

1 Locally Free Sheaves

2 Algebraic Vector Bundles

3 Derivations

Definition 3.0.1. Let \mathcal{A} be a sheaf of algebras and \mathcal{B} an \mathcal{A} -algebra and \mathcal{F} a \mathcal{B} -module. Then an \mathcal{A} -derivation $D : \mathcal{B} \rightarrow \mathcal{F}$ is a \mathcal{A} -module map such that on all local sections,

$$D(fg) = D(f)g + fD(g)$$

Furthermore, we write $\mathcal{D}er_{\mathcal{A}}(\mathcal{B}, \mathcal{F}) \subset \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{F})$ for the \mathcal{A} -submodule of derivations.

Definition 3.0.2. If the functor $\mathcal{F} \mapsto \mathcal{D}er_{\mathcal{A}}(\mathcal{B}, \mathcal{F})$ is representable on the category on \mathcal{B} -modules then we say the representing pair $(\Omega_{\mathcal{B}/\mathcal{A}}, d)$ is the \mathcal{B} -module of \mathcal{A} -differentials where,

$$\mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{B}/\mathcal{A}}, \mathcal{F}) = \mathcal{D}er_{\mathcal{A}}(\mathcal{B}, \mathcal{F})$$

and the derivation $d : \mathcal{B} \rightarrow \Omega_{\mathcal{B}/\mathcal{A}}$ is the universal element given by,

$$\text{id} \in \mathcal{H}om_{\mathcal{A}}(\Omega_{\mathcal{B}/\mathcal{A}}, \Omega_{\mathcal{B}/\mathcal{A}}) = \mathcal{D}er_{\mathcal{A}}(\mathcal{B}, \Omega_{\mathcal{B}/\mathcal{A}})$$

Definition 3.0.3. Given morphism of locally ringed spaces $f : X \rightarrow S$ we say that $(\Omega_{X/S}, d)$ is the \mathcal{O}_X -module of $f^{-1}\mathcal{O}_S$ -differentials viewing \mathcal{O}_X as a $f^{-1}\mathcal{O}_S$ -algebra via the map $f^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_X$.

4 Connections

Remark. Here we have a locally ringed space $X \rightarrow S$ over S . We write $\Omega_X = \Omega_{X/S}$ and

Definition 4.0.1. A connection on a vector bundle \mathcal{E} on X is a \mathcal{O}_S -linear derivation,

$$\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$$

Lemma 4.0.2. Suppose that $\nabla_1, \nabla_2 : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ are connections. Then,

$$\nabla_1 - \nabla_2 : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$$

is a \mathcal{O}_X -module map.

Proof. $(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 s - \nabla_2 s) + df \otimes s - df \otimes s = f(\nabla_1 - \nabla_2)s$. □

Remark. Therefore, the space of connections is an affine subspace of $\text{Hom}(\mathcal{E}, \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E})$. Then if \mathcal{E} is finite locally free,

$$\text{Hom}(\mathcal{E}, \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}) = H^0(X, \Omega_X^1 \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_S}(\mathcal{E}))$$

Definition 4.0.3. The first Chern class $c_1 : \text{Pic}(X) \rightarrow H^1(X, \Omega_X^1) \subset H_{\text{dR}}^2(X)$ is defined by $H^1(X, -)$ applied to the map $\text{dlog} : \mathcal{O}_X^\times \rightarrow \Omega_X^1$ defined as $\text{dlog}(f) = f^{-1}df$.

Proposition 4.0.4. A line bundle \mathcal{L} admits a connection $\nabla : \mathcal{L} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L}$ if and only if $c_1(\mathcal{L}) = 0$.

