Physics GR6037 Quantum Mechanics I
Assignment # 7

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Problem 22.

Consider a spin-j particle with $j = \frac{1}{2}$. A general state can be written in the form,

$$|\psi\rangle = a_+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + a_- \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

We first consider the eigenvectors of the operator $\hat{J}_{\hat{n}} = \vec{J} \cdot \hat{n}$ for some unit vector $\hat{n}$. For $j = \frac{1}{2}$, these eigenvectors can be easily found from the matrix representation of $\hat{J}_{\hat{n}}$.

$$\hat{J}_{\hat{n}} = \frac{\hbar}{2}\hat{n}_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\hat{n}_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\hbar}{2}\hat{n}_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

We know by rotational symmetry that the eigenvalues of this matrix are $\pm\frac{\hbar}{2}$. Thus, to find the eigenvectors, consider the matrix equations,

$$\frac{2}{\hbar}\left(\hat{J}_{\hat{n}} - I\frac{\hbar}{2}\right)|\hat{n}+\rangle = \begin{pmatrix} n_z - 1 & n_x - i n_y \\ n_x + i n_y & -n_z - 1 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = 0$$

Thus, $(n_z - 1)a_+ + (n_x - i n_y)a_- = 0$. Take $a_+ = \frac{1}{\sqrt{2(1-n_z)}}(n_x - i n_y)$ and $a_- = \frac{1}{\sqrt{2(1-n_z)}}(1 - n_z)$ such that $|a_+|^2 + |a_-|^2 = 1$. These also satisfy the second row because,

$$\begin{aligned} (n_x + i n_y)a_+ - (n_z + 1)a_- &= \frac{1}{\sqrt{2(1-n_z)}}\left[(n_x + i n_y)(n_x - i n_y) - (1 - n_z)(1 + n_z)\right] \\ &= \frac{1}{\sqrt{2(1-n_z)}}\left[n_x^2 + n_y^2 + n_z^2 - 1\right] = 0 \end{aligned}$$

Therefore,

$$|\hat{n}+\rangle = \frac{n_x - i n_y}{\sqrt{2(1-n_z)}} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \frac{1 - n_z}{\sqrt{2(1-n_z)}} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

Similarly for the spin down state, we can consider $\hat{n} \mapsto -\hat{n}$ and look at the spin up state. This state will equal the spin down state in the original direction up to phase. Thus,

$$|\hat{n}-\rangle = \frac{-n_x + i n_y}{\sqrt{2(1+n_z)}} \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \frac{1 + n_z}{\sqrt{2(1+n_z)}} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

(a)

Consider the state $|\psi\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle$ such that $\langle\psi|\,\hat{J}_3\,|\psi\rangle = \frac{\hbar}{2}$ and the operator $\hat{J}' = \hat{J}_3\cos\theta + \hat{J}_2\sin\theta$. The eigenvectors of $\hat{J}'$ correspond to $\hat{n} = (0, \sin\theta, \cos\theta)$. Thus,

$$|\hat{n}+\rangle = \frac{-i\sin\theta}{\sqrt{2(1-\cos\theta)}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \frac{1-\cos\theta}{\sqrt{2(1-\cos\theta)}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -i\cos\left(\tfrac{\theta}{2}\right)\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sin\left(\tfrac{\theta}{2}\right)\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

by half-angle formulae. Therefore, the probability to have spin up with respect to $\hat{J}'$ is

$$|\left\langle \hat{n}+\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle|^2 = \cos^2\left(\tfrac{\theta}{2}\right)$$

(b)

Since we can always find an eigenstate of $\hat{J}_{\hat{n}} = \vec{J}\cdot\hat{n}$, consider this state $|\hat{n}+\rangle$. By definition, $\hat{n}\cdot\vec{J}\,|\hat{n}+\rangle = \frac{\hbar}{2}\,|\hat{n}+\rangle$ and thus $\langle\hat{n}+|\,\hat{J}_{\hat{n}}\,|\hat{n}+\rangle = \frac{\hbar}{2}$. Futhermore,

