Physics GR8049 Quantum Field Theory III
Assignment # 1

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1 Unitarity Bound

1.1

Consider the conformal algebra with generators D, K_μ, P_μ, and $M_{\mu\nu}$ satisfying the standard (Euclidean) commutation relations of which I only write down the relevant ones,

$$\begin{aligned}
[D, P_\mu] &= P_\mu \\
[D, K_\mu] &= -K_\mu \\
[K_\mu, P_\nu] &= 2\left(\delta_{\mu\nu} D - M_{\mu\nu}\right) \\
[M_{\mu\nu}, P_\rho] &= \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu \\
[M_{\mu\nu}, K_\rho] &= \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu
\end{aligned}$$

Let $|\Delta\rangle = \mathcal{O}_\Delta(0)\,|\Omega\rangle$ be a spinless primary state corresponding to the spinles primary operator $\mathcal{O}_\Delta(x)$ with scaling dimension Δ i.e. $M_{\mu\nu}\,|\Delta\rangle = 0$ and $K_\mu\,|\Delta\rangle = 0$. This state has scaling dimension Δ i.e. $D\,|\Delta\rangle = \Delta\,|\Delta\rangle$. Then consider the combination,

$$\begin{aligned}
\langle\Delta|\, K_\mu K_\nu P_\rho P_\sigma\,|\Delta\rangle &= \langle\Delta|\, K_\mu [K_\nu, P_\rho] P_\sigma\,|\Delta\rangle + \langle\Delta|\, K_\mu P_\rho K_\nu P_\sigma\,|\Delta\rangle \\
&= \langle\Delta|\, K_\mu 2(\delta_{\nu\rho} D - M_{\nu\rho}) P_\sigma\,|\Delta\rangle + \langle\Delta|\, K_\mu P_\rho [K_\nu, P_\sigma]\,|\Delta\rangle + \langle\Delta|\, K_\mu P_\rho P_\sigma K_\nu\,|\Delta\rangle
\end{aligned}$$

Where I have used the fact that $K_\nu\,|\Delta\rangle = 0$ since $|\Delta\rangle$ is primary. Then we have,

$$\begin{aligned}
\langle\Delta|\, K_\mu K_\nu P_\rho P_\sigma\,|\Delta\rangle &= \langle\Delta|\, K_\mu 2(\delta_{\nu\rho} D - M_{\nu\rho}) P_\sigma\,|\Delta\rangle + \langle\Delta|\, K_\mu P_\rho 2(\delta_{\mu\sigma} D - M_{\mu\sigma})\,|\Delta\rangle \\
&= \langle\Delta|\, K_\mu 2[\delta_{\nu\rho} D - M_{\nu\rho}, P_\sigma]\,|\Delta\rangle + \langle\Delta|\, K_\mu P_\sigma 2(\delta_{\nu\rho} D - M_{\nu\rho})\,|\Delta\rangle + \langle\Delta|\, K_\mu P_\rho 2\delta_{\mu\sigma}\Delta\,|\Delta\rangle \\
&= 2\,\langle\Delta|\, K_\mu(\delta_{\nu\rho} P_\sigma - (\delta_{\rho\sigma} P_\nu - \delta_{\nu\sigma} P_\rho))\,|\Delta\rangle + 2(\delta_{\mu\sigma} + \delta_{\nu\sigma})\Delta\,\langle\Delta|\, K_\mu P_\rho\,|\Delta\rangle
\end{aligned}$$

Therefore, taking traces,

$$\begin{aligned}
\langle\Delta|\, K_\mu K^\mu P_\rho P^\rho\,|\Delta\rangle &= 2\,\langle\Delta|\, K_\mu(\delta^{\mu\rho} P_\rho - (d\delta^{\mu\nu} P_\nu - \delta^{\mu\rho} P_\rho))\,|\Delta\rangle + 4\delta^{\mu\rho}\Delta\,\langle\Delta|\, K_\mu P_\rho\,|\Delta\rangle \\
&= 2(2 - d + 2\Delta)\,\langle\Delta|\, K^\mu P_\mu\,|\Delta\rangle
\end{aligned}$$

