1 Picard Scheme

Theorem 1.0.1. Let X be a proper k-scheme. Then $\mathrm{Pic}_{X/k}$ is represented by a lft k-scheme.

Remark. However, this does not hold for proper flat families in general even for curves.

From here on let $f : X \to S$ be a flat, locally finitely presented, proper morphism where $S = \mathrm{Spec}\,(R)$ is a DVR.

Theorem 1.0.2 (8.3.2)**.** $\mathrm{Pic}_{X/S}$ is represented by an algebraic space if and only if f is cohomologically flat in degree 0.

Remark. In fact, the above holds when S is any reduced scheme.

This is a problem since we want to study non-cohomologically flat situations. We fix this in the next section.

1.1 Rigidified Picard Scheme

Proposition 1.1.1 (8.1.6)**.** f admits a rigidifying subscheme meaning a closed subscheme $Y \subset X$ which is flat, locally finitely presented, proper and such that for any $T \to S$ the map,

$$\Gamma(X_T, \mathcal{O}_{X_T}^\times) \to \Gamma(Y_T, \mathcal{O}_{Y_T}^\times)$$

is injective.

Definition 1.1.2. Let $Y \hookrightarrow X$ be a rigidifying subscheme. Then we define the rigidified Picard functor,

$$\mathrm{Pic}_{X/S|Y} : (T \to S) \mapsto \{(\mathcal{L}, \varphi) \mid \mathcal{L} \in \mathrm{Pic}\,(X_T) \text{ and } \varphi : \mathcal{L}|_Y \xrightarrow{\sim} \mathcal{O}_Y\}/\cong$$

The condition of being a rigidifying subscheme shows exactly that there are no nontrivial automorphism of $(\mathcal{L}, \varphi)$.

FGA shows that the functor,

$$(T \to S) \mapsto (f_T)_* \mathcal{O}_{X_T}$$

is representable by a linear scheme V_X over X. This is a vector bundle over X iff f is cohomologically flat in degree 0. Furthermore, the subsheaf of units,

$$(T \to S) \mapsto (f_T)_* \mathcal{O}_{X_T}^\times$$

is represented by an open subscheme,

$$V_X^\times \subset V_X$$

Now V_X is a ring scheme and $V_X^\times$ is a group scheme.

Proposition 1.1.3. Let $Y \hookrightarrow X$ be a rigidifier. There is an exact sequence of fppf sheaves of abelian groups,

$$0 \longrightarrow V_X^\times \longrightarrow V_Y^\times \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow \mathrm{Pic}_{X/S|Y} \longrightarrow 0$$

where the last map forgets the rigidification. It is surjective in the fppf topology because by definition any class in $\mathrm{Pic}_{X/S}$ is fppf locally represented by a line bundle.

Theorem 1.1.4 (8.3.3)**.** Let $Y \hookrightarrow X$ be a rigidifier. Then $\mathrm{Pic}_{X/S|Y}$ is representable by an algebraic space over S which admits a universal rigidified line bundle.

Proposition 1.1.5. Let $s \in S$ be a point such that $H^2(X_s, \mathcal{O}_{X_s}) = 0$. Then there is an open neighborhood $s \in U \subset S$ such that, both $\mathrm{Pic}_{X/S|Y}|_U$ and $\mathrm{Pic}_{X/S}|_U$ are formally smooth over U.

1.2 Relative Curves

Now suppose that f has relative dimension 1 and has geometrically connected fibers.

2 Overview of the Proof

Definition 2.0.1. Let C/k be an integral curve over a field k. Then the *gonality* of C is the smallest degree of a finite map $C \to \mathbb{P}^1$ *over* k. The *geometric gonality* of C is the maximum of the gonality over $\bar{k}$ of the irreducible components of $C_{\bar{k}}$.

Lemma 2.0.2. Let $f : X \to B$ be a proper morphism of relative dimension 1 between normal varities.

