Physics GR6037 Quantum Mechanics I
Assignment # 5

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Problem 16.

(a). Let $\hat{B}(\Delta p) = e^{i\Delta p \hat{x}/\hbar}$ then consider

$$\frac{\mathrm{d}}{\mathrm{d}p_0}\hat{B}^\dagger(p_0)\hat{p}\hat{B}(p_0) = \frac{i}{\hbar}\hat{B}^\dagger(p_0)\hat{p}\hat{x}\hat{B}(p_0)\hat{p} - \frac{i}{\hbar}\hat{B}^\dagger(p_0)\hat{x}\hat{p}\hat{B}(p_0) = \frac{-i}{\hbar}\hat{B}^\dagger(p_0)[\hat{x},\hat{p}]\hat{B}(p_0) = 1$$

Thus,

$$\hat{B}^\dagger(p_0)\hat{p}\hat{B}(p_0) = \hat{B}^\dagger(0)\hat{p}\hat{B}(0) + p_0 = \hat{p} + p_0$$

Therefore, applying the operator, $|\psi_{\Delta p}\rangle = \hat{B}(\Delta p)\,|\psi\rangle$ then

$$\langle\psi_{\Delta p}|\,\hat{p}\,|\psi_{\Delta p}\rangle = \langle\psi|\,\hat{B}^\dagger(\Delta p)\hat{p}\hat{B}(\Delta p)\,|\psi\rangle = \langle\psi|\,(\hat{p}+\Delta p)\,|\psi\rangle = \langle\hat{p}\rangle + \Delta p$$

(b). We can write $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$ so letting $\alpha = \sqrt{\frac{\hbar}{2m\omega}}\frac{\Delta p}{\hbar}$ we have,

$$\hat{B}(\Delta p) = e^{i\alpha(\hat{a}^\dagger + \hat{a})}$$

then because $[\hat{a},\hat{a}^\dagger] = 1$ which commutes with everything, so we can apply

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

Thus,

$$\hat{B}(\Delta p) = e^{i\alpha\hat{a}^\dagger}e^{i\alpha\hat{a}}e^{-\frac{1}{2}\alpha^2(\Delta p)^2}$$

However,

$$e^{i\alpha\hat{a}}\,|0\rangle = \left(1 + i\alpha\hat{a} + \frac{1}{2}(i\alpha\hat{a})^2 + \cdots\right)|0\rangle = |0\rangle$$

and therefore,

$$|\psi\rangle = \hat{B}(\Delta p)\,|0\rangle = e^{-\frac{1}{2}\alpha^2(\Delta p)^2}e^{i\alpha\hat{a}^\dagger}\,|0\rangle$$

Expanding the exponential,

$$|\psi\rangle = e^{-\frac{1}{2}\alpha^2(\Delta p)^2}\sum_{n=0}^{\infty}\frac{(i\alpha\Delta p)^n}{n!}(\hat{a}^\dagger)^n\,|0\rangle = e^{-\frac{1}{2}\alpha^2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{\sqrt{n!}}\,|n\rangle$$

Therefore, we can read off the propabilities,

$$P(0) = |\,\langle 0|\psi\rangle\,|^2 = e^{-\alpha^2} = \exp\left[-\frac{(\Delta p)^2}{2m\omega\hbar}\right]$$

(c). Simiarly,

$$P(n) = |\,\langle n|\psi\rangle\,|^2 = \frac{\alpha^{2n}e^{-\alpha^2}}{n!} = \frac{1}{n!}\left(\frac{(\Delta p)^2}{2m\omega\hbar}\right)^n\exp\left[-\frac{(\Delta p)^2}{2m\omega\hbar}\right]$$

Problem 17.

