1 Topics

(a). Basic homotopy theory

(b). Obstruction theory

(c). Characteristic Classes

(d). The Serre spectral sequence

(e). The Steenrod operations

(f). K-theory

References: Fuchs - Fomenko: homotopical topology, Hatcher's books
Six homeworks (one per topic)

2 Homotopy Theory

Basic Questions:

(a). given maps $f, g : X \to Y$ are they homotopy equivalent?

(b). given spaces X and Y are they homotopy equivalent?

Remark. All spaces will be connected and locally connected.

Definition 2.1. The set $[X, Y] = \text{Hom}\,(\mathbf{hTop}, X)\,Y$. Given based spaces X, Y we define $\langle X, Y\rangle = \text{Hom}\,(\mathbf{hTop}_{\bullet}, X)\,Y$ where morphisms in $\mathbf{hTop}_{\bullet}$ are continuous maps preserving the basepoint up to homotopy. Note that homotopies in $\mathbf{Top}_{\bullet}$ are basepoint preserving.

Example 2.2. Consider S^n. Given $f : S^n \to X$ we can construct, $X \sqcup_f D^{n+1}$ by gluing along f. This is the coproduct,

$$\begin{tikzcd} D^{n+1} \arrow[r] & X \sqcup_f D^{n+1} \\ S^n \arrow[u] \arrow[r, "f"] & X \arrow[u] \end{tikzcd}$$

Now if $f \sim f'$ then $X \sqcup_f D^{n+1} \sim X \sqcup_f D^{n+1}$.

Definition 2.3. Given a based space (X, x_0) we define the n^{th} homotopy group,

$$\pi_n(X, x_0) = \langle (S^n, p_0), (X, x_0)\rangle$$

The group structure is given by the equator squeezing map $s : S^n \to S^n \vee S^n$. Then we define $f * g = (f \vee g) \circ s$.

Proposition 2.4. $\pi_n(X, x_0)$ is abelian for $n \geq 2$.

Theorem 2.5. $\pi_n(S^m) = 0$ if $n < m$.

Theorem 2.6. $\pi_n(S^n) = \mathbb{Z}$

Theorem 2.7. $\pi_3(S^2) = \mathbb{Z}$ generated by the Hopf fibration $\eta : S^3 \to S^2$.

Theorem 2.8. For sufficiently large n,

$$\pi_{n+1}(S^n) = \mathbb{Z}/2\mathbb{Z} \qquad \pi_{n+2}(S^n) = \mathbb{Z}/2\mathbb{Z} \qquad \pi_{n+3}(S^3) = \mathbb{Z}/24\mathbb{Z}$$

Remark. Given $f : X \to Y$ we get $f_* : \pi_n(X) \to \pi_n(Y)$.

Theorem 2.9. Given a path $\gamma : x_1 \to x_2$ in X we get a map,

$$\gamma_\# : \pi_n(X, x_1) \to \pi_n(X, x_2)$$

depending only on the homotopy class of γ. In particular we have a $\pi_1(X, x_0)$-action on $\pi_n(X, x_0)$.

Remark. In the case $n = 1$ this is the conjugation action of $\pi_1(X, x_0)$ on itself.

Proposition 2.10. Given the previous proposition, we have,

$$[S^n, X] = \pi_n(X, x_0)/\pi_1(X, x_0)$$

Proposition 2.11. If $p : \tilde{X} \to X$ is a covering map then for $n \geq 2$ the induced map,

$$p_* : \pi_n(\tilde{X}) \to \pi_1(X)$$

is an isomorphism.

Proof. Injectivity is the homotopy lifting property. Furthermore given $f : S^n \to X$ we can lift it to $\tilde{f} : S^n \to \tilde{X}$ provided that $f_*(\pi_1(S^n)) \subset p_*(\pi_1(\tilde{X}))$. In the case $n \geq 2$, we have $\pi_1(S^n)$ thus such a lift always exists proving surjectivity. ☐

Example 2.12. Let Σ_g be a genus g surface. For $g \geq 1$ then Σ_g has universal cover $\mathbb{R}^2$ which is contractible and thus $\pi_n(\Sigma_g) = \pi_n(\mathbb{R}^2) = 0$ for $n \geq 2$.

Example 2.13. For $n \geq 2$ we have $\pi_n(\mathbb{RP}^k) = \pi_n(S^k)$.

2.1 Basic Operations on Spaces

Definition 2.14. The suspension of X is $\Sigma X = X \vee S^1$.

Definition 2.15. The loops space of X is $\Omega X = \mathrm{Hom}\,(\mathbf{Top}_\bullet, S^1)\,X$ with the compact-open topology.

Theorem 2.16 (Adjunction)**.**

$$\langle \Sigma X, Y \rangle = \langle X, \Omega Y \rangle$$

Example 2.17. $\Sigma S^n = S^{n+1}$

Proposition 2.18. $\pi_{n+1}(Y) = \langle S^{n+1}, Y \rangle = \langle \Sigma S^n, Y \rangle = \langle S^n, \Omega Y \rangle = \pi_n(\Omega Y)$

Proposition 2.19. The space ΩX is a group object in the category $\mathbf{hTop}_\bullet$.

Remark. The following definition is due to Hatcher.

Definition 2.20. A pointed space (X, e, μ) is an H-space is there is a map $\mu : X \times X \to X$ such that $\mu(-, e) \sim \mathrm{id}$ and $\mu(e, -) \sim \mathrm{id}$ as pointed maps (relative to the basepoint).

Remark. Any topological group (group object in **Top**) is an H-space (pointed at the identity element).

Remark. Loop spaces are H-spaces since they are group objects in $\mathbf{hTop}_\bullet$.

Theorem 2.21 (Adams)**.** The spheres S^n admitting an H-space structure are exactly S^0, S^1, S^3, S^7.

Corollary 2.22. $\mathbb{R}^n$ has a unital division $\mathbb{R}$-algebra structure iff $n = 1, 2, 4, 8$.

Proof. Consider the unit length elements $U = S^{n-1}$. Then a division algebra on $\mathbb{R}^n$ gives a multiplication $U \times U \to U$ (well defined since $xy = 0 \implies x = 0$ or $y = 0$ and thus the result can be scalled to lie in U). $\square$

3 Relative Groups

Definition 3.1. Given a space X a subspace $A \subset X$ and a point $x_0 \in A$ we denote the pointed pair as (X, A, x_0).

Definition 3.2. For a pointed pair (X, A, x_0) we define $\pi_n(X, A, x_0)$ as maps,

$$f : (D^n, S^{n-1}, p_0) \to (X, A, x_0)$$

modulo homotopy through maps of this form.

Remark. Suppose $[f] \in \pi_n(X, A, x_0)$ is zero if it is homotopic to a map with image inside A. In fact if this is the case then f may be homotoped relative to the boundary. Compression Lemma.

Theorem 3.3. There is a long exact sequence for the pointed pair (X, A, x_0),

$$\cdots \longrightarrow \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0) \longrightarrow \pi_{n-1}(X, x_0) \longrightarrow \pi_{n-1}(X, A, x_0) \longrightarrow \cdots$$

4 Results on CW Complexes

Definition 4.1. A CW pair is a CW complex X with a subcomplex $A \subset X$ (a closed subset which is a cunion of cells e.g.X^k the k-skelleton).

Theorem 4.2 (homotopy extension)**.** Let (X, A) be a CW pair. Then (X, A) has the homotopy extension property i.e. $\iota : A \to X$ is a cofibration.

Proof. Working cell-by-cell we can reduce to the case $(X, A) = (D^n, S^{n-1})$. In this case we are given a map on $D^n \times \{0\} \cup S^{n-1} \times I$ which is a deformation retract of $D^n \times I$ so any map can be extended. $\square$

Definition 4.3. A map $f : X \to Y$ between CW complexes is *cellular* if $f(X^k) \subset Y^k$.

Theorem 4.4 (cellular approximation)**.** Any map $f : X \to Y$ of CW complexes is homotopic to a cellular map.

Corollary 4.5. If $n < m$ then $\pi_n(S^m) = 0$.