$$\begin{aligned}
\langle\hat{n}+|\,\hat{J}_1\,|\hat{n}+\rangle &= \frac{\hbar}{2}\begin{pmatrix}a_+^* & a_-^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a_+\\ a_-\end{pmatrix} = \frac{\hbar}{2}(a_+^*a_- + a_-^*a_+)\\
&= \frac{\hbar}{2}\frac{1}{2(1-n_z)}\Big[(n_x+in_y)(1-n_z) + (1-n_z)(n_x-in_y)\Big]\\
&= \frac{\hbar}{2}\frac{1}{2(1-n_z)}\Big[2n_x(1-n_z)\Big] = \frac{\hbar}{2}n_x
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle\hat{n}+|\,\hat{J}_2\,|\hat{n}+\rangle &= \frac{\hbar}{2}\begin{pmatrix}a_+^* & a_-^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a_+\\ a_-\end{pmatrix} = -i\frac{\hbar}{2}(a_+^*a_- - a_-^*a_+)\\
&= \frac{\hbar}{2}\frac{-i}{2(1-n_z)}\Big[(n_x+in_y)(1-n_z) - (1-n_z)(n_x-in_y)\Big]\\
&= \frac{\hbar}{2}\frac{-i}{2(1-n_z)}\Big[2in_y(1-n_z)\Big] = \frac{\hbar}{2}n_y
\end{aligned}$$

And finally,

$$\begin{aligned}
\langle\hat{n}+|\,\hat{J}_2\,|\hat{n}+\rangle &= \frac{\hbar}{2}\begin{pmatrix}a_+^* & a_-^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_+\\ a_-\end{pmatrix} = \frac{\hbar}{2}(a_+^*a_+ - a_-^*a_-)\\
&= \frac{\hbar}{2}\frac{1}{2(1-n_z)}\Big[(n_x+in_y)(n_x-n_y) - (1-n_z)^2\Big]\\
&= \frac{\hbar}{2}\frac{1}{2(1-n_z)}\Big[n_x^2+n_y^2-1+2n_z-n_z^2\Big] = \frac{\hbar}{2}\frac{1}{2(1-n_z)}\Big[2n_z-2n_z^2\Big] = \frac{\hbar}{2}n_z
\end{aligned}$$

Therefore,

$$\langle\hat{n}+|\,\vec{J}\,|\hat{n}+\rangle = \frac{\hbar}{2}\hat{n}$$

Thus, writing $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we conclude that the desired state is,

$$\begin{aligned}
|\hat{n}+\rangle &= \frac{\sin\theta\cos\phi - i\sin\theta\sin\phi}{\sqrt{2(1-\cos\theta)}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \frac{1-\cos\theta}{\sqrt{2(1-\cos\theta)}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\\
&= \frac{\sin\theta}{\sqrt{2(1-\cos\theta)}}e^{-i\phi}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\frac{1-\cos\theta}{2}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\\
&= \cos\left(\tfrac{\theta}{2}\right)e^{-i\phi}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sin\left(\tfrac{\theta}{2}\right)\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle
\end{aligned}$$

For future convinience and to exhibit the symmety in the components, I will multiply this state by the total phase $e^{i\phi/2}$. Thus, $a_+ = \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2}$ and $a_- = \sin\left(\frac{\theta}{2}\right)e^{+i\phi/2}$ so we use the notation,

$$|\psi(\theta,\phi)\rangle = \cos\left(\tfrac{\theta}{2}\right)e^{-i\phi/2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i\phi/2}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

(c)

Suppose the Hamiltonian is given by,

$$\hat{H} = -\frac{ge}{2mc}\vec{J}\cdot\vec{B}$$

This Hamiltonian is time independent so the time evolution operator is given by,

$$\hat{U} = \exp\left[i\frac{ge}{2mc\hbar}\vec{J}\cdot\vec{B}t\right]$$

We can identify this operator as a rotation about $\hat{B}$ by angle $\theta = -\frac{ge|B|}{2mc}t$. Let $\omega_L = \frac{ge|B|}{2mc}$ so $\theta = \omega_L t$. For $j = \frac{1}{2}$ we can expand this matrix explicitly.

$$\begin{aligned}\hat{U} &= \exp\left[i\frac{\omega_L t}{\hbar}\vec{J}\cdot\hat{B}\right] = \exp\left[i\frac{\omega_L t}{2}\vec{\sigma}\cdot\hat{B}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\omega_L t}{2}\right)^n(\vec{\sigma}\cdot\hat{B})^n \\ &= I\cos\left(\tfrac{1}{2}\omega_L t\right) + i\vec{\sigma}\cdot\hat{B}\sin\left(\tfrac{1}{2}\omega_L t\right)\end{aligned}$$

which holds because $(\vec{\sigma}\cdot\hat{B})^2 = I$. Now, we write out the matrix,