Lemma 2.0.3. Let $f : X \to B$ a proper morphism of relative dimension 1 of varities over a perfect field k whose generic fiber is a smooth connected curve. Let $n = \dim X$. Suppose there is a line bundle $\mathcal{L} \hookrightarrow \Omega_X^{n-1}$ whose sections separate d general points on X. Then the general fiber of f has gonality $> d$.

Proof. We can shrink B such that the base and the map are smooth. Choose a general fiber $C \hookrightarrow X$ which is a smooth irreducible curve. Therefore, there is an exact sequence,

$$0 \to \mathcal{C}_{C|X} \to \Omega_X|_C \to \Omega_C \to 0$$

of vector bundles. Since Ω_C is a line bundle there is an exact sequence,

$$0 \to \mathcal{C}_{C|X}^{n-1} \to \Omega_X^{n-1}|_C \to (\wedge^{n-2}\mathcal{C}_{C|X}) \otimes \Omega_C \to 0$$

However, since C is a fiber of f we have $\mathcal{C}_{C|X} = \mathcal{O}_X^{n-1}$. Therefore, we get $n-1$ projection maps,

$$\mathcal{L} \to \Omega_X^{n-1}|_C \to \Omega_C$$

which are all zero exactly if $\mathcal{L} \hookrightarrow \Omega_X^{n-1}$ factors through $\mathcal{C}_{C|X}^{n-1}$ but these forms are constant along fibers so sections of $\mathcal{L}$ would not be able to separate any points on C. Therefore, one of the projections $\mathcal{L} \to \Omega_C$ is a nonzero map of line bundles hence injective meaning that,

$$H^0(C, \mathcal{L}) \to H^0(C, \Omega_C)$$

must be injective. Since we chose C generically $H^0(C, \mathcal{L})$ and hence $H^0(C, \Omega_C)$ can separate d general points on C. Therefore $\mathrm{gon}(C) > d$. $\square$

3 Meaning of Supersingular on even cohomology

What does it mean to have a Frob eigenvalue $\alpha = \zeta q^{i/2}$. This happens exactly when Frob^n has an eigenvalue $(q^n)^{i/2}$. In other words an eigenvector with eigenvalue $\alpha = \zeta q^{i/2}$ is the same as a vector fixed under $\text{Frob}^n / (q^n)^{i/2}$ for some n. By the Tate conjecture, these are classes should be algebraic cycles defined over $\mathbb{F}_{q^n}$. Therefore, for i even, the supersingular eigenspaces are exactly the set of "potentially algebraic cycles" meaning the cycles that are represented by cycle classes of varities defined over possibly lager fields.

4 Characters

We are considering the projective variety X defined by the polynomial,

$$f = a_0 x_0^{n_0} + \cdots + a_r x_r^{n_r}$$

Let $m = \text{lcm}(n_0, \ldots, n_r)$ and denote by μ_n the group of n^{th}-roots of unity in $\mathbb{F}_q$. Then there is an action of the group,

$$\mu_{n_0} \times \cdots \times \mu_{n_r}$$

on X. However, the map,

$$\mu_{n_0} \times \cdots \times \mu_{n_r} \to \text{Aut}\,(X)$$

is not injective since X is defined as the quotient under the action,

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{\frac{m}{n_0}} x_0, \ldots, \lambda^{\frac{m}{n_r}} x_r)$$

therefore the kernel of the map

$$\mu_{n_0} \times \cdots \times \mu_{n_r} \to \text{Aut}\,(X)$$

is exactly the image of

$$\mu_m \to \mu_{n_0} \times \cdots \times \mu_{n_r}$$

under the map

$$\lambda \mapsto (\lambda^{\frac{m}{n_0}}, \ldots, \lambda^{\frac{m}{n_r}})$$

Therefore, we get a map,

$$G = (\mu_{n_0} \times \cdots \times \mu_{n_r}) / \mu_m \to \text{Aut}\,(X)$$

Since $G \circlearrowright X$ by functoriality it also acts on the middle cohomology,

$$G \circlearrowright H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)$$

Then G is abelian so its irreducible representations are all one-dimensional characters. Therefore, we get a decomposition into spaces on which G acts through a given character,

$$H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell) = \bigoplus_{\chi \in \widehat{G}} H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)(\chi)$$

