1 Group Actions

Definition: Let G be a group acting on a set X, call X a G-set, then there eixsts a homomorphism $\phi : G \to \mathrm{Sym}(X)$ the group of bijections of X to itself.

For example, $\mathrm{GL}(n,k)$ acts on k^n for a field k. However, $\mathrm{GL}(n,k)$ also action on $(k^n)^*$ by the action $A \cdot f = f \circ A^{-1}$. Furthermore, $\mathrm{GL}(n,k)$ acts on $\mathrm{Hom}\,(k^n, k^n)$ by $A \cdot F = A \circ F \circ A^{-1}$.

2 Group Representations

Definition: A G-representation (V, ρ_V) is a group action on a vector space V with a homomorphism $\rho_V : G \to \mathrm{Aut}\,(V)$

Definition: A G-morphism between G-representations ρ_V and ρ_W is a linear map $F : V \to W$ satisfying $F \circ \rho_V(g) = \rho_W(g) \circ F$ for all $g \in G$. The set of all such G-morphisms is denoted $\mathrm{Hom}^G\,(V, W)$.

Definition: Let $\rho_V : G \to \mathrm{Aut}\,(V)$ be a G-representation, then $W \subset V$ is a G-invariant subspace if $\rho(g)(W) \subset W$ for all $g \in G$.

Definition: A G-representation (V, ρ_V) is irreducible if $V \neq \{0\}$ and the only invariant subspaces are $\{0\}$ and V.

Definition: Given G-representations (V, ρ_V) and (W, ρ_W), we can form the following additional G-representations,

1. (V^*, ρ_{V*}) given by $\rho_{V^*}(g) \cdot \varphi = \varphi \circ \rho_V(g)^{-1}$
2. $(V \oplus W, \rho_V \oplus \rho_W)$ given by,
$$(\rho_V \oplus \rho_W)(g) \cdot (v \oplus w) = (\rho_V(g) \cdot v) \oplus (\rho_W(g) \cdot w)$$
3. $(\mathrm{Hom}\,(V, W)\,, \rho_{\mathrm{Hom}(V,W)})$ given by, $\rho_{\mathrm{Hom}(V,W)} \cdot F = \rho_W(g) \circ F \circ \rho_V(g)^{-1}$. Note, the fixed points, $(\mathrm{Hom}\,(V, W))^G = \mathrm{Hom}^G\,(V, W)$ because $\rho_W(g) \circ F \circ \rho_V(g)^{-1} = F$ for every $g \in G$ if and only if F is a G-morphism.
4. $(V \otimes W, \rho_V \otimes \rho_W)$ given by,
$$(\rho_V \otimes \rho_W)(g) \cdot \left(\sum_{i=1}^{n} v_i \otimes w_i \right) = \sum_{i=1}^{n} (\rho_V(g) \cdot v_i) \otimes (\rho_W(g) \cdot w_i)$$

Lemma 2.1. If V is a G-representation such that $V \neq \{0\}$ then there exists a G-invariant subspace W which is an irreducible G-representation.

Lemma 2.2. Let $F : V \to W$ be a G-morphism then $\ker F$ and $\mathrm{Im}(F)$ are invariant subspaces.

Proof. Let V and W be G-representations and let $F : V \to W$ be a G-morphism. Take any $g \in G$. Take, $v \in \ker F$. Then, $F(v) = 0$ and thus, $\rho_W(g)(F(v)) = F(\rho_V(g)(v)) = 0$ so $\rho_V(g)(v) \in \ker F$. Therefore, $\ker F$ is invariant under the action of $\rho_V(g)$ for any $g \in G$. Therefore, $\ker K$ is a G-invariant subspace of V. Similarly, take $w \in \operatorname{Im}(F)$. Then there exists $v \in V$ such that $F(v) = w$. Therefore, $\rho_W(g)(w) = \rho_W(g)(F(v)) = F(\rho_V(g)(v)) \in \operatorname{Im}(F)$. Therefore, $\rho_V(g)(\operatorname{Im}(K)) \subset \operatorname{Im}(F)$ so $\operatorname{Im}(F)$ is a G-invariant subspace of W. $\square$

Lemma 2.3. Let $F : V \to W$ be a G-morphism then,

1. if V is irreducible then F is either 0 or injective.
2. if W if irreducible then F is either 0 or surjective.
3. if V and W are both irreducible then F is either 0 or an isomorphism.

Proof. Let V be irreducible. Since $\ker F$ is an invariant subspace, then $\ker F = \{0\}$ or $\ker F = V$ so F is either injective or the zero map. Likewise, let W be irreducible. Since $\operatorname{Im}(F)$ is an invariant subspace, then $\operatorname{Im}(F) = \{0\}$ or $\operatorname{Im}(F) = W$ so F is either the zero map or surjective. $\square$

Definition: The notation $(V, \rho_V) \cong (W, \rho_W)$ with shorthand $V \cong W$ mean that there exists a G-isomorphism $F : V \to W$ i.e. a bijective G-morphism.

Theorem 2.4 (Schur's Lemma)**.** If V is irreducible then $\operatorname{Hom}^G(V, V) \cong \mathbb{C} \cdot \mathrm{id}$. Also, if V and W are both irreducible then either $V \ncong W$ and $\operatorname{Hom}^G(V, W) = \{0\}$ or $V \cong W$ and $\dim \operatorname{Hom}^G(V, W) = 1$.

Proof. Let $F : V \to V$ be a G-morphism then F is either zero or an isomorphism because V is irreducible. Then F has an eigenvalue λ so consider the G-morphism $F - \lambda\mathrm{id}$. However, $\exists v \in V$ such that $F(v) = \lambda v$ so $(F - \lambda\mathrm{id})(v) = 0$ and therefore, $F - \lambda v$ is not injective. However, V is irreducible so F must be the zero map. Thus, $F = \lambda\mathrm{id}$. Furthermore, if every G-morphism $F \in \operatorname{Hom}^G(V, W)$ is not an isomorphism then because V and W are irreducible we must have $F = 0$. Thus, if $\operatorname{Hom}^G(V, W) \neq \{0\}$ then there must exist a G-isomorphism F. In particular, $V \cong W$. Therefore, $\operatorname{Hom}^G(V, W) \cong \operatorname{Hom}^G(V, V) \cong \mathbb{C} \cdot \mathrm{id}$ so $\dim \operatorname{Hom}^G(V, W) = 1$. $\square$

Corollary 2.5. $F \in \operatorname{Hom}^G(V, W)$ is either zero or an isomorphism and therefore invertible. Therefore, $\operatorname{Hom}^G(V, W)$ is a division ring.

Definition: A G-representation (V, ρ_V) is decomposable if $V \cong W_1 \oplus W_2$ where $W_i \neq \{0\}$

Definition: A G-representation is completely reducible if $V \cong W_1 \oplus \cdots \oplus W_n$ where W_i is irreducible.

