1 The Analytic Setting

Definition 1.0.1. An F-valued *local system* $\mathcal{L}$ on a topological space X is a locally-constant sheaf of finite dimensional F-vector spaces.

Proposition 1.0.2. Suppose that X is connected and admits a universal cover. Then the map,

$$\{F\text{-valued local systems on } X\} \to \{\pi_1(X, x)\text{-representations}\}$$

Given by sending a local system to its monodromy representation,

$$\mathcal{F} \mapsto \rho_{\mathcal{F}} : \pi_1(X, x) \to \mathrm{Aut}_F(\mathcal{F}_x) \cong \mathrm{GL}_n(F)$$

is an equivalence of categories.

https://people.maths.ox.ac.uk/liu/seminars/s20-category-o/cailan-notes2-part1.pdf. □

1.1 Local Monodromy

Remark. For the rest of the section, let X be a compact Riemann surface and $S \subset X$ a finite set of points. Let $U = X \setminus S$ and $j : U \hookrightarrow X$ the open immersion. For each $s \in X$ let $D(s) \subset$ be a small disk about s and $D^*(s) = D(s) \cap U$. Let $I(s) = \pi_1(D^*(s))$ and choose a generator γ_s such that $I(s) = \mathbb{Z}\gamma_s$.

Definition 1.1.1. Let $\mathcal{F}$ be a local system on U. The *local monodromy representation* at $s \in S$ is,

$$I(s) := \pi_1(D^*(s)) \to \pi_1(U) \xrightarrow{\rho_{\mathcal{F}}} \mathrm{GL}_n(F)$$

considered up to isomorphism. Explicitly, this is a conjugacy class $\gamma_s \mapsto A_s \in \mathrm{GL}_n(F)$.

Definition 1.1.2. We say that a local system $\mathcal{F}$ is *physically rigid* if for every local system $\mathcal{G}$ on U such that for each $s \in S$ the local monodromy data of $\mathcal{F}$ and $\mathcal{G}$ at s are equal. Explicitly, for each $s \in S$ there is an isomorphism of local systems $\mathcal{F}|_{D^*(s)} \cong \mathcal{G}|_{D^*(s)}$ or equivalently an isomorphism of representations $\rho_{\mathcal{F}}|_{I(s)} \cong \rho_{\mathcal{G}}|_{I(s)}$.

Remark. For $X = \mathbb{P}^1$ this is extremely explicit. For $\#S = r$ the fundamental group is $\pi_1(U) \cong F_{r-1}$ generated by $C_1, \dots, C_r$ sending $C_i \mapsto \gamma_i$ with one relation $C_1 \cdots C_r = 1$. A local system is a choice of matrices $A_1, \dots, A_r \in \mathrm{GL}_n(F)$ subject to $A_1 \dots A_r = I$ (and hence just the choice of $A_1, \dots, A_{r-1}$) up to overall conjugacy. The local monodromy is the conjugacy class $I(s_i) = [A_i]$. Given local monodromy data, $[B_i]$ we ask if there exists a local system $A_1, \dots, A_r$ such that $[A_i] = [B_i]$ and this is rigid if there is a unique such choice up to overall conjugacy.

Remark. If $X = \mathbb{P}^1$ and $S = \{0, \infty\}$ then every local system $\mathcal{F}$ on U is phyiscally rigid because $\mathcal{F}$ is completely determined by its monodromy data $I(0)$ since $D^*(0) \to U$ is a homotopy equivalence. Furthermore, rank 1 local systems on $\mathbb{P}^1 \setminus S$ are rigid because the monodromy directly determines the representation (there is no conjugacy).

Remark. NONRIGID EXAMPLE

Proposition 1.1.3. If $g(X) \geq 1$ there are no phyiscally rigid local systems.

