1 Unimodular Lattices

Definition 1.0.1. Let $(V, \langle -, - \rangle)$ be a real inner-product space with $n = \dim V$ finite. Then a *lattice* is a subgroup $\Lambda \subset V$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$.

Definition 1.0.2. Let Λ be a lattice. Then we define the *dual Lattice*,

$$\Lambda^* \subset V^* \quad \text{where } \Lambda^* = \{\varphi \in V^* \mid \forall \gamma \in \Lambda : \varphi(\gamma) \in \mathbb{Z}\}$$

However, V is equipped with an inner product and under the natural isomorphism $V \xrightarrow{\sim} V^*$ defined by $v \mapsto \langle v, - \rangle$ we can identify,

$$\Lambda^* \subset V \quad \text{via} \quad \Lambda^* = \{v \in V \mid \forall \gamma \in \Lambda \mid \langle v, \gamma \rangle \in \mathbb{Z}\}$$

Thus we can write,

$$\begin{tikzcd} \Lambda^* \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}\,(\Lambda, \mathbb{Z}) \arrow[d] \\ V^* \arrow[r] & \mathrm{Hom}\,(\Lambda, \mathbb{R}) \end{tikzcd}$$

Definition 1.0.3. The *covolume* or DEFINE

Proposition 1.0.4. $|\Lambda| \cdot |\Lambda^*| = 1$

Proof. DO THIS!! ☐

Definition 1.0.5. A lattice Λ is,

(a) *integral* if $\langle \gamma, \gamma' \rangle \in \mathbb{Z}$ for all $\gamma, \gamma' \in \Lambda$

(b) *unimodular* if $|\Lambda| = 1$

(c) *even* if $||\gamma||^2 \in 2\mathbb{Z}$ for all $\gamma \in \Lambda$

(d) *self-dual* if $\Lambda^* = \Lambda$ inside V.

Lemma 1.0.6. A lattice Λ is self-dual if and only if Λ is integral and unimodular.

Proof. If Λ is integral, $\Lambda \subset \Lambda^*$ and if Λ is unimodular then $|\Lambda^*| = |\Lambda| = 1$ proving that $\Lambda = \Lambda^*$. Conversely, if $\Lambda = \Lambda^*$ then $|\Lambda| = |\Lambda^*| = 1$ and $\Lambda \subset \Lambda^*$ proving that Λ is unimodular and integral. ☐

Definition 1.0.7. Let Λ be a lattice. We define the theta function,

$$\Theta_\Lambda : \mathfrak{h} \to \mathbb{C}$$

via the infinite summation,

$$\Theta_\Lambda(\tau) = \sum_{\gamma \in \Lambda} e^{i\pi\tau||\gamma||^2}$$

Proposition 1.0.8. The summation form of Θ_Λ is everywhere absolutely convergent on $\mathfrak{h}$.

Proof. Notice that,

$$|e^{i\pi\tau||\gamma||^2}| = e^{-\pi||\gamma||^2 \,\mathrm{Im}(\tau)}$$

Since $\mathrm{Im}\,(\tau) > 0$ we see that $0 < e^{-\pi \mathrm{Im}(\tau)} < 1$ and therefore because the number of lattice points of bounded norm grows polynomially the sum is convergent. ☐

Theorem 1.0.9 (Poisson Summation)**.** Let $f : V \to \mathbb{C}$ be a Schwartz function with Fourier transform $\hat{f} : V^* \to \mathbb{C}$. Then,

$$\sum_{\gamma \in \Lambda} f(\gamma) = \frac{1}{|\Lambda|} \sum_{\varphi \in \Lambda^*} \hat{f}(\varphi)$$

Proposition 1.0.10. Let Λ be a lattice. For any $\tau \in \mathfrak{h}$,

$$\Theta_{\Lambda^*}(-1/\tau) = \left(\frac{\tau}{i}\right)^{\frac{n}{2}} |\Lambda| \cdot \Theta_\Lambda(\tau)$$