Lemma 2.6. Let G be a finite group and V a G-representation, the map $p : V \to V$ given by,

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v)$$

is a G-invariant projection $p : V \to V^G$.

Proof. If $v \in V^G$ then,

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) = \frac{1}{|G|} \sum_{g \in G} v = v$$

Furthermore, for any $v \in V$ consider,

$$\rho_V(h) \circ p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(h) \circ \rho_V(g)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(hg)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) = p(v)$$

so $p(v) \in V^G$. Therefore, $\text{Im}(p) = V^G$. Furthermore,

$$p \circ \rho_V(g)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ \rho_V(h)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(gh)(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) = p(v)$$

Thus, $p \circ \rho_V(g) = \rho_V(g) \circ p$ for all $g \in G$. $\square$

Theorem 2.7 (Maschke)**.** If G is a finite group and $W \subset V$ are G-representations then there exists a G-invariant complement $W' \subset V$ of W and thus $V = W \oplus W'$.

Proof. Let $p_0 : V \to V$ be a projection onto W. Then, $p_0 \in \text{Hom}(V, V)$ so by the above lemma applied to the G-representation $(\text{Hom}(V, V), \rho_{\text{Hom}(V,V)})$, the map,

$$p_0 \mapsto p = \frac{1}{|G|} \sum_{g \in G} \rho_{\text{Hom}(V,V)}(g) \cdot p_0 = \frac{1}{|G|} \sum_{g \in G} \rho_V \circ p \circ \rho_V^{-1}$$

is a projection map $\text{Hom}(V, V) \to (\text{Hom}(V, V))^G = \text{Hom}^G(V, V)$. Thus, p is a G-invariant projection from V to W since $p(w) = w$. Therefore, $V \cong W \oplus \ker p$. $\square$

Corollary 2.8. If G is a finite group then every nonzero G-representation is completely reducible.

Corollary 2.9. If G is a finite abelian group then any G-representation is a sum of 1-dimensional representations.

Proof. It suffices to prove that every irreducible G-representation is 1-dimensional. Let W be an irreducible G-representation. However, since G is abelian, $\rho_W(g)$ is a G-morphism in $\text{Hom}^G(V, V) \cong \mathbb{C}$ so $\rho_W(g) = \lambda(g) \in \mathbb{C}$. Then, $\rho_W(g)(w) = \lambda(g)w$ so $\text{span}\{w\}$ is a nonempty G-invariant subspace. However W is irreducible so $W = \text{span}\{w\}$ which has dimension 1. $\square$

Corollary 2.10. Let $A \in \mathrm{GL}(n, \mathbb{C})$ and suppose that A has finite order then A is diagonalizable.

Proof. A defines a representation of $\mathbb{Z}/N\mathbb{Z}$ where N is the order of A. Therefore, $\mathbb{C}^n$ is the sum of 1-dimensional G-invariant subspaces which are eigenspaces. Therefore, the eigenvectors of A span $\mathbb{C}^n$. $\square$

Corollary 2.11. Let ρ_V be a G-representation of a finite group G then $\forall g \in G$ we can diagonalize $\rho_V(g)$ and its eigenvalues are roots of unity of order dividing $|G|$.

Proof. Because G is finite, and $g \in G$ has finite order and $\mathrm{ord}(g) \mid |G| =$ so $\rho_V(g)$ has order dividing n and is thus diagonalizable. Furthermore if v is an eigenvector, $\rho_V(g) \cdot v = \lambda v$ then $\rho_V(g)^n \cdot v = \lambda^n v$ but $\rho_V(g^n) = \rho_V(e) = \mathrm{id}$ so $\lambda^n v = v$ and thus $\lambda^n = 1$ since $v \neq 0$ so λ is a root of unity. $\square$

3 Group Characters

Definition: If (V, ρ_V) is a G-representation, the character is the map $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\rho_V(g))$.

Lemma 3.1. Let (V, ρ_V) be a G-representation with character χ then,

1. $\chi(e) = \mathrm{Tr}(\mathrm{id}) = \dim V$
2. $\chi(hgh^{-1}) = \mathrm{Tr}(\rho_V(h)\rho_V(g)\rho_V(h)^{-1}) = \mathrm{Tr}(\rho_V(h)) = \chi(g)$. Thus, χ is a function on conjugacy classes.
3. $\chi(g^{-1}) = \overline{\chi(g)}$ because $\rho(g)$ is diagonalizable with norm-1 eigenvalues.

Lemma 3.2. Let (V, ρ_V) and (W, ρ_W) be G-representations with character χ_V and χ_W then,

1. $\chi_{V \oplus W} = \chi_V + \chi_W$
2. $\chi_{V^*} = \overline{\chi_V}$
3. $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
4. $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W$

Lemma 3.3.

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

Proof. The map,

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v)$$

is a G-invariant projection $p : V \to V^G$ so $\mathrm{Tr}(p) = \dim V^G$. However,

$$\mathrm{Tr}(p) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho_V(g)) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

$\square$

Corollary 3.4. Applying this fact to $\mathrm{Hom}\,(V, W)$, then,

$$\dim\left(\mathrm{Hom}\,(V, W)^G\right) = \dim \mathrm{Hom}^G\,(V, W) = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi}_V(g) \chi_W(g)$$

Corollary 3.5. By Schur's lemma,

$$\dim \mathrm{Hom}^G\,(V, W) = \begin{cases} 1 & V \cong W \\ 0 & \text{else} \end{cases}$$

Therefore,

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi}_V(g) \chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \begin{cases} 1 & V \cong W \\ 0 & \text{else} \end{cases}$$

where I have used the fact that the sum is real because it is equal to an integer.

Definition: For $f_1, f_2 \in \mathbb{C}[G]$ define the Hermitian inner product,

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Proposition 3.6. Therefore, for irreducible representations (V, ρ_V) and (W, ρ_W) with characters χ_V and χ_W then,

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & V \cong W \\ 0 & \text{else} \end{cases}$$

Corollary 3.7. Let V be a completely reducible representation, $V = \bigoplus_{i=1}^{n} V_i^{m_i}$ with $V_i \cong V_j$ only if $i = j$ then,

$$\langle \chi_V, \chi_V \rangle = \sum_{i,j} m_i m_j \left\langle \chi_{V_i}, \chi_{V_j} \right\rangle = \sum_{i=1}^{n} m_i^2$$

Corollary 3.8. Let V be a completely reducible G-representation, $V = \bigoplus_{i=1}^{n} V_i^{m_i}$ with $V_i \cong V_j$ only if $i = j$ and W an irreducible G-representation then,

$$\langle \chi_W, \chi_V \rangle = \begin{cases} m_i & W \cong V_i \\ 0 & \text{else} \end{cases}$$

