

1 Local Cohomology

Definition 1.0.1. A local group or local system of groups \mathcal{L} is a locally-constant sheaf of abelian groups. We write $\mathfrak{Loc}(X)$ for the category of local systems on X .

Theorem 1.0.2. Let X be a locally-path-connected (AND) topological space. Then there is a equivalence of categories between the category of local groups on X and the category of actions of the fundamental groupoid $\Pi(X)$ on abelian groups.

Proof. There is a functor $\mathfrak{Loc}(X) \rightarrow \mathbf{AbGrp}^{\Pi(X)}$ sending a local system to its monodromy action. For any path $\gamma : I \rightarrow X$ and a point $\gamma(t)$ there is a open connected neighborhood $\gamma(t) \in U_t$ small enough such that $\mathcal{L}|_{U_t} \cong \underline{G}|_{U_t}$ for some abelian group G . Then $\gamma^{-1}(U_t)$ cover I which is compact so we may choose finitely many U_i which cover the path and we may assume that $U_i \cap U_{i+1} \neq \emptyset$. Then since both are connected and \mathcal{L} is constant on each we get isomorphisms,

$$\begin{array}{ccc} \mathcal{L}(U_i) & & \mathcal{L}(U_{i+1}) \\ & \searrow \sim & \swarrow \sim \\ & \mathcal{L}(W) & \end{array}$$

where W is a connected component of $U_i \cap U_{i+1}$. Thus $\mathcal{L}(U_i) \xrightarrow{\sim} \mathcal{L}(U_{i+1})$. Inductively, this gives $\mathcal{L}(U_0) \xrightarrow{\sim} \mathcal{L}(U_n)$ which, since it is well-defined after shrinking the neighborhoods admits restricting to stalks, gives the monodromy map $[\gamma] : \mathcal{L}_{\gamma(0)} \rightarrow \mathcal{L}_{\gamma(1)}$. Clearly this construction respects composition. Furthermore, we can do the exact same construction for maps $I^2 \rightarrow X$ showing that the identifications everywhere commute under homotopy. Explicitly, let $h : I^2 \rightarrow X$ be a path homotopy between $\gamma_1 : I \rightarrow X$ and $\gamma_2 : I \rightarrow X$ then for each t let $h(t, -) : I \rightarrow X$ be the path homotoping the point $\gamma_1(t)$ to $\gamma_2(t)$. Then $[h(t_2, -)] \circ [\gamma_1(t_1 \mapsto t_2)] = [\gamma_2(t_1 \mapsto t_2)] \circ [h(t_1, -)]$ as maps $\mathcal{L}_{\gamma_1(t_1)} \rightarrow \mathcal{L}_{\gamma_2(t_2)}$. Since at the endpoints $h(0, -) = h(1, -)$ is the constant path then we see that $[\gamma_1] = [\gamma_2]$. Therefore, monodromy defined a functor $M_{\mathcal{L}} : \Pi(X) \rightarrow \mathbf{AbGrp}$.

Now I claim this association $\mathcal{L} \mapsto M_{\mathcal{L}}$ is functorial. Given a morphism $\eta : \mathcal{L} \rightarrow \mathcal{L}'$ of local groups we get maps $\eta_x : \mathcal{L}_x \rightarrow \mathcal{L}'_x$ which commute with restriction and thus with the monodromy construction i.e. a natural transformation between functors $M_{\mathcal{L}}$ and $M_{\mathcal{L}'}$.

Now we need to show that $\mathcal{L} \mapsto M_{\mathcal{L}}$ is fully faithful.

Finally, $M : \mathfrak{Loc}(X) \rightarrow \mathbf{AbGrp}^{\Pi(X)}$ is essentially surjective. (PROVE THIS) □

Remark. When X is connected, then groupoid $\Pi(X)$ -representations are simply group representations of $\pi_1(X, x_0)$.

Definition 1.0.3. Let X be a locally-path-connected. For each $n > 1$ (for $n = 1$ the representation is simply the inner automorphism representation of a groupoid) there is a groupoid representation $\pi_n(X) : \Pi(X) \rightarrow \mathbf{AbGrp}$ which generalizes the action at each point $\pi_1(X, x_0) \curvearrowright \pi_n(X, x_0)$. By the above theorem, this corresponds to a local group $\underline{\pi_n(X)}$.

2 Action on Fibres of Fibration

Theorem 2.0.1. Let $F \hookrightarrow E \xrightarrow{\sim} B$ be a fibration. Then there is a groupoid action $\Pi(B)$ on the space of fibres and in particular $\pi_1(B, x_0) \rightarrow \text{Aut}(F)$.

Proof. Consider a path $\gamma : I \rightarrow B$ from x_1 to x_2 and then the diagram,

$$\begin{array}{ccc} F_{x_1} & \xhookrightarrow{\quad} & E \\ \downarrow & \nearrow \tilde{\gamma} & \downarrow p \\ F_{x_1} \times I & \xrightarrow{\quad \gamma \quad} & B \end{array}$$

By homotopy lifting we get a map $\tilde{\gamma} : F_{x_1} \times I \rightarrow E$ lifting $\gamma : F_{x_1} \times I \rightarrow B$. Then $p \circ \tilde{\gamma} = \gamma$ so $\tilde{\gamma}(-, 1) \subset F_{x_2}$ since $p \circ \tilde{\gamma}(-, 1) = \gamma(1) = x_2$. Therefore we get a map $[\gamma] : F_{x_1} \rightarrow F_{x_2}$ via $[\gamma](x) = \tilde{\gamma}(x, 1)$.

I claim that two lifts of homotopic paths are homotopic. Given two paths $\gamma_1, \gamma_2 : I \rightarrow B$ and a path homotopy $h : I^2 \rightarrow B$ and two lifts $\tilde{\gamma}_1, \tilde{\gamma}_2 : F_{x_1} \times I \rightarrow E$ we want a map $F_{x_1} \times I^2 \rightarrow E$ above $h : \times I^2 \rightarrow B$. This map is defined on $F_{x_1} \times (I \times \{0, 1\} \cup \{0\} \times I)$ via $\tilde{\gamma}_1$ on $F_{x_1} \times I \times \{0\}$ and $\tilde{\gamma}_2$ on $F_{x_1} \times I \times \{1\}$ any by inclusion of the fibre F_{x_1} on $F_{x_1} \times \{0\} \times I$ (constant on I) since $h_{\{0\} \times I}$ is constant since it is a path homotopy. Then by homotopy lifting, we get $\tilde{h} : F_{x_1} \times I \times I \rightarrow E$ such that $p \circ \tilde{h} = h$ and thus $\tilde{h}(-, 1, -) : F_{x_1} \times I \rightarrow F_{x_2}$ gives a homotopy from $[\gamma_1] : F_{x_1} \rightarrow F_{x_2}$ to $[\gamma_2] : F_{x_1} \rightarrow F_{x_2}$.

Therefore, we have a representation of $\Pi(B)$ on **hTop** sending $x \mapsto F_x$ and $\gamma \mapsto [\gamma]$. \square

3 Serre - Vanishing

Remark. First we prove the result for the case \mathbb{P}_R^n .

Theorem 3.0.1. Let $\mathbb{P}^n = \mathbb{P}_R^n$. For any coherent $\mathcal{O}_{\mathbb{P}^n}$ -module \mathcal{F} there is some $r > 0$ such that,

$$H^i(\mathbb{P}_R^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(s)) = 0$$

for all $i > 0$ and $s \geq r$.

Proof. Since this holds for $i > n$ we may apply reverse induction on i . Assume the theorem holds for $i + 1$ and let \mathcal{F} be some coherent sheaf. Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is ample, for some $\ell > 0$ the sheaf $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(\ell)$ is generated by global sections,

$$\bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(\ell)$$

and thus tensoring by $\mathcal{O}_{\mathbb{P}^n}(-\ell)$ we get a surjection,

$$\bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n}(-\ell) \twoheadrightarrow \mathcal{F}$$

which we may extend to an exact sequence,

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n}(-\ell) \longrightarrow \mathcal{F} \longrightarrow 0$$

Since $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free it is flat (exactness can be checked on stalks) so we get a short exact sequence,

$$0 \longrightarrow \mathcal{G}(d) \longrightarrow \bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^n}(d - \ell) \longrightarrow \mathcal{F}(d) \longrightarrow 0$$

Applying the LES of homology we get,

$$\bigoplus_{j=1}^N H^i(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}^n}(d - \ell)) \longrightarrow H^i(\mathbb{P}_R^n, \mathcal{F}(d)) \longrightarrow H^{i+1}(\mathbb{P}_R^n, \mathcal{G}(d))$$

By the induction hypothesis, for all sufficiently large $d \geq r_{\mathcal{G}}$ the cohomology $H^{i+1}(\mathbb{P}_R^n, \mathcal{G}(d)) = 0$ vanishes and furthermore by explicit calculation, $H^i(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}^n}(d - \ell)) = 0$ for $i > 0$ and $d \geq \ell$ so take $r_{\mathcal{F}} = \max\{\ell, r_{\mathcal{G}}\}$ and then for $d \geq r_{\mathcal{F}}$ we find,

$$H^i(\mathbb{P}_R^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)) = 0$$

proving the result by induction. \square

Theorem 3.0.2. Let R be a noetherian ring and $X \rightarrow \operatorname{Spec}(R)$ proper. Furthermore, let \mathcal{L} be an ample line bundle on X . Then for any coherent \mathcal{O}_X -module \mathcal{F} there is some $r > 0$ such that,

$$H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = 0$$

for all $i > 0$ and $s \geq r$.

Proof. Since $X \rightarrow \operatorname{Spec}(R)$ is finite type and X has an ample line bundle \mathcal{L} then X must be quasi-projective over R for some immersion $\iota : X \rightarrow \mathbb{P}_R^N$ where $\mathcal{L}^{\otimes d} = \iota^* \mathcal{O}_{\mathbb{P}^N}(1)$. Since $X \rightarrow \operatorname{Spec}(R)$ is proper and $\mathbb{P}_R^N \rightarrow \operatorname{Spec}(R)$ is separated then $\iota : X \rightarrow \mathbb{P}_R^N$ is automatically proper hence a closed immersion so X is projective.

Being a closed immersion $\iota : X \rightarrow \mathbb{P}_R^N$ is affine so we may compute (the Leray spectral sequence degenerates),

$$H^i(X, \mathcal{G}) = H^i(\mathbb{P}_R^N, \iota_* \mathcal{G})$$

for any quasi-coherent sheaf on X . Therefore, considering the coherent sheaf $\mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}$ it suffices to compute,

$$H^i(\mathbb{P}_R^N, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}))$$

We will apply the projection formula noting that writing $s = nd + r$ gives,

$$\mathcal{L}^{\otimes s} = (\mathcal{L}^{\otimes d})^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r} = (\iota^* \mathcal{O}_{\mathbb{P}^N}(1))^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r} = \iota^* \mathcal{O}_{\mathbb{P}^N}(n) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}$$

Therefore, let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^N}(n)$ in the projection formula to find that,

$$\iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r} \otimes_{\mathcal{O}_X} \iota^* \mathcal{O}_{\mathbb{P}^N}(n)) = \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(n)$$

Since $\iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r})$ is coherent the previous proposition allows us to choose n large enough (taking the maximum of the n large enough to kill the cohomology of each of $r = 0, 1, \dots, d-1$) so that,

$$H^i(\mathbb{P}_R^N, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(n)) = 0$$

for any $r = 0, 1, \dots, d-1$ and $n \gg 0$. Therefore, for all sufficiently large s we have,

$$H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = H^i(\mathbb{P}_R^N, \iota^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s})) = H^i(\mathbb{P}_R^N, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes r}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(n)) = 0$$

\square

Theorem 3.0.3 (projection formula). Let $f : X \rightarrow Y$ be a morphism of ringed spaces \mathcal{F} a \mathcal{O}_X -module and \mathcal{E} a finite locally free \mathcal{O}_Y -module. Then,

$$R^q f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{E}) = R^q f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$$

Theorem 3.0.4. Let X be projective, then the functors $\text{Ext}_{\mathcal{O}_X}^i(-, \mathcal{G}) : \mathbf{Coh}(\mathcal{O}_X) \rightarrow \mathbf{Mod}_{\Gamma(X, \mathcal{O}_X)}$ for a fixed quasi-coherent \mathcal{O}_X -module \mathcal{G} are universal contravariant δ -functors.

Proof. It suffices to show that $\text{Ext}_{\mathcal{O}_X}^i(-, \mathcal{G})$ are coexact for all $i > 0$. Since X is projective there is an ample line bundle \mathcal{L} on X and for the coherent \mathcal{O}_X -module \mathcal{G} there is some $r > 0$ such that,

$$H^i(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s}) = 0$$

for any $s \geq r$ and $i > 0$. Then since \mathcal{L} is ample, for any coherent \mathcal{O}_X -module \mathcal{F} for some n_0 such that for $n \geq n_0$ the coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections. Choosing $n \geq \max\{n_0, r\}$ we get a surjection,

$$\bigoplus_{j=1}^N \mathcal{O}_X \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

However, since \mathcal{L} is a line bundle we may tensor by $\mathcal{L}^{\otimes -n} = (\mathcal{L}^{\otimes n})^\vee$ to get a surjection,

$$\mathcal{H} = \bigoplus_{j=1}^N \mathcal{L}^{\otimes -n} \twoheadrightarrow \mathcal{F}$$

Furthermore, since \mathcal{L} is locally free of rank one,

$$\text{Ext}_{\mathcal{O}_X}^i(\mathcal{H}, \mathcal{G}) = \bigoplus_{j=1}^N \text{Ext}_{\mathcal{O}_X}^i((\mathcal{L}^{\otimes n})^\vee, \mathcal{G}) = \bigoplus_{j=1}^N \text{Ext}_{\mathcal{O}_X}^i(\mathcal{O}_X, \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{G}) = \bigoplus_{j=1}^N H^i(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$$

for $i > 0$ by Serre vanishing showing that $\text{Ext}_{\mathcal{O}_X}^i(-, \mathcal{G})$ is coexact for all $i > 0$. \square

4 Computing Ext and Tor in the Second Argument

4.1 Ext

Definition 4.1.1. Let \mathcal{C} be an abelian category (possibly enriched over another category \mathcal{D}). Then if \mathcal{C} has enough injectives, $\text{Ext}_{\mathcal{C}}^i(A, -) : \mathcal{C} \rightarrow \mathcal{D}$ are the right-derived functors of $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathcal{D}$.

Lemma 4.1.2. $\text{Ext}_{\mathcal{C}}^i(-, M) : \mathcal{C} \rightarrow \mathcal{D}$ is a contravariant functor.

Proof. Given an injective resolution $M \rightarrow \mathcal{I}^\bullet$ and a map $A \rightarrow B$ we get a morphism of complexes $\text{Hom}_{\mathcal{C}}(B, \mathcal{I}^\bullet) \rightarrow \text{Hom}_{\mathcal{C}}(A, \mathcal{I}^\bullet)$ and thus a morphism of cohomology,

$$\text{Ext}_{\mathcal{C}}^i(B, M) \rightarrow \text{Ext}_{\mathcal{C}}^i(A, M)$$

which clearly respects composition. \square

Lemma 4.1.3. If P is projective then $\text{Ext}_{\mathcal{C}}^i(P, -) = 0$ for $i > 0$.

Proof. This follow immediatly from the defining property that $\text{Hom}_{\mathcal{C}}(P, -)$ is exact. \square

Proposition 4.1.4. Given a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{C} and some $M \in \mathcal{C}$ then there is a long exact sequence,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, M) & \longrightarrow & \text{Hom}_{\mathcal{C}}(B, M) & \longrightarrow & \text{Hom}_{\mathcal{C}}(A, M) \\ & & & & & & \downarrow \\ & & \text{Ext}_{\mathcal{C}}^1(C, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^1(B, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^1(A, M) \\ & & & & & & \downarrow \\ & & \text{Ext}_{\mathcal{C}}^2(C, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^2(B, M) & \longrightarrow & \text{Ext}_{\mathcal{C}}^2(A, M) \longrightarrow \dots \end{array}$$

Proof. Take an injective resolution $M \rightarrow \mathcal{I}^\bullet$. Then since each \mathcal{I}^n is injective the functor $\text{Hom}_{\mathcal{C}}(-, \mathcal{I}^n)$ is exact so we get an exact sequence of complexes,

$$0 \longrightarrow \text{Hom}_{\mathcal{C}}(C, \mathcal{I}^\bullet) \longrightarrow \text{Hom}_{\mathcal{C}}(B, \mathcal{I}^\bullet) \longrightarrow \text{Hom}_{\mathcal{C}}(A, \mathcal{I}^\bullet) \longrightarrow 0$$

Taking the cohomology sequence of this short exact sequence of complexes gives the desired long exact sequence. \square

Lemma 4.1.5. If $P_\bullet \rightarrow A$ is a projective resolution then $\text{Ext}_{\mathcal{C}}^i(A, -) = H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$.

Proof. We may use the acyclicity lemma which may be proven by the above exact sequence for $\text{Hom}_{\mathcal{C}}(-, M)$ noting that $\text{Ext}_{\mathcal{C}}^i(P_n, M) = 0$. However, a more elegant argument goes as follows. Since P_\bullet is a complex of projectives the functor $\text{Hom}_{\mathcal{C}}(P_n, -)$ is exact so for any exact sequence,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$$

we get an exact sequence of complexes,

$$0 \longrightarrow \text{Hom}_{\mathcal{C}}(P_\bullet, M) \longrightarrow \text{Hom}_{\mathcal{C}}(P_\bullet, N) \longrightarrow \text{Hom}_{\mathcal{C}}(P_\bullet, K) \longrightarrow 0$$

which gives a long exact sequence in the cohomology functors $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$ which shows that $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$ form a δ -functor. Furthermore, since \mathcal{C} has enough injectives, for any $M \in \mathcal{C}$ we can embed $M \hookrightarrow I$ into an injective I and $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, I)) = 0$ since $\text{Hom}_{\mathcal{C}}(-, I)$ is exact. Therefore, $H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$ is an effaceable δ -functor and thus universal by Grothendieck. Furthermore, since $\text{Hom}_{\mathcal{C}}(-, M)$ is left-exact,

$$\begin{aligned} H^0(\text{Hom}_{\mathcal{C}}(P_\bullet, -)) &= \ker(\text{Hom}_{\mathcal{C}}(P^0, -) \rightarrow \text{Hom}_{\mathcal{C}}(P^1, -)) = \text{Hom}_{\mathcal{C}}(\text{coker}(P^1 \rightarrow P^0), -) \\ &= \text{Hom}_{\mathcal{C}}(A, -) \end{aligned}$$

However, $\text{Ext}_{\mathcal{C}}^i(A, -)$ are the derived functors of $\text{Hom}_{\mathcal{C}}(A, -)$ so they too form a universal δ -functor over $\text{Hom}_{\mathcal{C}}(A, -)$. Thus, since universal δ -functors with naturally isomorphic first terms are unique,

$$\text{Ext}_{\mathcal{C}}^i(A, -) = H^i(\text{Hom}_{\mathcal{C}}(P_\bullet, -))$$

\square

Remark. The above formalism applies exactly to any bifunctor $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ such that for any $A \in \mathcal{C}$ there are enough $F(A, -)$ -acyclics I for which $F(-, I)$ is exact and replacing ‘injective’ with this class of acyclics and ‘projective’ by any class of objects P such that $F(P, -)$ is exact. Furthermore we assume \mathcal{C} is abelian with enough injectives, \mathcal{D} is additive, and $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ is additive.

For example, in the category of \mathcal{O}_X -modules on a scheme, the bifunctor,

$$\mathcal{H}om_{\mathcal{O}_X}(-, -) : \mathcal{M}od(\mathcal{O}_X)^{\text{op}} \times \mathcal{M}od(\mathcal{O}_X) \rightarrow \mathcal{M}od(\mathcal{O}_X)$$

satisfies the following properties. For injective sheaves \mathcal{I} we have $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{I})$ is exact and there are enough injectives which are acyclic for $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$. If \mathcal{E} is a locally-free sheaf then,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, -) = \mathcal{E}^\vee \otimes_{\mathcal{O}_X} (-)$$

and \mathcal{E}^\vee is locally free and thus flat so $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, -)$ is exact. Therefore, we see that $\mathcal{E}xt_{\mathcal{O}_X}^i(-, \mathcal{G})$ is a contravariant δ -functor, vanishing on locally free sheaves, which may be computed via cohomology of locally-free complexes. Furthermore, whenever $\mathcal{M}od(\mathcal{O}_X)$ has enough locally frees (for example whenever X has an ample line bundle) then $\mathcal{E}xt_{\mathcal{O}_X}^i(-, \mathcal{G})$ forms a universal contravariant δ -functor.

4.2 Tor

Definition 4.2.1. When \mathcal{C} has a right-exact comonoid structure $- \otimes_{\mathcal{C}} -$ and \mathcal{C} has enough projectives then define $\text{Tor}_i^{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathcal{C}$ as the left-derived functors of $A \otimes_{\mathcal{C}} - : \mathcal{C} \rightarrow \mathcal{C}$.

Remark. Here it will be necessary to assume that \mathcal{C} has enough flat objects ($- \otimes_{\mathcal{C}} F$ is exact) which happens say when projectives are flat.

Lemma 4.2.2. $\text{Tor}_i^{\mathcal{C}}(-, M)$ is a covariant functor.

Proof. Given a map $A \rightarrow B$ and a projective resolution $P_\bullet \rightarrow M$ we get a morphism of complexes, $A \otimes_{\mathcal{C}} P_\bullet \rightarrow B \otimes_{\mathcal{C}} P_\bullet$ and thus a morphism of homology,

$$\text{Tor}_i^{\mathcal{C}}(A, M) \rightarrow \text{Tor}_i^{\mathcal{C}}(B, M)$$

□

Definition 4.2.3. We say an object $P \in \mathcal{C}$ is *flat* if $P \otimes_{\mathcal{C}} -$ is an exact functor.

Lemma 4.2.4. The following are equivalent,

- (a) P is flat
- (b) $\text{Tor}_i^{\mathcal{C}}(P, -) = 0$ for all $i > 0$
- (c) $\text{Tor}_1^{\mathcal{C}}(P, -) = 0$.

Proof. Clearly (a) \implies (b) \implies (c). Now, if $\text{Tor}_1^{\mathcal{C}}(P, -) = 0$ then for any exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we get an exact sequence,

$$\mathrm{Tor}_1^{\mathcal{C}}(P, C) \longrightarrow P \otimes_{\mathcal{C}} A \longrightarrow P \otimes_{\mathcal{C}} B \longrightarrow C \otimes_{\mathcal{C}} P \longrightarrow 0$$

so if $\mathrm{Tor}_1^{\mathcal{C}}(P, -) = 0$ then $P \otimes_{\mathcal{C}} -$ is exact i.e. P is flat. \square

Proposition 4.2.5. Given a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{C} and some $M \in \mathcal{C}$ then there is a long exact sequence,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{Tor}_2^{\mathcal{C}}(A, M) & \longrightarrow & \mathrm{Tor}_2^{\mathcal{C}}(B, M) & \longrightarrow & \mathrm{Tor}_2^{\mathcal{C}}(C, M) \\ & & & & & & \downarrow \\ & & & & & & \mathrm{Tor}_1^{\mathcal{C}}(A, M) \longrightarrow \mathrm{Tor}_1^{\mathcal{C}}(B, M) \longrightarrow \mathrm{Tor}_1^{\mathcal{C}}(C, M) \\ & & & & & & \downarrow \\ & & & & & & A \otimes_{\mathcal{C}} M \longrightarrow B \otimes_{\mathcal{C}} M \longrightarrow C \otimes_{\mathcal{C}} M \longrightarrow 0 \end{array}$$

Proof. Take a flat resolution $F_{\bullet} \rightarrow M$. Then since each F^n is flat the functor $F^n \otimes_{\mathcal{C}} -$ is exact so we get an exact sequence of complexes,

$$0 \longrightarrow A \otimes_{\mathcal{C}} F_{\bullet} \longrightarrow B \otimes_{\mathcal{C}} F_{\bullet} \longrightarrow C \otimes_{\mathcal{C}} F_{\bullet} \longrightarrow 0$$

Taking the homology sequence of this short exact sequence of complexes gives the desired long exact sequence since by the acyclicity lemma we may commute $\mathrm{Tor}_i^{\mathcal{C}}(A, M)$ via a flat resolution of M . \square

Lemma 4.2.6. If $F_{\bullet} \rightarrow A$ is a free resolution then $\mathrm{Tor}_i^{\mathcal{C}}(A, -) = H_i(F_{\bullet} \otimes_{\mathcal{C}} -)$.

Proof. We may use the acyclicity lemma which may be proven by the above exact sequence for $\mathrm{Tor}_i^{\mathcal{C}}(-, M)$ showing that $\mathrm{Tor}_i^{\mathcal{C}}(-, M)$ forms a δ -functor and noting that $\mathrm{Tor}_i^{\mathcal{C}}(F_n, M) = 0$. However, a more elegant argument goes as follows. Since F_{\bullet} is a complex of frees the functor $F_n \otimes -$ is exact so for any exact sequence,

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$$

we get an exact sequence of complexes,

$$0 \longrightarrow F_{\bullet} \otimes M \longrightarrow F_{\bullet} \otimes N \longrightarrow F_{\bullet} \otimes K \longrightarrow 0$$

which gives a long exact sequence in the homology functors $H_i(F_{\bullet} \otimes -)$ which shows that $H_i(F_{\bullet} \otimes -)$ form a (homological) δ -functor. Furthermore, since \mathcal{C} has enough frees, for any $M \in \mathcal{C}$ we have a surjection $F \twoheadrightarrow M$ for some free F and $H_i(F_{\bullet} \otimes_{\mathcal{C}} F) = 0$ since $- \otimes \mathcal{F}$ is exact (both rows and columns stay exact, it is the exactness of the columns here ensured by freeness of F which is needed for the vanishing). Therefore, $H_i(F_{\bullet} \otimes -)$ is a coeffaceable δ -functor and thus universal by Grothendieck. Furthermore, since $- \otimes_{\mathcal{C}} M$ is right-exact,

$$H_0(F_{\bullet} \otimes_{\mathcal{C}} -) = \mathrm{coker}([F_1 \otimes_{\mathcal{C}} -] \rightarrow [F_0 \otimes_{\mathcal{C}} -]) = \mathrm{coker}(F_1 \rightarrow F_0) \otimes_{\mathcal{C}} (-) = A \otimes_{\mathcal{C}} (-)$$

However, $\mathrm{Tor}_i^{\mathcal{C}}(A, -)$ are the derived functors of $A \otimes_{\mathcal{C}} -$ so they too form a universal δ -functor over $A \otimes_{\mathcal{C}} -$. Thus, since universal δ -functors with naturally isomorphic first terms are unique,

$$\mathrm{Tor}_i^{\mathcal{C}}(A, -) = H_i(F_{\bullet} \otimes_{\mathcal{C}} -)$$

\square

Proposition 4.2.7. Tor is symmetric: there is a natural isomorphism $\mathrm{Tor}_i^{\mathcal{C}}(A, B) = \mathrm{Tor}_i^{\mathcal{C}}(B, A)$.

Proof. Choose a flat resolution $F_{\bullet} \rightarrow A$. By the above lemma $\mathrm{Tor}_i^{\mathcal{C}}(A, B) = H_i(F_{\bullet} \otimes_{\mathcal{C}} B)$. However, by the symmetry of $-\otimes_{\mathcal{C}}-$ we have, $H_i(F_{\bullet} \otimes_{\mathcal{C}} B) = H_i(B \otimes_{\mathcal{C}} F_{\bullet})$. Furthermore, because $\mathrm{Tor}_i^{\mathcal{C}}(B, -)$ is the left-derived functor of $B \otimes_{\mathcal{C}} -$ we may compute it via acyclics (since it is a δ -functor) so $\mathrm{Tor}_i^{\mathcal{C}}(B, A) = H_i(B \otimes_{\mathcal{C}} F_{\bullet})$ and thus,

$$\mathrm{Tor}_i^{\mathcal{C}}(A, B) = H_i(F_{\bullet} \otimes_{\mathcal{C}} B) = H_i(B \otimes_{\mathcal{C}} F_{\bullet}) = \mathrm{Tor}_i^{\mathcal{C}}(B, A)$$

□

Remark. These arguments apply to the satellites of any symmetric bifunctor $F : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$ between abelian categories such that \mathcal{C} has enough objects A for which $F(A, -)$ is exact, in particular, if $F(P, -)$ is exact for projectives (as is the tensor product).

Remark. Symmetry follows directly from the following spectral sequence argument. Let $F_{\bullet}^A \rightarrow A$ and $F_{\bullet}^B \rightarrow B$ be free resolutions. Then consider the double complex $C_{p,q} = F_p^A \otimes_{\mathcal{C}} F_q^B$. There are two spectral sequences which compute the homology of the total complex $\mathrm{Tot}(C_{\bullet,\bullet})$. These two spectral sequences agree on their zeroth page, ${}^A E_{p,q}^0 = {}^B E_{p,q}^0 = F_p^A \otimes_{\mathcal{C}} F_q^B$. Now, the first pages are,

$$\begin{aligned} {}^A E_{p,q}^1 &= H_p(C_{\bullet,q}) = H_p(F_{\bullet}^A \otimes_{\mathcal{C}} F_q^B) = A \otimes_{\mathcal{C}} F_q^B & \text{in } p \text{ degree zero} \\ {}^B E_{p,q}^1 &= H_q(C_{p,\bullet}) = H_q(F_p^A \otimes_{\mathcal{C}} F_{\bullet}^B) = F_p^A \otimes_{\mathcal{C}} B & \text{in } q \text{ degree zero} \end{aligned}$$

where we have used the fact that $-\otimes_{\mathcal{C}} F_q^B$ and $F_p^A \otimes_{\mathcal{C}} -$ are exact (since the resolutions are free) and thus commute with taking homology. Then the second pages are,

$$\begin{aligned} {}^A E_{p,q}^2 &= H_q({}^A E_{p,\bullet}^1) = H_q(A \otimes_{\mathcal{C}} F_{\bullet}^B) = L^q(A \otimes_{\mathcal{C}} -)(B) & \text{in } p \text{ degree zero} \\ {}^B E_{p,q}^2 &= H_p({}^B E_{\bullet,q}^1) = H_p(F_{\bullet}^A \otimes_{\mathcal{C}} B) = L^p(- \otimes_{\mathcal{C}} B)(A) & \text{in } q \text{ degree zero} \end{aligned}$$

Since the second pages are supported in a single row or column both spectral sequences are converged. Therefore, we find,

$$H_n(\mathrm{Tot}(C_{\bullet,\bullet})) = {}^A E_{0,n}^2 = {}^B E_{n,0}^2 = L^n(A \otimes_{\mathcal{C}} -)(B) = L^n(- \otimes_{\mathcal{C}} B)(A)$$

Therefore, for a bifunctor we may derive in either component to get the same satellite functors. Furthermore, when $-\otimes_{\mathcal{C}}-$ is symmetric then,

$$\begin{aligned} L^n(A \otimes_{\mathcal{C}} -)(B) &= L^n(- \otimes_{\mathcal{C}} A)(B) = L^n(B \otimes_{\mathcal{C}} -)(A) \\ L^n(- \otimes_{\mathcal{C}} B)(A) &= L^n(B \otimes_{\mathcal{C}} -)(A) = L^n(- \otimes_{\mathcal{C}} A)(B) \end{aligned}$$

so the derived functors are symmetric.

4.3 Acyclicity

Lemma 4.3.1. Let F be a δ -functor. Suppose there is an exact sequence,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow K \longrightarrow 0$$

where I^i are F -acyclic. Then for $i > 0$,

$$F^{n+1+i}(A) = F^i(A)$$

and $F^{n+1}(A) = \mathrm{coker}(F^0(I^n) \rightarrow F^0(K))$.

Proof. We prove this by induction on n . For $n = 0$, we are given a short exact sequence,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow K \longrightarrow 0$$

Taking the long exact sequence,

$$0 \longrightarrow F^0(A) \longrightarrow F^0(I^0) \longrightarrow F^0(K) \longrightarrow F^1(A) \longrightarrow F^1(I^0)$$

and

$$F^i(I^0) \longrightarrow F^i(K) \longrightarrow F^{i+1}(A) \longrightarrow F^{i+1}(I^0)$$

However, I^0 is F -acyclic so $F^i(I^0) = 0$ for $i > 0$ and thus $F^{i+1}(A) = F^i(K)$ for $i > 0$. Furthermore, for the second sequence $F^1(A) = \text{coker}(F^0(I^0) \rightarrow F^0(K))$.

Now we assume the result holds for $n - 1$. We split the exact sequence into,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow \tilde{K} \longrightarrow 0$$

and

$$0 \longrightarrow \tilde{K} \longrightarrow I^n \longrightarrow K \longrightarrow 0$$

Applying the induction hypothesis we see that, $F^{n+i}(A) = F^i(\tilde{K})$ for $i > 0$. In particular, we will use, $F^{n+1}(A) = F^1(\tilde{K})$. Now, by the LES of the second exact sequence we find,

$$0 \longrightarrow F^0(\tilde{K}) \longrightarrow F^0(I^n) \longrightarrow F^0(K) \longrightarrow F^1(\tilde{K}) \longrightarrow F^1(I^n)$$

and

$$F^i(I^n) \longrightarrow F^i(K) \longrightarrow F^{i+1}(\tilde{K}) \longrightarrow F^{i+1}(I^n)$$

However, I^n is F -acyclic so for $i > 0$ we get,

$$F^i(K) = F^{i+1}(\tilde{K}) \quad \text{and} \quad F^1(\tilde{K}) = \text{coker}(F^0(I^n) \rightarrow F^0(K))$$

Therefore, we have $F^{n+i+1}(A) = F^{i+1}(\tilde{K}) = F^i(K)$ for $i > 0$. Furthermore,

$$F^{n+1}(A) = F^1(\tilde{K}) = \text{coker}(F^0(I^n) \rightarrow F^0(K))$$

proving the lemma. □

Theorem 4.3.2 (acyclicity). If F is a δ -functor and $A \rightarrow I^\bullet$ a resolution of F -acyclic objects,

$$F^n(A) = H^n(F^0(I^\bullet))$$

Proof. We may truncate the resolution by adding a cokernel K to give an exact sequence,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow K \longrightarrow 0$$

By the previous lemma, we can compute,

$$F^{n+1}(A) = \operatorname{coker}(F^0(I^n) \rightarrow F^0(K))$$

However, by exactness, $K = \operatorname{coker}(I^{n-1} \rightarrow I^n) = \ker(I^{n+1} \rightarrow I^{n+2})$. Furthermore, F^0 is left-exact so $F^0(K) = \ker(F(I^{n+1}) \rightarrow F(I^{n+2}))$. Therefore, for $n \geq 0$ we find,

$$F^{n+1}(A) = \operatorname{coker}(F^0(I^n) \rightarrow F^0(K)) = \operatorname{coker}(F^0(I^n) \rightarrow \ker(F(I^{n+1}) \rightarrow F(I^{n+2}))) = H^{n+1}(F^0(I^\bullet))$$

Furthermore, F^0 is left-exact so,

$$F^0(A) = F^0(\ker(I^0 \rightarrow I^1)) = \ker(F^0(I^0) \rightarrow F^0(I^1)) = H^0(F^0(I^\bullet))$$

□

4.4 Tor for Sheaves

Remark. Often the categories $\mathcal{Mod}(\mathcal{O}_X)$, $\mathfrak{Q}\mathfrak{Coh}(\mathcal{O}_X)$, and $\mathfrak{Coh}(\mathcal{O}_X)$ do not have enough projectives. Therefore, we cannot define Tor for sheaves as a left-derived functor we need an alternative definition.

Definition 4.4.1. Let X be a scheme such that $\mathfrak{Coh}(\mathcal{O}_X)$ has enough locally-frees (e.g. X has an ample line bundle). Given a coherent sheaf \mathcal{F} and a resolution $\mathcal{E}_\bullet \rightarrow \mathcal{F}$ by locally free coherent sheaves, we define,

$$\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, -) = H_i(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} -)$$

Proposition 4.4.2. $\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is a universal homological δ -functor.

Proof. First, given an exact sequence of coherent sheaves,

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3 \longrightarrow 0$$

we get an exact sequence of complexes,

$$0 \longrightarrow \mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{G}_1 \longrightarrow \mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{G}_2 \longrightarrow \mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{G}_3 \longrightarrow 0$$

since \mathcal{E}_\bullet is locally-free and thus flat. Taking homology gives a long exact sequence of $\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, -)$ sheaves making it a homological δ -functor. It suffices to show that $\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is coexactable. Since there are enough locally-free sheaves for any coherent \mathcal{G} we can find a locally-free and a surjection $\mathcal{E}' \twoheadrightarrow \mathcal{G}$. Then, since $- \otimes_{\mathcal{O}_X} \mathcal{E}'$ is exact then,

$$\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}) = H_i(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{E}) = 0$$

where $\mathcal{E}_\bullet \rightarrow \mathcal{F}$ is a locally-free resolution. Therefore, $\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is coexactable. □

Remark. Since $\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, -)$ is universal it will agree with any other reasonable definition (any definition which is a universal δ -functor) because there is a unique universal δ -functor over,

$$\mathcal{Tor}_0^{\mathcal{O}_X}(\mathcal{F}, -) = H_0(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} -) = \operatorname{coker}(\mathcal{E}^1 \rightarrow \mathcal{E}^0) \otimes_{\mathcal{O}_X} - = \mathcal{F} \otimes_{\mathcal{O}_X} -$$

where the second equality follows from right-exactness of $- \otimes_{\mathcal{O}_X} \mathcal{G}$.

Remark. Since $- \otimes_{\mathcal{O}_X} - : \mathcal{Mod}(\mathcal{O}_X) \times \mathcal{Mod}(\mathcal{O}_X) \rightarrow \mathcal{Mod}(\mathcal{O}_X)$ is a symmetric bifunctor with enough locally-frees which are flat. Then since $\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, -)$ are the left-satellite functors of $\mathcal{F} \otimes_{\mathcal{O}_X} -$ we can apply the acyclicity lemma to show that we map compute sheaf Tor from a locally free resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{F}$,

$$\mathcal{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H_i(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_\bullet)$$

which shows the symmetry of $\mathcal{Tor}_i^{\mathcal{O}_X}(-, -)$.

5 Depth of Field

First we calculate the size of the circle of confusion. Let the lense have aperature D and focal length f . The image distance is given by,

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f}$$

then,

$$i = \frac{fo}{o - f}$$

Therefore, we can compute the change in image distance as o changes,

$$\frac{di}{do} = \frac{f}{o - f} - \frac{fo}{(o - f)^2} = -\frac{f^2}{(o - f)^2}$$

For a depth of Δo we have a spread of image depths,

$$\Delta i \approx \frac{f^2 \Delta o}{(o - f)^2}$$

Then the width of the circle of confusion is given by,

$$\frac{C}{D} = \frac{\Delta i}{f + \Delta i} \approx \frac{\Delta i}{f}$$

Therefore,

$$C = \frac{fD}{(o - f)^2} \Delta o$$

For a fixed allowable circle of confusion C_{\max} for the desired resolution, we find the depth of field,

$$\text{DOF} = 2 \frac{C}{D} \cdot \frac{(o - f)^2}{f} = \frac{2(o - f)^2 NC}{f^2}$$

where $N = f/D$ is the focal ratio.

