Mathematics GU4053 Algebraic Topology
Assignment # 9

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Problem 1.

(a). Let X be a path-connected space and A a finite set of points of X. Consider the long exact sequence of relative homology generated by the pair (X, A),

$$\cdots \xrightarrow{\delta} \tilde{H}_1(A) \xrightarrow{\iota_*} \tilde{H}_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\delta} \tilde{H}_0(A) \xrightarrow{\iota_*} \tilde{H}_0(X) \xrightarrow{j_*} H_0(X, A) \longrightarrow 0$$

Because X is path-connected, we know that $\tilde{H}_0(X) = 0$ so the exactness at,

$$0 \longrightarrow H_0(X, A) \longrightarrow 0$$

implies that $H_0(X, A) = 0$. Furthermore, for $n > 1$ we know that $\tilde{H}_n(A) = 0$ since A is a collection of points. Therefore, the long exact sequence gives rise to the short exact sequences,

$$0 \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X, A) \longrightarrow 0$$

which implies that $H_n(X, A) \cong \tilde{H}_n(X)$. Finally, consider the case $n = 1$,

$$\begin{tikzcd} 0 \arrow[r, "\iota_*"] & \tilde{H}_1(X) \arrow[r, "j_*"] & H_1(X, A) \arrow[r, "\delta"] & \tilde{H}_0(A) \arrow[r, "\iota_*"] \arrow[l, bend left, "f"] & 0 \end{tikzcd}$$

We will construct a map $f : \tilde{H}_0(A) \to H_1(X, A)$ such that $\delta \circ f = \mathrm{id}_{\tilde{H}_0(A)}$. The relative homology groups is constructed as,

$$\tilde{H}_0(A) = \ker \epsilon / \mathrm{Im}(\partial_1)$$

However. A is a discrete set so any map $\sigma : \Delta^1 \to A$ is constant and therefore, $\partial_1 \sigma = 0$ so $\partial_1 = 0$. Furthermore,

$$\epsilon \left(\sum_{a \in A} n_a \, [a] \right) = \sum_{a \in A} n_a$$

so the kernel is the set generated by elements $[a_i] - [a_0]$. Thus, we can construct the map f by its action on these generators,

$$f([a_i] - [a_0]) = \sigma_i$$

where σ_i is some choice of path from a_0 to a_i which exists due to path-connectedness. This is a well-defined homomorphism $\tilde{H}_0(A) \to H_1(X, A)$ because σ_i has boundary in $C_0(A)$ so it is an element of the relative homology. Furthermore,

$$\delta \circ f([a_i] - [a_0]) = \delta(\sigma_i) = [a_i] - [a_0]$$

and therefore, extending f to a homomorphism, we see that $\delta \circ f = \mathrm{id}_{\tilde{H}_0(A)}$. Therefore, the sequence splits so,

$$H_1(X, A) \cong \tilde{H}_1(X) \oplus \tilde{H}_0(A) \cong \tilde{H}_1(X) \oplus \mathbb{Z}^{k-1}$$

where $|A| = k$ since $H_0(A) \cong \mathbb{Z}^k$, the number of path components, and relative homology reduces this factor by 1. In summary,

$$H_n(X, A) \cong \begin{cases} \tilde{H}_n(X) & n \neq 1 \\ \tilde{H}_1(X) \oplus \mathbb{Z}^{k-1} & n = 1 \end{cases}$$

Explicitly, for the case $X = S^2$ we know that,

$$\tilde{H}_n(X) \cong \begin{cases} \mathbb{Z} & n = 2 \\ 0 & n \neq 2 \end{cases}$$

so we can compute,

$$H_n(S^2, A) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}^{k-1} & n = 1 \\ 0 & n \neq 1, 2 \end{cases}$$

Likewise, for the case $X = T^2 = S^1 \times S^1$ we know that,

$$\tilde{H}_n(X) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \neq 1, 2 \end{cases}$$

so we can compute,

$$H_n(T^2, A) = \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}^{k+1} & n = 1 \\ 0 & n \neq 1, 2 \end{cases}$$

(b). Both (X, A) and (X, B) are good pairs. Therefore,

$$H_n(X, A) \cong \tilde{H}_n(X/A)$$

However, X/A is the wedge of two tori. Therefore,

$$H_n(X, A) \cong \tilde{H}_n(X/A) = \tilde{H}_n\left(T^2 \vee T^2\right) \cong \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \neq 1, 2 \end{cases}$$

Furthermore, X/B is homotopic to the wedge of a torus and a circle. Thus, again using the fact that $H_n(X,B) \cong \tilde{H}_n(X/B)$,

$$H_n(X,B) \cong \tilde{H}_n(X/B) = \tilde{H}_n\left(T^2 \vee S^1\right) \cong \tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \neq 1, 2 \end{cases}$$

Problem 2.

