1 Introduction

An affine scheme is the basic object of algebraic geometry. Associated to any ring A there is a scheme $\text{Spec}\,(A)$. If you haven't seen this before you should just think of it as a gadget which has the following data,

(a) a topological space (also called $\text{Spec}\,(A)$ with the Zariski topology) whose points are prime ideal $\mathfrak{p} \subset A$ and whose closed sets are of the form,

$$V(I) = \{\mathfrak{p} \subset A \mid \mathfrak{p} \supset I\}$$

where $I \subset A$ is an ideal of A. Equivalently, a basis of open sets is given by,

$$D(f) = \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}$$

where $f \in A$ ranges over elements of A

(b) which remembers the ring A thought of as the collection of "algebraic functions" on the topological space $\text{Spec}\,(A)$. We think of $a \in A$ as the function whose "value" at a point $\mathfrak{p} \in \text{Spec}\,(A)$ is the element $\bar{a} \in \kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ which is a field.

The most important property of affine schemes is that they are determined by their ring of global functions.

Definition 1.0.1. If A is a k-algebra then the set of *k-points* of $\text{Spec}\,(A)$ is the set of points $\mathfrak{p} \in \text{Spec}\,(A)$ such that the natural map $k \to \kappa(\mathfrak{p})$ is an isomorphism.

Example 1.0.2. $\mathbb{A}^2 = \text{Spec}\,(\mathbb{C}[x, y])$ is the affine plane. The points are prime ideal $\mathfrak{p} \subset \mathbb{C}[x, y]$ the $\mathbb{C}$-points correspond to the maximal ideal which are of the form $\mathfrak{p} = (x - a, y - b)$. Therefore, the set of $\mathbb{C}$-points is $\mathbb{C}^2$. We can describe the Zariski topology as the topology whose closed sets are the vanishing locus of polynomials on $\mathbb{C}^2$.

2 Smooth Manifolds

Associated to a smooth manifold M is a natural ring, the ring of smooth functions $C^\infty(M)$. We will show that M is an affine scheme in the sense that it can be recovered from the ring $C^\infty(M)$ of smooth functions. However, the correspondence goes far deeper. Indeed, M as a topological space is the set of $\mathbb{R}$-points of $\text{Spec}\,(C^\infty(M))$ with the Zariski topology. We will see even more is true.

Lemma 2.0.1. M has the Zariski topology, meaning that every closed set $Z \subset M$ is the zero locus of a smooth function.

Proof. Since M is separable, there exists a countable open cover $\{U_k\}$ of $M \backslash Z$ by balls so there are smooth functions $\rho_k : M \to \mathbb{R}$ such that $0 \le \rho_k(x) \le 1$ and $\rho_k^{-1}(0) = M \backslash U_k$. Then consider,

$$f(x) = \sum_{k \in 1}^{\infty} 2^{-k} \rho_k(x)$$

Since $|2^{-k}\rho_k| < 2^{-k}$ the series converges uniformly and absolutely so f is smooth. Furthermore,

$$f(x) = 0 \iff \forall k : \rho_k(x) = 0 \iff \forall k : x \notin U_k \iff xZ$$

$\square$

Remark. Some related mathoverflow posts,

(a) smooth function with given zero set

(b) separating open and closed set

(c) function whose zero set is the Cantor set

(d) closed set is set of zeros

(e) manifold as zero locus

2.1 Comparison map between M and $\operatorname{Spec}(C^\infty(M))$

Let X be a topological space and $A \subset C^0(X)$ be a subring of the continuous (real valued) functions. Consider the map $\varphi : X \to X^{\text{aff}} = \operatorname{Spec}(A)$ given by sending $x \mapsto \mathfrak{m}_x$ where,

$$\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}$$

which is a maximal ideal since it is the kernel of the evaluation map $\text{ev}_x : A \to \mathbb{R}$. We will later specialize to A being the ring of smooth functions but really all we will use about A is,

(a) A separates points meaning for all $x, y \in A$ with $x \neq y$ there exists $f_x \in A$ with $f_x(x) = 1$ and $f_x(y) = 0$

(b) A generates the topology of M meaning for every closed set $Z \subset M$ there is $f_Z \in A$ with $z = f_Z^{-1}(0)$

(c) if $f \in A$ and $f > 0$ then $\sqrt{f} \in A$ and $\frac{1}{f} \in A$

Proposition 2.1.1. The map $\Phi : X \to X^{\text{aff}}$ is continuous and injective if A separates points.

