Contents

1 Chapter 1.1

1.1 1.2

Consider the following conditions on a ring R,

(I) R satisfies the IBP (if $R^n \cong R^m$ then $n = m$).

(II) For all m, n and P if $R^m \cong R^n \oplus P$ then $m \geq n$.

(III) For all n and P if $R^n \cong R^n \oplus P$ then $P = 0$

We will show (III) $\implies$ (II) $\implies$ (I). First suppose R satisfies (III) and consider the situation that $R^m \cong R^n \oplus P$ and $m < n$. We can add R^{n-m} to each side to get,

$$R^n \cong R^n \oplus (P \oplus R^{n-m})$$

then applying (III) we find $P \oplus R^{n-m} = 0$ a contradiction proving (II).

Now assume property (II) and suppose that $R^m \cong R^n$. By applying (II) in the case $P = 0$ we find $m \geq n$ and $n \geq m$ and thus $m = n$ proving the IBP i.e. property (I).

1.2 1.3

We need to show that the following conditions on a ring R are equivalent,

(a) For all n, every surjection $R^n \to R^n$ is an isomorphism.

(b) For all n, and $f, g \in M_n(R)$ if $fg = \text{id}$ then $gf = \text{id}$ and $g \in \text{GL}_n(R)$.

(c) For all n and P if $R^n \cong R^n \oplus P$ then $P = 0$.

First suppose property (a) and let $fg = \text{id}$ for $f, g \in M_n(R) = \text{End}\,(R^n)$. Since $fg = \text{id}$ the map $g : R^n \to R^n$ is surjective and thus an isomorphism by property (a). so we find that $g \in \text{GL}_n(R)$ and there is some $h \in \text{GL}_n(R)$ such that $gh = hg = \text{id}$. However,

$$fgh = (fg)h = h = f(gh) = f$$

so $h = f$ and thus $fg = gf = \text{id}$ proving (b).

Now suppose (b) holds and suppose we have the situation $R^n \cong R^n \oplus P$. Then consider the maps $\iota : R^n \to R^n \oplus P$ and $\pi : R^n \oplus P$ which satisfiy $\pi \circ \iota = \text{id}$. Now let $f : R^n \to R^n \oplus P$ be the given isomorphism then define $\tilde{\iota} = f^{-1} \circ \iota : R^n \to R^n$ and $\tilde{\pi} = \pi \circ f : R^n \to R^n$ and thus $\tilde{\pi} \circ \tilde{\iota} = \text{id}$ and $\tilde{\pi}, \tilde{\iota} \in \text{End}\,(R^n) = M_n(R)$. Thus by (b), $\tilde{\iota} \circ \tilde{\pi} = \text{id}$ so $\tilde{\iota} = f^{-1} \circ \iota$ is an isomorphism which implies that $\iota : R^n \to R^n \oplus P$ is an isomorphism (since f^{-1} is) and thus $P = 0$ proving (c).

Finally, suppose (c) and suppose that $f : R^n \to R^n$ is a surjection. Then consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \longrightarrow R^n \longrightarrow 0$$

Then R^n is free and thus projective so the sequence is split,

$$R^n \cong R^n \oplus \ker f$$

so by (c) we have $\ker f = 0$ and thus f is an isomorphism proving (a).

Finally suppose that R is commutative and $f, g \in M_n(R)$ with $fg = \mathrm{id}$. Then $\det fg = \det f \det g = 1$ so $f, g \in \mathrm{GL}_n(R)$ and thus there exists a matrix (coefactors) h such that $gh = \mathrm{id}$ then $f = h$ by a previous argument. Therefore commutative rings satisfy all the above properties.

1.3 1.7

(NO IDEA)

1.4 1.8

(NO IDEA)

1.5 1.9

(NO IDEA)

2 Chapter 1.2

Remark. In this section R is a commutative (unital) ring.