Proof. A line bundle \mathcal{L} is represented by a Cech cocycle $(U_i, f_{ij}) \in H^1(X, \mathcal{O}_X^\times)$. Then a connection on a line bundle is represented by (U_i, ω_i) with $\omega_i \in \Omega_X^1(U_i)$ where (U_i, s_i) is a trivialization of \mathcal{L} with $\mathcal{O}_{U_i} \xrightarrow{s_i} \mathcal{L}|_{U_i}$ then $s_i|_{U_i \cap U_j} = f_{ij}s_j|_{U_i \cap U_j}$ and $\nabla s_i = \omega_i \otimes s_i$. However, we must have on $U_i \cap U_j$,

$$\nabla s_i = \nabla f_{ij}s_j = f_{ij}\nabla s_j + df_{ij} \otimes s_j$$

Therefore,

$$\omega_i \otimes f_{ij}s_j = f_{ij}\omega_j \otimes s_j + df_{ij} \otimes s_j$$

and thus,

$$(\omega_i - \omega_j)|_{U_i \cap U_j} = d\log(f_{ij})$$

Consider the Cech differential $d : \check{C}^0(\mathfrak{U}, \Omega_X^1) \rightarrow \check{C}^1(\mathfrak{U}, \Omega_X^1)$ which takes the sections (ω_i) to the coboundary $(\omega_i - \omega_j)|_{U_{ij}}$. Therefore, such a connection i.e. such a class exists iff the class,

$$c_1(\mathcal{L}) = [d\log(f_{ij})] \in \check{H}^1(X, \Omega_X^1)$$

is trivial since it is a coboundary. □

5 Curvature

Definition 5.0.1. The connection ∇ defines a corresponding curvature map,

$$\omega_\nabla = \nabla_1 \circ \nabla : \mathcal{E} \rightarrow \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$$

We say that ∇ is flat or integrable if the curvature vanishes $\omega_\nabla = \nabla_1 \circ \nabla = 0$.

Lemma 5.0.2. The curvature $\omega_\nabla : \mathcal{E} \rightarrow \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$ is a \mathcal{O}_X -module map.

Proof. Consider,

$$\begin{aligned} \omega_\nabla(fs) &= \nabla_1(df \otimes s + f\nabla s) = ddf \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla_1 \circ \nabla \\ &= f\nabla_1 \circ \nabla s = f\omega_\nabla(s) \end{aligned}$$

□

Remark. Therefore ω_∇ defines the curvature form $\omega_\nabla \in \Gamma(X, \Omega_X^2 \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E}))$.

Remark. If we write locally,

$$\nabla e = \sum_i f_i dg_i \otimes s_i$$

then the curvature takes the form,

$$\omega_\nabla(e) = \sum_i (df_i \wedge dg_i \otimes e - f_i dg_i \otimes \nabla s_i)$$

6 Differential Operators

Definition 6.0.1. Let \mathcal{A} be a sheaf of algebras and \mathcal{B} an \mathcal{A} -algebra and \mathcal{F}, \mathcal{G} be \mathcal{B} -modules. Then a differential operator $D : \mathcal{F} \rightarrow \mathcal{G}$ of order k is a \mathcal{A} -module map such that for all local sections $b \in \Gamma(U, \mathcal{B})$ the map, $D(b \cdot -) - b \cdot D : \Gamma(U, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{G})$ is a differential operator of order $k - 1$. Where a differential operator of order $k = 0$ is a \mathcal{B} -linear map $D : \mathcal{F} \rightarrow \mathcal{G}$. Furthermore, we write $\text{Diff}_{\mathcal{B}|\mathcal{A}}^k(\mathcal{F}, \mathcal{G}) \subset \text{Hom}_{\mathcal{B}}(\mathcal{F}, \mathcal{G})$ to denote the \mathcal{B} -submodule of differential operators of order k .

