1 The Construction

Consider a probability space $X = (W, \Sigma, \mathbb{P})$ where W is a set of worlds, Σ is a σ-algebra, $\mathbb{P}$ a probability function. We are going to define a lift to uncountably infinite sequences of worlds.

Definition 1.0.1. The probability space is the product space,

$$X_\infty = \prod_{\alpha < \omega_1} X$$

1.1 Products of Measure Spaces

Definition 1.1.1. The product of a (possibility infinite) collection of probability spaces $\{X_i = (W_i, \Sigma_i, \mathbb{P}_i)\}_{i \in I}$ has the following components. The underlying set is,

$$W = \prod_{i \in I} W_i$$

with a σ-algebra constructed from the following the basis sets,

$$\mathcal{B} = \left\{ \prod_{i \in I} S_i \;\middle|\; S_i \in \Sigma_i \text{ and } S_i = W_i \text{ for all but finitely many } i \right\}$$

where we let $\Sigma = \overline{\mathcal{B}}$ be the smallest σ-algebra containing $\mathcal{B}$ meaning the closure of $\mathcal{B}$ under complements and countable unions. Finally, we define the probability measure $\mathbb{P}$ on the basis sets,

$$\mathbb{P}\left(\prod_{i \in I} S_i \right) = \prod_{i \in I} \mathbb{P}_i(S_i)$$

which makes sense because $\mathbb{P}_i(S_i) = 1$ for all but finitely many $i \in I$. Then Caratheodory's theorem proves that $\mathbb{P}$ extends uniquely to Σ.

Remark. It is worth explaining why the product σ-algebra has this strange finiteness condition. One reason is that otherwise we would not know that,

$$\prod_{i \in I} \mathbb{P}_i(S_i)$$

is meaningful without worrying deeply about convergence issues. A more fundamental issue is the following. In general, product objects are distinguished by the following *universal property.* For any test space T the two sets of data are equivalent,

$$f : T > \prod_{i \in I} X_i \iff \{f_i : T > X_i\}_{i \in I}$$

Let's do an example to see why this requires some finiteness (really its a countability) restriction on the basis sets. Let I be an uncountable set. The identity function $\mathrm{id} : \mathbb{R} > \mathbb{R}$ is of course measurable. So this condition should tell us that these glue together to give a measurable function,

$$f : \mathbb{R} > \prod_{i \in I} \mathbb{R} \qquad f(x) = (x, x, x, \dots)$$

However, consider,

$$f^{-1}\left(\prod_{i\in I} S_i\right) = \{x \in \mathbb{R} \mid f(x) = (x, x, \dots) \in \prod_{i\in I} A_i\} = \bigcap_{i\in I} A_i$$

However, the sets A_i are arbitrary measurable sets so this can only be possibly if the intersection of arbitrarily many measurable sets is measurable. This is false (assuming the axiom of choice). Let $S \subset \mathbb{R}$ be any non-measurable set and consider,

$$S = \bigcap_{x\in(\mathbb{R}\setminus S)} (\mathbb{R}\setminus\{x\})$$

but clearly the sets $\mathbb{R}\setminus\{x\}$ are measurable.

1.2 Bacon's Selection Function

Now that we have a probability space $X_\infty = (W_\infty, \Sigma_\infty, \mathbb{P}_\infty)$ we can use the structure of ω_1 to define a selection function.

Remark. Here we equivocate between an ordinal α and its set representation as $\{\beta \mid \beta < \alpha\}$. For example,

$$\begin{aligned}
0 &\iff \{\} \\
1 &\iff \{0\} \\
2 &\iff \{0, 1\} \\
&\vdots \\
\omega &\iff \{0, 1, 2, \dots\} = \mathbb{N}
\end{aligned}$$

Definition 1.2.1. For $\alpha < \omega_1$ and $x, y \in W_\infty$ we view $\alpha \subset \omega_1$ and say $x \sim_\alpha y$ iff $x|_\alpha = y|_\alpha$. Say that A is *α-insensitive* if,

$$\forall x, y \in W_\infty : x \sim_\alpha y \implies (x \in A \iff y \in A)$$

Remark. Explicitly,

(a) $x \sim_0 y$ is vacuously true

(b) $x \sim_1 y \iff x(0) = y(0)$

(c) $x \sim_n y \iff x(m) = y(m)$ for all $m < n$

(d) $x \sim_\omega y \iff x(m) = y(m)$ for all $m \in \mathbb{N}$

(e) $x \sim_{\omega+1} y \iff x(m) = y(m)$ for all $m \in \mathbb{N}$ and $x(\omega) = y(\omega)$.