5.1 Hyperfocal Distance

At some focal distance H , all objects beyond H are in focus. This occurs when,

$$\frac{i - f}{f} = \frac{C}{D}$$

and

$$i = \frac{fH}{H - f}$$

Then,

$$\frac{H}{H - f} - 1 = \frac{f}{H - f} = \frac{C}{D}$$

Therefore,

$$H = \frac{f(D + C)}{C} = \frac{f^2}{CN} + f$$

Alternatively, if we focus at infinity and ask beyond which everything is in focus then,

$$\frac{i-f}{i} = \frac{C}{D}$$

and

$$i = \frac{fH}{H-f}$$

Then,

$$1 - \frac{H-f}{H} = \frac{f}{H} = \frac{C}{D}$$

Therefore,

$$H = \frac{fD}{C} = \frac{f^2}{NC}$$

6 Morphisms from Proper to Affine Schemes

Let $X \rightarrow \operatorname{Spec}(R)$ be proper and $\operatorname{Spec}(A) \rightarrow \operatorname{Spec}(R)$ be affine. Now,

$$\operatorname{Hom}_R(X, \operatorname{Spec}(A)) = \operatorname{Hom}_R(A, \Gamma(X, \mathcal{O}_X))$$

The map $X \rightarrow \operatorname{Spec}(A)$ is given as follows, consider $\varphi_x : A \rightarrow \Gamma(X, \mathcal{O}_X) \rightarrow \mathcal{O}_{X,x}$ then $x \mapsto \varphi_x^{-1}(\mathfrak{m}_x)$. Therefore, all maps $X \rightarrow \operatorname{Spec}(A)$ are constant if $x \mapsto \operatorname{res}_x^{-1}(\mathfrak{m}_x)$ is a fixed ideal independent of x .

7 Irreducible Polynomials over \mathbb{Z}

Consider the map $\operatorname{Spec}(\mathbb{Z}[X]) \rightarrow \operatorname{Spec}(\mathbb{Z})$. The fibres are, over the generic point (0) , we have $\operatorname{Spec}(\mathbb{Q}[X]) \rightarrow \operatorname{Spec}(\mathbb{Q})$ which corresponds to ideals of the form $(f(X))$ for f an irreducible polynomial $f \in \mathbb{Q}[X]$. The fibres over (p) are $\operatorname{Spec}(\mathbb{F}_p[X]) \rightarrow \operatorname{Spec}(\mathbb{F}_p)$ whose primes are of the form $(f(X))$ for f an irreducible polynomial $f \in \mathbb{F}_p[X]$. Therefore we get an explicit description of $\operatorname{Spec}(\mathbb{F}[X])$, we have the primes, $(f(X))$ for irreducible $f \in \mathbb{Q}[X]$ (for which we may clear denominators to get $f \in \mathbb{Z}[X]$) and $(p, f(X))$ for irreducible $f \in \mathbb{F}_p[X]$ (choosing some representative in $\mathbb{Z}[X]$) and finally of course (0) and (p) are prime (corresponding to the generic points of the fibres).

Suppose $f \in \mathbb{Z}[X]$ were irreducible then any prime (strictly) above (f) must be of the form $(p, f(X))$ otherwise f would be a nontrivial product. Then we have $\dim \mathbb{Z}[X]/(f) = 1$ furthermore, (COMPLETE THIS ARGUMENT ...)

8 Normalization

Example 8.0.1. Consider $X = \operatorname{Spec}(A)$ with $A = k[x, y]/(y^2 - x^2(x+1))$. Then consider,

$$A \rightarrow k[t] \quad x \mapsto t^2 - 1 \quad y \mapsto t(t^2 - 1)$$

Then $y^2 = t^2(t^2 - 1)$ and $x^2(x - 1) = t^2(t^2 - 1)^2$ so this map is well-defined. This gives a dominant map,

$$\mathbb{A}_k^1 \rightarrow \operatorname{Spec}(A)$$

Furthermore, I claim that,

$$\operatorname{Frac}(A) \hookrightarrow k(t)$$

is an isomorphism, clearly $\operatorname{Frac}(A) \rightarrow k(t)$ is injective. The inverse map is $t \mapsto y/x$ then $y/x \mapsto t \mapsto y/x$ and $t \mapsto y/x \mapsto t$. Furthermore, $x \mapsto (t^2 - 1) \mapsto (y^2/x^2 - 1) = x$ and $y \mapsto t(t^2 - 1) \mapsto y/x(y^2/x^2 - 1) = y$. Thus the map $\mathbb{A}^1 \rightarrow \operatorname{Spec}(A)$ is a dominant birational morphism. Furthermore,

$$\overline{A} = k[y/x] = k[t] \subset \operatorname{Frac}(A)$$

because $t = y/x$ satisfies the monic $t^2 - x - 1$ so $\mathbb{A}^1 \rightarrow \operatorname{Spec}(A)$ is the normalization.

Example 8.0.2. Consider the cusp $X = \operatorname{Spec}(A)$ with $A = k[x, y] = (y^2 - x^3)$. Then consider,

$$A \rightarrow k[t] \quad x \mapsto t^2 \quad y \mapsto t^3$$

Then $y^2 \mapsto t^6$ and $x^2 \mapsto t^6$ so this is well-defined. This gives a dominant map,

$$\mathbb{A}_k^1 \rightarrow \operatorname{Spec}(A)$$

Furthermore, I claim that,

$$\operatorname{Frac}(A) \hookrightarrow k(t)$$

is an isomorphism. Send $t \mapsto y/x$ then $t \mapsto y/x \mapsto t$ and $y/x \mapsto t \mapsto y/x$. Then $y \mapsto t^3 \mapsto y^3/x^3 = y$ and $x \mapsto t^2 \mapsto y^2/x^2 = x$. Therefore, $\mathbb{A}_k^1 \rightarrow \operatorname{Spec}(A)$ is a dominant birational morphism. Furthermore,

$$\overline{A} = k[y/x] = k[t] \subset k(t) = \operatorname{Frac}(A)$$

because $t = y/x$ satisfies the monic $t^2 - x$.

Example 8.0.3. Consider the tachnode $X = \operatorname{Spec}(A)$ with $A = k[x, y]/(x^2 - y^4)$. Then consider,

$$A \rightarrow k[t, s]/(s^2 - 1) \quad x \mapsto t \quad y \mapsto t^2 s$$

Then $x^4 \mapsto t^4$ and $y^2 \mapsto t^4$ so this is well-defined. this gives a dominant map,

$$\operatorname{Spec}(k[t, s]/(s^2 - 1)) = \mathbb{A}_k^1 \coprod \mathbb{A}_k^1 \rightarrow \operatorname{Spec}(A)$$

On the irreducible components $\mathfrak{p}_+ = (y - x^2)$ and $\mathfrak{p}_- = (y + x^2)$ of $\operatorname{Spec}(A)$ we have,

$$\mathcal{O}_{X, \mathfrak{p}_+} = \operatorname{Frac}(k[x, y]/(y - x^2)) \quad \mathcal{O}_{X, \mathfrak{p}_-} = \operatorname{Frac}(k[x, y]/(y + x^2))$$

and thus the map $\operatorname{Spec}(k[t, s]/(s^2 - 1)) \rightarrow \operatorname{Spec}(A)$ gives an isomorphism on each component and $\operatorname{Spec}(k[t, s]/(s^2 - 1))$ is normal.

9 A Very Werid Scheme

For finite products we have,

$$\mathrm{Spec}(A \times B) = \mathrm{Spec}(A) \coprod \mathrm{Spec}(B)$$

where we take the coproduct in the category of schemes. In particular, the primes of $A \times B$ are simply $\mathfrak{p}_1 \times B$ or $A \times \mathfrak{p}_2$ for primes $\mathfrak{p}_1 \subset A$ and $\mathfrak{p}_2 \subset B$. However, for infinite product this fails. Consider,

$$X = \mathrm{Spec} \left(\prod_{i=0}^{\infty} k \right) \quad R = \prod_{i=0}^{\infty} k$$

where k field. The prime ideals of this ring are not just the kernels of the projections $R \rightarrow k$ which are maximal ideals. To see this, consider the ideal I of functions $\mathbb{N} \rightarrow k$ which have finite support. Clearly $I \rightarrow R \rightarrow k$ is surjective for each projection so I is not contained in any of the described primes. It turns out that prime ideals of R correspond to ultrafilters \mathcal{F} of \mathbb{N} where $\mathfrak{p}(\mathcal{F})$ for some ultrafilter is the following,

$$\mathfrak{p}(\mathcal{F}) = \{(a_i) \mid \{i \mid a_i = 0\} \in \mathcal{F}\}$$

Therefore, the principal ultrafilter \mathcal{F}_i above $\{i\}$ gives exactly $\mathfrak{p}(\mathcal{F}_i) = \ker \pi_i$ but there are many more nonprincipal ultrafilters.

10 Coproducts in the Category of Schemes

Proposition 10.0.1. The forgetful functor $F : \mathbf{Sch} \rightarrow \mathbf{Top}$ preserves colimits.

Remark. Let $\mathrm{Hom}_{\mathbf{Top}}(F(X), S) = \mathrm{Hom}_{\mathbf{Sch}}(X, T(S))$

11 NOTE LOOK UP THE PROOF FOR PROJ -¿ LOCALLY FREE

12 Ravi Excercises

Remark. Maps $\mathrm{Spec}(k) \rightarrow \mathbb{P}_k^n$ are equivalent to giving a line bundle \mathcal{L} on $\mathrm{Spec}(k)$ i.e. a one-dimensional k -vectorspace $V \cong k$ and $n+1$ sections $s_i \in V$ not all zero. We call this point $[s_0, \dots, s_n] \in \mathbb{P}_k^n$ up to isomorphism $\varphi : V \cong V'$ and $\varphi(s_i) = s'_i$. This is simply global scalling by k^\times . Furthermore, for any extension K/k we can describe $\mathbb{P}_k^n(K)$ similarly but with $s_i \in K$.

Definition 12.0.1. Projection from a rational point $\mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$ given a projection point $p \in \mathbb{P}_k^n$. We define this as follows: by an automorphism of \mathbb{P}_k^n let $p = [1 : 0 : \dots : 0]$. Take the dense open $U = D(X_0) \setminus \{0\} = \mathrm{Spec}(x_1, \dots, x_n) \setminus \{(0)\}$. Then consider the map $U \rightarrow \mathbb{P}_k^{n-1}$ via $\mathcal{L} = \mathcal{O}_U$ and $s_i = x_i$. These global sections generate because we have removed the point at which they all vanish. This rational map $\mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$ has domain $\mathrm{Dom}(f) = \mathbb{P}_k^n \setminus \{p\}$.

12.1 6.5 F

Consider the conic $C = V(X^2 + Y^2 = Z^2) \subset \mathbb{P}_k^2$. Consider the map $\mathbb{P}_k^1 \rightarrow \mathbb{P}_k^2$ defined by the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2)$ and the sections $X_0^2 - X_1^2, 2X_0X_1, X_0^2 + X_1^2$. The image is exactly $C = V(X^2 + Y^2 = Z^2)$ and thus $C \cong \mathbb{P}_k^1$. However, if characteristic of $k = 2$ then these sections are $X_0^2 + X_1^2, 0, X_0^2 + X_1^2$ which does not define a map since these may all vanish simultaneously. In fact, $V(X^2 + Y^2 = Z^2)$ is not smooth in characteristic two since $X^2 + Y^2 = (X + Y)^2$ so we get $X + Y = \pm Z$ the union of two lines in \mathbb{P}_k^2 .

We can also describe an isomorphism as follows. First, let's do a change of coordinates $X \mapsto \frac{1}{2}(X+Z)$ and $Z \mapsto \frac{1}{2}(X-Z)$ then $C = V(XZ + Y^2)$. Take the point $p = [1 : 0 : 0]$ use the projection $\mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^1$ away from p . On the affine $D(X)$ this is the map $U = \text{Spec}(k[y, z]/(z + y^2)) \setminus \{0\} \rightarrow \mathbb{P}_k^1$ via $(y, z) \mapsto [y : z]$. Now $U = \text{Spec}(k[y, y^{-1}]) = \mathbb{G}_m^k$ and the map is $\mathbb{G}_m^k \rightarrow \mathbb{P}_k^1$ via $t \mapsto [t, t^2]$. This is a rational map $C \dashrightarrow \mathbb{P}_k^1$ of smooth projective curves so it extends to $C \rightarrow \mathbb{P}_k^1$ which is inverse to the previous map.

12.2 6.5 G

Consider $C = \text{Spec}(k[x, y]/(y^2 - x^3 - x^2))$. Then we construct a rational map $C \dashrightarrow \mathbb{A}_k^1$ via projecting from $p = (0, 0)$. Explicitly, consider $U = D(x)$ and consider, $f : U \rightarrow \mathbb{A}_k^1$ via $t \mapsto y/x$. Inversely we define $g : \mathbb{A}_k^1 \rightarrow C$ generated by the ring map $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$. Note that we have seen this is the normalization $\mathbb{A}_k^1 \rightarrow C$ of C . Then $g \circ f : U \rightarrow C$ is $x \mapsto y^2/x^2 - 1 = x$ and $y \mapsto y/x(y^2/x^2 - 1) = y$. Furthermore, $f \circ g : \mathbb{G}_m^k \rightarrow \mathbb{A}_k^1$ is $t \mapsto y/x \mapsto t$. Therefore, these are inverse rational maps showing that $C \xrightarrow{\sim} \mathbb{A}_k^1$ is birational. However we cannot extend this rational map to p since $\mathcal{O}_{C,p} = \text{Spec}((k[x, y]/(y^2 - x^2))_{(x,y)})$ is not a domain and thus not regular.

This gives a formula for the rational points of C by $\mathbb{A}_L^1 \dashrightarrow C_L$. Via $t \mapsto (t^2 - 1, t(t^2 - 1))$ which hit every L -rational point on C . Thus,

$$C(L) = \{(t^2 - 1, t(t^2 - 1)) \mid t \in L\}$$

We see that C is a rational curve i.e. $C \xrightarrow{\sim} \mathbb{P}_k^1$.

12.3 6.5 H

Consider the quadric surface,

$$Q = V(X^2 + Y^2 - Z^2 - W^2) \subset \mathbb{P}_k^3$$

First, we do a change of variables,

$$X \mapsto \frac{1}{2}(X + Z) \quad Z \mapsto \frac{1}{2}(X - Z) \quad Y \mapsto \frac{1}{2}(Y + W) \quad W \mapsto \frac{1}{2}(Y - W)$$

which gives,

$$Q = V(XZ + YW) \subset \mathbb{P}_k^3$$

Now we project from the point $p = [1 : 0 : 0 : 0]$ on $U = D(X) \setminus \{p\}$ this gives the map,

$$f : \text{Spec}(k[y, z, w]/(z + yw)) \setminus \{0\} \rightarrow \mathbb{P}_k^2$$

via sections y, z, w . We describe an inverse $\mathbb{P}_k^2 \dashrightarrow Q$ as follows, consider $\mathbb{P}_k^2 = \text{Proj}(k[T_0, T_1, T_2])$ then on $D(T_0 T_2)$ take $\text{Spec}(k[t_0, t_1]) \rightarrow \text{Spec}(k[y, z, w]/(z + yw))$ via $y \mapsto -t_1$ and $z \mapsto -t_1^2/t_0$ and $w \mapsto -t_1/t_0$ which is the map $(t_0, t_1) \mapsto (-t_1, -t_1^2/t_0, -t_1/t_0)$. This gives,

$$g : D(T_0 T_2) \rightarrow D(XW)$$

and thus $\mathbb{P}_k^2 \dashrightarrow Q$. Furthermore, $g \circ f : D(XW) \rightarrow U$ is,

$$(y, z, w) \mapsto [y : z : w] = [y/w : z/w : 1] \mapsto (-z/w, -z^2/wy, -w/y) = (y, z, w)$$

restriction of the identity since $z + wy = 0$. Furthermore, $f \circ g : D(T_0 T_1 T_2) \rightarrow D(T_0 T_1 T_2)$ is,

$$(t_0, t_1) \mapsto (-t_1, t_1^2/t_0, -t_1/t_0) \mapsto [-t_1 : -t_1^2/t_0 : -t_1/t_0] = [-t_0 t_1 : -t_1^2 : -t_1] = [t_0 : t_1 : 1] = (t_0, t_1)$$

Thus we have $\mathbb{P}_k^2 \xrightarrow{\sim} Q$ via $(t_0, t_1) \mapsto (-t_1, -t_1^2/t_0, -t_1/t_0)$ on $D(T_0 T_1 T_2) \cong D(XZW)$ and thus, clearing denominators and sending $t_1 \mapsto -t_1$, we get,

$$Q(L) = \{[t_0 : t_1 t_0 : -t_1^2 : t_1] \mid t_0, t_1 \in L^\times\} \cup \{[0 : t_0 : t_1 : 0] \mid t_1, t_2 \in L^\times\} \cup \{[0 : t_0 : 0 : t_1] \mid t_1, t_2 \in L^\times\}$$

12.4 6.5 I

Consider the rational map $c : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ given by $[x : y : z] \mapsto [1/x : 1/y : 1/z]$ on $D(xyz)$. Since \mathbb{P}_k^2 is smooth, we can extend over smooth codimension one irreducibles i.e. $V(x)$ and $V(y)$ and $V(z)$ such that c is defined on a dense open of each. In particular, on $D(yz)$ we have $[x : y : z] \mapsto [1 : x/y : x/z]$ is equivalent to c restricted to $D(xyz)$ and likewise on $D(xy)$ and $D(xz)$. Thus,

$$\text{Dom}(f) \supset D(xy) \cup D(yz) \cup D(xz) = \mathbb{P}_k^2 \setminus \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$$

The remaining closed set is codimension two so we generically will not be able to extend over it. Indeed, if we try $[x : y : z] \mapsto [y : x : xy/z]$ on $D(z)$ then at $[0 : 0 : 1]$ this is not defined so it does not work.

12.5 6.5 J

Show that there are no dominant rational maps $\mathbb{P}_k^1 \rightarrow F_k^n$ where $F_k^n = \text{Proj}(k[X, Y, Z]/(X^n + Y^n - Z^n))$ is the Fermat curve for $n > 2$.

13 Which Hypersurfaces are Isomorphic to Projective Space?

First, what is a hypersurface.

Definition 13.0.1. A hypersurface $H \subset \mathbb{P}_k^n$ is a codimension one integral closed subscheme i.e. a prime divisor on \mathbb{P}_k^n .

Theorem 13.0.2. Every hypersurface $H \subset \mathbb{P}_k^n$ is of the form $V(F)$ for some $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$.

Proof. Since H is a prime divisor and \mathbb{P}_k^n is locally factorial (in particular regular) then H is Cartier so its associated sheaf of ideals $\mathcal{I} \cong \mathcal{O}_{\mathbb{P}_k^n}(-d)$ is invertible. Then the inclusion map $\mathcal{O}_{\mathbb{P}_k^n}(-d) \hookrightarrow \mathcal{O}_{\mathbb{P}_k^n}$ is given by some regular section $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$ and thus $H = V(F)$. \square

Remark. In the case $n = 1$ hypersurfaces are exactly points and since $\mathbb{P}_L^0 = \text{Spec}(L)$ then for any finite extension L/k we can easily find $\text{Spec}(L) \rightarrow \mathbb{P}_k^1$ so hypersurfaces of \mathbb{P}_k^1 are exactly of the form \mathbb{P}_L^0 . We wonder how this generalizes to $n > 1$. Furthermore, note that we will use the fact that H is effective Cartier and argue, from the exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}_k^n} \longrightarrow \iota_* \mathcal{O}_H \longrightarrow 0$$

and the associated LES,

$$\begin{array}{ccccc} H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}) & \longrightarrow & H^0(H, \mathcal{O}_H) & \longrightarrow & H^1(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-d)) \\ \parallel & & & & \downarrow \\ k & & & & 0 \end{array}$$

to argue that for $n > 1$ we get a surjection $k \twoheadrightarrow H^0(H, \mathcal{O}_H)$ showing that we cannot have extensions of k . Note that this argument does not hold for $n = 1$ since $H^1(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}(-d)) \neq 0$ and we can, in fact, have extensions of the base field.

Theorem 13.0.3. Let $H \subset \mathbb{P}_k^n$ be a degree d hypersurface i.e. $H = V(F)$ for $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$ and $n > 1$. Then $H \cong \mathbb{P}_L^{n-1}$ for some L/k exactly when $L = k$ and either $d = 1$ or $n = 2$ and $d = 2$.

Proof. Suppose that $H \cong \mathbb{P}_L^{n-1}$ and consider the inclusion $\iota : H \hookrightarrow \mathbb{P}_k^n$ and let $X = \mathbb{P}_k^n$. Then for the ample sheaf $\mathcal{L} = \iota^* \mathcal{O}_X(1)$ we have $\mathcal{L} \in \text{Pic}(X) \cong \text{Pic}(\mathbb{P}_L^{n-1})$ so \mathcal{L} correspond to $\mathcal{O}_{\mathbb{P}_L^{n-1}}(k)$ for some $k \in \mathbb{Z}$. Therefore, we must have,

$$H^p(H, \mathcal{L}^{\otimes \ell}) = H^p(\mathbb{P}_L^{n-1}, \mathcal{O}_{\mathbb{P}_L^{n-1}}(k\ell))$$

In particular,

$$\dim_k H^p(H, \mathcal{L}^{\otimes \ell}) = (\dim_k L) \cdot \begin{cases} \binom{k\ell+n-1}{n-1} & p = 0 \\ 0 & p \neq 0, n-1 \\ \binom{-k\ell-1}{n-1} & p = n-1 \end{cases}$$

Furthermore, since ι is a closed immersion (and thus affine) we have,

$$H^p(H, \mathcal{L}^{\otimes \ell}) = H^p(X, \iota_* \mathcal{L}^{\otimes \ell}) = H^p(X, \iota_* \mathcal{O}_H \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell))$$

using the projection formula. Then, there is an exact sequence of sheaves,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \iota_* \mathcal{O}_H \longrightarrow 0 \\ & & \parallel & & & & \\ & & \mathcal{O}_X(-d) & & & & \end{array}$$

Twisting by $\mathcal{O}_X(\ell)$ gives,

$$0 \longrightarrow \mathcal{O}_X(\ell - d) \longrightarrow \mathcal{O}_X(\ell) \longrightarrow \iota_* \mathcal{O}_H \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell) \longrightarrow 0$$

Now denote $\mathcal{F} = \iota_* \mathcal{O}_H$ and $\mathcal{F}(\ell) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell)$ which is the sheaf whose cohomology we wish to compute. Taking the LES of cohomology we get,

$$0 \longrightarrow H^0(X, \mathcal{O}_X(\ell - d)) \longrightarrow H^0(X, \mathcal{O}_X(\ell)) \longrightarrow H^0(H, \mathcal{L}^{\otimes \ell}) \longrightarrow H^1(X, \mathcal{O}_X(\ell - d)) = 0$$

since $n > 1$. First, for $\ell = 0$ the first sequence gives $H^0(X, \mathcal{O}_X) \twoheadrightarrow H^0(H, \mathcal{O}_H)$ and thus $k \twoheadrightarrow L$ which is a k -morphism so $L = k$ since it is an extension. Furthermore, from the above short exact sequence, we see that,

$$h^0(H, \mathcal{L}^{\otimes \ell}) = h^0(X, \mathcal{O}_X(\ell)) - h^0(X, \mathcal{O}_X(\ell - d)) = \binom{\ell + n}{n} - \binom{\ell - d + n}{n}$$

In particular, for $d > 1$ and $\ell = 1$ we have,

$$h^0(H, \mathcal{L}) = h^0(X, \mathcal{O}_X(1)) = n + 1$$

This must equal (since $L = k$),

$$h^0(H, \mathcal{L}) = \binom{k + n - 1}{n - 1} = \binom{k + n - 1}{k} = r(k)$$

which is zero for $k < 0$ and monotonically increasing for $k > 0$. Note that $r(0) = 1$ and $r(1) = n$ and $r(2) = \frac{1}{2}(n+1)n$. Since $r(1) < r(2) < r(3)$ and $r(1) = n$ then either $r(2) = n+1$ or $r(k) \neq n+1$ for all k . However, $\frac{1}{2}n(n+1) = n+1$ exactly when $n = 2$ for $n > 0$ forcing the case $n = 2$ when $d > 1$. In particular for the case $n = 2$ and $d = 2$ we get a plane conic which we know is isomorphic to \mathbb{P}_k^1 . Also, we need to consider the case $d = 1$ in which H is a hyperplane and it is easy to see that $H \cong \mathbb{P}_k^{n-1}$ via the map $\mathbb{P}_k^{n-1} \hookrightarrow \mathbb{P}_k^n$ defined by $\mathcal{O}_{\mathbb{P}_k^{n-1}}(1)$ and the n sections perpendicular to $F \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1))$ which has image H proving the claim.

Note further that we get,

$$H^{n-1}(X, \mathcal{O}_X(\ell)) \longrightarrow H^{n-1}(H, \mathcal{L}^{\otimes \ell}) \longrightarrow H^n(X, \mathcal{O}_X(\ell - d)) \longrightarrow H^n(X, \mathcal{O}_X(\ell)) \longrightarrow H^n(H, \mathcal{O}_H)$$

and otherwise $H^p(X, \mathcal{O}_X(\ell)) = H^{p+1}(X, \mathcal{O}_X(\ell - d))$ so $H^p(H, \mathcal{O}_H) = 0$ for $p \neq 0, n-1$. Since $\dim H = n-1$ we have $H^n(H, \mathcal{O}_H) = 0$ and also $H^{n-1}(X, \mathcal{O}_X(\ell)) = 0$ so we find,

$$0 \longrightarrow H^{n-1}(H, \mathcal{L}^{\otimes \ell}) \longrightarrow H^n(X, \mathcal{O}_X(\ell - d)) \longrightarrow H^n(X, \mathcal{O}_X(\ell)) \longrightarrow 0$$

so we have,

$$h^{n-1}(H, \mathcal{L}^{\otimes \ell}) = h^{n-1}(X, \mathcal{O}_X(\ell - d)) - h^{n-1}(X, \mathcal{O}_X(\ell)) = \binom{d - \ell - 1}{n} - \binom{-\ell - 1}{n}$$

which does have the correct degree in $(-\ell)$ i.e. $n-1$ to be $h^{n-1}(\mathbb{P}_k^{n-1}, \mathcal{O}_{\mathbb{P}_k^{n-1}}(k\ell))$. \square

14 Random Comalg Facts

Lemma 14.0.1. Let (p_1) and (p_2) be incommensurable prime ideals. Then $(p_1) \cap (p_2) = (p_1 p_2)$.

Proof. Clearly $(p_1 p_2) \subset (p_1) \cap (p_2)$ so it suffices to show that if $a = p_1 x = p_2 y$ then $a \in (p_1 p_2)$. Since $a \in (p_1)$ and $p_2 \notin (p_1)$ we get $y \in (p_1)$ and likewise $x \in (p_2)$ showing that $a \in (p_1 p_2)$. \square

15 Open Questions

- (a) Coproducts in the Category of Schemes vs Affine Schemes why are they different but agree with LRS coproducts in the first case which agree with Top coproducts since the Forget : LRS \rightarrow Top has a right-adjoint (Raymond chat).
- (b) Which Hypersurfaces are Rational? GOOD QUESTION. I think all quadric hypersurfaces are rational even though only the conic $X^2 + Y^2 - Z^2$ is on the nose isomorphic to \mathbb{P}_k^1 . Can we prove this? Projection from a point?
- (c) Example of an affine curve which does not embed in \mathbb{A}_k^2
- (d) Does unirational imply finite domination by rational variety in general?

16 To Do on Thesis

- (a) Example of non-arithmetic curve with no Δ_ν -regular equation, try the think with weakly Δ -nondegenerate by never Δ -nondegenerate.
- (b) Is the elliptic curve example I gave toric?
- (c) find example which is toric: use the
- (d) Explicit example of curve not on toric surface?
- (e) Explicit example of curve not on a Hirzburch surface?
- (f) Example of curve which is toric but never weakly Δ -nondegenerate?
- (g)

17 When is a Sheaf a Pushforward

THE FOLLOWING IS NOT QUITE CORRECT BUT APPROXIMATELY

Lemma 17.0.1. Let $\iota : f : Z \hookrightarrow X$ be a closed embedding and $U = X \setminus Z$. Then if \mathcal{F} is a sheaf of \mathcal{O}_X -modules then $\mathcal{F} = \iota_* \iota^* \mathcal{F}$ if and only if $\mathcal{F}|_U = 0$. Furthermore, $\mathcal{F} = \iota_* \iota^* \mathcal{F}$ if and only if $\mathcal{I} \cdot \mathcal{F} = 0$ where \mathcal{I} is the ideal sheaf of $Z \hookrightarrow X$. Furthermore, if Z is reduced then these notions agree.

Proof.

□

Remark. Given simply topological maps, a sheaf \mathcal{F} is a pushforward of some sheaf on a closed subset exactly when it is zero on the complement. However, if we ask for this sheaf to be the pushforward of a sheaf of \mathcal{O}_Z -modules then we need the stronger $\mathcal{I} \cdot \mathcal{F} = 0$.

18 Cayley-Hamilton

Theorem 18.0.1. Let $A \in M_n(R)$ be a square matrix over a ring R and $p_A(\lambda) = \det(\lambda I - A)$ be its characteristic polynomial. Then $p_A(A) = 0$.

Proof. First, I argue in the case that $R = k$ is a field. Matrices $A \in M_n(k)$ correspond to closed points of $X = \mathbb{A}_k^{n^2} = \text{Spec}(k[a_{ij}])$. Now the fundamental observation is that $p_A(A)$ is a matrix of polynomials in a_{ij} and thus gives a morphism $p : X \rightarrow X$ via the ring map $k[a_{ij}] \rightarrow k[a_{ij}]$ sending a_{ij} to the i, j entry of the matrix $p_A(A)$ with $A = (a_{ij})$.

Now, if p_A is separable (i.e. has distinct roots over \bar{k}) then A is diagonalizable over \bar{k} (eigenvectors with distinct eigenvalues are independent). Then $A = BDB^{-1}$ with D diagonal (these matrices defined over \bar{k}) and it is clear that $p_A(BDB^{-1}) = Bp_A(D)B^{-1} = 0$ since $p_A(\lambda) = 0$ for each eigenvalue. Furthermore, this case occurs exactly when the discriminant $\Delta(p_A) \neq 0$ which is a polynomial in a_{ij} so $\Delta : X \rightarrow \mathbb{A}_k^1$ gives a global function. We have shown that for any closed point $A \in D(\Delta)$, i.e. some matrix over \bar{k} with $\Delta(p_A) \neq 0$, that $p_A(A) = 0$ so the map $p : X \rightarrow X$ vanishes on the closed points of $D(\Delta)$ which is dense since it is open and nontrivial (any diagonal matrix over \bar{k} with nonrepeated entries satisfies this, I guess I used \bar{k} is infinite here) in an irreducible variety X . Thus $p : X \rightarrow X$ is the zero map since it vanishes on a dense set (using that X is a variety). In particular p is the zero polynomial in a_{ij} .

Now, for an arbitrary ring R take a matrix $A \in M_n(R)$ then $p(a_{ij}) = p_A(A)$ is an integer coefficient polynomial in a_{ij} (meaning the coefficients are in the image $\mathbb{Z} \rightarrow R$). However, for each prime $\mathfrak{p} \in \text{Spec}(R)$, the above argument shows that $\overline{p_A(A)} \in \kappa(\mathfrak{p})$ is zero since it is the characteristic polynomial applied to the matrix $\bar{A} \in M_n(\kappa(\mathfrak{p}))$ over the field $\kappa(\mathfrak{p})$. Thus $p_A(A) \in \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$ so $p_A(A) \in \text{nilrad}(R)$ for any A thus the coefficients are in $\text{nilrad}(R)$ (we can see this because reducing p in $\kappa(\mathfrak{p})$ gives the zero polynomial). However, the coefficients are in the image of $\mathbb{Z} \rightarrow R$ then $\text{nilrad}(R) \cap \text{Im}(\mathbb{Z}) = \text{nilrad}(\mathbb{Z}/(n))$ where $n = \ker(\mathbb{Z} \rightarrow R)$ (DAMN DOESNT WORK) \square

19 Quasi-Compactness and Noetherian Spaces

Definition 19.0.1. A topological space X is Noetherian if every descending chain of closed sets stabilizes.

Lemma 19.0.2. Subspaces of Noetherian spaces are Noetherian.

Proof. Let $S \subset X$ with X noetherian. Then the closed sets of S are exactly $S \cap Z$ for $Z \subset X$ closed. Thus descending chains of closed sets in S stabilize. \square

Definition 19.0.3. A space is quasi-compact if every open cover has a finite subcover.

Lemma 19.0.4. Noetherian spaces are quasi-compact.

Proof. Let U_α be an open cover of X which is Noetherian. Then consider the poset A under inclusion of finite unions of the U_α all of which are open sets of X . Since X is Noetherian any ascending chain of opens must stabilize so any chain in A has a maximum. Then by Zorn's lemma A has a maximal element which must be X since the U_α form a cover. Therefore there exists a finite subcover. \square

Corollary 19.0.5. Every subset of a noetherian topological space is quasi-compact.

Definition 19.0.6. A continuous map $f : X \rightarrow Y$ is quasi-compact if for each quasi-compact open $U \subset Y$ then $f^{-1}(U)$ is quasi-compact open.

19.1 Irreducible Components

Lemma 19.1.1. Let $Z \subset X$ be irreducible. Then $\overline{Z} \subset X$ is irreducible.

Proof. Suppose that $\overline{Z} = Z_1 \cup Z_2$ with Z_1 and Z_2 closed. Then $Z \subset Z_1 \cup Z_2$ so either $Z \subset Z_1$ or $Z \subset Z_2$. But since Z_1 and Z_2 are closed, we get $\overline{Z} = Z_1$ or $\overline{Z} = Z_2$. \square

Lemma 19.1.2. Increasing unions of irreducible subsets are irreducible.

Proof. Consider a chain T of irreducible subsets and consider,

$$U = \bigcup_{S \in T} S$$

Suppose $U = Z_1 \cup Z_2$ for closed subsets Z_1 and Z_2 of U . Then for each $S \in T$ we have $S \subset Z_1$ or $S \subset Z_2$. If for some $S_0 \in T$ we have $S_0 \not\subset Z_2$ (otherwise $Z_2 \supset U$ and we are done) then $S_0 \subset Z_1$ and for any $S \in T$ with $S \supset S_0$ we cannot have $S \subset Z_2$ else $S_0 \subset Z_2$. Therefore, $S \subset Z_1$. For any $S \in T$, since T is totally ordered, either $S \subset S_0$ in which case $S \subset Z_1$ or $S \supset S_0$ in which case $S \subset Z_1$ (as we have just shown). Therefore, $U \subset Z_1$ so U is irreducible. \square

Definition 19.1.3. Let X be a topological space then its irreducible components are the maximal irreducible subsets of X .

Remark. The irreducible subsets of X form a poset under inclusion. Furthermore, since chains have a maximum, by Zorn's lemma X always has some irreducible component.

Lemma 19.1.4. Let X be a topological space. The following hold,

- (a) irreducible components are closed
- (b) every irreducible subset of X is contained in some irreducible component
- (c) the irreducible components of X cover X .

Proof. Let $C \subset X$ be an irreducible component. Then \overline{C} is irreducible and $C \subset \overline{C}$ so $\overline{C} = C$ by maximality. Thus, C is closed. For any irreducible set $S \subset X$, Zorn's Lemma gives a maximal element in the irreducible components above S i.e. $S \subset C$ is contained in some irreducible component. In particular, since any point $x \in X$ is irreducible so $x \in C$ is contained in some irreducible component. Thus the irreducible components cover X . \square

Lemma 19.1.5. Noetherian spaces have finitely many irreducible components.

Proof. Let S be the poset of closed subspaces with infinitely many components ordered by inclusion. By the Noetherian hypothesis, descending chains in S have minima so, by Zorn's lemma, S has a minimum Z which has infinitely many irreducible components. Clearly, Z cannot be irreducible so we can write $Z = Z_1 \cup Z_2$ with $Z_1, Z_2 \subsetneq Z$ are proper closed subsets. By minimality, $Z_1, Z_2 \notin S$ and thus Z_1, Z_2 have finitely many irreducible components. Thus, $Z = Z_1 \cup Z_2$ has finitely many irreducible components so S is empty. \square

19.2 The Case for Schemes

Lemma 19.2.1. Affine schemes are quasi-compact.

Proof. Let U_i be an open cover of $\text{Spec}(A_i)$. Since $D(f)$ for $f \in A$ forms a basis of the topology on $\text{Spec}(A_i)$ we can shrink to the case $U_i = D(f_i)$. Then,

$$X = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D(\{f_i \mid i \in I\})$$

And thus the ideal $I = (\{f_i \mid i \in I\})$ is not contained in any maximal ideal so $I = (1)$. Therefore, there are f_1, \dots, f_n such that $a_1 f_1 + \dots + a_n f_n = 1$ and thus $(f_1, \dots, f_n) = (1)$ which implies that,

$$X = D((f_1, \dots, f_n)) = \bigcup_{i=1}^n D(f_i)$$

so X is quasi-compact. \square

Definition 19.2.2. A scheme X is *locally Noetherian* if for every affine open U the ring $\mathcal{O}_X(U)$ is Noetherian. X is *Noetherian* if it is quasi-compact and locally-Noetherian.

Lemma 19.2.3. If $(f_1, \dots, f_n) = A$ and A_{f_i} is Noetherian then A is Noetherian.

Proof. For any ideal $I \subset A$ we know $I_{f_i} \subset A_{f_i}$ is finitely generated. Clearing denominators and collecting the finite union of these finite generators gives a map $A^N \rightarrow I$ which is surjective when localized $A_{f_i}^N \twoheadrightarrow I_{f_i}$. Consider the A -module $K = \text{coker}(A^N \rightarrow I)$ then for any $x \in K$ we have $f_i^{n_i} \cdot x = 0$ for each i but $f_i^{n_i}$ generate the unit ideal (since $D(f_i^{n_i}) = D(f_i)$ which cover $\text{Spec}(A)$) so $x = 0$ to $A^N \twoheadrightarrow I$ so I is finitely generated showing that A is Noetherian. \square

Lemma 19.2.4. If X has an open affine cover $U_i = \text{Spec}(A_i)$ with A_i noetherian then X is locally noetherian. Moreover, if the cover can be made finite then X is noetherian.

Proof. Let $V = \text{Spec}(B) \subset X$ be an affine open, Then $V \cap U_i \subset V$ is open so it may be covered by principal opens $D(f_{ij}) \subset V \cap U_i$ for $f_{ij} \in B$. Since V is quasi-compact we may find a finite subcover. We need to show that $B_{f_{ij}}$ is Noetherian then since $D(f_{ij})$ cover V we use the lemma to conclude that B is Noetherian. However, $D(f_{ij}) \subset V \cap U_i$ can be covered by principal opens (of $U_i = \text{Spec}(A_i)$) $W_{ijk} \subset D(f_{ij}) \subset U_i = \text{Spec}(A_i)$ and each $(A_i)_{f_{ijk}}$ is Noetherian since A_i is, so using the same lemma we find that $B_{f_{ij}}$ is Noetherian.

Now suppose the cover is finite and let V_j be any open cover of X . We need to show X is quasi-compact so we must show that V_i has a finite subcover. Consider $U_i \cap V_j$ which is open in the affine $U_i = \text{Spec}(A_i)$ so it may be covered by principal opens $D(f_{ijk}) \subset U_i \cap V_j$. Now,

$$U_i = \bigcup_{j,k} D(f_{ijk})$$

but U_i is affine and thus quasi-compact so we may find a finite subcover which only uses finitely many V_i but the cover U_i of X is also finite so only finitely many V_i are needed to cover X . \square

Corollary 19.2.5. $X = \text{Spec}(A)$ is Noetherian iff A is a Noetherian ring.