Theorem 4.6. If $\pi_i(X, x_0) = 0$ for $i \leq n$ (i.e. X is n-connected) then X is homotopic to a CW complex with a single zero 0-cell and no i-cells for $1 \leq i \leq n$.

Lemma 4.7. If (X, A) is a CW-pair and A is contractible then $X \to X/A$ is a homotopy equivalence.

5 More Results on CW Complexes (01/29)

Theorem 5.1 (Whitehead)**.** Let $f : X \to Y$ be a map of CW complexes such that $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for each n then f is a homotopy equivalence.

Example 5.2. If $\pi_n(X, x_0) = 0$ for all $n \geq 0$ and X is a CW complex then X is contractible. To see this consider the constant map $X \to *$.

Example 5.3. Consider $S^\infty = \varinjlim S^n$ where we consider $S^n \subset S^{n+1}$ as the equator. Then $\pi_n(S^\infty) = 0$ since any map $S^n \to S^\infty$ can be deformed to a point using the copy of S^{n+1}. Thus S^∞ is contractible.

Remark. In Whitehead's theorem, simply knowing $\pi_n(X) \cong \pi_n(Y)$ for each $n \geq 0$ does not imply $X \sim Y$ we need these isomorphisms to be induced by a single topological map $f : X \to Y$.

Example 5.4. Quotienting by the natural involution on S^∞ we get a double cover $p : S^\infty \to \mathbb{RP}^\infty$. Using covering theory we find,

$$\pi_n(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & n > 1 \end{cases}$$

Furthermore, consider $X = S^2 \times \mathbb{RP}^\infty$ whose universal cover is $\tilde{X} = S^2 \times S^\infty \sim S^2$ and thus,

$$\pi_n(X) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n > 1 \end{cases}$$

This has exactly the same homotopy groups as $Y = \mathbb{RP}^2$ whose universal vover is also $\tilde{X} = S^2$ and also has a two-fold cover. However, $H_*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$ is finite dimensional and $H_*(S^2 \times \mathbb{RP}^\infty, \mathbb{Z}/2\mathbb{Z})$ is infinite dimensional so they cannot be homotopy equivalent.

Definition 5.5. The mapping cylinder of a morphism $f : X \to Y$ is the pushout,

$$Mf = Y \coprod_f (X \times I)$$

There is a natural inclusion $\iota : X \hookrightarrow Mf$ and a deformation retract $j : Mf \to Y$.

Remark. If X and Y are CW complexes then we may homotope $f : X \to Y$ to a cellular map in which case Mf is a CW complex and $\iota : X \hookrightarrow M(f)$ makes (Mf, X) a CW pair.

Definition 5.6. If X and Y are any spaces $f : X \to Y$ is a *weak homotopy equivalence* if $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n \geq 0$.

Theorem 5.7. Any space is weakly homotopy equivalent to a CW complex.

Remark. Suspension is a functor: given $f : X \to Y$ we get $\Sigma f : \Sigma X \to \Sigma Y$ given by $\Sigma f(t, x) = (t, f(x))$.

Remark. The unit of the suspension-looping adjunction gives a map $X \to \Omega\Sigma X$ given by $x \mapsto (t \mapsto (t, x))$. Applying the functor π_n gives the Freudenthal map $\sigma_n : \pi_n(X) \to \pi_{n+1}(\Sigma X)$.

Theorem 5.8 (Freudenthal Suspension)**.** Let X be an n-connected pointed space. Then the Freudenthal map $\Sigma_k : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism if $k \leq 2n$ and an epimorphism if $k = 2n + 1$.

Corollary 5.9. $\pi_n(S^n) = \mathbb{Z}$.

Proof. We show this by induction. For $n = 1$ the result $\pi_1(S^1) = \mathbb{Z}$ is a simple application of covering space theory. Now we assume the result for S^n. Then since S^n is $(n-1)$-connected, by the Fruedenthal suspension theorem we get an isomorphism $\pi_k(S^n) \xrightarrow{\sim} \pi_{k+1}(S^{n+1})$ for $k < 2n - 1$. Setting $k = n$ we see that $\pi_{n+1}(S^{n+1}) \cong \pi_n(S^n)$ for $n > 1$. However, for the case $n = 1$ we only get an epimorphism $\pi_1(S^1) \to \pi_2(S^2)$ since $1 = 2 - 1$. However, there is a surjective degree map $\pi_2(S^2) \to \mathbb{Z}$ and thus $\pi_2(S^2) = \mathbb{Z}$. $\square$

6 Spectra

Definition 6.1. A spectrum is a sequence X_n of CW complexes along with structure maps $s_n : \Sigma X_n \to X_{n+1}$.

Definition 6.2. Let X be a spectrum then we define the homotopy groups of X via,

$$\pi_k(X) = \varinjlim_n \pi_{k+n}(X_n)$$

where the maps $\Sigma X_n \to X_{n+1}$ induce $\pi_{k+n}(X_n) \to \pi_{k+n+1}(X_{n+1})$ by adjunction making the groups $\pi_{k+n}(X_n)$ a directed system.

Remark. Spectra may have homotopy in negative dimension i.e. $\pi_k(X) \neq 0$ for $k \leq 0$ in general.

Definition 6.3. We say a spectrum is stable if the structure maps are eventually all weak homotopy equivalences.

Example 6.4. Given a CW complex X we can form the suspension specturm $X_n = \Sigma^n X = S^n \wedge X$ with identity maps $\Sigma X_n \to X_{n+1}$. This is clearly a stable spectrum.

Example 6.5. The suspension spectrum of S^0 is the sphere spectrum $\mathbf{S}$ given by $\mathbf{S}_n = S^n$ with the natural homeomorphisms $\Sigma S^n \to S^{n+1}$.

Definition 6.6. An Ω-spectrum is a specturm X such that the adjunction of the structue map $X_n \to \Omega X_{n+1}$ is a weak homotopy equivalence.

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Theorem 7.1. Two CW complexes of type $K(G,n)$ are homotopy equivalent.

Proof. Let X, Y be CW complexes. Assume that X has no $1, \dots, (n-1)$-cells (since it is $(n-1)$-connected) and one 0-cell (since it is connected). Then,

$$X^n = \bigvee_{i \in I} S^n$$

each of these spheres represents an element $\pi_n(X) = G$. Construct $f_n : X^n \to Y$ by sending each S^n to the corresponding element in $\pi_n(Y) = G$. Next construct $f_{n+1} : X^{n+1} \to Y$ so that each $\partial D^{n+1} = S^n \xrightarrow{f_n} Y$ represents $0 \in \pi_n(Y)$ (since the $(n+1)$-cells give the relations on G) then $\partial D^{n+2} = S^{n+1} \xrightarrow{f_{n+1}} Y$ is nullhomotopic because $\pi_{n+1}(Y) = 0$. Repeating, we can extend to all X. $\square$

Remark. Key point: $\pi_n(X)$ is generated by n-cells and has relations by $(n+1)$-cells. This is a first glimpse of obstruction theory. We ask the following questions:

Q1 Given a CW pair (X, A) and $f : A \to Y$ can we extend this to $\tilde{f} : X \to Y$?

Q2 Given a giber bundle $p : E \to B$ and a map $f : X \to B$ can we lift it to $\tilde{f} : X \to E$?

For Q1, assume that $\pi_1(Y) \circlearrowright \pi_n(Y)$ trivially (i.e. Y is simple so we need not worry about basepoints!). Given $f : X^n \to Y$ can we extend it to X^{n+1}? Gluing a disk D^{n+1} then f extends to D^{n+1} iff $f|_{S^n} : S^n \to Y$ is nullhomotopic i.e. is zero in $\pi_n(Y)$. In general, to each $(n+1)$-cell e, $[f_e] \in \pi_n(Y)$ then we can construct $c_f \in C^{n+1}(X, \pi_n(Y))$ a cellular cochain called the obstruction cochain. Then f extends to $X^{n+1} \iff c_f = 0$.

