1 Remedial Curve Theory

1.1 Geometric Irreducibility of Generic Fibers

Lemma 1.1.1 (Tag 0553)**.** Let $f : X \to Y$ be a morphism of schemes. Assume,

(a) Y is irreducible with generic point η,

(b) X_η is geometrically irreducible

(c) f is of finite type

then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibers.

Lemma 1.1.2. Let $f : X \to Y$ be a morphism of schemes. Suppose that,

(a) X and Y are integral

(b) X is normal

(c) the fibers of f are geometrically connected (e.g. $f_* \mathcal{O}_X = \mathcal{O}_Y$)

then the generic fiber $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically irreducible.

Proof. $X_\eta / \kappa(\eta)$ is geometrically irreducible iff $\kappa(\eta)$ is separable closed in $\kappa(\xi)$. This follows from Tag 054Q and Tag 0G33. Let $\alpha \in \kappa(\xi)$ be separably algebraic over $\kappa(\eta)$ i.e. a root of a separable polynomial $p \in \kappa(\eta)[x]$. There is a coordinate ring A of Y where all the denominators of p are invertible. We claim that $A[\alpha] \subset B$ where B is any coordinate ring of X containing A. Indeed, α is integral over A and hence over B so by normality $\alpha \in B$ so we get morphisms,

$$X_A \to \mathrm{Spec}\,(A[\alpha]) \to \mathrm{Spec}\,(A)$$

but the fibers of $X_A \to \mathrm{Spec}\,(A)$ are geometrically connected so we must have $\alpha \in A$ since otherwise the fibers of $\mathrm{Spec}\,(A[\alpha]) \to \mathrm{Spec}\,(A)$ and hence $X_A \to \mathrm{Spec}\,(A)$ are not geometrically irreducible. $\square$

Remark. If we only assumed that X/k is geometrically irreducible (which is weaker than X being normal) the result would not follow. Indeed, consider,

$$X = \mathrm{Proj}\left(k[t][X,Y,Z]/(X^2 - tY^2)\right) \to \mathrm{Spec}\,(k[t]) = Y$$

where k is algebraically closed. Then X and Y are geomtrically integral since they are integral. Indeed, we need to check that the polynomials on the charts,

$$\left(\frac{X}{Z}\right)^2 - t\left(\frac{X}{Y}\right)^2 \qquad \left(\frac{X}{Y}\right)^2 - t \qquad 1 - t\left(\frac{Y}{X}\right)^2$$

are irreducible. They are since t does not admit a square root. However, the generic fiber is,

$$X = \mathrm{Proj}\left(k(t)[X,Y,Z]/(X^2 - tY^2)\right) \to \mathrm{Spec}\,(k(t))$$

is not geometrically irreducible since after the extension $k(t^{\frac{1}{2}})/k(t)$ we can split the polynomial. However, X is not normal since $t^{\frac{1}{2}}$ is in the fraction ield (look at the second chart) but not in every chart since $H^0(X, \mathcal{O}_X) = k[t]$ and this does not contain $t^{\frac{1}{2}}$. The normalization of X is $\mathbb{P}^1 \times \mathrm{Spec}\left(k[t^{\frac{1}{2}}]\right)$ with the map,

$$[T_0 : T_1] \to [t^{\frac{1}{2}} T_0 : T_0 : T_1]$$

This "hits both branches" since $t^{\frac{1}{2}}$ "remembers which branch of the suqare root it is on" while still making $\widetilde{X}$ an integral scheme as it must be since it is the normalization of an integral schemes.

Remark. When the base has $\dim Y = 1$ and is over a perfect field then we can also ensure that the generic fiber is geometrically integral.

Proposition 1.1.3. Let $f : X \to Y$ be a proper morphism of schemes. Let X, Y be integral and finite type over a perfect field k. If X is normal and $\dim Y = 1$ then the following are equivalent,

(a) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically integral

(b) $\kappa(\eta)$ is algebraically closed in $\kappa(\xi)$

(c) $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism.