Definition 1.2.2. For $A \in \Sigma_\infty$ define the rank of A,

$$\text{rank}(A) = \min\{\alpha \mid A \text{ is } \omega^\alpha\text{-insensitive}\}$$

Proposition 1.2.3. For any $A \in \Sigma_\infty$ the rank exists and $\text{rank}\,(A) < \omega_1$.

Proof. Because any nonempty set of ordinals has a least element $\text{rank}(A) < \omega_1$ is equivalent to: there is some $\alpha < \omega_1$ such that A is α-insensitive.

We will prove that the set $\Sigma_{\text{rank}} \subset \mathcal{P}(W_\infty)$ of sets $A \subset W_\infty$ with $\text{rank}(A) < \omega_1$ is a σ-algebra containing $\mathcal{B}$ This suffices because then $\Sigma_\infty \subset \Sigma_{\text{rank}}$ as Σ_∞ is the *smallest* σ-algebra containing $\mathcal{B}$.

First, for any $A \in \mathcal{B}$ using the finiteness condition, take $\alpha < \omega_1$ to be the maximum of the nontrivial indices. If A is α-insensitive then A^C is α-insensitive so $A \in \Sigma_{\text{rank}} \iff A^C \in \Sigma_{\text{rank}}$. Finally, we need to show closure under countable unions.

For any sequence $\{A_i\}_{i \in I}$ if A_i is α-insensitive for each i then $A = \bigcup_{i \in I} A_i$ is α-insensitive therefore if A_i is α_i-insensitive then letting,

$$\alpha = \sup\{\alpha_i \mid i \in I\}$$

we see that each A_i is α-insensitive and hence A is α-insensitive the only issue is that $\alpha = \omega_1$ is possible. However, if I is countable then α is a supremum of a countable set of countable ordinals and hence $\alpha < \omega_1$. Thus if $A_i \in \Sigma_{\text{rank}}$ then $A \in \Sigma_{\text{rank}}$ proving that Σ_{rank} is a σ-algebra. $\square$

Definition 1.2.4. For $\pi \in W_\infty$ define $\pi[\alpha]$ to be the sequence $\pi[\alpha](\beta) = \pi(\alpha + \beta)$. This is the same sequence "forgetting the first α terms."

Definition 1.2.5. We say that π *occurs* in A if there is some integer $i \in \mathbb{N}$ such that $\pi[\omega^\alpha \cdot i] \in A$. If π occurs in A we call the minimal such value of i the *index* of π with respect to A.

Definition 1.2.6. The selection function $f : \Sigma_\infty \times W_\infty > W_\infty$ is defined by,

$$f(A, \pi) = \begin{cases} \bot & A = \varnothing \\ \pi[\omega^\alpha \cdot i] & \pi \text{ occurs in } A \text{ and } i \text{ is the index} \\ \tau_A & \pi \text{ does no occur in } A \end{cases}$$

where $\bot$ is the list of contradictory worlds

Definition 1.2.7. Given $A, B \in \Sigma_\infty$ define the implicative,

$$(A \Longrightarrow B) := \{\pi \in W_\infty \mid f(A, \pi) \in B\}$$

Theorem 1.2.8. For any $A, B \in \Sigma_\infty$ we have $(A \Longrightarrow B) \in \Sigma_\infty$ and if $\mathbb{P}(A) > 0$ then,

$$\mathbb{P}(A \Longrightarrow B) = \mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

Proof. Let $\alpha = \text{rank}(A)$. Then,

$$(A \Longrightarrow B) = \left[\bigcup_{i \in \mathbb{N}} \{\pi \mid \pi[\omega^\alpha \cdot i] \in A \cap B\} \right] \cup \{\pi \mid \text{not occurring in } A \text{ and } \tau_A \in B\}$$

