1 The Yoneda Embedding

Lemma 1.1. Let $\eta : \mathrm{Hom}\,(A, -) \to \mathrm{Hom}\,(B, -)$ be a natural transformation. Then η is uniquely determined by $\eta_A(\mathrm{id}_A)$ via $\eta_X(f) = f \circ \eta_A(\mathrm{id}_A)$ for any $f \in \mathrm{Hom}\,(A, X)$.

Proof. Let $f : A \to X$ be some map. Consider the naturality diagram,

$$\begin{tikzcd}
\mathrm{Hom}\,(A, A) \arrow[r, "f_*"] \arrow[d, "\eta_A"] & \mathrm{Hom}\,(A, X) \arrow[d, "\eta_X"] \\
\mathrm{Hom}\,(B, A) \arrow[r, "f_*"] & \mathrm{Hom}\,(B, X)
\end{tikzcd}$$

Consider the element $\mathrm{id}_A \in \mathrm{Hom}\,(A, A)$ which, under the upper path, maps to $\eta_X(f_*(\mathrm{id}_A)) = \eta_X(f \circ \mathrm{id}_A) = \eta_X(f)$ and, under the lower path, $f_*(\eta_A(\mathrm{id}_A)) = f \circ \eta_A(\mathrm{id}_A)$. Therefore,

$$\eta_X(f) = f \circ \eta_A(\mathrm{id}_A)$$

□

Corollary 1.2. Natural transformations $\eta : \mathrm{Hom}\,(A, -) \to \mathrm{Hom}\,(B, -)$ are in one-to-one correspondence with functions $\mathrm{Hom}\,(B, A)$. We say f^* is the natural transformation $f^*_X(g) = g \circ f$ for any $g \in \mathrm{Hom}\,(A, X)$.

Theorem 1.3. Let $\mathcal{C}$ be any category. The functor $Y : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$ sending $A \mapsto h^A$ where $h^A = \mathrm{Hom}\,(A, -)$ and $f \mapsto f^*$ described above is fully faithful.

Proof. Clearly $(\mathrm{id}_A)^* = \mathrm{id}_{h^A}$ since $(\mathrm{id}_A)^*(f) = f \circ \mathrm{id}_A = f$ and for $f \in \mathrm{Hom}\,(B, A)$ and $g \in \mathrm{Hom}\,(C, B)$ then $(f \circ g)^* = g^* \circ f^*$ since for any $q \in \mathrm{Hom}\,(A, X)$ we send,

$$(f \circ g)^*(q) = q \circ (f \circ g) = (q \circ f) \circ g = g^*(f^*(q))$$

The above corollary proves that Y is fully faithful. □

Lemma 1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be fully faithful then $X \cong Y \iff F(X) \cong F(Y)$.

Proof. If $F(X) \cong F(Y)$ then there are morphisms $f \in \mathrm{Hom}\,(F(X), F(Y))$ and $g \in \mathrm{Hom}\,(F(Y), F(X))$ which are inverses. However, since F is full there exist morphisms $\tilde{f} : \mathrm{Hom}\,(X, Y)$ and $g \in \mathrm{Hom}\,(Y, X)$ such that $F(\tilde{f}) = f$ and $F(\tilde{g}) = g$. Then,

$$F(\tilde{f} \circ \tilde{g}) = F(\tilde{f}) \circ F(\tilde{g}) = f \circ g = \mathrm{id}_{F(Y)} \quad \text{and} \quad F(\tilde{g} \circ \tilde{f}) = F(\tilde{g}) \circ F(\tilde{f}) = g \circ f = \mathrm{id}_{F(X)}$$

However, since F is faithful then,

$$\tilde{f} \circ \tilde{g} = \mathrm{id}_Y \quad \text{and} \quad \tilde{g} \circ \tilde{f} = \mathrm{id}_X$$

proving that $X \cong Y$. □

Definition: We say a functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if $F \cong h^A$ for some $A \in \mathcal{C}$.

2 Additive Categories

Definition: A category $\mathcal{C}$ is pre-additive if its hom sets have the structure of an abelian group and composition of maps distributes over addition. Explicitly, for $X, Y, Z \in \mathcal{C}$, there exits a binary operation,

$$+ : \mathrm{Hom}\,(X, Y) \times \mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(X, Y)$$

such that $(\mathrm{Hom}\,(X, Y)\,, +)$ is an abelian group and, for $f, g : X \to Y$ and $h, k : Y \to Z$ we have $h \circ (f + g) = h \circ f + h \circ g$ and $(h + k) \circ f = h \circ f + k \circ f$. This is equvalent to the requirement that hom is a functor,

$$\mathrm{Hom}\,(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$$

Lemma 2.1. In a pre-additive cateogory, there exists an identity element $0 \in \mathrm{Hom}\,(X, Y)$ such that $0 + f = f + 0 = f$ for $f \in \mathrm{Hom}\,(X, Y)$ and $f \circ 0 = 0$ for $f \in \mathrm{Hom}\,(Y, Z)$ and $0 \circ f = 0$ for $f \in \mathrm{Hom}\,(Z, X)$.

Proof. The hom sets are abelian groups by definiton and thus must have unique identiy elements satisfying $f + 0 = 0 + f = f$ for all $f \in \mathrm{Hom}\,(X, Y)$. Furthermore, for $f \in \mathrm{Hom}\,(Y, Z)$ we have $f \circ 0 = f \circ (0 + 0) = f \circ 0 + f \circ 0$ and thus $f \circ 0 = 0_{XZ}$. Furthermore for $f \in \mathrm{Hom}\,(Z, X)$ we know that $0 \circ f = (0 + 0) \circ f = 0 \circ f + 0 \circ f$ so $0 \circ f = 0_{ZY}$. □

Definition: A biproduct of an indexed set $\{X_i\}_I$ is an object $X = \bigoplus_I X_i$ along with projection maps $\pi_i : X \to X_i$ and inclusion maps $\iota_i : X_i \to X$ such that $(X, \{\pi_i\}_I)$ is the product of $\{X_i\}_I$ and $(X, \{\iota_i\}_I)$ is the coproduct of $\{X_i\}_I$.

Proposition 2.2. Let $\mathcal{C}$ be a pre-additive category. Every finite product and finite coproduct is a biproduct. In particular, finite products and coproducts are equal.

Proof. Let $X \times Y$ be the product of X and Y. Consider the diagram,

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "\iota_X"'] & & X \\
& X \times Y \arrow[ur, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
Y \arrow[ur, "\iota_Y"] \arrow[rr, "\mathrm{id}_Y"'] & & Y
\end{tikzcd}$$

where the maps $\iota_X : X \to X \times Y$ and $\iota_Y : Y \to X \times Y$ are defined via the universal property of the product applied to $(\mathrm{id}_X, 0)$ and $(0, \mathrm{id}_Y)$ respectivly where $0 \in \mathrm{Hom}\,(X, Y)$ is the identiy element of the abelian group. The universal property gives,

$$\pi_X \circ \iota_X = \mathrm{id}_X \quad \pi_Y \circ \iota_X = 0$$
$$\pi_X \circ \iota_Y = 0 \quad \pi_Y \circ \iota_Y = \mathrm{id}_Y$$

so the diagram commutes. We need to show that $X \times Y$ is universal with respect to the maps ι_X and ι_Y. Suppose we have maps $f_X : Z \to X$ and $f_Y : Z \to Y$ then define $\tilde{f} = f_X \circ \pi_X + f_Y \circ \pi_Y$.

$$\begin{tikzcd}
 & X \arrow[rr, "\mathrm{id}_X"] \arrow[dl, "f_X"'] \arrow[dr, "\iota_X"] & & X \\
Z & & X \times Y \arrow[ll, dashed, "\tilde{f}"'] \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"] & \\
 & Y \arrow[rr, "\mathrm{id}_Y"] \arrow[ul, "f_Y"] \arrow[ur, "\iota_Y"] & & Y
\end{tikzcd}$$

This map satisfies the required universal property because,

$$\tilde{f} \circ \iota_X = (f_X \circ \pi_X + f_Y \circ \pi_Y) \circ \iota_X = f_X \circ \pi_X \circ \iota_X + f_Y \circ \pi_Y \circ \iota_X = f_X + 0 = f_X$$

and likewse,

$$\tilde{f} \circ \iota_Y = (f_X \circ \pi_X + f_Y \circ \pi_Y) \circ \iota_Y = f_X \circ \pi_X \circ \iota_Y + f_Y \circ \pi_Y \circ \iota_Y = 0 + f_Y = f_Y$$

Lastly, we must show that $\tilde{f}$ is unique. Suppose there exits a map $\tilde{f} : X \times Y \to Z$ such that $\tilde{f} \circ \iota_X = f_X$ and $\tilde{f} \circ \iota_Y = f_Y$. Consider the map $I : X \times Y \to X \times Y$ given by,

$$I = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$$

Therefore,

$$\pi_X \circ I = \pi_X \circ \iota_X \circ \pi_X + \pi_X \circ \iota_Y \circ \pi_Y = \pi_X + 0 = \pi_X$$

and furthermore,

$$\pi_Y \circ I = \pi_Y \circ \iota_X \circ \pi_X + \pi_Y \circ \iota_Y \circ \pi_Y = 0 + \pi_Y = \pi_Y$$

However, by the universal property of the product, there exists a unique map, namely $\mathrm{id}_{X\times Y}$, satisfying these properties. Thus, $I = \mathrm{id}_{X\times Y}$. Thus,

$$\tilde{f} = \tilde{f} \circ \mathrm{id}_{X\times Y} = \tilde{f} \circ I = \tilde{f} \circ \iota_X \circ \pi_X + \tilde{f} \circ \iota_Y \circ \pi_Y = f_X \circ \pi_X + f_Y \circ \pi_Y$$

so the map we constructed earlier is unique.