Lemma 7.2. $\delta c_f = 0$ i.e. c_f is a cocycle. Therefore, $O_f := [c_f] \in H^{n+1}(X; \pi_n(Y))$ is the obstructuon class.

Theorem 7.3. $f|_{X^{n-1}}$ extends to X^{n+1} iff $O_f = 0$.

Proof. First we prove the Lemma. Consider the diagram,

$$\begin{tikzcd}
C_{n+2}(X) \arrow[r, equal] \arrow[ddd, "\partial"] & H_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, "h^{-1}"] & \pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[d, "\partial"] \\
 & & \pi_{n+1}(X^{n+1}) \arrow[d, "\iota"] \\
 & & \\
C_{n+1}(X) \arrow[r, equal] \arrow[rrdd, "c_f"] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "h^{-1}"] & \pi_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial"] \\
 & & \pi_n(X^n) \arrow[d, "f_*"] \\
 & & \pi_n(Y)
\end{tikzcd}$$

The piece of the LES,

$$\pi_{n+1}(X^{n+1}) \longrightarrow \pi_{n+1}(X^{n+1}, X^n) \longrightarrow \pi_n(X^n)$$

composes to zero so by the commutativity of the above diagram $c_f \circ \partial = 0$. ☐

Definition 7.4. Suppose there are two maps $f, g : X^n \to Y$ that agree on X^{n-1} then for each n-cell D^n if we glue two D^n along the boundary on which f, g agree then we get a map $(f, g) : S^n \to Y$ and thus an element $\pi_n(Y)$ for each n-cell. This gives a difference cochain $d_{f,g} \in C^n(X; \pi_n(Y))$ and $d_{f,g} = 0$ iff $f, g : X^n \to Y$ are homotopic relative to X^{n+1}.

Lemma 7.5. $\delta d_{f,g} = c_g - c_f$.

Lemma 7.6. Given $f : X^n \to Y$ for any $d \in C^n(X; \pi_n(Y))$ there is $g : X^n \to Y$ with $f|_{X^{n-1}} = g|_{X^{n-1}}$ s.t. $d_{f,g} = d$.

Proof. For $d \in C^n(X; \pi_n(Y))$ then for an n-cell e we have $d(e) \in \pi_n(Y)$ then consider the sum of maps f and $d(e)$ using the sum structure on e contracting the equator. ☐

Proof. Now we prove the theorem. Suppose that $O_f = 0$ then $c_f = \delta d$ for some $d \in C^n(X; \pi_n(Y))$. Now there exists $g : X^n \to Y$ with $f|_{X^{n-1}} = f|_{X^{n-1}}$ and $d_{f,g} = -d$. Also, $\delta d_{f,g} = c_g - c_f$ and thus $c_g = c_f + \delta d_{f,g} = c_f - \delta d = 0$ therefore $c_g = 0$ so g can extend to X^{n+1} and $f|_{X^{n-1}} = g|_{X^{n-1}}$. ☐

Theorem 7.7. Let $f, g : X^n \to Y$ be maps with $f|_{X^{n-2}} = g|_{X^{n-2}}$. Then $[d_{f,g}] = 0$ iff they are homotopic relative to X^{n-2}.

7.1 Cohomology of $K(G, n)$

Let $n \geq 2$ and G abelian. Consider a map $f : X \to K(G, n)$. By Hurewicz, $H_n(K(G, n), \mathbb{Z}) = \pi_n(K(G, n)) = G$ and $H_{n-1}(K(G, n), \mathbb{Z}) = 0$. Now, by the universal coefficient theorem,

$$H^n(K(G, n), G) = \mathrm{Hom}\,(H_n(K(G, n), \mathbb{Z}), G) = \mathrm{Hom}\,(G, G)$$

Therefore, there is a canonical element $\mathbb{1} \in H^n(K(G, n), G)$ which is the class of $\mathrm{id} : G \to G$.

Also, via $f : X \to K(G, n)$, we also get $f^*(\mathbb{1}) \in H^n(X; G)$, which depends only on the homotopy class of f.

Theorem 7.8. The map $[X, K(G, n)] \to H^n(X, G)$ sending $[f] \mapsto f^\times(\mathbb{1})$ is an isomorphism.

Remark. We say that $K(G, n)$ classifies $H^n(-, G)$ meaning that the functor,

$$H^n(-, G) : \{\text{CW-complexes}\} \to \mathbf{Set}$$

is represented by $[-, K(G, n)]$.

Definition 7.9. Given a contravariant functor $h : \{\text{CW-complexes}\} \to \mathbf{Set}$ we say that C classifies h if there is a natural isomorphism $h \cong [-, C]$ in this case we say that h is representable and the pair $(C, \mathrm{id} \in h(C))$ is a representation of h.

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Recall that the isomorphism $H^n(K(G,n),G) \cong \text{Hom}(G,G)$ gives a canonical element $\text{id} \in H^n(K(G,n),G)$ the fundamental class.

Theorem 8.1. We have $[X, K(G,n)] = H^n(X,G)$ given by $[f] \mapsto f^*(\text{id}) \in H^n(X,G)$.

Remark. We can describe the fundamental class explicitly as follows. Wrtting $K(G,n)$ as a wedge of n-spheres each for a generator $g \in G$. Then there is a class $c_0 \in C^n(K(G,n),G)$ sending each S^n to the corresponding element of g. Then, in terms of two important maps $K(G,n) \to K(G,n)$, the identity and the constant map, we get $c_0 = d_{\text{const, id}}$ and then $[c_0]$ is the fundamental class.

Proof. First we prove surjectivity. Given $[c] \in H^n(X,G)$ we want $f : X \to K(G,n)$ s.t. $f^*[c_0] = [c]$. Let $f|_{X^{n-1}}$ be constant. Last time we showed that given any $h : X^n \to K(G,n)$ and $d \in C^n(X,G)$ there is some f such that $d_{h,f} = d$ and $f|_{X^{n-1}} = h|_{X^{n-1}}$. Pick f such that $d_{\text{const},f} = c$ and we may assume $f : X^n \to K(G,n)$ is ceulluar. Now consider,

$$X^n \xrightarrow{f} K(G,n) \xrightarrow{\text{id}} K(G,n)$$

then,

$$f^*(d_{\text{const,id}}) = d_{\text{const}\circ f, \text{id}\circ f}$$

But $d_{\text{const,id}} = c_0 \in C^n(K(G,n),G)$ and thus,

$$f^*(c_0) = d_{\text{const},g} = c \in C^n(X,G)$$

Therefore, we get $f : X^n \to K(G,n)$ such that $f^*[c_0] = [c]$. Furthermore, we can extend to X^{n+1} since the homotopy of $K(G,n)$ vanishes above n.

Now we show injectivity. Suppose that $f^*[\text{id}] = g^*[\text{id}]$ for two maps $f, g : X \to K(G,n)$. We may assume that both maps are cellular. Then $[d_{\text{const},f}] - [d_{\text{const},g}] = \pm[d_{f,g}]$. But $d_{\text{const},f} = f^*[\text{id}]$ and likewise for g so $[d_{f,g}] = 0$. This implies that $f, g|_{X^n}$ are homotopic which may be extended a homotopy on X since $K(G,n)$ has no higher homotopy. □

Example 8.1. We have the following explicit calculations,

(a). $K(\mathbb{Z},1) = S^1 \implies H^1(X;\mathbb{Z}) = [X, S^1]$

(b). $K(\mathbb{Z},2) = \mathbb{CP}^\infty \implies H^2(X;\mathbb{Z}) = [X, \mathbb{CP}^\infty]$

(c). $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^\infty \implies H^1(X,\mathbb{Z}/2\mathbb{Z}) = [X, \mathbb{RP}^\infty]$

Theorem 8.2 (Hopf)**.** For every CW complex with $\dim X \le n$ we have $[X, S^n] \xrightarrow{\sim} H^n(X,\mathbb{Z})$ via $[f] \mapsto f^*(1)$ for $1 \in H^n(S^n,\mathbb{Z})$.

