \times \kappa \to A$ agree on R then we just need to show that φ, ψ agree on $\{0\} \times \kappa$ and $K \times \{0\}$. But these maps are obtained via applying $- \otimes_R K$ or $- \otimes_R \kappa$ as maps of R-modules since the images of these submodules are automatically K or κ-modules since φ, ψ are ring maps. Either functor kills one factor so the map from R beceomes an isomorphism and hence the two morphisms agree.

Another such example of a surjective nonflat monomorphism is $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k[\epsilon])$.

Definition 26.0.6. A G-pseduo-torsor is *split* or *trivial* if it has a section (meaning it is isomorphic to the trivial torsor). A pseudo-torsor is a *G-torsor* if it is fppf-locally split. This is equivalent to saying that for some fppf cover $U \to X$ we have $T(U) \neq \varnothing$.

Proposition 26.0.7. Let $\pi : T \to X$ be a G-torsor. If $\mathcal{P}$ is an fppf-local property and $G \to X$ has $\mathcal{P}$ then $T \to X$ has $\mathcal{P}$.

Proof. Obvious since after an fppf cover $U \to X$ we get that $T_U \to U$ is isomorphic to $G_U \to U$. $\square$

Corollary 26.0.8. A pseudo G-torsor $\pi : T \to X$ is a G-tosor if and only if $\pi : T \to X$ is fppf hence $T \to X$ is itself an fppf cover that splits T (torsors kill themselves).

Example 26.0.9. Let $G \circlearrowright X$ over a base scheme S. Then $\rho : G \times_S X \to X$ is a G-torsor over X if G is a fppf S-group. Furthermore, if $G \to S$ has property $\mathcal{P}$ (where $\mathcal{P}$ is true of isomorphisms and is preserved under composition and base change) then $\rho : G \times_S X \to X$ has $\mathcal{P}$.

This is because the following diagram commutes,

$$\begin{tikzcd} G \times_S X \arrow[rr, "(\pi_1\, \rho)"] \arrow[dr, "\rho"'] & & G \times_S X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

and $(\pi_1, \rho) : G \times_S X \to G \times_S X$ is an isomorphism because it has inverse $(\pi_1, \rho \circ (\iota \times \mathrm{id}))$. Therefore, if π has $\mathcal{P}$ so does ρ.

We saw that maps of pseudo torsors need not be isomorphisms (e.g. pullbacks of open or closed embeddings on the base). However maps of torsors are always isomorphisms.

Proposition 26.0.10. Let $f : T \to T'$ be a map of G-torsors. Then f is an isomorphism.

Proof. It suffices to check after an fppf cover so we may assume that T is split. Then the map f gives a section of T' so T' is also split. Hence there is a commutative diagram,

$$\begin{tikzcd} G \arrow[r, dashed] \arrow[d] & G \arrow[d] \\ T \arrow[r, "f"] & T' \end{tikzcd}$$

since the downward maps are G-equivariant isomorphisms we get a G-equivariant map $G \to G$. However, these are always of the form $m \circ (\mathrm{id} \times \sigma)$ for a section $\sigma : X \to G$ (indeed consider the image of $e : X \to G$) and hence are isomorphisms. Thus f is an isomorphism. $\square$

Now we ask the following question: suppose $\pi : T \to X$ has a G-action over X such that for each $x \in X$ then $G_x \circlearrowright T_x$ is isomorphic to G_x with its left translation action. Then is $\pi : T \to X$ a G-torsor?

Example 26.0.11. The answer is, in general, no. We saw an example,

$$G_K \sqcup G_\kappa \to \mathrm{Spec}\,(R)$$

for R a dvr with fraction field K and residue field κ which satisfies the hypothesis but is not flat. Another such example is $G_k \to \mathrm{Spec}\,(k[\epsilon])$.

Therefore, we should restrict to T and X varieties. However, if we don't controll the singularities, this can still go wrong.

Example 26.0.12. Let $X = \mathrm{Spec}\,(k[x,y]/(y^2 - x^3))$ and $T = \mathbb{A}^1 \backslash \{-1\}$ with the map $x \mapsto t^2$ and $y \mapsto t^3$. This map,

$$k[x,y]/(y^2 - x^3) \to k[t, (t+1)^{-1}]$$

of rings is an epimorphism and a monomorphism. Indeed, $(t^2 - 1)/(t+1) = t - 1$ and hence t is in the image. Furthermore, $k[t] \to k[t, (t+1)^{-1}]$ is an epimorphism since if you know where $(t+1)$

goes then you specify where $(t+1)^{-1}$ goes. Therefore, this gives an example of constant fiber degree 1 but not flat.

Base change along any G gives an example of a pseudo G-torsor

However, failure of flatness is the only issue as the following result shows.

Proposition 26.0.13. If $X' \to X$ is a finite type separated map of noetherian schemes with constant fiber degree 1 then,

$$\begin{tikzcd}
\text{proper} \arrow[r, leftrightarrow, "\text{quasi-finite}"] & \text{finite} \arrow[d] \\
\text{flat} \arrow[u, "(a)"] \arrow[ur, "(b)"] & \text{isomorphism} \arrow[l]
\end{tikzcd}$$

where the implication (a) is from the following result and (b) is from the lemma of Deligne-Rappoport (see Brian exercise done in previous section). Hence any of these properties implies it is an isomorphism.

Proposition 26.0.14 (EGA IV.15.7.10)**.** If $f : X \to Y$ is universally submersive (e.g. flat), finite type, separated and has proper and geometrically connected fibers then f is proper.

Therefore, miracle flatness works in the regular case.

Proposition 26.0.15. Let G be a smooth k-group and $T \to X$ a morphism of k-varities with X regular and a G-action on T relative to X. If each fiber $T_x \cong G_{\kappa(x)}$ with its left translation action then $T \to X$ is a G-torsor.

Proof. First we claim that T is regular. Since X is regular the fibers over closed points are cut out by $\dim X$ elements and since $G_{\kappa(x)}$ is regular the fibers are regular. Therefore, T is regular since the dimension of the fibers are $\dim T - \dim X$ (DO BETTER LOOK AT WHAT I WROTE EARLIER ABOUT MAPS WITH REGULAR FIBERS).

First, notice that $T \to X$ is flat by miracle flatness (FINISH PROOF, HOW TO GET PSEUDO-TORSOR CONDITION). $\square$

IS NORMALITY ENOUGH IN THE G IS sFINITE CASE?? NON-NORMAL EXAMPLE FOR G NOT FINITE

27 Applications of Zariski Connectedness

Proposition 27.0.1. Let $f : X \to S$ me a lfp morphism of schemes. Consider the function,

$$n(s) := \#\pi_0(X_{\bar{s}})$$

sending s to the number of geometric components of the fiber. If f is flat and proper then $n : S \to \mathbb{Z}$ is lower semi-continuous. If morover, the fibers of f are geometrically reduced then n is locally constant.

Remark. Without the reducedness assumption, n can certainally jump down e.g. consider a ramified covering of curves. Morover, all of the assumptions are necessary:

(a) flatness: consider $\mathbb{A}^1 \sqcup \{*\} \to \mathbb{A}^1$ mapping $* \to 0$

(b) properness: consider $X = \text{Proj}\,(k[t][X,Y,Z]/(X^2 - Y^2 + tZ^2)) \to \text{Spec}\,(k[t])$ which is the degeneration of a conic to two lines. Then consider $X \setminus \{(0,[0,0,0])\}$ where we remove the point connecting the two conics hence the number of connected components jumps up

Proof. Consider the Stein factorization $f : X \to S' \to S$ then the connected components of $X_{\bar{s}}$ are in bijection with the connected components of $S'_{\bar{s}}$ since $X \to S'$ has geometrically connected components. Hence if $S' \to S$ were flat this would be easy. However, this is not always the case[4]. Instead we use etale localization. Suppose there are n geometric points in the fiber over s. Then by Tag 0BSR there exists an étale neighborhood $U \to S$ of s such that $S'_U = V_1 \sqcup V_2 \sqcup \cdots V_m$ with $s_i \in V_i$ and $\kappa(s_i)/\kappa(s)$ purely inseparable. Then $X_{V_i} \to X$ are étale and $X \to S$ is flat and lfp so $X_{V_i} \to S$ has open image. Since there are finitely many, the intersection of their images is an open neighborhood of s on which there are at least m geometric points in each fiber.

Now suppose that $f : X \to S$ has geometrically reduced fibers. Then $n(s) = \dim_{\kappa(s)} H^0(X_s, \mathcal{O}_{X_s})$ because $H^0(X_s, \mathcal{O}_{X_s})$ is a finite product of local Artinain $\kappa(s)$-algebras corresponding to the connected components. By flat base change, we compute $H^0(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}})$ as the base change and this is reduced hence a product of $\kappa(\bar{s})$ since this is an algebraically closed fields and the number of products is eactly $n(s)$ and equal to the dimension of $H^0(X_s, \mathcal{O}_{X_s})$. However, the function $s \mapsto H^0(X_s, \mathcal{O}_{X_s})$ is upper semicontinuous hence we conclude. $\square$

Remark. It is interesting to me that "proper pushforward of coherent is coherent" is only considered in the locally noetherian case in EGA see [EGA III.1, Theorem 3.2.1]. Can you do anything by noetherian approximation? Maybe not if you can't make the base change from the noetherian setting flat in order to use flat base change. Hmm?

Theorem 27.0.2. Let $f : \mathscr{X} \to \mathscr{Y}$ me a lfp morphism of DM-stacks. Consider the function,

$$n(s) := \#\pi_0(\mathscr{X}_{\bar{s}})$$

sending s to the number of geometric components of the fiber. If f is flat and proper then $n : S \to \mathbb{Z}$ is lower semi-continuous. If morover, the fibers of f are geometrically reduced then n is locally constant.

Proof. The proof is identical up to some reductions. Taking an étale cover, we can assume $\mathscr{Y} = S$ is a scheme. Just as for schemes, there is a Stein factorization,

$$f : \mathscr{X} \to \mathbf{Spec}_S\,(f_* \mathcal{O}_{\mathscr{X}}) = S' \to S$$

and S' is affine over S is hence a scheme. Therefore, we conclude by applying étale localiation to $S' \to S$ exactly as above. Then we get opens $\mathscr{X}_{V_i}$ and the maps $\mathscr{X}_{V_i} \to S$ are flat and lpf hence open so taking the intersection of these images gives an open where the function n is at least $n(s)$. To conlude we need to prove upper semicontinuity for stacks. $\square$

Proposition 27.0.3. Let $\mathscr{X}$ be a normal noetherian connected DM-stack. Then $\mathscr{X}$ is irreducible.