Proof. This is a direct application of Poisson summation for $f(v) = e^{i\pi\tau||v||^2}$. A direct calculation shows that,

$$\hat{f}(v) = \left(\frac{i}{\tau}\right)^{\frac{n}{2}} e^{-i\pi||v||^2/\tau}$$

Then,

$$\Theta_\Lambda(\tau) = \sum_{\gamma \in \Lambda} f(\gamma) = \frac{1}{|\Lambda|} \sum_{\varphi \in \Lambda^*} \hat{f}(\varphi) = \frac{1}{|\Lambda|} \left(\frac{i}{\tau}\right)^{\frac{n}{2}} \Theta_{\Lambda^*}(-1/\tau)$$

□

Corollary 1.0.11. If Λ is self-dual then,

$$\Theta_\Lambda(-1/\tau) = \left(\frac{\tau}{i}\right)^{\frac{n}{2}} \Theta_\Lambda(\tau)$$

Proposition 1.0.12. If Λ is integral then $\Theta_\Lambda(\tau+2) = \Theta_\Lambda(\tau)$. If Λ is even then $\Theta_\Lambda(\tau+1) = \Theta_\Lambda(\tau)$.

Proof. If $||\gamma||^2 \in \mathbb{Z}$ then,

$$e^{i\pi(\tau+2)||\gamma||^2} = e^{2\pi i||\gamma||^2} e^{i\pi\tau||\gamma||^2} = e^{i\pi\tau||\gamma||^2}$$

Likewise, if $||\gamma||^2 \in 2\mathbb{Z}$ then,

$$e^{i\pi(\tau+1)||\gamma||^2} = e^{\pi i||\gamma||^2} e^{i\pi\tau||\gamma||^2} = e^{i\pi\tau||\gamma||^2}$$

□

Corollary 1.0.13. If Λ is self-dual and even then Θ_Λ is modular.

Theorem 1.0.14. Let Λ be an even integral unimodular lattice. Then $8 \mid \dim \Lambda$.

Proof. Since integral unimodular lattices are self-dual we see that Θ_Λ is modular. □

Proof. Let $S, T \in \mathrm{SL}_2(\mathbb{Z})$ describe $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$. The relation $(ST)^3 = \mathrm{id}$ describes the trajectory,

$$\tau \mapsto -\frac{1}{\tau} \mapsto \frac{\tau - 1}{\tau} \mapsto \frac{\tau}{1-\tau} \mapsto \frac{1}{1-\tau} \mapsto \tau - 1 \mapsto \tau$$

Using the modularity properties,

$$\Theta_\Lambda(\tau) = \left(\frac{i}{\tau - 1}\right)^{\frac{n}{2}} \left(\frac{\tau - 1}{i\tau}\right)^{\frac{n}{2}} \left(\frac{\tau}{i}\right)^{\frac{n}{2}} \Theta_\Lambda(\tau) = \left(\frac{1}{i}\right)^{\frac{n}{2}} \Theta_\Lambda(\tau)$$

Since $\Theta_\Lambda(\tau) \neq 0$ because,

$$\Theta_\Lambda(i) = \sum_\Lambda e^{-\pi||\gamma||^2} > 0$$

and therefore, we must have $i^{\frac{n}{2}} = 1$ and hence n is divisible by 8. □

Remark. All these numbers lie in a wedge on the complex plane (indeed $\mathrm{Re}(z) > 0$) and thus the function $(-)^{\frac{n}{2}}$ is well-defined and is multiplicative.

2 Introduction

Remark. Following F. BEUKERS Irrationality proofs using modular forms.

3 The Proof

3.1 The Main Lemma

Lemma 3.1.1. Consider power series $f_0, \dots, f_k \in \mathbb{Q}[[t]]$ and let $a_j(n)$ be the n^{th} coefficient of f_k. Suppose that there are positive integers r, d such that for all n, j,

$$d^n \text{lcm}(1, \dots, n)^r a_j(n) \in \mathbb{Z}$$

Then suppose there are real numbers $\theta_1, \dots, \theta_k \in \mathbb{R}$ such that,

$$F(t) = f_0(t) + \theta_1 f_1(t) + \cdots + \cdots + \theta_k f_k(t) \in \mathbb{R}[[t]]$$

is a power series with radius of convergence $\rho > de^r$. Then either a least one of $\theta_1, \dots, \theta_k$ is irrational or F is a polynomial.

