Math GR6262 Algebraic Geometry Assignment # 9

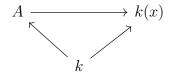
Benjamin Church

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1 Problem 1

Let X be a scheme over a field k and $x \in X$ have residue field k in the sense that the map $X \to \operatorname{Spec}(k)$ induces the identity at the stalk $\mathcal{O}_{\operatorname{Spec}(k),(0)} \to \mathcal{O}_{X,x} \to k(x)$.

Let $U \subset X$ be any affine open neighborhood $U = \operatorname{Spec}(A)$ and $x \in U$ corresponds to $\mathfrak{p} \subset A$ then $k(x) = k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = (A/\mathfrak{p})_{\mathfrak{p}}$. Furthermore, the map $\operatorname{Spec}(A) \to \operatorname{Spec}(k)$ makes A a k-algebra compatibly with the isomorphism k(x) = k i.e. the diagram commutes,



We may factor this map via,

$$k \longleftrightarrow A \longrightarrow A/\mathfrak{p} \longleftrightarrow (A/\mathfrak{p})_{\mathfrak{p}} \stackrel{\sim}{\longrightarrow} k(x)$$

which composes the the identity. Because A/\mathfrak{p} is a domain, the map $A/\mathfrak{p} \hookrightarrow (A/\mathfrak{p})_{\mathfrak{p}}$ is injective. Therefore, the tower of inclusions collapses showing $A/\mathfrak{p} = k(x) = k$ which implies that \mathfrak{p} is maximal since k is a field. Thus $\mathfrak{p} \in \operatorname{Spec}(A)$ is a closed point. Therefore, $x \in U$ is closed for each affine open neighborhood. Therefore there exists a closed $C \subset X$ such that $C \cap U = \{x\}$ and thus

$$U^C \cup \{x\} = (U \setminus \{x\})^C = (C^C \cap U)^C = C \cup U^C$$

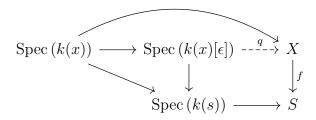
is closed. Now let $\{U_{\alpha}\}$ be an affine cover of X. If $x \in U_{\alpha}$ then we have shown that $U_{\alpha}^{C} \cup \{x\}$ is closed otherwise $x \in U_{\alpha}^{C}$ so $U_{\alpha}^{C} \cup \{x\}$ is closed. Therefore, using the fact that U_{α} cover X, the set

$$\bigcap_{\alpha} U_{\alpha}^{C} \cup \{x\} = \left(\bigcap_{\alpha} U_{\alpha}\right) \cup \{x\} = \emptyset \cup \{x\} = \{x\}$$

is closed.

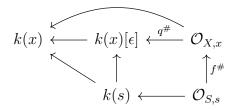
2 Tag: 029E

Let $f: X \to S$ be a morphism of schemes. Let $x \in X$ be a point and s = f(x). Note that $\operatorname{Spec}(k(x)[\epsilon]) = \{(\epsilon)\}$ and $\epsilon^2 = 0$. Consider the commutative diagram,



where Spec $(k(x)) \to \operatorname{Spec}(k(x)[\epsilon])$ is induced by the quotient map $k(x)[\epsilon] \to k(x)[\epsilon]/(\epsilon) = k(x)$ and Spec $(k(x)[\epsilon]) \to \operatorname{Spec}(k(s))$ is induced by the inclusion $k(s) \to k(x)[\epsilon]$ and the maps Spec $(k(x)) \to X$ and Spec $(k(s)) \to S$ are the canonical maps inducing the identity at the residue field.