[4]Suppose the schemes are noetherian, then $S' \to S$ finite and is moreover flat if and only if $f_* \mathcal{O}_X$ is a vector bundle (we need $S' \to S$ to be lfp for this equivalence) but there are examples of flat proper maps of noetherian schemes $f : X \to S$ such that $f_* \mathcal{O}_X$ is not a vector bundle. Whenever there is a failure of cohomological flatness this can occur for a thickening of the fiber via the theorem of formal functions e.g. here. DOD BETTER

Proof. Let $U \to \mathscr{X}$ be an étale neighborhood then U is normal. Shrinking so that U is connected (noetherianity implies that the connected components are clopen) we see that U is irreducible. Hence its image is open and irreducible. Therefore, $\mathscr{X}$ is locally irreducible and connected and hence irreducible. Indeed, for any irreducible component $T \subset |\mathscr{X}|$ then either $T \cap U_i$ is empty or $T \cap U_i$ is an irreducible component because if $T \cap U_i \subset Z \subset U_i$ is a larger irreducible then $\overline{Z}$ is irreducible and contains T since $T \subset \overline{Z} \cup U_i^C$ are closed but $T \not\subset U_i^C$ because $T \cap U_i \neq \varnothing$ hence $T \subset \overline{Z}$ and Z is irreducible so $T = \overline{Z}$ by maximality and hence $T \cap U_i = \overline{Z} \cap U_i = Z$ so we conclude that $T \cap U_i$ is an irreducible component of U_i. But U_i is irreducible so $U_i \supset T$ and hence $T = |\mathscr{X}|$ so we conclude. □

Theorem 27.0.4. The moduli space of curves $\mathcal{M}_{g,n}$ over any field is geometrically irreducible.

Proof. For $n' \geq n$ we have $\mathcal{M}_{g,n'} \to \mathcal{M}_{g,n}$ is surjective so it suffices to prove the claim for $n' \gg 0$ so we may assume that $\mathcal{M}_{g,n}$ is DM. It suffices to prove this over prime subfields. Then consider $\mathcal{M}_{g,n} \to \mathrm{Spec}\,(\mathbb{Z})$ and the DM-compactification of stable curves $\overline{\mathcal{M}}_{g,n} \to \mathrm{Spec}\,(\mathbb{Z})$ which is smooth and proper map of noetherian DM-stacks. Hence, we can apply Zariski connectedness to conclude that the number of connected components is constant and by smoothness (hence geometric normality) if the fiber is geometrically connected then it is geometrically irreducible. Therefore we reduce to $\mathbb{Q}$. Then we base change to $\mathbb{C}$ so we win by classical methods. Then $\mathcal{M}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}$ is open so we win. □

28 Smoothness of Pic over a field

Definition 28.0.1. Let $F : \mathbf{Art}_k \to \mathbf{Set}$ be a deformation functor. We say that F is *smooth* if for any small (hence any) extension $A' \twoheadrightarrow A$ in $\mathbf{Art}_k$ then $F(A') \twoheadrightarrow F(A)$ is surjective.

Remark. This is the same as saying that $F \to *$ is smooth as a morphism of functors.

Lemma 28.0.2. Let $R \in \widehat{\mathbf{Art}_k}$ then h_R is smooth if and only if $R \cong k[[x_1, \dots, x_n]]$.

Proof. Cohen structure theorem and infinitesimal criterion. LOOK AT BRIANS NOTES □

Lemma 28.0.3. Let $\varphi : F \to G$ be smooth then F is smooth if and only if G is smooth.

Proof. Indeed, if $A' \twoheadrightarrow A$ is an extension then the map,

$$F(A') \twoheadrightarrow F(A) \times_{G(A)} G(A')$$

is surjective. Suppose G is smooth then for any $x \in F(A)$ there is some $y \in G(A')$ lifting $\varphi(x)$ hence an element of the fiber product hence an element of $F(A')$ lifting x. Conversely, if F is smooth then for any $y \in G(A)$ by smoothness of φ applied to $A \to k$ there exists $x \in F(A)$ such that $\varphi : x \mapsto y$ then by smoothness of F there is $x' \in F(A')$ lifting x and $\varphi(x') \in G(A')$ is the required lift of y. □

Lemma 28.0.4. Let k be an infinite field. Let $F : \mathbf{Art}_k \to \mathbf{Set}$ be a deformation functor satisfying (H1) - (H3) i.e. admitting a hull. Then F is smooth if and only if $F(k[[t]]) \twoheadrightarrow F(k[t]/t^2)$ is surjective

Proof. It is clear we only need to show that if the lifting criterion holds then Let $\varphi : h_R \to F$ be a hull. Since φ is smooth then F is smooth if and only if h_R is smooth if and only if $R \cong k[[x_1, \dots, x_n]]$. Furthermore, by smoothness of φ we get the lifting criterion for h_R. Consider a minimal presentation,

$$0 \to I \to k[[x_1, \dots, x_n]] \twoheadrightarrow R \to 0$$

Suppose that $f \in I$ then we can write,

$$f = p(x_1, \dots, x_n) + O(\underline{x}^{d+1})$$

where $p \in k[x_1, \dots, x_n]$ is a homogeneous polynomial of degree d. By minimality, for any $v = (v_1, \dots, v_n)$ there is $\varphi_v : R \to k[t]/(t^2)$ such that $x_i \mapsto v_i t$. Then φ_v can be integrated to a local map $\widetilde{\varphi}_v : R \to k[[t]]$ by assumption. This means there are power series $g_1, \dots, g_n \in (t) \subset k[[t]]$ such that $f(g_1, \dots, g_r) = 0$ and $g_i = v_i t + O(t^2)$. Then,

$$f(g_1, \dots, g_r) = p(v)t^d + O(t^{d+1})$$

and hence $p(v) = 0$ for all $v \in k^n$. Since k is infinite this implies that $p = 0$ and hence $f = 0$ proving the claim. $\square$

Remark. This is false if k is finite. Indeed, let $k = \mathbb{F}_q$ and $R = k[[x, y]]/I$ where I is the ideal generated by $y \prod\limits_{a \in \mathbb{F}_q} (x - ay)$ so that $\operatorname{Spec}(R)$ is the completion at the origin of the union of all rational lines through the origin in $\mathbb{A}^2$. Then it is clear that h_R satisfies $h_R(k[[x]]) \twoheadrightarrow h_R(k[x]/x^2)$ because,

$$\operatorname{Hom}_{\text{loc}}\left(R, k[t]/(t^2)\right) = k\partial_x \oplus k\partial_y$$

where ∂_x is the map $x \mapsto t$ and $y \mapsto 0$ which is well-defined since $y \mapsto 0$ and ∂_y is the map $x \mapsto 0$ and $y \mapsto t$ which is well-defined since $x \mapsto 0$. Then $a\partial_x + b\partial_y$ integrates to $R \to k[[t]]$ given by $x \mapsto at$ and $y \mapsto bt$ which is well-defined since $bx - ay \mapsto 0$.

29 Miyaoka's Theorem

29.1 Harder-Narasimhan filtration

Remark. Note that it is not true that a nonzero map $\varphi : \mathcal{E} \to \mathscr{F}$ of vector bundles implies that $c_1(\mathcal{E}) \cdot H^{n-1} \le c_1(\mathscr{F}) \cdot H^{n-1}$ unless both have the same rank. For example, consider on $\mathbb{P}^1$ the map $\mathcal{O}_X(1) \to \mathcal{O}_X(1) \oplus \mathcal{O}_X(-1)$. However, if X is smooth $\varphi : \mathcal{E} \to \mathscr{F}$ is a nonzero map of torsion-free sheaves of the same rank r then there is a map $\det \varphi : \det \mathcal{E} \to \det \mathscr{F}$ and hence we get that $c_1(\mathscr{F}) - c_1(\mathcal{E}) = c_1(\det \mathscr{F}) - c_1(\det \mathcal{E})$ is effective.

References:

(a) Miyaoka, Higher Dimensional Algebraic Varities

(b)

Let X be a smooth projective variety of dimension n with ample divisor H. Then for any torsion-free coherent sheaf $\mathcal{E}$ define,

$$\mu(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\operatorname{rank} \mathcal{E}}$$

Then stability and semistablity are defined the usual way.

Proposition 29.1.1. Fix a torsion-free sheaf of rank r on the projective polarized variety (X, H). Then the set of slopes $\{\mu(\mathscr{F}) \mid 0 \neq \mathscr{F} \subset \mathcal{E}\} \subset \frac{1}{r!}\mathbb{Z}$ is bounded above. Let μ_1 be the maximum then $\{\mathscr{F} \subset \mathcal{E} \mid \mu(\mathscr{F}) = \mu_1\}$ contains the largest element with respect to the inclusion relation (the maximal destabilizer).

Proof. Because $\mathcal{E}$ is torsion-free there are injections,

$$\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee} \hookrightarrow \mathcal{O}_X(mH)^N$$

for some integers m, N. Therefore, it suffices to show that slopes of subsheaves of $\mathcal{O}_X(mH)^N$ are bounded. Let $\mathscr{F} \subset \mathcal{E}$ be a rank s subsheaf. At the generic point the matrix corresponding to $\mathscr{F} \hookrightarrow \mathcal{O}_X(mH)^N$ has s independent columns (because it is full rank) and hence we can choose $\mathscr{F} \hookrightarrow \mathcal{O}_X(mH)^N \to \mathcal{O}_X(mH)^s$ such that the composition is injective. Then taking determinants we get $\deg \mathscr{F} \le smH^n$ and hence $\mu(\mathscr{F}) \le mH^n$ proving a uniform bound.

