1 Stable Equivalence of Vector Bundles

Definition 1.1. Let E_1, E_2 be vector bundles on X and $\mathcal{E}^n$ be the rank n trivial vector bundle. Then we say that E_1 and E_2 are stably equivalent $E_1 \sim_s E_2$ if there is some $n \in \mathbb{Z}_+$ such that,

$$E_1 \oplus \mathcal{E}^n \cong E_2 \oplus \mathcal{E}^n$$

Furthermore we say that E_1 and E_2 are weakly equivalent $E_1 \sim E_2$ if there exist $n, m \in \mathbb{Z}_+$ such that,

$$E_1 \oplus \mathcal{E}^n \cong E_2 \oplus \mathcal{E}^m$$

Example 1.2. Consider $X = S^1$ and the Möbius bundle μ. Then $\mu \not\cong \mathcal{E}^1$ since μ has no nonvanishing global sections. However, I claim that,

$$\mu \oplus \mathcal{E}^1 \cong \mathcal{E}^2$$

so $\mu \cong_s \mathcal{E}^1$. To see this I must find two everywhere linearly independent sections.

We describe μ explicitly as the tautological bundle of $S^1 = \mathbb{RP}^1$,

$$\mu = \{(x, v) \mid x \in \mathbb{RP}^1, v \in \mathrm{Span}(x) \subset \mathbb{R}^2\} = \{(e^{2\pi i t}, v) \mid t \in [0,1] \;\; v \in \mathrm{Span}(e^{\pi i t})\}$$

Then take, $s_1, s_2 \in \Gamma(X, \mu \oplus \mathcal{E}^2)$ to be,

$$s_1(t) = (e^{2\pi i}, e^{\pi i t} \cos(\pi t)) \oplus (e^{2\pi i}, \sin(2\pi t)) \qquad s_2(t) = (e^{2\pi i t}, e^{\pi i t} \sin(\pi t)) \oplus (e^{2\pi i t}, \cos(2\pi t)$$

2 Definition of Topological K-Theory

3 Questions for March 3

(a). Split exact structure on a cateogry is a type of exact structure (right?)

(b). So if I have a split exact category C then $K_0(C) = K_0^{\oplus}(C)$ (correct?)

(c). However, if A is an abelian vategory then $K_0(A)$ is equivalent to the grothendieck group of A as an exact category while $K_0^{\oplus}(A)$ is the grothendieck group of A as a split exact category. These are not the same.

(d). Why is the category of vectorbundles on a scheme not split exact? Vector bundles are locally free and thus flat but I know not projective in $\mathscr{Mod}(\mathcal{O}_X)$. However, are they projective in $\mathfrak{QCoh}(\mathcal{O}_X)$?

(e). Okay so it seems that if we let $X = \mathbb{P}^1_R$ then,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$$

cannot split because there are no maps $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-1)$ since $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no global sections.