(a). Let

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x^2 + (\hat{p}_y - qA_y)^2 + \hat{p}_z^2\right) = \frac{1}{2m}\left(\hat{p}_x^2 + (\hat{p}_y - qB\hat{x})^2 + \hat{p}_z^2\right)$$

Because $[\hat{H}, \hat{p}_y] = 0$ and $[\hat{H}, \hat{p}_z] = 0$ and $[\hat{p}_y, \hat{p}_z] = 0$ we can find simultaneous eigenstates of the three operators. We search for solutions of the form $|\psi\rangle = |\psi_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle$ with $\hat{p}_y |p_y\rangle = p_y |p_y\rangle$ and $\hat{p}_z |p_z\rangle = p_z |p_z\rangle$. Then, using the commutation relations,

$$\hat{H}|\psi\rangle = \frac{1}{2m}\left(\hat{p}_x^2 + (p_y - qB\hat{x})^2 + p_z^2\right)|\psi_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle$$

Then

$$\hat{H}|\psi\rangle = \left[\frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega^2\left(\hat{x} - \frac{p_y}{qB}\right)^2 + \frac{p_z^2}{2m}\right]|\psi_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle$$

where $\omega = \frac{qB}{m}$. Then define the lowering operator,

$$\hat{a}_{p_y} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{p_y}{qB} + i\frac{\hat{p}_x}{m\omega}\right)$$

as before, this operators satisfies the commutation relation $[\hat{a}_{p_y}, \hat{a}_{p_y}^\dagger] = 1$ and,

$$\hat{a}_{p_y}^\dagger \hat{a}_{p_y} = \frac{1}{\hbar\omega}\left[\frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega^2\left(\hat{x} - \frac{p_y}{qB}\right)^2\right] - \frac{1}{2}$$

Thus,

$$\hat{H}|\psi\rangle = \left[\hbar\omega\left(\hat{a}_{p_y}^\dagger \hat{a}_{p_y} + \frac{1}{2}\right) + \frac{p_z^2}{2m}\right]|\psi_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle$$

The eigenstates of $\hat{a}_{p_y}^\dagger \hat{a}_{p_y}$ are $|\psi_x\rangle = |n\rangle$ with $\hat{a}_{p_y}^\dagger \hat{a}_{p_y}|n\rangle = n|n\rangle$ Therefore the spectrum is given by,

$$E_{n,p_z} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m}$$

which are independent of p_y.

(b). Consider the states: $\hat{a}_{p_y}|\alpha, p_y\rangle = \alpha|\alpha, p_y\rangle$ in the x-momentum basis:

$$\langle p_x|\hat{a}_{p_y}|\alpha, p_y\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(i\hbar\frac{\partial}{\partial p_x} - \frac{p_y}{qB} + \frac{ip_x}{m\omega}\right)\tilde{\psi}_\alpha(p_x) = \alpha\tilde{\psi}_\alpha(p_x)$$

where I have used the fact that, in the momentum basis,

$$\langle p_x|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p_x}\langle p_x|\psi\rangle = i\hbar\frac{\partial}{\partial p_x}\tilde{\psi}(p_x)$$

Furthermore, we can write:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_{p_y} + \hat{a}_{p_y}^\dagger\right) + \frac{p_y}{qB}$$

$$\hat{p} = \frac{m\omega}{2i}\sqrt{\frac{2\hbar}{m\omega}}\left(\hat{a}_{p_y} - \hat{a}_{p_y}^\dagger\right)$$

and thus,

$$
\begin{aligned}
\langle \alpha, p_y | \, \hat{x} \, | \alpha, p_y \rangle &= \sqrt{\frac{2\hbar}{m\omega}} \, \mathfrak{Re}[\alpha] + \frac{p_y}{qB} = x_0(p_y) \\
\langle \alpha, p_y | \, \hat{p}_x \, | \alpha, p_y \rangle &= \sqrt{2\hbar m\omega} \, \mathfrak{Im}[\alpha] = p_{x,0}
\end{aligned}
$$

