1 The Kobayashi Pseudodistance

Definition 1.0.1. A *directed pair* (X, V) is a pair of a complex mnifold X and a holomorphic subbundle $V \subset T_X$.

Here let Δ be the unit disk in $\mathbb{C}$ and ρ the Poincare metric on Δ.

Definition 1.0.2. Let X be a complex manifold. The *Kobayashi pseudodistance* is the pseduometric defined,

$$d_X(p, q) = \inf_{\alpha} \ell(\alpha)$$

where α is a chain of holomorphic disk $f_i : \Delta \to X$ and points $p = p_0, p_1, \dots, p_k = q$ of X and pairs $a_1, b_1, \dots, a_k, b_k \in \Delta$ such that,

$$f_i(a_i) = p_{i-1} \quad f_i(b_i) = p_i$$

and the length $\ell(\alpha)$ of the chain is defined as,

$$\ell(\alpha) := \rho(a_1, b_1) + \cdots + \rho(a_k, b_k)$$

where ρ is the Poincare metric on Δ.

Example 1.0.3. Let $X = \mathbb{C}$ then $d_X = 0$. Indeed, by choosing larger and larger discs containing p, q their pullback to the unit disk is then closer and closer to the origin and hence have vanishing Poincare distance.

Remark. Recall the Schwartz-Pick lemma says that any holomorphic map $f : \Delta \to \Delta$ is a contraction for the Poincare metric. Therefore, $d_\Delta = \rho$.

Lemma 1.0.4. Let $f : X \to Y$ be holomorphic. Then $d_Y(f(x), f(y)) \leq d_X(x, y)$

Proof. Indeed, choosing any chain of disks $g_i : \Delta \to X$ computing $d_X(x, y)$ we see that $f \circ g_i$ is a chain of disks connecting $f(x)$ and $f(y)$ of the same length. Therefore,

$$d_Y(f(x), f(y)) = \inf_{\alpha} \ell(\alpha) \leq d_X(x, y)$$

□

Corollary 1.0.5. If $f : \mathbb{C} \to X$ is an entire curve then for $x, y \in f(\mathbb{C})$ we have $d_X(x, y) = 0$ meaning if f is nonconstant then d_X is degenerate along the image of f.

Proof. Indeed, let $z_1, z_2 \in \mathbb{C}$ map to x, y respectively. Then,

$$d_X(x, y) \leq d_{\mathbb{C}}(z_1, z_2) = 0$$

□

Brody's theorem is a converse to this result. We start by considering an infinitesimal anlogue of the Kobayashi pseudodistance. Let $v \in T_{X,x_0}$ be a holomorphic tangent vector at $x_0 \in X$ and define,

$$\mathbf{k}_X(v) = \inf\{\lambda > 0 \mid \exists f : \Delta \to X \text{ such that } f(0) = x_0 \text{ and } \lambda f'(0) = v\}$$

where $f : \Delta \to X$ is holomorphic. It is easy to check that holomorphic maps contract this pseudometric and for $X = \Delta$ it agrees with the Poincare metric.

Theorem 1.0.6. Let X be a complex manifold. Then,

$$d_X(p,q) = \inf_{\gamma} \int_{\gamma} \mathbf{k}_X(\gamma'(t))\mathrm{d}t$$

where the infimum is taken over all piecewise smooth curves joining p and q.

Definition 1.0.7. A *Brody curve* $f : \mathbb{C} \to X$ is an entire curve which has bounded derivative (wrt to some/any hermitian metric).

Theorem 1.0.8 (Brody)**.** Let X be a compact complex manifold. If d_X is degenerate then there exists a Brody curve in X.

Remark. Of course, in the case that X is compact any entire curve is automatically Brody.

2 Definitions

Definition 2.0.1. We say a directed pair (X, V) is,

(a) *Brody hyperbolic* if there does not exist a nonconstant entire map $f : \mathbb{C} \to X$ tangent to V

(b) *Kobyashi hyperbolic* if the Kobayashi pseudodistance is nondegenerate (i.e. it is a metric).

Theorem 2.0.2 (Brody)**.** Let X be a compact complex manifold. Then X is Kobayashi hyperbolic if and only if it is Brody hyperbolic.

Remark. Therefore, we will call manifolds with this property just "hyperbolic" or "analytically hyperbolic" for emphasis.

Definition 2.0.3. If X is a complex projective algebraic variety we say (X, V) is

(a) *algebraically hyperbolic* if there exists $\epsilon > 0$ such that for every complete intergral curve $C \subset X$ we have,

$$2g(C) - 2 \geq \epsilon \deg_H C$$

where $g(C)$ is the geometric genus of C

(b) *algebraically quasi-hyperbolic* if X contains finitely many genus 0 and genus 1 curves.