Proof. We need to show that $\Phi^{-1}(D(f))$ is open and note that,

$$\Phi(x) \in D(f) \iff f \notin \mathfrak{m}_x \iff f(x) \neq 0$$

which is open since f is continuous. Since A separates points, $\mathfrak{m}_x \neq \mathfrak{m}_y$ for $x \neq y$ so Φ is injective. $\square$

Remark. This affinification is really a much more general construction. For any locally-ringed space X there is an affinification $X^{\text{aff}} = \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$ which is left-adjoint to the inclusion **AffSch** $\hookrightarrow$ **LRS**. Then the unit $X \to X^{\text{aff}}$ is a morphism of locally-ringed spaces and therefore is a continuous map.

Remark. First let's make some comments about the ring $C = C^\infty(M)$.

(a) First, there is a unique ring map $\mathbb{R} \to C$. Indeed, there is a unique map $\mathbb{Q} \to C$ since the nonzero images of the unique map $\mathbb{Z} \to C$ are invertible

(b) we define a partial order on C where $f \leq g$ iff there exists $h \in A$ such that $f + h^2 = g$. Consider the set of functions r with the property that for any $\epsilon > 0$ there exists $p, q \in \mathbb{Q}$ such that $|p - q| \leq \epsilon$ and $p \leq r \leq q$. This recovers precisely the copy of $\mathbb{R}$ inside C.

(c) Since any elements in the image of a map $\varphi : \mathbb{R} \to C$ satisfies this property so there is a unique map $\varphi : \mathbb{R} \to C$. Indeed if $x \in \mathbb{R}$ then for all $\epsilon > 0$ there are $p, q \in \mathbb{Q}$ with $|p - q| < \epsilon$ and $p \le x \le q$. Thus $p \le \varphi(x) \le q$ because $x \le y$ meaning $x + z^2 = y$ for some $z \in \mathbb{R}$ and thus $\varphi(x) + \varphi(z)^2 = \varphi(y)$ meaning $\varphi(x) \le \varphi(y)$.

Furthermore, the ring C knows the sup norm. Indeed let,

$$\sigma(f) = \{\lambda \in \mathbb{R} \mid f - \lambda \text{ is not invertible}\}$$

then $\sigma(f) = \operatorname{im} f$ (but we didn't need to know that f represented a function) and hence the spectral norm,

$$||f|| = \sup_{\lambda \in \sigma(f)} |\lambda|$$

which recovers the sup norm since $\sigma(f) = \operatorname{im} f$. Therefore, we recover $C^\infty(M)$ as a normed $\mathbb{R}$-vectorspace.

Remark. The $\mathbb{R}$-points of M^{aff} correspond to ring maps $\varphi : C^\varphi(M) \to \mathbb{R}$. Note that such maps are automatically $\mathbb{R}$-algebra morphisms since there is a unique ring map $\mathbb{R} \to C^\infty(M)$ and the composition $\mathbb{R} \to \mathbb{R}$ must be the identity since there is a unique ring map. We now want to classify these $\mathbb{R}$-algebra maps $\varphi : C^\infty(M) \to \mathbb{R}$.

Notice that if $f > 0$ then f is invertible so $\varphi(f) \neq 0$. Furthermore, $\sqrt{f}$ is smooth so $\varphi(f) = \varphi(\sqrt{f})^2 \ge 0$ and therefore $\varphi(f) > 0$.