7 Sheaves of Jets

8 The Atiyah Class

9 Riemann-Hilbert Correspondence

10 Connections on Real and Complex Manifolds

Remark. Let $\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes \mathcal{E}$ be a connection. For a vector field X we write $\nabla_X : \mathcal{E} \rightarrow \mathcal{E}$ for the map,

$$\mathcal{E} \xrightarrow{\nabla} \Omega_X^1 \otimes \mathcal{E} \xrightarrow{X \otimes \text{id}} \mathcal{O}_X \otimes \mathcal{E} \rightarrow \mathcal{E}$$

Therefore, in previous notation $\nabla_X = Q(X)$. Thus we see that, viewing $\omega_\nabla \in \Omega_X^2 \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E})$ that,

$$\omega_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

When ∇ is the Levi-Civita connection then ω_∇ is the Riemann tensor.

Definition 10.0.1. A form $\sigma \in \Gamma(X, \Omega_X \otimes \mathcal{E})$ is called a *solder form* if $\sigma : \mathcal{T}_X \rightarrow \mathcal{E}$ is an isomorphism. Given a connection $\nabla : \mathcal{E} \rightarrow \Omega_X \otimes \mathcal{E}$, the *torsion* is $T_{(\nabla, \sigma)} = \nabla_1 \sigma \in \Gamma(X, \Omega_X^2 \otimes \mathcal{E})$.

Remark. Choose a local frame $\{e_i\}$ of $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$ and $\{\sigma_i\}$ of Ω_X compatibly via σ . Then,

$$\sigma = \sum_i \sigma_i \otimes e_i$$

and write,

$$\nabla e_j = \sum_i \omega_{ij} \otimes e_i$$

for 1-forms $\omega_{ij} \in \Omega_X^1(U)$. Then we compute,

$$\begin{aligned} \nabla_1 \sigma &= \sum_i d\sigma_i \otimes e_i - \sum_j \sigma_j \wedge \nabla e_j \\ &= \sum_i \left(d\sigma_i + \sum_j \omega_{ij} \wedge \sigma_j \right) \otimes e_i \end{aligned}$$

Therefore,

$$T_{(\nabla, \sigma)} = 0 \iff \tau_i = d\sigma_i + \sum_j \omega_{ij} \wedge \sigma_j = 0$$

Remark. For $\mathcal{E} = \mathcal{T}_X$ we have a canonical solder form σ_{id} given by $\text{id} : \mathcal{T}_X \rightarrow \mathcal{T}_X$. Then $T_\nabla = T_{(\nabla, \sigma_{\text{id}})}$ is the torsion of ∇ . In local coordinates,

$$\sigma_{\text{id}} = \sum_j dx^j \otimes \frac{\partial}{\partial x^j} \quad \text{and} \quad \nabla \frac{\partial}{\partial x^j} = \sum_i \omega_{ij} \otimes \frac{\partial}{\partial x^i}$$

Then,

$$\nabla_1(\sigma_{\text{id}}) = - \sum_{i,j} (dx^j \wedge \omega_{ij}) \otimes \frac{\partial}{\partial x^j}$$

Therefore, if $X = v^i \frac{\partial}{\partial x^i}$ and $Y = u^i \frac{\partial}{\partial x^i}$ we find that,

$$T_{\nabla}(X, Y) = \sum_{i,j} \left(u^j v^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) - v^j u^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) \right) \otimes \frac{\partial}{\partial x^j}$$

However,

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= \sum_{i,j} \left(u^j v^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) - v^j u^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) \right) \otimes \frac{\partial}{\partial x^i} \\ &\quad + \left(v^k du^j \left(\frac{\partial}{\partial x^k} \right) \otimes \frac{\partial}{\partial x^j} - u^k dv^j \left(\frac{\partial}{\partial x^k} \right) \otimes \frac{\partial}{\partial x^j} \right) \\ &= T_{\nabla}(X, Y) + [X, Y] \end{aligned}$$

Therefore, we write down the following.