Now suppose that $\mathscr{F}_1, \mathscr{F}_2 \subset \mathcal{E}$ are two subsheaves with $\mu(\mathscr{F}_1) = \mu(\mathscr{F}_2) = \mu_1$. It suffices to show that $\mu(\mathscr{F}_1 + \mathscr{F}_2) = \mu_1$. Consider the exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \cap \mathscr{F}_2 \to \longrightarrow \mathscr{F}_1 \oplus \mathscr{F}_2 \longrightarrow \mathscr{F}_1 + \mathscr{F}_2 \longrightarrow 0$$

and the additivity of Chern classes,

$$r\mu(\mathscr{F}_1 + \mathscr{F}_2) = r_1\mu(\mathscr{F}_1) + r_2\mu(\mathscr{F}_2) - r'\mu(\mathscr{F}_1 \cap \mathscr{F}_2)$$

where $r = \text{rank}\,(\mathscr{F}_1 + \mathscr{F}_2)$ and $r_i = \text{rank}\,\mathscr{F}_i$ and $r' = \text{rank}\,(\mathscr{F}_1 \cap \mathscr{F}_2)$. By definition of μ_1 we have $\mu(\mathscr{F}_1 \cap \mathscr{F}_2) \le \mu_1$ and thus,

$$r\mu(\mathscr{F}_1 + \mathscr{F}_2) \ge (r_1 + r_2 - r')\mu_1$$

and thus $\mu(\mathscr{F}_1 + \mathscr{F}_2) \ge \mu_1$ but trivially $\mu(\mathscr{F}_1 + \mathscr{F}_2) \le \mu_1$ so we win. □

Definition 29.1.2. By the above result, setting $\mu_{\max}(\mathcal{E}) = \mu_1$ is a well-defined invariant of $(X, H, \mathcal{E})$ and so is the maximal destabilizer. By maximality, the maximal destabilizer is saturated and H-semistable.

Lemma 29.1.3. Let $\mathcal{E}$ be torsion-free and $\mathscr{F} \subset \mathcal{E}$ the maximal destabilizer. Then $\mathcal{E}$ is H-semistable iff $\mathscr{F} = \mathcal{E}$ iff $\mu(\mathcal{E}) = \mu_{\max}(\mathscr{F})$. If $\mathcal{E}$ is not H-semistable then $\mu_{\max}(\mathcal{E}/\mathscr{F}) < \mu_{\max}(\mathcal{E}) = \mu(\mathscr{F})$.

Proof. Indeed, $\mathcal{E}$ is H-semistable iff $\mu_{\max}(\mathcal{E}) = \mu(\mathcal{E})$ since this exactly means that every subsheaf has slope at most $\mu(\mathcal{E})$ but this is equivalent to $\mathscr{F} = \mathcal{E}$ since $\mathscr{F}$ is maximal amoung subsheaves with $\mu(\mathscr{F}) = \mu_{\max}(\mathcal{E})$.

Suppose that $\mu_{\max}(\mathcal{E}) > \mu(\mathcal{E})$. Then if $0 \neq \mathscr{F}' \subset (\mathcal{E}/\mathscr{F})$ is the maximal destabilizer then its preimage $\mathscr{F}'' \subset \mathcal{E}$ must satisfy $\mu(\mathscr{F}'') < \mu_{\max}(\mathcal{E})$ because $\mathscr{F}''$ strictly contains $\mathscr{F}$ then consider,

$$0 \to \mathscr{F} \to \mathscr{F}'' \to \mathscr{F}' \to 0$$

we have,

$$r\mu(\mathscr{F}) + r'\mu(\mathscr{F}') = r''\mu(\mathscr{F}'') < r''\mu(\mathscr{F})$$

and therefore,

$$r'\mu(\mathscr{F}') < (r'' - r)\mu(\mathscr{F})$$

but $r' = r'' - r$ so we conclude. □

Corollary 29.1.4. There exists a filtration,

$$0 = \mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \cdots \subsetneq \mathscr{F}_s = \mathcal{E}$$

where $\mathscr{F}_{i+1}$ is the preimage in $\mathcal{E}$ of the maximal destabilizer of $\mathcal{E}/\mathscr{F}_i$. Therefore, $\mathscr{F}_{i+1}/\mathscr{F}_i$ is H-semistable and the slopes satisfy,

$$\mu_{\max}(\mathcal{E}) = \mu(\mathscr{F}_1/\mathscr{F}_0) > \mu(\mathscr{F}_2/\mathscr{F}_1) > \cdots > \mu(\mathscr{F}_s/\mathscr{F}_{s-1}) = \mu_{\min}(\mathcal{E})$$

Furthermore, $\mu_{\min}(\mathcal{E}) = -\mu_{\max}(\mathcal{E}^\vee)$ is the minimal slope of a torsion-free quotient of $\mathcal{E}$.

Theorem 29.1.5 (Mehta-Ramanathan)**.** Let X be a normal projective variety of dimension ≥ 2 and H an ample divisor. Let $\mathcal{E}$ be torsion-free sheaf. Then for $m \gg 0$ the restriction of $\mathcal{E}$ to a gneral member $Y \in |mH|$ is $H|_Y$-semistable if and only if $\mathcal{E}$ is H-semistable.

Therefore, we can reduce to sufficently large degree complete intersection curves.

Proposition 29.1.6. Let C be a smooth projective curve over an algebraically closed field of characteristic zero. Let $\mathcal{E}$ be a locally free sheaf of rank r and $\pi : \mathbb{P}(\mathcal{E}) \to C$ the projective bundle. Let $M = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - \mu(\mathcal{E})\pi^* c_1(\mathcal{E})$. Then the following are equivalent,

(a) for any finite $f : C' \to C$ then $f^*\mathcal{E}$ is μ-semistable

(b) M is nef

(c) $M - \pi^* D$ is not pseduo-effective for any $\mathbb{Q}$-divisor D with $\deg D > 0$

(d) $M + \pi^* D$ is ample for some $\mathbb{Q}$-divisor D with $0 < \deg D < 1/r!$

(e) $M - \pi^* D$ is not pseudo-effective, where D is some $\mathbb{Q}$-divisor with $0 < \deg D < 1/r!$

(f) $\mathcal{E}$ is μ-semistable.

Corollary 29.1.7. Let (X, H) be a normal, projective, polarized scheme over a ring R of characteristic zero, finitely generated over $\mathbb{Z}$. Let $\mathcal{E}$ be a torsion free sheaf on X. Let $K = \overline{\operatorname{Frac}(R)}$. If $\mathcal{E}_K$ is H-semistable on X_K then $\mathcal{E}$ is H-semistalbe on reduction mod p for almost all p.

Proof. Let $C \sim mH^{n-1}$ be a general complete intersection curve on X of large degree. Then we may assume that $\mathcal{E}|_C$ is μ-semistable on C_K hence using the above notation $M + c\pi^* H$ is ample on $\mathbb{P}(\mathcal{E}_C)_K$ but ampleness is an open condition for projective morphisms so this is satisfied for $\mathcal{E}|_C$ modulo almost every p, which implies H-semistability modulo almost every prime. ☐

Lemma 29.1.8. Let C be a smooth curve and $\mathcal{E}$ a vector bundle. Then $\mathcal{E}$ is μ-semistable if and only if $\mathcal{E}(-\mu)$ is semipositive.

Proof. This is almost immediate from the definition. Semistable means that for any $\mathcal{E} \twoheadrightarrow \mathcal{L}$ we have $\mu(\mathcal{L}) \geq \mu(\mathcal{E})$ and semipositive means $\mu(\mathcal{L}) \geq 0$ so shifting by $-\mu(\mathcal{E})$ these are the same condition. ☐

Corollary 29.1.9. Over a field of characteristic zero, if $\mathcal{E}$ is H-semistable then $\mathcal{E}^{\otimes n}$ is H-semistable. Hence the direct summands $S^m\mathcal{E}$ and $\bigwedge^m \mathcal{E}$ are H-semistable. More generally, if $\mathcal{E}_1, \mathcal{E}_2$ are H-semistable then $\mathcal{E}_1 \otimes \mathcal{E}_2$ are H-semistable.

Proof. We can reduce to a complete intersection curve of sufficiently high degree. Then we just need to show that if $\mathcal{E}_1, \mathcal{E}_2$ are semipositive then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is semipositive. Consider $\mathcal{E}_1 \otimes \mathcal{E}_2 \twoheadrightarrow \mathcal{L}$ WHY? ☐

Definition 29.1.10. Let X be a projective variety and $\mathscr{F}$ a torsion-free coherent sheaf. We say that $\mathscr{F}$ is *generically H-semipositive* if $\mu_{\min}(\mathscr{F}) \geq 0$.

Remark. This is equivalent to "generically nef". WHY?

Proposition 29.1.11. Let X be a smooth projective variety over a field of characteristic $p > 0$. Assume there is a $\mathbb{Q}$-divisor D with $\deg D > 0$ such that when restricted to a generated complete intersection curve $\mathscr{F}(-D)$ ample and $(\mathcal{T}_X/F)(-D)$ negative. Then on the open U where $\mathscr{F} \subset \mathcal{T}_X$ is a subbundle we have that $\mathscr{F}$ is a p-closed foliation.

Proof. The bracket defines an $\mathcal{O}_X$-linear map $\bigwedge^2 \mathscr{F} \to \mathcal{T}_X/\mathscr{F}$. This must be zero because $(\wedge^2 \mathscr{F})(-D)$ is ample but $(\mathcal{T}_X/\mathscr{F})(-2D)$ is negative if restricted to a general curve. Hence $\mathscr{F}$ is a foliation.