2.1 2.2

2.2 2.4

Consider a continuous function $f : \mathrm{Spec}\,(R) \to \mathbb{Z}$. First, $\mathrm{Spec}\,(R)$ is quasi-compact. This is easily shown since every affine cover U_i can be refined to a cover by principal opens $D(f_i)$ then,

$$\mathrm{Spec}\,(R) = \bigcup_{i=1}^{\infty} D(f_i) = D(\langle f_i \rangle)$$

(since $f_i \notin \mathfrak{p}$ for some f_i iff $\langle f_i \rangle \not\supset \mathfrak{p}$) and thus $\langle f_i \rangle = R$ (otherwise it would be contained in a maximal ideal) but then $1 = r_1 f_1 + \cdots + r_n f_n$ for finitely many so,

$$\mathrm{Spec}\,(R) = D(\langle f_1, \dots, f_n \rangle) = \bigcup_{i=1}^{n} D(f_i)$$

so there is a finite subcover of U_i.

Therefore, $f(\mathrm{Spec}\,(R)) \subset \mathbb{Z}$ is compact and thus finite so it must take finitely many values $n_1, \dots, n_c$. Then $V_i = f^{-1}(n_i)$ is a closed subset of $\mathrm{Spec}\,(R)$ since $\mathbb{Z}$ is discrete.

If R is not reduced then consider $R_{\mathrm{red}} = R/\mathrm{nilrad}\,(R)$ and $\mathrm{Spec}\,(R) \cong \mathrm{Spec}\,(R_{\mathrm{red}})$ naturally so we may assume that R is reduced and we may use idempotent lifitng (2.2).

Since V_i is closed $V_i = V(I_i)$ for some ideal $I_i \subset R$. Furthermore,

$$\text{Spec}\,(R) = \bigcup_{i=1}^{n} V_i = \bigcup_{i=1}^{n} V(I_i) = \bigcup_{i=1}^{n} V(I_n) = V(I_1 \cdots I_n)$$

Thus $\sqrt{I_1 \cdots I_n} = \text{nilrad}\,(R) = (0)$ so $I_1 \cdots I_n = (0)$. Furthermore, the V_i are disjoint so,

$$\varnothing = V_i \cap V_j = V(I_i) \cap V(I_j) = V(I_i + I_j)$$

and thus $I_i + I_j = R$ so the ideals I_i and I_j are coprime. Therefore, by CRT,

$$R = R/(0) = R/(I_1 \cdots I_n) = (R/I_1) \times \cdots \times (R/I_n)$$

since these ideals are pairwise coprime. (Note, there is an error in the text, it has these two conditions backwards).

2.3 2.5

Conisder the following properties,

(a) $\text{Spec}\,(R)$ is connected.

(b) Every finitely generated projective R-module has constant rank.

(c) R has no idempotent elements except 0 and 1.

I claim that these are equivalent.

See the background material in Appendix A, but for any finitely-generated projective module If $\text{Spec}\,(A)$ is connected then since $\text{rank}(P)$ is continuous (see Appendix) then then its image must be connected in $\mathbb{Z}$ and thus constant.

Suppose $e \in R$ were a nontrivial idepotent. Then consider the module $P = (e)$ which I claim is f.g. (obvious) and projective. It suffices to show that P is free on some open cover. On the open set $D(e)$ we have $P_e \cong R_e$ so P is free on $D(e)$ of rank 1. Furthermore, on the open set $D(1-e)$ we have $P_{1-e} = (e)_{1-e} = (0)$ since $e^2 = e$ and thus P is free of rank 0. Since $e + (1-e) = 1$ these open sets cover $\text{Spec}\,(R)$. Therefore P is f.g. projective but does not have finite rank. Thus (b) $\implies$ (c).

Finally, if $\text{Spec}\,(R)$ is not connected then we can write $\text{Spec}\,(R) = V(I) \cup V(J)$ for two nontrivial disjoint closed sets in which case $IJ = (0)$ and $I + J = R$. Thus by CRT, $R = (R/I) \times (R/J)$. However, the element $(1, 0)$ in this product is a nontrivial idempotent in the ring. Thus (c) $\implies$ (a).

2.4 2.8

2.5 2.10

Let P, Q be R-modules and $P \otimes_R Q \cong R^n$ for $n > 0$. Then P and Q are f.g. projective R-modules.