Proof. Let $\mathcal{F}$ be a local system on U and $\mathcal{L}$ a rank 1 nontorsion (meaning no tensor power is trivial) local system on X which exists because $\pi_1(X) \neq 0$. Then $j^*\mathcal{L}$ is nontorsion because $j_* : \pi_1(U, u) \to \pi_1(X, u)$ is surjective. Therefore $j^*\mathcal{L}$ has trivial local monodromy so $\mathcal{F} \otimes j^*\mathcal{L}$ and $\mathcal{F}$ have the same local monodromy but are not isomorphic because $\det \mathcal{F}$ and $\det(\mathcal{F} \otimes j^*\mathcal{L}) = \det \mathcal{F} \otimes (j^*\mathcal{L})^{\mathrm{rank}\,\mathcal{F}}$ are nonisomorphic. □

1.2 Cohomological Rigidity

Proposition 1.2.1. Let X be a manifold and $\mathcal{F}$ a local system. Then,

$$\chi(X, \mathcal{F}) = \chi(X) \cdot \operatorname{rank} \mathcal{F} \quad \text{and} \quad \chi_c(X, \mathcal{F}) = \chi_c(X) \cdot \operatorname{rank} \mathcal{F}$$

Proof. DO MAYER VIETOREZ □

Proposition 1.2.2. Now we use our previous notation with a Riemann surface X. Let $\mathcal{F}$ be a local system on U then,

$$\chi(X, j_* \mathcal{F}) = \chi(X) \cdot \operatorname{rank} \mathcal{F} + \sum_{s \in S} \dim \mathcal{F}_s^{I(s)}$$

Proof. The Leray spectral sequence gives,

$$\chi(U, \mathcal{L}) = \chi(X, j_* \mathcal{L}) - \chi(X, R^1 f_* \mathcal{L})$$

Then $R^1 f_* \mathcal{L}$ is supported on S. For each disk $D^*(s)$ □

Proposition 1.2.3. Let $X = \mathbb{P}^1$ and $\mathcal{F}$ an irreducible local system on U. Then $\mathcal{F}$ is phyiscally rigid if and only if $H^1(X, j_* \operatorname{End}(\mathcal{F})) = 0$.

Proof. Apply the previous calculation to $\mathcal{L} = \operatorname{End}(\mathcal{F})$ and $\mathcal{L} = \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ which have isomorphic local monodromy. Therefore,

$$\chi(X, j_* \operatorname{Hom}(\mathcal{F}, \mathcal{G})) = \chi(X, j_* \operatorname{End}(\mathcal{F}))) = 2$$

Therefore,

$$h^0(X, j_* \operatorname{Hom}(\mathcal{F}, \mathcal{G})) + h^2(X, j_* \operatorname{Hom}(\mathcal{F}, \mathcal{G})) \geq 2$$

Furthermore,

$$h^2(X, j_* \operatorname{Hom}(\mathcal{F}, \mathcal{G})) = h_c^2(U, \operatorname{Hom}(\mathcal{F}, \mathcal{G})) = h^0(U, \operatorname{Hom}(\mathcal{G}, \mathcal{F}))$$

Therefore one of $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ or $\operatorname{Hom}(\mathcal{G}, \mathcal{F})$ has a nonzero global section. Because $\mathcal{F}$ and $\mathcal{G}$ are irreducible this must be an isomorphism. □

Remark. This justifies thinking of $H^1(X, j_* \operatorname{End}(\mathcal{F}))$ as the deformation space of local systems with fixed monodromy on S at $\mathcal{F}$. This is an idea we will explore further now.

DO THE MOTIVATION (3.2.2) IN THIS SETTING.

2 The étale Setting

Remark. For now, let k be any field and let U be a finite type scheme over k.

Definition 2.0.1. A *local system* on $U_{\text{ét}}$ is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf. The category $\operatorname{Loc}(U)$ is surprisingly difficult to define. First we define $\operatorname{Loc}(U, \mathbb{Z}/\ell^n\mathbb{Z})$ as the category of locally-constant finite locally-free étale sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. Then a lisse $\mathbb{Z}_\ell$-sheaf is a projective system $\{\mathcal{F}_n\}$ of $\mathbb{Z}/\ell^n\mathbb{Z}$-local systems such that,

$$\mathcal{F}_n \otimes \mathbb{Z}/\ell^{n-1}\mathbb{Z} \to \mathcal{F}_{n-1}$$

is an isomorphism. Thus we write,

$$\operatorname{Loc}(U, \mathbb{Z}_\ell) = \varprojlim \operatorname{Loc}(U, \mathbb{Z}/\ell^n\mathbb{Z})$$