Proof. We have, $\chi_V = \sum_{i=1}^n m_i \chi_{V_i}$. Thus,

$$\langle \chi_W, \chi_V \rangle = \sum_{i=1}^{n} m_i \langle \chi_W, \chi_{V_i} \rangle = \begin{cases} m_i & W \cong V_i \\ 0 & \text{else} \end{cases}$$

since by hypothesis $i \neq j \implies V_i \ncong V_j$. □

Corollary 3.9. *V is irrediucible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Theorem 3.10. *Let G be finite, then a G-representation V is determined up to isomorphism by χ_V. That is, $V \cong W \iff \chi_V = \chi_W$.*

Proof. If $V \cong W$ then there exists an isomorphism $F : V \to W$ such that $F \circ \rho_V(g) = \rho_W(g) \circ F$ and thus $\rho_V(g) = F^{-1} \circ \rho \circ F$. Thus,

$$\chi_V = \mathrm{Tr}(\rho_V(g)) = \mathrm{Tr}\big(F^{-1} \circ \rho \circ F\big) = \mathrm{Tr}(\rho_W(g)) = \chi_W(g)$$

Conversely, suppose that $\chi_V = \chi_W$. Then, because G is finite, we can write any G-representations as,

$$V = \bigoplus_{i=1}^{n} V_i^{m_i} \qquad W = \bigoplus_{i=1}^{n} W_i^{k_i}$$

Therefore, $\chi_V = \sum\limits_{i=1}^{n} m_i \, \chi_{V_i}$. Consider

$$\langle \chi_{V_i}, \chi_W \rangle = \langle \chi_{V_i}, \chi_V \rangle = \langle \chi_{V_i}, \chi_V \rangle = m_i$$

but V_i is irreducible so $\langle \chi_{V_i}, \chi_W \rangle = m_i$ implies that some factor $W_j^{k_j}$ is isomorphic to V_i and $m_i = k_j$. Therefore, up to order, the expansions of V and W are equal. Thus, $V \cong W$. □

Definition: The regular representation is $\rho_{reg} : G \to \mathbb{C}[G]$ given by $\rho(g)v = g \cdot v$. Call the character of this representation $\chi_{reg} = \chi_{\mathbb{C}[G]}$.

Lemma 3.11. *Let G act on X and let $(\mathbb{C}[X], \rho)$ be the permutation G-representation. Then,*

$$\chi_{\mathbb{C}[X]}(g) = \#(X^g)$$

Proof. We know that $\rho(g) \cdot x = g \cdot x$ so

$$\mathrm{Tr}(\rho(\sigma)) = \sum_{i=1}^{|X|} \mathbf{1}(g \cdot x = x) = \#(X^g)$$

□

Corollary 3.12.

$$\chi_{reg}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

Proof. A group acts freely on itself ($gh = h \implies g = e$) so there cannot be any fixed points of G for any map except $\rho(e)$ which fixes every element. $\square$

Lemma 3.13. $\langle \chi_V, \chi_{reg} \rangle = \dim V$

Proof.

$$\langle \chi_V, \chi_{reg} \rangle = \frac{\chi_V(e)|G|}{|G|} = \chi_V(e) = \dim V$$

$\square$

Theorem 3.14. Write,

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{n} V_i^{d_i}$$

If W is an irreducible G-representation then $W \cong V_i$ for some i. Furthermore, $\dim V_i = d_i$.

Proof. Let W be irreducible, then $\langle \chi_W, \chi_{reg} \rangle = \dim W > 0$ and therefore by corollary **??**, $W \cong V_i$ for a unique i. However, $\dim V_i = \langle \chi_{V_i}, \chi_{reg} \rangle = d_i$. $\square$

Corollary 3.15.

$$\dim \mathbb{C}[G] = |G| = \sum_{i=1}^{n} (d_i)^2$$

Corollary 3.16. For any $g \in G$,

$$\sum_{i=1}^{n} d_i \cdot \chi_{V_i}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

Proof. Because,

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{n} V_i^{d_i}$$

the character factors as,

$$\chi_{reg}(g) = \sum_{i=1}^{n} d_i \cdot \chi_{V_i}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

$\square$

Theorem 3.17. If G is a finite group, then there are finitely many irreducible G-representations.

Proof. Every irreducible G-representation must be isomorphic so a factor of the regular representation. Equivalently, the sum of the squares of the dimensions of all irreducible G-representations is $|G|$ which is, in particular, finite. $\square$

Proposition 3.18. Let G be abelian, then every representation is one-dimensional so $d_i = 1$. Thus, $\sum_{i=1}^{n} d_i^2 = n = |G|$. So there are exactly $|G|$ irreducible G-representations.

4 The Permutation Representation

5 Class Functions

Definition: $f : G \to \mathbb{C}$ is a class function if f is constant on conjugay classes or equivalently, $\forall g, h \in G : f(hgh^{-1}) = f(g)$.

Definition: $Z \subset \mathbb{C}[G]$ is the vectorspace of class functions.

Proposition 5.1. $f_{Cl(x)}$ is the characteristic function of $[x]$ which is,

$$f_{Cl(x)}(g) = \begin{cases} 1 & g \in Cl(x) \\ 0 & g \notin Cl(x) \end{cases}$$

form a basis of Z.

Proposition 5.2.

$$\left\langle f_{Cl(x)}, f_{Cl(y)} \right\rangle = \begin{cases} \frac{|Cl(x)|}{|G|} & Cl(x) = Cl(y) \\ 0 & \text{else} \end{cases}$$

Definition: For $f \in \mathbb{C}[G]$ the map, $F_{V,f} : V \to V$ is defined by,

$$F_{V,f} = \sum_{g \in G} f(g) \rho_V(g)$$

Lemma 5.3. If f is a class function, $F_{V,f}$ is a G-morphism. If in addition, V is irreducible, then $F_{V,f} = t \cdot \text{id}$ where,

$$t = \frac{|G| \cdot \langle f, \overline{\chi}_V \rangle}{\dim V}$$

Proof. $F_{V,f}$ if a G-morphism if and only if $\forall h \in G$ we have $\rho_V(h) \circ F_{V,f} \circ \rho_V(h)^{-1} = F_{v,f}$. Expanding,

$$\rho_V(h) \circ F_{V,f} \circ \rho_V(h)^{-1} = \sum_{g \in G} f(g) \rho_V(h) \circ \rho_g \circ \rho_V(h)^{-1} = \sum_{g \in G} f(g) \rho_V(hgh^{-1}) = \sum_{g \in G} f(h^{-1}gh) \rho_V(g) = F_{V,f}$$

because f is a class function.

Using Schur's Lemma, if V is irreducible then because $F_{V,f}$ is a G-morphism we know that $F_{V,f} = t \cdot \text{id}$. Thus, $\text{Tr}(F_{V,f}) = \text{Tr}(t \cdot \text{id}) = t \dim V$. However,

$$\text{Tr}(F_{V,f}) = \sum_{g \in G} f(g) \text{Tr}(\rho_V(g)) = \sum_{g \in G} f(g) \chi_V(g) = |G| \langle f, \overline{\chi_V} \rangle$$

Therefore, $t \dim V = |G| \langle f, \overline{\chi_V} \rangle$. ☐

Proposition 5.4. If f is a class function then $\langle f, \chi_V \rangle = 0$ for all irreducible V implies that $f = 0$. Furthermore, if $V_1, \cdots, V_n$ are the irreducible G-representations up to isomorphism then $\chi_{V_1}, \cdots, \chi_{V_n}$ are a basis for Z. Finally, n is the number of conugacy classes of G.

