Riemann Surfaces Final

Benjamin Church

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Remark 1. Whenever I have the complex coordinate z, I will denote the real coordinates by $x, y \in \mathbb{R}$ such that $z = x + iy$ and the derivatives $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$.

Problem 1

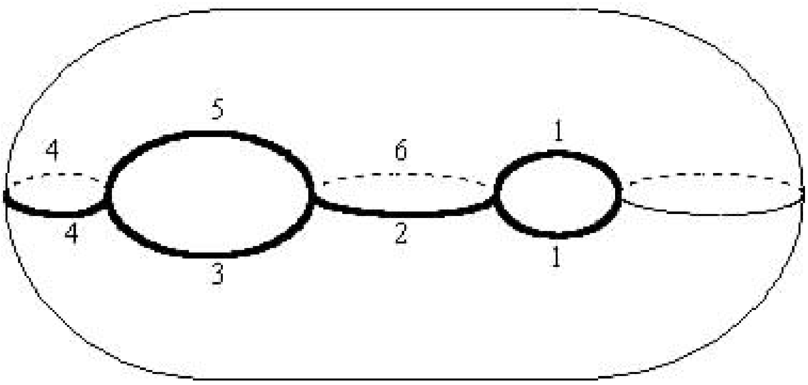
Take five pairwise distinct points $a_1, a_2, a_3, a_4, a_4, a_5 \in \mathbb{C}$ and consider the equation,

$$w^2 = \prod_{i=1}^{5} (x - a_i)$$

We wish to consider the compact Riemann surface $\hat{X}$ for this equation. I will assume, for convenience, that these points are all nonzero and not infinity although this assumption will not seriously affect my argument only remove degenerate cases.

(a)

Consider two copies of the complex plane, called sheets, labeled (I) and (II). We give each sheet three branch cuts, a_1-a_2 and a_3-a_4 and a_5-∞. When we glue these two sheets with cuts holes along the boundaries of the cuts we get a surface with two holes missing a point. Now we compactify the glued sheets by adding a point ∞ at infinity to get a two holed torus $\hat{X}$ which is the unique orientable compact surface of genus $g = 2$.



To make $\hat{X}$ a Riemann surface we need to choose a covering of $\hat{X}$ by holomorphic chars. For a point away from ∞ and any branch point a_i and any cut, we simply take a disk in the complex plane (I) or (II) with standard holomorphic coordinate. For a point near the cut, we take a half-disk on (I) which continues to a half-disk on (II) across the cut. At ∞ we take the holomorphic coordinate $z = \zeta^{-2}$ since ∞ is a branch point where the first half disk is mapped onto sheet (I) and the second onto sheet (II). At a branch point a_i we take the holomorphic coordinate $z = a_i + \zeta^2$ where the double cover of the disk about a_i by ζ^2 is taken to be injective by sending the first covering to sheet (I) and the second covering to sheet (II).

(b)

The function z on $\hat{X}$ has zeroes at the two $0_I \in (I)$ and $0_{II} \in (II)$ and has a double pole at ∞ because, in local coordinates, $z = \zeta^{-2}$ which is a pole of order two as $\zeta \to 0$ i.e. $z \to \infty$.

We need to show that w is well-defined and meromorphic on $\hat{X}$. Away from branch points and infinities w is a holomorphic function on $\mathbb{C}$ and thus on the holomorphic coordinate disks. Now we need to check the branch points. Near $z = a_i$ we have the holomorphic coordinate $z = \zeta^2 + a_i$ and thus,

$$w(\zeta) = \pm\sqrt{\prod_{j=1}^{5}(\zeta^2 + a_i - a_j)} = \zeta\sqrt{\prod_{j \neq i}^{5}(\zeta^2 + a_i - a_j)}$$

which is holomorphic on the ζ coordinate chart with a simple zero at $\zeta \to 0$ i.e. $z = a_i$. Next, near $z = \infty$ we have the coordinate chart $z = \frac{1}{\zeta^2}$ and thus,

$$w(\zeta) = \pm\sqrt{\prod_{j=1}^{5}(\zeta^{-2} - a_j)} = \zeta^{-5}\sqrt{\prod_{j=1}^{5}(1 - a_j\zeta)}$$

which has a fifth-order pole at ∞. I should remark that $\pm$ is shorthand notation for the fact that as ζ passes the negative imaginary axis we transition from (I) to (II) meaning that w changes sign. This negative sign allows the manipulation $\pm\sqrt{\zeta^2} = \zeta$ since the sign information that would usually be lost in the choice of square root is preserved by the $\pm$ on different sheets.