Therefore, I define the following quantities, x_0, y_0, z_0, $p_{x,0}$, $p_{y,0}$, and $p_{z,0}$ are the classical initial conditions about which the probability distributions will be peaked and the characteristic scalles: $x_c = \sqrt{\frac{\hbar}{m\omega}}$ and $p_c = \sqrt{\hbar m\omega}$. Now, let

$$
\alpha = \frac{1}{\sqrt{2}} \left[\frac{1}{x_c} \left(x_0 - \frac{p_{y,0}}{qB} \right) + i \frac{p_{x,0}}{p_c} \right]
$$

so that the expectation values of the coherent state corresponding to the classical initial value of p_y agree with the classical initial conditions. Now, we know that (see addendum)

$$
\begin{aligned}
e^{-i\hat{H}t/\hbar} \, |\alpha\rangle \otimes |p_y\rangle \otimes |p_z\rangle &= e^{-i\left(\frac{1}{2}\omega + \frac{p_z^2}{2\hbar m}\right)t} e^{-i\hat{a}^\dagger_{p_y}\hat{a}_{p_y}t/\hbar} \, |\alpha\rangle \otimes |p_y\rangle \otimes |p_z\rangle \\
&= e^{-i\left(\frac{1}{2}\omega + \frac{p_z^2}{2\hbar m}\right)t} \left| \alpha e^{-i\omega t} \right\rangle \otimes |p_y\rangle \otimes |p_z\rangle
\end{aligned}
$$

Therefore, we can solve for the time dependent momentum space wavefunction representing the state $|\alpha\rangle \otimes |p_y\rangle \otimes |p_z\rangle$ as follows,

$$
\left(i\hbar \frac{\partial}{\partial p_x} - \frac{p_y}{qB} + \frac{ip_x}{m\omega} - \sqrt{\frac{2\hbar}{m\omega}} \alpha e^{-i\omega t} \right) \tilde{\psi}_\alpha(p_x, t) = 0
$$

Expainding α,

$$
\left(\frac{\partial}{\partial p_x} + i \frac{p_y}{qB\hbar} + \frac{p_x}{\hbar m\omega} + i \left[\frac{1}{\hbar} \left(x_0 - \frac{p_{y,0}}{qB} \right) + i \frac{p_{x,0}}{\hbar m\omega} \right] e^{-i\omega t} \right) \tilde{\psi}_\alpha(p_x, t) = 0
$$

Let $g_0 = \left(x_0 - \frac{p_{y,0}}{qB} \right)$

$$
\frac{\partial}{\partial p_x} \tilde{\psi}_\alpha(p_x, t) = - \left(i \frac{p_y}{qB\hbar} + \frac{p_x}{p_c^2} + i \frac{g_0}{\hbar} \cos \omega t + \frac{g_0}{\hbar} \sin \omega t - \frac{p_{x,0}}{p_c^2} \cos \omega t + i \frac{p_{x,0}}{p_c^2} \sin \omega t \right) \tilde{\psi}_\alpha(p_x, t)
$$

Therefore,

$$
\begin{aligned}
\tilde{\psi}_\alpha(p_x, t) = N(t) \exp &\left[-\frac{i}{\hbar} \left(\frac{p_y}{qB} + g_0 \cos \omega t + \frac{p_{x,0}}{m\omega} \sin \omega t \right) (p_x - p_{x,0}) \right] \\
&\cdot \exp \left[-\frac{1}{2p_c^2} \Big(p_x - p_{x,0} \cos \omega t + g_0 m\omega \sin \omega t \Big)^2 \right]
\end{aligned}
$$

Where we have choosen the phase to be zero at the initial momentum as to not bias the initial positions. The first term is a pure momentum dependent phase and the second is a gaussian with $\sigma^2 = \frac{1}{2}p_c^2$ so matching the overall time dependent phase,

$$
N(t) = \frac{e^{-i\left(\frac{1}{2}\omega + \frac{p_z^2}{2\hbar m}\right)t}}{(\pi p_c^2)^{\frac{1}{4}}}
$$