Proof. We know that $H^n(X,\mathbb{Z}) \xrightarrow{\sim} [X, K(\mathbb{Z},n)]$. We can construct $K(\mathbb{Z},n)$ as one 0-cell, and one n-cell with no relations i.e. no $(n+1)$-cells. Thus, $K(\mathbb{Z},n)^{n+1} = S^n$. Therefore, if $\dim X = n$ then homotopy is controlled only by the $(n+1)$-skeleton. Explicitly, by cellular approximation $f : X \to K(G,n)$ is homotopic to $f : X \to K(G,n)^n = S^n$. Furthermore, if $f, g : X \to K(G,n)$ are homotopic then there is a homotopy $h : X \times I \to K(G,n)$ and again by cellular approximation a homotopy $h : X \times I \to K(G,n)^{n+1} = S^n$. Thus $[X, K(G,n)] = [X, S^n]$ proving the result. □

Definition 8.2. The cohomotopy groups of X are $\pi^n(X) = [X, S^n]$ which also have a pointed version $\pi^n(X, x_0) = \langle X, S^n \rangle$.

Remark. Since $H^n(X,\mathbb{Z}) = 0$ for $n > \dim X$ we immediately see that $\pi^n(X) = 0$ for $n > \dim X$ and $\pi^n(X) = H^n(X,\mathbb{Z})$ for $\dim X = n$. In particular, cohomotopy is bounded unlike homotopy.

8.1 Generalizations

Remark. Given a fibre bundle $F \hookrightarrow E \to B$ can we extend / find sections? For example, is there a nonvanishing vector field on S^n? Can you find k linearly independent vector field on S^n.

Remark. Here we make some simplifying assumptions: F is simple i.e. ($\pi_1 \circlearrowright \pi_n$ trivially) and B is simply connected (this can be weakened by using local coefficients).

The main questions will be, given a section $s : B^n \to E$ can you extend to B^{n+1} ?

Choose D^{n+1}, an $(n+1)$-cell of B then $E|_{D^{n+1}} \cong F \times D^{n+1}$ since D^{n+1} is contractible. The section s gives $\partial D^{n+1} \to F$ giving $[s] \in \pi_n(F)$. Then s extends to D^{n+1} iff $[s] = 0$. Since on each $(n+1)$-cell we get a $[s] \in \pi_n(F)$ and thus we patch these together to get a cochain,

$$c_s \in C^{n+1}(B, \pi_n(F))$$

To do this we require that B is simply connected such that there is a unique identification of the fibres $\pi_n(F_b)$.

Theorem 8.3. $O_s \in H^{n+1}(B, \pi_n(F))$ vanishes iff $s|_{B^{n-1}}$ extends to B^{n+1}.

8.1.1 Primary Obstruction

Assume that $\pi_0(F) = \pi_1(F) = \cdots = \pi_{n-1}(F) = 0$. Then there is no obstruction to finding a section $s : B^n \to E$ (since the obstructions are zero at the cochain level). Then $O_s \in H^{n+1}(B, \pi_n(F))$ we get the first nonzero obstruction cochain.

Proposition 8.3. This first nonvanish obstruction O_s does not depend on the choice of $s : B^n \to E$

Proof. It sufficies to show that the HEP holds for sections of bundles and that if $s \sim s'$ on B^n then $O_S = O_{S'}$.

By induction we will $\square$

Definition 8.4. The primary obstruction of a fibre bundle $F \hookrightarrow E \to B$ is the above obstruction class $O_B \in H^{n+1}(B, \pi_n(F))$ which an invariant of the bundle.

9 Vector Bundles

Definition 9.1. A vector bundle is a fibre bundle whose fibres are vectorspaces and fibre preserving maps are assumed to be linear.

Lemma 9.2. A rank n vector bundle is trivial iff there are n everywhere linearly independent sections.

Remark. Assuming B is paracompact (e.g. a CW complex) for any vector bundle $p : E \to B$ we can assume E comes with a fibrewise Euclidean structure i.e. a section of $E^* \otimes E^* \to B$.

Definition 9.3. Given a vector bundle $p : E \to B$ with a metric $g \in \Gamma(B, E^* \otimes E^*)$ then we define $p_k : E_k \to B$ to be the bundle of g-orthonormal k-frames. Thus $p_k^{-1}(x) \cong V(n, k)$ the Stiefl-manifold of orthonormal k-frames of $\mathbb{R}^n$ (whose $O(k)$-quotient is the Grasmannian $G(n, k)$). Then E has k linearly independent section iff E_k admits a section.

Lemma 9.4. $\pi_i(V(n,k)) = 0$ for $i < n-k$

Proof. Furthermore, for the base case, $V(n,1) = S^{n-1}$ so $\pi_i(V(n,1)) = \mathbb{Z}\delta_{i,n-1}$.

There is a fibre bundle $V(n,k) \to S^{n-1}$ via $(v_1, \dots, v_k) \mapsto v_k$ with fibre $V(n-1,k-1)$. Therefore, $V(n-1,k-1) \hookrightarrow V(n,k) \to S^{n-1}$. Then by the LES,

$$\cdots \longrightarrow \pi_{i+1}(S^{n-1}) \longrightarrow \pi_i(V(n-1,k-1)) \longrightarrow \pi_i(V(n,k)) \longrightarrow \pi_i(S^{n-1}) \longrightarrow \cdots$$

Then for $i < n-2$ we have $\pi_{i+1}(S^{n-1}) = \pi_i(S^{n-1}) = 0$ so $\pi_i(V(n,k)) = \pi_i(V(n-1,k-1))$.

Suppose that $i < n-k$ and $k \geq 2$ then $i < n-2$ and $i < (n-k+2)-2$ so we have,

$$\pi_i(V(n,k)) = \pi_i(V(n-1,k-1)) = \cdots \pi_i(V(n-k+2,2)) = \pi_i(V(n-k+1,1)) = \pi_i(S^{n-k}) = 0$$

For $i = n-k$ we get,

$$\pi_i(V(n,k)) = \pi_i(V(n-1,k-1)) = \cdots = \pi_i(V(n-k+2,2))$$

but $i = (n-k+2)-2$ so we cannot reduce further so we need to compute $\pi_{n-2}(V(n,2))$. However, $V(n,2)$ is the unit tangent bundle of S^{n-1}. Consider the fibration $S^{n-2} \hookrightarrow V(n,2) \to S^{n-1}$ gives the LES,

$$\begin{tikzcd}
\pi_{n-1}(S^{n-1}) \arrow[r, "\partial_*"] \arrow[d, equal] & \pi_{n-2}(S^{n-2}) \arrow[r] \arrow[d, equal] & \pi_{n-2}(V(n,2)) \arrow[r] & \pi_{n-2}(S^{n-1}) \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "d"] & \mathbb{Z} & & 0
\end{tikzcd}$$

Now the map ∂_* is $\pi_{n-1}(S^{n-1}) \cong \pi_{n-1}(T_1 S^{n-1}, S^{n-2}) \to \pi_{n-2}(S^{n-2})$. Then $d : \mathbb{Z} \to \mathbb{Z}$ computes the number of zeros of a generic vector field on S^{n-1} so,

$$d = \begin{cases} 0 & n-1 \text{ odd} \\ 2 & n-1 \text{ even} \end{cases}$$

Therefore,

$$\pi_{n-2}(V(n,2)) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & n \text{ odd} \end{cases}$$

$\square$

9.1 Stiefel-Whitney Classes

The primary obstruction is $W_i(E) \in H^i(B, \tilde{\mathbb{Z}})$ of the bundle E_k with $k = n+1-i$ is the obstruction to find k linearly independent sections on the i-skeleton.

Reduction mod 2 to get, $w_i(E) \in H^i(B, \mathbb{Z}/2\mathbb{Z})$ the Stiefel-Whitney classes. Furthermore, we define $w_0(E) = 1 \in \mathbb{Z}/2\mathbb{Z} = H^0(B, \mathbb{Z}_2)$.

Proposition 9.5. $w_1(E)$ is zero iff E is orientable.

