1 Basic Definitions and Examples

1.1 Genera

1.2 Riemann-Roch

1.3 Riemann–Hurwitz

2 Hyperelliptic Curves

Definition 2.0.1. A curve C is *hyperelliptic* if there exists a degree two map $f : C \to \mathbb{P}^1$.

Lemma 2.0.2. A curve C is hyperelliptic iff Ω^1_C is not very ample.

Proof. (DO THIS) $\square$

Proposition 2.0.3. Plane curves with $g > 1$ cannot be hyperelliptic.

Proof. Let $\iota : C \hookrightarrow \mathbb{P}^2$ be a plane curve. Then $\Omega^1_C = \iota^* \mathcal{O}_{\mathbb{P}^2}(d-3)$ where d is the degree of C. Since $g > 1$ we must have $d > 3$ and thus $\mathcal{O}_{\mathbb{P}^2}(d-3)$ is very ample defining the Veronese embedding $v : \mathbb{P}^2 \to \mathbb{P}^N$ s.t. $\mathcal{O}_{\mathbb{P}^2}(d-3) = v^* \mathcal{O}_{\mathbb{P}^N}(1)$. Then $v \circ \iota : C \to \mathbb{P}^N$ is an embedding such that $(v \circ \iota)^* \mathcal{O}_{\mathbb{P}^N}(1) = \Omega^1_C$. Thus Ω^1_C is very ample so C cannot be hyperelliptic. $\square$

Lemma 2.0.4. Let C have a $\mathfrak{g}^1_2$ then C is either hyperelliptic or rational.

Proof. Let D be a $\mathfrak{g}^1_2$ then $|D|$ defines a rational map $C \dashrightarrow \mathbb{P}^1$ of degree two. Suppose P were a basepoint of $|D|$ then $\dim |D - P| = 1$ which implies that C is rational because there is a rational degree one map $C \dashrightarrow \mathbb{P}^1$. $\square$

Proposition 2.0.5. Any genus 2 curve is hyperelliptic.

Proof. Consider the canonical divisor K_X which has $\deg K_X = 2g - 2 = 2$ and $\dim |K_X| = g - 1 = 1$ and thus gives a $\mathfrak{g}^1_2$. $\square$

3 Tangent Space

Definition 3.0.1. Let X be a scheme and $x \in X$ a point. Then we define:

(a) the geometric tangent space $T_x X = \mathrm{Spec}\left(\mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)\right)$

(b) the projectiveized tangent space $\mathbb{P}(T_x X) = \mathrm{Proj}\left(\mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)\right)$

(c) the geometric tangent cone $C_x X = \mathrm{Spec}\left(\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x})\right)$ where,

$$\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) = \bigoplus_{n=0}^{\infty} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$$

(d) the projectiveized tangent cone $\mathbb{P}(C_x X) = \mathrm{Proj}\left(\mathbf{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x})\right)$.

Remark. In particular, blowing up X at the sheaf of ideals $\mathscr{I}_x$ (defined as the subsheaf of $\mathcal{O}_X$ where evaluation in $\kappa(x)$ gives zero) gives the following,

$$\tilde{X} = \mathbf{Proj}_X \left(\bigoplus_{n=0}^{\infty} \mathscr{I}_x^n \right)$$

Choose an affine open neighborhood $x \in \mathrm{Spec}\,(A) = U \subset X$ then we see $\mathscr{I}_x|_U = \widetilde{\mathfrak{p}} \subset A$ is the prime corresponding to $x \in \mathrm{Spec}\,(A)$ and $\mathfrak{m}_x = \mathfrak{p} A_{\mathfrak{p}}$. Therefore, restricting $\pi : \tilde{X} \to X$ over U gives,

$$\mathrm{Proj} \left(\bigoplus_{n=0}^{\infty} \mathfrak{p}^n \right) \to \mathrm{Spec}\,(A)$$

and,

$$\mathrm{Bl}_{\mathfrak{p}}\,(A) = \bigoplus_{n=0}^{\infty} \mathfrak{p}^n$$

is the blowup algebra which is a graded A-algebra. Consider the fiber over x,

$$\mathrm{Proj}\,(\mathrm{Bl}_{\mathfrak{p}}\,(A) / \mathfrak{p} \mathrm{Bl}_{\mathfrak{p}}\,(A)) \to \mathrm{Spec}\,(\kappa(x))$$

where we see,

$$\mathrm{Bl}_{\mathfrak{p}}\,(A) / \mathfrak{p} \mathrm{Bl}_{\mathfrak{p}}\,(A) = \bigoplus_{n=0}^{\infty} \mathfrak{p}^n = \mathbf{gr}_{\mathfrak{p}}\,(A)$$

and therefore $\tilde{X}_x \to \mathrm{Spec}\,(\kappa(x))$ is $\mathrm{Proj}\left(\mathbf{gr}_{\mathfrak{p}}\,(A)\right) \to \mathrm{Spec}\,(\kappa(x))$. In particular, the tangent cone is the fiber over x in the blowup.

Remark. The exact same construction shows that given a ring A and ideal $I \subset A$ the blowup $\mathrm{Proj}\,(\mathrm{Bl}_I\,(A)) \to \mathrm{Spec}\,(A)$ where,

$$\mathrm{Bl}_I\,(A) = \bigoplus_{n=0}^{\infty} I^n$$

is the blowup algebra, has fiber over the closed subscheme $V(I)$ equal to,

$$\mathrm{Proj}\,(\mathbf{gr}_I\,(A)) \to \mathrm{Spec}\,(A/I)$$

which is the projectized tangent cone of I.

Remark. We can generalize this further. For a sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ we can form the blowup,

$$\tilde{X} = \mathbf{Proj}_X \left(\bigoplus_{n=0}^{\infty} \mathscr{I}^n \right) \to X$$

Restricting to the closed subscheme $Z = V(\mathscr{I}) \subset X$ we find,

$$\mathbf{Proj}_Z \left(\bigoplus_{n=0}^{\infty} \mathscr{I}^n / \mathscr{I}^{n+1} \right) \to Z$$

but notice that the graded algebra,

$$(\mathcal{O}_X / \mathscr{I}) \otimes_{\mathcal{O}_X} \bigoplus_{n=0}^{\infty} \mathscr{I}^n = \bigoplus_{n=0}^{\infty} \mathscr{I}^n / \mathscr{I}^{n+1} = \bigoplus_{n=0} (\mathscr{I} / \mathscr{I}^2)^{\otimes n} / K = \mathrm{Sym}_{\mathcal{O}_Z} \left(\mathscr{I} / \mathscr{I}^2 \right)$$

and $\mathcal{C}_{Z/X} = \mathscr{I} / \mathscr{I}^2$ is the conormal bundle (sheaf) so we find a pullback diagram,

$$\begin{tikzcd}
\mathbb{P}(\mathcal{C}_{Z/X}) \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\pi"] \\
Z \arrow[r] & X
\end{tikzcd}$$

and thus $\tilde{X} \to X$ is a projective bundle over Z and an isomorphism over $X \setminus Z$. We call $\mathbb{P}(\mathcal{C}_{Z/X})$ the projectiveized tangent cone of Z.

Proposition 3.0.2. In general $C_x X \hookrightarrow T_x X$ and this is an isomorphism if $x \in X$ is a regular point.

Proof. There is a surjective canonical map,

$$\mathrm{Sym}_{\kappa(x)}\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right) \to \mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)$$

giving a closed embedding,

$$C_x X \hookrightarrow T_x X$$

When $\mathcal{O}_{X,x}$ is regular then $\mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right) \cong \kappa(x)[x_1, \dots, x_r]$ where $x_1, \dots, x_r \in \mathfrak{m}_x$ are a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $\kappa(x)$-vectorspace. Therefore, $\mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2) \xrightarrow{\sim} \mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)$ as graded rings and thus $C_x X \to T_x X$ is an isomorphism. $\square$

Remark. Because the map is a graded map, the same holds projectivized $\mathbb{P}(C_x X) \hookrightarrow \mathbb{P}(T_x X)$.

(THIS IS WRONG BC COHEN ISOMORPHISM IS NOT CANONICAL) (MAKES SENSE BECAUSE ITS SUPPOSED TO BE LIKE NORMAL COORDINATES WHICH FOR MANIFOLDS REQUIRES A METRIC SAD)

Proposition 3.0.3. Let X be finite type over k and $x \in X$ be a closed point. There are canonical maps,

$$\widehat{T_x X} \leftarrow \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right) \to \mathrm{Spec}\left(\mathcal{O}_{X,x}\right) \to X$$

such that $\widehat{T_x X} \leftarrow \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right)$ is an isomorphism exactly when $x \in X$ is regular.

Proof. The canonical map,

$$A = \mathrm{Sym}_{\kappa(x)}\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right) \to \mathbf{gr}_{\mathfrak{m}_x}\left(\mathcal{O}_{X,x}\right)$$

is an isomorphism exactly when $x \in X$ is a regular point. By the Cohen structure theorem $\widehat{\mathcal{O}_{X,x}} = \kappa(x)[[x_1, \dots, x_r]]/I$ where $x_1, \dots, x_r \in \mathfrak{m}_x$ are a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$. Furthermore, $I = (0)$ exactly when $x \in X$ is regular. Therefore, consider the map,

$$\hat{A} \to \widehat{\mathcal{O}_{X,x}}$$

defined on finite levels by,

$$A/\mathfrak{m}^n = \bigoplus_{k \le n} \mathrm{Sym}^k\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right) \to \bigoplus_{k \le n} \mathfrak{m}$$

Taking spectra,

$$T_x X \leftarrow \mathrm{Spec}\left(\widehat{\mathcal{O}_{X,x}}\right)$$

and we see this is an isomorphism when $\mathcal{O}_{X,x}$ is regular. $\square$

4 Formal Schemes

Definition 4.0.1. Let A be a ring and $I \subset A$ an ideal. Then the completion of A along I is,

$$\hat{A} = \varprojlim_n A/I^n$$

Furthermore, for any A-module M we can complete M along I to get,

$$\hat{M} = \varprojlim_n M/I^n M = \varprojlim_n (M \otimes_A A/I^n) = M \otimes_A \hat{A}$$

Proposition 4.0.2. Let A be a ring and $I \subset A$ an ideal and M an A-module. Then $\hat{M}$ satisfies the following universal property. Any map $\varphi : M \to N$ to a complete A-module N facrors uniquely as $M \to \hat{M} \xrightarrow{\hat{\varphi}} N$.

Proof. The kernel of $M \to N/I^n N$ contains $I^n M$ and thus factors as $M \to M/I^n M \to N/I^n N$. Taking inverse limits gives $M \to \hat{M} \to N$. Uniqueness follows from the fact that a map $\hat{M} \to M$ is determined completely by $\hat{M} \to M/I^n M \to N/I^n N$. $\square$

Lemma 4.0.3. Let A be a ring and $I \subset A$ an ideal. Then the units of $\hat{A}$ are exactly those elements which map to units under $\hat{A} \to A/I$.

Proof. Suppose that $u \in \hat{A}$ is a unit. Then clearly its image under $\hat{A} \to A/I$ is a unit. Conversely, suppose that $u \mapsto u_1 \in A/I$ is a unit. Then there exists $v_1 \in A/I$ s.t. $u_1 v_1 = 1$ so lifting v_1 we get $u_2 \tilde{v}_1 = 1 + r$ for $r \in I$ so we can take $r u_2 \tilde{v}_1 = r + r^2 = r$ and thus $u_2(\tilde{v}_1 - r\tilde{v}_1) = 1$. Write $v_2 = \tilde{v}_1 - r\tilde{v}_1$ and we lift to see $u_3 \tilde{v}_2 = 1 + r'$ for $r' \in I^2$ etc giving by induction an element $v \in \hat{A}$ such that $uv = 1$ in each A/I^n and thus in $\hat{A}$. $\square$

Lemma 4.0.4. Let $\mathfrak{m} \subset A$ be a maximal ideal. Then $\hat{A} = \widehat{A_\mathfrak{m}}$ is local.

Proof. Consider,

$$\widehat{A_\mathfrak{m}} = \varprojlim_n (A_\mathfrak{m}/\mathfrak{m}^n A_\mathfrak{m}) = \varprojlim_n (A/\mathfrak{m}^n)_\mathfrak{m}$$

However, since $A/\mathfrak{m}^n$ is local with maximal ideal $\mathfrak{m}$ we see that $(A/\mathfrak{m}^n)_\mathfrak{m} = A/\mathfrak{m}^n$ and thus,

$$\widehat{A_\mathfrak{m}} = \varprojlim_n (A/\mathfrak{m}^n)_\mathfrak{m} = \varprojlim_n A/\mathfrak{m}^n = \hat{A}$$

$\square$

Remark. Localization does not, in general, behave nicely with completion. For example, let $A = \mathbb{Z}_p[x]$ and $\mathfrak{p} = (x)$. Then $\hat{A}_\mathfrak{p} = \widehat{\mathbb{Q}_p[x]_{(x)}} = \mathbb{Q}_p[[x]]$. However, $\hat{A} = \mathbb{Z}_p[[x]]$ and $\hat{A}_{\hat{\mathfrak{p}}} = \mathbb{Z}_p[[x]]_{(x)}$ which is a proper subring of $\mathbb{Q}_p[[x]]$ because it does not contain $1 + p^{-1}x + p^{-2}x^2 + \cdots$ and is this, in particular, not complete.

Lemma 4.0.5. Suppose that $(A, \mathfrak{m})$ is a local ring. Then $\mathrm{Spec}\left(\hat{A}\right) \to \mathrm{Spec}\,(A)$ is a homeomorphism.

