1 Cartier Divisors

1.1 Regular Sections

Definition 1.1.1. Let $(X, \mathcal{O}_X)$ be a ringed space. We say a section $f \in \Gamma(U, \mathcal{O}_X)$ is *regular* if the morphism $\mathcal{O}_X|_U \xrightarrow{f} \mathcal{O}_X|_U$ via $s \mapsto fs$ is injective.

Lemma 1.1.2. Let X be a locally ringed space and $f \in \Gamma(U, \mathcal{O}_X)$ a section. Then the following are equivalent,

(a) f is a regular section

(b) for any $x \in U$ the image $f \in \mathcal{O}_{X,x}$ is a non-zero divisor.

If X is a scheme there are also equivalent to,

(a) for any affine open $\mathrm{Spec}\,(A) = V \subset U$ the image $f \in A$ is a non-zero divisor

Proof. f is regular when for any open $V \subset U$ and $g \in \Gamma(V, \mathcal{O}_X)$ we have $f|_V g = 0 \implies g = 0$ which is exactly the condition that $f_x \in \mathcal{O}_{X,x}$ is a nonzero divisor for each $x \in U$ since $f_x \in \mathcal{O}_{X,x}$ is a zero divisor if there is some neighborhood $x \in V$ and nonzero $g \in \Gamma(V, \mathcal{O}_X)$ with $f|_V g = 0$.

The sheaf map $\mathcal{O}_X \to \mathcal{O}_X$ given by $f \mapsto fs$ is injective iff on stalks $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is injective i.e. $f \in \mathcal{O}_{X,x}$ is a non-zero divisor.

Now let X be a scheme. If on each affine open $\mathrm{Spec}\,(A) = V \subset U$ the image $f \in A$ is a non zero divisor then, since affine opens form a base for the topology on X, then $f_x \in \mathcal{O}_{X,x}$ is a non zero divisor since otherwise it would be a zero divisor on some open neighborhood containing an affine open. Conversely, if $f|_U$ is a zero divisor on some affine open then for each $x \in U$ the image $f_x \in \mathcal{O}_{X,x}$ is a zero divisor. □

Definition 1.1.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Then define the sheaf of regular sections $\mathcal{S}_X$ via,

$$\mathcal{S}_X(U) = \{f \in \Gamma(U, \mathcal{O}_X) \mid \text{regular}\}$$

Then $\mathcal{S}_X$ is a sheaf because a section is regular exactly if it is regular on a cover.

Definition 1.1.4. Let $(X, \mathcal{O}_X)$ be a ringed space. The sheaf $\mathscr{K}_X$ of *meromorphic functions* on X is the $\mathcal{O}_X$-module associated to the presheaf,

$$U \mapsto \mathcal{S}_X(U)^{-1}\mathcal{O}_X(U)$$

Lemma 1.1.5. The natural map $\mathcal{O}_X \to \mathscr{K}_X$ is injective.

Proof. Consider the map $\mathcal{O}_{X,x} \to \mathscr{K}_{X,x}$ where $\mathscr{K}_{X,x} = \mathcal{S}_{X,x}^{-1}\mathcal{O}_{X,x}$. Now $\mathcal{S}_{X,x} \subset \mathcal{O}_{X,x}$ is contained in the set of nonzerodivisors (although it may not be equal to the set of nonzerodivisors of $\mathcal{O}_{X,X}$). Therefore, the map $\mathcal{O}_{X,x} \to \mathcal{S}_{X,x}^{-1}\mathcal{O}_{X,x}$ is injective and further we have an inclusion $\mathscr{K}_{X,x} \subset Q(\mathcal{O}_{X,x})$ inside the total quotient ring of $\mathcal{O}_{X,x}$. □

Definition 1.1.6. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. We say that *pullbacks of meromorphic functions are defined* for f if for all opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ the pullback $f^\# : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ takes regular sections to regular sections i.e. for any $s \in \Gamma(V, \mathcal{S}_Y)$ the pullback $f^\#(s) \in \Gamma(U, \mathcal{O}_X)$ is an element of $\Gamma(U, \mathcal{S}_X)$.

In this case, there is a morphism $f^\# : \mathscr{K}_Y \to f_*\mathscr{K}_X$ and thus there is a morphism of ringed spaces,

$$\begin{tikzcd}
(X, \mathscr{K}_X) \arrow[r] \arrow[d, "f"] & (X, \mathcal{O}_X) \arrow[d, "f"] \\
(Y, \mathscr{K}_Y) \arrow[r] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

Proposition 1.1.7. Let $f : X \to Y$ be a morphism of schemes such that ether,

(a) X and Y are integral and f is dominant

(b) f is flat

then pullbacks of meromorphic functions are defined for f.

Proof. □

Lemma 1.1.8. Let X be an integral scheme X with generic point $\xi \in X$. Then for any open $U \subset X$, the map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\xi}$ is injective.

