

Contents

1	Chapter 1.1	2
1.1	1.2	2
1.2	1.3	2
1.3	1.7	3
1.4	1.8	3
1.5	1.9	3
2	Chapter 1.2	3
2.1	2.2	3
2.2	2.4	3
2.3	2.5	4
2.4	2.8	4
2.5	2.10	4
2.6	2.11	5
2.7	2.12	5
2.8	2.16	6
3	Chapter 1.3	6
3.1	3.1	6
3.2	3.4	7
3.3	3.15	7
3.4	3.18	7
4	Chapter 2.1	8
4.1	2.1	8
4.2	2.2	8
4.3	2.3	8
5	Chapter 2.3	8
5.1	3.3	8
5.2	3.4	8
5.3	3.5	8
5.4	3.7	8
6	Chapter 2.4	8
6.1	4.2	8
7	Chapter 2.5	8
7.1	5.2	8
7.2	5.7	8
7.3	5.8	8
8	Chapter 2.6-8	8
9	Appendix A. Rank Functions	8

1 Chapter 1.1

1.1 1.2

Consider the following conditions on a ring R ,

- (I) R satisfies the IBP (if $R^n \cong R^m$ then $n = m$).
- (II) For all m, n and P if $R^m \cong R^n \oplus P$ then $m \geq n$.
- (III) For all n and P if $R^n \cong R^n \oplus P$ then $P = 0$

We will show (III) \implies (II) \implies (I). First suppose R satisfies (III) and consider the situation that $R^m \cong R^n \oplus P$ and $m < n$. We can add R^{n-m} to each side to get,

$$R^n \cong R^n \oplus (P \oplus R^{n-m})$$

then applying (III) we find $P \oplus R^{n-m} = 0$ a contradiction proving (II).

Now assume property (II) and suppose that $R^m \cong R^n$. By applying (II) in the case $P = 0$ we find $m \geq n$ and $n \geq m$ and thus $m = n$ proving the IBP i.e. property (I).

1.2 1.3

We need to show that the following conditions on a ring R are equivalent,

- (a) For all n , every surjection $R^n \rightarrow R^n$ is an isomorphism.
- (b) For all n , and $f, g \in M_n(R)$ if $fg = \text{id}$ then $gf = \text{id}$ and $g \in \text{GL}_n(R)$.
- (c) For all n and P if $R^n \cong R^n \oplus P$ then $P = 0$.

First suppose property (a) and let $fg = \text{id}$ for $f, g \in M_n(R) = \text{End}(R^n)$. Since $fg = \text{id}$ the map $g : R^n \rightarrow R^n$ is surjective and thus an isomorphism by property (a). so we find that $g \in \text{GL}_n(R)$ and there is some $h \in \text{GL}_n(R)$ such that $gh = hg = \text{id}$. However,

$$fgh = (fg)h = h = f(gh) = f$$

so $h = f$ and thus $fg = gf = \text{id}$ proving (b).

Now suppose (b) holds and suppose we have the situation $R^n \cong R^n \oplus P$. Then consider the maps $\iota : R^n \rightarrow R^n \oplus P$ and $\pi : R^n \oplus P \rightarrow R^n$ which satisfy $\pi \circ \iota = \text{id}$. Now let $f : R^n \rightarrow R^n \oplus P$ be the given isomorphism then define $\tilde{\iota} = f^{-1} \circ \iota : R^n \rightarrow R^n$ and $\tilde{\pi} = \pi \circ f : R^n \rightarrow R^n$ and thus $\tilde{\pi} \circ \tilde{\iota} = \text{id}$ and $\tilde{\pi}, \tilde{\iota} \in \text{End}(R^n) = M_n(R)$. Thus by (b), $\tilde{\iota} \circ \tilde{\pi} = \text{id}$ so $\tilde{\iota} = f^{-1} \circ \iota$ is an isomorphism which implies that $\iota : R^n \rightarrow R^n \oplus P$ is an isomorphism (since f^{-1} is) and thus $P = 0$ proving (c).

Finally, suppose (c) and suppose that $f : R^n \rightarrow R^n$ is a surjection. Then consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \xrightarrow{f} R^n \longrightarrow 0$$

Then R^n is free and thus projective so the sequence is split,

$$R^n \cong R^n \oplus \ker f$$

so by (c) we have $\ker f = 0$ and thus f is an isomorphism proving (a).