Given a morphism $q: \operatorname{Spec}(k(x)[\epsilon]) \to X$ making the diagram commute we may consider the corresponding maps at stalks,



Consider the restriction $q^{\#}: \mathfrak{m}_{x} \to (\epsilon) \subset k(x)[\epsilon]$ since this map is local its image lies in (ϵ) the maximal ideal of $k(x)[\epsilon]$. Then $q^{\#}(\mathfrak{m}_{x}^{2}) \subset (\epsilon^{2}) = 0$ and thus $\mathfrak{m}_{x}^{2} \subset \ker q^{\#}$. Furthermore, by the commutativity of the diagram, the map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \xrightarrow{q^{\#}} k(x)[\epsilon]$ factors through k(s) and thus $q^{\#}(\mathfrak{m}_{s}\mathcal{O}_{X,x}) = 0$ so $\mathfrak{m}_{s}\mathcal{O}_{X,x} \subset \ker q^{\#}$. Thus we may factor,

Furthermore, $\mathcal{O}_{X,x} \to k(x)[\epsilon] \to k(x)$ is the identity so the induced map,

$$\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to (\epsilon)$$

is k(x)-linear.

Conversely, suppose that k(x) = k(s). Given the diagram, the doted morphism is uniquely determined on the underlying topological spaces since it must send the unique point of Spec $(k(x)[\epsilon])$ to x. Therefore it suffices to show that a local stalk map $q^{\#}: \mathcal{O}_{X,x} \to k(x)[\epsilon]$ is uniquely determined by a k(x)-linear map,

$$z: \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to k(x)$$

First, note that since $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is local we have maps,

$$\mathcal{O}_{S,s}/\mathfrak{m}_s \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x$$

whose composition gives the natural map $k(s) \to k(x)$ which we assume to be an isomorphism. Denote k(s) = k(x) = k then the above maps give $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ a natural k-algebra structure. The projection map (defined since $\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x} \subset \mathfrak{m}_x$),

$$\frac{(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x})}{\mathfrak{m}_x^2} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k$$

has kernel $\mathfrak{m}_x/(\mathfrak{m}_x^2+\mathfrak{m}_s\mathcal{O}_{X,x})$ giving a canonical decomposition as k-modules,

$$\frac{(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x})}{\mathfrak{m}_x^2} = \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}} = k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}}$$

Therefore, we get a map $q: \mathcal{O}_{X,x} \to k[\epsilon]$ via,

$$\mathcal{O}_{X,x} \to \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \xrightarrow{\mathrm{id} \oplus \epsilon z} k[\epsilon]$$

where the last map sends $(a, m) \mapsto a + z(m)\epsilon$. I claim that this map makes the diagram commute and is unique. First, it is clear that restructing q to \mathfrak{m}_x recovers the map z with image embdded as $k\epsilon \subset k[\epsilon]$. The constructed map $q: \mathcal{O}_{X,x} \to k[\epsilon]$ is a priori k-linear when it desends to a map $\mathcal{O}_{X,x}/(\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}) \to k[\epsilon]$ but we additionally need to show that q is a ring map. For $a, b \in \mathcal{O}_{X,x}$ then we write $a = \bar{a} + m_a$ and $b = \bar{b} + m_b$ in $\mathcal{O}_{X,x}/(\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x})$ where $m_a, m_b \in \mathfrak{m}_x$ and $\bar{a}, \bar{b} \in k$. Then,

$$ab = (\bar{a} + m_a)(\bar{b} + m_b) = \bar{a}\bar{b} + \bar{a}m_b + \bar{b}m_a + m_a m_b = \bar{a}\bar{b} + \bar{a}m_b + \bar{b}m_a$$

since $m_a m_b \in \mathfrak{m}_x^2$. Then applying q we get,

$$q(ab) = \bar{a}\bar{b} + \bar{a}z(m_b)\epsilon + \bar{b}z(m_a)\epsilon = \bar{a}\bar{b} + (\bar{a}z(m_b) + \bar{b}z(m_a))\epsilon$$

and furthermore,

$$q(a)q(b) = (\bar{a} + z(m_a)\epsilon)(\bar{b} + z(m_b)\epsilon) = \bar{a}\bar{b} + (\bar{a}z(m_b) + \bar{b}z(m_a))\epsilon + z(m_a)z(m_b)\epsilon^2$$
$$= \bar{a}\bar{b} + (\bar{a}z(m_b) + \bar{b}z(m_a))\epsilon$$
$$= q(ab)$$

and thus $q: \mathcal{O}_{X,x} \to k[\epsilon]$ is a ring map. Notice that,

$$\operatorname{Hom}_{\mathcal{O}_{S,s}}\left(\mathcal{O}_{X,x},k[\epsilon]\right) = \operatorname{Der}_{\mathcal{O}_{S,s}}\left(\mathcal{O}_{X,x},k\right) = \operatorname{Hom}_{k}\left(\mathfrak{m}_{x}/(\mathfrak{m}_{x}^{2} + \mathfrak{m}_{s}\mathcal{O}_{X,x}),k\right)$$

