1 Basic Definitions and Examples

1.1 Genera

1.2 Riemann-Roch

1.3 Riemann–Hurwitz

2 Hyperelliptic Curves

Definition 2.0.1. A curve C is *hyperelliptic* if there exists a degree two map $f : C \to \mathbb{P}^1$.

Lemma 2.0.2. A curve C is hyperelliptic iff Ω^1_C is not very ample.

Proof. (DO THIS) $\square$

Proposition 2.0.3. Plane curves with $g > 1$ cannot be hyperelliptic.

Proof. Let $\iota : C \hookrightarrow \mathbb{P}^2$ be a plane curve. Then $\Omega^1_C = \iota^* \mathcal{O}_{\mathbb{P}^2}(d-3)$ where d is the degree of C. Since $g > 1$ we must have $d > 3$ and thus $\mathcal{O}_{\mathbb{P}^2}(d-3)$ is very ample defining the Veronese embedding $v : \mathbb{P}^2 \to \mathbb{P}^N$ s.t. $\mathcal{O}_{\mathbb{P}^2}(d-3) = v^* \mathcal{O}_{\mathbb{P}^N}(1)$. Then $v \circ \iota : C \to \mathbb{P}^N$ is an embedding such that $(v \circ \iota)^* \mathcal{O}_{\mathbb{P}^N}(1) = \Omega^1_C$. Thus Ω^1_C is very ample so C cannot be hyperelliptic. $\square$

Lemma 2.0.4. Let C have a $\mathfrak{g}^1_2$ then C is either hyperelliptic or rational.

Proof. Let D be a $\mathfrak{g}^1_2$ then $|D|$ defines a rational map $C \dashrightarrow \mathbb{P}^1$ of degree two. Suppose P were a basepoint of $|D|$ then $\dim |D - P| = 1$ which implies that C is rational because there is a rational degree one map $C \dashrightarrow \mathbb{P}^1$. $\square$

Proposition 2.0.5. Any genus 2 curve is hyperelliptic.

Proof. Consider the canonical divisor K_X which has $\deg K_X = 2g - 2 = 2$ and $\dim |K_X| = g - 1 = 1$ and thus gives a $\mathfrak{g}^1_2$. $\square$

3 Tangent Space

Definition 3.0.1. Let X be a scheme and $x \in X$ a point. Then we define:

(a) the geometric tangent space $T_x X = \mathrm{Spec}\left(\mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_x / \mathfrak{m}_x^2)\right)$

(b) the projectiveized tangent space $\mathbb{P}(T_x X) = \mathrm{Proj}\left(\mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_x / \mathfrak{m}_x^2)\right)$

(c) the geometric tangent cone $C_x X = \mathrm{Spec}\left(\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x})\right)$ where,

$$\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) = \bigoplus_{n=0}^{\infty} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}$$

(d) the projectiveized tangent cone $\mathbb{P}(C_x X) = \mathrm{Proj}\left(\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x})\right)$.

Remark. In particular, blowing up X at the sheaf of ideals $\mathscr{I}_x$ (defined as the subsheaf of $\mathcal{O}_X$ where evaluation in $\kappa(x)$ gives zero) gives the following,

$$\tilde{X} = \mathbf{Proj}_X \left(\bigoplus_{n=0}^{\infty} \mathscr{I}_x^n \right)$$

Choose an affine open neighborhood $x \in \mathrm{Spec}\,(A) = U \subset X$ then we see $\mathscr{I}_x|_U = \widetilde{\mathfrak{p}} \subset A$ is the prime corresponding to $x \in \mathrm{Spec}\,(A)$ and $\mathfrak{m}_x = \mathfrak{p}A_{\mathfrak{p}}$. Therefore, restricting $\pi : \tilde{X} \to X$ over U gives,

$$\mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \mathfrak{p}^n \right) \to \mathrm{Spec}\,(A)$$

and,

$$\mathrm{Bl}\mathfrak{p}A = \bigoplus_{n=0}^{\infty} \mathfrak{p}^n$$

is the blowup algebra which is a graded A-algebra. Consider the fiber over x,

$$\mathrm{Proj}\,(\mathrm{Bl}\mathfrak{p}A/\mathfrak{p}\mathrm{Bl}\mathfrak{p}A) \to \mathrm{Spec}\,(\kappa(x))$$

where we see,

$$\mathrm{Bl}\mathfrak{p}A/\mathfrak{p}\mathrm{Bl}\mathfrak{p}A = \bigoplus_{n=0}^{\infty} \mathfrak{p}^n = \mathbf{gr}_{\mathfrak{p}}\,(A)$$

and therefore $\tilde{X}_x \to \mathrm{Spec}\,(\kappa(x))$ is $\mathrm{Proj}\left(\mathbf{gr}_{\mathfrak{p}}\,(A)\right) \to \mathrm{Spec}\,(\kappa(x))$. In particular, the tangent cone is the fiber over x in the blowup.

Remark. The exact same construction shows that given a ring A and ideal $I \subset A$ the blowup $\mathrm{Proj}\,(\mathrm{Bl}IA) \to \mathrm{Spec}\,(A)$ where,

$$\mathrm{Bl}IA = \bigoplus_{n=0}^{\infty} I^n$$

is the blowup algebra, has fiber over the closed subscheme $V(I)$ equal to,

$$\mathrm{Proj}\,(\mathbf{gr}_I\,(A)) \to \mathrm{Spec}\,(A/I)$$

which is the projectized tangent cone of I.

Remark. We can generalize this further. For a sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ we can form the blowup,

$$\tilde{X} = \mathbf{Proj}_X \left(\bigoplus_{n=0}^{\infty} \mathscr{I}^n \right) \to X$$

Restricting to the closed subscheme $Z = V(\mathscr{I}) \subset X$ we find,

$$\mathbf{Proj}_Z \left(\bigoplus_{n=0}^{\infty} \mathscr{I}^n/\mathscr{I}^{n+1} \right) \to Z$$

but notice that the graded algebra,

$$(\mathcal{O}_X/\mathscr{I}) \otimes_{\mathcal{O}_X} \bigoplus_{n=0}^{\infty} \mathscr{I}^n = \bigoplus_{n=0}^{\infty} \mathscr{I}^n/\mathscr{I}^{n+1} = \bigoplus_{n=0} (\mathscr{I}/\mathscr{I}^2)^{\otimes n}/K = \mathrm{Sym}_{\mathcal{O}_Z}\left(\mathscr{I}/\mathscr{I}^2\right)$$

and $\mathcal{C}_{Z/X} = \mathscr{I}/\mathscr{I}^2$ is the conormal bundle (sheaf) so we find a pullback diagram,

$$\begin{tikzcd}
\mathbb{P}(\mathcal{C}_{Z/X}) \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\pi"] \\
Z \arrow[r] & X
\end{tikzcd}$$

and thus $\tilde{X} \to X$ is a projective bundle over Z and an isomorphism over $X \setminus Z$. We call $\mathbb{P}(\mathcal{C}_{Z/X})$ the projectiveized tangent cone of Z.

Proposition 3.0.2. In general $C_x X \hookrightarrow T_x X$ and this is an isomorphism if $x \in X$ is a regular point.

Proof. There is a surjective canonical map,

$$\mathrm{Sym}_{\kappa(x)}\Big(\mathfrak{m}_x/\mathfrak{m}_x^2\Big) \to \mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)$$

giving a closed embedding,

$$C_x X \hookrightarrow T_x X$$

When $\mathcal{O}_{X,x}$ is regular then $\mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right) \cong \kappa(x)[x_1, \dots, x_r]$ where $x_1, \dots, x_r \in \mathfrak{m}_x$ are a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $\kappa(x)$-vectorspace. Therefore, $\mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2) \xrightarrow{\sim} \mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)$ as graded rings and thus $C_x X \to T_x X$ is an isomorphism. □

Remark. Because the map is a graded map, the same holds projectivized $\mathbb{P}(C_x X) \hookrightarrow \mathbb{P}(T_x X)$.

(THIS IS WRONG BC COHEN ISOMORPHISM IS NOT CANONICAL) (MAKES SENSE BECAUSE ITS SUPPOSED TO BE LIKE NORMAL COORDINATES WHICH FOR MANIFOLDS REQUIRES A METRIC SAD)

Proposition 3.0.3. Let X be finite type over k and $x \in X$ be a closed point. There are canonical maps,

$$\widehat{T_x X} \leftarrow \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right) \to \mathrm{Spec}\left(\mathcal{O}_{X,x}\right) \to X$$

such that $\widehat{T_x X} \leftarrow \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right)$ is an isomorphism exactly when $x \in X$ is regular.

Proof. The canonical map,

$$A = \mathrm{Sym}_{\kappa(x)}\Big(\mathfrak{m}_x/\mathfrak{m}_x^2\Big) \to \mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)$$

is an isomorphism exactly when $x \in X$ is a regular point. By the Cohen structure theorem $\widehat{\mathcal{O}_{X,x}} = \kappa(x)[[x_1, \dots, x_r]]/I$ where $x_1, \dots, x_r \in \mathfrak{m}_x$ are a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$. Furthermore, $I = (0)$ exactly when $x \in X$ is regular. Therefore, consider the map,

$$\widehat{A} \to \widehat{\mathcal{O}_{X,x}}$$

defined on finite levels by,

$$A/\mathfrak{m}^n = \bigoplus_{k \le n} \mathrm{Sym}^k\Big(\mathfrak{m}_x/\mathfrak{m}_x^2\Big) \to \bigoplus_{k \le n} \mathfrak{m}$$

Taking spectra,

$$T_x X \leftarrow \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right)$$

and we see this is an isomorphism when $\mathcal{O}_{X,x}$ is regular. □

4 Formal Schemes

Definition 4.0.1. Let A be a ring and $I \subset A$ an ideal. Then the completion of A along I is,

$$\hat{A} = \varprojlim_n A/I^n$$

Furthermore, for any A-module M we can complete M along I to get,

$$\hat{M} = \varprojlim_n M/I^n M = \varprojlim_n (M \otimes_A A/I^n) = M \otimes_A \hat{A}$$

Proposition 4.0.2. Let A be a ring and $I \subset A$ an ideal and M an A-module. Then $\hat{M}$ satisfies the following universal property. Any map $\varphi : M \to N$ to a complete A-module N facrors uniquely as $M \to \hat{M} \xrightarrow{\hat{\varphi}} N$.

Proof. The kernel of $M \to N/I^n N$ contains $I^n M$ and thus factors as $M \to M/I^n M \to N/I^n N$. Taking inverse limits gives $M \to \hat{M} \to N$. Uniqueness follows from the fact that a map $\hat{M} \to M$ is determined completely by $\hat{M} \to M/I^n M \to N/I^n N$. $\square$

Lemma 4.0.3. Let A be a ring and $I \subset A$ an ideal. Then the units of $\hat{A}$ are exactly those elements which map to units under $\hat{A} \to A/I$.

Proof. Suppose that $u \in \hat{A}$ is a unit. Then clearly its image under $\hat{A} \to A/I$ is a unit. Conversely, suppose that $u \mapsto u_1 \in A/I$ is a unit. Then there exists $v_1 \in A/I$ s.t. $u_1 v_1 = 1$ so lifting v_1 we get $u_2 \tilde{v}_1 = 1 + r$ for $r \in I$ so we can take $r u_2 \tilde{v}_1 = r + r^2 = r$ and thus $u_2(\tilde{v}_1 - r\tilde{v}_1) = 1$. Write $v_2 = \tilde{v}_1 - r\tilde{v}_1$ and we lift to see $u_3 \tilde{v}_2 = 1 + r'$ for $r' \in I^2$ etc giving by induction an element $v \in \hat{A}$ such that $uv = 1$ in each A/I^n and thus in $\hat{A}$. $\square$

Lemma 4.0.4. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then $\hat{A} = \widehat{A_{\mathfrak{m}}}$ is local.

Proof. Consider,

$$\widehat{A_{\mathfrak{m}}} = \varprojlim_n (A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}}) = \varprojlim_n (A/\mathfrak{m}^n)_{\mathfrak{m}}$$

However, since $A/\mathfrak{m}^n$ is local with maximal ideal $\mathfrak{m}$ we see that $(A/\mathfrak{m}^n)_{\mathfrak{m}} = A/\mathfrak{m}^n$ and thus,

$$\widehat{A_{\mathfrak{m}}} = \varprojlim_n (A/\mathfrak{m}^n)_{\mathfrak{m}} = \varprojlim_n A/\mathfrak{m}^n = \hat{A}$$

$\square$

Remark. Localization does not, in general, behave nicely with completion. For example, let $A = \mathbb{Z}_p[x]$ and $\mathfrak{p} = (x)$. Then $\hat{A}_{\mathfrak{p}} = \widehat{\mathbb{Q}_p[x]_{(x)}} = \mathbb{Q}_p[[x]]$. However, $\hat{A} = \mathbb{Z}_p[[x]]$ and $\hat{A}_{\hat{\mathfrak{p}}} = \mathbb{Z}_p[[x]]_{(x)}$ which is a proper subring of $\mathbb{Q}_p[[x]]$ because it does not contain $1 + p^{-1}x + p^{-2}x^2 + \cdots$ and is this, in particular, not complete.

Lemma 4.0.5. Suppose that $(A, \mathfrak{m})$ is a local ring. Then $\mathrm{Spec}\left(\hat{A}\right) \to \mathrm{Spec}\,(A)$ is a homeomorphism.

(THIS IS TOTALLY FALSE IMPLIES A IS UNABRANCH AT LEAST)

Proof. The units in $\hat{A}$ are everything except the preimage of zero under $\hat{A} \to A/\mathfrak{m} = \kappa$. Therefore $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$ is the unique maximal ideal of $\hat{A}$ making A local. I claim that $\mathfrak{p} \mapsto \mathfrak{p}\hat{A}$ is an inverse to $\mathrm{Spec}\left(\hat{A}\right) \to \mathrm{Spec}\,(A)$. (DO THIS!!) $\square$

Example 4.0.6. Consider $X = \mathrm{Spec}\,(k[x,y]/(y^2 - x^2(x-1))) \subset \mathbb{A}^2_k$. Take $p = (x, y)$. We know X is connected and thus $\mathcal{O}_{X,p} = A_{\mathfrak{p}}$ has a unique minimal prime. However,

$$\widehat{\mathcal{O}_{X,p}} = \hat{A} = k[[x,y]]/(y^2 - x^2(x+1)) \cong k[[x,y]]/(x^2 - y^2) = k[[x,y]]/((y-x)(x+y)) \cong k[[u]] \times k[[v]]$$

which has two minimal primes (branches).

5 Multiplicity of a Point

5.1 Hilbert-Samuel Polynomial

Definition 5.1.1. Let $(A, \mathfrak{m}, \kappa)$ be a noetherian local ring and M a finite A-module. Then define the Hilbert function,

$$\chi_M(n) = \mathrm{length}_A\,(M/\mathfrak{m}^n M)$$

Remark. We abreviate $\mathrm{length}_A\,(M) = \ell_A(M)$. We recall the following facts from dimension theory.

Proposition 5.1.2. There is a numerical polynomial $P_M \in \mathbb{Q}[x]$ of degree $d = \dim M$ such that,

$$P_M(n) = \chi_M(n)$$

for all $n \gg 0$.