Lemma 9.6. Naturality of the Stiefel-Whitney Classes. Given $f : B' \to B$ and $p : E \to B$ we have,

$$f^* w_i(E) = w_i(f^* E) \in H^i(B', \mathbb{Z}/2\mathbb{Z})$$

Theorem 9.7. If E, E' are bundles on B then,

$$w_i(E \oplus E') = \sum_{p+q=i} w_p(E) \smile w_q(E')$$

Definition 9.8. The full Stiefel-Whitney Class is $w(E) = \sum_{i=1}^{n} w_i(E) \in H^*(B, \mathbb{Z}/2\mathbb{Z})$.

Remark. The the sum formula reduces to,

$$w(E \oplus E') = w(E) \cdot w(E')$$

in the total cohomology ring.

Theorem 9.9. The previous properties,

(a). for the mobius bundle $\mu \to S^1$ then $w_1(\mu) = \mathbb{Z}/2\mathbb{Z}$

(b). for $f : B' \to B$ then $f^* w_i(E) = w_i(f^* E)$

(c). $w(E \oplus E') = w(E) \cdot w(E')$ in the cohomology ring

uniquely characterize w as a map from vector bundles to cohomology rings.

Example 9.10. Consider the tautological bundle $\gamma^n \to \mathbb{RP}^n$ defined by,

$$\gamma^n = \{(\ell, v) \mid v \in \ell\} \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$$

then $\gamma^1 \to S^1$ is the Mobius bundle.

10 Characteristic Classes

Example 10.1. Recall that $\gamma_1 \to \mathbb{RP}^1$ is the Modius bundle so $w(\gamma_1) = 1 + \alpha \in H^*(S^1)$ where $H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[\alpha]/(\alpha^{n+1})$.

Then for $\iota : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ we find $\iota^*(\gamma_n) = \gamma_1$. Also we can compute,

$$\iota^* : H^*(\mathbb{RP}^n) \to H^*(\mathbb{RP}^1)$$

must sent $\alpha \mapsto \alpha$. Since $w(\gamma_1) = w(\iota^*(\gamma_n)) = \iota^*(w(\gamma_n)) = 1 + \alpha$ we find that $w(\gamma_n) = 1 + \alpha$.

Example 10.2. Consider the orthogonal bundle to γ_n,

$$\gamma_n^\perp = \{(\ell, b) \mid v \in \ell^\perp\} \subset \mathbb{RP}^n \times \mathbb{R}^n$$

It is clear that $\gamma_n \oplus \gamma_n^\perp$ is the trivial rank-$n+1$ bundle over $\mathbb{RP}^n$. Thus,

$$w(\gamma_n \oplus \gamma_n^\perp) = w(\gamma_n) \cdot w(\gamma_n^\perp) = 1$$

But $w(\gamma_n) = 1 + \alpha$ so,

$$w(\gamma_1) = \frac{1}{1 + \alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^n$$

Example 10.3. Consider the tangent bundle $T\mathbb{RP}^n \to \mathbb{RP}^n$. First, consider,

$$TS^n = \{(x,v) \mid |x| = 1 \quad x \cdot v = 0\} \subset S^n \times \mathbb{R}^{n+1}$$

Then,

$$T\mathbb{RP}^n = TS^n/(x,v) \sim (-x,-v)$$

Then $x, -x \in \ell_x$ and $v, -v \in \ell_x^\perp$. Such an element $(x,v) \in T\mathbb{RP}^n$ is a pair (x, L) where $L : \ell_x \to \ell_x^\perp$ is a linear map. Therefore,

$$T\mathbb{RP}^n = \mathrm{Hom}\left(\gamma_n, \gamma_n^\perp\right)$$

First, note that,

$$\mathrm{Hom}\left(\gamma_n, \gamma_n\right) = \underline{\mathbb{R}}$$

since there is the section $\mathrm{id} : \gamma_n \to \gamma_n$. Therefore,

$$\begin{aligned} T\mathbb{RP}^n \oplus \underline{\mathbb{R}} &= \mathrm{Hom}\left(\gamma_n, \gamma_n^\perp\right) \oplus \mathrm{Hom}\left(\gamma_n, \gamma_n\right) = \mathrm{Hom}\left(\gamma_n, \gamma_n^\perp \oplus \gamma_n\right) = \mathrm{Hom}\left(\gamma_n, \underline{\mathbb{R}}^{n+1}\right) \\ &= \bigoplus_{i=1}^{n+1} \mathrm{Hom}\left(\gamma_n, \underline{\mathbb{R}}\right) \end{aligned}$$

However, for real bundles, a choice of metric gives an isomorphism $\mathrm{Hom}\left(\mathcal{E}, \underline{\mathbb{R}}\right) \cong \mathcal{E}$. Thus, we find,

$$T\mathbb{RP}^n \oplus \underline{\mathbb{R}} \cong \bigoplus_{i=1}^{n+1} \gamma_n$$

Then,

$$w(T\mathbb{RP}^n) = w(T\mathbb{RP}^n) \cdot w(\mathbb{R}) = (w(\gamma_n))^{n+1} = (1+\alpha)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \alpha^k \mod 2$$

Corollary 10.4. $w(T\mathbb{RP}^n) = 1$ iff $n+1$ is a power of 2 and thus if $T\mathbb{RP}^n$ is trivial then $n = 2^k - 1$.

11 Classifying Spaces

Consider the functor $V : \{\text{CW complex}\} \to \mathbf{Set}$ via $X \mapsto \mathrm{Vect} X$ to isomorphism classes of vector bundles on X. Is this functor representable in the homotopy category i.e. is $\mathrm{Vect}(X) = [X, C]$ for come classifying space C?

Recall the Grasmannian, $G(n,k)$ which classifies k-dimensional subspaces of $\mathbb{R}^n$. There is a natural inclusion $G(n,k) \to G(n+1,k)$ which alows us to construct,

$$G_k = \varinjlim_n G(n,k)$$

In particular, $G_1 = \mathbb{RP}^\infty$. We will see $G_n = BO(n)$. Furthermore, there is a tautological bundle $\gamma_n^k \to G(n,k)$ which is a rank k-vector bundle,

$$\gamma_n^k = \{(x,v) \mid x \in G(n,k) \; v \in V_x\} \subset G(n,k) \times \mathbb{R}^n$$

This gives a bundle $\gamma^k \to G_k$ which we call $EO(n) \to BO(n)$.

Theorem 11.1. Let X be a finite CW complex then,

$$[X, G_k] \to \mathrm{Vect}^k(X)$$

via $f : X \to G_k \mapsto f^*\gamma^k$ is an isomorphism.

Proof. Look at Milne. □

Theorem 11.2. $H^*(G_k, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[w_1, \dots, x_n]$ with w_i graded in degree i where $w_i = w_i(\gamma^k)$.

Corollary 11.3. If $E = f^*\gamma^n$ then $w_i(E) = f^*w_i(\gamma^n)$ so the Stiefel-Whitney classes of E detect nontriviality in $\mathbb{Z}/2\mathbb{Z}$ cohomology of the classifying map $f : X \to G_k$.

12 K-Theory

Proposition 12.1. Every class $\alpha \in K(X)$ can be represented as $[E] - [\varepsilon^n]$.

Proposition 12.2.

Definition 12.3. We say that vector bundles E and E' are *stably equivalent* if $Eoplus\varepsilon^n = E' \oplus \varepsilon^m$ for $n, m \in \mathbb{N}$. Then define reduced K-theory,

$$\tilde{K}(X, x_0) = \ker\left(K(X) \to K(x_0)\right)$$

for connected X we have $\tilde{K}(X) = \mathrm{Vect}_{\mathbb{C}}(X)/\text{stb. eq.}$.

Proposition 12.4. For connected X we have $\tilde{K}(X) = [X : BU]$ where $BU = \varinjlim_n BU(n)$ since homotopy equivalence to BU is equivalent to stable equivalence

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Let X be a finite CW complex and $\mathrm{Vect}_{\mathbb{C}}(X)$ complex vector-bundles on X. Then we define, $K(X)$ is the group completion of the monoid $\mathrm{Vect}_{\mathbb{C}}(X)$ under $\oplus$. Then $K(X)$ is a ring under $\oplus$ and $\otimes$.

