Physics C2801, Fall 2013 Prof. Brian Cole

Issued: **Nov. 27**	Problem Set # 9	Due: **Dec. 4**

Problem 1 Hyperbolic functions in Lorentz transformations and hyperbolic identities

We saw in lecture that hyperbolic trigonometric functions are valuable in expressing the Lorentz transformation. Remember that the hyperbolic sine and cosine functions are defined:

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} \\ \sinh x &= \frac{e^x - e^{-x}}{2} \end{aligned}$$

a. Prove the following identities:

1. $\cosh^2 y - \sinh^2 y = 1$.
2. $\cosh(y_1 + y_2) = \cosh y_1 \cosh y_2 + \sinh y_1 \sinh y_2$.
3. $\sinh(y_1 + y_2) = \sinh y_1 \cosh y_2 + \cosh y_1 \sinh y_2$.
4. Obtain the formula for $\tanh(y_1 + y_2)$.

b. As we saw in lecture, we can identify the Lorentz boost γ_B with $\cosh y_b$ where y_B is the rapidity of the boost and we can associate $\beta_B \gamma_B$ with $\sinh y_B$ with the result that $\beta_B = \tanh y_B$. Using the expression for $\tanh y_B$ in terms of exponentials, show that the boost rapidity can be obtained from the boost velocity by

$$y_B = \frac{1}{2} \ln \left(\frac{1 + \beta_B}{1 - \beta_B} \right)$$

c. Show that you obtain the expected result for the Lorentz transformation for $\beta_B = 0$. Calculate the values of the boost rapidity for $1 - \beta_B = 1 \times 10^{-n}$ where $n = 1, 2, 3, 4, 6, 8, 10$.

c. Using the Lorentz transformations expressed in terms of rapidity (in natural units),

$$\begin{aligned} x' &= x \cosh y_B - t \sinh y_B && (1) \\ t' &= t \cosh y_B - x \sinh y_B && (2) \\ & && (3) \end{aligned}$$

show explicitly that two successive boosts by rapidities y_{B1} and y_{B2} are equivalent to a single boost by a rapidity $y_{B12} = y_{B1} + y_{B2}$.

d. We can define the rapidity, y of a particle with respect to the x axis: $y \equiv \tanh^{-1} \beta_x$ or, similar to part b,

$$y = \frac{1}{2} \ln \left(\frac{1 + \beta_x}{1 - \beta_x} \right).$$

Show that under a Lorentz transformation by a rapidity y_B, $y' = y - y_B$.

Problem 2. Boosts along general direction

For convenience we have written the form for Lorentz transformations assuming Lorentz boosts along the x axis. Since we can generally choose which axis is the x axis, choosing the x axis to be the axis for boosts is typically not a problem. However, it is useful to know how to generalize the Lorentz transformation to be in other directions. Writing the LT for boosts along (e.g.) the z axis is straight-forward (in natural units)

$$
\begin{aligned}
t' &= \gamma_{\rm b}\, t - \beta_B \gamma_{\rm B}\, z \\
z' &= \gamma_{\rm b}\, z - \beta_B \gamma_{\rm B}\, t \\
x' &= x \\
y' &= y
\end{aligned}
$$

A more interesting question is how to write the LT for a boost in a direction that is not along one of the coordinate axes, i.e. that is in the $\vec{\beta}_{\rm B}$ direction. We can use a unit vector $\hat{\beta_B} \equiv \vec{\beta}_{\rm B}/|\vec{\beta}_{\rm B}|$ to select directions along the boost. So, if we use $\vec{r}$ as the 3-position vector, then $\vec{r}\cdot\ \hat{\beta_B}$ is the component of $\vec{r}$ in the direction of the velocity. The remainder of $\vec{r}$, namely $\vec{r} - (\vec{r}\cdot\ \hat{\beta_B})\hat{\beta_B}$ is orthogonal to $\vec{\beta}_B$.

a. Using this decomposition of $\vec{r}$ show that the Lorentz transformation can be written for a boost along the direction $\vec{\beta}_{\rm B}$,

$$
\begin{aligned}
t' &= \gamma_{\rm b}\, t - \gamma_{\rm b}(\vec{\beta}_{\rm B}\cdot\vec{r}) \\
\vec{r}' &= \vec{r} + (\gamma_{\rm b}-1)\frac{\vec{\beta}_{\rm B}(\vec{\beta}_{\rm B}\cdot\vec{r})}{\vec{\beta}_{\rm B}^{\,2}} - \gamma_{\rm b}\vec{\beta}_{\rm B}\, t
\end{aligned}
$$

b. Show that the general formula reduces to the result we have used for transformations along the x axis.