Mat, Theorem 13.4. ☐

Proposition 5.1.3. Given an exact sequence of finite A-modules,

$$0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$$

then,

$$d(M) = \max\{d(N), d(K)\}$$

and $P_M - P_N - P_K$ is a polynomial of degree strictly less than $d(N)$.

Proof. Consider the exact sequences,

$$0 \longrightarrow (\mathfrak{m}^n M + N)/\mathfrak{m}^n M \longrightarrow M/\mathfrak{m}^n M \longrightarrow K/\mathfrak{m}^n K \longrightarrow 0$$

$$0 \longrightarrow (\mathfrak{m}^n M \cap N)/\mathfrak{m}^n N \longrightarrow N/\mathfrak{m}^n N \longrightarrow (\mathfrak{m}^n M + N)/\mathfrak{m}^n M \longrightarrow 0$$

Therefore,

$$\chi_M(n) = \chi_K(n) + \ell((\mathfrak{m}^n M + N)/\mathfrak{m}^n M) = \chi_K(n) + \chi_N(n) - \ell((\mathfrak{m}^n M \cap N)/\mathfrak{m}^n N)$$

Furthermore, by Artin-Rees,

$$\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-c}(\mathfrak{m}^c M \cap N) \subset \mathfrak{m}^{n-c} N$$

for fixed c and $n \geq c$. Therefore,

$$\ell((\mathfrak{m}^n M \cap N)/\mathfrak{m}^n N) \leq \ell(\mathfrak{m}^{n-c} N/\mathfrak{m}^n N) = \chi_N(n) - \chi_N(n-c)$$

which is a polynomial φ of degree strictly less than $d(N)$. Furthermore, for $n \gg 0$ we have,

$$P_M(n) = P_N(n) + P_K(n) - \varphi(n)$$

and since φ has degree strictly smaller $d(N)$ we see that,

$$\mathrm{d}(M) = \max\{d(N), d(K)\}$$

$\square$

Definition 5.1.4. Let $a_d x^d$ be the leading term of P_M then the multiplicity of M is $e(M) = d!\, a_d$. Notice that,

$$e(M) = d! \lim_{n \to \infty} \frac{\chi_M(n)}{n^d}$$

We call $e(A)$ the multiplicity of the ring A. Furthermore, since,

$$\chi_M(n+1) - \chi_M(n) = \dim_\kappa (\mathfrak{m}^n M / \mathfrak{m}^{n+1} M)$$

therefore if $d > 0$

$$e(M) = (d-1)! \lim_{n \to \infty} \frac{\dim_\kappa (\mathfrak{m}^n M / \mathfrak{m}^{n+1} M)}{n^{d-1}}$$

Remark. If $d = 1$ then we find the following formula,

$$e(M) = \lim_{n \to \infty} \dim_\kappa (\mathfrak{m}^n M / \mathfrak{m}^{n+1} M)$$

and in particular,

$$e(A) = \lim_{n \to \infty} \dim_\kappa (\mathfrak{m}^n / \mathfrak{m}^{n+1})$$

Definition 5.1.5. Let X be a locally noetherian scheme and $x \in X$ a point. Then the multiplicity $m(x) = m(\mathcal{O}_{X,x})$ is defined as the multiplicity of the local ring $\mathcal{O}_{X,x}$,

$$m(x) = d! \lim_{n \to \infty} \frac{\ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n)}{n^d}$$

where $d = \dim \mathcal{O}_{X,x} = \operatorname{codim}\left(\overline{\{x\}}, X\right)$.

5.2 Schemes of Dimension One

Proposition 5.2.1. Let X be a connected noetherian T_0 space with $\dim X = 1$. Then X is equidimensional and for each $x \in X$ setting $Z_x = \overline{\{x\}}$ exactly one of the two equivalenes holds,

(a) $\operatorname{codim}(Z_x, X) = 0 \iff \dim Z_x = 1 \iff Z_x$ is maximal $\iff x$ is not closed

(b) $\operatorname{codim}(Z_x, X) = 1 \iff \dim Z_x = 0 \iff Z_x$ is not maximal $\iff x$ is closed

Proof. Let $Z_1, \dots, Z_n \subset X$ be the irreducible components of X. I claim that each irreducible component Z_i intersects some other irreducible component Z_j unless $n = 1$. Otherwise,

$$Z_i \cap \bigcup_{j \neq i} Z_j = \varnothing$$

but both sets are closed (critically using that the union is finite) and,

$$Z_i \cup \bigcup_{j \neq i} Z_j = X$$

which by the connectedness of X implies that one set is empty so $n = 1$. The case $n = 1$ is clear so suppose that $n > 1$. If $Z_i = \{x\}$ for a closed point $x \in X$ then there is some $Z_j \neq Z_i$ with $Z_i \cap Z_j \neq \emptyset$ and thus $x \in Z_j$ so $Z_i \subset Z_j$ so $Z_i = Z_j$ giving a contradiction. Then $\dim Z_i = 0$ or $\dim Z_i = 1$. However, if $\dim Z_i = 0$ then $Z_i = \{x\}$ since X is T_0 for a closed point $x \in X$ (since Z_i is closed). We showed this cannot happen so $\dim Z_i = 1$ proving that X is equidimensional.

Clearly, exacly one of $\operatorname{codim}(Z_x, X) = 0$ or $\operatorname{codim}(Z_x, X) = 1$ holds so we need to show the equivalences. We know $\operatorname{codim}(Z_x, X) = 0$ if and only if Z_x is an irreducible component (i.e. maximal). We showed that every irreducible component has dimension 1 so $\dim Z_x = 1$ iff Z_x is maximal and $\dim Z_x = 0$ iff Z_x is not maximal. If $\operatorname{codim}(Z_x, X) = 1$ then because,

$$\dim X \geq \operatorname{codim}(Z_x, X) + \dim Z_x$$

we see $\dim Z_x = 0$. Conversely, if $\dim Z_x = 0$ we have seen that Z_x is not maximal and thus $\operatorname{codim}(Z_x, X) \neq 0$ by the above equivalences so $\operatorname{codim}(Z_x, X) = 1$. If x is closed then $Z_x = \{x\}$ and we showed that Z_x is not maximal. Conversely, if Z_x is not maximal then $\dim Z_x = 0$ so x is closed. Finally, if Z_x is maximal then $\dim Z_x = 1$ so x is not closed and conversely if x is not closed then $\dim Z_x \neq 0$ by the previous equivalence so $\dim Z_x = 1$ so Z_x is maximal. □

Corollary 5.2.2. Let X be a connected noetherian scheme with $\dim X = 1$. Then for each closed point $x \in X$ we have $\dim \mathcal{O}_{X,x} = \operatorname{codim}(Z_x, X) = 1$ so the multiplicitly is,

$$m(x) = \lim_{n \to \infty} \dim_{\kappa(x)}(\mathfrak{m}_x^n / \mathfrak{m}_x^{n+1})$$

and for each non-closed point $\xi \in X$ or equivalently the generic point of an irreducible component, $\dim \mathcal{O}_{X,\xi} = \operatorname{codim}(Z_x, X) = 0$ and $\mathcal{O}_{X,\xi}$ is noetherian so $\mathcal{O}_{X,\xi}$ is Artin local meaning $\mathfrak{m}_\xi^n = 0$ for some n so the multiplicity satisfies,

$$m(\xi) = \ell_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi})$$

5.3 Normalization

(DO THIS FOR NOT INTEGRAL SCHEMES!!)

Proposition 5.3.1. Let X be a noetherian integral scheme with $\dim X = 1$ such that the normalization $\nu : \widetilde{X} \to X$ is a finite morphism. Then for each $x \in X$,

$$m(x) = \deg_x \nu = \dim_{\kappa(x)} \widetilde{X}_x$$

Proof. The generic point is clear since $m(\xi) = 1$ because A is reduced and ν is birational. Let $x \in X$ be a closed point and $A = \mathcal{O}_{X,x}$ then $\dim A = 1$. Let $K = \operatorname{Frac}(A)$ then $\widetilde{A} \subset K$ is the normalization. Consider the exact sequence of A-modules,

$$0 \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow Q \longrightarrow 0$$

Tensoring by K we see that $Q \otimes_A K = 0$. If $Q = 0$ we are immediately see that $P_A = P_{\widetilde{A}}$ as A-modules since $A = \widetilde{A}$. Otherwise, since $A \to \widetilde{A}$ is finite, Q is a finite A-module and thus $\mathrm{Supp}_A(Q)$ is closed and does not contain the generic point so $\dim \mathrm{Supp}_A(Q) = 0$. Indeed $B = A/\mathrm{Ann}_A(Q)$ is an Artinian ring and Q is a finite B-module and hence Artinian so $\ell_B(Q)$ is finite. Therefore, by Prop. 5.1.3, P_A and $P_{\widetilde{A}}$ have the same leading coefficient proving that,

$$m(x) = \lim_{n\to\infty} \frac{\ell_A(A/\mathfrak{m}^n)}{n} = \lim_{n\to\infty} \frac{\ell_A(\widetilde{A}/\mathfrak{m}^n\widetilde{A})}{n}$$

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals of $\widetilde{A}$ over $\mathfrak{m}$ and $B_i = \widetilde{A}_{\mathfrak{m}_i}$ then (Tag 02M0),

$$\ell_A(\widetilde{A}/\mathfrak{m}^n\widetilde{A}) = \sum_{i=1}^r \ell_{B_i}(B_i/\mathfrak{m}^n B_i) \cdot [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$$

However, B_i is a normal noetherian domain with $\dim B_i = 1$ and therefore a DVR. Let $\varpi_i \in \mathfrak{m}_i$ be a uniformizer and $\mathfrak{m}B_i = (\varpi_i^{e_i})$ for some integer e_i (if $\varpi = 0$ set $e_i = 1$) since B_i is a PID. Therefore, $\mathfrak{m}^n = \mathfrak{m}_i^{ne_i}$. In any DVR R with a uniformizer $\varpi \in R$ we have,

$$\ell_R(R/(\varpi^n)) = n$$

This is from the filtration,

$$(\varpi^n) \subset (\varpi^{n-1}) \subset \cdots \subset (\varpi) \subset R$$

and from the quotients we find,

$$\ell_R(R/(\varpi^n)) = \sum_{k=1}^n \ell_R((\varpi^k)/(\varpi^{k+1})) = n$$

because $(\varpi^k)/(\varpi^{k+1})$ is a one-dimensional κ-module. Therefore,

$$\ell_{B_i}(B_i/\mathfrak{m}^n) = ne_i$$

so we find,

$$\begin{aligned} m(x) &= \lim_{n\to\infty} \frac{\ell_A(\widetilde{A}/\mathfrak{m}^n)}{n} = \sum_{i=1}^r \lim_{n\to\infty} \frac{ne_i[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]}{n} = \sum_{i=1}^r e_i[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \\ &= \sum_{i=1}^r \mathrm{length}_{B_i}(B_i/\mathfrak{m})\,[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] = \mathrm{rank}_x(f_*\mathcal{O}_{\widetilde{X}}) = \dim_{\kappa(x)} \widetilde{X}_x = \deg_x \nu \end{aligned}$$

□

Proposition 5.3.2. Let $\widetilde{X} \to X$ be the normalization map as above. Then for nonzero $f \in \mathcal{O}_{X,x}$,

$$\mathrm{ord}_x(f) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f)) = \sum_{x' \in \nu^{-1}(x)} \mathrm{ord}_{x'}(f)[\kappa(x_i) : \kappa(x)]$$

Proof. Let $A = \mathcal{O}_{X,x}$ and $\widetilde{A}$ the normalization. Then $A \to \widetilde{A}$ is finite. Consider the submodules,

$$\begin{array}{ccc} A & \hookrightarrow & \widetilde{A} \\ \uparrow & & \uparrow \\ fA & \hookrightarrow & f\widetilde{A} \end{array}$$

where the maps are injective because A is a domain and $f \neq 0$. Furthermore, is an isomorphism of A-modules because $(f) \cong A$ is a flat A-module (easy to check directly). Therefore, from the diagram,

$$\ell_A(\widetilde{A}/f\widetilde{A}) = \ell_A(\widetilde{A}/fA) - \ell_A(f\widetilde{A}/fA) = \ell_A(\widetilde{A}/fA) - \ell_A(\widetilde{A}/A) = \ell_A(A/fA)$$

Finally, let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals of $\widetilde{A}$ over $\mathfrak{m}$ and $B_i = \widetilde{A}_{\mathfrak{m}_i}$ then by Tag 02M0,

$$\ell_A(\widetilde{A}/f\widetilde{A}) = \sum_{\mathfrak{m}_i} \ell_{B_i}(B_i/fB_i)[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] = \sum_{x' \in \nu^{-1}(x)} \mathrm{ord}_{x'}(f)[\kappa(x') : \kappa(x)]$$

□

Proposition 5.3.3. GENUS FORMULA

Proof. Consider $\nu : \widetilde{X} \to X$. Since the normalization is dominant there is an exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f_* \mathcal{O}_{\widetilde{X}} \longrightarrow \mathscr{C} \longrightarrow 0$$

Note that $f : S \to C$ induces an isomorphism $H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$ since it is a map of fields with the same (finite) dimension over k. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, \mathscr{C}) \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, \mathscr{C}) = 0$$

I claim that $H^1(S, \mathscr{C}) = 0$. Since f is birational, $\mathscr{C}$ is supported in codimension one. Thus, the map $H^1(C, \mathcal{O}_C) \twoheadrightarrow H^1(S, \mathcal{O}_S)$ is surjective but $g_a(C) = g_a(S)$ so these vectorspaces have the same dimension so $H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S)$ is an isomorphism. Thus, from the exact sequence we have $H^0(X, \mathscr{C}) = 0$. However, $\mathrm{Supp}_{\mathcal{O}_C}(\mathscr{C})$ is a closed ($\mathscr{C}$ is coherent) dimension zero subset i.e. finitely many discrete closed points. However, a sheaf supported on a discrete set of points is zero iff it has no global sections. Therefore, $\mathscr{C} = 0$ so $\mathcal{O}_C \xrightarrow{\sim} f_* \mathcal{O}_S$. In particular $\mathcal{O}_C(V) \xrightarrow{\sim} \mathcal{O}_S(U)$ is an isomorphism which implies that the map of affine schemes $f|_U : U \to V$ is an isomorphism. Since the affine opens V cover C we see that $f : S \to C$ is an isomorphism. In particular, C is smooth. □

6 Ramification

Definition 6.0.1. Given a map of schemes $f : X \to Y$ over S we get an exact sequence,

$$f^* \Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

We say that f is ramified at $x \in X$ if $(\Omega_{X/Y})_x \neq 0$ which is equivalent to the map $f_x : (\Omega_{Y/S})_{f(x)} \to (\Omega_{X/S})_x$ not being surjective. Furthermore define,

(a) The support of $\Omega_{X/Y}$ is called the *ramification locus*

(b) $f(\mathrm{Supp}_{\mathcal{O}_X}(\Omega_{X/Y}))$ is the *branch locus*

(c) if $\Omega_{X/Y} = 0$ then f is *formally unramified*

(d) f is *unramified* if f is formally unramified and locally of finite type

(e) f is *G-unramified* if f is formally unramified and locally of finite presentation

Lemma 6.0.2. Let $f : X \to Y$ be locally of finite type. Then the following are equivalent,

(a) the morphism $f : X \to Y$ is unramified at $x \in X$

(b) the stalk map $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces a finite seperable extension $\kappa(f(x))/\kappa(x)$ and $f_x^\#(\mathfrak{m}_{f(x)})\mathcal{O}_{X,x} = \mathfrak{m}_x$

Proof. (DO THIS PROOF) □

(DEFINE RAMIFICATION INDEX)

Remark. A ring map $\phi : A \to B$ corresponds to morphism of affine schemes,

$$\hat{\phi} : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$$

which is ramified at $\mathfrak{q} \subset B$ iff $(\Omega_{B/A})_{\mathfrak{q}} \neq 0$ or equivalently if $\mathfrak{q} \in \mathrm{Supp}_B\left(\Omega_{B/A}\right)$.