Definition 10.0.2. Let $\nabla : \mathcal{T}_X \rightarrow \Omega_X \otimes \mathcal{T}_X$ be a connection on the tangent bundle. The torsion $T_{\nabla} \in \Gamma(X, \Omega_X^2 \otimes \mathcal{T}_X)$ is defined via,

$$T_X(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

10.1 Metric Compatibility

Remark. A complex vector bundle $E \rightarrow M$ is equivalent to a pair (E, I) where $E \rightarrow M$ is a real vector bundle and $I : E \rightarrow E$ is a bundle endomorphism such that $I^2 = -\text{id}$. Therefore, an almost complex structure is the same as endowing the tangent bundle with a complex structure.

Remark. A holomorphic structure on a complex vector bundle $E \rightarrow X$ over a complex manifold is the structure of a complex manifold on E such that $E \rightarrow X$ is holomorphic and such that there exist biholomorphic linear charts for $E \rightarrow X$ as a bundle.

Definition 10.1.1. Let $E \rightarrow M$ be a real vector bundle. A metric on E is a positive-definite symmetric section $g \in \Gamma(M, \text{Sym}^2(E^*))$.

Definition 10.1.2. A connection $\nabla : E \rightarrow \mathcal{A}_X^1 \otimes E$ is compatible with the metric g if $\nabla g = 0$.

Remark. Explicitly,

$$(\nabla g)(s_1, s_2) = d(g(s_1, s_2)) - g(\nabla s_1, s_2) - g(s_1, \nabla s_2)$$

and thus $\nabla g = 0$ iff $d(g(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2)$.

Definition 10.1.3. Let $(E, I) \rightarrow M$ be a complex vector bundle. A hermitian structure on E is a section $h \in \Gamma(M, E^* \otimes \overline{E}^*)$ such that h_x is a hermitian metric on E_x .

Proposition 10.1.4. A hermitian structure on (E, I) is equivalent to a metric compatible with I .

Proof. The equivalence is given by $h = g - i\omega$ where $\omega(-, -) = g(I(-), -)$ is the fundamental form. (CHECK THIS) \square

Definition 10.1.5. Let $E \rightarrow M$ be complex. We say a connection $\nabla : E \rightarrow \mathcal{A}_M^1 \otimes E$ is complex if ∇ is complex linear. If E has a hermitian structure we say that ∇ is hermitian if $\nabla h = 0$.

Remark. Note that ∇ being complex linear is equivalent to $\nabla \circ I = I \circ \nabla$ is equivalent to $\nabla I = 0$ via the induced connection on $E^* \otimes E$. Explicitly,

$$(\nabla I)(s) = \nabla I(s) - I(\nabla(s)) = 0$$

Remark. Note that we need ∇ to be complex for $\nabla h = 0$ to make sense since we need ∇ to induce a connection on $E^* = \text{Hom}_{\mathbb{C}}(E, \mathcal{O}_X)$. To see why, consider a section $\varphi \in \Gamma(X, E^*)$ then, $\nabla \varphi$ should be complex linear. However,

$$\begin{aligned} (\nabla \varphi)(I(s)) &= d\varphi(I(s)) - \varphi(\nabla I(s)) = id\varphi(s) - i\varphi(\nabla s) + \varphi([I \circ \nabla - \nabla \circ I](s)) \\ &= i(\nabla \varphi)(s) + \varphi([I \circ \nabla - \nabla \circ I](s)) \end{aligned}$$

and therefore we need $\nabla \circ I = I \circ \nabla$.