Weil proved that in our case for each character χ we have,

$$\dim H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)(\chi) \leq 1$$

Furthermore, since G acts by automorphisms and the action of Frob is natural meaning that the action of Frob and G commute. Therefore, Frob preserves the irreducible decomposition of G. Since each factor is 1-dimensional,

$$\mathrm{Frob} \circlearrowright H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)$$

is just multiplication by a corresponding Frob eigenvalue α_χ. Furthermore, since if $[Z]$ is the class of a subvariety then $g \cdot [Z] = [g \cdot Z]$ so the action of G preserves the algebraic cycles. Therefore,

$$H^{2i}_{\text{alg}}(X, \mathbb{Q}_\ell) \subset H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell)$$

is a G-subrepresentation. Therefore, since each character space is 1-dimensional then the space of algebraic cycles is a sum of a subset of the characters. These are exactly the "algebraic characters". By the Tate conjecture, they are also the "supersingular characters" i.e. those characters such that $\alpha_\chi = \zeta q^{i/2}$.

Now we need to make the connection to the set $A_{\underline{n},q^f}$. To do this, we fix compatible isomorphisms $\mu_n \cong \mu_n(\mathbb{C})$ for each n dividing $q^f - 1$ (recall that $f = \mathrm{ord}_m(q)$. This just amounts to a choice of generator $g \in \mathbb{F}_{q^f}^\times$ which we identify with $\zeta_{q^f-1} = e^{\frac{2\pi i}{q^f-1}}$. Now for each i and a character,

$$\chi : \mu_{n_i} \to \mu_{n_i}(\mathbb{C})$$

consider the map,

$$\mathbb{F}_{q^f}^\times \to \mu_{n_i} \xrightarrow{\chi} \mu_{n_i}(\mathbb{C})$$

where the map is,

$$x \mapsto x^{\frac{q^f-1}{n_i}} \mapsto \chi(x^{\frac{q^f-1}{n_i}})$$

This gives a map,

$$\widehat{G} \to \mathrm{Hom}\left((\mathbb{F}_{q^f}^\times)^{r+1}, \mathbb{C}^\times\right)$$

The compatible isomorphism then enters when we identify,

$$\widehat{G} = \{(a_0, \dots, a_r) \mid a_i \in (\mathbb{Z}/n_i\mathbb{Z}) \text{ and } (m/n_0)a_0 + \cdots + (m/n_r)a_r \equiv 0 \mod m\}$$

By definition, the character corresponding to $a = 1$ is given by taking the generator of μ_{n_i} which is $g^{\frac{q^f-1}{n_i}}$ and sending to the generator of $\mu_{n_i}(\mathbb{C})$ which is ζ_{n_i} hence the corresponding character of $\mathbb{F}_{q^f}^\times$ is defined on the generator by

$$g \mapsto \zeta_{n_i}$$

which corresponds to $\alpha_i = \frac{1}{n_i}$ as we defined previously. Therefore, this identification of $\widehat{G}$ shows that its image in $\mathrm{Hom}\left((\mathbb{F}_{q^f}^\times)^{r+1}, \mathbb{C}^\times\right)$ is almost the set $A_{\underline{n},q^f}$. Notice we have "explained" where the sum condition comes from but not the conditional $0 < \alpha_i < 1$ i.e. corresponding to a condition that all $a_i \neq 0$. To do this let,

$$\widehat{G}^{\text{prim}} \subset \widehat{G}$$

be the subset where χ is nontrivial when restricted to each $\mu_{n_i} \to G$. The the image of,

$$\widehat{G}^{\text{prim}} \to \mathrm{Hom}\left((\mathbb{F}_{p^f}^\times)^{r+1}, \mathbb{C}^\times\right)$$

is exactly the set $A_{\underline{n},q^f}$. The reason geometrically for considering only primitive characters is, it turns out,

$$\dim H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)(\chi) = 1$$

exactly for the primitive characters and is zero otherwise.