Math GR6262 Algebraic Geometry
Assignment # 1

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1 027A

1.1 Exercise 103.6.3.

1.2 Exercise 103.6.5.

Let A be a ring and consider $X = \text{Spec}\,(A)$. Consider an open cover $\mathcal{U}$ of X so we have,

$$\bigcup_{U \in \mathcal{U}} U = X$$

The sets $D(f)$ form a base of the topology on X. Since $\mathcal{U}$ is a cover of X, for any $x \in X$ there exists $U \in \mathcal{U}$ such that $x \in U$ and then by the base property $\exists f \in A$ such that $x \in D(f) \subset U$. Therefore there exists a set $S \subset A$ such that,

$$\bigcup_{f \in S} D(f) = X \iff \bigcap_{f \in S} V(f) = \varnothing$$

and each $D(f)$ is contained in some $U \in \mathcal{U}$. However,

$$\bigcap_{f \in S} V(f) = V(S)$$

Since no primes lie above the ideal (S) then $(S) = A$ otherwise $A/(S)$ would be nonzero and thus contain a maximal ideal which would be a prime above (S). Thus, $1 \in S$ so there exist a finite set $\{f_1, \dots, f_n\} \subset S$ and coefficients $a_1, \dots, a_n$ such that,

$$a_1 f_1 + \cdots + a_n f_n = 1$$

Therefore, $V(\{f_1, \dots, f_n\}) = V(f_1) \cap \cdots \cap V(f_n) = \varnothing$ since no prime contains 1. This is equivalent to,

$$\bigcup_{i=1}^{n} D(f_i) = X$$

which implies that a finite subset of $\mathcal{U}$ covers X since each $D(f_i)$ is contained in some $U \in \mathcal{U}$.

1.3 Exercise 103.6.7.

Let A be a ring and consider $X = \operatorname{Spec}(A)$. Let $\mathfrak{p}, \mathfrak{q} \subset A$ be distinct prime ideals. By definition, if $\mathfrak{p} \neq \mathfrak{q}$ then one must contain an element that the other does not. WLOG, let $f \in \mathfrak{p}$ but $f \notin \mathfrak{q}$. Then $\mathfrak{q} \in D(f)$ but $\mathfrak{p} \notin D(f)$. Therefore, $D(f)$ is an open neighborhood of $\mathfrak{q}$ in X which does not contain $\mathfrak{p}$ so $\mathfrak{p}$ and $\mathfrak{q}$ are topologically distinguishable and thus $\operatorname{Spec}(A)$ is a T_0 space.

2 028P

Consider the one-point space X with the following sheaf $\mathcal{O}_X$ defined by $\mathcal{O}_X(X) = \mathbb{Z}_p$ where $\mathbb{Z}_p$ are the p-adic integers for some prime p (this may be replaced with any positive dimension local ring). At the unique point $x \in X = \{x\}$ there is only one open, namely X, containing x so

$$\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_X(X) = \mathbb{Z}_p$$

which is local so $(X, \mathcal{O}_X)$ is a locally ringed space. If $(X, \mathcal{O}_X)$ were a scheme then it must be covered by affine schemes but the one-point space can only be covered by itself so we must have $(X, \mathcal{O}_X)$ itself be an affine scheme. However, if $(X, \mathcal{O}_X) \cong \operatorname{Spec}(A)$ then $\Gamma(X, \mathcal{O}_X) \cong \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) = A$ which implies that $A \cong \mathcal{O}_X(X) = \mathbb{Z}_p$. Furthermore, $\operatorname{Spec}(\mathbb{Z}_p)$ is a two-point space because $\dim \mathbb{Z}_p = 1$ and thus X is not even homeomorphic to $\operatorname{Spec}(Z_\mathfrak{p})$ as topological spaces let alone as locally ringed spaces. Thus, $(X, \mathcal{O}_X)$ is not a scheme.

3 028Q

Let X be a scheme with underlying space having two points. Since X must be covered by affine schemes, the only possibility for X to be non-affine is if each of the one-point sets contained in $X = \{x, y\}$ are affine schemes individually and are open. This puts the discrete topology on X since all points are open. Thus we have $\mathcal{O}_X(\{x\}) = A$ and $\mathcal{O}_X(\{y\}) = B$ where A and B are local rings (they are stalks of a scheme) such that $\operatorname{Spec}(A) \cong (\{x\}, \mathcal{O}_X|_{\{x\}})$ and $\operatorname{Spec}(B) \cong (\{y\}, \mathcal{O}_X|_{\{y\}})$ which implies that A and B are artinian since they must be local rings of dimension 0. There are restiction maps $\mathcal{O}_X(X) \to A$ and $\mathcal{O}_X(X) \to B$ and since $\{x\}$ and $\{y\}$ is an open cover of X, the sheaf property implies that any pair of sections $(a, b) \in A \times B$ lifts uniquely to $\mathcal{O}_X(X)$ which implies that $\mathcal{O}_X(X) \cong A \times B$ since it satisfies the universal property of the product (a pair of maps $Z \to A$ and $Z \to B$ lift uniquely to $\mathcal{O}_X(X)$ by the sheaf property and the fact that uniqueness of the lift plus the projections being ring maps implies that the lift is a ring map). Furthermore,

$$\operatorname{Spec}(A \times B) \cong \operatorname{Spec}(A) \coprod \operatorname{Spec}(B) = (\{x\}, \mathcal{O}_X|_{\{x\}}) \coprod (\{y\}, \mathcal{O}_X|_{\{y\}}) = X$$

Thus, $X \cong \operatorname{Spec}(A \times B)$ is affine.

