1 Picard Scheme

Theorem 1.0.1. Let X be a proper k-scheme. Then $\mathrm{Pic}_{X/k}$ is represented by a lft k-scheme.

Remark. However, this does not hold for proper flat families in general even for curves.

From here on let $f : X \to S$ be a flat, locally finitely presented, proper morphism where $S = \mathrm{Spec}\,(R)$ is a DVR.

Theorem 1.0.2 (8.3.2)**.** $\mathrm{Pic}_{X/S}$ is represented by an algebraic space if and only if f is cohomologically flat in degree 0.

Remark. In fact, the above holds when S is any reduced scheme.

This is a problem since we want to study non-cohomologically flat situations. We fix this in the next section.

1.1 Rigidified Picard Scheme

Proposition 1.1.1 (8.1.6)**.** f admits a rigidifying subscheme meaning a closed subscheme $Y \subset X$ which is flat, locally finitely presented, proper and such that for any $T \to S$ the map,

$$\Gamma(X_T, \mathcal{O}_{X_T}^\times) \to \Gamma(Y_T, \mathcal{O}_{Y_T}^\times)$$

is injective.

Definition 1.1.2. Let $Y \hookrightarrow X$ be a rigidifying subscheme. Then we define the rigidified Picard functor,

$$\mathrm{Pic}_{X/S|Y} : (T \to S) \mapsto \{(\mathcal{L}, \varphi) \mid \mathcal{L} \in \mathrm{Pic}\,(X_T) \text{ and } \varphi : \mathcal{L}|_Y \xrightarrow{\sim} \mathcal{O}_Y\}/\cong$$

The condition of being a rigidifying subscheme shows exactly that there are no nontrivial automorphism of $(\mathcal{L}, \varphi)$.

FGA shows that the functor,

$$(T \to S) \mapsto (f_T)_* \mathcal{O}_{X_T}$$

is representable by a linear scheme V_X over X. This is a vector bundle over X iff f is cohomologically flat in degree 0. Furthermore, the subsheaf of units,

$$(T \to S) \mapsto (f_T)_* \mathcal{O}_{X_T}^\times$$

is represented by an open subscheme,

$$V_X^\times \subset V_X$$

Now V_X is a ring scheme and $V_X^\times$ is a group scheme.

Proposition 1.1.3. Let $Y \hookrightarrow X$ be a rigidifier. There is an exact sequence of fppf sheaves of abelian groups,

$$0 \longrightarrow V_X^\times \longrightarrow V_Y^\times \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow \mathrm{Pic}_{X/S|Y} \longrightarrow 0$$

where the last map forgets the rigidification. It is surjective in the fppf topology because by definition any class in $\mathrm{Pic}_{X/S}$ is fppf locally represented by a line bundle.

Theorem 1.1.4 (8.3.3)**.** Let $Y \hookrightarrow X$ be a rigidifier. Then $\mathrm{Pic}_{X/S|Y}$ is representable by an algebraic space over S which admits a universal rigidified line bundle.

Proposition 1.1.5. Let $s \in S$ be a point such that $H^2(X_s, \mathcal{O}_{X_s}) = 0$. Then there is an open neighborhood $s \in U \subset S$ such that, both $\mathrm{Pic}_{X/S|Y}|_U$ and $\mathrm{Pic}_{X/S}|_U$ are formally smooth over U.

1.2 Relative Curves

Now suppose that f has relative dimension 1 and has geometrically connected fibers.

2 Overview of the Proof

Definition 2.0.1. Let C/k be an integral curve over a field k. Then the *gonality* of C is the smallest degree of a finite map $C \to \mathbb{P}^1$ *over* k. The *geometric gonality* of C is the maximum of the gonality over $\bar{k}$ of the irreducible components of $C_{\bar{k}}$.

Lemma 2.0.2. Let $f : X \to B$ be a proper morphism of relative dimension 1 between normal varities.

Lemma 2.0.3. Let $f : X \to B$ a proper morphism of relative dimension 1 of varities over a perfect field k whose generic fiber is a smooth connected curve. Let $n = \dim X$. Suppose there is a line bundle $\mathcal{L} \hookrightarrow \Omega_X^{n-1}$ whose sections separate d general points on X. Then the general fiber of f has gonality $> d$.