Proof. If X is Noetherian then $\mathcal{O}_X(X) = A$ is a Noetherian ring (X is affine and thus quasi-compact). Conversely $\text{Spec}(A)$ is a finite Noetherian affine cover so X is Noetherian. \square

Remark. It is not the case that for a Noetherian scheme we must have $\mathcal{O}_X(X)$ a noetherian ring even for varieties. See <http://sma.epfl.ch/ojangure/nichtnoethersch.pdf>.

Corollary 19.2.6. A Noetherian ring has finitely many minimal primes.

Proof. Let A be Noetherian then primes $\mathfrak{p} \in \text{Spec}(A)$ correspond to irreducible closed subsets $V(\mathfrak{p})$ and thus minimal primes correspond to irreducible components of $\text{Spec}(A)$. Therefore, since $\text{Spec}(A)$ is Noetherian, we see that $\text{Spec}(A)$ has finitely many irreducible components and thus finitely many minimal primes. \square

Lemma 19.2.7. If A is Noetherian then $\text{Spec}(A)$ is a Noetherian topological space.

Proof. Every descending chain of subsets is of the form $V(I_1) \supsetneq V(I_2) \supsetneq V(I_3) \supsetneq \cdots$ but the ideals,

$$\sqrt{I_1} \subsetneq \sqrt{I_2} \subsetneq \sqrt{I_3} \subsetneq \cdots$$

stabilize since A is Noetherian and thus so does the chain of closed subsets. \square

Lemma 19.2.8. If X is a Noetherian scheme then its underlying topological space is Noetherian.

Proof. Choose a finite covering $U_i = \text{Spec}(A_i)$ by Noetherian rings. Then for any descending chain of closed subsets $Z_1 \supsetneq Z_2 \supsetneq Z_3 \supsetneq \cdots$ we know $Z \cap U_i$ stabilizes at n_i since $\text{Spec}(A_i)$ is a Noetherian space. Thus, Z stabilizes at $\max n_i$ which exists since the cover is finite. \square

Remark. The converses of the above are false and so is X Noetherian. Let R be a non-Noetherian valuation ring. Then $\text{Spec}(R)$ has two points and thus is Noetherian as a topological space but not as a scheme since R is not a Noetherian ring.

Lemma 19.2.9. If X is locally Noetherian then any immersion $\iota : Z \hookrightarrow X$ is quasi-compact.

Proof. Closed immersions are affine and thus quasi-compact so it suffices okay to show that open immersions are quasi-compact. Let $j : U \rightarrow X$ be an open immersion. It suffices to check that $j^{-1}(U_i)$ is quasi-compact on an affine open cover $U_i = \text{Spec}(A_i)$ with A_i Noetherian. But $j : j^{-1}(U_i) \rightarrow U_i \cap U$ is a homeomorphism and $\text{Spec}(A_i)$ is a Noetherian topological space so every subset is quasi-compact and, in particular, $U_i \cap U$ is quasi-compact so $j^{-1}(U_i)$ is also. \square

Remark. When X is Noetherian then it is a Noetherian space so any inclusion map $\iota : Z \hookrightarrow X$ for any subset $Z \subset X$ is quasi-compact since every subset is quasi-compact. In particular, every subset of X is retrocompact.

19.3 Quasi-Compact Morphisms

Lemma 19.3.1. A morphism $f : X \rightarrow Y$ is quasi-compact iff Y has a cover by affine opens V_i such that $f^{-1}(V_i)$ is quasi-compact.

Proof. Clearly if f is quasi-compact then any affine open cover V_i of Y satisfies $f^{-1}(V_i)$ is quasi-compact since V_i is a quasi-compact open by virtue of being affine open.

Now assume that such a cover exists. Let $U \subset Y$ be a quasi-compact open. Then U is covered by finitely many V_1, \dots, V_n . Then $U \cap V_i$ is open in V_i which is affine so it is covered by standard opens

W_{ij} . Since U is quasi-compact then we can choose finitely many W_{ij} . Now $f^{-1}(V_i)$ is quasi-compact by assumption so it has a finite cover by affine opens,

$$f^{-1}(V_i) = \bigcup_{j=1}^N \tilde{V}_{ij}$$

Then $f : \tilde{V}_{ik} \rightarrow V_i$ is a morphism of affine schemes so $f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$ is a principal affine. Therefore,

$$f^{-1}(U) = \bigcup_{i=1}^n f^{-1}(V_i \cap U) = \bigcup_{i,j} f^{-1}(W_{ij}) = \bigcup_{i,j,k} f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$$

is a finite union of principal affines each of which is quasi-compact so $f^{-1}(U)$ is quasi-compact. \square

Proposition 19.3.2. X is quasi-compact iff any morphism $X \rightarrow T$ for some affine scheme T is quasi-compact.

Proof. If X is quasi-compact then $f : X \rightarrow T$ is quasi-compact since T is an affine open cover of itself and $f^{-1}(T)$ is quasi-compact. Conversely, if $f : X \rightarrow T$ is quasi-compact with T affine then T is quasi-compact open in T so $X = f^{-1}(T)$ is quasi-compact. \square

Lemma 19.3.3. The base change of a quasi-compact morphism is quasi-compact.

Proof. (DO THIS) \square

20 Affine Morphisms

Definition 20.0.1. A morphism $f : X \rightarrow Y$ is *affine* if the preimage of every affine open is affine.

Lemma 20.0.2. Every morphism of affine schemes is affine and thus quasi-compact.

Proof. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ and $f : X \rightarrow Y$ be a morphism of affine schemes given by a ring map $\varphi : B \rightarrow A$. Then, any affine open $\text{Spec}(C) = V \subset Y$ can be covered by principal opens $D(f_i)$ for $f_i \in B$. Note that under $\psi : B \rightarrow C$ we see that $D(f_i) = D(\psi(f_i))$ since $D(f_i) \subset \text{Spec}(C)$. Since $D(\psi(f_i))$ cover $\text{Spec}(C)$ then $\psi(f_i) \in C$ generate the unit ideal. Then we have $f^{-1}(D(f_i)) = D(\varphi(f_i))$ which is affine and $\varphi(f_i)$ generate the unit ideal of $\Gamma(f^{-1}(V), \mathcal{O}_X)$ so f^{-1} is affine. \square

Remark. An alternative proof goes as follows. Consider the pullback diagram,

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

then open immersions are stable under base change so $f^{-1}(U) = U \times_Y X = \text{Spec}(C \otimes_B A)$ if affine.

Remark. In fact, by Tag 01S8, a morphism $f : X \rightarrow S$ is affine iff X is relatively affine over S meaning $X = \mathbf{Spec}_S(\mathcal{A})$ for some quasi-coherent \mathcal{O}_S -algebra \mathcal{A} .

Lemma 20.0.3. Let $f : X \rightarrow Y$ be a morphism and W_i an affine open cover of Y such that $f^{-1}(W_i)$ is affine. Then f is affine.

Proof. Let $\text{Spec}(A) = V \subset Y$ be affine open. Then $V_i = V \cap W_i$ is open in the affine open $V = \text{Spec}(A)$ so it can be covered by principal opens $D(f_{ij}) \subset V \cap W_i$ for $f_{ij} \in A$. Since $f : f^{-1}(W_i) \rightarrow W_i$ is a morphism of affine schemes, the preimage of the affine open $D(f_{ij}) \subset V \cap W_i$ is affine $f^{-1}(D(f_{ij}))$ (note that $D(f_{ij}) \subset V \cap W_i$ is not necessarily a principal affine open of W_i). But since $D(f_{ij})$ cover $\text{Spec}(A)$ the $f_{ij} \in A$ generate the unit ideal and thus $f^\#(f_{ij}) \in \Gamma(f^{-1}(V), \mathcal{O}_X)$ generate the unit ideal and $(f^{-1}(V))_{f_{ij}} = f^{-1}(D(f_{ij}))$ is affine so $f^{-1}(V)$ is affine. \square

Lemma 20.0.4. The base change of an affine morphism is affine.

Proof. (DO THIS) \square

Lemma 20.0.5. Affine morphisms are quasi-compact.

Proof. If $f : X \rightarrow Y$ is affine then any affine open cover V_i of Y gives $f^{-1}(V_i)$ is affine and thus quasi-compact so f is quasi-compact. \square

21 Separatedness

Definition 21.0.1. A morphism $f : X \rightarrow Y$ with diagonal $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is,

- (a) *separated* if the diagonal $\Delta_{X/Y}$ is a closed immersion
- (b) *affine-separated* if the diagonal $\Delta_{X/Y}$ is affine
- (c) *quasi-separated* if the diagonal $\Delta_{X/Y}$ is quasi-compact

Lemma 21.0.2. Any morphism of affine schemes is separated. Furthermore, affine morphisms are separated.

Proof. For a map $\text{Spec}(A) \rightarrow \text{Spec}(B)$ the diagonal is $\text{Spec}(A) \rightarrow \text{Spec}(A \otimes_B A)$ given by $A \otimes_B A \rightarrow A$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ which is surjective so the diagonal is a closed immersion. The second fact is Tag 01S7. \square

Lemma 21.0.3. The composition of (quasi/affine)-separated morphisms are (quasi/affine)-separated.

Proof. (DO THIS) \square

Lemma 21.0.4. For any morphism $f : X \rightarrow Y$ the diagonal $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is an immersion.

Proof. Let V_i be an affine cover of Y then choose an affine open cover U_{ij} of X with $f(U_{ij}) \subset V_i$. Then the diagonal of the affine map $U_{ij} \rightarrow V_j$ is $U_{ij} \rightarrow U_{ij} \times_{V_i} U_{ij}$ which is a closed immersion since it corresponds to $A_{ij} \otimes_{B_i} A_{ij} \rightarrow A_{ij}$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ is surjective. Therefore $f : X \rightarrow Y$ is locally on X a closed immersion and thus an immersion. \square

Remark. Therefore, to show that $f : X \rightarrow Y$ is separated, it suffices to show that the diagonal is closed (here equivalently meaning that the map or its image is closed).

Lemma 21.0.5. If X is Noetherian then every morphism $f : X \rightarrow S$ is quasi-compact and quasi-separated.

Proof. Every subset of X is quasi-compact since X is (topologically) Noetherian. Then apply the first part to the diagonal $\Delta_{X/S} : X \rightarrow X \times_S X$ which is then quasi-compact and thus $f : X \rightarrow S$ is quasi-separated. \square

Lemma 21.0.6. Let $f : X \rightarrow S$ be affine-separated/quasi-separated with $S = \operatorname{Spec}(A)$ affine. Then for any two affine opens $U, V \subset X$ the intersection $U \cap V$ is affine/quasi-compact.

Proof. Consider the pullback diagram,

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \times_S V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

where $U \cap V = \Delta_{X/S}^{-1}(U \times_S V)$ using the basechange of an open immersion is an open immersion. Then since S is affine, $U \times_S V$ is affine and thus quasi-compact open of $X \times_S X$. Then if f is affine-separated then $\Delta_{X/S}$ is affine so $U \cap V = \Delta_{X/S}^{-1}(U \times_S V)$ is affine. If f is quasi-separated then $\Delta_{X/S}$ is quasi-compact so $U \cap V = \Delta_{X/S}^{-1}(U \times_S V)$ is quasi-compact. \square

Remark. In the separated case, we see that $U \cap V$ is affine and $\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$ is surjective.

Remark. Tag 01KO gives a generalization of this lemma. For the separated case see Tag 01KP.

Lemma 21.0.7. Let $f : X \rightarrow Y$ be quasi-compact and quasi-separated and \mathcal{F} be a quasi-coherent \mathcal{O}_X -module then $f_*\mathcal{F}$ is a quasi-coherent \mathcal{O}_Y -module.

Proof. Since this is local on Y we can restrict to the case that Y is affine. Then $X = f^{-1}(Y)$ is quasi-compact (when Y is not affine $f^{-1}(V)$ will be quasi-compact) so take a finite affine open cover U_i and since $f : X \rightarrow Y$ is quasi-separated over an affine then by the above lemma $U_i \cap U_j$ is quasi-compact so it has a finite affine open cover U_{ijk} . Then, by the sheaf property, there is an exact sequence of sheaves on Y

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i f_*(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} f_*(\mathcal{F}|_{U_{ijk}})$$

which works because these are finite sums. However, $f : U_{ijk} \rightarrow Y$ is a morphism of affine schemes and since \mathcal{F} is quasi-coherent we have $\mathcal{F}|_{U_{ijk}} = \widetilde{M_{ijk}}$ so $f_*(\mathcal{F}|_{U_{ijk}}) = \widetilde{M_{ijk}}$ as an $\mathcal{O}_Y(Y)$ -module. Thus, $f_*\mathcal{F}$ is a kernel of quasi-coherent \mathcal{O}_Y -modules and thus is quasi-coherent. \square

Remark. If X is Noetherian then $f : X \rightarrow Y$ is automatically quasi-compact and quasi-separated so there is no issue in the above lemma.

22 Sets Cut Out By Some Function

Theorem 22.0.1. Every closed subset $E \subset \mathbb{R}^n$ is the vanishing of some smooth function.

Proof. Since \mathbb{R}^n is a metric space, it is hereditarily paracompact so the complement $E^C \subset \mathbb{R}^n$ is paracompact. Since \mathbb{R}^n is separable, E^C is covered by countably many balls $B_{r_i}(a_i)$ for $a_i \in E^C$ since it is open so, by paracompactness, we may shrink the radii such that this cover is locally finite. Choose a smooth bump function,

$$g : [0, \infty) \rightarrow [0, \infty)$$

such that $g([0, 1)) > 0$ and $g([1, \infty)) = 0$ e.g.

$$g(x) = \begin{cases} \exp\left(-\frac{1}{1-x}\right) & x < 1 \\ 0 & x \geq 1 \end{cases}$$

Then consider,

$$f(x) = \sum_{x \in X} g(|x - a_i|/r_i)$$

Since $g(|x - a_i|/r_i) = 0$ for $x \notin B_{r_i}(a_i)$ and the cover is locally finite, this is a finite sum so f is well-defined and smooth. Furthermore,

$$f(x) = 0 \iff x \notin \forall i \in I : x \notin B_{r_i}(a_i) \iff x \notin E^C \iff x \in E$$

□

Remark. This easily generalizes to show that any closed subset $Z \subset X$ of a smooth manifold is cut out by closed sets.

Our next question is what does the vanishing of analytic or holomorphic functions look like. We have one result.

Proposition 22.0.2. A nontrivial vanishing set of analytic functions in \mathbb{R}^n (or holomorphic functions in \mathbb{C}^n) has positive codimension. Explicitly, it does not contain any nonempty open.

Proof. This is clear because analytic and holomorphic functions which vanish on a nonempty open vanish everywhere. □

23 The Topology of Schemes

Here I want to ask what the topology of schemes “looks like” from the perspective of algebraic topology. The importance of the analytification functor $X \mapsto X^{\text{an}}$ is that it allows us to compute the “correct” topological invariants to complex varieties. However, what happens if we try to compute algebraic topology on the Zariski topology?

Lemma 23.0.1. Suppose X is a topological space with a dense point $\xi \in X$. Then X is contractible.

Proof. Consider the homotopy $h : X \times I \rightarrow X$ defined by,

$$h(x, t) = \begin{cases} x & t = 0 \\ \xi & t > 0 \end{cases}$$

This is continuous because no nontrivial closed set $Z \subset X$ contains ξ so $h^{-1}(Z) = Z \times \{0\}$ which is closed. Furthermore $h^{-1}(X) = X \times I$ so h is continuous. □

Remark. In particular, we see that every irreducible scheme is contractible.

However, there are example of varieties which have nontrivial homotopy type.

Example 23.0.2. <https://math.stackexchange.com/questions/2701914/connected-non-contractible-schemes>

24 Ample Invertible Sheaves

DO THIS!!!!

24.1 of Ample Divisor is Affine

Remark. Recall that $X_s = \{x \in X \mid s_x \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open since under a local trivialization this is $\tilde{s}_x \notin \mathfrak{m}_x$ and this happens exactly when s is locally invertible an open condition.

Remark. The following is Grothendieck's definition of Ampleness.

Definition 24.1.1. Let X be quasi-compact. Then an invertible \mathcal{O}_X -module \mathcal{L} is ample if for each $x \in X$ there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

Theorem 24.1.2. Let \mathcal{L} be ample on quasi-compact X and $s \in \Gamma(X, \mathcal{L})$ then X_s is affine.

Proof. We know that $s : \mathcal{O}_{X_s} \rightarrow \mathcal{L}|_{X_s}$ is an isomorphism. For each $x_i \in X_s$ we can choose $n_i \geq 1$ and $s_i \in \Gamma(X, \mathcal{L}^{\otimes n_i})$ such that X_{s_i} is affine and $x_i \in X_{s_i}$. \square

Remark. Since \mathcal{L} is ample iff $\mathcal{L}^{\otimes n}$ is ample for any $n \geq 1$ we see that X_s is affine for any $s \in \Gamma(X, \mathcal{L}^{\otimes n})$.

25 test

$$\begin{aligned} & \mathcal{O}_X \\ & A_{f_1} = A_{f_1} \\ & \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \\ & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \\ & \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \mathcal{G}) \\ & \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \\ & \text{End}_{\mathcal{O}_X}(\mathcal{F}) \\ & \mathcal{D}er(\mathcal{F}, \mathcal{G}) \end{aligned}$$

26 Backup

27 Rational Maps

Definition 27.0.1. Let X, Y be schemes over S . Consider the set of pairs of opens and S -morphisms,

$$\{(U, f_U) \mid U \subset X \text{ dense open } f_U : U \rightarrow Y\}$$

And an equivalence relation $(U, f_U) \sim (V, f_V)$ if for some dense (in X) open $W \subset U \cap V$ we have $(f_U)_W = (f_V)_W$. A rational S -morphism $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, f_U) .

The domain of the rational function $f : X \dashrightarrow Y$ is,

$$\text{Dom}(f) = \bigcup \{U \mid (U, f_U) \in f\}$$

The set of rational maps $X \dashrightarrow Y$ is exactly,

$$\text{Rat}(X, Y) = \varinjlim_{U \in \mathcal{D}(X)} \text{Hom}_{\mathbf{Sch}}(U, Y)$$

where $\mathcal{D}(X)$ is the set of dense open subset $U \subset X$.

Remark. This is an equivalence relation since if $(U_1, f_1) \sim (U_2, f_2) \sim (U_3, f_3)$ then there exist dense opens $V \subset U_1 \cap U_2$ and $W \subset U_2 \cap U_3$. Then $V \cap W \subset U_1 \cap U_2 \cap U_3 \subset U_1 \cap U_3$ and $V \cap W$ is a dense open. Furthermore,

$$(f_1|_V)_{V \cap W} = (f_2|_V)_{V \cap W} = (f_2|_W)_{V \cap W} = (f_3|_W)_{V \cap W}$$

Lemma 27.0.2. If $U, V \subset X$ are dense opens then $U \cap V$ is a dense open.

Proof. For any nonempty open $W \subset X$ we know $W \cap U$ is non empty open since U is dense and thus $W \cap U \cap V$ is nonempty since V is dense. Thus $U \cap V$ is dense. \square

27.1 Glueing Rational Maps

27.2 The Locus on Which Morphisms Agree

Lemma 27.2.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then for schemes X there is a natural bijection,

$$\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), X) \cong \{x \in X \text{ and local map } \mathcal{O}_{X,x} \rightarrow R\}$$

Proof. Given $\mathrm{Spec}(R) \rightarrow X$ we automatically get $\mathfrak{m} \mapsto x$ and $\mathcal{O}_{X,x} \rightarrow R_{\mathfrak{m}} = R$. Now, note that taking any affine open neighborhood $x \in \mathrm{Spec}(A) \subset X$ and then $A \rightarrow A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ to give $\mathrm{Spec}(\mathcal{O}_{X,x}) \rightarrow \mathrm{Spec}(A) \rightarrow X$. Clearly, this map sends $\mathfrak{m}_x \mapsto x$ and at \mathfrak{m}_x has stalk map $\mathrm{id} : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$ since it is the localization at \mathfrak{p} of $A \rightarrow A_{\mathfrak{p}}$.

Thus we get an inverse as follows. Given a point $x \in X$ and a local map $\phi : \mathcal{O}_{X,x} \rightarrow R$ then take,

$$\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(\mathcal{O}_{X,x}) \rightarrow X$$

This is inverse since $\mathfrak{m} \mapsto \mathfrak{m}_x$ (because $\mathcal{O}_{X,x} \rightarrow \mathfrak{m}_x$ is local) and $\mathfrak{m}_x \mapsto x$ and the stalk at \mathfrak{m} gives $\mathcal{O}_{X,x} \xrightarrow{\mathrm{id}} \mathcal{O}_{X,x} \xrightarrow{\phi} R$.

Finally, I claim that any $f : \mathrm{Spec}(R) \rightarrow X$ factors through $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ and thus is reconstructed from $x \in X$ and $\mathcal{O}_{X,x} \rightarrow R$. Choose an affine open neighborhood $x \in \mathrm{Spec}(A) \subset X$ then consider $f^{-1}(\mathrm{Spec}(A))$ which is open in $\mathrm{Spec}(R)$ and contains the unique closed point $\mathfrak{m} \in \mathrm{Spec}(R)$ so there is some $f \in R$ s.t. $\mathfrak{m} \in D(f) \subset f^{-1}(\mathrm{Spec}(A))$ so $f \notin \mathfrak{m}$ so $f \in R^\times$ and thus $D(f) = \mathrm{Spec}(R)$. Therefore, we get a map $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(A)$ and thus $\phi : A \rightarrow R$ where $\phi^{-1}(\mathfrak{m}) = \mathfrak{p} = x$ so $A \setminus \mathfrak{p}$ is mapped inside R^\times so this map factors through $A \rightarrow A_{\mathfrak{p}} \rightarrow R$ giving the desired factorization $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(\mathcal{O}_{X,x}) \rightarrow \mathrm{Spec}(A) \rightarrow X$. \square

Definition 27.2.2. The locus Z on which two maps $f, g : X \rightarrow Y$ over S agree is given as the pullback,

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \Delta_Y \\ X & \xrightarrow{F} & Y \times_S Y \end{array}$$

with $F = (f, g)$. Furthermore $Z \rightarrow X$ is an immersion.

Lemma 27.2.3. Topologically, the locus on which S -morphisms $f, g : X \rightarrow Y$ agree is,

$$Z = \{x \in X \mid f(x) = g(x) \text{ and } f_x = g_x : \kappa(f(x)) \rightarrow \kappa(x)\}$$

Proof. On some S -subscheme $G \subset X$, the maps $f|_G = g|_G$ agree iff there exists $G \rightarrow Y$ such that,

$$\begin{array}{ccc} G & \dashrightarrow & Y \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{F} & Y \times_S Y \end{array}$$

commutes. In particular, for any point $x \in X$ consider $\iota : \text{Spec}(\kappa(x)) \rightarrow X$ then $f \circ \iota = g \circ \iota$ iff $f(x) = g(x)$ and $f_x = g_x : \kappa(f(x)) \rightarrow \kappa(x)$. Consider a point $z \in Z$ and $\text{Spec}(\kappa(z)) \rightarrow Z$, such a point is equivalent to giving a diagram,

$$\begin{array}{ccccc} & & \text{Spec}(\kappa(z)) & & \\ & \searrow & & \nearrow & \\ & & Z & \xrightarrow{\quad} & Y \\ & & \downarrow \lrcorner & & \downarrow \Delta_Y \\ & & X & \xrightarrow{F} & Y \times_S Y \end{array}$$

However, $\iota : Z \rightarrow X$ is an immersion so $f_x : \kappa(f(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism. Therefore, points $\text{Spec}(\kappa(z)) \rightarrow Z$ of z , are exactly points of X for which a lift $\text{Spec}(\kappa(x)) \rightarrow Y$ exists i.e. points such that f and g agree in the required way. \square

Lemma 27.2.4. If $f : X \rightarrow Y$ is an immersion then $f_x : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is injective for each $x \in X$ and $f_x : \kappa(f(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism.

Proof. First note that $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is surjective by definition (surjective for the closed immersion factor and isomorphism for the open immersion factor). Thus we get an injection $f_x : \mathcal{O}_{Y,y} \rightarrow (f_*\mathcal{O}_X)_{f(x)}$. Furthermore, topologically, $f : X \rightarrow Y$ is a homeomorphism onto its image so for any open $U \subset X$ there exists an open $V \subset Y$ s.t. $U = f^{-1}(V)$ showing that,

$$(f_*\mathcal{O}_X)_{f(x)} = \varinjlim_{f(x) \in V} \mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}$$

Finally, since $f_x : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is local we get $f_x : \kappa(f(x)) \rightarrow \kappa(x)$ which is a surjection of fields and thus an isomorphism. \square

Lemma 27.2.5. If $Y \rightarrow S$ is separated then the locus on which $f, g : X \rightarrow Y$ over S agree is closed.

Proof. Since $X \rightarrow S$ is separated, $\Delta_{Y/S} : Y \rightarrow Y \times_S Y$ is a closed immersion. So $Z \rightarrow X$ is the base change of a closed immersion and thus a closed immersion. \square

Lemma 27.2.6. Let X be a reduced and Y be a separated scheme over S and $f, g : X \rightarrow Y$ be morphisms over S . If $f \circ j = g \circ j$ agree on a dense subscheme $j : G \hookrightarrow X$ then $f = g$.

Proof. Consider $F = (f, g) : X \rightarrow Y \times_S Y$. Since $\Delta : Y \rightarrow Y \times_S Y$ is a closed immersion (by separatedness). Then $F^{-1}(\Delta)$ is the locus on which $f = g$ which is closed because $\Delta : Y \rightarrow Y \times_S Y$ is a closed immersion. Since $f|_G = g|_G$ we get a diagram,

$$\begin{array}{ccccc}
G & & & & \\
& \searrow \text{dashed} & & \searrow & \\
& Z & \xrightarrow{\tilde{F}} & Y & \\
& \downarrow \iota & \lrcorner & \downarrow \Delta_Y & \\
& X & \xrightarrow{F} & Y \times_S Y &
\end{array}$$

Since $\iota : Z \hookrightarrow X$ is a closed immersion with dense image, $Z \hookrightarrow X$ is surjective. By the following, $\iota : Z \rightarrow X$ is an isomorphism. Thus, $F = F \circ \iota \circ \iota^{-1} = \Delta_Y \circ \tilde{F} \circ \iota^{-1}$. By the universal property of maps $X \rightarrow Y \times_S Y$ this implies that $f = g = \tilde{F} \circ \iota^{-1}$. \square

Lemma 27.2.7. Let X be a scheme and consider an exact sequence of quasi-coherent \mathcal{O}_X -modules,

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A} \longrightarrow 0$$

and \mathcal{A} is a sheaf of \mathcal{O}_X -algebras. Suppose that $\mathcal{A}_x \neq 0$ for each $x \in X$. Then $\mathcal{I} \hookrightarrow \mathcal{N}$ where \mathcal{N} is the sheaf of nilpotents.

Proof. Take an affine open $U = \text{Spec}(R) \subset X$ such that $\mathcal{A}|_U = \widetilde{A}$. Then we have a surjection of rings $R \twoheadrightarrow A$ giving $R/I = A$ for $I = \ker(R \rightarrow A)$. Now, for each $\mathfrak{p} \in \text{Spec}(R)$ we know $R_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}} \neq 0$. However, if $\mathfrak{p} \not\supset I$ then $(R/I)_{\mathfrak{p}} = A_{\mathfrak{p}} = 0$ so we must have $\mathfrak{p} \supset I$ for all $\mathfrak{p} \in \text{Spec}(R)$ i.e. $I \subset \text{nilrad}(R)$. Therefore, $\mathcal{I}|_U \hookrightarrow \mathcal{N}|_U$ for any affine open $U \subset X$ showing that \mathcal{I} is comprised of nilpotents. \square

Corollary 27.2.8. If X is reduced and $\iota : Z \hookrightarrow X$ is a surjective closed immersion then $\iota : Z \xrightarrow{\sim} X$ is an isomorphism.

Proof. Since $\iota : Z \hookrightarrow X$ is a homeomorphism onto its image X it suffices to show that the map of sheaves $\iota^{\#} : \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z$ is an isomorphism. Since $\iota : Z \rightarrow X$ is a closed immersion $\iota^{\#} : \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z$ is a surjection and \mathcal{O}_Z is a quasi-coherent sheaf of \mathcal{O}_X -algebras giving an exact sequence,

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Furthermore,

$$\text{Supp}_{\mathcal{O}_X}(\iota_* \mathcal{O}_Z) = \text{Im}(\iota) = X$$

since $(\iota_* \mathcal{O}_Z)_x = \mathcal{O}_{Z,x}$ when $x \in \text{Im}(\iota)$ (and zero elsewhere). by the above, $\mathcal{I} \hookrightarrow \mathcal{N} = 0$ since X is reduced so $\iota^{\#} : \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z$ is an isomorphism. \square

Lemma 27.2.9. A rational S -map $f : X \dashrightarrow Y$ with X reduced and $Y \rightarrow S$ separated is equivalent to a morphism $f : \text{Dom}(f) \rightarrow Y$.

Proof. For any (U, f_U) and (V, f_V) representing f there must be a dense (in X) open $W \subset U \cap V$ on which $f_U|_W = f_V|_W$ and thus $f_U|_{U \cap V} = f_V|_{U \cap V}$ since $f_U, f_V : U \cap V \rightarrow Y$ are morphisms from reduced to irreducible schemes. Now $\text{Dom}(f)$ has an open cover (U_i, f_i) for which $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ so these morphisms glue to give $f : \text{Dom}(f) \rightarrow Y$ ($\text{Hom}_S(-, Y)$ is a sheaf on the Zariski site). \square

27.3 Dominant Morphisms

Definition 27.3.1. A morphism $f : X \rightarrow Y$ is dominant if its image (topologically) is dense.

Lemma 27.3.2. If X and Y are irreducible with generic points $\xi \in X$ and $\eta \in Y$ then $f : X \rightarrow Y$ is dominant iff $f(\xi) = \eta$.

Proof. Clearly, if $f(\xi) = \eta$ then,

$$\overline{f(X)} \supset \overline{f(\xi)} = Y$$

so f is dominant. Conversely, suppose that $f : X \rightarrow Y$ is dominant. Then,

$$f(X) = f(\overline{\{\xi\}}) \subset \overline{f(\xi)}$$

but $f(X)$ is dense so $\overline{f(X)} = Y$ but Y has a unique generic point so $f(\xi) = \eta$. \square

Definition 27.3.3. Let X, Y be irreducible. A rational map $f : X \dashrightarrow Y$ is *dominant* if any representative $f : U \rightarrow Y$ is dominant.

Remark. Since, on an irreducible scheme X every nonempty open $W \subset X$ contains the generic point $\xi \in W \subset X$. Therefore, if $f_U : U \rightarrow Y$ and $f_V : V \rightarrow Y$ agree on some dense open $W \subset U \cap V$ then $f_U(\xi) = \eta \iff f_V(\xi) = \eta$ so some representative is dominant iff every representative is dominant.

Proposition 27.3.4. Irreducible schemes with dominant rational maps form a category.

Proof. It suffices to show how dominant rational maps may be composed. Given $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ and representatives $f_U : U \rightarrow Y$ and $g_V : V \rightarrow Z$. Then, consider $g \circ f : f^{-1}(V) \rightarrow Z$. Since f is dominant $\xi_X \in f^{-1}(\xi_Y) \subset f^{-1}(V)$ so $f^{-1}(V)$ is nonempty (since $\text{Im}(f) \cap V$ is nonempty because $\text{Im}(f)$ is dense). Furthermore, $f(\xi_X) = \xi_Y$ and $g(\xi_Y) = \xi_Z$ so $g \circ f(\xi_X) = \xi_Z$ and thus $g \circ f$ is dominant so it defines a dominant rational map $g \circ f : X \dashrightarrow Z$. Furthermore, if $(U_1, f_1) \sim (U_2, f_2)$ and $(V_1, g_1) \sim (V_2, g_2)$ then $f_1|_W = f_2|_W$ and $g_1|_{W'} = g_2|_{W'}$ for dense opens $W \subset U_1 \cap U_2$ and $W' \subset V_1 \cap V_2$. Then, $(g_1 \circ f_1)|_{f^{-1}(W') \cap W} = (g_2 \circ f_2)|_{f^{-1}(W') \cap W}$ so composition is well-defined. \square

Remark. We really need irreducibility to compose rational maps. Consider,

$$\text{Spec}(k[x, y]/(xy)) \xrightarrow{f} \text{Spec}(k[x]) \dashrightarrow \mathbb{G}_m$$

where $\text{Spec}(k[x]) \rightarrow \mathbb{G}_m$ is defined on the dense open $D(x)$. However, $f^{-1}(D(x)) \subset \text{Spec}(k[x, y]/(xy))$ is $\text{Spec}(k[x, x^{-1}]) \hookrightarrow \text{Spec}(k[x, y]/(xy))$ contained in the x -axis and thus is not dense.

27.4 Rational Functions

Definition 27.4.1. A *rational function* on a scheme X is a rational map $f : X \dashrightarrow \mathbb{A}_{\mathbb{Z}}^1$ or for $X \rightarrow S$ equivalently a rational S -map $f : X \dashrightarrow \mathbb{A}_S^1$. Since \mathbb{A}^1 is a ring object in the category of schemes and thus gives a ring structure on $\text{Hom}_{\mathbf{Sch}}(U, \mathbb{A}^1)$. This puts a ring structure on the set of rational functions forming the ring of rational functions,

$$R(X) = \varinjlim_{U \in \mathcal{D}(X)} \text{Hom}_{\mathbf{Sch}}(U, \mathbb{A}^1)$$

where $\mathcal{D}(X)$ is the set of dense open subsets $U \subset X$.

Proposition 27.4.2. Suppose that X has finitely many irreducible components with generic point ξ_i . Then,

$$R(X) = \mathcal{O}_{X,\xi_1} \times \cdots \times \mathcal{O}_{X,\xi_n}$$

Proof. For any dense open U there are finitely many irreducible components Z_i then $Z_i \cap U \neq \emptyset$ so $\xi_i \in U$ for each i since otherwise,

$$U \subset \bigcup_{i \neq j} Z_i$$

which is closed (since the union is finite) contradicting denseness of U . Now,

$$U_i = (Z_i \cap U) \setminus \bigcup_{j \neq i} Z_j$$

is open and $\xi \in U_i \subset Z_i$ and,

$$\bigcup_{i=1}^n U_i \subset U \subset X$$

is dense since it contains all ξ_i . However, $U_i \cap U_j = \emptyset$ and thus,

$$\begin{aligned} R(X) &= \varinjlim_{U \in \mathcal{D}(X)} \text{Hom}_{\mathbf{Sch}}(U, \mathbb{A}^1) \\ &= \varinjlim_{U \in \mathcal{D}(X)} \mathcal{O}_X(U) \\ &= \varinjlim_{U \in \mathcal{D}(X)} \prod_{i=1}^n \mathcal{O}_X(U_i) \\ &= \prod_{i=1}^n \varinjlim_{\xi_i \in U_i} \mathcal{O}_X(U_i) \\ &= \prod_{i=1}^n \mathcal{O}_{X,\xi_i} \end{aligned}$$

since all opens containing each generic point are dense. □

Corollary 27.4.3. If X is reduced then,

$$R(X) = \kappa(\xi_1) \times \cdots \times \kappa(\xi_n)$$

If X is irreducible then,

$$R(X) = \mathcal{O}_{X,\xi}$$

If X is integral then,

$$R(X) = \kappa(\xi) = K(X)$$

so the ring of rational functions is exactly the function field on an integral scheme.

Lemma 27.4.4. A dominant rational map $X \dashrightarrow Y$ (over S) between irreducible schemes induces a $\mathcal{O}_{S,s}$ -algebra map $\mathcal{O}_{Y,\xi_Y} \rightarrow \mathcal{O}_{X,\xi_X}$.

Proof. A morphism $X \dashrightarrow Y$ in the category of dominant rational S -maps gives by composition $R(Y) = \text{Hom}_{\mathbf{Rat}}(Y, \mathbb{A}_S^1) \rightarrow \text{Hom}_{\mathbf{Rat}}(X, \mathbb{A}_S^1) = R(X)$. Alternatively, since $X \dashrightarrow Y$ is defined on some nonempty open (dense is automatic for irreducible schemes) $U \rightarrow Y$ and $\xi_X \in U$. Since $X \dashrightarrow Y$ is dominant $\xi_X \mapsto \xi_Y$ and thus we get $\mathcal{O}_{X,\xi_Y} \rightarrow \mathcal{O}_{X,\xi_X}$ over $\mathcal{O}_{S,s}$ for $\xi_X, \xi_Y \mapsto s$. □

Corollary 27.4.5. A dominant rational k -map $X \dashrightarrow Y$ of integral schemes over $\mathrm{Spec}(k)$ induces an extension of function fields $K(Y) \hookrightarrow K(X)$ over k .

Remark. This is an extension of fields because a ring map $K(Y) \rightarrow K(X)$ is automatically injective on fields.

Corollary 27.4.6. There is a functor $\mathbf{Rat}_A^{\mathrm{op}} \rightarrow \mathbf{Ring}_A$ from the category of irreducible schemes over $\mathrm{Spec}(A)$ and dominant rational maps to the category of A -algebras sending $X \dashrightarrow Y$ to $R(Y) \rightarrow R(X)$.

Likewise, there is a functor $\mathbf{Rat}_{\mathrm{int},A}^{\mathrm{op}} \rightarrow \mathbf{Field}_A$ from the category of integral schemes over $\mathrm{Spec}(A)$ and dominant rational maps to the category of fields over A sending $X \dashrightarrow Y$ to $K(Y) \hookrightarrow K(X)$ over A .

27.5 Birational Maps

Definition 27.5.1. Irreducible S -schemes are *S -birational* if they are isomorphic in the category of irreducible S -schemes with dominant rational S -maps. We say that a rational S -map $f : X \dashrightarrow Y$ is a birational map if it is dominant and there is a dominant rational S -map $g : Y \dashrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ as rational maps.

Proposition 27.5.2. If irreducible schemes X and Y are birational then $R(X) = R(Y)$.

Proposition 27.5.3. In particular, if integral k -schemes X and Y are k -birational then $K(X) = K(Y)$ via a k -isomorphism.

Proposition 27.5.4. Let X and Y be irreducible S -schemes. Then X and Y are S -birational iff there are dense opens $U \subset X$ and $V \subset Y$ which are isomorphic $U \cong V$ over S .

Proof. If $f : U \rightarrow V$ and $g : V \rightarrow U$ are inverse S -isomorphisms then they represent inverse dominant (since they are surjective onto U, V which are dense) rational S -maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ so X and Y are birational.

The reverse direction is Tag 0BAA. □

Theorem 27.5.5. There is an equivalence of categories between the following,

- (a) the category of integral schemes locally of finite type over k with dominant rational maps
- (b) the category of affine integral schemes of finite type over k with dominant rational maps
- (c) the opposite category of finitely-generated k -algebra domains with dominant rational maps
- (d) the opposite category of finitely-generated fields over k with inclusions over k

Proof. We need to show that an embedding $K(Y) \hookrightarrow K(X)$ over k for integral scheme locally of finite type over k induces a rational map and for any finitely generated field K over k there is (DO THIS PROOF). □

Remark. The restriction on the category of schemes is necessary. $\mathrm{Spec}(k(x))$ is not finite type over k and there is no rational map $\mathrm{Spec}(k[x]) \dashrightarrow \mathrm{Spec}(k(x))$ induced by $k(x) \hookrightarrow k(x)$ since it would simply be a morphism and would be given by a ring map $k(x) \hookrightarrow k[x]$ inducing $\mathrm{id} : k(x) \rightarrow k(x)$ which is impossible since it needs to send $x \mapsto x$ which is a unit in the source but not the target.