Theorem 2.0.4 (Demailly)**.** Let X be a smooth projective varitety. Then the following hold,

$$X \text{ is hyperbolic} \implies X \text{ is algebraically hyperbolic}$$

Theorem 2.0.5. If X is algebraically hyperbolic then X admits no nonconstant morphisms from an abelian variety.

Some references:

(a) Xi Chen

(b) Javanpeykar

2.1 The Green-Griffiths Locus and Jets

Theorem 2.1.1 (Demilly's Notes Theorem 7.9)**.** Let (X, V) be a direct projective manifold and A an ample line bundle. Then for any entire curve $f : \mathbb{C} \to X$ tangent to V and any $P \in H^0(X, E^{GG}_{k,m}(V^*) \otimes A^{-1})$ we have $P(f', f'', \ldots, f^{(k)}) = 0$ identically.

Therefore, if we fix an ample line bundle we can consider the locus cut out by all these differential equations.

Definition 2.1.2. The *Green-Griffiths locus* $\mathrm{GG}_A(X, V)$ is the set $x \in X$ such that for all $k > 0$ there exists a k-jet $\varphi_k : (\mathbb{C}, 0) \to (X, x)$ tangent to V so that for all $m > 0$ every global jet differential $P \in H^0(X, E^{GG}_{k,m}(V^*) \otimes A^{-1})$ satisfies $P(\varphi_k) = 0$.

Remark. The locus $\mathrm{GG}_A(X, V)$ is independent of the choice of ample line bundle (see Diverio and Rousseau Lemma 2.2. This paper also gives many examples showing that $\mathrm{Exc}(X)$ can be strictly smaller than $\mathrm{GG}(X)$. However, it is conjectured that if X is general type then $\mathrm{GG}(X) \subsetneq X$.

LOOK AT THE HILBERT MODULAR SURFACES FOR WHICH THE GG LOCUS IS EVERYTHING

3 Conjectures

Conjecture 3.0.1 (Kobayashi)**.** For $n \geq 2$ and $D \subset \mathbb{P}^n$ a very general hypersurface of degree $\deg D \geq 2n + 1$ then,

(a) D is hyperbolic

(b) $\mathbb{P}^n \backslash D$ is hyperbolic.

Conjecture 3.0.2 (Green-Griffiths-Lang)**.** Let X be a projective variety of general type. Then there exists a proper algebraic subvariety containing all non-constant entire curves $f : \mathbb{C} \to X$.

Conjecture 3.0.3 (Demailly)**.** If X is algebraically hyperbolic then X is hyperbolic.

Proposition 3.0.4. The Green-Griffiths-Lang conjecture implies the Demailly conjecture.

WHY?

Proof. Suppose X is algebraically hyperbolic. If X is not of general type then X has a fibration over its canonical model by varities of Kodaira dimension 0. (I NEED THAT IF NOT GENERAL TYPE THEN NOT ALGEBRAICALLY HYPERBOLIC DOES THIS FOLLOW FROM MMP) $\square$

4 Theorems

Theorem 4.0.1 (Bogomolov)**.** Let X is a smooth projective surface with $s_2(X) = c_1(X)^2 - c_2(X) > 0$ then X has finitely many genus 0 or genus 1 curves (i.e. it is algebraically quasi-hyperbolic).

Theorem 4.0.2 (McQuillian)**.** Let X is a smooth projective surface with $s_2(X) = c_1(X)^2 - c_2(X) > 0$ and X has *no* genus 0 or genus 1 curves then X is hyperbolic.

5 Bogomolov's Theorem

The notion of stability of a point on a space of linear representations of a reductive group, due to Mumford [10], leads to a notion of stabilite for fiber bundles over a curve, whose properties were studied in [13] and [19].

Definition 5.0.1. Over a smooth proper integral curve, a vector bundle E of rank $r(E)$ and degree $d(E)$ is *stable* (resp. *semistable*) is for every nonzero proper subbundle $F \subsetneq E$ we have,

$$\frac{d(F)}{r(F)} < \frac{d(E)}{r(E)} \quad \left(\text{resp.} \frac{d(F)}{r(F)} \leq \frac{d(E)}{r(E)}\right)$$

A vector bundle is *unstable* if it is not semistable.

Now let X be a smooth proper surface over a field k, and E a vector bundle over rank 2 over X. Then a linear representation $\rho : \mathrm{GL}_2 \to \mathrm{GL}(V)$ produces an associated bundle $E^{(\rho)} := E \times_{\mathrm{GL}_2} V$ of rank $\dim V$.