Similarly, let $X \coprod Y$ be the coproduct of X and Y. A similar argument will hold reversing all arrows. $\square$

Definition: A category is additive if it is pre-additive, has a zero object, and has all finite biproducts. The preceding dicussion implies that it is enough to check that either all finite products or all finite coproducts exit.

Proposition 2.3. In an additive category, the zero map is the indentity obeject of the **Ab**-enriched hom-sets.

Proof. $\square$

Definition: A functor $T : \mathcal{C} \to \mathcal{D}$ is *additive* if it preserves finite biproducts.

Proposition 2.4. A functor $T : \mathcal{C} \to \mathcal{D}$ is additive iff the map on enriched hom-sets,

$$T_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(T(X), T(Y))$$

is a homomorphism in the category of abelian groups.

Proof. A biproduct $X \oplus Y$ with its projections and inclusions is completely characterized by the property $\mathrm{id}_{X \oplus Y} = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$. Thus T preserves the biproduct structure iff it preserves addition i.e. iff,

$$\mathrm{id}_{T(X \oplus Y)} = T(\mathrm{id}_{X \oplus Y}) = T(\iota_X \circ \pi_X + \iota_Y \circ \pi_Y) = T(\iota_X) \circ T(\pi_X) + T(\iota_Y) \circ T(\pi_Y)$$

□

DEF-COMPLEX PROP ADD-FUNC PRESERVE COMPLEXES

3 Abelian Categories

DEFINE NORMAL CATEGORY

Proposition 3.1. Let $\mathcal{A}$ be a binormal category. Then any morphism in $\mathcal{A}$ which is both monic and epic is an isomorphism.

Proof. Let $f : A \to B$ be both monic and epic. Since $\mathcal{A}$ is binormal, f must be a kernel and a cokernel of some maps,

$$K \xrightarrow{\quad a \quad} A \xrightarrow{\quad f \quad} B \xrightarrow{\quad b \quad} C$$

where $f : A \to B$ is the cokernel of $a : K \to A$ and the kernel of $b : B \to C$. Then $f \circ a = 0 = f \circ 0$ so, since f is monic, $a = 0$. Furthermore, $b \circ f = 0 = 0 \circ f$ so, sice f is an epic, $b = 0$. Therefore, consider the diagram,

$$\begin{tikzcd}
K \arrow[r, "0"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "0"] \arrow[dl, dashed] & C \\
& A & B \arrow[u, "\mathrm{id}_B"'] \arrow[ul, dashed] &
\end{tikzcd}$$

Where $\mathrm{id}_A \circ a = 0$ and $b \circ \mathrm{id}_B = 0$ (since $a = 0$ and $b = 0$) which implies that id_A factors through the cokernel $f : A \to B$ and id_B lifts over the kernel $f : A \to B$. Thus f has a left inverse $g_L : B \to A$ and right inverse $g_R : B \to A$ such that $g_L \circ f = \mathrm{id}_A$ and $f \circ g_R = \mathrm{id}_B$. Finally,

$$g_L = g_L \circ \mathrm{id}_B = g_L \circ (f \circ g_R) = (g_L \circ f) \circ g_R = \mathrm{id}_A \circ g_R = g_R$$

Thus f has a two-sided inverse $g = g_L = g_R$ so f is an isomorphism. □

DEFINITON OF AB-CAT
DEF OF IM AND COIM
IM = COIM

Definition: We say that a sequence,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a complex if $g \circ f = 0$ giving a monomorphism $\text{Im}(f) \to \ker g$. We say the sequence is *exact* if this morphism is also epic i.e. an isomorphism by the above lemma.

Proposition 3.2. $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$
is exact iff $(X \xrightarrow{f} Y) = \ker g$ and,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact iff $(Y \xrightarrow{g} Y) = \text{coker } f$.

Proof. DO THIS PROOF □

Definition: ABELIAN FUNCTOR

Definition: Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then we say that,

1. F is *left-exact* if F preserves kernels
2. F is *right-exact* if F preserves cokernels
3. F is *exact* if F preserves exact sequences

Proposition 3.3. *F is exact iff F is left and right-exact.*

Proof. □

Proposition 3.4. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{B}$ be an adjoint pair of additive functors between abelian categories. Then F is right-exact and G is left-exact.*

Proof. Left-adjoints preserve colimits and right-adjionts preserve limits. □

4 Homology

5 MISC

Lemma 5.1. *Let A be a ring and $\mathfrak{a} \subset A$ an ideal and M an A-module. Then,*

$$(A/\mathfrak{a}) \otimes_A M = M/\mathfrak{a}M$$

Proof. Consider the exact sequence,

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

Now applying the right-exact functor $(-) \otimes_A M$ we get an exact sequence,

$$\mathfrak{a} \otimes_A M \longrightarrow M \longrightarrow (A/\mathfrak{a}) \otimes_A M \longrightarrow 0$$

Clearly the image of $\mathfrak{a} \otimes_A M \to M$ is $\mathfrak{a}M \subset M$ and since $(A/\mathfrak{a}) \otimes_A M$ is the cokernel of this map by exactness we find,

$$(A/\mathfrak{a}) \otimes_A M = M/\mathfrak{a}M$$

□

Definition: We say a flat A-module M is *faithfully flat* iff the following equivalent conditions hold.

Proposition 5.2. Let M be a flat A-module. Then the following are equivalent.

1. for any maximal $\mathfrak{m} \subset A$ we have $\mathfrak{m}M \neq M$
2. for any A-module N we have $N \otimes_A M = 0 \implies N = 0$
3. for any map $f : N_1 \to N_2$ if $f \otimes \mathrm{id}_M : N_1 \otimes_A M \to N_2 \otimes_A M$ is an isomorphism then f is an isomorphism.
4. a sequence $N_1 \to N_2 \to N_3$ is exact iff $N_1 \otimes_A M \to N_2 \otimes_A M \to N_3 \otimes_A M$ is exact.

Theorem 5.3. Let A be a ring B an A-algebra which is faithfully flat as an A-module and M be an A-module. Then the following sequence is exact,

$$0 \longrightarrow M \xrightarrow{\partial_0} M \otimes_A B \xrightarrow{\partial_1} M \otimes_A B \otimes_A B$$

via $\partial_0(x) = x \otimes 1$ and $\partial_1(x \otimes b) = x \otimes b \otimes 1 - x \otimes 1 \otimes b$.

Proof. □

Theorem 5.4. The functor $\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRS}$ is right-adjoint to the global sections functor $\Gamma : \mathbf{LRS} \to \mathbf{Ring}^{\mathrm{op}}$ given by $\Gamma(X) = \mathcal{O}_X(X)$. Therefore,

$$\mathrm{Hom}_{\mathbf{LRS}}\left(X, \mathrm{Spec}\left(A\right)\right) \cong \mathrm{Hom}_{\mathrm{Ring}}\left(A, \mathcal{O}_X(X)\right)$$

Proof. □

Corollary 5.5. $\mathrm{Spec}\left(\mathbb{Z}\right)$ is the terminal object in the category of locally ringed spaces and $\mathrm{Spec}\left(\{0\}\right)$ is the initial object in the category of locally ringed spaces.

Corollary 5.6. Let A be a ring and $A \to B$ and $A \to C$ be A-algebras. Then,

$$\mathrm{Spec}\left(B \otimes_A C\right) \cong \mathrm{Spec}\left(B\right) \times_{\mathrm{Spec}(A)} \mathrm{Spec}\left(C\right)$$

Theorem 5.7. Let $U \subset \mathrm{Spec}\left(A\right)$ be an open subset. Suppose that $\forall \mathfrak{p} \in \mathrm{Spec}\left(A\right) : \mathfrak{p} \cap S_U = \varnothing$ or $\mathfrak{p} \in U$. Then $F_A(U) = \tilde{A}(U)$.

