1 Locally Free Sheaves

2 Algebraic Vector Bundles

3 Derivations

Definition 3.0.1. Let $\mathscr{A}$ be a sheaf of algebras and $\mathscr{B}$ an $\mathscr{A}$-algebra and $\mathscr{F}$ a $\mathscr{B}$-module. Then an $\mathscr{A}$-derivation $D : \mathscr{B} \to \mathscr{F}$ is a $\mathscr{A}$-module map such that on all local sections,

$$D(fg) = D(f)g + fD(g)$$

Furthermore, we write $\mathscr{D}er_{\mathscr{A}}(\mathscr{B}, \mathscr{F}) \subset \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$ for the $\mathscr{A}$-submodule of derivations.

Definition 3.0.2. If the functor $\mathscr{F} \mapsto \mathscr{D}er_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$ is representable on the category on $\mathscr{B}$-modules then we say the representing pair $(\Omega_{\mathscr{B}/\mathscr{A}}, \mathrm{d})$ is the $\mathscr{B}$-module of $\mathscr{A}$-differentials where,

$$\mathscr{H}om_{\mathscr{A}}\big(\Omega_{\mathscr{B}/\mathscr{A}}, \mathscr{F}\big) = \mathscr{D}er_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$$

and the derivation $\mathrm{d} : \mathscr{B} \to \Omega_{\mathscr{B}/\mathscr{A}}$ is the universal element given by,

$$\mathrm{id} \in \mathscr{H}om_{\mathscr{A}}\big(\Omega_{\mathscr{B}/\mathscr{A}}, \Omega_{\mathscr{B}/\mathscr{A}}\big) = \mathscr{D}er_{\mathscr{A}}\big(\mathscr{B}, \Omega_{\mathscr{B}/\mathscr{A}}\big)$$

Definition 3.0.3. Given morphism of locally ringed spaces $f : X \to S$ we say that $(\Omega_{X/S}, \mathrm{d})$ is the $\mathcal{O}_X$-module of $f^{-1}\mathcal{O}_S$-differentials viewing $\mathcal{O}_X$ as a $f^{-1}\mathcal{O}_S$-algebra via the map $f^{-1}\mathcal{O}_S \to \mathcal{O}_X$.

4 Connections

Remark. Here we have a locally ringed space $X \to S$ over S. We write $\Omega_X = \Omega_{X/S}$ and

Definition 4.0.1. A connection on a vector bundle $\mathcal{E}$ on X in a $\mathcal{O}_S$-linear derivation,

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

Lemma 4.0.2. Suppose that $\nabla_1, \nabla_2 : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ are connections. Then,

$$\nabla_1 - \nabla_2 : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

is a $\mathcal{O}_X$-module map.

Proof. $(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 s - \nabla_2 s) + \mathrm{d}f \otimes s - \mathrm{d}f \otimes s = f(\nabla_1 - \nabla_2)s$. □

Remark. Therefore, the space of connections is a affine subspace of $\mathrm{Hom}\,(\mathcal{E}, \Omega^1_X\mathcal{E})$. Then if $\mathcal{E}$ is finite locally free,

$$\mathrm{Hom}\,\big(\mathcal{E}, \Omega^1_X\mathcal{E}\big) = H^0(X, \Omega^1_X \otimes_{\mathcal{O}_X} \mathscr{E}nd_{\mathcal{O}_S}(\mathcal{E}))$$

Definition 4.0.3. The first Chern class $c_1 : \mathrm{Pic}\,(X) \to H^1(X, \Omega^1) \subset H^2_{\mathrm{dR}}(X)$ is defined by $H^1(X, -)$ applied to the map $\mathrm{dlog} : \mathcal{O}_X^\times \to \Omega^1_X$ defined as $\mathrm{dlog}(f) = f^{-1}\mathrm{d}f$.

Proposition 4.0.4. A line bundle $\mathcal{L}$ admits a connection $\nabla : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$ if and only if $c_1(\mathcal{L}) = 0$.