Definition 5.0.2. We say a rank 2 vector bundle is *instable* if there exists a representation $\rho : \mathrm{GL}_2 \to \mathrm{GL}(V)$ with $\det \rho = 1$ such that $E^{(\rho)}$ admits a nonzero section which vanishes at some point.

If the characteristic of k is zero, which we will assume for the remainder, then Bogomolov's instabilite criterion is simply expressed in terms of devissage of bundles of rank 2 (WHAT?). It is interesting to note that we can here short-circuit the theory and prove directly using these simpler methods.

There are many applications. We quote from memory a proof, elegant and algebraic, of the vanishing theorem of Kodaira-Ramanujan. In the remaning section we prove the following:

Theorem 5.0.3 (0.3)**.** Let X be a proper smooth surface of general type. Then Ω_X is not unstable.

As a consequence, we obtain the inequality $c_1^2 \leq 4c_2$ where c_1, c_2 are the Chern classes of the sheaf Ω_X^1 – improved by Miyaoka [9] which is the best form possible $c_1^2 \leq 3c_2$ - and a geometric result that we will develop here.

The problem is the following: can we show 'bound" the familly of curves of bounded geometric genus on a smooth proper surface X? We construct easily examples where the answer is negative. Bogomolov provides a partial solution in the case that X is a surface of general type. We summarize briefly the method.

Let $\pi : P = \mathbb{P}(\Omega_X^1) \to X$ be the canonical projection from the projectiviation of the canonical bundle. We construct on P a good linear system of divisors alowing it to be mapped to the projective space $\mathbb{P}^N$. If C is a smooth proper curve and $f : C \to X$ is a nonconstant morphism there is a lift $t_f : C \to P$ via the differential defined over points $\alpha \in P$ where f is unramified as $t_f(\alpha) = (f(\alpha), f(v_\alpha))$ where v_α is a nonzero tangent vector to C at α. We apply this to the normalizations of curves embedded in X and study their images in $\mathbb{P}^N$.

We prove the following result:

Theorem 5.0.4. Let X be a smooth proper surface minimal of general type.

(a) If $c_1^2 > c_2$ then the curves of bounded geomeric genus on X form a bounded family.

(b) If $c_1^2 \leq c_2$ and $\operatorname{rank} \mathrm{NS}(X) \geq 2$ then there exists a nonempty open cone $C \subset \mathrm{NS}(X)_{\mathbb{R}}$ containing the cone $\{z \mid z \in \mathrm{NS}(X)_{\mathbb{R}}, z^2 \leq 0\}$ such that for any closed cone C' contained in C the family of curves of bounded geometric grnus on X have image in $\mathrm{NS}(X)_{\mathbb{R}}$ contained in C' forms a bounded family. Moreover, any translate of C parallel to K_X has the same property.

As a corollary, we obtain finiteness of curves with negative self-intersection and bounded geometric genus on surfaces of general type. In particular a solution to Mordell's problem.

Let's point out finally that Bogomolov uses a powerful result of Deidenberg on differential equations [18] but a recent paper of Jouanalou [5] alows us to avoid the use of this sledgehammer.

5.1 Criteria for instability of vector bundles of rank 2 on surfaces

Considering the form of representations of PGL_2 we give a definition equivalent to above.

Definition 5.1.1. A vector bundle E of rank 2 on a surfaces is *unstable* if and only if there exists $n > 0$ such that $S^{2n}E \otimes (\det E)^{-n}$ has a nonzero section vanishing at some point of X.

5.1.1 Remark: devissage of vector bundles of rank 2

Let E be a vector bundle of rank 2 and L an invertible sheaf and $s : L \to E$ a nonzero map. The bidual M of E/L is reflexive and hence invertible (since X is a smooth surface), and the kernel L_1 of the homomorphism $E \to M$ is a larger invertible subsheaf of E contining L. We say that it is a saturated line bundle of E. The cokernel E/L_1 is torsion-free in rank 1, and hence of the form $I_Z \otimes M$ for M an invertible sheaf and I_Z a sheaf of ideals defining a closed subscheme Z of dimension 0 outside of which L' is a subbundle of E. We have a diagram of exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & E/L & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & E & \longrightarrow & I_Z \otimes M & \longrightarrow & 0 \end{array}$$

We will say that the second line is a devissage of E. We can deduce the Chern classes of E,

$$c_1(E) = c_1(L_1) + c_1(M) \qquad c_2(E) = c_1(L_1) \cdot c_1(M) + \deg Z$$

Theorem 5.1.2 (Bogomolov-Mumford)**.** A vector bundle E of rank 2 over a surface X is unstable if and only if there exists a devissage,

$$0 \to L \to E \to I_Z \otimes M \to 0$$

such that if $L' = L \otimes M^{-1} = L^2 \otimes (\det E)^{-1}$ then either,

(a) L' is in the cone $C_+ \subset \mathrm{NS}(X)_{\mathbb{Q}}$ generated by positive divisors (IS THIS NEF?)