In particular if ϕ is finite then $\Omega_{B/A}$ is a finitely generated B-module so,

$$\mathrm{Supp}_B\left(\Omega_{B/A}\right) = V(\mathrm{Ann}_B\left(\Omega_{B/A}\right))$$

and thus $\mathfrak{q}$ is ramified iff $\mathfrak{q} \supset \mathrm{Ann}_B\left(\Omega_{B/A}\right)$. This motivates the following definition.

Definition 6.0.3. Let $\phi : A \to B$ be finite. Then define the different $\delta_{B/A} = \mathrm{Ann}_B\left(\Omega_{B/A}\right)$.

Remark. The important fact about the different is that it classifies ramification in the sense that ϕ is ramified at $\mathfrak{q}$ (or we say $\mathfrak{q}$ ramifies) iff $\delta_{B/A} \subset \mathfrak{q}$.

Corollary 6.0.4. Let $\phi : A \to B$ be a finite map with B a Dedekind domain. Then only finitely many points ramify.

6.1 Ramification For Curves

Definition 6.1.1. We say a scheme X is *normal* if each point $x \in X$ that $\mathcal{O}_{X,x}$ is normal i.e. an integrally closed local domain.

Lemma 6.1.2. Any normal local ring of dimension one is a DVR.

Definition 6.1.3. If X is a normal curve then each $\mathcal{O}_{X,x}$ is a DVR so we may choose a uniformizer π_x. For a morphism of normal cuves $f : X \to Y$. Since $\mathfrak{m}_x = (\pi_x)$ clearly the ramification index is the power e such that $f_x^\#(\pi_{f(x)}) = u\pi_x^e$ for some $u \in \mathcal{O}_{X,x}^\times$.

Proof. □

6.2 Ramification for Dedekind Domains

Proof. Since B is Dedekind domain there is a finite unique factorization of $\delta_{B/A}$ into prime ideals. These are the only primes lying about $\delta_{B/A}$ and thus exactly the set of primes which ramify of which there are finitely many. □

Proposition 6.2.1. Let $\phi : A \to B$ be a finite inclusion of Dedekind domains with finite residue fields. Then $\mathfrak{q}$ ramifies iff the prime $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ extends to the ideal $\mathfrak{p}B$ with factorization,

$$\phi(\mathfrak{p})B = \prod_{i=1}^{n} \mathfrak{q}_i^{e_i}$$

with $\mathfrak{q}_i$ distinct, $\mathfrak{q}_0 = \mathfrak{q}$, and $e_i > 1$.

Proof. At a point $\mathfrak{p} \subset A$ the residue field $\kappa(\mathfrak{p}) = A/\mathfrak{p}$ is a finite field which is perfect so $\kappa(\mathfrak{q})/\kappa(\hat{\phi}(\mathfrak{q}))$ is automatically finite seperable. Thus $\mathfrak{q} \subset B$ is unramified iff,

$$\phi(\phi^{-1}(\mathfrak{q})A_{\mathfrak{p}})B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$$

Since $B_{\mathfrak{q}}$ is also Dedekind, by unique factorization of ideals this is equivalent to $e_0 = 1$ since localizing the above factorization gives,

$$\phi(\mathfrak{p})B_{\mathfrak{q}} = (\phi(\mathfrak{p})B)_{\mathfrak{q}} = \prod_{i=1}^{n} \mathfrak{q}_i^{e_i} B_{\mathfrak{q}} = \mathfrak{q}_0^{e_0} B_{\mathfrak{q}} = \mathfrak{q}^{e_0} B_{\mathfrak{q}}$$

$\square$

Proposition 6.2.2. DIFFERENT IN TERMS OF TRACE

Lemma 6.2.3. If $B = A[t]/(f(t))$ then,

$$\begin{aligned} \Omega_{B/A} &= (B \cdot \mathrm{d}t)/(f'(t) \cdot \mathrm{d}t) \\ \delta_{B/A} &= (f'(t)) \subset B \end{aligned}$$

Proof. $\Omega_{B/A}$ is generated by $\mathrm{d}x$ for $x \in B$. For any $g(t) \in A[t]/(f(t))$ then by the Leibniz relation, $\mathrm{d}g(t) = g'(t)\mathrm{d}t$. Thus, $\Omega_{B/A}$ is generated over B by $\mathrm{d}t$. Furthermore, $f(t) = 0$ so $f'(t)\mathrm{d}t = 0$. This is the only relation. Furthermore,

$$\delta_{B/A} = \mathrm{Ann}_B\left(\Omega_{B/A}\right) = (f'(t))$$

by the following lemma. $\square$

Lemma 6.2.4. Let A be a ring and B an A-algebra with structure map $\phi : A \to B$ then $\mathrm{Ann}_A(B) = \ker \phi$.

Proof. An element $a \in \mathrm{Ann}_A(B)$ iff $\phi(a)b = 0$ for all $b \in B$. In particular,

$$a \in \mathrm{Ann}_A(B) \iff \phi(a) \cdot 1_B = 0 \iff \phi(a) = 0 \iff a \in \ker \phi$$

$\square$

Corollary 6.2.5. If $K/\mathbb{Q}$ is a number field with $\mathcal{O}_K = \mathbb{Z}[\alpha]$ and let α have minimal polynomial $f \in \mathbb{Z}[X]$. Then a prime $\mathfrak{p} \subset \mathcal{O}_K$ ramifies iff $f'(\alpha) \in \mathfrak{p}$.

Proof. We have $\mathcal{O}_K = \mathbb{Z}[\alpha]/(f(\alpha))$ so $\delta_{\mathcal{O}_K/\mathbb{Z}} = (f'(\alpha)) \subset \mathbb{Z}[\alpha]/(f(\alpha))$. Then we know that $\mathfrak{p}$ ramifies iff $\mathfrak{p} \supset \delta_{\mathcal{O}_K/\mathbb{Z}}$. $\square$

6.3 The Discriminant of an Extension

Definition 6.3.1. Let $\phi : A \to B$ be a finite map of Dedekind domains with finite residue fields. Then we define the ideal norm as the homomorphism $\mathcal{N}_{B/A} : \mathscr{I}_B \to \mathscr{I}_A$ of the ideal groups which on the prime ideals which generate $\mathscr{I}_B$ acts via,

$$\mathcal{N}_{B/A}(\mathfrak{q}) = \mathfrak{p}^{[B/\mathfrak{q}:A/\mathfrak{p}]}$$

where $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.

Definition 6.3.2. Then we define the relative discriminant $\Delta_{B/A} = \mathcal{N}_{B/A}(\delta_{B/A}) \subset A$.

Proposition 6.3.3. Primes $\mathfrak{p} \subset A$ ramify iff $\mathfrak{p} \supset \Delta_{B/A}$.

Proof. We write,

$$\mathfrak{p}B = \prod_{i=1}^{n} \mathfrak{q}_i^{e_i}$$

$\mathfrak{p}$ ramifies exactly when some $e_i > 1$ in which case we know $\mathfrak{q}_i \supset \delta_{B/A}$ and thus,

$$\mathfrak{p} \supset \mathfrak{p}^{[B/\mathfrak{q}:A/\mathfrak{p}]} \supset \mathcal{N}_{B/A}(\delta_{B/A}) = \Delta_{B/A}$$

Conversely, suppose that $\mathfrak{p}$ does not ramify then we must have $e_i = 1$ for all i. Then $\mathfrak{q}_i \not\supset \delta_{B/A}$ so by unique factorization, no primes dividing $\delta_{B/A}$ lie above $\mathfrak{p}$ which implies that, $\Delta_{B/A} = \mathcal{N}_{B/A}(\delta_{B/A})$ does not contain $\mathfrak{p}$ in its factorization. Thus, by the uniqueness of factorization,

$$\mathfrak{p} \not\supset \Delta_{B/A}$$

□

7 Maps between Curves

7.1 Maps of a Proper Curve are Finite

Theorem 7.1.1. Let C be a proper curve over k and X is separated of finite type over k. Then any nonconstant map $f : C \to X$ over k is finite.

Proof. Since $C \to \mathrm{Spec}\,(k)$ is proper and $X \to \mathrm{Spec}\,(k)$ is separated then by Tag 01W6 the map $f : C \to X$ is proper. The fibres of closed points $x \in X$ are proper closed subschemes $C_x \hookrightarrow C$ (since if $C_x = C$ then $f : C \to X$ would be the constant map at $x \in X$) and thus finite since proper closed subsets of a curve are finite. Now I claim that if the fibres $f^{-1}(x)$ are finite at closed points $x \in X$ then all fibres are finite. Assuming this, $f : C \to X$ is proper with finite fibres and thus is finite by Tag 02OG.

To show the claim consider,

$$E = \{x \in X \mid \dim C_x = 0\}$$

Since C is Noetherian, $\dim C_x = 0$ iff C_x is finite (suffices to check for affine schemes since quasi-comact and dimension zero Noetherian rings are exactly Artinian rings which have finite spectrum). Then E is locally constructible by Tag 05F9 and contains all the closed points of X. Since X is finite type over k then X is Jacobinson which implies that E is dense in every closed set. Then for any point $\xi \in X$ then $Z = \overline{\{\xi\}}$ is closed and irreducible with generic point ξ and thus $E \cap Z$ is dense in Z. Then by Tag 005K we have $\xi \in E$ so $E = X$ proving that all fibres are finite. □

Remark. The only facts about C that I used were that $C \to \mathrm{Spec}(k)$ is proper and that C is irreducible of dimension one. The second two properties are needed for the following to hold.

Lemma 7.1.2. If X is an irreducible Noetherian scheme of dimension one then every nontrivial closed subset of X is finite.

Proof. Since X is quasi-compact it suffices to show this property for affine schemes $X = \mathrm{Spec}(A)$ with $\dim A = 1$ and prime nilradical. Any nontrivial closed subset is of the form $V(I)$ for some proper radical ideal $I \subset X$ with $I \supsetneq \mathrm{nilrad}(A)$. Then $\mathbf{ht}(I) = 1$ since any prime above I must properly contain $\mathrm{nilrad}(A)$ and thus have height at least one but $\dim A = 1$. Then,

$$\mathbf{ht}(I) + \dim A/I \le \dim A$$

so $\dim A/I = 0$. Since A is Noetherian so is A/I but $\dim A/I = 0$ and thus A/I is Artianian. Therefore $\mathrm{Spec}(A/I)$ is finite proving the proposition. □

Remark. Since $C \to \mathrm{Spec}(k)$ is proper it is finite type over k and thus C is Noetherian.

Remark. The condition that C be proper is necessary. Consider the map $\mathbb{G}_m^k \coprod \mathbb{A}_k^1 \to \mathbb{A}_k^1$ via $k[x] \to k[x, x^{-1}]$ and the identity. This is clearly surjective and finitely generated since on rings it is,

$$k[x] \to k[x, x^{-1}] \times k[x]$$

Furthermore, this map is quasi-finite since the fibers have at most two points. To see this, consider, $y = (x - a) \in \mathrm{Spec}(k[x])$ then $\kappa(y) = k[x]/(x - a)$ and the fibre is,

$$\begin{aligned}
X_y &= \mathrm{Spec}\left((k[xx^{-1}] \times k[x]) \otimes_{k[x]} k[x]/(x-a)\right) \\
&= \mathrm{Spec}\left(k[x, x^{-1}]/(x-a) \times k[x]/(x-a)\right) \\
&= \mathrm{Spec}\left(k[x, x^{-1}/(x-a)\right) \coprod \mathrm{Spec}(k[x]/(x-a)) \\
&= \begin{cases} \mathrm{Spec}(k) & a = 0 \\ \mathrm{Spec}(k) \coprod \mathrm{Spec}(k) & a \neq 0 \end{cases}
\end{aligned}$$

However, this map is not closed since $\mathbb{G}_m^k \subset \mathbb{G}_m^k \coprod \mathbb{A}_k^1$ is closed but its image is $\mathbb{A}_k^1 \setminus \{0\}$ which is not closed. Thus the map cannot be finite. In particular,

$$k[x, x^{-1}] = \bigoplus_{n \ge 0} x^{-n} k[x]$$

so $k[x, x^{-1}]$ is not a finitely-generated $k[x]$-module.

7.2 Maps of Normal Curves Are Flat

Lemma 7.2.1. Let X be an integral scheme with generic point $\xi \in X$ and $\mathscr{F} \to \mathscr{G}$ a map of $\mathcal{O}_X$-modules,

(a) if $\mathscr{F}$ is locally free then $\mathscr{F} \to \mathscr{G}$ is injective iff $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is injective

(b) if $\mathscr{F}$ is invertible then $\mathscr{F} \to \mathscr{G}$ is injective iff $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is nonzero.

Proof. Since $\xi \in U$ for each nonempty open we have a diagram,

$$\begin{CD} \mathscr{F}(U) @>>> \mathscr{G}(U) \\ @VVV @VVV \\ \mathscr{F}_\xi @>>> \mathscr{G}_\xi \end{CD}$$

therefore it suffices to show the map $\mathscr{F}(U) \to \mathscr{F}_\xi$ is injective since then injectivity of $\mathscr{F}_\xi \to \mathscr{G}_\xi$ will imply injectivity of $\mathscr{F}(U) \to \mathscr{G}(U)$ for each U. Choose an affine open cover $U_i = \mathrm{Spec}\,(A_i)$ trivializing $\mathscr{F}$. $\mathscr{F}|_{U_i \cap U} \cong \mathcal{O}_X^{\oplus n}|_{U_i \cap U}$ but X is integral so the restriction $\mathscr{F}(U_i \cap U) \to \mathscr{F}_\xi$ is simply $A_i^n \to \mathrm{Frac}\,(A)^n$ which is injective since A_i is a domain. Thus if $s \in \mathscr{F}(U)$ maps to zero in $\mathscr{F}_\xi$ then $s|_{U_i \cap U} = 0$ so $s = 0$ since U_i form a cover.

The second follows from the first since we need only to show that $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is injective. However, $\mathscr{F}_\xi$ is a rank-one free module over the field $K(X) = \mathcal{O}_{X,\xi}$. Thus every nonzero map $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is injective. $\square$

Lemma 7.2.2. Let $f : X \to Y$ be a conconstant map of curves. Then f is dominant.

Proof. Let $\xi \in X$ be the generic point and consider $f(\xi) \in Y$. Suppose that $f(\xi)$ is a closed point. Then $f(X) = f(\overline{\{\xi\}}) \subset \overline{f(\xi)} = f(\xi)$ so f is constant. Therefore, we must have $f(\xi)$ a nonclosed point. But $\dim Y = 1$ and irreducible so any point is either closed or the generic point of the unique irreducible component. Therefore, $f(\xi) = \eta$ the generic point so f is dominant. $\square$

Proposition 7.2.3. Let X and Y be curves over k with Y normal. Then any nonconstant map $f : X \to Y$ is flat.