However, in radial quantization we have the relation $P_\mu^\dagger = K_\mu$ and thus,

$$|P_\mu P^\mu\,|\Delta\rangle\,|^2 = \langle\Delta|\, P_\mu^\dagger (P^\mu)^\dagger P_\rho P^\rho \ker\Delta = \langle\Delta|\, K_\mu K^\mu P_\rho P^\rho \ker\Delta \geq 0$$

Furthermore,

$$\langle\Delta|\, K^\mu P_\mu\,|\Delta\rangle = \delta^{\mu\rho}\,\langle\Delta|\, P_\rho^\dagger P_\mu\,|\Delta\rangle = \sum_\mu |P_\mu\,|\Delta\rangle\,|^2 \geq 0$$

Since both sides of the derived expression must be positive and $\langle\Delta| K_\mu P_\rho |\Delta\rangle \geq 0$ we find that,

$$2 - d + 2\Delta \geq 0 \implies \Delta \geq \frac{d-2}{2}$$

giving the unitarity bound in arbitrary dimensions.

1.2

Using the scalar equations of motion, we find that, for a scalar of mass m, the primary scaling dimension must satisfy,

$$\Delta(\Delta - d) = m^2$$

which has solutions,

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{m^2 + \left(\frac{d}{2}\right)^2}$$

Since the modes have asymptotic behavior $r^{-\Delta}$ basic conditions on normalizability imply that $\Delta > 0$. The range in which normalization issue do not arise for either choice of Δ occurs when both Δ_+ and Δ_- are positive. Therefore, we must have,

$$\frac{d}{2} > \sqrt{m^2 + \left(\frac{d}{2}\right)^2} \implies m^2 < 0$$

However, the scaling dimension Δ cannot become complex since this will violate the Hermiticity of D whose eigenvalues are Δ. Therefore, we are limited by the minimum value of the quadratic $\Delta(\Delta - d)$ over the reals which occurs at $\Delta = \frac{d}{2}$ and gives,

$$m^2 \geq -\left(\frac{d}{2}\right)^2$$

Therefore, the consistent range of masses for two choices of boundary conditions is,

$$-\left(\frac{d}{2}\right)^2 \leq m^2 \leq 0$$

1.3

Suppose we choose to quantize both modes with "standard" and "alternate" boundary conditions simultaneously. Then we have a mode expansion containing both,

$$\phi_+(t,x) = \frac{e^{-i\Delta_+ t}}{(1+r^2)^{\Delta_+/2}} \quad \text{and} \quad \phi_-(t,x) = \frac{e^{-i\Delta_- t}}{(1+r^2)^{\Delta_-/2}}$$

However, we quantize with respect to the Klien-Gordon inner product,

$$(\phi_1, \phi_2) = -i \int \sqrt{-g} g^{tt} \mathrm{d}^3x \ (\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*)$$

However, applying this inner product to the plus and minus modes we find,

$$
\begin{aligned}
(\phi_+, \phi_-) &= -i \int \sqrt{-g} g^{tt} \mathrm{d}^3 x \, \left(\phi_+^* \partial_t \phi_- - \phi_- \partial_t \phi_+^* \right) \\
&= - \int \sqrt{-g} g^{tt} \mathrm{d}^3 x \, (1+r^2)^{-\frac{\Delta_+ + \Delta_-}{2}} \left(e^{i(\Delta_+ - \Delta_-)t} \Delta_- + e^{i(\Delta_+ - \Delta_-)t} \Delta_+ \right) \\
&= - \left(\Delta_- + \Delta_+ \right) e^{i(\Delta_+ - \Delta_-)t} \int \sqrt{-g} g^{tt} \mathrm{d}^3 x \, (1+r^2)^{-\frac{\Delta_+ + \Delta_-}{2}}
\end{aligned}
$$

which is both nonzero and time-dependent. Since ϕ_+ and ϕ_- are supposed to be energy modes with different energy eigenvalues, namely Δ_+ and Δ_- respectively, if $\hat{H}$ is Hermitian then they must be orthogonal. However, since they are not in fact orthogonal, this quantization prescription violates the Hermiticity of $\hat{H}$ and thus the unitarity of the time evolution operator $e^{-i\hat{H}t}$. More directly, unitary time evolution implies that inner products are invariant under time translation and therefore cannot be time-dependent as we observed the inner product (ϕ_+, ϕ_-) to be.