(c)

Consider the form $\mathrm{d}z$. Away from branch points, $\mathrm{d}z = \mathrm{d}\zeta$ in local coordinates because the sheets are given local coordinates by choosing standard disks in the complex planes of each sheet. Near the branch point a_i, in the local holomorphic coordinate, $z = \zeta^2 + a_i$ and thus $\mathrm{d}z = 2\zeta\,\mathrm{d}\zeta$ which has a simple zero at $\zeta \to 0$ i.e. $z \to a_i$. Furthermore, near ∞, in the local holomorphic coordinate $z = \zeta^{-2}$ and therefore $\mathrm{d}z = -2\zeta^{-3}\,\mathrm{d}\zeta$ which has a pole of order 3 at $\zeta \to 0$ i.e. $z \to \infty$.

Therefore, $\mathrm{d}z$ is a meromorphic $(1,0)$-form thus a meromorphic section of the line bundle K_X which has Chern class $c_1(K_X) = -\chi(\hat{X}) = 2g - 2 = 2$. Thus, for any meromorphic $(1,0)$-form φ we expect,

$$(\#\text{ zeros of }\varphi) - (\#\text{ poles of }\varphi) = 2$$

This here is true because $\mathrm{d}z$ has five simple zeros at the points a_i and a triple pole at ∞.

(d)

We define two holomorphic forms on $\hat{X}$,

$$\omega_1 = \frac{\mathrm{d}z}{w} = \begin{cases} \frac{\mathrm{d}z}{\sqrt{\prod_{i=1}^5 (z-a_i)}} & z \in (\mathrm{I}) \\ -\frac{\mathrm{d}z}{\sqrt{\prod_{i=1}^5 (z-a_i)}} & z \in (\mathrm{II}) \end{cases}$$
$$\omega_2 = \frac{z\,\mathrm{d}z}{w} = \begin{cases} \frac{z\,\mathrm{d}z}{\sqrt{\prod_{i=1}^5 (z-a_i)}} & z \in (\mathrm{I}) \\ -\frac{z\,\mathrm{d}z}{\sqrt{\prod_{i=1}^5 (z-a_i)}} & z \in (\mathrm{II}) \end{cases}$$

We need to check that these forms are holomorphic when expressed in terms of the local holomorphic variables on coordinate charts. At the branch point a_i, in terms of the local holomorphic coordinate $z = \zeta^2 + a_i$ we express the forms,

$$\omega_1 = \frac{\mathrm{d}z}{w} = \pm\frac{2\zeta\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^5 (\zeta^2 + a_i - a_j)}} = \frac{2\,\mathrm{d}\zeta}{\sqrt{\prod_{j\neq i}^5 (\zeta^2 + a_i - a_j)}}$$
$$\omega_2 = \frac{z\,\mathrm{d}z}{w} = \pm\frac{(\zeta^2 + a_i)2\zeta\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^5 (\zeta^2 + a_i - a_j)}} = \frac{2(\zeta^2 + a_i)\,\mathrm{d}\zeta}{\sqrt{\prod_{j\neq i}^5 (\zeta^2 + a_i - a_j)}}$$

which are well-defined and nonzero in the limit $\zeta \to 0$. Next, at ∞ with local holomorphic coordinate $z = \zeta^{-2}$ the forms are,

$$\omega_1 = \frac{\mathrm{d}z}{w} = -\frac{2\zeta^{-3}\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^5 (\zeta^{-2} - a_j)}} = -\frac{2\zeta^{-3}\,\mathrm{d}\zeta}{\zeta^{-5}\sqrt{\prod_{j=1}^5 (1 - a_j\zeta^2)}} = -\frac{2\zeta^{2}\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^5 (1 - a_j\zeta^2)}}$$
$$\omega_2 = \frac{z\,\mathrm{d}z}{w} = -\frac{2\zeta^{-5}\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^5 (\zeta^{-1} - a_j)}} = -\frac{2\zeta^{-5}\,\mathrm{d}\zeta}{\zeta^{-5}\sqrt{\prod_{j=1}^5 (1 - a_j\zeta^2)}} = -\frac{2\,\mathrm{d}\zeta}{\sqrt{\prod_{j=1}^5 (1 - a_j\zeta^2)}}$$

Both forms are well-defined and holomorphic in the limit $\zeta \to 0$. Furthermore, we see that ω_1 has a double zero at ∞ and ω_2 is nonzero in the limit $z \to \infty$. Therefore, ω_1 and ω_2 cannot be linearly dependent $(1,0)$-forms since they have different pole structures and are nonzero. Therefore, neither can be a multiple of the other.