(b) or $L' = \mathcal{O}_X$ and Z is nonempty

Moreoverm the devissage is unique.

We will prove this using only Mumford's theory of instability.

Let $P = \mathbb{P}(E)$ and $p : P \to X$ the projection and $\mathcal{O}_P(1)$ the canonical relatively ample bundle on P. A nonzero section $s \in H^0(X, S^{2n}E \otimes (\det E)^{-n})$ corresponds to a nonzero section $t \in H^0(P, \mathcal{O}_P(2n) \otimes p^*(\det E)^{-n}))$. Let $\xi \in X$ be the generic point and $K = \kappa(\xi)$. If we chose a basis of E_K then $s(\xi)$ corresponds to a homogeneous polynomial F of degree $2n$ in two variables. Since s is zero t some point of X, we know $s(\xi)$ is unstable for the action of PGL_2 on $S^{2n}E_K \otimes (\det E_K)^{-n}$ (WHY?). We deduce from the stability criterion using 1-parameter subgroups [11] that F has a root of order greater than n in the algebraic closure of K, so also in K (WHAT? WHY?), that's to say there exists an integer $r \geq 1$ and two polynomials G, H homogeneous of degrees 1 and $n - r$ respectively such that $F = G^{n+r}H$. Let D be the divisor of t and Δ the closure of the divisor defined over a generic point by G. We can write $D = (n+r)\Delta + \Delta'$ where has degree 1 and Δ' has degree $n - r$ on P. Therefore, there exist invertible modules L, L' on X such that,

$$\mathcal{O}_P(\Delta) = \mathcal{O}_P(1) \otimes p^*L \quad \mathcal{O}_P(\Delta') = \mathcal{O}_P(n-r) \otimes p^*L'$$

and hence,

$$(\det E)^{-n} = L^{n+r} \otimes L'$$

The divisor Δ corresponds to a section of $E \otimes L$ and thus an injection $L^{-1} \hookrightarrow E$ which by construction is saturated in E. We verify that it provides the desired devissage.

5.2 Operations on unstable bundles

Instability is preserved by passage to the dual and tensor product with a line bundle.

(a) Let $f : Y \to X$ be a surjective morphism of surfaces, E a vector bundle of rank 2 over X. Then E is unstable if and only if f^*E is.

(b) Let $f : Y \to X$ be a finite faithfully flat morphism of surfaces, F a fiber bundle of rank 2 on Y. Then if F is unstable so is f_*F.

5.3 Proof of Theorem 0.3

Suppose that Ω^1_X is unstable. Then there exists a devissage:

$$0 \to L \to \Omega^1_X \to I_Z \otimes M \to 0$$

and an integer $n > 0$ such that there is an injection $\mathcal{O}_X \hookrightarrow (L \otimes M^{-1})^{\otimes n}$. Note yhat $L \otimes M^{-1} = L^2 \otimes (\det \Omega^1_X)^{-1} = L^2 \otimes (\Omega^2_X)^{\otimes -1}$. Also, for $m \gg 0$,

$$h^0(L^{2m}) = h^0((L \otimes M^{-1})^{\otimes m} \otimes (\Omega^2_X)^{\otimes m}) \geq h^0((\Omega^2_X)^{\otimes m}) \in O(m^2)$$

Therefore, the theorem is a consequence of the following.

Theorem 5.3.1 (Bogomolov)**.** Let X be a smooth proper surface and $L \hookrightarrow \Omega^1_X$ an invertible subsheaf. Then $h^0(L^n) \in O(n)$.

First recall the pretty result of Castelnuovo and of Franchis which we will need for the proof.

Lemma 5.3.2 (4, 12)**.** Let ω_1, ω_2 be two holomorphic 1-forms on X which are linearly independent over k such that $\omega_1 \wedge \omega_2 = 0$. Then there exists a curve C which is proper and smooth over k of genus $g \geq 2$ and two holomorphic 1-forms θ_1, θ_2 on C and a morphism $u : X \to C$ such that $\omega_i = u^*\theta_i$ for $i = 1, 2$.