Proof. We need to check that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. Since Y is a normal curve $\mathcal{O}_{Y,y}$ is a Noetherian domain (Y is integral finite type over k) integrally closed (Y is normal) and dimension at most one ($\dim Y = 1$) therefore $\mathcal{O}_{Y,y}$ is a local Dedekind domain or a field so $\mathcal{O}_{Y,y}$ is a DVR or a field. Then by Tag 0539, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module iff it is torsion-free. However, $\mathcal{O}_{X,x}$ is a domain so it is a torsion-free $\mathcal{O}_{Y,f(x)}$-module iff $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.

Since f is dominant $f(\xi) = \eta$ (the generic points). Then $\mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ is a map of fields which is automatically injective so $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective because Y is integral proving that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective. $\square$

Remark. Morphisms of varieties are automatically finitely presented since curves are finite type over k so morphisms between them are locally finite type but Y is Noetherian so a locally finite type map is finitely presented. Furthermore, X is Noetherian so morphisms from it are automatrically quasi-compact and quasi-separated.

Proposition 7.2.4. Nonconstant maps of curves $f : X \to Y$ with Y normal are smooth iff unramified iff étale iff $\Omega_{X/Y} = 0$.

Proof. Maps of curves are automatically finitely presented. Furthermore, nonconstant maps of curves with Y normal are flat. Furthermore, we have seen that nonconstant maps of curves are quasi-finite so $\dim X_{f(x)} = 0$. Therefore, f is smooth iff $\Omega_{X/Y} = 0$ iff unramified but étale is smooth an unramified so we see smooth iff étale. $\square$

Lemma 7.2.5. Let $X \to Y$ be a nonconstant map of curves with $K(X)/K(Y)$ separable and Y smooth. Then there is an exact sequence,

$$0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Therefore, f is étale iff $f^*\Omega_Y \to \Omega_X$ is an isomorphism.

Proof. $K(X)/K(Y)$ is an extension of fields of transcendence degree one over k so it must be algebraic. Furthermore, both are finitely-generated field extensions of k so the algebraic extension $K(X)/K(Y)$ is finite. Then $(\Omega_{X/Y})_\xi = \Omega_{K(X)/K(Y)}$ which is zero iff $K(X)/K(Y)$ is separable. Thus, the standard exact sequence gives $(f^*\Omega_Y) \twoheadrightarrow (\Omega_X)_\xi$ because $(\Omega_{X/Y})_\xi = 0$. Furthermore, $f^*\Omega_Y$ is a line bundle since Y is smooth so we conclude that $f^*\Omega_Y \to \Omega_X$ is an injection since it is nonzero on the generic fiber (Lemma 7.2.1). $\square$

8 Finite Maps

Definition 8.0.1. A morphism $f : X \to Y$ of schemes if *finite* if it is affine and for every affine open $V \subset Y$ then $U = f^{-1}(V)$ is affine and the ring map associated to $U \to V$ is finite.

Proposition 8.0.2. Closed immersions are finite.

Proof. The map $A \to A/I$ is finite. $\square$

Proposition 8.0.3. Finite maps are preserved under base change.

Proposition 8.0.4. Finite maps are closed and thus universally closed.

Proposition 8.0.5. The following are equivalent for a map of schemes $f : X \to Y$

(a) f is finite

(b) f is affine and proper.

Proof. They are affine and thus separated, finite and thus finite type, and universally closed. $\square$

Proposition 8.0.6. Let $f : X \to Y$ be finite and $y \in Y$. Then the fiber is affine, zero dimensional, has finitely many points, and explicitly,

$$X_y = \mathrm{Spec}\left((f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)\right)$$

Furthermore,

$$\mathrm{rank}_y\left(f_*\mathcal{O}_X\right) = \sum_{x \in f^{-1}(y)} \mathrm{length}_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}\right) \cdot [\kappa(x) : \kappa(y)]$$

Proof. Let $f : X \to Y$ be finite then locally we have affine opens $V = \mathrm{Spec}\,(B) \subset Y$ and $U = f^{-1}(V) = \mathrm{Spec}\,(A)$ and the map $B \to A$ is finite. Then $(f_*\mathcal{O}_X)|_V = \widetilde{A}$ as a B-module. Choose a point $y \in Y$ corresponding to a prime $\mathfrak{p} \in \mathrm{Spec}\,(B)$. Consider the fiber $X_y = X \times_Y \mathrm{Spec}\,(\kappa(y))$. Because $U = f^{-1}(V)$ is affine, the fiber $X_y \subset \mathrm{Spec}\,(A)$ and thus,

$$X_y = \mathrm{Spec}\,(A) \times_{\mathrm{Spec}(B)} \mathrm{Spec}\,(\kappa(y)) = \mathrm{Spec}\,(A \otimes_B \kappa(y)) = \mathrm{Spec}\,((A/\mathfrak{p}A)_\mathfrak{p})$$

where $\kappa(y) = (B/\mathfrak{p}B)_\mathfrak{p}$. So set $R = A \otimes_B \kappa(y) = (A/\mathfrak{p}A)_\mathfrak{p}$ then,

$$R = A \otimes_B (B/\mathfrak{p}B)_\mathfrak{p} = A \otimes_B B_\mathfrak{p} \otimes_{B_\mathfrak{p}} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = A_\mathfrak{p} \otimes_{B_\mathfrak{p}} \kappa(y) = (f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$$

Since A is a finite B-module, R is a finite $\kappa(\mathfrak{p})$-module so R is an artinian ring. Thus $X_y = \mathrm{Spec}\,(R)$ has finitely many points and $\dim X_y = 0$. Furthermore,

$$\mathrm{rank}_y\,(f_*\mathcal{O}_X) = \dim_{\kappa(y)}\Big((f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)\Big)$$

and by our results on artinian k-algebras,

$$\dim_{\kappa(y)} R = \sum_{\mathfrak{m}_i \in \mathrm{Spec}(R)} \mathrm{length}_{R_{\mathfrak{m}_i}}\,(R_{\mathfrak{m}_i}) \cdot \dim_{\kappa(y)}(R/\mathfrak{m}_i)$$

However, the prime (maximal) ideals $\mathfrak{p}_x \in \mathrm{Spec}\,(R)$ correspond to points $x \in f^{-1}(y)$ furthermore,

$$R_{\mathfrak{m}_x} = (A_{\mathfrak{p}_x}/\mathfrak{p}A_{\mathfrak{p}_x}) = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$$

since $\mathfrak{p}A_{\mathfrak{p}_x} = \mathfrak{p}B_{\mathfrak{p}}A_{\mathfrak{p}_x} = \mathfrak{m}_y A_{\mathfrak{p}_x} = \mathfrak{m}_y\mathcal{O}_{X,x}$. Furthermore, since $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a surjection viewing $R_{\mathfrak{p}_x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ as a $\mathcal{O}_{X,x}$-module gives,

$$\mathrm{length}_{R_{\mathfrak{p}_x}}\,(R_{\mathfrak{p}_x}) = \mathrm{length}_{\mathcal{O}_{X,x}}\,(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})$$

Finally, $R/\mathfrak{p}_x = \mathcal{O}_{X,x}/\mathfrak{m}_x = \kappa(x)$ and thus we find,

$$\mathrm{rank}_y\,(f_*\mathcal{O}_X) = \sum_{x \in f^{-1}(y)} \mathrm{length}_{\mathcal{O}_{X,x}}\,(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) \cdot [\kappa(x) : \kappa(y)]$$

$\square$

Lemma 8.0.7. Let $A \hookrightarrow B$ be a finite inclusion of domains. Then $\mathrm{Frac}\,(B) = A^{-1}B$ and is a finite extension of $\mathrm{Frac}\,(A)$.

Proof. Since $A \to B$ is finite the map $\mathrm{Frac}\,(A) \to A^{-1}B$ is finite. However, $A^{-1}B$ is a domain finite dimensional over the field $\mathrm{Frac}\,(A)$ and thus $A^{-1}B$ is a field. However, $A^{-1}B \subset \mathrm{Frac}\,(B)$ so $\mathrm{Frac}\,(B) = A^{-1}B$. $\square$

Proposition 8.0.8. Let $f : X \to Y$ be a finite dominant map of integral schemes with generic points $\xi \in X$ and $\eta \in Y$. Then we have,

$$\deg f = \mathrm{rank}_\eta(f_*\mathcal{O}_X)$$

Proof. The map $\mathcal{O}_{Y,\eta} \to (f_*\mathcal{O}_X)_\eta$ is an injective finite map of domains because f is dominant. Therefore,

$$\mathrm{rank}_\eta(f_*\mathcal{O}_X) = \dim_{\kappa(\eta)}\Big((f_*\mathcal{O}_X) \otimes_{\mathcal{O}_{Y,\eta}} \kappa(\eta)\Big) = \dim_{K(Y)} K(Y)^{-1}(f_*\mathcal{O}_X)_\eta$$

However, the map $(f_*\mathcal{O}_X)_\eta \to \mathcal{O}_{X,\xi}$ is taking the fraction field $K(X) = \mathcal{O}_{X,\xi} = \mathrm{Frac}\,((f_*\mathcal{O}_X)_\eta)$ so by the previous lemma,

$$\mathrm{rank}_\eta(f_*\mathcal{O}_X) = \dim_{K(Y)} K(X) = [K(X) : K(Y)] = \deg f$$

$\square$

8.1 Finite Locally Free Morphisms

Definition 8.1.1. A morphism $f : X \to Y$ is *finite locally free* if f is affine and $f_* \mathcal{O}_X$ is a finite locally free as a $\mathcal{O}_Y$-module.

Proposition 8.1.2. A morphism $f : X \to Y$ is finite locally free iff f is finite, flat, and locally of finite presentation.

Proof. It suffices to show that if $A \to B$ is finite then B is locally free iff it is flat and finitely presented as an A-module. We know that finite locally free implies flat and locally finitely presented[1] (thus finitely presented). Conversely if B is flat and finitely presented[2] then it is projective (see Tag 00NX) and hence locally free. $\square$

Proposition 8.1.3. Let $f : X \to Y$ be a finite flat dominant map of integral schemes. Then for any $y \in Y$ we have,

$$\sum_{x \in f^{-1}(y)} \mathrm{length}_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x} \right) \cdot [\kappa(x) : \kappa(y)] = \deg f$$

we call $e_x = \mathrm{length}_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x} \right)$ the ramification degree and then,

$$\sum_{x \in f^{-1}(y)} e_x \cdot [\kappa(x) : \kappa(y)] = \deg f$$

Proof. Since $f_* \mathcal{O}_X$ is finite locally free and Y is connected, the sheaf $f_* \mathcal{O}_X$ has constant rank and thus $\mathrm{rank}_y(f_* \mathcal{O}_X) = \mathrm{rank}_\eta(f_* \mathcal{O}_X)$. Using our previous results proves the claim. $\square$

8.2 Ramification

9 Interesting Flasque Resolutions on Curves

9.1 Godement Resolution

For any abelian sheaf $\mathscr{F}$ on a space X we can consider it Godement resolution. Abstractly, take the continuous map $f : X_{\text{dis}} \to X$ from X given the discrete topology. Then the first stage of the Godement resolution is,

$$\mathscr{F} \to f_* f^* \mathscr{F}$$

Furthermore, since $f^* \mathscr{F}$ is an abelian sheaf on a discrete space it is flasque and f_* preserves flasqueness so $f_* f^* \mathscr{F}$ is flasque. Continuing gives a cosimplicial sheaf $\mathcal{G}^p(\mathscr{F}) = (f_* f^*)^p \mathscr{F}$ on X with coface maps given by the natural transformation $\mathrm{id} \to f_* f^*$ and codegeneracy maps given by contracting between pairs $(f_* f^*)(f_* f^*)$ via the natural transformation $f^* f_* \to \mathrm{id}$. The associated complex is then a flasque resolution of $\mathscr{F}$.

Remark. The above construction also works in the category of $\mathcal{O}_X$-modules on a ringed space by pulling back to $(X_{\text{dis}}, \mathcal{O}_{X_{\text{dis}}})$ where $\mathcal{O}_{X_{\text{dis}}} = f^{-1} \mathcal{O}_X$.

[1] it is finitely presented as an A-algebra because it is finitely presented as an A-module

[2] There is a subtlety there, B is finitely presented *as an A-algebra* not a priori as an A-module. However, B is a finite A-module so by Tag 0564 B is a finitely presented A-module since $A \to B$ is a finitely presented ring map and B is trivially a finitely presented B-module.

Lemma 9.1.1. Let $\mathscr{F}$ be a sheaf on a discrete space X. Then $\mathscr{F}$ is flasque and the canonical map,

$$\mathscr{F} \to \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

is an isomorphism.

Proof. Let $U \subset X$ be open (any set since X is discrete) then since points are open the set of points $x \in U$ forms an open cover. Then by the sheaf property,

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}(x)$$

is an isomorphims. Furthermore, clearly $\mathscr{F}(x) = \mathscr{F}_x$ since x is the inital object in the poset of open neighborhoods of x. Furthermore, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective because for any section $s \in \mathscr{F}(V)$ we may extend to a global section by setting $f_x = s_x$ for $x \in V$ and $f_x = 0$ for $x \notin V$. clearly $f_x = s_x$ on V so by the sheaf property $f|_V = s$. Then restricting $f|_U$ shows that $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective. $\square$

Thus, we can alternatively describe the Godement operation as follows. We can consider,

$$X_{\text{dis}} = \coprod_{x \in X} x$$

Then,

$$f^* X = \prod_{x \in X} \mathscr{F}_x$$

and $f : X_{\text{dis}} \to X$ is the bundled collection of the inclusions $\iota_x : x \to X$ giving,

$$f_* f^* \mathscr{F} = \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

reproducing the result on a discrete space.

9.2 Subsheaves of Godement

Now consider the diagram,

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, dashed] \arrow[d, hook] \\ \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x) \arrow[r] & \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x) \end{tikzcd}$$

We ask when the inclusion $\mathscr{F} \to \mathcal{G}^1(\mathscr{F})$ factors through the canonical map,

$$\bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x) \to \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

and when this sheaf or its image subsheaf is flasque.

First, note that direct sums commute with colimits (because they are colimts themselves) and thus denoting,

$$H(\mathscr{F}) = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

we have the stalks,

$$\begin{aligned} H(\mathscr{F})_x &= \varinjlim_{x \in U} H(\mathscr{F})(U) = \bigoplus_{y \in X} \varinjlim_{x \in U} \begin{cases} \mathscr{F}_y & y \in U \\ 0 & y \notin U \end{cases} \\ &= \bigoplus_{y \in X} \begin{cases} \mathscr{F}_y & x \in \overline{\{y\}} \\ 0 & x \notin \overline{\{y\}} \end{cases} \\ &= \bigoplus_{y \rightsquigarrow x} \mathscr{F}_y \end{aligned}$$

Therefore, if $\mathscr{F}$ is supported only on closed points of X we have,

$$H(\mathscr{F})_x = \mathscr{F}_x$$

However, in general there is not a sheaf map $\mathscr{F} \to H(\mathscr{F})$.