Now we return to the map, $\Phi : M \to M^{\text{aff}} = \operatorname{Spec}(C^\infty(M))$ given by sending $x \mapsto \mathfrak{m}_x$ where,

$$\mathfrak{m}_x = \{f \in C^\infty(M) \mid f(x) = 0\}$$

Proposition 2.1.2. $\varphi : M \to M^{\text{aff}}$ is continuous, and is an isomorphism onto the $\mathbb{R}$-points of M^{aff}.

Proof. We have seen that Φ is continuous and injective into $M^{\text{aff}}(\mathbb{R})$. We need to show that $\Phi : M \to M^{\text{aff}}(\mathbb{R})$ is surjective and closed.

Let $\varphi : C^\infty(M) \to \mathbb{R}$ be a ring map (hence an $\mathbb{R}$-algebra map) and consider,

$$Z(\varphi) = \bigcap_{f \in \ker \varphi} Z(f) = \{x \in M \mid \ker \varphi \subset \mathfrak{m}_x\}$$

Since φ is surjective $\ker \varphi$ is maximal and since the map $x \mapsto \mathfrak{m}_x$ is injective $\ker \varphi$ is contained in at most one $\mathfrak{m}_x$ so $Z(\varphi)$ has at most one point.

Thus it suffices to show that $Z \neq \varnothing$. Otherwise for each $x \in M$ we could find f_x such that $f_x \in \ker \varphi$ and $f_x(x) = 1$. Replacing f_x by f_x^2 we assume $f_x \ge 0$. The open sets $U_x = f_x^{-1}((0, \infty))$ are neighborhoods of x. Hence for any compact K we can find a finite set $x_1, \dots, x_n$ such that the U_{x_i} cover K. Then consider,

$$f_K = f_{x_1} + \cdots + f_{x_n}$$

and we see that $f_K > 0$ on K but $\varphi(f_K) = 0$. If $K = M$ we would be done by the remark that f_K is invertible. Otherwise, choose a positive exhaustion function $f : M \to \mathbb{R}$ meaning that $f^{-1}([0, c])$ is compact for all $c \in \mathbb{R}$. Let $\lambda = \varphi(f)$ and $K = f^{-1}([0, 2\lambda])$. Then we get f_K and there is some $\epsilon > 0$ with $f_K > \epsilon$ on K. Then consider,

$$h = f + \epsilon^{-1} \lambda f_K - \lambda$$

we see that $h > 0$ because for $x \notin K$ we have $f(x) > 2\lambda$ and for $x \in K$ we have $f_K > \epsilon$ and $f \geq 0$. Therefore h is invertible but,

$$\varphi(h) = \varphi(f) - \lambda = 0$$

giving a contradiction. Therefore $Z(\varphi) = \{x\}$ for some $x \in M$ and thus $\ker \varphi \subset \mathfrak{m}_x$ but $\ker \varphi$ is maximal so $\ker \varphi = \mathfrak{m}_x$.

Finally, Φ is closed since if $Z \subset M$ is a closed set there exists $f \in C^\infty(M)$ with $f^{-1}(0) = Z$ and then,

$$\Phi(Z) = \{\mathfrak{m}_x \mid x \in Z\}$$

however,

$$x \in Z \iff f(x) = 0 \iff f \in \mathfrak{m}_x$$

and therefore,

$$\varphi(Z) = \{\mathfrak{m}_x \mid f \in \mathfrak{m}_x\} = V(f) \cap M^{\text{aff}}(\mathbb{R})$$

is the closed in the subspace topology and is the "vanishing locus" of the ideal $(f) \subset C^\infty(M)$. $\square$

Remark. Notice that if M is compact, the proof that $Z(\varphi) \neq \varnothing$ did not use anything about the target field. Therefore, we have also shown that if M is compact then every map $\varphi : C^\infty(M) \to K$ for a field K must have $Z(\varphi) \neq \varnothing$.