There exists a meromorphic function $f : X \dashrightarrow \mathbb{P}^1$ such that $\omega_2 = f\omega_1$. This defines a morphism $f : X' \to \mathbb{P}^1$ where X' is a blowup of X. Let $u : X' \to C \to \mathbb{P}^1$ be the Stein factorization. We have an exact sequence of modules of differentials,

$$0 \to u^*\Omega^1_C \to \Omega^1_{X'} \to \Omega^1_{X'/C} \to 0$$

We know $\omega_2 = f\omega_1$ and $0 = \mathrm{d}\omega_2 = \mathrm{d}f \wedge \omega_1$ (since ω_i are global holomorphic forms they are closed by Hodge theory).

WHY DOES IT WORK ON AN OPEN

But $\mathrm{d}f$ is pulled back from an open of U so ω_1 is also as it is parallel to $\mathrm{d}f$ hence also $\omega_2 = f\omega_1$. So above an open $U \subset C$ the forms ω_1, ω_2 are in the image of,

$$H^0(u^{-1}(U), u^*\Omega^1_C) = H^0(U, \Omega^1_C) \to H^0(u^{-1}(U), \Omega^1_{X'})$$

so we choose θ_1, θ_2 holomorphic forms on U which pull back to ω_1, ω_2. However, $u_*\mathcal{O}_{X'} = \mathcal{O}_C$ so θ_1, θ_2 extend to global sections of Ω_C because ω_1, ω_2 are global sections of $\Omega_{X'}$. Indeed, (WHY DOES IT EXTEND??) THIS SEEMS WRONG

Since ω_1, ω_2 are k-independent so are θ_1, θ_2. Hence $g(C) \geq 2$ and therefore the map $u : X' \to C$ contracts all rational curves and hence factors through $X' \to X$ giving the requried map.

5.3.1 Interlude: regularizing meromorphic 1-forms via covers

WHATIS THE CORRECT DEFINITION OF TAME?

Lemma 5.3.3. Let $f : X \to Y$ be a morphism of locally noetherian schemes. If $Z \subset Y$ is an irreducible subset of codimension $\leq r$ then either f does not dominate Z or there is some closed $Z' \subset X$ of codimension $\leq r$.

Proof. Using that $\mathrm{codim}\,(Z, Y) = \dim \mathcal{O}_{Y,\xi}$ where $\xi \in Z$ is the generic point we immediately reduce to the affine case. Either $\xi \notin f(X)$ and we are done or we can choose $f : U \to V$ a mp of affine schemes sending $\xi' \in U$ to $\xi \in V$. Let $\varphi : A \to B$ be a map of noetherian rings and $\mathfrak{p} \subset A$ a prime of height $\leq r$ in the image of $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$. Passing to $A_\mathfrak{p} \to B_\mathfrak{p}$ we need to find a prime $\mathfrak{q}$ of $B_\mathfrak{p}$ of height $\leq r$ with $\varphi^{-1}(\mathfrak{q})$ maximal. Then $\mathfrak{p}$ is the unique minimal prime over an ideal of definition $(x_1, \dots, x_r) \subset A_\mathfrak{p}$ generated by at most r elements by Tag 00KQ. Since $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is nonzero (the fiber is nonempty) the ideal $(x_1, \dots, x_r)B_\mathfrak{p}$ is proper hence, by the Krull height theorem, there exists a prime $\mathfrak{q}$ containing it of height $\leq r$. Then each $x_i \in \varphi^{-1}(\mathfrak{q})$ so $\mathfrak{p} \subset \varphi^{-1}(\mathfrak{q})$ and we conclude. □

Example 5.3.4. Noetherianity is essential in the above. Indeed, we could take a domain D with every nonzero prime of infinite height (as constructed in "Anti-archimedean rings and power series rings" by D.D. Anderson). Then for any nonzero nonunit $t \in D$ the map $k[t] \to D$ certainly falisfies the claim that the divisor $V(t)$ is in the image of a divisor (since there are none) although it is in the image of some prime.

Proposition 5.3.5. Let $f : X \to Y$ be a proper dominant morphism of locally noetherian integral S-schemes that are smooth over S at the generic points of all divisors. If f is tame and $\omega \in (\Omega_{Y/S})_\eta$ is a meromorphic differential such that $f^*\omega \in H^0(X, \Omega^{\vee\vee}_{X/S})$ is a global reflexive differential then $\omega \in H^0(Y, \Omega^{\vee\vee}_{Y/S}$ is a global reflexive differential.