Proof. We can assume that $U = \mathrm{Spec}\,(A)$ (otherwise replace A with $S_U^{-1}A$). Let $(U_i)_{i\in I}$ be a finite open cover with $U_i = D(s_i)$ for $s_i \in A$. Then let,

$$B = \prod_{i\in I} A_{s_i}$$

which is a faithfully flat A-module (PROVE THIS). Therefore, we know there exits an exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow B \otimes B$$

or equivalently,

$$0 \longrightarrow A \longrightarrow \textstyle\prod_i A_{s_i} \longrightarrow \prod_{i,j} A_{s_i s_j}$$

is exact where the second map is the difference of the product of maps $A_{s_i} \to A_{s_i s_j}$ and $A_{s_j} \to A_{s_j s_i}$. This identifies A as the kernel of this map and thus the equalizer in the definition of a sheaf. Thus $\tilde{A}(U) = A$. □

6 Categories of Modules

Definition: RING Cat and Module Cat

Lemma 6.1. A ring homomorphism $f : R \to S$ induces an additive functor

$$F : \mathbf{Mod}_R \to \mathbf{Mod}_S$$

given by $(-) \otimes_R S$ where S is an R-module under the action $r \cdot s = f(r)s$.

Proof. □

Proposition 6.2. $\mathrm{GL}_n(-) : \mathbf{Ring} \to \mathbf{Grp}$ is a functor

Proof. □

Proposition 6.3. $\det : \mathrm{GL}_n(-) \Longrightarrow (-)^\times$ is a natrual transformation.

Proof. □

7 Derived Functors

7.1 Chain Complexes

Definition: A chain complex C is a diagram,

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$ or equivalently $\text{Im}(\partial_{n+1}) \subset \ker \partial_n$ for each n. We call ∂ the boundary map.

Similarly, a cochain complex D is equivalent but with increasing lables,

$$\cdots \longrightarrow D^{n-1} \xrightarrow{d^{n-1}} D^n \xrightarrow{d^n} D^{n+1} \xrightarrow{d^{n+1}} \cdots$$

such that $d^{n+1} \circ d^n = 0$ or equivalently $\text{Im}(d^n) \subset \ker d^{n+1}$ for each n. We call d the coboundary map.

Remark. Complexes are "half exact" sequences.

Definition: A map $f : C \to D$ of (co)chain complexes is a sequnces of maps, $f_n : C_n \to D_n$ such that the diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots
\end{tikzcd}$$

commutes.

Definition: Let $\mathcal{A}$ be an abelian cateogy then $\mathbf{Ch}(\mathcal{A})$ is the category of chain complexes with components in $\mathcal{A}$.

Remark. Since complexes are "half exact" sequences, we would like a way to measure how far a given complex is from being exact. This is accomplished via (co)homology.

Definition: Let C be a chain complex in $\mathbf{Ch}(\mathcal{A})$. The homology of the complex C is the sequence of $\mathcal{A}$ objects (usually abelian groups or R-modules),

$$H_n(C) = \ker \partial_n / \text{Im}(\partial_{n+1})$$

We can describe this categorically via,

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, dashed] & C_n \arrow[r, "\partial_n"] & C_{n-1} \\
& \ker \partial_n \arrow[u] \arrow[dr, dashed] & \\
& & H_n(C)
\end{tikzcd}$$

where $\partial_n \circ \partial_{n+1} = 0$ so ∂_{n+1} lifts to the kernel and $H_n(C)$ is the cokernel of this map.

Similarly, given a cochain complex D, the cohomology is the sequence

$$H^n(D) = \ker d^n / \text{Im}(d^{n-1}) = 0$$

which is constructed identically.

Proposition 7.1. Taking (co)homology is a functor $H_n : \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$.

Proof. A chain map $f : C \to D$ is a diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots
\end{tikzcd}$$

so if $x \in \ker \partial_n$ then $\partial_n \circ f(x) = f(\partial_n x) = 0$ so $f(x) \in \ker \partial_n$. Furthermore, if $x \in \mathrm{Im}(\partial_{n+1})$ then $f(x) \in f(\mathrm{Im}(\partial_{n+1})) = \mathrm{Im}(\partial_{n+1} \circ f_{n+1}) \subset \mathrm{Im}(\partial_{n+1})$. Therefore, $f_* : H_n(C) \to H_n(D)$ is a well-defined map taking $[x] \mapsto [f(x)]$. Clearly $\mathrm{id}_* = \mathrm{id}_{H_n}$ and $(f \circ g)_* = f_* \circ g_*$.

Categorically, (DO THIS) $\square$

Definition: Let $f, g : C \to D$ be morphisms of chain complexes. A *chain homotopy* $p : f \implies g$ is a sequence of maps $p_n : C_n \to D_{n+1}$ such that,

$$\partial \circ p + p \circ \partial = f - g$$

or more explicitly,

$$\partial^D_{n+1} \circ p_n + p_{n-1} \circ \partial^C_n = f_n - g_n$$

in the following diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[dl, "p_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[dl, "p_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "p_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \cdots \arrow[dl, "p_{n-2}"'] \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"] & D_n \arrow[r, "\partial^D_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 7.2. Let $f, g : C \to D$ be chain homotopic then $f_* = g_*$ on homology.

Proof. Let $p : f \implies g$ be a chain homotopy. It suffices to show that if $\alpha \in \ker \partial$ is a cycle then $(f_* - g_*)(\alpha) = 0$ which is equivalent to $(f - g)(\alpha) \in \mathrm{Im}(\partial)$ is a boundary. Suppose that $\partial \alpha = 0$. Then,

$$(f - g)(\alpha) = (\partial \circ p + p \circ \partial)(\alpha) = \partial(p(\alpha))$$

and therefore $(f - g)(\alpha)$ is a boundary. Therefore $f_* = g_*$. $\square$

Corollary 7.3. A chain homotopy equivalence is a quasi-isomorphism i.e. an isomorphism on homology.

Theorem 7.4. Given a short exact sequence of chain complexes,

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

we get a long exact sequence,

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(A) \arrow[r] & H_{n+1}(B) \arrow[r] & H_{n+1}(C) \arrow[r] & H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dllllll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r] & H_{n-2}(A) \arrow[r] & H_{n-2}(B) \arrow[r] & H_{n-2}(C) \arrow[r] & \cdots
\end{tikzcd}
$$

functorially.

Proof. Consider the diagram with exact rows,

$$
\begin{tikzcd}
& \vdots \arrow[d, "\partial^A_{n+2}"] & \vdots \arrow[d, "\partial^B_{n+2}"] & \vdots \arrow[d, "\partial^C_{n+2}"] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial^A_{n+1}"] & B_{n+1} \arrow[r, "j"] \arrow[d, "\partial^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial^A_n"] & B_n \arrow[r, "j"] \arrow[d, "\partial^B_n"] & C_n \arrow[r] \arrow[d, "\partial^C_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d, "\partial^A_{n-1}"] & B_{n-1} \arrow[r, "j"] \arrow[d, "\partial^B_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "\partial^C_{n-1}"] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}
$$

An application of the snake lemma gives an exact sequence,

$$
\begin{tikzcd}
0 \arrow[r] & \ker \partial^A_{n+2} \arrow[r] & \ker \partial^B_{n+2} \arrow[r] & \ker \partial^C_{n+2} \arrow[dll, "\delta", rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
& A_n/\mathrm{Im}(\partial^A_{n+1}) \arrow[r] & B_n/\mathrm{Im}(\partial^B_{n+1}) \arrow[r] & C_n/\mathrm{Im}(\partial^C_{n+1}) \arrow[r] & 0
\end{tikzcd}
$$

where I haved added the leading and trailing zeros by by the following observations. The map $B_{n+1}/\mathrm{Im}(\partial^B_{n+2}) \to C_{n+1}/\mathrm{Im}(\partial^C_{n+2})$ simply takes $[x] \mapsto [j(x)]$ and thus is clearly surjective because j is. Furthermore the map $\ker \partial^A_n \to \ker \partial^B_n$ is simply the restriction of ι which is still injective. Therefore, we can arrange these exact rows into a commutative diagram,

$$
\begin{tikzcd}
& A_{n+1}/\mathrm{Im}(\partial^A_{n+2}) \arrow[r] \arrow[d] & B_{n+1}/\mathrm{Im}(\partial^B_{n+2}) \arrow[r] \arrow[d] & C_{n+1}/\mathrm{Im}(\partial^C_{n+2}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \partial^A_n \arrow[r] & \ker \partial^B_n \arrow[r] & \ker \partial^C_n &
\end{tikzcd}
$$

where the vertical maps are simply restrictions of the boundary maps whose images lie inside the respective kernels since each column is a chain complex. Another application of the snake lemma gives the exact sequence,

$$\begin{tikzcd} \ker \partial^A_{n+1}/\mathrm{Im}(\partial^A_{n+2}) \arrow[r] & \ker \partial^B_{n+1}/\mathrm{Im}(\partial^B_{n+2}) \arrow[r] & \ker \partial^C_{n+1}/\mathrm{Im}(\partial^C_{n+2}) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[pos=0.75, above]{$\scriptstyle\delta$} ([yshift=-1.5ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ \ker \partial^A_{n}/\mathrm{Im}(\partial^A_{n+1}) \arrow[r] & \ker \partial^B_{n}.\mathrm{Im}(\partial^B_{n+1}) \arrow[r] & \ker \partial^C_{n}/\mathrm{Im}(\partial^C_{n+1}) \end{tikzcd}$$

Stringing together these long exact sequences (which we can do because they overlap at two points) gives the required long exact sequence. □

7.2 Injective and Projective Resolutions

Definition: P is a projective object if for any map $f : P \to X$ and epimorphism (surjection) $g : Y \to X$ the map f lifts to Y. This means there always exists a map such that the diagram,

$$\begin{tikzcd} & Y \arrow[d, two heads, "g"] \\ P \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

commutes. The slogan is: "projective objects lift over surjections".