Finally suppose that R is commutative and $f, g \in M_n(R)$ with $fg = \text{id}$. Then $\det fg = \det f \det g = 1$ so $f, g \in \text{GL}_n(R)$ and thus there exists a matrix (cofactors) h such that $gh = \text{id}$ then $f = h$ by a previous argument. Therefore commutative rings satisfy all the above properties.

1.3 1.7

(NO IDEA)

1.4 1.8

(NO IDEA)

1.5 1.9

(NO IDEA)

2 Chapter 1.2

Remark. In this section R is a commutative (unital) ring.

2.1 2.2

2.2 2.4

Consider a continuous function $f : \text{Spec}(R) \rightarrow \mathbb{Z}$. First, $\text{Spec}(R)$ is quasi-compact. This is easily shown since every affine cover U_i can be refined to a cover by principal opens $D(f_i)$ then,

$$\text{Spec}(R) = \bigcup_{i=1}^{\infty} D(f_i) = D(\langle f_i \rangle)$$

(since $f_i \notin \mathfrak{p}$ for some f_i iff $\langle f_i \rangle \not\subset \mathfrak{p}$) and thus $\langle f_i \rangle = R$ (otherwise it would be contained in a maximal ideal) but then $1 = r_1 f_1 + \cdots + r_n f_n$ for finitely many so,

$$\text{Spec}(R) = D(\langle f_1, \dots, f_n \rangle) = \bigcup_{i=1}^n D(f_i)$$

so there is a finite subcover of U_i .

Therefore, $f(\text{Spec}(R)) \subset \mathbb{Z}$ is compact and thus finite so it must take finitely many values n_1, \dots, n_c . Then $V_i = f^{-1}(n_i)$ is a closed subset of $\text{Spec}(R)$ since \mathbb{Z} is discrete.

If R is not reduced then consider $R_{\text{red}} = R/\text{nilrad}(R)$ and $\text{Spec}(R) \cong \text{Spec}(R_{\text{red}})$ naturally so we may assume that R is reduced and we may use idempotent lifting (2.2).

Since V_i is closed $V_i = V(I_i)$ for some ideal $I_i \subset R$. Furthermore,

$$\operatorname{Spec}(R) = \bigcup_{i=1}^n V_i = \bigcup_{i=1}^n V(I_i) = \bigcup_{i=1}^n V(I_n) = V(I_1 \cdots I_n)$$

Thus $\sqrt{I_1 \cdots I_n} = \operatorname{nilrad}(R) = (0)$ so $I_1 \cdots I_n = (0)$. Furthermore, the V_i are disjoint so,

$$\emptyset = V_i \cap V_j = V(I_i) \cap V(I_j) = V(I_i + I_j)$$

and thus $I_i + I_j = R$ so the ideals I_i and I_j are coprime. Therefore, by CRT,

$$R = R/(0) = R/(I_1 \cdots I_n) = (R/I_1) \times \cdots \times (R/I_n)$$

since these ideals are pairwise coprime. (Note, there is an error in the text, it has these two conditions backwards).

2.3 2.5

Consider the following properties,

- (a) $\operatorname{Spec}(R)$ is connected.
- (b) Every finitely generated projective R -module has constant rank.
- (c) R has no idempotent elements except 0 and 1.

I claim that these are equivalent.

See the background material in Appendix A, but for any finitely-generated projective module If $\operatorname{Spec}(A)$ is connected then since $\operatorname{rank}(P)$ is continuous (see Appendix) then its image must be connected in \mathbb{Z} and thus constant.

Suppose $e \in R$ were a nontrivial idempotent. Then consider the module $P = (e)$ which I claim is f.g. (obvious) and projective. It suffices to show that P is free on some open cover. On the open set $D(e)$ we have $P_e \cong R_e$ so P is free on $D(e)$ of rank 1. Furthermore, on the open set $D(1-e)$ we have $P_{1-e} = (e)_{1-e} = (0)$ since $e^2 = e$ and thus P is free of rank 0. Since $e + (1-e) = 1$ these open sets cover $\operatorname{Spec}(R)$. Therefore P is f.g. projective but does not have finite rank. Thus (b) \implies (c).

