1 The Tautological Bundle

Consider the fibre bundle, $\pi : S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$ given by consider ing $S^{2n+1} \subset \mathbb{C}^{n+1}$ and restricting the projection $\mathbb{C}^{n+1} \to \mathbb{P}^n_{\mathbb{C}}$. Then π is a principal S^1-bundle. Consider the tautological representation $\rho : U(1) \to \mathrm{GL}_1(\mathbb{C})$ which is the inclusion $U(1) \hookrightarrow \mathbb{C}^\times$, which gives an associated line bundle $S^{2n+1} \times_\rho \mathbb{C}$. We call this the tautological bundle since its fibre above a point is the line in $\mathbb{C}^{n+1}$ which that point on $\mathbb{P}^n_{\mathbb{C}}$ corresponds to.

To see this explicitly, consider the following bundle,

$$T = \{(L, v) \mid L \in \mathbb{P}^n_{\mathbb{C}} \text{ and } v \in L \subset \mathbb{C}^{n+1}\} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$$

with the projection $\pi : T \to \mathbb{P}^n_{\mathbb{C}}$ via $(L, v) \mapsto L$. I claim that this bundle is isomorphic to the tautological bundle constructed above.

Consider the map $f : S^{2n+1} \times_\rho \mathbb{C} \to T$ via $f : [x, \lambda] \mapsto (\mathrm{Span}(x), \lambda x)$. This is clearly a bundle map since $\pi([x, \lambda]) = \pi(x) = \mathrm{Span}\left(()\, x\right) = \pi(\mathrm{Span}\left(x\right), \lambda x)$. Furthermore it is well-defined because $f([x, \mu\lambda]) = (\mathrm{Span}\left(x\right), \mu\lambda x) = (\mathrm{Span}\left(\mu x\right), \lambda\mu x) = f([\mu x, \lambda])$. We need to check that this map is injective and surjective. First, if $f([x, \lambda]) = f([y, \mu])$ then $\mathrm{Span}\left(x\right) = \mathrm{Span}\left(y\right)$ so $y = \gamma x$ for $\gamma \in \mathbb{C}^\times$ and $\lambda x = \mu y$ so $\lambda = \mu\gamma$ (since these vectors are nonzero) and thus,

$$[x, \lambda] = [x, \gamma\mu] = [\gamma x, \mu] = [y, \mu]$$

For surjectivity note that given (L, v) with $v \in L$ then $L = \mathrm{Span}\left(x\right)$ for $x \in S^{2n+1}$ and $v = \lambda x$ with $\lambda \in \mathbb{C}$ since L is a line. Thus $f([x, \lambda]) = (L, v)$.

The tautological bundle has no nonzero (holomorphic) global sections. However, there are $n + 1$ independent glboal sections of its dual. To see this consder the global $\mathrm{Hom}(T, \mathcal{O}_{\mathbb{P}})$. There exist $n+1$ idependent functions defined by the $n+1$ projections $p_k : \mathbb{C}^{n+1} \to \mathbb{C}$ via the construction,

$$T \hookrightarrow \mathcal{O}_{\mathbb{P}}^{n+1} = \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \xrightarrow{p_k} \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C} = \mathcal{O}_{\mathbb{P}}$$

These sections are refered to as X_k ,the k^{th} coordinate function on $\mathbb{P}^n_{\mathbb{C}}$.

Producing the coordinate functions X_k as sections of the dual $X^\vee$ identifies the tautological bundle T with the algebraic twist $\mathcal{O}_{\mathbb{P}}(-1)$ and thus its dual is the Serre twisting sheaf $T^\vee = \mathcal{O}_{\mathbb{P}}(1)$.