Lemma 7.5. Any exact sequence ending in a projective object splits.

Proof. Consider the exact sequence where P is projective,

$$\begin{tikzcd} & & & P \arrow[dl, dashed] \arrow[d, "\mathrm{id}_P"] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

The induced map is a right inverse of f so the sequence is right-split. □

Definition: A projective resolution of A is an exact sequence,

$$\cdots \longrightarrow P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{p_0} A \longrightarrow 0$$

such that each P_i is projective. We will write this situation schematically as,

$$\mathbf{P}^A \xrightarrow{p_0} A \longrightarrow 0$$

Proposition 7.6. The category $\mathbf{Mod}_R$ has *enough* projectives. i.e. every R-module has a projective resolution

Proof. We will use the fact that for any R-module M there exists a free module F and a surjection $F \to M$ (take the free module on all the elements of M). Furthermore free modules are projective because any map can be defined by sending the generators to arbitrary lifts.

Let $P_0 = F$ and consider the kernel K_0 of $P_0 \to M$. Then, we can construct a free module surjecting onto K_0 call this P_1. We repeat this process inductivly to get the diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[rr, "\partial_3"] \arrow[dr] & & P_2 \arrow[rr, "\partial_2"] \arrow[dr] & & P_1 \arrow[rr, "\partial_1"] \arrow[dr] & & P_0 \arrow[r, "\epsilon"] & M \arrow[r] & 0 \\
& & K_2 \arrow[ur] \arrow[dr] & & K_1 \arrow[ur] \arrow[dr] & & K_0 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where the diagonals are exact. The map $P_{n+1} \to P_n$ factors through the kernel K_n and thus goes to zero under $P_n \to P_{n-1}$. Furthermore, $P_{n+1} \to K_n$ is a surjection so the map $P_{n+1} \to P_n$ surjects onto the kernel. Thus the top row is exact. $\square$

Proposition 7.7. Every projective module is a direct factor of a free module.

Proof. Let P be projective and F be a free module surjecting onto P. Then we know that the exact sequence,

$$0 \longrightarrow \ker \phi \longrightarrow F \longrightarrow P \longrightarrow 0$$

splits because P is projective. Thus, $F \cong \ker \phi \oplus P$. $\square$

Definition: I is an injective object if for any map $f : X \to I$ and monomorphism (injection) $g : X \to Y$ the map f extends to Y. This means there always exists a map such that the diagram,

$$\begin{tikzcd}
I & X \arrow[l, "f"'] \arrow[d, hook, "g"] \\
& Y \arrow[ul, dashed, "\tilde{f}"]
\end{tikzcd}$$

commutes. The slogan is: "injective objects extend over injections."

Definition: An injective resolution of A is an exact sequence,

$$0 \longrightarrow A \xrightarrow{\iota_0} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \cdots$$

such that each I^i is projective. We will write this situation schematically as,

$$0 \longrightarrow A \xrightarrow{\iota_0} \mathbf{I}_A$$

Lemma 7.8. Any exact sequence begining with an injective object splits.

Proof. Consider the exact sequence where I is projective,

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "f"] \arrow[d, "\mathrm{id}_I"'] & A \arrow[r, two heads] \arrow[dl, dashed] & B \arrow[r] & 0 \\ & I & & & \end{tikzcd}$$

The induced map is a left inverse of f so the sequence is left-split. □

Proposition 7.9. The category $\mathbf{Mod}_R$ has *enough* injectives i.e. every R-module has an injective resolution.

Proof. If for any module M we can find and injection $M \to I$ into an injective module than we can repeat the argument for the projective case. This is true but harder; a proof can be found in Godement. □

Lemma 7.10. Suppose we have the diagram,

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed] & \\ A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \end{tikzcd}$$

such that P is projective $\beta \circ f = 0$ and the bottom row is exact. Then there is a map $P \to A$ which makes the diagram commute.

Similarly, suppose we have the diagram,

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] \arrow[d, "f"] & C \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

such that I is injective, $f \circ \alpha = 0$, and the top row is exact. Then there is a map $C \to I$ which makes the diagram commute.

Proof. In the first case, since $\beta \circ f = 0$ we have $\mathrm{Im}(f) \subset \ker \beta = \mathrm{Im}(\alpha)$ so we may replace B with $\mathrm{Im}(\alpha)$,

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "\tilde{f}"] & \\ A \arrow[r, "\alpha'"] & \mathrm{Im}(\alpha) \arrow[r] & 0 \end{tikzcd}$$

and α' is surjective so we get a lift $\tilde{f}$ to A of f and $\alpha \circ \tilde{f} = f$.

Similarly, since $f \circ \alpha = 0$ then $\ker \beta = \mathrm{Im}(\alpha) \subset \ker f$. Thus, f factors through the quotient $B/\ker \alpha$ to get,

$$\begin{tikzcd}
 & B \arrow[d, "\pi"] \arrow[dr, "\beta"] & \\
0 \arrow[r] & B/\ker\beta \arrow[r, "\beta'"] \arrow[d, "\bar{f}"] & C \arrow[dl, dashed] \\
 & I &
\end{tikzcd}$$

where β' is injective so we can extend f to C over β'. Thus, f lifts over β. $\square$

Lemma 7.11. Given projective or injective resolutions of both objects A and B and a map $f : A \to B$ there exists a unque lift up to chain homotopy to a chain map on the resolutions.

Proof. Let P^A and P^B be projective resolutions of A and B respectivly. We will construct the chain map inductivly. First, we have the diagram,

$$\begin{tikzcd}
P_0^A \arrow[r] \arrow[d, dashed, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
P_0^B \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

so we have a map $P_0^A \to B$ which lifts over the surjective map $P_0^B \to B$ since P_0^A is projective. Now assume we have constructed the map up to $n-1$,

$$\begin{tikzcd}
P_{n+1}^A \arrow[r, "\partial_{n+1}^A"] \arrow[d, dashed, "f_{n+1}"] & P_n^A \arrow[r, "\partial_n^A"] \arrow[d, "f_n"] & P_{n-1}^A \arrow[d, "f_{n-1}"] \\
P_{n+1}^B \arrow[r, "\partial_{n+1}^B"] & P_n^B \arrow[r, "\partial_n^B"] & P_{n-1}^B
\end{tikzcd}$$

However, the map $(f_n \circ \partial_{n+1}^A)$ satisfies $\partial_n^B \circ (f_n \circ \partial_{n+1}^A) = f_{n-1} \circ \partial_n^A \circ \partial_{n+1}^A = 0$ by commutativity and exactness of the top row. Since the bottom row is also exact, by Lemma 7.10, we get a lift to P_{n+1}^A such that the diagram commutes. Thus, we get a chain map $\mathbf{P}^A \to \mathbf{P}^B$.

Now, suppose we have two chain maps $f, g : \mathbf{P}^A \to \mathbf{P}^B$ which are lifts of f. At first, we have,

$$\begin{tikzcd}
P_1^A \arrow[r, "\partial_1^A"] & P_0^A \arrow[r, "\epsilon"] \arrow[dl, dashed, "s_0"'] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
P_1^B \arrow[r, "\partial_1^B"] & P_0^B \arrow[r, "\epsilon'"] & B \arrow[r] & 0
\end{tikzcd}$$

Because $\epsilon' \circ (f_0 - g_0) = 0$, the bottom row is exact, and P_0^A is projective, we get a lift s_0 such that $\partial_1^B \circ s_0 = f_0 - g_0$. Let $\Delta_n = f_n - g_n$. Now, suppose we have a chain homotopy up to position n and consider the diagram,

$$\begin{tikzcd}
P_{n+1}^A \arrow[r, "\partial_{n+1}^A"] \arrow[d, "\Delta_{n+1}"] & P_n^A \arrow[r, "\partial_n^A"] \arrow[d, "\Delta_n"] \arrow[dl, dashed, "s_n"] & P_{n-1}^A \arrow[r, "\partial_{n-1}^A"] \arrow[d, "\Delta_{n-1}"] \arrow[dl, "s_{n-1}"] & P_{n-2}^A \arrow[d, "\Delta_{n-2}"] \arrow[dl, "s_{n-2}"] \\
P_{n+1}^B \arrow[r, "\partial_{n+1}^B"] & P_n^B \arrow[r, "\partial_n^B"] & P_{n-1}^B \arrow[r, "\partial_{n-2}"] & P_{n-2}^B
\end{tikzcd}$$