Suppose that $\mathscr{F}$ has finitely supported sections meaning that for any $s \in \mathscr{F}(U)$ its support,

$$\text{Supp}\,(s) = \{x \in X \mid s_x \neq 0\}$$

is finite. Then we get an injection,

$$\mathscr{F} \hookrightarrow \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

by mapping for each s,

$$s \in \mathscr{F} \hookrightarrow \prod_{x \in \text{Supp}(s)} (\iota_x)_*(\mathscr{F}_x) = \bigoplus_{x \in \text{Supp}(s)} (\iota_x)_*(\mathscr{F}_x) \subset \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

Furthermore, notice that if $\mathscr{F}$ is only supported at closed points then,

$$H(\mathscr{F})_x = \bigoplus_{y \rightsquigarrow x} \mathscr{F}_y = \mathscr{F}_x$$

since $\mathscr{F}_y = 0$ for any generalization of x. Therefore, in this case the map $\mathscr{F} \to H(\mathscr{F})_x$ defined by virtue of sections having finite support is an isomorphism. Thus if $\mathscr{F}$ is a abelian sheaf whose sections have finite support which is supported on the closed points then,

$$\mathscr{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

9.3 The Case for Curves

Let X be a curve (separated integral Noetherian scheme of dimension one) with generic point $\xi \in X$. Then I claim any torsion sheaf $\mathscr{F}$ satisfies,

$$\mathscr{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

By the previous discussion, it suffices to show that $\mathscr{F}$ is supported at closed points any every section has finite support. The only nonclosed point is ξ and we assumed that $\mathscr{F}_\xi = 0$. Furthermore, consider $s \in \mathscr{F}(U)$. We know s_ξ so there is some open V such that $\xi \in V \subset U$ on which $s|_U = 0$. Therefore $\text{Supp}\,(\xi) \subset V^C$. I claim that $V^C \subset X$ is finite. Since X is quasi-compact, we can choose an affine open cover $U_i = \mathrm{Spec}\,(A_i)$ and $V^C \cap U_i = V(I_i)$ for some ideal $I_i \subset A_i$. It suffices to show that $V(I_i)$ is finite. Note that $\dim A_i \le 1$ and X is irreducible so $\mathrm{codim}\left(V^C, X\right) \ge 1$ and therefore $\dim V^C = 0$ because,

$$\dim X \ge \mathrm{codim}\left(V^C, X\right) + \dim V^C$$

This shows that $\dim A_i/I_i = 0$ and it is Noetherian so A_i/I_i is Artinian and thus $V(I_i) = \mathrm{Spec}\,(A_i/I_i)$ is finite.

Therefore, each section has finite support so we have demonstrated the equality,

$$\mathscr{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

for any torsion sheaf ($\mathscr{F}_\xi = 0$).

9.4 Resolutions on Curves

Consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \mathscr{K}_X/\mathcal{O}_X \longrightarrow 0$$

Notice that $(\mathscr{K}_X/\mathcal{O}_X)_\xi = K(X)/\mathcal{O}_{X,\xi} = 0$ so $\mathscr{K}_X/\mathcal{O}_X$ is torsion. Therefore, we get a sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \bigoplus_{x \in X} (\iota_x)_*(K(X)/\mathcal{O}_{X,x}) \longrightarrow 0$$

Since X is integral $\mathscr{K}_X$ is constant (since all opens are connected it is truely constant) and thus we get a flasque resolution of $\mathcal{O}_X$. Then the long exact sequence gives,

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow K(X) \longrightarrow \bigoplus_{x \in X} K(X)/\mathcal{O}_{X,x} \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

and $H^i(X, \mathcal{O}_X) = 0$ for $i > 1$. Furthermore, for any flat sheaf $\mathscr{F}$, we can tensor the above exact sequence to get,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X \longrightarrow \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_\xi/\mathscr{F}_x) \longrightarrow 0$$

Where $(\iota_x)_*(K(X)) \otimes_{\mathcal{O}_X} \mathscr{F} =$

Lemma 9.4.1. Let X be an irreducible scheme with generic point $\xi \in X$ and $\mathscr{F}$ an abelian sheaf on X. Then the natural map,

$$\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X \to (\iota_\xi)_*(\mathscr{F}_\xi)$$

is an isomorphism.

Proof. Locally, on affine opens $\square$

10 Appendix

10.1 Curves and Genera

Lemma 10.1.1. Let X be a integral scheme proper over k then $K = H^0(X, \mathcal{O}_X)$ is a finite field extension of k and for any coherent $\mathcal{O}_X$-module $\mathscr{F}$, the cohomology $H^p(X, \mathscr{F})$ is a finite-dimensional $H^0(X, \mathcal{O}_X)$-module.

Proof. Since $\mathcal{O}_X$ is coherent, and X is proper over k so $K = H^0(X, \mathcal{O}_X)$ is a finite k-module. However, since X is integral $H^0(X, \mathcal{O}_X)$ is a domain but a finite k-algebra domain is a field and we see K/k is a finite extension of fields. Furthermore, the $\mathcal{O}_X(X)$-module structure on $H^p(X, \mathscr{F})$ gives it a K-module structure. Since X is proper over k then $H^p(X, \mathscr{F})$ is a finite k-module and thus finite as a K-module. $\square$

Remark. Unfortunately, when k is not algebraically closed then we may not have $H^0(X, \mathcal{O}_X) = k$ even for smooth projective varieties. Therefore, some caution must be taken in defining numerical invariants of the curve such as genus. However, by [**?**, Tag 0BUG], whenever X is proper geometrically integral then indeed $H^0(X, \mathcal{O}_X) = k$. Furthermore, for proper X if $H^0(X, \mathcal{O}_X) \neq k$ then X cannot be geometrically connected by [**?**, Tag 0FD1].

Definition 10.1.2. Let C be a smooth proper curve over k with $H^0(C, \mathcal{O}_C) = K$. Then we define $g(C) := \dim_K H^0(X, \Omega_{C/k})$. If C is any curve over k then there is a unique smooth proper curve S over k which is k-birational to C. Then we define $g(C) := g(S)$.

Remark. By definition, the genus of a curve is clearly a birational invariant since there is a unique smooth complete curve in every birational equivalence class of curves.

Remark. There is a slight subtlety in this definition in the case of a non-perfect base field. It it always true that we can find a proper *regular* curve C in each birational equivalence class however when k is non-perfect the curve C may not be smooth. However, under a finite purely separable extension K/k, we can ensure that C_K admits a smooth proper model. Then we define $g(C) := g(C_K)$ in the case that C_K is a curve. The only thing that can go wrong is when C is not geometrically irreducible since then C_K will not be integral.

Definition 10.1.3. The *arithmetic genus* $g_a(C)$ of a proper curve C over k with $H^0(C, \mathcal{O}_C) = K$ is,

$$g_a(C) := \dim_K H^1(X, \mathcal{O}_C)$$

By Serre duality, if C is smooth then $H^0(C, \Omega_C) = H^1(C, \mathcal{O}_X)^\vee$ meaning that $g_a(C) = g(C)$.

Remark. The arithmetic genus depends on the projective compactification and singularities meaning it will not be a birational invariant unlike the (geometric) genus.

Example 10.1.4. Let $k = \mathbb{F}_p(t)$ for an odd prime $p = 2k + 1$ and consider the curve,

$$C = \mathrm{Spec}\left(k[x, y]/(y^2 - x^p - t)\right)$$

which is regular but not smooth at $P = (y, x^p - t)$. Consider the purely inseperable extension $K = \mathbb{F}(t^{1/p})$. Then $C_K = \mathrm{Spec}\left(K[x, y]/(y^2 - (x - t^{1/p})^p)\right) \cong \mathrm{Spec}\left(K[x, y]/(y^2 - x^p)\right)$. Taking the normalization of C_K gives $\mathbb{A}^1_K \to C_K$ via $t \mapsto (t^p, t^2)$. This is birational since the following ring map is an isomorphism,

$$(K[x, y]/(y^2 - x^p))_x \to K[t]_t$$

sending $x \mapsto t^2$ and $y \mapsto t^p$ which has an inverse $t \mapsto y/x^k$ since $x \mapsto t^2 \mapsto y^2/x^{2k} = x$ and $y \mapsto t^p \mapsto y^p/x^{kp} = y(y^{2k}/x^{pk}) = y$ and $t \mapsto y/x^k \mapsto t^{p-2k} = t$.

Therefore, $C_K \overset{\sim}{\dashrightarrow} \mathbb{P}^1_K$ so $g(C) = g(C_K) = 0$. However, consider the projective closure,

$$\overline{C} = \text{Proj}\left(k[X,Y,Z]/(Y^2Z^{p-2} - X^p - tZ^p)\right)$$

then $\overline{C} \hookrightarrow \mathbb{P}^2_k$ is a Cartier divisor (since $\mathbb{P}^2_k$ is locally factorial) so we find that $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = k$ and $\dim_k H^1(\overline{C}, \mathcal{O}_{\overline{C}}) = \frac{1}{2}(p-1)(p-2) = k(2k-1)$ since its sheaf of ideals is $\mathcal{O}_{\mathbb{P}^2_k}(-p)$. Then $p = 3$ we expect this to be an elliptic curve and we do see $g_a(\overline{C}) = 1$. However, $g(\overline{C}) = 0$ and correspondingly C is not smooth due to the positive characteristic phenomenon.

Lemma 10.1.5. Suppose that $f : X \to Y$ is a finite birational morphism of n-dimensional irreducible Noetherian schemes. Then $H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(X, \mathcal{O}_X)$ is surjective.

Proof. The map f must restrict on some open subset $U \subset X$ to an isomorphism $f|_U : U \to V$. Thus, the sheaf map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ restricts on V to an isomorphism $\mathcal{O}_Y|_V \xrightarrow{\sim} (f_*\mathcal{O}_X)|_V$. We factor this map into two exact sequences,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_Y \longrightarrow \mathscr{I} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{I} \longrightarrow f_*\mathcal{O}_X \longrightarrow \mathscr{C} \longrightarrow 0$$

with $\mathscr{K} = \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{C} = \text{coker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{I} = \text{Im}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. Taking cohomology and using that it vanishes in degree above n we get,

$$H^{n-1}(Y, \mathscr{I}) \longrightarrow H^n(Y, \mathscr{K}) \longrightarrow H^n(Y, \mathcal{O}_Y) \longrightarrow\!\!\!\!\!\rightarrow H^n(Y, \mathscr{I}) \longrightarrow 0$$

$$H^{n-1}(Y, \mathscr{C}) \longrightarrow H^n(Y, \mathscr{I}) \longrightarrow H^n(X, \mathcal{O}_X) \longrightarrow\!\!\!\!\!\rightarrow H^n(X, \mathscr{C}) \longrightarrow 0$$

where we have used that $f : X \to Y$ is affine to conclude that $H^p(Y, f_*\mathscr{F}) = H^p(Y, \mathscr{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$. Furthermore, $\mathscr{C}|_V = 0$ so $\text{Supp}_{\mathcal{O}_Y}(\mathscr{C}) \subset X \setminus V$ but $\mathscr{C}$ is coherent so the support is closed. Since V is dense open, $\mathscr{C}$ is supported in positive codimension so $H^n(Y, \mathscr{C}) = 0$ (since $H^n(S, \mathscr{C})$ vanishes due to dimension on the closed subscheme $S = \text{Supp}_{\mathcal{O}_X}(\mathscr{C})$ on which $\mathscr{C}$ is supported). Thus we have,

$$H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(X, \mathcal{O}_X)$$

proving the proposition. ☐

Corollary 10.1.6. Let S and C be proper curves over k where S is smooth which are birationally equivalent and $H^0(S, \mathcal{O}_S) \cong H^0(C, \mathcal{O}_C)$. Then the genera satisfy,

(a) $g_a(C) \geq g_a(S)$

(b) $g(C) = g(S)$

(c) $g(C) \leq g_a(C)$ with equality if and only if C is smooth.

Proof. Given a birational map $S \overset{\sim}{\dashrightarrow} C$ we can extend it to a birational morphism $S \to C$ since S is regular. The morphism $S \to C$ is automatically finite since it is a non-constant map of proper curves. Then the previous lemma implies that $g_a(S) \le g_a(C)$. (b). follows from the definition of $g(C)$. The third follows from the fact that $g(S) = g_a(S)$ because of Serre duality,

$$H^1(S, \mathcal{O}_S) \cong H^0(S, \Omega_{S/k})^\vee$$

using that S is smooth. Then we see that $g(C) = g(S) = g_a(S) \le g_a(C)$ proving the inequality part of (c). Finally, if C is smooth we see by Serre duality that $g(C) = g_a(C)$. Conversely, suppose that $g(C) = g_a(C)$ then $g_a(C) = g(C) = g(S) = g_a(S)$ and consider the map $f : S \to C$ which is finite birational map of integral schemes over k. In particular, f is affine so for each $y \in C$ we may choose an affine open $y \in V \subset C$ whose preimage $U = f^{-1}(V)$ is also affine. On sheaves, this gives a map of domains $\mathcal{O}_C(V) \to \mathcal{O}_S(U)$ which localizes to an isomorphism on the fraction fields. However, the localization map of a domain is injective so $\mathcal{O}_C(V) \hookrightarrow \mathcal{O}_S(U)$ is an injection. This shows that $\mathcal{O}_C \to f_*\mathcal{O}_S$ is an injection of sheaves which we extend to an exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_*\mathcal{O}_S \longrightarrow \mathscr{C} \longrightarrow 0$$

Note that $f : S \to C$ induces an isomorphism $H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$ since it is a map of fields with the same (finite) dimension over k. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, \mathscr{C}) \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, \mathscr{C}) = 0$$

I claim that $H^1(S, \mathscr{C}) = 0$. Since f is birational, $\mathscr{C}$ is supported in codimension one. Thus, the map $H^1(C, \mathcal{O}_C) \twoheadrightarrow H^1(S, \mathcal{O}_S)$ is surjective but $g_a(C) = g_a(S)$ so these vectorspaces have the same dimension so $H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S)$ is an isomorphism. Thus, from the exact sequence we have $H^0(X, \mathscr{C}) = 0$. However, $\mathrm{Supp}_{\mathcal{O}_C}(\mathscr{C})$ is a closed ($\mathscr{C}$ is coherent) dimension zero subset i.e. finitely many discrete closed points. However, a sheaf supported on a discrete set of points is zero iff it has no global sections. Therefore, $\mathscr{C} = 0$ so $\mathcal{O}_C \xrightarrow{\sim} f_*\mathcal{O}_S$. In particular $\mathcal{O}_C(V) \xrightarrow{\sim} \mathcal{O}_S(U)$ is an isomorphism which implies that the map of affine schemes $f|_U : U \to V$ is an isomorphism. Since the affine opens V cover C we see that $f : S \to C$ is an isomorphism. In particular, C is smooth. ☐

10.2 The Locus on Which Morphisms Agree

Lemma 10.2.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then for schemes X there is a natural bijection,

$$\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(R), X) \cong \{x \in X \text{ and local map } \mathcal{O}_{X,x} \to R\}$$

Proof. Given $\mathrm{Spec}(R) \to X$ we automatically get $\mathfrak{m} \mapsto x$ and $\mathcal{O}_{X,x} \to R_\mathfrak{m} = R$. Now, note that taking any affine open neighborhood $x \in \mathrm{Spec}(A) \subset X$ and then $A \to A_\mathfrak{p} = \mathcal{O}_{X,x}$ to give $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(A) \to X$. Clearly, this map sends $\mathfrak{m}_x \mapsto x$ and at $\mathfrak{m}_x$ has stalk map $\mathrm{id} : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ since it is the localization at $\mathfrak{p}$ of $A \to A_\mathfrak{p}$.