Proof. We can shrink B such that the base and the map are smooth. Choose a general fiber $C \hookrightarrow X$ which is a smooth irreducible curve. Therefore, there is an exact sequence,

$$0 \to \mathcal{C}_{C|X} \to \Omega_X|_C \to \Omega_C \to 0$$

of vector bundles. Since Ω_C is a line bundle there is an exact sequence,

$$0 \to \mathcal{C}_{C|X}^{n-1} \to \Omega_X^{n-1}|_C \to (\wedge^{n-2}\mathcal{C}_{C|X}) \otimes \Omega_C \to 0$$

However, since C is a fiber of f we have $\mathcal{C}_{C|X} = \mathcal{O}_X^{n-1}$. Therefore, we get $n-1$ projection maps,

$$\mathcal{L} \to \Omega_X^{n-1}|_C \to \Omega_C$$

which are all zero exactly if $\mathcal{L} \hookrightarrow \Omega_X^{n-1}$ factors through $\mathcal{C}_{C|X}^{n-1}$ but these forms are constant along fibers so sections of $\mathcal{L}$ would not be able to separate any points on C. Therefore, one of the projections $\mathcal{L} \to \Omega_C$ is a nonzero map of line bundles hence injective meaning that,

$$H^0(C, \mathcal{L}) \to H^0(C, \Omega_C)$$

must be injective. Since we chose C generically $H^0(C, \mathcal{L})$ and hence $H^0(C, \Omega_C)$ can separate d general points on C. Therefore $\mathrm{gon}(C) > d$. □

3 Meaning of Supersingular on even cohomology

What does it mean to have a Frob eigenvalue $\alpha = \zeta q^{i/2}$. This happens exactly when Frob^n has an eigenvalue $(q^n)^{i/2}$. In other words an eigenvector with eigenvalue $\alpha = \zeta q^{i/2}$ is the same as a vector fixed under $\mathrm{Frob}^n / (q^n)^{i/2}$ for some n. By the Tate conjecture, these are classes should be algebraic cycles defined over $\mathbb{F}_{q^n}$. Therefore, for i even, the supersingular eigenspaces are exactly the set of "potentially algebraic cycles" meaning the cycles that are represented by cycle classes of varities defined over possibly lager fields.

4 Characters

We are considering the projective variety X defined by the polynomial,

$$f = a_0 x_0^{n_0} + \cdots + a_r x_r^{n_r}$$

Let $m = \mathrm{lcm}(n_0, \dots, n_r)$ and denote by μ_n the group of n^{th}-roots of unity in $\mathbb{F}_q$. Then there is an action of the group,

$$\mu_{n_0} \times \cdots \times \mu_{n_r}$$

on X. However, the map,

$$\mu_{n_0} \times \cdots \times \mu_{n_r} \to \mathrm{Aut}\,(X)$$

is not injective since X is defined as the quotient under the action,

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda^{\frac{m}{n_0}} x_0, \dots, \lambda^{\frac{m}{n_r}} x_r)$$

therefore the kernel of the map

$$\mu_{n_0} \times \cdots \times \mu_{n_r} \to \mathrm{Aut}\,(X)$$

is exactly the image of

$$\mu_m \to \mu_{n_0} \times \cdots \times \mu_{n_r}$$

under the map

$$\lambda \mapsto (\lambda^{\frac{m}{n_0}}, \dots, \lambda^{\frac{m}{n_r}})$$

Therefore, we get a map,

$$G = (\mu_{n_0} \times \cdots \times \mu_{n_r})/\mu_m \to \mathrm{Aut}\,(X)$$

Since $G \circlearrowright X$ by functoriality it also acts on the middle cohomology,

$$G \circlearrowright H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)$$

Then G is abelian so its irreducible representations are all one-dimensional characters. Therefore, we get a decomposition into spaces on which G acts through a given character,

$$H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell) = \bigoplus_{\chi \in \widehat{G}} H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)(\chi)$$