Now, the Fourier transform of a definite momentum state is a delta function in p-space so the full wavefunction in momentum space is:

$$\begin{aligned}\tilde{\psi}_{\bar{p}_y,\bar{p}_z}(p_x,p_y,p_z,t) = \frac{1}{(\pi p_c^2)^{\frac{1}{4}}} \exp\left[-\frac{i}{\hbar}\left(\frac{p_y}{qB} + g_0\cos\omega t + \frac{p_{x,0}}{m\omega}\sin\omega t\right)(p_x - p_{x,0})\right] \\ \cdot \exp\left[-\frac{i}{\hbar}\left(\frac{1}{2}\hbar\omega + \frac{p_z^2}{2m}\right)t\right]\exp\left[-\frac{1}{2p_c^2}\Big(p_x - p_{x,0}\cos\omega t + g_0 m\omega\sin\omega t\Big)^2\right] \\ \cdot\,\delta(p_y - \bar{p}_y)\delta(p_z - \bar{p}_z)\end{aligned}$$

Now, we construct a wavepacket by superimposing these wavefunctions with coeficients peaked about $\bar{p}_x = p_{x,0}$ and $\bar{p}_y = p_{y,0}$ and with phases $e^{-i\bar{p}_y y_0/\hbar}$ and $e^{-i\bar{p}_z z_0/\hbar}$ to shift the distribution to the initial positions. Thus,

$$\begin{aligned}\tilde{\psi}(p_x,p_y,p_z,t) &= \int \tilde{\psi}_\alpha(p_x,t)\delta(p_y-\bar{p}_y)\delta(p_z-\bar{p}_z)C(\bar{p}_y,\bar{p}_z)e^{-i\bar{p}_y y_0/\hbar}e^{-i\bar{p}_z z_0/\hbar}\,\mathrm{d}\bar{p}_y\mathrm{d}\bar{p}_z \\ &= \psi_\alpha(p_x,t)C(p_y,p_z)e^{-ip_y y_0/\hbar}e^{-ip_z z_0/\hbar}\end{aligned}$$

Therefore, in all its horrifying glory,

$$\begin{aligned}\tilde{\psi}(p_x,p_y,p_z,t) = \frac{1}{(\pi p_c^2)^{\frac{1}{4}}} \exp\left[-\frac{i}{\hbar}\left(\frac{p_y}{qB} + g_0\cos\omega t + \frac{p_{x,0}}{m\omega}\sin\omega t\right)(p_x - p_{x,0})\right] \\ \cdot \exp\left[-\frac{i}{\hbar}\left(\frac{1}{2}\hbar\omega + \frac{p_z^2}{2m}\right)t\right]\exp\left[-\frac{1}{2p_c^2}\Big(p_x - p_{x,0}\cos\omega t + g_0 m\omega\sin\omega t\Big)^2\right] \\ \cdot\, C(p_y,p_z)e^{-ip_y y_0/\hbar}e^{-ip_z z_0/\hbar}\end{aligned}$$

Finally, we apply the inverse Fourier transform,

$$\begin{aligned}&\psi(x,y,z,t) = \\ &\int \frac{1}{(\pi p_c^2)^{\frac{1}{4}}\sqrt{2\pi\hbar}} \exp\left[-\frac{i}{\hbar}\left(\frac{p_y}{qB} + g_0\cos\omega t + \frac{p_{x,0}}{m\omega}\sin\omega t\right)(p_x - p_{x,0})\right]\exp\left[-\frac{i}{\hbar}\left(\frac{1}{2}\hbar\omega + \frac{p_z^2}{2m}\right)t\right] \\ &\quad\cdot \exp\left[-\frac{1}{2p_c^2}\Big(p_x - p_{x,0}\cos\omega t + g_0 m\omega\sin\omega t\Big)^2\right]C(p_y,p_z)e^{-ip_y y_0/\hbar}e^{-ip_z z_0/\hbar}e^{i(p_x x + p_y y + p_z z)/\hbar}\mathrm{d}^3p\end{aligned}$$