2 Boundary Correlators of EAdS and Conformal Symmetry

2.1

The boundary two-point function has the expression,

$$
\langle \mathcal{O}(\vec{u}) \mathcal{O}(\vec{u}') \rangle = \frac{c_\Delta}{|\vec{u} - \vec{u}'|^{2\Delta}}
$$

where the constant c_Δ is determined by matching the propagator in Minkowski space in the short distance limit to be,

$$
c_\Delta = \frac{\Gamma(\Delta)}{2\pi^{d/2} \Gamma(\Delta - 1 - \frac{d}{2})}
$$

We have show that there is a unitarity bound,

$$
\Delta \geq \frac{d-2}{2}
$$

When Δ crosses this threshold, the argument of the denominator Gamma function becomes negative. Furthermore, $\Gamma(x)$ as x crosses zero diverges to $+\infty$ and then comes back from $-\infty$ as x becomes negative. Thus $\Gamma(x)$ crosses from being positive to negative as its argument does. This is more clearly summarized by taylor expanding,

$$
\frac{1}{\Gamma(x)} = x + O(x^2)
$$

Thus, we have, about $\epsilon = 0$ where $\epsilon = \Delta - \frac{d-2}{2}$ we have,

$$
c_\Delta = \frac{\Gamma\left(\frac{d-2}{2}\right)}{2\pi^{d/2}} \epsilon + O(\epsilon^2)
$$

Thus as we lower Δ past the unitarity bound $\frac{d-2}{2}$ (i.e. take ϵ negative) the coefficient c_Δ changes sign (passing though zero) from positive to negative and, since all other terms remain well-defined in this transition, the two-point function must also become negative. Intreguingly, exactly at the unitarity bound $\Delta = \frac{d-2}{2}$ the constant c_Δ vanishes and thus,

$$
\langle \mathcal{O}(\vec{u}) \mathcal{O}(\vec{u}') \rangle = 0
$$

2.2

First, consider EAdS in Euclidean global coordinates,

$$X^0 = \sqrt{1+r^2}\cosh t \qquad X^d = \sqrt{1+r^2}\sinh t \qquad X^i = x^i$$

in which the metric becomes,

$$\mathrm{d}s^2 = (1+r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\Omega^2$$

In Euclidean EAdS the length inner product becomes,

$$P = X^I Y_I = -X^0 Y^0 + X^I Y^I = -\sqrt{(1+r^2)(1+r'^2)}\cosh(t-t') + rr'\cos\eta$$

where η is the angle between x^i and y^i on the boundary sphere. We have shown that the two-point function in the large-distance limit becomes,

$$G_\Delta \to c_\Delta(-2p)^{-\Delta}$$

Therefore, setting $r = r'$ and taking the limit $r \to \infty$ gives,

$$P = r^2\left(-\cosh(t-t') + \cos\eta\right)$$

Thus, define the boundary operator via,

$$\mathcal{O}(t,\Omega) = \lim_{r\to\infty} r^\Delta \phi(r,t,\Omega)$$

such that the boundary two-point function becomes,

$$\begin{aligned}\langle \mathcal{O}(t,\Omega)\mathcal{O}(t',\Omega')\rangle = \lim_{r\to\infty} r^{2\Delta}\langle\phi(r,t,\Omega)\phi(r,t',\Omega')\rangle &= \lim_{r\to\infty}\frac{r^{2\Delta}c_\Delta}{2^\Delta r^{2\Delta}(\cosh(t-t')-\cos\eta)^\Delta} \\ &= \frac{c_\Delta}{2^\Delta(\cosh(t-t')-\cos\eta)^\Delta}\end{aligned}$$