(THIS IS TOTALLY FALSE IMPLIES A IS UNABRANCH AT LEAST)

Proof. The units in $\hat{A}$ are everything except the preimage of zero under $\hat{A} \to A/\mathfrak{m} = \kappa$. Therefore $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$ is the unique maximal ideal of $\hat{A}$ making A local. I claim that $\mathfrak{p} \mapsto \mathfrak{p}\hat{A}$ is an inverse to $\mathrm{Spec}\left(\hat{A}\right) \to \mathrm{Spec}\,(A)$. (DO THIS!!) $\square$

Example 4.0.6. Consider $X = \operatorname{Spec}(k[x,y]/(y^2 - x^2(x-1))) \subset \mathbb{A}^2_k$. Take $p = (x,y)$. We know X is connected and thus $\mathcal{O}_{X,p} = A_{\mathfrak{p}}$ has a unique minimal prime. However,

$$\widehat{\mathcal{O}_{X,p}} = \hat{A} = k[[x,y]]/(y^2 - x^2(x+1)) \cong k[[x,y]]/(x^2 - y^2) = k[[x,y]]/((y-x)(x+y)) \cong k[[u]] \times k[[v]]$$

which has two minimal primes (branches).

5 Multiplicity of a Point

5.1 Hilbert-Samuel Polynomial

Definition 5.1.1. Let $(A, \mathfrak{m}, \kappa)$ be a noetherian local ring and M a finite A-module. Then define the Hilbert function,

$$\chi_M(n) = \operatorname{length}_A(M/\mathfrak{m}^n M)$$

Remark. We abreviate $\operatorname{length}_A(M) = \ell_A(M)$. We recall the following facts from dimension theory.

Proposition 5.1.2. There is a numerical polynomial $P_M \in \mathbb{Q}[x]$ of degree $d = \dim M$ such that,

$$P_M(n) = \chi_M(n)$$

for all $n \gg 0$.

Mat, Theorem 13.4. □

Proposition 5.1.3. Given an exact sequence of finite A-modules,

$$0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$$

then,

$$d(M) = \max\{d(N), d(K)\}$$

and $P_M - P_N - P_K$ is a polynomial of degree strictly less than $d(N)$.

Proof. Consider the exact sequences,

$$0 \longrightarrow (\mathfrak{m}^n M + N)/\mathfrak{m}^n M \longrightarrow M/\mathfrak{m}^n M \longrightarrow K/\mathfrak{m}^n K \longrightarrow 0$$

$$0 \longrightarrow (\mathfrak{m}^n M \cap N)/\mathfrak{m}^n N \longrightarrow N/\mathfrak{m}^n N \longrightarrow (\mathfrak{m}^n M + N)/\mathfrak{m}^n M \longrightarrow 0$$

Therefore,

$$\chi_M(n) = \chi_K(n) + \ell((\mathfrak{m}^n M + N)/\mathfrak{m}^n M) = \chi_K(n) + \chi_N(n) - \ell((\mathfrak{m}^n M \cap N)/\mathfrak{m}^n N)$$

Furthermore, by Artin-Rees,

$$\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-c}(\mathfrak{m}^c M \cap N) \subset \mathfrak{m}^{n-c} N$$

for fixed c and $n \geq c$. Therefore,

$$\ell((\mathfrak{m}^n M \cap N)/\mathfrak{m}^n N) \leq \ell(\mathfrak{m}^{n-c} N/\mathfrak{m}^n N) = \chi_N(n) - \chi_N(n-c)$$

which is a polynomial φ of degree strictly less than $d(N)$. Furthermore, for $n \gg 0$ we have,

$$P_M(n) = P_N(n) + P_K(n) - \varphi(n)$$

and since φ has degree strictly smaller $d(N)$ we see that,

$$\mathrm{d}(M) = \max\{d(N), d(K)\}$$

$\square$

Definition 5.1.4. Let $a_d x^d$ be the leading term of P_M then the multiplicity of M is $e(M) = d!\, a_d$. Notice that,

$$e(M) = d! \lim_{n \to \infty} \frac{\chi_M(n)}{n^d}$$

We call $e(A)$ the multiplicity of the ring A. Furthermore, since,

$$\chi_M(n+1) - \chi_M(n) = \dim_\kappa \left(\mathfrak{m}^n M / \mathfrak{m}^{n+1} M\right)$$

therefore if $d > 0$

$$e(M) = (d-1)! \lim_{n \to \infty} \frac{\dim_\kappa \left(\mathfrak{m}^n M / \mathfrak{m}^{n+1} M\right)}{n^{d-1}}$$

Remark. If $d = 1$ then we find the following formula,

$$e(M) = \lim_{n \to \infty} \dim_\kappa \left(\mathfrak{m}^n M / \mathfrak{m}^{n+1} M\right)$$

and in particular,

$$e(A) = \lim_{n \to \infty} \dim_\kappa \left(\mathfrak{m}^n / \mathfrak{m}^{n+1}\right)$$

Definition 5.1.5. Let X be a locally noetherian scheme and $x \in X$ a point. Then the multiplicity $m(x) = m(\mathcal{O}_{X,x})$ is defined as the multiplicity of the local ring $\mathcal{O}_{X,x}$,

$$m(x) = d! \lim_{n \to \infty} \frac{\ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x} / \mathfrak{m}_x^n)}{n^d}$$

where $d = \dim \mathcal{O}_{X,x} = \operatorname{codim}\left(\overline{\{x\}}, X\right)$.

5.2 Schemes of Dimension One

Proposition 5.2.1. Let X be a connected noetherian T_0 space with $\dim X = 1$. Then X is equidimensional and for each $x \in X$ setting $Z_x = \overline{\{x\}}$ exactly one of the two equivalenes holds,

(a) $\operatorname{codim}(Z_x, X) = 0 \iff \dim Z_x = 1 \iff Z_x$ is maximal $\iff x$ is not closed

(b) $\operatorname{codim}(Z_x, X) = 1 \iff \dim Z_x = 0 \iff Z_x$ is not maximal $\iff x$ is closed

Proof. Let $Z_1, \dots, Z_n \subset X$ be the irreducible components of X. I claim that each irreducible component Z_i intersects some other irreducible component Z_j unless $n = 1$. Otherwise,

$$Z_i \cap \bigcup_{j \neq i} Z_j = \varnothing$$

but both sets are closed (critically using that the union is finite) and,

$$Z_i \cup \bigcup_{j \neq i} Z_j = X$$

which by the connectedness of X implies that one set is empty so $n = 1$. The case $n = 1$ is clear so suppose that $n > 1$. If $Z_i = \{x\}$ for a closed point $x \in X$ then there is some $Z_j \neq Z_i$ with $Z_i \cap Z_j \neq \emptyset$ and thus $x \in Z_j$ so $Z_i \subset Z_j$ so $Z_i = Z_j$ giving a contradiction. Then $\dim Z_i = 0$ or $\dim Z_i = 1$. However, if $\dim Z_i = 0$ then $Z_i = \{x\}$ since X is T_0 for a closed point $x \in X$ (since Z_i is closed). We showed this cannot happen so $\dim Z_i = 1$ proving that X is equidimensional.

Clearly, exacly one of $\text{codim}\,(Z_x, X) = 0$ or $\text{codim}\,(Z_x, X) = 1$ holds so we need to show the equivalences. We know $\text{codim}\,(Z_x, X) = 0$ if and only if Z_x is an irreducible component (i.e. maximal). We showed that every irreducible component has dimension 1 so $\dim Z_x = 1$ iff Z_x is maximal and $\dim Z_x = 0$ iff Z_x is not maximal. If $\text{codim}\,(Z_x, X) = 1$ then because,

$$\dim X \geq \text{codim}\,(Z_x, X) + \dim Z_x$$

we see $\dim Z_x = 0$. Conversely, if $\dim Z_x = 0$ we have seen that Z_x is not maximal and thus $\text{codim}\,(Z_x, X) \neq 0$ by the above equivalences so $\text{codim}\,(Z_x, X) = 1$. If x is closed then $Z_x = \{x\}$ and we showed that Z_x is not maximal. Conversely, if Z_x is not maximal then $\dim Z_x = 0$ so x is closed. Finally, if Z_x is maximal then $\dim Z_x = 1$ so x is not closed and conversely if x is not closed then $\dim Z_x \neq 0$ by the previous equivalence so $\dim Z_x = 1$ so Z_x is maximal. □

Corollary 5.2.2. Let X be a connected noetherian scheme with $\dim X = 1$. Then for each closed point $x \in X$ we have $\dim \mathcal{O}_{X,x} = \text{codim}\,(Z_x, X) = 1$ so the multiplicitly is,

$$m(x) = \lim_{n \to \infty} \dim_{\kappa(x)}(\mathfrak{m}_x^n / \mathfrak{m}_x^{n+1})$$

and for each non-closed point $\xi \in X$ or equivalently the generic point of an irreducible component, $\dim \mathcal{O}_{X,\xi} = \text{codim}\,(Z_x, X) = 0$ and $\mathcal{O}_{X,\xi}$ is noetherian so $\mathcal{O}_{X,\xi}$ is Artin local meaning $\mathfrak{m}_\xi^n = 0$ for some n so the multiplicity satisfies,

$$m(\xi) = \ell_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{X,\xi})$$

5.3 Normalization

(DO THIS FOR NOT INTEGRAL SCHEMES!!)

Proposition 5.3.1. Let X be a noetherian integral scheme with $\dim X = 1$ such that the normalization $\nu : \widetilde{X} \to X$ is a finite morphism. Then for each $x \in X$,

$$m(x) = \deg_x \nu = \dim_{\kappa(x)} \widetilde{X}_x$$

Proof. The generic point is clear since $m(\xi) = 1$ because A is reduced and ν is birational. Let $x \in X$ be a closed point and $A = \mathcal{O}_{X,x}$ then $\dim A = 1$. Let $K = \text{Frac}\,(A)$ then $\widetilde{A} \subset K$ is the normalization. Consider the exact sequence of A-modules,

$$0 \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow Q \longrightarrow 0$$

Tensoring by K we see that $Q \otimes_A K = 0$. If $Q = 0$ we are immediately see that $P_A = P_{\widetilde{A}}$ as A-modules since $A = \widetilde{A}$. Otherwise, since $A \to \widetilde{A}$ is finite, Q is a finite A-module and thus $\mathrm{Supp}_A(Q)$ is closed and does not contain the generic point so $\dim \mathrm{Supp}_A(Q) = 0$. Indeed $B = A/\mathrm{Ann}_A(Q)$ is an Artinian ring and Q is a finite B-module and hence Artinian so $\ell_B(Q)$ is finite. Therefore, by Prop. 5.1.3, P_A and $P_{\widetilde{A}}$ have the same leading coefficient proving that,

$$m(x) = \lim_{n \to \infty} \frac{\ell_A(A/\mathfrak{m}^n)}{n} = \lim_{n \to \infty} \frac{\ell_A(\widetilde{A}/\mathfrak{m}^n \widetilde{A})}{n}$$

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals of $\widetilde{A}$ over $\mathfrak{m}$ and $B_i = \widetilde{A}_{\mathfrak{m}_i}$ then (Tag 02M0),

$$\ell_A(\widetilde{A}/\mathfrak{m}^n \widetilde{A}) = \sum_{i=1}^r \ell_{B_i}(B_i/\mathfrak{m}^n B_i) \cdot [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$$

However, B_i is a normal noetherian domain with $\dim B_i = 1$ and therefore a DVR. Let $\varpi_i \in \mathfrak{m}_i$ be a uniformizer and $\mathfrak{m} B_i = (\varpi_i^{e_i})$ for some integer e_i (if $\varpi = 0$ set $e_i = 1$) since B_i is a PID. Therefore, $\mathfrak{m}^n = \mathfrak{m}_i^{n e_i}$. In any DVR R with a uniformizer $\varpi \in R$ we have,

$$\ell_R(R/(\varpi^n)) = n$$

This is from the filtration,

$$(\varpi^n) \subset (\varpi^{n-1}) \subset \cdots \subset (\varpi) \subset R$$

and from the quotients we find,

$$\ell_R(R/(\varpi^n)) = \sum_{k=1}^n \ell_R((\varpi^k)/(\varpi^{k+1})) = n$$

because $(\varpi^k)/(\varpi^{k+1})$ is a one-dimensional κ-module. Therefore,

$$\ell_{B_i}(B_i/\mathfrak{m}^n) = n e_i$$

so we find,

$$\begin{aligned} m(x) &= \lim_{n \to \infty} \frac{\ell_A(\widetilde{A}/\mathfrak{m}^n)}{n} = \sum_{i=1}^r \lim_{n \to \infty} \frac{n e_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]}{n} = \sum_{i=1}^r e_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \\ &= \sum_{i=1}^r \mathrm{length}_{B_i}(B_i/\mathfrak{m}) [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] = \mathrm{rank}_x(f_* \mathcal{O}_{\widetilde{X}}) = \dim_{\kappa(x)} \widetilde{X}_x = \deg_x \nu \end{aligned}$$

□

Proposition 5.3.2. Let $\widetilde{X} \to X$ be the normalization map as above. Then for nonzero $f \in \mathcal{O}_{X,x}$,

$$\mathrm{ord}_x(f) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f)) = \sum_{x' \in \nu^{-1}(x)} \mathrm{ord}_{x'}(f)[\kappa(x_i) : \kappa(x)]$$