Proof. A line bundle $\mathcal{L}$ is represented by a Cech cocycle $(U_i, f_{ij}) \in H^1(X, \mathcal{O}_X^\times)$. Then a connection on a line bundle is represented by (U_i, ω_i) with $\omega_i \in \Omega_X^1(U_i)$ where (U_i, s_i) is a trivialization of $\mathcal{L}$ with $\mathcal{O}_{U_i} \xrightarrow{s_i} \mathcal{L}|_{U_i}$ then $s_i|_{U_i \cap U_j} = f_{ij} s_j|_{U_i \cap U_j}$ and $\nabla s_i = \omega_i \otimes s_i$. However, we must have on $U_i \cap U_j$,

$$\nabla s_i = \nabla f_{ij} s_j = f_{ij} \nabla s_j + \mathrm{d} f_{ij} \otimes s_j$$

Therefore,

$$\omega_i \otimes f_{ij} s_j = f_{ij} \omega_j \otimes s_j + \mathrm{d} f_{ij} \otimes s_j$$

and thus,

$$(\omega_i - \omega_j)|_{U_i \cap U_j} = \mathrm{dlog}(f_{ij})$$

Consider the Cech differential $\mathrm{d} : \check{C}^0(\mathfrak{U}, \Omega_X^1) \to \check{C}^1(\mathfrak{U}, \Omega_X^1)$ which takes the sections (ω_i) to the coboundary $(\omega_i - \omega_j)|_{U_{ij}}$. Therefore, such a connection i.e. such a class exists iff the class,

$$c_1(\mathcal{L}) = [\mathrm{dlog}(f_{ij})] \in \check{H}^1(X, \Omega_X^1)$$

is trivial since it is a coboundary. □

5 Curvature

Definition 5.0.1. The connection ∇ defines a corresponding curvature map,

$$\omega_\nabla = \nabla_1 \circ \nabla : \mathcal{E} \to \Omega_X^2 \otimes_{\mathcal{O}_S} \mathcal{E}$$

We say that ∇ is flat or integrable if the curvature vanishes $\omega_\nabla = \nabla_1 \circ \nabla = 0$.

Lemma 5.0.2. The curvature $\omega_\nabla : \mathcal{E} \to \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$ is a $\mathcal{O}_X$-module map.

Proof. Consider,

$$\begin{aligned} \omega_\nabla(fs) &= \nabla_1(\mathrm{d}f \otimes s + f \nabla s) = \mathrm{dd}f \otimes s - \mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + f \nabla_1 \circ \nabla \\ &= f \nabla_1 \circ \nabla s = f\, \omega_\nabla(s) \end{aligned}$$

□

Remark. Therefore ω_∇ defines the curvature form $\omega_\nabla \in \Gamma(X, \Omega_X^2 \otimes \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{E}))$.

Remark. If we write locally,

$$\nabla e = \sum_i f_i \mathrm{d} g_i \otimes s_i$$

then the curvature takes the form,

$$\omega_\nabla(e) = \sum_i (\mathrm{d} f_i \wedge \mathrm{d} g_i \otimes e - f_i \mathrm{d} g_i \otimes \nabla s_i)$$

6 Differential Operators

Definition 6.0.1. Let $\mathscr{A}$ be a sheaf of algebras and $\mathscr{B}$ an $\mathscr{A}$-algebra and $\mathscr{F}, \mathscr{G}$ be $\mathscr{B}$-modules. Then a differential operator $D : \mathscr{F} \to \mathscr{G}$ of order k is a $\mathscr{A}$-module map such that for all local sections $b \in \Gamma(U, \mathscr{B})$ the map, $D(b \cdot -) - b \cdot D : \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G})$ is a differential operator of order $k-1$. Where a differential operator of order $k = 0$ is a $\mathscr{B}$-linear map $D : \mathscr{F} \to \mathscr{G}$. Furthermore, we write $\mathscr{D}iff^k_{\mathscr{B}/\mathscr{A}}(\mathscr{F}, \mathscr{G}) \subset \mathscr{H}om_{\mathscr{B}}(\mathscr{F}, \mathscr{G})$ to denote the $\mathscr{B}$-submodule of differential operators of order k.