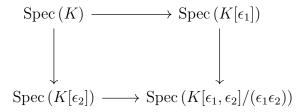
Next, the diagram commutes because q composed with $k[\epsilon] \to k$ via $\epsilon \mapsto 0$ sends $a \mapsto \bar{a}$ and thus equals the projection $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k$. Furthermore, $\mathcal{O}_{S,s} \to k(s) \to k(x)[\epsilon]$ is exactly given by $\mathcal{O}_{S,s} \to \mathcal{O}_{S,s}/\mathfrak{m}_s \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x) \subset k(x)[\epsilon]$ which equals $q \circ f^{\#}$ because the image of $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x}$ is $k = \mathcal{O}_{S,s}/\mathfrak{m}_s \subset \mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x}$ so q sends the image of $\mathcal{O}_{S,s}$ to $k(s) = k(x) \subset k(x)[\epsilon]$ under the projection. Since the diagram commutes, it suffices to show that such a construction will recover the original map $q^{\#}: \mathcal{O}_{X,x} \to k[\epsilon]$. The difference $\tilde{q} = q - q^{\#}$ is a map $\mathcal{O}_{X,s} \to k[\epsilon]$ which factors through,

$$\frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} = k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}}$$

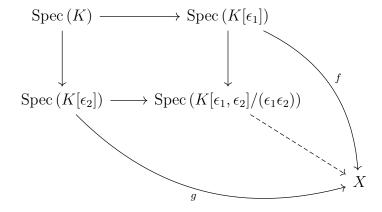
but is zero on each factor because q and $q^{\#}$ agree on $\mathcal{O}_{X,x} \to k$ and on $\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x})$ by construction. Thus $\tilde{q} = 0$ since it factors though the zero map on each factor of the quotient. Therefore, $q = q^{\#}$ proving the result.

3 Tag: 029G

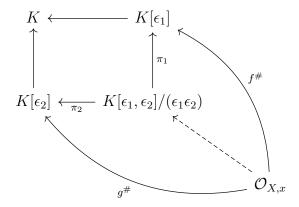
Let K be a field then consider the diagram of schemes,



we are asked to show that this diagram is a pushout in the category of schemes. Let X be any scheme and consider a commutative diagram,



Each affine scheme has one point so a map Spec $(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \to X$ is given by choosing a point $x \in X$ and map $\mathcal{O}_{X,x} \to K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)$. We chose the point $x \in X$ as the image of f which equals the image of g. The sheaf maps (which on a one point space are equivalent to the maps on the stalk) must satisfy the diagram,



However, $K[\epsilon_1, \epsilon_2]/(\epsilon_1 \epsilon_2) = K[\epsilon_1] \times_K K[\epsilon_2]$ is the pullback in the category of rings and thus there exists a unique map $\mathcal{O}_{X,x} \to K[\epsilon_1, \epsilon_2]/(\epsilon_1 \epsilon_2)$ making the diagram commute. Since the topological part is fixed this is equivalent to a giving a unque morphism of schemes Spec $(K[\epsilon_1, \epsilon_2]/(\epsilon_1 \epsilon_2)) \to X$ such that the first diagram commutes. This proof works because $K[\epsilon_1] \times_K K[\epsilon_2]$ is the pullback in the category of rings making (by the antiequivalence of the Spec functor) the origional diagram a pushout in the category of affine schemes. However, any morphism Spec $(k[\epsilon_i]) \to X$ factors through an open immersion of some affine patch because the image is a single point which must lie in some affine open. Therefore, this pushout diagram in the category of affine schemes is a pushout in the category of schemes.