Now the category of lisse $\mathbb{Q}_\ell$-sheaves is,

$$\mathrm{Loc}(U, \mathbb{Q}_\ell) = \mathrm{Loc}(U, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

where we invert ℓ in the Hom. Similarly, if $L/\mathbb{Q}_\ell$ is a finite extensions we define $\mathrm{Loc}(U, \mathcal{O}_L)$ and $\mathrm{Loc}(U, L)$ in the same way. Finally, we define,

$$\mathrm{Loc}(U) := \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell) = \varinjlim \mathrm{Loc}(U, L)$$

Theorem 2.0.2. Let U be normal and connected and $\bar{u} \in U$ a geometric point. Then there is an equivalence of categories,

$$\mathrm{Loc}(U) \xrightarrow{\sim} \{\rho : \pi_1^{\text{ét}}(U, \bar{u}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \text{ continuous}\}$$

defined by evaluating on the fiber over $\bar{u}$,

$$\mathcal{F} \mapsto \rho_{\mathcal{F}} : \pi_1(U, \bar{u}) \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}_{\bar{u}}) \cong \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

Remark. The “correct“ statement is PROETALE AND FINITENESS

Remark. Let $\pi_1^{\text{geom}}(U, \bar{u}) = \pi_1(U_{\bar{k}}, \bar{u})$. Then there is a short exact sequence,

$$1 \longrightarrow \pi_1^{\text{geom}}(U, \bar{u}) \longrightarrow \pi_1(U, \bar{u}) \longrightarrow \mathrm{Gal}\left(k^{\text{sep}}/k\right) \longrightarrow 1$$

2.1 H-Local Systems

Remark. Local systems correspond to continuous representations,

$$\rho : \pi_1(U, \bar{u}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

Given an affine algebraic group H, we want a geometric object that corresponds to a continuous homomorphism,

$$\rho : \pi_1(U, \bar{u}) \to H(\overline{\mathbb{Q}}_\ell)$$

which form a category under intertwining by elements of $H(\overline{\mathbb{Q}}_\ell)$.

Definition 2.1.1. Let $\mathrm{Rep}(H)$ be the tensor category of algebraic repreesntations of H on finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces. An H-local system is a tensor-preserving functor $\mathcal{F} : \mathrm{Rep}(H) \to \mathrm{Loc}(U)$. Thus the category of H-local systems is,

$$\mathrm{Loc}_H(U) = \mathrm{Fun}^{\otimes}(\mathrm{Rep}(H), \mathrm{Loc}(U))$$

Theorem 2.1.2. Let U be normal and connected. Then there is an equivalence of categories,

$$\mathrm{Loc}_H(U) \xrightarrow{\sim} \{\rho : \pi_1(U, \bar{u}) \to H(\overline{\mathbb{Q}}_\ell)\}$$

Defined by sending ρ to the functor,

$$\mathcal{F}_\rho : V \in \mathrm{Rep}(H) \mapsto [\rho_V : \pi_1(U, \bar{u}) \xrightarrow{\rho} H(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}(V)]$$

Conversely, $\mathcal{F} \in \mathrm{Loc}_H(U)$ can be viewed as a functor $\mathcal{F} : \mathrm{Rep}(H) \to \mathrm{Rep}(\pi_1(U, \bar{u}))$ and hence defines a continuous homomorphism $\rho_{\mathcal{F}} : \pi_1(U, \bar{u}) \to H(\overline{\mathbb{Q}}_\ell)$ well-defined up to conjugacy.

Definition 2.1.3. Let $\mathcal{F} \in \mathrm{Loc}_H(U)$ with corresponding $\rho_{\mathcal{F}} : \pi_1(U, \bar{u}) \to H(\overline{\mathbb{Q}}_\ell)$. The *global geometric monodromy group* $H_{\mathcal{F}}^{\text{geom}}$ is the Zariski closure,

$$H_{\mathcal{F}}^{\text{geom}} = \overline{\rho(\pi_1^{\text{geom}}(U, \bar{u}))} \subset H$$

Theorem 2.1.4. DELIGNE??

2.2 Local Monodromy

Remark. In this section, we let X be a projective, smooth geomertically connected curve over a perfect field k and $S \subset X(k)$ a finite set of rational points. Let $U = X \setminus S$ be the open complement and $j : U \hookrightarrow X$ the open immersion.

