1 Examples of stable ∞-cagtegories

1.1 dg-Categories

Review, let $\mathcal{A}$ be an abelian cagtegory with enough projectives. Then there is a category $D^-(\mathcal{A})$ the derived category of bounded-above.

$\mathbf{Ch}\left(()\,\mathcal{A}\right)$ is naturally a dg-Category meaning it is enriched in $\mathbf{Ch}\left(()\,\mathbf{Ab}\right)$. Indeed, if $A, B \in \mathbf{Ch}\left(()\,\mathcal{A}\right)$ then we define,

$$\mathrm{Hom}\left(A_\bullet, B_\bullet\right)_n = \prod_k \mathrm{Hom}\left(A_k, B_{k-n}\right)$$

with differential,

$$(\mathrm{d}f)_k = f_{k+1} \circ \mathrm{d}_B + (-1)^{n+1} \mathrm{d}_B \circ f_k$$

Then we check that,

$$H^0(\mathrm{Hom}\left(A, B\right)_\bullet) = \mathrm{Hom}_{\mathbf{Ch}(()\mathcal{A})}\left(A, B\right)$$

1.2 Program

Given a dg-category, we get a simplicially enriched category by truncating and then applying Dold-Kan and then apply the homotopy-coherent nerve.

However, this requires checking many steps. We can instead go directly with the following construction.

1.3 Construction

Definition 1.3.1. A *dg-category* is a category enriched in $\mathbf{Ch}\left(()\,\mathcal{A}\right)$. This includes the requirement that,

$$\mathrm{d}(g \circ f) = \mathrm{d}g \circ f + (-1)^{\deg g} g \circ \mathrm{d}f$$

which arises from needing to preserve the monoidal structure on $\mathbf{Ch}\left(()\,\mathcal{A}\right)$ given by graded tensor $A \otimes B$.

We will define a dg-Nerve which goes directly from dg-categories to ∞-categories which does not require passing through simplically-enriched categories.

Definition 1.3.2. Let C be a dg-category. Then the *dg-Nerve* is the simplicial set $N_{\mathrm{dg}}(C)$ where $N_{\mathrm{dg}}(C)_n$ is the set of objects X_i for $i \in \{0, \dots, n\}$ and for each ordered set $I = \{i_- < i_m < \cdots < i_1 < i_+\}$ for $m \geq 0$ of elements in $\{0, \dots, n\}$ we have map,

$$f_I \in X_{i_-} X_{i_+}$$

such that,

$$\mathrm{d}f_I = \sum_{1 \leq j \leq m} (-1)^i (f_{I \setminus \{i_j\}} - f_{\{i_j < \cdots < i_+\}} \circ f_{\{i_- < \cdots < i_j\}})$$

and the maps work as follow. If $\alpha : [m] \to [n]$ is monotone. Then the induced map α^* is given by,

$$\alpha^*(\{X_i\}, \{f_I\}) = (\{X_{\alpha(i)}\}, \{g_I\})$$

where,

$$g_J = \begin{cases} f_{\alpha(J)} & \alpha|_J \text{ injective} \\ \mathrm{id}_{X_i} & J = \{j, j'\} \text{ and } \alpha(j) = j' \\ 0 & \text{else} \end{cases}$$

Proposition 1.3.3. For any dg-category C the dg-Nerve $N_{\mathrm{dg}}(C)$ is an ∞-category.

Proof. We need to show that inner horns can be filled. However, $\Lambda^n_i \to N_{\mathrm{dg}}(C)$ this is the same data as specifying $\Delta^n \to N_{\mathrm{dg}}(C)$ except we haven't specified all the maps f_I. In fact, this has specified the maps for all I except for $I = [n]$ and $I = [n]\backslash\{i\}$. Then we set $f_{[n]} = 0$ and,

$$f_{[n]\backslash\{i\}} = \sum_{0<p<n} (-1)^{p-i} f_{\{p,\dots,n\}} \circ f_{\{0,\dots,p\}} - \sum_{p<p<n, p\neq i} (-1)^{p-i} f_{[n]\backslash\{0\}}$$

$\square$