The p^{th}-power map induces $F^*\mathscr{F} \to (\mathcal{T}_X/\mathscr{F})$ then $F^*\mathscr{F}(-D)$ is ample on a generic curve but $(\mathcal{T}_X/\mathscr{F})(-D)$ is negative so the map is zero. $\square$

Theorem 29.1.12. Let (X, H) be a smooth, polarized projective variety over a field of characteristic $p > 0$. Assume that there is a p-closed foliation $\mathscr{F} \subset \mathcal{T}_X$ such that,

$$(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1} > 0$$

Then X contains a rational curve C through a general point of X such that,

$$C \cdot H \le \frac{2pH^n}{(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1}}$$

Proof. Let $\pi : X \to Y$ be the quotient by $\mathscr{F}$. Let $H^{(1)}$ be an ample divisor on $X^{(1)}$ such that $\varphi^* H^{(1)} = pH$. Let $mH^{(1)}$ be very ample and $\Gamma^{(1)} \subset X^{(1)}$ be a general complete intersection curve cut out by $mH^{(1)}$ and $\Gamma^* \subset Y$ and $\Gamma \subset X$ its inversel image with reduced structure. The natural projection $\Gamma \to \Gamma^{(1)}$ is Frobenius and Γ is numerically equivalent to $m^{n-1}H^{n-1}$ as a 1-cycle on X. Let d be the degree of $\pi : \Gamma \to \Gamma^*$ which is either 1 or p. Then we have,

$$d(\Gamma^* \cdot (-K_Y)) = \Gamma \cdot (-\pi^* K_Y) = \Gamma \cdot (-K_X + (p-1)\det \mathscr{F}) = m^{n-1}H^{n-1} \cdot (-K_X + (p-1)\det \mathscr{F})$$

Since this is positive, by Bend-and-Break through a general point of Y there exists a rational curve C' such that,

$$C' \cdot \pi_* H \le 2\frac{\Gamma^* \cdot \pi_* H}{\Gamma^* \cdot (-K_Y)}$$

Then its image under $Y^{(-1)} \to X$ produces a rational curve C through a general point of X of degree at most,

$$C \cdot H \le \frac{2d(\Gamma \cdot H)}{\Gamma \cdot (-\pi^* K_Y)} = \frac{2pm^{n-1}H^n}{m^{n-1}(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1}} = \frac{2pH^n}{(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1}}$$

$\square$

Theorem 29.1.13. Let X be a normal projective variety over an algebraically closed field of characteristic zero. If $\mathcal{T}_X$ is not generically semi-negative then X is uniruled.

Proof. Let $\mathscr{F} \subset \mathcal{T}_X$ be the maximal destabilizer and we assume $\mu(\mathscr{F}) > 0$. Then let $D = cH$ with $\mu(\mathscr{F}) > c > \mu_{\max}(\mathcal{T}_X/\mathscr{F})$ so that $\mathscr{F}(-D)$ is ample and $(\mathcal{T}_X/\mathscr{F})(-D)$ is negative on the generic complete intersection curve. Then applying the previous result we get modulo almost all primes a p-closed foliation $\mathscr{F} \subset \mathcal{T}_X$. Then we apply the previous theorem so for almost all p the reduction of X is uniruled by rational curves C of degree bounded uniformly by,

$$C \cdot H \le \frac{3H^n}{(\det \mathscr{F}) \cdot K_X}$$

because $\mu(\mathscr{F}) > 0$ so the denominator is nonzero. Therefore, because the Hom scheme is finite type X must be uniruled. $\square$

30 t-Connections and the complex C_X

For two complexes $(C^\bullet, \mathrm{d}^A)$ and $(C^\bullet, \mathrm{d}^B)$ with the same terms C^i then,

$$\mathrm{d}^n = \mathrm{d}^A + (-1)^n \mathrm{d}^B$$

is also a differential. Indeed,

$$\mathrm{d}^{n+1} \circ \mathrm{d}^n = (-1)^{n+1}\mathrm{d}^A \circ \mathrm{d}^B + (-1)^n \mathrm{d}^B \circ \mathrm{d}^A = 0$$

Can we do this with the complex C_X and the Hodge complex? Do the twists mess this up?

31 Refining Vanishing of 1-forms

31.1 Notes to Self

Consider the complex, $C_{X,\bullet}$ of graded modules. The bullet denotes the grading not the indexing of the complex which is written in the top as a homological grading. The complex is supported in degrees $-\dim X$ to 0.

I want to claim that if $\phi : X' \to X$ is a birational map of smooth projective varieties then the map,

$$Rp_{2*}C_X \to Rp_{2*}C_{X'}$$

is a quasi-isomorphism. Indeed, I think that, this should follow from some spectral sequence and the projection formula and $\phi_*\mathcal{O}_{X'} = \mathcal{O}_X$. We should get, NOO

$$E_1^{p,q} = R^q p_{2*} C_X^p \implies R^{p+q} p_{2*} C_X$$

and $C_X^p = \Omega^{p-n} \otimes S_{\bullet - g + p - n}$ and the pushforward is the cohomology. Only the R^0 term is invariant I guess under the blowup. Actually, look at Kollar's results on this says we get vanishing for ω_X and higher R.

CAN SHOW DIRECTLY THAT THE SECTIONS WE BUILD IN X' WILL FACTOR THOUGH X SINCE THEY COME FROM GLOBAL FORMS.

WE WANT TO TAKE k or m graded part and also R^{-k} OR SUCH TO GET THE RIGHT DEGREES. GIVEN THIS SHOULD GET EQUALITY. Indeed, I think the $g - r$ graded part of,

$$R^{r-n} p_{2*} C_X \to R^{r-n} p_{2*} C_{X'}$$

is probably an isomorphism. CAN USE INVARIANCE OF THIS JUST NEED TO SHOW ITS NONZERO ON SOME MODEL!

31.2 The Complex

Definition 31.2.1. $\mathbf{Var}_A$ is the slice category of smooth projective k-varieties over a fixed abelian variety A. Explicitly, the objects are smooth projective k-varieties X equipped with a map $f : X \to A$ and the morphisms in $\mathbf{Var}_A$ are morphisms of varieties compatible with the map to A.

Each $[X \xrightarrow{f} A] \in \mathbf{Var}_A$ is equipped with a complex of graded $S_\bullet$-modules,

$$0 \longrightarrow \mathcal{O}_X \otimes S_\bullet(-g) \longrightarrow \Omega^1_X \otimes S_\bullet(-g+1) \longrightarrow \cdots \longrightarrow \Omega^n_X \otimes S_\bullet(-g+n) \longrightarrow 0$$

where $S(i)$ is the graded $S_\bullet$-module with $S(i)_\bullet = S_{\bullet+i}$ and the differential is defined via,

$$\theta \otimes s \mapsto \sum_{i=1}^{g} (\theta \wedge f^*\omega_i) \otimes s_i s$$

where $\omega_1, \dots, \omega_g \in H^0(A, \Omega_A)$ is a basis and $s_1, \dots, s_g$ is the dual basis. We view this as a complex C_X of $\mathcal{O}_X \otimes S$-modules on the scheme $X \times V$ supported in degrees $[-\dim X, 0]$.

Remark. This complex naturally computes the associated graded of the f-pushforward of the trivial Hodge module on X,

$$\mathrm{gr}^F_\bullet \mathscr{H}^i(f_+(\mathcal{O}_X, F)) \cong R^i f_* C_{X,\bullet}$$

is an isomorphism of sheaves on $T^*A = A \times V$ whih by abuse of notation we write $\mathbf{R}f_*C_X$ for the complex of sheaves on T^*A which more accurately would be writen as $\mathbf{R}(f \times \mathrm{id}_V)_* C_X$.

Remark. Write $C^\alpha_{X,\bullet} := C_{X,\bullet} \otimes f^*\alpha$ for $\alpha \in \mathrm{Pic}^0_A$. We will often consider the sheaves $R^i p_{2*} C^\alpha_{X,\bullet}$ on V where $p_2 : X \times V \to V$ is the projection. Varying over α these can be packaged together into the following Fourier-Mukai transform,

$$E_X = \mathbf{R}\Phi_P(\mathbf{R}f_*C_X) = \mathbf{R}(p_{23})_* \left([p_{13}^* \mathbf{R} f_* C_X] \otimes^{\mathbb{L}} p_{12}^* \mathcal{P} \right)$$

using the projections on $A \times \hat{A} \times V$ where $\mathcal{P}$ is the Poincare bundle. Then E_X is a complex of sheaves on $\hat{A} \times V$. Since $p_{12}^*\mathcal{P}$ is flat for p_{23} this complex commutes with arbitrary base change. In particular, for $g_\alpha : \{\alpha\} \times V \to \hat{A} \times V$ we get,

$$\mathbf{L}g_\alpha^* E_X = \mathbf{R}(p_2)_* \left(\mathbf{R}f_* C_X \otimes^{\mathbf{L}} p_1^*\alpha \right) = \mathbf{R}(p_2)_* \mathbf{R}f_* \left(C_X \otimes^{\mathbf{L}} f^*\alpha \right) = \mathbf{R}p_{2*} C_X^\alpha$$

where p_1, p_2 are the projections of $A \times V$. Generic vanishing results or Simpson's description of the cohomology jump loci on the moduli space of rank 1 higgs bundles shows that $\mathbf{L}g_\alpha^* E_X$ has locally free cohomology sheaves for general $\alpha \in \mathrm{Pic}^0_A$. Therefore, $\mathscr{H}^i(E_X)$ are locally free away from codimension 2 on $\hat{A} \times V$ hence to show that $\mathrm{Supp}\,(E_X)$ dominates V it suffices to show that any $\mathscr{H}^i(E_X)$ is nonzero at the generic point and to do this we must simply show that any graded part of $R^i p_{2*} C^\alpha_{X,\bullet}$ is nonzero for generically chosen α.

Proposition 31.2.2. Let $\phi : X' \to X$ be a dominant (resp. birational) morphism in $\mathbf{Var}_A$ then the natural map,

$$C_X \to \mathbf{R}\phi_* C_{X'}$$

induces an injection (resp. isomorphism) for each $k \geq 0$,

$$(R^{k-n} p_{2*} C^\alpha_X)_{g-k} \to (R^{k-n} p_{2*} C^\alpha_{X'})_{g-k}$$

of the graded parts in degree $g - k$.

Proof. The $g-k$ graded part of the complex looks like,

$$0 \longrightarrow \Omega_X^k \otimes S_0 \otimes f^*\alpha \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes S_{n-k} \otimes f^*\alpha \longrightarrow 0$$

supported in degrees $[-(n-k), 0]$. Therefore, $R^{k-n}p_{2*}$ is computed by taking the kernel of global sections in degree $-(n-k)$. Explicitly,

$$(R^{k-n}p_{2*}C_X^\alpha)_{g-k} = \ker\left(H^0(X, \Omega_X^k \otimes f^*\alpha) \otimes S_0 \to H^0(X, \Omega_X^{k+1} \otimes f^*\alpha) \otimes S_1\right)$$

By generic smoothness, the pullback of forms is injective along ϕ so the natural pullback is injective. If ϕ is birational, inverse to the above map is defined by the usual Hartog argument. $\square$

Definition 31.2.3. We say that a smooth projective variety X of dimension n with a map $f : X \to A$ has *nonvanishing property* NV_k if,

$$(R^{k-n}p_{2*}C_X^\alpha)_{g-k} \neq 0$$

for a generic choice of $\alpha \in \mathrm{Pic}^0(A)$.