Proof. Let $A = \mathcal{O}_{X,x}$ and $\widetilde{A}$ the normalization. Then $A \to \widetilde{A}$ is finite. Consider the submodules,

$$\begin{tikzcd} A \arrow[r, hook] & \widetilde{A} \\ fA \arrow[u, hook] \arrow[r, hook] & f\widetilde{A} \arrow[u, hook] \end{tikzcd}$$

where the maps are injective because A is a domain and $f \neq 0$. Furthermore, is an isomorphism of A-modules because $(f) \cong A$ is a flat A-module (easy to check directly). Therefore, from the diagram,

$$\ell_A(\widetilde{A}/f\widetilde{A}) = \ell_A(\widetilde{A}/fA) - \ell_A(f\widetilde{A}/fA) = \ell_A(\widetilde{A}/fA) - \ell_A(\widetilde{A}/A) = \ell_A(A/fA)$$

Finally, let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals of $\widetilde{A}$ over $\mathfrak{m}$ and $B_i = \widetilde{A}_{\mathfrak{m}_i}$ then by Tag 02M0,

$$\ell_A(\widetilde{A}/f\widetilde{A}) = \sum_{\mathfrak{m}_i} \ell_{B_i}(B_i/fB_i)[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] = \sum_{x' \in \nu^{-1}(x)} \mathrm{ord}_{x'}(f)[\kappa(x') : \kappa(x)]$$

□

Proposition 5.3.3. GENUS FORMULA

Proof. Consider $\nu : \widetilde{X} \to X$. Since the normalization is dominant there is an exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f_*\mathcal{O}_{\widetilde{X}} \longrightarrow \mathscr{C} \longrightarrow 0$$

Note that $f : S \to C$ induces an isomorphism $H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$ since it is a map of fields with the same (finite) dimension over k. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, \mathscr{C}) \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, \mathscr{C}) = 0$$

I claim that $H^1(S, \mathscr{C}) = 0$. Since f is birational, $\mathscr{C}$ is supported in codimension one. Thus, the map $H^1(C, \mathcal{O}_C) \twoheadrightarrow H^1(S, \mathcal{O}_S)$ is surjective but $g_a(C) = g_a(S)$ so these vectorspaces have the same dimension so $H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S)$ is an isomorphism. Thus, from the exact sequence we have $H^0(X, \mathscr{C}) = 0$. However, $\mathrm{Supp}_{\mathcal{O}_C}(\mathscr{C})$ is a closed ($\mathscr{C}$ is coherent) dimension zero subset i.e. finitely many discrete closed points. However, a sheaf supported on a discrete set of points is zero iff it has no global sections. Therefore, $\mathscr{C} = 0$ so $\mathcal{O}_C \xrightarrow{\sim} f_*\mathcal{O}_S$. In particular $\mathcal{O}_C(V) \xrightarrow{\sim} \mathcal{O}_S(U)$ is an isomorphism which implies that the map of affine schemes $f|_U : U \to V$ is an isomorphism. Since the affine opens V cover C we see that $f : S \to C$ is an isomorphism. In particular, C is smooth. □

6 Ramification

Definition 6.0.1. Given a map of schemes $f : X \to Y$ over S we get an exact sequence,

$$f^*\Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

We say that f is ramified at $x \in X$ if $(\Omega_{X/Y})_x \neq 0$ which is equivalent to the map $f_x : (\Omega_{Y/S})_{f(x)} \to (\Omega_{X/S})_x$ not being surjective. Furthermore define,

(a) The support of $\Omega_{X/Y}$ is called the *ramification locus*

(b) $f(\mathrm{Supp}_{\mathcal{O}_X}(\Omega_{X/Y}))$ is the *branch locus*

(c) if $\Omega_{X/Y} = 0$ then f is *formally unramified*

(d) f is *unramified* if f is formally unramified and locally of finite type

(e) f is *G-unramified* if f is formally unramified and locally of finite presentation

Lemma 6.0.2. Let $f : X \to Y$ be locally of finite type. Then the following are equivalent,

(a) the morphism $f : X \to Y$ is unramified at $x \in X$

(b) the stalk map $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces a finite seperable extension $\kappa(f(x))/\kappa(x)$ and $f_x^\#(\mathfrak{m}_{f(x)})\mathcal{O}_{X,x} = \mathfrak{m}_x$

Proof. (DO THIS PROOF) □

(DEFINE RAMIFICATION INDEX)

Remark. A ring map $\phi : A \to B$ corresponds to morphism of affine schemes,

$$\hat{\phi} : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$$

which is ramified at $\mathfrak{q} \subset B$ iff $(\Omega_{B/A})_{\mathfrak{q}} \neq 0$ or equivalently if $\mathfrak{q} \in \mathrm{Supp}_B\left(\Omega_{B/A}\right)$.

In particular if ϕ is finite then $\Omega_{B/A}$ is a finitely generated B-module so,

$$\mathrm{Supp}_B\left(\Omega_{B/A}\right) = V(\mathrm{Ann}_B\left(\Omega_{B/A}\right))$$

and thus $\mathfrak{q}$ is ramified iff $\mathfrak{q} \supset \mathrm{Ann}_B\left(\Omega_{B/A}\right)$. This motivates the following definition.

Definition 6.0.3. Let $\phi : A \to B$ be finite. Then define the different $\delta_{B/A} = \mathrm{Ann}_B\left(\Omega_{B/A}\right)$.

Remark. The important fact about the different is that it classifies ramification in the sense that ϕ is ramified at $\mathfrak{q}$ (or we say $\mathfrak{q}$ ramifies) iff $\delta_{B/A} \subset \mathfrak{q}$.

Corollary 6.0.4. Let $\phi : A \to B$ be a finite map with B a Dedekind domain. Then only finitely many points ramify.

6.1 Ramification For Curves

Definition 6.1.1. We say a scheme X is *normal* if each point $x \in X$ that $\mathcal{O}_{X,x}$ is normal i.e. an integrally closed local domain.

Lemma 6.1.2. Any normal local ring of dimension one is a DVR.

Definition 6.1.3. If X is a normal curve then each $\mathcal{O}_{X,x}$ is a DVR so we may choose a uniformizer π_x. For a morphism of normal cuves $f : X \to Y$. Since $\mathfrak{m}_x = (\pi_x)$ clearly the ramification index is the power e such that $f_x^\#(\pi_{f(x)}) = u\pi_x^e$ for some $u \in \mathcal{O}_{X,x}^\times$.

Proof. □

6.2 Ramification for Dedekind Domains

Proof. Since B is Dedekind domain there is a finite unique factorization of $\delta_{B/A}$ into prime ideals. These are the only primes lying about $\delta_{B/A}$ and thus exactly the set of primes which ramify of which there are finitely many. □

Proposition 6.2.1. Let $\phi : A \to B$ be a finite inclusion of Dedekind domains with finite residue fields. Then $\mathfrak{q}$ ramifies iff the prime $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ extends to the ideal $\mathfrak{p}B$ with factorization,

$$\phi(\mathfrak{p})B = \prod_{i=1}^{n} \mathfrak{q}_i^{e_i}$$

with $\mathfrak{q}_i$ distinct, $\mathfrak{q}_0 = \mathfrak{q}$, and $e_i > 1$.

Proof. At a point $\mathfrak{p} \subset A$ the residue field $\kappa(\mathfrak{p}) = A/\mathfrak{p}$ is a finite field which is perfect so $\kappa(\mathfrak{q})/\kappa(\hat{\phi}(\mathfrak{q}))$ is automatically finite seperable. Thus $\mathfrak{q} \subset B$ is unramified iff,

$$\phi(\phi^{-1}(\mathfrak{q})A_{\mathfrak{p}})B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$$

Since $B_{\mathfrak{q}}$ is also Dedekind, by unique factorization of ideals this is equivalent to $e_0 = 1$ since localizing the above factorization gives,

$$\phi(\mathfrak{p})B_{\mathfrak{q}} = (\phi(\mathfrak{p})B)_{\mathfrak{q}} = \prod_{i=1}^{n} \mathfrak{q}_i^{e_i} B_{\mathfrak{q}} = \mathfrak{q}_0^{e_0} B_{\mathfrak{q}} = \mathfrak{q}^{e_0} B_{\mathfrak{q}}$$

□

Proposition 6.2.2. DIFFERENT IN TERMS OF TRACE

Lemma 6.2.3. If $B = A[t]/(f(t))$ then,

$$\begin{aligned} \Omega_{B/A} &= (B \cdot \mathrm{d}t)/(f'(t) \cdot \mathrm{d}t) \\ \delta_{B/A} &= (f'(t)) \subset B \end{aligned}$$

Proof. $\Omega_{B/A}$ is generated by $\mathrm{d}x$ for $x \in B$. For any $g(t) \in A[t]/(f(t))$ then by the Leibniz relation, $\mathrm{d}g(t) = g'(t)\mathrm{d}t$. Thus, $\Omega_{B/A}$ is generated over B by $\mathrm{d}t$. Furthermore, $f(t) = 0$ so $f'(t)\mathrm{d}t = 0$. This is the only relation. Furthermore,

$$\delta_{B/A} = \mathrm{Ann}_B\left(\Omega_{B/A}\right) = (f'(t))$$

by the following lemma. □

Lemma 6.2.4. Let A be a ring and B an A-algebra with structure map $\phi : A \to B$ then $\mathrm{Ann}_A(B) = \ker \phi$.

Proof. An element $a \in \mathrm{Ann}_A(B)$ iff $\phi(a)b = 0$ for all $b \in B$. In particular,

$$a \in \mathrm{Ann}_A(B) \iff \phi(a) \cdot 1_B = 0 \iff \phi(a) = 0 \iff a \in \ker \phi$$

□

Corollary 6.2.5. If $K/\mathbb{Q}$ is a number field with $\mathcal{O}_K = \mathbb{Z}[\alpha]$ and let α have minimal polynomial $f \in \mathbb{Z}[X]$. Then a prime $\mathfrak{p} \subset \mathcal{O}_K$ ramifies iff $f'(\alpha) \in \mathfrak{p}$.

Proof. We have $\mathcal{O}_K = \mathbb{Z}[\alpha]/(f(\alpha))$ so $\delta_{\mathcal{O}_K/\mathbb{Z}} = (f'(\alpha)) \subset \mathbb{Z}[\alpha]/(f(\alpha))$. Then we know that $\mathfrak{p}$ ramifies iff $\mathfrak{p} \supset \delta_{\mathcal{O}_K/\mathbb{Z}}$. □

6.3 The Discriminant of an Extension

Definition 6.3.1. Let $\phi : A \to B$ be a finite map of Dedekind domains with finite residue fields. Then we define the ideal norm as the homomorphism $\mathcal{N}_{B/A} : \mathscr{I}_B \to \mathscr{I}_A$ of the ideal groups which on the prime ideals which generate $\mathscr{I}_B$ acts via,

$$\mathcal{N}_{B/A}(\mathfrak{q}) = \mathfrak{p}^{[B/\mathfrak{q}:A/\mathfrak{p}]}$$

where $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.

Definition 6.3.2. Then we define the relative discriminant $\Delta_{B/A} = \mathcal{N}_{B/A}(\delta_{B/A}) \subset A$.

Proposition 6.3.3. Primes $\mathfrak{p} \subset A$ ramify iff $\mathfrak{p} \supset \Delta_{B/A}$.

Proof. We write,

$$\mathfrak{p}B = \prod_{i=1}^{n} \mathfrak{q}_i^{e_i}$$

$\mathfrak{p}$ ramifies exactly when some $e_i > 1$ in which case we know $\mathfrak{q}_i \supset \delta_{B/A}$ and thus,

$$\mathfrak{p} \supset \mathfrak{p}^{[B/\mathfrak{q}:A/\mathfrak{p}]} \supset \mathcal{N}_{B/A}(\delta_{B/A}) = \Delta_{B/A}$$

Conversely, suppose that $\mathfrak{p}$ does not ramify then we must have $e_i = 1$ for all i. Then $\mathfrak{q}_i \not\supset \delta_{B/A}$ so by unique factorization, no primes dividing $\delta_{B/A}$ lie above $\mathfrak{p}$ which implies that, $\Delta_{B/A} = \mathcal{N}_{B/A}(\delta_{B/A})$ does not contain $\mathfrak{p}$ in its factorization. Thus, by the uniqueness of factorization,

$$\mathfrak{p} \not\supset \Delta_{B/A}$$

□

7 Maps between Curves

7.1 Maps of a Proper Curve are Finite

Theorem 7.1.1. Let C be a proper curve over k and X is separated of finite type over k. Then any nonconstant map $f : C \to X$ over k is finite.

Proof. Since $C \to \mathrm{Spec}\,(k)$ is proper and $X \to \mathrm{Spec}\,(k)$ is separated then by Tag 01W6 the map $f : C \to X$ is proper. The fibres of closed points $x \in X$ are proper closed subschemes $C_x \hookrightarrow C$ (since if $C_x = C$ then $f : C \to X$ would be the constant map at $x \in X$) and thus finite since proper closed subsets of a curve are finite. Now I claim that if the fibres $f^{-1}(x)$ are finite at closed points $x \in X$ then all fibres are finite. Assuming this, $f : C \to X$ is proper with finite fibres and thus is finite by Tag 02OG.

To show the claim consider,

$$E = \{x \in X \mid \dim C_x = 0\}$$

Since C is Noetherian, $\dim C_x = 0$ iff C_x is finite (suffices to check for affine schemes since quasi-comact and dimension zero Noetherian rings are exactly Artinian rings which have finite spectrum). Then E is locally constructible by Tag 05F9 and contains all the closed points of X. Since X is finite type over k then X is Jacobinson which implies that E is dense in every closed set. Then for any point $\xi \in X$ then $Z = \overline{\{\xi\}}$ is closed and irreducible with generic point ξ and thus $E \cap Z$ is dense in Z. Then by Tag 005K we have $\xi \in E$ so $E = X$ proving that all fibres are finite. □

Remark. The only facts about C that I used were that $C \to \mathrm{Spec}\,(k)$ is proper and that C is irreducible of dimension one. The second two properties are needed for the following to hold.

