1 Introduction and Definitions

Let k be a finite field and K a dimension 1 function field over k (i.e. a field extension K/k of transcendence degree 1). Let $\bar{k}$ be a fixed algberaic closure of k and $L = K\bar{k}$ the compositum inside a fixed alebgraic closure $\bar{K}$. Let X denote the unique regular projective curve over k with $K(X) = K$. Note that because k is perfect, X is smooth. We assume that X is geometrically integral over k so that $k = \Gamma(X, \mathcal{O}_X)$ is the field of constants, otherwise we replace k by $\Gamma(X, \mathcal{O}_X)$. Throughout we denote $q = p^e = \# k$ where $p = \operatorname{char} k$.

1.1 Background Results

Here we collect some results on the class group which we will try to reprove using adelic techniques.

Remark. Because X is smooth we can freely use isomorphisms $\operatorname{Cl}(X) \cong \operatorname{CaCl}(X) \cong \operatorname{Pic}(X)$. Furthermore, there is a degree map, $\deg : \operatorname{Cl}(X) \to \mathbb{Z}$ sending

$$[P] \mapsto [\kappa(P) : k] = \log_q \# \kappa(P)$$

Lemma 1.1.1. $\operatorname{Pic}^0(X)$ is finite and $\operatorname{Pic}(X) \cong \operatorname{Pic}^0(X) \times \mathbb{Z}$ noncanonically.

Proof. Choose some prime divisor D_0 (meaning a point $P \in X$) and let $d = \deg D$. Then the map $D \mapsto D - nD_0$ gives an isomorphism $\operatorname{Pic}^{nd}(X) \xrightarrow{\sim} \operatorname{Pic}^0(X)$ so it suffices to show that $\operatorname{Pic}^{nd}(X)$ is finite for $n \gg 0$. However, by Riemann-Roch, if $\deg D = nd \geq 2g$ then

$$H^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g \geq g + 1 \geq 1$$

so there is an effective divisor $D' \sim D$. Then fixing n large enough so that $nd \geq 2d$ for any $D \in \operatorname{Pic}^{nd}(X)$ there is $D' \sim D$ with D effective and $\deg D' = nd$ however there are finitely many prime divisors of bounded degree because $X(k')$ is finite for each finite extension k'/k and thus there are finitely many effective divisors with fixed degree so $\operatorname{Pic}^{nd}(X)$ is finite.

There is a canonical exact sequence,

$$0 \longrightarrow \operatorname{Pic}^0(X) \longrightarrow \operatorname{Pic}(X) \longrightarrow \mathbb{Z}$$

Surjectivity of $\deg : \operatorname{Pic}(X) \to \mathbb{Z}$ is obvious if X has a k-point P because $\deg [P] = 1$. In general, surjectivity is a consequence of X being geometrically integral (otherwise suppose that X is a k'-scheme then every divisor will have residue field containing k' so $\operatorname{im} \deg$ will have index at least $[k' : k]$). Because X is geometrically integral, the Weil conjectures gives,

$$\# X(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{i=1}^{2g} \beta_i^n$$

with $|\beta_i| = q^{\frac{1}{2}}$. Therefore,

$$|\# X(\mathscr{F}_{q^n}) - 1 - q^n| \leq 2g\, q^{\frac{n}{2}}$$

and thus for $n \gg 0$ we have $q^n + 1 > 2gq^{\frac{n}{2}}$ so $X(\mathbb{F}_{q^n}) \neq \varnothing$. In paricular there are points $P, Q \in X$ with $\deg P = n$ and $\deg Q = n+1$ so $D = [Q] - [P]$ is a divisor with $\deg D = 1$ proving surjectivity. Then because $\mathbb{Z}$ is projective the sequence splits. $\square$

Remark. The point counting formula requires X to be geometrically integral such that $X_{\bar{k}}$ is an (in particular connected) variety so that $H^0_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ and $H^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ have the expected dimension and Galois representations. To see what can go wrong, consider $X = \mathbb{P}^1_{\mathbb{F}_{q^2}}$ over $\mathrm{Spec}\,(\mathbb{F}_q)$. Then,

$$X(\mathbb{F}_{q^n}) = \begin{cases} 2(1+q^n) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

which does not following the counting formula nor does it have a divisor of degree 1.