There is a map $(\Delta_n - s_{n-1}) \circ \partial_n^A : P_n^A \to P_n^B$. Furthermore,

$$\partial_n^B \circ (\Delta_n - s_{n-1} \circ \partial_n^A) = \Delta_{n-1} \circ \partial_n^A - \partial_n^B \circ s_{n-1} \circ \partial_n^A$$

where I have used commutativity to show,

$$\partial_n^B \circ \Delta_n = \partial_n^B \circ (f_n - g_n) = f_{n-1} \circ \partial_n^A - g_{n-1} \circ \partial_n^A = \Delta_{n-1} \circ \partial_n^A$$

By the induction hypothesis, $\Delta_{n-1} = s_{n-2} \circ \partial_{n-1}^A + \partial_n^B \circ s_{n-1}$. Therefore,

$$\partial_n^B \circ (\Delta_n - s_{n-1} \circ \partial_n^A) = s_{n-2} \circ \partial_{n-1}^A \circ \partial_n^A + \partial_n^B \circ s_{n-1} \circ \partial_n^A - \partial_n^B \circ s_{n-1} \circ \partial_n^A = 0$$

because $\partial_{n-1}^A \circ \partial_n^A = 0$. Thus, we get a lift s_n of this map to P_{n+1}^B. Furthermore,

$$\partial_{n+1}^B \circ s_n + s_{n-1} \circ \partial_n^A = \Delta_n - s_{n-1} \circ \partial_n^A + s_{n-1} \circ \partial_n^A = \Delta_n$$

so we have constructed a chain homotopy up to position n. By induction, $s : \mathbf{P}^A \to \mathbf{P}^B$ is a chain homotopy between f, g. The proof for the injective case is very similar. $\square$

Corollary 7.12. All projective resolutions of a given object are chain homotopic. Likewise, all injective resolutions of a given object are chain homotpic.

Proof. Let $\mathbf{P}^A \longrightarrow A \longrightarrow 0$ and $\mathbf{Q}^A \longrightarrow A \longrightarrow 0$ be two projective resolutions of A. Then the identitiy map $\mathrm{id}_A : A \to A$ gives lifts to chain maps $f : \mathbf{P}^A \to \mathbf{Q}^A$ and $g : \mathbf{Q}^A \to \mathbf{P}^A$. Then, the compositions $g \circ f : \mathbf{P}^A \to \mathbf{P}^A$ and $f \circ g : \mathbf{Q}^A \to \mathbf{Q}^A$ are lifts of the identity. The idenity chain maps are also lifts of the identiy from each resolution to itself so we must have $g \circ f \sim \mathrm{id}_{\mathbf{P}^A}$ and $f \circ g \sim \mathrm{id}_{\mathbf{Q}^A}$ via chain homotopies. Thus the two complexes are chain homotopic. $\square$

Lemma 7.13 (Horseshoe)**.** If we have an exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and projective resolutions $\mathbf{P}^A \longrightarrow A \longrightarrow 0$ and $\mathbf{P}^C \longrightarrow C \longrightarrow 0$ then there exists a projective resolution $\mathbf{P}^B \longrightarrow B \longrightarrow 0$ and chain maps lifting the short exact sequnce such that,

$$0 \longrightarrow \mathbf{P}^A \longrightarrow \mathbf{P}^B \longrightarrow \mathbf{P}^C \longrightarrow 0$$

is an exact sequence of chain complexes. The same is true of injective resolutions.

Proof. The proof follows from the nine lemma and can be found in Rotman. $\square$

7.3 Derived Functors

Definition: Let $T : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories with enough projectives and injectives. Then for any object A in the category $\mathcal{A}$ we first take a projective resolution $\mathbf{P}^A \to A \to 0$ of A and also an injective resolution $0 \to A \to \mathbf{I}_A$. Then we can form two chain complexes by applying the functor T,

$$\cdots \longrightarrow T(P_3^A) \longrightarrow T(P_2^A) \longrightarrow T(P_1^A) \longrightarrow T(P_0^A) \longrightarrow 0$$

and

$$0 \longrightarrow T(I_A^0) \longrightarrow T(I_A^1) \longrightarrow T(I_A^2) \longrightarrow T(I_A^2) \longrightarrow \cdots$$

note that I have conventionally removed the A term and sent the last map to zero. These are chain complexes because additive functors preserve the zero map so the composition of two maps remains zero after we apply T. Thus we can take the (co)homology of these complexes. We define, the left and right derived functors of T,

$$L_n T(A) = H_n(T(\mathbf{P}^A)) \quad \text{and} \quad R^n T(A) = H^n(T(\mathbf{I}_A))$$

Given a map $f : A \to B$ we can lift this map to any two projective or injective resolutions of A and B,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I_A^0 \arrow[r] \arrow[d] & I_A^1 \arrow[r] \arrow[d] & I_A^2 \arrow[r] \arrow[d] & I_A^3 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B \arrow[r] & I_B^0 \arrow[r] & I_B^1 \arrow[r] & I_B^2 \arrow[r] & I_B^3 \arrow[r] & \cdots
\end{tikzcd}$$

If we hit this diagram with T and replace the first column with 0 then we get a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & T(I_A^0) \arrow[r] \arrow[d] & T(I_A^1) \arrow[r] \arrow[d] & T(I_A^2) \arrow[r] \arrow[d] & T(I_A^3) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & T(I_B^0) \arrow[r] & T(I_B^1) \arrow[r] & T(I_B^2) \arrow[r] & T(I_B^3) \arrow[r] & \cdots
\end{tikzcd}$$

which gives a chain map $T(\mathbf{I}_A) \to T(\mathbf{I}_B)$. Such a chain map induces a map on the homology $H^n(T(\mathbf{I}_A)) \to H^n(T(\mathbf{I}_B))$ which we call the induced map

$$f_* : R^n T(A) \to R^n T(B)$$

on the derived functors.

Proposition 7.14. Derived functors are indeed functors and are well-defined up to natural isomorphism with respect to choices of resolution.

Proof. Given A we know that any two projective or injective resolutions of A are chain homotopy equivalent. Since T is an additive functior, applying T to a chain homotopy diagram gives a chain homotopy of the new complexes. Therefore, the two resolutions have isomorphic homology so $L_n T(A) = H_n(T(\mathbf{P}^A)$ and $R^n T(A) = H^n(T(\mathbf{I}_A))$ are well-defined up to isomorphisms which, one can show with far too much notation, are natural in A. Furthermore, given a map $f : A \to B$ and resolutions of both A and B we know that any two lifts of f to chain maps are chain homotopic and therefore induce the same map on homology. Thus, the induced maps,

$$f_* : L_n T(A) \to L_n T(B) \quad \text{and} \quad f_* : R^n T(A) \to R^n T(B)$$

are well-defined with respect to the choice of lift.

If we have two maps $f : A \to B$ and $g : B \to C$ then the composition of the lifted chain maps of f and g to the respective resolutions clearly compose to give a lift of $g \circ f$. Therefore, $(g \circ f)_* = g_* \circ f_*$. Furthermore, $\mathrm{id}_{\mathbf{P}^A}$ is a lift of $\mathrm{id}_A : A \to A$ so $(\mathrm{id}_A)_* = \mathrm{id}$. ☐

Proposition 7.15. If T is left-exact then $R^0 T \cong T$ and if T is right exact then $L_0 T \cong T$ naturally.

Proof. Suppose T is left-exact and take an injective resolution of A,

$$0 \longrightarrow A \longrightarrow \mathbf{I}_A$$

which is an exact sequence. Applying T and envoking left-exactness we get the exact seqeunce,

$$0 \longrightarrow T(A) \longrightarrow T(I_A^0) \xrightarrow{T(d^0)} T(I_A^1)$$

Thus, $\ker T(d^0) = T(A)$. However,

$$R^0 T(A) = \ker T(d^0)/\mathrm{Im}(0) = T(A)$$

Furthermore given a map $f : A \to B$ we get a lift,

$$\begin{tikzcd}
0 \arrow[r] & T(A) \arrow[r] \arrow[d, "T(f)"] & T(I_A^0) \arrow[r, "T(d_A^0)"] \arrow[d] & T(I_A^1) \arrow[d] \\
0 \arrow[r] & T(B) \arrow[r] & T(I_B^0) \arrow[r, "T(d_B^0)"] & T(I_B^1)
\end{tikzcd}$$

Thus, taking kernels we have the commutative square,

$$\begin{tikzcd}
T(A) \arrow[r, "\sim"] \arrow[d, "T(f)"] & \ker T(d_A^0) \subset T(I_A^0) \arrow[d] \\
T(A) \arrow[r, "\sim"] & \ker T(d_B^0) \subset T(I_A^0)
\end{tikzcd}$$

and thus $f_* : R^0T(A) \to R^0T(B)$ is identified with $T(f)$ under the isomorphisms $R^0T(A) \cong T(A)$ and $R^0T(B) \cong T(B)$.