Remark. See Tag 0BAD for a generalization.

27.6 Rational Varieties

27.7 Extending Rational Maps

Lemma 27.7.1. Regular local rings of dimension 1 exactly correspond to DVRs.

Proof. Any DVR R has a uniformizer $\varpi \in R$ then $\dim R = 1$ and $\mathfrak{m}/\mathfrak{m}^2 = (\varpi)/(\varpi^2) = \varpi\kappa$ which also has $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 1$ so R is regular.

Conversely, if R is a regular local ring of dimension $\dim R = 1$ then, by regularity, R is a normal noetherian domain so by $\dim R = 1$ then R is Dedekind but also local and thus is a DVR. \square

Proposition 27.7.2. Let X be a Noetherian S -scheme and $Z \subset X$ a closed irreducible codimension 1 generically nonsingular subset (with generic point $\eta \in Z$ such that $\mathcal{O}_{X,\eta}$ is regular). Let $f : X \dashrightarrow Y$ be a rational map with Y proper over S . Then $Z \cap \text{Dom}(f)$ is a dense open of Z .

Proof. Choose some representative (U, f_U) for $f : X \dashrightarrow Y$. Note that $\mathcal{O}_{X,\eta}$ is a regular dimension one (see Lemma ??) ring and thus a DVR. Consider the generic point $\xi \in X$ of X then, by localizing, we get an inclusion of the generic point $\text{Spec}(\mathcal{O}_{X,\xi}) \rightarrow \text{Spec}(\mathcal{O}_{X,\eta}) \rightarrow X$ and $\mathcal{O}_{X,\xi} = K(X) = \text{Frac}(\mathcal{O}_{X,\eta})$. Furthermore, the inclusion of the generic point gives $\text{Spec}(K(X)) \rightarrow U \xrightarrow{f_U} Y$ and thus we get a diagram,

$$\begin{array}{ccc} \text{Spec}(K(X)) & \xrightarrow{\quad} & Y \\ \downarrow & \searrow \ell & \downarrow \\ \text{Spec}(\mathcal{O}_{X,\eta}) & \longrightarrow & \text{Spec}(k) \end{array}$$

and a lift $\text{Spec}(\mathcal{O}_{X,\eta}) \rightarrow Y$ by the valuative criterion for properness applied to $Y \rightarrow \text{Spec}(k)$ since $\mathcal{O}_{X,\eta}$ is a DVR. Choose an affine open $\text{Spec}(R) \subset Y$ containing the image of $\text{Spec}(\mathcal{O}_{X,\eta}) \rightarrow Y$ (i.e. choose a neighborhood of the image of η which automatically contains $f(\xi)$ since the map factors $\text{Spec}(\mathcal{O}_{X,\eta}) \rightarrow \text{Spec}(\mathcal{O}_{Y,f(\eta)}) \rightarrow \text{Spec}(R) \rightarrow Y$) and let $\eta \in V = \text{Spec}(A) \subset X$ be an affine open neighborhood of ξ mapping onto $\text{Spec}(R)$. By Lemma 27.8.6, since $\mathcal{O}_{X,\eta}$ is a domain, we may shrink V so that A is a domain. Since X is irreducible $U \cap V$ is a dense open. Note that if $\eta \in U$ then $\eta \in \text{Dom}(f)$ and thus $Z \cap \text{Dom}(f)$ is a nonempty open of the irreducible space Z and therefore a dense open so we are done. Otherwise, let $\mathfrak{p} \in \text{Spec}(A)$ correspond to $\eta \in Z$ then $A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$ is a DVR. Take some principal affine open $D(f) \subset U \cap V$ for $f \in A$ so $f \in \mathfrak{p}$ since $\mathfrak{p} \notin D(f) \subset U \cap V$. Since $A_{\mathfrak{p}}$ is a DVR we may choose a uniformizer $\varpi \in \mathfrak{p}$ so the map $A \rightarrow \mathfrak{p}$ via $1 \mapsto \varpi$ is an isomorphism when localized at \mathfrak{p} . Since A is Noetherian both are f.g. A -modules so there must be some $s \in A \setminus \mathfrak{p}$ such that $A_s \rightarrow \mathfrak{p}_s$ is an isomorphism. Replacing A by A_s we may assume $\mathfrak{p} = (\varpi) \subset A$ is principal. Since $f \in \mathfrak{p}$ we can write $f = t\varpi^k$ for some $a \in A \setminus \mathfrak{p}$ (see Lemma 27.8.1). Then consider $\tilde{V} = \text{Spec}(A_t)$. Since $t \notin \mathfrak{p}$ then $\eta \in \tilde{V}$ and since $f = t\varpi^k$ we have $D(f) \subset D(t) = \tilde{V}$. Now we get the following diagram,

$$\begin{array}{ccc}
& & \text{Spec}(R) \\
& \nearrow \ell & \\
\text{Spec}(A_{\mathfrak{p}}) & \longrightarrow & \text{Spec}(A_t) \\
\uparrow & & \uparrow \\
\text{Spec}(\text{Frac}(A)) & \longrightarrow & \text{Spec}(A_f)
\end{array}
\begin{array}{l}
\text{---} f_V \text{---} \\
\text{---} f_U \text{---}
\end{array}$$

I claim the square is a pushout in the category of affine schemes because maps $R \rightarrow A_{\mathfrak{p}}$ and $R \rightarrow A_f$ which agree under the inclusion to $\text{Frac}(A)$ gives a map $R \rightarrow A_{\mathfrak{p}} \cap A_f \subset \text{Frac}(A)$. However, consider,

$$x \in A_{\mathfrak{p}} \cap A_t \implies x = \frac{u\varpi^r}{s} = \frac{a}{f^n}$$

for $u, s, t \in A \setminus \mathfrak{p}$ and $a \in A$. Thus we get,

$$ut^n \varpi^{r+nk} = sa$$

so $a \in \mathfrak{p}^{r+nk} \setminus \mathfrak{p}^{r+nk+1}$ ($s \notin \mathfrak{p}$ which is prime) and thus $a = u'\varpi^{r+nk}$ for $u' \in A \setminus \mathfrak{p}$. Therefore,

$$x = \frac{u'\varpi^{r+nk}}{t^n \varpi^{nk}} = \frac{u'\varpi^r}{t^n} \in A_t$$

Thus, $A_{\mathfrak{p}} \cap A_f \subset A_t$ so we get a map $R \rightarrow A_t$. Therefore we get a map $f_{\tilde{V}} : \tilde{V} \rightarrow Y$ such that $(f|_{\tilde{V}})|_{D(f)} = (f_U)|_{D(f)}$ which implies that $\eta \in \tilde{V} \subset \text{Dom}(f)$ so $Z \cap \text{Dom}(f)$ is a dense open of Z . \square

Proposition 27.7.3. Let $C \rightarrow S$ be a proper regular noetherian scheme with $\dim C = 1$ and $f : C \dashrightarrow Y$ a rational S -map with $Y \rightarrow S$ proper. Then f extends uniquely to a morphism $f : C \rightarrow Y$.

Proof. For any point $x \notin \text{Dom}(f)$ let $Z = \overline{\{x\}} \subset D$ for $D = C \setminus \text{Dom}(f)$. Since $\text{Dom}(f)$ is a dense open, by lemma 27.8.4, we have $\text{codim}(Z, C) \geq \text{codim}(D, C) \geq 1$ but $\dim C = 1$ so $\text{codim}(Z, C) = 1$. Furthermore, since C is regular, $\mathcal{O}_{C,x}$ is regular and thus, by the previous proposition, $Z \cap \text{Dom}(f)$ is a dense open and in particular $x \in \text{Dom}(f)$ meaning that $\text{Dom}(f) = C$ so we get a morphism $C \rightarrow Y$. This is unique because C is reduced (it is regular) and Y is separated (it is proper over S) so morphisms $C \rightarrow Y$ are uniquely determined on a dense open which any representative for $f : C \dashrightarrow Y$ is defined on. \square

Definition 27.7.4. A *curve* over k is an integral separated dimension one scheme finite type over $\text{Spec}(k)$.

Corollary 27.7.5. Rational maps between normal proper curves are morphisms.

Corollary 27.7.6. Birational maps between normal proper curves are isomorphisms.

Proof. Let $f : C_1 \dashrightarrow C_2$ and $g : C_2 \dashrightarrow C_1$ be birational inverses of smooth proper curves. Then we know that these extend to morphisms $f : C_1 \rightarrow C_2$ and $g : C_2 \rightarrow C_1$. Furthermore, the maps $g \circ f : C_1 \rightarrow C_1$ must extend the identity on some dense open. However, since curves are separated and reduced there is a unique extension of this map so $g \circ f = \text{id}_{C_1}$ and likewise $f \circ g = \text{id}_{C_2}$. \square

Theorem 27.7.7. If k is perfect then there exists a unique normal curve in each birational equivalence class of curves.

Proof. It suffices to show existence. Given a curve X , we consider the projective closure $X \rightarrow \overline{X}$ (WHY THIS EXISTS) which is birational and $\overline{X} \rightarrow \operatorname{Spec}(k)$ is proper. Then take the normalization $\overline{X}^\nu \rightarrow \overline{X}$ which remains proper over $\operatorname{Spec}(k)$ (CHECK THIS) and is birational. Then \overline{X}^ν is regular and thus smooth over k since k is perfect and $\overline{X}^\nu \rightarrow X$ is birational. \square

27.8 Lemmas

Lemma 27.8.1. Let A be a Noetherian domain and $\mathfrak{p} = (\varpi)$ a principal prime. Then any $f \in \mathfrak{p}$ can be written as $f = t\varpi^k$ for $f \in A \setminus \mathfrak{p}$.

Proof. From Krull intersection,

$$\bigcap_{n \geq 0} \mathfrak{p}^n = (0)$$

so there is some n such that $f \in \mathfrak{p}^n \setminus \mathfrak{p}^{n+1}$. Thus $f = t\varpi^n$ for some $f \in A$ but if $t \in \mathfrak{p}$ then $f \in \mathfrak{p}^{n+1}$ so the result follows. \square

Lemma 27.8.2. Let X be a Noetherian scheme then the nonreduced locus,

$$Z = \{x \in X \mid \operatorname{nilrad}(\mathcal{O}_{X,x}) \neq 0\}$$

is closed.

Proof. The subsheaf $\mathcal{N} \subset \mathcal{O}_X$ is coherent since X is Noetherian. Thus $Z = \operatorname{Supp}_{\mathcal{O}_X}(\mathcal{N})$ is closed and $\mathcal{N}_x = \operatorname{nilrad}(\mathcal{O}_{X,x})$. Locally, on $U = \operatorname{Spec}(A)$ we have $\widetilde{\mathcal{N}}|_U = \operatorname{nilrad}(A)$ and $\operatorname{nilrad}(A)$ is a f.g. A -module since A is Noetherian so,

$$\operatorname{Supp}_{\mathcal{O}_X}(\mathcal{N}) \cap U = \operatorname{Supp}_A(\operatorname{nilrad}(A)) = V(\operatorname{Ann}_A(\operatorname{nilrad}(A)))$$

is closed in $\operatorname{Spec}(A)$. \square

Lemma 27.8.3. Let X be a Noetherian scheme then X has finitly many irreducible components.

Proof. First let $X = \operatorname{Spec}(A)$ for a Noetherian ring A . Then the irreducible components of A correspond to minimal primes $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $\dim A_{\mathfrak{p}} = 0$ and $A_{\mathfrak{p}}$ is Noetherian so $A_{\mathfrak{p}}$ is artinian. $A_{\mathfrak{p}}$ must have some associated prime so $\operatorname{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}\}$. By Tag 05BZ,

$$\operatorname{Ass}_A(A) \cap \operatorname{Spec}(A_{\mathfrak{p}}) = \operatorname{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \{\mathfrak{p}\}$$

so every minimal prime is an associated prime. However, for A noetherian then A admits a finite composition series so there are finitely many associated primes.

Now let X be a Noetherian scheme. For any affine open $U \subset X$ we have shown that U has finitely many irreducible components. However, since X is quasi-compact there is a finite cover of affine opens and thus X must have finitely many irreducible components. \square

Lemma 27.8.4. Let X be a Noetherian scheme and Y is the complement of some dense open U . Then $\operatorname{codim}(Y, X) \geq 1$.

Proof. It suffices to show that Y does not contain any irreducible component since then any irreducible contained in Y cannot be maximal. Since X is Noetherian, it has finitely many irreducible components Z_i . Then if $Z_j \subset Y$ for some i we would have $Z_i \cap U = \emptyset$ but then,

$$U \subset \bigcup_{i \neq j} Z_i$$

which is closed so $\overline{U} \subsetneq X$ contradicting our assumption that U is dense. \square

Example 27.8.5. This may not hold when X is not Noetherian. For example, (FIND EXAMPLE)

$$X = \bigcup_{i=1}^{\infty} V(x_i) \subset k[x_1, x_2, \dots]$$

Lemma 27.8.6. Let X be a Noetherian scheme and $x \in X$ such that $\mathcal{O}_{X,x}$ is a domain. Then there is an affine open neighborhood $x \in U \subset X$ with $U = \text{Spec}(A)$ and A is a domain.

Proof. Take any affine open neighborhood $x \in U \subset X$ with $U = \text{Spec}(A)$ and $\mathfrak{p} \in \text{Spec}(A)$ corresponding to x . Then $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is a domain. Since X is Noetherian then A is Noetherian so it has finitely many minimal primes \mathfrak{p}_i (corresponding to the generic points of irreducible components of U) with $\mathfrak{p}_0 \subset \mathfrak{p}$. Since $A_{\mathfrak{p}}$ is a domain, it has a unique minimal prime and thus \mathfrak{p}_0 is the only minimal prime contained in \mathfrak{p} (geometrically $A_{\mathfrak{p}}$ being a domain corresponds to the fact that \mathfrak{p} is the generic point of a generically reduced irreducible subset which lies in only one irreducible component)

Now for any $i \neq 0$ take $f_i \in \mathfrak{p} \setminus \mathfrak{p}_0$. This is always possible else $\mathfrak{p} \subset \mathfrak{p}_0$ contradicting the minimality of \mathfrak{p}_0 . If $f \notin \mathfrak{q}$ then $\mathfrak{q} \not\supset \mathfrak{p}_i$ for any $i \neq 0$ so $\mathfrak{q} \supset \mathfrak{p}_0$ since it must lie above some minimal prime. Thus $\text{nilrad}(A_f) = \mathfrak{p}_0 A_f$ is prime and $f \notin \mathfrak{p}$ since else $\mathfrak{p} \supset \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$ which is impossible since $\mathfrak{p} \not\supset \mathfrak{p}_i$ for any i . Now we know that $\text{nilrad}(A_{\mathfrak{p}}) = 0$ and A_f is Noetherian so $\text{nilrad}(A_{\mathfrak{p}})$ is finitely generated. Thus, there is some $g \notin \mathfrak{p}$ such that $\text{nilrad}(A_{fg}) = (\text{nilrad}(A_f))_g = 0$. Thus A_{fg} is a domain since $\text{nilrad}(A_{fg}) = (0)$ and is prime and $\mathfrak{p} \in A_{fg}$ because $fg \notin \mathfrak{p}$. Therefore, $x \in \text{Spec}(A_{fg}) \subset U$ is an affine open satisfying the requirements. \square

Remark. This does not imply that X is integral if $\mathcal{O}_{X,x}$ is a domain for each $x \in X$ (which is false, consider $\text{Spec}(k \times k)$) because it only shows there is an integral cover of X not that $\mathcal{O}_X(U)$ is a domain for each U .

Example 27.8.7. Let $X = \text{Spec}(k[x, y]/(xy, y^2))$. Then for the bad point $\mathfrak{p} = (x, y)$ we have $\text{nilrad}(\mathcal{O}_{X,\mathfrak{p}}) = (y)$. Away from the bad point, say $\mathfrak{p} = (x - 1, y)$ we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}(k[x]_{(x-1)})$ so $\text{nilrad}(\mathcal{O}_{X,\mathfrak{p}}) = (0)$. Furthermore, at the generic point $\mathfrak{p} = (y)$, we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}(k(x))$ so $\text{nilrad}(\mathcal{O}_{X,\mathfrak{p}}) = (0)$.

Example 27.8.8. Consider $X = \text{Spec}(k[x, y, z]/(yz))$ which is the union of the x - y and x - z planes. Consider the generic point of the z -axis $\mathfrak{p} = (x, y)$ then $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}(k[x, z]_{(x)})$ is a domain since the z -axis only lies in one irreducible component. However, at the generic point of the x -axis, $\mathfrak{p} = (y, z)$ we get $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}((k[x, y, z]/(yz))_{(y,z)})$ has zero divisors $yz = 0$ so is not a domain since the x -axis lives in two irreducible components.

28 Reflexive Sheaves

Definition 28.0.1. Recall the dual of a \mathcal{O}_X module \mathcal{F} is the sheaf $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. We say that a coherent \mathcal{O}_X -module \mathcal{F} is *reflexive* if the natural map $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism.

Lemma 28.0.2. Let X be an integral locally Noetherian scheme and \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules. If \mathcal{G} is reflexive then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is reflexive.

Tag. 0AY4. □

In particular, since \mathcal{O}_X is clearly reflexive, this lemma shows that for any coherent \mathcal{O}_X -module then \mathcal{F}^\vee is a reflexive coherent sheaf. We say the map $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$ gives the reflexive hull $\mathcal{F}^{\vee\vee}$ of \mathcal{F} .

Definition 28.0.3. Let \mathcal{R} be the full subcategory $\mathcal{Coh}(\mathcal{O}_X)$ of coherent reflexive \mathcal{O}_X -modules. \mathcal{R} is an additive category and in fact has all kernels and cokernels defined by taking reflexive hulls of the sheaf kernel and cokernel. Furthermore, \mathcal{R} inherits a monoidal structure from the tensor product defined using the reflexive hull as follows,

$$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\vee\vee}$$

Finally, we define $\mathrm{RPic}(X)$ to be group of constant rank one reflexives induced by the monoidal structure on \mathcal{R} . Explicitly, $\mathrm{RPic}(X)$ is the group of isomorphism classes of constant rank one reflexive coherent \mathcal{O}_X -modules with multiplication $(\mathcal{F}, \mathcal{G}) \mapsto (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\vee\vee}$ and inverse $\mathcal{F} \mapsto \mathcal{F}^\vee$.

The importance of reflexive sheaves derives from their correspondence to Weil divisors. Here we let X be a normal integral separated Noetherian scheme.

Proposition 28.0.4. If D is a Weil divisor then $\mathcal{O}_X(D)$ is reflexive of constant rank one.

Proof. (CITE OR DO). □

Theorem 28.0.5. Let X be a normal integral separated Noetherian scheme. There is an isomorphism of groups $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{RPic}(X)$ defined by $D \mapsto \mathcal{O}_X(D)$.

Proof. (DO OR CITE) □

We summarize the important results as follows.

Theorem 28.0.6. Let X be a Noetherian normal integral scheme. Then for any Weil divisors D, E ,

- (a) $\mathcal{O}_X(D + E) = (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee}$
- (b) $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^\vee$
- (c) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(E)) = \mathcal{O}_X(E - D)$
- (d) if E is Cartier then $\mathcal{O}_X(D + E) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$

Proof. (DO OR CITE) □

Finally, we have a result which controls when the dualizing sheaf can be expressed in terms of a divisor.

Proposition 28.0.7. Let X be a projective variety over k . Then,

- (a) if X is normal then its dualizing sheaf ω_X is reflexive of rank 1 and thus X admits a canonical divisor K_X s.t. $\omega_X = \mathcal{O}_X(K_X)$
- (b) if X is Gorenstein then ω_X is an invertible module so K_X is Cartier.

Proof. (FIND CITATION OR DO). □

29 Smooth Morphisms

29.1 Kahler Differentials

Proposition 29.1.1. We have the following general facts about Kahler differentials.

Given ring maps $R \rightarrow A \rightarrow B$ we have an exact sequence,

$$\Omega_{A/R} \otimes_A B \longrightarrow \Omega_{B/R} \longrightarrow \Omega_{B/A} \longrightarrow 0$$

Given $R \rightarrow A$ and $B = A/I$ we have an exact sequence,

$$I/I^2 \longrightarrow \Omega_{A/R} \otimes_A B \longrightarrow \Omega_{B/R} \longrightarrow 0$$

Commutates with tensor product, given $R \rightarrow R'$ and setting $A' = A \otimes_R R'$ we have,

$$\Omega_{A'/R'} = \Omega_{A/R} \otimes_R R'$$

Commutates with localization.

$$\Omega_{S^{-1}A/R} = S^{-1}\Omega_{A/R}$$

Proposition 29.1.2. Let A be a local k -algebra with $A/\mathfrak{m} \cong k$. Then, the map,

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \Omega_{A/k} \otimes_A k$$

is an isomorphism.

29.2 Smooth Ring Maps

Definition 29.2.1. Let $R \rightarrow S$ be a ring map. Consider the surjection $R[S] \rightarrow S$ and let I be its kernel. We define the *naive cotangent complex* as a complex supported in degree -1 and 0 ,

$$NL_{S/R} = (I/I^2 \rightarrow \Omega_{R[S]/R} \otimes_R S)$$

The second exact sequence shows that $H^0(NL_{S/R}) = \Omega_{S/R}$.

Definition 29.2.2. We say a ring map $R \rightarrow S$ is smooth if it is finitely presented and $NL_{S/R}$ is quasi-isomorphic to a finite projective S -module placed in degree zero.

Example 29.2.3. We say a morphism $R \rightarrow S$ is *standard smooth* if $S = R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ and the polynomial,

$$g = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i=1, \dots, c \\ j=1, \dots, c}}$$

in $R[x_1, \dots, x_n]$ maps to a unit in S . Such a ring map is smooth.

Definition 29.2.4. A ring map $R \rightarrow A$ is *formally smooth* if every diagram of the form,

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \searrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I^2 = 0$ admits a map $A \rightarrow B$ making the diagram commute.

Proposition 29.2.5. Standard smooth ring maps are formally smooth.

Proof. Let $A = R[x_1, \dots, x_n]/(f_1, \dots, f_c)$. Consider the map $\varphi : A \rightarrow B/I$ defined by $\varphi(x_i) = \bar{b}_i$. We may choose lifts $b_i + q_i \in B$ for $q_i \in I$ which define a morphism exactly when all $f_i(b + q) = 0$. Since $\varphi : A \rightarrow B/I$ is a ring map, we know that $\pi(f_i(b + q)) = f_i(\bar{b}) = 0$ meaning that $f_i(b + q) \in I$. Then, Taylor expanding,

$$f_i(b + q) = f_i(b) + \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(b) q_j + O(q_j^2)$$

but $I^2 = 0$ and thus the order q_j^2 terms vanish so we find,

$$f_i(b + q) = f_i(b) + \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} q_j$$

Therefore, we can make the map $A \rightarrow B$ defined by the lifts $x_i \mapsto b_i + q_i$ well-defined if we can solve the matrix equation,

$$\left(\frac{\partial f_i}{\partial x_j}(b) \right) \cdot q = -f(b)$$

Note that $I/I^2 = I$ so I is naturally a B/I -module. Therefore, we may replace this with,

$$\left(\frac{\partial f_i}{\partial x_j}(\bar{b}) \right) \cdot q = -f(b)$$

where $\frac{\partial f_i}{\partial x_j}(\bar{b})$ is the image of the matrix under $\pi : B \rightarrow B/I$. Finally, the determinant of the first $c \times c$ minor is a unit in A and thus this matrix is a unit in B/I so the above matrix equation admits a solution. Furthermore, the solution is unique if $n = c$ in which case the matrix is square and nonsingular so we get a unique map and we call $R \rightarrow A$ formally étale. \square

Proposition 29.2.6. $R \rightarrow A$ is smooth iff it is of finite presentation and formally smooth.

Proof. Tag 00TN. \square

Proposition 29.2.7. Consider ring maps $R \rightarrow A \rightarrow B$ with $A \rightarrow B$ formally smooth then the first exact sequence,

$$0 \longrightarrow \Omega_{A/R} \otimes_A B \longrightarrow \Omega_{B/R} \longrightarrow 0$$

is a split short exact sequence of B -modules.

Proposition 29.2.8. Let $R \rightarrow A$ be a ring map and $B = A/I$ with $R \rightarrow B$ formally smooth. Then the second exact sequence,

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{A/R} \otimes_A B \longrightarrow \Omega_{B/R} \longrightarrow 0$$

is a split short exact sequence of B -modules.

29.3 Smooth Morphisms of Schemes

Definition 29.3.1. We say that a morphism of schemes $f : X \rightarrow Y$ is *smooth* at x if there are affine opens $x \in U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f^\# : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U)$ is smooth. We say f is smooth if it is smooth at every point.

Remark. The smooth locus of a morphism $f : X \rightarrow Y$ is automatically open by definition.

Remark. Since $R \rightarrow A$ being smooth is a local property smoothness of a morphism implies smoothness on all affine ring maps.

Lemma 29.3.2. Let $f : X \rightarrow S$ be locally of finite presentation. Then f is smooth at x iff $\mathcal{O}_{S,f(x)} \rightarrow \mathcal{O}_{X,x}$ is flat and $X_{f(x)} \rightarrow \text{Spec}(\kappa(f(x)))$ is smooth at x .

Proof. Tag 01V9. □

Lemma 29.3.3. Let $f : X \rightarrow \text{Spec}(k)$ be locally of finite type. Then f is smooth iff X is geometrically regular over k .

Proof. Tag 038X. □

Remark. The preceding two facts gives an alternative description of a smooth morphism.

Proposition 29.3.4. A morphism $f : X \rightarrow Y$ is smooth iff it is

- (a) locally of finite presentation
- (b) flat
- (c) has geometrically regular fibers i.e. $X_y \rightarrow \text{Spec}(\kappa(y))$ is geometrically regular over $\kappa(y)$ for each $y \in Y$.

Proposition 29.3.5. Given a morphism $f : X \rightarrow Y$ of schemes over S , there is a canonical exact sequence,

$$f^* \Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

which when f is smooth is short exact i.e. $f^* \Omega_{Y/S} \rightarrow \Omega_{X/S}$ is injective.

Definition 29.3.6. Given an immersion $\iota : Z \hookrightarrow X$ with sheaf of ideals $\mathcal{I} = \ker(\mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z)$ the nonnormal sheaf is $\mathcal{C}_{Z/X} = \iota^* \mathcal{I}$ and $\iota_* \mathcal{C}_{Z/X} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Z = \mathcal{I} / \mathcal{I}^2$

Proposition 29.3.7. Given an immersion $\iota : Z \hookrightarrow X$ of schemes over S with sheaf of ideals \mathcal{I} there is a canonical exact sequence,

$$\mathcal{C}_{Z/X} \longrightarrow \iota^* \Omega_{X/S} \longrightarrow \Omega_{Z/S} \longrightarrow 0$$

which, when $Z \rightarrow S$ is smooth, is short exact i.e. $\mathcal{C}_{Z/X} \rightarrow \iota^* \Omega_{X/S}$ is injective.

Proposition 29.3.8. Let $f : X \rightarrow S$ be smooth. Then $\Omega_{X/S}$ is finite locally free and,

$$\text{rank}_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$$

Definition 29.3.9. $f : X \rightarrow S$ is smooth of relative dimension n if f is smooth and $\Omega_{X/S}$ is locally free of constant rank n .

Proposition 29.3.10. A morphism $f : X \rightarrow S$ is smooth iff

- (a) f is locally of finite presentation
- (b) f is flat
- (c) $\Omega_{X/S}$ is locally free with $\text{rank}_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$.

Definition 29.3.11. A morphism $f : X \rightarrow Y$ is *étale* if it is smooth of relative dimension zero. Therefore, we see that étale is equivalent to.

- (a) locally of finite presentation
- (b) flat
- (c) $\Omega_{X/Y} = 0$

which is the same as saying $f : X \rightarrow Y$ is smooth and unramified or flat and G-unramified.

Proposition 29.3.12. When $f : X \rightarrow Y$ is over S is étale the induced map $f^* \Omega_{Y/S} \xrightarrow{\sim} \Omega_{X/S}$ is an isomorphism.

Proof. By smoothness, there is an exact sequence,

$$0 \longrightarrow f^* \Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

but $\Omega_{X/Y} = 0$ since $f : X \rightarrow Y$ is étale. □

29.4 Formal Maps

Definition 29.4.1. A morphism $f : X \rightarrow S$ consider every diagram,

$$\begin{array}{ccc} X & \xleftarrow{\quad} & \text{Spec}(A/I) \\ f \downarrow & \nwarrow \text{dashed} & \downarrow \\ S & \xleftarrow{\quad} & \text{Spec}(A) \end{array}$$

with $I^2 = 0$. Searching for a lift $\text{Spec}(A) \rightarrow X$ making the diagram commute, we say that f is,

- (a) *formally smooth* if a lift exists

(b) *formally unramified* if at most one lift exists

(c) *formally étale* if a unique lift exists.

Remark. Clearly f is formally étale iff it is formally smooth and formally unramified.

Remark. In particular, if X and S are affine then $X \rightarrow S$ is formally smooth (resp. unramified resp. étale) iff $\mathcal{O}_S(S) \rightarrow \mathcal{O}_X(X)$ is formally smooth (rep. unramified resp. étale) by definition and the anti-equivalence of categories between affine schemes and rings.

Theorem 29.4.2. The following are equivalent for a morphism of schemes $f : X \rightarrow Y$,

- (a) f is smooth
- (b) f is formally smooth and locally of finite type.

Proof. Tag 02H6. □

Proposition 29.4.3. Let $f : X \rightarrow Y$ over S be formally smooth. Then the canonical exact sequence is short exact,

$$0 \longrightarrow f^*\Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Furthermore, if $\iota : Z \rightarrow X$ is formally unramified over S with $Z \rightarrow S$ formally smooth then the canonical exact sequence is short exact,

$$0 \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \iota^*\Omega_{X/S} \longrightarrow \Omega_{Z/S} \longrightarrow 0$$

Proof. Tag 06B6 and Tag 06B7. □

29.5 Intuition

Remark. We give intuition for why it is correct to think of the following correspondences between properties in algebraic and in differential geometry,

- (a) smooth morphisms \iff submersions
- (b) unramified morphisms \iff immersions
- (c) étale morphisms \iff local diffeomorphisms

We now justify this intuition as follows. The important facts about submersions follows from the constant rank theorem,

Theorem 29.5.1. Let $f : M \rightarrow N$ be a morphism such that $df : T_p M \rightarrow T_{f(p)} N$ has constant rank in a neighborhood of $p \in M$. Then there exists opens $U \subset M$ and $V \subset N$ with $p \in U$ and $f(p) \in V$ and diffeomorphisms $u : T_p X \rightarrow U$ and $v : T_{f(p)} Y \rightarrow V$ making the diagram commute,

$$\begin{array}{ccc} T_p M & \xrightarrow{u} & U \subset M \\ \downarrow df & & \downarrow f \\ T_{f(p)} N & \xrightarrow{v} & V \subset N \end{array}$$

Corollary 29.5.2. If $df : T_p M \rightarrow T_{f(p)} N$ is an isomorphism then f is locally at p a diffeomorphism.

Corollary 29.5.3. Local immersions are locally the inclusion of a subspace and local submersion are locally a projection to a subspace.

Definition 29.5.4. Let $f : M \rightarrow N$ be a smooth map. Then $x \in M$ is a *regular point* if $df_x : T_x M \rightarrow T_x N$ is surjective. We say that $y \in N$ is a *regular value* if each $x \in f^{-1}(y)$ is a regular point. If every point (equivalently value) is regular then $f : M \rightarrow N$ is a *submersion*.

Theorem 29.5.5 (Preimage). Let $f : M \rightarrow N$ be smooth and $y \in N$ a regular value. Then $Y = f^{-1}(y)$ is an embedded submanifold of M of codimension $\dim N$. Furthermore, for any $x \in Y$ the tangent space is $T_x Y = \ker df_x \subset T_x M$.

The preimage theorem is analogous to the criterion that (let f be locally of finite presentation and flat) $f : X \rightarrow Y$ is smooth at every point over y (i.e. y is a regular value) iff the fiber $X_y \rightarrow \operatorname{Spec}(\kappa(y))$ is smooth i.e. X_y is nonsingular. Therefore, a smooth map is one with smooth fibers just as a submersion is a map whose fibers are smooth submanifolds.

Furthermore, we can directly relate the algebraic geometry conditions to conditions about the pullback map on differential forms dual to the derivative map on tangent vectors which will directly relate these properties to the definitions of the associated differential geometry concepts.

We have the following,

- (a) smooth $\implies f^* \Omega_Y \hookrightarrow \Omega_X$ is injective
- (b) unramified $\implies f^* \Omega_Y \twoheadrightarrow \Omega_X$ is surjective
- (c) étale $\implies f^* \Omega_Y \xrightarrow{\sim} \Omega_X$ is an isomorphism.

Then dualizing we see that on tangent sheaves,

- (a) smooth $\implies \mathcal{T}_X \twoheadrightarrow f^* \mathcal{T}_Y$ is surjective
- (b) unramified $\implies \mathcal{T}_X \hookrightarrow f^* \mathcal{T}_Y$ is surjective
- (c) étale $\implies \mathcal{T}_Y \xrightarrow{\sim} f^* \mathcal{T}_X$ is an isomorphism.

These exactly correspond to the differential geometry definitions of submersion, immersion, and local diffeomorphism.

Finally, we may give intuition for the formal versions of these properties and how they relate to the condition on the differential map on tangent vectors. Let $f : X \rightarrow Y$ be formally smooth / unramified / étale map of varieties over an algebraically closed field k . Then for each closed point $y \in Y$ and tangent vector at y there is an associated map $\operatorname{Spec}(k[x]/(x^2)) \rightarrow Y$. Furthermore, for any closed point $\operatorname{Spec}(k) \rightarrow X$ above y , taking the ideal (x) which has square zero we get a diagram,

$$\begin{array}{ccc} X & \xleftarrow{\quad} & \operatorname{Spec}(k) \\ f \downarrow & \swarrow \text{dashed} & \downarrow \\ S & \xleftarrow{\quad} & \operatorname{Spec}(k[x]/(x^2)) \end{array}$$

- (a) If f is formally smooth there exists a lift $\operatorname{Spec}(k[x]/(x^2)) \rightarrow X$ so we may lift tangent vectors i.e. the differential $df : T_x X \rightarrow T_y Y$ is surjective (f is a submersion).

- (b) If f is formally unramified there is at most one lift i.e. the differential $df : T_x X \rightarrow T_y Y$ is injective (f is an immersion).
- (c) If f is formally étale then there is a unique lift i.e. the differential $df : T_x X \rightarrow T_y Y$ is an isomorphism (f is a localy diffeomorphism).

30 Maps between Curves

30.1 Maps of a Proper Curve are Finite

Theorem 30.1.1. Let C be a proper curve over k and X is separated of finite type over k . Then any nonconstant map $f : C \rightarrow X$ over k is finite.

Proof. Since $C \rightarrow \text{Spec}(k)$ is proper and $X \rightarrow \text{Spec}(k)$ is separated then by Tag 01W6 the map $f : C \rightarrow X$ is proper. The fibres of closed points $x \in X$ are proper closed subschemes $C_x \hookrightarrow C$ (since if $C_x = C$ then $f : C \rightarrow X$ would be the constant map at $x \in X$) and thus finite since proper closed subsets of a curve are finite. Now I claim that if the fibres $f^{-1}(x)$ are finite at closed points $x \in X$ then all fibres are finite. Assuming this, $f : C \rightarrow X$ is proper with finite fibres and thus is finite by Tag 02OG.

To show the claim consider,

$$E = \{x \in X \mid \dim C_x = 0\}$$

Since C is Noetherian, $\dim C_x = 0$ iff C_x is finite (suffices to check for affine schemes since quasi-compact and dimension zero Noetherian rings are exactly Artinian rings which have finite spectrum). Then E is locally constructible by Tag 05F9 and contains all the closed points of X . Since X is finite type over k then X is Jacobson which implies that E is dense in every closed set. Then for any point $\xi \in X$ then $Z = \overline{\{\xi\}}$ is closed and irreducible with generic point ξ and thus $E \cap Z$ is dense in Z . Then by Tag 005K we have $\xi \in E$ so $E = X$ proving that all fibres are finite. \square

Remark. The only facts about C that I used were that $C \rightarrow \text{Spec}(k)$ is proper and that C is irreducible of dimension one. The second two properties are needed for the following to hold.

Lemma 30.1.2. If X is an irreducible Noetherian scheme of dimension one then every nontrivial closed subset of X is finite.

Proof. Since X is quasi-compact it suffices to show this property for affine schemes $X = \text{Spec}(A)$ with $\dim A = 1$ and prime nilradical. Any nontrivial closed subset is of the form $V(I)$ for some proper radical ideal $I \subset A$ with $I \supsetneq \text{nilrad}(A)$. Then $\text{ht}(I) = 1$ since any prime above I must properly contain $\text{nilrad}(A)$ and thus have height at least one but $\dim A = 1$. Then,

$$\text{ht}(I) + \dim A/I \leq \dim A$$

so $\dim A/I = 0$. Since A is Noetherian so is A/I but $\dim A/I = 0$ and thus A/I is Artinian. Therefore $\text{Spec}(A/I)$ is finite proving the proposition. \square

Remark. Since $C \rightarrow \text{Spec}(k)$ is proper it is finite type over k and thus C is Noetherian.

Remark. The condition that C be proper is necessary. Consider the map $\mathbb{G}_m^k \coprod \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ via $k[x] \rightarrow k[x, x^{-1}]$ and the identity. This is clearly surjective and finitely generated since on rings it is,

$$k[x] \rightarrow k[x, x^{-1}] \times k[x]$$

Furthermore, this map is quasi-finite since the fibers have at most two points. To see this, consider, $y = (x - a) \in \text{Spec}(k[x])$ then $\kappa(y) = k[x]/(x - a)$ and the fibre is,

$$\begin{aligned} X_y &= \text{Spec}((k[x, x^{-1}] \times k[x]) \otimes_{k[x]} k[x]/(x - a)) \\ &= \text{Spec}(k[x, x^{-1}]/(x - a) \times k[x]/(x - a)) \\ &= \text{Spec}(k[x, x^{-1}]/(x - a)) \coprod \text{Spec}(k[x]/(x - a)) \\ &= \begin{cases} \text{Spec}(k) & a = 0 \\ \text{Spec}(k) \coprod \text{Spec}(k) & a \neq 0 \end{cases} \end{aligned}$$

However, this map is not closed since $\mathbb{G}_m^k \subset \mathbb{G}_m^k \coprod \mathbb{A}_k^1$ is closed but its image is $\mathbb{A}_k^1 \setminus \{0\}$ which is not closed. Thus the map cannot be finite. In particular,

$$k[x, x^{-1}] = \bigoplus_{n \geq 0} x^{-n} k[x]$$

so $k[x, x^{-1}]$ is not a finitely-generated $k[x]$ -module.