Proof. If V is irreducible then V^* is irreducible so $\langle f, \chi_{\overline{V}} \rangle = 0$ and thus $F_{V,f} = 0 \cdot \mathrm{id} = 0$ for all irreducible V. However, $F_{V_1 \oplus V_2, f} = F_{V_1,f} + F_{V_2,f} = 0$ so by induction $F_{W,f} = 0$ for all G-representations. In particular, $F_{\mathbb{C}[G],f} = 0$ that is,

$$F_{\mathbb{C}[G],f} = \sum_{g \in G} f(g) \rho_{reg}(g) = 0$$

so applied to 1,

$$F_{\mathbb{C}[G],f} = \sum_{g \in G} f(g) \rho_{reg}(g)(1) = \sum_{g \in G} f(g) \cdot g = 0$$

and therefore $f = 0$ because $\mathbb{C}[G]$ is a free vectorspace over G.

By orthogonality conditions, $\left\langle \chi_{V_i}, \chi_{V_j} \right\rangle = \delta_{ij}$ and thus these characters are linearly independent. Consider the subspace of Z orthogonal to all χ_{V_i}. However, we have shown that if $\langle f, \chi_{V_i} \rangle = 0$ for all irreducible representations V_i then $f = 0$. Thus, the orthogonal complement is empty so the set $\{\chi_{V_1}, \ldots, \chi_{V_n}\}$ spans Z and thus $\dim V = n$.

However, the functions $f_{Cl(x)}$ form a basis of Z. Therefore, $\dim Z = n$ is the number of conjugacy classes of G. ☐

Proposition 5.5. G is abelian if and only if every irreducible G-representation is one-dimensional.

Proof. If $d_i = 1$ then $\sum_{i=1}^{n} d_i^2 = n = |G|$ so there are $|G|$ conjugacy classes and thus G is abelian. We have already proved the converse. ☐

Proposition 5.6. We having the following orthogonality relationship on G over the set of irreducible characters,

- $$\forall x \in G : \quad \sum_{i=1}^{h} |\chi_{V_i}(x)|^2 = \frac{|G|}{|Cl(x)|}$$

- $$\forall x, y \in G : y \notin Cl(x) : \quad \sum_{i=1}^{h} \chi_{V_i}(x) \overline{\chi_{V_j}}(y) = 0$$

6 Fourier Inversion on Groups

6.1 The Structure of $\mathbb{C}[G]$

Definition: A K-algebra is a K-vectorspace A together with a K-bilinear map donoted by $B : A \times A \to A$ where $B(a, b) \mapsto ab$.

Proposition 6.1. If A is an *associative unital* K-algebra, then A has a ring structure.

Proof. $(a_1+a_2)b = B(a_1+a_2, b) = B(a_1, b)+B(a_2, b) = a_1b+a_2b$. The other properties are similar. □

Definition: A homomorphism of K-algebras is a K-linear map $F : A \to A'$ such that $F(B(a, b)) = B'(F(a), F(b))$. In particular, if A is an associative unital algebra then F is a linear ring homomorphism.

Proposition 6.2. A G-representation (V, ρ_V) induces a homomorphism of $\mathbb{C}$-algebras $\rho_V : \mathbb{C}[G] \to \mathrm{End}\,(V) = \mathrm{Hom}\,(V, V)$ given by,

$$\rho_V\left(\sum_{g\in G} t_g \cdot g\right) = \sum_{g\in G} t_g \cdot \rho_V(g)$$

or alternatively given a map $f : G \to \mathbb{C}$ define,

$$\rho_V(f) = \sum_{g\in G} f(g)\rho_V(g)$$

Proposition 6.3. Let $V = \mathbb{C}[G]$ then the regular representation induces a homomorphism $\rho_{\mathbb{C}[G]} : \mathbb{C}[G] \to \mathrm{End}\,(\mathbb{C}[G])$. This map is given by $\rho_{\mathbb{C}[G]}(\alpha)(\beta) = \alpha\beta$.

Theorem 6.4 (Weddenburn)**.** Define $\rho : \mathbb{C}[G] \to \mathrm{End}\,(V_1) \times \cdots \times \mathrm{End}\,(V_h)$ where $V_1, \cdots, V_h$ enumerates all the irreducible G-representations by the map,

$$\rho(\alpha) = (\rho_{V_1}(\alpha), \cdots, \rho_{V_h}(\alpha))$$

where $\rho_{V_i}(\alpha) = \sum\limits_{g\in G} \alpha(g)\rho_V(g)$ for $\alpha \in \mathbb{C}[G]$. Then, ρ is an isomorphism of $\mathbb{C}$-algebras.

Proof. $\dim \mathbb{C}[G] = |G|$ and $\dim\,(\mathrm{End}\,(V_1) \times \cdots \times \mathrm{End}\,(V_h)) = \dim \mathrm{End}\,(V_1) + \cdots + \dim \mathrm{End}\,(V_h) = (\dim V_1)^2 + \cdots + (\dim V_h)^2 = d_1 + \cdots + d_h^2 = |G|$. Therefore, to prove that ρ is an isomorphism of $\mathbb{C}$-algebras it suffices to prove that ρ is an injective $\mathbb{C}$-algebra homomorphism. Suppose that $\rho(\alpha) = 0$ then $\rho_{V_i}(\alpha) = 0$ for all i. Therefore, $\rho_V(\alpha) = 0$ for every representation because we have shown this for every irreducible component. In particular, $\rho_{\mathbb{C}[G]}(\alpha) = 0$ and in particular $\rho_{\mathbb{C}[G]}(\alpha)(1) = \alpha = 0$ so $\alpha = 0$. Therefore ρ is injective and thus an isomorphism. □

Theorem 6.5 (Hard)**.** Suppose K is a field of characteristic zero then,

$$K[G] \cong \mathrm{End}\,(D_1) \times \cdots \times \mathrm{End}\,(D_h)$$

where D_i is not necessarily a field but a division ring.

Lemma 6.6. The center $Z(\mathbb{C}[G]) \cong Z$ the set of class functions.

Proof. Suppose $g \in Z(\mathbb{C}[G])$ if and only if $\forall g \in \mathbb{C}[G]$ we have $f * g = g * f$. Thus,

$$f \in Z(\mathbb{C}[G]) \iff \delta_x * f = f * \delta_x \iff f(x^{-1}y) = f(yx^{-1}) \iff f(h) = f(xhx^{-1}) \iff f \in Z$$

$\square$

Remark 1. We will sometimes refer to $\rho : \mathbb{C}[G] \to \mathrm{End}\,(V_1) \times \cdots \times \mathrm{End}\,(V_h)$ as the Fourier transform.

