Mathematics GU4051 Topology
Assignment # 2

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Problem 1.

Let $f : X \to Y$ be constant. Then $\exists c \in Y : \forall x \in X : f(x) = c$. Then for any $U \subset Y$, $f^{-1}(U) = \begin{cases} \emptyset & c \notin U \\ X & c \in U \end{cases}$. Since both $\emptyset$ and X are open in any topology on X, for any choice of $U \subset Y$, $f^{-1}(U)$ is open in X. Thus, f is continuous regardless of the topologies on X and Y.

Let $f : X \to Y$ be continuous for every topology on X and on Y. In particular, let X have the indiscrete topology and Y have the discrete topology. Since every set is open in Y, for some $x_0 \in X$ take $U = \{f(x_0)\}$ which is open in Y. Then $f^{-1}(U) \neq \emptyset$ is open by continuity. However, X has the indiscrete topology so the only non-empty open set is X. Thus $f^{-1}(U) = X$ so $\forall x \in X : f(x) \in \{f(x_0)\}$ therefore $f(x) = f(x_0)$.

Problem 2.

Let the cofinite topology on X be:

$$\mathcal{T} = \{U \subset X \mid \text{ either } U = \emptyset \text{ or } X \setminus U \text{ is finite}\}$$

Clearly, $\emptyset, X \in \mathcal{T}$. Now suppose that Λ is an index set s.t. $V_\lambda \in \mathcal{T}$.
Consider,

$$X \setminus \bigcup_{\lambda \in \Lambda} V_\lambda = \bigcap_{\lambda \in \Lambda} X \setminus V_\lambda$$

which is finite because each $X \setminus V_\lambda$ is finite and subsets of finite sets are finite. Therefore,

$$X \setminus \bigcup_{\lambda \in \Lambda} V_\lambda \in \mathcal{T}$$

Let Λ be finite and consider

$$X \setminus \bigcap_{\lambda \in \Lambda} V_\lambda = \bigcup_{\lambda \in \Lambda} X \setminus V_\lambda$$

which is finite because each $X \setminus V_\lambda$ is finite and finite unions of finite sets are finite.
Thus if Λ is finite,

$$X \setminus \bigcap_{\lambda \in \Lambda} V_\lambda \in \mathcal{T}$$

Therefore, $(X, \mathcal{T})$ is a topological space.

Problem 3.

Let $(X, \mathcal{T})$ be a topological space with the cofinite topology. In the cofinite topology, $U \subset X$ is closed $\iff$ $X \backslash U$ is open $\iff$ $X \backslash (X \backslash U) = U$ is finite or $U = X$. Therefore, by the closed set formulation of continuity and since $f^{-1}(X) = X$ for any function:

$f : X \to X$ is continuous $\iff$ (U is finite $\implies$ $f^{-1}(U)$ is finite or equal to X)

Thus, if f is continuous, then since $\forall x \in X : \{x\}$ is finite then $f^{-1}(\{x\})$ is finite or equals X. If for any $x_0 \in X$, $f^{-1}(\{x_0\}) = X$ then f is constant because $\forall x \in X : f(x) \in \{x_0\}$ therfore $f(x) = x_0$. Otherwise, $f^{-1}(\{x\})$ is finite for every $x \in X$.

Conversely, if f is constant then it is continuous on every topology. Otherwise, let $f^{-1}(\{x\})$ be finite for every $x \in X$. If $U = X$ then $f^{-1}(U) = X$. If $U \subset X$ is finite then

$$U = \{x_1, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\}$$

thus,

$$f^{-1}(U) = \bigcup_{i=1}^{n} f^{-1}(\{x_i\})$$

is finite because it is a finite union of finite sets. Therefore, if U is closed under $\mathcal{T}$ then $f^{-1}(U)$ is closed under $\mathcal{T}$ thus f is continuous.

Problem 4.

Let $i : \mathbb{R} \to \mathbb{R}$ take $i : x \mapsto x$. Then $i^{-1}(U) = U$ because $x \in U \iff i(x) \in U$. Therefore, i is continuous iff $U \in \mathcal{T}_{codom} \implies i^{-1}(U) = U \in \mathcal{T}_{dom}$ i.e. iff $\mathcal{T}_{codom} \subset \mathcal{T}_{dom}$. In particular, if $\mathcal{T}_{codom} = \mathcal{T}_{dom}$ then i is continuous.

Any $f : X \to Y$ is continuous when Y has the indiscrete toplogy because both $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ are always open in X. Conversely, the indescrete topology is the smallest possible topology on a given set so if the domain has the indiscrete topology and the codomain has any other topology then $\mathcal{T}_{codom} \supsetneq \mathcal{T}_{dom}$ so i is not continuous.

Also, any $f : X \to Y$ is continuous when X has the discrete toplogy because every set in X and thus every preimage is automatically open in X. Conversely, the descrete topology is the largest possible topology on a given set so if the codomain has the discrete topology and the domain has any other topology then $\mathcal{T}_{codom} \supsetneq \mathcal{T}_{dom}$ so i is not continuous.

There remain two possiblities not covered by the cases above which are: both topologies are equal, one is the discrete topology, or one is the indiscrete topology. These are: the standard topology mapping to the cofinite and the cofinite topology mapping to the standard topology. If $U \in \mathcal{T}_{cofin.}$ then $\mathbb{R} \backslash U$ is finite so $\mathbb{R} \backslash U$ is closed in $\mathcal{T}_{stand.}$. However, most open sets in $\mathcal{T}_{stand.}$ do not have finite complement so $\mathcal{T}_{cofin.} \subsetneq \mathcal{T}_{stand.}$. Therefore, i is continuous from standard to cofinite but not from cofinite to standard.

