ASTR GR6001 Radiative Processes
Assignment # 10

Benjamin Church

November 25, 2019

1 Problem 1

(a)

Recall that we split the Crab spectrum into a broken power law with two segments in frequency, $10^7\,\mathrm{Hz} - 10^{14}\,\mathrm{Hz}$ with spectral index $s_1 \approx 0.25$ and $10^{14}\,\mathrm{Hz} - 10^{24}\,\mathrm{Hz}$. The end points of these segments gives the minimum and maximum γ-factors and their slopes give the values of s and p. We need to do a linear (in log-log) fit to each segment. Define the parameter,

$$g = \left(\frac{f_\nu}{10^{-20}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}} \right) \cdot \left(\frac{\nu}{4 \cdot 10^6\,\mathrm{Hz}} \right)^s$$

with s chosen to fit a segment. This number is approximately constant on each segment. From linear fits to the data (note that the data is given in $\mathrm{W\,m^{-2}\,Hz^{-1}} = 10^{-3}\,\mathrm{erg\,cm^{-2}\,Hz^{-1}}$), I find,

segment	s	p	$a(p)$	g	$\gamma_{\min}$	$\gamma_{\max}$
1	0.25	1.5	0.085	7.07	79	$2.5 \cdot 10^5$
2	1.20	3.4	0.074	$7.54 \cdot 10^7$	$2.5 \cdot 10^5$	$2.5 \cdot 10^{10}$

Then the total luminosity of the Crab is,

$$\begin{aligned} L &= 4\pi d^2 \int_0^\infty f_\nu \, \mathrm{d}\nu \\ &= 4\pi d^2 \left(10^{-20}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}\right) \sum_{\text{segment}} g \left[\int_{\nu_{\min}}^{\nu_{\max}} \left(\frac{\nu}{4 \cdot 10^6\,\mathrm{Hz}} \right)^{-s} \mathrm{d}\nu \right] \end{aligned}$$

The Crab is at a distance of $d = 2000\,\mathrm{pc}$ so define,

$$x = \frac{\nu}{4 \cdot 10^6\,\mathrm{Hz}}$$

and then,

$$\begin{aligned} L &= 4\pi d^2 \left(10^{-20}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}\right) \sum_{\text{segment}} g \left[\int_{x_{\min}}^{x_{\max}} x^{-s} \, \mathrm{d}x \right] \\ &= \left(1.91 \cdot 10^{31}\,\mathrm{erg\,s^{-1}}\right) \sum_{\text{segment}} \frac{g}{s-1} \left[\frac{1}{x_{\min}^{s-1}} - \frac{1}{x_{\max}^{s-1}} \right] \\ &= 3.858 \cdot 10^{38}\,\mathrm{erg\,s^{-1}} \end{aligned}$$

(b)

The spectrum of electron energies can be computed from the synchrotron emission spectrum of the nebula. The emission spectrum is computed in the notes as,

$$\mathcal{E}_\nu = (1.7 \cdot 10^{-21}\ \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}) \left(\frac{n_0}{1\ \mathrm{cm^{-3}}}\right) a(p) \left(\frac{B}{1\ \mathrm{G}}\right)^{s+1} \cdot \left(\frac{\nu}{4 \cdot 10^6\ \mathrm{Hz}}\right)^{-s}$$

where s is the spectral index of the radiation and $a(p)$ is a numerical value depending on the index $p = 2s + 1$. Furthermore, the n_0 which appears above is the specific number density appearing in the power-law distribution for electron energy distribution,

$$n(\gamma) = n_0 \gamma^{-p}$$

and therefore, the energy in emitting electrons is,

$$E_{\mathrm{seg}} = V \int_{\gamma_{\mathrm{min}}}^{\gamma_{\mathrm{max}}} (\gamma mc^2) n(\gamma)\, \mathrm{d}\gamma \;= V \int_{\gamma_{\mathrm{min}}}^{\gamma_{\mathrm{max}}} (\gamma mc^2) n_0 \gamma^{-p}\, \mathrm{d}\gamma \;= V \frac{n_0 mc^2}{p-2} \left[\frac{1}{\gamma_{\mathrm{min}}^{p-2}} - \frac{1}{\gamma_{\mathrm{max}}^{p-2}}\right]$$

Therefore, it suffices to find p, n_0, and γ_{min} and γ_{max}.