Proof. Choose an open cover $U_i = \text{Spec}\,(A_i) \subset X$ where A_i is a domain then $K(X) = \mathcal{O}_{X,\xi} = \text{Frac}\,(A_i)$ since $\xi \in \text{Spec}\,(A_i)$ is the generic point. Thus, $\mathcal{O}_X(U) \to \mathcal{O}_{X,\xi}$ is an injection because, if $f_\xi = 0$ then consider $f|_{U \cap U_i} \in A_i$ but A_i is a domain so if $f_\xi \in \text{Frac}\,(A_i)$ is zero then $f|_{U \cap U_i} = 0$ for each U_i so $f = 0$. □

Remark. The above lemma alows us to view all functions on X as elements of $K(X)$. In fact, the meromorphic functions on X are exactly $K(X)$.

Proposition 1.1.9. Let X be a integral scheme. Then $\mathscr{K}_X = \underline{K(X)}$.

Proof. Let $\xi \in X$ be the generic point and $U \subset X$ an open set. Consider the presheaf map $\mathcal{S}_X(U)^{-1}\mathcal{O}_X(U) \to K(X)$ sending $f \mapsto f_\xi$ which is well-defined because regular sections have $f_\xi \neq 0$ and $K(X)$ is a field so regular sections are invertible in $K(X)$. Sheafifying, gives a map $\mathscr{K}_X \to \underline{K(X)}$. To show this map is an isomorphism it suffices to check on the stalks which can be computed from the above presheaves. By above, the map $\mathcal{S}_X(U)^{-1}\mathcal{O}_{X,(}U) \to K(X)$ is always injective. Furthermore, for any $x \in X$ choose an affine open neighborhood $U = \text{Spec}\,(A)$ with A a domain. Then $\mathcal{S}_X(U) = A \setminus \{0\}$ since $A \to A_\mathfrak{p}$ is injective and $A_\mathfrak{p}$ is a domain for each prime $\mathfrak{p}$ so evey nonzero $f \in A$ is regular. Thus, $\mathcal{S}_X(U)^{-1}\mathcal{O}_X(U) = \text{Frac}\,(A)$ and the map $\mathcal{S}_X(U)^{-1}\mathcal{O}_X(U) \to K(X) = A_{(0)} = \text{Frac}\,(A)$ is an isomorphism. □

1.2 Meromorphic Sections

Definition 1.2.1. Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module on a ringed space $(X, \mathcal{O}_X)$. Then the *sheaf of meromorphic sections of* $\mathscr{F}$ is $\mathscr{K}_X(\mathscr{F}) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X$. A *meromorphic section* is a global section $\eta \in \Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X)$.

Remark. The sheaf $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X$ is the sheaf associated to the presheaf,

$$U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{K}_X^{\text{ps}}(U) = \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{S}_X(U)^{-1}\mathcal{O}_X(U) = \mathcal{S}_X(U)^{-1}\mathscr{F}(U)$$

explaining the notation.

Proposition 1.2.2. Let X be a Noetherian scheme and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $\mathscr{F}$ has a meromorphic section i.e. $\Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X) \neq 0$.

Proposition 1.2.3. Let X be an integral scheme with generic point ξ. Then $\mathscr{K}_X = \underline{\mathscr{F}_\xi}$.

Proof. □

1.3 Cartier Divisors

Definition 1.3.1. Let X be a ringed space. The *sheaf of Cartier divisors* on X is $\mathfrak{Div}_X = \mathscr{K}_X^\times / \mathcal{O}_X^\times$. The group of Cartier divisors is $\text{Ca}(X) = H^0(X, \mathfrak{Div}_X)$ and the Cartier class group is,

$$\text{CaCl}(X) = \text{coker}\,(H^0(X, \mathscr{K}_X^\times) \to H^0(X, \mathfrak{Div}_X))$$

Proposition 1.3.2. There is a natural embedding $\text{CaCl}(X) \hookrightarrow \text{Pic}(X)$ which is an isomorphism when $H^1(X, \mathscr{K}_X^\times) = 0$.

Proof. Consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathscr{K}_X^\times \longrightarrow \mathfrak{Div}_X \longrightarrow 0$$

Taking cohomology gives,

$$0 \longrightarrow H^0(X, \mathcal{O}_X^\times) \longrightarrow H^0(X, \mathscr{K}_X^\times) \longrightarrow H^0(X, \mathfrak{Div}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathscr{K}_X^\times)$$

But $H^1(X, \mathcal{O}_X^\times) = \text{Pic}(X)$ and by exactness, we get an exact sequence,

$$0 \longrightarrow \text{CaCl}(X) \longrightarrow \text{Pic}(X) \longrightarrow H^1(X, \mathscr{K}_X^\times)$$

□

Remark. The condition $H^1(X, \mathscr{K}_X^\times) = 0$ occurs when X is an integral scheme. Then $\mathscr{K}_X^\times = \underline{K(X)^\times}$ is a constant sheaf and X is irreducible so its higher cohomology vanishes.

1.4 Cousins Problems

Here we let X be a complex manifold and $\mathcal{O}_X$ be its sheaf of holomorphic functions and $\mathscr{K}_X$ be its sheaf of meromorphic functions. The Cousins problems are the following questions given a cover U_i and a meromorphic function $f_i \in \Gamma(U_i, \mathscr{K}_X)$ on U_i.

Definition 1.4.1. The Cousins problems ask the following.

(a) (First or additive Cousin Problem) if $(f_i - f_j)|_{U_i \cap U_j}$ is holomorphic for each pair i, j then does there exist a global meromorphic function $f \in \Gamma(X, \mathscr{K}_X)$ such that $f|_{U_i} - f_i$ is holomorphic?

