Math 56: Proofs and Modern Mathematics
Homework 4 Solutions

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November 8, 2021

Problem 1 (Axler 3.A.1). Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (2x?4y + 3z + b, 6x + cxyz)$. Show that T is linear if and only if $b = c = 0$.

Solution. Suppose first that $b = c = 0$, so $T(x, y, z) = (2x - 4y + 3z, 6x)$. Let (x_1, y_1, z_1), (x_2, y_2, z_2) be two arbitrary elements of $\mathbb{R}^2$: we have

$$\begin{aligned}
T((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) && \text{addition in } \mathbb{R}^3 \\
&= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2), 6(x_1 + x_2)) \\
&= (2x_1 + 2x_2 - 4y_1 - 4y_2 + 3z_1 + 3z_2, 6x_1 + 6x_2) \\
& && \text{distribution in } \mathbb{R} \\
&= (2x_1 - 4y_1 + 3z_1, 6x_1) + (2x_2 - 4y_2 + 3z_2, 6x_2) \\
& && \text{addition in } \mathbb{R}^2 \\
&= T(x_1, y_1, z_1) + T(x_2, y_2, z_2).
\end{aligned}$$

So T commutes with addition. Similarly, let (x, y, z) be an arbitrary element of $\mathbb{R}^3$ and λ an arbitrary scalar in $\mathbb{R}$: we have

$$\begin{aligned}
T(\lambda(x, y, z)) &= T(\lambda x, \lambda y, \lambda z) && \text{scalar multiplication in } \mathbb{R}^3 \\
&= (2\lambda x - 4\lambda y + 3\lambda z, 6\lambda x) \\
&= \lambda(2x - 4y + 3z, 6x) && \text{scalar multiplication in } \mathbb{R}^2 \\
&= \lambda T(x, y, z).
\end{aligned}$$

So T also commutes with scalar multiplication and is therefore linear.

Conversely, suppose that $a \neq 0$ or $b \neq 0$. If $b \neq 0$, then $T(0) = (b, 0) \neq 0$, so T is not linear, therefore b has to be zero if T is linear. Similarly, if $b = 0$ but $c \neq 0$, then

$$T(1, 1, 1) = (1, 6 + c), \qquad T(2, 2, 2) = (2, 12 + 8c),$$

so $T(2, 2, 2) \neq 2T(1, 1, 1)$, since if $c \neq 0$, then $8c \neq 2c$, so T is not linear. Hence c must also be 0 if T is linear.

Problem 2 (Axler 3.A.3)**.** Suppose that $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $a_{jk} \in \mathbb{F}$, $j = 1, \ldots, m, k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{m1}x_1 + \cdots + a_{mn}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Solution. To prove this, we use the following basis for $\mathbb{F}^n$, called the *standard basis*:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, \ldots, 0, 1) \end{aligned}$$

where each e_j has zero co-ordinates except for the jth co-ordinate, which is 1. There are n of these, and we have

$$(x_1, \ldots, x_n) = x_1e_1 + \cdots + x_ne_n,$$

so these indeed form a basis.

Now, for each e_j, the image $T(e_j)$ is an element of $\mathbb{F}^m$, so for every $j = 1, \ldots, n$, we have $T(e_j) = (a_{1j}, a_{2j}, \ldots, a_{mj})$ for some scalars $a_{1j}, \ldots, a_{mj} \in \mathbb{F}$. Using these image vectors, we have

$$\begin{aligned} T(x_1, \ldots, x_n) &= T(x_1e_1 + \cdots + x_ne_n) && \text{(using the standard basis)} \\ &= x_1T(e_1) + \cdots + x_nT(e_n) && \text{(since } T \text{ is linear)} \\ &= x_1(a_{11}, a_{21}, \ldots, a_{m1}) + \cdots + x_n(a_{1n} + a_{2n} + \cdots + a_{nn}) \\ &= (a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{m1}x_1 + \cdots + a_{mn}x_n) \\ & && \text{(taking linear combinations in } \mathbb{F}^m.) \end{aligned}$$

This proves the statement.

(As an aside, if you're familiar with matrices, this is basically how they work, with respect to the standard bases for both $\mathbb{F}^n$ and $\mathbb{F}^m$.)

Problem 3 (Axler 3.A.4)**.** Suppose that $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Solution. Consider the equation $a_1v_1 + \cdots + a_mv_m = 0$. We can apply T to both sides of this equation, and since T is linear, this gives us

$$a_1Tv_1 + \cdots + a_mTv_m = T(0) = 0.$$

Since the vectors $Tv_1, \ldots, Tv_m$ are linearly independent, this means that $a_1, \ldots, a_m = 0$. Hence $v_1, \ldots, v_m$ are linearly independent as required.

Problem 4 (Axler 3.B.2)**.** Suppose that V is a vector space, $S, T \in \mathcal{L}(V, V)$ are such that $\operatorname{range} S \subset \operatorname{null} T$. Prove that $(ST)^2 = 0$.