With respect to various toplogies, i is (C. = continuous, N.C. = not continuous):

from \ to	stand.	disc.	indisc.	cofin.
stand.	C.	N.C.	C.	C.
disc.	C.	C.	C.	C.
indisc.	N.C.	N.C.	C.	N.C.
cofin.	N.C	N.C.	C.	C.

Problem 5.

Let $S \in \mathcal{T} \iff \forall x \in S : \exists x \in [a, b) \subset S$. Then vacuously, $\emptyset \in \mathcal{T}$ and $\forall x \in \mathbb{R} : x \in [x-1, x+1) \subset \mathbb{R}$ so $\mathbb{R} \in \mathcal{T}$. Now take an index set Λ s.t. V_λ is open. Then if

$$x \in \bigcup_{\lambda \in \Lambda} V_\lambda$$

then $\exists V_\lambda$ s.t. $x \in V_\lambda$ so because V_λ is open,

$$x \in [a, b) \subset V_\lambda \subset \bigcup_{\lambda \in \Lambda} V_\lambda$$

Thus, $\bigcup_{\lambda \in \Lambda} V_\lambda$ is open in $\mathcal{T}$. Now if Λ is a finite index set, take any

$$x \in \bigcap_{\lambda \in \Lambda} V_\lambda$$

Then for each $\lambda \in \Lambda$, $x \in [a_\lambda, b_\lambda) \subset V_\lambda$ and since Λ is finite then $\max_{\lambda \in \Lambda} a_\lambda \leq x < \min_{\lambda \in \Lambda} b_\lambda$ so $x \in [\max_{\lambda \in \Lambda} a_\lambda, \min_{\lambda \in \Lambda} b_\lambda) \subset [a_\lambda, b_\lambda) \subset V_\lambda$. Therefore,

$$x \in [\max_{\lambda \in \Lambda} a_\lambda, \min_{\lambda \in \Lambda} b_\lambda) \subset \bigcap_{\lambda \in \Lambda} V_\lambda$$

Thus, $\bigcap_{\lambda \in \Lambda} V_\lambda$ is open in $\mathcal{T}$. Therefore, $(X, \mathcal{T})$ is a topological space.

Problem 6.

Take the subset topology $\mathcal{T}_\mathbb{Z}$ on $\mathbb{Z} \subset \mathbb{R}$ under the standard (Euclidean) topology. For any $n \in \mathbb{Z}$, the interval $B_{\frac{1}{2}}(n)$ contains only one integer namely n. Thus, $\mathbb{Z} \cap B_{\frac{1}{2}}(n) = \{n\}$.

Let $S \subset \mathbb{Z}$ be any set of integers. Then

$$S = \bigcup_{n \in S} \{n\} = \bigcup_{n \in S} \mathbb{Z} \cap B_{\frac{1}{2}}(n) = \mathbb{Z} \cap \left(\bigcup_{n \in S} B_{\frac{1}{2}}(n) \right)$$

But each $B_{\frac{1}{2}}(n)$ is open in $\mathbb{R}$ so therefore, $\bigcup_{n \in S} B_{\frac{1}{2}}(n)$ is open in R. Thus, S is open in the subspace topology on $\mathbb{Z}$. $\mathcal{T}_\mathbb{Z}$ is the discrete topology because every set is open under $\mathcal{T}_\mathbb{Z}$.

Problem 7.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $\pi : (x, y) \mapsto x$. Let

$$S = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$

Now let $f : \mathbb{R} \to S$ be given by $f : x \mapsto (x, 0)$ is the inverse of $\pi|_S$ because $f \circ \pi|_S(x, 0) = f(x) = (x, 0)$ and $\pi|_S \circ f(x) = \pi|_S(x, 0) = x$. Therefore, $\pi|_S$ is a bijection. Since $\pi : \mathbb{R}^2 \to \mathbb{R}$ is linear, it is continuous by Lemma **??** and thus, by Lemma **??**, $\pi|_S$ is continuous. Also, $f : \mathbb{R} \to S$ is linear and thus, by Lemma **??**, continuous. So $\pi|_S$ is a continuous bijection with continuous inverse and thus a homeomorphism. Therefore $\mathbb{R}$ with the standard topology and S with the subspace topology are homeomorphic.

Lemmas

Lemma 0.1. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear then f is uniformly continuous which makes f a continuous map with respect the standard topologies of R^n and R^m.

Proof. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $g(\mathbf{x}) = \begin{cases} |f(\mathbf{x})|/|\mathbf{x}| & \mathbf{x} \neq \vec{0} \\ 0 & \mathbf{x} = \vec{0} \end{cases}$ is bounded (proven in Honors Math). Thus $\exists M \in \mathbb{R}^+ : \forall \mathbf{v} \in \mathbb{R}^n : |f(\mathbf{v})| < M|\mathbf{v}|$ so f is Lipschitz.

Given $\epsilon > 0$ take $\delta = \frac{1}{M}\epsilon$.
If $|\mathbf{x} - \mathbf{y}| < \delta$ then $|f(\mathbf{x}) - f(\mathbf{y})| = |f(\mathbf{x} - \mathbf{y})| < M|\mathbf{x} - \mathbf{y}| < M\delta = \epsilon$

Therefore, $|\mathbf{x} - \mathbf{y}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$ □

Lemma 0.2. If $f : X \to Y$ is continuous and $S \subset X$ then $f|_S : S \to Y$ is continuous under the subspace topology on S.

Proof. Let U be open in Y then $x \in f|_S^{-1}(U) \iff f(x) \in U$ and $x \in S \iff x \in f^{-1}(U) \cap S$. But $f^{-1}(U)$ is open in X so $f|_S^{-1}(U) = f^{-1}(U) \cap S$ is open in S under the subsapce topology. □