Weil proved that in our case for each character χ we have,

$$\dim H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)(\chi) \le 1$$

Furthermore, since G acts by automorphisms and the action of Frob is natural meaning that the action of Frob and G commute. Therefore, Frob preserves the irreducible decomposition of G. Since each factor is 1-dimensional,

$$\mathrm{Frob} \circlearrowright H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)$$

is just multiplication by a corresponding Frob eigenvalue α_χ. Furthermore, since if $[Z]$ is the class of a subvariety then $g \cdot [Z] = [g \cdot Z]$ so the action of G preserves the algebraic cycles. Therefore,

$$H^{2i}_{\text{alg}}(X, \mathbb{Q}_\ell) \subset H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell)$$

is a G-subrepresentation. Therefore, since each character space is 1-dimensional then the space of algebraic cycles is a sum of a subset of the characters. These are exactly the "algebraic characters". By the Tate conjecture, they are also the "supersingular characters" i.e. those characters such that $\alpha_\chi = \zeta q^{i/2}$.

Now we need to make the connection to the set $A_{\underline{n},q^f}$. To do this, we fix compatible isomorphisms $\mu_n \cong \mu_n(\mathbb{C})$ for each n dividing $q^f - 1$ (recall that $f = \mathrm{ord}_m(q)$. This just amounts to a choice of generator $g \in \mathbb{F}_{q^f}^\times$ which we identify with $\zeta_{q^f-1} = e^{\frac{2\pi i}{q^f-1}}$. Now for each i and a character,

$$\chi : \mu_{n_i} \to \mu_{n_i}(\mathbb{C})$$

consider the map,

$$\mathbb{F}_{q^f}^\times \to \mu_{n_i} \xrightarrow{\chi} \mu_{n_i}(\mathbb{C})$$

where the map is,

$$x \mapsto x^{\frac{q^f-1}{n_i}} \mapsto \chi(x^{\frac{q^f-1}{n_i}})$$

This gives a map,

$$\widehat{G} \to \mathrm{Hom}\left((\mathbb{F}_{q^f}^\times)^{r+1}, \mathbb{C}^\times\right)$$

The compatible isomorphism then enters when we identify,

$$\widehat{G} = \{(a_0, \dots, a_r) \mid a_i \in (\mathbb{Z}/n_i\mathbb{Z}) \text{ and } (m/n_0)a_0 + \cdots + (m/n_r)a_r \equiv 0 \mod m\}$$

By definition, the character corresponding to $a = 1$ is given by taking the generator of μ_{n_i} which is $g^{\frac{q^f-1}{n_i}}$ and sending to the generator of $\mu_{n_i}(\mathbb{C})$ which is ζ_{n_i} hence the corresponding character of $\mathbb{F}_{q^f}^\times$ is defined on the generator by

$$g \mapsto \zeta_{n_i}$$

which corresponds to $\alpha_i = \frac{1}{n_i}$ as we defined previously. Therefore, this identification of $\widehat{G}$ shows that its image in $\mathrm{Hom}\left((\mathbb{F}_{q^f}^\times)^{r+1}, \mathbb{C}^\times\right)$ is almost the set $A_{\underline{n},q^f}$. Notice we have "explained" where the sum condition comes from but not the conditional $0 < \alpha_i < 1$ i.e. corresponding to a condition that all $a_i \neq 0$. To do this let,

$$\widehat{G}^{\text{prim}} \subset \widehat{G}$$

be the subset where χ is nontrivial when restricted to each $\mu_{n_i} \to G$. The the image of,

$$\widehat{G}^{\text{prim}} \to \mathrm{Hom}\left((\mathbb{F}_{p^f}^\times)^{r+1}, \mathbb{C}^\times\right)$$

is exactly the set $A_{\underline{n},q^f}$. The reason geometrically for considering only primitive characters is, it turns out,

$$\dim H^{r-1}_{\text{ét}}(X, \mathbb{Q}_\ell)(\chi) = 1$$

exactly for the primitive characters and is zero otherwise.

