1 Bertini over Finite Fields

Bertini smoothness. Let k be a field. We want $X \subset \mathbb{P}^n_k$ quasi-projective, has a property P then "most" hyperplanes $H_f \in (\mathbb{P}^n_k)^*$ have $X \cap H_f$ has property P.

1.1 Classical Bertini

Theorem 1.1.1. (Bertini for the case $k = \bar{k}$) and (k infinite Jouanolou). Let k be an infinite field and say that $X \subset \mathbb{P}^n_k$ is quasiprojective, smooth over k, and of dimension m. Then there exists a dense open $U \subset (\mathbb{P}^n_k)^*$ such that for all open $H \in U$ we have $X \cap U$ is smooth of dimension $m - 1$.

Proof. When $k = \bar{k}$. There are two steps. Step (1), smoothness if local, so characterize algebraically when $H \in (\mathbb{P}^n_k)^*$ has $H \cap X$ smooth of dim $n - 1$. First we consider the set of bad hyperplanes,

$B_x = \{H \in (\mathbb{P}^n_k)^* \mid H \cap X \text{ not smooth of dim } n-1 \text{ at } x\} = \{f \in H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \mid \mathrm{Proj}\,(S/(x)) \cap X \text{ ... }\}$

Set $V := H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ for $f_0 \in V$ with $x \notin H_{f_0}$ to dehomogenize then define,

$$\varphi_x : V \to \mathcal{O}_X x/\mathfrak{m}_x^2 \quad \text{via} \quad f \mapsto (f/f_0)_x$$

Then $x \in H_f \cap X$ iff $\varphi_x(f) \in \mathfrak{m}_x$ and $H_g \cap X$ is not regular at x iff $\varphi_x(f) = 0$ because then $\mathcal{O}_{X,x}/\varphi_x(f)$ would not be regular. Therefore $B_x = \mathbb{P}(\ker \varphi_x)$.

Step (2). Because $k = \bar{k}$ then φ_x is surjective because $\kappa(x) = k$. Then $\dim_k(V) = n + 1$ and,

$$\dim_k \mathcal{O}_{X,x}/\mathfrak{m}_x^2 = 1 + m$$

where $m = \dim X$. Therefore,

$$\dim B_x = \dim \ker \varphi_x - 1 = n - m - 1$$

Then let,

$$B = \{(x, H) \mid x \in X \text{ closed point and } H \in B_x\}$$

Then $B \subset X \times (\mathbb{P}^n_k)^*$ defines a closed subscheme. Then step (1) implies that $p_1 : B \to X$ has fibers isomorphic to $\mathbb{P}^{n-m-1}$ and thus is surjective and B is irreducible and $\dim B = (n-m-1)+m = n-1$. Therefore, the image of $p_2(B)$ cannot be dense because it has dimension $n - 1$ so the complement of its image is a dense open.

Now if k is infinite, a dense open $U \subset (\mathbb{P}^n_k)^*$ has a k-point so there exists $H_f \in (\mathbb{P}^n_k)^*(k)$ with $H_f \cap X$ smooth of dimension $m - 1$.

Therefore we pass to $\bar{k}$ and then U will be defined over some finite extension so the image of its complement is closed. □

Remark. However over $\mathbb{F}_q$ the set $\mathfrak{p}^n_{\mathbb{F}_q}(\mathbb{F}_q)$ is finite so there exsits a dense open containing no rational point.

Example 1.1.2 (Katz)**.** Bertini for hyperplanes fails over $\mathbb{F}_q$. Consider,

$$f = \sum_{i=1}^{n+1} (X_i Y_i^q - X_i^q Y_i)$$

and take $X = V(f) \subset \mathbb{P}^{2n+1}_{\mathbb{F}_q}$. Note that,

$$X(\mathbb{F}_q) = \mathbb{P}^{2n+1}_{\mathbb{F}_q}(\mathbb{F}_q)$$

because every rational point satisfies f. Now all $H_f \in (\mathbb{P}^{2n+1}_{\mathbb{F}_q})(\mathbb{F}_q)$ have $H_f \cap X$ not transverse.

Ideal: the dual variety $X \to (\mathbb{P}^{2n+1}_{\mathbb{F}_q})^*$ via $x \mapsto T_x X \subset T_x \mathbb{P}^n$ then its scheme theoretic image is the dual variety.

In our case,

$$\varphi : (X_i, Y_i) \mapsto (Y_i^q, -X_i^q) = \mathrm{Frob}(Y_i, -X_i)$$

so $\varphi : X \to X^*$ is an isomorphism. Therefore, since X contains every $\mathbb{F}_q$-rational point X^* also contains every $\mathbb{F}_q$-rational point and therefore it is tangent to every $\mathbb{F}_q$ hyperplane because it is isomorphic to its dual variety.