Solution. Since $\operatorname{range} S \subset \operatorname{null} T$, we have $T(S(v)) = 0$ for any $v \in V$, so that $TS = 0$. Note that "multiplication" here is composition of linear maps, which is associative, but *not* commutative! This gives us $(ST)^2 = (ST)(ST) = S(TS)T = S0T = 0$, as required.

Problem 5 (Axler 3.B.13)**.** Suppose that T is a linear map from $\mathbb{F}^4$ to $\mathbb{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 = 5x_2, x_3 = 7x_4\}.$$

Prove that T is surjective.

Solution. We have a subspace of $\mathbb{F}^4$ defined by the solutions to two linearly independent equations, so we expect that $\dim \operatorname{null} T = 4 - 2 = 2$. Let's prove this rigorously: using the equations, we can say that a general element in $\operatorname{null} T$ is of the form

$$(5x_2, x_2, 7x_4, x_4) = x_2(5, 1, 0, 0) + x_4(0, 0, 7, 1)$$

for some scalars $x_2, x_4 \in \mathbb{F}$. Hence the vectors $(5, 1, 0, 0)$ and $(0, 0, 7, 1)$ span $\operatorname{null} T$. Moreover, they are linearly independent, as if

$$a(5, 1, 0, 0) + b(0, 0, 7, 1) = (0, 0, 0, 0),$$

then looking at the first or second co-ordinate gives us $a = 0$, and looking at the third or fourth co-ordinate gives us $b = 0$. Hence $(5, 1, 0, 0)$ and $(0, 0, 7, 1)$ form a basis of $\operatorname{null} T$, so $\dim \operatorname{null} T = 2$. By the rank-nullity theorem, we have $\dim \operatorname{null} T + \dim \operatorname{range} T = \dim \mathbb{F}^4$. We know that $\dim \operatorname{null} T = 2$ and $\dim \mathbb{F}^4 = 4$, so we have $\dim \operatorname{range} T = 2$. Now $\operatorname{range} T \subset \mathbb{F}^2$, which also has dimension 2, so, by Homework 4 Problem 2, we have $\operatorname{range} T = \mathbb{F}^2$. By definition, this is the same as saying that T is surjective.

Problem 6 (Axler 3.B.20)**.** Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Show that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Solution. ($\Leftarrow$): Suppose that there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V. Suppose that we have $v \in V$ such that $Tv = 0$. Since ST is the identity, we have $STv = v$, and since S is linear, we also have $STv = S(0) = 0$. Hence $v = 0$, so $\operatorname{null} T$ is trivial and T is injective.

($\Rightarrow$): Suppose that T is injective. We first prove that T preserves linear independence, i.e. if $v_1, \ldots, v_n$ are linearly independent in V, then $Tv_1, \ldots, Tv_n$ are linearly independent in W. (NOTE: This is NOT true if T is not injective!) Suppose that $v_1, \ldots, v_n$ is a linearly independent list, and suppose that $a_1Tv_1 + \cdots + a_nTv_n = 0$. Using the linearity of T, we can rewrite this as $T(a_1v_1 + \cdots + a_nv_n) = 0$. Since T is injective, $Tv = 0$ if and only if $v = 0$, so this means that $a_1v_1 + \cdots + a_nv_n = 0$, so $a_1, \ldots, a_n = 0$ by linear independence of $v_1, \ldots, v_n$.

This means that if we have a basis for V, the image of that basis in W is a linearly independent set: in particular, V must be finite-dimensional and satisfy $\dim V \leq \dim W$. Let $v_1, \ldots, v_n$ be a basis for V; we know that $Tv_1, \ldots, Tv_n$ is a linearly independent list in W. Let us extend this to a basis of W, which will be of the form $Tv_1, \ldots, Tv_n, w_1, \ldots, w_m$, for some $w_1, \ldots, w_m \in W$. Define the linear map $S : W \to V$ by setting $S(Tv_i) = v_i$ and $S(w_j) = 0$ (note: you can make $S(w_j)$ anything in V, this is just simplest), and extending this to a linear map on all of W. This is a linear map, so let's check that ST is the identity on V. Let v be an arbitrary vector in V, so $v = a_1v_1 + \cdots + a_nv_n$, using our basis. Then

$$\begin{aligned}
STv &= ST(a_1v_1 + \cdots + a_nv_n) \\
&= S(a_1Tv_1 + \cdots + a_nTv_n) && \text{(using linearity of } T\text{)} \\
&= a_1S(Tv_1) + \cdots + a_nS(Tv_n) && \text{(using linearity of } S\text{)} \\
&= a_1v_1 + \cdots + a_nv_n && \text{(using definition of } S \text{ on basis elements)} \\
&= v.
\end{aligned}$$

Hence we have a linear map $S \in \mathcal{L}(W, V)$ such that ST is the identity on V, as required.

(Additional notes for your interest: first, the map S is not unique unless $\dim V = \dim W$, because I can set $S(w_j)$ to be anything I want (it didn't have to be 0) and it will still be a linear map such that ST is the identity on V. Second, there is a similar statement for *surjectivity*: let V be a finite dimensional vector space, and let $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity on W.)