Finally, if $\operatorname{Spec}(R)$ is not connected then we can write $\operatorname{Spec}(R) = V(I) \cup V(J)$ for two nontrivial disjoint closed sets in which case $IJ = (0)$ and $I + J = R$. Thus by CRT, $R = (R/I) \times (R/J)$. However, the element $(1, 0)$ in this product is a nontrivial idempotent in the ring. Thus (c) \implies (a).

2.4 2.8

2.5 2.10

Let P, Q be R -modules and $P \otimes_R Q \cong R^n$ for $n > 0$. Then P and Q are f.g. projective R -modules.

Proof. (DO) □

2.6 2.11

Let M be a finitely generated module over a commutative ring R . I claim that the following are equivalent for every n ,

- (a) M is f.g. projective of constant rank n
- (b) $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$ for every prime ideal \mathfrak{p} of R .

Clearly (a) \implies (b) so we assume that $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$ at each prime \mathfrak{p} . By Lemma 2.4 it suffices to show that M is finitely presented since then freeness of the stalks implies projectivity and M is automatically of constant rank n by definition.

Lift the basis map $R_{\mathfrak{p}}^n \rightarrow M_{\mathfrak{p}}^n$ to a map $f : R^n \rightarrow M$ by clearing denominators. Now consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \longrightarrow M \longrightarrow \operatorname{coker} f \longrightarrow 0$$

Since M is finitely generated then so is $\operatorname{coker} f$. Furthermore, when we localize at \mathfrak{p} we get,

$$0 \longrightarrow (\ker f)_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}^n \longrightarrow M_{\mathfrak{p}} \longrightarrow (\operatorname{coker} f)_{\mathfrak{p}} \longrightarrow 0$$

but we know $R_{\mathfrak{p}}^n \rightarrow M_{\mathfrak{p}}$ is an isomorphism so $(\operatorname{coker} f)_{\mathfrak{p}} = (\ker f)_{\mathfrak{p}} = 0$. Since $\operatorname{coker} f$ is f.g. there exists $g \in R$ such that $(\operatorname{coker} f)_g = 0$. Then localizing at g instead we find,

$$0 \longrightarrow (\ker f)_g \longrightarrow R_g^n \longrightarrow M_g \longrightarrow 0$$

Then for any prime $\mathfrak{q} \in D(g)$ we may localize again to find,

$$0 \longrightarrow (\ker f)_{\mathfrak{q}} \longrightarrow R_{\mathfrak{q}}^n \longrightarrow M_{\mathfrak{q}} \longrightarrow 0$$

so $R_{\mathfrak{q}}^n \rightarrow M_{\mathfrak{q}}$ is a surjection. However, by assumption $M_{\mathfrak{q}}$ is free of rank n and R is commutative so by 1.3 property (a). we know $R_{\mathfrak{q}}^n \rightarrow M_{\mathfrak{q}}$ is an isomorphism and thus $(\ker f)_{\mathfrak{q}} = 0$. Therefore $(\ker f)_g$ is an A_g -module with empty support so $(\ker f)_g = 0$. Therefore, $M_g \cong R_g^n$ so M is locally free and thus projective.

Therefore, suppose that M is finitely generated free at each stalk with $\operatorname{rank}(M)$ continuous. Then $\operatorname{Spec}(R)$ has a finite open cover $U_i = (\operatorname{rank}(M))^{-1}(n_i)$ such that $M|_{U_i}$ is f.g. with $M_{\mathfrak{p}} = R_{\mathfrak{p}}^{n_i}$ for fixed n_i on each U_i . Thus we have shown that M is locally free on U_i and thus locally free on $\operatorname{Spec}(R)$ and thus projective. Conversely if M is f.g. projective then we know (by Lemma 2.4) that M is locally free and thus $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{n_{\mathfrak{p}}}$ and has continuous rank function.