Lemma 7.1.2. If X is an irreducible Noetherian scheme of dimension one then every nontrivial closed subset of X is finite.

Proof. Since X is quasi-compact it suffices to show this property for affine schemes $X = \mathrm{Spec}\,(A)$ with $\dim A = 1$ and prime nilradical. Any nontrivial closed subset is of the form $V(I)$ for some proper radical ideal $I \subset X$ with $I \supsetneq \mathrm{nilrad}\,(A)$. Then $\mathbf{ht}\,(I) = 1$ since any prime above I must properly contain $\mathrm{nilrad}\,(A)$ and thus have height at least one but $\dim A = 1$. Then,

$$\mathbf{ht}\,(I) + \dim A/I \le \dim A$$

so $\dim A/I = 0$. Since A is Noetherian so is A/I but $\dim A/I = 0$ and thus A/I is Artianian. Therefore $\mathrm{Spec}\,(A/I)$ is finite proving the proposition. ☐

Remark. Since $C \to \mathrm{Spec}\,(k)$ is proper it is finite type over k and thus C is Noetherian.

Remark. The condition that C be proper is necessary. Consider the map $\mathbb{G}_m^k \coprod \mathbb{A}_k^1 \to \mathbb{A}_k^1$ via $k[x] \to k[x, x^{-1}]$ and the identity. This is clearly surjective and finitely generated since on rings it is,

$$k[x] \to k[x, x^{-1}] \times k[x]$$

Furthermore, this map is quasi-finite since the fibers have at most two points. To see this, consider, $y = (x - a) \in \mathrm{Spec}\,(k[x])$ then $\kappa(y) = k[x]/(x - a)$ and the fibre is,

$$\begin{aligned}
X_y &= \mathrm{Spec}\left((k[xx^{-1}] \times k[x]) \otimes_{k[x]} k[x]/(x-a)\right) \\
&= \mathrm{Spec}\left(k[x, x^{-1}]/(x-a) \times k[x]/(x-a)\right) \\
&= \mathrm{Spec}\left(k[x, x^{-1}/(x-a)\right) \coprod \mathrm{Spec}\,(k[x]/(x-a)) \\
&= \begin{cases} \mathrm{Spec}\,(k) & a = 0 \\ \mathrm{Spec}\,(k) \coprod \mathrm{Spec}\,(k) & a \neq 0 \end{cases}
\end{aligned}$$

However, this map is not closed since $\mathbb{G}_m^k \subset \mathbb{G}_m^k \coprod \mathbb{A}_k^1$ is closed but its image is $\mathbb{A}_k^1 \setminus \{0\}$ which is not closed. Thus the map cannot be finite. In particular,

$$k[x, x^{-1}] = \bigoplus_{n \ge 0} x^{-n} k[x]$$

so $k[x, x^{-1}]$ is not a finitely-generated $k[x]$-module.

7.2 Maps of Normal Curves Are Flat

Lemma 7.2.1. Let X be an integral scheme with generic point $\xi \in X$ and $\mathscr{F} \to \mathscr{G}$ a map of $\mathcal{O}_X$-modules,

(a) if $\mathscr{F}$ is locally free then $\mathscr{F} \to \mathscr{G}$ is injective iff $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is injective

(b) if $\mathscr{F}$ is invertible then $\mathscr{F} \to \mathscr{G}$ is injective iff $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is nonzero.

Proof. Since $\xi \in U$ for each nonempty open we have a diagram,

$$\begin{array}{ccc} \mathscr{F}(U) & \longrightarrow & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}_\xi & \longrightarrow & \mathscr{G}_\xi \end{array}$$

therefore it suffices to show the map $\mathscr{F}(U) \to \mathscr{F}_\xi$ is injective since then injectivity of $\mathscr{F}_\xi \to \mathscr{G}_\xi$ will imply injectivity of $\mathscr{F}(U) \to \mathscr{G}(U)$ for each U. Choose an affine open cover $U_i = \mathrm{Spec}\,(A_i)$ trivializing $\mathscr{F}$. $\mathscr{F}|_{U_i \cap U} \cong \mathcal{O}_X^{\oplus n}|_{U_i \cap U}$ but X is integral so the restriction $\mathscr{F}(U_i \cap U) \to \mathscr{F}_\xi$ is simply $A_i^n \to \mathrm{Frac}\,(A)^n$ which is injective since A_i is a domain. Thus if $s \in \mathscr{F}(U)$ maps to zero in $\mathscr{F}_\xi$ then $s|_{U_i \cap U} = 0$ so $s = 0$ since U_i form a cover.

The second follows from the first since we need only to show that $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is injective. However, $\mathscr{F}_\xi$ is a rank-one free module over the field $K(X) = \mathcal{O}_{X,\xi}$. Thus every nonzero map $\mathscr{F}_\xi \to \mathscr{G}_\xi$ is injective. ☐

Lemma 7.2.2. Let $f : X \to Y$ be a conconstant map of curves. Then f is dominant.

Proof. Let $\xi \in X$ be the generic point and consider $f(\xi) \in Y$. Suppose that $f(\xi)$ is a closed point. Then $f(X) = f(\overline{\{\xi\}}) \subset \overline{f(\xi)} = f(\xi)$ so f is constant. Therefore, we must have $f(\xi)$ a nonclosed point. But $\dim Y = 1$ and irreducible so any point is either closed or the generic point of the unique irreducible component. Therefore, $f(\xi) = \eta$ the generic point so f is dominant. ☐

Proposition 7.2.3. Let X and Y be curves over k with Y normal. Then any nonconstant map $f : X \to Y$ is flat.

Proof. We need to check that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. Since Y is a normal curve $\mathcal{O}_{Y,y}$ is a Noetherian domain (Y is integral finite type over k) integrally closed (Y is normal) and dimension at most one ($\dim Y = 1$) therefore $\mathcal{O}_{Y,y}$ is a local Dedekind domain or a field so $\mathcal{O}_{Y,y}$ is a DVR or a field. Then by Tag 0539, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module iff it is torsion-free. However, $\mathcal{O}_{X,x}$ is a domain so it is a torsion-free $\mathcal{O}_{Y,f(x)}$-module iff $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.

Since f is dominant $f(\xi) = \eta$ (the generic points). Then $\mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ is a map of fields which is automatically injective so $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective because Y is integral proving that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective. ☐

Remark. Morphisms of varieties are automatically finitely presented since curves are finite type over k so morphisms between them are locally finite type but Y is Noetherian so a locally finite type map is finitely presented. Furthermore, X is Noetherian so morphisms from it are automatrically quasi-compact and quasi-separated.

Proposition 7.2.4. Nonconstant maps of curves $f : X \to Y$ with Y normal are smooth iff unramified iff étale iff $\Omega_{X/Y} = 0$.

Proof. Maps of curves are automatically finitely presented. Furthermore, nonconstant maps of curves with Y normal are flat. Furthermore, we have seen that nonconstant maps of curves are quasi-finite so $\dim X_{f(x)} = 0$. Therefore, f is smooth iff $\Omega_{X/Y} = 0$ iff unramified but étale is smooth an unramified so we see smooth iff étale. ☐

Lemma 7.2.5. Let $X \to Y$ be a nonconstant map of curves with $K(X)/K(Y)$ separable and Y smooth. Then there is an exact sequence,

$$0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Therefore, f is étale iff $f^*\Omega_Y \to \Omega_X$ is an isomorphism.

Proof. $K(X)/K(Y)$ is an extension of fields of transcendence degree one over k so it must be algebraic. Furthermore, both are finitely-generated field extensions of k so the algebraic extension $K(X)/K(Y)$ is finite. Then $(\Omega_{X/Y})_\xi = \Omega_{K(X)/K(Y)}$ which is zero iff $K(X)/K(Y)$ is separable. Thus, the standard exact sequence gives $(f^*\Omega_Y) \twoheadrightarrow (\Omega_X)_\xi$ because $(\Omega_{X/Y})_\xi = 0$. Furthermore, $f^*\Omega_Y$ is a line bundle since Y is smooth so we conclude that $f^*\Omega_Y \to \Omega_X$ is an injection since it is nonzero on the generic fiber (Lemma 7.2.1). $\square$

8 Finite Maps

Definition 8.0.1. A morphism $f : X \to Y$ of schemes if *finite* if it is affine and for every affine open $V \subset Y$ then $U = f^{-1}(V)$ is affine and the ring map associated to $U \to V$ is finite.

Proposition 8.0.2. Closed immersions are finite.

Proof. The map $A \to A/I$ is finite. $\square$

Proposition 8.0.3. Finite maps are preserved under base change.

Proposition 8.0.4. Finite maps are closed and thus universally closed.

Proposition 8.0.5. The following are equivalent for a map of schemes $f : X \to Y$

(a) f is finite

(b) f is affine and proper.

Proof. They are affine and thus separated, finite and thus finite type, and universally closed. $\square$

Proposition 8.0.6. Let $f : X \to Y$ be finite and $y \in Y$. Then the fiber is affine, zero dimensional, has finitely many points, and explicitly,

$$X_y = \mathrm{Spec}\left((f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)\right)$$

Furthermore,

$$\mathrm{rank}_y\left(f_*\mathcal{O}_X\right) = \sum_{x \in f^{-1}(y)} \mathrm{length}_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}\right) \cdot [\kappa(x) : \kappa(y)]$$

Proof. Let $f : X \to Y$ be finite then locally we have affine opens $V = \mathrm{Spec}\,(B) \subset Y$ and $U = f^{-1}(V) = \mathrm{Spec}\,(A)$ and the map $B \to A$ is finite. Then $(f_*\mathcal{O}_X)|_V = \widetilde{A}$ as a B-module. Choose a point $y \in Y$ corresponding to a prime $\mathfrak{p} \in \mathrm{Spec}\,(B)$. Consider the fiber $X_y = X \times_Y \mathrm{Spec}\,(\kappa(y))$. Because $U = f^{-1}(V)$ is affine, the fiber $X_y \subset \mathrm{Spec}\,(A)$ and thus,

$$X_y = \mathrm{Spec}\,(A) \times_{\mathrm{Spec}(B)} \mathrm{Spec}\,(\kappa(y)) = \mathrm{Spec}\,(A \otimes_B \kappa(y)) = \mathrm{Spec}\,((A/\mathfrak{p}A)_\mathfrak{p})$$

where $\kappa(y) = (B/\mathfrak{p}B)_\mathfrak{p}$. So set $R = A \otimes_B \kappa(y) = (A/\mathfrak{p}A)_\mathfrak{p}$ then,

$$R = A \otimes_B (B/\mathfrak{p}B)_\mathfrak{p} = A \otimes_B B_\mathfrak{p} \otimes_{B_\mathfrak{p}} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = A_\mathfrak{p} \otimes_{B_\mathfrak{p}} \kappa(y) = (f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$$

Since A is a finite B-module, R is a finite $\kappa(\mathfrak{p})$-module so R is an artinian ring. Thus $X_y = \text{Spec}\,(R)$ has finitely many points and $\dim X_y = 0$. Furthermore,

$$\text{rank}_y\,(f_* \mathcal{O}_X) = \dim_{\kappa(y)} \Big((f_* \mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y) \Big)$$

and by our results on artinian k-algebras,

$$\dim_{\kappa(y)} R = \sum_{\mathfrak{m}_i \in \text{Spec}(R)} \text{length}_{R_{\mathfrak{m}_i}}\,(R_{\mathfrak{m}_i}) \cdot \dim_{\kappa(y)}(R/\mathfrak{m}_i)$$

However, the prime (maximal) ideals $\mathfrak{p}_x \in \text{Spec}\,(R)$ correspond to points $x \in f^{-1}(y)$ furthermore,

$$R_{\mathfrak{m}_x} = (A_{\mathfrak{p}_x}/\mathfrak{p}A_{\mathfrak{p}_x}) = \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$$

since $\mathfrak{p}A_{\mathfrak{p}_x} = \mathfrak{p}B_{\mathfrak{p}}A_{\mathfrak{p}_x} = \mathfrak{m}_y A_{\mathfrak{p}_x} = \mathfrak{m}_y \mathcal{O}_{X,x}$. Furthermore, since $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ is a surjection viewing $R_{\mathfrak{p}_x} = \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ as a $\mathcal{O}_{X,x}$-module gives,

$$\text{length}_{R_{\mathfrak{p}_x}}\,(R_{\mathfrak{p}_x}) = \text{length}_{\mathcal{O}_{X,x}}\,(\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x})$$

Finally, $R/\mathfrak{p}_x = \mathcal{O}_{X,x}/\mathfrak{m}_x = \kappa(x)$ and thus we find,

$$\text{rank}_y\,(f_* \mathcal{O}_X) = \sum_{x \in f^{-1}(y)} \text{length}_{\mathcal{O}_{X,x}}\,(\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}) \cdot [\kappa(x) : \kappa(y)]$$

□

Lemma 8.0.7. Let $A \hookrightarrow B$ be a finite inclusion of domains. Then $\text{Frac}\,(B) = A^{-1}B$ and is a finite extension of $\text{Frac}\,(A)$.

