Mathematics GU4051 Topology
Assignment # 4

Benjamin Church

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Problem 1.

Let X, Y be topological spaces and $X \times Y$ have the product topology. Take $(x, y) \in \overline{A \times B}$ then for any open sets U, V such that $x \in U \in \mathcal{T}_X$ and $y \in V \in \mathcal{T}_Y$. Then $(x, y) \in U \times V$ so because (x, y) is in the closure, $A \times B \cap U \times V \neq \emptyset$. Then $A \cap U \neq \emptyset$ and $B \cap V \neq \emptyset$. Thus, $x \in U \implies A \cap U \neq \emptyset$ and $y \in V \implies B \cap V \neq \emptyset$ so $x \in \bar{A}$ and $y \in \bar{B}$ thus, $(x, y) \in \bar{A} \times \bar{B}$. Therefore, $\overline{A \times B} \subset \bar{A} \times \bar{B}$.

Alternatively, by Lemma **??**, $\bar{A} \times \bar{B}$ is a closed subset of $X \times Y$ and $A \subset \bar{A}$ and $B \subset \bar{B}$. Therefore, $A \times B \subset \bar{A} \times \bar{B}$ which is closed so $\overline{A \times B} \subset \bar{A} \times \bar{B}$.

Conversely, if $(x, y) \in \bar{A} \times \bar{B}$ and $(x, y) \in W \in \mathcal{T}_{X \times Y}$ then by the definition of the product topology, $\exists U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ s.t. $(x, y) \in U \times V \subset A \times B$ but $x \in \bar{A}$ so $U \cap A \neq \emptyset$ and similarly, $y \in \bar{B}$ so $V \cap B \neq \emptyset$. Thus, $U \times V \cap A \times B \neq \emptyset$ but $U \times V \subset W$ so $W \cap A \times B \neq \emptyset$ so $(x, y) \in \overline{A \times B}$. Thus, $\bar{A} \times \bar{B} \subset \overline{A \times B}$,

Problem 2.

Take $A = (-2, 1) \cup \{2\} \subset \mathbb{R}$ and $B = \{-2\} \cup (-1, 2) \subset \mathbb{R}$ Then, $A \cap B = (-1, 1)$ and $\bar{A} \cap B = \{-2\} \cup (-1, 1]$ and $A \cap \bar{B} = [-1, 1) \cup \{2\}$ and $\overline{A \cap B} = [-1, 1]$ and $\bar{A} \cap \bar{B} = [-1, 1] \cup \{-2, 2\}$. No two of these are equal.

Problem 3.

Let $(X, \mathcal{T})$ be a Hausdorff space. Consider $x \in X$ and any $y \in X \setminus \{x\}$. Now, since $x \neq y$, by the Haudorff property, there exist $U_y, V_y \in \mathcal{T}$ s.t. $x \in U_y$ and $y \in V_y$ and $U_y \cap V_y = \emptyset$. Thus, since $x \in U_y$ then $x \notin V_y$. Now take

$$V = \bigcup_{y \in X \setminus \{x\}} V_y$$

Because $x \notin V_y$ we have $x \notin V$ so $V \subset X \setminus \{x\}$. However, for any $y \in X \setminus \{x\}$ we have $y \in V_y$ thus $y \in V$ so $V = X \setminus \{x\}$. But each V_y is open thus $V = X \setminus \{x\}$ is open so $\{x\}$ is closed.

Problem 4.

Let the *diagonal* of X be the set $\Delta = \{(x, x) \in X \times X \mid x \in X\}$. Let Δ be closed in the product topology $X \times X$. Then $\Delta^C = (X \times X)\backslash\Delta$ is open. Take $x \neq y$ then $(x, y) \in \Delta^C$ so by openness, $\exists : U, V \in \mathcal{T}$ s.t. $(x, y) \in U \times V \subset \Delta^C$. For any $z \in U$ if $z \in V$ then $(z, z) \in U \times V \subset \Delta^C$ but $(z, z) \in \Delta$ which is a contradiction. Thus, $U \cap V = \emptyset$ which gives the Hausdorff condition.

Conversely, let X be Hausdorff then if $(x, y) \in \Delta^C$ then $x \neq y$ so by the Hausdorff property, $\exists U, V \in \mathcal{T}$ s.t. $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. If $(z, z) \in \Delta$ then $(z, z) \notin U \times V$ else $z \in U$ and $z \in V$. Therefore, $(x, y) \in U \times V \subset \Delta^C$. Therefore, Δ^C is open in the product topology which implies that Δ is closed.

Problem 5.

Let $f : X \to Y$ be continuous and $C \subset Y$ be closed and $D \subset X$ be dense. Let $f(D) \subset C$ then by continuity and Lemma **??**, $f(\overline{D}) \subset \overline{f(D)} \subset \overline{C}$. However, D is dense so $\overline{D} = X$ and C is closed so $\overline{C} = C$. Thus, $f(X) \subset C$.

Problem 6.

