1 Prismatic Cohomology

Our goal will be the following theorem about the topology of algebraic varities.

Theorem 1.0.1. et X be a smooth, proper, $\mathbb{C}$-variety with unramified good reduction at p. Let $i < p - 2$ and $W \subset X$ and Zariki open. Then the image of the restriction map,

$$H^i(X, \mathbb{F}_p) \to H^i(W, \mathbb{F}_p)$$

has dimension at least $h^{0,i}_X := \dim H^0(X, \Omega^i_X)$.

This statement amounts to showing that certain cohomology classes are not p-divisible.

There is a version with $\mathbb{Q}$-coefficients that follows from Hodge theory.

Theorem 1.0.2. Let X be a smooth, proper, complex variety and $W \subset X$ any Zariki open. Then the image of the restiction map,

$$H^i(X, \mathbb{Q}) \to H^i(W, \mathbb{Q})$$

has dimension at least $h^{0,i}_X := \dim H^0(X, \Omega^i_X)$.

Proof. The map $H^i(X, \mathbb{Q}) \to H^i(W, \mathbb{Q})$ is a morphism of mixed hodge structures. Posibly passing to a log resolution $\pi : \widetilde{X} \to X$ of $Z = X \backslash W$ we may assume that $\pi^{-1}(Z) = D$ is an snc divisor (note the birational modification does not change $h^{0,i}_X$ and the map $H^i(\widetilde{X}, \mathbb{Q}) \to H^i(W, \mathbb{Q})$ factors through $H^i(X, \mathbb{Q})$ so its image is the same). Then there is a commutative diagram,

$$\begin{tikzcd}
H^0(\widetilde{X}, \Omega^i_{\widetilde{X}}) \arrow[r] \arrow[d] & H^0(\widetilde{X}, \Omega^i_{\widetilde{X}}(\log D)) \arrow[d] \\
H^i(\widetilde{X}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r] & \mathrm{Gr}^W_i H^i(W, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}
\end{tikzcd}$$

where the top map is injective and the downward maps are injective. This immediately implies the claim. ☐

The real power of our main result is that it works integrally. This has applications to essential dimension to be discussed later.

1.1 Mod-p Cohomology

We need the following about Delinge-Illusie's treatment of de Rham cohomology and basics of prismatic cohomology.

1.1.1 Log de Rham cohomology

Let k be a perfect field of characteritic p, and let X be a smooth k-scheme. Suppose that X is equipped with a normal crossings divisor $D \subset X$. Let $\Omega^\bullet_{X/k}(\log D)$ denote the de Rham complex with log poles in D.

Let (X^1, D^1) be the base change by Frobenius $F_k : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k)$ and $F_{X/k} : X \to X^1$ denote the relative Frobenius. It is a finite flat map (since X is smooth) of k-schemes such that $F_{X/k} : D \to D^1$.

Lemma 1.1.1. Suppose that (X, D) admits a lift to $W_2(k)$ called $(\widetilde{X}, \widetilde{D})$ with $\widetilde{D}$ a snc divisor flat over $W_2(k)$. Then for $j < p$,

$$H^0(X^1, \Omega^j_{X^1/k}(\log D^1)) \hookrightarrow H^j(X, \Omega^\bullet_{X/k}(\log D))$$

is canonically a direct summand.

Proof. This follows from the existence of the Cartier operator in the same way as in Deligne-Illusie. $\square$

1.1.2 Prisms

Let K be a field of characteristic 0. By a *p-adic valuation* on K we mean a rank one valuation ν on K, with $\nu(p) > 0$. We suppose that K is complete with respect to ν with ring of integers $\mathcal{O}_K$ and perfect residue field k. We will only recall exactly as much about prismatic cohomology as necessary.

Definition 1.1.2. A *δ-ring* is a pair (R, δ) where R is a commutative ring and $\delta : R \to R$ is a set map such that,

(a) $\delta(0) = \delta(1) = 0$

(b) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$

(c) $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$

Note that the last term exists as some universal polynomial with integer coefficents.

We think of this as a sort of "derivation along the p-direction". It is also related to lifting Frobenius on R/p. Indeed, if $\phi(x) = x^p + p\delta(x)$ then $\phi : R \to R$ is a ring map by property (c) and obviously it lifts $x \mapsto x^p$ on R/p. In fact, if R is p-torsionfree then lifts of Frobenius are exactly the same as δ-ring structures.

Definition 1.1.3. Let (A, I) be a pair where A is a δ-ring and $I \subset A$ is an ideal. The pair is a *prism* if

(a) $I \subset A$ is invertible (defines a Cartier divisor on $\mathrm{Spec}\,(A)$)

(b) A is derived (p, I)-complete

(c) $p \in I + \phi(I)A$

Example 1.1.4. Let A be a p-torsionfree and p-complete δ-ring then $(A, (p))$ is a prism.

Example 1.1.5. The *Breuil-Kisin* prism. Assume that ν on K is discrete. Set $A = W(k)[[u]]$ equipped with Frobenius φ extending Frobenius on $W(k)$ by $u \mapsto u^p$. Equip A with the map $A \to \mathcal{O}_K$ sending $u \mapsto \pi$ some uniformizer. It kernel is generated by an Eisenstein polynomial $E(u) \in W(k)[u]$ for π. In fact, in applications we will assume $\mathcal{O}_K = W(k)$ and $\pi = p$. Then $(A, E(u)A)$ is the Breuil-Kisin prism.

Example 1.1.6. Suppose that K is algebraically closed. Let $R = \varprojlim \mathcal{O}_K/p$ taking the limit over Frobenius. We take $A = W(R)$. Any element $(x_0, x_1, \dots) \in R$ lifts uniquely to a sequence $(\hat{x}_0, \hat{x}_1, \dots,) \in \mathcal{O}_K$ with $\hat{x}_i^p = \hat{x}_{i-1}$. Then there is a natural surjective map of rings $\theta : A \to \mathcal{O}_K$ sending a Teichmuller element x as above to $\hat{x}_0$. The kernel of θ is principal, generated by $\xi = p - [\underline{p}]$ where $\underline{p} = (p, p^{1/p}, \dots)$ then $(A, \xi A)$ is an example of a perfect prism.

1.1.3 Logarithmic Cohomology

We will use logarithmic formal schemes over $\mathcal{O}_K$. We will consider logarithmic étale cohomology meaning the natural cohomology on the site of log étale covers of logarithmic schemes. The main fact we will use is the following comparison result:

Theorem 1.1.7. Let k be an algebraically closed field and X a smooth k-scheme. Let $D \subset X$ be an snc divisor and $X_D^{\log}$ the log structure induced by D. Then there is a canonical isomorphism,

$$H^i_{\text{ét}}(X_D^{\log}, \mu) \xrightarrow{\sim} H^i(X \backslash D, \mu)$$

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Proof. Idea: show that any finite étale map $Y \to X \backslash D$ extends canonically to a finite log-étale map $\overline{Y} \to X_D$ which proves the statment for $i = 1$ then use dimension shifting and some spectral sequence. To show the claim, take the normalization of Y in X which gives a finite map $Y \to X$ ramified only over D by Zariski nagata purity. Then a local check shows that this map is log-étaleWHY? □

1.1.4 Prismatic Cohomology

Let K be either discretely valued or algebraically closed. Let X be a formal smooth $\mathcal{O}_K$-scheme equipped with a relative normal crossings divisor D. Write X_D for log structure induced by D. We will denote by $X_{D,K}$ the associated log adic space giving by analytification.

The *prismatic cohomology* of X_D is the complex of A-modules $R\Gamma_{\Delta}(X_D/A)$ equipped with a φ-semi-linear map φ. The mod p cohomology is given by setting,

$$\overline{R\Gamma_{\Delta}(X_D/A)} = R\Gamma_{\Delta}(X_D/A) \otimes^{\mathbb{L}}_A A/pA$$

and we will denote by $\overline{H^i_{\Delta}(X_D/A)}$ the cohomology of $\overline{R\Gamma_{\Delta}(X_D/A)}$. Then we have the following properties:

(a) There is a canonical isomorphism of commutative algebras in $D(A)$

$$R\Gamma(\Omega^\bullet_{X_k/k}(\log D_k)) \cong \overline{R\Gamma_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_{A/pA,\varphi} l$$

(b) If K is algebraically closed then there is an isomorphism of commutative algebras in $D(A)$

$$R\Gamma_{\text{ét}}(X_{D,K}, \mathbb{F}_p) \cong \overline{R\Gamma_{\Delta}(X_D/A)}[1/\xi]^{\varphi=1}$$

(c) the linear map,

$$\varphi^* \overline{R\Gamma_{\Delta}(X_D/A)} \to \overline{R\Gamma_{\Delta}(X_D/A)}$$

becomes an isomorphism in $D(A)$ after inverting u (resp ξ) if K i discrete (resp. algebraically closed). For each $i \geq 0$, there is a canonical map,

$$V_i : \overline{H^i_{\Delta}(X_D/A)} \to H^i(\varphi^* \overline{R\Gamma_{\Delta}(X_D/A)}$$

(d) Let K' be a field complete with respect to a p-adic valuation, and which is either discrete or algebraically closed. Let $B \to \mathcal{O}_{K'}$ be the corresponding prism, as defined above. Suppose $K \to K'$ is a map of valued field and $A \to B$ is compatbile with the projection to $\mathcal{O}_K \to \mathcal{O}_{K'}$ and Frobenius. Then there is a canonical isomorphism

$$\overline{R\Gamma_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A B \cong \overline{R\Gamma_{\Delta}(X_{D,\mathcal{O}_{K'}}/B)}$$

(e) When X is proper over $\mathcal{O}_K$ then $\overline{R\Gamma_{\Delta}(X_D/A)}$ is a perfect complex of A/p-modules.

(f) Suppose that K is algebraically closed, and that X is proper over $\mathcal{O}_K$ then for each $i \geq 0$ there are natural isomorphisms

$$H^i_{\text{ét}}(X_{D,K}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} A/pA[1/\xi] \cong \overline{H^i_{\Delta}(X_D/A)}[1/\xi]$$

1.2 Main Result

Let k be a perfect field of characteristic p. Here we can take K to be a complete p-adic field with discrete valuation such that $\mathcal{O}_K = W(k)$.

Proposition 1.2.1. Let X be a proper smooth scheme over $\mathcal{O}_K$ equipped with a relative normal crossings divisor $D \subset X$. Set $U = X \backslash D$ and $W \subset U_C$ be a dense open subscheme. If $0 \leq i < p-2$ then,

$$\dim_{\mathbb{F}_p} \mathrm{im}\,(H^i_{\text{ét}}(U_C, \mathbb{F}_p) \to H^i_{\text{ét}}(W, \mathbb{F}_p)) \geq h^{0,i}_{(X_C, D_C)}$$

Let's see how this implies the theorem. Let Y be a proper smooth scheme over $\mathbb{C}$ and $D \subset Y$ a normal crossings divisor. We say that (Y, D) has *good reduction at p* if there exists an algebraically closed field $C \hookrightarrow \mathbb{C}$ over which (Y, D) is defined and a p-adic valuation on C with ring of integers $\mathcal{O}_C$ and an extension to a smooth proper $\mathcal{O}_C$-scheme Y° with a relative normal crossings divisor $D^\circ \subset Y^\circ$ over $\mathcal{O}_C$ extending D. We say that (Y, D) has *unramified good reduction* at p if in addition (Y°, D°) can be chosen so that it descends to an absolutely unramified[1] dvr $\mathcal{O} \subset \mathcal{O}_C$.