Proof. Since $A \to B$ is finite the map $\text{Frac}\,(A) \to A^{-1}B$ is finite. However, $A^{-1}B$ is a domain finite dimensional over the field $\text{Frac}\,(A)$ and thus $A^{-1}B$ is a field. However, $A^{-1}B \subset \text{Frac}\,(B)$ so $\text{Frac}\,(B) = A^{-1}B$. □

Proposition 8.0.8. Let $f : X \to Y$ be a finite dominant map of integral schemes with generic points $\xi \in X$ and $\eta \in Y$. Then we have,

$$\deg f = \text{rank}_\eta(f_* \mathcal{O}_X)$$

Proof. The map $\mathcal{O}_{Y,\eta} \to (f_* \mathcal{O}_X)_\eta$ is an injective finite map of domains because f is dominant. Therefore,

$$\text{rank}_\eta(f_* \mathcal{O}_X) = \dim_{\kappa(\eta)} \Big((f_* \mathcal{O}_X) \otimes_{\mathcal{O}_{Y,\eta}} \kappa(\eta) \Big) = \dim_{K(Y)} K(Y)^{-1} (f_* \mathcal{O}_X)_\eta$$

However, the map $(f_* \mathcal{O}_X)_\eta \to \mathcal{O}_{X,\xi}$ is taking the fraction field $K(X) = \mathcal{O}_{X,\xi} = \text{Frac}\,((f_* \mathcal{O}_X)_\eta)$ so by the previous lemma,

$$\text{rank}_\eta(f_* \mathcal{O}_X) = \dim_{K(Y)} K(X) = [K(X) : K(Y)] = \deg f$$

□

8.1 Finite Locally Free Morphisms

Definition 8.1.1. A morphism $f : X \to Y$ is *finite locally free* if f is affine and $f_* \mathcal{O}_X$ is a finite locally free as a $\mathcal{O}_Y$-module.

Proposition 8.1.2. A morphism $f : X \to Y$ is finite locally free iff f is finite, flat, and locally of finite presentation.

Proof. It suffices to show that if $A \to B$ is finite then B is locally free iff it is flat and finitely presented as an A-module. We know that finite locally free implies flat and locally finitely presented[1] (thus finitely presented). Conversely if B is flat and finitely presented[2] then it is projective (see Tag 00NX) and hence locally free. □

Proposition 8.1.3. Let $f : X \to Y$ be a finite flat dominant map of integral schemes. Then for any $y \in Y$ we have,

$$\sum_{x \in f^{-1}(y)} \mathrm{length}_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x} \right) \cdot [\kappa(x) : \kappa(y)] = \deg f$$

we call $e_x = \mathrm{length}_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x} / \mathfrak{m}_y \mathcal{O}_{X,x} \right)$ the ramification degree and then,

$$\sum_{x \in f^{-1}(y)} e_x \cdot [\kappa(x) : \kappa(y)] = \deg f$$

Proof. Since $f_* \mathcal{O}_X$ is finite locally free and Y is connected, the sheaf $f_* \mathcal{O}_X$ has constant rank and thus $\mathrm{rank}_y(f_* \mathcal{O}_X) = \mathrm{rank}_\eta(f_* \mathcal{O}_X)$. Using our previous results proves the claim. □

8.2 Ramification

9 Interesting Flasque Resolutions on Curves

9.1 Godement Resolution

For any abelian sheaf $\mathscr{F}$ on a space X we can consider it Godement resolution. Abstractly, take the continuous map $f : X_{\mathrm{dis}} \to X$ from X given the discrete topology. Then the first stage of the Godement resolution is,

$$\mathscr{F} \to f_* f^* \mathscr{F}$$

Furthermore, since $f^* \mathscr{F}$ is an abelian sheaf on a discrete space it is flasque and f_* preserves flasqueness so $f_* f^* \mathscr{F}$ is flasque. Continuing gives a cosimplicial sheaf $\mathcal{G}^p(\mathscr{F}) = (f_* f^*)^p \mathscr{F}$ on X with coface maps given by the natural transformation $\mathrm{id} \to f_* f^*$ and codegeneracy maps given by contracting between pairs $(f_* f^*)(f_* f^*)$ via the natural transformation $f^* f_* \to \mathrm{id}$. The associated complex is then a flasque resolution of $\mathscr{F}$.

Remark. The above construction also works in the category of $\mathcal{O}_X$-modules on a ringed space by pulling back to $(X_{\mathrm{dis}}, \mathcal{O}_{X_{\mathrm{dis}}})$ where $\mathcal{O}_{X_{\mathrm{dis}}} = f^{-1} \mathcal{O}_X$.

[1] it is finitely presented as an A-algebra because it is finitely presented as an A-module

[2] There is a subtlety there, B is finitely presented *as an A-algebra* not a priori as an A-module. However, B is a finite A-module so by Tag 0564 B is a finitely presented A-module since $A \to B$ is a finitely presented ring map and B is trivially a finitely presented B-module.

Lemma 9.1.1. Let $\mathscr{F}$ be a sheaf on a discrete space X. Then $\mathscr{F}$ is flasque and the canonical map,

$$\mathscr{F} \to \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

is an isomorphism.

Proof. Let $U \subset X$ be open (any set since X is discrete) then since points are open the set of points $x \in U$ forms an open cover. Then by the sheaf property,

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}(x)$$

is an isomorphims. Furthermore, clearly $\mathscr{F}(x) = \mathscr{F}_x$ since x is the inital object in the poset of open neighborhoods of x. Furthermore, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective because for any section $s \in \mathscr{F}(V)$ we may extend to a global section by setting $f_x = s_x$ for $x \in V$ and $f_x = 0$ for $x \notin V$. clearly $f_x = s_x$ on V so by the sheaf property $f|_V = s$. Then restricting $f|_U$ shows that $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective. $\square$

Thus, we can alternatively describe the Godement operation as follows. We can consider,

$$X_{\text{dis}} = \coprod_{x \in X} x$$

Then,

$$f^* X = \prod_{x \in X} \mathscr{F}_x$$

and $f : X_{\text{dis}} \to X$ is the bundled collection of the inclusions $\iota_x : x \to X$ giving,

$$f_* f^* \mathscr{F} = \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

reproducing the result on a discrete space.

9.2 Subsheaves of Godement

Now consider the diagram,

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, dashed] \arrow[d, hook] \\ \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x) \arrow[r] & \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x) \end{tikzcd}$$

We ask when the inclusion $\mathscr{F} \to \mathcal{G}^1(\mathscr{F})$ factors through the canonical map,

$$\bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x) \to \prod_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

and when this sheaf or its image subsheaf is flasque.

First, note that direct sums commute with colimits (because they are colimts themselves) and thus denoting,

$$H(\mathscr{F}) = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

we have the stalks,

$$
\begin{aligned}
H(\mathscr{F})_x = \varinjlim_{x \in U} H(\mathscr{F})(U) &= \bigoplus_{y \in X} \varinjlim_{x \in U} \begin{cases} \mathscr{F}_y & y \in U \\ 0 & y \notin U \end{cases} \\
&= \bigoplus_{y \in X} \begin{cases} \mathscr{F}_y & x \in \overline{\{y\}} \\ 0 & x \notin \overline{\{y\}} \end{cases} \\
&= \bigoplus_{y \rightsquigarrow x} \mathscr{F}_y
\end{aligned}
$$

Therefore, if $\mathscr{F}$ is supported only on closed points of X we have,

$$
H(\mathscr{F})_x = \mathscr{F}_x
$$

However, in general there is not a sheaf map $\mathscr{F} \to H(\mathscr{F})$.

Suppose that $\mathscr{F}$ has finitely supported sections meaning that for any $s \in \mathscr{F}(U)$ its support,

$$
\mathrm{Supp}\,(s) = \{x \in X \mid s_x \neq 0\}
$$

is finite. Then we get an injection,

$$
\mathscr{F} \hookrightarrow \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)
$$

by mapping for each s,

$$
s \in \mathscr{F} \hookrightarrow \prod_{x \in \mathrm{Supp}(s)} (\iota_x)_*(\mathscr{F}_x) = \bigoplus_{x \in \mathrm{Supp}(s)} (\iota_x)_*(\mathscr{F}_x) \subset \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)
$$

Furthermore, notice that if $\mathscr{F}$ is only supported at closed points then,

$$
H(\mathscr{F})_x = \bigoplus_{y \rightsquigarrow x} \mathscr{F}_y = \mathscr{F}_x
$$

since $\mathscr{F}_y = 0$ for any generalization of x. Therefore, in this case the map $\mathscr{F} \to H(\mathscr{F})_x$ defined by virtue of sections having finite support is an isomorphism. Thus if $\mathscr{F}$ is a abelian sheaf whose sections have finite support which is supported on the closed points then,

$$
\mathscr{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)
$$

9.3 The Case for Curves

Let X be a curve (separated integral Noetherian scheme of dimension one) with generic point $\xi \in X$. Then I claim any torsion sheaf $\mathscr{F}$ satisfies,

$$
\mathscr{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)
$$

By the previous discussion, it suffices to show that $\mathscr{F}$ is supported at closed points any every section has finite support. The only nonclosed point is ξ and we assumed that $\mathscr{F}_\xi = 0$. Furthermore, consider $s \in \mathscr{F}(U)$. We know s_ξ so there is some open V such that $\xi \in V \subset U$ on which $s|_U = 0$. Therefore $\mathrm{Supp}\,(\xi) \subset V^C$. I claim that $V^C \subset X$ is finite. Since X is quasi-compact, we can choose an affine open cover $U_i = \text{Spec}\,(A_i)$ and $V^C \cap U_i = V(I_i)$ for some ideal $I_i \subset A_i$. It suffices to show that $V(I_i)$ is finite. Note that $\dim A_i \le 1$ and X is irreducible so $\text{codim}\left(V^C, X\right) \ge 1$ and therefore $\dim V^C = 0$ because,

$$\dim X \ge \text{codim}\left(V^C, X\right) + \dim V^C$$

This shows that $\dim A_i/I_i = 0$ and it is Noetherian so A_i/I_i is Artinian and thus $V(I_i) = \text{Spec}\,(A_i/I_i)$ is finite.

Therefore, each section has finite support so we have demonstrated the equality,

$$\mathscr{F} = \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_x)$$

for any torsion sheaf ($\mathscr{F}_\xi = 0$).

9.4 Resolutions on Curves

Consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \mathscr{K}_X/\mathcal{O}_X \longrightarrow 0$$

Notice that $(\mathscr{K}_X/\mathcal{O}_X)_\xi = K(X)/\mathcal{O}_{X,\xi} = 0$ so $\mathscr{K}_X/\mathcal{O}_X$ is torsion. Therefore, we get a sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \bigoplus_{x \in X} (\iota_x)_*(K(X)/\mathcal{O}_{X,x}) \longrightarrow 0$$

Since X is integral $\mathscr{K}_X$ is constant (since all opens are connected it is truely constant) and thus we get a flasque resolution of $\mathcal{O}_X$. Then the long exact sequence gives,

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow K(X) \longrightarrow \bigoplus_{x \in X} K(X)/\mathcal{O}_{X,x} \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

and $H^i(X, \mathcal{O}_X) = 0$ for $i > 1$. Furthermore, for any flat sheaf $\mathscr{F}$, we can tensor the above exact sequence to get,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X \longrightarrow \bigoplus_{x \in X} (\iota_x)_*(\mathscr{F}_\xi/\mathscr{F}_x) \longrightarrow 0$$

Where $(\iota_x)_*(K(X)) \otimes_{\mathcal{O}_X} \mathscr{F} =$

Lemma 9.4.1. Let X be an irreducible scheme with generic point $\xi \in X$ and $\mathscr{F}$ an abelian sheaf on X. Then the natural map,

$$\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{K}_X \to (\iota_\xi)_*(\mathscr{F}_\xi)$$

is an isomorphism.

Proof. Locally, on affine opens □

10 Appendix

10.1 Curves and Genera

Lemma 10.1.1. Let X be a integral scheme proper over k then $K = H^0(X, \mathcal{O}_X)$ is a finite field extension of k and for any coherent $\mathcal{O}_X$-module $\mathscr{F}$, the cohomology $H^p(X, \mathscr{F})$ is a finite-dimensional $H^0(X, \mathcal{O}_X)$-module.

Proof. Since $\mathcal{O}_X$ is coherent, and X is proper over k so $K = H^0(X, \mathcal{O}_X)$ is a finite k-module. However, since X is integral $H^0(X, \mathcal{O}_X)$ is a domain but a finite k-algebra domain is a field and we see K/k is a finite extension of fields. Furthermore, the $\mathcal{O}_X(X)$-module structure on $H^p(X, \mathscr{F})$ gives it a K-module structure. Since X is proper over k then $H^p(X, \mathscr{F})$ is a finite k-module and thus finite as a K-module. $\square$

Remark. Unfortunately, when k is not algebraically closed then we may not have $H^0(X, \mathcal{O}_X) = k$ even for smooth projective varieties. Therefore, some caution must be taken in defining numerical invariants of the curve such as genus. However, by [**?**, Tag 0BUG], whenever X is proper geometrically integral then indeed $H^0(X, \mathcal{O}_X) = k$. Furthermore, for proper X if $H^0(X, \mathcal{O}_X) \neq k$ then X cannot be geometrically connected by [**?**, Tag 0FD1].

Definition 10.1.2. Let C be a smooth proper curve over k with $H^0(C, \mathcal{O}_C) = K$. Then we define $g(C) := \dim_K H^0(X, \Omega_{C/k})$. If C is any curve over k then there is a unique smooth proper curve S over k which is k-birational to C. Then we define $g(C) := g(S)$.

Remark. By definition, the genus of a curve is clearly a birational invariant since there is a unique smooth complete curve in every birational equivalence class of curves.

Remark. There is a slight subtlety in this definition in the case of a non-perfect base field. It it always true that we can find a proper *regular* curve C in each birational equivalence class however when k is non-perfect the curve C may not be smooth. However, under a finite purely separable extension K/k, we can ensure that C_K admits a smooth proper model. Then we define $g(C) := g(C_K)$ in the case that C_K is a curve. The only thing that can go wrong is when C is not geometrically irreducible since then C_K will not be integral.