Proof. Suppose $\theta_1, \dots, \theta_k \in \mathbb{Q}$ then let D be the product of their denominators. Then $F \in \mathbb{Q}[[t]]$ and its coefficients satisfy,

$$c_n(F) = a_0(n) + \theta_1 a_1(n) + \cdots + \theta_k a_k(n)$$

and thus,

$$D\, d^n \, \text{lcm}(1, \dots, n)^r c_n(F) \in \mathbb{Z}$$

However, since the power series is convergent for $|t| = de^r$ we see that,

$$|c_n(F) t^n| = (de^r)^n c_n(F) \to 0$$

Furthermore, $\text{lcm}(1, \dots, n) \sim e^n$ and therefore, we see that,

$$D\, d^n \, \text{lcm}(1, \dots, n)^r c_n(F) \to 0$$

but it is an integer so we conclude that $c_n(F) = 0$ for $n \gg 0$ meaning that F is a polynomial. $\square$

Remark. To see why $\text{lcm}(1, \dots, n) \sim e^n$ we use the Chebyshev function,

$$\psi(x) = \sum_{p^k \le x} \log p = \sum_{n \le x} \Lambda(n)$$

where Λ is the von Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{else} \end{cases}$$

Then we see that,

$$\psi(x) = \sum_{p \le x} \lfloor \log_p(n) \rfloor \log p$$

$$e^{\psi(n)} = \prod_{p^k \le n} p = \prod_{\text{primes}} p^{\lfloor \log_p(n) \rfloor} = \text{lcm}(1, \dots, n)$$

Then the prime number theorem is equivalent to,

$$\psi(x) \sim x$$

3.2 Some Modualar Forms

The Eisenstein series,

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(n,m)\neq(0,0)} \frac{1}{(n\tau+m)^k}$$

are modular meaning that for any matrix,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

we have,

$$E_k(\gamma \cdot \tau) = E_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k E_k(\tau)$$

Define,

$$F(\tau) = \tfrac{1}{40}\left[E_4(\tau) - 36E_4(6\tau) - 7(4E_4(2\tau) - 9E_4(3\tau))\right]$$

(note that the 6τ is written as 36τ but I think this is a typo). and also we define,

$$E(\tau) = \tfrac{1}{24}\left[-5(E_2(\tau) - 6E_2(6\tau)) + 2E_2(2\tau) - 3E_2(3\tau)\right]$$

these are modular forms of weight 4 and 2 respectively and of level $\Gamma_1(6)$ meaning they transform under $\tau \mapsto -1/(6\tau)$ as,

$$F(-1/(6\tau)) = (6\tau)^4 F(\tau) \quad \text{ and } F(-1/(6\tau)) =$$

(FUCK WHAT!!!) (ETA FUNCTION!!) We also define the function,

$$t(\tau) = \left(\frac{\Delta(6\tau)\Delta(\tau)}{\Delta(3\tau)\Delta(2\tau)}\right)^{\frac{1}{2}} = q\prod_{0}^{\infty}(1-q^{6n+1})^{12}(1-q^{6n+5})^{-12}$$

which can more correctly be expressed in terms if the Dedekind η function (to remove the square-root ambiguity) is invariant under $\tau \mapsto -1/(6\tau)$ (SHOW THIS) DO THIS PROPERLY!! Finally, we define $f(\tau)$ as the solution to the differential equation,

$$f^{(3)}(\tau) = (2\pi i)^3 F(\tau)$$

with boundary condition $f(i\infty) = 0$.