Proposition 6.7. For $(A_1, \cdots, A_n) \in \mathrm{End}\,(V_1) \times \cdots \times \mathrm{End}\,(V_h)$ we have,

$$\rho^{-1}(A_1, \cdots, A_n) = \sum_{g \in g} t_g \cdot g$$

where

$$t_g = \frac{1}{|G|} \sum_{i=1}^{h} d_i \mathrm{Tr}\big(\rho_{V_i}(g^{-1}) \cdot A_i\big)$$

Proof. We know that ρ is an isomorphism so ρ takes any basis of $\mathbb{C}[G]$ to an basis of $End V_1 \times \cdots \times \mathrm{End}\,(V_h)$. $\square$

Classical Finite Fourier Analysis

Let G be an abelian group.

Definition: The dual group is $\hat{G} = \{\lambda : G \to \mathbb{C}^\times \mid \lambda \text{ is a homo.}\}$ with pointwise multiplication.

Proposition 6.8. $|\hat{G}| = |G|$

Proof. Suppose the group G is cyclic, all its irreducible representations are finite. Therefore, there is a one-to-one correspondence between irreducible representations and homomorphisms $\lambda : G \to \mathbb{C}^\times$. However, there are exactly $|G|$ irreducible representations because in an abelian group every element defines a distinct conjugacy class. $\square$

Proposition 6.9. For a finite group $G \cong \hat{G}$ (but not naturally) and $G \cong \hat{\hat{G}}$ naturally.

Definition: The Fourier transform is a map $\mathbb{C}[G] \to \mathbb{C}[\hat{G}]$ given by $f \mapsto \hat{f}$ where,

$$\hat{f}(\lambda) = |G| \langle f, \lambda \rangle = \sum_{g \in G} f(g)\lambda(g)$$

Proposition 6.10. The Fourier transform satisfies,

- $\widehat{f_1 * f_2} = \hat{f}_1 \cdot \hat{f}_2$

- Inversion: $f = \frac{1}{|G|} \sum\limits_{\lambda \in G} \hat{f}(\lambda) \cdot \lambda$ such that $f = \hat{\hat{f}}$ up to normalization.
- $\langle f_1, f_2 \rangle = \frac{1}{|G|} \left\langle \hat{f}_1, \hat{f}_2 \right\rangle$

Proof. Because λ forms a unitary basis,

$$f = \sum_{\lambda \in \hat{G}} \langle f, \lambda \rangle \cdot \lambda = \frac{1}{|G|} \sum_{\lambda} \hat{f}(\lambda) \cdot \lambda$$

Furthermore,

$$\langle f_1, f_2 \rangle = \sum_{\lambda \in \hat{G}} \langle f_1, \lambda \rangle \langle \lambda, f_2 \rangle = \sum_{\lambda \in \hat{G}} \langle f_1, \lambda \rangle \langle \lambda, f_2 \rangle = \frac{1}{|G|^2} \sum_{\lambda \in \hat{G}} \hat{f}_1(\lambda) \overline{\hat{f}_2}(\lambda) = \frac{1}{|G|} \left\langle \hat{f}_1, \hat{f}_2 \right\rangle$$

$\square$

Theorem 6.11. Let G be a finite abelian group then the map,

$$ev : G \to \hat{\hat{G}}$$

is an isomorphism and $ev : f \mapsto \hat{\hat{f}} = |G| f(g^{-1})$.

7 One-Dimensional Representations

Theorem 7.1. Let G be finite. The number of one-dimensional representations of G is the order of G^{ab}.

Proof. Any one-dimensional representation is given by a homomorphism $\lambda : G \to \mathbb{C}^\times$. However, $\mathbb{C}^\times$ is abelian so such homomorphisms are in one-to-one correspondence with homomorphisms $G^{ab} \to \mathbb{C}^\times$ i.e. to the group $\widehat{G^{ab}}$. Therefore, the number of one-dimensional representations is $|G^{ab}|$ and thus this number divides $|G|$. $\square$

Lemma 7.2. A subgroup $N \lhd G$ such that $N \subset G'$ and G/N is abelian then $N = G'$

Proof. We know that G/N is abelian and $\pi : G \to G/N$ is a homomorphism so $G' \subset \ker \pi = N$. Thus, $N = G'$. $\square$

8 Product Groups

Theorem 8.1. Let ρ_{V_1} be an irreducible G_1-representation and ρ_{V_2} be an irreducible G_2-representation then $\rho_{V_1 \otimes V_2} : G_1 \times G_2 \to \mathrm{Aut}\,(V_1 \otimes V_2)$ given by,

$$\rho_{V_1 \otimes V_2}(g_1, g_2) = \rho_{V_1}(g_1) \otimes \rho_{V_2}(g_2)$$

is an irreducible $G_1 \times G_2$ representation and every irreducible $G_1 \times G_2$ representation is of this form.

Proof. The chracter is given by,

$$\chi_{V_1 \otimes V_2}(g_1, g_2) = \mathrm{Tr}(\rho_{V_1 \otimes V_2}(g_1, g_2))) = \mathrm{Tr}(\rho_{V_1}(g_1)) \cdot \mathrm{Tr}(\rho_{V_2}(g_2)) = \chi_{V_1}(g_1) \cdot \chi_{V_2}(g_2)$$

Therefore,

$$\begin{aligned}\langle \chi_{V_1 \otimes V_2}, \chi_{V_1 \otimes V_2} \rangle &= \frac{1}{|G_1 \times G_2|} \sum_{(g_1, g_2) \in G_1 \times G_2} |\chi_{V_1 \otimes V_2}(g_1, g_2)|^2 \\ &= \frac{1}{|G_1||G_2|} \sum_{g_1 \in G} |\chi_{V_1}(g_1)|^2 \sum_{g_2 \in G} |\chi_{V_2}(g_2)|^2 = \langle \chi_{V_1}, \chi_{V_1} \rangle \cdot \langle \chi_{V_2}, \chi_{V_2} \rangle = 1\end{aligned}$$

and therefore $\rho_{V_1 \otimes V_2}$ is irreducible.

Furthermore, (WIP) □

9 Burnside's Theorem

Definition: $c(x) = |Cl(x)|$ is the size of the conjugacy class of x.

Lemma 9.1. If G is finite and ρ_V is a G-representation, then $\chi_V(g)$ is an algebraic integer.

Proof. We know that $\rho_V(g)$ is diagonalizable and each eigenvalue is a root of unity because $\rho_V(g)^n = \rho_V(g^n) = \rho_V(e) = \mathrm{id}$. Therefore, $\chi_V(g) = \mathrm{Tr}(\rho_V(g))$ is the sum of roots of unity which is an algebraic integer. □

Theorem 9.2. Let V be an irreducible G-representation with $\dim V = d_V$ then for all $g \in G$ the number $\frac{c(g)}{d_V}\chi_V(g)$ is an algebraic integer.