5 Potential Examples

General references,

(a) Beauville

(b) cuboids

(c) review of surfaces

5.1 Desingularizing Branched covers

5.2 Hilbert Modular Surfaces

Let X be a desingularization of the Baliey-Borel compactification.

(a) $\pi_1(X) = 0$ Minimal Models

5.3 Explicit Complete intersection

We need a complete intersection $X = X_{d_1,d_2} \subset \mathbb{P}^4$ of degrees $d_1, d_2 \geq 9$ which is CM. This seems hard to compute.

(a) $\pi_1(X) = 0$ by Lefschetz

(b) Ω_X is big by results of Debarre I think or explicit calculation

(c) H^2 CM is unknown

Some references,

(a) Brotbek explicit

(b) Debarre

(c) weak torelli

(d) debarre thesis

5.4 Special Fermat Complete intersections

Consider a complete intersection of Fermat 3-folds in $\mathbb{P}^4$. This is dual to a complete intersection Fermat curve in $\mathbb{P}^4$ but of rather high genus.

(a) $\pi_1(X) = 0$ by Lefschetz

(b) Ω_X is big by results of Debarre I think or explicit calculation

(c) H^2 CM is unknown

References,

(a) Terasoma

(b) some are unirational (Does this work for only degree $q+1$ or for any degree such that $p^v \equiv -1 \mod d$).

5.5 Torodorov Surfaces

(a) TODOROV

(b) Beauville

(c) counterexamples to Global Torelli

(d) usui

5.6 Moduli of Vector bundles

Are these simply connected?

(a) beauville

(b) parabolic rank 2 on $\mathbb{P}^1$

6 Jets References

(a) Two Applications of Algebraic Geometry to Entire Holomorphic Mappings MARK GREEN AND PHILLIP GRIFFITHS

(b) exceptional set

(c) demailly

(d) log jet bundles

(e) green-griffiths-lang

(f) semple jets

7 Relative BB

References:

(a) Campana

(b) Higgs

(c) algebraic foliations

(d) schnell-singular metrics

(e) Campana - without tears

(f) Horing and Peternell - Algebraic integrability

8 Shioda

We develop a novel obstruction to unirationality in positive characteristic and apply it to produce a counterexample to the Shioda conjecture. The construction of this obstruction is based on jet-bundle techniques.

8.1 Introduction

We say that a variety over $\mathbb{F}_q$ is *supersingular* if the Newton polygon of $\mathrm{Frob}_q \circlearrowleft H^i(X, \mathbb{Q}_\ell)$ has a single slope for each i.

(CHECK EQUIVALENCE)

If X is smooth and projective, then supersingularity is equivalent to the eigenvalues of $\mathrm{Frob}_q \circlearrowleft H^i(X, \mathbb{Q}_\ell)$ are all of the form $\zeta q^{i/2}$ where ζ is a root of unity. Sawin and [OTHER AUTHOR] (CITE) introduced a cohomological birational invariant H^i_{tdF} based on divisibility of Frobenius eigenvalues in the ring of integers. If X is a smooth projective simply-connected surface then X is supersingular if and only if $H^2_{\mathrm{tdF}}(X) = 0$.

In 1973 Shioda made the following conjecture (CITE)

Conjecture 8.1.1 (Shioda)**.** Let X be a smooth projective surface over $\overline{\mathbb{F}}_p$ with $\pi_1^{\text{ét}}(X) = 1$. Then X is supersingular if and only if X is unirational.

He remarked that "if false, this conjecture would be very difficult to verfiy" (FIND QUOTE). His reasoning was presumably based on the dearth of known obstructions to unirationality since the setup of his conjecture ensures that the only as-yet known obstructions, $\pi_1^{\text{ét}}$ and the Galois representation of $H^2(X, \mathbb{Q}_\ell)$ do not suffice. To provide a counterexample therefore, we must develop a novel obstruction.

8.2 Jet Bundles

Jet bundles capture higher-order differential information. There are unfortunately two dual notions which both go under the title "jets". Our jet bundles are jets of maps $X \to \mathbb{A}^1$ – known in EGA as "bundles of principal parts" because they can be interpreted as symbols for differential operators – while the "jets" more commonly studied in the complex geometry literature are jets of maps $\mathbb{A}^1 \to X$. Here we take the former notion.