2.7 2.12

Let $\phi : R \rightarrow S$ be a morphism of rings then let $f = \phi^{-1} : \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$ be the associated morphism of affine schemes. Now there is a functor,

$$f^* : \mathfrak{QCo}\mathfrak{h}(\operatorname{Spec}(R)) \rightarrow \mathfrak{QCo}\mathfrak{h}(\operatorname{Spec}(S))$$

given explicitly by $M \mapsto M \otimes_R S$. I claim that if P is f.g. projective then f^*P is f.g. projective. This is clear using the following property and noting that $(-)\otimes_R S$ is left adjoint to restriction of an S module to an R module which is clearly exact.

Lemma 2.7.1. If a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is left adjoint to $G : \mathcal{D} \rightarrow \mathcal{C}$ between abelian categories and G is exact then F preserves projectives.

Proof. $F(P)$ is projective iff $\text{Hom}_{\mathcal{C}}(F(P), -)$ is exact but,

$$\text{Hom}_{\mathcal{D}}(F(P), -) \cong \text{Hom}_{\mathcal{C}}(P, G(-))$$

which is exact since G and $\text{Hom}_{\mathcal{C}}(P, -)$ are for projective P . □

Now I claim that $\text{rank}(f^*P) = \text{rank}(P) \circ f$. This is because,

$$(f^*P) \otimes_{S_{\mathfrak{p}}} \kappa(\mathfrak{p}) = P \otimes_R S \otimes_{S_{\mathfrak{p}}} \kappa(\mathfrak{p}) = P \otimes_R \kappa(\mathfrak{p})$$

Via the map $R \rightarrow S \rightarrow \kappa(\mathfrak{p})$. Now we get an inclusion of fields, $\kappa(f(\mathfrak{p})) \rightarrow \kappa(\mathfrak{p})$ which $R \rightarrow \kappa(\mathfrak{p})$ factors through. Thus,

$$P \otimes_R \kappa(\mathfrak{p}) = P \otimes_R \kappa(f(\mathfrak{p})) \otimes_{\kappa(f(\mathfrak{p}))} \kappa(\mathfrak{p})$$

In particular, these vectorspaces have equal rank i.e.

$$\begin{aligned} \text{rank}_{\mathfrak{p}}(f^*P) &= \dim_{\kappa(\mathfrak{p})}(P \otimes_R \kappa(\mathfrak{p})) = \dim_{\kappa(\mathfrak{p})}(P \otimes_R \kappa(f(\mathfrak{p})) \otimes_{\kappa(f(\mathfrak{p}))} \kappa(\mathfrak{p})) \\ &= \dim_{\kappa(f(\mathfrak{p}))}(P \otimes_R \kappa(f(\mathfrak{p}))) = \text{rank}_{f(\mathfrak{p})}(P) \end{aligned}$$

2.8 2.16

Fix a small category of rings \mathcal{R} . A big projective R -module is a choice of a finitely generated projective S -module P_S for each S over R in \mathcal{R} equipped with an isomorphism $P_S \otimes_S T \rightarrow P_T$ for every $S \rightarrow T$ over R which satisfies the following properties,

- (a) the identity $\text{id} : S \rightarrow S$ induces $\text{id} : P_S \rightarrow P_S$
- (b) to each commutative triangle of R -algebras we have a commutative triangle of modules.

Now let $\mathbb{P}'(R)$ denote the category of big R -modules and $\mathbb{P}'(R) \rightarrow \mathbb{P}(R)$ be the forgetful functor sending P to P_R . (FINISH THIS)

3 Chapter 1.3

Remark. Here R is a commutative (unital) ring.

3.1 3.1

We need to show that the following are equivalent properties of an R -module L ,

- (a) there is an R -module M such that $L \otimes M \cong R$
- (b) L is an algebraic line bundle (a f.g. projective module of constant rank 1)
- (c) L is a finitely generated R -module and $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ for each prime \mathfrak{p} .

Proof. Assuming (a) then by 2.10 we have L and M are finitely generated projective. Thus $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$ and $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^m$ for some n, m but then $L_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{mn}$ so $m = n = 1$ proving (b).

(b) \implies (c) is a trivial consequence of Lemma 2.4.