Thus we get an inverse as follows. Given a point $x \in X$ and a local map $\phi : \mathcal{O}_{X,x} \to R$ then take,

$$\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$$

This is inverse since $\mathfrak{m} \mapsto \mathfrak{m}_x$ (because $\mathcal{O}_{X,x} \to \mathfrak{m}_x$ is local) and $\mathfrak{m}_x \mapsto x$ and the stalk at $\mathfrak{m}$ gives $\mathcal{O}_{X,x} \xrightarrow{\mathrm{id}} \mathcal{O}_{X,x} \xrightarrow{\phi} R$.

Finally, I claim that any $f : \mathrm{Spec}(R) \to X$ factors through $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ and thus is reconstructed from $x \in X$ and $\mathcal{O}_{X,x} \to R$. Choose an affine open neighborhood $x \in \mathrm{Spec}\,(A) \subset X$ then consider $f^{-1}(\mathrm{Spec}\,(A))$ which is open in $\mathrm{Spec}\,(R)$ and contains the unique closed point $\mathfrak{m} \in \mathrm{Spec}\,(R)$ so there is some $f \in R$ s.t. $\mathfrak{m} \in D(f) \subset f^{-1}(\mathrm{Spec}\,(A))$ so $f \notin \mathfrak{m}$ so $f \in R^\times$ and thus $D(f) = \mathrm{Spec}\,(R)$. Therefore, we get a map $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(A)$ and thus $\phi : A \to R$ where $\phi^{-1}(\mathfrak{m}) = \mathfrak{p} = x$ so $A \setminus \mathfrak{p}$ is mapped inside $R^\times$ so this map factors through $A \to A_\mathfrak{p} \to R$ giving the desired factorization $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}) \to \mathrm{Spec}\,(A) \to X$. $\square$

Definition 10.2.2. The locus Z on which two maps $f, g : X \to Y$ over S agree is given as the pullback,

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

with $F = (f, g)$. This is the equalizer of $f, g : X \to Y$. Furthermore $Z \to X$ is an immersion since it is the base change of $\Delta_{Y/S}$ which is an immersion.

Lemma 10.2.3. Topologically, the locus on which S-morphisms $f, g : X \to Y$ agree is,

$$Z = \{x \in X \mid f(x) = g(x) \text{ and } f_x = g_x : \kappa(f(x)) \to \kappa(x)\}$$

Proof. On some S-subscheme $G \subset X$, the maps $f|_G = g|_G$ agree iff there exists $G \to Y$ such that,

$$\begin{tikzcd} G \arrow[r, dashed] \arrow[d, hook] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

commutes. In particular, for any point $x \in X$ consider $\iota : \mathrm{Spec}\,(\kappa(x)) \to X$ then $f \circ \iota = g \circ \iota$ iff $f(x) = g(x)$ and $f_x = g_x : \kappa(f(x)) \to \kappa(x)$. Consider a point $z \in Z$ and $\mathrm{Spec}\,(\kappa(z)) \to Z$, such a point is equivalent to giving a diagram,

$$\begin{tikzcd} \mathrm{Spec}\,(\kappa(z)) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ & X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

However, $\iota : Z \to X$ is an immersion so $\iota_x : \kappa(\iota(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism. Therefore, points $\mathrm{Spec}\,(\kappa(z)) \to Z$, are exactly points of X for which a lift $\mathrm{Spec}\,(\kappa(x)) \to Y$ exists i.e. points such that f and g agree in the required way. $\square$

Lemma 10.2.4. If $f : X \to Y$ is an immersion then $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective for each $x \in X$ and $f_x : \kappa(f(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism.

Proof. For closed immersions, $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective by definition. Thus we get a surjection $f_x : \mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_{f(x)}$. Furthermore, topologically, $f : X \to Y$ is a homomorphism onto its image so for any open $U \subset X$ there exists an open $V \subset Y$ s.t. $U = f^{-1}(V)$ showing that,

$$(f_* \mathcal{O}_X)_{f(x)} = \varinjlim_{f(x) \in V} \mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}$$

Furthermore, for an open immersion, $f^\flat : f^{-1}\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism so $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. Thus the composition, $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective. Furthermore, f_x is local we get $f_x : \kappa(f(x)) \twoheadrightarrow \kappa(x)$ which is a surjection of fields and thus an isomorphism. $\square$

Lemma 10.2.5. If $Y \to S$ is separated then the locus on which $f, g : X \to Y$ over S agree is closed.

Proof. Since $X \to S$ is separated, $\Delta_{Y/S} : Y \to Y \times_S Y$ is a closed immersion. So $Z \to X$ is the base change of a closed immersion and thus a closed immersion. $\square$

Lemma 10.2.6. Let X be a reduced and Y be a separated scheme over S and $f, g : X \to Y$ be morphism over S. If $f \circ j = g \circ j$ agree on a dense subscheme $j : G \hookrightarrow X$ then $f = g$.

Proof. Consider $F = (f, g) : X \to Y \times_S Y$. Since $\Delta : Y \to Y \times_S Y$ is a closed immersion (by separateness). Then $F^{-1}(\Delta)$ is the locus on which $f = g$ which is closed because $\Delta : Y \to Y \times_S Y$ is a closed immersion. Since $f|_G = g|_G$ we get a diagram,

$$\begin{tikzcd}
G \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, hook, bend right] & & \\
& Z \arrow[r, "\tilde{F}"] \arrow[d, hook, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\
& X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}$$

Since $\iota : Z \hookrightarrow X$ is a closed immersion with dense image, $Z \hookrightarrow X$ is surjective. By the following, $\iota : Z \to X$ is an isomorphism. Thus, $F = F \circ \iota \circ \iota^{-1} = \Delta_Y \circ \tilde{F} \circ \iota^{-1}$. By the universal property of maps $X \to Y \times_S Y$ this implies that $f = g = \tilde{F} \circ \iota^{-1}$. $\square$

Lemma 10.2.7. Let X be a scheme and consider an exact sequence of quasi-coherent $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A} \longrightarrow 0$$

and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebra. Suppose that $\mathscr{F}_x \neq 0$ for each $x \in X$. Then $\mathscr{I} \hookrightarrow \mathcal{N}$ where $\mathcal{N}$ is the sheaf of nilpotent.

Proof. Take an affine open $U = \mathrm{Spec}\,(R) \subset X$ such that $\mathcal{A}|_U = \widetilde{A}$. Then we have an surjection of rings $R \twoheadrightarrow A$ giving $R/I = A$ for $I = \ker(R \to A)$. Now, for each $\mathfrak{p} \in \mathrm{Spec}\,(R)$ we know $R_\mathfrak{p} = \mathcal{O}_{X,\mathfrak{p}} \neq 0$. However, if $\mathfrak{p} \not\supset I$ then $(R/I)_\mathfrak{p} = A_\mathfrak{p} = 0$ so we must have $\mathfrak{p} \supset I$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ i.e. $I \subset \mathrm{nilrad}\,(R)$. Therefore, $\mathscr{I}|_U \hookrightarrow \mathcal{N}|_U$ for any affine open $U \subset X$ showing that $\mathscr{I}$ is comprised of nilpotents. $\square$

Corollary 10.2.8. If X is reduced and $\iota : Z \hookrightarrow X$ is a surjective closed immersion then $\iota : Z \xrightarrow{\sim} X$ is an isomorphism.

Proof. Since $\iota : Z \hookrightarrow X$ is a homeomorphism onto its image X it suffices to show that the map of sheaves $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. Since $\iota : Z \to X$ is a closed immersion $\iota^\# : \mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Z$ is a surjection and $\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras giving an exact sequence,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Furthermore,

$$\mathrm{Supp}_{\mathcal{O}_X}(\iota_* \mathcal{O}_Z) = \mathrm{Im}(\iota) = X$$

since $(\iota_* \mathcal{O}_Z)_x = \mathcal{O}_{Z,x}$ when $x \in \mathrm{Im}(\iota)$ (and zero elsewhere). by the above, $\mathscr{I} \hookrightarrow \mathcal{N} = 0$ since X is reduced to $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. $\square$

Lemma 10.2.9. A rational S-map $f : X \dashrightarrow Y$ with X reduced and $Y \to S$ separated is equivalent to a morphism $f : \mathrm{Dom}(f) \to Y$.

Proof. For any (U, f_U) and (V, f_V) representing f there must be a dense (in X) open $W \subset U \cap V$ on which $f_U|_W = f_V|_W$ and thus $f_U|_{U \cap V} = f_V|_{U \cap V}$ since $f_U, f_V : U \cap V \to Y$ are morphisms from reduced to irreducible schemes. Now $\mathrm{Dom}(f)$ has an open cover (U_i, f_i) for which $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ so these morphisms glue to give $f : \mathrm{Dom}(f) \to Y$ ($\mathrm{Hom}_S(-, Y)$ is a sheaf on the Zariski site). $\square$

10.3 Extending Rational Maps

Lemma 10.3.1. Regular local rings of dimension 1 exactly correspond to DVRs.

Proof. Any DVR R has a uniformizer $\varpi \in R$ then $\dim R = 1$ and $\mathfrak{m}/\mathfrak{m}^2 = (\varpi)/(\varpi^2) = \varpi \kappa$ which also has $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 1$ so R is regular. Conversely, if R is a regular local ring of dimension $\dim R = 1$ then, by regularity, R is a normal Noetherian domain so by $\dim R = 1$ then R is Dedekind but also local and thus is a DVR. $\square$

Proposition 10.3.2. Let X be a Noetherian S-scheme and $Z \subset X$ a closed irreducible codimension 1 generically nonsingular subset (with generic point $\eta \in Z$ such that $\mathcal{O}_{X,\eta}$ is regular). Let $f : X \dashrightarrow Y$ be a rational map with Y proper over S. Then $Z \cap \mathrm{Dom}(f)$ is a dense open of Z.

Proof. Choose some representative (U, f_U) for $f : X \dashrightarrow Y$. Note that $\mathcal{O}_{X,\eta}$ is a regular dimension one (see Lemma 10.4.3) ring and thus a DVR. Consider the generic point $\xi \in X$ of X then, by localizing, we get an inclusion of the generic point $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ and $\mathcal{O}_{X,\xi} = K(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta})$. Furthermore, the inclusion of the generic point gives $\mathrm{Spec}(K(X)) \to U \xrightarrow{f_U} Y$ and thus we get a diagram,

$$\begin{tikzcd} \mathrm{Spec}(K(X)) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{X,\eta}) \arrow[r] \arrow[ur, dashed, "\ell"] & \mathrm{Spec}(k) \end{tikzcd}$$

and a lift $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to Y$ by the valuative criterion for properness applied to $Y \to \mathrm{Spec}(k)$ since $\mathcal{O}_{X,\eta}$ is a DVR. Choose an affine open $\mathrm{Spec}(R) \subset Y$ containing the image of $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to Y$ (i.e. choose a neighborhood of the image of η which automatically contains $f(\xi)$ since the map factors $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to \mathrm{Spec}\left(\mathcal{O}_{Y,f(\eta)}\right) \to \mathrm{Spec}(R) \to Y$) and let $\eta \in V = \mathrm{Spec}(A) \subset X$ be an affine open neighborhood of ξ mapping into $\mathrm{Spec}(R)$. By Lemma 10.4.7, since $\mathcal{O}_{X,\eta}$ is a domain, we may shrink V so that A is a domain. Since X is irreducible $U \cap V$ is a dense open. Note that if $\eta \in U$ then $\eta \in \mathrm{Dom}(f)$ and thus $Z \cap \mathrm{Dom}(f)$ is a nonempty open of the irreducible space Z

and therefore a dense open so we are done. Otherwise, let $\mathfrak{p} \in \operatorname{Spec}(A)$ correspond to $\eta \in Z$ then $A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$ is a DVR. Take some principal affine open $D(f) \subset U \cap V$ for $f \in A$ so $f \in \mathfrak{p}$ since $\mathfrak{p} \notin D(f) \subset U \cap V$. Since $A_{\mathfrak{p}}$ is a DVR we may choose a uniformizer $\varpi \in \mathfrak{p}$ so the map $A \to \mathfrak{p}$ via $1 \mapsto \varpi$ is as isomorphism when localized at $\mathfrak{p}$. Since A is Noetherian both are f.g. A-modules so there must be some $s \in A \setminus \mathfrak{p}$ such that $A_s \to \mathfrak{p}_s$ is an isomorphism. Replacing A by A_s we may assume $\mathfrak{p} = (\varpi) \subset A$ is principal. Since $f \in \mathfrak{p}$ we can write $f = t\varpi^k$ for some $a \in A \setminus \mathfrak{p}$ (see Lemma 10.4.1). Then consider $\tilde{V} = \operatorname{Spec}(A_t)$. Since $t \notin \mathfrak{p}$ then $\eta \in \tilde{V}$ and since $f = t\varpi^k$ we have $D(f) \subset D(t) = \tilde{V}$. Now we get the following diagram,

$$\begin{tikzcd}
 & & \operatorname{Spec}(R) \\
\operatorname{Spec}(A_{\mathfrak{p}}) \arrow[urr, bend left, "\ell"] \arrow[r] & \operatorname{Spec}(A_t) \arrow[ur, dashed, "f_V"] & \\
\operatorname{Spec}(\operatorname{Frac}(A)) \arrow[u] \arrow[r] & \operatorname{Spec}(A_f) \arrow[u] \arrow[uur, bend right, "f_U"'] &
\end{tikzcd}$$

I claim the square is a pushout in the category of affine schemes because maps $R \to A_{\mathfrak{p}}$ and $R \to A_f$ which agree under the inclusion to $\operatorname{Frac}(A)$ gives a map $R \to A_{\mathfrak{p}} \cap A_f \subset \operatorname{Frac}(A)$. However, consider,

$$x \in A_{\mathfrak{p}} \cap A_t \implies x = \frac{u\varpi^r}{s} = \frac{a}{f^n}$$

for $u, s, t \in A \setminus \mathfrak{p}$ and $a \in A$. Thus we get,

$$ut^n\varpi^{r+nk} = sa$$

so $a \in \mathfrak{p}^{r+nk} \setminus \mathfrak{p}^{r+nk+1}$ ($s \notin \mathfrak{p}$ which is prime) and thus $a = u'\varpi^{r+nk}$ for $u' \in A \setminus \mathfrak{p}$. Therefore,

$$x = \frac{u'\varpi^{r+nk}}{t^n\varpi^{nk}} = \frac{u'\varpi^r}{t^n} \in A_t$$

Thus, $A_{\mathfrak{p}} \cap A_f \subset A_t$ so we get a map $R \to A_t$. Therefore we get a map $f_{\tilde{V}} : \tilde{V} \to Y$ such that $(f|_{\tilde{V}})|_{D(f)} = (f_U)|_{D(f)}$ showing that $\eta \in \tilde{V} \subset \operatorname{Dom}(f)$ so $Z \cap \operatorname{Dom}(f)$ is a dense open of Z. $\square$

Proposition 10.3.3. Let $C \to S$ be a proper regular Noetherian scheme with $\dim C = 1$ and $f : C \dashrightarrow Y$ a rational S-map with $Y \to S$ proper. Then f extends uniquely to a morphism $f : C \to Y$.