30.2 Maps of Normal Curves Are Flat

Lemma 30.2.1. Let X be an integral scheme with generic point $\xi \in X$ and $\mathcal{F} \rightarrow \mathcal{G}$ a map of \mathcal{O}_X -modules,

- (a) if \mathcal{F} is locally free then $\mathcal{F} \rightarrow \mathcal{G}$ is injective iff $\mathcal{F}_\xi \rightarrow \mathcal{G}_\xi$ is injective
- (b) if \mathcal{F} is invertible then $\mathcal{F} \rightarrow \mathcal{G}$ is injective iff $\mathcal{F}_\xi \rightarrow \mathcal{G}_\xi$ is nonzero.

Proof. Since $\xi \in U$ for each nonempty open we have a diagram,

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_\xi & \longrightarrow & \mathcal{G}_\xi \end{array}$$

therefore it suffices to show the map $\mathcal{F}(U) \rightarrow \mathcal{F}_\xi$ is injective since then injectivity of $\mathcal{F}_\xi \rightarrow \mathcal{G}_\xi$ will imply injectivity of $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each U . Choose an affine open cover $U_i = \text{Spec}(A_i)$ trivializing \mathcal{F} . For each $s \in \mathcal{F}(U)$ then $\mathcal{F}|_{U_i \cap U} \cong \mathcal{O}_X^{\oplus n}|_{U_i \cap U}$ but X is integral so the restriction $\mathcal{F}(U_i \cap U) \rightarrow \mathcal{F}_\xi$ is simply $A_i^n \rightarrow \text{Frac}(A_i)^n$ which is injective since A_i is a domain. Thus if $s|_{U_i \cap U} \in \mathcal{F}(U \cap U_i)$ maps to zero in \mathcal{F}_ξ then $s|_{U_i \cap U} = 0$ so $s = 0$ since U_i form a cover.

The second follows from the first since we need only to show that $\mathcal{F}_\xi \rightarrow \mathcal{G}_\xi$ is injective. However, \mathcal{F}_ξ is a rank-one free module over the field $K(X) = \mathcal{O}_{X, \xi}$. Thus every nonzero map $\mathcal{F}_\xi \rightarrow \mathcal{G}_\xi$ is injective. \square

Lemma 30.2.2. Let $f : X \rightarrow Y$ be a conconstant map of curves. Then f is dominant.

Proof. Let $\xi \in X$ be the generic point and consider $f(\xi) \in Y$. Suppose that $f(\xi)$ is a closed point. Then $f(X) = f(\{\xi\}) \subset \overline{f(\xi)} = f(\xi)$ so f is constant. Therefore, we must have $f(\xi)$ a nonclosed point. But $\dim Y = 1$ and irreducible so any point is either closed or the generic point of the unique irreducible component. Therefore, $f(\xi) = \eta$ the generic point so f is dominant. \square

Proposition 30.2.3. Let X and Y be curves over k with Y normal. Then any nonconstant map $f : X \rightarrow Y$ is flat.

Proof. We need to check that $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is flat. Since Y is a normal curve $\mathcal{O}_{Y,y}$ is a Noetherian domain (Y is integral finite type over k) integrally closed (Y is normal) and dimension at most one ($\dim Y = 1$) therefore $\mathcal{O}_{Y,y}$ is a local Dedekind domain or a field so $\mathcal{O}_{Y,y}$ is a DVR or a field. Then by Tag 0539, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$ -module iff it is torsion-free. However, $\mathcal{O}_{X,x}$ is a domain so it is a torsion-free $\mathcal{O}_{Y,f(x)}$ -module iff $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is injective.

Since f is dominant $f(\xi) = \eta$ (the generic points). Then $\mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$ is a map of fields which is automatically injective so $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is injective because Y is integral proving that $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is injective. \square

Remark. Morphisms of varieties are automatically finitely presented since curves are finite type over k so morphisms between them are locally finite type but Y is Noetherian so a locally finite type map is finitely presented. Furthermore, X is Noetherian so morphisms from it are automatically quasi-compact and quasi-separated.

Proposition 30.2.4. Nonconstant maps of curves $f : X \rightarrow Y$ with Y normal are smooth iff unramified iff étale iff $\Omega_{X/Y} = 0$.

Proof. Maps of curves are automatically finitely presented. Furthermore, nonconstant maps of curves with Y normal are flat. Furthermore, we have seen that nonconstant maps of curves are quasi-finite so $\dim X_{f(x)} = 0$. Therefore, f is smooth iff $\Omega_{X/Y} = 0$ iff unramified but étale is smooth an unramified so we see smooth iff étale. \square

Lemma 30.2.5. Let $X \rightarrow Y$ be a nonconstant map of curves with $K(X)/K(Y)$ separable and Y smooth. Then there is an exact sequence,

$$0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Therefore, f is étale iff $f^*\Omega_Y \rightarrow \Omega_X$ is an isomorphism.

Proof. $K(X)/K(Y)$ is an extension of fields of transcendence degree one over k so it must be algebraic. Furthermore, both are finitely-generated field extensions of k so the algebraic extension $K(X)/K(Y)$ is finite. Then $(\Omega_{X/Y})_\xi = \Omega_{K(X)/K(Y)}$ which is zero iff $K(X)/K(Y)$ is separable. Thus, the standard exact sequence gives $(f^*\Omega_Y) \rightarrow (\Omega_X)_\xi$ because $(\Omega_{X/Y})_\xi = 0$. Furthermore, $f^*\Omega_Y$ is a line bundle since Y is smooth so we conclude that $f^*\Omega_Y \rightarrow \Omega_X$ is an injection since it is nonzero on the generic fiber (Lemma 30.2.1). \square

31 Serre Duality

31.1 Some Homological Algebra

Remark. Here we work in abelian categories \mathcal{A}, \mathcal{B} with a left-exact additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$.

Lemma 31.1.1. Every summand of an injective object is injective.

Proof. Suppose that $I = K \oplus L$ is injective. Given an injection $A \hookrightarrow B$ and a map $A \rightarrow K$ we get a diagram,

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & K \\
 \downarrow & \searrow & \swarrow \text{id} \\
 & K & \\
 \downarrow & \nearrow & \downarrow \\
 B & \xrightarrow{\quad} & I
 \end{array}$$

so $A \rightarrow K$ extends to $B \rightarrow K$ so K is injective. □

Corollary 31.1.2. Suppose $K \hookrightarrow I$ is an embedding of injective modules. Then the quotient I/K is injective.

Proof. There is an exact sequence,

$$0 \longrightarrow K \longrightarrow I \longrightarrow I/K \longrightarrow 0$$

But K is injective so this splits giving $I = K \oplus I/K$ and thus I/K is injective. □

Lemma 31.1.3. Let I^\bullet be a complex of injective objects. Define,

$$B^n = \text{Im}(I^{n-1} \rightarrow I^n) \quad \text{and} \quad Z^n = \ker(I^n \rightarrow I^{n+1})$$

Fix n such that Z^{n-1} is injective. Then $H^n(F(I^\bullet)) = F(H^n(I^\bullet))$. If further Z^n is injective then $H^n(I^\bullet)$ is also injective.

Proof. There are exact sequences,

$$0 \longrightarrow Z^{n-1} \longrightarrow I^{n-1} \longrightarrow B^n \longrightarrow 0$$

$$0 \longrightarrow B^n \longrightarrow Z^n \longrightarrow H^n(I^\bullet) \longrightarrow 0$$

since Z^{n-1} and I^{n-1} are injective then B^n is injective. Thus, the sequences split because Z^{n-1} and B^n are injective. Furthermore, if Z^n is injective then $H^n(I^\bullet)$ is also injective. Furthermore, because F is additive, it preserves split exact sequences so,

$$0 \longrightarrow F(Z^{n-1}) \longrightarrow F(I^{n-1}) \longrightarrow F(B^n) \longrightarrow 0$$

$$0 \longrightarrow F(B^n) \longrightarrow F(Z^n) \longrightarrow F(H^n(I^\bullet)) \longrightarrow 0$$

are exact. From the first exact sequence, $F(B^n) = \text{Im}(F(I^{n-1}) \rightarrow F(I^n))$. Since F preserves kernels $F(Z^n) = \ker(F(I^n) \rightarrow F(I^{n+1}))$. Therefore, the second sequence gives $H^n(F(I^\bullet)) = F(H^n(I^\bullet))$. □

Lemma 31.1.4. Let I^\bullet be a bounded below complex of injectives and c the smallest integer with $H^c(I^\bullet) \neq 0$. Then, $H^i(F(I^\bullet)) = 0$ for $i < c$ and $H^c(F(I^\bullet)) = F(H^c(I^\bullet))$.

Proof. Since I^\bullet is bounded below it begins with $0 \rightarrow I^0 \rightarrow I^1 \rightarrow I^2$. If $c = 0$ then the first condition is trivially true and the second condition says $\ker(F(I^0) \rightarrow F(I^1)) = F(\ker(I^0 \rightarrow I^1))$ which holds because F is left-exact. Otherwise $c > 0$ so I^\bullet is exact at I^0 . Thus, $Z^0 = \ker(I^1 \rightarrow I^2) = I^0$ is injective. Then, by the exact sequence,

$$0 \longrightarrow Z^{n-1} \longrightarrow I^{n-1} \longrightarrow B^n \longrightarrow 0$$

we find that if Z^{n-1} is injective then so is B^n . If $n < c$ then I^\bullet is exact at I^n so $B^n = Z^n$ so Z^n is injective. Therefore, $Z^n = B^n$ is injective for $n < c$. By the previous lemma, for $n \leq c$ we have $H^n(F(I^\bullet)) = F(H^n(I^\bullet))$ which is zero for $n < c$. \square

Remark. Alternatively, there is a spectral sequence proof. There is a spectral sequence computing the hyperderived functors,

$$E_2^{p,q} = RF^p(H^q(I^\bullet)) \implies \mathbb{R}F^{p+q}(I^\bullet)$$

However, since I^\bullet is injective, $\mathbb{R}F^{p+q}(I^\bullet) = H^{p+q}(F(I^\bullet))$. Then,

$$E_2^{p,q} = RF^p(H^q(I^\bullet)) = \begin{cases} 0 & q < c \\ RF^q(H^c(I^\bullet)) & q \geq 0 \end{cases}$$

Therefore, for $p + q < c$ all $E_2^{p,q} = 0$ so $E_\infty^{p,q} = H^{p+q}(F(I^\bullet)) = 0$ for $p + q < c$. Furthermore, for $p + q = c$ the only nonzero term is $q = c$ and $p = 0$. However, for $r \geq 2$ all differentials starting at $(0, c)$ end outside the positive quadrant so are zero and all differentials ending at $(0, c)$ must start with $q < c$ and thus must be zero. Therefore, $E_\infty^{0,c} = F(H^c(I^\bullet))$ and $E_\infty^{p,q} = 0$ for $p + q = c$ and $q \neq c$. Thus, $H^c(F(I^\bullet)) = E_2^{0,c} = F(H^c(I^\bullet))$.

31.2 Tensor-Hom Adjunction Done Right

Theorem 31.2.1. Let A, B, C, D be (non-commutative) rings and M be an (A, B) -bimodule, N be a (B, C) -bimodule, and K be a (D, C) -bimodule. Then there is a natural adjunction,

$$\mathrm{Hom}_C(M \otimes_B N, K) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_C(N, K))$$

as (D, A) -bimodules. Furthermore, let M be an (A, B) -bimodule, N be a (B, C) -bimodule, and K be a (A, D) -bimodule. Then there is a natural adjunction,

$$\mathrm{Hom}_A(M \otimes_B N, K) \cong \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, K))$$

as (C, D) -bimodules.

Proof. Let $\varphi : M \otimes_B N \rightarrow K$ be a right C -module map. Send this to the map $\tilde{\varphi} : M \rightarrow \mathrm{Hom}_C(N, K)$ via $\tilde{\varphi} : m \mapsto (n \mapsto \varphi(m \otimes n))$ and $\tilde{\varphi}$ is a map of right B -modules since,

$$m \cdot b \mapsto (n \mapsto \varphi(m \otimes bn)) = (n \mapsto \varphi(m \otimes n)) \cdot b$$

We must check this is an isomorphism of (C, A) -bimodules. \square

Remark. From here on we assume all rings are commutative.

Corollary 31.2.2. Let $A \rightarrow B$ be a map of rings, M, N be B -modules and K an A -module. Then there is a natural adjunction of B -modules,

$$\mathrm{Hom}_A(M \otimes_B N, K) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(N, K))$$

Proof. We replace A, B, C, D in the theorem by B, B, A, A . Via the ring map, we view M as an (B, B) -bimodule, N as an (B, A) -bimodule, and K as an (A, A) -bimodule. Then we get a map of (A, B) -bimodules,

$$\mathrm{Hom}_A(M \otimes_B N, K) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(N, K))$$

□

Corollary 31.2.3. Let $A \rightarrow B$ be a map of rings, M be an A -module and N, K be B -modules. Then there is a natural adjunction of B -modules,

$$\mathrm{Hom}_B(M \otimes_A N, K) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, K))$$

Proof. We replace A, B, C, D in the theorem by A, A, B, B . Via the ring map, we view M as an (A, A) -bimodule, N as an (A, B) -bimodule, and K as an (B, B) -bimodule. Then we get a map of (B, A) -bimodules,

$$\mathrm{Hom}_B(M \otimes_A N, K) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, K))$$

□

Corollary 31.2.4. Let A be a ring, M, N, K be A -modules. Then there is a natural adjunction of A -modules,

$$\mathrm{Hom}_A(M \otimes_A N, K) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, K))$$

Proof. Viewing M, N, K as (A, A) -bimodules we find,

$$\mathrm{Hom}_A(M \otimes_A N, K) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, K))$$

as (A, A) -bimodules.

□

Corollary 31.2.5. Let $A \rightarrow B$ be a map of rings. The restriction $(-)_A : \mathbf{Mod}_B \rightarrow \mathbf{Mod}_A$ functor and internal hom $\mathrm{Hom}_A(B, -) : \mathbf{Mod}_A \rightarrow \mathbf{Mod}_B$ are adjoint via,

$$\mathrm{Hom}_A(M_A, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N))$$

as B -modules.

Proof. We replace A, B, C, D in the theorem by B, B, A, A . View M as an (B, B) -bimodule, B as a (B, A) -bimodule, and N as a (A, A) -bimodule. Then we get,

$$\mathrm{Hom}_A(M \otimes_B B, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N))$$

as (A, B) -bimodules.

□

Corollary 31.2.6. Let $A \rightarrow B$ be a map of rings. The tensor product $- \otimes_A B : \mathbf{Mod}_A \rightarrow \mathbf{Mod}_B$ and restriction $(-)_A : \mathbf{Mod}_B \rightarrow \mathbf{Mod}_A$ are adjoint via,

$$\mathrm{Hom}_B(M \otimes_A B, N) \cong \mathrm{Hom}_A(M, N_A)$$

as B -modules.

Proof. We replace A, B, C, D in the theorem by A, A, B, B . View M as an (A, A) -bimodule, B as a (A, B) -bimodule, and N as a (B, B) -bimodule. Then we get,

$$\mathrm{Hom}_B(M \otimes_A B, N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, N))$$

as (B, A) -bimodules. \square

Corollary 31.2.7. Let $A \rightarrow B$ be a ring map, N a flat B -module, and I an injective A -module. Then $\mathrm{Hom}_A(N, I)$ is an injective B -module.

Proof. The functor $\mathrm{Hom}_A(N, -)$ is right-adjoint to $-\otimes_B N$ which is exact so $\mathrm{Hom}_A(B, -)$ preserves injectives. Explicitly, the functor $\mathrm{Hom}_B(-, \mathrm{Hom}_A(N, I)) = \mathrm{Hom}_A(-\otimes_B N, I)$ is exact on \mathbf{Mod}_B since $\mathrm{Hom}_A(-, I)$ and $-\otimes_B N$ are exact. \square

Corollary 31.2.8. Let $A \rightarrow B$ be a ring map, P a projective A -module, Q a projective B -module. Then $P \otimes_A Q$ is a projective B -module.

Proof. The functor $-\otimes_A Q$ is left-adjoint to $\mathrm{Hom}_B(Q, -)$ which is exact since Q is projective so $-\otimes_A Q$ preserves projectives. Explicitly, the functor $\mathrm{Hom}_B(P \otimes_A N, -) = \mathrm{Hom}_A(P, \mathrm{Hom}_B(N, -))$ is exact on \mathbf{Mod}_B since $\mathrm{Hom}_A(P, -)$ and $\mathrm{Hom}_B(Q, -)$ are exact. \square

Theorem 31.2.9. Let $A \rightarrow B$ be a map of rings. Let M, N be B -modules with N flat over B and K an A -module. Then there is a spectral sequence,

$$E_2^{p,q} = \mathrm{Ext}_B^p(M, \mathrm{Ext}_A^q(N, K)) \implies \mathrm{Ext}_A^{p+q}(M \otimes_B N, K)$$

Proof. The functors $\mathrm{Hom}_B(M, -)$ and $\mathrm{Hom}_A(N, -)$ satisfy,

$$\mathrm{Hom}_B(M, \mathrm{Hom}_A(N, -)) = \mathrm{Hom}_A(M \otimes_B N, -)$$

Furthermore, $\mathrm{Hom}_A(N, -)$ preserves injectives because N is flat so $\mathrm{Hom}_A(N, -)$ is right-adjoint to $-\otimes_B N$ which is exact. Therefore, the Grothendieck spectral sequence applies to this composition of functors giving a spectral sequence,

$$E_2^{p,q} = \mathrm{Ext}_B^p(M, \mathrm{Ext}_A^q(N, K)) \implies \mathrm{Ext}_A^{p+q}(M \otimes_B N, K)$$

\square

31.3 Tensor-Hom Adjunction for Sheaves

Proposition 31.3.1. Let X be a site and $\mathcal{A} \rightarrow \mathcal{B}$ be a map of sheaves of rings. Let \mathcal{F}, \mathcal{H} be \mathcal{B} -modules and \mathcal{G} be a \mathcal{A} -module. Then there is a natural isomorphism,

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}, \mathcal{H}) \cong \mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{H}))$$

Proposition 31.3.2. Let (X, \mathcal{O}_X) be a ringed space, $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be \mathcal{O}_X -modules. Then there is a canonical isomorphism,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

Proposition 31.3.3. Let $\mathcal{A} \rightarrow \mathcal{B}$ be a map of sheaves of rings. Let \mathcal{K} be a flat (e.g. locally free) \mathcal{B} -module and \mathcal{I} an injective \mathcal{A} -module. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{K}, \mathcal{I})$ is an injective \mathcal{B} -module.

Proof. The functor

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{K}, -) : \mathbf{Mod}_{\mathcal{A}} \rightarrow \mathbf{Mod}_{\mathcal{B}}$$

is right-adjoint to tensor product $- \otimes_{\mathcal{B}} \mathcal{K} : \mathbf{Mod}_{\mathcal{B}} \rightarrow \mathbf{Mod}_{\mathcal{A}}$ which is exact because \mathcal{K} is flat. Explicitly, the functor,

$$\mathrm{Hom}_{\mathcal{B}}(-, \mathcal{H}om_{\mathcal{A}}(\mathcal{K}, \mathcal{I})) = \mathrm{Hom}_{\mathcal{A}}(- \otimes_{\mathcal{B}} \mathcal{K}, \mathcal{I})$$

is exact since \mathcal{I} is an injective \mathcal{A} -module and \mathcal{K} is \mathcal{B} -flat. \square

Theorem 31.3.4. Let $\mathcal{A} \rightarrow \mathcal{B}$ be a map of sheaves of rings, \mathcal{F} a \mathcal{B} -module, \mathcal{K} a flat \mathcal{B} -module, and \mathcal{G} a \mathcal{A} -module. Then there exist two spectral sequences,

$$\begin{aligned} E_2^{p,q} &= \mathcal{E}xt_{\mathcal{B}}^p(\mathcal{F}, \mathcal{E}xt_{\mathcal{A}}^q(\mathcal{K}, \mathcal{G})) \implies \mathcal{E}xt_{\mathcal{A}}^{p+q}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{K}, \mathcal{G}) \\ E_2^{p,q} &= \mathrm{Ext}_{\mathcal{B}}^p(\mathcal{F}, \mathcal{E}xt_{\mathcal{A}}^q(\mathcal{K}, \mathcal{G})) \implies \mathrm{Ext}_{\mathcal{A}}^{p+q}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{K}, \mathcal{G}) \end{aligned}$$

Proof. The functors $\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, -)$ and $\mathcal{H}om_{\mathcal{A}}(\mathcal{K}, -)$ satisfy,

$$\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{H}om_{\mathcal{A}}(\mathcal{K}, -)) = \mathcal{H}om_{\mathcal{A}}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{K}, -)$$

Furthermore, by the previous lemma, $\mathcal{H}om_{\mathcal{A}}(\mathcal{K}, -)$ preserves injectives which are acyclic for $\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, -)$ so we may apply the Grothendieck spectral sequence to this composition of functors to get a spectral sequence,

$$E_2^{p,q} = \mathcal{E}xt_{\mathcal{B}}^p(\mathcal{F}, \mathcal{E}xt_{\mathcal{A}}^q(\mathcal{K}, \mathcal{G})) \implies \mathcal{E}xt_{\mathcal{A}}^{p+q}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{K}, \mathcal{G})$$

Furthermore, taking the global section of the first equation gives,

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, \mathcal{H}om_{\mathcal{A}}(\mathcal{K}, -)) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{K}, -)$$

and $\mathcal{H}om_{\mathcal{A}}(\mathcal{K}, -)$ preserves injectives which are acyclic for $\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, -)$ so we may apply the Grothendieck spectral sequence to this composition of functors to get a spectral sequence,

$$E_2^{p,q} = \mathrm{Ext}_{\mathcal{B}}^p(\mathcal{F}, \mathcal{E}xt_{\mathcal{A}}^q(\mathcal{K}, \mathcal{G})) \implies \mathrm{Ext}_{\mathcal{A}}^{p+q}(\mathcal{F} \otimes_{\mathcal{B}} \mathcal{K}, \mathcal{G})$$

\square

31.4 Depth and Cohen-Macaulayness

Proposition 31.4.1. Let A be a regular local ring and $B = A/I$. Then $\mathrm{Ext}_A^p(B, -) = 0$ for $p > \dim A - \dim B$.

Proposition 31.4.2. Let A be a Cohen-Macaulay local ring and $B = A/I$. Then $\mathrm{Ext}_A^p(B, -) = 0$ for $p < \dim A - \dim B$.

31.5 Serre Duality

(CHECK THE HYPOTHESES!!)

(DEFINE DUALIZING SHEAF!)

(UNIQUENESS OF DUALIZING SHEAF!)

(EXISTENCE OF QUASI-COHERENT DUALIZING SHEAF)

Theorem 31.5.1. Let X be projective, Cohen-Macaulay, k -scheme of equidimension n . Further, suppose X has a dualizing sheaf ω_X . Then, there is a natural isomorphism for any coherent sheaf \mathcal{F} and $i \geq 0$,

$$\theta^i : \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^\vee$$

agreeing in $i = 0$ with $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee$.

Proof. (DO THISS!!!!) □

Theorem 31.5.2. (DOOO!!!) Serre Duality for \mathbb{P}^n .

Proposition 31.5.3. Let $\iota : X \hookrightarrow Y$ be a closed immersion of projective k -schemes. Let $c = \dim X - \dim Y$ be the codimension. Suppose that Y is Cohen-Macaulay, equidimensional and has a dualizing sheaf ω_Y . Furthermore, suppose that $\mathcal{E}xt_{\mathcal{O}_Y}^p(\iota_*\mathcal{O}_X, \omega_Y) = 0$ for $p < c$. Then $\omega_X = \iota^* \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y)$ is a dualizing sheaf for X .

Proof. Let $n = \dim X$ and $m = \dim Y$. We need to show there is a natural isomorphism,

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = H^n(X, \mathcal{F})^\vee$$

We compute,

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = \text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y))$$

Now I claim that,

$$\text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y)) = \text{Ext}_{\mathcal{O}_Y}^c(\iota_*\mathcal{F}, \omega_Y)$$

To do this, we can use Prop. 31.1.4. Choose an injective resolution $\omega_Y \rightarrow \mathcal{I}^\bullet$. Then we take the complex $\mathcal{H}om_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{I}^\bullet)$ which is a complex of injective $\iota_*\mathcal{O}_X$ -modules since

$$\mathcal{H}om_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, -) : \mathbf{Mod}_{\mathcal{O}_Y} \rightarrow \mathbf{Mod}_{\iota_*\mathcal{O}_X}$$

is right-adjoint to restriction $\mathbf{Mod}_{\iota_*\mathcal{O}_X} \rightarrow \mathbf{Mod}_{\mathcal{O}_Y}$ which is exact (see Lemma 31.3.3). Furthermore,

$$\mathcal{E}xt_{\mathcal{O}_Y}^p(\iota_*\mathcal{O}_X, \omega_Y) = H^p(\mathcal{H}om_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{I}^\bullet))$$

so by assumption the complex is exact for $p < c$. Taking the right-exact functor $\text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*, -)$ we can apply Prop. 31.1.4 to get,

$$\begin{aligned} \text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y)) &= \text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, H^c(\mathcal{H}om_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{I}^\bullet))) \\ &= H^c(\text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{H}om_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{I}^\bullet))) \end{aligned}$$

However, by Prop. 31.3.1, we get,

$$\text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{H}om_{\mathcal{O}_Y}(\iota_*\mathcal{O}_X, \mathcal{I}^\bullet)) = \text{Hom}_{\mathcal{O}_Y}(\iota_*\mathcal{F}, \mathcal{I}^\bullet)$$

Therefore,

$$\text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y)) = H^c(\text{Hom}_{\mathcal{O}_Y}(\iota_*\mathcal{F}, \mathcal{I}^\bullet)) = \text{Ext}_{\mathcal{O}_Y}^c(\iota_*\mathcal{F}, \omega_Y)$$

Furthermore, since Y is Cohen-Macaulay and equidimensional, by Serre Duality, we have,

$$\text{Ext}_{\mathcal{O}_Y}^c(\iota_*\mathcal{F}, \omega_Y) = H^{m-c}(Y, \iota_*\mathcal{F})^\vee = H^n(X, \mathcal{F})^\vee$$

Putting everything together we find,

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = \text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y)) = H^n(X, \mathcal{F})^\vee$$

as required. □

Remark. Alternatively, we may give a spectral sequence proof. The ring map $\mathcal{O}_Y \rightarrow \iota_*\mathcal{O}_X$ gives a spectral sequence,

$$E_2^{p,q} = \text{Ext}_{\iota_*\mathcal{O}_X}^p(\iota_*\mathcal{F}, \mathcal{E}xt_{\mathcal{O}_Y}^q(\iota_*\mathcal{O}_X, \omega_Y)) \implies \text{Ext}_{\mathcal{O}_Y}^{p+q}(\iota_*\mathcal{F}, \omega_Y)$$

where we have used that $\iota_*\mathcal{O}_X$ is trivially flat as a $\iota_*\mathcal{O}_X$ -module. Now, $E_2^{p,q} = 0$ for $q < c$ because $\mathcal{E}xt_{\mathcal{O}_Y}^q(\iota_*\mathcal{O}_X, \omega_Y) = 0$ for $q < c$. Furthermore, any differential with $r \geq 2$ ending or beginning at $(0, c)$ hits zero since $d_r^{0,c} : E_r^{0,c} \rightarrow E_r^{r,c-r+1}$ but $c-r+1 < c$ so $E_r^{r,c-r+1} = 0$ also $d_r^{-r,c+r-1} : E_r^{-r,c+r-1} \rightarrow E_r^{0,c}$ has $E_r^{-r,c+r-1} = 0$. Therefore, $E_\infty^{0,c} = E_2^{0,c}$. Furthermore, $E_\infty^{p,q} = 0$ if $p < c$ so the $p+q = c$ diagonal of $E_\infty^{p,q}$ is zero except $E_\infty^{0,c}$ and thus,

$$\text{Ext}_{\mathcal{O}_Y}^c(\iota_*\mathcal{F}, \omega_Y) = E_\infty^{0,c} = E_2^{0,c} = \text{Hom}_{\iota_*\mathcal{O}_X}(\iota_*\mathcal{F}, \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y))$$

proving the claim. This is just the spectral sequence proof of Prop. 31.1.4.

Lemma 31.5.4. Let $\iota : X \hookrightarrow Y$ be a closed immersion of projective k -schemes. Suppose that Y is regular, equidimensional and has a locally free dualizing sheaf ω_Y . Then $\mathcal{E}xt_{\mathcal{O}_Y}^p(\iota_*\mathcal{O}_X, \omega_Y) = 0$ for $p \neq \dim X - \dim Y$.

Proof. (PROVE THIS!!) □

Proposition 31.5.5. Let $\iota : X \hookrightarrow Y$ be a closed immersion of projective k -schemes. Let $c = \dim X - \dim Y$ be the codimension. Suppose that Y is regular, equidimensional and has a locally free dualizing module ω_Y . Then $\omega_X = \iota^* \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_*\mathcal{O}_X, \omega_Y)$ is a dualizing sheaf for X .

Proof. This follows from the previous two propositions. □

Proposition 31.5.6. Let $X \xrightarrow{\iota_1} Y \xrightarrow{\iota_2} P$ be closed immersion of projective k -schemes. Define,

$$\begin{aligned} \iota &= \iota_2 \circ \iota_1 \\ n &= \dim X \\ m &= \dim Y \\ c_1 &= \dim P - \dim X \\ c_2 &= \dim P - \dim Y \end{aligned}$$

Suppose that P is regular, equidimensional and has a locally free dualizing module ω_P then we have seen $\omega_X = \iota^* \mathcal{E}xt_{\mathcal{O}_P}^{c_1}(\iota_*\mathcal{O}_X, \omega_P)$ and $\omega_Y = \iota_2^* \mathcal{E}xt_{\mathcal{O}_P}^{c_2}(\iota_{2*}\mathcal{O}_Y, \omega_P)$ are dualizing modules for X and Y . Then, if Y is Cohen-Macaulay and equidimensional we have,

$$\omega_X = \iota_2^* \mathcal{E}xt_{\mathcal{O}_Y}^{n-m}(\iota_{1*}\mathcal{O}_X, \omega_Y)$$

Proof. Consider the spectral sequence of sheaves on P ,

$$E_2^{p,q} = \mathcal{E}xt_{\iota_{2*}\mathcal{O}_Y}^p(\iota_*\mathcal{O}_X, \mathcal{E}xt_{\mathcal{O}_P}^q(\iota_{2*}\mathcal{O}_Y, \omega_P)) \implies \mathcal{E}xt_{\mathcal{O}_P}^{p+q}(\iota_*\mathcal{O}_X, \omega_P)$$

Futhermore, by Lemma 31.5.4, $\mathcal{E}xt_{\mathcal{O}_P}^q(\iota_{2*}\mathcal{O}_Y, \omega_P) = 0$ for $q \neq c_2$ so the spectral sequence degenerates on the second page. Therefore,

$$\mathcal{E}xt_{\mathcal{O}_P}^{p+c_2}(\iota_*\mathcal{O}_X, \omega_P) = E_\infty^{p,c_2} = \mathcal{E}xt_{\iota_{2*}\mathcal{O}_Y}^p(\iota_*\mathcal{O}_X, \iota_{2*}\omega_Y)$$

In particular, taking $p = c_1 - c_2 = n - m$ we find,

$$\iota_* \omega_X = \mathcal{E}xt_{\mathcal{O}_P}^{c_1}(\iota_* \mathcal{O}_X, \omega_P) = \mathcal{E}xt_{\iota_{2*} \mathcal{O}_Y}^{n-m}(\iota_* \mathcal{O}_X, \iota_{2*} \omega_Y) = \iota_{2*} \mathcal{E}xt_{\mathcal{O}_Y}^{n-m}(\iota_{1*} \mathcal{O}_X, \omega_Y)$$

with the last equality using that ι_2 is affine. Therefore, using the equivalence of categories induced by ι_* we find,

$$\omega_X = \iota_1^* \mathcal{E}xt_{\mathcal{O}_Y}^{n-m}(\iota_{1*} \mathcal{O}_X, \omega_Y)$$

□

Corollary 31.5.7. Let $\iota : X \rightarrow Y$ be closed immersion of projective k -schemes. If Y is Cohen-Macaulay and equidimensional then X and Y have dualizing modules ω_X and ω_Y satisfying,

$$\omega_X = \iota^* \mathcal{E}xt_{\mathcal{O}_Y}^c(\iota_* \mathcal{O}_X, \omega_Y)$$

Proof. Consider $X \hookrightarrow Y \hookrightarrow \mathbb{P}^n$ and apply the previous proposition. □

32 The Koszul Complex

Definition 32.0.1. Let E be an R -module and $s : E \rightarrow R$ an R -linear map. Then the *Koszul Complex* $K_\bullet(E, s)$ is the chain complex,

$$0 \longrightarrow \bigwedge^r E \xrightarrow{d_r} \bigwedge^{r-1} E \xrightarrow{d_{r-1}} \cdots \longrightarrow \bigwedge^1 E \xrightarrow{s} R \longrightarrow 0$$

with $K_\ell(s) = \bigwedge^\ell E$ and the differential,

$$d_\ell(e_1 \wedge \cdots \wedge e_\ell) = \sum_{i=1}^{\ell} (-1)^{i+1} s(e_i) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_\ell$$

It is straightforward to check from the definition that $K_\bullet(s)$ is, in fact, a chain complex. Consider,

$$\begin{aligned} d_{\ell-1}(d_\ell(e_1 \wedge \cdots \wedge e_\ell)) &= d_{\ell-1} \left(\sum_{i=1}^{\ell} (-1)^{i+1} s(e_i) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_\ell \right) \\ &= \sum_{j < i}^{\ell} (-1)^{i+j} s(e_j) s(e_i) e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_\ell \\ &\quad + \sum_{j > i}^{\ell} (-1)^{i+j+1} s(e_j) s(e_i) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_\ell \\ &= 0 \end{aligned}$$

Example 32.0.2. Let $E = R^{\oplus n}$ be a free module. Then the map $s : E \rightarrow R$ is given by a vector $(f_1, \dots, f_n) \in R^n$ under the dot product. Then the Koszul complex $K_\bullet(R, f_1, \dots, f_n) = K_\bullet(E, s)$ becomes,

$$0 \longrightarrow R^{\binom{n}{n}} \longrightarrow R^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow R^{\binom{n}{2}} \longrightarrow R^{\binom{n}{1}} \longrightarrow R \longrightarrow 0$$

with $R^{\binom{n}{k}} = \bigwedge^k R^n$ given the basis $e_1 \wedge \cdots \wedge e_k$.

Definition 32.0.3. Let $f_1, \dots, f_n \in R$ be a sequence of elements and M an R -module. We say the sequence is,

- (a) *M-regular* if f_i is a nonzero divisor in $M/(f_1, \dots, f_{i-1})M$ and $M/(f_1, \dots, f_n)M$ is nonzero
- (b) *M-Koszul regular* if $H_i(K_\bullet(R, f_1, \dots, f_n) \otimes_R M) = 0$ for $i > 0$
- (c) *M- H_1 -regular* if $H_1(K_\bullet(R, f_1, \dots, f_n) \otimes_R M) = 0$
- (d) *regular* if it is *R-regular*
- (e) *Koszul regular* if it is *R-Koszul regular*
- (f) *H_1 -regular* if it is *R- H_1 -regular*.

Lemma 32.0.4. Given a sequence $f_1, \dots, f_n \in R$ then regular implies Koszul regular implies H_1 -regular. Furthermore these implications are reversible if R is a local Noetherian ring with $f_1, \dots, f_n \in \mathfrak{m}$.

Proof. See Tag 062D. □

Lemma 32.0.5. Let

32.1 Complete Intersections

33 Degree Zoo

33.1 Degree of a Finite Map

Definition 33.1.1. Let $f : X \rightarrow Y$ be a finite dominant map of integral schemes. Then if $\xi \in X$ and $\eta \in Y$ are the generic points, the map $\mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$ gives a finite extension of fields $K(X)/K(Y)$. We say the *degree* of f is the degree of this extension,

$$\deg f = [K(X) : K(Y)]$$

Lemma 33.1.2. Let $f : X \rightarrow Y$

33.2 The Case for Complete Varieties

Lemma 33.2.1. Let $f : X \rightarrow Y$ be a morphism of proper k -varieties and \mathcal{F} a coherent \mathcal{O}_X -module. Then,

$$\chi(X, \mathcal{F}) = \sum_{q=0}^c (-1)^q \chi(Y, R^q f_* \mathcal{F})$$

Proof. Since X and Y are proper k -varieties, we have finiteness of cohomology and vanishing above $\dim X$ and $\dim Y$. The Leray spectral sequence,

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \implies H^{p+q}(X, \mathcal{F})$$

converges. Notice that the integers,

$$\ell_r = \sum (-1)^{p+q} \dim_k E_r^{p,q}$$

are well-defined since each $E_r^{p,q}$ is a finite k -module and $E_r^{p,q}$ is bounded in p and q . Furthermore, ℓ_r is independent of r since,

$$E_{r+1}^{p,q} = \ker(d_r^{p,q}) / \operatorname{Im}(d_r^{p-r,q+r-1})$$

Then,

$$\begin{aligned} \ell_{r+1} &= \sum (-1)^{p+q} \dim_k \ker(d_r^{p,q}) - \sum (-1)^{p+q} \dim_k \operatorname{Im}(d_r^{p-r,q+r-1}) \\ &= \sum (-1)^{p+q} \dim_k \ker(d_r^{p,q}) - \sum (-1)^{p+r+q-r+1} \dim_k \operatorname{Im}(d_r^{p,q}) \\ &= \sum (-1)^{p+q} \dim_k \ker(d_r^{p,q}) - \sum (-1)^{p+q+1} \dim_k \operatorname{Im}(d_r^{p,q}) \\ &= \sum (-1)^{p+q} (\dim_k \ker(d_r^{p,q}) + \dim_k \operatorname{Im}(d_r^{p,q})) \\ &= \sum (-1)^{p+q} \dim_k E_r^{p,q} = \ell_r \end{aligned}$$

Furthermore,

$$\begin{aligned} \ell_2 &= \sum (-1)^{p+q} \dim_k E_2^{p,q} = \sum (-1)^{p+q} \dim_k H^p(Y, R^q f_* \mathcal{F}) \\ &= \sum (-1)^q \sum (-1)^p \dim_k H^p(Y, R^q f_* \mathcal{F}) = \sum (-1)^q \chi(Y, R^q f_* \mathcal{F}) \end{aligned}$$

Finally, since $E_\infty^{p,q}$ is a filtration of $H^{p+q}(X, \mathcal{F})$ we see that,

$$\dim_k H^n(X, \mathcal{F}) = \sum_{p+q=n} \dim_k E_\infty^{p,q}$$

and thus,

$$\ell_\infty = \sum (-1)^{p+q} \dim_k E_\infty^{p,q} = \sum_{n=0}^c (-1)^n \dim_k H^n(X, \mathcal{F}) = \chi(X, \mathcal{F})$$

so we find that,

$$\chi(X, \mathcal{F}) = \ell_\infty = \ell_2 = \sum_{q=0}^c (-1)^q \chi(Y, R^q f_* \mathcal{F})$$

□

Corollary 33.2.2. Let $f : X \rightarrow Y$ be an affine morphism of proper k -varieties and \mathcal{F} a coherent \mathcal{O}_X -module. Then,

$$\chi(Y, f_* \mathcal{F}) = \chi(X, \mathcal{F})$$

Proof. This follows from the former because $R^q f_* = 0$ for $q > 0$ when f is affine. Alternatively, we may use the fact that $H^p(X, \mathcal{F}) = H^p(Y, f_* \mathcal{F})$ since f is affine. □

Lemma 33.2.3.