c. Show that the general transformation in part a preserves the interval $t^2 - \vec{r}\cdot\vec{r}$.

d. Write out the transformation matrix for a boost velocity that is in the $x-y$ plane (i.e. for which $\vec{\beta}_{\rm B}\cdot\hat{k} = 0$). We will use this result in the next part of the problem.

e. I mentioned in class a couple of times in response to questions that special relativity and angular momenta are intimately related. One way to see this is to look at two Lorentz boosts in directions that are not parallel. We will take a special case for which the algebra is tractable (for the general case the algebra is downright hideous), namely a boost along the x direction by velocity β_B, followed by a boost with a small velocity in the $x'-y'$ plane – $\vec{\beta}_{\rm B}{}' = \beta'_{Bx}\hat{\imath}' + \beta'_{By}\hat{\jmath}'$. Note: we specify the components of the velocity for the second boost in the coordinates of the primed frame (resulting from the first boost) not in terms of the original coordinates. Write out the product of the transformation matrices (keep the order straight!) keeping terms only to linear power in β'_{Bx} and β'_{By}. This transformation will transform positions and time (t, x, y, z) from the unprimed to the double-primed frame $(t,'', x'', y'', z'')$.

f. We can evaluate the relative velocity between the unprimed and double-primed frame by transforming the four-vector representing the origin of the unprimed frame $(t, 0, 0, 0)$ to the double-primed frame. The three components of the resulting four-vector give the velocity of the unprimed frame relative to the double-primed frame. When evaluating the relative velocity, ignore any components of β'_B when they are added to β_B since we are taking those components to be small.

g. Now, suppose we use the velocity obtained form part f to (nominally) boost back to the original unprimed frame. We need to boost in the direction of the relative velocity vector. Write the product of the Lorentz boost in the direction and the product transformation matrix from part e. Naively we might expect to get the identity transformation matrix since we have "undone" the original two boosts. But you should find, in fact, that the resulting transformation is a rotation around the z axis. This means that the original product transformation in part e is a combination of a pure Lorentz boost in the direction of (opposite to) the relative velocity from part f and a rotation around the z axis.

Problem 3. Four-velocity, four-acceleration, four-force

In lecture we have seen that the four-velocity can be obtained from the coordinate four vector $\mathbf{X}$, using $\mathbf{U} = \dfrac{d\mathbf{X}}{d\tau}$, where $d\tau$ is the differential change in proper time – i.e. the differential time elapsed in the rest frame of the particle. We saw that writing $d\tau = dt/\gamma$ for a particle moving with velocity $\vec{\beta}$ $\gamma = 1/\sqrt{1-\vec{\beta}^2}$ gives $U = (\gamma, \beta_x\gamma, \beta_y\gamma, \beta_z\gamma)$. Since X is a four-vector and $d\tau$ is a Lorentz invariant, we saw that U also transforms as a four-vector under Lorentz transformations.

a. Similar to the definition of the four-velocity, we saw that we could obtain a four-acceleration from, $\mathbf{a} = \dfrac{d\mathbf{U}}{d\tau}$. Using the transformation properties of the four-acceleration (and the four-velocity), evaluate the non-relativistic three-acceleration ($\dot{\vec{\beta}}$) of a particle in a frame where it is moving with velocity $\vec{\beta} = \beta\hat{i}$ when the *proper* acceleration is $\vec{a}_p$. Your result for the x component should be consistent with Kleppner and Kolenkow problem 12.11.

b. We can also define a four-force, $\mathbf{F} = \dfrac{d\mathbf{p}}{d\tau}$ where $\mathbf{p}$ is the four-momentum. The four-force also transforms like a four-vector. In lecture I told you that the ordinary force $\vec{F}$ satisfies $\vec{F} = \dfrac{d\vec{p}}{dt}$ where $\vec{p}$ is the spatial part of the relativistic momentum, $\vec{p} = m\vec{\beta}\gamma$. From that starting point show that the spatial components of the four force are given by $\mathbf{F}_x = \gamma\vec{F}_x$ (and the same for the y and z components).