Consider the subspace $\mathbb{Q} \subset \mathbb{R}$. The pair $(\mathbb{R}, \mathbb{Q})$ gives rise to the long exact sequence,

$$\cdots \xrightarrow{\delta} H_1(\mathbb{Q}) \xrightarrow{\iota_*} H_1(\mathbb{R}) \xrightarrow{j_*} H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\delta} H_0(\mathbb{Q}) \xrightarrow{\iota_*} H_0(\mathbb{R}) \xrightarrow{j_*} H_0(\mathbb{R}, \mathbb{Q}) \longrightarrow 0$$

However, $H_1(\mathbb{R}) = 0$ and $H_0(\mathbb{R}) \cong \mathbb{Z}$ because $\mathbb{R}$ is contractible. Furthermore,

$$H_0(\mathbb{Q}) = \ker \partial_0 / \mathrm{Im}(\partial_1) = C_0(\mathbb{Q}) / \mathrm{Im}(\partial_1)$$

However, if $\sigma : \Delta^1 \to \mathbb{Q}$ is continuous then $\mathrm{Im}(\sigma)$ is connected and thus $\mathrm{Im}(\sigma) = \{x_0\}$ so σ is constant. Thus, $\partial_1 \sigma = 0$ so $\partial_1 = 0$. Therefore, $H_0(\mathbb{Q}) = C_0(\mathbb{Q}) = \mathbb{Z}^{\mathbb{Q}}$. Therefore, we have the exact sequence,

$$0 \longrightarrow H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\ \delta\ } \mathbb{Z}^{\mathbb{Q}} \xrightarrow{\ i_*\ } \mathbb{Z}$$

The map $i_\# : C_0(\mathbb{Q}) \to C_0(\mathbb{R})$ acts as the inclusion on generators. Therefore, $i_* : H_0(\mathbb{Q}) \to H_0(\mathbb{R})$ takes generators to generators. However, $H_0(\mathbb{R}) \cong \mathbb{Z}$ so there is a single generator. Therefore,

$$i_*\left(\sum_{q \in \mathbb{Q}} n_q\, [q]\right) = \sum_{q \in \mathbb{Q}} n$$

where $n_q = 0$ for all but finitely many values. Thus,

$$\ker i_* = \left\{ \sum_{q \in \mathbb{Q}} n_q\, [q] \;\middle|\; \sum_{q \in \mathbb{Q}} n_q = 0 \right\}$$

From the exact sequence, we see that $\mathrm{Im}(\delta) = \ker i_*$ and $\ker \delta = 0$ so $\mathrm{Im}(\delta) \cong H_1(\mathbb{R}, \mathbb{Q})$. Therefore,

$$H_1(\mathbb{R}, \mathbb{Q}) \cong \mathrm{Im}(\delta) = \ker i_* = \left\{ \sum_{q \in \mathbb{Q}} n_q\, [q] \;\middle|\; \sum_{q \in \mathbb{Q}} n_q = 0 \right\} \subset \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}$$

We can give an explicity basis,

$$\{([q] - [0]) \mid q \in \mathbb{Q} \setminus \{0\}\}$$

Because given an element,

$$\sum_{q \in \mathbb{Q}} n_q\, [q] \quad \text{such that} \qquad \sum_{q \in \mathbb{Q}} n_q = 0$$

then we can write,

$$\sum_{q \in \mathbb{Q}} n_q\, [q] = \sum_{q \in \mathbb{Q}} n_q\, ([q] - [0]) + \sum_{q \in \mathbb{Q}} n_q\, [0] = \sum_{q \in \mathbb{Q}} n_q\, ([q] - [0])$$

Clearly, any linear combination of these basis elements is in the kernel of i_*.

Problem 3.