$$\hat{U} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\left(\tfrac{1}{2}\omega_L t\right) + i\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\hat{B}_x + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\hat{B}_y + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\hat{B}_z\right]\sin\left(\tfrac{1}{2}\omega_L t\right)$$

$$= \begin{pmatrix}\cos\left(\frac{1}{2}\omega_L t\right) + i\hat{B}_z\sin\left(\frac{1}{2}\omega_L t\right) & (i\hat{B}_x + \hat{B}_y)\sin\left(\frac{1}{2}\omega_L t\right)\\ (i\hat{B}_x - \hat{B}_y)\sin\left(\frac{1}{2}\omega_L t\right) & \cos\left(\frac{1}{2}\omega_L t\right) - i\hat{B}_z\sin\left(\frac{1}{2}\omega_L t\right)\end{pmatrix}$$

With the initial state $|\psi\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle$, we get the time evolved state,

$$|\psi(t)\rangle = \hat{U}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \left[\cos\left(\tfrac{1}{2}\omega_L t\right) + i\hat{B}_z\sin\left(\tfrac{1}{2}\omega_L t\right)\right]\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \left[(i\hat{B}_x - \hat{B}_y)\sin\left(\tfrac{1}{2}\omega_L t\right)\right]\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

(d)

Let $\vec{B} = B\hat{z}$ then $\hat{B}_z = B$ and $\hat{B}_x = \hat{B}_y = 0$ so

$$\hat{U} = \begin{pmatrix}e^{i\omega_L t/2} & 0\\ 0 & e^{-i\omega_L t/2}\end{pmatrix}$$

Let the initial state be,

$$|\psi_0\rangle = |\psi(\theta,\phi)\rangle = \cos\left(\tfrac{\theta}{2}\right)e^{-i\phi/2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i\phi/2}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

Then the evolved state is,

$$
\begin{aligned}
|\psi(t)\rangle = \hat{U}\,|\psi_0\rangle &= \cos\left(\tfrac{\theta}{2}\right)e^{-i\phi/2}e^{i\omega_L t/2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i\phi/2}e^{-i\omega_L t/2}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \\
&= \cos\left(\tfrac{\theta}{2}\right)e^{-i(\phi-\omega_L t)/2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i(\phi-\omega_L t)/2}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \\
&= |\psi(\theta,\phi-\omega_L t)\rangle
\end{aligned}
$$

This is exactly the classical motion. The state rotates clockwise about the z-axis with rate given by the Larmor precession frequency $\omega_L = \frac{ge|B|}{2mc}$

Problem 23.

(a)

Consider any normalized $j = \frac{1}{2}$ spin state. Define $\hat{n} = \langle\psi|\,\vec{J}\,|\psi\rangle\,\frac{2}{\hbar}$.

$$
\langle\psi|\,\hat{J}_1\,|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} a_+^* & a_-^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{\hbar}{2}(a_+^* a_- + a_-^* a_+) = \frac{\hbar}{2}n_x
$$

Similarly,

$$
\langle\psi|\,\hat{J}_2\,|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} a_+^* & a_-^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = -i\frac{\hbar}{2}(a_+^* a_- - a_-^* a_+) = \frac{\hbar}{2}n_y
$$

And finally,

$$
\langle\psi|\,\hat{J}_2\,|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} a_+^* & a_-^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{\hbar}{2}(a_+^* a_+ - a_-^* a_-) = \frac{\hbar}{2}n_z
$$

Now, consider the length of $\hat{n}$,

$$
\begin{aligned}
\hat{n}^2 &= n_x^2 + n_y^2 + n_z^2 = (a_+^* a_- + a_-^* a_+)^2 + (\tfrac{1}{i}\left[a_+^* a_- - a_-^* a_+\right])^2 + (a_+^* a_+ - a_-^* a_-)^2 \\
&= \left(2\mathfrak{Re}\left[a_+^* a_-\right]\right)^2 + \left(2\mathfrak{Im}\left[a_+^* a_-\right]\right)^2 + |a_+|^4 - 2|a_+|^2|a_-|^2 + |a_-|^4 \\
&= 4|a_+^* a_-|^2 + |a_+|^4 - 2|a_+|^2|a_-|^2 + |a_-|^4 = 4|a_+|^2|a_-|^2 + |a_+|^4 - 2|a_+|^2|a_-|^2 + |a_-|^4 \\
&= |a_+|^4 + 2|a_+|^2|a_-|^2 + |a_-|^4 = (|a_+|^2 + |a_-|^2)^2 = 1
\end{aligned}
$$

where the last equality holds by the fact that $|\psi\rangle$ is normalized so $|a_+|^2 + |a_-|^2 = 1$.