1.2 Poonen's Theorem

Remark. We need to increase somthing either the size of the field or the degree of the hypersurface. It is clear by passing to the algebraic closure that by doing a finite field extension we can get such a hyperplane. However, we want to stay over $\mathbb{F}_q$. Therefore we want to ask if Bertini works over $\mathbb{F}_q$ for large enough degree hyperplanes.

Setup,

$$S = \bigoplus_{d \ge 0} S_d = \mathbb{F}_q[x_0, \dots, x_n]$$

and let

$$S_{\text{homog}} = \bigcup_{d \ge 0} S_d$$

For each $f \in S_{\text{homog}}$ we get $H_f = \mathrm{Proj}\,(S/(f))$ a hypersurface. For $\mathcal{P} \subset S_{\text{homog}}$ we define a notion of density,

$$\mu(\mathcal{P}) = \lim_{d \to \infty} \frac{\#(\mathcal{P} \cap S_d)}{\# S_d}$$

Furthermore we define the upper and lower denity,

$$\overline{\mu}(\mathcal{P}) = \limsup_{d \to \infty} \frac{\#(\mathcal{P} \cap S_d)}{\# S_d}$$

and

$$\underline{\mu}(\mathcal{P}) = \liminf_{d \to \infty} \frac{\#(\mathcal{P} \cap S_d)}{\# S_d}$$

Furthermore, we define the Zeta function,

$$\zeta_X(s) = Z_X(q^{-s}) = \prod_{x \in X_{\text{closed}}} (1 - q^{-s \deg x})^{-1} = \exp\left(\sum_{r \ge 1} \frac{\# X(\mathbb{F}_{q^r})}{r} q^{-rs} \right)$$

Theorem 1.2.1 (Poonen)**.** Let $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ quasi-projective, smooth of dimension $n \ge 0$. Then,

$$\mathcal{P}_{\text{sm}} = \{ f \in S_{\text{homog}} \mid H_f \cap X \text{ smooth of dimension } n-1 \}$$

then,

$$\mu(\mathcal{P}_{\text{sm}}) = \zeta_X(m+1)^{-1}$$

Remark. The number $\zeta_X(m+1)^{-1}$ is nonzero and rational because Z_X is a rational function whose polls have $|\alpha| = p^{i/2}$ for $i \leq m$. This is why is converges.

Example 1.2.2. Let $X = \mathbb{A}^1_{\mathbb{F}_q}$. When if $V(f)$ smooth? It is smooth exactly when f is not squarefree. For each irred. poly g need $g^2 \nmid f$. This should happen with probability,

$$\left(1 - \left(\frac{1}{\#(k[x]/g)}\right)^2\right)$$

because under $k[x] \twoheadrightarrow k[x]/(g^2)$ we expect every residue to be equally likely and then,

$$\#k[x]/(g^2) = (\#k[x]/(g))^2$$

Therefore, we guess that,

$$\mu(\mathcal{P}) = \prod_{g \text{ irred}} \left(1 - (\#k[x]/(g))^{-2}\right) = \zeta_{\mathbb{A}^1}(2)^{-1}$$

Remark. More generally, let $f \in S_d$. Then for each closed $x \in X$ we have,

$$f \text{ singular} \iff m+1 \text{ linear conditions vanish over } \kappa(p)$$

Therefore, we expect this to happen with probability,

$$\left(q^{-\deg x}\right)^{m+1}$$

Therefore we guess,

$$\mu(f) = \prod_{\text{closed} x \in X} \left(1 - q^{-\deg x (m+1)}\right) = \zeta_X(m+1)^{-1}$$

To prove this we need "Sieve techniques" to rigorize, we need to "handle error terms".

Theorem 1.2.3 (Bertini with Taylor conditions)**.** For $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ as above. Together with the data $Z \subset \mathbb{P}^n_{\mathbb{F}_q}$ a finite subscheme such that $U = X \setminus (Z \cap Z)$. Fix $T \subset H^0(Z, \mathcal{O}_Z)$. For $f \in S_d$ we define $f|_Z \in H^0(Z, \mathcal{O}_Z)$ so that on each component $Z_i \subset Z$ we let $f|_Z$ is restriction of $x_j^{-d} f$ with j the smallest index of x_j invertible on Z_i. Then let,

$$\mathcal{P}_T = \{f \in S_{\text{homog}} \mid H_f \cap U \text{ smooth of dim } n-1 \text{ and } f|_Z \in T\}$$

Then we conclude,

$$\mu(\mathcal{P}) = \frac{\#T}{H^0(Z, \mathcal{O}_Z)} \zeta_U(m+1)^{-1}$$

Proof. Singularities of low degree (main term). Let,

$$U_{<r} = \{x \in U \text{ closed} \mid \deg x < r\}$$

likewise for $U_{>r}$ and $U_{=r}$.