The zeros of these forms are interesting. We have shown that ω_1 has a double zero at ∞ and is nonzero elsewhere while ω_2 is nonzero at all branch points and ∞ but clearly has simple zeros at $z = 0_{\mathrm{I}}$ and $z = 0_{\mathrm{II}}$. We should expect this because as holomorphic $(1,0)$-forms, ω_1 and ω_2 are holomorphic sections of the canonical bundle K_X and thus (since holomorphic forms have no poles) the number of zeros of each must equal $c_1(K_X) = -\chi(\hat{X}) = 2g - 2 = 2$ where $g = 2$ is the genus of our Riemann surface.

Problem 2

Let τ be a complex number with $\mathrm{Im}(\tau) > 0$. Consider the lattice

$$\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$$

(a)

The Weierstrass $\wp$-function is defined by,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

Furthermore, the function $\zeta(z)$ is defined by,

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda^\times} \left(\frac{1}{z+\omega} - \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

and satisfies,

$$\frac{\mathrm{d}}{\mathrm{d}z} \zeta(z) = -\wp(z)$$

Finally, the function $\sigma(z)$ is defined by,

$$\sigma(z) = \exp\left(\int^z \zeta(z') \, \mathrm{d}z' \right) = z \prod_{\omega \in \Lambda^\times} \left(1 + \frac{z}{\omega} \right) e^{-\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}$$

which satisfies,

$$\frac{\mathrm{d}}{\mathrm{d}z} \log \sigma(z) = \zeta(z)$$

The integral here is formal with coefficients chosen on the right hand side such that the product converges. We need to derive the transformation properties of these functions.

First,

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\omega \in \Lambda^\times} \left(\frac{2}{(z+\omega)^3} \right) = -2 \sum_{\omega \in \Lambda} \left(\frac{1}{(z+\omega)^3} \right)$$

Clearly $\wp'$ is invariant under shifts by lattice points. Thus, for $\omega \in \Lambda$,

$$\frac{\mathrm{d}}{\mathrm{d}z} \left(\wp(z+\omega) - \wp(z) \right) = 0 \implies \wp(z+\omega) = \wp(z) + c_\omega$$

In particular, take $z = -\frac{1}{2}\omega$ so $\wp(\frac{1}{2}\omega) = \wp(-\frac{1}{2}\omega) + c_\omega$ but $\wp$ is even so $c_\omega = 0$. Thus, $\wp(z+\omega) = \wp(z)$ so $\wp$ is doubly periodic.

Since $\zeta'(z) = -\wp(z)$ which is doubly periodic, we have that,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\zeta(z+\omega) - \zeta(z)\right) = 0$$

which implies that $\zeta(z+\omega) = \zeta(z) + c_\omega$. In particular, write,

$$\zeta(z+1) = \zeta(z) + \eta_1$$
$$\zeta(z+\tau) = \zeta(z) + \eta_2$$

If we integrate ζ around a contour C containing its unique pole at $z = 0$ then we can deform this contour to the boundary of X_{cut} (making sure to enclose the point $z = 0$ and not any other lattice points). Thus, since $\operatorname{Res}_0 \zeta = 1$, the residue theorem gives,

$$\int_{\partial X_{\text{cut}}} \zeta(z)\,\mathrm{d}z = 2\pi i$$

We compute this integral along the cycles,

$$\begin{aligned}
\int_{\partial X_{\text{cut}}} \zeta(z)\,\mathrm{d}z &= \int_A [\zeta(z) - \zeta(z+\tau)]\,\mathrm{d}z + \int_B [\zeta(z+1) - \zeta(z)]\,\mathrm{d}z \\
&= \int_A (-\eta_2)\,\mathrm{d}z + \int_B \eta_1\,\mathrm{d}z = -\eta_2 + \eta_1\tau
\end{aligned}$$

Therefore,

$$\eta_1\tau - \eta_2 = 2\pi i$$

Lastly, consider the transformation properties of σ. We have,

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\sigma(z) = \frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\sigma(z+1) - \log\sigma(z)\right) &= \zeta(z+1) - \zeta(z) = \eta_1 \\
\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\sigma(z+\tau) - \log\sigma(z)\right) &= \zeta(z+\tau) - \zeta(z) = \eta_2
\end{aligned}$$

This implies that,

$$\log\left(\frac{\sigma(z+1)}{\sigma(z)}\right) = \eta_1 z + \mu_1$$
$$\log\left(\frac{\sigma(z+\tau)}{\sigma(z)}\right) = \eta_2 z + \mu_2$$

and hence,

$$\sigma(z+1) = \sigma(z)e^{\eta_1 z+\mu_1}$$
$$\sigma(z+\tau) = \sigma(z)e^{\eta_2 z+\mu_2}$$