Proof. (DO) $\square$

2.6 2.11

Let M be a finitely generated module over a commuative ring R. I claim that the following are equivalent for every n,

(a) M is f.g. projective of constant rank n

(b) $M_\mathfrak{p} \cong R_\mathfrak{p}^n$ for every prime ideal $\mathfrak{p}$ of R.

Clearly (a) $\implies$ (b) so we assume that $M_\mathfrak{p} \cong R_\mathfrak{p}^n$ at each prime $\mathfrak{p}$. By Lemma 2.4 it suffices to show that M is finitely presented since then freeness of the stalks implies projectivity and M is automatically of constant rank n by definition.

Lift the basis map $R_\mathfrak{p}^n \to M_\mathfrak{p}^n$ to a map $f : R^n \to M$ by clearing denominators. Now consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \longrightarrow M \longrightarrow \operatorname{coker} f \longrightarrow 0$$

Since M is finitely generated then so is $\operatorname{coker} f$. Furthermore, when we localize at $\mathfrak{p}$ we get,

$$0 \longrightarrow (\ker f)_\mathfrak{p} \longrightarrow R_\mathfrak{p}^n \longrightarrow M_\mathfrak{p} \longrightarrow (\operatorname{coker} f)_\mathfrak{p} \longrightarrow 0$$

but we know $R_\mathfrak{p}^n \to M_\mathfrak{p}$ is an isomorphism so $(\operatorname{coker} f)_\mathfrak{p} = (\ker f)_\mathfrak{p} = 0$. Since $\operatorname{coker} f$ is f.g. there exists $g \in R$ such that $(\operatorname{coker} f)_g = 0$. Then localizing at g instead we find,

$$0 \longrightarrow (\ker f)_g \longrightarrow R_g^n \longrightarrow M_g \longrightarrow 0$$

Then for any prime $\mathfrak{q} \in D(g)$ we may localize again to find,

$$0 \longrightarrow (\ker f)_\mathfrak{q} \longrightarrow R_\mathfrak{q}^n \longrightarrow M_\mathfrak{q} \longrightarrow 0$$

so $R_\mathfrak{q}^n \to M_\mathfrak{q}$ is a surjection. However, by assumption $M_\mathfrak{q}$ is free of rank n and R is commutative so by 1.3 property (a). we know $R_\mathfrak{q}^n \to M_\mathfrak{q}$ is an isomorphism and thus $(\ker f)_\mathfrak{q} = 0$. Therefore $(\ker f)_g$ is an A_g-module with empty support so $(\ker f)_g = 0$. Therefore, $M_g \cong R_g^n$ so M is locally free and thus projective.

Therefore, suppose that M is finitely generated free at each stalk with $\operatorname{rank}(M)$ continuous. Then $\operatorname{Spec}(R)$ has a finite open cover $U_i = (\operatorname{rank}(M))^{-1}(n_i)$ such that $M|_{U_i}$ is f.g. with $M_\mathfrak{p} = R_\mathfrak{p}^{n_i}$ for fixed n_i on each U_i. Thus we have shown that M is locally free on U_i and thus locally free on $\operatorname{Spec}(R)$ and thus projective. Conversely if M is f.g. projective then we know (by Lemma 2.4) that M is locally free and thus $M_\mathfrak{p} \cong R_\mathfrak{p}^{n_\mathfrak{p}}$ and has continuous rank funciton.

2.7 2.12

Let $\phi : R \to S$ be a morphism of rings then let $f = \phi^{-1} : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ be the associated morphism of affine schemes. Now there is a functor,

$$f^* : \mathfrak{QCoh}(\operatorname{Spec}(R)) \to \mathfrak{QCoh}(\operatorname{Spec}(S))$$

given explicitly by $M \mapsto M \otimes_R S$. I claim that if P is f.g. projective then f^*P is f.g. projective. This is clear using the following property and noting that $(-) \otimes_R S$ is left adjoint to restriction of an S module to an R module which is clearly exact.

Lemma 2.7.1. If a functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$ between abelian categories and G is exact then F preserves projectives.