Remark. For any smooth projective variety, we will say that X has property NV_k if $X \to \mathrm{Alb}_X$ does. It is clear that having property NV_k wrt to $X \to \mathrm{Alb}_X$ is equivalent to having property V_k with respect to all maps $X \to A$.

The above discussion has proven that,

Proposition 31.2.4. If X has property NV_k for any k then every 1-forms $f^*\omega$ has a nonempty vanishing locus.

Proposition 31.2.5. The properties NV_k are birationally invariant for smooth projective varieties.

Proof. This is immediate from Proposition 31.2.2 and resolution of a birational map by morphisms of smooth projective varities. $\square$

Remark. Therefore, the properties NV_k refine the (definitionally) birationally invariant property "on every smooth projective model of X every 1-form has a nonempty zero locus" and have the additional benifit of actually being computable!

Lemma 31.2.6. Suppose that $g : X \to Y$ is a dominant morphism in $\mathbf{Var}_k$ and $m = \dim Y$. Then there is an injection of complexes,

$$g^*\omega_Y[n-m] \hookrightarrow C_{X,g-m}$$

Proof. We use the map $g^*\omega_Y \hookrightarrow \Omega_X$. To show it fits into the $g-m$ graded part of C_X we need to show that the differential applied to its image is zero. This is clear because the forms $f^*\omega_i$ are pulled back through g and therefore,

$$f^*\theta \wedge f^*\omega_i = f^*(\theta \wedge \omega_i) = 0$$

since θ is a top form. $\square$

Proposition 31.2.7. Let $\phi : X \dashrightarrow Y$ be a dominant rational map in $\mathbf{Var}_A$. If Y has NV_k then X has NV_k.

Proof. This follows from applying Proposition 31.2.2 twice to the diagram $X \leftarrow X' \rightarrow Y$ given by resolving the rational map by morphisms in $\mathbf{Var}_A$. $\square$

Now we recall the Main theorem of [PS14]. In our language this says:

Theorem 31.2.8 (PS14)**.** If $H^0(X, \omega_X^{\otimes d} \otimes f^*\mathcal{L}^{-1}) \neq 0$ for some ample $\mathcal{L} \in \mathrm{Pic}\,(A)$ then X has property NV_n where $n = \dim X$.

Corollary 31.2.9. Suppose that $X \dashrightarrow Y$ is a dominant rational map in $\mathbf{Var}_A$ such that Y is general type (or more generally $Y \rightarrow A$ factors birationally through the Iitaka fibration) then X has property NV_m where $m = \dim Y$.

Corollary 31.2.10. Suppose that $X \dashrightarrow Y$ is an MRC fibration over a smooth projective variety Y of general type. Then every 1-form on X has a nonempty zero locus.

32 NV_k Properties Write Up

Proposition 32.0.1. If $g : X \to Y$ is a dominant map in $\mathbf{Var}_A$ and $f^*\mathcal{L} \hookrightarrow g^*\omega_Y$ for some ample $\mathcal{L}$ on A then X satisfies NV_m.

Proof. Indeed, we have the following,

$$f^*(L \otimes \alpha)[(n-m)] \hookrightarrow g^*\omega_Y[(n-m)] \otimes f^*\alpha \hookrightarrow (C_X^\alpha)_{g-m}$$

and then taking the derived pushforward,

$$R^{m-n}p_{2*}f^*(L \otimes \alpha)[-(n-m)] \hookrightarrow R^{m-n}p_{2*}g^*\omega_Y[(n-m)] \otimes f^*\alpha \hookrightarrow R^{m-n}p_{2*}(C_X^\alpha)_{g-m}$$

However,

$$R^{m-n}p_{2*}f^*(L \otimes \alpha)[(n-m)] = H^0(X, f^*(L \otimes \alpha)) = H^0(A, L \otimes \alpha \otimes f_*\mathcal{O}_X) \neq \varnothing$$

for generic α since otherwise $f(X)$ would be contained in a general translate of a hyperplane section of A which cannot happen. Therefore, we conclude that,

$$R^{m-n}p_{2*}(C_X^\alpha)_{g-m} \neq \varnothing$$

□

Using the same

Now we need to do the case that just $f^*\mathcal{L}^{\otimes d} \hookrightarrow g^*\omega_Y^{\otimes d}$. To do this we write,

$$B = f^*\mathcal{L}^{\otimes -1} \otimes g^*\omega_Y$$

Then choose nonzero $s \in H^0(X, B^{\otimes k})$ and consider the k-cyclic cover defined by s and a resolution $\widetilde{X} \to X_d \to X$. Now we should consider the image $\mathscr{F}$ of,

$$Rp_{2*}(p_1^*B^{-1} \otimes C_X^\alpha) \to Rp_2^*C_{\widetilde{X}}^\alpha$$

We want to show that $\mathscr{F}$ is nonzero. Recall that if $\psi : \widetilde{X} \to X$ is the generically finite morphism from the covering and then resolution we get a morphism $\psi^*B^{-1} \to \mathcal{O}_{\widetilde{X}}$ which is an isomorphism away from the singular locus of X_d. We need to show that the morphism,

$$B^{-1} \otimes C_{X,k}^\alpha \to \mathbf{R}\psi_*C_{\widetilde{X},g-k}^\alpha$$

is nonzero after applying $R^{k-n}p_{2*}$ for some k. Indeed we will apply this for $k = m$. We want to have a map,

$$f^*(\mathcal{L} \otimes \alpha)[-(n-m)] \hookrightarrow R\psi_*C_{\widetilde{X},g-m}^\alpha$$

and we need injectivity. We just need to check there is a map,

$$f^*(\mathcal{L} \otimes \alpha)[-(n-m)] \hookrightarrow B^{-1} \otimes C_{X,g-m}$$

this is the same as making a map,

$$g^*\omega_Y[-(n-m)] \hookrightarrow C_{X,g-m}$$

which we already constructed. Okay therefore it suffices to show injectivity. Let's look at the degree $g-m$ part,

$$B^{-1} \otimes C_{X,g-m}^\alpha \to \mathbf{R}\phi_*C_{\widetilde{X},g-m}^\alpha$$

Proposition 32.0.2. Let $\phi : X \dashrightarrow Y$ be a dominant rational map in $\mathbf{Var}_A$. If Y has V_k then X has V_k.

(EXAMPLE TO SHOW THAT THE CONVERSE IS FALSE!!)
(FUNCTORIALITY ARISING FROM MIXED HODGE MODULE PERSPECTIVE!)

Example 32.0.3. Consider X the blowup of $S \times A$ along C where A is a simple abelian surface and S is a general type surface with a nonzero 1-form and C is a general type curve on both A and S. Then any 1-form can be written as,

$$\omega = \pi_2^* \omega_1 + \pi_2^2 \omega_2$$

What is the vanishing locus?

Corollary 32.0.4. Let $X \dashrightarrow Y$ be a dominant rational map. If $Y \to A$ factors through its Iitaka fibration then any form,

$$f^* \omega \in f^* H^0(A, \Omega_A) \subset H^0(X, \Omega_X)$$

has a nonempty zero locus.

Proof. By (REF THM) Y has property V_m with $\dim Y = m$. Thus X has property V_m by Proposition 32.0.2 $\square$

Theorem 32.0.5. Let $P\Omega_X^k$ denote the kernel of,

$$\Omega_X^k \to \Omega_X^{k+1} \otimes S_1$$

given by the differential of C_X. Suppose that,

$$H^0(X, \mathrm{Sym}_d\left(P\Omega_X^k\right) \otimes f^* \mathcal{L}^{-1}) \neq 0$$

for some $k \geq 0$ and $d \geq 0$ with $\mathcal{L} \in \mathrm{Pic}\,(A)$ ample. Then X has property V_k.

Remark. Using that $f^* \mathcal{L} \hookrightarrow \omega_Y^{\otimes d} \hookrightarrow \mathrm{Sym}_d\left(P\Omega_X^k\right)$ we recover Corollary 32.0.4.

Proof. DO THIS!! $\square$

32.1 PROBLEM (FIXED)

We needed to pass to a finite etale cover first to get NV_m in PS14. But if $X \dashrightarrow Y$ is dominant rational map and $Y' \to Y$ finite étale then because X is smooth and $X \dashrightarrow Y$ is defined away from codimension 2 we see that there exists an étale cover $X' \to X$ such that the diagram,

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, dashed] & Y \end{tikzcd}$$

Because we consider $U \to Y$ with $U \subset X$ codimension ≥ 2 and $U' = U \times_Y Y'$ then extends to a finite étale cover $X' \to X$ and $U' \subset X'$ is an open so the above diagram exists. Then we win because Y has NV_m.

Proposition 32.1.1. Suppose that there is a finite étale cover $X' \to X$ such that X' has property NV_k then every 1-form on X has a nonempty zero locus.

33 étale vs finite étale

For adic spaces one defines a map $f : X \to Y$ to be étale if around each $x \in X$ there are opens $f|_U : U \to V$ and a factorization $U \to \overline{U} \to V$ such that $U \hookrightarrow \overline{U}$ is an open embedding and $\overline{U} \to V$ is finite étale.

First we should note that the corresponding property in the category of schemes is *much* more restrictive than begin étale.

Example 33.0.1. Consider $f : \mathbb{A}^1 \backslash \{0\} \to \mathbb{A}^1$ given by $t \mapsto x^2(x-1)$ then f is étale but does not satisfy the property around 1. Indeed, even after Zariski shrinking the target around 0 this map cannot be factored as a composition of any number of Zariski opens and finite étale maps. Indeed, there can only be one finite étale cover since $\deg f = 3$ is prime. We can shrink the opens so that both are principal opens. Therefore we get,

$$k[t]_f \to R \to k[x]_g$$

However, according the Brian the adic space definition is "natural" from the perspective of analytic geometry. Some remarks about this,

(a) since an étale map is the same as a local biholomorphism, in the analytic topology, around each closed point there is a disk which is isomorphic onto its image.

(b) However, to make the definition correct for non-closed points an non-algebraically closed fields which is very important for rigid geometry we need to have an inclusion of an open into a finite étale map which alows for the finite extension of fields that often arise for non-closed points or non algebraically closed residue fields.

Q: what is the topology generated by Zariski opens and finite étale covers? Apparently this is closed to Grothendieck's original definition until – according to Brian according to Oort – Artin talked him out of it with a computation of a variety with the incorrect cohomology groups.