and collct the totall phase,

$$\Phi = p_x x + p_y(y - y_0) + p_z(z - z_0) - \left(\frac{1}{2}\hbar\omega + \frac{p_z^2}{2m}\right)t - \left(\frac{p_y}{qB} + g_0\cos\omega t + \frac{p_{x,0}}{m\omega}\sin\omega t\right)(p_x - p_{x,0})$$

so the integral can be written as,

$$\psi(x,y,z,t) = \frac{1}{(\pi p_c^2)^{\frac{1}{4}}\sqrt{2\pi\hbar}}\int e^{i\Phi/\hbar}\exp\left[-\frac{1}{2p_c^2}\Big(p_x - p_{x,0}\cos\omega t + g_0 m\omega\sin\omega t\Big)^2\right]C(p_y,p_z)\mathrm{d}^3p$$

which is maximized at points where the phase is stationary at the peak values of the weigting coeficients. That is, the wavefunction is maximized at points where,

$$\nabla_{\vec{p}}\,\Phi\Big|_{\vec{p}_m(t)} = 0$$

Thus,

$$\frac{\partial\Phi}{\partial p_x} = x - \left(\frac{p_{y,0}}{qB} + \left(x_0 - \frac{p_{y,0}}{qB}\right)\cos\omega t + \frac{p_{x,0}}{m\omega}\sin\omega t\right) = 0$$
$$\frac{\partial\Phi}{\partial p_y} = y - y_0 - \frac{p_{x,m}(t) - p_{x,0}}{qB} = 0$$
$$\frac{\partial\Phi}{\partial p_z} = z - z_0 - \frac{p_{z,0}}{m}t = 0$$

Where $p_{x,m}(t)$ maximizes the gaussian, i.e. $p_{x,m}(t) = p_{x,0}\cos\omega t - \left(x_0 - \frac{p_{y,0}}{qB}\right) m\omega\sin\omega t$
Therefore, the maximum of the wavefunction evolves as:

$$x = \frac{p_{y,0}}{qB} + \left(x_0 - \frac{p_{y,0}}{qB}\right)\cos\omega t + \frac{p_{x,0}}{m\omega}\sin\omega t$$
$$y = y_0 - \frac{p_{x,0}}{qB}(1-\cos\omega t) - \left(x_0 - \frac{p_{y,0}}{qB}\right)\frac{m\omega}{qB}\sin\omega t$$
$$z = z_0 + \frac{p_{z,0}}{m}t$$

Notice that $\frac{m\omega}{qB} = 1$ because $\omega = \frac{qB}{m}$ by definition. Also, if we make the identifications,

$$v_{x,0} = \frac{p_{x,0}}{m}$$
$$v_{y,0} = \frac{p_{y,0} - qBx_0}{m}$$
$$v_{z,0} = \frac{p_{z,0}}{m}$$

then we have,

$$x = \frac{m}{qB}v_{y,0} + x_0 - \frac{m}{qB}v_{y,0}\cos\omega t + \frac{v_{x,0}}{\omega}\sin\omega t$$
$$y = y_0 - \frac{m}{qB}v_{x,0}(1-\cos\omega t) + \frac{m}{qB}v_{y,0}\sin\omega t$$
$$z = z_0 + v_{z,0}t$$

and therefore, simplifying,

$$x = x_0 + \frac{v_{y,0}}{\omega}(1-\cos\omega t) + \frac{v_{x,0}}{\omega}\sin\omega t$$
$$y = y_0 - \frac{v_{x,0}}{\omega}(1-\cos\omega t) + \frac{v_{y,0}}{\omega}\sin\omega t$$
$$z = z_0 + v_{z,0}t$$

which is exactly a clockwise (from above) circular helix, the motion of a classical charged particle in a constant magnetic field. The frequency of the circular orbit is

$$\omega = \frac{qB}{m}$$

which is the Larmor precession frequency.

Problem 18.