Finally assume (c) then I claim that $L \otimes_R L^{\vee} \cong R$ where $L^{\vee} = \text{Hom}_R(L, R)$. First, not there is a natural map $L \otimes L^{\vee} \rightarrow R$ by evaluation. We may check this map is an isomorphism locally on stalks,

$$L_{\mathfrak{p}} \otimes \text{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, R_{\mathfrak{p}}) \rightarrow R_{\mathfrak{p}}$$

(note that $(\text{Hom}_R(L, R))_{\mathfrak{p}} = \text{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, R_{\mathfrak{p}})$ holds since L is finitely presented which holds because it is f.g. projective using criterion (4) proved in 2.11). However, $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ so this above map is clearly an isomorphism with $1 \otimes \text{id} \mapsto 1$. \square

3.2 3.4

3.3 3.15

3.4 3.18

Consider the following sequence,

$$1 \longrightarrow \text{Pic}(R) \longrightarrow \text{Pic}(R[t]) \times \text{Pic}(R[t^{-1}]) \longrightarrow \text{Pic}(R[t, t^{-1}])$$

the first map is induced by the inclusions $R \rightarrow R[t]$ and $R[t^{-1}]$ and the second by the difference of the maps induced by the inclusion $R[t] \rightarrow R[t, t^{-1}]$ and $R[t^{-1}] \rightarrow R[t, t^{-1}]$. Since $\text{Pic}(-)$ is a covariant functor on the category of commutative rings the above sequence is a complex since,

$$R \longrightarrow R[t] \times R[t^{-1}] \longrightarrow R[t, t^{-1}]$$

is exact (this is the computation showing that $\Gamma(\mathbb{P}_R^1, \mathcal{O}_{\mathbb{P}_R^1}) = R$).

Now, given $P \in \text{Pic}(R[t])$ and $Q \in \text{Pic}(R[t^{-1}])$ suppose that $P \otimes_{R[t]} R[t, t^{-1}]$ and $Q \otimes_{R[t^{-1}]} R[t, t^{-1}]$ are isomorphic as $R[t, t^{-1}]$ -modules.

(USE UNITS-PIC sequence and snake lemma)

4 Chapter 2.1

4.1 2.1

4.2 2.2

4.3 2.3

5 Chapter 2.3

5.1 3.3

5.2 3.4

5.3 3.5

5.4 3.7

6 Chapter 2.4

6.1 4.2

7 Chapter 2.5

7.1 5.2

7.2 5.7

7.3 5.8

8 Chapter 2.6-8

9 Appendix A. Rank Functions

Remark. Here R is a commutative (unital) ring.

Definition 9.0.1. Let M be an R -module. Then there is a function $\text{rank}(M) : \text{Spec}(R) \rightarrow \mathbb{Z}$ defined by $x \mapsto \text{rank}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}))$.

Proposition 9.0.2. $\text{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators of $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$ -module.

Proof. If $M_{\mathfrak{p}}$ is generated by m_1, \dots, m_n then $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is generated by $\bar{m}_1, \dots, \bar{m}_n$ over $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ since surjectivity of $R_{\mathfrak{p}}^n \rightarrow M_{\mathfrak{p}}$ is preserved after applying $(-) \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p})$. Thus, $\text{rank}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})} M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) \leq n$.

Now suppose that v_1, \dots, v_n is a $\kappa(\mathfrak{p})$ -basis of $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ then choose lifts $m_1, \dots, m_n \in M_{\mathfrak{p}}$. I claim that m_1, \dots, m_n generated $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$ -module. Let $N \subset M_{\mathfrak{p}}$ be the $R_{\mathfrak{p}}$ -submodule generated by the m_1, \dots, m_n and let $K = M_{\mathfrak{p}}/N$. Then I claim that $\mathfrak{p}K = K$. To see this it suffices

to show that $K \subset \mathfrak{p}K$. For any $m \in M_{\mathfrak{p}}$ we know that its image $\bar{m} \in M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is in the span of the basis v_1, \dots, v_n so,

$$\bar{m} = r_1 v_1 + \dots + r_n v_n$$

for $r_i \in R_{\mathfrak{p}}$. Thus,

$$m - (r_1 m_1 + \dots + r_n m_n) \in \mathfrak{p}M$$

This implies that in K we have $m \in \mathfrak{p}K$ so $K = \mathfrak{p}K$. Then since $\text{Jac}(R_{\mathfrak{p}}) = \mathfrak{p}$ (because $R_{\mathfrak{p}}$ is local) by Nakayama $K = 0$ so $M_{\mathfrak{p}}$ is generated by m_1, \dots, m_n . \square

Theorem 9.0.3. The following are equivalent:

- (a) M is a finitely-generated projective R -module
- (b) M is a locally-free R -module of finite rank $\text{rank}_x(M) < \infty$
- (c) M is a finitely-presented R -module and for each $\mathfrak{p} \in \text{Spec}(R)$, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module.