Proof. Lemma 7.2 of Badescu. □

Example 1.1.4. If the base has dimension > 1 this isfalse. For example,

$$X = \mathrm{Proj}\,(\mathbb{F}_p[s,t][X,Y,Z]/(X^p + sY^p + tZ^p)) \to \mathrm{Spec}\,(\mathbb{F}_p[s,t]) = Y$$

satisfies $f_* \mathcal{O}_X = \mathcal{O}_Y$ and X is normal but the generic fiber,

$$X = \mathrm{Proj}\,(\mathbb{F}_p(s,t)[X,Y,Z]/(X^p + sY^p + tZ^p)) \to \mathrm{Spec}\,(\mathbb{F}_p(s,t))$$

is not geometrically reduced. Indeed, alhough $\mathbb{F}_p(s,t)$ is algebraically closed in,

$$\mathrm{Frac}\,(\mathbb{F}_p(s,t)[x,y]/(x^p + sy^p + t))$$

it is not separable since separability implies reducedness fo the base change by the field extension $\mathbb{F}_p(s^{\frac{1}{p}}, t^{\frac{1}{p}})$.

Remark. Note that if X is any of,

(a) reduced

(b) integral

(c) normal

(d) regular

then the same is true of X_η for any map $f : X \to Y$ by localization. However, unlike the case for irreducibiliy above, the corresponding geometric versions do *not* hold as the following and previous examples show.

Example 1.1.5. Quasi-elliptic fibrations $\mathrm{Bl}\mathbb{P}^2 \to \mathbb{P}^1$ have fibers which are not geometrically normal or regular.

Theorem 1.1.6 (Fujita, 1982)**.** Let $f : X \to Y$ be a proper dominant morphism of integral locally noetherian schemes. Consider the following properties,

(a) $\kappa(\xi_Y)$ is algebraically closed in $\kappa(\xi_X)$

(b) $\mathrm{rank}_Y(f_* \mathcal{O}_X) = 1$

(c) the general fiber satisfies $h^0(X_y, \mathcal{O}_{X_y}) = 1$

(d) $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism.

Then the following implications hold,

$$\begin{tikzcd}
(a) \arrow[r] & (b) \arrow[l, bend right, "X \text{ normal}"'] \arrow[r, bend left, "Y \text{ normal}"] \arrow[d, leftrightarrow] & (d) \arrow[l] \\
& (c) &
\end{tikzcd}$$

Proof. DO IT!!! □

Example 1.1.7. Consider,

$$X = \mathrm{Proj}\,(k[t][X, Y, Z]/(X^p + cY^p + tZ^p)) \to \mathrm{Spec}\,(k[t])$$

where $c \in k$ is not a p^{th}-power. Then X_η is a smooth genus $\frac{(p-1)(p-2)}{2}$ curve but X_0 is integral and $H^0(X_0, \mathcal{O}_{X_0}) = k$ but X_0 is not geometrically reduced. The arithmetic genus is still constant but the geometric genus drops to zero.

1.2 Genera of Curves

Definition 1.2.1. A *curve* C over k is a separated finite type scheme over k of pure dimension 1.

Definition 1.2.2. Let X be a proper curve over k. The *arithmetic genus* of X is,

$$p_a(X/k) := \dim_k H^1(X, \mathcal{O}_X)$$

If $H^0(X, \mathcal{O}_X) = K$ is a field then we write,

$$p_a(X) := \dim_K H^1(X, \mathcal{O}_X)$$

Remark. The arithmetic genus is stable under field extension by flat base change. However, if X admits $X \to \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ then the arithmetic genus of X viewed over k is $[k' : k]$ times the arithmetic genus of X viewed over k'. The point of the second definition is that when it it applies the base field is unambigious.