Remark. This condition is actually easily checkable. Indeed if Y is a smooth proper finite type $\mathbb{C}$-scheme then it spreads out to a smooth proper scheme $\mathcal{Y} \to \mathrm{Spec}\,(A)$ over some finite type $\mathbb{Z}$-algebra $A \subset \mathbb{C}$. Now suppose there exists $\mathfrak{p} \subset A$ such that $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth at $\mathfrak{p}$ and $\mathfrak{p} \mapsto (p)$. This is nothing more than saying that p is not contained in the Jacobian ideal. Then choose a minimal prime ξ over pA since $\xi \rightsquigarrow \mathfrak{p}$ we see that $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth at ξ and hence $A_\xi \subset \mathbb{C}$ is a p-adic dvr unramified over $\mathbb{Z}_{(p)}$ by smoothness. Then we extend this p-adic valuation to $\mathbb{C}$ and $\mathcal{O} = A_\xi$ is our requisite unramified dvr.

Corollary 1.2.2. Let Y be a proper smooth connected $\mathbb{C}$-scheme and $D \subset Y$ a normal crossing divisor and $W \subset U := Y \backslash D$ a dense ope nsubscheme. Suppose that (Y, D) has unramified good reduction at p. If $0 \leq i < p-2$ then,

$$\dim_{\mathbb{F}_p} \mathrm{im}\,(H^i_{\text{ét}}(U, \mathbb{F}_p) \to H^i_{\text{ét}}(W, \mathbb{F}_p)) \geq h^{0,i}_{(X,D)}$$

This proves the main theorem if we take $D = \emptyset$.

Proof. Since the étale cohomology groups do not change upon base change to algebraically closed fields. By assumption, we may assume that (Y, D) is defined over $\mathcal{O}$ unramified. Then taking the p-adic completion $C \subset C'$ we get $\mathcal{O} \subset \mathcal{O}'$ which is unramified and p-adically complete so we reduce to the previous case. ☐

Proof of Proposition 4.4.1. Let k_C be the residue field of C. We may replace X by it base change to $W(k_C)$ and assume that C and K have the ame residue field. Denote by $\widehat{X}$ and $\widehat{D}$ the formal completions of X and D. Let $\widehat{W} \subset \widehat{X}$ be the formal open subschem, which is the complement of Z_k. Note that we have $\widehat{W}_C \subset W^{\mathrm{ad}}$ so there is a commutative diagram,

[1] meaning unramified over $\mathbb{Z}_{(p)}$

$$\begin{tikzcd}
H^i_{\text{ét}}(X_{D,C},\mathbb{F}_p) \arrow[r] \arrow[dd, "\alpha"] & H^i_{\text{ét}}(W,\mathbb{F}_p) \arrow[d] \\
& H^i_{\text{ét}}(W^{\text{ad}},\mathbb{F}_p) \arrow[d] \\
H^i(\widehat{X}_{D,C},\mathbb{F}_p) \arrow[r, "\beta"] & H^i_{\text{ét}}(\widetilde{X}_C,\mathbb{F}_p)
\end{tikzcd}$$

We need to show the following facts,

(a) α is an isomorphism

(b) $\dim_{\mathbb{F}_p} \operatorname{im} \beta \geq h^{0,i}_{(X,D)}$

(c) $H^i_{\text{ét}}(X_{D,C},\mathbb{F}_p) \cong H^i_{\text{ét}}(U_C,\mathbb{F}_p)$

□

WHY IS THE FIRST LEMMA 2.2.10

Now we will prove these three facts.

The only hard one:

Let X be a proper, smooth formal scheem over $\mathcal{O}_K$ equipped with a relative normal crossing divisor $D \subset X$. Let,

$$h^{0,i}_{(X,D)} := \dim_K H^0(X_K, \Omega^i_{X_K/K}(\log D))$$

Proposition 1.2.3. Let $W \subset X \backslash D$ be a dense open formal subscheme. Then for $0 \leq i < p-2$

$$\dim_{\mathbb{F}_p} \operatorname{im}\left(H^i_{\text{ét}}(X_{D,C},\mathbb{F}_p) \to H^i_{\text{ét}}(W_C,\mathbb{F}_p) \geq h^{0,i}_{(X,D)}\right.$$

Proof. Take the prism A to be $W(k)[[u]]$ with $E(u) = u - p$. We obtain a prism $A_C \to \mathcal{O}_C$. There is a Frobenius compatible map $A \to A_C$ sending $u \mapsto [\underline{p}]$. Set,

$$M_{\Delta} = \operatorname{im}\left(\overline{H^i_{\Delta}(X_D/A)} \to \overline{H^i_{\Delta}(W/A)}\right)$$

which is a finitely generated $A/pA = k[[u]]$-module. There is an isomorphism,

$$\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C \xrightarrow{\sim} \overline{H^i_{\Delta}(X_{D,\mathcal{O}_C}/A_C)}$$

and similarly for the open W. Therefore, by PROPERTY there is an isomorphism

$$\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C \xrightarrow{\sim} \overline{H^i_{\Delta}(X_{D,\mathcal{O}_C}/A_C)}$$

Then there are maps,

$$\begin{aligned}
\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C[1/\xi] &\cong H^i_{\text{ét}}(X_{D,C},\mathbb{F}_p) \otimes_{\mathbb{F}_p} A_C/pA_C[1/\xi] \\
&\to H^i_{\text{ét}}(W_C,\mathbb{F}_p) \otimes_{\mathbb{F}_p} A_C/pA_C[1/\xi] \to \overline{H^i_{\Delta}(W/A)} \otimes_A A_C[1/\xi]
\end{aligned}$$

the composite is the natural map. Hence,

$$\dim_{\mathbb{F}_p} \operatorname{im}\left(H^i_{\text{ét}}(X_{D,C},\mathbb{F}_p) \to H^i_{\text{ét}}(W_C,\mathbb{F}_p)\right) \geq \dim_{k((u))} M_{\Delta}[1/u]$$

By LEMMA M_{Δ} is a finitely generated free $k[[u]]$-module. Hence it suffices to show $\dim_k M_{\Delta}/uM_{\Delta} \geq h^{0,i}_{(X,D)}$.

Hence using Lemma 2.2.1 again, we see that $\overline{H^j_{\Delta}(X_D/A)}$ is u-torsion free for $0 \leq j \leq i+1$. Hence there are maps,

$$\begin{aligned} H^i(X_k, \Omega^\bullet_{X_k/k}(\log D_k)) \cong \overline{H^i_{\Delta}(X_D/A)} \otimes_{A,\varphi} k \to M_{\Delta} \otimes_{A,\varphi} k \\ \to \overline{H^i_{\Delta}(W/A)} \otimes_{A,\varphi} k \to H^i(W_k, \Omega^\bullet_{W_k/K}(\log D)) \end{aligned}$$

where the composition is the natural map. This shows that the image has dimension $\leq \dim_k M_{\Delta}/uM_{\Delta}$ and it suffices to how that this dimension is $\geq h^{0,i}_{(X,D)}$. Since $W \subset X$ is dense, the map,

$$H^0(X_k, \Omega^i_{X_k/k}(\log D)) \to H^0(W_k, \Omega^i_{X_k/k})$$

is injective. Hence the image has dimension at leat $\dim_k H^0(X_k, \Omega^i_{X_k/k}(\log D_k)) \geq h^{0,i}_{(X,D)}$ I THINK THIS WAS AN ERROR IN THE PAPER NEED LOG D where the last inequality follows from the upper semi-continuity of h^0. $\square$

2 Talk 1

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has dimension at least $h^{0,i}_X := \dim H^0(X, \Omega^i_X)$.

This statement amounts to showing that certain cohomology classes are not p-divisible.

There is a version with $\mathbb{Q}$-coefficients that follows from Hodge theory.

Theorem 2.0.2. Let X be a smooth, proper, complex variety and $W \subset X$ any Zariki open. Then the image of the restiction map,

$$H^i(X, \mathbb{Q}) \to H^i(W, \mathbb{Q})$$

has dimension at least $h^{0,i}_X := \dim H^0(X, \Omega^i_X)$.

Proof. The map $H^i(X, \mathbb{Q}) \to H^i(W, \mathbb{Q})$ is a morphism of mixed hodge structures. Posibly passing to a log resolution $\pi : \widetilde{X} \to X$ of $Z = X \setminus W$ we may assume that $\pi^{-1}(Z) = D$ is an snc divisor (note the birational modification does not change $h^{0,i}_X$ and the map $H^i(\widetilde{X}, \mathbb{Q}) \to H^i(W, \mathbb{Q})$ factors through $H^i(X, \mathbb{Q})$ so its image is the same). Then there is a commutative diagram,

$$\begin{tikzcd} H^0(\widetilde{X}, \Omega^i_{\widetilde{X}}) \arrow[r] \arrow[d] & H^0(\widetilde{X}, \Omega^i_{\widetilde{X}}(\log D)) \arrow[d] \\ H^i(\widetilde{X}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r] & \mathrm{Gr}^W_i H^i(W, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \end{tikzcd}$$

where the top map is injective and the downward maps are injective. This immediately implies the claim. $\square$

The real power of our main result is that it works integrally. This has applications to essential dimension to be discussed later.

2.1 Main Result

Proposition 2.1.1. Let X be a proper smooth scheme over $\mathcal{O}_K$ equipped with a relative normal crossings divisor $D \subset X$. Set $U = X \backslash D$ and $W \subset U_C$ be a dense open subscheme. If $0 \le i < p-2$ then,

$$\dim_{\mathbb{F}_p} \mathrm{im}\,(H^i_{\text{ét}}(U_C, \mathbb{F}_p) \to H^i_{\text{ét}}(W, \mathbb{F}_p)) \ge h^{0,i}_{(X_C, D_C)}$$

Let's see how this implies the theorem. Let Y be a proper smooth scheme over $\mathbb{C}$ and $D \subset Y$ a normal crossings divisor. We say that (Y, D) has *good reduction at p* if there exists an algebraically closed field $C \hookrightarrow \mathbb{C}$ over which (Y, D) is defined and a p-adic valuation on C with ring of integers $\mathcal{O}_C$ and an extension to a smooth proper $\mathcal{O}_C$-scheme Y° with a relative normal crossings divisor $D^\circ \subset Y^\circ$ over $\mathcal{O}_C$ extending D. We say that (Y, D) has *unramified good reduction* at p if in addition (Y°, D°) can be chosen so that it descends to an absolutely unramified[2] dvr $\mathcal{O} \subset \mathcal{O}_C$.