(b)

No! Consider the the $j = 1$ and $m = 0$ state. Then,

$$
\begin{aligned}
&\langle 1,0|\,\hat{J}_3\,|1,0\rangle = 0 \\
&\langle 1,0|\,\hat{J}_2\,|1,0\rangle = \langle 1,0|\,\frac{1}{2i}(\hat{J}_+ - \hat{J}_-)\,|1,0\rangle = \frac{1}{2i}\left(\sqrt{2}\,\langle 1,0|1,1\rangle - \sqrt{2}\,\langle 1,0|1,-1\rangle\right) = 0 \\
&\langle 1,0|\,\hat{J}_1\,|1,0\rangle = \langle 1,0|\,\frac{1}{2}(\hat{J}_+ + \hat{J}_-)\,|1,0\rangle = \frac{1}{2}\left(\sqrt{2}\,\langle 1,0|1,1\rangle + \sqrt{2}\,\langle 1,0|1,-1\rangle\right) = 0
\end{aligned}
$$

Therefore, $\langle 1,0|\,\vec{J}\,|1,0\rangle$ has zero length and therefore no multiple of it is a unit vector.

Problem 24.

Let $F(x,y,z) = \sum_{i,j,k}^{N} c_{ijk} x^i y^j z^k$ where the sum runs over values such that $i+j+k=N$.

(a)

$\vec{L} = -i\hbar \vec{r} \times \nabla$ and thus, $\hat{L}_i = -i\hbar \epsilon_{ijk} r_j \partial_k$. If we act on a monomial with the operator $\hat{L}_i$,

$$\hat{L}_1 x^a y^b z^c = -i\hbar(y\partial_z - z\partial_y)\partial_k x^a y^b z^c = -i\hbar(cyx^a y^b z^{c-1} - bzx^a y^{a-1} z^c) = -i\hbar(cx^a y^{b+1} z^{c-1} - bx^a y^{b-1} z^{c+1})$$

If $a+b+c=N$ then the final polynomial will have each term of overall order N because $a+(b+1)+(c-1) = a+(b-1)+(c+1) = a+b+c = N$. Therefore, $\hat{L}_1$ acting on monomials produces another polynomial in this subspace. The other components of $\hat{L}$ and thus $\hat{L}^2$ act similarly to produce homogeneous polynomials from monomials of the same order. Therefore, since a homogeneous polynomial is a sum of monomials of equal order, each component of $\hat{L}_i$ acts on each term to generate two terms of the same order. Thus, the overall order of the polynomial is preserved so order N homogeneous polynomials are mapped into other order N homogeneous polynomials. Furthermore, the rotation operator about $\hat{n}$ is given by

$$R(\hat{n},\theta) = e^{-\frac{i}{\hbar}\vec{L}\cdot\hat{n}\theta} = I - \frac{i}{\hbar}\vec{L}\cdot\hat{n}\theta + \frac{1}{2}\left(\frac{i}{\hbar}\vec{L}\cdot\hat{n}\theta\right)^2 + \frac{1}{3!}\left(\frac{i}{\hbar}\vec{L}\cdot\hat{n}\theta\right)^3 + \cdots$$

Therefore, acting with $R(\hat{n},\theta)$ preserves the order and homogeneity of the polynomials because each term in the series preserves the property.

(b)

Consider the function,

$$\psi_{0,0}(r,\theta,\phi) = r^N Y_{0,0}(\theta,\phi)$$

Because $\hat{L}_i$ only acts on angular functions, we can move the $\hat{L}_i$ operators through the r^N to act only on the spherical harmonic. Thus, $\psi_{0,0}$ is a state with $\ell = 0$. Now, $Y_{0,0} = \frac{1}{2\sqrt{\pi}}$ so

$$\psi_{0,0}(x,y,z) = \frac{1}{2\sqrt{\pi}}(x^2+y^2+z^2)^{N/2}$$

If N is even, we can expand this expession as a homogeneous polynomial of order N using the trinomial coeficients.