Then $X \mapsto K(X)$ is a functor: $f : X \to Y$ gives $f^* : K(Y) \to K(X)$ from pulling back bundles. Then f^* depends only on the homotopy class $f \in [X : Y]$.

Remark. Is X is contractible then $K(X) \cong K(*) = \mathbb{Z}$.

Proposition 12.5. For any vector bundle E on X there exists a vector bundle E' on X with $E \oplus E' = \underline{\mathbb{C}}^N$.

Corollary 12.6. In the K-group we have $[E] - [F] = [E'] - [\mathbb{C}^N]$ and furthermore $[E] - [\mathbb{C}^N] = [E'] - [\mathbb{C}^M]$ if and only if $\dim E - N = \dim E' - M$ and $E \sim_s E'$ are stably equivalent i.e. $E \oplus \underline{\mathbb{C}}^n \cong E' \oplus \underline{\mathbb{C}}^m$.

Definition 12.7. For a point $x_0 \in X$ we have $\tilde{K}(X) = \ker\left(K(X) \to K(x_0)\right)$ this gives bundles up to stable equivalence (assume that X is connected).

Proposition 12.8. $\tilde{K}(X) = [X : BU]$ with $BU = \varinjlim_n BU(n)$.

Remark. Consider the Grassmannian,

$$G_{\mathbb{C}}(N,n) = U(N)/U(n) \times U(N-n)$$

and the Steifl manifold,

$$V_{\mathbb{C}}(N,n) = U(N)/U(N-n)$$

then there is a fibration,

$$U(n) \hookrightarrow V_{\mathbb{C}}(N,n) \to G_{\mathbb{C}}(N,n)$$

Proposition 12.9. The homotopy groups of the Stiefl manifold satisfies,

$$\pi_r(V_{\mathbb{C}}(N,n)) = 0 \quad \text{for} \quad r \le 2(N-n)$$

Furthermore, the fibration,

$$U(u) \hookrightarrow V_{\mathbb{C}}(\infty,n) \to G_{\mathbb{C}}(\infty,n) = BU(n)$$

Then $\pi_r(V_{\mathbb{C}}(\infty,n)) = 0$ by above and thus $V_{\mathbb{C}}(\infty,n)$ is contractible. Therefore, from the long exact sequence,

$$\pi_r(U(n)) \cong \pi_{n+1}(BU(n))$$

Definition 12.10. Now we have $BU = \varinjlim_n BU(n)$ and $U = \varinjlim_n U(n)$

Remark. Our previous proposition says that $\pi_r(U) \cong \pi_{r+1}(BU)$.

Proposition 12.11. Then $\tilde{K}(S^r) = [S^r, BU] = \pi_r(BU) = \pi_{r-1}(U)$.

Proof. There is no issuse with based vs unbased maps because $\pi_1(BU) \circlearrowright \pi_n(BU)$ is trivial since the above is a fibration and $\pi_1(V_{\mathbb{C}}(\infty,n)) = 0$ (use the homework). Alos $\pi_1(BU) = 0$ so we can conclude. $\square$

Example 12.12. $\tilde{K}(S^1) = \pi_0(U) = 0$ and $\tilde{K}(S^2) = \pi_1(U) = \pi_1(U(1)) = \mathbb{Z}$

Remark. The generator of $\tilde{K}(S^2) = \mathbb{Z}$ is the tautological bundle $\gamma \to \mathbb{CP}^1 = S^2$ and we write its reduced class as $[\gamma] - [\mathbb{C}]$. To see this, note that $\tilde{K}(S^2) \xrightarrow{\sim} \mathbb{Z}$ is given by the Chern class c_1 and $c_1(\gamma) = -1 \in H^2(S^2;\mathbb{Z})$.

Remark. We can describe $K(S^2) = \tilde{K}(S^2) \oplus \mathbb{Z} = \mathbb{Z}oplus\mathbb{Z}$ via $\xi \mapsto (c_1(\xi), \dim \xi)$. Consider in particular,

$$(\gamma \otimes \gamma) \oplus \mathbb{C} \mapsto (-2,2) \quad \text{and thus} \quad (\gamma \otimes \gamma) \mapsto (-2,2)$$

Then in $K(S^2)$ we have $\gamma^2 \oplus \underline{\mathbb{C}} = 2\gamma$ and thus $\gamma^2 + 1 = 2\gamma$ therefore $([\gamma]-1)^2 = 0$ in $K(S^2)$. Therefore, $K(S^2) = \mathbb{Z}[x]/x^2$.

12.2 Long Exact Sequence of a Pair

Lemma 12.13. Let (X,A) be a finite CW pair. Then consider the sequence $A \hookrightarrow X \to X/A$. Then we get induced maps,

$$\tilde{K}(X/A) \xrightarrow{p^*} \tilde{K}(X) \xrightarrow{\iota^*} \tilde{K}(A)$$

which is exact meaning $\ker \iota^* = \operatorname{Im}(p^*)$.

Proof. We know that $\tilde{K}(-) = [-, BU]$ so the above sequence is simply,

$$[X/A, BU] \to [X, BU] \to [A, BU]$$

if a map $f : X \to BU$ restricts to a nullhomotopic map $f|_A : [A, BU]$ then by the homotopy extension property of the pair (X, A) we get a homotopy on $f : X \to BU$ making $f|_A$ trivial i.e. f descends to the quotient $\tilde{f} : X/A \to BU$. $\square$

Remark. Using the homotopy equivalence $X \cup CA \simeq X/A$ we get a diagram,

$$\begin{tikzcd}
A \arrow[r, hook] & X \arrow[r, two heads] \arrow[d, equal] & X/A \arrow[d, "\simeq"] & & \\
& X \arrow[r, hook] & X \cup CA \arrow[r, two heads] \arrow[d, equal] & \Sigma A \arrow[d, "\simeq"] & \\
& & X \cup CA \arrow[r, hook] & (X \cup CA) \cup CX \arrow[r, two heads] & \Sigma X
\end{tikzcd}$$

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Definition 13.1. We define $\tilde{K}^{-q}(X) = \tilde{K}(\Sigma^q X)$. Furthermore, $K(X, A) = \tilde{K}(X/A)$ and $K(X) =$.

Theorem 13.1 (Bott)**.** The map $K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism. Explicitly, this is,

$$\begin{aligned}
K(X) \otimes K(S^2) = K(X) \otimes \mathbb{Z}[H]/(H-1)^2 &\cong K(X) \oplus K(X) \\
K(X) \oplus K(X) &\xrightarrow{\sim} K(X \times S^2) \\
(\alpha_1, \alpha_2) &\mapsto p_1^*\alpha_1 \oplus p_1^*\alpha_2 \otimes p_2^*\gamma
\end{aligned}$$

Corollary 13.2. $\tilde{K}(X) = \tilde{K}(\Sigma^2 X) = \tilde{K}^{-1}(X)$.

Proof. Consider,

$$\Sigma^2 X = X \times S^2 / X \vee S^2$$

Now we consider the pair $(X \times S^2, X \vee S^2)$ which gives an exact sequence,

$$K^{-1}(X \times S^2) \twoheadrightarrow K^{-1}(X \vee S^2) \longrightarrow K(X \times S^2, X \vee S^2) \longrightarrow K(X \times S^2) \xrightarrow{\varphi} K(X \vee S^2)$$

where the maps $K(X \times Y) \to K(X \vee Y)$ are surjective since $\tilde{K}(X \vee Y) = \tilde{K}(X) \oplus \tilde{K}(Y)$ and there is a pullback $K(X) \to K(X \times Y)$. Note that there is a diagram,

$$\begin{tikzcd}
K(X \times S^2) \arrow[r, two heads] & K(X \vee S^2) \arrow[d, equal] \\
\tilde{K}(X) \oplus \tilde{K}(S^2) \oplus \mathbb{Z} \arrow[u, dashed] \arrow[r, equal] & \tilde{K}(X \vee S^2) \oplus \mathbb{Z}
\end{tikzcd}$$

Therefore, $K(X \times S^2, X \vee S^2) = \ker \varphi$. By Bott's theorem, we see that $\alpha_1 \otimes 1 + \alpha_2 \otimes \gamma = \alpha \otimes (\gamma - 1) + \beta \otimes \gamma$ then $\varphi(x) = 0$ iff both restrictions to X and S^2 are zero.