(b) (Second or multiplicative Cousin Problem) if $(f_i/f_j)|_{U_i \cap U_j}$ is non-vanishing holomoprhic for each pair i, j then does there exist a global meromorphic function $f \in \Gamma(X, \mathscr{K}_X)$ such that $f|_{U_i}/f_i$ is holomorphic and non-vanishing?

Notice that set of pairs $\{(U_i, f_i)\}$ in the first Cousin problem defines a global section of the sheaf $\mathscr{K}_X/\mathcal{O}_X$ exactly because $(f_i - f_j)|_{U_i \cap U_j} \in \mathcal{O}_X(U_i \cap U_j)$ is holomorphic. Likewsie, the set of pairs $\{(U_i, f_i)\}$ in the second Cousin problem defined a global section of the sheaf $\mathscr{K}_X^\times / \mathcal{O}_X^\times$ exactly because $(f_i/f_j)|_{U_i \cap U_j} \in \mathcal{O}_X^\times(U_i \cap U_j)$ is holomorphic and nonvanishing. Therefore, we can restate the Cousins problems as follows.

Definition 1.4.2. The Cousins problems ask the following.

(a) (First Cousin Problem) is the map $H^0(X, \mathscr{K}_X) \to H^0(X, \mathscr{K}_X/\mathcal{O}_X)$ surjective?

(b) (Second Cousin Problem) is the map $H^0(X, \mathscr{K}_X^\times) \to H^0(X, \mathscr{K}_X^\times/\mathcal{O}_X^\times)$ surjective?

Now we can solve these problems using the following two exact sequences of sheaves,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \mathscr{K}_X/\mathcal{O}_X \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathscr{K}_X^\times \longrightarrow \mathscr{K}_X^\times/\mathcal{O}_X^\times \longrightarrow 0$$

and we can relate the sheaf cohomology needed in the two problems via the exponential exact sequence,

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

Theorem 1.4.3. The first cousin problem is solvable when $H^1(X, \mathcal{O}_X) = 0$.

Proof. The first exact sequence gives a cohomology exact sequence,

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathscr{K}_X) \longrightarrow H^0(X, \mathscr{K}_X/\mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathscr{K}_X)$$

Clearly, if $H^1(X, \mathcal{O}_X) = 0$ then, by exactness, $H^0(X, \mathscr{K}_X) \to H^0(X, \mathscr{K}_X/\mathcal{O}_X)$ is surjective. $\square$

Remark. By Cartan's theorem B, we know $H^1(X, \mathcal{O}_X) = 0$ for any Stein manifold. So the first Cousin problem is always solvable for Stein manifolds.

Theorem 1.4.4. The second cousin problem is solvable when $H^1(X, \mathcal{O}_X^\times) = 0$ or when $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ and $H^2(X; \mathbb{Z}) = 0$.

Proof. The second exact sequence gives a cohomology exact sequence,

$$0 \longrightarrow H^0(X, \mathcal{O}_X^\times) \longrightarrow H^0(X, \mathscr{K}_X^\times) \longrightarrow H^0(X, \mathscr{K}_X^\times/\mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathscr{K}_X^\times)$$

Clearly, if $H^1(X, \mathcal{O}_X^\times) = 0$ then, by exactness, $H^0(X, \mathscr{K}_X) \to H^0(X, \mathscr{K}_X/\mathcal{O}_X)$ is surjective. Now consider the cohomology of the exponential sequence,

$$H^1(X; \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X; \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X)$$

Then if $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathcal{O}_X) = 0$ we get an isomorphism (the first Chern class) $H^1(X, \mathcal{O}_X^\times) = H^2(X; \mathbb{Z})$ so if $H^2(X; \mathbb{Z}) = 0$ then $H^1(X, \mathcal{O}_X^\times) = 0$ giving the surjection. $\square$

Remark. For Stein manifolds we always have $H^p(X, \mathcal{O}_X) = 0$ for $p > 0$ by Cartan's theorem B. Therefore, the second cousin problem is solvable for Stein manifolds when $H^2(X; \mathbb{Z}) = 0$.

2 Effective Cartier Divisors

2.1 Closed Subschemes

Definition 2.1.1. A *closed subscheme* $Z \subset X$ is an equivalence class of closed immersions $Z \hookrightarrow X$ where we say two closed immersions $\iota_1 : Z_1 \hookrightarrow X$ and $\iota_2 : Z_2 \hookrightarrow X$ are equivalent if there exists an isomorphism $f : Z_1 \to Z_2$ making the diagram,

$$\begin{tikzcd} Z_1 \arrow[rr, "f"] \arrow[dr, hook, "\iota_1"'] & & Z_2 \arrow[dl, hook', "\iota_2"] \\ & X & \end{tikzcd}$$

Theorem 2.1.2. There is a correspondence between closed subschemes Z of X and quasi-coherent sheaves of ideals $\mathscr{I} \subset \mathcal{O}_X$ i.e. injections of quasi-coherent $\mathcal{O}_X$-modules up to isomorphism,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X$$

Via the correspondence: given $\iota : Z \to X$ the map of sheaves $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is surjective take $\mathscr{I} = \ker (\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z)$ which thus fits into an exact sequence,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Conversely, given a sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ then take $Z = (\mathrm{Supp}_{\mathcal{O}_X} (\mathcal{O}_X / \mathscr{I}), \mathcal{O}_X / \mathscr{I})$.