Remark. We require that k is perfect so that the residue fields of X are also all perfect which leads to good behavior of the unramified extensions of the local fields.

Definition 2.2.1. Let $x \in X$ be a closed point let $\widehat{\mathcal{O}_{X,x}}$ be the completed local ring and F_x its fraction field and k_x its residue field. Choose an algebraic closure $\overline{F_x}$ which defines a geometric generic point,

$$\begin{tikzcd} \eta_x : \mathrm{Spec}\left(\overline{F_x}\right) \arrow[r] & \mathrm{Spec}\left(F_x\right) \arrow[r] \arrow[rrd] & \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right) \arrow[r] & X \\ & & & U \arrow[u, hook] \end{tikzcd}$$

This map gives a homomorphism of fundamental groups,

$$\Gamma_x = \mathrm{Gal}\left(F_x^{\mathrm{sep}}/F_x\right) \xrightarrow{\eta_x} \pi_1(U, \eta_x) \cong \pi_1(U, \bar{u})$$

where the second isomorphism is well-defined upt to conjugacy.

Proposition 2.2.2. If $x \in S$ then $\eta_x : \Gamma_x \to \pi_1(U, \bar{u})$ is injective.

Definition 2.2.3. Consider the diagram,

$$\begin{tikzcd} & \mathrm{Spec}\left(k_x\right) \arrow[d] \\ \mathrm{Spec}\left(F_x\right) \arrow[r] & \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right) \end{tikzcd}$$

which induces a diagram of fundamental groups,

$$\begin{tikzcd} & \mathrm{Gal}\left(\bar{k}_x/k_x\right) \arrow[d] \arrow[dr, "\sim"] & \\ \mathrm{Gal}\left(F_x^{\mathrm{sep}}/F_x\right) \arrow[r, two heads] \arrow[ur, dashed] & \pi_1(\mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right), \eta_x) \arrow[r, equal] & \mathrm{Gal}\left(F_x^{\mathrm{ur}}/F_x\right) \end{tikzcd}$$

using that k_x/k is finite and hence k_x is perfect. Then because F_x is a local field with perfect residue field k_x the map $\mathrm{Gal}\left(F_x^{\mathrm{ur}}/F_x\right) \to \mathrm{Gal}\left(\bar{k}_x/k_x\right)$ is an isomorphism. We define the kernel,

$$1 \longrightarrow I_x \longrightarrow \mathrm{Gal}\left(F_x^{\mathrm{sep}}/F_x\right) \longrightarrow \mathrm{Gal}\left(\bar{k}_x/k_x\right) \longrightarrow 1$$

to be the *inertia group* at $x \in U$.

Proposition 2.2.4. Under the map $\Gamma_x \to \pi_1(U, \bar{u})$ the subgroup I_x lands in $\pi_1^{\mathrm{geom}}(U, \bar{u}) \triangleleft \pi_1(U, \bar{u})$.

Proof. This is immediate from the fact that the previous diagram is in the category of k-schemes. Explicitly,

$$\begin{tikzcd}
\mathrm{Spec}\,(F_x) \arrow[r] \arrow[d] & \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right) \arrow[d] & \mathrm{Spec}\,(k_x) \arrow[l] \arrow[d] \\
U \arrow[r, hook] & X \arrow[r] & \mathrm{Spec}\,(k)
\end{tikzcd}$$

commutes. Therefore, we get a diagram of exact sequences,

$$\begin{tikzcd}
1 \arrow[r] & I_x \arrow[r] \arrow[d, dashed] & \mathrm{Gal}\,(F_x^{\mathrm{sep}}/F_x) \arrow[r] \arrow[d] & \mathrm{Gal}\left(\bar{k}_x/k_x\right) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \pi_1^{\mathrm{geom}}(U, \bar{u}) \arrow[r] & \pi_1(U, \bar{u}) \arrow[r] & \mathrm{Gal}\left(\bar{k}/k\right) \arrow[r] & 1
\end{tikzcd}$$

□

Remark. Furthermore, if $x \in U$ then $\eta_x : \Gamma_x \to \pi_1(U, \bar{u})$ factors through $\mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right) \to U$ which means it factors through $\mathrm{Gal}\,(F_x^{\mathrm{ur}}/F_x)$ and hence sends the monodromy to zero.