Proof. Since Y is regular in codimension 1 it suffices to show that for each $\xi \in Y$ of height 1 that $\omega_\xi \in (\Omega_Y)_\xi$. Since f is proper and dominant it is surjective so we may choose $\xi' \in X$ mapping to ξ. The fiber over a divisor must contain a divisor of X so we can choose ξ' in the smooth locus. locus hence $f^*\omega$ is a well-defined differential form over $\mathcal{O}_{X,\xi'}$. Since $\mathcal{O}_{X,\xi'}$ is a noetherian local domain by [Hartshorne, Ex.4.11] there exists a DVR $R \subset \mathrm{Frac}\,(\mathcal{O}_{X,\xi'})$ dominating $\mathcal{O}_{X,\xi'}$.

FINISH

□

Remark. For example, this holds for any tame dominant map of normal proper varities over a perfect field.

COUNTEREXAMPLES

5.3.2 Completion of the Theorem

Either, for all $n > 0$ we have $h^0(X, L^{\otimes n}) \leq 1$ or there exists $n > 0$ such that $h^0(X, L^{\otimes n}) \geq 2$. In the second case, there is a standard method of extracting an n^{th}-root (WHAT THE HELL DOES THIS MEAN) to get $h^0(X, L) \geq 2$. In this case, there are two forms $\omega_1, \omega_2 \in H^0(X, \Omega^1_X)$ such that $\omega_1 \wedge \omega_2 = 0$ since they arise from the same subsheaf of rank 1. Therefore, by the lemma, there exists a curve C and a morphism $u : X \to C$ and an invertible sheaf L_0 on C such that,

$$L \subset u^*(L_0)$$

AGAIN WHY? so we can conclude that,

$$h^0(X, L^n) \leq h^0(C, L_0^n) \in O(n)$$

Corollary 5.3.6. If c_1 and c_2 are the Chern classes of Ω^1_X then $c_1^2 \leq 4c_2$.

5.3.3 Curves of bounded genus on a minimal surface of general type

We provide a few examples showing that X being general type plays an essential role, and that in the contrary case, there can be unbounded families of curves of fixed geometric genus.

Example 5.3.7. Let $X = \mathbb{P}^2$ then $\mathrm{NS}\,(X) = \mathbb{Z}$. There exist in the projective plane curves of bounded geometric genus but arbitrarily large degree.

Example 5.3.8. Let E be an elliptic curve without complex multiplication and let $X = E \times E$. Then $\mathrm{NS}\,(X) = \mathbb{Z} f_1 \oplus \mathbb{Z} f_2 \oplus \mathbb{Z}\Delta$ where f_i are the fiber classes and Δ is the diagonal. For every pair of integers (m, n) the image in X od the morphism $f_{m,n} : E \to X$ given by $f_{m,n}(\alpha) = (m\alpha, n\alpha)$ is a curve of class $m^2 f_1 + n^2 f_2 + (m-n)^2 \Delta$ and genus 1.

Example 5.3.9. Let B be a smooth proper curve and $\pi : X \to B$ a nonisotrivial (that is to say it does not become trivial after some finite base change $B' \to B$) minimal elliptic fibration admitting a section $\sigma : B \to X$ of infinite order. Let ω be the conormal bundle of σ. Then there exist global sections $g_2 \in H^0(X, \omega^4)$ and $g_3 \in H^0(X, \omega^6)$ such that X is the minimal resolution of the surface $Y \subset \mathbb{P}_B(\omega^2 \oplus \omega^3 \oplus \mathcal{O}_B)$ defined by the Weierstrass equation,

$$y^2 z = x^3 - g_2 x z^2 - g_3 z^3$$

Morover, ω is independent of the section σ as has degree $-\sigma(B)^2$. If the degree is zero, then g_2 and g_3 are constant and the fibration π is isotrivial. There exist infinitely many sections of negative self-intersection and the classes are algebraically distict.

Notation: we write K_X for a canonical divisor of X and T_X the tangent bundle and $\pi : \mathbb{P}(\Omega_X) \to X$ the canonical projection and $L = \mathcal{O}_P(1)$ the relatively ample bundle for π.

Let F be a invertible bundle on X. We note that F is a divisor of some linear system (DOES HE MEAN F IS THE ZERO LOCUS OF SOME SECTION). Moreover, for any rational number $\ell \in \mathbb{Q}$, we allow ourselves to form the sheaf ℓF, extending that we consider the tensor powers $(\ell F)^{\otimes m}$ for which m is such that $m\ell$ is an integer.