Each $\{\pi \mid \pi[\omega^\alpha \cdot i] \in A \cap B\} \in \Sigma_\infty$ because $\omega^\alpha \cdot i$ is countable. Thus the first term and the second,

$$\{\pi \mid \text{not occurring in } A\} = \bigcap_{i \in \mathbb{N}} \{\pi \mid \pi[\omega^\alpha \cdot i] \notin A\}$$

are both measurable. Now since $\text{rank}(A) = \alpha$ the sets $S_i = \{\pi \mid \pi[\omega^\alpha \cdot i] \notin A\}$ are independent with equal probability (they are each the set of π when restricted to the range $\omega^\alpha \cdot i$ to $\omega^\alpha \cdot (i+1)$ lands in the interesting part of A) so,

$$\mathbb{P}(\{\pi \mid \text{not occurring in } A\}) = \lim_{n > \infty} \mathbb{P}(A^C)^n = 0$$

because $P(A) > 0$. Likewise, we can expand the first term as a disjoint union,

$$\bigcup_{i \in \mathbb{N}} \{\pi \mid \pi[\omega^\alpha \cdot i] \in A \cap B\} = \bigcup_{i \in \mathbb{N}} \bigcap_{j < i} S_j \cap \{\pi \mid \pi[\omega^\alpha \cdot i] \in A \cap B\}$$

The union is disjoint and the intersection is over independent sets so,

$$\mathbb{P}(A \Longrightarrow B) = \sum_{i=1}^{\infty} \mathbb{P}(S_0) \cdots \mathbb{P}(S_{i-1}) \mathbb{P}(A \cap B) = \sum_{i=1}^{\infty} \mathbb{P}(A^C)^i \mathbb{P}(A \cap B) = \frac{\mathbb{P}(A \cap B)}{1 - \mathbb{P}(A^C)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

□

2 Triviality Results

2.1 Probabilistic Triviality

Definition 2.1.1. We say a probability space $X = (W, \Sigma, \mathbb{P})$ is *non-trivial* if there exist events $A, B \in \Sigma$ such that,

(a) $\mathbb{P}(A) > 0$

(b) $\mathbb{P}(B) > 0$

(c) $\mathbb{P}(A \wedge B) = 0$

(d) $\mathbb{P}(A \vee B) < 1$

otherwise we say that X is *trivial*.

Lemma 2.1.2. The following are equivalent,

(a) X is trivial

(b) $\mathbb{P}(B|A) = \mathbb{P}(B)$ for all $A, B \in \Sigma$ with $\mathbb{P}(A \wedge B) > 0$ and $\mathbb{P}(A \wedge \neg B) > 0$

(c) all $A, B \in \Sigma$ are independent or $\mathbb{P}(A \wedge B) = 0$ or $\mathbb{P}(A \wedge \neg B) = 0$

Proof. Suppose X is trivial. If $\mathbb{P}(A \wedge B) > 0$ and $\mathbb{P}(A \wedge \neg B)$ then applying triviality $\mathbb{P}(A) = \mathbb{P}((A \wedge B) \vee (A \wedge \neg B)) = 1$. Hence, $\mathbb{P}(B|A) = \mathbb{P}(A \wedge B) = \mathbb{P}(B)$. Also (b) and (c) are equivalent since $\mathbb{P}(B|A) = \mathbb{P}(B)$ iff $\mathbb{P}(A \wedge B) = \mathbb{P}(A)\mathbb{P}(B)$ iff A and B are independent. Finally, assume (c) then we claim that X is trivial. Indeed, if $A, B \in \Sigma$ with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$ and $\mathbb{P}(A \wedge B) = 0$ then let $C = A \vee B$. Then $\mathbb{P}(C \wedge A) = \mathbb{P}(A) > 0$ and $\mathbb{P}(C \wedge \neg A) = \mathbb{P}(B) > 0$ so by (c) $\mathbb{P}(C \wedge A) = \mathbb{P}(C)\mathbb{P}(A)$ but $\mathbb{P}(C \wedge A) = \mathbb{P}(A)$ and thus $\mathbb{P}(C) = 1$ proving triviality. □

Lemma 2.1.3 (Lewis)**.** Let $0 < \mathbb{P}(B) < 1$. The following cannot all be true,

(a) $\mathbb{P}(A > B|B) = 1$

(b) $\mathbb{P}(A > B | \neg B) = 0$

(c) $\mathbb{P}(A > B) \neq \mathbb{P}(B)$

Proof. Suppose (a) and (b) then the law of total probability gives,

$$\mathbb{P}(A > B) = \mathbb{P}(A > B | B)\mathbb{P}(B) + \mathbb{P}(A > B | \neg B)\mathbb{P}(\neg B) = \mathbb{P}(B)$$

□

Corollary 2.1.4. If the following hold,

(L1) for all $A, B \in \Sigma$ with $\mathbb{P}(A \wedge B) > 0$ and $\mathbb{P}(A \wedge \neg B) > 0$ we have $\mathbb{P}(A > B | B) = 1$

(L2) for all $A, B \in \Sigma$ with $\mathbb{P}(A \wedge B) > 0$ and $\mathbb{P}(A \wedge \neg B) > 0$ we have $\mathbb{P}(A > B | \neg B) = 0$

then X is trivial.