Proposition 10.1.6. Let (E, I, h) be a complex bundle with a hermitian structure and g the associated compatible metric with funamental form ω . A complex connection $\nabla : E \rightarrow \mathcal{A}_X^1 \otimes E$ is hermitian iff

$$\nabla h = 0 \iff \nabla g = 0 \iff \nabla \omega = 0$$

Proof. Because $\nabla I = 0$ we see that $(\nabla \omega)(-, -) = (\nabla g)(I(-), -)$ and thus $\nabla g = 0 \iff \nabla \omega = 0$. Furthermore, $h = g - i\omega$ so if $\nabla h = 0$ then the real and imaginary parts must indiviually vanish so $\nabla g = \nabla \omega = 0$. Explicitly,

$$\begin{aligned} d(h(s_1, s_2)) &= h(\nabla s_1, s_2) + h(s_1, \nabla s_2) \\ &\iff \\ d(g(s_1, s_2)) - id(\omega(s_1, s_2)) &= g(\nabla s_1, s_2) + g(s_1, \nabla s_2) - i\omega(\nabla s_1, s_2) - i\omega(s_1, \nabla s_2) \end{aligned}$$

and therefore,

$$d(g(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2) \quad \text{and} \quad d(\omega(s_1, s_2)) = \omega(\nabla s_1, s_2) + \omega(s_1, \nabla s_2)$$

□

10.2 The Levi-Civita and Chern Connections

Proposition 10.2.1. Let (E, g, σ) be a real vector bundle on M with a metric and solder form $\sigma : T_M \rightarrow E$. Then there exists a unique torsion-free connection ∇ compatible with the metric called the Levi-Civita connection.

Proof. DO THIS!!

□

Definition 10.2.2. Let $E \rightarrow X$ be a holomorphic vector bundle. We say a complex connection $\nabla : E \rightarrow \mathcal{A}_X^1 \otimes E$ is compatible if $\nabla^{0,1} = \bar{\partial}_E$ where $\nabla^{0,1} = (\Pi^{0,1} \otimes \text{id}_E) \circ \nabla$.

Proposition 10.2.3. Let (E, h) be a holomorphic vector bundle with a hermitian structure. Then there exists a unique compatible hermitian connection ∇ called the Chern connection.

Proof. (DO THIS)

□

Remark. Now we consider the tangent bundle of a hermitian manifold (X, g) that is a Riemannian manifold (M, g) with a compatible almost complex structure $X = (M, I)$. There may be obstructions to the Levi-Civita connection being complex

Proposition 10.2.4.

Proposition 10.2.5. Let (X, g) be a hermitian manifold. Let ∇_{LC} be the Levi-Civita connection on TM of the underlying Riemannian manifold (M, g) . Then,

$$\nabla_{\text{LC}}(I) =$$

Proposition 10.2.6. Let (X, g) be a hermitian complex manifold. Let ∇ be a torsion-free complex hermitian connection. Then the following hold,

- (a) ∇ is the Levi-Civita connection for the underlying Riemannian structure
- (b) ∇ is the Chern connection of $(T^{1,0}X, g_{\mathbb{C}})$
- (c) (X, g) is Kähler.

10.3 Ricci Curvature**11 Conventions**

Symmetric and exterior algebras are *quotients* not subspaces. The subspaces of symmetric and alternating tensors are a distinct notion. In characteristic zero $V^{\otimes n} \twoheadrightarrow \wedge^n V$ is split and the image is the alternating tensors and similarly for symmetric tensors and $V^{\otimes n} \twoheadrightarrow \text{Sym}^n(V)$.

To identify $\wedge^k V^* \cong (\wedge^k V)^*$ we need to choose a perfect pairing $\wedge^k V \times \wedge^k V^* \rightarrow k$. We do this in the only natural way that works in all characteristics,

$$(v_1 \wedge \cdots \wedge v_k, \varphi^1 \wedge \cdots \wedge \varphi^k) \mapsto \det \varphi^i(v^j)$$

Note that $(\varphi \wedge \psi)(v, u) = (\varphi \wedge \psi)(v \wedge u) = \varphi(v)\psi(u) - \varphi(u)\psi(v)$. There are NO factors of $\frac{1}{2}$ anywhere to be seen! The natural map $\wedge^k V^* \xrightarrow{\sim} (\wedge^k V)^* \hookrightarrow (V^{\otimes k})^* \xrightarrow{\sim} (V^*)^{\otimes k} \twoheadrightarrow \wedge^k V^*$ is thus multiplication by $k!$.