Definition 2.2.5. When $\mathrm{char}\, k = p$ is positive there is a normal subgroup $I_x^w \lhd I_x$ called the *wild interia* subgroup suhc that its quotient $I_x^t = I_x/I_x^w$ the *tame inertia group* is the maximal prime-to-p quotient of I_x.

Proposition 2.2.6. There is a canonical isomorphism of $\mathrm{Gal}\left(\bar{k}_x/k_x\right)$-modules,

$$I_x^t \xrightarrow{\sim} \varprojlim_{(n,p)=1} \mu_n(\bar{k}) = \hat{\mathbb{Z}}^{(p)}(1)$$

Definition 2.2.7. Let $\rho : \pi_1(U, \bar{i}) \to H(\overline{\mathbb{Q}}_\ell)$ be an H-local system. The *local monodromy* of ρ at $x \in S$ is the homomorphism $\rho_x := \rho|_{I_x} I_x \to H(\overline{\mathbb{Q}}_\ell)$. The local system ρ is *tame* at $x \in S$ if $\rho_x(I_x^w) = 0$ and hence if ρ_x factors through the tame inertia group I_x^t.

Remark. In the case $H = \mathrm{GL}_n$ the map ρ_x is just the representation of $\pi_1(U, \bar{u})$ restricted to the subgroup $\eta_x(I_x) \subset \pi_1(U, \bar{u})$. For some reason, Zhiwei intermittently calls this the "local geometric monodromy".

2.3 Ramification Conductors

Definition 2.3.1. Let $\sigma : I_x \to \mathrm{GL}(V)$ be a continuous representation of inertia on a $\overline{\mathbb{Q}}_\ell$-vector space V such that $D = \sigma(I_x)$ is finite[1]. There is some finite Galois extension L/F_x^{ur} such that $D = \mathrm{Gal}\,(L/F_x^{\mathrm{ur}})$ and then we define a filtration,

$$D = D_0 \rhd D_1 \rhd D_2 \rhd \cdots$$

where,

$$D_i = \{\sigma \in D \mid \forall x \in \mathcal{O}_L : \sigma(x) \equiv x \mod \mathfrak{m}_L^{i+1}\}$$

is the subgroup of D acting trivially on $\mathcal{O}_L/\mathfrak{m}_L^{i+1}$. Then the Swan conductor is defined as,

$$\mathrm{Sw}(\sigma) = \sum_{i \geq 1} \frac{\dim\,(V/V^{D_i})}{[D : D_i]}$$

[1]This will be the case for those arising from Galois representations (WHY!??)

Likewise, the Artin conductor is,

$$a(\sigma) := \sum_{i \ge 0} \frac{\dim(V/V^{D_i})}{[D : D_i]} = \dim(V/V^{I_x}) + \mathrm{Sw}(\sigma)$$

Remark. I think there is a typo in Zhiwei's notes here with i and $i+1$.

Remark. Since $D_1 = \sigma(I_x^w)$ if σ is tamely ramified then $\mathrm{Sw}(\sigma) = 0$ because there is no $i = 0$ term in $\mathrm{Sw}(\sigma)$. Indeed σ is tamely ramified if and only if $\mathrm{Sw}(\sigma) = 0$. Likewise, σ is unramified (i.e. trivial because we are only considering $\sigma = \rho|_{I_x}$) if and only if $a(\sigma) = 0$.

2.4 Rigidity

Remark. In this section, we assume that S is nonempty so that U is nonproper.

Definition 2.4.1. An H-local system $\mathcal{F} \in \mathrm{Loc}_H(U)$ is *physically rigid* if for any other $\mathcal{F}' \in \mathrm{Loc}_H(U)$ such that for each $x \in S$ the local

Definition 2.4.2. Let $\mathcal{F} \in \mathrm{Loc}_H(U)$ be an H-local system and $n = \dim H$. We define a GL_n-local system (i.e. a local system in the standard sense) $\mathrm{Ad}(\mathcal{F})$ via,

$$\mathrm{Ad}(\mathcal{F}) = \mathcal{F}_{\mathrm{Ad}} \in \mathrm{Loc}(U)$$

Furthermore, $\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})$ is the GL_{n-1}-local system,

$$\mathrm{Ad}(\mathcal{F}) = \mathcal{F}_{\mathrm{Ad}^{\mathrm{der}}}$$

where $\mathrm{Ad}^{\mathrm{der}}$ is the representation of H on $\mathfrak{h}^{\mathrm{der}} = \ker(\mathfrak{h} \to \mathfrak{h}^{\mathrm{ab}})$ is the Lie algebra of the derived subgroup.