2 MATH 275A 2021 Lecture 2

Using the Stern-Gerlach boxes we define spin operators $\hat{S}_i$ on our Hilbert space $H = \mathbb{C}^2$. These have eigenstate $\ker \pm$ along each axis. Furthermore, we have a Hamiltonian $\hat{H}$. For a constant magnetic field, up to a constant,

$$\hat{H} = \hat{S} \cdot \vec{B}$$

For B along the z-direction,

$$\hat{H} = \hat{S}_z B$$

Then the evolution follows the Schrodinger equation,

$$i\partial_t \left|\psi\right\rangle = \hat{H} \left|\psi\right\rangle$$

For any observable (i.e. operator $\hat{A}$) we can define the expected value,

$$\left\langle \hat{A} \right\rangle_\psi = \left\langle \psi \right| \hat{A} \left|\psi\right\rangle$$

Then,

$$i\partial_t \left\langle \hat{A} \right\rangle_\psi = \left\langle [\hat{A}, \hat{H}] \right\rangle_\psi$$

Now for example, we choose $\left|\psi(0)\right\rangle = \left|+_x\right\rangle$. Then we expand,

$$\left|\psi(0)\right\rangle = \frac{1}{\sqrt{2}} \left(\left|+_z\right\rangle + \left|-_z\right\rangle\right)$$

Then applying the evolution operator,

$$\left|\psi(t)\right\rangle = e^{-iHt} \frac{1}{\sqrt{2}} \left(\left|+_z\right\rangle + \left|-_z\right\rangle\right) = \frac{1}{\sqrt{2}} \left(e^{-i\frac{B}{2}t} \left|+_z\right\rangle + e^{i\frac{B}{2}t} \left|-_z\right\rangle\right)$$

Now we consider,

$$i\partial_t \left\langle \hat{S}_x \right\rangle = \left\langle \psi \right| \hat{S}_x \left|\psi\right\rangle = \left\langle [\hat{S}_x, \hat{H}] \right\rangle = B \left\langle [\hat{S}_x, \hat{S}_z] \right\rangle = -iB\hat{S}_y$$

and therefore,

$$\partial_t \left\langle \hat{S}_x \right\rangle = -B \left\langle \hat{S}_y \right\rangle$$

Likewise,

$$\partial_t \left\langle \hat{S}_y \right\rangle = B \left\langle \hat{S}_x \right\rangle$$

This coupled system has solution,

$$\left\langle \hat{S}_x \right\rangle = \cos(Bt) \quad \text{and} \quad \left\langle \hat{S}_y \right\rangle = \sin(Bt)$$

2.0.1 Operators

Infinite dimensional space $H = L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \int |f|^2 < \infty\}$. We take observables to be "self-adjoint" operators on $H = L^2(\mathbb{R})$. For example, $\hat{x} = x\cdot$ and $\hat{p} = -\partial_x$. However, the eigenfunctions of these operators are not L^2 they are tempered distributions. We say,

$$\left\langle \frac{1}{\sqrt{2\pi}} e^{ipx} \middle| \frac{1}{\sqrt{2\pi}} e^{iqx} \right\rangle = \delta(p - q)$$

2.0.2 Uncertainty Principle

Define,

$$\Delta\hat{A} = \hat{A} = - \langle \hat{x} \rangle I$$

and likewise for B two self-adjoint operators A, B. Then,

$$\left\langle (\Delta\hat{x})^2 \right\rangle_\psi \left\langle (\Delta\hat{p})^2 \right\rangle_\psi \geq \frac{1}{4} \left| \langle\psi| \, [\hat{A}, \hat{B}] \, |\psi\rangle \right|^2$$

For example,

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = iI$$

because,

$$(\hat{x}\hat{p} - \hat{p}\hat{x})\psi = x(-i\partial_x\psi) + i\partial_x(x\psi) = -i\partial_x\psi + i\psi + x\partial_x\psi = i\psi$$

Therefore,

$$\sigma_x^2\sigma_p^2 \geq \frac{1}{4}$$

2.0.3 Angular Momentum

Classical angular momentum $\vec{L} = \vec{x} \times \vec{p}$. We upgrade these to quantum self-adjoint operators. Thus we get, for example,

$$\hat{L}_z = -i(x\partial_y - y\partial_x)$$

Then $L^2 = L_x^2 + L_y^2 + L_z^2$.