Proof. Define the map $\rho_V : \mathbb{C}[G] \to \mathrm{End}\,(V)$ by,

$$\rho_V(f) = \sum_{g \in G} f(g)\rho_V(g)$$

We know that since V is irreducible if f is a class function then,

$$\rho_V(g) = \frac{|G| \langle f, \overline{\chi_V} \rangle}{\dim V} \cdot \mathrm{id}$$

Since $\delta_{Cl(x)}$ is a class function,

$$\rho_V(\delta_{Cl(x)}) = \frac{|G| \left\langle \delta_{Cl(x)}, \overline{\chi_V} \right\rangle}{d_V} \cdot \mathrm{id}$$

but we know that,

$$\left\langle \delta_{Cl(x)}, \overline{\chi_V} \right\rangle = \frac{1}{|G|} \sum_{g \in G} \delta_{Cl(x)}(g)\chi_V(g) = \frac{1}{|G|} \sum_{g \in Cl(x)} \chi_V(g) = \frac{c(x)}{|G|}\chi_V(x)$$

since χ_V is a class function. Therefore,

$$\rho_V(\delta_{Cl(x)}) = \frac{c(x)}{d_V}\chi_V(x)\cdot\mathrm{id}$$

Therefore,

$$\frac{c(x)}{d_V}\chi_V(x)$$

is the eigenvalue of the map $\rho_V(\delta_{Cl(x)}$ which must be an algebraic integer. $\square$

Theorem 9.3 (Frobenius)**.** If V is irreducible then $d_V \mid |G|$.

Proof. $\langle \chi_V, \chi_V\rangle = 1$ so $|G| = \sum_{g\in G}\chi_V(g)\overline{\chi_V(g)}$. We wrtite G as the disjoint union over conjugacy classes. Thus,

$$|G| = \sum_{i=1}^{n}\sum_{g\in Cl(x_i)}\chi_V(g)\overline{\chi_V(g)} = \sum_{i=1}^{h}c(x_i)\chi_V(x_i)\overline{\chi_V(x_i)}$$

Therefore,

$$\frac{|G|}{d_V} = \sum_{i=1}^{h}\left(\frac{c(x_i)\chi_V(x_i)}{d_V}\right)\overline{\chi_V(x_i)}$$

is the sum of products of algebraic integers and thus an algebraic integer. Therefore, $|G|/d_V$ is an algebraic integer but also rational. therefore $|G|/d_V \in \mathbb{Z}$ so $d_V \mid |G|$. $\square$

Lemma 9.4. Let $\lambda_1, \cdots, \lambda_d$ be roots of unity. Then,

1. $|\lambda_1 + \cdots + \lambda_d| \leq d$ with equality iff $\lambda_1 = \cdots = \lambda_d$.
2. $\alpha = \frac{1}{d}(\lambda_1 + \cdots + \lambda_d)$ is an algebraic integer if and only if $\alpha = 0$ or $\lambda_1 = \cdots = \lambda_d$.

Proof. $\square$

Lemma 9.5. Let G be finite and V any G-representation of dimension $d = d_V$ then,

1. $\forall g \in G : |\chi_V(g)| \leq d_V$ with equality iff $\rho_V(g) = \frac{\chi_V(g)}{d_V}\mathrm{id}$
2. $\forall g \in G : \chi_V(g) = d_V \iff \rho_V(g) = \mathrm{id} \iff g \in \ker \rho_V$.

Proof. We know that $\rho_V(g)$ is diagonalizable with eigenvalues which are roots of unity. Therefore $\chi_V(g) = \lambda_1 + \cdots + \lambda_d$. Thus, $|\chi_V(g)| \leq d_V$ with equality iff $\lambda_1 = \cdots = \lambda_d = \frac{\chi_V(g)}{d_V}$ so $\rho_V(g) = \frac{\chi_V(g)}{d_V}\mathrm{id}$. Furthermore,

$$\chi_V(g) = d_V \implies |\chi_V(g)| = d_V \implies \rho_V(g) = \frac{\chi_V(g)}{d_v}\mathrm{id} = \mathrm{id}$$

And clearly if $\rho_V(g) = \mathrm{id}$ then $\chi_V(g) = \mathrm{Tr}(\mathrm{id}) = d_V$. $\square$

Corollary 9.6. A finite group G is not simple iff there exists a nontrivial irreducible G-representation V such that $\exists g \in G \backslash \{e\} : \chi_V(g) = \chi_V(e) = d_V$.

Proof. G is not simple if there exists $N \triangleleft G$ such that N is nontrivial and proper. Therefore, G/N is not isomorphic to G or $\{e\}$. Therefore, there must exist a nontrivial representation $\rho_V : G/N \to \mathrm{Aut}\,(V)$ of G/N which lifts under $\pi : G \to G/N$ to a representation $\pi^* \rho_V = \rho_V \circ \pi : G \to \mathrm{Aut}\,(V)$.

Converseley, choose ρ_V which is a nontrivial irreducible G-representation such that $\exists g \in G \backslash \{e\} : \chi_V(g) = \chi_V(e) = d_V$. Then, $\ker \rho_V \triangleleft G$ but $\ker \rho_V \neq G$ since ρ_V is nontrivial. However, there exists $g \in G \backslash \{e\}$ such that $\chi_V(g) = d_V$ which implies that $g \in \ker \rho_V$ so $\ker \rho_V$ is nontrivial. Thus, G is not simple becuase $\ker \rho_V$ is a nontrivial proper subgroup. □

Proposition 9.7. Let G be a finite group, let V be an irreducible G-representation suppose that $\gcd(c(g), d_V) = 1$ then $\chi_V(g) = 0$ or $\rho_V(g) = \lambda \cdot \mathrm{id}$.

Proof. Since $gcd(c(x), d_V) = 1$ we know that $\exists a, b \in \mathbb{Z}$ such that $ac(x) + bd_V = 1$ but,

$$\frac{\chi_V(g)}{d_V} = (ac(x) + bd_V) \frac{\chi_V(g)}{d_V} = a \left(\frac{c(x)\chi_V(g)}{d_V} \right) + b\chi_V(g)$$

which is the sum of algebraic integers. Thus, $\frac{\chi_V(g)}{d_V}$ is an algebraic integer. However, $\chi_V(g) = \lambda_1 + \cdots + \lambda_d$ is a sum of roots of unity. Therefore, since $\frac{1}{d}(\lambda_1 + \cdots + \lambda_d)$ is an algebraic integer, we know that $\lambda_1 + \cdots + \lambda_d = 0$ so $\chi_V(g) = 0$ or $\lambda_1 = \cdots = \lambda_d$ so $\chi_V(g) = \lambda \cdot \mathrm{id}$. □

Corollary 9.8. Let G be a finite simple nonabelian group and V a nontrivial irreducible G-representation then $\gcd(c(g), d_V) = 1 \implies \chi_V(g) = 0$.