Proof. Given a quasi-coherent sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ then we must show that,

$$Z = (\mathrm{Supp}_{\mathcal{O}_X} (\mathcal{O}_X / \mathscr{I}), \mathcal{O}_X / \mathscr{I})$$

is a closed subscheme. This is a local property so take an affine open $U \subset X$ on which $U = \mathrm{Spec}\,(A)$ and $\mathscr{I} = \widetilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \subset A$. Then in the induced supspace topology $U \cap Z = \mathrm{Supp}_A (A/\mathfrak{a}) = V(\mathfrak{a})$ and the sheaf $\mathcal{O}_Z|_{U \cap Z} = \widetilde{A/\mathfrak{a}}$ so locally $Z \cap U = \mathrm{Spec}\,(A/\mathfrak{a})$ as schemes. Furthermore, the map $\iota : Z \hookrightarrow X$ is given locally by the ring map $A \to A/\mathfrak{a}$ which gives a closed immersion. Finally, it is clear that the sheaf of ideals corresponding to this Z is exactly $\mathscr{I}$ since it is the kernel of the map $\mathcal{O}_X \to \mathcal{O}_X / \mathscr{I}$.

Given a closed subscheme $\iota : Z \hookrightarrow X$ we need to check that the corresponding ideal sheaf $\mathscr{I}$ generates Z. Since closed immersions are separated and quasi-compact then $\iota_* \mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module which implies that $\mathscr{I}$ is also quasi-coherent. In this case there is an isomorphism $\iota_* \mathcal{O}_Z \cong \mathcal{O}_X / \mathscr{I}$. Note that $\iota(Z)$ is closed and thus if $x \notin \iota(Z)$ then any open neighborhood of x contains some $U \subset X \setminus \iota(Z)$ open neighborhood of x on which,

$$(\iota_* \mathcal{O}_Z)(U) = \mathcal{O}_Z(f^{-1}(U)) = \mathcal{O}_Z(\varnothing) = 0$$

Thus if $x \notin \iota(Z)$ then $(\iota_* \mathcal{O}_Z)_x = 0$ Furthermore, if $\iota(z) \in \iota(Z)$ then because ι is a homeomorphism onto its image, every open neighborhood of z is of the form $\iota^{-1}(U)$ for some open $U \subset X$ and thus,

$$(\iota_* \mathcal{O}_Z)_{\iota(z)} = \varinjlim_{\iota(z) \in U} \mathcal{O}_Z(\iota^{-1}(U)) = \varinjlim_{z \in V} \mathcal{O}_Z(V) = \mathcal{O}_{Z,z}$$

In particular, if $\iota(z) \in \iota(Z)$ then $(\iota_* \mathcal{O}_Z)_{\iota(z)} = \mathcal{O}_{Z,z} \neq 0$. Therefore we have shown that,

$$x \in \iota(Z) \iff (\iota_* \mathcal{O}_Z)_x \neq 0 \iff x \in \mathrm{Supp}_{\mathcal{O}_X} (\mathcal{O}_X / \mathscr{I})$$

Thus let $Z' = (\mathrm{Supp}_{\mathcal{O}_X} (\mathcal{O}_X / \mathscr{I}), \mathcal{O}_X / \mathscr{I})$ then there is an isomorphism $\iota : Z \to Z'$ which has $\iota^\# : \mathcal{O}_X / \mathscr{I} \to \iota_* \mathcal{O}_X$ which makes the diagram commute,

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\sim"] & X \arrow[d, "\mathrm{id}_X"] \\ Z' \arrow[r, hook] & X \end{tikzcd}$$

$\square$

2.2 Sheaves Defining Closed Subschemes

Definition 2.2.1. Let $\mathscr{G} \subset \mathscr{F}$ be a subsheaf of a coherent sheaf $\mathcal{O}_X$-module. Then $Z(\mathscr{G})$ is the closed subscheme associated to the sheaf of ideals, $\mathscr{I} = \mathrm{Im}\,(\mathscr{G} \otimes_{\mathcal{O}_X} \mathscr{F}^\vee \to \mathcal{O}_X)$.

DO!!

What about defining $I = \mathrm{Ann}_A\,(M/N)$. Which is correct? When do these give the same results??

2.3 Effective Cartier Divisors as Closed Subschemes

Definition 2.3.1. Let X be a scheme then *a locally principal closed subscheme* of X is a closed subsheme $Z \subset X$ such that the sheaf of ideals $\mathscr{I}_Z$ is locally generated by a single element.

Definition 2.3.2. An *effective Carier divisor* on X is a closed subscheme $D \subset X$ whose ideal sheaf $\mathscr{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Definition 2.3.3. Let X be a scheme and $D \subset X$ a closed subscheme then the following are equivalent,

(a) D is an effective Cartier divisor on X

(b) for each $x \in D$ there exists an affine open neighborhood $x \in U \subset X$ with $U = \mathrm{Spec}\,(A)$ such that $U \cap D = \mathrm{Spec}\,(A/(f))$ for $f \in A$ a nonzerodivisor.