Remark. We can also give a much fancier proof. There is an exact sequence of group schemes,

$$0 \longrightarrow \mathbf{Pic}^0_{X/k} \longrightarrow \mathbf{Pic}_{X/k} \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0$$

where $\mathbf{Pic}^0_{X/k}$ is finite type over k and thus $\mathrm{Pic}^0\,(X) = \mathbf{Pic}^0_{X/k}(k)$ is finite because k is a finite field. Surjectivity of $\mathbf{Pic}_{X/k} \to \underline{\mathbb{Z}}$ is clear in the étale topology on $\mathrm{Spec}\,(k)$ because X has a degree 1 prime divisor after a finite extension of k (e.g. take the residue field of any closed point). Therefore, we get an exact sequence,

$$0 \longrightarrow \mathbf{Pic}^0_{X/k}(k) \longrightarrow \mathbf{Pic}_{X/k}(k) \longrightarrow \underline{\mathbb{Z}}(k) \longrightarrow H^1(k, \mathbf{Pic}^0_{X/k})$$

However, $\mathbf{Pic}^0_{X/k}$ is an abelian variety so by Lang's theorem on H^1-vanishing, $H^1(k, \mathbf{Pic}^0_{X/k}) = 0$ and therefore,

$$0 \longrightarrow \mathrm{Pic}^0\,(X) \longrightarrow \mathrm{Pic}\,(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact and $\mathbb{Z}$ is projective so it splits.

Remark. We needed to assume X was geometrically integral over k for representability of the relative Picard functor [FGA V, Thm. 3.1]. In general, there is an abelian variety J called the *Jacobian* but $J(k) \neq \mathrm{Pic}^0\,(X)$ in general (see Poonen 5.7).

Theorem 1.1.2 (Lang)**.** Let A be a smooth connected finite type k-group with k finite. Then,

$$H^1(k, A) = 0$$

Proof. FIND BETTER REFERENCE? Can Look at Chapter VI of Serre's *Algebraic Groups and Class Fields*. Or Poonen Rational Points Thm. 5.12.19. □

Remark. In the case that A is an elliptic curve over k, I have a cheeky proof. A class $X \in H^1(k, A)$ represents an A-torsor on $\mathrm{Spec}\,(k)_{\text{ét}}$ and thus is trivial if and only if X has a k-point (a "global section"). However, X is a form of A and thus,

$$H^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell) = H^i_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell)$$

so by the Lefschetz trace formula $\#X(k) = 1 + q - \alpha - \bar{\alpha}$ with $|\alpha| = \sqrt{q}$. Thus,

$$\#X(k) \geq 1 + q - 2\sqrt{q} = (\sqrt{q} - 1)^2 > 0$$

so $X(k) \neq \varnothing$. Maybe I can make this work for abelian varieties.

1.1.1 The Adèles and Idèles

Lemma 1.1.3. Every valuation ring of K/k is a DVR and is $\mathcal{O}_{X,x}$ for some unique point $x \in K$.

Proof. See [H, Ex. 4.12(a)]. □

Remark. We write v_x for the associated valuation which we normalize so that,

$$v_x(\varpi_x) = \deg x = [\kappa(x) : k] = \log_q \#\kappa(x)$$

This normalization is chosen such that the associated norm $|a|_x = q^{-v_x(a)}$ satisfies $|\varpi_x|_x = (\#\kappa(x))^{-1}$. We also write ord_x for the valuation normalized such that $\text{ord}_x(\varpi) = 1$ so that,

$$\text{div} f = \sum_{x \in X} \text{ord}_x(f)[x]$$

Definition 1.1.4. The adeles and ideles of a function field are,

$$\mathbb{A}_K = \prod_{x \in X}{}' (K_x, \mathcal{O}_{K,x}) \quad \text{and} \quad \mathbb{I}_K = \prod_{x \in X}{}' (K_x^\times, \mathcal{O}_{K,x}^\times)$$

where $\mathcal{O}_{K,x}$ is the completed local ring,

$$\mathcal{O}_{K,x} = \widehat{\mathcal{O}_{X,x}} = \varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^n$$

and $K_x = \text{Frac}\,(\mathcal{O}_{K,x})$ is the local field at $x \in X$. The valuations and norms extend to $v_x : K_x^\times \to \mathbb{Z}$ making it a non-archimedean local field with discrete valuation ring $\mathcal{O}_{K,x}$.