Second, consider EAdS in Euclidean hyperbolic ball coordinates,

$$X^I = \frac{2y^I}{1-|y|^2} \qquad X^0 = \frac{1+|y|^2}{1-|y|^2}$$

where I restrict,

$$|y|^2 = \sum_{I=1}^{d+1}(y^I)^2 < 1$$

In these coordinates, the ambiant coordinate dot product becomes,

$$P = X^I X_I = -\frac{1+|y|^2}{1-|y|^2}\frac{1+|y'|^2}{1-|y'|^2} + \frac{4y^I y'_I}{(1-|y|^2)(1-|y'|^2)}$$

In the case that $|y| = |y'|$ (preparing to take the limits of both to the boundary). Then we have,

$$P = -\left(\frac{1+|y|^2}{1-|y|^2}\right)^2 + \frac{4y^I y'_I}{(1-|y'|^2)} = -\frac{1+2|y|^2-4y^I y'_I+|y|^4}{(1-|y|^2)^2} = -1 + \frac{4(y^I y'_I - |y|^2)}{(1-|y|^2)^2}$$

Thus, in the limit $|y|^2 \to 1$ we find,

$$P = \frac{4(y^I y'_I - |y|^2)}{(1-|y|^2)^2}$$

Thus, define the boundary operator via,

$$\mathcal{O}(y^I) = \lim_{|y|\to 1} (1-|y|^2)^{-\Delta} \phi(y^I)$$

such that the boundary two-point function becomes,

$$\begin{aligned}\left\langle \mathcal{O}(y^I)\mathcal{O}(y'^I)\right\rangle = \lim_{|y|\to 1}(1-|y|^2)^{-2\Delta}\left\langle \phi(y^I)\phi(y'^I)\right\rangle &= \lim_{|y|\to 1}(1-|y|^2)^{-2\Delta}\frac{(1-|y|^2)^{2\Delta}c_\Delta}{8^\Delta(1-y^I y'^I)^\Delta} \\ &= \frac{c_\Delta}{8^\Delta(1-y^I y'_I)^\Delta}\end{aligned}$$

2.3

In Euclidean Poincare coodinates, consider a change of coordinates $z = \lambda\tilde{z}$ and $\vec{u} = \lambda\vec{\tilde{u}}$ with $\lambda > 0$ some constant. Then the bulk metric becomes,

$$\mathrm{d}s^2 = \frac{\mathrm{d}\vec{u}^2 + \mathrm{d}z^2}{z^2} = \frac{\mathrm{d}\vec{\tilde{u}}^2 + \mathrm{d}\tilde{z}^2}{\tilde{z}^2}$$

and thus this scaling is an isometry. In these new coordinates, the boundary metric and operators are defined via,

$$\begin{aligned}\mathrm{d}\tilde{\sigma}^2 &= \lim_{\tilde{z}\to 0}\tilde{z}^2\mathrm{d}s^2\,|_{\tilde{z}} = \mathrm{d}\vec{\tilde{u}}^2 \\ \tilde{\mathcal{O}}(\vec{\tilde{u}}^2) &= \lim_{\tilde{z}\to 0}\tilde{z}^{-\Delta}\tilde{\phi}(\tilde{z},\vec{\tilde{u}}) = \lim_{\tilde{z}\to 0}(z/\lambda)^{-\Delta}\phi(z,u) = \lambda^\Delta\mathcal{O}(\vec{u}) = \lambda^\Delta\mathcal{O}(\lambda\vec{\tilde{u}})\end{aligned}$$

Now consider the two-point function expressed in terms of these new boundary operators in the new coordinates.

$$\left\langle \tilde{\mathcal{O}}(\vec{\tilde{u}})\tilde{\mathcal{O}}(\vec{\tilde{u}}')\right\rangle = \left\langle \lambda^\Delta\mathcal{O}(\lambda\vec{\tilde{u}})\lambda^\Delta\mathcal{O}(\lambda\vec{\tilde{u}}')\right\rangle = \lambda^{2\Delta}\left\langle \mathcal{O}(\lambda\vec{\tilde{u}})\mathcal{O}(\lambda\vec{\tilde{u}}')\right\rangle = \frac{\lambda^{2\Delta}c_\Delta}{|\lambda^2\vec{\tilde{v}} - \lambda^2\vec{\tilde{v}}'|^2} = \frac{c_\Delta}{|\vec{\tilde{u}} - \vec{\tilde{u}}'|^2}$$

Therefore, the two-point function of the new boundary operators in the new boundary coordinates takes exactly the same form as the old operators in the old coordinates.