(c)

Consider the functions,

$$\psi_{1,m}(r,\theta,\phi) = r^N Y_{1,m}(\theta,\phi)$$

for values $m = +1, 0, -1$. These are $\ell = 1$ multiplet eigenstates of angular momentum because the angular momentum operators commute with the radial distance r. We write these functions out explicitly in angular functions,

$$\psi_{1,1}(r,\theta,\phi) = -r^N \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi} = -r^{N-1}\frac{1}{2}\sqrt{\frac{3}{2\pi}}\, r\sin\theta(\cos\phi + i\sin\phi)$$
$$\psi_{1,0}(r,\theta,\phi) = r^N \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta = r^{N-1}\frac{1}{2}\sqrt{\frac{3}{\pi}}\, r\cos\theta$$
$$\psi_{1,-1}(r,\theta,\phi) = r^N \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi} = r^{N-1}\frac{1}{2}\sqrt{\frac{3}{2\pi}}\, r\sin\theta(\cos\phi - i\sin\phi)$$

If N is odd so $N-1$ is even, we can write these functions as homogeneous polynomials in Cartesian coordinates,

$$F_{1,1}(x,y,z) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}(x^2+y^2+z^2)^{(N-1)/2}\,(x+iy)$$
$$F_{1,0}(r,\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}(x^2+y^2+z^2)^{(N-1)/2} z$$
$$F_{1,-1}(r,\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}(x^2+y^2+z^2)^{(N-1)/2}(x-iy)$$

The function $(x^2+y^2+z^2)^{(N-1)/2}$ is a homogeneous polynomial of degree $N-1$ if $N-1$ is an even number i.e. if N is odd. This is multiplied by a homogeneous polynomial of degree 1 to get in total a homogeneous polynomial of degree N.

(d)

The spherical harmonics for $m = \ell$ are given by $Y_{l,l} \propto e^{i\ell\phi}\sin^l\theta$. Now, consider the function,

$$\psi_{N,N}(r,\theta,\phi) = r^N e^{iN\phi}\sin^N\theta = r^N(\cos\phi + i\sin\phi)^N \sin^N\theta = (r\cos\phi\sin\theta + i\, r\sin\phi\cos\theta)^N$$

However, $x = r\cos\phi\sin\theta$ and $y = r\sin\phi\sin\theta$ therefore, we can write this function as a degree N homogenous polynomial,

$$F_{N,N} = (x+iy)^N = \sum_{n=0}^{N} i^k \binom{N}{k} x^{N-k} y^k$$

Since the angular dependence of this function is identical to $Y_{N,N}$, this must be a state with $\ell = N$ and $m = N$.

Problem 25.

(a)

Consider the Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

We apply the standard factorization to this Hamiltonian by first splitting it into 1D factors. Specifically, define the lowering operators,

$$\hat{a}_i = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{r}_i + \frac{i}{m\omega}\hat{p}_i\right)$$

These operators satisfy the commutation relations,

$$[\hat{a}_i, \hat{a}_j] = 0 \qquad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \qquad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

Therefore, we can rewrite the Hamiltonian as,

$$\begin{aligned}
\hat{H} &= \left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2 x^2\right) + \left(\frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2 y^2\right) + \left(\frac{\hat{p}_z^2}{2m} + \frac{1}{2}m\omega^2 z^2\right) \\
&= \hbar\omega\left(\hat{a}_x^\dagger \hat{a}_x + \frac{1}{2}\right) + \hbar\omega\left(\hat{a}_y^\dagger \hat{a}_y + \frac{1}{2}\right) + \hbar\omega\left(\hat{a}_z^\dagger \hat{a}_z + \frac{1}{2}\right) \\
&= \hbar\omega\left(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + \hat{a}_z^\dagger \hat{a}_z + \frac{3}{2}\right)
\end{aligned}$$

Using the above commutation relations,

$$[\hat{H}, \hat{a}_i^\dagger] = \hbar\omega\hat{a}_i^\dagger \qquad [\hat{H}, \hat{a}_i] = -\hbar\omega\hat{a}_i$$

Therefore, we can describe any energy eigenstate as,

$$|n_x, n_y, n_z\rangle = \frac{(\hat{a}_x^\dagger)^{n_x}(\hat{a}_y^\dagger)^{n_y}(\hat{a}_z^\dagger)^{n_z}}{\sqrt{n_x!}\sqrt{n_y!}\sqrt{n_z!}}\,|0\rangle$$