Our available information is the observed specific flux spectrum which is related to $\mathcal{E}_\nu$ via the distance to the cluster and the total volume via,

$$f_\nu = \frac{\mathcal{E}_\nu V}{4\pi d^2}$$

since $\mathcal{E}_\nu V$ is the total power emitted by the nebula. Therefore, we have,

$$n_0 = (1\ \mathrm{cm^{-3}}) \cdot \frac{4\pi d^2 f_\nu}{V} \cdot (1.7 \cdot 10^{-21}\ \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}})^{-1} a(p)^{-1} \cdot \left(\frac{B}{1\ \mathrm{G}}\right)^{-(s+1)} \cdot \left(\frac{\nu}{4 \cdot 10^6\ \mathrm{Hz}}\right)^{s}$$

Computing this we will find the total energy in emitting electrons,

$$\begin{aligned} E &= V \sum_{\mathrm{segments}} \frac{n_0 mc^2}{p-2} \left[\frac{1}{\gamma_{\mathrm{min}}^{p-2}} - \frac{1}{\gamma_{\mathrm{max}}^{p-2}}\right] \\ &= (1\ \mathrm{cm^{-3}}) \cdot (4\pi d^2 f_\nu) \cdot (1.7 \cdot 10^{-21}\ \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}})^{-1} a(p)^{-1} \cdot \left(\frac{B}{1\ \mathrm{G}}\right)^{-(s+1)} \cdot \left(\frac{\nu}{4 \cdot 10^6\ \mathrm{Hz}}\right)^{s} \\ &\quad \cdot \sum_{\mathrm{segments}} \frac{mc^2}{p-2} \left[\frac{1}{\gamma_{\mathrm{min}}^{p-2}} - \frac{1}{\gamma_{\mathrm{max}}^{p-2}}\right] \end{aligned}$$

Again, we split the Crab spectrum into a broken power law with two segments in frequency, 10^7 Hz – 10^{14} Hz with spectral index $s_1 \approx 0.25$ and 10^{14} Hz – 10^{24} Hz. The end points of these segments gives the minimum and maximum γ-factors and their slopes give the values of s and p. We need to do a linear (in log-log) fit to each segment. Define the parameter,

$$g = \left(\frac{f_\nu}{10^{-20}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}}}\right) \cdot \left(\frac{\nu}{4 \cdot 10^6\ \mathrm{Hz}}\right)^{s}$$

with s chosen to fit a segment. This number is approximately constant on each segment. In terms of the parameters, g and s and plugging in for $B = 5 \cdot 10^{-4}$ G and $d = 2000$ pc we find,

$$\begin{aligned} E &= (2.8 \cdot 10^{45}) \sum_{\text{segments}} \left(5 \cdot 10^{-4}\right)^{-(s+1)} \cdot \frac{gmc^2}{a(p)(p-2)} \cdot \left[\frac{1}{\gamma_{\text{min}}^{p-2}} - \frac{1}{\gamma_{\text{max}}^{p-2}}\right] \\ &= (2.29 \cdot 10^{39}) \sum_{\text{segments}} \left(5 \cdot 10^{-4}\right)^{-(s+1)} \cdot \frac{g}{a(p)(p-2)} \cdot \left[\frac{1}{\gamma_{\text{min}}^{p-2}} - \frac{1}{\gamma_{\text{max}}^{p-2}}\right] \end{aligned}$$

From linear fits to the data (note that the data is given in W m^{-2} Hz^{-1} = 10^{-3} erg cm^{-2} Hz^{-1}), I find,

segment	s	p	$a(p)$	g	γ_{min}	γ_{max}
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Therefore, plugging in,

$$E_{\text{el}} = 3.35 \cdot 10^{48} \text{ erg}$$

Furthermore, the energy in the magnetic field is,

$$E_{\text{mag}} = V \cdot \left(\frac{B^2}{8\pi}\right)$$

Plugging in,

$$E_{\text{mag}} = 2.98 \cdot 10^{48} \text{ erg}$$

and therefore the cluster is approximately in equipartition between the emitting electrons and the magnetic field energy.

2 Problem 2

Consider a cluster of galaxies containing $10^{13}\,M_\odot$ of ionized plasma in the intracluster medium, in a volume of 0.1 Mpc^3, at a temperature of 5×10^7 K. We assume the composition is 90% H and 10% He.

(a)

We can use the formula for Bremsstrahlung luminosity,

$$\mathcal{E} = (1.4 \times 10^{-27} \text{ erg cm}^{-3}\text{ s}^{-1}) \left(\frac{T}{1\text{ K}}\right)^{\frac{1}{2}} (1 \text{ cm}^6) \sum_i n_e n_i Z^2 \overline{g}_{ff}(T)$$

For $T > 10^6$K we may take the Gaunt factor to be $\overline{g}_{ff}(T) \approx 1.2$. Furthermore, for the given abundances,

$$\sum_i n_e n_i Z^2 = 1.4\, n_H^2$$

Therefore, we find,

$$\mathcal{E} = (2.4 \cdot 10^{-27} \text{ erg cm}^{-3}\text{ s}^{-1}) \left(\frac{T}{1\text{ K}}\right)^{\frac{1}{2}} \cdot \left(\frac{n_H}{1\text{ cm}}\right)^2$$