Remark. Unlike the number field case, all the local fields K_x are isomorphic to $k'((t))$ because X is regular so $\widehat{\mathcal{O}_{X,x}} \cong k'[[t]]$ for $k' = \kappa(s)$. We require k to be finite in order that K_x is a local field, in particular so that K_x is locally compact. Indeed, a fundamental system of neighborhoods of $0 \in k((t))$ are given by groups isomorphic to $k[[t]]$ which is compact if and only if k is finite. Indeed, $k[[t]] \twoheadrightarrow k[t]/(t^n)$ so if $k[[t]]$ is compact then its image $k[t]/(t^n)$ is compact but also discrete and thus finite. Conversely, if k is finite then $k[t]/(t^n)$ is finite and thus discrete so Tychonoff's theorem shows that.

$$k[[t]] = \varprojlim_n k[t]/(t^n)$$

is compact as well. Therefore, it is essential that we restric to function fields over *finite* fields if we want to have a good local theory.

Definition 1.1.5. The idèle class group is,

$$C_K = \mathbb{I}_K / K^\times$$

where $K^\times \hookrightarrow C_K$ via the diagonal embedding $K^\times \hookrightarrow K_x^\times$. This makes sense because each $f \in K$ has only finitely many poles meaning $f \in \mathcal{O}_{X,x}$ and thus $f \in \widehat{\mathcal{O}_{X,x}}$ for all but finitely many $x \in X$.

Definition 1.1.6. There is a degree map $\deg : C_K \to \mathbb{Z}$ defined by taking,

$$\deg(a_v) = \sum_v v(a_v)$$

which is well-defined because $a_v \in \mathcal{O}_{K,v}$ so $v(a_v) = 0$ for all but finitely many v and a norm,

$$|a| = \prod_v |a_v|_v$$

Now define the open subgroup $C_K^0 = \ker \deg = \mathbb{I}^1 / K^\times$ where

$$\mathbb{I}_K^1 = \left\{ (a_v) \;\middle|\; a_v \in K_v \text{ and } a_v \in \mathcal{O}_{K,v} \text{ for all but finitely many } v \text{ and } \prod_v |a_v|_v = 1 \right\}$$

There is another open subgroup,

$$U_K = \left(\prod_v \mathcal{O}_{K,v} \right) / K^\times$$

Proposition 1.1.7. There is a surjection $C_K \twoheadrightarrow \text{Pic}\,(X)$ with kernel U such that the diagram,

$$\begin{tikzcd} C_K \arrow[r, two heads] \arrow[dr, "\text{ord}"'] & \text{Pic}\,(X) \arrow[d, two heads, "\deg"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes giving isomorphisms $C_K / C_K^0 \xrightarrow{\sim} \mathbb{Z}$ and $C_K^0 / U_K \xrightarrow{\sim} \text{Pic}^0\,(X)$.

Proof. For $(a_x) \in \mathbb{I}_K$ we know $\text{ord}_x(a_x) = 0$ for all by finitely many x so there is a map,

$$(a_x) \mapsto \sum_{x \in X} \text{ord}_x(a_x)[x]$$

which is well-defined because $f \mapsto \text{div} f$ for $f \in K^\times$. This is surjective since divisors are finite sums and $(\varpi_{x_0}) \mapsto [x_0]$. Furthermore,

$$\deg \left(\sum_{x \in X} \text{ord}_x(a_x)[x] \right) = \sum_{x \in X} \text{ord}_x(a_x) \deg x = \sum_{x \in X} v_x(a_x) = \deg\,(a_v)$$