Corollary 2.1.5. If for all $A, B, C \in \Sigma$ with $\mathbb{P}(A \wedge C) > 0$,

$$\mathbb{P}(A > B | C) = \mathbb{P}(B | A \wedge C)$$

then X is trivial.

Proposition 2.1.6 (Lewis)**.** The set of events for which the thesis holds for all conditionalized probability measures is trivial.

Proof. We verify the property,

$$\mathbb{P}(A > B | C) = \mathbb{P}(B | A \wedge C)$$

when $\mathbb{P}(A \wedge C) > 0$. Indeed,

$$\mathbb{P}_C(A > B) := \mathbb{P}(A > B | C)$$

and since the thesis holds for $\mathbb{P}_C$ we have,

$$\mathbb{P}_C(A > B) = \mathbb{P}_C(B | A) = \frac{\mathbb{P}_C(B \wedge A)}{\mathbb{P}_C(A)} = \frac{\mathbb{P}(B \wedge A | C)}{\mathbb{P}(A | C)} = \frac{\mathbb{P}(B \wedge A \wedge C)\mathbb{P}(C)}{\mathbb{P}(A \wedge C)\mathbb{P}(C)} = \mathbb{P}(B | A \wedge C)$$

□

Remark. Stalnaker claims that the above argument relies on the hidden assumption: "metaphysical realism" that the proposition expressed by a conditional sentence is independent of the probability function. This he claims is implicit in that the content of $A > B$ is the same in the contexts $\mathbb{P}(A > B)$ and $\mathbb{P}_C(A > B)$ which is needed in claiming that,

$$\mathbb{P}_C(A > B) = \mathbb{P}(A > B | C)$$

The following argument of Fitelson avoids conditionalization and therefore removes this assumption as well as the assumption that the thesis need be preserved under conditionalization.

Proposition 2.1.7 (Fitelson)**.** The set of events $A, B \in \Sigma$ such that,

(PIE) $\mathbb{P}(A > (B > C)) = \mathbb{P}((A \wedge B) > C)$

(T) $\mathbb{P}(A > B) = \mathbb{P}(B | A)$ for $\mathbb{P}(A) > 0$

is trivial.

Proof. We verify the property,

$$\mathbb{P}(A > B|C) = \mathbb{P}(B|A \wedge C)$$

as follows: using $\mathbb{P}(C) > 0$ the thesis says,

$$\mathbb{P}(A > B|C) = \mathbb{P}(C > (A > B))$$

using PIE,

$$\mathbb{P}(C > (A > B)) = \mathbb{P}((A \wedge C) > B)$$

then because $\mathbb{P}(A \wedge C) > 0$ we can apply the thesis to conclude,

$$\mathbb{P}((A \wedge C) > B) = \mathbb{P}(B|A \wedge C)$$

□

2.2 Stalnaker's Triviality Result

Remark. Stalnaker isolates the fragment of the propositional calculus of the conditional which is problematic when combined with the thesis. (CITE BACON's OTHER PAPER FOR MORE DETAIL)

Definition 2.2.1. Stalnaker's logic **C2** for the conditional is generated on the language $\mathcal{L}$ with signature $\sigma = \{\wedge, \vee, \neg, \rightarrow, >\}$ via the standard Hilbert-style calculus for the fragment $\{\wedge, \vee, \neg, \rightarrow\}$ and additional axioms,

LT $A > \top$

ID $A > A$

AND $((A > B) \wedge (A > C)) \rightarrow (A > (B \wedge C))$

OR $((A > C) \wedge (B > C)) \rightarrow ((A \vee B) > C)$

CCut $((A > V) \wedge ((A \wedge B) > C)) \rightarrow (A > C)$

CMon $((A > B) \wedge (A > C)) \rightarrow ((A \wedge B) > C)$

CSO $((A > B) \wedge (B > A)) \rightarrow ((A > C) \equiv (B > C))$

SM $(A > B) \rightarrow (A \rightarrow B)$

CS $(A \wedge B) \rightarrow (A \rightarrow B)$

RMon $((A \rightarrow B) \wedge \neg(A > \neg C)) \rightarrow ((A \wedge C) > B)$