Proof. Assume that D is an effective Cartier divisor then for each $x \in X$ there exists an affine open $x \in U \subset X$ such that $\mathscr{I}_D|_U \cong \mathcal{O}_X|_U$. Since $\mathscr{I}_D$ is quasi-coherent, we may further shrink U such that $\mathscr{I}_D|_U = \widetilde{\mathfrak{a}}$ for some ideal of A where $U = \mathrm{Spec}\,(A)$. The isomorphism $A \to \mathfrak{a}$ is uniquely determined by the image of $1 \in \mathfrak{a} \subset A$ say $1 \mapsto f$ then $\mathfrak{a} = (f)$. Therefore, $\mathscr{I}_D|_U = \widetilde{(f)}$ meaning that locally $D \cap U = \mathrm{Supp}_A\,(A/(f)) = \mathrm{Spec}\,(A/(f))$. Furthermore, suppose that $\exists g \in A$ such that $fg = 0$. Consider the preimage $\tilde{g} \mapsto g$ under the isomorphism $A \to \tilde{\mathfrak{a}}$ and thus $\tilde{g} = 1\tilde{g} \mapsto fg = 0$ so $\tilde{g}$ is in the kernel of the map so $g = 0$ implying that f cannot be a zero divisor.

Conversely, we have $U \cap D = \mathrm{Spec}\,(A/(f))$ then locally the map $D \to X$ is given by the ring map $A \to A/(f)$ so $\mathscr{I}_D|_U = \widetilde{(f)}$. Since f is a non-zero divisor, the map $f : A \to (f)$ is an isomorphism proving that $\mathscr{I}_D$ is an invertible sheaf since $\mathcal{O}_X|_U = \widetilde{A}$. $\square$

Definition 2.3.4. Let X be a scheme. Given effective Carteir divisors D_1 and D_2 on X we set $D = D_1 + D_2$ to be the closed subscheme of X corresponding o the quasi-coherent sheaf of ideals $\mathscr{I}_{D_1} \cdot \mathscr{I}_{D_2} \subset \mathcal{O}_X$.

Proposition 2.3.5. The sum of effective Cartier divisors is an effective Cartier divisor.

Proof. The product of non-zero divisors is a non-zero divisor and thus the product of these ideals is locally invertible. $\square$

Definition 2.3.6. Let X be a scheme and $D \subset X$ an effective Cartier divisor with an ideal sheaf $\mathscr{I}_D$. Recall that $\mathscr{I}_D$ is an invertible $\mathcal{O}_X$-module so we may define,

(a) The invertible sheaf $\mathcal{O}_X(D)$ associated to D is defined by,

$$\mathcal{O}_X(D) = \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_D, \mathcal{O}_X) = \mathscr{I}_D^{\otimes -1}$$

(b) The canonical section, $1_D \in \mathcal{O}_X(D)$ is the inclusion morphism $\mathscr{I}_D \to \mathcal{O}_X$.

(c) We write $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\otimes -1} = \mathscr{I}_D$.

(d) Given a second effective Cartier divisor $D' \subset X$ we define,

$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D')$$

Remark. By definition, for any effective Cartier divisor $D \subset X$ there is an exact sequence,

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Lemma 2.3.7. Let X be a scheme and $D, C \subset X$ be effective Cartier divisors with $C \subset D$ and let $D' = D + C$. Then there exists a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{O}_X(-D)|_C \longrightarrow \mathcal{O}_{D'} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Proof. Let $\mathscr{I}$ be the ideal sheaf of $D \to D'$. Then there is a short exact sequence,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_{D'} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Now I claim that $\mathcal{O}_X(-D)|_C = \mathscr{I}_D|_C = \mathscr{I}$. $\square$

Lemma 2.3.8. Let X be a scheme and $D_1, D_2 \subset X$ be effective Cartier divisors on X. Let $D = D_1 + D_2$. Then there is a unique isomorphism,

$$\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \to \mathcal{O}_X(D)$$

which maps $1_{D_1} \otimes 1_{D_2} \to 1_D$.

Proof. By definition $\mathscr{I}_D = \mathscr{I}_{D_1} \cdot \mathscr{I}_{D_2}$. Consider the map,

$$\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_{D_1}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_{D_2}, \mathcal{O}_X) \to \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_D, \mathcal{O}_X)$$

via $f_1 \otimes f_2 \mapsto f_1 \cdot f_2$. Clearly, this map sends $1_{D_1} \otimes 1_{D_2}$ to 1_D. Thus, it is sufficient to prove that this map is the unique isomorphism. Because these sheaves are invertible, on stalks, this map becomes the isomorphism,

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}((f_1), \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} \mathrm{Hom}_{\mathcal{O}_{X,x}}((f_2), \mathcal{O}_{X,x}) \to \mathrm{Hom}_{\mathcal{O}_{X,x}}((f_1 f_2), \mathcal{O}_{X,x})$$

This is unique because each side is abstractly isomorphic to $\mathcal{O}_X x$ and the map abstractly the identity since it sends $(f_1 \mapsto 1) \otimes (f_2 \mapsto 1) \mapsto (f_1 f_2 \mapsto 1)$. $\square$

Corollary 2.3.9. Let G be the group completion of the monoid of effective Cartier divisors. Then $D \mapsto \mathcal{O}_X(D)$ induces a well-defined group homomorphism $G \to \mathrm{Pic}\,(X)$.