3.3 The L-Function

Given a modular form f with a q-expansion,

$$f(\tau) = \sum_{n\geq 0} a_n q^n$$

we associate an L-function,

$$L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

For example, for the Eisenstein series E_{2k} we get,

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

Therefore,

$$L(E_{2k}, s) = -\frac{4k}{B_{2k}} \sum_{n \geq 1} \frac{\sigma_{2k-1}(n)}{n^s}$$

In particular, for $k = 2$ we get,

$$L(E_4, s) = 240 \sum_{n \geq 1} \frac{\sigma_3(n)}{n^s}$$

Let's try to understand this function. Like ζ is has an Euler product expression because σ_3 is multiplicative over coprime inputs. If $n = p_1^{e_1} \cdots p_r^{e_r}$ then we just need to understand,

$$\frac{\sigma_k(p^e)}{p^{es}} = p^{-es}\left(1 + p^k + p^{2k} + \cdots + p^{ek}\right) = p^{-es}\frac{p^{(e+1)k} - 1}{p^k - 1}$$

Therefore,

$$\begin{aligned}
L(E_4, s) &= 240 \prod_p \sum_{e \geq 0} p^{-es}\frac{p^{(e+1)k} - 1}{p^k - 1} \\
&= 240 \prod_p \frac{1}{p^k - 1} \sum_{e \geq 0} (p^{(e+1)k - es} - p^{-es}) \\
&= 240 \prod_p \frac{1}{p^k - 1} \cdot \left(\frac{p^k}{1 - p^{-(s-k)}} - \frac{1}{1 - p^{-s}} \right) = 240 \prod_p \frac{1}{p^k - 1} \cdot \frac{p^k - 1}{(1 - p^{-(s-k)})(1 - p^{-s})} \\
&= 240 \prod_p \frac{1}{1 - p^{-(s-k)}} \cdot \frac{1}{1 - p^{-s}} = 240\zeta(s-k)\zeta(s)
\end{aligned}$$

Therefore, for our modular form E we get the following L-function just by plugging in,

$$\begin{aligned}
L(F, s) &= 6 \sum_{n \geq 1} \left(\frac{\sigma_3(n)}{n^s} - 36\frac{\sigma_3(n)}{(6n)^s} - 28\frac{\sigma_3(n)}{(2n)^s} + 63\frac{\sigma_3(n)}{(3n)^s} \right) \\
&= 6(1 - 6^{2-s} - 7 \cdot 2^{2-s} + 7 \cdot 3^{2-s})\zeta(s)\zeta(s-3)
\end{aligned}$$

This 3 appearing in the ζ will be the key.

3.4 Some Computations

Proposition 3.4.1. Let,

$$F(\tau) = \sum_{n \geq 1} a_n q^n$$

be convergence for $|q| < 1$ such that for some positive integers k, N,

$$F(-1/(N\tau)) = -(-i\tau\sqrt{N})^{k+1} F(\tau)$$

for $\epsilon = \pm 1$. Let $f(\tau)$ be defined by the following Fourier series,

$$f(\tau) = \sum_{n \geq 1} \frac{a_n}{n^k} q^n$$

meaning exactly that f satisfies the differential equation,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^k f(\tau) = (2\pi i)^k F(\tau)$$

and define,

$$h(\tau) = f(\tau) - \sum_{0 \le r < \frac{1}{2}(k-1)} (2\pi i \tau)^r \frac{L(F, k-r)}{r!}$$

Then,

$$h(\tau) = (-1)^{k+1}(-i\tau\sqrt{N})^{k-1} h(-1/(N\tau))$$

We apply this with $k = 3$ and $N = 6$ because we have,

$$F(-1/(6\tau)) = -(6\tau^2)^2 F(\tau)$$

In this case,

$$h(\tau) = f(\tau) - L(F, 3) = f(\tau) - 6[1 - 6^{-1} - 7 \cdot 2^{-1} + 7 \cdot 3^{-1})\zeta(3)\zeta(0) = f(\tau) - \zeta(3)$$

which hopefully explains why some of these strange constants were chosen.

$$E(-1/(6\tau))\Big[f(-1/(6\tau)) - \zeta(3)\Big] = E(\tau)\Big[f(\tau) - \zeta(3)\Big]$$

this implies that $E(\tau)f(\tau)$ satisfies the hypothesis that $\mathrm{lcm}(1, \ldots, n)^3 E(\tau)f(\tau) \in \mathbb{Z}[[q]]$. (WHY IS THIS!!!)

Finally, I claim using some nasty calculations that,

$$G(\tau) := E(\tau)\Big[f(\tau) - \zeta(3)\Big]$$

has radius of convergence $\rho > e^3$ and hence by the main lemma one of 1 and $\zeta(3)$ must be irrational.