We know that the suspension is a union of cones $SX = C_+X \cup C_-X$ whose intersection is X. Take $A = C_+X$ and $B = C_-X$. Since C_+X is contractible, by Lemma **??** we know that $\tilde{H}_n(SX) \cong \tilde{H}_n(SX, C_+X)$. However, by Excision, we know that $\tilde{H}_n(B, A \cap B) \cong \tilde{H}_n(X, A)$ and therefore,

$$\tilde{H}_n(C_-X, X) \cong \tilde{H}_n(SX, C_+) \cong \tilde{H}_n(SX)$$

Furthermore, consider the pair (C_-X, X). Since C_-X is contractible, by Lemma **??**, we know that,

$$\tilde{H}_{n+1}(C_-X, X) \cong \tilde{H}_n(X)$$

Putting these results together, we find that,

$$\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$$

Now consider the problem when Y is the union of k cones of X,

$$Y = \bigcup_{i=1}^{k} C_i X$$

which all intersect at the base to form $X \subset Y$. I claim that,

$$\tilde{H}_{n+1}(Y) \cong \bigoplus_{i=1}^{k-1} \tilde{H}_{n+1}(S_i X) \cong \bigoplus_{i=1}^{k-1} \tilde{H}_k(X)$$

By Excision,

$$\tilde{H}_{n+1}(Y, C_k X) \cong \tilde{H}_{n+1}\left(\bigcup_{i=1}^{k-1} C_i X, X\right)$$

However, the relative homology in the last line is of a good pair so,

$$\tilde{H}_{n+1}\left(\bigcup_{i=1}^{k-1} C_i X, X\right) \cong \tilde{H}_{n+1}\left(\left[\bigcup_{i=1}^{k-1} C_i X\right]/X\right) \cong \tilde{H}_{n+1}\left(\bigvee_{i=1}^{k-1} S_i X\right) = \bigoplus_{i=1}^{k-1} \tilde{H}_{n+1}(S_i X)$$

However, by Lemma **??**, since $C_k X$ is contractible, we know that $\tilde{H}_{n+1}(Y, C_k X) \cong \tilde{H}_{n+1}(Y)$. Furthermore, using our previous result that $\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X)$ we get that,

$$\tilde{H}_{n+1}(Y) \cong \tilde{H}_{n+1}(Y, C_k X) \cong \bigoplus_{i=1}^{k-1} \tilde{H}_{n+1}(S_i X) \cong \bigoplus_{i=1}^{k-1} \tilde{H}_{n+1}(X)$$

proving the claim.

Problem 4.

(a). Suppose we have a morphism of pairs $f : (X, A) \to (Y, B)$ such that $f : X \to Y$ and $f : A \to B$ are homotopy equivalences. The long exact sequence of pairs is natural. Therefore, given a map of pairs $f : (X, A) \to (Y, B)$ we get a morphism of long exact sequences $f_\#$ such that the following diagram commutes,

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(A) \arrow[r] \arrow[d, "f_*"] & H_{n+1}(X) \arrow[r] \arrow[d, "f_*"] & H_{n+1}(X, A) \arrow[r] \arrow[d, "f_*"] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X, A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(B) \arrow[r] & H_{n+1}(Y) \arrow[r] & H_{n+1}(Y, B) \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, A) \arrow[r] & \cdots
\end{tikzcd}$$

For the current situation, because $f : X \to Y$ and $f : A \to B$ are homotopy equivalences we know that $f_* : H_n(X) \to H_n(Y)$ and $f_* : H_n(A) \to H_n(B)$ are isomorphisms. Consider the section of the long exact sequence,

$$\begin{tikzcd}
H_{n+1}(A) \arrow[r] \arrow[d, "f_*", "\wr"'] & H_{n+1}(X) \arrow[r] \arrow[d, "f_*", "\wr"'] & H_{n+1}(X, A) \arrow[r] \arrow[d, "f_*"] & H_n(A) \arrow[r] \arrow[d, "f_*", "\wr"'] & H_n(X) \arrow[d, "f_*", "\wr"'] \\
H_{n+1}(B) \arrow[r] & H_{n+1}(Y) \arrow[r] & H_{n+1}(Y, B) \arrow[r] & H_n(B) \arrow[r] & H_n(Y)
\end{tikzcd}$$

Therefore, by the five-lemma, we know that $f_* : H_{n+1}(X, A) \to H_{n+1}(Y, B)$ is an isomorphism for each n. This argument also holds for $n = 0$ because the right half of the diagram is just zeros which still satisfies the isomorphism conditions.

(b). Suppose that $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n \setminus \{0\})$ is a homotopy equivalence of pairs. Then, by Lemma **??** we know that $f : (D^n, S^{n-1}) \to (D^n, D^n)$ is a homotopy equivalence of pairs. However, since D^n is contractible, by Lemma **??** we know that $\tilde{H}_k(D^n, S^{n-1}) \cong \tilde{H}_{k-1}(S^{n-1})$ and $\tilde{H}_k(D^n, D^n) \cong \tilde{H}_{k-1}(D^n)$. However, $\tilde{H}_{k-1}(D^n) = 0$ for all k since D^n is contractible but $\tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$ is nontrivial. Therefore, $f : (D^n, S^{n-1}) \to (D^n, D^n)$ cannot be a homotopy equivalence and thus $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n \setminus \{0\})$ cannot be a homotopy equivalence.