It turns out that the fppf topology is generated by Zariski opens and surjective finitely locally free maps (see Tag 05WM). I want to know a corresponding statement for the étale and syntomic topologies.

34 Nathan References

(a) Junyan Cao

(b) BB in char p

(c) Moduli of products

35 Kodaira Vanishing In the Sense of Deligne-Illusie

Theorem 35.0.1. Let k be a field of characteristic p, and let X be a smooth projective k-scheme. Let L be an ample line bundle on X. If X has pure dimension $d < p$ and lifts over $W_2(k)$ then,

$$H^j(X, L \otimes \Omega^i_X) = 0 \quad i + j > d$$

Proof. By Serre vanishing we have,

$$H^j(X, L^{p^\ell} \otimes \Omega^i_X) = 0$$

for all $j > 0$ and $\ell \gg 0$. Therefore, by induction it suffices to prove the following

if M is a line bundle such that $H^j(X, M^{\otimes p} \otimes \Omega^i_X) = 0$ for all $i + j > d$ then $H^j(X, M \otimes \Omega^i_X) = 0$ for all $i + j > d$.

Note that $\mathrm{Frob}_X^* M \cong M^{\otimes p}$ via the p-linear map $M \to M^{\otimes p}$ given by $m \mapsto m^{\otimes p}$. Now $\mathrm{Frob}_X = F \circ F_k$ where $F_k : X^{(p)} \to X$ is the base change to X of $\mathrm{Frob}_k : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k)$. Therefore $F^* M' \cong M^{\otimes p}$ where $M' = F_k^* M$. Therefore,

$$M' \otimes F_* \Omega^i_X = F_*(F^* M' \otimes \Omega^i_X) = F_*(M^{\otimes p} \otimes \Omega^i_X)$$

hence since F is finite (alternatively because it is a homeomorphism),

$$H^j(X, M^{\otimes p} \otimes \Omega^i_X) = H^j(X^{(p)}, F_*(M^{\otimes p} \otimes \Omega^i_X)) = H^j(X^{(p)}, M' \otimes F_* \Omega^i_X)$$

and by assumption these are zero for $i + j > d$. However, there is a spectal sequence computing the hypercohomology of $M' \otimes F_* \Omega^\bullet_X$,

$$E_1^{i,j} = H^j(X^{(p)}, M' \otimes F_* \Omega^i_X) \implies H^{i+j}(X^{(p)}, M' \otimes F_* \Omega^\bullet_X)$$

and the vanishing hence implies the vanishing of $H^n(X^{(p)}, M' \otimes F_* \Omega^\bullet_X) = 0$ for $n > d$. But the Cartier operator induces a decomposition,

$$F_* \Omega^\bullet_X \cong \bigoplus_i \Omega^i_{X^{(p)}}[-i]$$

therefore

$$H^n(X^{(p)}, M' \otimes F_* \Omega^\bullet_X) \cong \bigoplus_{i+j=n} H^j(X^{(p)}, M' \otimes \Omega^i_{X^{(p)}})$$

and hence we get the vanishing,

$$H^j(X^{(p)}, M' \otimes \Omega^i_{X^{(p)}}) = 0 \quad i + j > d$$

Hence it just remains to transfer this information to X. Since F_k is flat, by flat base change,

$$F_k^* H^i(X, \mathscr{G}) \xrightarrow{\sim} H^i(X^{(p)}, F_k^* \mathscr{G})$$

for any coherent sheaf $\mathscr{G}$. Furthemore, by compatibility of Ω^i_X with base change we get,

$$F_k^*(M \otimes \Omega^i_X) = M' \otimes \Omega^i_{X^{(p)}}$$

and therefore we see,

$$F_k^* H^j(X, M \otimes \Omega^i_X) \xrightarrow{\sim} H^j(X^{(p)}, M' \otimes \Omega^i_{X^{(p)}}) = 0$$

completing the proof. $\square$

35.1 Frobenius Splitting

Definition 35.1.1. Let X be a scheme of characteristic $p > 0$. Then X is *globally F-split* if the Frobenius map $\mathcal{O}_X \to F_{X*}\mathcal{O}_X$ is left split.

Proposition 35.1.2. Let X be a globally F-split scheme and $\mathcal{L}$ an ample line bundle on X. Then $H^i(X, \mathcal{L}) = 0$ for all $i > 0$.

Proof. Indeed, by the projection formula the unit map $\mathcal{L} \to F_{X*}F_X^*\mathcal{L}$ is $- \otimes \mathcal{L}$ applied to $F_X^\#$ and hence is split. Therefore, there are injections,

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, F_{X*}F_X^*\mathcal{L}) = H^i(X, F_X^*\mathcal{L})$$

because F_X is affine. Furthermore $F_X^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$ via $s \mapsto s^{\otimes p}$. Iterating we get injections,

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L}^{\otimes p}) \hookrightarrow H^i(X, \mathcal{L}^{\otimes p^2}) \hookrightarrow \cdots$$

By Serre vanishing we see that $H^i(X, \mathcal{L}^{p^n}) = 0$ for $n \gg 0$ and hence $H^i(X, \mathcal{L}) = 0$. $\square$

Remark. The same argument shows that if X has Serre duality then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim X$.

Remark. If $f : X \to S$ is a morphism of schemes of characteristic p then $F_X = F_S \circ F_{X/S}$ factors as relative Frobenius $F_{X/S} : X \to X'$ followed by the Frobenius on S base changed to $F_S : X' \to X$. Therefore,

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr, "F_X^\#"] \arrow[d, equal] & & F_{X*}\mathcal{O}_X \arrow[d, equal] \\
\mathcal{O}_X \arrow[r, "F_S^\#"] & F_{S*}\mathcal{O}_{X'} \arrow[r, "F_{S*}F_{X/S}^\#"] & F_{S*}F_{X/S*}\mathcal{O}_X
\end{tikzcd}$$

Therefore, if $F_S^\#$ and $F_{X/S}^\#$ are split then X is globally F-split. Suppose that $F_{X/S}^\#$ is split and S is globally F-split. Let $r : F_{S*}\mathcal{O}_S \to \mathcal{O}_S$ be a retraction on S and consider,

$$\begin{tikzcd}
X' \arrow[r, "F_S"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S \arrow[r, "F_S"] & S
\end{tikzcd}$$

so $f^*r : f^*F_{S*}\mathcal{O}_S \to \mathcal{O}_X$ is a section of $f^*F_S^\# : \mathcal{O}_X \to f^*F_{S*}\mathcal{O}_S$ consider the diagram (the left square is just given by commutativity of the above square via adjunction f)

$$\begin{tikzcd}
f^*\mathcal{O}_S \arrow[r, "f^*F_S^\#"] \arrow[d, equal] & f^*F_{S*}\mathcal{O}_S \arrow[r, "f^*r"] \arrow[d] & f^*\mathcal{O}_S \arrow[d, equal] \\
\mathcal{O}_X \arrow[r, "F_{S*}^\#"] & F_{S*}\mathcal{O}_{X'} \arrow[r, dashed] & \mathcal{O}_X
\end{tikzcd}$$

the composition along the top row is id so if the dashed map exists then X is globally F-split. This happens when the natural map $f^*F_{S*}\mathcal{O}_S \to F_{S*}\mathcal{O}_{X'}$ is an isomorphism which occurs when f is flat or S is regular so F_S is finite locally free.

35.2 Bott Vanishing

Proposition 35.2.1. Let $f : X \to S$ be smooth of relative dimension n with X, S schemes of characteristic $p > 0$. Suppose that X is DO I NEED F-lifting and $\mathcal{L}$ an ample line bundle on X. Then $H^j(X, \Omega^i_X \otimes \mathcal{L}) = 0$ for all $j > 0$ and $i \geq 0$.

Proof. $\square$

36 Total Complex

Given a double complex $C^{\bullet,\bullet}$ with differentials $\mathrm{d}_A^{p,q}$ increasing p and $\mathrm{d}_B^{p,q}$ increasing q such that,

(a) $\mathrm{d}_A^{p+1,q} \circ \mathrm{d}_A^{p,q} = 0$

(b) $\mathrm{d}_B^{p,q+1} \circ \mathrm{d}_B^{p,q} = 0$

(c) $\mathrm{d}_A^{p,q+1} \circ \mathrm{d}_B^{p,q} = \mathrm{d}_B^{p+1,q} \circ \mathrm{d}_A^{p,q}$

then we construct the Total complex,

$$\mathrm{Tot}(C^{\bullet,\bullet})^k = \bigoplus_{p+q=k} C^{p,q}$$

The differential is given by,

$$\mathrm{d}_C^k = \sum_{p+q=k} [\mathrm{d}_A^p \otimes \mathrm{id}_B + (-1)^p \mathrm{id}_A \otimes \mathrm{d}_B^q]$$

such that,

$$\begin{aligned}(\mathrm{d}_C^{k+1} \circ \mathrm{d}_C^k)|_{C^{p,q}} &= [\mathrm{d}_A^{p+1,q} + (-1)^{p+1}\mathrm{d}_B^{p+1,q}] \circ \mathrm{d}_A^{p,q} \\ &\quad + [\mathrm{d}_A^{p,q} + (-1)^p \mathrm{d}_B^{p,q+1}] \circ (-1)^p \mathrm{d}_B^{p,q}] \\ &= (-1)^{p+1}\mathrm{d}_B^{p+1,q} \circ \mathrm{d}_A^{p,q} + (-1)^p \mathrm{d}_A^{p,q+1} \circ \mathrm{d}_B^{p,q} \\ &= 0\end{aligned}$$

Given a morphism $\varphi^{\bullet,\bullet} : C^{\bullet,\bullet} \to D^{\bullet,\bullet}$ of double complexes then there is a morphism,

$$\varphi^\bullet : \mathrm{Tot}(C^{\bullet,\bullet}) \to \mathrm{Tot}(D^{\bullet,\bullet})$$

given by φ^k is the sum over $\varphi^{p,q}$ for $p + q = k$. It is clear that,

$$\mathrm{d}_C^k \circ \varphi^k|_{C^{p,q}} = \mathrm{d}_A^{p,q} \circ \varphi^{p,q} + (-1)^p \mathrm{d}_B^{p,q} \circ \varphi^{p,q} = \varphi^{p+1,q} \circ \mathrm{d}_A^{p,q} + (-1)^p \varphi^{p,q+1} \circ \mathrm{d}_B^{p,q} = \varphi^k \circ \mathrm{d}_C^k|_{C^{p,q}}$$

so that this determines a map of complexes between the total complexes. A bounded double complex induces a spectral sequence,

$$E_0^{p,q} = C^{p,q} \implies H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$$

Flipping the indices produces a second spectral sequence as well. Then the morphism of double complexes induces a morphism of spectral sequences as well,

$$\varphi : {}^C E_\bullet^{\bullet,\bullet} \to {}^D E_\bullet^{\bullet,\bullet}$$

convering to the map induced on the cohomology by $\varphi^\bullet$.