Proof. Sending $D - D' \mapsto \mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D')$ as before gives a well-defined map because $D + D' \mapsto \mathcal{O}_X(D + D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')$ so this is a homomorphism where $\otimes$ is multiplication in $\text{Pic}(X)$. $\square$

Remark. Recall that the conormal sheaf is the $\mathcal{O}_D$-module, $\mathcal{C}_{D/X} = \mathscr{I}_D/\mathscr{I}_D^2 = \iota^*\mathscr{I}_D$. Therefore, the normal bundle is,

$$\mathcal{N}_{D/X} = \iota^*\mathscr{I}_D^\vee = \mathscr{H}\!om_{\mathcal{O}_Z}(\iota^*\mathscr{I}_D, \mathcal{O}_Z) = \iota^*\,\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_D, \mathcal{O}_X) = \iota^*\mathscr{I}_D^{\otimes -1} = \iota^*\mathcal{O}_X(D)$$

Furthermore, from the exact sequence,

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_D \longrightarrow 0$$

tensoring with $\mathcal{O}_X(D)$ and using the projection formula $\iota_*\mathcal{O}_D \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) = \iota_*\iota^*\mathcal{O}_X(D) = \iota_*(\mathcal{N}_{D/X})$ we get an exact sequence,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{1_D} \mathcal{O}_X(D) \longrightarrow \iota_*(\mathcal{N}_{D/X}) \longrightarrow 0$$

2.4 Checking Effective Cartier Divisors on Noetherian Schemes

Lemma 2.4.1. Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme corresponding to the quasi-coherent sheaf $\mathscr{I} \subset \mathcal{O}_X$. Then,

(a) if $\mathscr{I}_x \subset \mathcal{O}_{X,x}$ for all $x \in D$ can be generated by a single element then D is locally prinipal

(b) if $\mathscr{I}_x \subset \mathcal{O}_{X,x}$ for all $x \in D$ can be generated by a single nonzerodivisor then D is an effective Cartier divisor.

Proof. Let $U = \text{Spec}(A)$ be an affine open neighborhood of $x \in D$ and $\mathfrak{p} \subset A$ correspond to x. Then $U \cap D = V(I)$ for some ideal $I \subset A$. Since A is Noetherian $I = (f_1, \dots, f_r)$ is finitely generated. In the first case $I_\mathfrak{p} = (f)$ for some $f \in A_\mathfrak{p}$ thus $f_i = g_i f$ for $g_i \in A_\mathfrak{p}$. We may write $g_i = a_i/h_i$ and $f = f'/h$ for $a_i, h_i, f', h \in A$ and $h, h_i \notin \mathfrak{p}$. Then $I_{h_1 \dots h_r h} \subset A_{h_1 \dots h_r h}$ is generagted by f' so $\mathscr{I}_D|_{D(h_1 \dots h_r h)} = (f')$ is principal proving the first claim. If furthermore, $f \in A_\mathfrak{p}$ is a nonzerodivisor then it must be a nonzerodivisor on some open $\tilde{U} \subset U$ thus $\mathscr{I}_D|_{\tilde{U} \cap D(h_1 \dots h_r h)} = (f')$ is generated by a single nonzerodivisor so D is an effective Cartier divisor. $\square$

Lemma 2.4.2. Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme with,

(a) $\text{codim}(D, X) = 1$

(b) $\forall x \in D : \mathcal{O}_{X,x}$ is a UFD

then D is an effective Cartier divisor.

Proof. Let $x \in D$ and let $A = \mathcal{O}_{X,x}$ with $\mathfrak{p} \subset A$ corresponding to the generic point $\eta \in D$. Then,

$$\textbf{ht}(\mathfrak{p}) = \dim A_\mathfrak{p} = \dim \mathcal{O}_{X,\eta} = \text{codim}(D, X) = 1$$

Furthermore, since A is a UFD, every height one prime is principal so $\mathfrak{p} = (f)$ for some nonzerodivisor[1] $f \in A$. Therefore, by the previous lemma D is an effective Cartier divisor since $(\mathscr{I}_D)_x = \mathfrak{p} = (f)$. To see the last equality, choose an affine open $U = \text{Spec}(R)$ with $x \in U$ corresponding to a prime $\mathfrak{q}$. Then $U \cap D = V(\mathfrak{p})$ where $\mathscr{I}_D = \widetilde{\mathfrak{p}}$ which is prime since D is closed irreducible and $\mathfrak{p} \subset \mathfrak{q}$ and $A = R_\mathfrak{q}$ and $\mathfrak{p} \in \text{Spec}(R_\mathfrak{q})$ thus $(\mathscr{I}_D)_x = \mathfrak{p}R_\mathfrak{p} = \mathfrak{p}A$. $\square$

[1] A is a domain

Corollary 2.4.3. Let X be a Noetherian locally factorial (e.g. regular) scheme. Then every integral codimension one closed subscheme is an effective Cartier divisor.

Lemma 2.4.4. Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme which is also an effective Cartier divisor. Let $\eta \in D$ be its generic point then $\mathcal{O}_{X,\eta}$ is a DVR.