Some obnoxious assholes define the pairing with a factor of $\frac{1}{k!}$ to agree with alternating tensors but then they also define the wedge product with a strange coefficient to make everything work out. Explicitly,

$$\text{Alt}(\varphi \otimes \psi)(v, u) = \frac{1}{2} (\varphi \otimes \psi - \psi \otimes \varphi)(v, u) = \frac{1}{2} (\varphi(v)\psi(u) - \psi(v)\varphi(u))$$

and likewise this means that,

$$\langle \text{Alt}(\varphi \otimes \psi), \text{Alt}(v \otimes u) \rangle = \frac{1}{4} (\varphi(v)\psi(u) - \psi(v)\varphi(u) - \varphi(u)\psi(v) + \psi(u)\varphi(v)) = \frac{1}{2} (\varphi(v)\psi(u) - \varphi(u)\psi(v))$$

But then they define $v \wedge u = 2\text{Alt}(v \otimes u) = v \otimes u - u \otimes v$ to “fix” everything so that,

$$(\varphi \wedge \psi)(v, u) = \varphi(v)\psi(u) - \psi(v)\varphi(u)$$

so in fact $v \wedge u$ has the same image in $V^{\otimes 2}$ as previously.

12 Flatness

Proposition 12.0.1. Let $\varphi : A \rightarrow B$ be a flat local map of local rings. Let M be a finitely presented B -module which is flat over A . Suppose that $M/\mathfrak{m}_A M$ is a free $B/\mathfrak{m}_A B$ -module. Then M is a free M -module.

Proof. Choose an isomorphism,

$$(B/\mathfrak{m}_A B)^n \xrightarrow{\sim} M/\mathfrak{m}_A M$$

and choose a lift to a map $B^n \rightarrow M$ inducing a sequence,

$$0 \longrightarrow K \longrightarrow B^n \longrightarrow M \longrightarrow C \longrightarrow 0$$

Since M is finitely-presented, K and C are finite B -modules. From the exact sequence, $C/\mathfrak{m}_A C = 0$ and thus,

$$C/\mathfrak{m}_A C \twoheadrightarrow C/\mathfrak{m}_B C$$

proves that $C = \mathfrak{m}_B C$ and thus by Nakayama's lemma $C = 0$. Therefore, we have a short exact sequence,

$$0 \longrightarrow K \longrightarrow B^n \longrightarrow M \longrightarrow 0$$

Since M is flat over A this sequence remains exact after applying $-\otimes_A (A/\mathfrak{m}_A)$ and thus $K/\mathfrak{m}_A K = 0$ and hence $K/\mathfrak{m}_B K = 0$. Since K is a finite B -module, by Nakayama, we see that $K = 0$ and hence $B^n \xrightarrow{\sim} M$. \square

Corollary 12.0.2. Let $f : X \rightarrow Y$ be a flat map of schemes and \mathcal{F} a coherent \mathcal{O}_X -module flat over Y . Suppose that $\mathcal{F}|_{X_y}$ is a vector bundle on X_y for some y . Then there is an open neighborhood $U \subset X$ of X_y such that $\mathcal{F}|_U$ is a vector bundle.

Proof. Since \mathcal{F} is coherent, it suffices to show that \mathcal{F}_x is a free $\mathcal{O}_{X,x}$ -module for each $x \in X_y$ which follows immediately from the previous result. \square

Example 12.0.3. Consider $X = \mathbb{A}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{A}^1 = \text{Spec}(k[z])$ and $\mathcal{F} = \widetilde{(x, y)}$. This sheaf is obviously flat but its fiber over $z = 0$ is a vector bundle since it is \mathcal{O}_X away from $x = y = 0$. However, it is not a vector bundle on any other fiber.