Similarly, the mass density of the plasma is $\rho = 1.4\, m_H n_H$. Therefore,

$$n_H = \frac{M}{1.4\, V m_H}$$

Plugging in for the mass and volume of the plasma,

$$n_H = 2.89 \cdot 10^{-3}\ \mathrm{cm}^{-3}$$

Then plugging in to our formula,

$$L = \mathcal{E} V = 2.43 \cdot 10^{44}\ \mathrm{erg\ s}^{-1}$$

(b)

The Luminosity is proportional to $T^{\frac{1}{2}}$ i.e. we have,

$$L = A T^{\frac{1}{2}}$$

Note that the energy for a relativistic gas is,

$$E = 3NkT$$

and thus,

$$L = \dot{E} = 3Nk\dot{T} = -AT^{\frac{1}{2}}$$

This has a solution,

$$T(t) = \left(T_0^{\frac{1}{2}} - \frac{At}{6Nk}\right)^2$$

Therefore, the lifetime is,

$$t = \frac{6NkT_0^{\frac{1}{2}}}{A} = \frac{6NkT_0}{AT_0^{\frac{1}{2}}} = \frac{2E}{L}$$

Plugging in we find,

$$t = 4.59 \cdot 10^{10}\ \mathrm{yr}$$

(c)

The Compton y parameter is defined as,

$$y = \int n_e \frac{kT_e}{mc^2} \sigma_\tau\, \mathrm{d}\ell \approx \frac{kT}{mc^2} n_e \sigma_\tau L$$

where L is the line of sight, σ_τ is the Thomson cross section, T_e is the electron density, and n_e is the electron density. The density of electrons is, $n_e = 1.1\, n_H$. Assuming the cloud is approximately spherical, the line of sight is,

$$L = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$$

Therefore, plugging in, we find,

$$y = 4.02 \cdot 10^{-5}$$

The magnitude of the Sunyaev-Zel'dovich effect is given by,

$$\frac{\Delta T_{\mathrm{SZE}}}{T_{\mathrm{CMB}}} = \left(x \left(\frac{e^x + 1}{e^x - 1} \right) - 4 \right) \int n_e \frac{kT_e}{mc^2} \sigma_\tau \, \mathrm{d}\ell$$

Where,

$$x = \frac{h\nu}{kT_{\mathrm{CMB}}} = \left(\frac{\nu}{57\,\mathrm{GHz}} \right)$$

Then consider a frequency of $\nu = 120\,\mathrm{GHz}$ we have $x = 2.11$. Therefore,

$$\frac{\Delta T_{\mathrm{SZE}}}{T_{\mathrm{CMB}}} = -2.30 \cdot 10^{-4}$$

3 Problem 3

Consider Photo-ionized broad emission-line "clouds" in quasars are expected to have density $n \approx 10^{10}\ \mathrm{cm}^{-3}$, column density $N \approx 10^{22}\ \mathrm{cm}^{-2}$, and temperature $T = 15,000\ {}^\circ\mathrm{K}$. Now we consider the effects of free-free absorption. The specific optical depth is approximately,

$$\tau_\nu = N\alpha_\nu$$

Furthermore, in the Rayleigh - Jeans limit we can compute the absorption from free-free processes from the bremsstrahlung emission. This gives,

$$\begin{aligned}
\alpha_\nu &= \frac{4e^6}{3mhc} \left(\frac{2\pi}{3km} \right)^{\frac{1}{2}} \cdot \left(\frac{T}{1\,\mathrm{K}} \right)^{-\frac{1}{2}} \sum_i \frac{n_e n_i Z^2}{\nu^3} \left(1 - e^{-h\nu/kT}\right) \bar{g}_{ff} \\
&= (0.018\,\mathrm{cm}^2) \left(\frac{T}{1\,\mathrm{K}} \right)^{-\frac{3}{2}} \left(\frac{\nu}{1\,\mathrm{Hz}} \right)^{-2} (1\,\mathrm{cm}^6) \sum_i n_e n_i Z^2 \, \bar{g}_{ff}
\end{aligned}$$

We know that $n_e = 0.9\,n + 0.2\,n = 1.1\,n$ and we have,

$$\sum_i n_e n_i Z^2 = 1.4\,n^2$$

Furthermore, since the temperature is much higher than $10^6\ {}^\circ\mathrm{K}$ then $\bar{g}_{ff} \approx 1.2$. Therefore, we find,

$$\tau_\nu = (1.65 \cdot 10^{34}) \left(\frac{\nu}{1\,\mathrm{Hz}} \right)^{-2}$$

This implies that the cloud is optically thick due to free-free absorption for frequencies less than,

$$\nu_0 = 1.28 \cdot 10^{17}\ \mathrm{Hz}$$