3 What are the other points?

Proposition 3.0.1. $M^{\text{aff}}_{\text{closed}}$ is larger than $M^{\text{aff}}(\mathbb{R})$ if and only if M is non-compact in which case it contains "hyperreal valuations".

Proof. Consider a sequence $s = \{z_i \in M\}_{i \in I}$ then we get a map,

$$\text{ev}_s : C^\infty(M) \to \prod_{i \in I} \mathbb{R}$$

by evaluating at the sequence. Choosing an ultrafilter $\mathcal{U}$ on I we get,

$$\text{ev}_{s,\mathcal{U}} : C^\infty(M) \to \prod_{i \in I} \mathbb{R} \to \left(\prod_{i \in I} \mathbb{R} \right) / \mathcal{U} = \mathbb{R}^{\mathcal{U}}$$

which is the field of hyperreal numbers (if $\mathcal{U}$ is non-principal). If $\mathcal{U}$ is principal on the index i then $\text{ev}_{\mathcal{U}} = \text{ev}_{z_i}$ is just ordinary evaluation at a point. Likewise, if the sequence is constant at z then we get

$$C^\infty(M) \xrightarrow{\text{ev}_z} \mathbb{R} \hookrightarrow \mathbb{R}^{\mathcal{U}}$$

via the constant embedding which does not give a new point. However, otherwise we can get new interesting points.

Let $z_i \to z$ be a convergent sequence. Then if $\text{ev}_{s,\mathcal{U}}(f) = 0$ we see that $\{i \in I \mid f(z_i) = 0\} \in \mathcal{U}$ and is, in particular, infinite. Thus by continuity $f(z) = 0$. Therefore, $\ker \text{ev}_{s,\mathcal{U}} \subsetneq \mathfrak{m}_x$ so we find nontrivial prime ideals inside $\mathfrak{m}_x$. These are not closed points.

If M is not compact then we can choose a sequence $s = \{z_i\}_{i \in I}$ with no limit points. Then I claim that $\text{ev}_{s,\mathcal{U}}$ is surjective meaning $\ker \text{ev}_{s,\mathcal{U}}$ is a new maximal ideal. Indeed, there is a countable cover

$\{U_i\}_{i\in I}$ such that $z_i \in U_i$ is the only element of the sequence. We may refine this cover so that it is locally finite and use a partition of unity to construct a function f having any specified sequence of values $(a_i)_{i\in I}$ at the points z_i and thus ev_s is already surjective.

The most interesting case is when M is compact. We have seen that any map $\varphi : C^\infty(M) \twoheadrightarrow K$ satisfies $\ker\varphi = \mathfrak{m}_x$ for some x since both are maximal ideal. Why does the previous construction not work? Any ultrafilter $\mathcal{U}$ on I pushes forward to M and has a unique limit point $\mathcal{U} \to z$. This is the extension from the Stone-Cech compactification,

$$\begin{tikzcd} I \arrow[r] \arrow[dr] & \beta I \arrow[d, dashed] \\ & M \end{tikzcd}$$

By definition we say that $\mathcal{U} \to z$ if every neighborhood of z contains an element of $\mathcal{U}$. Suppose that $f \in \ker \mathrm{ev}_{s,\mathcal{U}}$ then for every neighborhood V of z there is an infinite index set $J \in \mathcal{U}$ so that $z_i \in V$ thus $f(z_i) = 0$ for some element of V so by continuity $f(z) = 0$. Hence $\ker \mathrm{ev}_{s,\mathcal{U}} \subset \mathfrak{m}_{\lim \mathcal{U}}$ so these do not contribute new closed points. ☐

4 Recovering the Smooth Structure

The smooth structure on a manifold is determined by its ring of smooth functions (as these determine the sheaf of smooth functions). Therefore, we can see that the topological space $M^{\mathrm{aff}}(\mathbb{R})$ inherits a natural smooth structure from the ring $C^\infty(M)$ which is the ring of "algebraic functions" on this scheme. Furthermore, this is not just an abstract ring, its elements are canonically functions on $M^{\mathrm{aff}}(\mathbb{R})$ in the following way. For any $x \in M^{\mathrm{aff}}(\mathbb{R})$ this corresponds to an ideal $\mathfrak{m}_x$ and a function $\varphi_x : C^\infty(\mathbb{R}) \to \mathbb{R}$ which we call evaluation at x. Then f becomes a function where $f(x) = \varphi_x(f)$.