CEM $(A > B) \vee (A > \neg B)$

along with additional rules of inference,

LLE $B \equiv C \vdash (B > A) \equiv (C > A)$

RW $(B \rightarrow C) \vdash (A > B) \rightarrow (A > C)$

RCK $(B_1 \wedge \cdots \wedge B_n) \to C \vdash (A > B_1) \wedge \cdots \wedge (A > B_n) \to (A > C)$.

Lemma 2.2.2. The schema *Strict Centering* follows from **C2**,

C2 $A \wedge (A \to B) \equiv A \wedge B$

Proof. DO THIS IT'S CLEAR $\square$

Remark. Here, $\to$ is the material conditional so $A \to B :\equiv \neg A \vee B$.

Definition 2.2.3. A *credence-conditional space* $X = (W, \Sigma, \mathbb{P}, \Longrightarrow)$ is a probability space $(W, \Sigma, \mathbb{P})$ along with a binary operation $\Longrightarrow$ on Σ. An *interpretation function* is an assignment of propositional variables, $p \mapsto [\![p]\!] \in \Sigma$ extended to $\mathcal{L}$ via,

(a) $[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$

(b) $[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$

(c) $[\![\neg A]\!] = W \setminus [\![A]\!]$

(d) $[\![A > B]\!] = [\![A]\!] \Longrightarrow [\![B]\!]$.

We say X *verifies almost surely* a sentence $A \in \mathcal{L}$ if $\mathbb{P}([\![A]\!]) = 1$ for any interpretation function $[\![\bullet]\!]$ which we write as $X \models_{\text{a.s.}} A$.

Definition 2.2.4. A credence-conditional space X is a *model* of a theory Γ over σ if $X \models_{\text{a.s.}} \Gamma$.

(WHAT IS THE CORRECT THING TO SAY ABOUT THE RULES OF INFERENCE OR CAN I MAKE THEM JUST AXIOMS?)

Theorem 2.2.5 (Stalnaker)**.** Suppose that X is a credence-conditional space such that,

(a) X is a model of **C2** i.e. $X \models_{\text{a.s.}}$ **C2**

(b) X verifies the thesis i.e. $\mathbb{P}(A \Longrightarrow B) = \mathbb{P}(B|A)$ for all $A, B \in \Sigma$ with $\mathbb{P}(A) > 0$.

then X is a trivial probability space.

Proof. Step 1, we show that if $\mathbb{P}(\neg A) > 0$ then,

$$\mathbb{P}(A > B|\neg A) = \mathbb{P}(A > B)$$

Indeed,

$$\mathbb{P}(A > B|\neg A) = \frac{\mathbb{P}(A > B) - \mathbb{P}(A \wedge (A > B))}{\mathbb{P}(\neg A)} = \frac{\mathbb{P}(A > B) - \mathbb{P}(A \wedge B)}{\mathbb{P}(\neg A)}$$

If $\mathbb{P}(A) = 0$ then we conclude $\mathbb{P}(A > B|\neg A) = \mathbb{P}(A > B)$. Otherwise, we may apply the thesis,

$$\mathbb{P}(A > B|\neg A) = \frac{\mathbb{P}(A \wedge B) - \mathbb{P}(A \wedge B)\mathbb{P}(A)}{\mathbb{P}(A)\mathbb{P}(\neg A)} = \frac{\mathbb{P}(A \wedge B)}{\mathbb{P}(A)} = \mathbb{P}(B|A) = \mathbb{P}(A > B)$$

$\square$

Theorem 2.2.6. Bacon's Model X_∞ satisfies,

(a) $X_\infty \models_{\text{a.s.}}$ **C2**$\setminus$CSO

(b) X_∞ verifies the thesis: $\mathbb{P}_\infty(A \Longrightarrow B) = \mathbb{P}(B|A)$ for $A, B \in \Sigma_\infty$ with $\mathbb{P}_\infty(A) > 0$.

Proof. WORK IN PROGRESS $\square$

Corollary 2.2.7. Bacon's model X_∞ must violate CSO.