Proof. $F(P)$ is projective iff $\mathrm{Hom}_{\mathcal{C}}(F(P), -)$ is exact but,

$$\mathrm{Hom}_{\mathcal{D}}(F(P), -) \cong \mathrm{Hom}_{\mathcal{C}}(P, G(-))$$

which is exact since G and $\mathrm{Hom}_{\mathcal{C}}(P, -)$ are for projective P. $\square$

Now I claim that $\mathrm{rank}(f^*P) = \mathrm{rank}(P) \circ f$. This is because,

$$(f^*P) \otimes_{S_{\mathfrak{p}}} \kappa(\mathfrak{p}) = P \otimes_R S \otimes_{S_{\mathfrak{p}}} \kappa(\mathfrak{p}) = P \otimes_R \kappa(\mathfrak{p})$$

Via the map $R \to S \to \kappa(\mathfrak{p})$. Now we get an inclusion of fields, $\kappa(f(\mathfrak{p})) \to \kappa(\mathfrak{p})$ which $R \to \kappa(\mathfrak{p})$ factors through. Thus,

$$P \otimes_R \kappa(\mathfrak{p}) = P \otimes_R \kappa(f(\mathfrak{p})) \otimes_{\kappa(f(\mathfrak{p}))} \kappa(\mathfrak{p})$$

In particular, these vectorspaces have equal rank i.e.

$$\begin{aligned}\mathrm{rank}_{\mathfrak{p}}(f^*P) &= \dim_{\kappa(\mathfrak{p})}(P \otimes_R \kappa(\mathfrak{p})) = \dim_{\kappa(\mathfrak{p})}(P \otimes_R \kappa(f(\mathfrak{p})) \otimes_{\kappa(f(\mathfrak{p}))} \kappa(\mathfrak{p})) \\ &= \dim_{\kappa(f(\mathfrak{p}))}(P \otimes_R \kappa(f(\mathfrak{p})) = \mathrm{rank}_{f(\mathfrak{p})}(P)\end{aligned}$$

2.8 2.16

Fix a small category of rings $\mathcal{R}$. A big projective R-module is a choice of a finitely generated projective S-module P_S for each S over R in $\mathcal{R}$ equiped with an isomrophism $P_S \otimes_S T \to P_T$ for every $S \to Y$ over R which satisfies the following properties,

(a) the identity $\mathrm{id} : S \to S$ induces $\mathrm{id} : P_S \to P_S$

(b) to each commutative triangle of R-algebras we have a commutative triangle of modules.

Now let $\mathbb{P}'(R)$ denote the category of big R-modules and $\mathbb{P}'(R) \to \mathbb{P}(R)$ be the forgetful functor sending P to P_R. (FINISH THIS)

3 Chapter 1.3

Remark. Here R is a commutative (unital) ring.

3.1 3.1

We need to show that the following are equivalent properties of an R-module L,

(a) there is an R-module M such that $L \otimes M \cong R$

(b) L is an algebraic line bundle (a f.g. projective module of constant rank 1)

(c) L is a finitely generated R-module and $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ for each prime $\mathfrak{p}$.

Proof. Assuming (a) then by 2.10 we have L and M are finitely generated projective. Thus $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$ and $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^m$ for some n, m but then $L_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{mn}$ so $m = n = 1$ proving (b).

(b) $\implies$ (c) is a trivial consequence of Lemma 2.4.

Finally assume (c) then I claim that $L \otimes_R L^\vee \cong R$ where $L^\vee = \mathrm{Hom}_R(L, R)$. First, not there is a natural map $L \otimes L^\vee \to R$ by evaluation. We may check this map is an isomorphism locally on stalks,

$$L_{\mathfrak{p}} \otimes \mathrm{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, R_{\mathfrak{p}}) \to R_{\mathfrak{p}}$$

(note that $(\mathrm{Hom}_R(L, R))_{\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(L_{\mathfrak{p}}, R_{\mathfrak{p}})$ holds since L is finitely presented which holds because it is f.g. projective using criterion (4) proved in 2.11). However, $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ so this above map is clearly an isomorphism with $1 \otimes \mathrm{id} \mapsto 1$. $\square$

