1 The Embedding Theorem in Hurwitz-Brill-Noether Theory

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Definition 1.0.1. $W^r_d(C) = \{L \in \mathrm{Pic}^d(C) \mid h^0(C, L)r + 1\}$ line bundles with enough sections to give nondegenerate map to $\mathbb{P}^r$ of degree d. These fit together to a universal Brill-Noether stack,

$$^r_d \to \mathcal{M}_g$$

Definition 1.0.2. Let $\rho(g, r, d) = g - (r + 1)(g - d + r)$.

Remark. When we say a "general map" to $\mathbb{P}^r$ we mean a general line bundle in $^r_{d,}$.

Theorem 1.0.4 (Eisenbud-Harris)**.** Assume $\rho \geq 0$. If $r \geq 3$ then a general degree d map $C \to \mathbb{P}^r$ is an embedding.

Remark. Also necessary away from,

(a) $(g, r, d,) \in \{(0, 1, 1), (0, 2, 2), (1, 2, 3), (3, 2, 4)\}$

Definition 1.0.5. We say that $\mathcal{L}$ is p-very ample if for all divisors $D \subset C$ of degree $p + 1$,

$$h^0(L(-D)) = h^0(L) - (p + 1)$$

Remark. (a) base-point free $\iff$ 0-very ample

(b) very ample $\iff$ 1-very ample.

Remark. Geometrically, this means the span of the image of D is dimension p.

Theorem 1.0.6 (Farkas)**.** A general $L \in^r_{d,}$ is p-very ample if $r \geq 2p + 1$.

We want to extend this to Hurwitz Brill-Noether Theory,

$$^{\vec{e}} \to \mathscr{H}_{k,g}$$

where $\mathscr{H}_{k,g}$ parametrizes degree k genus g covers $C \to \mathbb{P}^1$.

1.1 Hurwitz-Brill-Noether Theory

If $f : C \to \mathbb{P}^1$ has degree k. Then $W^r_d(C)$ is often reducible (meaning for varying over curves of fixed gonality I guess). But we can refine the space as follows. If L is a line bundle on C, then f_*L is a rank k vector bundle so we can decompose it,

$$f_*L = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k)$$

with $e_1 \leq \cdots \leq e_k$.

Definition 1.1.1. We say that $\vec{e}'$ specializes $\vec{e}$ if,

$$e'_1 + \cdot + e'_j \leq e_1 + \cdots + e_j$$

for all j. Then,

$$W^{\vec{e}}(C) = \{L \in \mathrm{Pic}\,(C) \mid f_*L \cong \mathcal{O}(\vec{e}) \text{ or a specialization }\}$$

Proposition 1.1.2. We see,

$$W_d^r(C) = \bigcup_{h^0(\mathcal{O}(\vec{e})\geq r+1//e_1+\cdots+e_k=d-g+1-k} W^{\vec{e}}(C)$$

Definition 1.1.3. $\rho'(g,\vec{e}) = g - \sum_{i,j} \max\{0, e_i \cdots e_j - 1\}$.

Remark. Given $\vec{e}$ we recover the pair (r,d) as,

$$r = h^0(\mathcal{O}(\vec{e})) - 1 \quad \text{and} \quad d = e_1 + \cdots + e_k + g - 1 + k$$

Theorem 1.1.4. If $\rho'(g,\vec{e}) \geq 0$ then $^{\vec{e}}$ has a unique irreducible component dominating $\mathscr{H}_{k,g}$. Call it $^{\vec{e}}$.