Therefore,

$$\tilde{K}(X) = K(X \times S^2, X \vee S^2) \cong \tilde{K}(\Sigma^2 X)$$

$\square$

13.1 Proof of Bott's Theorem

First we consider vector bundles on S^2. Such vector bundles must be trivial on the upper and lower hemispheres but can have a nontrivial attaching function $f : S^1 \to \mathrm{GL}_n(\mathbb{C})$. Then, noting that $\pi_1(\mathrm{GL}_n(\mathbb{C})) = \mathbb{Z}$ we see immediately that $\tilde{K}(S^2) = \mathbb{Z}$.

Now consider $E|_{X \times D^2_\pm}$ is the pullback of a bundle E_0 on X since D^2 is contractible. Then E is obtained by gluing these together via,

$$f : X \times S^1 \to \mathrm{Aut}(E_0)$$

so E can be described as a pair $[E, f]$.

Example 13.3. The tautological bundle $\gamma \to S^2$ is represented by the pair $[\underline{\mathbb{C}}, z^{-1}]$.

We have some basic facts,

$$[E_1, f_1] \oplus [E_2, f_2] = [E_1 \oplus E_2, f_1 \oplus f_2]$$

and

$$[E_1, f_1] \otimes [E_2, f_2] = [E_1 \otimes E_2, f_1 \otimes f_2]$$

in particular,

$$[E_0, z^n f] = [E_0, f] \otimes \gamma^{-n}$$

Bott's theorem is that every bundle on $X \times S^2$ can be stabily represented as $\alpha_1 \otimes 1 + \alpha_2 \otimes \gamma$ i.e. every element in $K(X \times S^2)$ is stabily represented by,

$$[E_0, \mathrm{id}] \oplus [E_1, z^{-1}]$$

First, given $[E_0, f]$ we have $f : X \times S^1 \to \mathrm{Aut}(E_0)$ so fixing $x \in X$ we get $f(x, -) : S^1 \to \mathrm{Aut}((E_0)_x) = \mathrm{GL}_n(\mathbb{C})$ and thus we can use Fourier analysis to conclude that there is a convergent series,

$$f(x, z) = \sum_{n \in \mathbb{Z}} a_n(x) z^n$$

for each $x \in X$. Furthermore, since f is continuous on $X \times S^1$ we get that $a_n : X \to \mathbb{C}$ are continuous since they are computed by a continuous family of integrals. However, X is compact we get that,

$$\sum_{n=-N}^{N} a_n(x) z^n \to f(x, z)$$

converges uniformly. Then we can choose sufficiently large M such that,

$$f_M = \sum_{n=-M}^{M} a_n(x) z^n$$

is uniformly close to f and thus are homotopic by a linear homotopy. Thus we can assume that f is a Laruent series. After multiplying by z^M we can assume that f is a polynomial,

$$f(x) = p(x) = \sum_{n=0}^{N} a_n(x) z^n$$

so we can assume that our bundle is of the form $[E_0, p]$ for a polynomial function.

Now we claim that,

$$[E_0, p] \oplus [E_0^n, \mathrm{id}] \cong [E_0^{\oplus(n+1)}, b(x) + za(x)]$$

and show this with an explicit matrix computation.

Now we have our bundle of the form $[F, b(x) + za(z)]$ and we need to show that,

$$[F, b(x) + za(z)] \sim [F_+, \mathrm{id}] \oplus [F_-, z]$$

with $F = F_+ \oplus F_-$. In fact, we can assume $a(z) = \mathrm{id}$ so we have $[F, b(x) + z]$. Fix $x \in X$ then for any $z \in S^1$ and we know that $b(x) + z \in \mathrm{GL}_n(\mathbb{C})$ and thus $b(x)$ has no eigenvalue of norm $|\lambda| = 1$ else $b(x) + z$ for $z = \lambda$ would not be invertible.

Lemma 13.4. Given $b(x)$ acting on $F \to x$ with no eigenvalue $|\lambda| = 1$ then,

$$F = F_+ \oplus F_-$$

such that $b(F_\pm) \subset F_\pm$ and $b|_{F_+}$ has eigenvalues $|\lambda| > 1$ and $b|_{F_-}$ has eigenvalues $|\lambda| < 1$.

Finally,

$$F, b + z] \sim [F_+, b + z] \oplus [F_-, b + z]$$

Now we may homotope $[F_+, b + z] \sim [F_+, \mathrm{id}]$ using the homotopy $b + t \xrightarrow{b+tz} b \to \mathrm{id}$ where the first map is a homotopy since $b + tz$ is always invertible since b has all eigenvalues $|\lambda| > 1$ and the second map is just a change of coordinates on fibers over X which is fine since it is constant on S^1. Furthermore, we homotopy $[F_-, b + z] \sim [F_-, z]$ via the homotopy $b + z \xrightarrow{tb+z} z$ which gives a homotopy since each $bt + z$ is invertable since b has all eigenvalues $|\lambda| < 1$.

13.2 K-Theory in the Real Case

Definition 13.5. $KO(X) = K(\mathrm{Vect}_{\mathbb{R}}(X))$

Example 13.6. For spheres we can compute $\widetilde{KO}(S^n)$,

n	0	1	2	3	4	5	6	7
$\widetilde{KO}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

and these groups are 8-periodic.

14 K-Theory and Cohomology

Let X be a finite CW complex then we have cohomology theories $H^*(X)$ and $K(X)$ which are both rings.

We have the total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E) \in H^*(X; \mathbb{Z})$. Then given line bundles L_1, L_2 we have $c(L_i) = 1 + c_1(L_i)$ so,

$$c(L_1 \oplus L_2) = 1 + c_1(L_1) + c_1(L_2) + c_1(L_1) \smile c_1(L_2) \neq c(L_1) + c(L_2)$$

and

$$c(L_1 \otimes L_2) = 1 + c_1(L_1) + c_1(L_2) \neq c(L_1) \cdot c(L_2)$$

Therefore, the total Chen class is poorly behaved as a ring map.

Theorem 14.1 (Splitting)**.** Given a vectorbundle $E \to X$ then there exists $f : X' \to X$ s.t. $f^*E = L_1 \oplus \cdots \oplus L_n$ is a direct sum of line bundles with $n = \text{rank}E$. Furthermore, $f^*H^*(X;\mathbb{Z}) \to H^*(X';\mathbb{Z})$ and $f^* : K(X) \to K(X')$ are injective.

Remark. In particular,

$$f^*c(E) = c(f^*E) = c(L_1 \oplus \cdots \oplus L_n) = (1 + c_1(L_1)) \cdots (1 + c_1(L_n))$$

Remark. Thus we can suppose that $E = L_1 \oplus \cdots \oplus L_n$ and write $c_1(L_i) = x_i \in H^2(X;\mathbb{Z})$. Therefore, we have,

$$c(E) = (1 + x_1) \cdots (1 + x_n) = \sum_{i=0}^{n} e_i(x_1, \dots, x_n)$$

where $e_i(x_1, \dots, x_n)$ are the elementary symmetric polynomials in $x_1, \cdots, x_n$.