5.4 COnstruction of a good linear system of divisors on P

Proposition 5.4.1. Let F be an invertible sheaf on X and ℓ a rational positive number such that,

(a) $K \cdot F \geq 0$

(b) $(K + 2\ell F)^2 > 0$

(c) $c_1^2(\Omega_X \otimes \ell F) - c_2(\Omega_X \otimes \ell F) > 0$

Then for $m \gg 0$ the linear system $(L \otimes \pi^*(\ell F))^m$ defines a rational map $u_F : \mathbb{P}(\Omega^1_X) \dashrightarrow \mathbb{P}^N$ birational onto its image.

IT SEEMS WRONG THAT ℓF IS INSIDE THE S^m THIS GIVES $(\ell F)^{2m}$ NOT $(\ell F)^m$ AS SHOULD BE FROM PROJECTION FORMULA

Proof. By the theorem of Iitaka [20], it suffices to show that for $m \gg 0$,

$$h^0(P, (L \otimes \pi^*(\ell F))^m) = h^0(X, S^m(\Omega_X \otimes \ell F)) \geq O(m^3)$$

The Riemann-Roch formula for E shows that,

$$\chi(S^m E) = \frac{m(m+1)(m+2)}{24}(c_1^2(E) - 4c_2(E)) + \frac{m+1}{2}\left[\frac{m^2}{4}c_1^2(E) - \frac{m}{2}K_X \cdot c_1(E)\right] + (m+1)\chi(\mathcal{O}_X)$$

and hence for $m \gg 0$,

$$h^0(S^m(\Omega^1_X \otimes \ell F)) + h^2(S^m(\Omega^1_X \otimes \ell F)) \sim h^1(S^m(\Omega^1_X \otimes \ell F)) + \frac{m^3}{6}\left[c_1^2(\Omega_X \otimes \ell F) - c_2(\Omega^1_X \otimes \ell F)\right] \geq O(m^3)$$

By Serre duality, HOW DO I FIX THE DUAL AND S^m IN POSITIVE CHAR

$$h^2(S^m(\Omega_X \otimes \ell F)) = h^0(K \otimes S^m(T_X \otimes -\ell F))$$

Chosing some divisors D and D' ample and smooth such that,

$$\mathcal{O}_X(-D') \subset K \subset \mathcal{O}_X(D)$$

we find that,

$$\left| h^0(K \otimes S^m(T_X \otimes -\ell F)) - h^0(S^m(T_X \otimes -\ell F)) \right| \in O(m^2)$$

Therefore, we conclude by appealing to the following lemma. □

Lemma 5.4.2. For any $m > 0$ we have $H^0(S^m(T_X \otimes -\ell F)) = 0$.

THE m VS $2m$ DOESNT MAKE SENSE

Proof. We showed that $T_X \otimes -\ell F$ is not unstable. Hence, the only sections of $H^0(S^{2m}(T_X \otimes -\ell F) \otimes (\det(T_X \otimes -\ell F))^{-m})$ are nowhere vanishing. If we show for $m \gg 0$ that $H^0(\det(T_X \otimes -\ell F)^{-m})$ has a nonzero section with a zero at some point $x \in X$ then its product with a section $H^0(S^{2m}(T_X \otimes -\ell F))$ will give a contradiction. Thus, the result will follow from the definition,

$$\det(T_X \otimes -\ell F)^{-m} = m(K + 2\ell F)$$

and Riemann-Roch,

$$\chi(m(K + 2\ell F)) \sim \frac{m^2}{2}(K + 2\ell F)^2 \in O(m^2)$$

and therefore,

$$h^0(m(K + 2\ell F)) + h^2(m(K + 2\ell F)) \geq O(m^2)$$

by Serre duality,

$$h^0(m(K + 2\ell F)) = h^0(K - m(K + 2\ell F))$$

Since $K \cdot (K - m(K + 2\ell F)) = K^2 - mK \cdot (K + 2\ell F) < 0$ and K is nef (we assumed that X is minimal) $h^0(K - m(K + 2\ell F)) = 0$ for $m \gg 0$ giving the result. $\square$

Our any bundle F verifying the conditions of the properosition, we fix, once and for all, m and ℓ and let Z_F be the closed subset of $\mathbb{P}(\Omega_X)$ outside of which u_F is defined.

Definition 5.4.3. Let C be a curve embedded in C and $f : \widetilde{C} \to C$ its normalization. If $t_f(\widetilde{C})$ is not contained in (resp. is contained in) Z_F, we say that C is F-regular (resp. F-irregular).