With energy $E = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)$. The degeneracy of a state with energy

$$E_N = \hbar\omega(N + \frac{3}{2})$$

is given by the number of nonnegative integer solutions to $n_x + n_y + n_z = N$. For each of the $N+1$ possible values of n_x, there are $N+1-n_x$ possible values of n_y and, for n_x and n_y given, n_z is fixed. Therefore, the degeneracy of the state E_N is,

$$\begin{aligned}
D_N &= \sum_{n=0}^{N}(N+1) - n = (N+1)^2 - \sum_{n=0}^{N} n = (N+1)^2 - \frac{N(N+1)}{2} = N^2 + 2N + 1 - \frac{1}{2}(N^2 + N) \\
&= \frac{1}{2}(N^2 + 3N + 2) = \frac{(N+1)(N+2)}{2}
\end{aligned}$$

(b)

First, notice that $\hat{p}^2$ and $\hat{r}^2$ are dot products of vectors under rotation and therefore commute with every component of $\vec{L}$. Therefore,

$$[\hat{H}, \hat{L}_z] = 0 \qquad [\hat{H}, \hat{L}^2] = 0$$

I introduce the right and left circular ladder operators,

$$\hat{a}_L = \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y) \qquad \hat{a}_R = \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y)$$

These operators have the expected commutation relations:

$$\begin{aligned}
[\hat{a}_R, \hat{a}_R^\dagger] &= 1 \qquad [\hat{a}_L, \hat{a}_L^\dagger] = 1 \\
[\hat{a}_R, \hat{a}_L] &= 0 \qquad [\hat{a}_R^\dagger, \hat{a}_L^\dagger] = 0 \\
[\hat{a}_R^\dagger, \hat{a}_L] &= 0 \qquad [\hat{a}_R, \hat{a}_L^\dagger] = 0
\end{aligned}$$

which are simple yet tedious to check. Furthermore, these operators commute with $\hat{a}_z$ and its adjoint because both $\hat{a}_x$ and $\hat{a}_y$ do. Now the Hamiltonian can be rewitten using the fact that,

$$\begin{aligned}\hat{a}_L^\dagger \hat{a}_L + \hat{a}_R^\dagger \hat{a}_R &= \frac{1}{2}\Big(\hat{a}_x^\dagger \hat{a}_x - i\hat{a}_y^\dagger \hat{a}_x i\hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger \hat{a}_y + \hat{a}_x^\dagger \hat{a}_x + i\hat{a}_y^\dagger \hat{a}_x - i\hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger \hat{a}_y\Big) \\ &= \frac{1}{2}\Big(2\hat{a}_x^\dagger \hat{a}_x + 2\hat{a}_y^\dagger \hat{a}_y\Big) = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y\end{aligned}$$

So therefore,

$$\hat{H} = \hbar\omega\left(\hat{a}_L^\dagger \hat{a}_L + \hat{a}_R^\dagger \hat{a}_R + \hat{a}_z^\dagger \hat{a}_z + \frac{3}{2}\right)$$

Furthermore, we can express the angular momentum operators in terms of these operators by expressing the coordinates and momenta. For the sake of sanity, I will omit these calculations and simply state the results.

$$\begin{aligned}\hat{L}_x &= \frac{\hbar}{\sqrt{2}}\left[(\hat{a}_R^\dagger - \hat{a}_L^\dagger)\hat{a}_z + (\hat{a}_R - \hat{a}_L)\hat{a}_z^\dagger\right] \\ \hat{L}_y &= \frac{i\hbar}{\sqrt{2}}\left[(\hat{a}_R^\dagger + \hat{a}_L^\dagger)\hat{a}_z - (\hat{a}_R + \hat{a}_L)\hat{a}_z^\dagger\right] \\ \hat{L}_z &= \hbar\left[\hat{a}_R^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_L\right]\end{aligned}$$

Therefore,

$$\begin{aligned}[\hat{L}_z, \hat{a}_R^\dagger] &= \hbar\hat{a}_R^\dagger \qquad [\hat{L}_z, \hat{a}_R] = -\hbar\hat{a}_R \\ [\hat{L}_z, \hat{a}_L^\dagger] &= -\hbar\hat{a}_L^\dagger \qquad [\hat{L}_z, \hat{a}_L] = \hbar\hat{a}_L\end{aligned}$$

so the right and left rasing and lowering operators act as ladder operators for $\hat{L}_z$. Now, we can exhibit the angular momentum ladder operators.

