1 Motivation

Example 1.0.1. Complex K-theory. $K^0(X)$ is Stable isomorphism classes of complex vector bundles on X. With ring structure. $K^{-n}(S^n X)$ and $K^n(\Omega^n X)$. Then K-theory is exceptional because,

$$K(*) = \mathbb{Z}[\beta, \beta^{-1}]$$

We see that,

$$K^n(*) = \begin{cases} \mathbb{Z} & n \text{ is even} \\ 0 & \text{else} \end{cases}$$

K-theory is even, periodic, and multiplicative (we say "nice"). We can make singular cohomology have these properties by definitin,

$$H^n(X) := \prod_{k \in \mathbb{Z}} H^{n+2k}_{\text{sing}}(X)$$

If A is a cohomology theory that is "nice" we should look at,

$$A(\mathbb{CP}^\infty) \cong A(*)[[t]]$$

this is a noncanonical isomorphism. This comes from Chern classes.

Example 1.0.2. $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[[t]]$ because $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$ and thus,

$$H^2(X, \mathbb{Z}) = [X, \mathbb{CP}^\infty]$$

so $\mathbb{CP}^\infty$ is also the universal object for line bundles. Given a line bundle L on X there is a map $f : X \to \mathbb{CP}^\infty$ unique up to homotopy and therefore by the above identification we get f^*t the first chern class.

Analogously, if we have come cohomology theory A, we define the first Chern class $c_1^A(\mathcal{O}(1)) = t \in A(\mathbb{CP}^\infty)$. Then we get, $c_1^A(L) = f^* c_1^A(\mathcal{O}(1)) = t \in A^2(X)$.

What happens if I consider,

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

This does not hold for generalized cohomology theories. For example, in Complex K-theory,

$$c_1^K(L_1 \otimes L_2) = c_1^K(L_1) + c_1^K(L_2) + c_1(L_1) \cdot c_1(L_2)$$

The way to see this is to consider,

$$A(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = A(*)[[t_1, t_2]]$$

Then we can compute,

$$c_1^A(\pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1)) = f(t_1, t_2) \in A(*)[[t_1, t_2]]$$

By naturality this gives the formula for tensor products. We have the following properties,

(a) trivial bundles have trivial chern classes so $f(t, 0) = f(0, t) = t$.

(b) tensor product is symmetric $f(u,v) = f(v,u)$

(c) tensor product is associative $f(f(u,v),w) = f(u, f(v,w))$

This is a formal group law over $A(*)$.

Given a formal group law over $A(*)$ I get a formal group,

$$\mathrm{Spf}(A(*)[[t]])$$

which does not depend on any choices of isomorphisms.

Example 1.0.3. The formal group laws,

(a) $f(u,v) = u + v$ for singular cohomology

(b) $f(u,v) = u + v + uv$ for complex K-theory

Quillen showed that there exists a cohomology theory called MP such that,

$$MP(\mathbb{CP}^\infty) = MP(*)[[t]]$$

canonically such that,

$$\mathrm{Hom}\,(MP(*), R) = \{\text{formal group laws over } R\}$$

In many cases, given some $\mathbb{G} = \mathrm{Spf}(A(*)[[t]])$ and this should have some formal group law and this gives a map $f : MP(*) \to R$ and then define,

$$A^n_{\mathbb{G}}(X) = MP^n(X) \otimes_{MP(*)} R$$

but this only works if R is flat.

Let $\mathcal{M}_{\mathrm{FLG}}$ be the moduli stack of formual group laws. By a theorem of Quillen $\mathcal{M}_{\mathrm{FGL}} = \mathrm{Spec}\,(MP(*))$. Let G be the group scheme of automorphisms of the formal affine line $\mathrm{Spf}(\mathbb{Z}[[x]])$. Then we consider the moduli stack of formal groups,

$$\mathcal{M}_{\mathrm{FG}} = [\mathcal{M}_{\mathrm{FGL}}/G]$$

Using this stack, we can give a correspondence between $MP(X)$ and quasi-coherent sheaves on $\mathcal{M}_{\mathrm{FG}}$. This gives better criteria for $A^n_{\mathbb{G}}$ to be a cohomology theory.

The formal groups that we get from singular cohomology and K-theory are the formal completions of $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively.

Maybe to find more cohomology theories we should look at formal completions of 1-dimensional commutative algebraic groups. However, over an algebraically closed field there are not very many,

(a) $\mathbb{G}_a$

(b) $\mathbb{G}_m$

(c) elliptic curves

1.1 Elliptic Cohomology

We now want to consider elliptic curves. Let $\mathcal{M}_{1,1}$ be the moduli stack of elliptic curves.

Theorem 1.1.1. There exists a unique up to homotopy lift,

$$\begin{tikzcd} & \{\text{spectra}\} \arrow[d] \\ \{\text{elliptic curves}\} \arrow[r] \arrow[ur, dashed] & \{\text{cohomology theories}\} \end{tikzcd}$$

2 Lurie's A Survey of Elliptic Cohomology

Hopkins' and Miller's insight is to lift to E_∞-rings and give a proper sheaf of E_∞-rings on $\mathcal{M}_{1,1}$.

2.1 E_∞-rings

Let A be a space equiped with a multiplication map $a : A \times A \to A$. There are some axioms you might want to hold of A to make it a ring object. However, we only consider these diagrams to commute "up to homotopy". However, just up to homotopy is not enough. You really want to include the homotopies with higher coherence.

Consequences: $\pi_*(A)$ is a graded ring. $\pi_n(A)$ has a natural addition structure and also Eckman-Hilton gives that addition on A gives the same operation on $\pi_n(A)$ then multiplication on A gives multiplication on $\pi_*(A)$.

A map $A \to B$ of E_∞-rings is an equivalence if it induces an isomorphism on the homotopy groups preserving the addition and multiplications thus gives a map of graded rings $\pi_*(A) \to \pi_*(B)$.

This tells me that $\pi_0(A)$ is an ordinary ring. We want to think of higher homotopy groups $\pi_n(A)$ as measuring the failure of A to be a ring.

2.2 Spectral AG

Definition 2.2.1. If A is an E_∞-ring then $\operatorname{Spec}(A) = (\operatorname{Spec}(\pi_0(A)), \mathcal{O})$ with a sheaf of E_∞-rings.