Remark. Bacon proposes a number of potential counter-examples to CSO in natural language [**Bacon**].

3 Appendix: relation of my notation to Bacon's

In Bacon's paper, Stalnaker's Thesis in Context, he formulates the construction of his semantics in a somewhat different form. He writes $W_\alpha = W^{\omega^\alpha}$ which has the inductive form,

(a) $W_0 = W$

(b) $W_{\alpha+1} \cong W_\alpha^\omega$

Then abusing notation to write $A \times W_\infty$ for $A \subset W_\alpha$ to mean the set of sequences starting π such that $\pi|_{\omega^\alpha} \in A$ (with the remainder arbitrary), he defines inductively a filtered σ-algebra,

(a) $\Sigma_0 = \{A \times W_\infty \mid A \in \Sigma\}$

(b) $\Sigma_{\alpha+1} = \overline{\{A_0 \times \cdots \times A_n \times W_\infty \mid n \in \mathbb{N} \text{ and } A_i \times W_\infty \in \Sigma_\alpha \text{ for } 0 \le i \le n\}}$

(c) $\Sigma_\gamma = \overline{\bigcup_{\alpha<\gamma} \Sigma_\alpha}$

Then Bacon's definition of $\Sigma_\infty^{(B)}$ is the union of this filtered sequence of σ-algebras.

Proposition 3.0.1. These definitions agree, meaning $\Sigma_\infty^{(B)} = \Sigma_\infty$ and moreover,

$$A \in \Sigma_\alpha \iff \text{rank}(A) \le \alpha$$

Proof. We prove by transfinite induction that $\Sigma_\alpha \subset \Sigma_\infty$. It is clear that $\Sigma_0 \subset \Sigma_\infty$. If $\Sigma_\alpha \subset \Sigma_\infty$ then for $A_i \in \Sigma_\alpha$ the set $A_0 \times \cdots \times A_n \times W_\infty$ is a finite intersection of shifts of elements of Σ_∞ and thus an element of Σ_∞ so $\Sigma_{\alpha+1} \subset \Sigma_\infty$. Finally, the limit ordinal case is clear because if each $\Sigma_\alpha \subset \Sigma_\infty$ then the union and closure (since Σ_∞ is a σ-algebra) also lies inside Σ_∞ proving by transfinite induction that all $\Sigma_\alpha \subset \Sigma_\infty$ and hence $\Sigma_\infty^{(B)} \subset \Sigma_\infty$.

To prove that Σ_∞ is exhausted by the Σ_α it suffices to prove the second claim since every $A \in \Sigma_\infty$ has some rank and hence would lie in some $\Sigma_\alpha \subset \Sigma_\infty^{(B)}$.

Now we prove,

$$A \in \Sigma_\alpha \iff \text{rank}(A) \le \alpha$$

by transfinite induction. Let $\Sigma_{\text{rank}\le\alpha} = \{A \in \Sigma_\infty \mid \text{rank}(A) \le \alpha\}$. I claim that $\Sigma_{\text{rank}\le\alpha}$ is the σ-algebra generated by $\mathcal{B} \cap \Sigma_{\text{rank}\le\alpha}$. It is clear that $\mathcal{B} \cap \Sigma_{\text{rank}\le\alpha} \subset \Sigma_\alpha$ so $\Sigma_{\text{rank}\le\alpha} \subset \Sigma_\alpha$. Now, $A \in \Sigma_0 \iff \text{rank}(A) = 0$ because both describe 1-insensitive sets. Now we show the inductive steps. Suppose $\Sigma_\alpha = \Sigma_{\text{rank}\le\alpha}$. For any set of the form,

$$A = A_0 \times \cdots \times A_n \times W_\infty$$

with $A_i \times W_\infty \in \Sigma_\alpha$ then by definition $\text{rank}(A) \le \alpha + 1$. Since $\Sigma_{\alpha+1}$ is generated by sets of this form, $\Sigma_{\alpha+1} \subset \Sigma_{\text{rank}\le\alpha+1}$. Finally, we do the limit step. For $\alpha < \gamma$ we have $\Sigma_\alpha = \Sigma_{\text{rank}\le\alpha} \subset \Sigma_{\text{rank}\le\gamma}$ so $\Sigma_\gamma \subset \Sigma_{\text{rank}\le\gamma}$. $\square$