Definition 8.2.1. Let X/S be a separated S-scheme. We denote by $\Delta_X^n \hookrightarrow X \times_S X$ the n-th order thickening of the diagonal. If $\mathscr{I}_\Delta \subset \mathcal{O}_{X \times_S X}$ is the ideal corresponding to the closed embedding $\Delta_{X/S} : X \hookrightarrow X \times_S X$ then Δ^n is defined by $\mathscr{I}^{n+1}$. The closed subscheme $\Delta_X^n \subset X \times_S X$ is then equipped with projection maps $\pi_i : \Delta_X^n \to X$ for $i = 1, 2$.

Definition 8.2.2. Let X/S be a smooth separated scheme. Let $\mathcal{E}$ be a vector bundle on X. Then the n-th *jet bundle* of $\mathcal{E}$ is defined as the $\mathcal{O}_X$-module,

$$J^n(\mathcal{E}) := \pi_{1*}\pi_2^*\mathcal{E} = (\mathcal{O}_{X \times_S X}/\mathscr{I}_\Delta^{n+1}) \otimes_{\mathcal{O}_X} \mathcal{E}$$

using the projections $\pi_i : \Delta_X^n \to X$. We write $J^n(X) = J^n(\mathcal{O}_X)$.

Proposition 8.2.3. There are exact sequences,

$$0 \longrightarrow \mathrm{Sym}^n(\Omega_X) \otimes \mathcal{E} \longrightarrow J^n(\mathcal{E}) \longrightarrow J^{n-1}(\mathcal{E}) \longrightarrow 0$$

Proof. DO IT. ☐

Proposition 8.2.4. Let $f : X \to Y$ be morphism of smooth varities and $\mathcal{E}$ a vector bundle on Y. Then there is a pullback map,

$$f^* J^n(\mathcal{E}) \to J^n(f^*\mathcal{E})$$

compatible with the pullback on Ω_X and the projection maps.

Proof. This follows immediately from the map $\Delta_X^n \to \Delta_Y^n$ which induces the natural pullback $f^*\mathrm{Sym}^n(\Omega_Y) \to \mathrm{Sym}^n(\Omega_X)$ DO THIS!! ☐

Finally, we show that sections of jet bundles are birational invariants in the category of smooth projective varities as are symmetric differentials.

Proposition 8.2.5. Let $f : X \dashrightarrow Y$ be a rational map of smooth projective varieties. Then there are functorial pullback maps on global sections,

$$H^0(Y, J^m(Y)) \to H^0(X, J^m(X))$$

comptatible with the pullback on global symmetric differentials and the projection maps.

Proof. Let $U \subset X$ be the domain of f. Since X is smooth and Y is proper then $\mathrm{codim}\left(U^C, X\right) \geq 2$. Since $J^m(X)$ is a vector bundle we then get by Harthog,

$$H^0(Y, J^m(Y)) \to H^0(U, J^m(U)) = H^0(X, J^m(X))$$

This is also how the pullback on $\mathrm{Sym}^n(\Omega_Y)$ is defined and is compatible with projections because the pullback along $U \to Y$ is. ☐

8.3 The Obstruction

The obstruction is based on a higher-order generalization of the observation that a unirational variety in characteristic zero cannot carry any nonzero symmetric differentials.

Theorem 8.3.1. Suppose that $\omega \in H^0(X, \mathrm{Sym}^r(\Omega_X))$ is a symmetric differential form that admits a lift to some section,

$$\widetilde{\omega} \in H^0(X, J^{p^h r}(X))$$

meaning $\widetilde{\omega} \mapsto \omega$ under the projection $J^{p^h r}(X) \to J^r(X)$ and the inclusion $\mathrm{Sym}^r(\Omega_X) \hookrightarrow J^r(X)$. Then there does not exist a dominant purely-inseparable map $\mathbb{P}^n \dashrightarrow X$ of height h.