4 028R

Let $X = \{x, y, z\}$ have the toplogy defined by declaring $\varnothing$, $\{x\}$, $\{x,y\}$, $\{x,z\}$, $\{x,y,z\}$ to be the open sets. These open sets form the inclusion category (suppressing identity morphisms and initial objects),

$$\begin{tikzcd}
& X & \\
\{x,y\} \arrow[ur] & & \{x,z\} \arrow[ul] \\
& \{x\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Next, we define the sheaf, $\mathcal{O}_X$ via,

$$\begin{tikzcd}
& X & & & & \mathbb{Z}_p \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
\{x,y\} \arrow[ur] & & \{x,z\} \arrow[ul] & \overset{\mathcal{O}_X}{\Longrightarrow} & \mathbb{Z}_p \arrow[dr] & & \mathbb{Z}_p \arrow[dl] \\
& \{x\} \arrow[ul] \arrow[ur] & & & & \mathbb{Q}_p &
\end{tikzcd}$$

Then X is a locally ringed space because the stalk at each point is $\mathbb{Z}_p$ or $\mathbb{Q}_p$ which are local. Furthermore, $(\{x,y\}, \mathcal{O}_X|_{\{x,y\}}) \cong \mathrm{Spec}\,(\mathbb{Z}_p)$ and $(\{x,z\}, \mathcal{O}_X|_{\{x,z\}}) \cong \mathrm{Spec}\,(\mathbb{Z}_p)$ is a cover of X by affine schemes. Therefore, X is a scheme. However, if X were affine with $X \cong \mathrm{Spec}\,(A)$ then $\Gamma(X, \mathcal{O}_X) \cong \Gamma(\mathrm{Spec}\,(A), \mathcal{O}_{\mathrm{Spec}(A)}) = A$ and thus $A \cong \mathcal{O}_X(X) \cong \mathbb{Z}_p$. However, $\mathrm{Spec}\,(\mathbb{Z}_p)$ which has two points so it cannot be isomorphic to X. Thus X is a scheme but not an affine scheme.

5 028W

Let X be a scheme and a field K with a morphism $\mathrm{Spec}\,(K) \to X$ *as ringed spaces*. We need to examine the data required to define such a map. Since $\mathrm{Spec}\,(K)$ has one point (0_K) the structure sheaf is defined simply via $\mathcal{O}_{\mathrm{Spec}(K)}(0_K) = K$. This morphism is a pair $(f, f^\#)$ with f an inclusion of the one-point space into X at the point $f(0_K) = x_0 \in X$. Furthermore, the map $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_{\mathrm{Spec}(K)}$ is a collection of maps,

$$f^\# : \mathcal{O}_X(U) \to \mathcal{O}_{\mathrm{Spec}(K)}(f^{-1}(U)) = \begin{cases} K & x_0 \in U \\ 0 & x_0 \notin U \end{cases}$$

We need to find a case in which the induced map on stalks is *not* local. In particular, if the map,

$$f^\#_{x_0} : \mathcal{O}_{X,x_0} \to K$$

is nontrivial on $\mathfrak{m}_x$ then the map will not be local since K has maximal ideal (0) and then $(f, f^{\#})$ will not be a map of schemes. To accomplish this, let $X = \mathrm{Spec}\,(\mathbb{Z})$ and $K = \mathbb{Q}$ and send $f(0_{\mathbb{Q}}) = (p)$ for any nonzero prime $p \in \mathbb{Z}$ and take the maps $f^{\#} : \mathbb{Z}_f \to \mathbb{Q}$ defined on standrd opens $D(f)$ to be inclusions if $f \notin (p)$ and zero maps otherwise. Then the stalk map $f^{\#}_p : \mathbb{Z}_{(p)} \to \mathbb{Q}$ is the inclusion since it must be the inclusion on each localization $\mathbb{Z}_f$ building up the stalk. Thus the maximal ideal $p\mathbb{Z}_{(p)}$ does not map to zero in $\mathbb{Q}$ so this is not a morphism of schemes. We could immediately tell this is the case because $\mathrm{Spec}\,(Z)$ is terminal in the category of affine schemes so there is a unique morphism of schemes $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(Z)$ but this map sends the ideal (0_K) to $(0_{\mathbb{Z}})$ so it is not the map we defined.