7 Sheaves of Jets

8 The Atiyah Class

9 Riemann-Hilbert Correspondence

10 Connections on Real and Complex Manifolds

Remark. Let $\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ be a connection. For a vector field X we write $\nabla_X : \mathcal{E} \to \mathcal{E}$ for the map,

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{E} \xrightarrow{X \otimes \mathrm{id}} \mathcal{O}_X \otimes \mathcal{E} \to \mathcal{E}$$

Therefore, in previous notation $\nabla_X = Q(X)$. Thus we see that, viewing $\omega_\nabla \in \Omega^2_X \otimes \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{E})$ that,

$$\omega_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

When ∇ is the Levi-Civita connection then ω_∇ is the Riemann tensor.

Definition 10.0.1. A form $\sigma \in \Gamma(X, \Omega_X \otimes \mathcal{E})$ is called a *solder form* if $\sigma : \mathscr{T}_X \to \mathcal{E}$ is an isomorphism. Given a connection $\nabla : \mathcal{E} \to \Omega_X \otimes \mathcal{E}$, the *torsion* is $T_{(\nabla,\sigma)} = \nabla_1 \sigma \in \Gamma(X, \Omega^2_X \otimes \mathcal{E})$.

Remark. Choose a local frame $\{e_i\}$ of $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$ and $\{\sigma_i\}$ of Ω_X compatibly via σ. Then,

$$\sigma = \sum_i \sigma_i \otimes e_i$$

and write,

$$\nabla e_j = \sum_i \omega_{ij} \otimes e_i$$

for 1-forms $\omega_{ij} \in \Omega^1_X(U)$. Then we compute,

$$\begin{aligned} \nabla_1 \sigma &= \sum_i \mathrm{d}\sigma_i \otimes e_i - \sum_j \sigma_j \wedge \nabla e_j \\ &= \sum_i \left(\mathrm{d}\sigma_i + \sum_j \omega_{ij} \wedge \sigma_j \right) \otimes e_i \end{aligned}$$

Therefore,

$$T_{(\nabla,\sigma)} = 0 \iff \tau_i = \mathrm{d}\sigma_i + \sum_j \omega_{ij} \wedge \sigma_j = 0$$

Remark. For $\mathcal{E} = \mathcal{T}_X$ we have a canonical solder form σ_{id} given by $\mathrm{id} : \mathcal{T}_X \to \mathcal{T}_X$. Then $T_\nabla = T_{(\nabla,\sigma_{\mathrm{id}})}$ is the torsion of ∇. In local coodinates,

$$\sigma_{\mathrm{id}} = \sum_j \mathrm{d}x^j \otimes \frac{\partial}{\partial x^j} \quad \text{and} \quad \nabla \frac{\partial}{\partial x^j} = \sum_i \omega_{ij} \otimes \frac{\partial}{\partial x^i}$$

Then,

$$\nabla_1(\sigma_{\mathrm{id}}) = -\sum_{i,j} (\mathrm{d}x^j \wedge \omega_{ij}) \otimes \frac{\partial}{\partial x^j}$$

Therefore, if $X = v^i \frac{\partial}{\partial x^i}$ and $Y = u^i \frac{\partial}{\partial x^i}$ we find that,

$$T_\nabla(X, Y) = \sum_{i,j} \left(u^j v^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) - v^j u^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) \right) \otimes \frac{\partial}{\partial x^j}$$

However,

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= \sum_{i,j} \left(u^j v^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) - v^j u^k \omega_{ij} \left(\frac{\partial}{\partial x^k} \right) \right) \otimes \frac{\partial}{\partial x^i} \\ &\quad + \left(v^k \mathrm{d}u^j \left(\frac{\partial}{\partial x^k} \right) \otimes \frac{\partial}{\partial x^j} - u^k \mathrm{d}v^j \left(\frac{\partial}{\partial x^k} \right) \otimes \frac{\partial}{\partial x^j} \right) \\ &= T_\nabla(X, Y) + [X, Y] \end{aligned}$$

Therefore, we write down the following.

Definition 10.0.2. Let $\nabla : \mathcal{T}_X \to \Omega_X \otimes \mathcal{T}_X$ be a connection on the tangent bundle. The torsion $T_\nabla \in \Gamma(X, \Omega_X^2 \otimes \mathcal{T}_X)$ is defined via,

$$T_X(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

10.1 Metric Compatibility

Remark. A complex vector bundle $E \to M$ is equivalent to a pair (E, I) where $E \to M$ is a real vector bundle and $I : E \to E$ is a bundle endomorphism such that $I^2 = -\mathrm{id}$. Therefore, an almost complex structure is the same as endowing the tangent bundle with a complex structure.