Proof. We may choose an affine open neighborhood $U = \mathrm{Spec}\,(A)$ of $x \in D$ such that $D \cap U = \mathrm{Spec}\,(A/(f))$ for a nonzerodivisor $f \in A$. Furthermore, D is irreducible so $D \cap U = V(\mathfrak{p})$ for a prime $\mathfrak{p} \subset A$ and thus $\sqrt{(f)} = \mathfrak{p}$ but furthermore, D is reduced so (f) is radical i.e. $(f) = \mathfrak{p}$ is prime. Then $D \cap U = V(\mathfrak{p})$ has generic point $\eta = \mathfrak{p} \in U$. Thus, $\mathcal{O}_{X,\eta} = A_{\mathfrak{p}}$ is a local Noetherian PID[2] and thus a DVR. $\square$

2.5 Effective Cartier Divisors Defined by Global Sections

Remark. Recall the definition of a regular global section.

Definition 2.5.1. Let X be a locally ringed space and $\mathcal{L}$ an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a regular section in the map $\mathcal{O}_X \to \mathcal{L}$ via $f \mapsto fs$ is injective.

Remark. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and $s \in \Gamma(X, \mathcal{L})$ is a global section. We may realize s as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual then gives a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Definition 2.5.2. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$ be a global section. The *zero scheme* of s is the closed subscheme $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ defined by $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Remark. Let $f : X \to Y$ be a morphism of locally ringed spaces and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. A global section $s \in \Gamma(Y, \mathscr{F})$ can be realized as a morphism $s : \mathcal{O}_Y \to \mathscr{F}$. Applying the functor f^* gives a morphism $f^*s : f^*\mathcal{O}_Y \to f^*\mathscr{F}$ which is equivalent to a section $f^*s : \mathcal{O}_X \to f^*\mathscr{F}$ since $f^*\mathcal{O}_Y = \mathcal{O}_X$.

Lemma 2.5.3. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X and $s \in \Gamma(X, \mathcal{L})$ a global section. Then,

(a) Consider the closed immersions $\iota : Z \hookrightarrow X$ such that $\iota^* s \in \Gamma(Z, \iota^*\mathcal{L})$ is zero, ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this poset.

(b) The zero scheme $Z(s)$ is a locally principal closed subscheme.

(c) a morphism of schemes $f : X' \to X$ factors through $Z(s) \hookrightarrow X$ iff $f^*s = 0$.

(d) $Z(s)$ is an effective Cartier divisor iff s is a regular section of $\mathcal{L}$.

[2]First $A_{\mathfrak{p}}$ is a principal ideal ring since its unique maximal ideal is principal. Furthermore, $A_{\mathfrak{p}}$ is a domain because if $g \in A_{\mathfrak{p}}$ is a zero divisor then $\mathrm{Ann}_A((g)) \subset (f)$ (else $g = 0$ in $A_{\mathfrak{p}}$) then let $\mathfrak{q}$ be a maximal anihilator and thus a prime above $\mathrm{Ann}_A((g))$ but $\mathfrak{q} \subset (f)$ because $A_{\mathfrak{p}}$ is local so $\mathfrak{q} = (a)$ since $A_{\mathfrak{q}}$ is a principal ideal ring. Thus $a = a'f$ is a zero divisor so a' is a zero divisor since f is not but $(a'f)$ is prime so either $a \in (af)$ or $f \in (a'f)$ but $f \notin (a'f)$ since f is not a zero divisor and thus $a' \in (a'f)$. We can write $a' = ra'f$ and thus $a'(rf - 1) = 0$ but $rf - 1 \notin (f)$ and thus $rf - 1$ is a unit so $a' = 0$ and thus $g = 0$ showing that $A_{\mathfrak{p}}$ is a domain.

Proof. Suppose that $\iota : Z \hookrightarrow X$ is a closed subscheme such that $\iota^* s \in \Gamma(Z, \iota^* \mathcal{L})$ is zero. It suffices to show that $\mathscr{I}_{Z(s)} \subset \mathscr{I}_Z$. However, $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is zero because $\iota^* s = 0$ and thus $\mathcal{J}_{Z(s)} = \mathrm{Im}\,(s) \subset \ker(\mathcal{O}_X \to \iota_* \mathcal{O}_Z) = \mathcal{J}$.

Since $\mathcal{L}$ is invertible, there is an affine open cover such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$ on each open $\mathrm{Spec}\,(A) = U \subset X$. Thus, $\mathcal{L}|_U = \widetilde{M}$ for some A-module M such that $M \cong A$ as A-modules i.e. M is free of rank 1. Then consider the map $s : \mathcal{O}_X \to \mathcal{L}$ which restricts to the map $s|_U : A \to M$ given by $a \mapsto as|_U$ whose dual is $s|_U : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$ takes $(f : M \to A) \mapsto f(s|_U)$. Since M is free of rank 1 we may write $s|_U = s_A m$ for $s_A \in A$ and $m \in M$ the basis element. Then every A-module map $f : M \to A$ is determined by the image of $m \mapsto f(m)$ so $f(s|_U) = s_A f(m)$. In particular, there exists an isomorphism $M \to A$ which has $f(m) = 1$ so $\mathrm{Hom}_A(M, A) \cong A$ via $f \mapsto f(m)$ so $\mathrm{Im}\,(s|_U) = \{s_A f(m) \mid f \in \mathrm{Hom}_A(M, A)\} = (s_A) \subset A$. Thus the sheaf of ideals of $Z(s)$ is locally generated by a single element.