Let $f, g : X \to Y$ be continuous with Y Hausdorff and let $D \subset X$ be dense. Also let $\forall z \in D$: $f(z) = g(z)$. Now suppose that $\exists x \in X : f(x) \neq g(x)$. Because Y is Hausdorff, $\exists U, V \in \mathcal{T}_Y$ s.t. $f(x) \in U$ and $g(y) \in V$ and $U \cap V = \emptyset$. Since U, V are open and f, g are continuous then $f^{-1}(U)$ and $g^{-1}(V)$ are open. Thus, $f^{-1}(U) \cap g^{-1}(V)$ is also open. However, $x \in f^{-1}(U)$ and $y \in g^{-1}(V)$ so $x \in f^{-1}(U) \cap g^{-1}(V)$. Thus, $f^{-1}(U) \cap g^{-1}(V) \neq \emptyset$. By Lemma **??**, $\exists d \in D$ s.t. $d \in f^{-1}(U) \cap g^{-1}(V)$ but $f(d) = g(d)$ because $d \in D$. However, $d \in f^{-1}(U)$ and $d \in g^{-1}(V)$ so $f(d) \in U$ and $g(d) \in V$ so $f(d) = g(d) \in U \cap V$ which is a contradiction because $U \cap V = \emptyset$. Thus, $\forall x \in X : f(x) = g(x)$ so $f = g$.

An alternative solution is given by considering the map $F : X \to Y \times Y$ given by

$$F(x) = (f(x), g(x))$$

This function is continuous by a previous homework problem because f and g are continuous. Now, $\forall x \in D : f(x) = g(x)$ so $F(D) \subset \Delta$ but Δ is closed in $Y \times Y$ because Y is Hausdorff and $D \subset X$ is dense so by the previous problem, $F(X) \subset \Delta$. Therefore, $\forall x \in X : (f(x), g(x)) \in \Delta$ which gives $f(x) = g(x)$ for every $x \in X$.

Problem 7.

(a). Suppose that A contains no limit points of itself. Take any $x \in A$ then $x \notin \overline{A\backslash\{x\}}$ so $\exists U \in \mathcal{T}$ s.t. $x \in U$ and $U \cap (A \setminus \{x\}) = \emptyset$. However, $x \in A$ and $x \in U$ so $x \in U \cap A$. Thus, $U \cap A = \{x\}$. But U is open in X so $U \cap A$ is open in A. Thus, every $\{x\}$ is open in A. For any $S \subset X$, $S = \bigcup_{x \in S} \{x\}$ is open because each $\{x\}$ is open so every set is open in A. Conversely, if the subset topology on A in X is discrete then for any $x \in A$ there must exist

$U \in \mathcal{T}$ s.t. $U \cap A = \{x\}$ because $\{x\}$ is open in A. Thus, $U \cap (A \setminus \{x\}) = \emptyset$ so x is not a limit point of A so A contains no limit points.

(b). Take $S = \left\{\frac{1}{n} \mid n \in \mathbb{Z}^+\right\}$ then for any $\delta > 0$ we have that $\exists n \in \mathbb{Z}^+$ s.t. $0 < \frac{1}{n} < \delta$ so $\frac{1}{n} \in B_\delta(0)$ so 0 is a limit point of S. However, for any $\frac{1}{n} \in S$ take $\delta = \frac{1}{n(n+1)}$ and $U = B_\delta\left(\frac{1}{n}\right)$. Then $\frac{1}{k} - \frac{1}{n} = \frac{n-k}{nk} \geq \frac{1}{n(n+1)}$ so $U \cap S = \{\frac{1}{n}\}$ thus S is discrete.

Lemmas

Lemma 0.1. If $A \subset X$ and $B \subset Y$ are closed in X and Y respectivly, then $A \times B$ is closed in the product topology on $X \times Y$.

Proof. Let $A = X \backslash C$ with $C \in \mathcal{T}_X$ and $B = Y \backslash D$ with $D \in \mathcal{T}_Y$ then

$$A \times B = (X \backslash C) \times (Y \backslash D) = (X \times Y) \backslash ((C \times Y) \cup (X \times D))$$

but $C \times Y$ and $X \times D$ are open in the product so $(C \times Y) \cup (X \times D)$ is also open and thus $A \times B$ is closed. $\square$

Lemma 0.2. Let $f : X \to Y$ be continuous and $A \subset X$ then $f(\bar{A}) \subset \overline{f(A)}$.

Proof. Let $y \in f(\bar{A})$ then $y = f(x)$ and $x \in \bar{A}$ thus for any open $U \subset X$, if $x \in U$ then $U \cap A \neq \emptyset$. Take a open $V \subset Y$ and $y \in V$ so $x \in f^{-1}(V)$. But f is continous so $f^{-1}(V)$ is open and $x \in f^{-1}(V)$ so $\exists z \in f^{-1}(V) \cap A$ then $f(z) \in V$ and $z \in A$ thus $f(z) \in f(A)$. Thus, $f(z) \in V \cap f(A)$ so $V \cap f(A) \neq \emptyset$ thus $x \in \overline{f(A)}$. $\square$

Lemma 0.3. Let $(X, \mathcal{T})$ be a topological space and $D \subset X$ be dense then $\forall U \in \mathcal{T} \backslash \{\emptyset\} : \exists d \in U \cap D$.

Proof. If U is a nonempty open set then $\exists x \in U$. D is dense so $x \in \overline{D}$ thus because $x \in U$ and U is open then we have $\implies U \cap D \neq \emptyset$. Thus, $\exists d \in U \cap D$. $\square$