5.5 Proof of Theorem 0.4

We suppose that $\mathcal{L} \hookrightarrow \Omega^1_X$ is a invertible subsheaf. If $h^0(X, \mathcal{L}^{\otimes n}) \leq 1$ for all n then we are done. Otherwise, there is some $n > 0$ such that $h^0(X, \mathcal{L}^{\otimes n}) \geq 2$. In this case, by passing to a cyclic cover we may assume that $h^0(X, \mathcal{L}) \geq 2$. Therefore, there are two independent 1-forms $\omega_1, \omega_2 \in H^0(X, \mathcal{L}) \subset H^0(X, \Omega^1_X)$ such that $\omega_1 \wedge \omega_2 = 0$ because they lie in the same 1-dimensional subspace at the generic point $\mathcal{L}_\eta \subset \Omega^1_{X,\eta}$. Therefore, we may apply Castelnuovo's lemma to produce a morphism $f : X \to C$ to some curve of genus $g \geq 2$ with ω_1, ω_2 pulled back along f. By the proof of this lemma, we see that any local section of $\mathcal{L}$ is pulled back along f hence $\mathcal{L} \hookrightarrow f^*\Omega_C$.

5.5.1 Ramified Cyclic Covers

Let X be a scheme and $\mathcal{L} \in \mathrm{Pic}(X)$ a line bundle and $s \in H^0(X, \mathcal{L}^{\otimes n})$ a nonzero section of some tensor power. Then we may form a finitely-presented sheaf of $\mathcal{O}_X$-algebras,

$$\mathcal{A} = \mathcal{O}_X \oplus t\mathcal{L}^{\otimes -1} \oplus \cdots \oplus t^{n-1}\mathcal{L}^{\otimes -(n-1)}$$

where multiplication is defined in the obvious manner,

$$(t^a f_1)(t^b t_2) = \begin{cases} t^{a+b} f_1 f_2 & a + b < n \\ t^{a+b-nk}[(s^\vee)^{\otimes k} \otimes \mathrm{id}](f_1 f_2) & nk \leq a + b < (n+1)k \end{cases}$$

where $[(s^\vee)^{\otimes k} \otimes \mathrm{id}] : \mathcal{L}^{\otimes -(a+b)} \to \mathcal{L}^{\otimes -(a+b-nk)}$. Then we define $X_{\mathcal{L},s} := \mathbf{Spec}_X(\mathcal{A})$. Over the locus where s is nonvanishing it is clear that $X_{\mathcal{L},s} \to X$ is a degree n cyclic cover which is étale for n nonzero in the base scheme.

Note that $\mathcal{A}$ can also be described as follows. Consider the symmetric algebra,

$$\mathrm{Sym}_{\bullet}(\mathcal{L}^{\vee}) = \bigoplus_{n=0}^{\infty} t^n \mathcal{L}^{\otimes -n} \twoheadrightarrow \mathcal{A}$$

which is the quotient as a sheaf of algebras by the ideal generated by $(t^n f - s^{\vee}(f))$ for local sections f of $\mathcal{L}^{\otimes -n}$. Therefore, $X_{\mathcal{L},s} \hookrightarrow \mathbb{V}_X(\mathcal{L})$ is a closed subscheme of the total space of the line bundle $\mathcal{L}$ which can be described as the locus of points (x, v) such that $v^n = s(x)$.

Note that under $\pi : \mathbb{V}_X(\mathcal{L}) \to X$ we get a canonical section $t \in H^0(\mathbb{V}_X(\mathcal{L}), \pi^*\mathcal{L})$ and hence for $f : X_{\mathcal{L},s} \to X$ there is a canonical section $t \in H^0(X_{\mathcal{L},s}, f^*\mathcal{L})$ such that $t^n = f^*s$.

Now suppose that $s_1, s_2 \in H^0(X, \mathcal{L}^{\otimes n})$ are two independent sections. Then by passing to the iterating cyclic cover, $X' = (X_{\mathcal{L},s_1})_{f^*\mathcal{L}, f^*s_2} \to X_{\mathcal{L},s_1} \to X$ we get $\mathcal{L}' = f^*\mathcal{L}$ and two canonical sections $t_1, t_2 \in H^0(X', \mathcal{L}')$ such that $t_i^n = f^*s_i$ for $i = 1, 2$.

Furthermore, suppose that n is invertible on the base and there is an injection $\mathcal{L} \hookrightarrow \Omega^1_X$. Then passing to the cyclic cover (which is generically étale) we get $f^*\mathcal{L} \hookrightarrow f^*\Omega^1_X \hookrightarrow \Omega^1_{X'}$ which is injective because it is at the generic point. Hence we reduce to the situation that $h^0(X, \mathcal{L}) \geq 2$.