Definition 10.1.3. The *arithmetic genus* $g_a(C)$ of a proper curve C over k with $H^0(C, \mathcal{O}_C) = K$ is,

$$g_a(C) := \dim_K H^1(X, \mathcal{O}_C)$$

By Serre duality, if C is smooth then $H^0(C, \Omega_C) = H^1(C, \mathcal{O}_X)^\vee$ meaning that $g_a(C) = g(C)$.

Remark. The arithmetic genus depends on the projective compactification and singularities meaning it will not be a birational invariant unlike the (geometric) genus.

Example 10.1.4. Let $k = \mathbb{F}_p(t)$ for an odd prime $p = 2k + 1$ and consider the curve,

$$C = \mathrm{Spec}\left(k[x, y]/(y^2 - x^p - t)\right)$$

which is regular but not smooth at $P = (y, x^p - t)$. Consider the purely inseperable extension $K = \mathbb{F}(t^{1/p})$. Then $C_K = \mathrm{Spec}\left(K[x, y]/(y^2 - (x - t^{1/p})^p)\right) \cong \mathrm{Spec}\,(K[x, y]/(y^2 - x^p))$. Taking the normalization of C_K gives $\mathbb{A}^1_K \to C_K$ via $t \mapsto (t^p, t^2)$. This is birational since the following ring map is an isomorphism,

$$(K[x, y]/(y^2 - x^p))_x \to K[t]_t$$

sending $x \mapsto t^2$ and $y \mapsto t^p$ which has an inverse $t \mapsto y/x^k$ since $x \mapsto t^2 \mapsto y^2/x^{2k} = x$ and $y \mapsto t^p \mapsto y^p/x^{kp} = y(y^{2k}/x^{pk}) = y$ and $t \mapsto y/x^k \mapsto t^{p-2k} = t$.

Therefore, $C_K \overset{\sim}{\dashrightarrow} \mathbb{P}^1_K$ so $g(C) = g(C_K) = 0$. However, consider the projective closure,

$$\overline{C} = \text{Proj}\left(k[X,Y,Z]/(Y^2Z^{p-2} - X^p - tZ^p)\right)$$

then $\overline{C} \hookrightarrow \mathbb{P}^2_k$ is a Cartier divisor (since $\mathbb{P}^2_k$ is locally factorial) so we find that $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = k$ and $\dim_k H^1(\overline{C}, \mathcal{O}_{\overline{C}}) = \frac{1}{2}(p-1)(p-2) = k(2k-1)$ since its sheaf of ideals is $\mathcal{O}_{\mathbb{P}^2_k}(-p)$. Then $p = 3$ we expect this to be an elliptic curve and we do see $g_a(\overline{C}) = 1$. However, $g(\overline{C}) = 0$ and correspondingly C is not smooth due to the positive characteristic phenomenon.

Lemma 10.1.5. Suppose that $f : X \to Y$ is a finite birational morphism of n-dimensional irreducible Noetherian schemes. Then $H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(X, \mathcal{O}_X)$ is surjective.

Proof. The map f must restrict on some open subset $U \subset X$ to an isomorphism $f|_U : U \to V$. Thus, the sheaf map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ restricts on V to an isomorphism $\mathcal{O}_Y|_V \xrightarrow{\sim} (f_*\mathcal{O}_X)|_V$. We factor this map into two exact sequences,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_Y \longrightarrow \mathscr{I} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{I} \longrightarrow f_*\mathcal{O}_X \longrightarrow \mathscr{C} \longrightarrow 0$$

with $\mathscr{K} = \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{C} = \text{coker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ and $\mathscr{I} = \text{Im}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. Taking cohomology and using that it vanishes in degree above n we get,

$$H^{n-1}(Y, \mathscr{I}) \longrightarrow H^n(Y, \mathscr{K}) \longrightarrow H^n(Y, \mathcal{O}_Y) \longrightarrow\!\!\!\!\!\rightarrow H^n(Y, \mathscr{I}) \longrightarrow 0$$

$$H^{n-1}(Y, \mathscr{C}) \longrightarrow H^n(Y, \mathscr{I}) \longrightarrow H^n(X, \mathcal{O}_X) \longrightarrow\!\!\!\!\!\rightarrow H^n(X, \mathscr{C}) \longrightarrow 0$$

where we have used that $f : X \to Y$ is affine to conclude that $H^p(Y, f_*\mathscr{F}) = H^p(Y, \mathscr{F})$ for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$. Furthermore, $\mathscr{C}|_V = 0$ so $\text{Supp}_{\mathcal{O}_Y}(\mathscr{C}) \subset X \setminus V$ but $\mathscr{C}$ is coherent so the support is closed. Since V is dense open, $\mathscr{C}$ is supported in positive codimension so $H^n(Y, \mathscr{C}) = 0$ (since $H^n(S, \mathscr{C})$ vanishes due to dimension on the closed subscheme $S = \text{Supp}_{\mathcal{O}_X}(\mathscr{C})$ on which $\mathscr{C}$ is supported). Thus we have,

$$H^n(Y, \mathcal{O}_Y) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(Y, \mathscr{I}) \twoheadrightarrow H^n(X, \mathcal{O}_X)$$

proving the proposition. □

Corollary 10.1.6. Let S and C be proper curves over k where S is smooth which are birationally equivalent and $H^0(S, \mathcal{O}_S) \cong H^0(C, \mathcal{O}_C)$. Then the genera satisfy,

(a) $g_a(C) \geq g_a(S)$

(b) $g(C) = g(S)$

(c) $g(C) \leq g_a(C)$ with equality if and only if C is smooth.

Proof. Given a birational map $S \overset{\sim}{\dashrightarrow} C$ we can extend it to a birational morphism $S \to C$ since S is regular. The morphism $S \to C$ is automatically finite since it is a non-constant map of proper curves. Then the previous lemma implies that $g_a(S) \le g_a(C)$. (b). follows from the definition of $g(C)$. The third follows from the fact that $g(S) = g_a(S)$ because of Serre duality,

$$H^1(S, \mathcal{O}_S) \cong H^0(S, \Omega_{S/k})^\vee$$

using that S is smooth. Then we see that $g(C) = g(S) = g_a(S) \le g_a(C)$ proving the inequality part of (c). Finally, if C is smooth we see by Serre duality that $g(C) = g_a(C)$. Conversely, suppose that $g(C) = g_a(C)$ then $g_a(C) = g(C) = g(S) = g_a(S)$ and consider the map $f : S \to C$ which is finite birational map of integral schemes over k. In particular, f is affine so for each $y \in C$ we may choose an affine open $y \in V \subset C$ whose preimage $U = f^{-1}(V)$ is also affine. On sheaves, this gives a map of domains $\mathcal{O}_C(V) \to \mathcal{O}_S(U)$ which localizes to an isomorphism on the fraction fields. However, the localization map of a domain is injective so $\mathcal{O}_C(V) \hookrightarrow \mathcal{O}_S(U)$ is an injection. This shows that $\mathcal{O}_C \to f_*\mathcal{O}_S$ is an injection of sheaves which we extend to an exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_*\mathcal{O}_S \longrightarrow \mathscr{C} \longrightarrow 0$$

Note that $f : S \to C$ induces an isomorphism $H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$ since it is a map of fields with the same (finite) dimension over k. Then the long exact sequence of cohomology gives,

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(S, \mathcal{O}_S) \longrightarrow H^0(X, \mathscr{C}) \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, \mathscr{C}) = 0$$

I claim that $H^1(S, \mathscr{C}) = 0$. Since f is birational, $\mathscr{C}$ is supported in codimension one. Thus, the map $H^1(C, \mathcal{O}_C) \twoheadrightarrow H^1(S, \mathcal{O}_S)$ is surjective but $g_a(C) = g_a(S)$ so these vectorspaces have the same dimension so $H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1(S, \mathcal{O}_S)$ is an isomorphism. Thus, from the exact sequence we have $H^0(X, \mathscr{C}) = 0$. However, $\mathrm{Supp}_{\mathcal{O}_C}(\mathscr{C})$ is a closed ($\mathscr{C}$ is coherent) dimension zero subset i.e. finitely many discrete closed points. However, a sheaf supported on a discrete set of points is zero iff it has no global sections. Therefore, $\mathscr{C} = 0$ so $\mathcal{O}_C \xrightarrow{\sim} f_*\mathcal{O}_S$. In particular $\mathcal{O}_C(V) \xrightarrow{\sim} \mathcal{O}_S(U)$ is an isomorphism which implies that the map of affine schemes $f|_U : U \to V$ is an isomorphism. Since the affine opens V cover C we see that $f : S \to C$ is an isomorphism. In particular, C is smooth. ☐

10.2 The Locus on Which Morphisms Agree

Lemma 10.2.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then for schemes X there is a natural bijection,

$$\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(R), X) \cong \{x \in X \text{ and local map } \mathcal{O}_{X,x} \to R\}$$

Proof. Given $\mathrm{Spec}(R) \to X$ we automatically get $\mathfrak{m} \mapsto x$ and $\mathcal{O}_{X,x} \to R_\mathfrak{m} = R$. Now, note that taking any affine open neighborhood $x \in \mathrm{Spec}(A) \subset X$ and then $A \to A_\mathfrak{p} = \mathcal{O}_{X,x}$ to give $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(A) \to X$. Clearly, this map sends $\mathfrak{m}_x \mapsto x$ and at $\mathfrak{m}_x$ has stalk map $\mathrm{id} : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ since it is the localization at $\mathfrak{p}$ of $A \to A_\mathfrak{p}$.

Thus we get an inverse as follows. Given a point $x \in X$ and a local map $\phi : \mathcal{O}_{X,x} \to R$ then take,

$$\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$$

This is inverse since $\mathfrak{m} \mapsto \mathfrak{m}_x$ (because $\mathcal{O}_{X,x} \to \mathfrak{m}_x$ is local) and $\mathfrak{m}_x \mapsto x$ and the stalk at $\mathfrak{m}$ gives $\mathcal{O}_{X,x} \xrightarrow{\mathrm{id}} \mathcal{O}_{X,x} \xrightarrow{\phi} R$.

Finally, I claim that any $f : \mathrm{Spec}\,(R) \to X$ factors through $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}) \to X$ and thus is reconstructed from $x \in X$ and $\mathcal{O}_{X,x} \to R$. Choose an affine open neighborhood $x \in \mathrm{Spec}\,(A) \subset X$ then consider $f^{-1}(\mathrm{Spec}\,(A))$ which is open in $\mathrm{Spec}\,(R)$ and contains the unique closed point $\mathfrak{m} \in \mathrm{Spec}\,(R)$ so there is some $f \in R$ s.t. $\mathfrak{m} \in D(f) \subset f^{-1}(\mathrm{Spec}\,(A))$ so $f \notin \mathfrak{m}$ so $f \in R^\times$ and thus $D(f) = \mathrm{Spec}\,(R)$. Therefore, we get a map $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(A)$ and thus $\phi : A \to R$ where $\phi^{-1}(\mathfrak{m}) = \mathfrak{p} = x$ so $A \setminus \mathfrak{p}$ is mapped inside $R^\times$ so this map factors through $A \to A_\mathfrak{p} \to R$ giving the desired factorization $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}) \to \mathrm{Spec}\,(A) \to X$. □

Definition 10.2.2. The locus Z on which two maps $f, g : X \to Y$ over S agree is given as the pullback,

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

with $F = (f, g)$. This is the equalizer of $f, g : X \to Y$. Furthermore $Z \to X$ is an immersion since it is the base change of $\Delta_{Y/S}$ which is an immersion.

Lemma 10.2.3. Topologically, the locus on which S-morphisms $f, g : X \to Y$ agree is,

$$Z = \{x \in X \mid f(x) = g(x) \text{ and } f_x = g_x : \kappa(f(x)) \to \kappa(x)\}$$

Proof. On some S-subscheme $G \subset X$, the maps $f|_G = g|_G$ agree iff there exists $G \to Y$ such that,

$$\begin{tikzcd} G \arrow[r, dashed] \arrow[d, hook] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

commutes. In particular, for any point $x \in X$ consider $\iota : \mathrm{Spec}\,(\kappa(x)) \to X$ then $f \circ \iota = g \circ \iota$ iff $f(x) = g(x)$ and $f_x = g_x : \kappa(f(x)) \to \kappa(x)$. Consider a point $z \in Z$ and $\mathrm{Spec}\,(\kappa(z)) \to Z$, such a point is equivalent to giving a diagram,

$$\begin{tikzcd} \mathrm{Spec}\,(\kappa(z)) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\ & X \arrow[r, "F"] & Y \times_S Y \end{tikzcd}$$

However, $\iota : Z \to X$ is an immersion so $\iota_x : \kappa(\iota(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism. Therefore, points $\mathrm{Spec}\,(\kappa(z)) \to Z$, are exactly points of X for which a lift $\mathrm{Spec}\,(\kappa(x)) \to Y$ exists i.e. points such that f and g agree in the required way. □

Lemma 10.2.4. If $f : X \to Y$ is an immersion then $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective for each $x \in X$ and $f_x : \kappa(f(x)) \xrightarrow{\sim} \kappa(x)$ is an isomorphism.

Proof. For closed immersions, $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective by definition. Thus we get a surjection $f_x : \mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_{f(x)}$. Furthermore, topologically, $f : X \to Y$ is a homomorphism onto its image so for any open $U \subset X$ there exists an open $V \subset Y$ s.t. $U = f^{-1}(V)$ showing that,

$$(f_* \mathcal{O}_X)_{f(x)} = \varinjlim_{f(x) \in V} \mathcal{O}_X(f^{-1}(V)) = \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}$$

Furthermore, for an open immersion, $f^\flat : f^{-1}\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism so $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. Thus the composition, $f_x : \mathcal{O}_{Y,f(x)} \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective. Furthermore, f_x is local we get $f_x : \kappa(f(x)) \twoheadrightarrow \kappa(x)$ which is a surjection of fields and thus an isomorphism. $\square$

Lemma 10.2.5. If $Y \to S$ is separated then the locus on which $f, g : X \to Y$ over S agree is closed.