Problem 5.

We define the homotopy category of chain complexes, $\mathbf{K}(\mathbf{Ab})$ as the category with objects as chain complexes of abelian groups and morphisms which are chain homotopy classes of morphisms of chain complexes. To show that this is well-defined, we need to show that chain homotopy is an equivalence relation and that chain homotopy respects composition.

First, if $f : C \to D$ is a morphism of chain complexes then $p_n = 0 : C_n \to D_{n+1}$ is a chain homotopy from f to f since,

$$\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n = 0 = f_n - f_n$$

Therefore $f \simeq f$ so chain homotopy is reflexive. Furthermore, if $f, g : C \to D$ are chain homotopic morphisms of chain complexes such that $f \sim g$ and thus there exists a chain homotopy, $p_n : C_n \to D_{n+1}$ such that,

$$\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n = g_n - f_n$$

Then consider the map $(-p_n) : C_n \to D_n$ such that,

$$\partial_{n+1} \circ (-p_n) + (-p_{n-1}) \circ \partial_n = -(\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n) = f_n - g_n$$

so $g \simeq f$. Therefore, chain homotopy is symmetric. Finally, suppose that $f, g, h : C \to D$ are morphisms of chain complexes such that $f \simeq g$ and $g \simeq h$. Then, we have chain homotopies, $p_n : C_n \to D_{n+1}$ and $q_n : C_n \to D_{n+1}$ such that,

$$\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n = g_n - f_n$$

and likewise,

$$\partial_{n+1} \circ q_n + q_{n-1} \circ \partial_n = h_n - g_n$$

Then, consider the map $p_n + q_n : C_n \to D_{n+1}$. Using the above relations,

$$\begin{aligned}\partial_{n+1} \circ (p_n + q_n) + (p_{n-1} + q_{n-1}) \circ \partial_n &= \partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n + \partial_{n+1} \circ q_n + q_{n-1} \circ \partial_n \\ &= (g_n - f_n) + (h_n - g_n) = h_n - f_n\end{aligned}$$

Therefore, $f \simeq h$ since $p + q$ is a chain homotopy between them. Therefore, chain homotopy is an equivalence relation. We much further check that chain homotopy respects composition. Suppose that, $f, f' : C \to D$ are chain homotopy morphisms of chain complexes and $g, g' : D \to E$ are also chain homotopic morphisms of chain complexes. Then, there exist chain homotopies, $p_n : C_n \to D_{n+1}$ and $q_n : D_n \to E_{n+1}$ such that,

$$\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n = f'_n - f_n$$

and likewise,

$$\partial_{n+1} \circ q_n + q_{n-1} \circ \partial_n = g'_n - g_n$$

Using the fact that the maps f, f', g, g' are all chain maps, we can simplify,

$$\begin{aligned}g'_n \circ f'_n - g_n \circ f_n &= g'_n \circ f'_n - g'_n \circ f_n + g'_n \circ f_n - g_n \circ f_n = g'_n \circ (f'_n - f_n) + (g'_n - g_n) \circ f_n \\ &= g'_n \circ (\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n) + (\partial_{n+1} \circ q_n + q_{n-1} \circ \partial_n) \circ f_n \\ &= \partial_{n+1} \circ g'_{n+1} \circ p_n + \partial_{n+1} \circ q_n \circ f_n + g'_n \circ p_{n-1} \circ \partial_n + q_{n-1} \circ f_{n-1} \circ \partial_n \\ &= \partial_{n+1} \circ (g'_{n+1} \circ p_n + q_n \circ f_n) + (g'_n \circ p_{n-1} + q_{n-1} \circ f_{n-1}) \circ \partial_n\end{aligned}$$

Which shows that $r_n = g'_{n+1} \circ p_n + q_n \circ f_n : C_n \to E_{n+1}$ is a chain homotopy between $g_n \circ f_n$ and $g'_n \circ f'_n$. Therefore, $g_n \circ f_n \simeq g'_n \circ f'_n$ so chain homotopy respects composition. Therefore, the composition in the category $\mathbf{K}(\mathbf{Ab})$ is well defined since if $[f] = [f']$ and $[g] = [g']$ then, $[g] \circ [f] = [g \circ f]$ and $[g'] \circ [f'] = [g' \circ f']$ but since $f \simeq f'$ and $g \simeq g'$ we know that $g \circ f \simeq g' \circ f'$ and thus, $[g \circ f] = [g' \circ f']$. So finally,

$$[g] \circ [f] = [g'] \circ [f']$$

so composition does not depend on representative.

