Math 56: Proofs and Modern Mathematics
Homework 9 Solutions

Naomi Kraushar and Ben Church

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Problem 1 (Abbott, Exercise 2.3.11: Cesàro means). (a) Show that if (x_n) is a convergent sequence then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Given an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution. (a) Let $x = \lim_{n\to\infty} x_n$. Fix arbitrary $\varepsilon > 0$. By definition, there exists some $N_\varepsilon \in \mathbb{N}$ such that $n \geq N_\varepsilon$ implies $|x_n - x| < \varepsilon/2$. Nnow consider the sequence of averages for $n \geq N_\varepsilon$: we have

$$\begin{aligned}
|y_n - x| &= \left| \frac{\sum_{k=1}^n x_k}{n} - x \right| && \text{(definition of the average } y_n\text{)} \\
&= \left| \frac{\sum_{k=1}^n (x_k - x)}{n} \right| && \text{(moving } x \text{ inside the sum)} \\
&\leq \frac{1}{n} \sum_{k=1}^{N_\varepsilon} |x_k - x| + \frac{1}{n} \sum_{k=N_\varepsilon+1}^{n} |x_k - x|
\end{aligned}$$

(using the triangle inequality and taking $1/n$ out of the sums.)

The first sum is finite, and depends only on N_ε, and hence on ε, so let $S_\varepsilon = \sum_{k=1}^{N_\varepsilon} |x_k - x|$. Every term in the second sum is less than $\varepsilon/2$, so we have

$$\begin{aligned}
|y_n - x| &\leq \frac{1}{n} \sum_{k=1}^{N_\varepsilon} |x_k - x| + \frac{1}{n} \sum_{k=N_\varepsilon+1}^{n} |x_k - x| \\
&< \frac{S_\varepsilon}{n} + \frac{\varepsilon(n - N_\varepsilon + 1)}{2n} \\
&\leq \frac{S_\varepsilon}{n} + \frac{\varepsilon}{2} && \text{(since } n \geq N_\varepsilon \text{, so } n - N_\varepsilon + 1 \leq n.\text{)}
\end{aligned}$$

Recall that S_ε depends on ε, not n, so we can now choose n so large that $\frac{S_\varepsilon}{n} < \frac{\varepsilon}{2}$, which then gives $|y_n - x| < \varepsilon$. Hence the sequence of averages has the same limit as the original sequence.

(b) Let $x_n = (-1)^n$, so odd terms are -1 and even terms are 1. The sum of the first n terms is 0 if n is even, and -1 if n is odd. This gives us the sequence of averages

$$y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1/n & \text{if } n \text{ is odd} \end{cases}$$

Then for any $\varepsilon > 0$, if we choose N sufficiently large so that $1/N < \varepsilon$, then $|y_n| \leq 1/N < \varepsilon$ for all $n \geq N$, so $y_n \to 0$. Hence it is possible for a divergent sequence to have its sequence of averages converge.

Problem 2 (Abbott, Exercise 2.6.3)**.** If (x_n) and (y_n) are Cauchy sequences then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem implies that $(x_n + y_n)$ is convergent, hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

Solution. (a) We have that (x_n) and (y_n) are Cauchy sequence, so, by definition, there exists $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $|x_n - x_m| < \varepsilon/2$ and there exists $N_2 \in \mathbb{N}$ such that $m, n \geq N_2$ implies $|y_n - y_m| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$, so that for $m, n \geq N$, we have

$$\begin{aligned}
|(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\
&\leq |x_n - x_m| + |y_n - y_m| \qquad \text{(by the triangle inequality)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $(x_n + y_n)$ is a Cauchy sequence.

(b) For this part of the problem, we use Lemma 2.6.3: every Cauchy sequence is bounded. Hence there exists some $X, Y \in \mathbb{R}$ such that $|x_n| \leq X$ and $|y_n| \leq Y$ for all $n \in \mathbb{N}$. By definition, there exists $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $|x_n - x_m| < \varepsilon/2Y$ and there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|y_n - y_m| < \varepsilon/2X$. Let $N = \max\{N_1, N_2\}$, so that for $m, n \geq N$, we have

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \\
&\leq |x_n y_n - x_m y_n| + |x_m y_n - x_m y_m| \qquad \text{(by the triangle inequality)} \\
&= |x_n - x_m||y_n| + |x_m||y_n - y_m| \\
&< \left(\frac{\varepsilon}{2Y}\right) Y + X \left(\frac{\varepsilon}{2X}\right) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $(x_n y_n)$ is a Cauchy sequence.