Remark. A holomorphic structue on a complex vector bundle $E \to X$ over a complex manifold is the structue of a complex manifold on E such that $E \to X$ is holomorphic and such that there exist biholomorphic linear charts for $E \to X$ as a bundle.

Definition 10.1.1. Let $E \to M$ be a real vector bundle. A metric on E is a positive-definite symetric section $g \in \Gamma(M, \mathrm{Sym}^2(E^*))$.

Definition 10.1.2. A connection $\nabla : E \to \mathcal{A}_X^1 \otimes E$ is compatible with the metric g if $\nabla g = 0$.

Remark. Explicitly,

$$(\nabla g)(s_1, s_2) = \mathrm{d}(g(s_1, s_2)) - g(\nabla s_1, s_2) - g(s_1, \nabla s_2)$$

and thus $\nabla g = 0$ iff $\mathrm{d}(g(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2)$.

Definition 10.1.3. Let $(E, I) \to M$ be a complex vector bundle. A hermitian structure on E is a section $h \in \Gamma(M, E^* \otimes \overline{E}^*)$ such that h_x is a hermitian metric on E_x.

Proposition 10.1.4. A hermitian structure on (E, I) is equivalent to a metric compatible with I.

Proof. The equivalence is given by $h = g - i\omega$ where $\omega(-, -) = g(I(-), -)$ is the fundamental form. (CHECK THIS) □

Definition 10.1.5. Let $E \to M$ be complex. We say a connection $\nabla : E \to \mathcal{A}_M^1 \otimes E$ is complex if ∇ is complex linear. If E has a hermitian structure we say that ∇ is hermitian if $\nabla h = 0$.

Remark. Note that ∇ being complex linear is equivalent to $\nabla \circ I = I \circ \nabla$ is equivalent to $\nabla I = 0$ via the induced connection on $E^* \otimes E$. Explicitly,

$$(\nabla I)(s) = \nabla I(s) - I(\nabla(s)) = 0$$

Remark. Note that we need ∇ to be complex for $\nabla h = 0$ to make sense since we need ∇ to induce a connection on $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathcal{O}_X)$. To see why, consider a section $\varphi \in \Gamma(X, E^*)$ then, $\nabla\varphi$ should be complex linear. However,

$$\begin{aligned}(\nabla\varphi)(I(s)) &= \mathrm{d}\varphi(I(s)) - \varphi(\nabla I(s)) = i\mathrm{d}\varphi(s) - i\varphi(\nabla s) + \varphi([I \circ \nabla - \nabla \circ I](s)) \\ &= i(\nabla\varphi)(s) + \varphi([I \circ \nabla - \nabla \circ I](s))\end{aligned}$$

and therefore we need $\nabla \circ I = I \circ \nabla$.

Proposition 10.1.6. Let (E, I, h) be a complex bundle with a hermitian structure and g the associated compatible metric with funamental form ω. A complex connection $\nabla : E \to \mathcal{A}^1_X \otimes E$ is hermitian iff

$$\nabla h = 0 \iff \nabla g = 0 \iff \nabla\omega = 0$$

Proof. Because $\nabla I = 0$ we see that $(\nabla\omega)(-,-) = (\nabla g)(I(-),-)$ and thus $\nabla g = 0 \iff \nabla\omega = 0$. Furthermore, $h = g - i\omega$ so if $\nabla h = 0$ then the real and imaginary parts must indiviually vanish so $\nabla g = \nabla\omega = 0$. Explicitly,

$$\begin{gathered}\mathrm{d}(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2) \\ \iff \\ \mathrm{d}(g(s_1, s_2)) - i\mathrm{d}(\omega(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2) - i\omega(\nabla s_1, s_2) - i\omega(s_1, \nabla s_2)\end{gathered}$$

and therefore,

$$\mathrm{d}(g(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2) \quad \text{and} \quad \mathrm{d}(\omega(s_1, s_2)) = \omega(\nabla s_1, s_2) + \omega(s_1, \nabla s_2)$$

□

10.2 The Levi-Civita and Chern Connections

Proposition 10.2.1. Let (E, g, σ) be a real vector bundle on M with a metric and solder form $\sigma : T_M \to E$. Then there exists a unique torsion-free connection ∇ compatible with the metric called the Levi-Civita connection.