Proposition 9.0.4. If P is a finitely-generated projective module then $\text{rank}(P) : \text{Spec}(R) \rightarrow \mathbb{Z}$ is continuous.

Proof. It suffices to prove for $f = \text{rank}(P)$ that $f^{-1}(n) = V$ is open. For any $\mathfrak{p} \in V$ we know that $P_{\mathfrak{p}}$ is free of rank n . Lift a basis (by clearing denominators) to a map $f : R^n \rightarrow P$ and consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \xrightarrow{f} P \longrightarrow \text{coker } f \longrightarrow 0$$

Since P is finitely generated then $\text{coker } P$ is also finitely generated. Localizing this exact sequence at \mathfrak{p} we get an exact sequence,

$$0 \longrightarrow (\ker f)_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}^n \xrightarrow{f} P_{\mathfrak{p}} \longrightarrow (\text{coker } f)_{\mathfrak{p}} \longrightarrow 0$$

but $f : R_{\mathfrak{p}}^n \rightarrow P_{\mathfrak{p}}$ is an isomorphism so $(\text{coker } f)_{\mathfrak{p}} = \ker f_{\mathfrak{p}} = 0$. Since $\text{coker } f_{\mathfrak{p}}$ is finitely generated there is some $g \notin \mathfrak{p}$ such that $\text{coker } f_g = 0$. Thus we have.

$$0 \longrightarrow (\ker f)_g \longrightarrow R_g^n \longrightarrow P_g \longrightarrow 0$$

We have yet to use projectivity of P so, in particular, we see that $\forall \mathfrak{q} \in D(g) : \text{rank}_{\mathfrak{q}}(M) \leq n$ for any finitely-generated R -module M . We call this upper-semicontinuity of $\text{rank}(M) : \text{Spec}(R) \rightarrow \mathbb{Z}$.

Now applying projectivity of P (and thus P_g as a R_g -module) the above exact sequence splits to give,

$$R^n \cong P_g \oplus \ker f_g$$

so the projection $R^n \twoheadrightarrow (\ker f)_g$ shows that $\ker f_g$ is finitely generated and $((\ker f)_g)_{\mathfrak{p}} = 0$ so there is some $h \notin \mathfrak{p}$ such that $(\ker f)_{gh} = 0$. Then, by exactness of localization we get $R_{gh}^n \xrightarrow{\sim} P_{gh}$ so P is free of rank n on $D(gh)$ and thus $\forall \mathfrak{q} \in D(gh) : \text{rank}_{\mathfrak{q}}(P) = n$ so $\mathfrak{p} \in D(gh) \subset V$. Therefore, V is open so this function is continuous. \square

Definition 9.0.5. Let X be a scheme and \mathcal{F} a coherent \mathcal{O}_X -module then there is a function $\text{rank}(\mathcal{F}) : X \rightarrow \mathbb{Z}$ defined by $x \mapsto \text{rank}_x(\mathcal{F}) = \dim_{\kappa(x)}(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x))$.

Remark. Since \mathcal{F} is coherent then locally $\mathcal{F}|_U = \widetilde{M}$ for some finitely generated A -module with $U = \operatorname{Spec}(A)$. (Note that this is necessary for coherence but only sufficient when X is locally noetherian). Thus, \mathcal{F}_x is a finitely-generated $\mathcal{O}_{X,x}$ -module and thus $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is finite dimensional.

Theorem 9.0.6. If \mathcal{F} is a projective coherent \mathcal{O}_X -module then $\operatorname{rank}(\mathcal{F}) : X \rightarrow \mathbb{Z}$ is continuous.

Proof. □

Proposition 9.0.7. Projective coherent \mathcal{O}_X -modules on a scheme X are exactly locally-free \mathcal{O}_X -modules of finite type. (CHECK).