Problem 6.

Suppose C is a contractible complex i.e. such that the identity map is chain homotopic to the zero map through a chain homotopy, $p : C_n \to C_{n+1}$ such that $\partial_{n+1} \circ p_n + p_{n-1} \circ \partial_n = \mathrm{id}_n$. Then, take any cycle $a \in C_n$ such that $\partial_n a = 0$. Using the above result,

$$\partial_{n+1} \circ p_n(a) + p_{n-1} \circ \partial_n(a) = a \implies \partial_{n+1}(p_n(a)) = a$$

so $a \in \operatorname{Im}(\partial_{n+1})$ is a boundary. Therefore, the complex is exact and therefore has trivial homology which, by definition, means that the complex is acyclic.

However, consider the sequence,

$$0 \longrightarrow 2\mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

which is exact with the inclusion and quotient maps. Since this sequence is exact, it is a complex with trivial homology and thus acyclic. However, this complex is not contractible. To see this, suppose there were a chain homotopy p between the identity and the zero map,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 2\mathbb{Z} \arrow[r, "\iota"] \arrow[d] \arrow[dl] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d] \arrow[dl, "p_1"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] \arrow[dl, "p_2"] & 0 \arrow[d] \arrow[dl] \\
0 \arrow[r] & 2\mathbb{Z} \arrow[r, "\iota"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

For this sequence of maps to give a chain homotopy, we need to have,

$$\iota \circ p_1 + p_2 \circ \pi = \mathrm{id}_{\mathbb{Z}}$$

However, the map $p_2 : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must be trivial because $\operatorname{Im}(p_2)$ is a torsion group but $\mathbb{Z}$ has trivial torsion. Therefore, $p_2 = 0$ so we must have,

$$\iota \circ p_1 = \mathrm{id}_{\mathbb{Z}}$$

which is clearly impossible because $\operatorname{Im}(\iota) = 2\mathbb{Z} \subsetneq \mathbb{Z}$.

1 Lemmas

Lemma 1.1. Let (X, A) be a pair such that A is contractible then $\tilde{H}_n(X, A) \cong \tilde{H}_n(X)$.

Proof. Consider the long exact sequence,

$$\cdots \xrightarrow{\delta} \tilde{H}_{n+1}(A) \xrightarrow{\iota_*} \tilde{H}_{n+1}(X) \xrightarrow{j_*} \tilde{H}_{n+1}(X, A) \xrightarrow{\delta} H_n(A) \xrightarrow{\iota_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X, A) \longrightarrow \cdots$$

However, since A is contractible, we know that it has isomorphic homology to a point and thus $\tilde{H}_n(A) = 0$. Therefore, the long exact sequence gives short exact sequences,

$$0 \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X, A) \longrightarrow 0$$

and therefore $\tilde{H}_n(X) \cong \tilde{H}_n(X, A)$ for each n. □

Lemma 1.2. Let (X, A) be a pair such that X is contractible then $\tilde{H}_{n+1}(X, A) \cong \tilde{H}_n(A)$.

Proof. Consider the long exact sequence,

$$\cdots \xrightarrow{\delta} \tilde{H}_{n+1}(A) \xrightarrow{\iota_*} \tilde{H}_{n+1}(X) \xrightarrow{j_*} \tilde{H}_{n+1}(X, A) \xrightarrow{\delta} \tilde{H}_n(A) \xrightarrow{\iota_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X, A) \longrightarrow \cdots$$

However, since X is contractible, we know that it has isomorphic homology to a point and thus $\tilde{H}_n(X) = 0$. Therefore, the long exact sequence gives short exact sequences,

$$0 \longrightarrow \tilde{H}_{n+1}(X, A) \longrightarrow \tilde{H}_n(A) \longrightarrow 0$$

and therefore $\tilde{H}_{n+1}(X, A) \cong \tilde{H}_n(A)$ for each n. $\square$

Lemma 1.3. If $f : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs then $f : (X, \overline{A}) \to (Y, \overline{B})$ is a homotopy equivalence of pairs.

Proof. Let $H : X \times I \to Y$ be a homotopy such that $H(A \times \{t\}) \subset B$. Then, because H is continuous, $H(\overline{A \times \{t\}}) \subset \overline{H(A \times \{t\})} \subset \overline{B}$. Therefore, H is a homotopy of pairs $(X, \overline{A})$ to $(Y, \overline{B})$. $\square$