Proof. G is simple so ρ_V is injective since $\ker \rho_V$ is normal and ρ_V is nontrivial. Therefore, take g as in the condition, if $\chi_V(g) \neq 0$ then $\rho_V(g) = \lambda \cdot \mathrm{id}$. Therefore, $\rho_V(g) \in Z(\mathrm{Aut}\,(V))$ so $\mathrm{Im}(\rho_V)$ is abelian so $G' \subset \ker \rho_V = \{e\}$. Therefore $G' = \{e\}$ which implies that $G/G' \cong G$ is abelian which contradicts the assumption that G is nonabelian. Thus, $\chi_V(g) = 0$. □

Theorem 9.9. Let G be a nonabelian finite simple group let $g \in G \backslash \{e\}$ then $c(g)$ is not a prime power.

Proof. Suppose that $|Cl(g)| = p^a$ for some prime p. If $a = 0$ then $a \in Z(G)$ but $Z(G) \neq G$ because G is nonabelian so $Z(G)$ is a nontrivial proper normal subgroup contradicting simplicity. Let V be an irreducible G-representation. If $\gcd(c(x), d_V) = 1$ then $\chi_V(g) = 0$. Therefore, if $p \not| \, d_V$ then $\chi_V(g) = 0$ so either $p \mid d_V$ or $\chi_V(g) = 0$. Consider,

$$\chi_{\mathrm{reg}}(g) = 0 = \sum_{i=1}^{h} d_i \chi_{V_i}(g) = 1 + \sum_{i=2}^{h} d_i \chi_{V_i}(g)$$

However, $\chi_V(g) = 0$ or $p \mid d_i$ so $\frac{d_i \chi_{V_i}(g)}{p}$ is an algebraic integer. Therefore,

$$\frac{1}{p} \sum_{i=2}^{h} d_i \chi_{V_i}(g) = -\frac{1}{p}$$

is an algebraic integer but $\frac{-1}{p}$ is rational so it would need to be in $\mathbb{Z}$ which is clearly false. Thus, $|Cl(g)| = p^a$ is false. □

Theorem 9.10 (Burnside)**.** If $|G| = p^a q^b$ for primes p, q and $a, b \geq 1$ then G is not simple.

Proof. Assume that G is simple. We know that G cannot be abelian because G does not have prime order. However, for all $g \in G$ we know that $c(g)$ is not a prime power. However,

$$|G| = p^a q^b = \sum_{i=1}^{h} |Cl(x_i)| = 1 + \sum_{i \geq 2}^{h} |Cl(x_i)|$$

However, the nontrivial conjugacy classes divide $p^a q^b$ and cannot be prime powers so they each must be divisible by pq. Thus,

$$p^a q^b = 1 + \sum_{i \geq 2}^{h} |Cl(x_i)| \equiv 1 \bmod p \quad \text{and} \quad p^a q^b = 1 + \sum_{i \geq 2}^{h} |Cl(x_i)| \equiv 1 \bmod q$$

which are clearly contradictions. □

10 Induced Representations

Definition: Let G be a finite group and $H \subset G$ a subgroup then the induced representation,

$$\text{Ind}_H^G(W) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

as a left $\mathbb{C}[G]$ module thus a G-representation. Alternatively,

$$\text{Ind}_H^G(W) = \text{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W) = \{f : G \to W \mid f(hg) = \rho_W(h) f(g)\}$$

Proposition 10.1. Properties of the induced representation.

1. $$\text{Ind}_H^G(\mathbb{C}) \cong \mathbb{C}[G/H]$$

2. $$\text{Ind}_G^G(V) \cong V$$

Remark 2 (Notation)**.** Let $x_1, \cdots, x_n$ be representatives for G/H. Then, $gx_i \in gx_iH = x_{j(i,g)}H$ so $gx_i = x_{j(i,g)} h_i(g)$

We want to determine the structure $\text{Ind}_H^G(W)$.

Definition: For $w \in W$, let $F_{i,w} : G \to W$ be given by,

$$F_{i,w}(g) = \rho_W(h)^{-1}(w)$$

where $g = x_i h \in x_i H$ and zero otherwise.

Proposition 10.2. Properties of $F_{i,w}$,

1. $F_{i,w} \in \mathrm{Ind}_H^G(W)$
2. $F_{i,w_1+w_2} = F_{i,w_1} + F_{i,w_2}$
3. $F_{i,t\cdot w} = t \cdot F_{i,w}$
4. $W^{(i)} = \{F_{i,w} \mid w \in W\}$ is a vector subspace of $\mathrm{Ind}_H^G(W)$ and,
$$W^{(i)} = \{F \in \mathrm{Ind}_H^G(W) \mid F(g) = 0 \text{ if } g \notin x_i H\}$$
5. $\forall F \in \mathrm{Ind}_H^G(W)$ we have $F = \sum_{i=1}^{k} F_{i,w_i}$ where $w_i = F(x_i)$.
6. We have the isomorphism of vectorspaces,
$$\mathrm{Ind}_H^G(W) \cong \bigoplus_{i=1}^{k} W^{(i)}$$
Therefore,
$$\dim \mathrm{Ind}_H^G(W) = k \dim W = [G : H] \dim W$$

Proposition 10.3.

$$\rho_{\mathrm{Ind}_H^G(W)}(g) \cdot F_{i,w} = F_{j(i,g),\rho_W(h_i(g))\cdot w}$$

Proof. Consider, $\rho(g)\cdot F_{i,w}(x_\ell) = F_{i,w}(g^{-1}x_\ell)$. Now, $g^{-1}x_\ell \in x_i H$ so $x_\ell \in g x_i H = x_j H$ therefore zero unless $\ell = j$. Assume that $\ell = j$ then $\rho(g) \cdot F_{i,w}(x) = F_{i,w}(g^{-1}x$ but $x \in x_j H$ so $x = x_j h$ □

Theorem 10.4 (Frobenius Reciprocity)**.**

$$\mathrm{Hom}^H\left(W, \mathrm{Res}_H^G(U)\right) \cong \mathrm{Hom}^G\left(\mathrm{Ind}_H^G(W), U\right)$$

Theorem 10.5. For any class functions $f_1 : H \to \mathbb{C}$ and $f_2 : G \to \mathbb{C}$ we have,

$$\left\langle f_1, \mathrm{Res}_H^G(f_2)\right\rangle_H = \left\langle \mathrm{Ind}_H^G(f_1), f_2\right\rangle_G$$

Proof. and the right hand side is,

$$\left\langle \mathrm{Ind}_H^G(f_1), f_2\right\rangle_G = \frac{1}{|G|}\sum_{g\in G} \mathrm{Ind}_H^G(f_1)(g)\overline{f_2(g)} = \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{x\in G}\tilde{f}_1(x^{-1}gx)\overline{f_2(g)}$$