Proof. For any point $x \notin \operatorname{Dom}(f)$ let $Z = \overline{\{x\}} \subset D$ for $D = C \setminus \operatorname{Dom}(f)$. Since $\operatorname{Dom}(f)$ is a dense open, by lemma 10.4.2, we have $\operatorname{codim}(Z, C) \geq \operatorname{codim}(D, C) \geq 1$ but $\dim C = 1$ so $\operatorname{codim}(Z, C) = 1$. Furthermore, since C is regular $\mathcal{O}_{C,x}$ is regular and thus, by the previous proposition, $Z \cap \operatorname{Dom}(f)$ is a dense open and in particular $x \in \operatorname{Dom}(f)$ meaning that $\operatorname{Dom}(f) = C$ so we get a morphism $C \to Y$. This is unique because C is reduced (it is regular) and Y is separated (it is proper over S) so morphisms $C \to Y$ are uniquely determined on a dense open which any representative for $f : C \dashrightarrow Y$ is defined on. $\square$

Corollary 10.3.4. Rational maps between normal proper curves are morphisms.

Corollary 10.3.5. Birational maps between normal proper curves are isomorphisms.

Proof. Let $f : C_1 \dashrightarrow C_2$ and $g : C_2 \dashrightarrow C_1$ be birational inverses of smooth proper curves. Then we know that these extend to morphisms $f : C_1 \to C_2$ and $g : C_2 \to C_1$. Furthermore, the maps $g \circ f : C_1 \to C_1$ must extend the identity on some dense open. However, since curves are separated and reduced there is a unique extension of this map so $g \circ f = \mathrm{id}_{C_1}$ and likewise $f \circ g = \mathrm{id}_{C_2}$. ☐

Theorem 10.3.6. If k is perfect then there exists a unique normal curve in each birational equivalence class of curves.

Proof. It suffices to show existence. Given a curve X, we consider the projective closure $X \hookrightarrow \overline{X}$ which is birational and $\overline{X} \to \mathrm{Spec}\,(k)$ is proper. Then take the normalization $\overline{X}^{\nu} \to \overline{X}$ which remains proper over $\mathrm{Spec}\,(k)$ and is birational. Then $\overline{X}^{\nu}$ is regular and thus smooth over k since k is perfect and $\overline{X}^{\nu} \to X$ is birational. ☐

10.4 Lemmas

Lemma 10.4.1. Let A be a Noetherian domain and $\mathfrak{p} = (\varpi)$ a principal prime. Then any $f \in \mathfrak{p}$ can be written as $f = t\varpi^k$ for $f \in A \setminus \mathfrak{p}$.

Proof. From Krull intersection,

$$\bigcap_{n \ge 0}^{\infty} \mathfrak{p}^n = (0)$$

so there is some n such that $f \in \mathfrak{p}^n \setminus \mathfrak{p}^{n+1}$. Thus $f = t\varpi^n$ for some $f \in A$ but if $t \in \mathfrak{p}$ then $f \in \mathfrak{p}^{n+1}$ so the result follows. ☐

Lemma 10.4.2. Consider a closed subset $Y \subset X$ and an open $U \subset X$ with $U \cap Z \neq \varnothing$. Then $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Y \cap U, U)$.

Proof. Consider a chain of irreducible $Z_i \supsetneq Z_{i+1}$ with $Z_0 \subset Y$. I claim that $Z_i \mapsto Z_i \cap U$ and $Z_i \mapsto \overline{Z_i}$ are inverse functions giving a bijection between closed irreducible chains in X with final terms contained in Y and closed irreducible chains in U with final term contained in $Y \cap U$. Note, if $Z_i \subset Y \cap U$ then $\overline{Z_i} \subset Y$ since Y is closed in X.

First, $\overline{Z_i \cap U} \subset Z_i$ and is closed in X. Then $\overline{Z_i \cap U} \cup U^C \supset Z_i$ so because Z_i is irreducible $\overline{Z_i \cap U} = Z_i$ since by assumption $Z_i \not\subset U^C$. Conversely, if $Z_i \subset U$ is a closed irreducible subset then $\overline{Z_i}$ is closed and irreducible in X and $Z_i \subset \overline{Z_i} \cap U$ but $Z_i = C \cap U$ for closed $C \subset X$ so $Z_i \subset C$ and thus $\overline{Z_i} \subset C$ so $\overline{Z_i} \cap U \subset C \cap U = Z_i$ meaning $Z_i = \overline{Z_i} \cap U$. Thus we have shown these operations are inverse to each other.

Finally, if $Z_i \cap U - Z_{i+1} \cap U$ then $\overline{Z_i \cap U} = \overline{Z_i \cap U}$ so $Z_i = Z_{i+1}$ so the chain does not degenerate. Likewise, if $\overline{Z_i} = \overline{Z_{i+1}}$ then $\overline{Z_i} \cap U = \overline{Z_{i+1}} \cap U$ so $Z_i = Z_{i+1}$. Therefore, we get a length-preserving bijection between the chains defining $\mathrm{codim}\,(Y, X)$ and $\mathrm{codim}\,(Y \cap U, U)$. ☐

Lemma 10.4.3. Let $Z \subset X$ be a closed irreducible subset with generic point $\eta \in Z$. Then $\mathrm{codim}\,(Z, X) = \dim \mathcal{O}_{X,\eta}$.

Proof. Take affine open neighborhood $\eta \in U = \mathrm{Spec}\,(A) \subset X$. Then for $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponding to η we get $A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$. However, $\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U)$ and $Z \cap U = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Therefore,

$$\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U) = \mathbf{ht}\,(\mathfrak{p}) = \dim A_{\mathfrak{p}} = \dim \mathcal{O}_{X,\eta}$$

☐

Lemma 10.4.4. Let X be a Noetherian scheme then the nonreduced locus,

$$Z = \{x \in X \mid \mathrm{nilrad}\,(\mathcal{O}_{X,x}) \neq 0\}$$

is closed.

Proof. The subsheaf $\mathcal{N} \subset \mathcal{O}_X$ is coherent since X is Noetherian. Thus $Z = \mathrm{Supp}_{\mathcal{O}_X}(\mathcal{N})$ is closed and $\mathcal{N}_x = \mathrm{nilrad}\,(\mathcal{O}_X x)$. Locally, on $U = \mathrm{Spec}\,(A)$ we have $\mathcal{N}|_U = \widetilde{\mathrm{nilrad}\,(A)}$ and $\mathrm{nilrad}\,(A)$ is a f.g. A-module since A is Noetherian so,

$$\mathrm{Supp}_{\mathcal{O}_X}(\mathcal{N}) \cap U = \mathrm{Supp}_A\,(\mathrm{nilrad}\,(A)) = V(\mathrm{Ann}_A\,(\mathrm{nilrad}\,(A))$$

is closed in $\mathrm{Spec}\,(A)$. ☐

Lemma 10.4.5. Let X be a Noetherian scheme then X has finitely many irreducible components.

Proof. First let $X = \mathrm{Spec}\,(A)$ for a Noetherian ring A. Then the irreducible components of A correspond to minimal primes $\mathfrak{p} \in \mathrm{Spec}\,(A)$. Then $\dim A_{\mathfrak{p}} = 0$ and $A_{\mathfrak{p}}$ is Noetherian so $A_{\mathfrak{p}}$ is Artinian. $A_{\mathfrak{p}}$ must have some associated prime so $\mathrm{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}\}$. By [**?**, Tag 05BZ], then $\mathrm{Ass}_A\,(A) \cap \mathrm{Spec}\,(\mathbb{A}_{\mathfrak{p}}) = \mathrm{Ass}_{\mathbb{A}_{\mathfrak{p}}}(\mathbb{A}_{\mathfrak{p}}) = \{\mathfrak{p}\}$ so every minimal prime is an associated prime. However, for A Noetherian then A admits a finite composition series so there are finitely many associated primes.

Now let X be a Noetherian scheme. For any affine open $U \subset X$ we have shown that U has finitely many irreducible components. However, since X is quasi-compact there is a finite cover of affine opens and thus X must have finitely many irreducible components. ☐

Lemma 10.4.6. Let X be a Noetherian scheme and Y is the complement of some dense open U. Then $\mathrm{codim}\,(Y, X) \geq 1$.

Proof. It suffices to show that Y does not contain any irreducible component since then any irreducible contained in Y cannot be maximal. Since X is Noetherian, it has finitely many irreducible components Z_i. Then if $Z_j \subset Y$ for some i we would have $Z_i \cap U = \varnothing$ but then,

$$U = \bigcup_{i \neq j} Z_i$$

which is closed so $\overline{U} \subsetneq X$ contradicting our assumption that U is dense. ☐

Lemma 10.4.7. Let X be a Noetherian scheme and $x \in X$ such that $\mathcal{O}_{X,x}$ is a domain. Then there is an affine open neighborhood $x \in U \subset X$ with $U = \mathrm{Spec}\,(A)$ and A is a domain.

Proof. Take any affine open neighborhood $x \in U \subset X$ with $U = \mathrm{Spec}\,(A)$ and $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponding to x. Then $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is a domain. Since X is Noetherian then A is Noetherian so it has finitely many minimal primes $\mathfrak{p}_i$ (corresponding to the generic points of irreducible components of U) with $\mathfrak{p}_0 \subset \mathfrak{p}$. Since $A_{\mathfrak{p}}$ is a domain, it has a unique minimal prime and thus $\mathfrak{p}_0$ is the only minimal prime contained in $\mathfrak{p}$ (geometrically $A_{\mathfrak{p}}$ being a domain corresponds to the fact that $\mathfrak{p}$ is the generic point of a generically reduced irreducible subset which lies in only one irreducible component)

Now for any $i \neq 0$ take $f_i \in \mathfrak{p} \setminus \mathfrak{p}_0$. This is always possible else $\mathfrak{p} \subset \mathfrak{p}_0$ contradicting the minimality of $\mathfrak{p}_0$. If $f \notin \mathfrak{q}$ then $\mathfrak{q} \not\supset \mathfrak{p}_i$ for any $i \neq 0$ so $\mathfrak{q} \supset \mathfrak{p}_0$ since it must lie above some minimal prime. Thus

$\operatorname{nilrad}(A_f) = \mathfrak{p}_0 A_f$ is prime and $f \notin \mathfrak{p}$ since else $\mathfrak{p} \supset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ which is impossible since $\mathfrak{p} \not\supset \mathfrak{p}_i$ for any i. Now we know that $\operatorname{nilrad}(A_{\mathfrak{p}}) = 0$ and A_f is Noetherian so $\operatorname{nilrad}(A_{\mathfrak{p}})$ is finitely generated. Thus, there is some $g \notin \mathfrak{p}$ such that $\operatorname{nilrad}(A_{fg}) = (\operatorname{nilrad}(A_f))_g = 0$. Thus A_{fg} is a domain since $\operatorname{nilrad}(A_{fg}) = (0)$ and is prime and $\mathfrak{p} \in A_{fg}$ because $fg \notin \mathfrak{p}$. Therefore, $x \in \operatorname{Spec}(A_{fg}) \subset U$ is an affine open satisfying the requirements. □

Remark. This does not imply that X is integral if $\mathcal{O}_{X,x}$ is a domain for each $x \in X$ (which is false, consider $\operatorname{Spec}(k \times k)$) because it only shows there is an integral cover of X not that $\mathcal{O}_X(U)$ is a domain for each U.

Example 10.4.8. Let $X = \operatorname{Spec}(k[x,y]/(xy, y^2))$. Then for the bad point $\mathfrak{p} = (x, y)$ we have $\operatorname{nilrad}(\mathcal{O}_{X,\mathfrak{p}}) = (y)$. Away from the bad point, say $\mathfrak{p} = (x-1, y)$ we have, $\mathcal{O}_{X,\mathfrak{p}} = \operatorname{Spec}\left(k[x]_{(x-1)}\right)$ so $\operatorname{nilrad}(\mathcal{O}_{X,\mathfrak{p}}) = (0)$. Furthermore, at the generic point $\mathfrak{p} = (y)$, we have, $\mathcal{O}_{X,\mathfrak{p}} = \operatorname{Spec}(k(x))$ so $\operatorname{nilrad}(\mathcal{O}_{X,\mathfrak{p}}) = (0)$.

Example 10.4.9. Consider $X = \operatorname{Spec}(k[x,y,z]/(yz))$ which is the union of the x-y and x-z planes. Consider the generic point of the z-axis $\mathfrak{p} = (x, y)$ then $\mathcal{O}_{X,\mathfrak{p}} = \operatorname{Spec}\left(k[x,z]_{(x)}\right)$ is a domain since the z-axis only lies in one irreducible component. However, at the generic point of the x-axis, $\mathfrak{p} = (y, z)$ we get $\mathcal{O}_{X,\mathfrak{p}} = \operatorname{Spec}\left((k[x,y,z]/(yz))_{(y,z)}\right)$ has zero divisors $yz = 0$ so is not a domain since the x-axis lives in two irreducible components.

10.5 Reflexive Sheaves (WIP)

Definition 10.5.1. Recall the dual of a $\mathcal{O}_X$ module $\mathscr{F}$ is the sheaf $\mathscr{F}^\vee = \mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. We say that a coherent $\mathcal{O}_X$-module $\mathscr{F}$ is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism.

Lemma 10.5.2. Let X be an integral locally Noetherian scheme and $\mathscr{F}, \mathscr{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathscr{G}$ is reflexive then $\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is reflexive.

Proof. See [**?**, Tag 0AY4]. □

In particular, since $\mathcal{O}_X$ is clearly reflexive, this lemma shows that for any coherent $\mathcal{O}_X$-module then $\mathscr{F}^\vee$ is a reflexive coherent sheaf. We say the map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ gives the reflexive hull $\mathscr{F}^{\vee\vee}$ of $\mathscr{F}$.

Definition 10.5.3. Let $\mathcal{R}$ be the full subcategory $\mathfrak{Coh}(\mathcal{O}_X)$ of coherent reflexive $\mathcal{O}_X$-modules. $\mathcal{R}$ is an additive category and in fact has all kernels and cokernels defined by taking reflexive hulls of the sheaf kernel and cokernel. Furthermore, $\mathcal{R}$ inherits a monoidal structure from the tensor product defined using the reflexive hull as follows,

$$\mathscr{F} \otimes_{\mathcal{R}} \mathscr{G} = (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$$

Finally, we define $\operatorname{RPic}(X)$ to be group of constant rank one reflexives induced by the monoidal structure on $\mathcal{R}$. Explicitly, $\operatorname{RPic}(X)$ is the group of isomorphism classes of constant rank one reflexive coherent $\mathcal{O}_X$-modules with multiplication $(\mathscr{F}, \mathscr{G}) \mapsto (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$ and inverse $\mathscr{F} \mapsto \mathscr{F}^\vee$.

The importance of reflexive sheaves derives from their correspondence to Weil divisors. Here we let X be a normal integral separated Noetherian scheme.

Proposition 10.5.4. If D is a Weil divisor then $\mathcal{O}_X(D)$ is reflexive of constant rank one.

Proof. (CITE OR DO). $\square$

Theorem 10.5.5. Let X be a normal integral separated Noetherian scheme. There is an isomorphism of groups $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{RPic}(X)$ defined by $D \mapsto \mathcal{O}_X(D)$.