2.4

In Poincare coordinates, consider a coordinate transformation $z = Z(\tilde{z},\tilde{u})$ and $u^i = U^i(\tilde{z},\tilde{u})$ such that we preserve the metric,

$$\mathrm{d}s^2 = \frac{(\mathrm{d}u^2 + \mathrm{d}z^2)}{z^2} = \frac{(\mathrm{d}\tilde{u}^2 + \mathrm{d}\tilde{z}^2)}{\tilde{z}^2}$$

We care about the metric near the boundary so we taylor expand about $\tilde{z}0$. Since such isometries must preserve the boundary, they map $z=0$ to $\tilde{z}=0$. Therefore we may expand,

$$z = \left(\partial_{\tilde{z}}Z(\tilde{z},\tilde{u})|_{\tilde{z}=0}\right)\tilde{z} + O(\tilde{z}^2)$$

Let $\gamma(\tilde{u}) = \partial_{\tilde{z}} Z(\tilde{z},\tilde{u})|_{\tilde{z}=0}$. Then plugging in with $\mathrm{d}\tilde{z} = 0$ we find,

$$\mathrm{d}s^2 = \frac{\mathrm{d}u^2 + \mathrm{d}\gamma(\tilde{u})^2\tilde{z}^2 + O(\tilde{z})}{\gamma(\tilde{u})^2\tilde{z}^2 + O(\tilde{z}^3)} = \frac{\mathrm{d}u^2 + \mathrm{d}\gamma(\tilde{u})^2\,\tilde{z}^2}{\gamma(\tilde{u})^2\tilde{z}^2} + O(\tilde{z}^{-1}) = \frac{\mathrm{d}u^2}{\gamma(\tilde{u})^2\tilde{z}^2} + O(1)$$

Furthermore, using the fact that this transformation preserves the metric we find (recalling that we set $\mathrm{d}\tilde{z} = 0$),

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{u}^2}{\tilde{z}^2} = \frac{\mathrm{d}u^2}{\gamma(\tilde{u})^2\tilde{z}^2} + O(1)$$

and thus,

$$\gamma(\tilde{u})^2\mathrm{d}\tilde{u}^2 = \mathrm{d}u^2 + O(\tilde{z}^2)$$

and therefore, in the limit $z \to 0$ we find,

$$\gamma(\tilde{u})^2\mathrm{d}\tilde{u}^2 = \mathrm{d}u^2$$

or equivalently,

$$\mathrm{d}\tilde{\sigma}^2 = \lim_{\tilde{z}\to 0}\tilde{z}^2\mathrm{d}s^2 = \lim_{\tilde{z}\to 0}\mathrm{d}\tilde{u}^2 = \lim_{\tilde{z}\to 0}\left[\frac{\mathrm{d}u^2}{\gamma(\tilde{u})^2} + O(\tilde{z}^2)\right] = \frac{\mathrm{d}u^2}{\gamma(\tilde{u})^2} = \frac{\mathrm{d}\sigma^2}{\gamma(\tilde{u}^2)}$$

Therefore the conformal factor is

$$\rho(\tilde{u}) = \gamma(\tilde{u}) = (\partial_{\tilde{z}} Z(\tilde{z},\tilde{u})|_{\tilde{z}=0})$$

Furthermore, the boundary operators can be written as,

$$\begin{aligned}\mathcal{O}(u) &= \lim_{z\to 0} z^{-\Delta}\phi(z,u) = \lim_{z\to 0}(z/\tilde{z})^{-\Delta}\tilde{z}^{-\Delta}\tilde{\phi}(\tilde{z},\tilde{u}) = \lim_{z\to 0}(Z(\tilde{z},\tilde{u})/\tilde{z})^{-\Delta}\lim_{z\to 0}\tilde{z}^{-\Delta}\tilde{\phi}(\tilde{z},\tilde{u})\\ &= (\partial_{\tilde{z}} Z(\tilde{z},\tilde{u})|_{\tilde{z}=0})^{-\Delta}\tilde{\mathcal{O}}(\tilde{u}) = \gamma(\tilde{u})^{-\Delta}\tilde{\mathcal{O}}(\tilde{u})\end{aligned}$$