3.2 3.4

3.3 3.15

3.4 3.18

Consider the following sequence,

$$1 \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R[t]) \times \mathrm{Pic}(R[t^{-1}]) \longrightarrow \mathrm{Pic}(R[t, t^{-1}])$$

the first map is induced by the inclusions $R \to R[t]$ and $R[t^{-1}]$ and the second by the difference of the maps induced by the inclusion $R[t] \to R[t, t^{-1}]$ and $R[t^{-1}] \to R[t, t^{-1}]$. Since $\mathrm{Pic}(-)$ is a covariant functor on the category of commutative rings the above sequence is a complex since,

$$R \longrightarrow R[t] \times R[t^{-1}] \longrightarrow R[t, t^{-1}]$$

is exact (this is the computation showing that $\Gamma(\mathbb{P}^1_R, \mathcal{O}_{\mathbb{P}^1_R}) = R$).

Now, given $P \in \mathrm{Pic}(R[t])$ and $Q \in \mathrm{Pic}(R[t^{-1}])$ suppose that $P \otimes_{R[t]} R[t, t^{-1}]$ and $Q \otimes_{R[t^{-1}]} R[t, t^{-1}]$ are isomorphic as $R[t, t^{-1}]$-modules.

(USE UNITS-PIC sequence and snake lemma)

4 Chapter 2.1

4.1 2.1

4.2 2.2

4.3 2.3

5 Chapter 2.3

5.1 3.3

5.2 3.4

5.3 3.5

5.4 3.7

6 Chapter 2.4

6.1 4.2

7 Chapter 2.5

7.1 5.2

7.2 5.7

7.3 5.8

8 Chapter 2.6-8

9 Appendix A. Rank Functions

Remark. Here R is a commutative (unital) ring.

Definition 9.0.1. Let M be an R-module. Then there is a function $\text{rank}(M) : \text{Spec}\,(R) \to \mathbb{Z}$ defined by $x \mapsto \text{rank}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}))$.

Proposition 9.0.2. $\text{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators of M_p as an $R_{\mathfrak{p}}$-module.

Proof. If $M_{\mathfrak{p}}$ is generated by $m_1, \dots, m_n$ then $M_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{p}) = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is generated by $\bar{m}_1, \dots, \bar{m}_n$ over $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ since surjectivity of $R_{\mathfrak{p}}^n \to M_{\mathfrak{p}}$ is preserved after applying $(-) \otimes_R \kappa(\mathfrak{p})$. Thus, $\text{rank}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})} M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) \le n$.

Now suppose that $v_1, \dots, v_n$ is a $\kappa(\mathfrak{p})$-basis of $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ then choose lifts $m_1, \dots, m_n \in M_{\mathfrak{p}}$. I claim that $m_1, \dots, m_n$ generated $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module. Let $N \subset M_{\mathfrak{p}}$ be the $R_{\mathfrak{p}}$-submodule generated by the $m_1, \dots, m_n$ and let $K = M_{\mathfrak{p}}/N$. Then I claim that $\mathfrak{p}K = K$. To see this it suffices to show that $K \subset \mathfrak{p}K$. For any $m \in M_{\mathfrak{p}}$ we know that its image $\bar{m} \in M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is in the span of the basis $v_1, \ldots, v_n$ so,

$$\bar{m} = r_1 v_1 + \cdots r_n v_n$$

for $r_i \in R_{\mathfrak{p}}$. Thus,

$$m - (r_1 m_1 + \cdots r_n m_n) \in \mathfrak{p}M$$

This implies that in K we have $m \in \mathfrak{p}K$ so $K = \mathfrak{p}K$. Then since $\mathrm{Jac}\,(R_{\mathfrak{p}}) = \mathfrak{p}$ (because $R_{\mathfrak{p}}$ is local) by Nakayama $K = 0$ so $M_{\mathfrak{p}}$ is generated by $m_1, \ldots, m_n$. □

Theorem 9.0.3. The following are equivalent:

(a) M is a finitely-generated projective R-module

(b) M is a locally-free R-module of finite rank $\mathrm{rank}_x(M) < \infty$

(c) M is a finitely-presented R-module and for each $\mathfrak{p} \in \mathrm{Spec}\,(R)$, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module.