Definition 1.2.3. Let X be a curve which is a disjoint union of finitely many smooth curves over an algebraically closed field k. Then the *geometric genus* (or just *genus*) of X is,

$$g(X) := p_a(X/k) = \sum_{i=1}^{n} p_a(C_i/k)$$

Definition 1.2.4. Let X be a curve over a field k. Consider $\widetilde{X}$ which is the normalization of $(X_{\bar{k}})_{\mathrm{red}}$. This is a disjoint union of finitely many smooth curves C_i over $\bar{k}$. Thus we can define,

$$g(X/k) := g(\widetilde{X})$$

If $H^0(X, \mathcal{O}_X) = K$ is a field then we set,

$$g(X) := g(X/k)$$

Remark. The geometric genus is stable under field extension by definition. However, notice that $g(X/k)$ does depend on the base field. If X admits $X \to \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ then the geometric genus of X viewed over k is $[k' : k]$ times the geometric genus of X viewed over k'. The point of the second definition is that when it it applies the base field is unambigious.

PUT IN THE RELATIONSHIP BETWEEN THE TWO

Lemma 1.2.5. Let $f : X \to Y$ be a nonconstant map of proper regular curves over an algebrcially closed field k. Then $g(X) \geq g(Y)$.

Proof. Riemann-Hurwitz and Frobenius tricks CITE [H] $\square$

Proposition 1.2.6. Let $f : X \to Y$ be a dominant map of proper curves over a field k. Then $g(X/k) \geq g(Y/k)$.

Proof. By definition, we set $\widetilde{X}$ to be the normalization of $(X_{\bar{k}})_{\mathrm{red}}$ and then $g(X/k) = g(\widetilde{X})$. Then the induced map $f : \widetilde{X} \to \widetilde{Y}$ is also surjective since it is dominant (because this is preserved by base change and reduction and normalization) and proper. Therefore, each component of $\widetilde{Y}$ is hit by some component of $\widetilde{X}$ so we reduce to the previous lemma and conclude,

$$g(X/k) \geq g(Y/k)$$

$\square$

Example 1.2.7. Say $E = \mathrm{Proj}\,(\mathbb{R}[X, Y, Z]/(Y^2 Z - X^3 - xZ^2))$ is an elliptic curve over $\mathbb{R}$. It is important that we consider the genus of $E_{\mathbb{C}}$ *as a curve over* $\mathbb{R}$ as 2 and not 1 because,

$$X = \mathrm{Proj}\left(\mathbb{R}[X, Y, Z]/((Y^2 Z - X^3)^2 + (XZ^2)^2)\right)$$

has normalization $E_{\mathbb{C}}$. However, X has genus 2 since $H^0(X, \mathcal{O}_X) = \mathbb{R}$ so we must view it over $\mathbb{R}$ and to compute its genus we base change to $X_{\mathbb{C}}$ then our definition will give genus 2. If we want the map $E_{\mathbb{C}} \to X$ to satisfy the above lemma we must have $g(E_{\mathbb{C}}/\mathbb{R}) = 2$.

Proposition 1.2.8. Let $f : X \to Y$ be a dominant map of proper curves over k with,

$$k \to H^0(Y, \mathcal{O}_Y) \to H(X, \mathcal{O}_X)$$

. Then $g(X) \geq g(Y)$.

1.3 Degenerations of Curves

Definition 1.3.1. A *degeneration of curves* is a proper flat family $X \to S = \mathrm{Spec}\,(R)$ over a DVR R where X_η is an integral normal projective curve over $K = \mathrm{Frac}\,(R)$. If X is normal we say that X is a *model* of X_η over R.

Lemma 1.3.2. The total space X of a degeneration of curves is integral.