Furthermore, $s \in \Gamma(X, \mathcal{L})$ is a regular section iff $s|_U$ is regular for each affine open U i.e. the map $a \mapsto as_A$ is injective meaning $A \cong (s_A)$. Thus, since locally the sheaf of ideals of $Z(s)$ is (s_A), the section s is regular iff $Z(s)$ is an effective Cartier divisor. $\square$

Theorem 2.5.4. Let X be a scheme.

(a) If $D \subset X$ is an effective Cartier divisor then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.

(b) Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$ then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ sending $1_D \mapsto s$.

The construction $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ are inverse giving a bijective correspondence between effective Cartier divisors on X and isomorphism classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf of $\mathcal{O}_X$-modules and $s \in \Gamma(X, \mathcal{L})$ is a regular global section.

Proof. Let $D \subset X$ be an effective Cartier divisor and consider the canonical section 1_D of $\mathcal{O}_X(D) = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{I}_D, \mathcal{O}_X)$. Consider the map $\mathcal{O}_X \to \mathcal{O}_X(D)$ given by $f \mapsto f \cdot 1_D$. On stalks, we know that the ideal $(\mathscr{I}_D)_x \cong \mathcal{O}_{X,x}$ so $(\mathscr{I}_D)_x = (f)$ where $f \in \mathcal{O}_{X,x}$ is the preimage of 1. Given any section $g \in \mathcal{O}_{X,x}$ if $g_x(1_D)_x = 0$ then $g \cdot f = 0$ meaning that either $g_x = 0$ or f is a zero divisor. However, since $\mathscr{I}_D$ is invertible, f is a nonzerodivisor thus $g_x = 0$. Therefore this map $1_D : \mathcal{O}_X \to \mathcal{O}_X(D)$ is injective at the stalks and therefore injective.

Now suppose that $\mathcal{L}$ is an invertible sheaf and $s \in \Gamma(X, \mathcal{L})$ a regular secton. Consider $D = Z(s) \subset X$. Since s is regular, we have shown that $Z(s)$ is an effective Cartier divisor. Furthermore, $\mathscr{I}_D = \mathrm{Im}\,(s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X) = \mathcal{L}^{\otimes -1}$ where s is regular so this is injective. Thus, $\mathcal{O}_X(D) = \mathscr{I}_D^{\otimes -1} = \mathcal{L}$. Finally, given an effective Cartier divisor we know that $(\mathcal{O}_X(D), 1_D)$ is an invertible sheaf with a regular section. Consider $Z(s)$ which is the closed subscheme uniquely defined by the sheaf of ideals given by the image of $1_D : \mathcal{O}_X(D)^{\otimes -1} \to \mathcal{O}_X$ which is exactly the inclusion map $\mathscr{I}_D \to \mathcal{O}_X$ since $\mathcal{O}_X(D) = \mathscr{I}_D^{\otimes -1}$. Therefore, we find that $Z(s) \cong Z$. $\square$

Remark. Let $(\mathcal{L}, s)$ be a invertible module and a global regular section. Then there are exact sequences,

$$0 \longrightarrow \mathcal{L}^{\otimes -1} \xrightarrow{s^\vee} \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_D \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{L} \longrightarrow \iota_*(\mathcal{L}|_D) \longrightarrow 0$$

where $\iota : D \hookrightarrow \mathcal{L}$ is the inclusion of the effective Cartier divisor $D = Z(s)$.

2.6 Relationship to the Previous Definition

Theorem 2.6.1. There is a natural bijection $G \xrightarrow{\sim} \mathrm{Ca}X$ between the group completion of effective Cartier divisors and the group of Cartier divisors.

Proof. Given D we can find a open affine cover $U_i = \mathrm{Spec}\,(A_i)$ such that $\mathscr{I}_D|_{U_i} = \widetilde{(f_i)}$ so we send $D \mapsto \{(U_i, f_i)\}$ the Cartier divisor. Since $\mathscr{I}_D$ is a sheaf, we must have $(f_i)|_{U_i \cap U_j} = (f_j)|_{U_i \cap U_j}$ on the overlaps and thus f_i/f_j is a unit on the overlap so $\{(U_i, f_i)\}$ defines a Cartier divisor. We say that $\{(U_i, f_i)\}$ is effective because each $f_i \in \mathcal{O}_X(U_i)$ has no poles. Furthermore, any such divisor $\{(U_i, f_i)\}$ defines an invertible sheaf $\mathcal{L}$ (OKAY WE NEED EVERY BUNDLE IS THE DIFFERENCE OF BUNDLES!! Tag 0B3Q) □

3 Weil Divisors

We only consider Weil divisors for sufficiently nice schemes. (DEFINE)

3.1 Reflexive Sheaves

3.2 The Sheaf Associated to a Weil Divisor

3.3 The Relationship between Weil Divisors and Cartier Divisors

4 Linear Systems of Divisors

5 The Chow Ring

6 Pushforward and Pullback of Divisors

7 Divisors on Curves