Remark. This condition is actually easily checkable. Indeed if Y is a smooth proper finite type $\mathbb{C}$-scheme then it spreads out to a smooth proper scheme $\mathcal{Y} \to \mathrm{Spec}\,(A)$ over some finite type $\mathbb{Z}$-algebra $A \subset \mathbb{C}$. Now suppose there exists $\mathfrak{p} \subset A$ such that $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth at $\mathfrak{p}$ and $\mathfrak{p} \mapsto (p)$. This is nothing more than saying that p is not contained in the Jacobian ideal. Then choose a minimal prime ξ over pA since $\xi \rightsquigarrow \mathfrak{p}$ we see that $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth at ξ and hence $A_\xi \subset \mathbb{C}$ is a p-adic dvr unramified over $\mathbb{Z}_{(p)}$ by smoothness. Then we extend this p-adic valuation to $\mathbb{C}$ and $\mathcal{O} = A_\xi$ is our requisite unramified dvr.

Corollary 2.1.2. Let Y be a proper smooth connected $\mathbb{C}$-scheme and $D \subset Y$ a normal crossing divisor and $W \subset U := Y \backslash D$ a dense ope nsubscheme. Suppose that (Y, D) has unramified good reduction at p. If $0 \le i < p-2$ then,

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This proves the main theorem if we take $D = \emptyset$.

Proof. Since the étale cohomology groups do not change upon base change to algebraically closed fields. By assumption, we may assume that (Y, D) is defined over $\mathcal{O}$ unramified. Then taking the p-adic completion $C \subset C'$ we get $\mathcal{O} \subset \mathcal{O}'$ which is unramified and p-adically complete so we reduce to the previous case. ☐

Proof of Proposition 4.4.1. We just need something that lives between $H^i_{\text{ét}}(-, \mathbb{F}_p)$ and H^i_{dR}. ☐

Proof of Proposition 4.4.1. Let k_C be the residue field of C. We may replace X by it base change to $W(k_C)$ and assume that C and K have the ame residue field. Denote by $\widehat{X}$ and $\widehat{D}$ the formal completions of X and D. Let $\widehat{W} \subset \widehat{X}$ be the formal open subschem, which is the complement of Z_k. Note that we have $\widehat{W}_C \subset W^{\mathrm{ad}}$ so there is a commutative diagram,

$$\begin{tikzcd}
H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \arrow[r] \arrow[dd, "\alpha"] & H^i_{\text{ét}}(W, \mathbb{F}_p) \arrow[d] \\
 & H^i_{\text{ét}}(W^{\mathrm{ad}}, \mathbb{F}_p) \arrow[d] \\
H^i(\widehat{X}_{D,C}, \mathbb{F}_p) \arrow[r, "\beta"] & H^i_{\text{ét}}(\widetilde{X}_C, \mathbb{F}_p)
\end{tikzcd}$$

[2] meaning unramified over $\mathbb{Z}_{(p)}$

We need to show the following facts,

(a) α is an isomorphism

(b) $\dim_{\mathbb{F}_p} \operatorname{im} \beta \geq h^{0,i}_{(X,D)}$

(c) $H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \cong H^i_{\text{ét}}(U_C, \mathbb{F}_p)$

$\square$

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Now we will prove these three facts.

The only hard one:

Let X be a proper, smooth formal scheme over $\mathcal{O}_K$ equipped with a relative normal crossing divisor $D \subset X$. Let,

$$h^{0,i}_{(X,D)} := \dim_K H^0(X_K, \Omega^i_{X_K/K}(\log D))$$

Proposition 2.1.3. Let $W \subset X \backslash D$ be a dense open formal subscheme. Then for $0 \leq i < p-2$

$$\dim_{\mathbb{F}_p} \operatorname{im}(H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \to H^i_{\text{ét}}(W_C, \mathbb{F}_p) \geq h^{0,i}_{(X,D)}$$

Proof. Take the prism A to be $W(k)[[u]]$ with $E(u) = u - p$. We obtain a prism $A_C \to \mathcal{O}_C$. There is a Frobenius compatible map $A \to A_C$ sending $u \mapsto [\underline{p}]$. Set,

$$M_{\Delta} = \operatorname{im}(\overline{H^i_{\Delta}(X_D/A)} \to \overline{H^i_{\Delta}(W/A)})$$

which is a finitely generated $A/pA = k[[u]]$-module. There is an isomorphism,

$$\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C \xrightarrow{\sim} \overline{H^i_{\Delta}(X_{D,\mathcal{O}_C}/A_C)}$$

and similarly for the open W. Therefore, by PROPERTY there is an isomorphism

$$\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C \xrightarrow{\sim} \overline{H^i_{\Delta}(X_{D,\mathcal{O}_C}/A_C)}$$

Then there are maps,

$$\begin{aligned}\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C[1/\xi] &\cong H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} A_C/pA_C[1/\xi] \\ &\to H^i_{\text{ét}}(W_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} A_C/pA_C[1/\xi] \to \overline{H^i_{\Delta}(W/A)} \otimes_A A_C[1/\xi]\end{aligned}$$

the composite is the natural map. Hence,

$$\dim_{\mathbb{F}_p} \operatorname{im}(H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \to H^i_{\text{ét}}(W_C, \mathbb{F}_p)) \geq \dim_{k((u))} M_{\Delta}[1/u]$$

By LEMMA M_{Δ} is a finitely generated free $k[[u]]$-module. Hence it suffices to show $\dim_k M_{\Delta}/uM_{\Delta} \geq h^{0,i}_{(X,D)}$.

Hence using Lemma 2.2.1 again, we see that $\overline{H^j_{\Delta}(X_D/A)}$ is u-torsion free for $0 \leq j \leq i+1$. Hence there are maps,

$$\begin{aligned}H^i(X_k, \Omega^\bullet_{X_k/k}(\log D_k)) &\cong \overline{H^i_{\Delta}(X_D/A)} \otimes_{A,\varphi} k \to M_{\Delta} \otimes_{A,\varphi} k \\ &\to \overline{H^i_{\Delta}(W/A)} \otimes_{A,\varphi} k \to H^i(W_k, \Omega^\bullet_{W_k/K}(\log D))\end{aligned}$$

where the composition is the natural map. This shows that the image has dimension $\leq \dim_k M_{\Delta}/uM_{\Delta}$ and it suffices to how that this dimension is $\geq h^{0,i}_{(X,D)}$. Since $W \subset X$ is dense, the map,

$$H^0(X_k, \Omega^i_{X_k/k}(\log D)) \to H^0(W_k, \Omega^i_{X_k/k})$$

is injective. Hence the image has dimension at leat $\dim_k H^0(X_k, \Omega^i_{X_k/k}(\log D_k)) \geq h^{0,i}_{(X,D)}$ I THINK THIS WAS AN ERROR IN THE PAPER NEED LOG D where the last inequality follows from the upper semi-continuity of h^0. □

2.2 Prismatic Cohomology

2.2.1 Prisms

Let K be a field of characteristic 0. By a *p-adic valuation* on K we mean a rank one valuation ν on K, with $\nu(p) > 0$. We suppose that K is complete with respect to ν with ring of integers $\mathcal{O}_K$ and perfect residue field k. We will only recall exactly as much about prismatic cohomology as necessary.

Definition 2.2.1. A *δ-ring* is a pair (R, δ) where R is a commutative ring and $\delta : R \to R$ is a set map such that,

(a) $\delta(0) = \delta(1) = 0$

(b) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$

(c) $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$

Note that the last term exists as some universal polynomial with integer coefficents.

We think of this as a sort of "derivation along the p-direction". It is also related to lifting Frobenius on R/p. Indeed, if $\phi(x) = x^p + p\delta(x)$ then $\phi : R \to R$ is a ring map by property (c) and obviously it lifts $x \mapsto x^p$ on R/p. In fact, if R is p-torsionfree then lifts of Frobenius are exactly the same as δ-ring structures.

Definition 2.2.2. Let (A, I) be a pair where A is a δ-ring and $I \subset A$ is an ideal. The pair is a *prism* if

(a) $I \subset A$ is invertible (defines a Cartier divisor on $\mathrm{Spec}\,(A)$)

(b) A is derived (p, I)-complete

(c) $p \in I + \phi(I)A$

Example 2.2.3. Let A be a p-torsionfree and p-complete δ-ring then $(A, (p))$ is a prism.

Example 2.2.4. The *Breuil-Kisin* prism. Assume that ν on K is discrete. Set $A = W(k)[[u]]$ equipped with Frobenius φ extending Frobenius on $W(k)$ by $u \mapsto u^p$. Equip A with the map $A \to \mathcal{O}_K$ sending $u \mapsto \pi$ some uniformizer. It kernel is generated by an Eisenstein polynomial $E(u) \in W(k)[u]$ for π. In fact, in applications we will assume $\mathcal{O}_K = W(k)$ and $\pi = p$. Then $(A, E(u)A)$ is the Breuil-Kisin prism.

Example 2.2.5. Suppose that K is algebraically closed. Let $R = \varprojlim \mathcal{O}_K/p$ taking the limit over Frobenius. We take $A = W(R)$. Any element $(x_0, x_1, \dots) \in R$ lifts uniquely to a sequence $(\hat{x}_0, \hat{x}_1, \dots,) \in \mathcal{O}_K$ with $\hat{x}_i^p = \hat{x}_{i-1}$. Then there is a natural surjective map of rings $\theta : A \to \mathcal{O}_K$ sending a Teichmuller element x as above to $\hat{x}_0$. The kernel of θ is principal, generated by $\xi = p - [\underline{p}]$ where $\underline{p} = (p, p^{1/p}, \dots)$ then $(A, \xi A)$ is an example of a perfect prism.

2.2.2 Logarithmic Cohomology

We will use logarithmic formal schemes over $\mathcal{O}_K$. We will consider logarithmic étale cohomology meaning the natural cohomology on the site of log étale covers of logarithmic schemes. The main fact we will use is the following comparison result:

Theorem 2.2.6. Let k be an algebraically closed field and X a smooth k-scheme. Let $D \subset X$ be an snc divisor and $X_D^{\log}$ the log structure induced by D. Then there is a canonical isomorphism,

$$H^i_{\text{ét}}(X_D^{\log}, \mu) \xrightarrow{\sim} H^i(X \backslash D, \mu)$$

COEFFICIENTS

Proof. Idea: show that any finite étale map $Y \to X \backslash D$ extends canonically to a finite log-étale map $\overline{Y} \to X_D$ which proves the statment for $i = 1$ then use dimension shifting and some spectral sequence. To show the claim, take the normalization of Y in X which gives a finite map $Y \to X$ ramified only over D by Zariski nagata purity. Then a local check shows that this map is log-étaleWHY? □

2.2.3 Prismatic Cohomology

Let K be either discretely valued or algebraically closed. Let X be a formal smooth $\mathcal{O}_K$-scheme equipped with a relative normal crossings divisor D. Write X_D for log structure induced by D. We will denote by $X_{D,K}$ the associated log adic space giving by analytification.