(a). Define $e^{-i\hat{\phi}} = \frac{1}{\sqrt{1+\hat{a}^\dagger\hat{a}}}\hat{a}$ then

$$e^{-i\hat{\phi}}\left|n\right\rangle = \frac{1}{\sqrt{1+\hat{a}^\dagger\hat{a}}}\sqrt{n}\left|n-1\right\rangle$$

but $\left|n-1\right\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$ so

$$e^{-i\hat{\phi}}\left|n\right\rangle = \sqrt{n}\frac{1}{\sqrt{1+(n-1)}}\left|n-1\right\rangle = \left|n-1\right\rangle$$

(b). $e^{i\hat{\phi}} = \left(e^{-i\hat{\phi}}\right)^\dagger = \hat{a}^\dagger\frac{1}{\sqrt{1+\hat{a}^\dagger\hat{a}}}$ because $\hat{a}^\dagger\hat{a}$ is Hermitian. Then,

$$e^{i\hat{\phi}}\left|n\right\rangle = \hat{a}^\dagger\frac{1}{\sqrt{1+\hat{a}^\dagger\hat{a}}}\left|n\right\rangle = \hat{a}^\dagger\frac{1}{\sqrt{1+n}}\left|n\right\rangle = \left|n+1\right\rangle$$

(c). Since $\frac{1}{\sqrt{1+\hat{a}^\dagger\hat{a}}}$ is a function of $\hat{a}^\dagger\hat{a}$ alone, it commutes with $\hat{a}^\dagger\hat{a}$. Thus,

$$\begin{aligned}[\hat{a}^\dagger\hat{a}, e^{-\hat{\phi}}] &= \hat{a}^\dagger\hat{a}\frac{1}{1+\hat{a}^\dagger\hat{a}}\hat{a} - \frac{1}{1+\hat{a}^\dagger\hat{a}}\hat{a}\hat{a}^\dagger\hat{a} = \frac{1}{1+\hat{a}^\dagger\hat{a}}\hat{a}^\dagger\hat{a}\hat{a} - \frac{1}{1+\hat{a}^\dagger\hat{a}}(\hat{a}^\dagger\hat{a} + [\hat{a},\hat{a}^\dagger])\hat{a} \\ &= -\frac{1}{1+\hat{a}^\dagger\hat{a}}[\hat{a},\hat{a}^\dagger] = -\frac{1}{1+\hat{a}^\dagger\hat{a}} = -e^{-i\hat{\phi}}\end{aligned}$$

(d). Define:

$$\sin\hat{\phi} = \frac{1}{2i}\left(e^{i\hat{\phi}} - e^{-i\hat{\phi}}\right)$$

and let,

$$\left|\phi_0\right\rangle = \sum_{n=0}^{\infty} c_n(\phi_0)\left|n\right\rangle$$

such that,

$$\begin{aligned}\sin\hat{\phi}\left|\phi_0\right\rangle &= \frac{1}{2i}\sum_{n=0}^{\infty}\left(e^{i\hat{\phi}} - e^{-i\hat{\phi}}\right)c_n(\phi_0)\left|n\right\rangle = \sin\phi_0\left|\phi_0\right\rangle \\ \sin\hat{\phi}\left|\phi_0\right\rangle &= \frac{1}{2i}\sum_{n=0}^{\infty}\left(\left|n+1\right\rangle - \left|n-1\right\rangle\right)c_n(\phi_0) = \sin\phi_0\sum_{n=0}^{\infty}c_n(\phi_0)\left|n\right\rangle \\ &= \frac{1}{2i}\left(\sum_{n=1}^{\infty}c_{n-1}(\phi_0)\left|n\right\rangle - \sum_{n=0}^{\infty}c_{n+1}(\phi_0)\left|n\right\rangle\right) \\ &= \frac{1}{2i}\sum_{n=1}^{\infty}(c_{n-1}(\phi_0) - c_{n+1}(\phi_0))\left|n\right\rangle - \frac{1}{2i}c_1(\phi_0)\left|0\right\rangle\end{aligned}$$

matching terms,

$$-\frac{1}{2i}c_1(\phi_0) = c_0(\phi_0)\sin\phi_0$$

and

$$c_{n+1}(\phi_0) = c_{n-1}(\phi_0) - (2i\sin\phi_0)c_n(\phi_0)$$

this recurrence relation can be easily solved by finding characteristic roots or using generating functions. I will not waste your time with that here; I will simply write out the solution.