Likewise, suppose that T is right-exact and take a projective resolution of A,

$$\mathbf{P}_0^A \longrightarrow A \longrightarrow 0$$

which is an exact sequence. Applying T and envoking right-exactness we get the exact sequence,

$$T(P_1^A) \xrightarrow{T(\partial_1)} T(P_0^A) \longrightarrow T(A) \longrightarrow 0$$

Thus, $T(A) = T(P_0^A)/\mathrm{Im}(T(P_1^A))$. However,

$$L_0T(A) = \ker T(\partial_0)/\mathrm{Im}(T(\partial_1)) = \ker T(P_0^A)/\mathrm{Im}(T(P_1^A)) = T(A)$$

Furthermore given a map $f : A \to B$ we get a lift,

$$\begin{tikzcd}
T(P_1^A) \arrow[r, "T(\partial_1^A)"] \arrow[d] & T(P_0^A) \arrow[r] \arrow[d] & T(A) \arrow[r] \arrow[d, "T(f)"] & 0 \\
T(P_1^B) \arrow[r, "T(\partial_1^B)"] & T(P_0^B) \arrow[r] & T(B) \arrow[r] & 0
\end{tikzcd}$$

Thus, taking cokernels we have the commutative square,

$$\begin{tikzcd}
T(P_0^A)/\mathrm{Im}(T(P_1^A)) \arrow[r, "\sim"] \arrow[d] & T(A) \arrow[d, "T(f)"] \\
T(P_0^A)/\mathrm{Im}(T(P_1^A)) \arrow[r, "\sim"] & T(B)
\end{tikzcd}$$

and thus $f_* : L_0T(A) \to L_0T(B)$ is identified with $T(f)$ under the isomorphisms $L_0T(A) \cong T(A)$ and $L_0T(B) \cong T(B)$. $\square$

Theorem 7.16. Given an exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and and additive functor $T : \mathcal{A} \to \mathcal{B}$ we get a long exact sequence,

$$\begin{tikzcd}
\cdots \arrow[r] & L_3T(A) \arrow[r] & L_3T(B) \arrow[r] & L_3T(C) \arrow[r] & L_2T(A) \arrow[r] & L_2T(B) \arrow[r] & L_2T(C) \arrow[dlllll, rounded corners] \\
& L_1T(A) \arrow[r] & L_1T(B) \arrow[r] & L_1T(C) \arrow[r] & L_0T(A) \arrow[r] & L_0T(B) \arrow[r] & L_0T(C) \arrow[r] & 0
\end{tikzcd}$$

of left-derived functors and a long exact sequence,

$$\begin{tikzcd}
0 \arrow[r] & R^0T(A) \arrow[r] & R^0T(B) \arrow[r] & R^0T(C) \arrow[r] & R^1T(A) \arrow[r] & R^1T(B) \arrow[r] & R^1T(C) \arrow[dlllll, rounded corners] \\
& R^2T(A) \arrow[r] & R^2T(B) \arrow[r] & R^2T(C) \arrow[r] & R^3T(A) \arrow[r] & R^3T(B) \arrow[r] & R^3T(C) \arrow[r] & \cdots
\end{tikzcd}$$

of right-derived functors. Furthermore, a morphism of short exact sequences will induce a morphisms of the long exact sequences.

Proof. By the Horseshoe lemma, there exists an exact sequence of projective resolutions of A, B, and C respectivly,

$$0 \longrightarrow \mathbf{P}^A \longrightarrow \mathbf{P}^B \longrightarrow \mathbf{P}^C \longrightarrow 0$$

Each row of this sequence of chain maps is a short exact sequence of projectives and thus split. However, additive functors preserve splitting so the sequence of chain complexes,

$$0 \longrightarrow T(\mathbf{P}^A) \longrightarrow T(\mathbf{P}^B) \longrightarrow T(\mathbf{P}^C) \longrightarrow 0$$

is short exact. Finally, this short exact sequence of chain complexes gives rise to a long exact sequence of homology which are exactly the left-derived functors.

Similarly, by the Horseshoe lemma, there exists an exact sequence of injective resolutions of A, B, and C respectivly,

$$0 \longrightarrow \mathbf{I}^A \longrightarrow \mathbf{I}^B \longrightarrow \mathbf{I}^C \longrightarrow 0$$

Each row of this sequence of chain maps is a short exact sequence of injectives and thus split. However, additive functors preserve splitting so the sequence of chain complexes,

$$0 \longrightarrow T(\mathbf{I}^A) \longrightarrow T(\mathbf{I}^B) \longrightarrow T(\mathbf{I}^C) \longrightarrow 0$$

is short exact. Finally, this short exact sequence of chain complexes gives rise to a long exact sequence of homology which are exactly the right-derived functors. □

Remark. In practice, we will only are about left-derived functors of right-exact functors and right-derived functors of left-exact functors because for the long exact sequences to be of use we need to have T applied to the original objects appear in it somewhere.

Proposition 7.17. Let $T : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories with enough projectives and injectives. If I is projective then $R^nT(I) = 0$ and if P is projective then $L_nT(P) = 0$ for all $n > 0$.

Proof. If I is injective then

$$0 \longrightarrow I \longrightarrow I \longrightarrow 0$$

is an injective resolution of I where $I^0 = I$ and $I^n = 0$ for $n > 0$. Thus, for $n > 0$, $R^nT(I) = \ker T(d^n)/\mathrm{Im}(T(d^{n-1})) = 0$ because $d^n = 0$.

If P is projective then,

$$0 \longrightarrow P \longrightarrow P \longrightarrow 0$$

is an injective resolution of P where $P_0 = P$ and $P_n = 0$ for $n > 0$. Thus, for $n > 0$, $L_nT(P) = \ker T(\partial_n)/\mathrm{Im}(T(\partial_{n+1})) = 0$ because $\partial_n = 0$. □

7.4 Ext and Tor

Proposition 7.18 (Tensor-Hom Adjunction)**.**

$$\mathrm{Hom}_A\left(M\otimes N, P\right) = \mathrm{Hom}_A\left(M, \mathrm{Hom}_A\left(N, P\right)\right)$$

That is, the functor $(-)\otimes_R N$ is a left-adjoint of the functor $\mathrm{Hom}_R\left(N, -\right)$.

Remark. Since $(-)\otimes_R N$ is a left-adjoint it is cocontinuous and thus right-exact. Furthermore, $\mathrm{Hom}\left(R, N\right) -$ is a right-adjoint so it is continuous and thus left-exact. However, we will prove these facts explicitly without too much appeal to abstract nonsense.

Lemma 7.19. The functor $(-)\otimes_R N$ is right-exact.

Proof. Let

$$K \xrightarrow{i} L \xrightarrow{j} M \longrightarrow 0$$

be exact. Consider the sequence,

$$K\otimes N \xrightarrow{i\otimes \mathrm{id}_N} L\otimes N \xrightarrow{j\otimes \mathrm{id}_N} M\otimes N \longrightarrow 0$$

Construct a map $\phi : M\times N \to L\otimes N/(i\otimes \mathrm{id}_N)(K\otimes M)$ by $\phi(m,n) = \ell\otimes n$ where $j(\ell) = m$ where I have used the fact that j is surjective. If $\ell, \ell' \in L$ where $j(\ell) = j(\ell')$ then,

$$\ell\otimes n - \ell'\otimes n = (\ell - \ell')\otimes n$$

However, $\ell - \ell' \in \ker j = \mathrm{Im}(i)$ so take $k\in K$ such that $i(k) = \ell - \ell'$. Thus,

$$\ell\otimes n - \ell'\otimes n = i(k)\otimes n = (i\otimes \mathrm{id}_N)(k\otimes n) = 0$$

in the quotient. By the universal property of the tensor product, there exists a linear map,

$$\tilde{\phi} : M\otimes N \to L\otimes N/(i\otimes \mathrm{id}_N)(K\otimes M)$$

Furthermore, $\tilde{\phi}$ is the inverse map to $\jmath\otimes \mathrm{id}_N$ on the quotient. Therefore, $\ker j\otimes \mathrm{id}_N$ is exactly $\mathrm{Im}(i\otimes \mathrm{id})$. ☐

Definition: Define, $\mathrm{Tor}_n^R\left(-, N\right)$ to be the n^{th} left-derived functor of $(-)\otimes_R N$.

Proposition 7.20. Tor is symmetric, $\mathrm{Tor}_n^R\left(M, N\right) \cong \mathrm{Tor}_n^R\left(N, M\right)$.

Proposition 7.21. Properties of the Tor functor,

1. If M or N is projective then $\mathrm{Tor}_n^R\left(M, N\right) = 0$ for $n > 0$.
2. $\mathrm{Tor}_n^R\left(\bigoplus_\alpha M_\alpha, N\right) \cong \bigoplus_\alpha \mathrm{Tor}_n^R\left(M_\alpha, N\right)$

3. If $r \in R$ is not a zero divisor, then,

$$\mathrm{Tor}_1^R(R/(r), N) \cong \{n \in N \mid rn = 0\}$$

   the r-torsion of N and,

$$\mathrm{Tor}_n^R(R/(r), N) = 0$$

   for $n > 1$.