The *prismatic cohomology* of X_D is the complex of A-modules $R\Gamma_{\Delta}(X_D/A)$ equipped with a φ-semi-linear map φ. The mod p cohomology is given by setting,

$$\overline{R\Gamma_{\Delta}(X_D/A)} = R\Gamma_{\Delta}(X_D/A) \otimes^{\mathbb{L}}_A A/pA$$

and we will denote by $\overline{H^i_{\Delta}(X_D/A)}$ the cohomology of $\overline{R\Gamma_{\Delta}(X_D/A)}$. Then we have the following properties:

(a) There is a canonical isomorphism of commutative algebras in $D(A)$

$$R\Gamma(\Omega^\bullet_{X_k/k}(\log D_k)) \cong \overline{R\Gamma_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_{A/pA,\varphi} l$$

(b) If K is algebraically closed then there is an isomorphism of commutative algebras in $D(A)$

$$R\Gamma_{\text{ét}}(X_{D,K}, \mathbb{F}_p) \cong \overline{R\Gamma_{\Delta}(X_D/A)}[1/\xi]^{\varphi=1}$$

(c) the linear map,

$$\varphi^* \overline{R\Gamma_{\Delta}(X_D/A)} \to \overline{R\Gamma_{\Delta}(X_D/A)}$$

becomes an isomorphism in $D(A)$ after inverting u (resp ξ) if K i discrete (resp. algebraically closed). For each $i \geq 0$, there is a canonical map,

$$V_i : \overline{H^i_{\Delta}(X_D/A)} \to H^i(\varphi^* \overline{R\Gamma_{\Delta}(X_D/A)}$$

(d) Let K' be a field complete with respect to a p-adic valuation, and which is either discrete or algebraically closed. Let $B \to \mathcal{O}_{K'}$ be the corresponding prism, as defined above. Suppose $K \to K'$ is a map of valued field and $A \to B$ is compatbile with the projection to $\mathcal{O}_K \to \mathcal{O}_{K'}$ and Frobenius. Then there is a canonical isomorphism

$$\overline{R\Gamma_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A B \cong \overline{R\Gamma_{\Delta}(X_{D,\mathcal{O}_{K'}}/B)}$$

(e) When X is proper over $\mathcal{O}_K$ then $\overline{R\Gamma_{\Delta}(X_D/A)}$ is a perfect complex of A/p-modules.

(f) Suppose that K is algebraically closed, and that X is proper over $\mathcal{O}_K$ then for each $i \geq 0$ there are natural isomorphisms

$$H^i_{\text{ét}}(X_{D,K}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} A/pA[1/\xi] \cong \overline{H^i_{\Delta}(X_D/A)}[1/\xi]$$

2.3 Proof For Real

Proof of Proposition 4.4.1. Let k_C be the residue field of C. We may replace X by it base change to $W(k_C)$ and assume that C and K have the ame residue field. Denote by $\widehat{X}$ and $\widehat{D}$ the formal completions of X and D. Let $\widehat{W} \subset \widehat{X}$ be the formal open subschem, which is the complement of Z_k. Note that we have $\widehat{W}_C \subset W^{\text{ad}}$ so there is a commutative diagram,

$$\begin{tikzcd}
H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \arrow[r] \arrow[dd, "\alpha"] & H^i_{\text{ét}}(W, \mathbb{F}_p) \arrow[d] \\
& H^i_{\text{ét}}(W^{\text{ad}}, \mathbb{F}_p) \arrow[d] \\
H^i(\widehat{X}_{D,C}, \mathbb{F}_p) \arrow[r, "\beta"] & H^i_{\text{ét}}(\widetilde{X}_C, \mathbb{F}_p)
\end{tikzcd}$$

We need to show the following facts,

(a) α is an isomorphism

(b) $\dim_{\mathbb{F}_p} \text{im}\, \beta \geq h^{0,i}_{(X,D)}$

(c) $H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \cong H^i_{\text{ét}}(U_C, \mathbb{F}_p)$

$\square$

WHY IS THE FIRST LEMMA 2.2.10

Now we will prove these three facts.

The only hard one:

Let X be a proper, smooth formal scheem over $\mathcal{O}_K$ equipped with a relative normal crossing divisor $D \subset X$. Let,

$$h^{0,i}_{(X,D)} := \dim_K H^0(X_K, \Omega^i_{X_K/K}(\log D))$$

Proposition 2.3.1. Let $W \subset X \backslash D$ be a dense open formal subscheme. Then for $0 \leq i < p-2$

$$\dim_{\mathbb{F}_p} \text{im}\,(H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \to H^i_{\text{ét}}(W_C, \mathbb{F}_p) \geq h^{0,i}_{(X,D)}$$

Proof. Take the prism A to be $W(k)[[u]]$ with $E(u) = u - p$. We obtain a prism $A_C \to \mathcal{O}_C$. There is a Frobenius compatible map $A \to A_C$ sending $u \mapsto [\underline{p}]$. Set,

$$M_{\Delta} = \text{im}\,(\overline{H^i_{\Delta}(X_D/A)} \to \overline{H^i_{\Delta}(W/A)})$$

which is a finitely generated $A/pA = k[[u]]$-module. There is an isomorphism,

$$\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C \xrightarrow{\sim} \overline{H^i_{\Delta}(X_{D,\mathcal{O}_C}/A_C)}$$

and similarly for the open W. Therefore, by PROPERTY there is an isomorphism

$$\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C \xrightarrow{\sim} \overline{H^i_{\Delta}(X_{D,\mathcal{O}_C}/A_C)}$$

Then there are maps,

$$\begin{aligned}\overline{H^i_{\Delta}(X_D/A)} \otimes^{\mathbb{L}}_A A_C[1/\xi] &\cong H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} A_C/pA_C[1/\xi] \\ &\to H^i_{\text{ét}}(W_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} A_C/pA_C[1/\xi] \to \overline{H^i_{\Delta}(W/A)} \otimes_A A_C[1/\xi]\end{aligned}$$

the composite is the natural map. Hence,

$$\dim_{\mathbb{F}_p} \operatorname{im}\left(H^i_{\text{ét}}(X_{D,C}, \mathbb{F}_p) \to H^i_{\text{ét}}(W_C, \mathbb{F}_p)\right) \geq \dim_{k((u))} M_{\Delta}[1/u]$$

By LEMMA M_{Δ} is a finitely generated free $k[[u]]$-module. Hence it suffices to show $\dim_k M_{\Delta}/uM_{\Delta} \geq h^{0,i}_{(X,D)}$.

Hence using Lemma 2.2.1 again, we see that $\overline{H^j_{\Delta}(X_D/A)}$ is u-torsion free for $0 \leq j \leq i+1$. Hence there are maps,

$$\begin{aligned}H^i(X_k, \Omega^{\bullet}_{X_k/k}(\log D_k)) &\cong \overline{H^i_{\Delta}(X_D/A)} \otimes_{A,\varphi} k \to M_{\Delta} \otimes_{A,\varphi} k \\ &\to \overline{H^i_{\Delta}(W/A)} \otimes_{A,\varphi} k \to H^i(W_k, \Omega^{\bullet}_{W_k/K}(\log D))\end{aligned}$$

where the composition is the natural map. This shows that the image has dimension $\leq \dim_k M_{\Delta}/uM_{\Delta}$ and it suffices to how that this dimension is $\geq h^{0,i}_{(X,D)}$. Since $W \subset X$ is dense, the map,

$$H^0(X_k, \Omega^i_{X_k/k}(\log D)) \to H^0(W_k, \Omega^i_{X_k/k})$$

is injective. Hence the image has dimension at leat $\dim_k H^0(X_k, \Omega^i_{X_k/k}(\log D_k)) \geq h^{0,i}_{(X,D)}$ I THINK THIS WAS AN ERROR IN THE PAPER NEED LOG D where the last inequality follows from the upper semi-continuity of h^0. ☐

Therefore we conclude using the following lemma:

Lemma 2.3.2. Suppose that (X, D) admits a lift to $W_2(k)$ called $(\widetilde{X}, \widetilde{D})$ with $\widetilde{D}$ a snc divisor flat over $W_2(k)$. Then for $j < p$,

$$H^0(X^1, \Omega^j_{X^1/k}(\log D^1)) \hookrightarrow H^j(X, \Omega^{\bullet}_{X/k}(\log D))$$

is canonically a direct summand.

Proof. This follows from the existence of the Cartier operator in the same way as in Deligne-Illusie. ☐

3 Talk 2

3.1 The Prismatic Site

Lemma 3.1.1. If $(A, I) \to (B, J)$ is a map of prismis then the natural map induces an isomorphism $I \otimes_A B \cong J$. In particular, $IB = J$.

Proof. Lemma 3.5 in Scholze $\square$

Fix a (bounded) prism (A, I) and a formuall smooth A/I-algebra R. The *prismatic site of* R relatve to A, dentoed $(R/A)_{\Delta}$, is the category whose objects are prisims (B, IB) over (A, I) together with an A/I-algebra map $R \to B/IB$

$$\begin{tikzcd} B \arrow[r] & B/I & R \arrow[l] \arrow[d] \\ A \arrow[u] \arrow[rr] & & A/I \end{tikzcd}$$

these are the diagrams. Covers are *faithfully flat* maps of prisms.

Definition 3.1.2. A map $(A, I) \to (B, IB)$ of prisms is *(faithfully) flat* if $A/(p, I) \to B \otimes^{\mathbb{L}}_A A/(p, I)$ is (faithfully) flat.

Definition 3.1.3. The structure sheaf of $(R/A)_{\Delta}$ is the sheaf,

$$\mathcal{O}_{\Delta} : (B, IB) \mapsto B$$

Likewise we define a sheaf $\overline{\mathcal{O}_{\Delta}}$ on $(R/A)_{\Delta}$ defined by,

$$\overline{\mathcal{O}}_{\Delta} : (B, IB) \mapsto B/IB$$

Definition 3.1.4. $\Delta_{R/A} := R\Gamma_{\Delta}(X/A) := R\Gamma_{\Delta}((R/A)_{\Delta}, \mathcal{O}_{\Delta})$

3.1.1 The non-affine case

Definition 3.1.5. Let (A, I) be a bounded prism and $X \to \mathrm{Spec}\,(A/I)$ be a scheme. Then the *prismatic site* of X relative to A, denoted $(X/A)_{\Delta}$, is the category of objects,

$$\begin{tikzcd} \mathrm{Spec}\,(B) \arrow[d] & \mathrm{Spec}\,(B/IB) \arrow[l] \arrow[r] & X \arrow[d] \\ \mathrm{Spec}\,(A) & & \mathrm{Spec}\,(A/I) \arrow[ll] \end{tikzcd}$$

We endow $(X/A)_{\Delta}$ by the Grothendieck topology given by faithfully flat covers of prisms and there are sheaves,

$$\mathcal{O}_{\Delta} : (B, IB) \mapsto B$$

and

$$\overline{\mathcal{O}}_{\Delta} : (B, IB) \mapsto B/I$$

Note that $\mathcal{O}_{\Delta}$ is valued in (p, I)-complete A-δ-algebras while $\overline{\mathcal{O}}_{\Delta}$ is valued in p-complete R-algebras.