Therefore,

$$\tilde{\mathcal{O}}(\tilde{u}) = \gamma(\tilde{u})^{\Delta}\mathcal{O}(U(\tilde{z},\tilde{u}))$$

2.5

Consider the coordinate transformation from Euclidean Poincare coordinates to Euclidean global coordinates given by,

$$t = \log\sqrt{\vec{u}^2 + z^2} \xrightarrow{z\to 0} \log|u| \qquad x^i = \frac{u^i}{z}$$

Now consider the tranformation between boundary operators,

$$\mathcal{O}_{\mathrm{cyl}}(t,\Omega) = \lim_{r\to\infty} r^{\Delta}\phi(r,t,\Omega) = \lim_{z\to 0}\frac{|u|^{\Delta}}{z^{\Delta}}\phi(z,u) = |u|^{\Delta}\lim_{z\to 0} z^{-\Delta}\phi(z,u) = |u|^{\Delta}\mathcal{O}_{\mathrm{pl}}(u)$$

Using the inverse transformation,

$$z = \frac{e^t}{\sqrt{1+r^2}} \qquad \vec{u} = \frac{\vec{x}e^t}{\sqrt{1+r^2}} \xrightarrow{r\to\infty} \vec{\Omega}e^t$$

we can also go the opposite direction to find,

$$\mathcal{O}_{\mathrm{pl}}(u) = \lim_{z\to 0} z^{-\Delta}\phi(z,u) = \lim_{r\to\infty} e^{-t\Delta}(1+r^2)^{\Delta/2}\phi(r,t,\Omega) = e^{-t\Delta}\lim_{r\to\infty} r^{\Delta}\phi(r,t,\Omega) = e^{-t\Delta}\mathcal{O}_{\mathrm{cyl}}(t,\Omega)$$

Furthermore, these are compatible because $e^t = |u|$. Now consider the conformal two-point function,

$$\begin{aligned}\langle \mathcal{O}_{\text{cyl}}(t,\Omega)\mathcal{O}_{\text{cyl}}(t',\Omega')\rangle = (|u||u'|)^{\Delta} \langle \mathcal{O}_{\text{pl}}(u)\mathcal{O}_{\text{pl}}(u')\rangle &= \frac{c_\Delta (|u||u'|)^{\Delta}}{|\vec{u}-\vec{u}'|^{2\Delta}} \\ &= \frac{c_\Delta e^{(t+t')\Delta}}{|\vec{\Omega}e^t - \vec{\Omega}' e^{t'}|^{2\Delta}} = \frac{c_\Delta}{\left|\vec{\Omega}e^{\frac{1}{2}(t-t')\Delta} - \vec{\Omega}' e^{\frac{1}{2}(t'-t)\Delta}\right|^{2\Delta}}\end{aligned}$$

However,

$$\begin{aligned}\left|\vec{\Omega}e^{\frac{1}{2}(t-t')\Delta} - \vec{\Omega}' e^{\frac{1}{2}(t'-t)\Delta}\right|^2 &= |\vec{\Omega}|^2 e^{(t-t')\Delta} + |\vec{\Omega}'|^2 e^{(t'-t)\Delta} - 2\vec{\Omega}\cdot\vec{\Omega}' \\ &= (e^{(t-t')\Delta} + e^{(t'-t)}) - 2\cos\eta = 2\left(\cosh\left(t-t'\right) - \cos\eta\right)\end{aligned}$$