Proof. We need to show that every affine open $\mathrm{Spec}\,(A) = U \subset X$ has A a domain. Indeed, $R \to A$ is flat so $A \hookrightarrow A_K$ is injective but A_K is an affine open of X_K which in integral so A_K and hence A is a domain. □

Lemma 1.3.3. Let $f : X \to S$ be a proper flat map of integral schemes with S normal. Then the following are equivalent,

(a) $f_* \mathcal{O}_X = \mathcal{O}_S$

(b) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = K$

Proof. Indeed, $f_* \mathcal{O}_X$ is a finite $\mathcal{O}_S$-algebra and since X is integral it is a sheaf of domains. We need to show that $\mathcal{O}_S \to f_* \mathcal{O}_X$ is an isomorphism which is a local question so we reduce to $\mathrm{Spec}\,(A) \subset S$ and $\mathrm{Spec}\,(B) \subset X$ such that $A \to B$. Then we have maps $A \to (f_* \mathcal{O}_X)(A) \to B$ and $A \to B$ is flat hence injective since they are domains. Hence $\mathcal{O}_S \to f_* \mathcal{O}_X$ is injective. Furthermore, by flat base change,

$$H^0(X_\eta, \mathcal{O}_{X_\eta}) = (f_* \mathcal{O}_X)_\eta$$

so if (b) holds then $(f_* \mathcal{O}_X)_\eta = \kappa(\eta)$. Since $\mathcal{O}_S$ is normal and $f_* \mathcal{O}_X$ is integral over $\mathcal{O}_S$ we see that $\mathcal{O}_S \to f_* \mathcal{O}_X$ is an isomorphism since it is contained in the fraction field. □

Proposition 1.3.4. Let $X \to S$ be a degeneration of curves. Consider the following properties,

(a) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically integral

(b) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically irreducible

(c) $X_\eta \to \mathrm{Spec}\,(\kappa(\eta))$ is geometrically connected

(d) $H^0(X_\eta, \mathcal{O}_{X_\eta}) = \kappa(\eta)$

(e) $f_* \mathcal{O}_X = \mathcal{O}_S$

then the following implications hold,

$$\begin{tikzcd}
(a) \arrow[r] \arrow[d, shift left] & (d) \arrow[d, leftrightarrow] \\
(b) \arrow[u, bend left, "X_\eta \text{ geom.red.}"] \arrow[d, shift left] & (e) \arrow[dl] \\
(c) \arrow[u, bend left, "X \text{ normal}"] &
\end{tikzcd}$$

In particular, if X is normal and X_η is geometrically reduced all the properties are equivalent.

Proof. The only nontrivial implications are:

- $(a) \implies (d)$ is Tag 0BUG (8)
- $(d) \implies (e)$ is exactly Lemma 1.3.3
- $(c) \implies (b)$ is Lemma 1.1.2 and the fact that geometric connectedness of fibers can be checked generically in universally open (e.g. flat finitely presented) families [EGA IV, Cor. 15.5.4].

□

Remark. Even if $f_* \mathcal{O}_X = \mathcal{O}_S$ we don't necessarily have that X_η is geometrically reduced e.g. Example 1.1.7.

1.4 Examples

Suppose that we have a flat proper family $f : X \to S$ with $f_* \mathcal{O}_X = \mathcal{O}_S$. Formation of this pushforward my fail to be compatible with basechange (this is failure of cohomological flatness in degree zero). When this happens we can have jumping up of $h^0(X_s, \mathcal{O}_{X_s})$. Consider the finite $\kappa(s)$-algebra,

$$A = H^0(X_s, \mathcal{O}_{X_s})$$

There are three ways we could imagine A jumping up:

(a) A is a finite separable extension of $\kappa(s)$

(b) A is a finite purely-inseparable extension of $\kappa(s)$

(c) A is nonreduced.

The first cannot happen because $f : X \to S$ has geometrically connected fibers but if there is a factorization $X \to \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ with k' separable then it is geometrically disconnected. Therefore, any field inside A must be purely inseparable over k. However both (b) and (c) can happen as we will now see.

DEGENERATE GENUS 1 TO PURELY INSEP EXTN

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