Remark. Notice that if $H = \mathrm{GL}_n$ then $\mathrm{Ad}(\mathcal{F}) = \mathrm{End}(\mathcal{F})$ and $\mathrm{Ad}^{\mathrm{der}}\mathcal{F} = \mathrm{End}^0(\mathcal{F})$ the subsheaf of traceless endomorphisms.

Remark. Following Zhiewei, we denote by $j_!$ and j_* the *derived* extension by zero and pushforward respectively. Furthermore we denote by $j_{!*}$ the usually pushforward operation on sheaves (what sane people would call j_*) because for a local system $\mathcal{F}$ the sheaf $j_{!*}\mathcal{F}$ agrees with the middle extension of the perverse sheaf $\mathcal{F}[1]$.

Definition 2.4.3. An object $\mathcal{F} \in \mathrm{Loc}_H(U)$ is *cohomolocally rigid* if,

$$\mathrm{Rig}(\mathcal{F}) := H^1(X, j_{!*}\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})) = 0$$

Remark. Since $\mathfrak{h}^{\mathrm{der}}$ carries the Ad-invariant symmetric bilinear Killing form then $j_{!*}\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})$ is Verdier self-dual and $\mathrm{Rig}(\mathcal{F})$ is a symplectic space and hence has even dimension. Furthermore,

$$\dim H^0(X, j_{!*}\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})) = \dim H^2(X, j_{!*}\mathrm{Ad}^{\mathrm{der}}(\mathcal{F}))$$

which says that $\mathcal{F}$ is unobstructed if and only if it has no automorphisms.

Remark. EXPLAIN FIXING THE CHARACTER!!!

Remark. Because $j_{!*}\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})$ does not change if we shrink U and pull back $\mathcal{F}$ we see that cohomological rigidity is also insensitive to U (there is of course a largest U on which $\mathcal{F}$ is defined).

Lemma 2.4.4. For any local system $\mathcal{L}$ on U there is an exact sequence,

$$0 \longrightarrow H^0(U,\mathcal{L}) \longrightarrow \bigoplus_{s\in S}(\mathcal{L}_x)^{I_x} \longrightarrow H^1_c(U,\mathcal{L}) \longrightarrow H^1(U,\mathcal{L}) \longrightarrow \bigoplus_{s\in S}(\mathcal{L}_x)_{I_x}(-1) \longrightarrow H^2_c(U,\mathcal{L}) \longrightarrow 0$$

Proof. This should follow from an exact sequence of sheaves,

$$0 \longrightarrow j_!\mathcal{L} \longrightarrow j_{!*}\mathcal{L} \longrightarrow \bigoplus_{x\in S}\mathcal{L}_x \longrightarrow 0$$

Taking the associated long exact sequence gives the desired result noting that $H^q(X, j_!\mathcal{L}) = H^q_c(U,\mathcal{L})$ and $H^0_c(U,\mathcal{L}) = 0$ for $S \neq \varnothing$ along with the following identifications,

$$\begin{aligned} H^0(X, j_{!*}\mathcal{L}) &= H^0(U,\mathcal{L}) \cong (\mathcal{L}_{\bar{u}})^{\pi_1(U,\bar{u})} \\ H^1(X, j_{!*}\mathcal{L}) &= \mathrm{im}\,(H^1_c(U,\mathcal{L}) \to H^1(U,\mathcal{L})) \\ H^2(X, j_{!*}\mathcal{L}) &= H^2_c(U,\mathcal{L}) \cong (\mathcal{L}_{\bar{u}})_{\pi_1(U,\bar{u})}(-1) \end{aligned}$$

□

Theorem 2.4.5 (Grothendieck-Ogg-Shafarevich)**.** Let $\mathcal{L}$ be a local system. Then,

$$\chi_c(U,\mathcal{L}) = \chi_c(U)\cdot \mathrm{rank}\,\mathcal{L} - \sum_{x\in S}\mathrm{Sw}_x(\mathcal{L})$$

Example 2.4.6. DO THE ARTIN-SCRIER COVER!!