Lemma 33.2.4. Let $f : X \rightarrow Y$ be a finite dominant étale morphism of proper k -varieties and \mathcal{F} a coherent \mathcal{O}_Y -modules. Then,

$$\chi(X, f^* \mathcal{F}) = \deg f \cdot \chi(Y, \mathcal{F})$$

Proof. (IS THIS TRUE!!!!!!)

□

33.3 Degree on Curves

(DO I NEED SMOOTH FOR THIS!!)

Proposition 33.3.1. Let C be a proper curve over k . There exists a map $c : G(C) \rightarrow \text{Cl}(C)$ on the Grothendieck group of C which is the unique such map satisfying the following properties,

- (a) $c : \mathcal{O}_C(D) \mapsto D$
- (b) if \mathcal{F} is torsion (i.e. $\mathcal{F}_\xi = 0$) then,

$$c(\mathcal{F}) = \sum_{x \in C} \ell_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \cdot [x]$$

Lemma 33.3.2.

Proposition 33.3.3. Let $f : X \rightarrow Y$ be a nonconstant map of normal proper curves and \mathcal{L} a line bundle on Y . Then $\deg f^* \mathcal{L} = \deg f \cdot \deg \mathcal{L}$.

Proof. Let η be a meromorphic section of \mathcal{L} . Then $\deg \mathcal{L} = \deg \text{div}(\eta)$ and $\deg f^* \mathcal{L} = \deg \text{div}(f^* \eta)$. Therefore, we write,

$$\text{div}(\eta) = \sum_{y \in Y} n_y [y]$$

then we get,

$$\text{div}(f^* \eta) = f^* \text{div}(\eta) = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} n_y e_x [x]$$

and thus,

$$\deg(f^* \mathcal{L}) = \deg \text{div}(f^* \eta) = \sum_{y \in Y} \left(\sum_{x \in f^{-1}(y)} e_x \right) n_y = (\deg f) \sum_{y \in Y} n_y = \deg f \cdot \deg \mathcal{L}$$

□

33.4 Scheme Theoretic Image

33.5 Projective Degree

Definition 33.5.1. Let \mathcal{F} be a coherent sheaf on \mathbb{P}_k^n . Then we define the Hilbert function,

$$p_{\mathcal{F}}(d) = \chi(\mathbb{P}^n, \mathcal{F}(d))$$

Proposition 33.5.2. For any coherent sheaf \mathcal{F} on \mathbb{P}_k^n the Hilbert function is a rational polynomial. Explicitly, there exists a rational polynomial $p_{\mathcal{F}} \in \mathbb{Q}[x]$ such that,

$$p_{\mathcal{F}}(d) = \chi(\mathbb{P}^n, \mathcal{F}(d))$$

for all $d \in \mathbb{Z}$.

Proof. We proceed by induction on n . For $n = 0$ the function is constant. Now suppose the theorem holds for n . Consider a coherent sheaf \mathcal{F} on \mathbb{P}_k^{n+1} . Now, we can choose a hyperplane H such that there is an exact sequence,

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

where the middle map is $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X(-1) \rightarrow \mathcal{O}_X)$ determining the hyperplane H . To do this see Tag 08A0 (if k is finite we need to pass to some extension but by flat base change this does not change the dimensions of the cohomology). Then $\mathcal{J} = \text{Im}(\mathcal{O}_X(-1) \rightarrow \mathcal{O}_X)$ annihilates \mathcal{G} because $\mathcal{J} \cdot \mathcal{F}$ is the image of $\mathcal{F}(-1) \rightarrow \mathcal{F}$. Thus we can write $\mathcal{G} = \iota_* \tilde{\mathcal{G}}$ for the inclusion of the hyperplane $\iota : H \hookrightarrow X$ and some sheaf of \mathcal{O}_H -modules $\tilde{\mathcal{G}}$. Then, twisting we get,

$$0 \longrightarrow \mathcal{F}(d-1) \longrightarrow \mathcal{F}(d) \longrightarrow \iota_* \tilde{\mathcal{G}}(d) \longrightarrow 0$$

and taking Euler characteristics gives,

$$\chi(\mathbb{P}^n, \mathcal{F}(d)) - \chi(\mathbb{P}^n, \mathcal{F}(d-1)) = \chi(H, \tilde{\mathcal{G}}(d))$$

but $H \cong \mathbb{P}_k^{n-1}$ and thus $\chi(H, \tilde{\mathcal{G}}(d))$ is a polynomial in d by induction. Thus we find that,

$$P_{\mathcal{F}}(d) - P_{\mathcal{F}}(d-1) = \chi(\mathbb{P}^n, \mathcal{F}(d)) - \chi(\mathbb{P}^n, \mathcal{F}(d-1)) = \chi(H, \tilde{\mathcal{G}}(d))$$

is a polynomial in d for all $d \in \mathbb{Z}$ and thus $P_{\mathcal{F}}(d)$ is a polynomial in d . □

Definition 33.5.3. Let \mathcal{F} be a coherent sheaf on \mathbb{P}_k^n . Then we define the following quantities,

- (a) $\dim \mathcal{F} = \dim \text{Supp}_{\mathcal{O}_X}(\mathcal{F})$
- (b) $\deg \mathcal{F} = \lim_{d \rightarrow \infty} n! \cdot \frac{\chi(\mathbb{P}^n, \mathcal{F}(d))}{d^\ell}$ where $\ell = \dim \mathcal{F}$
- (c) $p_a(\mathcal{F}) = (-1)^\ell (\chi(\mathbb{P}^n, \mathcal{F}) - 1)$

Lemma 33.5.4. We can read of the above information from the Hilbert polynomials as follows,

- (a) $\dim \mathcal{F} = \deg P_{\mathcal{F}}$
- (b) $\deg \mathcal{F} = n! \cdot a_\ell$ where $P_{\mathcal{F}}(t) = a_\ell t^\ell + \dots + a_0$
- (c) $p_a(\mathcal{F}) = (-1)^\ell (a_0 - 1)$

Proof. The second two properties follow directly from the definition. It remains to show that $\deg P_{\mathcal{F}} = \dim \text{Supp}_{\mathcal{O}_X}(\mathcal{F})$. □

Lemma 33.5.5. Let $\iota : X \hookrightarrow \mathbb{P}^n$ be a closed immersion. Let $\mathcal{F} = \iota_* \mathcal{O}_X$ and the associated Hilbert polynomial, $P_\iota(d) = \chi(\mathbb{P}^n, \iota_* \mathcal{O}_X(d))$ ¹ Then we interpret the above data as follows,

- (a) $\deg \mathbb{P}_\iota = \dim \mathcal{F} = \dim X$
- (b) for a generic $(n - \ell)$ -dimensional plane $H \subset \mathbb{P}^n$ we have $\dim X \cap H = 0$ and $\dim_k H^0(X \cap H, \mathcal{O}_{X \cap H}) = \deg X$
- (c) $(-1)^\ell (a_0 - 1) = p_a(\mathcal{F}) = p_a(X)$.

Proof. The second □

¹Note that $\chi(\mathbb{P}^n, \iota_* \mathcal{O}_X(d)) = \chi(X, \iota^* \mathcal{O}_{\mathbb{P}^n}(d))$ by the projection formula.

33.6 Relations Between Degrees

Proposition 33.6.1. Let $\iota : C \hookrightarrow \mathbb{P}_k^n$ be an embedding of a curve. Then $d = \deg \iota^* \mathcal{O}_{\mathbb{P}^n}(1)$ is the degree of the embedding.

Proof. □

Definition 33.6.2. Let $f : X \rightarrow \mathbb{P}^n$ be a morphism which is finite onto its image. Factoring $X \xrightarrow{g} Z \hookrightarrow \mathbb{P}^n$ by taking the scheme theoretic image we define $\deg f = \deg g \cdot \deg (Z \hookrightarrow \mathbb{P}^n)$.

Proposition 33.6.3. Let $f : C \rightarrow \mathbb{P}_k^n$ be a nonconstant morphism from a curve C . Then $\deg f = \deg f^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Proof. Factoring $C \rightarrow \tilde{C} \hookrightarrow \mathbb{P}_k^n$ we know that $\deg f = \deg g \cdot \deg \iota$ but $\deg \iota = \deg \iota^* \mathcal{O}_{\mathbb{P}^n}(1)$. Furthermore, $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^n}(1) = g^* \iota^* \mathcal{O}_{\mathbb{P}^n}(1)$ and thus,

$$\deg \mathcal{L} = \deg g^* \iota^* \mathcal{O}_{\mathbb{P}^n}(1) = \deg g^* \cdot \deg \iota^* \mathcal{O}_{\mathbb{P}^n}(1) = \deg g^* \cdot \deg \iota^* = \deg f$$
□

34 Some Counter Examples

Example 34.0.1. If $\iota : Z \hookrightarrow X$ is a closed immersion then ι_* is exact and ι^* is a left inverse. However, this does not mean that ι^* is exact. For example, we have the exact sequence,

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Then pulling back we get the right-exact sequence,

$$\mathcal{C}_{Z/X} \longrightarrow \mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_Z \longrightarrow 0$$

but the second map is an isomorphism so $\mathcal{C}_{Z/X} \rightarrow \mathcal{O}_Z$ must be zero (it is because the map $\mathcal{I} \rightarrow \mathcal{O}_X$ vanishes on the curve) but $\mathcal{C}_{Z/X}$ is not generally zero so ι^* cannot be left-exact.

Example 34.0.2. Consider the normalization map $\nu : \mathbb{P}^1 \rightarrow \text{Proj}(k[X, Y, Z]/(Y^2Z - (X+1)X^2))$ of a nodal cubic. Locally, about the origin, this is the map,

$$\text{Spec}(k[t]) \rightarrow \text{Spec}(k[x, y]/(y^2 - (x+1)x^2))$$

given by $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$. However, ν is birational because localizing at x gives,

$$k[x, y, x^{-1}]/(y^2 - (x+1)x^2) \rightarrow k[t, t^{-1}]$$

via $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$ and its inverse $t \mapsto y/x$ which are mutually inverse because $t \mapsto y/x \mapsto t(t^2 - 1)/(t^2 - 1) = t$ and $x \mapsto t^2 - 1 \mapsto y^2/x^2 - 1 = x$ and $y \mapsto t(t^2 - 1) \mapsto y/x \cdot (y^2/x^2 - 1) = y$. However, consider the fiber above the origin,

$$\nu^*[0] = [-1] + [1]$$

which has degree 2 contradicting the sum formula for fibers. This is because the nodal cubic is not normal.

Example 34.0.3. when the map def by line bundle is not morphism (FORMULA!!)

35 Stein Factorization

Remark. Let S be affine. For any $\mathcal{O}_S(S)$ -algebra A , we know that,

$$\mathrm{Hom}_S(X, \mathrm{Spec}(A)) = \mathrm{Hom}_{\mathcal{O}_S(S)}(A, \Gamma(X, \mathcal{O}_X))$$

We want to generalize this to \mathcal{O}_S -algebras \mathcal{A} on any scheme S .

Definition 35.0.1. For a quasi-coherent sheaf of \mathcal{O}_S -algebras \mathcal{A} , the scheme $\mathbf{Spec}_S(\mathcal{A}) \rightarrow S$ represents the functor $(f : X \rightarrow S) \mapsto \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, f_*\mathcal{O}_X)$ i.e.

$$\mathrm{Hom}_S(X, \mathbf{Spec}_S(\mathcal{A})) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, f_*\mathcal{O}_X)$$

Remark. In the affine case, $S = \mathrm{Spec}(R)$, we see that $\mathcal{A} = \widetilde{A}$ for some R -module A and then,

$$\mathrm{Hom}_S(X, \mathrm{Spec}(A)) = \mathrm{Hom}_R(A, \Gamma(X, \mathcal{O}_X)) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, f_*\mathcal{O}_X)$$

and therefore, $\mathbf{Spec}_S(\mathcal{A}) = \mathrm{Spec}(A)$.

Theorem 35.0.2. For a quasi-coherent sheaf of \mathcal{O}_S -algebras \mathcal{A} , the scheme $\mathbf{Spec}_S(\mathcal{A}) \rightarrow S$ exists.

Proof. For an open affine cover $U \subset X$ with $U = \mathrm{Spec}(R)$ we get $\mathcal{A}|_U = \widetilde{A}$ for some R -module A . Then by the previous remark $\pi^{-1}(U) = \mathrm{Spec}(A)$ represents the correct functor. These glue to form $\mathbf{Spec}_S(\mathcal{A})$, see Tag 01LL for details. \square

Remark. Since $\mathbf{Spec}_S(\mathcal{A})$ represents the functor $(f : X \rightarrow S) \mapsto \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, f_*\mathcal{O}_X)$ we get a universal element for $\pi : \mathbf{Spec}_S(\mathcal{A}) \rightarrow S$,

$$\alpha : \mathcal{A} \rightarrow \pi_*\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}$$

Proposition 35.0.3. The universal map $\alpha : \mathcal{A} \rightarrow \pi_*\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}$ is an isomorphism of \mathcal{O}_S -algebras.

Proof. This is local on S so choose an affine open cover $U \subset S$ with $U = \mathrm{Spec}(R)$ then we have $\mathcal{A}|_U = \widetilde{A}$ for some R -module A . Then we get $\pi^{-1}(U) = \mathrm{Spec}(A)$ so $\pi_*\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}|_U = \widetilde{A}$ because $\Gamma(\pi^{-1}(U), \mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}) = A$. Thus, locally the map $\mathcal{A} \rightarrow \pi_*\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}$ is an isomorphism. \square

Remark. We can explicitly describe the natural transformation given a morphism $f : X \rightarrow \mathbf{Spec}_S(\mathcal{A})$,

$$\begin{array}{ccc} \mathrm{Hom}_S(\mathbf{Spec}_S(\mathcal{A}), \mathbf{Spec}_S(\mathcal{A})) & \xlongequal{\quad} & \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, \pi_*\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})}) \\ \downarrow f^* & & \downarrow f_* \\ \mathrm{Hom}_S(X, \mathbf{Spec}_S(\mathcal{A})) & \xlongequal{\quad} & \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, \pi_*f_*\mathcal{O}_X) \end{array}$$

Then we send $f \mapsto (\mathcal{A} \xrightarrow{\sim} \pi_*(\mathcal{O}_{\mathbf{Spec}_S(\mathcal{A})} \rightarrow f_*\mathcal{O}_X))$.

Theorem 35.0.4. Let $f : X \rightarrow S$ be proper. Then there exists a factorization,

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & T \\ & \searrow f & \swarrow g \\ & S & \end{array}$$

where $g : T \rightarrow S$ is finite and $\tilde{f} : X \rightarrow T$ is proper with $\tilde{f}_*\mathcal{O}_X = \mathcal{O}_T$.

Proof. Let $T = \mathbf{Spec}_S(f_*\mathcal{O}_X)$ then $T \rightarrow S$ is finite since $f_*\mathcal{O}_X$ is coherent. The identity map $f_*\mathcal{O}_X \rightarrow f_*\mathcal{O}_X$ gives, by the universal property, an S -morphism $\tilde{f} : X \rightarrow \mathbf{Spec}_S(f_*\mathcal{O}_X)$ so we have a diagram,

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & T \\ & \searrow f & \swarrow g \\ & S & \end{array}$$

Since g is finite it is separated and f is proper so \tilde{f} is proper. Furthermore, by construction, the map $\tilde{f} : X \rightarrow T$ corresponds to $f_*\mathcal{O}_X \xrightarrow{\sim} g_*\mathcal{O}_{\mathbf{Spec}_S(f_*\mathcal{O}_X)} \rightarrow g_*\tilde{f}_*\mathcal{O}_X$ which is $\text{id} : f_*\mathcal{O}_X \rightarrow f_*\mathcal{O}_X$ by construction. Thus $g_*(\mathcal{O}_{\mathbf{Spec}_S(f_*\mathcal{O}_X)} \rightarrow \tilde{f}_*\mathcal{O}_X)$ is an isomorphism but $g : \mathbf{Spec}_S(f_*\mathcal{O}_X) \rightarrow S$ is affine so it reflects isomorphism² and therefore, $\mathcal{O}_T \xrightarrow{\sim} \tilde{f}_*\mathcal{O}_X$. \square

Theorem 35.0.5 (Zariski). Let $f : X \rightarrow Y$ be a proper morphism of schemes with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then f is surjective and has geometrically connected fibers.

Proof. DO THIS!! \square

36 Zariski's Main Theorem

37 When is Pullback on Cohomology Injective

Lemma 37.0.1. Let $f : X \rightarrow Y$ be a dominant quasi-compact morphism of schemes with Y reduced. Then $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is injective.

Proof. DO THIS!!!! \square

Remark. When X and Y are integral, the map $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is automatically injective because we have a diagram,

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \longrightarrow & \mathcal{O}_X(U) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & K(X) \end{array}$$

where the map $K(Y) \rightarrow K(X)$ is injective since it is a map of fields. Since X and Y are integral, the downward maps are injective so $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U)$ is injective as well.

Proposition 37.0.2.

38 Birational Invariance of Plurigenera

Proposition 38.0.1. DO!!

²Suppose that $f : X \rightarrow Y$ is affine and $f_*(\mathcal{F} \rightarrow \mathcal{G})$ is an isomorphism then on any affine open $U \subset Y$, since f is affine, $V = f^{-1}(U) = \text{Spec}(S)$. Then $\mathcal{F}|_V = \widetilde{M}$ and $\mathcal{G}|_V = \widetilde{N}$ for S -modules, M, N then $f_*\mathcal{F}|_U = \widetilde{M_R}$ and $f_*\mathcal{G}|_U = \widetilde{N_R}$ and we have $M_R \xrightarrow{\sim} N_R$ is an isomorphism of R -modules so $M \rightarrow N$ is an isomorphism of S -modules. Therefore, the map $\mathcal{F} \rightarrow \mathcal{G}$ is locally an isomorphism

39 Automorphisms and Closed Subschemes

DO!!!!

39.1 Proof of Picard Scheme Action Relation

DO!!!

40 Finite Maps

Definition 40.0.1. A morphism $f : X \rightarrow Y$ of schemes is *finite* if it is affine and for every affine open $V \subset Y$ then $U = f^{-1}(V)$ is affine and the ring map associated to $U \rightarrow V$ is finite.

Proposition 40.0.2. Closed immersions are finite.

Proof. The map $A \rightarrow A/I$ is finite. □

Proposition 40.0.3. Finite maps are preserved under base change.

Proposition 40.0.4. Finite maps are closed and thus universally closed.

Proposition 40.0.5. The following are equivalent for a map of schemes $f : X \rightarrow Y$

- (a) f is finite
- (b) f is affine and proper.

Proof. They are affine and thus separated, finite and thus finite type, and universally closed. □

Proposition 40.0.6. Let $f : X \rightarrow Y$ be finite and $y \in Y$. Then the fiber is affine, zero dimensional, has finitely many points, and explicitly,

$$X_y = \operatorname{Spec} ((f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y))$$

Furthermore,

$$\operatorname{rank}_y(f_* \mathcal{O}_X) = \sum_{x \in f^{-1}(y)} \operatorname{length}_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x}) \cdot [\kappa(x) : \kappa(y)]$$

Proof. Let $f : X \rightarrow Y$ be finite then locally we have affine opens $V = \operatorname{Spec}(B) \subset Y$ and $U = f^{-1}(V) = \operatorname{Spec}(A)$ and the map $B \rightarrow A$ is finite. Then $(f_* \mathcal{O}_X)|_V = \widetilde{A}$ as a B -module. Choose a point $y \in Y$ corresponding to a prime $\mathfrak{p} \in \operatorname{Spec}(B)$. Consider the fiber $X_y = X \times_Y \operatorname{Spec}(\kappa(y))$. Because $U = f^{-1}(V)$ is affine, the fiber $X_y \subset \operatorname{Spec}(A)$ and thus,

$$X_y = \operatorname{Spec}(A) \times_{\operatorname{Spec}(B)} \operatorname{Spec}(\kappa(y)) = \operatorname{Spec}(A \otimes_B \kappa(y)) = \operatorname{Spec}((A/\mathfrak{p}A)_{\mathfrak{p}})$$

where $\kappa(y) = (B/\mathfrak{p}B)_{\mathfrak{p}}$. So set $R = A \otimes_B \kappa(y) = (A/\mathfrak{p}A)_{\mathfrak{p}}$ then,

$$R = A \otimes_B (B/\mathfrak{p}B)_{\mathfrak{p}} = A \otimes_B B_{\mathfrak{p}} \otimes_{B_{\mathfrak{p}}} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_{B_{\mathfrak{p}}} \kappa(y) = (f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$$

Since A is a finite B -module, R is a finite $\kappa(\mathfrak{p})$ -module so R is an artinian ring. Thus $X_y = \operatorname{Spec}(R)$ has finitely many points and $\dim X_y = 0$. Furthermore,

$$\operatorname{rank}_y(f_* \mathcal{O}_X) = \dim_{\kappa(y)} ((f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y))$$

and by our results on artinian k -algebras,

$$\dim_{\kappa(y)} R = \sum_{\mathfrak{m}_i \in \text{Spec}(R)} \text{length}_{R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i}) \cdot \dim_{\kappa(y)}(R/\mathfrak{m}_i)$$

However, the prime (maximal) ideals $\mathfrak{p}_x \in \text{Spec}(R)$ correspond to points $x \in f^{-1}(y)$ furthermore,

$$R_{\mathfrak{p}_x} = (A_{\mathfrak{p}_x}/\mathfrak{p}A_{\mathfrak{p}_x}) = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$$

since $\mathfrak{p}A_{\mathfrak{p}_x} = \mathfrak{p}B_{\mathfrak{p}}A_{\mathfrak{p}_x} = \mathfrak{m}_yA_{\mathfrak{p}_x} = \mathfrak{m}_y\mathcal{O}_{X,x}$. Furthermore, since $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a surjection viewing $R_{\mathfrak{p}_x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ as a $\mathcal{O}_{X,x}$ -module gives,

$$\text{length}_{R_{\mathfrak{p}_x}}(R_{\mathfrak{p}_x}) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})$$

Finally, $R/\mathfrak{p}_x = \mathcal{O}_{X,x}/\mathfrak{m}_x = \kappa(x)$ and thus we find,

$$\text{rank}_y(f_*\mathcal{O}_X) = \sum_{x \in f^{-1}(y)} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) \cdot [\kappa(x) : \kappa(y)]$$

□

Lemma 40.0.7. Let $A \hookrightarrow B$ be a finite inclusion of domains. Then $\text{Frac}(B) = A^{-1}B$ and $[\text{Frac}(B) : \text{Frac}(A)]$ is a finite extension.

Proof. Since $A \rightarrow B$ is finite the map $\text{Frac}(A) \rightarrow A^{-1}B$ is finite. However, $A^{-1}B$ is a domain finite dimensional over the field $\text{Frac}(A)$ and thus $A^{-1}B$ is a field. However, $A^{-1}B \subset \text{Frac}(B)$ so $\text{Frac}(B) = A^{-1}B$. □

Proposition 40.0.8. Let $f : X \rightarrow Y$ be a finite dominant map of integral schemes with generic points $\xi \in X$ and $\eta \in Y$. Then we have,

$$\deg f = \text{rank}_{\eta}(f_*\mathcal{O}_X)$$

Proof. The map $\mathcal{O}_{Y,\eta} \rightarrow (f_*\mathcal{O}_X)_{\eta}$ is an injective finite map of domains because f is dominant. Therefore,

$$\text{rank}_{\eta}(f_*\mathcal{O}_X) = \dim_{\kappa(\eta)}((f_*\mathcal{O}_X) \otimes_{\mathcal{O}_{Y,\eta}} \kappa(\eta)) = \dim_{K(Y)} K(Y)^{-1}(f_*\mathcal{O}_X)_{\eta}$$

However, the map $(f_*\mathcal{O}_X)_{\eta} \rightarrow \mathcal{O}_{X,\xi}$ is taking the fraction field $K(X) = \mathcal{O}_{X,\xi} = \text{Frac}((f_*\mathcal{O}_X)_{\eta})$ so by the previous lemma,

$$\text{rank}_{\eta}(f_*\mathcal{O}_X) = \dim_{K(Y)} K(X) = [K(X) : K(Y)] = \deg f$$

□

40.1 Finite Locally Free Morphisms

Definition 40.1.1. A morphism $f : X \rightarrow Y$ is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free as a \mathcal{O}_Y -module.

Proposition 40.1.2. A morphism $f : X \rightarrow Y$ is finite locally free iff f is finite, flat, and locally of finite presentation.

Proof. It suffices to show that if $A \rightarrow B$ is finite then B is locally free iff it is flat and finitely presented as an A -module. We know that finite locally free implies flat and locally finitely presented³ (thus finitely presented). Conversely if B is flat and finitely presented⁴ then it is projective (see Tag 00NX) and hence locally free. \square

Proposition 40.1.3. Let $f : X \rightarrow Y$ be a finite flat dominant map of integral schemes. Then for any $y \in Y$ we have,

$$\sum_{x \in f^{-1}(y)} \text{length}_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x}) \cdot [\kappa(x) : \kappa(y)] = \deg f$$

we call $e_x = \text{length}_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x})$ the ramification degree and then,

$$\sum_{x \in f^{-1}(y)} e_x \cdot [\kappa(x) : \kappa(y)] = \deg f$$

Proof. Since $f_* \mathcal{O}_X$ is finite locally free and Y is connected, the sheaf $f_* \mathcal{O}_X$ has constant rank and thus $\text{rank}_y(f_* \mathcal{O}_X) = \text{rank}_\eta(f_* \mathcal{O}_X)$. Using our previous results proves the claim. \square

40.2 Ramification

41 Some General Properties of Morphisms

41.1 Monomorphisms

Remark. Here we work in any category with fiber products.

Definition 41.1.1. A morphism $f : X \rightarrow Y$ is *monic* if $f \circ g_1 = f \circ g_2 \implies g_1 = g_2$.

Proposition 41.1.2. A morphism $f : X \rightarrow Y$ is monic iff $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is an isomorphism.

Proof. Let $f : X \rightarrow Y$ be monic. We have a diagram,

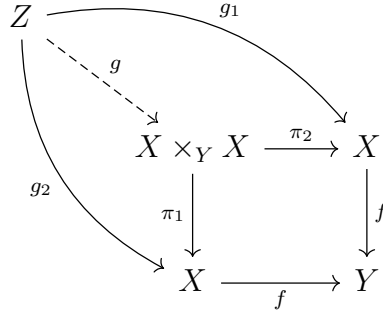
$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\pi_2} & X \\ \pi_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

so $f \circ \pi_1 = f \circ \pi_2$ and thus $\pi_1 = \pi_2$ since f is monic. Furthermore, we know $\pi_i \circ \Delta_{X/Y} = \text{id}_X$ so consider $\Delta_{X/Y} \circ \pi_1 : X \times_Y X \rightarrow X \times_Y X$. Then $\pi_i \circ \Delta_{X/Y} \circ \pi_1 = \pi_1$ because $\pi_i \circ \Delta_{X/Y} = \text{id}_X$ but $\pi_1 = \pi_i$ so we see, $\pi_i \circ (\Delta_{X/Y} \circ \pi_1) = \pi_i$ and thus by the universal property of the product, $\Delta_{X/Y} \circ \pi_1 = \text{id}_{X \times_Y X}$ since they agree after projection. Thus $\Delta_{X/Y}$ is an isomorphism.

Now assume that $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is an isomorphism. Let $g_1, g_2 : Z \rightarrow X$ be morphisms such that $f \circ g_1 = f \circ g_2$. Consider the diagram,

³it is finitely presented as an A -algebra because it is finitely presented as an A -module

⁴There is a subtlety there, B is finitely presented *as an A -algebra* not a priori as an A -module. However, B is a finite A -module so by Tag 0564 B is a finitely presented A -module since $A \rightarrow B$ is a finitely presented ring map and B is trivially a finitely presented B -module.



so we get a unique map $g : Z \rightarrow X \times_Y X$. Consider $\Delta_{Y/S}^{-1} \circ g : Z \rightarrow X$. Since $\pi_i \circ \Delta_{Y/S} = \text{id}_X$,

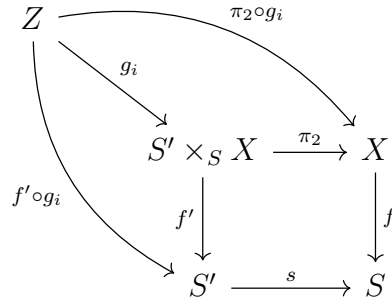
$$\Delta_{Y/S}^{-1} \circ g = (\pi_i \circ \Delta_{Y/S}) \circ (\Delta_{Y/S}^{-1} \circ g) = \pi_i \circ g = g_i$$

and thus $g_1 = g_2 = g$ so f is a mono. □

Lemma 41.1.3. Monomorphisms are preserved under composition and base change.

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be monomorphisms. Suppose that $h_1, h_2 : W \rightarrow X$ are any two morphisms such that $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$. Then $g \circ (f \circ h_1) = g \circ (f \circ h_2)$ so $f \circ h_1 = f \circ h_2$ since g is monic. Then $h_1 = h_2$ since f is monic. Thus $g \circ f$ is monic.

Let $f : X \rightarrow S$ be monic and $s : S' \rightarrow S$ any morphism. Consider the base change,



and two maps $g_1, g_2 : Z \rightarrow S' \times_S X$ such that $f' \circ g_1 = f' \circ g_2$. Then $f \circ \pi_2 \circ g_i = s \circ f' \circ g_i$ and we assume $f' \circ g_1 = f' \circ g_2$ so $f \circ \pi_2 \circ g_1 = f' \circ \pi_2 \circ g_2$. Since f is monic we know $\pi_2 \circ g_1 = \pi_2 \circ g_2$. However, also $f' \circ g_1 = f' \circ g_2$ so $g_1 = g_2$ by uniqueness in the universal property so f' is monic. □

Remark. From now on we work in the category of schemes over S .

Lemma 41.1.4. A monomorphism is separated.

Proof. Let $f : X \rightarrow Y$ be monic. Then $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is an isomorphism and in particular a closed immersion. □

Lemma 41.1.5. Embeddings (immersions) are monomorphisms.

Proof. Both open and closed immersions satisfy the cancellation property (see Tag 01L7). □

Proposition 41.1.6. Let $f : X \rightarrow S$ be separated. Then any immersed subscheme $Z \hookrightarrow X$ (via a locally closed embedding) is separated over S .

Proof. Immersions are monic and thus separated. Furthermore, composition of separated morphisms are separated. □

41.2 Implication of Properties

41.2.1 Categorical Pullback Properties

Remark. These lemmas are categorical results hold in any category with fiber products.

Lemma 41.2.1 (Magic Square). Let $X_1, X_2 \rightarrow Y$ and $Y \rightarrow Z$ be morphisms. The following diagram is a pullback,

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/Z}} & Y \times_Z Y \end{array}$$

Proof. Consider a diagram,

$$\begin{array}{ccc} T & & \\ \downarrow & \searrow \text{dashed} & \downarrow \\ X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/Z}} & Y \times_Z Y \end{array}$$

A map $T \rightarrow X_1 \times_Z X_2$ is a pair of maps $T \rightarrow X_1$ and $T \rightarrow X_2$ agreeing on Z . However, composing with $X_1 \times_Z X_2 \rightarrow Y \times_Z Y$ must agree with $T \rightarrow Y \rightarrow Y \times_Z Y$ and thus the maps $T \rightarrow X_1$ and $T \rightarrow X_2$ must agree on Y so this defines a map $Z \rightarrow X_1 \times_Y X_2$. \square

Lemma 41.2.2. Let $f, g : X \rightarrow Y$ be morphisms over S . Then $Z \rightarrow X$ in the following pullback diagram,

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \Delta_{X/S} \\ X & \xrightarrow{(f,g)} & Y \times_S Y \end{array}$$

is the equalizer of $f, g : X \rightarrow Y$.

Proof. Consider a morphism $h : T \rightarrow X$ such that $f \circ h = g \circ h$. Then $(f \circ h, g \circ h) : T \rightarrow Y \times_S Y$ equals $(f \circ h, f \circ h) = \Delta_{X/S} \circ f \circ h$ giving a commutative diagram,

$$\begin{array}{ccccc}
T & & & & \\
\swarrow \text{---} & & & \searrow & \\
& Z & \xrightarrow{\quad} & Y & \\
& \downarrow \lrcorner & & \downarrow \Delta_{X/S} & \\
& X & \xrightarrow{(f,g)} & Y \times_S Y &
\end{array}$$

There is thus a unique map $T \rightarrow Z$ such that $T \rightarrow Z \rightarrow X$ is h proving that Z is the equalizer. \square

Corollary 41.2.3. Given S -morphisms $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ the morphism $Z \rightarrow X_1 \times_S X_2$ in the pullback diagram,

$$\begin{array}{ccc}
Z & \longrightarrow & X_1 \times_S X_2 \\
\downarrow \lrcorner & & \downarrow \\
Y & \xrightarrow{\Delta_{Y/Z}} & Y \times_S Y
\end{array}$$

is the equalizer of $f_1 \circ \pi_1, f_2 \circ \pi_2 : X_1 \times_S X_2 \rightarrow Y$. Giving three descriptions of $Z \rightarrow X_1 \times_S X_2$,

- (a) the natural map $X_1 \times_Y X_2 \rightarrow X_1 \times_S X_2$
- (b) the base change / pullback of $\Delta_{Y/S}$ along $f_1 \times f_2$
- (c) the equalizer of $f \circ \pi_1, g \circ \pi_2 : X_1 \times_S X_2 \rightarrow Y$.

Corollary 41.2.4. Let $f : X \rightarrow Y$ be a morphism over S . Then the graph $\Gamma_f : X \rightarrow X \times_S Y$ is the base change of the diagonal $\Delta_{Y/S} : Y \rightarrow Y \times_S Y$ by $f \times \text{id}_Y$.

Proof. The diagram,

$$\begin{array}{ccc}
X & \xrightarrow{\Gamma_f} & X \times_S Y \\
\downarrow f & & \downarrow f \times \text{id}_Y \\
Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y
\end{array}$$

is cartesian (a pullback) by the magic diagram applied to $f : X \rightarrow Y$ and $\text{id}_Y : Y \rightarrow Y$ over S . \square

Remark. Furthermore, the graph $\Gamma_f : X \rightarrow X \times_S Y$ is the equalizer of $f \circ \pi_1, \pi_2 : X \times_S Y \rightarrow Y$.

41.2.2 Properties of The Graph Morphism of Schemes

Lemma 41.2.5. Let $f : X \rightarrow Y$ be unramified over S . Then $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is an open immersion.

Proof. Tag 02GE. \square

Corollary 41.2.6. Let $f : X \rightarrow Y$ be a morphism over S . In general, $\Gamma_f : X \rightarrow X \times_S Y$ is a locally closed immersion. Furthermore,

- (a) if $Y \rightarrow S$ is quasi-separated then $\Gamma_f : X \rightarrow X \times_S Y$ is quasi-compact
- (b) if $Y \rightarrow S$ is affine-separated then $\Gamma_f : X \rightarrow X \times_S Y$ is affine
- (c) if $Y \rightarrow S$ is separated then $\Gamma_f : X \rightarrow X \times_S Y$ is a closed immersion
- (d) if $Y \rightarrow S$ is unramified then $\Gamma_f : X \rightarrow X \times_S Y$ is an open immersion.

Proof. These properties are preserved under base change and the diagonal $\Delta_{Y/S} : Y \rightarrow Y \times_S Y$ has the corresponding property for each case and is, in general, a locally closed immersion. \square

41.2.3 Implication of Properties

Remark. The following hold in any category with fiber products.

Proposition 41.2.7. Let \mathcal{P} be a property of morphisms which is preserved under composition and base change. Then the product of two morphisms with property \mathcal{P} has property \mathcal{P} .

Proof. Let $f : X_1 \rightarrow Y_1$ and $g : X_2 \rightarrow Y_2$ be morphisms over S with \mathcal{P} . We factor the product, $f \times g : X_1 \times_S X_2 \rightarrow Y_1 \times_S Y_2$ as,

$$X_1 \times_S Y_1 \xrightarrow{f \times \text{id}_{Y_1}} X_2 \times_S Y_1 \xrightarrow{\text{id}_{X_2} \times g} X_2 \times_S Y_2$$

Both morphisms have \mathcal{P} by base change so $f \times g$ does as well by composition. \square

Proposition 41.2.8. Let \mathcal{P} be a property of morphisms which is preserved under composition and base change. Consider a morphism $f : X \rightarrow Y$ of S -objects giving a diagram,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & S & \end{array}$$

If p has property \mathcal{P} and the diagonal $\Delta_{Y/S}$ of q has property \mathcal{P} then f has property \mathcal{P} .

Proof. The morphism $f : X \rightarrow Y$ factors through the graph $\Gamma_f : X \rightarrow X \times_S Y$ giving the following diagram,

$$\begin{array}{ccccc} & & X \times_S Y & & \\ & \nearrow \Gamma_f & & \searrow \pi_2 & \\ X & & & & Y \\ & \searrow p & \xrightarrow{f} & \swarrow q & \\ & & S & & \end{array}$$

Since Γ_f is a base change of $\Delta_{Y/S} : Y \rightarrow Y \times_S Y$, it has property \mathcal{P} . Since $\pi_2 : X \times_S Y \rightarrow Y$ is the base change of $p : X \rightarrow S$ by $q : Y \rightarrow S$ so it has property \mathcal{P} . Composing, f has property \mathcal{P} . \square

41.2.4 Application to the Category of Schemes

Remark. From now on we work in the category of schemes over S .

Proposition 41.2.9. Let \mathcal{P} be a property of morphisms such that,

- (a) \mathcal{P} is preserved under composition
- (b) \mathcal{P} is preserved under base change
- (c) any closed immersion has \mathcal{P}

then if $f : X \rightarrow Y$ has \mathcal{P} its reduction $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ also has \mathcal{P} .

Proof. Consider the diagram,

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ \pi_X \downarrow & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

where the downward maps are closed immersions which exists because $X_{\text{red}} \rightarrow X \rightarrow Y$ is a map from a reduced scheme to Y and thus factors through $Y_{\text{red}} \rightarrow Y$. Then consider,

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ & \searrow f \circ \pi_X & \swarrow \pi_Y \\ & Y & \end{array}$$

Since π_Y is a closed immersion and thus separated so $\Delta_{Y_{\text{red}}/Y}$ is a closed immersion so it has \mathcal{P} . Furthermore, $f \circ \pi_X$ has \mathcal{P} because f does by hypothesis and π_X does because it is a closed immersion and \mathcal{P} is closed under composition. Therefore, by the main lemma, f_{red} has \mathcal{P} . \square

Corollary 41.2.10. Let Y be an S -scheme and X a Y -scheme.

- (a) if $X \rightarrow S$ is separated then $X \rightarrow Y$ is separated.
- (b) if $Y \rightarrow S$ is separated then $X \rightarrow Y$ is separated $\iff X \rightarrow S$ is separated.

Proof. We have,

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & S & \end{array}$$

The diagonal $\Delta_{Y/S} : Y \rightarrow Y \times_S Y$ is, in general, an immersion and thus monic and thus separated. Furthermore, $X \rightarrow S$ is separated so by the main lemma $X \rightarrow Y$ is separated.

Conversely, if $Y \rightarrow S$ is separated and $X \rightarrow Y$ is separated then $X \rightarrow S$ is separated by composition. \square

Corollary 41.2.11. Let $f : X \rightarrow Y$ be a morphism of S -schemes. Then, if the structue morphism $X \rightarrow S$ is any of the following,

- (a) an immersion

- (b) locally of finite type
- (c) quasi-separated

then $f : X \rightarrow Y$ satisfies those same properties.