Using this definition, the map $M \to M^{\mathrm{aff}}(\mathbb{R})$ is a diffeomorphism. Indeed, it is a homeomorphism and is an isomorphism on rings of smooth functions. This is a general fact, for any locally-ringed space, the map $X \to X^{\mathrm{aff}}$ is an isomorphism on global functions by definition.

Proposition 4.0.1. The functor $M \mapsto M^{\mathrm{aff}}$ gives a fully faithful embedding $\mathrm{SmMfd} \hookrightarrow \mathrm{AffSch}$.

Proof. Concretely this means the map,

$$\{\text{smooth map } f : M \to N\} \to \{\text{ring maps } C^\infty(N) \to C^\infty(M)\}$$

via sending,

$$f \mapsto f^* \text{ where } f^* : \varphi \mapsto \varphi \circ f$$

is a bijection. This is well-known but here we can give a slick proof. The inverse is given as follows. A ring map $g : C^\infty(N) \to C^\infty(M)$ gives a morphism of schemes $g : M^{\mathrm{aff}} \to N^{\mathrm{aff}}$ and therefore we get a continuous map,

$$\begin{tikzcd} M^{\mathrm{aff}}(\mathbb{R}) \arrow[r, "g"] & N^{\mathrm{aff}}(\mathbb{R}) \\ M \arrow[u, "\sim"] \arrow[r, dashed] & N \arrow[u, "\sim"] \end{tikzcd}$$

giving a continuous map $f : M \to N$ such that $\mathfrak{m}_{f(x)} = g^{-1}(\mathfrak{m}_x)$ therefore, $\varphi_{f(x)} = \varphi_x \circ g$ (because any ring map g is an $\mathbb{R}$-algebra map since there is a unique map $\mathbb{R} \to C^\infty(M)$) meaning for any $h \in C^\infty(N)$ then,

$$(f^* h)(x) = h(f(x)) = \varphi_{f(x)}(h) = \varphi_x(g(h)) = (g(h))(x)$$

and therefore $g = f^*$. □

5 Kahler Differentials

Remark. In this section we critically use that $A = C^\infty(M)$.

There is an algebraic description of the differential forms of an $\mathbb{R}$-algebra (or any ring) called the Kahler differentials. Defined by choosing a surjection $\mathbb{R}[S] \twoheadrightarrow A$ from a free algebra. Then let J be the kernel and define,

$$\Omega_{A/\mathbb{R}} := \left(\bigoplus_{s \in S} A \mathrm{d}s \right) / \left(\sum_{s \in S} \frac{\partial f}{\partial s} \mathrm{d}s \right)_{f \in J}$$

This satisfies the universal property, for any A-module M,

$$\mathrm{Der}_{\mathbb{R}}(A, M) = \mathrm{Hom}_A \left(\Omega_{A/\mathbb{R}}, M \right)$$

Therefore the universal derivation $\mathrm{d} : C^\infty(M) \to \Omega^1(M)$ to global 1-forms defines a comparison morphism,

$$a : \Omega^1_{C^\infty(M)/\mathbb{R}} \to \Omega^1(M)$$

I claim that a is surjective but not injective. Surjectivity is clear from the construction of Kahler differentials.

Proposition 5.0.1. The map $a : \Omega^1_{C^\infty(M)/\mathbb{R}} \to \Omega^1(M)$ is not injective.