Corollary 12.0.4. Let $f : X \rightarrow Y$ be a flat and proper map of schemes and \mathcal{F} a coherent \mathcal{O}_X -module flat over Y . Suppose that $\mathcal{F}|_{X_{y_0}}$ is a vector bundle on X_{y_0} for some $y_0 \in Y$. Then there is an open $y_0 \in V \subset Y$ such that $\mathcal{F}|_{X_V}$ is a vector bundle. In particular for all $y \in V$ we have that $\mathcal{F}|_{X_y}$ is a vector bundle.

Proof. Using the previous result, it suffices to show that the set,

$$V = \{y \in Y \mid \mathcal{F}|_{X_y} \text{ is a vector bundle}\}$$

is open. For any $y \in V$ there is an open neighborhood $X_y \subset U \subset X$ so that $\mathcal{F}|_U$ is a vector bundle and thus $y \in f(U^c)^c \subset V$ is open because f is closed. \square

Example 12.0.5. Let $\pi_1 : X = \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 = S$ be the projection. Let $x = X$ be a point and $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of $x = (0, 0) \in X$. For each fiber X_t with $t \neq 0$ we have $\mathcal{I}|_{X_t} = \mathcal{O}_{X_t}$ is a vector bundle. However, \mathcal{I} is not a vector bundle so we cannot have $\mathcal{I}|_{X_0}$ be a vector bundle

by the above result. I claim that \mathcal{S} is π_1 -flat. This is clear on $X \setminus \{x\}$ so I we consider the local structure around x . On a dense open we have the following algebra problem,

$$A = k[x]_{(x)} \rightarrow k[x, y]_{(x, y)} = B \quad \text{with the ideal} \quad I = \mathfrak{m}_B = (x, y) \subset k[x, y]_{(x, y)}$$

I claim that I is flat over A . There is an exact sequence,

$$0 \longrightarrow B \xrightarrow{(y-x)} B^2 \xrightarrow{(x \ y)} I \longrightarrow 0$$

Then applying Tag 00MK we just need to show that $B/\mathfrak{m}_A B \rightarrow (B/\mathfrak{m}_A B)^2$ is injective which is true because y is a non zero-divisor on $B/\mathfrak{m}_A B$. Thus I is A -flat. Furthermore, there is an exact sequence,

$$0 \longrightarrow (B/\mathfrak{m}_A B) \xrightarrow{(y \ 0)} (B/\mathfrak{m}_A B)^2 \xrightarrow{(0 \ y)} I/\mathfrak{m}_A I \longrightarrow 0$$

Therefore, we get the local structure,

$$I/\mathfrak{m}_A I \cong k \oplus k[y]_{(y)}$$

but its image in $B/\mathfrak{m}_A B$ is just (y) which is locally free. This we see that $\mathcal{S}|_{X_0} \cong \mathcal{O}_{X_0}(-1) \oplus \iota_* k$ which has degree zero as it must because $\mathcal{S}|_{X_t} \cong \mathcal{O}_{X_t}$ for $t \neq 0$ and degree is constant in flat families.

Example 12.0.6. Consider a degeneration,

$$f : X = \text{Proj} \left(k[t][X, Y, Z]/(XY - tZ^2) \right) \rightarrow \text{Spec} (k[t]) = S$$

with X smooth and f flat and proper but f has a singular fiber over $t = 0$. Then there is a sequence,

$$0 \longrightarrow f^* \Omega_S^1 \longrightarrow \Omega_X \longrightarrow \Omega_{X/S} \longrightarrow 0$$

Now $\Omega_{X/S}|_{X_t} = \Omega_{X_t}$ is a vector bundle for the smooth fibers ($t \neq 0$). However, $\Omega_{X/S}|_{X_0} = \Omega_{X_0}$ is not a vector bundle since X_0 is singular. I claim that $\Omega_{X/S}$ is flat over S . We consider the local structure, on the chart $D_+(Z)$. Let $A = k[t]$ and $B = k[t][x, y]/(xy - t)$ then the above exact sequence becomes,

$$0 \longrightarrow B dt \xrightarrow{xdy+ydx} B dx \oplus B dy \longrightarrow \Omega_{D_+(Z)/S} \longrightarrow 0$$