4. If R is a PID then $\mathrm{Tor}_n^R(M, N) = 0$ for $n > 1$.

Proof. I will sketch each:

1. If M is projective then $\mathrm{Tor}_n^R(M, N) = 0$ for $n > 0$ by Proposition 7.17. Otherwise use symmetry.

2. This follows from the fact that direct sum and tensor product commute.

3. (DO THIS)

4. If R is a PID then submodules of free modules are free. Therefore given any R-module M we can chose a projective resolution,

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

   where $F \to M$ is the surjection of a free R-module and $K \to F$ is the inclusion of the kernel which is also free since $K \subset F$ and F is a free R-module. Thus, the left derived functors vanish after $n = 1$ since $P_n^M = 0$ for $n > 1$ and thus the kernels of the boundary maps are zero.

$\square$

Proposition 7.22. Given a short exact sequence of R-modules,

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

then we get a long exact sequence,

$$\cdots \longrightarrow \mathrm{Tor}_1^R(K, N) \longrightarrow \mathrm{Tor}_1^R(L, N) \longrightarrow \mathrm{Tor}_1^R(M, N) \longrightarrow K \otimes N \longrightarrow L \otimes N \longrightarrow M \otimes N \longrightarrow 0$$

Lemma 7.23. The functor $\mathrm{Hom}(A, -)$ is left-exact.

Proof. $\mathrm{Hom}(A, -)$ is a continuous functor and therefore preserves kernels. $\square$

Lemma 7.24. The functor $\mathrm{Hom}(P, -)$ is exact if and only if P is projective. Similarly, the functor $\mathrm{Hom}(-, I)$ is exact if and only if I in injective.

Proof. Since $\mathrm{Hom}(P, -)$ is always left-exact, we need only that $\mathrm{Hom}(P, -)$ takes surjections to surjections. Thus if $f : A \to B$ is a surjection, we need that any map $g : P \to B$ can lift to a map $\tilde{g} : P \to A$ such that $f \circ \tilde{g} = g$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{g}"'] \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

This is exactly the definition of P being projective. The injective case is similar. □

Definition: Let M be an R-module. Define $\operatorname{Ext}_R^n(M,-)$ to be the n^{th} right-derived functor of $\operatorname{Hom}_R(M,-)$.

Proposition 7.25. Properties of the Ext functor,

1. $\operatorname{Ext}_R^n(A,B)=0$ for $n>0$ if either A is projective or B is injective.

2. $$\operatorname{Ext}_R^n\left(\bigoplus_\alpha A_\alpha, B\right) \cong \prod_\alpha \operatorname{Ext}_R^n(A_\alpha,B)$$
$$\operatorname{Ext}_R^n\left(A, \prod_\beta B_\beta\right) \cong \prod_\beta \operatorname{Ext}_R^n(A,B_\beta)$$

3. If R is a PID then $\operatorname{Ext}_R^n(A,B)=0$ for $n>1$.

Proof. I will sketch each:

1. If P is projective then $\operatorname{Hom}_R(P,-)$ is exact so its derived functors are trivial. If I is injective then $\operatorname{Ext}_R^n(A,I)=0$ by Lemma 7.17.

2. This follows from the fact that $\operatorname{Hom}(A,-)$ is contious and thus commutes with products so a resolution of the product is sent to a complex of products. Furthermore, $\operatorname{Hom}(-,B)$ takes colimts to limits and thus
$$\operatorname{Hom}\left(\bigoplus_\alpha A_\alpha, -\right) \cong \prod_\alpha \operatorname{Hom}(A_\alpha,-)$$
and its derived functors will also be products since it takes each injective to a product.

3. (DO THIS)

□

Proposition 7.26. Given a short exact sequence of R-modules,

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

then we get a long exact sequence,

$$0 \to \operatorname{Hom}_R(N,K) \to \operatorname{Hom}_R(N,L) \to \operatorname{Hom}_R(N,M) \to \operatorname{Ext}_R^1(N,K) \to \operatorname{Ext}_R^1(N,L) \to \cdots$$

8 Flatness

Definition: An A-module Q is said to to be A-flat if $(-) \otimes_A Q$ is exact. Thus, Q is A-flat iff $\mathrm{Tor}_n^A(-, Q) = 0$ for $n > 0$. Furthermore if $\mathrm{Tor}_1^A(-, Q) = 0$ then $(-) \otimes_A Q$ is exact by the long exact sequence. Thus, Q is A-flat iff $\mathrm{Tor}_1^A(-, Q) = 0$.

Proposition 8.1.

$$\mathrm{Tor}_n^A\left(\varinjlim M_i, P\right) = \varinjlim \mathrm{Tor}_n^A(M_i, P)$$

Proof. The functor $\varinjlim$ is exact. Furthermore,

$$\begin{aligned}\mathrm{Hom}_A\left((\varinjlim M_i) \otimes_A P, N\right) &= \mathrm{Hom}_A\left(\varinjlim M_i, \mathrm{Hom}_A(P, N)\right) = \varprojlim \mathrm{Hom}_A(M_i, \mathrm{Hom}_A(P, N)) \\ &= \varprojlim \mathrm{Hom}_A(M_i \otimes_A P, N) = \mathrm{Hom}_A\left(\varinjlim(M_i \otimes_A P), N\right)\end{aligned}$$

Then since the Yonenda embedding is injective,

$$(\varinjlim M_i) \otimes_A P = \varinjlim(M_i \otimes_A P)$$

□

Proposition 8.2. If Q is projective then Q is A-flat.

Proof. Since Q is projective $\mathrm{Tor}_n^A(-, Q) = 0$ for $n > 0$. □

Proposition 8.3. Let M be an A-module then the following are equivalent.

1. The A-module M is A-flat.
2. The functor $(-) \otimes_A M$ preserves monomorphisms.
3. Every finitely generated ideal $I \subset A$ satisfies $I \otimes_A M = IM$.
4. $\mathrm{Tor}_1^A(M, A/I) = 0$ for all finitely generated ideals $I \subset A$.
5. $\mathrm{Tor}_1^A(M, N) = 0$ for any finitely generated A-module N.
6. For all $a_i \in A$ and $x_i \in M$ with $\sum_{i=1}^r a_i x_i = 0$ there exists $b_{ij} \in A$ such that $\sum_{i=1}^r b_{ij} = 0$ for all j and there exist $y_i \in M$ such that $x_i = \sum_{j=1}^s b_{ij} y_j$.

Proposition 8.4. Let B be an A-algebra which is flat as an A-module and M is a B-flat B-module then M is an A-flat A-module.

Proof. Let S be an A-module. Then,

$$S \otimes_A M = S \otimes_A (B \otimes_B M) = (S \otimes_A B) \otimes_B M$$

However, $(-) \otimes_A B$ and $(-) \otimes_B M$ are exact so the composition $(-) \otimes_A M$ is exact. □

Proposition 8.5. Suppose B is an A-algebra then if M is A-flat then $B \otimes_A M$ is B-flat.

Proof. Suppose S is a B-module then,

$$S \otimes_B (B \otimes_A M) = (S \otimes_B B) \otimes_A M = S \otimes_A M$$

However, $(-) \otimes_A M$ is exact so $(-) \otimes_B (B \otimes_A M)$ is exact. $\square$

Proposition 8.6. If $S \subset A$ is multiplicative then $S^{-1}A$ is A-flat.

Proof. Notice that if M is an A-module then $S^{-1}M \cong M \otimes_A S^{-1}A$ and localization is exact so $(-) \otimes_A S^{-1}A$ is exact. $\square$

Proposition 8.7. Let M, N be A-modules and assume B is a flat A-algebra then,

$$\mathrm{Tor}_i^B (M \otimes_A B, N \otimes_A B) \cong \mathrm{Tor}_i^A (M, N) \otimes_A B$$

and similarly,

$$\mathrm{Ext}_B^i (M \otimes_A B, N \otimes_A B) \cong \mathrm{Ext}_A^i (M, N) \otimes_A B$$

Proof. Let $\mathbf{P} \to N \to 0$ be a projective resolution of N. Because B is A-flat then $\mathbf{P} \otimes_A B \to N \otimes_A B \to 0$ is a projective resolution. Thus,

$$\begin{aligned} \mathrm{Tor}_i^B (M \otimes_A B, N \otimes_A B) &= H_i((M \otimes_A B) \otimes_B (\mathbf{P} \otimes_A B)) \\ &= H_i((M \otimes_A \mathbf{P}) \otimes_A B) = \mathrm{Tor}_i^A (M, N) \otimes_A B \end{aligned}$$

where again I have used the exactness of $(-) \otimes_A B$ to pull it out of the homology since it preserves kernels and images. $\square$

Proposition 8.8. Let A be a local ring and M a finitely generated A-module. Then the following are equivalent,