Proof. Consider a map,

$$C^\infty(M) \xrightarrow{\pi} C^\infty(I) \xrightarrow{q} \mathbb{R}[[t]]$$

where the first map is restriction to some interval I in a coordinate chart of M and q is the map taking a smooth function on I to its Taylor series at 0. Note that an amazing theorem is that q is surjective. Now I claim that in $\Omega_{\mathbb{R}[[t]]/\mathbb{R}}$ we have $\mathrm{d}e^x \neq e^x \mathrm{d}x$. Indeed, consider the map,

$$\Omega_{\mathbb{R}[[t]]/\mathbb{R}} \to \Omega_{\mathbb{R}((t))/\mathbb{R}}$$

then we see that if L/K is a field extension of characteristic zero fields and $\alpha, \beta \in L$ are transcendentally independent over K then $\mathrm{d}\alpha, \mathrm{d}\beta \in \Omega_{L/K}$ are L-independent. Then if A is a K-algebra domain and $\alpha, \beta \in A$ transcendentally independent then there are derivations,

$$A \to L \to L$$

which send $\alpha \mapsto 0$ or $\beta \mapsto 0$. In particular we apply this to $A = \mathbb{R}[[t]]$ with t, e^t. Suppose these satisfy a polynomial relation $f \in \mathbb{R}[X, Y]$ so $f(t, e^t) = 0$ but since $f \neq 0$ there is some integer n such that $f(n, Y)$ is a nonzero polynomial and $f(n, e^n) = 0$ contradicting the transcendence of e. □

The problem, as seen from the definition, is that Kahler differentials are a very "free" construction. It captures all derivations not just the "smooth" ones we want to consider. Really, $C^\infty(M)$ is a topological ring and we should consider the topology in the set of differentials. This makes us consider the category of smooth algebras also called C^∞-rings. However, what is really interesting is that when we take the dual, we get the right thing. The following is standard but is very surprising from this perspective.

Proposition 5.0.2. Let $A = C^\infty(M)$ then the natural map,

$$\mathscr{X}(M) = \mathrm{Hom}_A\left(\Omega^1(M), A\right) \to \mathrm{Hom}_A\left(\Omega_{A/\mathbb{R}}, A\right) = \mathrm{Der}_{\mathbb{R}}(A, A)$$

is an isomorphism.

Proof. This is the dual of a surjective map and therefore injective. Thus it suffices to prove that any derivation $D : A \to A$ arises from a smooth vector field. We do this for $M = \mathbb{R}^n$ first. Using the Hadamard lemma, for any $f \in A$ and $p \in \mathbb{R}^n$ we write,

$$f(x) = f(p) + \sum_{i=1}^{n} (x^i - p^i) h_i(x)$$

where h_i is a function such that $h_i(p) = \frac{\mathrm{d}f}{\mathrm{d}x_i}\Big|_{x=p}$. Then applying the derivation,

$$D(f) = \sum_{i=1}^{n} D((x^i - p^i)) h_i(x) + \sum_{i=1}^{n} (x^i - p^i) D(h_i)$$

then evaluating the output at $x = p$ show that,

$$D(f)(p) = \sum_{i=1}^{n} v^i \frac{\mathrm{d}f}{\mathrm{d}x_i}\Big|_{x=p}$$

where $v^i(p) = D(x^i)(p)$ is a smooth function of p. Doing this for all p, we see that,

$$D = \sum_{i=1}^{n} v^i \frac{\mathrm{d}}{\mathrm{d}x_i}$$

which exactly means that D is a smooth vector field. We can apply the same argument to M using partitions of unity. $\square$

6 Vector Bundles

Theorem 6.0.1 (Serre-Swan)**.** Let M be a smooth manifold. There is an equivalence of categories,

$$\{\text{vector bundles on } M\} \xrightarrow{\sim} \{\text{finite projective } C^\infty(M)\text{-modules}\}$$

given by,

$$E \mapsto \Gamma(M, E)$$

Remark. There is a similar result for any compact Hausdorff space X with the ring $C^0(X, \mathbb{R})$ of continuous real functions.