By definition, $C_K^0 = \ker \text{ord}$ giving the first isomorphism. Then $C_K^0 \to \text{Pic}\,(X)$ surjects onto $\text{Pic}^0\,(X) = \ker\,(\text{Pic}\,(X) \to \mathbb{Z})$ and $\ker\,(C_K^0 \to \text{Pic}\,(X)) = U_K$ because if $(a_v) \mapsto D$ and $D = \text{div} f$ then $v(a_v/f) = 0$ so $a_v/f \in \mathcal{O}_{K,v}$ proving that $(a_v) \in U_K$. $\square$

Theorem 1.1.8. C_K^0 is compact.

Proof. DO THIS!!! $\square$

Corollary 1.1.9. $\text{Pic}^0\,(X)$ is finite. Indeed, because C_K^0 is compact we see that $\text{Pic}^0\,(X)$ is compact. Furthermore, $U_K \subset C_K^0$ is open so $C_K^0/U \xrightarrow{\sim} \text{Pic}^0\,(X)$ is also discrete and thus finite.

2 The First Inequality

(WHAT THE HELL IS CURLY H)

We first recall some facts about the Herbrand quotient. Define,

$$h^i(G, M) = \dim_k H^i(G, M)$$

then the Herbrand quotient is,

$$h_{2/1}(G, A) = h^2(G, A) / h^1(G, A)$$

(DO I NEED G TO BE CYCLIC HERE!!)

Proposition 2.0.1. The index $h_{2/1}$ is multiplicative. Given an exact sequence,

$$0 \longrightarrow M_1 \longrightarrow M_2 M_3 \longrightarrow 0$$

of G-modules then,

$$h_{2/1}(M_2) = h_{2/1}(M_1) h_{2/1}(M_3)$$

Proposition 2.0.2. If A is finite then $h_{2/1}(A) = 1$.

Proposition 2.0.3. $h_{2/1}(\mathbb{Z}) = |G|$ where $\mathbb{Z}$ has a trivial $\mathbb{Z}$-action.

Proposition 2.0.4. Let L/K be an extension of local fields then,

$$h_1(\mathrm{Gal}\,(L/K)\,, U) = h_2(\mathrm{Gal}\,(L/K)\,, U) = e(L/K)$$

where $U \subset L$ are the units of the ring of integers.

Definition 2.0.5. Let L/K be a finite cyclic extension of order n. Then,

$$h_{2/1}(G, C_L) = n$$

Theorem 2.0.6. Let L/K be a cyclic extension of degree n with Galois group G. Then,

$$h_{2/1}(G, C_L) = n$$

Proof. We have,

$$h_{2/1}(C_L) = h_{2/1}(C_L/C_L^0) h_{2/1}(C_L^0/U) h_{2/1}(U)$$

First, $C_L/C_L^0 \xrightarrow{\sim} \mathbb{Z}$ and thus $h_{2/1}(C_L/C_L^0) = n$ and $h_{2/1}(C_L^0/U_L) \xrightarrow{\sim} \mathrm{Pic}^0\,(X)$ which is finite so $h_{2/1}(C_L^0/U_L) = 1$. Now,

$$h_{2/1}(U_L) = h_{2/1}(W) h_{2/1}(L^\times \cap W)^{-1}$$

where,

$$W = \prod_w \mathcal{O}_{L,w}^\times = \prod_v \left(\prod_{w|v} \mathcal{O}_{L,w}^\times \right)$$

Now,

$$H^r(G, W) = \prod_v H^r \left(G, \prod_{w|v} \mathcal{O}_{L,w}^\times \right) = \prod_v H^r(G_\nu, \mathcal{O}_{L_\nu}^\times)$$

by Shapiro's lemma since,

$$\prod_{w|v} \mathcal{O}_{L,w}^\times = \mathrm{Ind}_{G_\nu}^G \left(\mathcal{O}_{L_v}^\times \right)$$