$$c_n = N(\phi_0)\left[(-1)^n e^{i\phi_0(n+1)} + e^{-i\phi_0(n+1)}\right]$$

And thus, writting the series explicitly,

$$|\phi_0\rangle = N(\phi_0)\sum_{n=0}^{\infty}\left[(-1)^n e^{i\phi_0(n+1)} + e^{-i\phi_0(n+1)}\right]|n\rangle$$

(e). In an exactly analogous manner, define:

$$\cos\hat{\phi} = \frac{1}{2}\left(e^{i\hat{\phi}} + e^{-i\hat{\phi}}\right)$$

and let,

$$|\phi_0\rangle = \sum_{n=0}^{\infty} c_n(\phi_0)\,|n\rangle$$

such that,

$$\begin{aligned}
\cos\hat{\phi}\,|\phi_0\rangle &= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{i\hat{\phi}} + e^{-i\hat{\phi}}\right)c_n(\phi_0)\,|n\rangle = \cos\phi_0\,|\phi_0\rangle \\
\cos\hat{\phi}\,|\phi_0\rangle &= \frac{1}{2}\sum_{n=0}^{\infty}\left(|n+1\rangle + |n-1\rangle\right)c_n(\phi_0) = \cos\phi_0\sum_{n=0}^{\infty}c_n(\phi_0)\,|n\rangle \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty}c_{n-1}(\phi_0)\,|n\rangle + \sum_{n=0}^{\infty}c_{n+1}(\phi_0)\,|n\rangle\right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty}(c_{n-1}(\phi_0) + c_{n+1}(\phi_0))\,|n\rangle + \frac{1}{2}c_1(\phi_0)\,|0\rangle
\end{aligned}$$

matching terms,

$$\frac{1}{2}c_1(\phi_0) = c_0(\phi_0)\cos\phi_0$$

and

$$c_{n+1}(\phi_0) = (2\cos\phi_0)c_n(\phi_0) - c_{n-1}(\phi_0)$$

this recurrence relation can be easily solved by finding characteristic roots or using generating functions. I will not waste your time with that here; I will simply write out the solution.

$$c_n = N(\phi_0)\left[e^{i\phi_0(n+1)} - e^{-i\phi_0(n+1)}\right]$$

And thus, writting the series explicitly,

$$|\phi_0\rangle = N(\phi_0)\sum_{n=0}^{\infty}\left[e^{i\phi_0(n+1)} + e^{-i\phi_0(n+1)}\right]|n\rangle$$

Addendum

We want to show that for $\hat{H} = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$ the time evolution operator acts as,

$$e^{-i\hat{H}t/\hbar}\left|\alpha\right\rangle = e^{-\frac{1}{2}i\omega t}\left|\alpha e^{-i\omega t}\right\rangle$$

In the energy basis,

$$\left|\alpha\right\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle$$

but each eigenstate evolves nicely,

$$e^{-i\hat{H}t/\hbar}\left|n\right\rangle = e^{-i\omega\hat{a}^\dagger\hat{a}t}e^{-\frac{1}{2}i\omega t}\left|n\right\rangle = e^{-i\omega nt}e^{-\frac{1}{2}i\omega t}\left|n\right\rangle$$

Therefore,

$$\begin{aligned} e^{-i\hat{H}t/\hbar}\left|\alpha\right\rangle &= e^{-\frac{1}{2}i\omega t}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega\hat{a}^\dagger\hat{a}t}\left|n\right\rangle = e^{-\frac{1}{2}i\omega t}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\omega nt}}{\sqrt{n!}}\left|n\right\rangle \\ &= e^{-\frac{1}{2}i\omega t}\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\left|n\right\rangle = e^{-\frac{1}{2}i\omega t}\left|\alpha e^{-i\omega t}\right\rangle \end{aligned}$$