However, σ is an odd function and thus, taking $z = -\frac{1}{2}$,

$$\sigma(\tfrac{1}{2}) = -\sigma(\tfrac{1}{2})e^{-\frac{1}{2}\eta_1+\mu_1}$$

which implies that $-\frac{1}{2}\eta_1 + \mu_1 = \pi i$ up to factors of $2\pi i$ which do not change the exponential. Thus, take

$$\mu_1 = \tfrac{1}{2}\eta_1 + \pi i$$

Furthermore, taking $z = -\frac{1}{2}\tau$ we find that,

$$\sigma(\tfrac{1}{2}\tau) = -\sigma(\tfrac{1}{2}\tau)e^{-\frac{1}{2}\tau\eta_2+\mu_2}$$

and thus $-\frac{1}{2}\tau\eta_2 + \mu_2 = \pi i + 2\pi i$ meaning that we can take,

$$\mu_2 = \tfrac{1}{2}\tau\eta_2 + \pi i$$

Putting everything together,

$$\sigma(z+1) = \sigma(z)e^{\eta_1(z+\frac{1}{2})+\pi i}$$
$$\sigma(z+\tau) = \sigma(z)e^{\eta_2(z+\frac{1}{2}z)+\pi i}$$

(b)

Define the Jacobi $\theta_1(z|\tau)$ function by,

$$\theta_1(z|\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\tfrac{1}{2})^2\tau + 2\pi i(n+\tfrac{1}{2})(z+\tfrac{1}{2})\right)$$

The transformation properties of this function are as follows,

$$\begin{aligned}
\theta_1(z+1|\tau) &= \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\tfrac{1}{2})^2\tau + 2\pi i(n+\tfrac{1}{2})(z+1+\tfrac{1}{2})\right) \\
&= \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\tfrac{1}{2})^2\tau + 2\pi i(n+\tfrac{1}{2})(z+\tfrac{1}{2}) + 2\pi i(n+\tfrac{1}{2})\right) \\
&= \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\tfrac{1}{2})^2\tau + 2\pi i(n+\tfrac{1}{2})(z+\tfrac{1}{2}) + \pi i\right) \\
&= -\theta_1(z|\tau)
\end{aligned}$$

since $e^{2\pi i} = 1$ and $e^{\pi i} = -1$. Likewise,

$$\begin{aligned}
\theta_1(z+\tau|\tau) &= \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\tfrac{1}{2})^2\tau + 2\pi i(n+\tfrac{1}{2})(z+\tau+\tfrac{1}{2})\right) \\
&= \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\tfrac{1}{2})^2\tau + 2\pi i(n+\tfrac{1}{2})(z+\tfrac{1}{2}) + 2\pi i\tau(n+\tfrac{1}{2})\right) \\
&= \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+1+\tfrac{1}{2})^2\tau - \pi i\tau + 2\pi i(n+\tfrac{1}{2})(z+\tfrac{1}{2})\right) \\
&= e^{-\pi i\tau} \sum_{m\in\mathbb{Z}} \exp\left(\pi i(m+\tfrac{1}{2})^2\tau + 2\pi i(m-1+\tfrac{1}{2})(z+\tfrac{1}{2})\right) \\
&= e^{-\pi i\tau} \sum_{m\in\mathbb{Z}} \exp\left(\pi i(m+\tfrac{1}{2})^2\tau + 2\pi i(m+\tfrac{1}{2})(z+\tfrac{1}{2})\right) - 2\pi i(z+\tfrac{1}{2}) \\
&= e^{-\pi i\tau - 2\pi iz - \pi i} \sum_{m\in\mathbb{Z}} \exp\left(\pi i(m+\tfrac{1}{2})^2\tau + 2\pi i(m+\tfrac{1}{2})(z+\tfrac{1}{2})\right) \\
&= -e^{-\pi i\tau - 2\pi iz}\theta_1(z|\tau)
\end{aligned}$$

(c)

First, define,

$$f(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z|\tau) + \eta_1 z = \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} + \eta_1 z$$

The function $f(z)$ transforms as,

$$\begin{aligned}
f(z+1) &= \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z+1|\tau) + \eta_1 z + \eta_1 \\
&= \frac{\mathrm{d}}{\mathrm{d}z}(\log\theta_1(z|\tau) + \pi i) + \eta_1 z + \eta_1 = \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z|\tau) + \eta_1 z + \eta_1 \\
&= f(z) + \eta_1
\end{aligned}$$