By the local theory,

$$h_2(G_\nu, \mathcal{O}_{L_v}^\times) = h_1(G_\nu, \mathcal{O}_{L_v}^\times) = e_\nu$$

and since $e_\nu = 1$ all but finitely often we see that,

$$h^1(G, W) = h^2(G, W) = \prod_v e_v$$

and therefore $h_{2/1}(W) = 1$. Finally, $L^\times \cap W$ is the field of constants which is finite (this is where we're using the function field setting! otherwise we need to do more work) so $h_{2/1}(L^\times \cap W) = 1$ proving that,

$$h_{2/1}(C_L) = n \cdot 1 \cdot 1 \cdot 1 = n$$

$\square$

3 The Second Inequality

4 The Existence Theorem

Definition 4.0.1. Let L/K be a finite extension and $f : X' \to X$ the corresponding finite map of nonsingular curves. Then L_w/K_v is a finite extension so there is a local norm $\mathrm{N}_{L_w/K_v} : L_w^\times \to K_v^\times$. Then we define the norm,

$$\mathrm{N}_{L/K} : C_L \to C_K \quad (a_w) \mapsto (b_v) \quad \text{where} \quad b_v = \prod_{w \mapsto v} \mathrm{N}_{L_w/K_v}(a_w)$$

Definition 4.0.2. What is ω?????????????

Theorem 4.0.3. Let $N \subset C_K$ be a finite index open subgroup. Then there exists a finite abelian extension L/K such that $\mathrm{N}_{L/K}(C_L) = N$ and K is the fixed field of $\omega(N)$.

Theorem 4.0.4. Let L/K be a finite abelian extension with $N = \mathrm{N}_{L/K}(C_L)$. Then $x \in X$ is uniramified if and only if $\mathcal{O}_{K,x}^\times \subset N$ and x splits completely if and only if $K_x^\times \subset N$.

5 The Hilbert Class Field

In the number field case, we consider the open subgroup $U_K = (K^\times \cdot \mathbb{I}_{K,S_\infty})/K^\times$ with $C_K/U_K \xrightarrow{\sim} \mathrm{Cl}\,(K)$. Then by the global existence theorem there is a finite abelian extension H_K/K with $\mathrm{N}_{H_K/K}(C_{H_K}) = U_K$. Therefore, we see that H_K is unramified everywhere because $\mathcal{O}_{K,\nu}^\times \subset U_K$ and also for any L/K such that L/K is everywhere unramified then $\mathcal{O}_{X,\nu}^\times \subset \mathrm{N}_{L/K}(C_L)$ and therefore $U_K \subset \mathrm{N}_{L/K}(C_L)$ which implies that $L \subset H_K$ so H_K is the maximal abelian unramified extension of K.

Proposition 5.0.1. Let K be a number field and H_K/K be its Hilbert class field. Then $\mathfrak{p}$ is prinicpal iff $\mathfrak{p}$ splits completely in H_K.

Proof. The isomorphism $C_K/\mathrm{N}_{H_K/K}(H_K) \xrightarrow{\sim} \mathrm{Gal}\,(H_K/K)$ and $C_K/\mathrm{N}_{H_K/K}(H_K) \xrightarrow{\sim} \mathrm{Cl}\,(K)$ send the uniformizer $\varpi_\mathfrak{p}$ to $\mathrm{Frob}_\mathfrak{p}$ and $[\mathfrak{p}]$ respectively. Therefore, $[\mathfrak{p}] = [0]$ iff $\mathrm{Frob}_\mathfrak{p}$ is trivial iff $\mathfrak{p}$ splits completely in H_K/K. ☐

However, in the function field case we have,

$$C_K/U_K \xrightarrow{\sim} \mathrm{Pic}\,(X) \cong \mathrm{Pic}^0\,(X) \times \mathbb{Z}$$

which is not finite. Therefore, to apply the global existence theorem and thus get an analogue of the Hilbert Class field we need to choose a different open subgroup that does have finite index.

The issue is essentially due to extensions of the constant field k which are all abelian and unramified. This should somehow correspond to the factor $\mathbb{Z}$ in $\mathrm{Pic}\,(X)$ which should relate to $\mathrm{Gal}\left(\bar{k}/k\right) \cong \hat{\mathbb{Z}}$. We will now make these correspondences precise.