Proof. All properties here are preserved under composition and base change. Furthermore, the diagonal is an immersion (in particular locally of finite type and quasi-separated) giving the first two. The diagonal is an immersion and thus monic and thus separated. In particular the diagonal is generally quasi-separated so quasi-separated maps satisfy the cancellation property. \square

Corollary 41.2.12. Let $f : X \rightarrow Y$ be a morphism of S -schemes with $Y \rightarrow S$ quasi-separated. Then, if the structure morphism $X \rightarrow S$ is any of the following,

- (a) quasi-compact
- (b) finite type
- (c) quasi-projective

then $f : X \rightarrow Y$ satisfies those same properties.

Proof. If $Y \rightarrow S$ is quasi-separated then (by definition) the diagonal is a quasi-compact immersion and thus finite type and quasi-projective so the cancellation property holds for quasi-compact, finite type, and quasi-projective morphisms. \square

Corollary 41.2.13. Morphism of varieties over k are finite type and separated.

Proof. A morphism $f : X \rightarrow Y$ of varieties over $\text{Spec}(k)$ is finite type since X is finite type and separated since X is separated. \square

Corollary 41.2.14. Let $f : X \rightarrow Y$ be a morphism of S -schemes with $Y \rightarrow S$ separated. Then, if the structure morphism $X \rightarrow S$ is any of the following,

- (a) a closed immersion
- (b) proper
- (c) projective
- (d) finite
- (e) universally closed

then $f : X \rightarrow Y$ satisfies those same properties.

Proof. Since $Y \rightarrow S$ is separated, the diagonal $\Delta : Y \rightarrow Y \times_S Y$ is a closed immersion. Furthermore, closed immersions satisfy all the properties listed above and thus give the cancellation property for such morphisms $X \rightarrow S$. \square

Corollary 41.2.15. Let $f : X \rightarrow S$ be a morphism and $\sigma : S \rightarrow X$ a section ($f \circ \sigma = \text{id}_S$). Then the section $\sigma : S \rightarrow X$ is an immersion. Furthermore,

- (a) if $f : X \rightarrow S$ is quasi-separated then $\sigma : S \rightarrow X$ is a quasi-compact immersion

- (b) if $f : X \rightarrow S$ is separated then $\sigma : S \rightarrow X$ is a closed immersion
- (c) if $f : X \rightarrow S$ is unramified then $\sigma : S \rightarrow X$ is an open immersion

Proof. Consider the diagram,

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & X \\ & \searrow \text{id} & \swarrow f \\ & S & \end{array}$$

Then the result follows from above. \square

Corollary 41.2.16. Let $f : X \rightarrow S$ be separated and unramified with S connected. Let $\sigma : S \rightarrow X$ be a section, $f \circ \sigma = \text{id}_S$, then $\sigma : S \rightarrow X$ is an isomorphism onto a connected component of X .

Proof. By above, $\sigma : S \rightarrow X$ is an open and closed embedding so σ is an isomorphism onto a clopen connected (since S is connected) subscheme which must be a connected component. \square

Proposition 41.2.17. Let $q : Y \rightarrow S$ be unramified and separated. Given two S -morphisms $f, g : X \rightarrow Y$ from a connected scheme X and some point $x \in X$ such that,

- (a) $f(x) = g(x) = y$
- (b) $f^\#, g^\# : \kappa(y) \rightarrow \kappa(x)$ agree

then $f = g$.

Proof. Consider the locus Z on which $f, g : X \rightarrow Y$ agree i.e. the equalizer of $f, g : X \rightarrow Y$,

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow \iota & & \downarrow \Delta_{Y/S} \\ X & \xrightarrow{(f,g)} & Y \times_S Y \end{array}$$

Since $q : Y \rightarrow S$ is unramified and separated the diagonal $\Delta_{Y/S} : Y \rightarrow Y \times_S Y$ is an open and closed immersion. Thus, by base change $\iota : Z \hookrightarrow X$ is an open and closed immersion. Since X is connected, if Z is nonempty then $\iota : Z \rightarrow X$ is an isomorphism which shows that $f = g$. Consider the point $\text{Spec}(\kappa(x)) \rightarrow X$. But Z is the equalizer of $f, g : X \rightarrow Y$ so $\text{Spec}(\kappa(x)) \rightarrow X$ factors through $Z \rightarrow X$ iff $f, g : \text{Spec}(\kappa(x)) \rightarrow Y$ are equal which is ensured by the hypotheses so Z has a point and thus $\iota : Z \xrightarrow{\sim} X$ is an isomorphism showing that $f = g$. \square

42 Quasi-Projective Morphisms

Definition 42.0.1. A morphism $f : X \rightarrow S$ is *quasi-projective* / *projective* if there exists an quasi-compact immersion / closed immersion $X \hookrightarrow \mathbb{P}_S^n$ over S .

Lemma 42.0.2. Projective and quasi-projective morphisms are preserved under composition and base change.

Proof. \square

Definition 42.0.3. We say a k -variety is *quasi-projective* / *projective* if $X \rightarrow \text{Spec}(k)$ is.

Proposition 42.0.4. Let $X \rightarrow S$ be quasi-projective. Then $X \rightarrow S$ is separated and quasi-compact.

Proof. Follows from the fact that $\mathbb{P}_S^n \rightarrow S$ is separated and quasi-compact and that quasi-compact immersions are as well. \square

Proposition 42.0.5. Let $X \rightarrow S$ be projective. Then $X \rightarrow S$ is proper.

Proof. Follows from the fact that $\mathbb{P}_S^n \rightarrow S$ is proper and that closed immersions are proper as well. \square

42.1 Proper and Universally Closed Morphisms

Definition 42.1.1. A morphism $f : X \rightarrow S$ is *universally closed* if for any $T \rightarrow S$ the base change $f_T : X \times_S T \rightarrow T$ is closed.

Definition 42.1.2. A morphism $f : X \rightarrow Y$ is *proper* if it is finite type, separated, and universally closed.

Lemma 42.1.3. Universally closed morphisms are preserved under composition and under arbitrary base change.

Proof. The composition of closed morphisms is closed and composition is preserved under base change. Furthermore, universally closed morphisms are obviously preserved under base change. \square

Proposition 42.1.4. Consider a morphism $f : X \rightarrow Y$ of S -schemes giving a diagram,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & S & \end{array}$$

If $p : X \rightarrow S$ is universally closed and $f : X \rightarrow Y$ is surjective then $q : Y \rightarrow S$ is universally closed.

Proof. The base change of a surjective morphism is surjective (Tag 01S1). Then under any base change $T \rightarrow S$ we get the diagram,

$$\begin{array}{ccc} X_T & \xrightarrow{f_T} & Y_T \\ & \searrow p_T & \swarrow q_T \\ & T & \end{array}$$

satisfying the hypotheses. Therefore, it suffices to show that $q : Y \rightarrow S$ is closed. Take a closed $C \subset Y$ then, because f is surjective, $q(C) = q(f^{-1}(C))$ is closed. \square

Corollary 42.1.5. Consider a morphism $f : X \rightarrow Y$ of S -schemes giving a diagram,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & S & \end{array}$$

If we have,

- (a) $p : X \rightarrow S$ is proper
- (b) $f : X \rightarrow Y$ is surjective
- (c) $q : Y \rightarrow S$ is finite type and separated

then both $f : X \rightarrow Y$ and $q : Y \rightarrow S$ are proper.

Proof. By above, $q : Y \rightarrow S$ is universally closed and, by assumption, finite type and separated and thus proper. Furthermore, since $q : Y \rightarrow S$ is separated, by the previous discussion $f : X \rightarrow Y$ is proper. \square

Remark. This gives a proof of Hartshorne II.4.4.

43 Interesting Flasque Resolutions on Curves

43.1 Godement Resolution

For any abelian sheaf \mathcal{F} on a space X we can consider its Godement resolution. Abstractly, take the continuous map $f : X_{\text{dis}} \rightarrow X$ from X given the discrete topology. Then the first stage of the Godement resolution is,

$$\mathcal{F} \rightarrow f_* f^* \mathcal{F}$$

Furthermore, since $f^* \mathcal{F}$ is an abelian sheaf on a discrete space it is flasque and f_* preserves flasqueness so $f_* f^* \mathcal{F}$ is flasque. Continuing gives a cosimplicial sheaf $\mathcal{G}^p(\mathcal{F}) = (f_* f^*)^p \mathcal{F}$ on X with coface maps given by the natural transformation $\text{id} \rightarrow f_* f^*$ and codegeneracy maps given by contracting between pairs $(f_* f^*)(f_* f^*)$ via the natural transformation $f^* f_* \rightarrow \text{id}$. The associated complex is then a flasque resolution of \mathcal{F} .

Remark. The above construction also works in the category of \mathcal{O}_X -modules on a ringed space by pulling back to $(X_{\text{dis}}, \mathcal{O}_{X_{\text{dis}}})$ where $\mathcal{O}_{X_{\text{dis}}} = f^{-1} \mathcal{O}_X$.

Lemma 43.1.1. Let \mathcal{F} be a sheaf on a discrete space X . Then \mathcal{F} is flasque and the canonical map,

$$\mathcal{F} \rightarrow \prod_{x \in X} (\iota_x)_* (\mathcal{F}_x)$$

is an isomorphism.

Proof. Let $U \subset X$ be open (any set since X is discrete) then since points are open the set of points $x \in U$ forms an open cover. Then by the sheaf property,

$$\mathcal{F}(U) \rightarrow \prod_{x \in U} \mathcal{F}(x)$$

is an isomorphism. Furthermore, clearly $\mathcal{F}(x) = \mathcal{F}_x$ since x is the initial object in the poset of open neighborhoods of x . Furthermore, the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective because for any section $s \in \mathcal{F}(V)$ we may extend to a global section by setting $f_x = s_x$ for $x \in V$ and $f_x = 0$ for $x \notin V$. clearly $f_x = s_x$ on V so by the sheaf property $f|_V = s$. Then restricting $f|_U$ shows that $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective. \square

Thus, we can alternatively describe the Godement operation as follows. We can consider,

$$X_{\text{dis}} = \coprod_{x \in X} x$$

Then,

$$f^* X = \prod_{x \in X} \mathcal{F}_x$$

and $f : X_{\text{dis}} \rightarrow X$ is the bundled collection of the inclusions $\iota_x : x \rightarrow X$ giving,

$$f_* f^* \mathcal{F} = \prod_{x \in X} (\iota_x)_* (\mathcal{F}_x)$$

reproducing the result on a discrete space.

43.2 Subsheaves of Godement

Now consider the diagram,

$$\begin{array}{ccc} & & \mathcal{F} \\ & \swarrow \text{dashed} & \downarrow \\ \bigoplus_{x \in X} (\iota_x)_* (\mathcal{F}_x) & \longrightarrow & \prod_{x \in X} (\iota_x)_* (\mathcal{F}_x) \end{array}$$

We ask when the inclusion $\mathcal{F} \rightarrow \mathcal{G}^1(\mathcal{F})$ factors through the canonical map,

$$\bigoplus_{x \in X} (\iota_x)_* (\mathcal{F}_x) \rightarrow \prod_{x \in X} (\iota_x)_* (\mathcal{F}_x)$$

and when this sheaf or its image subsheaf is flasque.

First, note that direct sums commute with colimits (because they are colimits themselves) and thus denoting,

$$H(\mathcal{F}) = \bigoplus_{x \in X} (\iota_x)_* (\mathcal{F}_x)$$

we have the stalks,

$$\begin{aligned} H(\mathcal{F})_x &= \varinjlim_{x \in U} H(\mathcal{F})(U) = \bigoplus_{y \in X} \varinjlim_{x \in U} \begin{cases} \mathcal{F}_y & y \in U \\ 0 & y \notin U \end{cases} \\ &= \bigoplus_{y \in X} \begin{cases} \mathcal{F}_y & x \in \overline{\{y\}} \\ 0 & x \notin \overline{\{y\}} \end{cases} \\ &= \bigoplus_{y \rightsquigarrow x} \mathcal{F}_y \end{aligned}$$

Therefore, if \mathcal{F} is supported only on closed points of X we have,

$$H(\mathcal{F})_x = \mathcal{F}_x$$

However, in general there is not a sheaf map $\mathcal{F} \rightarrow H(\mathcal{F})$.

Suppose that \mathcal{F} has finitely supported sections meaning that for any $s \in \mathcal{F}(U)$ its support,

$$\text{Supp}(s) = \{x \in X \mid s_x \neq 0\}$$

is finite. Then we get an injection,

$$\mathcal{F} \hookrightarrow \bigoplus_{x \in X} (\iota_x)_*(\mathcal{F}_x)$$

by mapping for each s ,

$$s \in \mathcal{F} \hookrightarrow \prod_{x \in \text{Supp}(s)} (\iota_x)_*(\mathcal{F}_x) = \bigoplus_{x \in \text{Supp}(s)} (\iota_x)_*(\mathcal{F}_x) \subset \bigoplus_{x \in X} (\iota_x)_*(\mathcal{F}_x)$$

Furthermore, notice that if \mathcal{F} is only supported at closed points then,

$$H(\mathcal{F})_x = \bigoplus_{y \rightsquigarrow x} \mathcal{F}_y = \mathcal{F}_x$$

since $\mathcal{F}_y = 0$ for any generalization of x . Therefore, in this case the map $\mathcal{F} \rightarrow H(\mathcal{F})_x$ defined by virtue of sections having finite support is an isomorphism. Thus if \mathcal{F} is a abelian sheaf whose sections have finite support which is supported on the closed points then,

$$\mathcal{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathcal{F}_x)$$

43.3 The Case for Curves

Let X be a curve (separated integral Noetherian scheme of dimension one) with generic point $\xi \in X$. Then I claim any torsion sheaf \mathcal{F} satisfies,

$$\mathcal{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathcal{F}_x)$$

By the previous discussion, it suffices to show that \mathcal{F} is supported at closed points any every section has finite support. The only nonclosed point is ξ and we assumed that $\mathcal{F}_\xi = 0$. Furthermore, consider $s \in \mathcal{F}(U)$. We know $s_\xi = 0$ so there is some open V such that $\xi \in V \subset U$ on which $s|_V = 0$. Therefore $\text{Supp}(s) \subset V^c$. I claim that $V^c \subset X$ is finite. Since X is quasi-compact, we can choose an affine open cover $U_i = \text{Spec}(A_i)$ and $V^c \cap U_i = V(I_i)$ for some ideal $I_i \subset A_i$. It suffices to show that $V(I_i)$ is finite. Note that $\dim A_i \leq 1$ and X is irreducible so $\text{codim}(V^c, X) \geq 1$ and therefore $\dim V^c = 0$ because,

$$\dim X \geq \text{codim}(V^c, X) + \dim V^c$$

This shows that $\dim A_i/I_i = 0$ and it is Noetherian so A_i/I_i is Artinian and thus $V(I_i) = \text{Spec}(A_i/I_i)$ is finite.

Therefore, each section has finite support so we have demonstrated the equality,

$$\mathcal{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathcal{F}_x)$$

for any torsion sheaf ($\mathcal{F}_\xi = 0$).

43.4 Resolutions on Curves

Consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{K}_X/\mathcal{O}_X \longrightarrow 0$$

Notice that $(\mathcal{K}_X/\mathcal{O}_X)_\xi = K(X)/\mathcal{O}_{X,\xi} = 0$ so $\mathcal{K}_X/\mathcal{O}_X$ is torsion. Therefore, we get a sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K}_X \longrightarrow \bigoplus_{x \in X} (\iota_x)_*(K(X)/\mathcal{O}_{X,x}) \longrightarrow 0$$

Since X is integral \mathcal{K}_X is constant (since all opens are connected it is truly constant) and thus we get a flasque resolution of \mathcal{O}_X . Then the long exact sequence gives,

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow K(X) \longrightarrow \bigoplus_{x \in X} K(X)/\mathcal{O}_{X,x} \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

and $H^i(X, \mathcal{O}_X) = 0$ for $i > 1$. Furthermore, for any flat sheaf \mathcal{F} , we can tensor the above exact sequence to get,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X \longrightarrow \bigoplus_{x \in X} (\iota_x)_*(\mathcal{F}_\xi/\mathcal{F}_x) \longrightarrow 0$$

Where $(\iota_x)_*(K(X)) \otimes_{\mathcal{O}_X} \mathcal{F} =$

Lemma 43.4.1. Let X be an irreducible scheme with generic point $\xi \in X$ and \mathcal{F} an abelian sheaf on X . Then the natural map,

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X \rightarrow (\iota_\xi)_*(\mathcal{F}_\xi)$$

is an isomorphism.

Proof. Locally, on affine opens □

44 Submodules of Locally Free Modules

Lemma 44.0.1. Let M be a finitely presented R -module such that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module for each prime $\mathfrak{p} \in \text{Spec}(R)$. Then M is a finite locally free R -module.

Proof. Lifting a basis of $M_{\mathfrak{p}}$ gives an exact sequence,

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow C \longrightarrow 0$$

Since M is finitely presented, both K and C are finite R -modules. Furthermore, localizing the sequence at \mathfrak{p} gives,

$$0 \longrightarrow K_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}^n \xrightarrow{\sim} M_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0$$

so $K_{\mathfrak{p}} = 0$ and $C_{\mathfrak{p}} = 0$ by exactness. However, since both are finite R -modules there exists some $f \notin \mathfrak{p}$ such that $K_f = 0$ and $C_f = 0$ and thus by the exact sequence,

$$0 \longrightarrow K_f \longrightarrow R_f^n \longrightarrow M_f \longrightarrow C_f \longrightarrow 0$$

we get $R_f^n \xrightarrow{\sim} M_f$ showing that M is locally free. □

Lemma 44.0.2. Let X be a scheme and \mathcal{F} be a coherent \mathcal{O}_X -module such that \mathcal{F}_x is a free $\mathcal{O}_{X,x}$ -module for each $x \in X$. Then \mathcal{F} is finite locally free.

Proof. On an affine open cover $U = \operatorname{Spec}(A)$ we know $\mathcal{F}|_U = \widetilde{M}$ for some finitely presented A -module M . Then for $x = \mathfrak{p} \in \operatorname{Spec}(A)$, we have $\mathcal{F}_x = M_{\mathfrak{p}}$ is a free (and finite since M is finite) $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ -module. Thus by the previous lemma, M is a locally free module and thus $\mathcal{F}|_U$ is finite locally free showing that \mathcal{F} is finite locally free. \square

Example 44.0.3. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves with \mathcal{F} finite locally free. Then $\mathcal{K} = \ker \varphi$ need not be finite locally free. This is because stalkwise $\mathcal{K}_x \subset \mathcal{F}_x$ and \mathcal{F}_x is finite free but if the local ring $\mathcal{O}_{X,x}$ is not a PID in general \mathcal{K}_x need not be a free module. However, if X is a regular curve then each $\mathcal{O}_{X,x}$ is a DVR and thus a PID so indeed submodules of finite free modules are finite free. Furthermore, since X is noetherian \mathcal{K} is coherent and thus finite locally free since \mathcal{K}_x is a finite free $\mathcal{O}_{X,x}$ -module.

Remark. From now on let (X, \mathcal{O}_X) be a locally ringed space such that \mathcal{O}_X is coherent. On such a space, \mathcal{F} is coherent iff it is locally of finite presentation as an \mathcal{O}_X -module (see Tag 01BZ).

Proposition 44.0.4. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ an epimorphism of finite locally free \mathcal{O}_X -modules. Then the kernel $\mathcal{K} = \ker \varphi$ is finite locally free.

Proof. Since \mathcal{K} is a kernel of coherent sheaves, \mathcal{K} is coherent. The exact sequence of sheaves,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

gives an exact sequence on stalks,

$$0 \longrightarrow \mathcal{K}_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow 0$$

Since \mathcal{F}_x and \mathcal{G}_x are finite free $\mathcal{O}_{X,x}$ -modules they are projective and thus \mathcal{K}_x is also finite projective (see next lemma). Since $\mathcal{O}_{X,x}$ is local, finite projective $\mathcal{O}_{X,x}$ -modules are finite free. Thus \mathcal{F} is a coherent \mathcal{O}_X -module with \mathcal{F}_x a free $\mathcal{O}_{X,x}$ -module so \mathcal{F} is finite locally free by the previous lemma. \square

Lemma 44.0.5. Every summand of a projective object is projective.

Proof. Suppose that $P = K \oplus L$ is projective. Given a surjection $A \twoheadrightarrow B$ and a map $K \rightarrow B$ we get a diagram,

$$\begin{array}{ccc} P & \overset{\quad}{\dashrightarrow} & A \\ & \swarrow \iota & \nearrow \\ & K & \\ & \swarrow \text{id} & \searrow \\ K & \xrightarrow{\quad} & B \end{array}$$

so $K \rightarrow B$ lifts to $K \rightarrow A$ showing that K is projective. \square

Corollary 44.0.6. Suppose that $P \twoheadrightarrow Q$ is an epimorphism of projective modules then the kernel $K = \ker(P \rightarrow Q)$ is projective.

Proof. There is an exact sequence,

$$0 \longrightarrow K \longrightarrow P \longrightarrow Q \longrightarrow 0$$

But Q is projective so this sequence splits giving $P = K \oplus Q$. But P is projective and thus K is projective. \square

Proposition 44.0.7. Let $\iota : X \hookrightarrow Y$ be a closed immersion of schemes smooth over locally noetherian S . Then $\mathcal{C}_{X/Y}$ is finite locally free (i.e. a vector bundle).

Proof. There is an exact sequence of sheaves,

$$0 \longrightarrow \mathcal{C}_{X/Y} \longrightarrow \iota^* \Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

which is exact on the left because $X \rightarrow S$ is smooth. Furthermore, since X and Y are smooth over S we know $\Omega_{Y/S}$ and $\Omega_{X/S}$ are finite locally free. Furthermore, X and Y are finitely presented over S which is locally noetherian so X and Y are locally noetherian and thus \mathcal{O}_X and \mathcal{O}_Y are coherent. Thus applying the previous results, $\mathcal{C}_{X/Y}$ is finite locally free. \square

44.1 Computing (Co)Normal Bundles

45 Canonical Forms

Definition 45.0.1. Let M be a smooth manifold and let $P = TM$ and $Q = T^*M$ denote its tangent bundle and cotangent bundle. Then there are canonical objects:

- (a) a vector field $V \in \Gamma(P, TP)$
- (b) a 1-form $\theta \in \Gamma(Q, T^*Q)$

defined as follows. Choose $(x, v) \in P$ where $x \in M$ and $v \in T_x M$ then the projection $\pi : T^*P \rightarrow P$ induces $d\pi : TP \rightarrow TM$ so

$$v \in TP$$

For the canonical 1-form, consider a point $(x, \varphi) \in Q$ where $x \in M$ and $\varphi : T_x M \rightarrow \mathbb{R}$. Using the projection $\pi : Q \rightarrow M$ we define $\theta_{(x, \varphi)} = \varphi \circ d\pi_{(x, \varphi)} : T_m Q \rightarrow \mathbb{R}$ giving a smooth section $\theta : Q \rightarrow T^*Q$. We describe this as the 1-form given at (x, φ) by pulling back φ along π .

Remark. For a chart $\varphi : U \rightarrow M$ we get coordinate functions $x_i = \varphi^{-1} \circ \pi_i : M \rightarrow \mathbb{R}$ and a basis of vector fields $\frac{\partial}{\partial x^i} = d\varphi \left(\frac{\partial}{\partial x^i} \right)$ and a basis of 1-forms $dx^i = \varphi^* dx^i$.

Remark. Therefore, we get a chart for P ,

$$\varphi : U \times \mathbb{R}^n \rightarrow P \quad \text{via} \quad (\{x_i\}, \{v^i\}) \mapsto \left(\varphi(\{x_i\}), v^i \frac{\partial}{\partial x^i} \right)$$

and a chart for Q ,

$$\varphi : U \times \mathbb{R}^n \rightarrow Q \quad \text{via} \quad (\{x_i\}, \{p_i\}) \mapsto (\varphi(\{x_i\}), p_i dx^i)$$

Consider two charts $\varphi : U \rightarrow M$ and

46 There is only one conic in \mathbb{P}^2

Theorem 46.0.1. All irreducible conics in \mathbb{P}^2 are related by an automorphism of \mathbb{P}^2 .

Proof. By definition, a conic is a degree 2 subscheme $C \hookrightarrow \mathbb{P}^2$ thus defined by a section of $\mathcal{O}_{\mathbb{P}^2}(2)$. Thus, if irreducible it is a curve of arithmetic genus 0. Normalizing gives a degree two map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ with $f^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_{\mathbb{P}^1}(2)$ (since $\mathcal{O}_{\mathbb{P}^1}(2)$ is the unique degree 2 line bundle). Now I claim the pullback map $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ is an isomorphism. Indeed, pushing forward we get a map $\mathcal{O}_{\mathbb{P}^2}(1) \rightarrow f_*f^*\mathcal{O}_{\mathbb{P}^2}(1)$ and by the projection formula $f_*f^*\mathcal{O}_{\mathbb{P}^2}(1) = f_*\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{\mathbb{P}^2}(1)$ which fits into the exact sequence,

$$0\mathcal{O}_{\mathbb{P}^2}(1-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_C(1) \longrightarrow 0$$

giving a long exact sequence containing,

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1-d)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow H^0(C, \mathcal{O}_C(1)) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1-d))$$

for $d > 1$ we get $H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1-d)) = 0$ for $i \leq 1$ and thus an isomorphism,

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \xrightarrow{\sim} H^0(C, \mathcal{O}_C(1))$$

given by pulling back. Any map $\mathbb{P}^1 \rightarrow \mathbb{P}^2$ of degree two is defined by a basis of sections of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ (since we showed the above map is surjective and has the correct dimension 3) and any two bases are related by PGL_3 giving an automorphism of \mathbb{P}^2 . \square

47 Connected Etale Covers

Remark. Integrality is not purely local in the sense that $\mathcal{O}_{X,x}$ being a domain does not imply that X is integral nor does having an affine open cover by domains. For example, take $\mathbb{A}^1 \coprod \mathbb{A}^1$.

Lemma 47.0.1. Let X be a connected scheme with finitely many irreducible components such that $\mathcal{O}_{X,x}$ is a domain at each $x \in X$. Then X is integral.

Proof. Since X is automatically reduced we must show X is irreducible. Let $Z_1, \dots, Z_n \subset X$ be the irreducible components. It suffices to show that $Z_i \cap Z_j = \emptyset$ for $i \neq j$ since in that case,

$$Z_1 \cap \left(\bigcup_{j=2}^n Z_j \right) = \emptyset$$

and because there are finitely many,

$$Y = \bigcup_{j=2}^n Z_j$$

is closed and $Z_1 \cup Y = X$. Thus either $Z_1 = X$ or $Y = X$ by connectedness. Proceeding we see that $Z_i = X$ for some Z_i so X has a single irreducible component i.e. X is irreducible.

Suppose otherwise that $p \in Z_i \cap Z_j$. Then Z_i and Z_j correspond to distinct minimal primes in $\mathcal{O}_{X,p}$ contradicting the fact that $\mathcal{O}_{X,p}$ is a domain and thus has a unique minimal prime. \square

Lemma 47.0.2. Let $f : X \rightarrow Y$ be an étale morphism of locally Noetherian schemes and $f(x) = y$. Then $\mathcal{O}_{X,x}$ is a normal domain iff $\mathcal{O}_{Y,y}$ is a normal domain.

Proof. See Tag 025P and note that $f : X \rightarrow Y$ is étale at x iff $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is an étale homomorphism of local rings. \square

Proposition 47.0.3. Let Y be a normal Noetherian integral scheme and $f : X \rightarrow Y$ a quasi-compact étale cover with X connected. Then X is a normal Noetherian integral scheme.

Proof. Since $X \rightarrow Y$ is finite presentation and Y is Noetherian then X is Noetherian. Furthermore, by the previous result we have $\mathcal{O}_{X,x}$ is a normal domain for each $x \in X$, in particular a domain. Therefore, since X is connected and Noetherian we find that X is integral by previous results and normal by definition. \square

Corollary 47.0.4. Let X be a normal Noetherian integral scheme. Then quasi-compact étale covers by normal Noetherian integral schemes are cofinal in $X_{\text{ét}}$.

Proof. For any quasi-compact étale cover $f : \tilde{X} \rightarrow X$ since X is Noetherian and f is finitely presented \tilde{X} is Noetherian and thus has finitely many connected components U_i which are thus clopen. Then $U_i \hookrightarrow \tilde{X}$ are open immersions and thus étale. Furthermore, \tilde{X} is Noetherian so $U_i \hookrightarrow \tilde{X}$ is quasi-compact. Thus, composing gives a quasi-compact étale cover $U_i \rightarrow X$ with U_i connected so, by the previous proposition, the U_i are normal Noetherian integral schemes. \square

Example 47.0.5. The normality assumption is necessary. Indeed let,

$$X = \text{Spec}(k[x, y]/(y^2 - x^2(x + 1)))$$

be an affine nodal cubic which is a curve (in particular Noetherian and integral). Then consider the cover $\tilde{X} = \mathbb{A}^1 \coprod_{\{\pm 1\}} \mathbb{A}^1$ (glued at 1 and -1) where $\mathbb{A}^1 \rightarrow X$ is the normalization map. We need to show that this map is étale. Notice that $\mathbb{A}^1 \rightarrow X$ is not flat at $1 \mapsto (0, 0)$ and $-1 \mapsto (0, 0)$. However, $\mathcal{O}_{X,(0,0)} \rightarrow \mathcal{O}_{\tilde{X},1}$ and $\mathcal{O}_{X,(0,0)} \rightarrow \mathcal{O}_{\tilde{X},-1}$ are isomorphisms because these local rings are both isomorphic to $k[x, y]/(xy)$ (note the change of variables $y^2 - x^2 = (u + v)^2 - (u - v)^2 = 4uv$). However, this cover is not irreducible.

48 Hilbert Spaces

48.1 Basic Definitions

Definition 48.1.1. A *norm* on a \mathbb{R} -vector space V is a function $\|\bullet\| : V \rightarrow \mathbb{R}$ such that,

- (a) $\|v\| \geq 0$
- (b) $\|v\| = 0 \iff v = 0$
- (c) $\|v + u\| \leq \|v\| + \|u\|$
- (d) $\|\alpha v\| = \alpha\|v\|$ for any $\alpha \in \mathbb{R}$.

Remark. These imply that $d(u, v) = \|u - v\|$ is a metric and thus a normed vector space is a topological vector space.

Definition 48.1.2. A Banach space is a normed vector space $\|\bullet\| : V \rightarrow \mathbb{R}$ which is complete with respect to its metric.

Definition 48.1.3. An Hermitian inner product on a \mathbb{C} -vector space is a conjugate-symmetric bilinear form $\langle -, - \rangle : V \times \overline{V} \rightarrow \mathbb{C}$ such that $\|v\| = \sqrt{\langle v, v \rangle}$ is a norm. Explicitly,

- (a) $\langle \alpha v + \beta u, w \rangle = \alpha \langle v, w \rangle + \beta \langle u, w \rangle$
- (b) $\langle u, v \rangle = \overline{\langle v, u \rangle}$
- (c) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \iff v = 0$

Definition 48.1.4. A *Hilbert Space* \mathcal{H} is a Hermitian inner product space which is complete with respect to the induced norm.

Remark. Therefore a Hilbert space is a Banach space whose norm is induced by an inner product.

Proposition 48.1.5. A norm $\|\bullet\|$ is induced by a (real) inner product if and only if the real polarization identity holds:

$$\frac{1}{2} (\|v + u\|^2 + \|v - u\|^2) = \|v\|^2 + \|u\|^2$$

and by a Hermitian inner product if and only if the complex polarization identities hold:

$$\frac{1}{2} (\|v + iu\|^2 + \|v - iu\|^2) = \|v\|^2 + \|u\|^2$$

Proof. Suppose that $\|v\|^2 = \langle v, v \rangle$ then,

$$\begin{aligned} \frac{1}{2} (\|v + u\|^2 + \|v - u\|^2) &= \frac{1}{2} (\langle v + u, v + u \rangle + \langle v - u, v - u \rangle) \\ &= \frac{1}{2} (\langle v, v \rangle + 2\langle v, u \rangle + \langle u, u \rangle + \langle v, v \rangle - 2\langle v, u \rangle + \langle u, u \rangle) \\ &= \langle v, v \rangle + \langle u, u \rangle = \|v\|^2 + \|u\|^2 \end{aligned}$$

Conversely, we define $\langle v, u \rangle = \frac{1}{2} (\|v + u\|^2 - \|v\|^2 - \|u\|^2)$. Then clearly $\langle v, u \rangle = \langle u, v \rangle$ and,

$$\langle v, v \rangle = \frac{1}{2} (\|2v\|^2 - \|v\|^2 - \|v\|^2) = \frac{1}{2} (4\|v\|^2 - 2\|v\|^2) = \|v\|^2$$

Finally,

$$\begin{aligned} \langle \alpha v + \beta u, w \rangle - \alpha \langle v, w \rangle - \beta \langle u, w \rangle &= \frac{1}{2} (\|\alpha v + \beta u + w\|^2 - \|\alpha v + \beta u\|^2 - \|w\|^2) \\ &\quad - \frac{1}{2} \alpha (\|v + w\|^2 - \|v\|^2 - \|w\|^2) - \frac{1}{2} \beta (\|u + w\|^2 - \|u\|^2 - \|w\|^2) \\ &= \end{aligned}$$

(WTFF!!)

□

Proposition 48.1.6 (Cauchy-Schwarz). Let $(V, \langle -, - \rangle)$ be a Hermitian inner product space. Then, for any $v, u \in V$,

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

With equality exactly when u, v are linearly dependent.

Proof. If $v = 0$ equality is obvious. Consider,

$$u' = u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v$$

Then we have,

$$\langle u', v \rangle = \langle u, v \rangle - \frac{\langle u, v \rangle}{\langle v, v \rangle} \langle v, v \rangle = \langle u, v \rangle - \langle u, v \rangle = 0$$

Therefore, writing

$$u = u' + \frac{\langle u, v \rangle}{\langle v, v \rangle} v$$

and thus we get

$$\|u\|^2 = \|u'\|^2 + \left| \frac{\langle u, v \rangle}{\langle v, v \rangle} \right|^2 \|v\|^2 + \langle u', v \rangle \frac{\langle v, u \rangle}{\langle v, v \rangle} + \langle v, u' \rangle \frac{\langle u, v \rangle}{\langle v, v \rangle} = \|u'\|^2 + \left| \frac{\langle u, v \rangle}{\langle v, v \rangle} \right|^2 \|v\|^2$$

Therefore,

$$\|u\|^2 \geq \left| \frac{\langle u, v \rangle}{\langle v, v \rangle} \right|^2 \|v\|^2$$

or equivalently

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

with equality exactly when,

$$\|u'\| = 0 \iff u' = 0 \iff u \langle v, v \rangle = \langle u, v \rangle v$$

which is exactly when u and v are dependent (recally we assumed $v \neq 0$). □

48.2 Operators

Definition 48.2.1. A *densely defined* operator $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ of Hilbert spaces is a linear map defined on a dense subspace $D(T) \subset \mathcal{H}_1$. An operator is everywhere defined if $D(T) = \mathcal{H}_1$ which is generically called an operator.

Definition 48.2.2. An operator $T : X \rightarrow Y$ is *bounded* if the operator norm,

$$\|T\| = \sup_{v \in X} \frac{\|T(v)\|_Y}{\|v\|_X} = \sup_{v \in X} \frac{\langle T(v), T(v) \rangle_Y}{\langle v, v \rangle_X} = \sup_{\|v\|_X=1} \langle T(v), T(v) \rangle_Y$$

exists i.e. there exists some $M \in \mathbb{R}$ such that $\|T(v)\|_Y \leq M\|v\|_X$ for all $v \in X$.

Proposition 48.2.3. An operator $T : X \rightarrow Y$ the following are equivalent,

- (a) T is bounded
- (b) T is Lipschitz
- (c) T is continuous

Proof. Suppose that $T : X \rightarrow Y$ is bounded by M . Then consider,

$$\|T(v) - T(u)\| = \|T(v - u)\| \leq M\|v - u\|$$

and thus $T : X \rightarrow Y$ is Lipschitz which implies continuous. Now suppose that T is continuous. At $v = 0$, for $\epsilon = 1$ there exists $\delta > 0$ such that,

$$\|v\| \leq \delta \implies \|T(v)\| \leq 1$$

Then for any nonzero v we set $u = \frac{\delta}{\|v\|}v$ and then,

$$\|u\| = \delta \implies \|T(u)\| \leq 1$$

and thus,

$$\|T(v)\| \leq \frac{1}{\delta}\|v\|$$

proving that T is bounded. \square

48.3 Basic Topological Properties

Proposition 48.3.1. Let (M, d) be a metric space. Then $d : M \times M \rightarrow \mathbb{R}$ is continuous with respect to the metric topology.

Proof. It suffices to show that for any $x, y \in X$ and $\epsilon > 0$ we have $|d(x', y') - d(x, y)| < 2\epsilon$ for all $(x', y') \in B_\epsilon(x) \times B_\epsilon(y)$. From the triangle inequality,

$$d(x', y') \leq d(x, y') + d(x', x) \quad \text{and} \quad d(x, y') \leq d(x', y') + d(x', x) \implies |d(x', y') - d(x, y')| \leq d(x', x)$$

likewise $|d(x, y') - d(x, y)| \leq d(y', y)$ and therefore,

$$|d(x', y') - d(x, y)| \leq d(x', x) + d(y', y) < 2\epsilon$$

\square

Proposition 48.3.2. Let \mathcal{H} be a Hilbert space then $\langle -, - \rangle : \mathcal{H} \rightarrow \mathbb{C}$ is continuous.

Proof. This follows from the complex polarization identities and the continuity of $\|v\|^2 = \langle v, v \rangle$. \square

Lemma 48.3.3. A subspace $S \subset M$ of a metric space is closed iff every convergent sequence $x_n \rightarrow x$ with $x_n \in S$ converges to $x \in S$.

Proof. If W is closed then $x \in \overline{W} = W$ since every open neighborhood of x contains points in the sequence $x_n \in W$. Conversely, suppose that $x \in \overline{W}$ we need to show $x \in W$. By hypothesis it suffices to find a sequence $x_n \in W$ converging to x . For any $n \in \mathbb{N}$ there exists $x_n \in W \cap B_{\frac{1}{n}}(x)$ and clearly $x_n \rightarrow x$. \square

Proposition 48.3.4. Let $W \subset \mathcal{H}$ be a subspace of a Hilbert space. Then,

$$W^\perp = \{v \in \mathcal{H} \mid \forall u \in W : \langle v, u \rangle = 0\}$$

is a closed subspace.

Proof. Consider a sequence $x_n \in W^\perp$ such that $x_n \rightarrow x$ converges. Then $\forall v \in W : \langle x_n, v \rangle = 0$ so,

$$\langle x, v \rangle = \left\langle \lim_n x_n, v \right\rangle = \lim_n \langle x_n, v \rangle = 0$$

because $\langle -, - \rangle$ is continuous with respect to the metric topology. Thus $x \in W$ and thus W is closed. \square

Remark. Note we do not assume W is closed. Warning: when W is not closed we may not have $\mathcal{H} = W \oplus W^\perp$. For example, take ℓ_2 and the subset W of finite sequences. Then $W^\perp = (0)$. We now see that this cannot happen if W is closed.

Proposition 48.3.5. Let $W \subset \mathcal{H}$ be a closed subspace. Then there is a direct sum decomposition of the Hilbert space into $\mathcal{H} \cong W \oplus W^\perp$.