$$\begin{aligned}\hat{L}_+ &= \hat{L}_x + i\hat{L}_y = \hbar\sqrt{2}\left[\hat{a}_R^\dagger \hat{a}_z - \hat{a}_z^\dagger \hat{a}_L\right] \\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y = \hbar\sqrt{2}\left[\hat{a}_z^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_z\right]\end{aligned}$$

And we will use the identity,

$$\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z$$

Using the commutation relations,

$$\hat{L}_z(\hat{a}_R^\dagger)^n |0\rangle = (\hat{a}_R^\dagger)^n(n\hbar + \hat{L}_z)|0\rangle = \hbar n(\hat{a}_R^\dagger)^n |0\rangle$$

Therefore, this state has $m = n$. Furthermore,

$$\hat{L}_+(\hat{a}_R^\dagger)^n |0\rangle = \hbar\sqrt{2}\left[\hat{a}_R^\dagger \hat{a}_z - \hat{a}_z^\dagger \hat{a}_L\right](\hat{a}_R^\dagger)^n |0\rangle = \hbar\sqrt{2}(\hat{a}_R^\dagger)^n\left[\hat{a}_R^\dagger \hat{a}_z - \hat{a}_z^\dagger \hat{a}_L\right]|0\rangle = 0$$

where I have used the fact that $[\hat{a}_z, \hat{a}_R^\dagger] = [\hat{a}_L, \hat{a}_R^\dagger] = 0$. Therefore, this state must have $\ell = m = n$ because it is annihilated by the rasing operator. We can check this condition explicitly by considering the action of $\hat{L}^2$,

$$\begin{aligned}\hat{L}^2(\hat{a}_R^\dagger)^n |0\rangle &= \left(\hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z\right)(\hat{a}_R^\dagger)^n |0\rangle = \hat{L}_-\hat{L}_+(\hat{a}_R^\dagger)^n |0\rangle + \hat{L}_z\left(\hat{L}_z + \hbar\right)(\hat{a}_R^\dagger)^n |0\rangle \\ &= \hbar^2\ell(\ell+1)(\hat{a}_R^\dagger)^n |0\rangle\end{aligned}$$

Thus, $(\hat{a}_R^\dagger)^n |0\rangle$ is an eigenstate of $\hat{L}^2$ with eigenvalue $\hbar^2\ell(\ell+1) = \hbar^2 n(n+1)$ and thus the top state in an $\ell = n$ multiplet. By acting with L_- we recover every $2n+1$ state in the multiplet. Consider the operator,

$$(\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger$$

We will show that this operator commutes with $\hat{L}^2$ by calculating the following commutators,

$$\begin{aligned}
[\hat{L}_-, (\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger] &= [\hat{L}_-, (\hat{a}_z^\dagger)^2] + [\hat{L}_-, 2\hat{a}_R^\dagger \hat{a}_L^\dagger] = -\hbar\sqrt{2}\left[\hat{a}_L^\dagger \hat{a}_z, (\hat{a}_z^\dagger)^2\right] + \hbar\sqrt{2}\left[\hat{a}_z^\dagger \hat{a}_R, 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right] \\
&= -\hbar\sqrt{2}\left[2\hat{a}_L^\dagger \hat{a}_z^\dagger\right] + \hbar\sqrt{2}\left[2\hat{a}_z^\dagger \hat{a}_L^\dagger\right] = 0 \\
[\hat{L}_+, (\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger] &= [\hat{L}_+, (\hat{a}_z^\dagger)^2] + [\hat{L}_+, \hat{a}_R^\dagger \hat{a}_L^\dagger] = \hbar\sqrt{2}\left[\hat{a}_R^\dagger \hat{a}_z, (\hat{a}_z^\dagger)^2\right] + \hbar\sqrt{2}\left[\hat{a}_z^\dagger \hat{a}_L, 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right] \\
&= \hbar\sqrt{2}\left[2\hat{a}_R^\dagger \hat{a}_z^\dagger\right] - \hbar\sqrt{2}\left[2\hat{a}_z^\dagger \hat{a}_R^\dagger\right] = 0 \\
[\hat{L}_z, (\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger] &= [\hat{L}_z, (\hat{a}_z^\dagger)^2] + [\hat{L}_z, 2\hat{a}_R^\dagger \hat{a}_L^\dagger] = 2\hbar[\hat{a}_R^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_L, \hat{a}_R^\dagger \hat{a}_L^\dagger] = 2\hbar\left[\hat{a}_R^\dagger \hat{a}_L^\dagger - \hat{a}_L^\dagger \hat{a}_R^\dagger\right] = 0
\end{aligned}$$