Proposition 2.4.7. Let $\mathcal{F} \in \mathrm{Loc}_H(U)$. Then $\mathcal{F}$ is cohomologically rigid if and only if,

$$\frac{1}{2}\sum_{x\in S} a_x(\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})) = (1-g_X)\dim\mathfrak{h}^{\mathrm{der}} - \dim H^0(U,\mathrm{Ad}^{\mathrm{der}}(\mathcal{F}))$$

where a_x is the Artin conductor at $x \in S$ and g_X is the genus of X.

Proof. We apply the Grothendieck-Ogg-Shafarevich formula,

$$\chi_c(U,\mathcal{L}) = \chi_c(U)\cdot \mathrm{rank}\,\mathcal{L} - \sum_{x\in S}\mathrm{Sw}_x(\mathcal{L})$$

And $\chi_c(U) = 2 - 2g_X - \#S$. However, by the previous lemma,

$$\dim H^1_c(X, j_{!*}\mathcal{L}) = \dim H^1_c(U,\mathcal{L}) - \sum_{x\in S}\dim(\mathcal{L}_x)^{I_x} + \dim H^0(U,\mathcal{L})$$

Adding the RHS - LHS of the GOS formula on the RHS we get [2]

$$\dim H^1_c(X, j_{!*}\mathcal{L}) = \sum_{x\in S}\Big(\dim(\mathcal{L}_x/\mathcal{L}_x^{I_x}) + \mathrm{Sw}_x(\mathcal{L})\Big) + (2g_X - 2)\cdot\mathrm{rank}\,\mathcal{L} + \dim H^2_c(U,\mathcal{L}) + \dim H^0(U,\mathcal{L})$$

By the definition of the Artin condutor and Poincare duality if $\mathcal{L}$ is self-dual,

$$\dim H^1_c(X, j_{!*}\mathcal{L}) = \sum_{x\in S} a_x(\mathcal{L}) + (2g_X - 2)\cdot\mathrm{rank}\,\mathcal{L} + 2\dim H^0(U,\mathcal{L})$$

[2] The first term comes from

$$\#S\cdot\mathrm{rank}\,\mathcal{L} - \sum_{x\in S}\dim\mathcal{L}_x^{I_x} = \sum_{x\in S}\dim(\mathcal{L}_x/\mathcal{L}_x^{I_x})$$

and $\chi_c(X,\mathcal{L}) + \dim H^1_c(U,\mathcal{L}) = \dim H^0_c(U,\mathcal{L}) + \dim H^2_c(U,\mathcal{L}) = \dim H^2_c(U,\mathcal{L})$ since $H^0_c(U,\mathcal{L}) = 0$.

Applying this to $\mathcal{L} = \mathrm{Ad}^{\mathrm{der}}(\mathcal{F})$ we conclude that,

$$\frac{1}{2}\mathrm{Rig}(\mathcal{F}) = \frac{1}{2}\sum_{x \in S} a_x(\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})) - \Big[(1 - g_X)\dim \mathfrak{h}^{\mathrm{der}} - \dim H^0(U, \mathrm{Ad}^{\mathrm{der}}(\mathcal{F}))\Big]$$

proving the claim. □

Corollary 2.4.8. Cohomologically rigid H-local systems exist only when $g_X \leq 1$. When $g_X = 1$ and $\mathcal{F} \in \mathrm{Loc}_H(U)$ is cohomologically rigid then $\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})$ must be everywhere unramified and have no global sections.

Proof. For $g_X > 1$ the RHS of the above is negative but the LHS is by definition non-negative giving a contradiction. For $g_X = 1$ the RHS is only non-negative if $H^0(U, \mathrm{Ad}^{\mathrm{der}}(\mathcal{F})) = 0$ in which case both sides are zero and thus each Artin conductor $a_x(\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})) = 0$ meaning that $\mathrm{Ad}^{\mathrm{der}}(\mathcal{F})$ is everywhere unramified. □

Theorem 2.4.9 (Katz)**.** For $X = \mathbb{P}^1$ and $H = \mathrm{GL}_n$ the notions of physical rigidty and cohomological rigidity coincide.