We will then split up into low degree $U_{<r}$ and medium degree ($r < \deg x < \frac{d}{m+1}$) and high degree ($\frac{d}{n+1} < \deg x$). $\square$

Lemma 1.2.4. Define,

$$\mathcal{P}_r = \{f \in S_{\text{homog}} \mid H_f \cap U \text{ smooth of dim } m-1 \text{ at } x \text{ and } f|_Z \in T \text{ for all } x \in U_{<r}\}$$

Then,

$$\mu(\mathcal{P}_r) = \frac{\#T}{\#H^0(Z, \mathcal{O}_Z)} \prod \left(1 - q^{-(m+1)\deg x}\right)$$

Proof. Let $U_{<r} = \{x_1, \dots, x_s\}$. let $\mathfrak{m}_i \subset \mathcal{O}_U$ be the ideal sheaf of x_i. Let $Y_i = V(\mathfrak{m}_i^2)$. Set $Y = \bigcup Y_i$. Then,

$$\phi_d : S_d = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(Y, \mathcal{O}_Y(d))$$

Then $H_f \cap Y$ not smooth of dim $m-1$ at x iff $f \in \ker \phi_d$. Furthermore, ϕ_d is surjective for,

$$d \geq \dim H^0(Y, \mathcal{O}_Y) - 1$$

To see this, $\operatorname{coker} \phi_d \subset H^1(\mathbb{P}^n, \mathscr{I}_Y(d))$ this will vanish for $d \gg 0$ so its surjective for large enough d. Furthermore, if $B_j = \operatorname{im} \phi_j$ then,

$$B_{j+1} = B_j + \sum x_i B_j$$

so $B_{j+1} \supset B_j$. By the formula if $B_{j+1} = B_j$ for some j then it stabilizes at j. Therefore, it must stabilize after $j \geq \dim H^0(Y, \mathcal{O}_Y) - 1$ since it cant jump more times than dimensions in the codomain.

Then $\mathcal{P}_r \cap S_d$,

$$R_d : S_d \to H^0(Y \cap Z, \mathscr{I}_{Y \cup Z}(d)) \cong H^0(Z, \mathcal{O}_Z) \times \prod_{i=1}^{s} H^0(Y_i, \mathcal{O}_{Y_i})$$

Then,

$$\mathcal{P}_r \cap S_d = R_d^{-1}(T \times (H^0(Y_i, \mathcal{O}_{Y_i}) \setminus \{0\})$$

Therefore,

$$\mu(\mathcal{P}_r) = \frac{\#(\mathcal{P}_r \cap S_d)}{\#S_d} = \frac{\#R_d^{-1}(\mathcal{P}_d \cap S_d)}{\#H^0(Z, \mathcal{O}_Z) \times \prod H^0(Y_i, \mathcal{O}_{Y_i})} = \frac{\#T \cdot (\#H^0(Y_i, \mathcal{O}_{Y_i}) - 1)}{\#H^0(Z, \mathcal{O}_Z) \times \prod \#H^0(Y_i, \mathcal{O}_{Y_i})}$$

Note that,

$$H^0(Y_i, \mathcal{O}_{Y_i}) = \mathcal{O}_{X, x_i} / \mathfrak{m}_{x_i}^2$$

which has dimension $m+1$ over $\kappa(x_i)$. Therefore,

$$\mu(\mathcal{P}_r) = \frac{\#T}{\#H^0(Z, \mathcal{O}_Z)}$$

□

1.3 Consequences

Theorem 1.3.1 (Anti-Bertini)**.** There exists $X \subset \mathbb{P}_k^n$ such that $H_f \cap X$ is not smooth for any $f \in S_1 \cup \cdots \cup S_d$ for fixed d.

Theorem 1.3.2 (Space-Filling Curves)**.** For $X/\mathbb{F}_q$ nice (smooth projective geometrically integral) and $E/\mathbb{F}_q$ any finite extension there is a nice curve $Y \subset X$ s.t. $Y(E) = X(E)$.

Theorem 1.3.3. For any nice X there is a nice curve $Y \subset X$ such that $\operatorname{Alb}(Y) \to \operatorname{Alb}(X)$ is surjective