Therefore,

$$[\hat{L}^2, (\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger] = [\hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z, (\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger] = 0$$

We have constructed a rasing operator of $\hat{H}$ which commutes with $\hat{L}^2$ and $\hat{L}_z$ and therefore preserves the angular momentum state. Consider the states,

$$|k, \ell\rangle = \left((\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right)^k (\hat{a}_R^\dagger)^\ell |0\rangle$$

This state has $2k+\ell$ powers acting evenly on $|0\rangle$ so the state has energy raised by $(2k+\ell)\hbar\omega$ above the groundstate. Thus,

$$\hat{H}|k, \ell\rangle = \hbar\omega\left(2k + \ell + \frac{3}{2}\right)|k, \ell\rangle$$

so this state has energy level $N = 2k + \ell$. Furthermore, because $\hat{L}_z$ and $\hat{L}^2$ commute with the first operator, we can explitly find the angular momentum eigenvalues of these states.

$$\begin{aligned}
\hat{L}_z |k, \ell\rangle &= \hat{L}_z \left((\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right)^k (\hat{a}_R^\dagger)^\ell |0\rangle = \left((\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right)^k \hat{L}_z (\hat{a}_R^\dagger)^\ell |0\rangle \\
&= \hbar\ell \left((\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right)^k (\hat{a}_R^\dagger)^\ell |0\rangle = \hbar\ell |k, \ell\rangle \\
\hat{L}^2 |k, \ell\rangle &= \hat{L}_z \left((\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right)^k (\hat{a}_R^\dagger)^\ell |0\rangle = \left((\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right)^k \hat{L}^2 (\hat{a}_R^\dagger)^\ell |0\rangle \\
&= \hbar^2\ell(\ell+1) \left((\hat{a}_z^\dagger)^2 + 2\hat{a}_R^\dagger \hat{a}_L^\dagger\right)^k (\hat{a}_R^\dagger)^\ell |0\rangle = \hbar^2\ell(\ell+1) |k, \ell\rangle
\end{aligned}$$

Therefore, $|k, \ell\rangle$ is a state with energy level $N = 2k + \ell$, total angular momentum ℓ, and maximum z angular momentum $m = \ell$. Therefore, we get every state in a ℓ multiplet at this energy level because $\hat{L}_-$ commutes with $\hat{H}$. Therefore, we have found simultaneous eigenvectors of $\hat{H}$, $\hat{L}_z$, and $\hat{L}^2$ which we knew must have been possible from the start because these operators commute. Fixing N, there is an angular momentum multiplet for each ℓ for which $N = 2k + \ell$ has nonnegative integer solutions. Thus, $N \geq \ell \geq 0$ and $N \equiv \ell \bmod 2$. For even N there are $N/2 + 1$ possible values of ℓ and for odd N there are $(N+1)/2$ possible values. Counting the total degeneracy, the multiplicity of states in each E_N energy eigenspace is for even N,

$$D_N = \sum_{i=0}^{N/2} (2\ell_i + 1) = \sum_{i=0}^{N/2} (4i + 1) = 4\frac{(N/2)(N/2+1)}{2} + (N/2+1) = (N+1)(N/2+1) = \frac{(N+1)(N+2)}{2}$$

and for odd N,

$$D_N = \sum_{i=1}^{(N+1)/2}(2\ell_i+1) = \sum_{i=1}^{(N+1)/2}(2(2i-1)+1) = \sum_{i=1}^{(N+1)/2}(4i-1) = 4\frac{(N+1)((N+1)/2+1)}{4} - (N+1)/2$$
$$= \frac{(N+1)^2}{2} + (N+1) - (N+1)/2 = \frac{(N+1)^2+(N+1)}{2} = \frac{(N+1)(N+2)}{2}$$

However, this is exactly the degeneracy of the energy eigenspace corresponding to energy level N caluculated in part (a) which means that the decomposition in terms of angular momentum multiplets has covered all of the states. Therefore, there is no multiplicity of fixed ℓ angular momentum representations within a given energy level. In summary, the states with energy level N have a full ℓ-multiplet of states for ℓ starting at N and decreasing by 2 i.e. the eigenspace is spanned by angular momentum states for $\ell = N, N-2, N-4, \ldots, 0$ if N is even which give $N/2+1$ full angular momentum multiplets and $\ell = N, N-2, N-4, \ldots, 1$ if N is odd, giving $(N+1)/2$ such multiplets.