Definition 14.2. In the above case, the Chern character of E is,

$$\text{ch}(E) = \sum_{i=0}^{n} e^{x_i} \in H^*(X;\mathbb{Q})$$

This gives,

$$\text{ch}(E) = \text{rank}E + x_1 + \cdots x_n + \tfrac{1}{2!}(x_1^2 + \cdots + x_n^2) + \tfrac{1}{3!}(x_1^3 + \cdots x_n^3) + \cdots$$

Therefore, we may write this in terms of the elementary symmetric polynomials. We have Newton sums,

$$x_1^2 + \cdots + x_n^2 = (x_1 + \cdots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n)$$

In genereral, there is a formula for,

$$s_k(x_1, \dots, x_n) = x_1^k + \cdots + x_n^k$$

giving,

$$s_k(x_1, \dots, x_n) = \sum_{i=0}^{\ell_k} f_{ki} e_i(x_1, \dots, x_n)$$

these may be computed via Newton formulae. Then we see that,

$$\text{ch}(E) = \sum_{k=0}^{n} \frac{1}{n!} s_k(x_1, \cdots, x_n) = \sum_{k=0}^{n} \sum_{i=0}^{\ell_k} f_{ki} \frac{1}{n!} e_i(x_1, \dots, x_n)$$

but $c_i(E) = e_i(x_1, \dots, x_n)$ so we have,

$$\text{ch}(E) = \sum_{k=0}^{n} \sum_{i=0}^{\ell_k} f_{ki} \frac{1}{n!} c_i(E)$$

this is the definition we take for an arbitrary bundle.

Lemma 14.3. $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ and $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$ in $H^{\text{even}}(X;\mathbb{Q}) \subset H^*(X;\mathbb{Q})$. Therefore, we get a ring map,

$$\mathrm{ch} : K(X) \to H^{\text{even}}(X;\mathbb{Q})$$

via $\mathrm{ch}([E] - [F]) = \mathrm{ch}(E) - \mathrm{ch}(F)$.

Proof. Use the splitting principle and properties of $\sum e^{x_i}$. □

Theorem 14.4. Let X be a finite CW complex. Then, $\mathrm{ch}_{\mathbb{Q}} : K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} H^{\text{even}}(X;\mathbb{Q})$ is a ring isomorphism.

Remark. Thus $K(X)$ and $H^{\text{even}}(X;\mathbb{Z})$ agree up to torsion.

Example 14.5 (Atiyah-K-Theory)**.**

$$K(\mathbb{RP}^{2m-1}) = \mathbb{Z} \oplus \mathbb{Z}/2^{m-1}\mathbb{Z}$$

but

$$H^{\text{even}}(\mathbb{RP}^{2m-1};\mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus(m-1)}$$

so the torsion of these groups may not agree.

Theorem 14.6. There is a diagram,

$$\begin{tikzcd}
K(\Sigma X) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\mathrm{ch}"] \arrow[d, "\sim"] & H^{\text{even}}(\Sigma X;\mathbb{Q}) \arrow[d, equal] \\
K^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\mathrm{ch}"] & H^{\text{odd}}(X;\mathbb{Q})
\end{tikzcd}$$

Proof. We proceed by induction on $n = \dim X$. Suppose it holds for CW complexes of $\dim Y < n$ and let $\dim X = n$ then $X = X^n$ and look at the pair (X^n, X^{n-1}) which gives a morphism of long exact sequences,

$$\begin{tikzcd}
K^{p-1}(X^n, X^{n-1}) \arrow[r] \arrow[d, "\mathrm{ch}"] & K^p(X^{n-1}) \arrow[r] \arrow[d, "\mathrm{ch}"] & K^p(X^n) \arrow[r] \arrow[d, "\mathrm{ch}"] & K^p(X^n, X^{n-1}) \arrow[r] \arrow[d, "\mathrm{ch}"] & K^{p+1}(X^{n-1}) \arrow[d, "\mathrm{ch}"] \\
H^{p-1}(X^n, X^{n-1}) \arrow[r] & H^p(X^{n-1}) \arrow[r] & H^p(X^n) \arrow[r] & H^p(X^n, X^{n-1}) \arrow[r] & H^{p+1}(X^{n-1})
\end{tikzcd}$$

where $p =$ even or $p =$ odd and $H^p(X) = H^{\text{even}}(Y,\mathbb{Q})$ or $H^{\text{odd}}(X;\mathbb{Q})$ and the K-groups are given rational coefficnents. The outer maps are isomorphisms by indunction and the two inner maps $\mathrm{ch} : K^p(X^n, X^{n-1}) \to H^p(X^n, X^{n-1})$ are isomorphism by our computation for S^n since $X^n/X^{n-1} = \bigvee_i S^n$. Thus the central map is an isomorphism by the five lemma proving the proposition by induction. □

Example 14.7. $K(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^n)$ so $K(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1}$ as a $\mathbb{Z}$-module.

14.1 The Splitting Principle

Consider a vector bundle $b : E \to X$ with rank$E = n$. Consider the projective bundle $p : \mathbb{P}(E) \to X$. Then we get a pullback bundle,

$$\begin{tikzcd} p^*E \arrow[r] \arrow[d] & E \arrow[d, "b"] \\ \mathbb{P}(E) \arrow[r, "p"] & X \end{tikzcd}$$

Then p^*E has a natural line sub-bundle, namely the tautological bundle of $\mathbb{P}(E)$. A point in $\mathbb{P}(E)$ is a pair $x \in X$ and $[L] \in \mathbb{P}(E_x)$ thus the tautological bundle is,

$$\gamma = \{(x, [L], v) \mid L \subset E_x \text{ and } v \in L\}$$

Then $\gamma \subset p^*E$ since we get a map $(x, [L], v) \mapsto v \subset L \subset E_x$. This defines a line sub-bundle.

We then apply this proceedure to $L^\perp \subset p^*\mathbb{P}(E)$ to decompose E into line bundles.

15 The Hopf Invariant

Hatcher ch. 4

For a map $f : S^{2n-1} \to S^n$ we define:

Definition 15.1. $C_f = S^n \cup_f D^{2n}$ so there is no difference in Cellular cohomology. Then $H^n(C_f; \mathbb{Z}) = \mathbb{Z}$ and $H^{2n}(C_f; \mathbb{Z}) = \mathbb{Z}$. Pick generators α and β. Then,

$$\alpha^2 \in H^{2n}(C_f; \mathbb{Z})$$

which implies that $\alpha^2 = h(f)\beta$ and we call $h(f) \in \mathbb{Z}$ the Hopf invariant.

Proposition 15.2. The Hopf invariant gives a homomorphism $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$ with the following properties,

(a). if n is odd then $h = 0$ (since $\alpha \smile \alpha = 0$ in odd n).

(b). for the Hopf fibration $H : S^3 \to S^2$ then $C_f = S^2 \cup_H D^4 = \mathbb{CP}^2$ and $H^*(\mathbb{CP}^2; \mathbb{Z}) = \mathbb{Z}[x]/(x^3)$ so the generator of $H^2(\mathbb{CP}^2; \mathbb{Z})$ squares to the generator of $H^4(\mathbb{CP}^2; \mathbb{Z})$ which implies that $h(H) = 1$. In particular, $h : \pi_3(S^2) \xrightarrow{\sim} \mathbb{Z}$ sending $H \mapsto 1$.

(c). There is a generalized Hopf fibration $f : S^7 \to S^4 = \mathbb{HP}^1$ then $\mathbb{HP}^2 = S^4 \cup_f D^8$ but $H^*(\mathbb{HP}^2; \mathbb{Z}) = \mathbb{Z}[x]/(x^3)$ so $h(f) = 1$ giving $h : \pi_7(S^4) \xrightarrow{\sim} \mathbb{Z}$ sending $f \mapsto 1$.

(d). Furthermore, $\mathbb{OP}^2 = S^8 \cup_f D^{18}$ via a sphere fibration $f : S^{15} \to S^8$ then $h(f) = 1$ giving $h : \pi_{15}(S^8) \xrightarrow{\sim} \mathbb{Z}$ sending $f \mapsto 1$.

Remark. The map $h : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$ is never trivial. It is easy to construct maps $f : S^{4n-1} \to S^{2n}$ with $h(f) = 2$. However we have the following theorem,

Theorem 15.3 (Adams)**.** Suppose there exists $f : S^{4n-1} \to S^{2n}$ with $h(f) = 1$ then $n = 1, 2, 4$.

Remark. This is related to the following fact.

Theorem 15.4. Real division algebras has dimensions $n = 1, 2, 4, 8$.