3.2 Breuil-Kisin and Breuil-Kisin-Fargues Prisms

As pointed out last time, to make the étale comparison theorem work we need an algebraically closed field but we want to work over $K = \mathrm{Frac}\,(W(k))$ to set up our Breuil-Kisin prism but this is not algebraically closed. Therefore, we will need to work with two different prisms and a comparison between them.

3.2.1 Breuil-Kisin Prism

Recall our construction. Let k be a perfect field of characteristic p and $K = \mathrm{Frac}\,(W(k))$ which is a complete p-adic field with $\mathcal{O}_K = W(k)$. You should think of the example $k = \mathbb{F}_p$ and $K = \mathbb{Q}_p$ but we might need k to be the perfection of the function field of a variety over $\mathbb{F}_p$ as we discussed last time. Then we define.

Definition 3.2.1. The *Breuil-Kisin prism* for K is $A = W(k)[[u]]$ with $I = (u-p) = (E(u))$ so we get a map $A \to A/I = W(k) = \mathcal{O}_K$.

3.2.2 Breuil-Kisin-Fargues Prism

Let C be an algebraically closed complete p-adic field (we will later take C to be the completion of the algebraic closure of K). Then we set,

$$R = \varprojlim_{x \mapsto x^p} \mathcal{O}_K/p$$

Definition 3.2.2. The *Breuil-Kisin-Fargues prism* is $B = W(R)$ with its canonical Frobenius. Note there is an isomorphism of commutative monoids:

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_K \to \varprojlim_{x \mapsto x^p} \mathcal{O}_K/p$$
$$x \mapsto [x]$$

There is a surjective map of rings

$$\theta : B \to \mathcal{O}_C$$

which sends

$$[x] \mapsto x \mapsto x_0$$

Then $\ker \theta$ is generated by

$$\xi := p - [\underline{p}]$$

where $\underline{p} = (p, p^{1/p}, \dots)$. Then $(B, \xi B)$ is a perfect prism.

We will always work with $A = W(k)[[u]]$ the Breuil-Kisin prism for a scheme over $\mathcal{O}_K = W(k)$ and the Breuil-Kisin-Fargues prism B for a scheme over $\mathcal{O}_C$.

Let $K \to C$ be a map of p-adic fields with K and C as above. Then there is a comparison map,

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \mathcal{O}_K \arrow[r] & \mathcal{O}_C \end{tikzcd}$$

where the map $A \to B$ is given by $u \mapsto [\underline{p}]$ and therefore $E(u) = u - p \mapsto -\xi$.

It will be useful to record the following fact:

$$k[[u]] = A/pA \to B/pB$$

is flat. Since $k[[u]]$ is a DVR this amounts to showing that $u \mapsto [\underline{p}] \in B/pB = R$ is a non-zerodivisor. Since $[\underline{p}]$ lists along the monoid map to $\underline{p}$ which is nonzero this is clear because $\mathcal{O}_C$ is a domain.

3.3 Comparison Results

We need the following comparison theorems.

3.3.1 de Rham Comparison

Let k be the residue field of $\mathcal{O}_K$. Let $X \to \mathrm{Spec}\,(\mathcal{O}_K)$ be a smooth scheme. Then for any bounded prism (A, I) (we will always take the Breuil-Kisin prism) with $A/I \xrightarrow{\sim} \mathcal{O}_K$ there are canonical isomorphisms,

$$R\Gamma(X, \Omega^\bullet_X) \xrightarrow{\sim} R\Gamma_{\Delta}(X/A)\hat{\otimes}^{\mathbb{L}}_{A,\phi_A}\mathcal{O}_K$$

and therefore canonical isomorphisms,

$$R\Gamma(X_k, \Omega^\bullet_{X_k}) \xrightarrow{\sim} R\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_{A,\varphi} k \xrightarrow{\sim} \overline{R\Gamma_{\Delta}(X/A)} \otimes^{\mathbb{L}}_{A/pA,\varphi} k$$

3.3.2 étale Comparison

Let $(B, \xi B)$ be a perfect prism and $B/I \xrightarrow{\sim} \mathcal{O}_C$ for C an algebraically closed p-adically complete field (we will always take $(B, \xi B)$ to be the Breuil-Kisin-Fargues prisim associated to C). Let $X \to \mathrm{Spec}\,(\mathcal{O}_C)$ be a smooth scheme. Then there are canonical isomorphisms,

$$R\Gamma_{\text{ét}}(X_C, \mathbb{F}_p) \xrightarrow{\sim} \overline{R\Gamma_{\Delta}(X/B)}[1/\xi]^{\varphi=1}$$

where $\varphi = 1$ means taking the fiber of the semilinear endomorphism $\varphi - 1$.

Lemma 3.3.1. This comparison theorem gives an exact triangle,

$$R\Gamma_{\text{ét}}(X_C, \mathbb{F}_p) \to R\overline{R\Gamma_{\Delta}(X/B)}[1/\xi] \xrightarrow{1-\varphi} \overline{R\Gamma_{\Delta}(X/B)}[1/\xi] \to +1$$

and hence (because the target is a B/pB-module) morphisms,

$$H^i_{\text{ét}}(X_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} B/pB \to \overline{H^i_{\Delta}(X/B)}[1/\xi]$$

If $X \to \mathrm{Spec}\,(\mathcal{O}_C)$ is proper these are isomorphisms.

3.3.3 Base Change

Because we are working with two different prisms, we need some sort of base change result. Luckily the following very general comparison theorem holds.

Theorem 3.3.2. Let $(A, I) \to (B, J)$ be a map of bounded prisms and $Y = X \times_{\mathrm{Spec}(A/I)} \mathrm{Spec}\,(B/J)$. Then the natural map,

$$R\Gamma_{\Delta}(X/A)\hat{\otimes}^{\mathbb{L}}_A B \xrightarrow{\sim} R\Gamma_{\Delta}(Y/B)$$

is an isomorphism.

This implies the following,

$$\begin{aligned}
\overline{R\Gamma_{\Delta}(Y/B)} &= (R\Gamma_{\Delta}(X/A)\hat{\otimes}^{\mathbb{L}}_A B) \otimes^{\mathbb{L}}_B B/pB \\
&= R\Gamma_{\Delta}(X/A)\hat{\otimes}^{\mathbb{L}}_A B/pB \\
&= R\Gamma_{\Delta}(X/A)\hat{\otimes}^{\mathbb{L}}_A (A/pA)\hat{\otimes}^{\mathbb{L}}_{A/pA} B/pB \\
&= \overline{R\Gamma_{\Delta}(X/A)}\hat{\otimes}^{\mathbb{L}}_{A/pA} B/pB
\end{aligned}$$

In particular, if $A/pA \to B/pB$ is flat then we get comparison isomorphisms,

$$\overline{H^i_{\Delta}(Y/B)} \xrightarrow{\sim} \overline{H^i_{\Delta}(X/A)}\hat{\otimes}_A B/pB = \overline{H^i_{\Delta}(X/A)}\hat{\otimes}_A B$$

3.3.4 Finiteness of cohomology

Theorem 3.3.3. Let (A, I) be a bounded prism. Let $X \to \mathrm{Spec}\,(A/I)$ be a smooth proper scheme. Then $R\Gamma_{\Delta}(X/A)$ is a perfect complex of A-modules.

In particular, applying $- \otimes^{\mathbb{L}} A/pA$ and taking cohomology we see that $\overline{H^i_{\Delta}(X/A)}$ is a finite A/pA-module.

3.4 Proof of the Main Theorem

As before let $K = \mathrm{Frac}\,(W(k))$ for k a perfect field. Let C be the completion of the algebraic closure.

Theorem 3.4.1. Let $X \to \mathrm{Spec}\,(\mathcal{O}_K)$ be a smooth proper scheme and $W \subset X$ an open which is dense in the special fiber. Then for $0 \leq i < p-2$

$$\dim_{\mathbb{F}_p} \mathrm{im}\,(H^i_{\text{ét}}(X_C, \mathbb{F}_p) \to H^i_{\text{ét}}(W_C, \mathbb{F}_p)) \geq h^{i,0}_X := \dim_K H^0(X_K, \Omega^i_{X_K})$$

Proof. As before, we set A to be the Breuil-Kisin prisim for K and B to be the Breuil-Kisin-Fargues prism for C. Now set,

$$M_{\Delta} := \mathrm{im}\,(\overline{H^i_{\Delta}(X/A)} \to \overline{H^i_{\Delta}(W/A)})$$

Because X is proper the first term is finite and hence M_{Δ} is a finite $A/pA = k[[u]]$-module. By the comparison theorem and the fact that $A/pA \to B/pB$ is flat,

$$\overline{H^i_{\Delta}(X/A)}\hat{\otimes}_A B \xrightarrow{\sim} \overline{H^i_{\Delta}(X_{\mathcal{O}_C}/B)}$$

The proof will then proceed by the following steps. $\square$

3.4.1 The étale Comparison Diagram

Consider the diagram,

$$\begin{tikzcd}
H^i_{\text{ét}}(X_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} B/pB[1/\xi] \arrow[r, "\sim"] \arrow[dd, "\mathrm{res}^{\text{ét}}_W"'] & \overline{H^i_{\Delta}(X_{\mathcal{O}_C}/B)} \arrow[r, "\sim"] & \overline{H^i_{\Delta}(X/A)}\hat{\otimes}_A B[1/\xi] \arrow[d, two heads] \\
& & M_{\Delta}\hat{\otimes}_{k[[u]]} B/pB[1/\xi] \arrow[d, hook] \\
H^i_{\text{ét}}(W_C, \mathbb{F}_p) \otimes_{\mathbb{F}_p} B/pB[1/\xi] \arrow[r] & \overline{H^i_{\Delta}(W_{\mathcal{O}_C}/B)} \arrow[r, "\sim"] & \overline{H^i_{\Delta}(W/A)}\hat{\otimes}_A B[1/\xi]
\end{tikzcd}$$

The top maps are isomorphisms because X is proper (using the lemma after the étale comparison theorem). Furthermore, since B/pB is flat over A/pA the map

$$M_{\Delta}\hat{\otimes}_{k[[u]]} B/pB[1/\xi] \to \overline{H^i_{\Delta}(W/A)} \otimes_A B/pB[1/\xi]$$

is injective. Therefore,

$$\dim_{\mathbb{F}_p} \mathrm{im}\,\mathrm{res}^{\text{ét}}_W \geq \dim_{k((u))} M_{\Delta}[1/u]$$

note that mod p we have $u \mapsto -\xi$.