Proof. (DO OR CITE) $\square$

We summarize the important results as follows.

Theorem 10.5.6. Let X be a Noetherian normal integral scheme. Then for any Weil divisors D, E,

(a) $\mathcal{O}_X(D+E) = (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee}$

(b) $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\vee}$

(c) $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(E)) = \mathcal{O}_X(E-D)$

(d) if E is Cartier then $\mathcal{O}_X(D+E) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$

Proof. (DO OR CITE) $\square$

Finally, we have a result which controls when the dualizing sheaf can be expressed in terms of a divisor.

Proposition 10.5.7. Let X be a projective variety over k. Then,

(a) if X is normal then its dualizing sheaf ω_X is reflexive of rank 1 and thus X admits a canonical divisor K_X s.t. $\omega_X = \mathcal{O}_X(K_X)$

(b) if X is Gorenstein then ω_X is an invertible module so K_X is Cartier.

Proof. (FIND CITATION OR DO). $\square$

11 On a Conjecture of Griffiths and Harris

In [4], Grifiths and Harrised proposed the following range of conjectures concerning curves contained in a general hypersurface in $\mathbb{P}^4$ of degree $d \geq 6$ (such that the canonical bundle is ample)

(a) we have d divides $\deg C$

(b) the image of the Abel-Jacobi map

$$\varphi : \mathrm{Hom}^2(X)/\mathrm{Rat}^2(X) \to J^2(X)$$

of C is trivial

(c) the group

$$\mathrm{Hom}^2(X)/\mathrm{Alg}^2(X)$$

is trivial

(d) the group

$$\mathrm{Alg}^2(X)/\mathrm{Rat}^2(X)$$

is trivial

(e) if C is smooth, then C is a complete intersection with a surface $\Sigma \subset \mathbb{P}^4$.

Mark Green has explained some recent progress on (b), this note proposes to show that (e) is false, we show this by showing that the variant (e'), is also false,

the normal bundle sequence for $C \subset X \subset \mathbb{P}^4$ is split

I thank the CIRM and the university of Trento for excellent welcome given to us during this conference as well as C. Ciliberto and E.Ballico for having allowed me to include these remarks in their proceedings.

11.1 The counterexample to (e)

We suppose $d > 2$ since the case $d = 2$ is trivial since every quadric contains a line.

Let $X \subset \mathbb{P}^n$ for $n \geq 4$ be a smooth hypersurface of degree d, and let $C \subset X$ be a smooth curve. We suppose there exists a surface $\Sigma \subset \mathbb{P}^n$ such that C is the complete intersection of X and Σ. Since C is smooth, Σ is smooth along C, so Σ^{sing} is comprised of isolated points located outside of C (I guess because $C \subset \Sigma$ is ample). Let $\tau : \overline{\Sigma} \to \Sigma$ be the designgularization of Σ. We get a natural inclusion $C \subset \overline{\Sigma}$ and C is a member of the linear system $|\tau^*\mathcal{O}_\Sigma(d)|$ on $\overline{\Sigma}$. The class of C in $H^2(\overline{\Sigma}, \mathbb{Z})$ is divisible by d which implies

(a) d divides $C \cdot_{\overline{\Sigma}} C$

(b) d divides $K_{\overline{\Sigma}} \cdot_{\overline{\Sigma}} C$

and therefore the adjunction formula implies:

$$d \text{ divides } \deg K_C$$

11.2 •

We consider now a curve with an ordinary double point D constructed via two smooth tangent plances $P_1 \cap X = C_1$ and $P_2 \cap X = C_2$ of X meeting transversely at a point p. One such curve exists when $n \geq 4$.

We have, for $i = 1, 2$

(a) d divides $\deg C_i$

(b) d divides $\deg K_{C_i}$

from which we have

$$\deg K_D = \deg K_{C_1} + \deg K_{C_2} + 2 \equiv 2 \mod d$$

Then if S is a complete intersection surface $X \cap X_1 \cap \cdots \cap X_{n-3}$ containing D and let $D' \subset S$ be a smooth member of the linear system $|mH + D|$ on S (D' exists for m sufficiently large).

We have

$$\begin{aligned}\deg K_{D'} &= (D')^2 + K_S \cdot D' = (D + mH)^2 + K_S \cdot (D + mH) \\ &= \deg K_D + 2mH \cdot D + m^2 H^2 + mK_S \cdot H\end{aligned}$$

the last two terms are divisible by d; according to (a) $\deg D$ is also 2 mod d hence $\deg K_{D'}$ is 2 mod d.

Since $d > 2$, D' does not satisfy condition 1.2, and provides a counterexample to (v).

11.3 Counterexample to (e)

We supppose from now on (for simplicity) that $n = 3$. We start again with the curve $D = C_1 \cup_p C_2$.

We will show the following facts

(A) the normal bundle exact sequence of $D \subset X \subset \mathbb{P}^4$ is not split

(B) if $S = X \cap X'$ is a smooth surface containing D with $\deg X' = k$ sufficiently large; let D' be a smooth curve on S in the linear system $|mH + D|$ for m sufficiently large; then the normal bundle exact sequence of $D' \subset X \subset \mathbb{P}^4$ is not split.

Lemma 11.3.1. (A) $\implies$ (b)

Proof. We denote $e_D \in H^1(C, N_{D|X}(-d))$ the class of the extension

$$0 \to N_{D|X} \to N_{D|\mathbb{P}^4} \to \mathcal{O}_D(d) \to 0$$

and for any hypersurface X' such that $X' \cap X = S$, we denote $F_D^{X'} \in H^1(D, N_{D|S}(-d))$ the class of the extension

$$0 \to N_{D|S} \to N_{D|X'} \to \mathcal{O}_D(d) \to 0$$

and the same notations for D'

We consider the exact sequence

$$0 \to N_{D|S} \to N_{D|X} \to \mathcal{O}_D(k) \to 0$$

which provides the map

$$\alpha : H^1(D, N_{D|S}(-d)) \to H^1(D, N_{D|X}(-d)) \quad \text{such that} \quad \alpha(F_D^{X'}) = e_D$$

(i) We suppose S is fixed, and let m be such that $H^1(S, \mathcal{O}_S(-D)(k-d-m)) = 0$. This implies: $H^0(S, \mathcal{O}_S(k-d)) \to H^0(D', \mathcal{O}_{D'}(k-d))$ is surjective, pour any curve D' in the liniear system $|mH + D|$ on S. We immediately deduce: if $e_{D'} - 0$, there exists a hypersurface X'' of degree k, such that $X \cap X'' = S$ and $F_{D'}^{X''} = 0$.

(ii) We consider the exact sequence

$$0 \to \mathcal{O}_S(-d) \to \mathcal{O}_S(D')(-d) \to N_{D'|S}(-d) \to 0$$

This gives a map

$$\delta_{D'} : H^1(D', N_{D'|S}(-d)) \to H^2(S, \mathcal{O}_S(-d))$$

we also have the natural map given by cup-product:

$$\beta : H^2(S, \mathcal{O}_S(-d)) \to \mathrm{Hom}\left(H^0(S, \mathcal{O}_S(d)), H^2(S, \mathcal{O}_S)\right)$$

we verify easily that β is injective as soon as $K_S \geq 0$.

It is then well known that the image $\beta \circ \delta_{D'}(F_{D'}^{X''}) \in \mathrm{Hom}\,(H^0(\mathcal{O}_S(d)), H^2(\mathcal{O}_S))$ is identified as the composite

$$H^0(\mathcal{O}_S(d)) \to H^1(T_S) \xrightarrow{\smile \lambda_{D'}} H^2(\mathcal{O}_S)$$

this map is the cup product with the class $\lambda_{D'} \in H^1(\Omega_S)$ corresponding to D, and the map $H^0(\mathcal{O}_S(d)) \to H^1(T_S)$ is provided by the exact sequence

$$0 \to T_S \to T_{X''|S} \to \mathcal{O}_S(d) \to 0$$

It is also easy to verify that $\smile \lambda_{D'}$ only depends on the "class in primitive cohomology" of D' i.e. if $\smile \lambda_{D'} = \smile \lambda_{D''}$ if $\lambda_{D'} = \lambda_{D''} + n\lambda_H$ for $n \in \mathbb{Z}$. We thus deduce $\smile \lambda_{D'} = \smile \lambda_D$.

(iii) We choose some k (and S) such that we have $H^1(\mathcal{O}_S(D)(-d)) = 0$ (it is easily seen that this condition is satisfied for k big enough).

(iv) We suppose towards a contradiciton that $e_{D'} = 0$, or D' is a smooth and chosen as in (i): there exists some X'' such that $F_{D'}^{X''} = 0$. We then deduce $\beta \circ \delta_{D'}(F_{D'}^{X''}) = 0$, and applying (ii) that $\smile \lambda_{D'} = 0$. Still according to (ii), it comes from $\smile \lambda_D$, where $\beta \circ \delta_D(F_D^{X''}) = 0$. But the choice of k, made in (iii), implies that δ_D is injective. Since β is also injective, we have deduced $F_D^{X''} = 0$, an immediately $e_D = 0$, which contradicts (A).

□

11.3.1 Proof of (A)

The normal exact sequence of $D \subset X \subset \mathbb{P}^4$ write as

$$0 \to N_{D|X} \to N_{D|\mathbb{P}^4} \to \mathcal{O}_D(d) \to 0$$

it is clear that it is enough to prove that

$$h^0(N_{D|\mathbb{P}^4}(-d)) = 0$$

We consider the following exact sequences (E_1) and (E_2):

$$0 \to N_{D|\mathbb{P}^4}(-d) \to N_{D|\mathbb{P}^4}(-d)|_{C_1} \oplus N_{D|\mathbb{P}^4}(-d)|_{C_2} \to N_{D|\mathbb{P}^4}(-d)|_p \to 0$$

$$0 \to N_{C_1|\mathbb{P}^4}(-d) \to N_{D|\mathbb{P}^4}(-d)|_{C_1} \to \mathcal{O}_p \to 0$$

We have $h^0(N_{C_i|\mathbb{P}^4}(-d)) = 1$.

It suffices to show

(i) $H^0(N_{D|\mathbb{P}^4}(-d)|_{C_1}) \cong H^0(N_{C_i|\mathbb{P}^4}(-d))$

(a) $H^0(N_{C_1|\mathbb{P}^4}(-d)) \oplus H^0(N_{C_2|\mathbb{P}^4}(-d)) \hookrightarrow H^0(N_{D|\mathbb{P}^4}(-d)_p)$

(i) By Riemann-Roch an duality, $H^0(N_{D|\mathbb{P}^4}(-d)|_{C_i}) = H^0(N_{C_i|\mathbb{P}^4}(-d))$ if and only if the inclusion

$$H^0(N^\vee_{D|\mathbb{P}^4}(-d) \otimes K_{C_i}) \hookrightarrow H^0(N^\vee_{C_i|\mathbb{P}^4}(d) \otimes K_{C_i})$$

is strict, or for $d \geq 3$, the sheaf $N^\vee_{C_i|\mathbb{P}^4}(d) \otimes K_{C_i}$ is generated by global sections. The conslusion is then immediate, using the dual of E_2.

(ii) the section of $H^0(N_{C_i|\mathbb{P}^4}(-d))$ comes from the canonical section of $N_{C_i|P_i}(-d)$ (where P_i are the planes defining C_i) for $i = 1, 2$. The assertion results immediately from the fact that the tangent spaces of P_1 and P_2 are transverse at the point p, and the local description of $N_{D|\mathbb{P}^4}$.

Remark. It is natural to think that a curve of type $C_1 \cup_p C_2$ furnishes counterexamples to (e) and (e'), indeed we consider the reducible surface $P = P_1 \cup_p P_2$, the union of planes P_1 and P_2 meeting transversally at the point p. Then its scheme-theoretic intersection with X is not the reducible curve $C_1 \cup_p C_2$, but has an embedded point, so that D is not a global complete intersection $P \cap X$.

Conserning the other points of the conjecture of Griffiths and Harris, we can make the (perhaps obvious) remark:

Lemma 11.3.2. (ii) $\implies$ (i)

Proof. We suppose that the general hypersurface X contains a curve of degree m, and that its image under the Abel-Jacobi map φ_X is not zero.

There exists an irreducible variety W equipped with a proper map $p : W \to \mathscr{X}$ where $\mathscr{X} = \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}(d))$, such that the fiber over X parametrizes curves of degree m inside X; two such curves are homologous and for general X, we hace $\forall C, C' \in p^{-1}(X) : \varphi_X(C - C') = 0$; in fact, this remains true for all smooth X. In effect, if H denotes a plane section of X, we have, for general X, $\varphi_X(dC - mH) = 0$ for all $C \in p^{-1}(X)$. By irreducibility of W, this remains true for all $X \in \mathscr{X}$. Then $\varphi_X(C - D')$ is a torsion point, constant over the connected components of $p^{-1}(X) \times p^{-1}(X)$. But the normality of $\mathscr{X}$ and irreducibility of W imply that if $W \to W_1 \to X$ is the Stein factorization of p, each irreducible component of the product $W_1 \times_{\mathscr{X}} W_1$ dominates $\mathscr{X}$. This leads easily to the result.

We fix a line Δ in $\mathbb{P}^4$ and we write $\mathscr{X}_\Delta$ for the family of hypersufraces of degree d containig Δ. We notation $W_\Delta := p^{-1}(\mathscr{X}_\Delta)$; we have then a normal function ν_Δ defined as follows on $\mathscr{X}_\Delta$: let X be a smooth point of $\mathscr{X}_\Delta$ and let $C \in p^{-1}(X)$; then $\deg(m\Delta - C) = 0$ and we can set $\nu_\Delta(X) = \varphi_X(m\Delta - C)$.

But it is known that the group of normal functions on $\mathscr{X}_\Delta$ is cyclic generated by the normal unction ν^H_Δ defined by: $\nu^H_\Delta(X) = \varphi_X(d\Delta - H)$ (it suffices to generalize the arugment of [3], section 3). We then deduce that there exist a k such that for all smooth X in $\mathscr{X}_\Delta$ we have $\Phi_X(m\Delta - C) = k\varphi_X(d\Delta - H)$ for $C \in p^{-1}(X)$.

As Δ is deformation equivalent to Δ' we have in fact $k = k'$; on $\mathscr{X}_\Delta \cap \mathscr{X}_{\Delta'}$, so $(m - kd)\Phi_X(\Delta - \Delta') = 0$. But by Griffiths [3], if X is general in $\mathscr{X}_\Delta \cap \mathscr{X}_{\Delta'}$ then $\Phi_X(\Delta - \Delta') \in J(X)$ is not a torsion point. Then $m - kd = 0$ which proves (i). $\square$

11.4 Conclusion

In paragraph 1 we cleared the necessary condition 1.2 for the curve C to be a complete intersection $X \cap \Sigma$ of X with a surface $\Sigma \subset \mathbb{P}^4$. If d divides the degree of C, this condition is automatically saisfied when C is sub-canical (i.e. $K_C = \mathcal{O}_C(m)$ for some m). Likewise, it seems difficult to construct by an analogous process to the one described in paragraphs 1 and 2, a sub-canonical curve for which the normal bundle sequence is not eact. It does therefore not exclude that (e) or (e') might hold for sub-canonical curves.