Therefore,

$$M = \Omega_{D_+(Z)/S} = (B dx \oplus B dy)/(x dy + x dy)$$

Thus the rank jumps at $\mathfrak{m} = (x, y)$. However, I claim that M is flat over A . Applying Tag 00MK we just need to show that,

$$(B/tB)_{\mathfrak{m}} dt \rightarrow (B/tB)_{\mathfrak{m}} dx \oplus (B/tB)_{\mathfrak{m}} dy$$

is injective. Indeed, if $f dt \mapsto 0$ then $fx = 0$ and $fy = 0$ in $(B/tB)_{\mathfrak{m}} = (k[x, y]/(xy))_{\mathfrak{m}}$. Then $f \in \text{Ann}(x) \cap \text{Ann}(y) = (y) \cap (x) = (xy)$ so $f = 0$ in $(B/tB)_{\mathfrak{m}}$. Thus the map is injective.

Remark. We saw in the first example that a smooth proper map can have a flat ideal sheaf fail to be a vector bundle. However, this does not happen for flat families of Cartier divisors in smooth families.

Proposition 12.0.7. Let $f : X \rightarrow Y$ be a smooth proper map of schemes and $Z \subset X$ a closed subscheme flat over Y . Then the locus,

$$V = \{y \in Y \mid Z_y \subset X_y \text{ is Cartier}\}$$

is clopen.

Proof. Consider the ideal sheaf sequence,

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Because $Z \rightarrow Y$ is flat, $\mathcal{I}|_{X_y}$ is the ideal sheaf of $Z_y \subset X_y$. By the previous result, the locus where $\mathcal{I}|_{X_y}$ is a vector bundle (and hence a line bundle since it embeds in \mathcal{O}_X) is open. Thus we just need to prove closedness. It suffices to show that V is stable under specialization. (REDUCE TO THE DVR CASE, 1 NOETHERIAN, 2 BLOW UP, 3 NORMALIZE) Thus we can assume that $Y = \text{Spec}(R)$ where R is a DVR and $D_K \subset X_K$ is a Cartier divisor. We need to show that $D_0 \subset X_0$ is Cartier. For each $x \in X_0$ let $A = \mathcal{O}_{X,x}$ and we have the following: a flat ring map $R \rightarrow A$ with A regular, an ideal $I \subset A$ with $R \rightarrow A/I$ flat such that $I \otimes_R K \subset A \otimes_R K$ is principal. Since $R \rightarrow A/I$ is flat A/I can only have associated points in the generic fiber thus A/I is unmixed since in the generic fiber I is principal and A is regular so I has no embedded primes by the unmixedness theorem. Consider the primary decomposition,

$$I = Q_1 \cap \cdots \cap Q_r$$

where Q_i is \mathfrak{p}_i -primary where $\text{ht}(\mathfrak{p}_i) = 1$ by unmixedness. Since A is a UDF we have $\mathfrak{p}_i = (p_i)$ are principal. Therefore, FINISH THIS!!! \square

Remark. The following example shows that smoothness really is necessary.

Example 12.0.8. Consider,

$$f : X = \text{Proj} \left(k[t][X, Y, Z] / (X^3 - Y^2 Z) \right) \rightarrow S = \text{Spec}(k[t])$$

and the divisor

$$D = \text{Proj} \left(k[t][X, Y, Z] / (X^3 - Y^2 Z, X - t^2 Z, Y - t^3 Z) \right)$$

which is the image of a section of f and hence flat. For $t \neq 0$ we have $D_t \subset X_t$ a Cartier divisor but $D_0 \subset X_0$ is not a Cartier divisor.