1 Kodaira Vanishing Revisited

1.1 Classical Kodaira Vanishing

Theorem 1.1.1. Let $X/\mathbb{C}$ be smooth projective $d = \dim X$ and $\mathcal{L} \in \mathrm{Pic}\,(X)$ ample. Then,

$$H^{<d}(X, \mathcal{L}^{\otimes -1}) = 0$$

or equivalently,

$$H^{>0}(X, \Omega_X \otimes \mathcal{L}) = 0$$

Remark. There is a version for $\mathcal{L}$ big and nef (Kawamata-Viehweg). Useful because big + nef is stable under pullback along proper birational maps unlike ampleness (e.g. consider blowups).

1.2 Applications

Say $H = V(s) \subset X$ with $\mathcal{L} = \mathcal{O}_X(H)$ is a hyperplane section. Then,

$$H^0(X, \omega_X(2H)) \to H^0(H, \omega_H(H))$$

is surjective.

Proof. There is an exact sequence,

$$0 \longrightarrow \omega_X \longrightarrow \omega_X(H) \longrightarrow \omega_H \longrightarrow 0$$

and then tensor by $\mathcal{O}_X(H)$ and apply Kodaira vanishing to get $H^1(X, \omega_X(H)) = 0$. $\square$

1.3 Proof of Kodaira Vanishing

Say $\mathcal{L} = \mathcal{O}_X(H)$ for $H \subset X$ meaning $\mathcal{L}$ is effective. We want to show that $H^{<d}(X, \mathcal{L}^{\otimes -1}) = 0$. Hodge theory,

$$H^{<d}(X, \mathcal{L}^{\otimes -1}) \subset_{\text{summand}} H_c^{<d}(X \setminus H, \mathbb{C}) \cong H^{>d}(X \setminus H, \mathbb{C})^\vee = 0$$

using Poincare duality and Artin vanishing because $X \setminus H$ is affine.

1.4 What Happens over $\mathbb{F}_p$ or $\mathbb{Z}$

(a) False in general over $\mathbb{F}_p$

  (a) Mumford (singular surface over $\mathbb{F}_p$)

  (b) Raynaud (over $\mathbb{F}_p$)

  (c) Totaro (over $\mathbb{Z}$)

(b) Deligne-Illusie: it is true in dim $< p$ in the liftable case.

Salvage: work up to finite covers. (In characteristic zero, vanishing after a finite cover implies vanishing but not in positive characteristic).

Example 1.4.1. $X = S^1$ and $H^1(S^1, \mathbb{F}_p) = \mathbb{F}_p$ but $[p] : S^1 \to S^1$ annhilates this class,

$$[p]^* : H^1(S^1, \mathbb{F}_p) \to H^1(S^1, \mathbb{F}_p)$$

is zero.

2 Kodaira Vanishing in Mixed Characteristic

Theorem 2.0.1. Say $X/\mathbb{Z}_p$ is proper flat variety of relative dimension d and $\mathcal{L} \in \operatorname{Pic}(X)$ is semiample and big. Then there exist a finite surjective map $\pi : Y \to X$ such that,

$$\pi^* : H^\bullet(X, \mathcal{L}^{\otimes a})_{\text{tors}} \to H^\bullet(Y, \pi^*\mathcal{L}^{\otimes a})_{\text{tors}}$$

is 0 for $a \in \{-1, 0, 1\}$.

Remark. Analog over $\mathbb{F}_p$ is also true (Hochster-Huneke Smith 90s) for $H^{<d}(X, \mathcal{L}^{\otimes -1})$ and $H^{>0}(X, \mathcal{L}^{\otimes a})$ for $a \in \{0, 1\}$.

Example 2.0.2. $X = E$ elliptic curve over $\mathbb{F}_p$ and $H^1(X, \mathcal{O}_X) = \mathbb{F}_p$ then $[p] : E \to E$ does the job.