This amazing theorem tells us that $M \mapsto M^{\text{aff}}$ induces an equivalence of categories between smooth vector bundles on M and algebraic vector bundles (locally-free coherent $\mathcal{O}_{M^{\text{aff}}}$-modules) on M^{aff}.

Corollary 6.0.2. We can recover a smooth manifold M from its category $\mathbf{Vect}(M)$ of smooth vector bundles.

Proof. By Serre-Swan $\mathbf{Vect}(M)$ is equivalent to $\mathbf{Mod}_{C^\infty(M)}$. If R is any commutative ring then $R \cong \text{End}\left(\text{id}_{\text{Proj}_R^{\text{fin}}}\right)$ so we recover $C^\infty(M)$ as a commutative ring. From $C^\infty(M)$ we have seen we can recover M. $\square$

Even more amazingly this theorem can be applied to produce interesting commutative algebra examples leveraging our knowledge of topology.

Remark. A finite projective modules P admit a finite projective complement Q such that $P \oplus Q$ is free. However, it is tricky to find a nontrivial projective which is *stably-free* meaning that we can take Q to be free. The following example shows a remarkable application of Serre-Swan.

Corollary 6.0.3. Let $A = \mathbb{R}[x_0, \dots, x_n]/(x_0^2 + \cdots + x_n^2 - 1)$. Let P be the A-module with generators $s_0, \dots, s_n$ and relation,

$$\sum_i x_i s_i = 0$$

Then $P \oplus A$ is free but P is not free for $n \neq 1, 3, 7$ and indecomposable if n is even.

Proof. The $n \neq 1, 3, 7$ should make you immediately suspicious. Consider $A \subset C^\infty(S^n)$ as the subring of polynomial functions. Then $P \otimes_A C^\infty(S^n) \cong \Gamma(TS^n)$ which is trivial if and only if $n = 1, 3, 7$ by Adam's theorem and indecomposable if n is even. Therefore, P is not free if $n \neq 1, 3, 7$ and indecomposable if n is even. Furthermore, we know TS^n is stably trivial. This does not immediately descent to A-module results (we could tensor a non-free module and have it become free) but we can easily deduce the rest algebraically.

It remains to show that P is projective and stably trivial. Indeed, let $F = A^{\oplus n}$ be a free module with basis $s_0, \dots, s_n$ and consider the map,

$$g : F \to F$$

via,

$$g(s_i) = x_i \sum_j x_j s_j$$

Then notice,

$$g^2(s_i) = x_i \sum_j x_j^2 \sum_k x_k s_k = x_i \sum_k x_k s_k$$

and thus g is idempotent. Thus g splits F (consider $1 - g$),

$$F \cong \ker g \oplus \operatorname{im} g$$

so both $\ker g$ and $\operatorname{im} g$ are projective. Moreover,

$$g\left(\sum_i x_i s_i\right) = \sum_i x_i^2 \sum_j x_j s_j = \sum_j x_j s_j$$

which clearly generates $\operatorname{im} g$ so $\operatorname{im} g \cong A$. Then $P = F / \operatorname{im} g$ so via the splitting,

$$\ker g \cong P$$

and hence P is projective and stably free. $\square$

References:

(a) The above example and more can be found in Swan's original paper.

(b) can we recover a compact manifold from its rings of functions

(c) smooth manifold determined by ring

(d) can we recover a space from its continuous functions

(e) manifolds from sheaves

(f) recover a compact smooth manifold from its ring of smooth functions

(g) ring of smooth functions vs continuous functions

(h) characterize differentiation

(i) Kahler and ordinary differentials

(j) algebraic description of compact manifolds

(k) functor of points for manifolds

(l) AG over smooth rings

(m) model theory and Kahler geometry

(n) introduction to smooth schemes