3.4.2 The de Rham Comparison Diagram

Consider the diagram,

$$\begin{tikzcd}
H^i(X_k, \Omega^\bullet_{X_k}) \arrow[dd, "\mathrm{res}^{\mathrm{dR}}_W"'] & H^i(\overline{R\Gamma_{\Delta}(X/A)} \otimes^{\mathbb{L}}_{A,\varphi} k) \arrow[l, "\sim"'] & \overline{H^i_{\Delta}(X/A)} \otimes_{A,\varphi} k \arrow[l, "\sim"'] \arrow[d] \\
& & M_{\Delta} \otimes_{A,\varphi} k \arrow[d] \\
H^i(W_k, \Omega^\bullet_{W_k}) & H^i(\overline{R\Gamma_{\Delta}(W/A} \otimes^{\mathbb{L}}_{A,\varphi} k) \arrow[l, "\sim"'] & \overline{H^i_{\Delta}(W/A)} \otimes_{A,\varphi} k \arrow[l]
\end{tikzcd}$$

The leftmost maps are given by the subs in the Tor-spectral sequence. To show the map,

$$\overline{H^i_{\Delta}(X/A)} \otimes_{A,\varphi} k \xrightarrow{\sim} H^i(\overline{R\Gamma_{\Delta}(X/A)} \otimes^{\mathbb{L}}_{A,\varphi} k)$$

is an isomorphism we need to prove the following claim:

$$\text{For } 0 \le j \le i+1 \text{ the } A/pA = k[[u]]\text{-modules } \overline{H^i_{\Delta}(X/A)} \text{ are } u\text{-torsion free.}$$

Given this claim, since $\mathrm{res}^{\mathrm{dR}}_W$ factors through the k-module $M_{\Delta} \otimes_{A,\varphi} k$ we see that,

$$\dim_k \mathrm{im}\, \mathrm{res}^{\mathrm{dR}}_W \le \dim_k M_{\Delta} \otimes_{A,\varphi} k = \dim_k M_{\Delta}/uM_{\Delta}$$

WHAT ABOUT THE FROB HERE?

Therefore if we can show the next claim:

$$M_{\Delta} \text{ is a finitely generated free } k[[u]]\text{-module.}$$

Then we conclude that,

$$\dim_{\mathbb{F}_p} \mathrm{im}\, \mathrm{res}^{\text{ét}}_W \ge \dim_{k((u))} M_{\Delta}[1/u] = \dim_k M_{\Delta}/uM_{\Delta} \ge \dim_k \mathrm{im}\, \mathrm{res}^{\mathrm{dR}}_W$$

Therefore it suffices to bound $\mathrm{res}^{\mathrm{dR}}_W$.

3.4.3 Cartier Isomorphism

Recall that because X_k is a smooth scheme over a perfect field k which lifts over $W_2(k)$ there is an isomorphism in the derived category,

$$\bigoplus_{i<p} \Omega^i_{X_k^{(p)}}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_X$$

in the derived category where $F : X_k \to X_k^{(p)}$ is the relative Frobenius. This decomposition is natural so we get a commutative diagram,

$$\begin{tikzcd}
H^0(X_k^{(p)}, \Omega^i_{X_k^{(p)}}) \arrow[r, hook] \arrow[d] & H^0(W_k^{(p)}, \Omega_{W_k^{(p)}}) \arrow[d, hook] \\
H^i(X_k, \Omega^\bullet_{X_k}) \arrow[r, "\mathrm{res}^{\mathrm{dR}}_W"] & H^i(W_k, \Omega^\bullet_{W_k})
\end{tikzcd}$$

Since the maps along the top are injective we see that,

$$\dim_k \operatorname{im} \operatorname{res}_W^{\mathrm{dR}} \geq \dim_k H^0(X_k^{(p)}, \Omega^i_{X_k^{(p)}}) = \dim_k H^0(X_k, \Omega^i_{X_k})$$

The last equality follows from the φ-semilinear isomorphism of schemes $X_k \to X_k^{(p)}$. Finally,

$$\dim_k \operatorname{im} \operatorname{res}_W^{\mathrm{dR}} \geq \dim_k H^0(X_k, \Omega^i_{X_k}) \geq \dim_K H^0(X_K, \Omega^i_{X_K})$$

by upper semicontinuity which completes the proof (modulo the claims).

4 Talk 3

Definition 4.0.1. Let $f : Y \to X$ be a finite map of complex algebraic varities. The *essential dimension* $\mathrm{ed}(Y/X)$ of f is the smallest integer e such that, over some dense open of X, the map f arises as the pullback of a map of varities of dimension e.

Example 4.0.2. Note that if $g : Y \to X$ is a cyclic cover, meaning the extension of fields is Galois with a cyclic Galois group, then because the base field $\mathbb{C}$ contains all roots of unity we see that $g : Y \to X$ is generically the extraction of an n^{th}-root of some rational function x on X. Then the map $Y \to X$ is generically pulled back from $z^n : \mathbb{A}^1 \to \mathbb{A}^1$ hence $\mathrm{ed}(Y/X) = 1$.

Example 4.0.3. The S_n-quotient $f_n : \mathbb{A}^n \to \mathbb{A}^n$ is the example that motivated the development of the study of essential dimenison. Note that if the generic degree n polynomial is solvable in radicals then f_n is a composition of cyclic covers and hence has $\mathrm{ed}(f_n) = 1$. For $n = 5$ we know $\mathrm{ed}(f_5) = 2$ so given radicals and one other function (determined by the essential dimension covering) we can solve degree 5 polynomials. Working out $\mathrm{ed}(f_n)$ is a major open problem.

Definition 4.0.4. Let $f : Y \to X$ be a finite map of complex algebraic varities. The *p-essential dimension* $\mathrm{ed}(Y/X; p)$ of f is the minimum over $\mathrm{ed}(Y'/X'; p)$ of all generically-finite maps $X' \to X$ of degree coprime to p and $Y' = Y \times_X X'$.

Definition 4.0.5. Recall that the mod p-homology cover of a space X is the étale cover $Y \to X$ corresponding to the maximal $(\mathbb{Z}/p\mathbb{Z})^n$ quotient of $\pi_1(X)$.

4.1 Theorems

Theorem 4.1.1 (A)**.** Let X be a smooth proper complex variety, and $Y \to X$ its mod p homology cover. Suppose that X has good unramified reduction at p, and let b_1 denote the first betti number of X. Then for $p > \max\{\frac{1}{2}b, 3\}$,

$$\mathrm{ed}(Y/X; p) \geq \min\{\dim X, \tfrac{1}{2}b_1\}$$

In the following cases, this theorem shows that the mod p homology cover is *p-imcompressible* meaning $\mathrm{ed}(Y/X; p) = \dim X$

(a) X is an abelian variety

(b) $X = C_1 \times \cdots \times C_r$ for curve of genus $g(C_i) \geq 1$

(c) locally symmetric varities associated to cocompact lattices in $\mathrm{SU}(n, 1)$

Theorem 4.1.2 (B)**.** Let X be a smooth, proper complex variety, G a finite group, and $Y \to X$ a G-cover. Suppose that X has unramified good reduction at p and let $i < p-2$. If $H^0(X, \Omega_X^i) \neq 0$ and the map $H^i(G, \mathbb{F}_p) \to H^i(X, \mathbb{F}_p)$ is surjective then

$$\mathrm{ed}(Y/X; p) \geq i$$

Note the the map is defined by the map $\pi_1(X) \to G$ and the natural maps

$$H^i(G, \mathbb{F}_p) \to H^i(\pi_1(X), \mathbb{F}_p) \to H^i(X, \mathbb{F}_p)$$

4.2 Abelian Varieties

4.3 Idea

We will use the following theorem

Theorem 4.3.1 (C)**.** Let X be a smooth, proper, complex variety, with unramified good reduction at p and $W \subset X$ a Zariski open. Then the following hold

(a) if $i < p-2$ then

$$\dim_{\mathbb{F}_p} \mathrm{im}\,(H^i(X, \mathbb{F}_p) \to H^i(W, \mathbb{F}_p)) \geq h_X^{i,0}$$

(b) if X is an abelian variety then the above also holds for $i = p-2$

(c) if $p > \max\{i+1, 3\}$ and $i \leq \dim X$ then

$$\dim_{\mathbb{F}_p} \mathrm{im}\,(\wedge^i H^1_{\text{ét}}(X, \mathbb{F}_p) \to H^i_{\text{ét}}(W, \mathbb{F}_p)) \geq \binom{h_X^{1,0}}{i}$$

The proof uses prismatic cohomology. Then we will deduce Theorem B as follows. Consider the composite,

$$H^i(G, \mathbb{F}_p) \to H^i(X, \mathbb{F}_p) \to H^i(W, \mathbb{F}_p)$$

The assumptions ensure that this map is nonzero. However, if $Y|_W \to W$ arises from a covering of varieties $Y' \to Z'$ of dimension $< i$ then the map factors as

$$H^i(G, \mathbb{F}_p) \to H^i(Z, \mathbb{F}_p) \to H^i(W, \mathbb{F}_p)$$

By possibily shrinking W and then Z we may assume that Z is affine, and it follows that the above map must vanish since the cohomological dimension of affine varities is at most their dimension.

For theorem A we instead we need a result for $\wedge^i H^1(X, \mathbb{F}_p) \to H^i(W, \mathbb{F}_p)$.

4.4 Proofs

Let k be a perfect field and $K = \mathrm{Frac}\,(W(k))$.

Proposition 4.4.1. Let X be a smooth proper scheme over $\mathcal{O}_K$ let $W \subset X$ be a dense open subscheme. If $0 \leq i < p-2$ then

(a) if $i < p-2$ then

$$\dim_{\mathbb{F}_p} \mathrm{im}\,(H^i(X, \mathbb{F}_p) \to H^i(W, \mathbb{F}_p)) \geq h_X^{i,0}$$

(b) if X is an abelian variety then the above also holds for $i = p - 2$

(c) if $p > \max\{i+1, 3\}$ and $i \le \dim X$ then

$$\dim_{\mathbb{F}_p} \operatorname{im}(\wedge^i H^1_{\text{ét}}(X, \mathbb{F}_p) \to H^i_{\text{ét}}(W, \mathbb{F}_p)) \ge \binom{h^{1,0}_X}{i}$$

Let's see how this implies the theorem. Let Y be a proper smooth scheme over $\mathbb{C}$. We say that Y has *good reduction at p* if there exists an algebraically closed field $C \hookrightarrow \mathbb{C}$ over which Y is defined and a p-adic valuation on C with ring of integers $\mathcal{O}_C$ and an extension to a smooth proper $\mathcal{O}_C$-scheme Y° over $\mathcal{O}_C$. We say that Y has *unramified good reduction* at p if in addition (Y°, D°) can be chosen so that it descends to an absolutely unramified[3] dvr $\mathcal{O} \subset \mathcal{O}_C$.