Remark. In positive characteristic, the $\mathcal{L}$ in positive degrees case is easy because we can take, $H^i(X, (\text{Frob}_p^n)^*\mathcal{L}) = H^i(X, \mathcal{L}^{\otimes p^n}) = 0$ by Serre vanishing.

Remark. Ramification is necessary (ex consider K3 surface).

Remark. There is a relative varient over complete noetherian local domains $(R, \mathfrak{m})$ with $p \in \mathfrak{m}$.

Remark. Try to prove the following: say $X/\mathbb{Z}_p$ is a relative curve. Then there exists $\pi : Y \to X$ finite cover such that $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_Y)$ is divisible by p.

2.1 Application

Theorem 2.1.1 (BMPSTWW, TY)**.** One can run the MMP for arithmetic 3-folds (relative dimension 2) over $\mathbb{Z}[30^{-2}]$.

2.2 Local Kodaira Vanishing

Let $X \subset \mathbb{P}^n$ be a projective variety then the affine cone $C(X)$ preserves nice properties of X.

Principle: projective geometry of $X \subset \mathbb{P}^n$ is equivalent to local geometry of Cone at 0.

Theorem 2.2.1 (Local Kodaira)**.** Let $(R, \mathfrak{m})$ be an excellent noetherian local domain. Let R^+ be the absolute integral closure (integral closure of R in $\overline{\operatorname{Frac}(R)}$). Then R^+/p is a Cohen-Macalay module over R/p.

Remark. Say we have $\mathbb{Z}_p[[x_1, \ldots, x_n]] \hookrightarrow R$ is finite. Then there exists a fintie extension $R \hookrightarrow S$ with the following feature. Any relation $\sum a_i x_i = 0$ in R/p is trivial in S/p.

Remark. This thm implies "homological conjectures" in commutative algebra (e.g. direct summand conjecture). Easy to deduce the direct summand conjecture from CM theorem.

2.3 What Goes Into the Proof?

p-adic Riemann-Hilbert correspondences (joint with Lurie)

Theorem 2.3.1. Say $C/\mathbb{Q}_p$ is complete and algebraically closed and $X/\mathcal{O}_C$ is proper flat scheme. There exists a natural exact functor,

$$RH : D(X_C, \mathbb{F}_p) \to D_{qc}(X_{p=0})$$

such that perverse $\mathbb{F}_p$-sheaves are taken to almost CM complexes.

Why is this useful. Say $X/\mathcal{O}_C$ as before and consdier the absolute integral closure $\pi : X^+ \to X$. Fact,

$$RH(\pi_* \mathbb{F}_p|_{X_C}) = \pi_* \mathcal{O}_{X^+}/p$$

Using this, can "almost" prove theorem via:

Lemma 2.3.2. $Y/\mathbb{C}$ any variety of $\dim Y = d$ and $\pi : Y^+ \to Y$ absolute integral closure. Then $\pi_* \underline{\mathbb{F}_p}[d]$ is perverse.

To go to honest statement (not almost) use prismatic cohomology.

2.4 Question

Say $(R, \mathfrak{m})$ is a p-adically complete excellent domain.

Definition 2.4.1. The *test ideal* is,

$$\tau(\omega_R) = \bigcap_{R \hookrightarrow S \hookrightarrow R^+} \operatorname{im}(\operatorname{tr} : \omega_S \to \omega_R) \subset \omega_R$$

with $R \hookrightarrow S$ finite.

Question: Does $R \mapsto \tau(\omega_R)$ commute with localization?

Evidence:

(a) True if $p = 0$ in R (Smith)

(b) True "up to p-perturbation".

Corollary 2.4.2. $\tau(\omega_R)[p^{-1}]$ is the Grauert-Riemenschneider sheaf in $\omega_{R[p^{-1}]}$.

Remark. $\operatorname{Spec}(\mathcal{O}_C)$ has two points with residue fields C and $\overline{\mathbb{F}_p}$ respectively.