Proof. Since $X \to S$ is separated, $\Delta_{Y/S} : Y \to Y \times_S Y$ is a closed immersion. So $Z \to X$ is the base change of a closed immersion and thus a closed immersion. $\square$

Lemma 10.2.6. Let X be a reduced and Y be a separated scheme over S and $f, g : X \to Y$ be morphism over S. If $f \circ j = g \circ j$ agree on a dense subscheme $j : G \hookrightarrow X$ then $f = g$.

Proof. Consider $F = (f, g) : X \to Y \times_S Y$. Since $\Delta : Y \to Y \times_S Y$ is a closed immersion (by separateness). Then $F^{-1}(\Delta)$ is the locus on which $f = g$ which is closed because $\Delta : Y \to Y \times_S Y$ is a closed immersion. Since $f|_G = g|_G$ we get a diagram,

$$\begin{tikzcd}
G \arrow[drr, bend left] \arrow[dr, dashed] \arrow[ddr, hook, bend right] & & \\
& Z \arrow[r, "\tilde{F}"] \arrow[d, hook, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\Delta_Y"] \\
& X \arrow[r, "F"] & Y \times_S Y
\end{tikzcd}$$

Since $\iota : Z \hookrightarrow X$ is a closed immersion with dense image, $Z \hookrightarrow X$ is surjective. By the following, $\iota : Z \to X$ is an isomorphism. Thus, $F = F \circ \iota \circ \iota^{-1} = \Delta_Y \circ \tilde{F} \circ \iota^{-1}$. By the universal property of maps $X \to Y \times_S Y$ this implies that $f = g = \tilde{F} \circ \iota^{-1}$. $\square$

Lemma 10.2.7. Let X be a scheme and consider an exact sequence of quasi-coherent $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A} \longrightarrow 0$$

and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebra. Suppose that $\mathscr{F}_x \neq 0$ for each $x \in X$. Then $\mathscr{I} \hookrightarrow \mathcal{N}$ where $\mathcal{N}$ is the sheaf of nilpotent.

Proof. Take an affine open $U = \mathrm{Spec}\,(R) \subset X$ such that $\mathcal{A}|_U = \widetilde{A}$. Then we have an surjection of rings $R \twoheadrightarrow A$ giving $R/I = A$ for $I = \ker(R \to A)$. Now, for each $\mathfrak{p} \in \mathrm{Spec}\,(R)$ we know $R_\mathfrak{p} = \mathcal{O}_{X,\mathfrak{p}} \neq 0$. However, if $\mathfrak{p} \not\supset I$ then $(R/I)_\mathfrak{p} = A_\mathfrak{p} = 0$ so we must have $\mathfrak{p} \supset I$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ i.e. $I \subset \mathrm{nilrad}\,(R)$. Therefore, $\mathscr{I}|_U \hookrightarrow \mathcal{N}|_U$ for any affine open $U \subset X$ showing that $\mathscr{I}$ is comprised of nilpotents. $\square$

Corollary 10.2.8. If X is reduced and $\iota : Z \hookrightarrow X$ is a surjective closed immersion then $\iota : Z \xrightarrow{\sim} X$ is an isomorphism.

Proof. Since $\iota : Z \hookrightarrow X$ is a homeomorphism onto its image X it suffices to show that the map of sheaves $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. Since $\iota : Z \to X$ is a closed immersion $\iota^\# : \mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Z$ is a surjection and $\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras giving an exact sequence,

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

Furthermore,

$$\mathrm{Supp}_{\mathcal{O}_X}(\iota_* \mathcal{O}_Z) = \mathrm{Im}(\iota) = X$$

since $(\iota_* \mathcal{O}_Z)_x = \mathcal{O}_{Z,x}$ when $x \in \mathrm{Im}(\iota)$ (and zero elsewhere). by the above, $\mathscr{I} \hookrightarrow \mathcal{N} = 0$ since X is reduced to $\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is an isomorphism. ☐

Lemma 10.2.9. A rational S-map $f : X \dashrightarrow Y$ with X reduced and $Y \to S$ separated is equivalent to a morphism $f : \mathrm{Dom}(f) \to Y$.

Proof. For any (U, f_U) and (V, f_V) representing f there must be a dense (in X) open $W \subset U \cap V$ on which $f_U|_W = f_V|_W$ and thus $f_U|_{U \cap V} = f_V|_{U \cap V}$ since $f_U, f_V : U \cap V \to Y$ are morphisms from reduced to irreducible schemes. Now $\mathrm{Dom}(f)$ has an open cover (U_i, f_i) for which $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ so these morphisms glue to give $f : \mathrm{Dom}(f) \to Y$ ($\mathrm{Hom}_S(-, Y)$ is a sheaf on the Zariski site). ☐

10.3 Extending Rational Maps

Lemma 10.3.1. Regular local rings of dimension 1 exactly correspond to DVRs.

Proof. Any DVR R has a uniformizer $\varpi \in R$ then $\dim R = 1$ and $\mathfrak{m}/\mathfrak{m}^2 = (\varpi)/(\varpi^2) = \varpi \kappa$ which also has $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = 1$ so R is regular. Conversely, if R is a regular local ring of dimension $\dim R = 1$ then, by regularity, R is a normal Noetherian domain so by $\dim R = 1$ then R is Dedekind but also local and thus is a DVR. ☐

Proposition 10.3.2. Let X be a Noetherian S-scheme and $Z \subset X$ a closed irreducible codimension 1 generically nonsingular subset (with generic point $\eta \in Z$ such that $\mathcal{O}_{X,\eta}$ is regular). Let $f : X \dashrightarrow Y$ be a rational map with Y proper over S. Then $Z \cap \mathrm{Dom}(f)$ is a dense open of Z.

Proof. Choose some representative (U, f_U) for $f : X \dashrightarrow Y$. Note that $\mathcal{O}_{X,\eta}$ is a regular dimension one (see Lemma 10.4.3) ring and thus a DVR. Consider the generic point $\xi \in X$ of X then, by localizing, we get an inclusion of the generic point $\mathrm{Spec}(\mathcal{O}_{X,\xi}) \to \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ and $\mathcal{O}_{X,\xi} = K(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta})$. Furthermore, the inclusion of the generic point gives $\mathrm{Spec}(K(X)) \to U \xrightarrow{f_U} Y$ and thus we get a diagram,

$$\begin{tikzcd} \mathrm{Spec}(K(X)) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{X,\eta}) \arrow[r] \arrow[ur, dashed, "\ell"] & \mathrm{Spec}(k) \end{tikzcd}$$

and a lift $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to Y$ by the valuative criterion for properness applied to $Y \to \mathrm{Spec}(k)$ since $\mathcal{O}_{X,\eta}$ is a DVR. Choose an affine open $\mathrm{Spec}(R) \subset Y$ containing the image of $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to Y$ (i.e. choose a neighborhood of the image of η which automatically contains $f(\xi)$ since the map factors $\mathrm{Spec}(\mathcal{O}_{X,\eta}) \to \mathrm{Spec}\left(\mathcal{O}_{Y,f(\eta)}\right) \to \mathrm{Spec}(R) \to Y$) and let $\eta \in V = \mathrm{Spec}(A) \subset X$ be an affine open neighborhood of ξ mapping into $\mathrm{Spec}(R)$. By Lemma 10.4.7, since $\mathcal{O}_{X,\eta}$ is a domain, we may shrink V so that A is a domain. Since X is irreducible $U \cap V$ is a dense open. Note that if $\eta \in U$ then $\eta \in \mathrm{Dom}(f)$ and thus $Z \cap \mathrm{Dom}(f)$ is a nonempty open of the irreducible space Z

and therefore a dense open so we are done. Otherwise, let $\mathfrak{p} \in \mathrm{Spec}\,(A)$ correspond to $\eta \in Z$ then $A_\mathfrak{p} = \mathcal{O}_{X,\eta}$ is a DVR. Take some principal affine open $D(f) \subset U \cap V$ for $f \in A$ so $f \in \mathfrak{p}$ since $\mathfrak{p} \notin D(f) \subset U \cap V$. Since $A_\mathfrak{p}$ is a DVR we may choose a uniformizer $\varpi \in \mathfrak{p}$ so the map $A \to \mathfrak{p}$ via $1 \mapsto \varpi$ is as isomorphism when localized at $\mathfrak{p}$. Since A is Noetherian both are f.g. A-modules so there must be some $s \in A \setminus \mathfrak{p}$ such that $A_s \to \mathfrak{p}_s$ is an isomorphism. Replacing A by A_s we may assume $\mathfrak{p} = (\varpi) \subset A$ is principal. Since $f \in \mathfrak{p}$ we can write $f = t\varpi^k$ for some $a \in A \setminus \mathfrak{p}$ (see Lemma 10.4.1). Then consider $\tilde{V} = \mathrm{Spec}\,(A_t)$. Since $t \notin \mathfrak{p}$ then $\eta \in \tilde{V}$ and since $f = t\varpi^k$ we have $D(f) \subset D(t) = \tilde{V}$. Now we get the following diagram,

$$\begin{tikzcd}
& & \mathrm{Spec}\,(R) \\
\mathrm{Spec}\,(A_\mathfrak{p}) \arrow[r] \arrow[urr, bend left, "\ell"] & \mathrm{Spec}\,(A_t) \arrow[ur, dashed, "f_V"] & \\
\mathrm{Spec}\,(\mathrm{Frac}\,(A)) \arrow[r] \arrow[u] & \mathrm{Spec}\,(A_f) \arrow[u] \arrow[uur, bend right, "f_U"'] &
\end{tikzcd}$$

I claim the square is a pushout in the category of affine schemes because maps $R \to A_\mathfrak{p}$ and $R \to A_f$ which agree under the inclusion to $\mathrm{Frac}\,(A)$ gives a map $R \to A_\mathfrak{p} \cap A_f \subset \mathrm{Frac}\,(A)$. However, consider,

$$x \in A_\mathfrak{p} \cap A_t \implies x = \frac{u\varpi^r}{s} = \frac{a}{f^n}$$

for $u, s, t \in A \setminus \mathfrak{p}$ and $a \in A$. Thus we get,

$$ut^n\varpi^{r+nk} = sa$$

so $a \in \mathfrak{p}^{r+nk} \setminus \mathfrak{p}^{r+nk+1}$ ($s \notin \mathfrak{p}$ which is prime) and thus $a = u'\varpi^{r+nk}$ for $u' \in A \setminus \mathfrak{p}$. Therefore,

$$x = \frac{u'\varpi^{r+nk}}{t^n\varpi^{nk}} = \frac{u'\varpi^r}{t^n} \in A_t$$

Thus, $A_\mathfrak{p} \cap A_f \subset A_t$ so we get a map $R \to A_t$. Therefore we get a map $f_{\tilde{V}} : \tilde{V} \to Y$ such that $(f|_{\tilde{V}})|_{D(f)} = (f_U)|_{D(f)}$ showing that $\eta \in \tilde{V} \subset \mathrm{Dom}\,(f)$ so $Z \cap \mathrm{Dom}\,(f)$ is a dense open of Z. ☐

Proposition 10.3.3. Let $C \to S$ be a proper regular Noetherian scheme with $\dim C = 1$ and $f : C \dashrightarrow Y$ a rational S-map with $Y \to S$ proper. Then f extends uniquely to a morphism $f : C \to Y$.

Proof. For any point $x \notin \mathrm{Dom}\,(f)$ let $Z = \overline{\{x\}} \subset D$ for $D = C \setminus \mathrm{Dom}\,(f)$. Since $\mathrm{Dom}\,(f)$ is a dense open, by lemma 10.4.2, we have $\mathrm{codim}\,(Z, C) \geq \mathrm{codim}\,(D, C) \geq 1$ but $\dim C = 1$ so $\mathrm{codim}\,(Z, C) = 1$. Furthermore, since C is regular $\mathcal{O}_{C,x}$ is regular and thus, by the previous proposition, $Z \cap \mathrm{Dom}\,(f)$ is a dense open and in particular $x \in \mathrm{Dom}\,(f)$ meaning that $\mathrm{Dom}\,(f) = C$ so we get a morphism $C \to Y$. This is unique because C is reduced (it is regular) and Y is separated (it is proper over S) so morphisms $C \to Y$ are uniquely determined on a dense open which any representative for $f : C \dashrightarrow Y$ is defined on. ☐

Corollary 10.3.4. Rational maps between normal proper curves are morphisms.

Corollary 10.3.5. Birational maps between normal proper curves are isomorphisms.

Proof. Let $f : C_1 \dashrightarrow C_2$ and $g : C_2 \dashrightarrow C_1$ be birational inverses of smooth proper curves. Then we know that these extend to morphisms $f : C_1 \to C_2$ and $g : C_2 \to C_1$. Furthermore, the maps $g \circ f : C_1 \to C_1$ must extend the identity on some dense open. However, since curves are separated and reduced there is a unique extension of this map so $g \circ f = \mathrm{id}_{C_1}$ and likewise $f \circ g = \mathrm{id}_{C_2}$. □

Theorem 10.3.6. If k is perfect then there exists a unique normal curve in each birational equivalence class of curves.