Problem 3 (From the writing assignment)**.** Prove that if all the elements $b_n \geq 0$, then

$$\sum_{n=1}^{\infty} b_n = \sup \left\{ \sum_{i \in A} b_i : A \subset \mathbb{N}, A \text{ finite} \right\},$$

where the supremum is taken over all finite subsets of $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Solution. Let (s_N) be the sequence of partial sums, i.e. $s_N = \sum_{n=1}^{N} b_n$. Let X be the set of all partial sums, i.e. $X = \{s_N\}_{N \in \mathbb{N}}$ and let $Y = \left\{\sum_{i \in A} b_i : A \subset \mathbb{N}, A \text{ finite}\right\}$ be the set in the question. We will prove that $\sup X = \sup Y$. First, we observe that $\{1, 2, \ldots, N\}$ is a finite subset of $\mathbb{N}$, so that

$$s_N = \sum_{i \in \{1, \ldots, N\}} b_i$$

is in Y. Hence every partial sum is in the set Y, so $X \subset Y$, which means that $\sup X \leq \sup Y$.

Next, let A be an arbitrary finite subset of $\mathbb{N}$. Since A is finite, it has a maximal element N. Since $b_i \geq 0$ for all $i \in \mathbb{N}$, we have

$$\sum_{i \in A} b_i \leq \sum_{i \in A} b_i + \sum_{\substack{i \leq N \\ i \notin A}} b_i = \sum_{i=1}^{N} b_i = s_N.$$

Hence every element in Y is bounded above by some partial sum, which is an element in X, so $\sup Y \leq \sup X$. Since we also have $\sup X \leq \sup Y$, this means that $\sup X = \sup Y$.

Finally, we want to show that $\sum_{n=1}^{\infty} b_n = \sup X$. By definition, $\sum_{n=1}^{\infty} b_n = \lim_{N \to \infty} s_N$. Since $b_n \geq 0$ for all n, the sequence of partial sums is a monotone increasing sequence. From the proof of Theorem 2.4.2 (Monotone Convergence Theorem), if this sequence is bounded above, it converges to its supremum, which is $\sup X$. Hence in the case where the series converges, we have $\sum_{n=1}^{\infty} b_n = \sup X = \sup Y$ as required. If the sequence (s_N) is not bounded above, the supremum $\sup X = \infty$, and also $\lim s_N = \sum_{n=1}^{\infty} b_n = \infty$, so we again have $\sum_{n=1}^{\infty} = \sup X = \sup Y (= \infty)$, as required.

Problem 4. Recall that A is closed if whenever (a_n) is a sequence in A with $x = \lim a_n$ existing in $\mathbb{R}$, in fact $x \in A$. Show directly from the definition of being closed that

(a) Any intersection of any collection of closed sets is closed.

(b) The finite union of closed sets is closed.

(Hint: if (a_n) is a sequence in $C_1 \cup \cdots \cup C_N$, show that there is a subsequence (a_{n_k}) and an index j such that $a_{n_k} \in C_j$ for all k.)

Solution. (a) Let $\{C_\alpha\}_{\alpha \in A}$ be an arbitrary collection of closed sets in $\mathbb{R}$, with A some indexing set, and suppose that (a_n) is a sequence in $\bigcap_\alpha C_\alpha$ for all $\alpha \in A$, with $x = \lim a_n$ existing in $\mathbb{R}$. Since $a_n \in \bigcap_{\alpha \in A} C_\alpha$, we have $a_n \in C_\alpha$ for all $\alpha \in A$. Since every C_α is closed, we have $x \in C_\alpha$ for all $\alpha \in A$. Hence $x \in \bigcap_{\alpha \in A} C_\alpha$. Hence the intersection of closed sets is closed.

(b) Let $C_1, \ldots, C_N$ be a finite collection of closed sets in $\mathbb{R}$, and suppose that a_n is a sequence in $\bigcup_{j=1}^N C_j$ for all n, with $x = \lim a_n$ existing in $\mathbb{R}$. We now use the hint: suppose that there is no subsequence (a_{n_k}) in C_j for some j. This means that there are only finitely many terms of the original sequence in C_j. This means that if there is no subsequence of (a_n) in any C_j, then each C_j only has finitely many terms of the sequence, so the union $\bigcup_{j=1}^N C_j$ has only finitely many terms in the sequence, which contradicts the assumption that (a_n) is a sequence in $\bigcup_{j=1}^N C_j$. Hence there must be some index j such that C_j contains some subsequence (a_{n_k}) of the sequence (a_n). Since subsequences of a convergent sequence converge to the same limit, a_{n_k} converges to x, and since C_j is closed, this means that $x \in C_j \subset \bigcup_{j=1}^N C_j$. Hence the union does contain the limit, so finite unions of closed sets are closed.

Problem 5 (Abbott, Exercise 3.3.3)**.** Suppose that $K \subset \mathbb{R}$ is closed and bounded. Show that every sequence (a_n) in K has a subsequence converging to a point in K, i.e. that K is compact.

Solution. Because K is bounded and $a_n \in K$ this means that the sequence a_n is bounded. Therefore, by the Bolzano-Weierstrass theorem, there exists a subsequence $a_{n(j)}$ that coverges to some $a \in \mathbb{R}$. Therefore it suffices to show that $a \in K$. However, K is closed and each $a_{n(j)} \in K$ which implies that the limit $a \in K$ by Theorem 3.2.5.