c. Use the result from part c to evaluate the Newtonian force on a particle in a frame where it is moving with velocity $\vec{\beta} = \beta\hat{i}$ when the *proper* force (the force in the rest frame of the particle) is $\vec{F}_p$. You may be surprised by the results as they may seem counter-intuitive to you. **We will use this result extensively next semester.**

Problem 4. Kinematics of protons in the Large Hadron Collider

In 2012, the LHC accelerated beams of protons to an energy of 4 TeV (4×10^{12} eV). The LHC uses two such beams that propagate in opposite directions around a approximately circular ring that has a circumference of 27 km. The beams "cross" each other at four intersection points. Cross is a bit of a misnomer: magnets placed on both sides of the intersection regions direct the beams so that they are nearly (anti)parallel over a limited distance. Thus, proton-proton collisions when one proton propagating in the $+\hat{i}$ direction collides with a proton propagating in the $-\hat{i}$ direction. Because the total momentum of the collisions is zero, the laboratory frame is also the center-of-mass frame for the proton-proton collisions. Perform the following calculations or answer the following questions taking the mass of the proton to be 0.94 GeV/c^2.

a. Calculate the momentum of the protons in each beam expressing your result in TeV/c units.

b. Calculate the velocities, Lorentz factor (γ), and the rapidities of the protons in the two beams.

c. What is the total energy of the proton-proton collisions?

d. Determine the boost rapidity needed to boost from the laboratory frame to the rest frame of the protons moving in the $-\hat{\imath}$ direction.

e. Calculate the magnitude of the momentum and the energy of the protons in the beam moving in the $\hat{\imath}$ direction as observed form the rest frame of the protons moving in the $-\hat{\imath}$ direction.

f. Calculate the velocity, Lorentz factor (γ), and the (absolute value) rapidity of the protons under the conditions in part d.

g. Calculate the total energy of the proton-proton collisions as observed from the rest frame of the beam moving in the $-\hat{\imath}$ direction. How do you understand the difference between this result and the result in part c?

h. Calculate the magnitude of the force needed to keep the protons in the LHC ring. Make sure your calculation is relativistically correct. That force is provided by the magnetic fields generated in the LHC dipole magnets.

Problem 5. Cosmic ray muons

The average energy of muons produced in high-energy cosmic ray collisions in the earth's atmosphere is 6 GeV (6×10^9 eV) as seen in the frame of the earth's atmoshpere. Muons have a mass of approximately 106 MeV (million eV) or 0.106 GeV. For convenience we will take the direction at which the muons are propagating to be the $\hat{\imath}$ direction.

a. Calculate the magnitude of the momentum for such muons.

b. Calculate the magnitude of the velocity, the Lorentz factor, and the rapidity for such muons.

c. Muons can decay to electrons (mass 0.511 MeV/c^2) and anti-neutrinos (approximately massless). In the rest frame of the muon, what are the energies and momenta of the electrons and neutrinos produced by such muon decays.

d. Suppose the muon decays such that the electron is emitted in exactly the same direction as the muon (i.e. the $\hat{\imath}$ direction). Calculate the momenta (magnitude) and energies of the electron and (anti)-neutrino in the frame of the earth's atmosphere?

e. Suppose the muon decays such that as observed in its rest frame, the electron is emitted perpendicular to the $\hat{\imath}$ direction. What are the momentum (magnitude) and energies of the electron and the neutrino in the frame of the earths atmosphere?

f. Continuing with part e, what are the angles of the electron and neutrino with respect to the $\hat{\imath}$ axis as observed in the frame of the earths atmoshpere?

Problem 6. Kleppner and Kolenkow, problem 13.2

Problem 7. Kleppner and Kolenkow, problem 13.4

I would like you to do this problem two ways. First, do it the "brute force" way using the Lorentz transformation to transform the quantities from one frame to another. Then I would like you to do it using Lorentz invariants. In particular, use the invariance of $s \equiv \mathbf{p} \cdot \mathbf{p}$ where $\mathbf{p} = \mathbf{p_1} + \mathbf{p_2}$ with $\mathbf{p_1}$ and $\mathbf{p_2}$ the four-momenta of the two particles. If you express $p \cdot p$ in the two frames and apply the relevant kinematic constraints in the two frames you should obtain the same result as the brute force approach, but more simply.

Problem 8. Kleppner and Kolenkow, problem 13.8