Rewriting,

$$\left\langle \mathrm{Ind}_H^G\left(f_1\right), f_2\right\rangle_G = \frac{1}{|G|\cdot|H|}\sum_{g,x\in G}\tilde{f}_1(x^{-1}gx)\overline{f}_2(g)$$
$$= \frac{1}{|G|\cdot|H|}\sum_{g,x\in G}\tilde{f}_1(g)\overline{f}_2(xgx^{-1}) = \frac{1}{|H|}\sum_{g\in G}\tilde{f}_1(g)\overline{f}_2(g)$$

where I have used the fact that f_2 is a G-class function. However, $\tilde{f}(g) = 0$ unless $g \in h$ so the left hand side becomes,

$$\left\langle \mathrm{Ind}_H^G\left(f_1\right), f_2\right\rangle_G = \frac{1}{|H|}\sum_{h\in H} f_1(h)\overline{f_2(h)} = \left\langle f_1, \mathrm{Res}_H^G\left(f_2\right)\right\rangle_H$$

□

Corollary 10.6.

$$\left\langle \chi_W, \chi_{\mathrm{Res}_H^G(U)}\right\rangle_H = \left\langle \chi_{\mathrm{Ind}_H^G(W)}, \chi_U\right\rangle_G$$

Theorem 10.7 (Projection Formula)**.**

$$\mathrm{Ind}_H^G\left(W \otimes \mathrm{Res}_H^G\left(U\right)\right) = \left(\mathrm{Ind}_H^G\left(W\right)\right) \otimes U$$

Corollary 10.8.

$$\mathrm{Ind}_H^G\left(\mathrm{Res}_H^G\left(V\right)\right) = \mathrm{Ind}_H^G\left(\mathrm{Res}_H^G\left(\mathbb{C}\otimes V\right)\right) = \mathbb{C}[G/H]\otimes V$$

Definition:

Theorem 10.9. Suppose that W is irreducible then $\mathrm{Ind}_H^G\left(W\right)$ is irreducible if and only if $\forall x \in G\backslash H$ the representations W and W_x are not isomorphic G-representations.

Proof.

$$\left\langle \chi_{\mathrm{Ind}_H^G(W)}, \chi_{\mathrm{Ind}_H^G(W)}\right\rangle_G = \left\langle \chi_W, \chi_{\mathrm{Res}_H^G\left(\mathrm{Ind}_H^G(W)\right)}\right\rangle_H$$

□

Definition: Let $H \subset G$ and $[G : H] = 2$ then define the homomorphism $\epsilon : G \to \{\pm 1\} \subset \mathbb{C}^\times$ by,

$$\epsilon(g) = \begin{cases} 1 & g \in H \\ 0 & g \notin H \end{cases}$$

Theorem 10.10. Let V be an irreducible G-representation, $W = \mathrm{Res}_H^G\left(V\right)$ and let $V \otimes \epsilon$ correspond to $\epsilon\rho_V$. Then, exactly one of the following holds,

1. $V \cong V \otimes \epsilon$ and $W \cong W' \oplus W'_x$ where W' is irreducible and $W' \ncong W'_x$ and $V \cong \mathrm{Ind}_H^G\left(W'\right) \cong \mathrm{Ind}_H^G\left(W'_x\right)$.

2. $V \ncong V \otimes \epsilon$ and $W \cong W_x$ is irreducible and $\mathrm{Ind}_H^G\left(W\right) \cong V \otimes \left(V \otimes \epsilon\right)$.

11 Real Representations

Definition: A G-representation $\rho_V : G \to \mathrm{Aut}(V)$ is real if V is an $\mathbb{R}$-vectorspace.

Proposition 11.1. If ρ_V is a real representation then $V \cong V^*$ as a G-reprsentation.

Proof. If ρ_V is real then χ_V is real so $\chi_V = \overline{\chi_V}$ and thus $V \cong V^*$. $\square$

Remark 3. The condition $V \cong V^*$ is not sufficient to show that ρ_V is the complexification of a real representation.

Theorem 11.2. Let V be an irreducible G-representation then,

1. $V \ncong V^*$ and V cannot be defined over $\mathbb{R}$ if and only if $(\mathrm{Bil}\, V)^G = 0$.
2. $V \cong V^*$ and V cannot be defined over $\mathbb{R}$ if and only if $\dim \left(\bigwedge^2 V^*\right)^G = 1$.
3. $V \cong V^*$ and V can be defined over $\mathbb{R}$ if and only if $\dim (\mathrm{Sym}\, V)^G = 1$.

Proof. We know that $\mathrm{Bil}\, V \cong \mathrm{Hom}\,(V, V^*)$ so $(\mathrm{Bil}\, V)^G = \mathrm{Hom}^G\,(V, V^*) = 0$ if and only if $V \ncong V^*$.

Furthermore, $\square$

12 Representations of the Symmetric Group

Remark 4. For any n we always have the 1-dimensional (irreducible) representations $\mathbb{C}$ and $\mathbb{C}(\epsilon)$ and the n-dimensional permutation representation $\mathbb{C}^n \cong \mathbb{C} \oplus V$ where V is an $(n-1)$-dimensional irreducible S_n-representation.

Lemma 12.1. Any $\sigma \in S_n$ can be written as a unique product of disjoint nontrivial cycles $\gamma_1 \cdots \gamma_k$ ordered by length. The cycle type of σ is $(n_1, \cdots, n_k)$ where n_i is the length of γ_i. Futhermore, there is a one-to-one correspondence between cycle types and conjugacy classes.

Definition: λ is a partition of n written as $\lambda \vdash n$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ such that,

$$\sum_{i=1}^{\ell} \lambda_i = n$$

Proposition 12.2. Every $\sigma \in S_n$ determines a partition of n. Furthermore, the action of $\langle \sigma \rangle$ on S_n by partition S_n into orbits of sizes $\lambda_i, \ldots, \lambda_\ell$.

Proposition 12.3. Conjugacy classes of S_n are indexed by partitions $\lambda \vdash n$.

Definition: The Young Subgroup of a partition $\lambda \vdash n$ is the group $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ where $\sigma \in S_\lambda$ means that σ preserves the partition λ of the set $\{1, \cdots, n\}$.

Definition: For each $\lambda \vdash n$ we get an S_n-representation,

$$M^\lambda = \mathbb{C}[S_n/S_\lambda] = \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C})$$

For example, for the extreme partitions $\lambda = (n)$ we have $S_\lambda = S_n$ so $M^{(n)} = \mathbb{C}[S_n/S_n] = \mathbb{C}$. Furthermore, if $\lambda = (1, \cdots, 1)$ then $S_\lambda = \{e\}$ so $M^{(1,\cdots,1)} = \mathbb{C}[S_n]$ the regular representation.

Definition: Given two partitions $\lambda, \mu \vdash n$ then λ dominates μ writen as $\lambda \trianglerighteq \mu$ if,

$$\forall i : \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

Proposition 12.4. Domination is a partial order on the set of paritions of n and for any $\lambda \vdash n$ we have $(n) \trianglerighteq \lambda \trianglerighteq (1, \cdots, 1)$