Proof. DO THIS!! □

Definition 10.2.2. Let $E \to X$ be a holomorphic vector bundle. We say a complex connection $\nabla : E \to \mathcal{A}^1_X \otimes E$ is compatible if $\nabla^{0,1} = \bar{\partial}_E$ where $\nabla^{0,1} = (\Pi^{0,1} \otimes \mathrm{id}_E) \circ \nabla$.

Proposition 10.2.3. Let (E, h) be a holomorphic vector bundle with a hermitian structure. Then there exists a unique compatible hermitian connection ∇ called the Chern connection.

Proof. (DO THIS) □

Remark. Now we consider the tangent bundle of a hermitian manifold (X, g) that is a Riemannian manifold (M, g) with a compatible almost complex structure $X = (M, I)$. There may be obstructions to the Levi-Civita connection being complex

Proposition 10.2.4.

Proposition 10.2.5. Let (X, g) be a hermitian manifold. Let ∇_{LC} be the Levi-Civita connection on TM of the underlying Riemannian manifold (M, g). Then,

$$\nabla_{\text{LC}}(I) =$$

Proposition 10.2.6. Let (X, g) be a hermitian complex manifold. Let ∇ be a torsion-free complex hermitian connection. Then the following hold,

(a) ∇ is the Levi-Civita connection for the underlying Riemannian structure

(b) ∇ is the Chern connection of $(T^{1,0}X, g_{\mathbb{C}})$

(c) (X, g) is Kähler.

10.3 Ricci Curvature

11 Conventions

Symmetric and exterior algebras are *quotients* not subspaces. The subspaces of symmetric and alternating tensors are a distinct notion. In characteristic zero $V^{\otimes n} \twoheadrightarrow \bigwedge^n V$ is split and the image is the alternating tensors and similarly for symmetric tensors and $V^{\otimes n} \twoheadrightarrow \text{Sym}^n(V)$.

To identify $\bigwedge^k V^* \cong (\bigwedge^k V)^*$ we need to choose a perfect pairing $\bigwedge^k V \times \bigwedge^k V^* \to k$. We do this in the only natural way that works in all characteristics,

$$(v_1 \wedge \cdots \wedge v_k, \varphi^1 \wedge \cdots \varphi^k) \mapsto \det \varphi^i(v^j)$$

Note that $(\varphi \wedge \psi)(v, u) = (\varphi \wedge \psi)(v \wedge u) = \varphi(v)\psi(u) - \varphi(u)\psi(v)$. There are NO factors of $\frac{1}{2}$ anywhere to be seen! The natural map $\bigwedge^k V^* \xrightarrow{\sim} (\bigwedge^k V)^* \hookrightarrow (V^{\otimes k})^* \xrightarrow{\sim} (V^*)^{\otimes k} \twoheadrightarrow \bigwedge^k V^*$ is thus multipliciation by $k!$.

Some obnoxious assholes define the pairing with a factor of $\frac{1}{k!}$ to agree with alternating tensors but then they also define the wedge product with a strange coefficient to make everything work out. Explicitly,

$$\text{Alt}(\varphi \otimes \psi)(v, u) = \tfrac{1}{2} (\varphi \otimes \psi - \psi \otimes \varphi)(v, u) = \tfrac{1}{2} (\varphi(v)\psi(u) - \psi(v)\varphi(u))$$

and likewise this means that,

$$\langle \text{Alt}(\varphi \otimes \psi), \text{Alt}(v \otimes u) \rangle = \tfrac{1}{4} (\varphi(v)\psi(u) - \psi(v)\varphi(u) - \varphi(u)\psi(v) + \psi(u)\varphi(v)) = \tfrac{1}{2} (\varphi(v)\psi(u) - \varphi(u)\psi(v))$$

But then they define $v \wedge u = 2\text{Alt}(v \otimes u) = v \otimes u - u \otimes v$ to "fix" everything so that,

$$(\varphi \wedge \psi)(v, u) = \varphi(v)\psi(u) - \psi(v)\varphi(u)$$

so in fact $v \wedge u$ has the same image in $V^{\otimes 2}$ as previously.