1. M is free
2. M is projective
3. M is flat

Proof. The first and second implications are true in general. Suppose $\mathfrak{m} \subset A$ is the maximal ideal and $k = A/\mathfrak{m}$. Then $M \otimes_A k = M/(\mathfrak{m}M)$ is a finite-dimensional k-vectorspace. There exist $x_1, \dots, x_r \in M$ such that their image $\bar{x}_1, \dots, \bar{x}_r \in M$ is a basis of $M \otimes_A k$. Consider the span map $\phi : A^r \to M$ then $\phi \otimes \mathrm{id} : k^r \to M \otimes_A k = M/(\mathfrak{m}M)$ is surjective so $\mathrm{Im}(\phi) + \mathfrak{m}M = M$. By Nakayama, $M = \mathrm{Im}(\phi)$. $\square$

Lemma 8.9. Let $\phi : A \to B$ be a ring map. Take $\mathfrak{P} \in \mathrm{Spec}\,(A)$ and $\mathfrak{p} = \phi^{-1}(\mathfrak{P})$ and N an A-module. Then,

$$\mathrm{Tor}_i^{A_\mathfrak{p}} (B_\mathfrak{P}, N_\mathfrak{p}) = \mathrm{Tor}_i^A (B, N)_\mathfrak{P}$$

Proposition 8.10. Let $\phi : A \to B$ be a ring map then the following are equivalent,

1. B is A-flat

2. $B_{\mathfrak{P}}$ is $A_{\mathfrak{p}}$-flat for all primes $\mathfrak{p} = \phi^{-1}(\mathfrak{P})$

3. $B_{\mathfrak{P}}$ is $A_{\mathfrak{p}}$-flat for all maximal ideals $\mathfrak{p} = \phi^{-1}(\mathfrak{P})$

Proof. First, $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$ which is clearly flat over $A_{\mathfrak{p}}$ by change of base. Furthermore, $B_{\mathfrak{P}}$ is flat over $B_{\mathfrak{p}}$ because $B_{\mathfrak{P}} = S^{-1}B_{\mathfrak{p}}$ for $S = B_{\mathfrak{p}} \setminus \mathfrak{P}B_{\mathfrak{p}}$. By transitivity, $B_{\mathfrak{P}}$ is $A_{\mathfrak{p}}$-flat. Clearly, the second implies the third. Take $Q = \operatorname{Tor}_i^A(B, N)$ using the above lemma,

$$Q_{\mathfrak{P}} = \operatorname{Tor}_i^{A_{\mathfrak{p}}}(B_{\mathfrak{P}}, N_{\mathfrak{p}}) = 0$$

because $B_{\mathfrak{P}}$ is $A_{\mathfrak{p}}$-flat. Thus, $\forall \mathfrak{P} \in \operatorname{Spec}(A)$ which are maximal we have $Q_{\mathfrak{P}} = 0$ which implies that $Q = 0$. $\square$

Definition: Let M be an A-module. We say that M is *faithfully flat* over A if the sequence,

$$N \longrightarrow P \longrightarrow Q$$

is exact if and only if the sequence,

$$N \otimes_A M \longrightarrow P \otimes_A M \longrightarrow Q \otimes_A M$$

is exact.

Theorem 8.11. Let M be an A-module. Then the following are equivalent,

1. M is faithfully flat over A

2. M is A-flat and for any A-module $N \neq 0$ we have $N \otimes_A M \neq 0$.

3. M is A-flat an $\forall m \subset A$ maximal we have $M \neq \mathfrak{m}M$.

Proof. Faithfully flat implies flatness. Furhtermore, consider the sequence

$$0 \to N \to 0$$

If $M \otimes_A N = 0$ then clearly the sequence

$$0 \to M \otimes_A N \to 0$$

is exact. Thus,

$$0 \to N \to 0$$

must be exact so $N = 0$.

Now suppose 2. and let,

$$N \xrightarrow{f} P \xrightarrow{g} Q$$

be a sequence such that,

$$N \otimes_A M \longrightarrow P \otimes_A M \longrightarrow Q \otimes_A M$$

is exact. However, $g \circ f = 0$ by exactness and the flatness of M. Furthermore,

$$\ker g \otimes_A \mathrm{id}_M = \ker g \otimes_A M \qquad \mathrm{Im}(f \otimes_A \mathrm{id}_M) = \mathrm{Im}(f) \otimes_A M$$

by flatness. However, exactness implies that $\ker g \otimes_A M = \mathrm{Im}(f) \otimes_A M$ which implies that $(\ker g / \mathrm{Im}(f)) \otimes_A M = 0$ so $\ker g = \mathrm{Im}(f)$ because $(-) \otimes_A M$ is injective. Furthermore, assuming 2. take $\mathfrak{m} \subset A$ maximal then $M \otimes A/\mathfrak{m} \neq 0$ implies that $M \neq \mathfrak{m}M$. Now assume 3. and take $N \neq 0$ with $x \in N$ nonzero. Let $I = \mathrm{Ann}_A(x) \subset \mathfrak{m}$ for some maximal ideal. Consider the map $\iota : A/I \xrightarrow{\sim} Ax \subset N$. Then $A/\mathfrak{m} \otimes_A M \neq 0$ implies tht $A/I \otimes M \neq 0$ so $Ax \otimes_A M \neq 0$ by 3. Then $Ax \otimes_A M$ embedds inside $N \otimes_A M$ because M is A-flat. Thus $N \otimes_A M \neq 0$. $\square$

Corollary 8.12. Let A and B be local rings and $A \to B$ a local map. Let M be a nontrivial finitely generated B-module, then M if A-flat $\iff$ M of faithfully flat over A.

Proof. Consider the maximal ideal $\mathfrak{m}_B \subset B$ then M is A-flat implies that $M \otimes_B B/\mathfrak{m}_B \neq 0$ by Nakayama, $M \otimes_B B/\mathfrak{m}_A B \neq 0$. However, this equals $M \otimes_A A/\mathfrak{m}_A$ which must be nonzero so $M \neq \mathfrak{m}_A M$. Thus, by above, M is faithfully flat. $\square$

Proposition 8.13. Let $A \to B$ be a map of rings. If M is faithfully flat over A then $M_B = M \otimes_A B$ is faithfully flat over B.

Proposition 8.14. Let M be a B-module and $A \to B$ a map of rings. Suppose that M is faithfully flat over B and faithfully flat over A then B is faithfully flat over A.

Proposition 8.15. Let $\phi : A \to B$ be a map of rings with B faithfully flat over A then,

1. For any A-module N, the canonical map,

$$N \to N \otimes_A B$$

   is injective. In particular, ϕ is injective.

2. For any ideal $I \subset A$, we have $IB \cap A = I$.

3. $\phi^{-1} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Proof. Let $x \neq 0$ take $x \otimes 1 \neq 0$ since $Ax \otimes_A B \neq 0$ because B is faithfully flat. Thus, $x \mapsto x \otimes 1$ is injective, Now consider the map,

$$A/I \to A/I \otimes_A B = B/IB$$

which is injective by the above argument. Thus we have a diagram,

$$\begin{array}{ccc} A/I & \xrightarrow{\tilde{\phi}} & A/I \otimes_A B \\ \uparrow & & \downarrow \\ A & \xrightarrow{\bar{\phi}} & B/IB \end{array}$$

Then $IB \cap A = \ker \bar{\phi}$ and $\ker \tilde{\phi} = \ker \bar{\phi}/I = IB \cap A/I = 0$. Thus $IB \cap A = I$. Furthermore, consider $\phi^{-1} : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ and take $\mathfrak{p} \in \mathrm{Spec}\,(A)$. Consider,

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A B \neq 0$$

which is nonzero because B is faithfully flat. Thus $B_{\mathfrak{p}} \supsetneq \mathfrak{p}B_{\mathfrak{p}}$ which implies that there exists $\mathfrak{m}$ a maximal ideal of $B_{\mathfrak{p}}$ containing $\mathfrak{p}B_{\mathfrak{p}}$. Furthermore, $\mathfrak{m} \cap A_{\mathfrak{p}} \supset \mathfrak{p}A_{\mathfrak{p}}$ which implies that $\mathfrak{m} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Then $\mathfrak{P} = \mathfrak{m} \cap B$ so

$$\mathfrak{P} \cap A = \mathfrak{m} \cap A = (\mathfrak{m} \cap A_{\mathfrak{p}}) \cap A = (\mathfrak{p}A_{\mathfrak{p}}) \cap A = \mathfrak{p}$$

□

Proposition 8.16. Let B be a faithfully flat A-algebra and M an A-module then,

1. M is flat (resp. faithfully flat) over $A \iff M_B$ is flat (resp. faithfully flat) over B.
2. If A is local and M is a finitly generated A-module then M is free over $A \iff M_B$ is free over B.

Proof. The □

Theorem 8.17. Let $\varphi : A \to B$ be a ring map thn the following are equivalent,

1. B is faithfully flat over A i.e. φ is faithfully flat.
2. φ is flat and $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is a surjection.
3. φ is flat and for any maximal ideal $\mathfrak{m}$ of A there exists a maximal ideal $\mathfrak{m}'$ of B such that $\varphi^{-1}(\mathfrak{m}') = \mathfrak{m}$.