Remark. This condition is actually easily checkable. Indeed if Y is a smooth proper finite type $\mathbb{C}$-scheme then it spreads out to a smooth proper scheme $\mathcal{Y} \to \operatorname{Spec}(A)$ over some finite type $\mathbb{Z}$-algebra $A \subset \mathbb{C}$. Now suppose there exists $\mathfrak{p} \subset A$ such that $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathbb{Z})$ is smooth at $\mathfrak{p}$ and $\mathfrak{p} \mapsto (p)$. This is nothing more than saying that p is not contained in the Jacobian ideal. Then choose a minimal prime ξ over pA since $\xi \rightsquigarrow \mathfrak{p}$ we see that $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathbb{Z})$ is smooth at ξ and hence $A_\xi \subset \mathbb{C}$ is a p-adic dvr unramified over $\mathbb{Z}_{(p)}$ by smoothness. Then we extend this p-adic valuation to $\mathbb{C}$ and $\mathcal{O} = A_\xi$ is our requisite unramified dvr.

Proof. $\square$

Remark. Given a variety over $\mathbb{C}$, it has unramified good reduction at all but finitely many primes p because if we spread out to some finite type $\mathbb{Z}$-algebra $A \subset \mathbb{C}$ then $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathbb{Z})$ is smooth over all but finitely many primes.

DO Corollary 2.2.13 and discussion following and the discussion following 2.2.15 but we.

4.5 Characeristic Classes

Let C be an algebraically closed field of characteristic 0.

Let X be a proper, connected, smooth C-scheme, equipped with a normal crossings divisor D. Fix a geometric point $\bar{\eta}$ mapping to the generic point $\eta \in X$. Let $\pi_1^{\text{ét}}(X, \bar{\eta}) \twoheadrightarrow G$ be a finite quotient. For any i there are canoincal maps,

$$H^i(G, \mathbb{F}_p) \to H^i(\pi_1^{\text{ét}}(X, \bar{\eta}), \mathbb{F}_p) \to H^i_{\text{ét}}(X, \mathbb{F}_p)$$

where the first map is inflation and the second is induced by the comparison map between the finite étale and étale sites.

Theorem 4.5.1 (D)**.** Suppose that $i < p - 2$ and X has unramified good reduction at p. Let G be a finite group and $Y \to X$ a connected G-cover. Suppose that $h^{i,0}_X \neq 0$ and that the map

$$H^i(G, \mathbb{F}_p) \to H^i_{\text{ét}}(X, \mathbb{F}_p)$$

is surjective. Then $\operatorname{ed}(Y/X; p) \ge i$. If X is an abelain variety the above also holds for $i = p - 2$.

[3]meaning unramified over $\mathbb{Z}_{(p)}$

Proof. Let $X' \to X$ be a finite connected covering which has prime to p degree over η, and let $\eta' \in X'$ be the generic point. We need to show that $\mathrm{ed}(Y'/X' \geq i$ where $Y' = Y \times_X X'$. Consider the composite

$$H^i(G, \mathbb{F}_p) \to H^i(\pi_1^{\text{ét}}(X, \bar{\eta}), \mathbb{F}_p) \to H^i_{\text{ét}}(X, \mathbb{F}_p) \to H^i(\eta, \mathbb{F}_p) \to H^i(\eta', \mathbb{F}_p)$$

Our assumptions imply that the composition of the first two maps is surjective. Since $h^{i,0}_X \neq 0$ then Theorem B implies that the third map is nonzero. The composite of the fourth map with trace $H^i(\eta', \mathbb{F}_p) \to H^i(\eta, \mathbb{F}_p)$ is multiplication by $\deg X'/X$ which is coprime to p and hence invertible. Therefore, the fourth map must be injective so the composite is nonzero.

Suppose $\mathrm{ed}(Y'/X') < i$. Then for some dense open $W \subset X'$ there is a map of C-schemes $W \to Z$ with $\dim Z < i$ and a G-cover $Y'_Z \to Z$ such that $Y'|_W \cong Y'_Z \times_Z W$ as G-torsors. Shrinking Z and W if necessary, we may assume that Z is affine. The above constructions give a diagram

$$\begin{tikzcd}
H^i(G, \mathbb{F}_p) \arrow[r] \arrow[d, equal] & H^i_{\text{ét}}(Z, \mathbb{F}_p) \arrow[d] & \\
H^i(G, \mathbb{F}_p) \arrow[r] & H^i_{\text{ét}}(W, \mathbb{F}_p) \arrow[r] & H^i_{\text{ét}}(\eta', \mathbb{F}_p)
\end{tikzcd}$$

Since Z is affine of dimension $< i$ it follows that $H^i_{\text{ét}}(Z, \mathbb{F}_p) = 0$. This implies that the composite of the maps in the bottom from is zero contradicting what we previously demonstrated. □

Corollary 4.5.2. Let A/C be an abelian variety of dimension g. Let $p \geq g + 2$ and suppose that X has unramified good reduction at p. Then $[p] : A \to A$ as a $(\mathbb{Z}/p\mathbb{Z})^{2g}$-cover has $\mathrm{ed}([p]; p) = g$. In particular, this equality holds for almost all p.

Proof. By definition $g = \dim X \geq \mathrm{ed}([p]; p)$ so it suffices to prove that $\mathrm{ed}([p]; p) \geq g$. Let $G = (\mathbb{Z}/p\mathbb{Z})^{2g}$ be the quotient of $\pi_1^{\text{ét}}(A, \bar{\eta})$ corresponding to $[p] : A \to A$. The map

$$H^i(G, \mathbb{F}_p) \to H^i_{\text{ét}}(A, \mathbb{F}_p)$$

is surjective because it is surjective on $i = 1$ and $H^\bullet(A, \mathbb{F}_p)$ is the exterior algebra generated in $H^1(A, \mathbb{F}_p)$ by cup product. Since $h^{g,0} = 1$ we conclude that $\mathrm{ed}([p]; p) \geq g$ by the previous theorem. □

4.6 Mod p homology covers

We now specify our attention to when the G-cover $Y \to X$ is the mod p homology cover of X. Recall that the mod p homology cover is given by the maximal quotient $\pi_1^{\text{ét}}(X) \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^{2g}$. This is the same as the quotient by p of the abelianization of $\pi_1^{\text{ét}}(X)$. Note that this arises as follows,

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alb}_X \arrow[d, "\times p"] \\
X \arrow[r] & \mathrm{Alb}_X
\end{tikzcd}$$

because $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(\mathrm{Alb}_X)$ IS THIS TRUE IF THERE IS TORSION IN H^1

Theorem 4.6.1 (E)**.** Suppose X has unramified good reduction at p. Suppose that $i \leq \min\{h_X^{1,0}, \dim X\}$ and that $p > \max\{i+1, 3\}$. Then the mod p homology cover $Y \to X$ satisifes $\mathrm{ed}(Y/X; p) \geq i$. In particular, if $p > \max\{h_X^{1,0} + 1, 3\}$ then

$$\mathrm{ed}(Y/X; p) \geq \min\{h_X^{1,0}, \dim X\}$$

Remark. Note the bounds are exactly those in Theorem C part (c).

Proof. An in the proof of Theorem D, let $X' \to X$ be a finite connecting covering of degree prime to p over η and let $\eta' \in X'$ be the generic point. Let $G = \mathrm{Gal}\,(Y/X)$, and consider the composite map

$$\wedge^i H^1(G, \mathbb{F}_p) \xrightarrow{\sim} \wedge^i H^1_{\text{ét}}(X, \mathbb{F}_p) \to H^i(\eta, \mathbb{F}_p) \to H^i(\eta', \mathbb{F}_p)$$

By Theorem C, the second map is nonzero assuming $i \leq h_X^{1,0}$. The last map is injective since $X' \to X$ has degree coprime to p over η. Since the compositie map factors through $H^i(G, \mathbb{F}_p)$, it foloows that

$$H^i(G, \mathbb{F}_p) \to H^i(\eta', \mathbb{F}_p)$$

is nonzero, which implies that $\mathrm{ed}(Y/X; p) \geq i$ as in the proof of Theorem D. □

Corollary 4.6.2. Let X be a projective C-scheme with unramified good reduction at p. Let $b_1 = \dim_{\mathbb{Q}} H^1(X, \mathbb{Q})$ and suppose $p > \max\{\frac{1}{2}b_1 + 1, 3\}$. Then the mod p homology cover $Y \to X$ satisfies

$$\mathrm{ed}(Y/X; p) \geq \min\{\tfrac{1}{2}b_1, \dim X\}$$

Proof. Since X is projective, we have $h_X^{1,0} = h_X^{0,1} = \frac{1}{2}b_1$. Thus we reduce to the previous result. □

5 Mar 8

Example 5.0.1. Let $(A, I) = (\mathbb{Z}_p, p)$ and $X = \mathbb{A}^1_{\mathbb{F}_p}$. Consider $(B, IB) = (\mathbb{Z}_p\langle x\rangle, (p))$. Where $\mathbb{Z}_p\langle x\rangle$ is power series $\sum_{i\geq 0} a_i x^i$ whose coefficients $a_i \to 0$ in the p-adic topology. We need to do this because $\mathbb{Z}_p[x]$ is *not* p-adically complete. This defines an object in $(X/A)_{\Delta}$.

If we want to invert x in (B, IB) we have to take $\mathbb{Z}_p\langle x\rangle[x^{-1}]^\wedge = \mathbb{Z}_p\langle x^{\pm 1}\rangle$ is given by power series $\sum_{i\in\mathbb{Z}} a_i x^i$ where $a_i \to 0$ as $|i| \to \infty$.

Theorem 5.0.2. (a) When $X = \mathrm{Spf}\,(R)$ then $H^i(R\Gamma_{\Delta}(X/A)\hat{\otimes}^{\mathbb{L}} A/I) \cong \Omega^i_{X/(A/I)}\{-i\}$

(b) in general

$$R\Gamma_{\Delta}(X/A)\hat{\otimes}^{\mathbb{L}}_{A,\phi_A} A/I \cong R\Gamma_{\mathrm{dR}}(X/(A/I))$$

Definition 5.0.3. The *Breuil-Kisin twist* of an A/I-module M is

$$M\{1\} = M \otimes_{A/I} I/I^2$$

Note that since I is a Cartier divisor I/I^2 is a rank 1 locally free module.

5.1 Proof

First define a morphism of topoi

$$\nu : \mathfrak{Sh}((X/A)_{\mathbbl{\Delta}}) \to \mathfrak{Sh}(X_{\text{ét}})$$

Lemma 5.1.1 (BS, 2.18)**.** When (B, IB) is a bounded prism and $B/IB \to \bar{C}$ is p-completely étale then it lifts uniquely to $(B, IB) \to (C, IC)$ a map of prisms such that $\bar{C} = C/IC$ recovering the original map.