Proof. Fix $v \in \mathcal{H}$. Now consider a sequence $w_k \in W$ such that $\|v - w_k\|^2 \rightarrow \inf_{w \in W} \|v - w\|^2$. Notice that,

$$\|w - v\|^2 = \|v\|^2 + \|w\|^2 - 2\Re\langle w, v \rangle$$

Now for v_n and v_m consider,

$$\begin{aligned} \|w_n - v\|^2 + \|w_m - v\|^2 - 2\|\tfrac{1}{2}(w_n + w_m) - v\|^2 \\ = \|w_n\|^2 + \|w_m\|^2 - 2\|\tfrac{1}{2}(w_n + w_m)\|^2 - 2\Re\langle w_n, v \rangle - 2\Re\langle w_m, v \rangle + 2\Re\langle w_n + w_m, v \rangle \\ = \|w_n\|^2 + \|w_m\|^2 - \tfrac{1}{2}\|w_n + w_m\|^2 \\ = \tfrac{1}{2}\|w_n - w_m\|^2 \end{aligned}$$

However, as $n, m \rightarrow \infty$ I claim that,

$$\|w_n - v\|^2 + \|w_m - v\|^2 - 2\|\tfrac{1}{2}(w_n + w_m) - v\|^2 \rightarrow 0$$

Let $L = \inf_{w \in W} \|v - w\|^2$ which bounds $\|\tfrac{1}{2}(w_n + w_m) - v\| \geq L$ meaning that,

$$0 \leq \|w_n - v\|^2 + \|w_m - v\|^2 - 2\|\tfrac{1}{2}(w_n + w_m) - v\|^2 \leq \|w_n - v\|^2 + \|w_m - v\|^2 - 2L$$

which is nonnegative because it equals $\tfrac{1}{2}\|w_n - w_m\|^2$. Furthermore, both $\|w_n - v\|^2 \rightarrow L$ and $\|w_m - v\|^2 \rightarrow L$ so in the limit this quantity is zero. Thus $\|w_n - w_m\| \rightarrow 0$ so the sequence w_n is Cauchy and thus converges to a limit $w_n \rightarrow w$. Since W is closed and $w_n \in W$ we have $w \in W$. Furthermore, $\|v - w\|^2 \rightarrow \inf_{w \in W} \|v - w\|^2$. And thus for any $\tilde{w} \in W$ we must have,

$$\|v - \tilde{w}\|^2 \geq \|v - w\|^2$$

In particular, take,

$$\|v - (w + t\tilde{w})\|^2 = \|v - w\|^2 - t\Re\langle v - w, \tilde{w} \rangle + t^2\|\tilde{w}\|^2$$

Thus, $-t\Re\langle v - w, \tilde{w} \rangle + t^2\|\tilde{w}\|^2$ is positive for all t which implies that $\Re\langle v - w, \tilde{w} \rangle = 0$. Changing $\tilde{w} \mapsto i\tilde{w}$ we see the imaginary part must vanish as well so $\langle v - w, \tilde{w} \rangle = 0$. Since this holds for all $\tilde{w} \in W$ we find $v - w \in W^\perp$. Thus we can write $v = w + (v - w)$ for $w \in W$ and $v - w \in W^\perp$. Furthermore, clearly $W^\perp \cap W = (0)$ giving an internal direct sum decomposition $\mathcal{H} = W \oplus W^\perp$. \square

Proposition 48.3.6. For any subspace $W \subset \mathcal{H}$ of a Hilbert space, we have $(W^\perp)^\perp = \overline{W}$.

Proof. $W \subset (W^\perp)^\perp$ since for any $w \in W$ if $v \in W^\perp$ then $\langle w, v \rangle = 0$ so $w \in (W^\perp)^\perp$. Since $(W^\perp)^\perp$ is closed, $\overline{W} \subset (W^\perp)^\perp$. Conversely, \overline{W} is closed so $\mathcal{H} = \overline{W} \oplus \overline{W}^\perp$ and thus any $v \in (\overline{W}^\perp)^\perp$ decomposes as $v = v_1 + v_2$ with $v_1 \in \overline{W}$ and $v_2 \in \overline{W}^\perp$ so $\langle v_1, v_2 \rangle = 0$. However, $v \in (\overline{W}^\perp)^\perp$ so $\langle v, v_2 \rangle = 0$ and thus $\langle v_2, v_2 \rangle = 0$ which implies that $v_2 = 0$ and thus $v = v_1 \in \overline{W}$. Therefore, $(W^\perp)^\perp \subset (\overline{W}^\perp)^\perp \subset \overline{W}$. \square

Corollary 48.3.7. A subspace $W \subset \mathcal{H}$ of a Hilbert space is trivial iff $W^\perp = \mathcal{H}$.

Proof. Suppose $W^\perp = \mathcal{H}$. Then $W \subset W^\perp$ but $W \cap W^\perp = (0)$ so $W = (0)$. \square

Corollary 48.3.8. A subspace $W \subset \mathcal{H}$ of a Hilbert space is dense iff $W^\perp = (0)$.

Proof. From before $\overline{W} = \mathcal{H} \iff (W^\perp)^\perp = \mathcal{H} \iff W^\perp = (0)$. \square

48.4 Dual Spaces

Definition 48.4.1. Let \mathcal{H} be a Hilbert space. A linear functional is a continuous linear map $\varphi : \mathcal{H} \rightarrow \mathbb{R}$. The dual space \mathcal{H}^* is the Hilbert space of linear functionals along with the inner product,

$$\langle \varphi, \psi \rangle = \sup_{v \in \mathcal{H}} \frac{\langle \varphi(v), \psi(v) \rangle}{\langle v, v \rangle}$$

Remark. We need to justify that this inner product exists. We need to show that $\langle \varphi(v), \psi(v) \rangle$ is bounded on the unit sphere $\|v\| = 1$. This holds by continuity and the compactness of $S(\mathcal{H})$. We need to show this is an inner product. Linearity and conjugate-symmetry are clear. Furthermore positivity for $\langle \varphi, \varphi \rangle$ is clear. Finally suppose that,

$$\langle \varphi, \varphi \rangle = \sup_{v \in \mathcal{H}^*} \frac{\langle \varphi(v), \varphi(v) \rangle}{\langle v, v \rangle} = 0 \implies \forall v \in \mathcal{H} : \langle \varphi(v), \varphi(v) \rangle = 0 \implies \forall v \in \mathcal{H} : \varphi(v) = 0 \implies \varphi = 0$$

Theorem 48.4.2 (Riesz). Let \mathcal{H} be a Hilbert space⁵ For any linear functional $\varphi \in \mathcal{H}^*$ there exists a unique $v_\varphi \in \mathcal{H}$ such that $\forall u \in \mathcal{H} : \varphi(u) = \langle u, v_\varphi \rangle$. Furthermore, $\|v_\varphi\|_{\mathcal{H}} = \|\varphi\|_{\mathcal{H}^*}$.

Proof. Consider $K = \ker \varphi$ which is closed (since φ is continuous). If $K = \mathcal{H}$ then $\varphi = 0$ so we may take $v_\varphi = 0$ and conclude. If $K \subsetneq \mathcal{H}$ then because K is closed and \mathcal{H} is complete K^\perp is nonempty. I claim, K^\perp is one dimensional since if $v_1, v_2 \in K^\perp$ are nonzero then $\varphi(v_1)$ and $\varphi(v_2)$ are nonzero so there exists $\lambda \in \mathbb{R}$ such that $\varphi(v_1) = \lambda \varphi(v_2)$ and thus $\varphi(v_1 - \lambda v_2) = 0$ so $v_1 - \lambda v_2 \in K^\perp \cap K$ and thus $v_1 = \lambda v_2$ so we may choose a unit vector $u \in K^\perp$. Therefore, for any $v \in \mathcal{H}$ the vector $v - \langle v, u \rangle u \in K$ therefore,

$$\varphi(v) = \langle v, u \rangle \varphi(u) = \langle v, u \overline{\varphi(u)} \rangle$$

Hence we take $v_\varphi = \overline{\varphi(u)}u$. Furthermore,

$$\|v_\varphi\|_{\mathcal{H}}^2 = \langle v_\varphi, v_\varphi \rangle = \langle u \overline{\varphi(u)}, u \overline{\varphi(u)} \rangle = |\varphi(u)|^2 \langle u, u \rangle = |\varphi(u)|^2$$

and likewise,

$$\|\varphi\|_{\mathcal{H}^*} = \sup_{v \in \mathcal{H}} \frac{|\varphi(v)|}{\|v\|} = \sup_{v \in \mathcal{H}} \frac{|\langle v, u \rangle \varphi(u)|}{\|v\|} = \sup_{v \in \mathcal{H}} \frac{|\langle v, u \rangle|}{\|v\|} |\varphi(u)| = |\varphi(u)|$$

Because $\|u\| = 1$ so by Cauchy-Schwarz,

$$|\langle v, u \rangle| \leq \|v\|$$

with equality when $v \in \text{span}\{u\}$. Therefore, $\|v_\varphi\| = \|\varphi\|_{\mathcal{H}^*}$. □

Corollary 48.4.3. Let \mathcal{H} be a Hilbert space. Then the map $\overline{\mathcal{H}} \rightarrow \mathcal{H}^*$ defined by $v \mapsto \langle -, v \rangle$ is an isometric isomorphism.

Proof. This map is linear because $\lambda \cdot v = \bar{\lambda}v \mapsto \langle -, \bar{\lambda}v \rangle = \lambda \langle -, v \rangle$. Furthermore, if $\langle u, v \rangle = 0$ for all u then $\langle v, v \rangle = 0$ and thus $v = 0$ so the map is injective. By the Riesz representation theorem, this map is surjective and isometric. □

Corollary 48.4.4. The natural map $\mathcal{H} \rightarrow \mathcal{H}^{**}$ is an isomorphism of Hilbert spaces.

⁵we require that $\langle -, - \rangle$ be linear in its first argument.

48.5 Adjoints

48.6 Weak Convergence

Definition 48.6.1. We say a sequence $x_n \in \mathcal{H}$ converges weakly to $x \in \mathcal{H}$ if for all $y \in \mathcal{H}$ we get a convergent sequence,

$$\langle x_n, y \rangle \rightarrow \langle x, y \rangle$$

49 Soft Sheaves

Definition 49.0.1. Let X be a topological space. A sheaf \mathcal{F} is *soft* if for any closed $Z \subset X$ the induced map $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(Z, \mathcal{F}|_Z)$ is surjective.

Remark. Let \mathcal{F} be a sheaf. For any open $U \subset X$ the sheaf takes values $\mathcal{F}(U)$ and we know $\mathcal{F}(U) = \Gamma(U, \mathcal{F}|_U)$. Analogously, for a closed $Z \subset X$ we define,

$$\mathcal{F}(Z) = \Gamma(Z, \mathcal{F}|_Z)$$

Lemma 49.0.2. Let X be paracompact and $Z \subset X$ closed. Then for any sheaf \mathcal{F} the natural map,

$$\lim_{\substack{\longrightarrow \\ U \supset Z}} \mathcal{F}(U) \rightarrow \mathcal{F}(Z) = \Gamma(Z, \mathcal{F}|_Z)$$

is an isomorphism.

Proof. □

Lemma 49.0.3. If \mathcal{F} is flasque then \mathcal{F} is soft.

Proof. The restriction maps $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ are surjective and thus the induced maps,

$$\mathcal{F}(X) \rightarrow \lim_{\substack{\longrightarrow \\ U \supset Z}} \mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(Z)$$

are surjective. □

Corollary 49.0.4. Any injective sheaf is soft.

Proposition 49.0.5. Let \mathcal{P} be a class of sheaves on X such that,

- (a) injectives are in \mathcal{P}
- (b) for any exact sequence where $\mathcal{F} \in \mathcal{P}$ and \mathcal{I} is injective,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{K} \longrightarrow 0$$

then also $\mathcal{K} \in \mathcal{P}$ and the sequence of sections,

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{K}) \longrightarrow 0$$

is exact.

Then all sheaves in \mathcal{P} are acyclic for $H^i(X, -)$ i.e. if $\mathcal{F} \in \mathcal{P}$ then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. We show by induction on n that $H^n(X, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathcal{P}$. Take $\mathcal{F} \in \mathcal{P}$ and embed \mathcal{F} into an injective,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{K} \longrightarrow 0$$

Since $\mathcal{F}, \mathcal{I} \in \mathcal{P}$ then $\mathcal{K} \in \mathcal{P}$. Since \mathcal{I} is injective $H^i(X, \mathcal{I}) = 0$ for all $i > 0$. Thus, from the long exact sequence we find,

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{K}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow 0$$

$$0 \longrightarrow H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}) \longrightarrow 0$$

However, by assumption the global sections sequence is exact so $H^1(X, \mathcal{F}) = 0$ proving the $i = 1$ base case. Now for induction assume that $H^n(X, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{P}$. Then because $\mathcal{K} \in \mathcal{P}$ using the long exact sequence,

$$H^{n+1}(X, \mathcal{F}) = H^n(X, \mathcal{K}) = 0$$

proving the statement for $n + 1$ and any $\mathcal{F} \in \mathcal{P}$. □

Lemma 49.0.6. Let \mathcal{F} be a sheaf such that for any closed $Z \subset X$,

$$\mathcal{F}(X) \rightarrow \varinjlim_{U \supset Z} \mathcal{F}(U)$$

is surjective. Then for any exact sequence of sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{K} \longrightarrow 0$$

then for any open $U \subset X$, the sequence of sections,

$$0 \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{G}) \longrightarrow \Gamma(U, \mathcal{K}) \longrightarrow 0$$

is exact.

Proof. Fix a global section $s \in \mathcal{K}(U)$. Now we consider the poset of (local) sections mapping to s ,

$$P = \{(V, t) \mid V \subset U \text{ and } t \in \mathcal{G}(V) \text{ with } t|_V = s|_V\}$$

with the order $(V, t) \leq (V', t') \iff V \subset V' \text{ and } t'|_V = t$. We want to show P contains (U, t) for some t . Notice that for any chain $\{(V_i, t_i) \mid i \in I\}$ in P with a totally ordered index set I we can define,

$$V = \bigcup_{i \in I} V_i$$

and the t_i glue to $t \in \mathcal{G}(V)$ because the t_i agree on overlaps. Then (V, t) is an upper bound of $\{(V_i, t_i) \mid i \in I\}$ so by Zorn's Lemma P contains a maximal element (V, t) . Suppose that $V \neq U$ (i.e. $V \subset U$ is a proper subset) and take $Z = V^c$. □

Lemma 49.0.7. Let \mathcal{F}, \mathcal{G} be soft. Then for any exact sequence of sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{K} \longrightarrow 0$$

the sheaf \mathcal{K} is also soft.

Proof. For any closed subspace $Z \subset X$ the diagram of sections,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(X, \mathcal{G}) & \longrightarrow & \Gamma(X, \mathcal{H}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma(Z, \mathcal{F}|_Z) & \longrightarrow & \Gamma(X, \mathcal{G}|_Z) & \longrightarrow & \Gamma(X, \mathcal{H}|_Z) \longrightarrow 0 \end{array}$$

with exact rows because restrictions of soft sheaves remain soft so we may apply the previous lemma to both exact sequences. The first two vertical maps are surjections and thus so is the last by exactness. \square

50 Acyclicity

Proposition 50.0.1. Let \mathcal{C} be an abelian category and \mathcal{P} a class of objects such that,

- (a) for every injective $I \in \mathcal{C}$ there exists a monic $I \hookrightarrow B$ with $B \in \mathcal{P}$
- (b) if $A \oplus B \in \mathcal{P}$ then $A, B \in \mathcal{P}$.

Then every injective object of \mathcal{C} is in \mathcal{P} .

Proof. Let I be injective. Take a monic $I \hookrightarrow B$ with $B \in \mathcal{P}$. Taking the cokernel gives a sequence,

$$0 \longrightarrow I \longrightarrow B \longrightarrow K \longrightarrow 0$$

which splits because I is injective. Thus $B = I \oplus K$ and $B \in \mathcal{P}$ so $I, K \in \mathcal{P}$. \square

Proposition 50.0.2. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor of abelian categories where \mathcal{A} has enough injectives and \mathcal{P} be a class of \mathcal{A} -objects such that

- (a) injectives are in \mathcal{P}
- (b) for any exact sequence where $A \in \mathcal{P}$ and I is injective,

$$0 \longrightarrow A \longrightarrow I \longrightarrow K \longrightarrow 0$$

then also $K \in \mathcal{P}$ and the sequence,

$$0 \longrightarrow F(A) \longrightarrow F(I) \longrightarrow F(K) \longrightarrow 0$$

is exact.

Then all objects in \mathcal{P} are F -acyclic i.e. if $A \in \mathcal{P}$ then $R^p F(A) = 0$ for all $p > 0$.

Proof. We show by induction on n that $R^n F(A) = 0$ for all $A \in \mathcal{P}$. Take $A \in \mathcal{P}$ and embed A into an injective,

$$0 \longrightarrow A \longrightarrow I \longrightarrow K \longrightarrow 0$$

Since $F, I \in \mathcal{P}$ then $K \in \mathcal{P}$. Since I is injective $R^i F(I) = 0$ for all $i > 0$. Thus, from the long exact sequence we find,

$$0 \longrightarrow F(A) \longrightarrow F(I) \longrightarrow F(K) \longrightarrow R^1 F(A) \longrightarrow 0$$

$$0 \longrightarrow R^i F(K) \longrightarrow R^{i+1} F(A) \longrightarrow 0$$

However, by assumption the global sections sequence is exact so $R^1 F(A) = 0$ proving the $i = 1$ base case. Now for induction assume that $R^n F(B) = 0$ for all $B \in \mathcal{P}$. Then because $K \in \mathcal{P}$ using the long exact sequence,

$$R^{n+1} F(A) = R^n F(K) = 0$$

proving the statement for $n + 1$ and any $A \in \mathcal{P}$. \square

Proposition 50.0.3. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor of abelian categories where \mathcal{A} has enough injectives and \mathcal{P} be a class of \mathcal{A} -objects such that,

- (a) for every $A \in \mathcal{A}$ there exists a monic $A \hookrightarrow B$ with $B \in \mathcal{P}$
- (b) if $A \oplus B \in \mathcal{P}$ then $A, B \in \mathcal{P}$
- (c) for any exact sequence where $A \in \mathcal{P}$ and I is injective,

$$0 \longrightarrow A \longrightarrow I \longrightarrow K \longrightarrow 0$$

then also $K \in \mathcal{P}$ and the sequence,

$$0 \longrightarrow F(A) \longrightarrow F(I) \longrightarrow F(K) \longrightarrow 0$$

is exact.

Then the derived functors $R^p F$ may be computed on \mathcal{P} -resolutions. Explicitly, for any $A \in \mathcal{C}$ there exists an exact sequence $0 \rightarrow A \rightarrow P^\bullet$ with $P^i \in \mathcal{P}$ and for any such resolution,

$$R^p F(A) = H^p(F(P^\bullet))$$

Proof. The first two properties imply that all injectives are in \mathcal{P} . The second two properties then imply that objects in \mathcal{P} are F -acyclic. Finally (a) implies that \mathcal{P} -resolutions exist and that cohomology (e.g. derived functors) is computed from them follows from the Leray acyclicity lemma. \square

51 Classifying Principal G -bundles

Recall that $K(G, n)$ is the classifying space for $H^n(-; G)$. Explicitly,

$$H^n(X, G) = [X, K(G, n)]$$

given by pulling back along a function $f : X \rightarrow K(G, n)$ the class,

$$\text{id} \in \text{Hom}(G, G) = \text{Hom}(\pi_n(X), G) = H^n(X, G)$$

When G is a discrete abelian group, we want to justify the claim that $BG = K(G, 1)$ is the classifying space for principal G -bundles. Since G is discrete, these are nearly Galois covering spaces except they need not be connected. Let X be a connected CW complex then the universal coefficient theorem gives an exact sequence,

$$0 \longrightarrow \text{Ext}_{\mathbb{Z}}^1(H_0(X, \mathbb{Z}), G) \longrightarrow H^1(X, G) \longrightarrow \text{Hom}(H_1(X, \mathbb{Z}), G) \longrightarrow 0$$

However, since X is connected $H_0(X, \mathbb{Z}) = \mathbb{Z}$ and from the Hurewicz theorem $\pi_1(X) \rightarrow H_1(X, \mathbb{Z})$ is the abelianization map. Therefore, there is an isomorphism,

$$H^1(X, G) = \text{Hom}(H_1(X, \mathbb{Z}), G) = \text{Hom}(\pi_1(X)^{\text{ab}}, G) = \text{Hom}(\pi_1(X), G)$$

Since $K(G, 1)$ classifies $H^1(X, G)$, we need to show that principal G -bundles are classified by $\text{Hom}(\pi_1(X), G)$ which implies that the functor sending X to its set of principal G -bundles up to isomorphism is represented by $K(G, 1)$.

51.1 The Connected Case

Notice that since G is discrete a connected principal G -bundle is exactly a Galois cover of X with deck transformation group G . Furthermore, such Galois covers are classified by normal subgroups $N \triangleleft \pi_1(X)$ such that $\pi_1(X)/N \cong G$ or equivalent surjective morphisms $\pi_1(X) \twoheadrightarrow G$. Therefore the surjective subset of $\text{Hom}(\pi_1(X), G)$ corresponds to the connected G -bundles.

51.2 Group Cohomology and Cocycles

Definition 51.2.1. Let G be a group consider the category \mathbf{Mod}_G of left $\mathbb{Z}[G]$ -modules. The functor $M \mapsto M^G$ where $M^G = \{m \in M \mid \sigma(m) = m \forall \sigma \in G\}$ is left exact. Then we define the group cohomology $H^n(G, -)$ as the right-derived functors of $(-)^G$.

Remark. Notice that $(-)^G = \text{Hom}(\mathbb{Z}, -)$ where \mathbb{Z} is a trivial G -module. Therefore,

$$H^n(G, -) = \text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, -)$$

By general homological algebra, we can compute $\text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M)$ via projective resolutions in the first argument.

Definition 51.2.2. The complex $P_n(G) = \mathbb{Z}[G^{n+1}]$ with the diagonal G -action where the boundary map $d : P_{n+1}(G) \rightarrow P_n(G)$ is defined by,

$$d(g_0, \dots, g_n) = \sum_{i=0}^n (-1)^i (g_1, \dots, \hat{g}_i, \dots, g_n)$$

Proposition 51.2.3. $P_\bullet(G) \rightarrow \mathbb{Z}$ with $\epsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$ sending $g \mapsto 1$ is a projective resolution.

Proof. The modules $P_n = \mathbb{Z}[G^{n+1}]$ is a free $\mathbb{Z}[G]$ -module with basis $\{(1, g_1, \dots, g_n) \mid g_i \in G\}$ and thus projective. Furthermore, we see,

$$\begin{aligned} d^2(g_0, \dots, g_n) &= \sum_{i=0}^n (-1)^i d(g_0, \dots, \hat{g}_i, \dots, g_n) \\ &= \sum_{i=0}^n \sum_{j=0}^{i-1} (-1)^{i+j} (g_0, \dots, \hat{g}_j, \dots, \hat{g}_i, \dots, g_n) + \sum_{i=0}^n \sum_{j=i+1}^n (-1)^{i+j} (g_0, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_n) \\ &= 0 \end{aligned}$$

Furthermore, this complex is exact because there exists a map $k_r : P_r \rightarrow P_{r+1}$ defined by sending $(g_0, \dots, g_r) \mapsto (1, g_0, \dots, g_r)$ which satisfies,

$$k_{r-1} \circ d_r + d_{r+1} \circ k_r = \text{id}$$

because,

$$k_{r-1} \circ d_r(g_0, \dots, g_r) = \sum_{i=0}^r (-1)^i (1, g_0, \dots, \hat{g}_i, \dots, g_r)$$

and likewise,

$$d_{r+1} \circ k_r(g_0, \dots, g_r) = (g_0, \dots, g_r) + \sum_{i=0}^r (-1)^{i+1} (1, g_0, \dots, \hat{g}_i, \dots, g_r)$$

proving that,

$$(k_{r-1} \circ d_r + d_{r+1} \circ k_r)(g_0, \dots, g_r) = (g_0, \dots, g_r)$$

Therefore, P_\bullet is homotopic to the zero complex in positive degrees and is thus exact. Explicitly, if $d_r \gamma = 0$ for $\gamma \in \mathbb{Z}[G^{r+1}]$ for $r > 0$ then,

$$\gamma = (k_{r-1} \circ d_r + d_{r+1} \circ k_r)(\gamma) = d_{r+1}(k_r(\gamma))$$

so $\gamma \in \text{Im}(d : P_{r+1} \rightarrow P_r)$. Finally, notice that $\ker \epsilon$ is the submodule generated by $g_i - g_j$ which is exactly the image of $d : P^1 \rightarrow P^0$. \square

Proposition 51.2.4. Let G be a group then,

$$H^n(G, M) = H^n(\text{Hom}(P_\bullet, M))$$

Proof. Because $P_\bullet \rightarrow \mathbb{Z}$ is a projective resolution this computes $\text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M) = H^n(G, M)$. \square

Definition 51.2.5. The homogeneous n -cochains $\tilde{C}^n(G, M) = \text{Hom}(P_n, M)$ are $\varphi : G^{n+1} \rightarrow M$ (as sets) satisfying $\varphi(gg_0, \dots, gg_r) = g\varphi(g_0, \dots, g_r)$ with boundary map $d^r : \tilde{C}^r(G, M) \rightarrow \tilde{C}^{r+1}(G, M)$,

$$d^r \varphi(g_0, \dots, g_{r+1}) = \sum_{i=0}^r (-1)^i \varphi(g_0, \dots, \hat{g}_i, \dots, g_{r+1})$$

Remark. Notice that such a map $\varphi : G^{n+1} \rightarrow M$ is uniquely determined on $(1, g_1, \dots, g_n)$ since we may pull out g_0 (and replace $g_1 \mapsto g_0^{-1}g_1$). However, restricting to elements of this sort does not play nicely with the boundary map because,

$$d^r \varphi(1, g_1, \dots, g_{r+1}) = \varphi(g_1, \dots, g_{r+1}) + \sum_{i=1}^{r+1} (-1)^i \varphi(1, g_1, \dots, \hat{g}_i, \dots, g_{r+1})$$

but the first term,

$$\varphi(g_1, \dots, g_{r+1}) = g_1 \varphi(1, g_1^{-1}g_2, \dots, g_1^{-1}g_{r+1})$$

which is a bit nasty because of all the g_1^{-1} . We can use the following trick to put the differential in a nicer form on our basis. Because multiplication is bijective elements of the form

$(1, g_1, g_1g_2, \dots, g_1 \cdots g_r)$ are exactly the same as the elements of the form $(1, g_1, \dots, g_r)$. However, furthermore the differential acts via,

$$\begin{aligned} d\varphi(1, g_1, g_1g_2, \dots, g_1 \cdots g_{r+1}) &= \varphi(g_1, g_1g_2, \dots, g_1 \cdots g_{r+1}) + \sum_{i=1}^{r+1} (-1)^i \varphi(1, g_1, \dots, \widehat{g_1 \cdots g_i}, \dots, g_1 \cdots g_{r+1}) \\ &= g_1 \varphi(g_2, \dots, g_2 \cdots g_{r+1}) + \sum_{i=1}^r \varphi(1, g_1, \dots, g_1 \cdots \tilde{g}_i, \dots, g_1 \cdots \tilde{g}_{r+1}) \\ &\quad + \varphi(1, g_1, \dots, g_1 \cdots g_r) \end{aligned}$$

where $\tilde{g}_i = g_i g_{i+1}$ and $\tilde{g}_j = g_{j+1}$ for $j > i$. Therefore, if we define $\varphi' : G^n \rightarrow M$ via,

$$\varphi'(g_1, \dots, g_r) = \varphi(1, g_1, g_1g_2, \dots, g_1 \cdots g_r)$$

we find that,

$$d\varphi(1, g_1, g_1g_2, \dots, g_1 \cdots g_{r+1}) = g_1 \varphi'(g_2, \dots, g_{r+1}) + \sum_{i=1}^r \varphi'(g_1, \dots, g_i g_{i+1}, \dots, g_{r+1}) + (-1)^{r+1} \varphi(g_1, \dots, g_r)$$

Thus we define the following equivalent description of the complex $P_\bullet \rightarrow \mathbb{Z}$.

Definition 51.2.6. The inhomogeneous n -cochains $C^n(G, M)$ are (set) maps $\varphi : G^n \rightarrow M$ with corresponding boundary maps $d^r : C^r(G, M) \rightarrow C^{r+1}(G, M)$ defined via,

$$d\varphi(g_1, \dots, g_{r+1}) = g_1 \varphi(g_2, \dots, g_{r+1}) + \sum_{i=1}^r (-1)^i \varphi(g_1, \dots, g_i g_{i+1}, \dots, g_{r+1}) + (-1)^{r+1} \varphi(g_1, \dots, g_r)$$

Proposition 51.2.7. The group cohomology may be computed as,

$$H^n(G, M) = H^n(C^\bullet(G, M))$$

Proof. We have shown the map $\tilde{C}^\bullet(G, M) \rightarrow C^\bullet(G, M)$ via sending $\varphi \mapsto \varphi'$ is an isomorphism commuting with the boundary maps by the preceding remarks. \square

51.2.1 Low Degree Computations

Proposition 51.2.8. Let M be a G -module. Then,

$$H^1(G, M) = \frac{\{\varphi : G \rightarrow M \mid \varphi(g_1g_2) = g_1\varphi(g_2) + \varphi(g_1)\}}{\{\varphi : G \rightarrow M \mid \varphi(g) = g \cdot m - m\}}$$

and likewise,

$$H^2(G, M) = \frac{\{\varphi : G^2 \rightarrow M \mid g_1\varphi(g_2, g_3) - \varphi(g_1g_2, g_3) + \varphi(g_1, g_2g_3) - \varphi(g_1, g_2) = 0\}}{\{\varphi : G^2 \rightarrow M \mid \varphi(g_1, g_2) = g_1\eta(g_2) - \eta(g_1g_2) + \eta(g_1)\}}$$

Remark. These 2-cocycles $G^2 \rightarrow M$ have an important interpretation.

Definition 51.2.9. Let M be a G -module. Then an M -extension is an exact sequence,

$$1 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

where for any $g \in G$ any preimage e_g acts on $m \in M$ via $e_g m e_g^{-1} = g \cdot m$.

Remark. Notice that the formula $e_g m e_g^{-1} = g \cdot m$ is well-defined because for any other lift e'_g of g we must have $e'_g = e_g m'$ for some $m' \in M$ and,

$$e'_g m e'^{-1}_g = e_g m' m m'^{-1} e_g^{-1} = e_g m e_g^{-1}$$

because M is commutative.

Proposition 51.2.10. Let M be a G -module. Then isomorphism classes of M -extensions are bijective to $H^2(G, M)$ where the split extension corresponds to the trivial cocycle.

Proof. Given an M -extension,

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 0$$

Then we consider a (set-theoretic) section $s : G \rightarrow E$. This may not be a group homomorphism so we can measure its failure to be a homomorphism by a map $\varphi(g_1, g_2) = s(g_1 g_2) s(g_1)^{-1} s(g_2)^{-1}$. However, $\pi(\varphi(g_1, g_2)) = (g_1 g_2) g_1^{-1} g_2^{-1} = 1$ (because $\pi : E \rightarrow G$ is a group homomorphism). Therefore, this defines a cochain $\varphi : G^2 \rightarrow M$. Furthermore, consider the tripple products,

$$s(g_1 g_2 g_3) = \varphi(g_1 g_2, g_3) s(g_1 g_2) s(g_3) = \varphi(g_1 g_2, g_3) \varphi(g_1, g_2) s(g_1) s(g_2) s(g_3)$$

but we can also write,

$$\begin{aligned} s(g_1 g_2 g_3) &= \varphi(g_1, g_2 g_3) s(g_1) s(g_2 g_3) = \varphi(g_1, g_2 g_3) s(g_1) \varphi(g_2, g_3) s(g_2) s(g_3) \\ &= \varphi(g_1, g_2 g_3) [s(g_1) \varphi(g_2, g_3) s(g_1)^{-1}] s(g_1) s(g_2) s(g_3) \end{aligned}$$

Therefore, writing in additive notation, we find,

$$\varphi(g_1 g_2, g_3) + \varphi(g_1, g_2) = \varphi(g_1, g_2 g_3) + g_1 \cdot \varphi(g_2, g_3)$$

where, by definition, $s(g_1) m s(g_1)^{-1} = g_1 \cdot m$ for all $m \in M$. Therefore $\varphi : G^2 \rightarrow M$ is a cocycle. This defines a map from M -extensions to $H^1(G, M)$. It suffices to construct an inverse. Given a 2-cocycle $\varphi : G^2 \rightarrow M$. We define an extension,

$$1 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

where $E = G \times M$ with a “twisted” product structure,

$$(g_1, m_1) \cdot (g_2, m_2) = (g_1 g_2, m_1 + g_1 \cdot m_2 + \varphi(g_1, g_2))$$

□

51.3 The General Case

Consider a cocycle $\varphi : N^2 \rightarrow G/N$

We want to construct a principal G -bundle $P(\varphi) = \tilde{X} \times G/N$ with the following action,

$$g_1 \cdot (x, g_2 N) = (g_1 \cdots x, g_1 g_2 N)$$

51.4 Examples

52 Low Degree Terms of Spectral Sequences

Suppose we have a bounded spectral sequence $E_r^{p,q} \implies H^{p+q}(E^\bullet)$ converging to H^n . Explicitly the differentials are maps $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$. Then there is a filtration $F^p H^n(E^\bullet)$ of $H^n(E^\bullet)$ with $F^0 H^n(E^\bullet) = H^n(E^\bullet)$ and $F^{n+1} H^n(E^\bullet) = 0$ such that,

$$F^p H^{p+q}(E^\bullet) / F^{p+1} H^{p+q}(E^\bullet) = E_\infty^{p,q}$$

Proposition 52.0.1. Given a spectral sequence $E_r^{p,q} \implies H^{p+q}(E^\bullet)$ then $H^0(E^\bullet) = E_2^{0,0}$ and there is a spectral sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(E^\bullet)$$

Proof. First, notice that differentials $d_r : E_r^{1,0} \rightarrow E_r^{1+r,1-r}$ and $d_r : E_r^{1-r,r-1} \rightarrow E_r^{1,0}$ both contain terms outside the support of the spectral sequence when $r > 1$. Therefore, $E_2^{1,0} = E_\infty^{1,0}$. Furthermore, since $F^2 H^1(E^\bullet) = 0$ we get,

$$F^1 H^1(E^\bullet) = E_\infty^{1,0} = E_2^{1,0}$$

Furthermore, since $F^0 H^1(E^\bullet) / F^1 H^1(E^\bullet) = E_\infty^{0,1}$ we get an exact sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(E^\bullet) \longrightarrow E_\infty^{0,1} \longrightarrow 0$$

Furthermore, consider the differentials $d_r : E_r^{-r,r} \rightarrow E_r^{0,1}$ and $d_r : E_r^{0,1} \rightarrow E_r^{r,2-r}$. The first lies outside the support and thus is zero when $r > 1$ and the second is outside the support and thus zero when $r > 2$. Therefore, $E_\infty^{0,1} = E_3^{0,1} = \ker(d_r : E_2^{0,1} \rightarrow E_2^{2,0})$. This allows us to refine the exact sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}$$

Finally, since $F^3 H^2(E^\bullet) = 0$ we find $F^2 H^2(E^\bullet) = E_\infty^{2,0}$ embeds inside $H^2(E^\bullet)$. However, the differentials $d_r : E_r^{2,0} \rightarrow E_r^{2+r,1-r}$ is outside the support when $r > 1$ and $d_r : E_r^{2-r,r-1} \rightarrow E_r^{2,0}$ is zero for $r > 2$ so $E_\infty^{2,0} = E_2^{2,0} / \text{Im}(d_r : E_2^{0,1} \rightarrow E_2^{2,0})$. Therefore, the kernel of $E_2^{2,0} \rightarrow E_\infty^{2,0} \rightarrow H^2(E^\bullet)$ is $\text{Im}(d_r : E_2^{0,1} \rightarrow E_2^{2,0})$ and thus the sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(E^\bullet)$$

is exact. Finally, the differentials $d_r : E_r^{0,0} \rightarrow E_r^{r,1-r}$ and $d_r : E_r^{-r,r-1} \rightarrow E_r^{0,0}$ are zero for $r > 1$ so $E_\infty^{0,0} = E_2^{0,0}$ and $F^1 H^0(E^\bullet) = 0$ so

$$H^0(E^\bullet) = F^0 H^0(E^\bullet) = E_\infty^{0,0} = E_2^{0,0}$$

□

Example 52.0.2. The Hodge to de Rham spectral sequence,

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H_{\text{dR}}^{p+q}(X)$$

for curves gives a low degree exact sequence,

$$0 \longrightarrow H^0(X, \Omega_X^1) \longrightarrow H_{\text{dR}}^1(X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

because $E_1^{p,q} = 0$ if $p > 1$ or $q > 1$ and thus $E_2^{p,q} = E_1^{p,q}$. Analytically, we can compare this to the differential sequence,

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \longrightarrow 0$$

since this gives an exact sequence,

$$0 \longrightarrow H^0(X, \Omega_X^1) \longrightarrow H^1(X, \mathbb{C}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

notice that the map $d : H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \Omega_X^1)$ is zero for proper curves since the global functions are constant and similarly $d : H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \Omega_X^1)$ by Serre duality.

Example 52.0.3. There is a dual Hodge to de Rham spectral sequence,

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\Omega^\bullet)) \implies H_{\text{dR}}^{p+q}(X)$$

which gives an exact sequence of low degree terms,

$$0 \longrightarrow H^1(X, \ker d) \longrightarrow H_{\text{dR}}^1(X) \longrightarrow H^0(X, \mathcal{H}^1(\Omega^\bullet)) \xrightarrow{d_2} H^1(X, \ker d) \longrightarrow H_{\text{dR}}^2(X)$$

Often we will have $H^i(X, \ker d) = 0$ for $i > 0$ in which case $H_{\text{dR}}^1(X) = H^0(X, \mathcal{H}^1(\Omega^\bullet))$.

Example 52.0.4. The Leray spectral sequence for a morphism $f : X \rightarrow Y$ and a sheaf \mathcal{F} on X ,

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

gives an exact sequence of low degree terms,

$$0 \longrightarrow H^1(Y, f_* \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^0(Y, R^1 f_* \mathcal{F}) \xrightarrow{d_2} H^2(Y, f_* \mathcal{F}) \longrightarrow H^2(X, \mathcal{F})$$

A particular example is when $f : X \rightarrow Y$ has $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $\mathcal{F} = \mathcal{O}_X^\times$ in which case,

$$0 \longrightarrow \text{Pic}(Y) \longrightarrow \text{Pic}(X) \longrightarrow H^0(Y, R^1 f_* \mathcal{O}_X^\times) \xrightarrow{d_2} H^2(Y, f_* \mathcal{O}_X^\times) \longrightarrow H^2(X, \mathcal{O}_X^\times)$$

which is why we define $\text{Pic}(X/Y) = H^0(Y, R^1 f_* \mathcal{O}_X^\times)$.

Example 52.0.5. The Grothendieck spectral sequence,

$$E_2^{p,q} = R^p F \circ R^q G(A) \implies R^{p+q}(F \circ G)(A)$$

gives an exact sequence of low degree terms,

$$0 \longrightarrow R^1 F \circ G(A) \longrightarrow R^1(F \circ G)(A) \longrightarrow F \circ R^1 G(A) \xrightarrow{d_2} R^2 F \circ G(A) \longrightarrow R^2(F \circ G)(A)$$