1 Splitting Types of Finite Monodromy Vector Bundles

Q: Let C be a general genus g curve. Does there exists a smooth nonisotrivial curve of relative genus h over C.

Let H be a finite group, $\#H = d$. Consider a cover of curves $f : X \to \mathbb{P}^1$ with Galois group H. What is the splitting type of,

$$f_* \mathcal{O}_X = \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

where we require $a_1 \geq a_2 \geq \cdots \geq a_d$. What can we say about the numbers a_i?

First Naive guess 1: maybe the a_i are equal? This is impossible from,

$$H^i(\mathbb{P}^1, f_* \mathcal{O}_X) = H^i(X, \mathcal{O}_X)$$

Therefore,

$$\deg f_* \mathcal{O}_X = \sum_i a_i$$

Naive guess 2: maybe they all differ by 1.

Remark. $a_1 = 0$ meaning there is a copy of $\mathcal{O}_{\mathbb{P}^1}$. Indeed,

$$h^0(\mathbb{P}^1, f_* \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$$

But $\mathcal{O}_{\mathbb{P}^1}$ is the only line bundle $\mathcal{L}$ on $\mathbb{P}^1$ with $h^0(\mathbb{P}^1, \mathcal{L}) = 1$. Therefore, $a_i < 0$ for $i > 1$.

Remark. Observe, guess 2 is also wrong because,

$$\sum a_i = 1 - d - g$$

so some $a_i < -1$ when $g > 0$ but $a_0 = 0$ and thus guess 2 is wrong.

1.1 Decomposition

$$f_* \mathcal{O}_X = \bigoplus_{\rho} E_{\rho}^{\oplus \dim \rho}$$

where we sum over irreps of H. This is because H acts on the fiber via the regular representation.

Next guess: maybe for fixed ρ the E_ρ is balanced.

Example 1.1.1. If $H = S_m$ choose a finite cover $f : X \to \mathbb{P}^1$ then E_{std} is called the Tschirhousen bundle. Coskun-Larson-Vogt showed E_ρ is balanced after deforming the cover.

Example 1.1.2. Let $H = S_5$ and $\rho = \text{std}$ choose $f : X \to \mathbb{P}^1$ such that,

$$E_\rho = \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}(-2)^{\oplus 2}$$

balanced. Then if f is the galois cover of a simply branched can show,

$$E_{\wedge^2 \rho} \cong \wedge^2(E_\rho)$$

Given,

$$\begin{tikzcd} & X \arrow[dl] \arrow[dd, "f"] \\ Z \arrow[dr, "h"] & \\ & \mathbb{P}^1 \end{tikzcd}$$

where $\deg h = 5$ simply branched and f is an S_4 which is its Galois cover. Then concretely,

$$E_\rho = h_* \mathcal{O}_Z / \mathcal{O}_{\mathbb{P}^1}$$

However, then,

$$E_{\wedge^2 \rho} \cong \mathcal{O}(-2) \oplus \mathcal{O}(-3)^{\oplus 3} \oplus \mathcal{O}(-4)$$

Theorem 1.1.3 (L-Litt)**.** For a general $f : X \to \mathbb{P}^1$ if we decompose,

$$E_\rho = \bigoplus_{i=1}^{r} \mathcal{O}(b_i)$$

then $|b_i - b_{i+1}| \le 1$ so the b_i are consecutive.

Definition 1.1.4. If E is a vector bundle on a curve Y, the *slope* of E is,

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E}$$

we say that,

(a) E is *semi-stable* if for all subbundles $F \subset E$ then,

$$\mu(F) \le \mu(E)$$

(b) E is *stable* if for all subbundles $0 \subsetneq F \subsetneq E$

$$\mu(F) < \mu(E)$$

Example 1.1.5. If V is semistable on $\mathbb{P}^1$ then,

$$V \cong \mathcal{O}(a)^{\oplus b}$$

Theorem 1.1.6. Any E on Y has a filtration,

$$0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = E$$

where E_i / E_{i+1} is semistalbe and,

$$\mu(E_i / E_{i-1}) > \mu(E_{i+1} / E_i)$$

called the Harder-Narishiman filtration.

Example 1.1.7. If $Y = \mathbb{P}^1$ then,

$$E \cong \mathcal{O}(3)^{\oplus 2} \oplus \mathcal{O}(5)^{\oplus 7}$$

then the HN filtration is,

$$0 \subset \mathcal{O}(5)^{\oplus 7} \subset E$$

Theorem 1.1.8 (L-Litt)**.** If $f : X \to Y$ is an H-cover general. Then,

$$f_*\mathcal{O}_X \cong \bigoplus_\rho E_\rho^{\oplus \dim \rho}$$

then,

(a) the slopes of consecutive HN graded parts of E_ρ differ by ≤ 1

(b) if $\operatorname{rank} \rho < 2\sqrt{g+1}$ then E_ρ is semistable.

Theorem 1.1.9. Riemann-Hilbert gives,

$$\{\rho : \pi_1 \to \mathrm{GL}_r \text{ with finite image}\} \iff \{E_\rho \subset f_*\mathcal{O}_X \mid f \text{ unramified}\}$$

Theorem 1.1.10. Given irrep $\rho : \pi_1(Y) \to \mathrm{GL}_r(\mathbb{C})$ after deforming the complex structure on Y we can arrange that E_ρ satisfies,

(a) the slopes of consecutive HN graded parts of E_ρ differ by ≤ 1

(b) if $\operatorname{rank} \rho < 2\sqrt{g+1}$ then E_ρ is semistable.

Definition 1.1.11. A local system $\mathcal{L}$ on Y is *of geometric origin* if there is a dense Zariski open $U \subset Y$ and smooth proper $f : Z \to U$ such that $\mathcal{L}|_U \subset R^j f_*\mathcal{C}$.

Theorem 1.1.12 (L-Litt)**.** If $\operatorname{rank} \mathcal{L} < 2\sqrt{g+1}$ and Y is a general curve of genus g and $\mathcal{L}$ is of geometric origin then $\mathcal{L}$ has finite monodromy.

Proof. Idea: have compact image as reps by some correspondence. Then finite is compact and discrete and the discreteness comes from integral structure on cohomology. ☐

Theorem 1.1.13 (L-Litt)**.** There is no family in the question if $h < \sqrt{g+1}$.

Proof. Suppose we had $f : S \to C$ nonisotrivial. Then we get $\mathcal{L} = R^1 f_*\mathcal{C}$ gives a local system on C which is of genometric origin. So by the theorem it has finite monodromy. However, this implies that the family is isotrivial, is trivialized by a finite cover trivializing $\mathcal{L}$. ☐

Theorem 1.1.14. If $(C, x_1, \ldots, x_n)$ are general in $\mathcal{M}_{g,n}$ then there are no nonisotrivial smooth families $S \to C \backslash \{x_1, \ldots, x_n\}$ for $h < \sqrt{g+1}$.

Remark. However, there are smooth covers of relative genus $h \sim e^g$. Therefore, we don't really know if these bounds are sharp.