Given a sheaf $\mathscr{F}$ on $(X/A)_{\mathbbl{\Delta}}$ and an étale map $U \to X$ we can define $\mathscr{F}|_{(U/A)_{\mathbbl{\Delta}}}$ via the restriction functor $(U/A)_{\mathbbl{\Delta}} \to (X/A)_{\mathbbl{\Delta}}$. We define the map of topoi,

(a) ν_* is restriction to $(U/A)_{\mathbbl{\Delta}}$ and then take global section. This means

$$(\nu_* \mathscr{F})(U \to X) = \Gamma(\mathscr{F}|_{(U/A)_{\mathbbl{\Delta}}})$$

(b) ν^*

so that

$$\begin{tikzcd}
\mathrm{Spf}\,(B) & U \arrow[d] \\
 & X \arrow[d] \\
\mathrm{Spf}\,(A) & \mathrm{Spf}\,(A/I) \arrow[l]
\end{tikzcd}$$

Definition 5.1.2. $\mathbbl{\Delta}_{X/A} = R\nu_* \mathcal{O}_{\mathbbl{\Delta}} \in D(X_{\text{ét}}, A)$. Likewise $\overline{\mathbbl{\Delta}}_{X/A} = R\nu_* \mathcal{O}_{\overline{\mathbbl{\Delta}}} \in D(X_{\text{ét}}, \mathcal{O}_X)$.

By definition $\mathrm{R}\Gamma(X_{\text{ét}}, \mathbbl{\Delta}_{X/A}) \cong R\Gamma_{\mathbbl{\Delta}}(X/A)$ so the strategy is

(a) produce some comparison maps between $\mathbbl{\Delta}_{X/A}$ and $\Omega^{\bullet}_{X/(A/I)}\{\bullet\}$

(b) work étale-locally

(c) do the computation for affine spaces.

6 Hodge-Tate II

Setup (A, I) is a bounded prism. Let $X/\mathrm{Spf}\,(A/I)$ be a smooth relative scheme. Then there is a site $(X/A)_{\mathbbl{\Delta}}$ given by diagrams

$$\begin{tikzcd}
\mathrm{Spf}\,(B) \arrow[dd] & \mathrm{Spf}\,(B/IB) \arrow[l, hook'] \arrow[d] \\
 & X \arrow[d] \\
\mathrm{Spf}\,(A) & \mathrm{Spf}\,(A/I) \arrow[l, hook']
\end{tikzcd}$$

We construct a morphism of topoi

$$\nu : \mathfrak{Sh}(X/A)_{\Delta} \to \mathfrak{Sh}X_{\text{ét}}$$

via

(a) $(\nu_*\mathscr{F})(U \xrightarrow{\text{ét}} X) = H^0((U/A)_{\Delta}, \mathscr{F}|_{(U/A)_{\Delta}})$ which is a sheaf by a lemma from last time

(b) $(\nu^{-1}\mathscr{G})^{\text{pre}}(B, IB) = \varinjlim_{\text{Spf}(B/IB)\to U} \mathscr{G}(U \to X)$ this is a filtered colimit.

Definition 6.0.1. $\Delta_{X/A} = \mathrm{R}\nu_*\mathcal{O}_{\Delta} \in D(X_{\text{ét}}, A)$ and $\overline{\Delta}_{X/A} = \Delta_{X/A} \otimes^{\mathbb{L}}_A A/I = \mathbb{R}\nu_*\mathcal{O}_{\overline{\Delta}} \in D(X_{\text{ét}}, \mathcal{O}_X)$.

Theorem 6.0.2 (Hodge-Tate)**.** $\mathscr{H}^i(\overline{\Delta}_{X/A}) = \Omega^i_{X/(A/I)}\{-i\}$

6.1 Construction of the Map

Note there is a distinguished triangle

$$\overline{\Delta}_{X/A}\{i+1\} \to \Delta_{X/A} \otimes^{\mathbb{L}}_A I^i/I^{i+2} \to \overline{\Delta}_{X/A}\{i\} \to +1$$

from this we obtain a Bockstein

$$\mathscr{H}^i(\overline{\Delta}_{X/A}\{i\}) \xrightarrow{\beta} \mathscr{H}^{i+1}(\overline{\Delta}_{A/I}\{i+1\})$$

this is only A/I-linear not $\mathcal{O}_X$-linear.

Proposition 6.1.1. The sequence

$$\mathscr{H}^0(\overline{\Delta}_{X/A}) \xrightarrow{\beta} \mathscr{H}^1(\overline{\Delta}_{X/A}\{1\}) \xrightarrow{\beta} \mathscr{H}^2(\overline{\Delta}_{X/A}\{2\}) \to \cdots$$

is a differential graded algebra.

There is a canonical map

$$\mathcal{O}_X \to \mathscr{H}^0(\overline{\Delta}_{X/A})$$

induced by $\text{Spf}\,(A/I) \to X$. Since it is a DGA for $f, g \in \mathscr{H}^0(\overline{\Delta}_{X/A})$ then $\beta(fg) = f\beta(g) + g\beta(f)$ (everything in degree zero commutes with everything else so there is no order ambiguity). This induces a map

$$\Omega_{X/A} \to \mathscr{H}^1(\overline{\Delta}_{X/A})\{1\}$$

then some graded commutativity implies the existence of

$$\Omega^i_{X/A} \to \mathscr{H}^i(\overline{\Delta}_{X/A})\{i\}$$

6.2 Prismatic Envelopes

Lemma 6.2.1. For A a δ-ring there is a "free δ-A-algebra" on one generator called $A\{x\}$ which as a ring is polynomial algebrain $x, \delta(x), \delta^2(x), \ldots$

Proposition 6.2.2. Let (A, I) be a prism. Let $A/I \to R$ be a p-completely smooth A/I-algebra. Let $B_0 \twoheadrightarrow R$ be a surjection where B_0 is (p, I)-completely smooth over A

$$\begin{tikzcd}
\mathrm{Spf}\,(B) \arrow[r] & \mathrm{Spf}\,(B_0) \arrow[d, "smooth"] & \mathrm{Spf}\,(R) \arrow[l, hook'] \arrow[d, "smooth"] \\
& \mathrm{Spf}\,(A) & \mathrm{Spf}\,(A/I) \arrow[l, hook']
\end{tikzcd}$$

where (A, I) is a prism but B_0 is not.

Let $B_0 \to B$ be a (p, I)-faithfully flat map. Let B be a δ-A-algebra, let $J = \ker(B \to B \otimes^{\mathbb{L}}_{B_0} R)$ then there exists a prism

$$(B, J) \to (B\{\tfrac{J}{I}\}^{\vee}, IB\{\tfrac{J}{I}\}^{\vee})$$

where the second is a universal prism over (A, I) which is initial for maps $(B, J) \to (A, I)$.

Coming back to our calculaton, to check that the map $\Omega^i_X \to \mathscr{H}^i(\overline{\Delta}_{X/A})\{i\}$ is an isomorphism we may work locally on X. Let $X = \mathrm{Spf}\,(R)$ with a diagram

$$\begin{tikzcd}
& \mathrm{Spf}\,(R) \arrow[d, "smooth"] \\
\mathrm{Spf}\,(A) & \mathrm{Spf}\,(A/I) \arrow[l]
\end{tikzcd}$$

Choose an embedding $\mathrm{Spf}\,(R) \hookrightarrow \mathrm{Spf}\,(A[x_1, \dots, x_n])$. Then there is a diagram

$$\begin{tikzcd}
\mathrm{Spf}\,(C) \arrow[d] & \mathrm{Spf}\,(C/IC) \arrow[l, hook'] \arrow[d] \\
\mathrm{Spf}\,(A\{\underline{x}\}) \arrow[d] & \mathrm{Spf}\,(A\{\underline{x}\}/J) \arrow[l] \arrow[d] \\
\mathrm{Spf}\,(A[\underline{x}]) \arrow[d, "smooth"] & \mathrm{Spf}\,(R) \arrow[l, hook'] \arrow[d, "smooth"] \\
\mathrm{Spf}\,(A) & \mathrm{Spf}\,(A/I) \arrow[l]
\end{tikzcd}$$

where $A[\underline{x}] \to A\{\underline{x}\}$ is just a ring map but $A \to A\{\underline{x}\}$ is a δ-ring map. Here J is defined via the fiber product and C is the prismatic envenlope of the map $(A, I) \to (A\{\underline{x}\}, J)$

Proposition 6.2.3. $(C, IC) \in (X/A)_{\Delta}$ is weakly final.

Proof. Want for any

$$\begin{tikzcd}
\mathrm{Spf}\,(D) \arrow[d, dashed, "3"] \arrow[dd, dashed, bend right, "2"'] \arrow[ddd, dashed, bend right=50, "1"'] \arrow[dddd, bend right=70] & \mathrm{Spf}\,(D/ID) \arrow[l, hook'] \arrow[d, dashed, "3"] \arrow[ddd, bend left=60] \\
\mathrm{Spf}\,(C) \arrow[d] & \mathrm{Spf}\,(C/IC) \arrow[l, hook'] \arrow[d] \\
\mathrm{Spf}\,(A\{\underline{x}\}) \arrow[d] & \mathrm{Spf}\,(A\{\underline{x}\}/J) \arrow[l] \arrow[d] \\
\mathrm{Spf}\,(A[\underline{x}]) \arrow[d, "smooth"] & \mathrm{Spf}\,(R) \arrow[l, hook'] \arrow[d, "smooth"] \\
\mathrm{Spf}\,(A) & \mathrm{Spf}\,(A/I) \arrow[l]
\end{tikzcd}$$

we want to fill in to get a map $(C, I) \to (D, I)$.

(a) $A[\underline{A}] \to R \to D/ID$ then there is a lift along $D \twoheadrightarrow D/ID$ since $A[\underline{x}]$ is free.

(b) Since $A[\underline{x}]$ is free we can uniquely lift to a δ-map $A\{\underline{x}\} \to D$

(c) use the universal property of the prismatic envelope.

$\square$

Note that there is nonuniquness in the construction arising from step 1 only. This construction is partially functorial. Indeed given $A[\underline{x}] \twoheadrightarrow R$ the construction of (C, IC) is functorial. Morover, carries $A[\underline{x}, \underline{y}] \twoheadrightarrow R$ to coproducts in $(X/A)_{\mathbb{\Delta}}$. So choose one surjection $B_0 = A[\underline{x}] \twoheadrightarrow R$ then take the Cech nerve

$$B_0 \to B_0 \otimes_A B_0 \to B_0 \otimes_A B_0 \otimes_A B_0 \to \cdots$$

which all have maps to R. Therefore we get a cosimplicial object $(C^\bullet, IC^\bullet)$ in $(X/A)_{\mathbb{\Delta}}$.

Lemma 6.2.4. Suppose that (A, IA) is a prism then $((A/I)/A)_{\mathbb{\Delta}}$ produces $\mathbb{\Delta}_{(A/I)/A}$ which is the complex A supported in degree zero. This means the higher cohomology of the structure sheaf of a prism is zero.

Proposition 6.2.5. The cochain complex associated to $C^\bullet$ computes $\Delta_{X/A}$. The "ech-Alexander complex".