Proposition 36.0.1. Let $\varphi_A : A_1^\bullet \to A_2^\bullet$ and $\varphi_B : B_1^\bullet \to B_2^\bullet$ be quasi-isomorphisms where the $A_i^\bullet$ are bounded above complexes of flat modules. Then there is a quasi-isomorphism,

$$(\varphi_A \otimes \varphi_B) : \mathrm{Tot}(A_1 \otimes B_1) \to \mathrm{Tot}(A_2 \otimes B_2)$$

Proof. Indeed, let $C_i = \mathrm{Tot}(A_i \otimes B_i)$. Then consider the map $C_1^k \to C_2^k$ given by,

$$\bigoplus_{p+q=k} A_1^p \otimes B_1^q \to \bigoplus_{p+q=k} A_2^p \otimes B_2^q$$

which is the diagonal matrix given by $\varphi_A^p \otimes \varphi_B^q$. The differential is given by,

$$\mathrm{d}_C^k = \sum_{p+q=k} [\mathrm{d}_A^p \otimes \mathrm{id}_B + (-1)^p \mathrm{id}_A \otimes \mathrm{d}_B^q]$$

Furthermore, it is clear that,

$$\mathrm{d}_C^k|_{C^{p,q}} \circ (\varphi_A^p \otimes \varphi_B^q) = (\varphi_A^{p+1} \otimes \varphi_B^q) \circ \mathrm{d}_C^k|_{C^{p,q}} + (\varphi_A^p \otimes \varphi_B^{q+1}) \circ \mathrm{d}_C^k|_{C^{p,q}}$$

We apply the spectral sequences, computing the map on cohomology of the total complex,

$${}^{C_1}E_2^{p,q} \to {}^{C_2}E_2^{p,q}$$

is given by the map,

$$H^p(H^q(A_1^\bullet \otimes B_1^\bullet)) \to H^p(H^q(A_2^\bullet \otimes B_2^\bullet))$$

since A_1^p and A_2^p are flat we can pull them out of the homology to get,

$$H^p(A_1^\bullet \otimes H^q(B_1^\bullet)) \to H^p(A_2^\bullet \otimes H^q(B_2^\bullet))$$

but the map $\varphi_B^q : H^q(B_1^\bullet) \to H^q(B_2^\bullet)$ is an isomorphism by assumption. Then we conclude this is an isomorphism of spectral sequences on the E_2-page and hence gives an isomorphism of the filtered parts on the E_∞-page. □

Lemma 36.0.2. Let $\varphi : A_1^\bullet \to A_2^\bullet$ be a quasi-isomorphism of bounded above complexes of flat modules and B any module. Then $\varphi \otimes \mathrm{id} : A_1^\bullet \otimes B \to A_2^\bullet \otimes B$ is a quasi-isomorphism.

Proof. By considering the cone construction, it suffices to take $A_2^\bullet = 0$ and $A_1^\bullet$ exact. Let $C^\bullet = A_1^\bullet$ be an exact complex. We first show that all kernels and images are flat by induction on sequences of the form,

$$0 \to K^n \to C^n \to C^{n+1} \to 0$$

to get that K^n is flat and then,

$$0 \to K^i \to C^i \to K^{i+1} \to 0$$

implies that K^i is flat assuming K^{i+1} is flat. Therefore, $- \otimes B$ preserves exactness of each sequence and hence of $C^\bullet$. □

Remark. The same argument shows that this holds for any bounded below (resp. above) complex of F-acyclic modules if F is a left (resp. right) exact additive functor.

37 Jumping image dimension of fibers

The general setup is the following. Let S be a scheme and X, Y finite presentation S-schemes. Given a morphism $f : X \to Y$ of S-schemes, how does the function $I_f : s \mapsto \dim \operatorname{im} f_s$ behave?

Example 37.0.1. In general, this function can jump either up or down.

(a) consider $\mathbb{A}^2 \to \mathbb{A}^2$ over $\mathbb{A}^1$ given by $(x, y) \mapsto (x, xy)$ then clearly the image is 1-dimensional over $x \neq 0$ and 0-dimensional over $x = 0$.

(b) $\mathrm{id} : X \to X$ for any nonflat scheme over S will usually have jumping dimension of the image $X \to S$.

Proposition 37.0.2. If $X \to S$ is flat then I_f is lower semi-continuous.

Proof. Since $Y \to S$ is finitely presented so by definition qcqs its diagonal is finitely presented thus since $X \to S$ is fintiely presented by the usual lemma $f : X \to Y$ is finite presentation. Since we need to prove that certain preimages of I_f are open we can reduce to checking on affine opens of S. Therefore by noetherian approximation and spreading out we reduce to the setting that $S = \operatorname{Spec}(A)$ with A a noetherian ring (since the fiber dimension does not change under a base change of fields). By Chevallay's theorem $f(X) \subset Y$ is constructible hence I_f is constructible (the level sets are images of the set of points where $\dim_x(f(X) \cap Y_s) = n$ which is constructible since $f(X)$ is a union of locally closed subschemes). Hence it suffices to show that I_f drops under specialization. Since S is noetherian we can base change to a dvr R and reduce to checking that if $f : X_R \to Y_R$ is an morphism of flat finite type R-schemes then $\dim \operatorname{im} f_K \geq \dim \operatorname{im} f_\kappa$. We can assume that X_R and Y_R are irreducible since by flatness X_R has no component living only in the special fiber and once X_R is irreducible it must land in an irreducible component of Y_R. Then by taking the scheme theoretic image we may assume that $X_R \to Y'$ is dominant and taking the reduction we arrive at the case $X \to Y$ over R is a map of integral schemes. Thus $Y \to R$ is flat because its generic point is the image of the generic point of X which lives over the generic point of $\operatorname{Spec}(R)$ and Y is integral. Thus either one of the generic points of X or Y only hits the generic point of $\operatorname{Spec}(R)$ in which case $\operatorname{im} f_\kappa = \varnothing$ and we win or $\dim Y_K = \dim Y_\kappa$ and $\dim X_K = \dim X_\kappa$ and $\operatorname{im} f_K = \dim Y_K$ and trivially $\dim \operatorname{im} f_\kappa \leq \dim Y_\kappa = \dim Y_K = \operatorname{im} f_K$ so we win. □

Proposition 37.0.3. If $X \to S$ is proper and $Y \to S$ is separated then I_f is upper semicontinuous.

Proof. By the standard lemma, $f : X \to Y$ is proper so $f(X)$ is closed. I claim that endowing $f(X)$ with the reduced structure, $f(X) \to S$ is proper. Given the claim we simply apply the following semi-continuity result to $f(X) \to S$. To show the claim, $f(X) \to S$ is universally closed because its formation commutes with base change (since it is the set-theoretic image, see here) and it is separated as a closed subscheme of Y. □

Now we use,

Theorem 37.0.4 (EGA IV 13.1.3)**.** Let $f : X \to Y$ be a morphism of schemes locally of finite type. Then,

$$x \mapsto \dim_x(X_{f(x)})$$

is upper semi-continuous.

Corollary 37.0.5. Let $f : X \to Y$ be closed and locally of finite type. Then

$$y \mapsto \dim(X_y)$$

is upper semi-continuous.

Proof. Indeed, consider the sets,

$$E_n = \{y \in Y \mid \dim(X_y) \geq n\}$$

we need to show that these are closed. Notice that $E_n = f(D_n)$ where,

$$D_n = \{x \in X \mid \dim_x(X_{f(x)}) \geq n\}$$

because $y \in E_n$ if and only if there is some $x \in X_y$ so that $\dim_x(X_y) \geq n$. Therefore, by closedness of f we conclude. $\square$

Corollary 37.0.6. If $f : X \to Y$ is map of finitely presented S-schemes with $X \to S$ flat and proper and $Y \to S$ separated then I_f is locally constant.

38 Stuff about Group Schemes

Proposition 38.0.1. Let G/S be a locally finite type group scheme. Then,

$$s \mapsto \dim(G_s)$$

is upper semi-continuous.

Proof. Let $e : S \to G$ be the identity section. Since $\dim(G_s) = \dim_{e(s)}(G_s)$ because over a field the dimension of a group scheme is the dimension of the identity component we have that the desired function is the composition of the continuous function $s \mapsto e(s)$ and the upper semi-continuous function $g \mapsto \dim_g(G_{\pi(g)})$. $\square$

38.1 Andres' results

Proposition 38.1.1. Let $m \circ f : X \to Y \to T$ be morphisms of schemes. We assume that,

(a) X is qcqs and Y is noetherian

(b) X and Y are integral, Y is normal

(c) $f : X \to Y$ is separated, of finite type, surjective, and with geometrically connected fibers

(d) for every[5] point $t \in T$, the morphism $f_t : X_t \to Y_t$ is proper

then f is proper.

Proof. This paper Proposition 1.6. $\square$

[5]In the paper it says *closed* points $t \in T$ but Andres says we really need all points in this level of generality (indeed otherwise we get nonsense over a DVR)

Definition 38.1.2. Let $G \to S$ be a group scheme locally of finite type. Define $G^\circ \subset G$ to be the subbundlector of morphisms of S-schemes $T \to G$ such that for all $s \in S$ the image of $T_s \to G_s$ lands in the neutral component.

Theorem 38.1.3 (SGA3 Exp. 6b, Cor. 4.4)**.** Let $G \to S$ be a group scheme locally of finite presentation. If all the fibers G_s are smooth and the dimension of G_s is locally constant, then the subfunctor G° of connected components is represented by an open subgroup scheme of G. Furthermore, if S is reduced then $G^\circ \to S$ is smooth.