9 Every Abelian variety over a finite field has "CM"

This is because $\mathrm{End}\,(A)$ is always bigger than $\mathbb{Z}$ since there is relative $\mathrm{Frob}_q : A \to A$ if A is defined over $\mathbb{F}_q$. However, this does not work if A is defined over an infinite field k. This is because we have a relative Frobenius $\mathrm{Frob}_q : A \to A^{(q)}$ but these are not isomorphic k-schemes unless A is defined over $\mathbb{F}_q$.

10 Cartier Operator

Setting: let $X \to S$ be a morphism of schemes of characteristic p. Then there is absolute Frobenius $F_S : S \to S$ which is identity on the underlying spaces and $a \mapsto a^p$ as a map of sheaves $\mathcal{O}_S \to \mathcal{O}_S$. Consider the diagram,

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[r, bend left, "F_X"] \arrow[dr] & X' \arrow[r] \arrow[d, phantom, "\lrcorner"] & X \arrow[d] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

then the dashed map is called the relative Frobenius $\mathrm{Frob}_{X/S} : X \to X'$. Then $\mathrm{Frob}_{X/S}$ is finite if $X \to S$ is lft and is finite locally free if $X \to S$ is smooth.

Example 10.0.1. Let $X = \mathbb{A}^n$ and $S = \mathrm{Spec}\,(k)$ then $\mathrm{Frob}_{X/S} : x_i \mapsto x_i^p$ is a k-linear map while $F_X : f \mapsto f^p$ is not k-linear since $af \mapsto a^p f^p$.

Theorem 10.0.2 (Cartier)**.** There exists a unique morphism of graded $\mathcal{O}_{X'}$-algebras.

$$C^{-1} : \bigoplus_{i \ge 0} \Omega^i_{X'/S} \to \bigoplus_{i \ge 0} \mathcal{H}^i(F_* \Omega^\bullet_{X/S})$$

such that,

(a) $C^{-1}(1) = 1$

(b) $C^{-1}(\alpha \wedge \beta) = C^{-1}(\alpha) \wedge C^{-1}(\beta)$

(c) $C^{-1}(1 \otimes \mathrm{d}f) = [f^{p-1}\mathrm{d}f]$ where $1 \otimes \mathrm{d}f$ denotes the pullback of $\mathrm{d}f \in \Gamma(X', \Omega^1_{X'/S})$ along $U : X' \to X$

Furthermore, if $\pi : X \to S$ is smooth then C^{-1} is an isomorphism.

Proof. It suffices to construct the degree 1 part of C^{-1} via the universal property of exterior algebras. To get,

$$C^{-1} : \Omega^1_{X'/S} \to \mathcal{H}^1(F_* \Omega^\bullet_{X/S})$$

Suffices to construct,

$$\Omega^1_{X/S} \to u_* \Omega^1_{X'/S} \to \mathcal{H}^1(F_* \Omega^\bullet_X)$$

which is the same data as a derivation,

$$\delta : \mathcal{O}_X \to \mathcal{H}^1(F_* \Omega^\bullet_{X/S})$$

meaning,

$$\delta(fg) = f^p \delta(g) + g^p \delta(f)$$

Try $\delta(f) = [f^{p-1}\mathrm{d}f]$. It satisfies Leibnitz law and additivity since,

$$\delta(f+g) - \delta(f) - \delta(g) = \left[\mathrm{d} \left(\frac{(f+g)^p - f^p - g^p}{p} \right) \right] = 0$$

This takes care of existence and also uniqueness. Then for smoothness we reduce to $\mathbb{A}^n_S$ and thus to $\mathbb{A}^1_k$. Then we set $X = \mathrm{Spec}\,(k[x])$ and $X' = \mathrm{Spec}\,(k[y])$. Now $F_* \Omega^1_{X/S}$ is the complex,

$$0 \to k[x^p] \left\langle 1, x, \dots, x^{p-1} \right\rangle \to k[x^p] \left\langle \mathrm{d}x, x\mathrm{d}x, \dots, x^{p-1}\mathrm{d}x \right\rangle \to 0$$

$\square$