Proof. It suffices to show existence. Given a curve X, we consider the projective closure $X \hookrightarrow \overline{X}$ which is birational and $\overline{X} \to \mathrm{Spec}\,(k)$ is proper. Then take the normalization $\overline{X}^{\nu} \to \overline{X}$ which remains proper over $\mathrm{Spec}\,(k)$ and is birational. Then $\overline{X}^{\nu}$ is regular and thus smooth over k since k is perfect and $\overline{X}^{\nu} \to X$ is birational. □

10.4 Lemmas

Lemma 10.4.1. Let A be a Noetherian domain and $\mathfrak{p} = (\varpi)$ a principal prime. Then any $f \in \mathfrak{p}$ can be written as $f = t\varpi^k$ for $f \in A \setminus \mathfrak{p}$.

Proof. From Krull intersection,

$$\bigcap_{n \ge 0}^{\infty} \mathfrak{p}^n = (0)$$

so there is some n such that $f \in \mathfrak{p}^n \setminus \mathfrak{p}^{n+1}$. Thus $f = t\varpi^n$ for some $f \in A$ but if $t \in \mathfrak{p}$ then $f \in \mathfrak{p}^{n+1}$ so the result follows. □

Lemma 10.4.2. Consider a closed subset $Y \subset X$ and an open $U \subset X$ with $U \cap Z \neq \varnothing$. Then $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Y \cap U, U)$.

Proof. Consider a chain of irreducible $Z_i \supsetneq Z_{i+1}$ with $Z_0 \subset Y$. I claim that $Z_i \mapsto Z_i \cap U$ and $Z_i \mapsto \overline{Z_i}$ are inverse functions giving a bijection between closed irreducible chains in X with final terms contained in Y and closed irreducible chains in U with final term contained in $Y \cap U$. Note, if $Z_i \subset Y \cap U$ then $\overline{Z_i} \subset Y$ since Y is closed in X.

First, $\overline{Z_i \cap U} \subset Z_i$ and is closed in X. Then $\overline{Z_i \cap U} \cup U^C \supset Z_i$ so because Z_i is irreducible $\overline{Z_i \cap U} = Z_i$ since by assumption $Z_i \not\subset U^C$. Conversely, if $Z_i \subset U$ is a closed irreducible subset then $\overline{Z_i}$ is closed and irreducible in X and $Z_i \subset \overline{Z_i} \cap U$ but $Z_i = C \cap U$ for closed $C \subset X$ so $Z_i \subset C$ and thus $\overline{Z_i} \subset C$ so $\overline{Z_i} \cap U \subset C \cap U = Z_i$ meaning $Z_i = \overline{Z_i} \cap U$. Thus we have shown these operations are inverse to each other.

Finally, if $Z_i \cap U - Z_{i+1} \cap U$ then $\overline{Z_i \cap U} = \overline{Z_i \cap U}$ so $Z_i = Z_{i+1}$ so the chain does not degenerate. Likewise, if $\overline{Z_i} = \overline{Z_{i+1}}$ then $\overline{Z_i} \cap U = \overline{Z_{i+1}} \cap U$ so $Z_i = Z_{i+1}$. Therefore, we get a length-preserving bijection between the chains defining $\mathrm{codim}\,(Y, X)$ and $\mathrm{codim}\,(Y \cap U, U)$. □

Lemma 10.4.3. Let $Z \subset X$ be a closed irreducible subset with generic point $\eta \in Z$. Then $\mathrm{codim}\,(Z, X) = \dim \mathcal{O}_{X,\eta}$.

Proof. Take affine open neighborhood $\eta \in U = \mathrm{Spec}\,(A) \subset X$. Then for $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponding to η we get $A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$. However, $\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U)$ and $Z \cap U = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Therefore,

$$\mathrm{codim}\,(Z, X) = \mathrm{codim}\,(Z \cap U, U) = \mathbf{ht}\,(\mathfrak{p}) = \dim A_{\mathfrak{p}} = \dim \mathcal{O}_{X,\eta}$$

□

Lemma 10.4.4. Let X be a Noetherian scheme then the nonreduced locus,

$$Z = \{x \in X \mid \text{nilrad}(\mathcal{O}_{X,x}) \neq 0\}$$

is closed.

Proof. The subsheaf $\mathcal{N} \subset \mathcal{O}_X$ is coherent since X is Noetherian. Thus $Z = \text{Supp}_{\mathcal{O}_X}(\mathcal{N})$ is closed and $\mathcal{N}_x = \text{nilrad}(\mathcal{O}_X x)$. Locally, on $U = \text{Spec}(A)$ we have $\mathcal{N}|_U = \widetilde{\text{nilrad}(A)}$ and $\text{nilrad}(A)$ is a f.g. A-module since A is Noetherian so,

$$\text{Supp}_{\mathcal{O}_X}(\mathcal{N}) \cap U = \text{Supp}_A(\text{nilrad}(A)) = V(\text{Ann}_A(\text{nilrad}(A))$$

is closed in $\text{Spec}(A)$. $\square$

Lemma 10.4.5. Let X be a Noetherian scheme then X has finitely many irreducible components.

Proof. First let $X = \text{Spec}(A)$ for a Noetherian ring A. Then the irreducible components of A correspond to minimal primes $\mathfrak{p} \in \text{Spec}(A)$. Then $\dim A_{\mathfrak{p}} = 0$ and $A_{\mathfrak{p}}$ is Noetherian so $A_{\mathfrak{p}}$ is Artinian. $A_{\mathfrak{p}}$ must have some associated prime so $\text{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}\}$. By [**?**, Tag 05BZ], then $\text{Ass}_A(A) \cap \text{Spec}(\mathbb{A}_{\mathfrak{p}}) = \text{Ass}_{\mathbb{A}_{\mathfrak{p}}}(\mathbb{A}_{\mathfrak{p}}) = \{\mathfrak{p}\}$ so every minimal prime is an associated prime. However, for A Noetherian then A admits a finite composition series so there are finitely many associated primes.

Now let X be a Noetherian scheme. For any affine open $U \subset X$ we have shown that U has finitely many irreducible components. However, since X is quasi-compact there is a finite cover of affine opens and thus X must have finitely many irreducible components. $\square$

Lemma 10.4.6. Let X be a Noetherian scheme and Y is the complement of some dense open U. Then $\text{codim}(Y, X) \geq 1$.

Proof. It suffices to show that Y does not contain any irreducible component since then any irreducible contained in Y cannot be maximal. Since X is Noetherian, it has finitely many irreducible components Z_i. Then if $Z_j \subset Y$ for some i we would have $Z_i \cap U = \varnothing$ but then,

$$U = \bigcup_{i \neq j} Z_i$$

which is closed so $\overline{U} \subsetneq X$ contradicting our assumption that U is dense. $\square$

Lemma 10.4.7. Let X be a Noetherian scheme and $x \in X$ such that $\mathcal{O}_{X,x}$ is a domain. Then there is an affine open neighborhood $x \in U \subset X$ with $U = \text{Spec}(A)$ and A is a domain.

Proof. Take any affine open neighborhood $x \in U \subset X$ with $U = \text{Spec}(A)$ and $\mathfrak{p} \in \text{Spec}(A)$ corresponding to x. Then $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is a domain. Since X is Noetherian then A is Noetherian so it has finitely many minimal primes $\mathfrak{p}_i$ (corresponding to the generic points of irreducible components of U) with $\mathfrak{p}_0 \subset \mathfrak{p}$. Since $A_{\mathfrak{p}}$ is a domain, it has a unique minimal prime and thus $\mathfrak{p}_0$ is the only minimal prime contained in $\mathfrak{p}$ (geometrically $A_{\mathfrak{p}}$ being a domain corresponds to the fact that $\mathfrak{p}$ is the generic point of a generically reduced irreducible subset which lies in only one irreducible component)

Now for any $i \neq 0$ take $f_i \in \mathfrak{p} \setminus \mathfrak{p}_0$. This is always possible else $\mathfrak{p} \subset \mathfrak{p}_0$ contradicting the minimality of $\mathfrak{p}_0$. If $f \notin \mathfrak{q}$ then $\mathfrak{q} \not\supset \mathfrak{p}_i$ for any $i \neq 0$ so $\mathfrak{q} \supset \mathfrak{p}_0$ since it must lie above some minimal prime. Thus nilrad $(A_f) = \mathfrak{p}_0 A_f$ is prime and $f \notin \mathfrak{p}$ since else $\mathfrak{p} \supset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ which is impossible since $\mathfrak{p} \not\supset \mathfrak{p}_i$ for any i. Now we know that nilrad $(A_\mathfrak{p}) = 0$ and A_f is Noetherian so nilrad $(A_\mathfrak{p})$ is finitely generated. Thus, there is some $g \notin \mathfrak{p}$ such that nilrad $(A_{fg}) = (\text{nilrad}\,(A_f))_g = 0$. Thus A_{fg} is a domain since nilrad $(A_{fg}) = (0)$ and is prime and $\mathfrak{p} \in A_{fg}$ because $fg \notin \mathfrak{p}$. Therefore, $x \in \text{Spec}\,(A_{fg}) \subset U$ is an affine open satisfying the requirements. ☐

Remark. This does not imply that X is integral if $\mathcal{O}_{X,x}$ is a domain for each $x \in X$ (which is false, consider Spec $(k \times k)$) because it only shows there is an integral cover of X not that $\mathcal{O}_X(U)$ is a domain for each U.

Example 10.4.8. Let $X = \text{Spec}\,(k[x,y]/(xy,y^2))$. Then for the bad point $\mathfrak{p} = (x,y)$ we have nilrad $(\mathcal{O}_{X,\mathfrak{p}}) = (y)$. Away from the bad point, say $\mathfrak{p} = (x-1,y)$ we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left(k[x]_{(x-1)}\right)$ so nilrad $(\mathcal{O}_{X,\mathfrak{p}}) = (0)$. Furthermore, at the generic point $\mathfrak{p} = (y)$, we have, $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\,(k(x))$ so nilrad $(\mathcal{O}_{X,\mathfrak{p}}) = (0)$.

Example 10.4.9. Consider $X = \text{Spec}\,(k[x,y,z]/(yz))$ which is the union of the x-y and x-z planes. Consider the generic point of the z-axis $\mathfrak{p} = (x,y)$ then $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left(k[x,z]_{(x)}\right)$ is a domain since the z-axis only lies in one irreducible component. However, at the generic point of the x-axis, $\mathfrak{p} = (y,z)$ we get $\mathcal{O}_{X,\mathfrak{p}} = \text{Spec}\left((k[x,y,z]/(yz))_{(y,z)}\right)$ has zero divisors $yz = 0$ so is not a domain since the x-axis lives in two irreducible components.

10.5 Reflexive Sheaves (WIP)

Definition 10.5.1. Recall the dual of a $\mathcal{O}_X$ module $\mathscr{F}$ is the sheaf $\mathscr{F}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. We say that a coherent $\mathcal{O}_X$-module $\mathscr{F}$ is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism.

Lemma 10.5.2. Let X be an integral locally Noetherian scheme and $\mathscr{F}, \mathscr{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathscr{G}$ is reflexive then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is reflexive.

Proof. See [**?**, Tag 0AY4]. ☐

In particular, since $\mathcal{O}_X$ is clearly reflexive, this lemma shows that for any coherent $\mathcal{O}_X$-module then $\mathscr{F}^\vee$ is a reflexive coherent sheaf. We say the map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ gives the reflexive hull $\mathscr{F}^{\vee\vee}$ of $\mathscr{F}$.

Definition 10.5.3. Let $\mathcal{R}$ be the full subcategory $\mathfrak{Coh}\,(\mathcal{O}_X)$ of coherent reflexive $\mathcal{O}_X$-modules. $\mathcal{R}$ is an additive category and in fact has all kernels and cokernels defined by taking reflexive hulls of the sheaf kernel and cokernel. Furthermore, $\mathcal{R}$ inherits a monoidal structure from the tensor product defined using the reflexive hull as follows,

$$\mathscr{F} \otimes_\mathcal{R} \mathscr{G} = (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$$

Finally, we define RPic (X) to be group of constant rank one reflexives induced by the monoidal structure on $\mathcal{R}$. Explicitly, RPic (X) is the group of isomorphism classes of constant rank one reflexive coherent $\mathcal{O}_X$-modules with multiplication $(\mathscr{F}, \mathscr{G}) \mapsto (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})^{\vee\vee}$ and inverse $\mathscr{F} \mapsto \mathscr{F}^\vee$.

The importance of reflexive sheaves derives from their correspondence to Weil divisors. Here we let X be a normal integral separated Noetherian scheme.

Proposition 10.5.4. If D is a Weil divisor then $\mathcal{O}_X(D)$ is reflexive of constant rank one.

Proof. (CITE OR DO). □

Theorem 10.5.5. Let X be a normal integral separated Noetherian scheme. There is an isomorphism of groups $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{RPic}(X)$ defined by $D \mapsto \mathcal{O}_X(D)$.

Proof. (DO OR CITE) □

We summarize the important results as follows.

Theorem 10.5.6. Let X be a Noetherian normal integral scheme. Then for any Weil divisors D, E,

(a) $\mathcal{O}_X(D+E) = (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee}$

(b) $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{\vee}$

(c) $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(E)) = \mathcal{O}_X(E-D)$

(d) if E is Cartier then $\mathcal{O}_X(D+E) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$

Proof. (DO OR CITE) □

Finally, we have a result which controls when the dualizing sheaf can be expressed in terms of a divisor.

Proposition 10.5.7. Let X be a projective variety over k. Then,

(a) if X is normal then its dualizing sheaf ω_X is reflexive of rank 1 and thus X admits a canonical divisor K_X s.t. $\omega_X = \mathcal{O}_X(K_X)$

(b) if X is Gorenstein then ω_X is an invertible module so K_X is Cartier.

Proof. (FIND CITATION OR DO). □