1 Connections on Principle Bundles

Proposition 1.0.1. Let $\pi: G \to S$ be a group scheme. Then,

$$\Omega_{G/S} = \pi^* e^* \Omega_{G/S}$$

so if we set,

$$\omega_{G/S} = e^* \Omega_{G/S}$$

then,

$$\Omega_{G/S} = \pi^* \omega_{G/S}$$

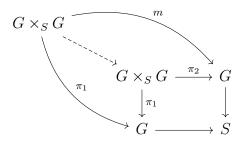
Furthermore, by the projection form,

$$\pi_*\Omega_{G/S} = \omega_{G/S} \otimes_{\mathcal{O}_S} \mathcal{O}_G$$

and thus if π is \mathcal{O} -connected (e.g. for G/S an abelian scheme) then,

$$\pi_*\Omega_{G/S} = \omega_{G/S}$$

Proof. Consider the Cartesian diagram,



because the dashed arrow is an isomorphism, the outside square is Cartesian so $m^*\Omega_{G/S} = \pi_1^*\Omega_{G/S}$. Then,

$$\pi^* e^* \Omega_{G/S} = (e \circ \pi, id)^* \pi_1^* \Omega_{G/S} = (e \circ \pi, id)^* m^* \Omega_{G/S} = id^* \Omega_{G/S} = \Omega_{G/S}$$

Remark. If S = k then $\omega_{G/S} = \mathfrak{g}$ is the Lie algebra.

Proposition 1.0.2. Let $G \to S$ be a smooth group scheme and $\pi: P \to X$ be a G-bundle. Consider the sequence,

$$0 \longrightarrow \pi^*\Omega_X \longrightarrow \Omega_P \longrightarrow \Omega_{P/X} \longrightarrow 0$$

This is an exact sequence of descent data and therefore arises as π^* of an exact sequence,

$$0 \longrightarrow \Omega_X \longrightarrow Q \longrightarrow \operatorname{ad}(P) \longrightarrow 0$$

where ad(P) is the adjoint bundle,

$$ad(P) = P \times_G \mathfrak{g}$$