Proposition 9.0.4. If P is a finitely-generated projective module then $\mathrm{rank}(P) : \mathrm{Spec}\,(R) \to \mathbb{Z}$ is continuous.

Proof. It suffices to prove for $f = \mathrm{rank}(P)$ that $f^{-1}(n) = V$ is open. For any $\mathfrak{p} \in V$ we know that $P_{\mathfrak{p}}$ is free of rank n. Lift a basis (by clearing demoninators) to a map $f : R^n \to P$ and consider the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow R^n \xrightarrow{f} P \longrightarrow \mathrm{coker}\, f \longrightarrow 0$$

Since P is fintely generated then $\mathrm{coker}\, P$ is also finitely generated. Localizing this exact sequence at $\mathfrak{p}$ we get an exact sequence,

$$0 \longrightarrow (\ker f)_{\mathfrak{p}} \longrightarrow R^n_{\mathfrak{p}} \xrightarrow{f} P_{\mathfrak{p}} \longrightarrow (\mathrm{coker}\, f)_{\mathfrak{p}} \longrightarrow 0$$

but $f : R^n_{\mathfrak{p}} \to P_{\mathfrak{p}}$ is an isomorphism so $(\mathrm{coker}\, f)_{\mathfrak{p}} = (\ker f)_{\mathfrak{p}} = 0$. Since $(\mathrm{coker}\, f)_{\mathfrak{p}}$ is finitely generated there is some $g \notin \mathfrak{p}$ such that $\mathrm{coker}\, f_{\mathfrak{p}} = 0$. Thus we have.

$$0 \longrightarrow (\ker f)_g \longrightarrow R^n_g \longrightarrow P_g \longrightarrow 0$$

We have yet to use projectivity of P so, in particular, we see that $\forall \mathfrak{q} \in D(g) : \mathrm{rank}_{\mathfrak{q}}(M) \le n$ for any finitely-generated R-module M. We call this upper-semicontinuity of $\mathrm{rank}(M) : \mathrm{Spec}\,(R) \to \mathbb{Z}$.

Now applying projectivity of P (and thus P_g as a R_g-module) the above exact sequence splits to give,

$$R^n \cong P_g \oplus (\ker f)_g$$

so the projection $R^n \twoheadrightarrow (\ker f)_g$ shows that $(\ker f)_g$ is finitely generated and $((\ker f_) g)_{\mathfrak{p}} = 0$ so there is some $h \notin \mathfrak{p}$ such that $(\ker f)_{gh} = 0$. Then, by exactness of localization we get $R^n_{gh} \xrightarrow{\sim} P_{gh}$ so P is free of rank n on $D(gh)$ and thus $\forall \mathfrak{q} \in D(gh) : \mathrm{rank}_{\mathfrak{q}}(P) = n$ so $\mathfrak{p} \in D(gh) \subset V$. Therefore, V is open so this function is continuous. □

Definition 9.0.5. Let X be a scheme and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module then there is a function $\mathrm{rank}(\mathscr{F}) : X \to \mathbb{Z}$ defined by $x \mapsto \mathrm{rank}_x(\mathscr{F}) = \dim_{\kappa(x)}(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x))$.

Remark. Since $\mathscr{F}$ is coherent then locally $\mathscr{F}|_U = \widetilde{M}$ for some finitely generated A-module with $U = \operatorname{Spec}(A)$. (Note that this is necessary for coherence but only sufficient when X is locally noetherian). Thus, $\mathscr{F}_x$ is a finitely-generated $\mathcal{O}_{X,x}$-module and thus $\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is finite dimensional.

Theorem 9.0.6. If $\mathscr{F}$ is a projective coherent $\mathcal{O}_X$-module then $\operatorname{rank}(\mathscr{F}) : X \to \mathbb{Z}$ is continuous.

Proof. $\square$

Proposition 9.0.7. Projective coherent $\mathcal{O}_X$-modules on a scheme X are exactly locally-free $\mathcal{O}_X$-modules of finite type. (CHECK).