Therefore,

$$\langle \mathcal{O}_{\text{cyl}}(t,\Omega)\mathcal{O}_{\text{cyl}}(t',\Omega')\rangle = \frac{c_\Delta}{2^\Delta \left(\cosh\left(t-t'\right) - \cos\eta\right)^\Delta}$$

which is exactly the two-point function we computed earlier in Euclidean global coordinates. Likewise, we may check this result using the forward transformation,

$$\begin{aligned}\langle \mathcal{O}_{\text{pl}}(u)\mathcal{O}_{\text{pl}}(u')\rangle = (|u||u'|)^{-\Delta} \langle \mathcal{O}_{\text{cyl}}(t,\Omega)\mathcal{O}_{\text{cyl}}(t',\Omega')\rangle &= \frac{c_\Delta (|u||u'|)^{-\Delta}}{2^\Delta \left(\cosh\left(t-t'\right) - \cos\eta\right)^\Delta} \\ &= \frac{c_\Delta (|u||u'|)^{-\Delta}}{2^\Delta \left(\cosh\left(\log|u| - \log|u'|\right) - (|u||u'|)^{-1}\vec{u}\cdot\vec{u}'\right)^\Delta} \\ &= \frac{c_\Delta (|u||u'|)^{-\Delta}}{\left(\frac{|u|}{|u'|} + \frac{|u'|}{|u|} - 2(|u||u'|)^{-1}\vec{u}\cdot\vec{u}'\right)^\Delta} = \frac{c_\Delta}{(|u|^2 + |u'|^2 - 2\vec{u}\cdot\vec{u}')^\Delta} \\ &= \frac{c_\Delta}{|\vec{u}-\vec{u}'|^{2\Delta}}\end{aligned}$$

which is exactly the conformal two-point function in Poincare coordinates.

2.6

Consider the Eulcidean generators M_{ij}, P_i, K_i and D which are given in Poincare coordinates as,

$$\begin{aligned}M_{ij} &= -i(u^i\partial_{u^j} - u^j\partial_{u^i}) \\ D &= -i(u^i\partial_{u^i} + z\partial_z) \\ P_i &= -i\partial_{u^i} \\ K_i &= -i(u^2+z^2)\partial_{u^i} + 2iu^i(u^j\partial_{u^j} + z\partial_z)\end{aligned}$$

Such transformations act on the conformal boundary operators under a transformation $\delta\phi = i\epsilon G\phi$ via,

$$\delta\mathcal{O} = \lim_{z\to 0} z^{-\Delta}\delta\phi = i\epsilon \lim_{z\to 0} z^{-\Delta} G\phi = i\epsilon\left(G\lim z^{-\Delta}\phi - \lim_{z\to 0}[G, z^{-\Delta}]\phi\right) = \mathcal{G}\mathcal{O}$$

Therefore, first we need to compute the commutators,

$$\begin{aligned}
[M_{ij}, z^{-\Delta}] &= 0 \\
[D, z^{-\Delta}] &= i\Delta z^{-\Delta} \\
[P_i, z^{-\Delta}] &= 0 \\
[K_i, z^{-\Delta}] &= -2iu^i\Delta z^{-\Delta}
\end{aligned}$$

Therefore, we can write,

$$[G, z^{-\Delta}] = He^{-\Delta}$$

for some function H simplifying,

$$\delta\mathcal{O} = i\epsilon\left(G\lim_{z\to 0} z^{-\Delta}\phi - \lim_{z\to 0} Hz^{-\Delta}\phi\right) = i\epsilon\left(G\mathcal{O} - H\mathcal{O}\right)$$

Therefore, plugging in for the given generators,

$$\begin{aligned}
M_{ij} &\implies \delta\mathcal{O} = \epsilon\left(u^i\partial_{u^j} - u^j\partial_{u^i}\right)\mathcal{O}(u) \\
D &\implies \delta\mathcal{O} = \epsilon\left(u^i\partial_{u^i} + \Delta\right)\mathcal{O} \\
P_i &\implies \delta\mathcal{O} = \epsilon\partial_{u^i}\mathcal{O} \\
K_i &\implies \delta\mathcal{O} = \epsilon\left(u^2\partial_{u^i} - 2u^iu^j\partial_{u^j} - 2u^i\Delta\right)\mathcal{O}
\end{aligned}$$