1 Basic Definitions and Examples

1.1 Genera

1.2 Riemann-Roch

1.3 Riemann–Hurwitz

2 Hyperelliptic Curves

Definition 2.0.1. A curve C is *hyperelliptic* if there exists a degree two map $f : C \to \mathbb{P}^1$.

Lemma 2.0.2. A curve C is hyperelliptic iff Ω^1_C is not very ample.

Proof. (DO THIS) $\square$

Proposition 2.0.3. Plane curves with $g > 1$ cannot be hyperelliptic.

Proof. Let $\iota : C \hookrightarrow \mathbb{P}^2$ be a plane curve. Then $\Omega^1_C = \iota^*\mathcal{O}_{\mathbb{P}^2}(d-3)$ where d is the degree of C. Since $g > 1$ we must have $d > 3$ and thus $\mathcal{O}_{\mathbb{P}^2}(d-3)$ is very ample defining the Veronese embedding $v : \mathbb{P}^2 \to \mathbb{P}^N$ s.t. $\mathcal{O}_{\mathbb{P}^2}(d-3) = v^*\mathcal{O}_{\mathbb{P}^N}(1)$. Then $v \circ \iota : C \to \mathbb{P}^N$ is an embedding such that $(v \circ \iota)^*\mathcal{O}_{\mathbb{P}^N}(1) = \Omega^1_C$. Thus Ω^1_C is very ample so C cannot be hyperelliptic. $\square$

Lemma 2.0.4. Let C have a $\mathfrak{g}^1_2$ then C is either hyperelliptic or rational.

Proof. Let D be a $\mathfrak{g}^1_2$ then $|D|$ defines a rational map $C \dashrightarrow \mathbb{P}^1$ of degree two. Suppose P were a basepoint of $|D|$ then $\dim |D - P| = 1$ which implies that C is rational because there is a rational degree one map $C \dashrightarrow \mathbb{P}^1$. $\square$

Proposition 2.0.5. Any genus 2 curve is hyperelliptic.

Proof. Consider the canonical divisor K_X which has $\deg K_X = 2g-2 = 2$ and $\dim |K_X| = g-1 = 1$ and thus gives a $\mathfrak{g}^1_2$. $\square$