Corollary 38.1.4. This immediately implies the smoothness of $\mathrm{Pic}^\circ_{X/S}$ over a reduced base if we know existence of $\mathrm{Pic}_{X/S}$ as a scheme and $R^1 f_* \mathcal{O}_X$ is a vector bundle whose formation commutes with base change (eg when the fibers have du Bois singularities). Does this need normality anywhere, I think we just need that the fibers are geometrically reduced and connected.

Theorem 38.1.5. Let $f : X \to S$ be a morphism of finite type schemes over an algebraically closed field k. Supose that,

(a) S is geometrically unibranch

(b) X is equidimensional of dimension $\dim S + r$

(c) $\Omega_{X/S}$ is locally free of rank r

then f is smooth.

What does he mean: "in the absense of properness or having a section"

Proof. □

39 Pseudo-effective

Definition 39.0.1. A divisor class $D \in N^1(X)_{\mathbb{R}}$ is *pseudo-effective* if it is in the closure of the cone of effective divisors.

Definition 39.0.2. A class $\alpha \in N_1(X)_{\mathbb{R}}$ is *movable* if $\alpha \cdot D \geq 0$ for any effective Cartier divisor D.

Proposition 39.0.3. If D is pseudo-effective if and only if $D \cdot \alpha \geq 0$ for all movable classes α.

Proof. If D is pseudo-effective then by definition,

$$D = \lim_{t \to 0} D_t$$

for D_t effective $\mathbb{R}$-divisors. If α is movable then by definition $D_t \cdot \alpha \geq 0$ for $t > 0$. Since intersection products are continuous (they are really polynomials in the coefficients) we have $D \cdot \alpha \geq 0$. The coverse holds for duals of cones in finite-dimensional vector spaces. Indeed, if D is not pseudo-effective, the separating hyperplane theorem ensures the existence of a numerical curve class α such that $E \cdot \alpha \geq 0$ on all effective divisors, i.e. α is movable, but $D \cdot \alpha < 0$. □

40 TODO

(a) Ambro: Prop. 4.4 does this extension property give an alternative proof the Isom step in relative BB

(b) read Ambro Prop. 3.5 also for relative trivial.

(c)

41 Galois Reps

Huristic: let a_p^i be the coefficients of the characteristic polynomial of Frob_p of a d-dimensional weight 1 representation. Then RH implies,

$$-\binom{d}{i} p^{i/2} \le a_p^i \le \binom{d}{i} p^{i/2}$$

For the newton polygon to be a straight line we are asking that $p^{\lceil i/2 \rceil} \mid a_p^i$ for all $1 \le i \le d$.

42 Rational Singularities (WIP)

Proposition 42.0.1. Let $f : X \to Y$ be a finite surjective morphism. Let Y be normal and X have rational singularities. Then Y has rational singularities.

Proof. Consider the resolutions,

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "a"] & Y' \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\mathbf{R}a_* \mathcal{O}_{X'} = \mathcal{O}_X$ because X has rational singularities. Then,

$$\mathbf{R}b_* \mathbf{R}f'_* \mathcal{O}_{X'} = \mathbf{R}f_* \mathcal{O}_X$$

Since f is finite $\mathbf{R}f_* \mathcal{O}_X = f_* \mathcal{O}_X$ and the natural map,

$$\mathcal{O}_Y \to \mathbf{R}b_* \mathbf{R}f'_* \mathcal{O}_{X'} = \mathbf{R}f_* \mathcal{O}_X$$

splits because of the trace map why if not flat?. We build f' as a finite map compose WHY? $\square$

43 Picard-Lefschetz Theorem

Theorem 43.0.1. Let $f : X \to \mathbb{P}^1$ be a holomorphic map from a $(k+1)$-dimensional complex manifold such that all critical points are *non-degenerate* and lie in different fibers. Let $x_1, \ldots, x_n$ denote the images in $\mathbb{P}^1$. The monodromy action $\pi_1(\mathbb{P}^1 \backslash \{x_1, \ldots, x_n\}) \circlearrowright H_k(Y_k)$ is described by the generator w_i of the loop around x_i acting via,

$$w_i \cdot \gamma = \gamma + (-1)^{(k+1)(k+2)/2} \langle \gamma, \delta_i \rangle \, \delta_i$$

where δ_i is the vanishing cycle of x_i. The action of monodromy on $H^{\neq k}(Y_x)$ is trivial.

The non-degeneracy is important. If we take a ramified cover $\mathbb{P}^1 \to \mathbb{P}^1$ of the base then the monodromy can be multiplied by the ramification order but the pullback of f will not be non-degenerate.

Example 43.0.2. Consider the Legendre family

$$y^2 = x(x-1)(x-\lambda)$$

This is *not* a non-degenerate family since we can write it as

$$y^2 = (x-1)[(x - \tfrac{1}{2}\lambda)^2 + \tfrac{1}{4}\lambda^2]$$

so when $\lambda \mapsto 0$ we get behavior $uv - \lambda^2$ which is semistable but the total space is not regular hence it is not a nondegenerate critical point. Correspondingly, we have to consider the pullback of this family locally for the map $\lambda = t^2$ and then multiply the monodromy by 2. Let α be the nearby cycle and δ the vanishing cycle. Then we have,

$$h(\delta) = \delta \qquad h(\alpha) = \alpha - 2\delta$$

44 Abelian Varities Theorems

44.1 Theorem of the Cube

Theorem 44.1.1. Let X, Y, Z be proper geometrically integral k-schemes with points $x_0 \in X(k)$ and $y_0 \in Y(k)$ and $z_0 \in Z(k)$. Suppose $\mathcal{L}$ is a line bundle on $X \times Y \times Y$ is trivial on $\{x_0\} \times Y \times Z$ and $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ then $\mathcal{L}$ is trivial.

Proof. Idea:

(a) we show there is a universal closed subscheme $Z' \subset Z$ where $\mathcal{L}$ is trivial on $X \times Y \times Z'$.

(b) show that Z' is open since the map

$$H^1(X \times Y, \mathcal{O}_{X \times Y}) \to H^1(X, \mathcal{O}_X) \oplus H^1(Y, \mathcal{O}_Y)$$

is injective so the deformation of $\mathcal{L}|_z$ over $X \times Y \times \{z\}$ are controlled by the deformation of $\mathcal{L}|_{X \times \{y_0\} \times \{z\}}$ which is trivial and the deformation of $\mathcal{L}|_{\{x_0\} \times Y \times Z}$ which is also trivial

(c) by assumption $z_0 \in Z'$ so $Z' = Z$ because Z is irreducible and Z' is clopen.

□

References

(a) Pedro A. Castillejo

(b) Stacks project

44.2 Theorem of the Square and Pullbacks of Line Bundles

Theorem 44.2.1 (See-Saw)**.** Let X and Y be geometrically integral proper k-schemes and $\mathcal{L}, \mathcal{M}$ be line bundles on $X \times Y$ which are isomorphic on all fibers over Y. If $\mathcal{L}_x \cong \mathcal{M}_x$ for some $x \in X(k)$ then $\mathcal{L} \cong \mathcal{M}$.

Proof. The first assumption implies that $\mathcal{L} \cong \mathcal{M} \otimes \pi_2^* \mathcal{N}$ but then $\mathcal{M}_x \cong \mathcal{L}_x \cong \mathcal{M}_x \otimes \mathcal{N}$ so $\mathcal{N} \cong \mathcal{O}_Y$. □

Proposition 44.2.2. Let A be an abelian variety. Let $I = \{i_1, \dots, i_r\} \subset \{1, 2, 3\}$. Denote $p_I : A^3 \to A$ be the morphism $(x_1, x_2, z_3) \mapsto x_{i_1} + \cdots + x_{i_r}$. Let $\mathcal{L}$ be a line bundle on A. Then,

$$\Theta(\mathcal{L}) := \bigoplus_{I \subset \{1,2,3\}} p_I^I \mathcal{L}^{\otimes (-1)^{1 + \# I}} = p_{123}^* \mathcal{L} \otimes p_{12}^* \mathcal{L}^{-1} \otimes p_{12}^* \mathcal{L}^{-1} \otimes p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L}$$

is trivial.

Proof. We just need to check that $\Theta(\mathcal{L})$ is trivial on $\{0\} \times A \times A$ etc. This is basically immediate. Let $j : \{0\} \times A \times A \to A^3$ then $j^* p_{123}^* = p_{23}^*$ etc so we get,

$$j^* \Theta(\mathcal{L}) = p_{23}^* \mathcal{L} \otimes p_2^* \mathcal{L}^{-1} \otimes p_3^* \mathcal{L}^{-1} \otimes p_{23}^* \mathcal{L}^{-1} \otimes \mathcal{O} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L} \cong \mathcal{O}$$

the other faces are idential. □

Corollary 44.2.3. Let Y be a scheme and A an abelian variety. For any $f, g, h : Y \to A$ and line bundle $\mathcal{L}$ on A the bundle

$$(f + g + h)^* \mathcal{L} \otimes (f + g)^* \mathcal{L}^{-1} \otimes (f + h)^* \mathcal{L}^{-1} \otimes (g + h)^* \mathcal{L}^{-1} \otimes f^* \mathcal{L} \otimes g^* \mathcal{L} \otimes h^* \mathcal{L}$$

on Y is trivial.

Proof. Consider $(f, g, h) : Y \to A^3$ then this bundle is $(f, g, h)^* \Theta(\mathcal{L})$ which is trivial. □

Corollary 44.2.4 (Theorem of the Square)**.** Let A be an aelian variety and $\mathcal{L}$ an line bundle. Then for all $x, y \in X(k)$

$$t_{x+y}^* \mathcal{L} \otimes \mathcal{L} \cong t_x^* \mathcal{L} \otimes t_y^* \mathcal{L}$$

Proof. Take $f = \mathrm{id}_X$ and g, h the constant maps at x, y respectively. □

Corollary 44.2.5. Consider the bundle on $A \times A$ given by,

$$\mathcal{M}_{\mathcal{L}} := m^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}$$

for some line bundle $\mathcal{L}$ on A. Then $\mathcal{M}_{\mathcal{L}}$ is trivial on $\{0\} \times A$ and $A \times \{0\}$ and hence defines a map $A \to A^\vee$ sending $0 \mapsto 0$. This is a homomorphism.

Proof. This follows immediately from before. But doesn't it also follow from the fact that any map $A \to B$ sending $0 \mapsto 0$ of abelian varities is automatically a group homomorphism? This is easy to prove by rigidity. □