Furthermore,

$$\begin{aligned}
f(z+\tau) &= \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z+\tau|\tau) + \eta_1 z + \eta_1\tau \\
&= \frac{\mathrm{d}}{\mathrm{d}z}(\log\theta_1(z|\tau) - \pi i - \pi i\tau - 2\pi iz) + \eta_1 z + \eta_1\tau \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z|\tau) - 2\pi i + \eta_1 z + \eta_1\tau \\
&= f(z) + \eta_1\tau - 2\pi i
\end{aligned}$$

I claim that $\zeta(z) - f(z)$ is doubly periodic. Consider,

$$\zeta(z+1) - f(z+1) = \zeta(z) + \eta_1 - f(z) - \eta_1 = \zeta(z) - f(z)$$

and likewise,

$$\zeta(z+\tau) - f(z+\tau) = \zeta(z) + \eta_2 - f(z) + 2\pi i - \eta_1\tau = \zeta(z) - f(z)$$

because $\eta_1\tau - \eta_2 = 2\pi i$. Furthermore, $\theta_1(z|\tau)$ has a simple pole at $z = 0$ meaning that near $z = 0$,

$$f(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z|\tau) + \eta_1 z = \frac{1}{z} + O(z)$$

Similarly, near $z = 0$,

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in\Lambda^\times}\left(\frac{z^2}{\omega^2(z+\omega)}\right) = \frac{1}{z} + O(z^2)$$

Thus, $\zeta(z) - f(z)$ is holomorphic near $z = 0$. Furthermore, $\zeta(z)$ and $\theta_1(z|\tau)$ have no other poles or zeros so $\zeta(z) - f(z)$ is a doubly-periodic holomorphic function that vanishes at $z = 0$. Thus, by Liouville's theorem,

$$\zeta(z) = f(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z|\tau) + \eta_1 z$$

(d)

Given the above equality, we find,

$$\begin{aligned}\sigma(z) = \exp\left(\int^z \zeta(z')\,\mathrm{d}z'\right) &= \exp\left(\int^z \left[\frac{\mathrm{d}}{\mathrm{d}z}\log\theta_1(z'|\tau) + \eta_1 z'\right]\,\mathrm{d}z'\right)\\ &= \exp\left(\log\left(\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right) + \frac{1}{2}\eta_1 z^2\right) = e^{\frac{1}{2}\eta_1 z^2}\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\end{aligned}$$

However, I have employed some slight-of-hand to make the coefficients work out correctly. We must check this explicitly. Consider the function,

$$f(z) = \log\left(\frac{\sigma(z)}{e^{\frac{1}{2}\eta_1 z^2}\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}}\right) = \log\sigma(z) - \frac{1}{2}\eta_1 z^2 - \log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}$$

Consider the transformation properties of f. First,

$$\begin{aligned}f(z+1) &= \log\sigma(z+1) - \frac{1}{2}\eta_1(z+1)^2 - \log\frac{\theta_1(z+1|\tau)}{\theta_1'(0|\tau)}\\ &= \log\sigma(z) + \eta_1(z+\tfrac{1}{2}) + \pi i - \frac{1}{2}\eta_1 z^2 - \eta_1(z+\tfrac{1}{2}) - \log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} - \pi i\\ &= \log\sigma(z) - \frac{1}{2}\eta_1 z^2 - \log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\\ &= f(z)\end{aligned}$$

Likewise,

$$\begin{aligned}
f(z+\tau) &= \log\sigma(z+\tau) - \frac{1}{2}\eta_1(z+\tau)^2 - \log\frac{\theta_1(z+\tau|\tau)}{\theta_1'(0|\tau)} \\
&= \log\sigma(z) + \eta_2(z+\tfrac{1}{2}\tau) + \pi i - \frac{1}{2}\eta_1 z^2 - \eta_1\tau(z+\tfrac{1}{2}\tau) - \log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} - \pi i - \pi i\tau - 2\pi i z \\
&= \log\sigma(z) - \frac{1}{2}\eta_1 z^2 - \log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} + [\eta_2 - \eta_1\tau - 2\pi i](z+\tfrac{1}{2}\tau) \\
&= f(z)
\end{aligned}$$

because $\eta_2 - \eta_1\tau = 2\pi i$. Therefore, f is doubly periodic. However, both σ and θ_1 are holomorphic and both have one simple zero at $z=0$ which means that their ratio is holomorphic and non-vanishing. Thus, f is holomorphic and doubly periodic and thus constant. Near $z=0$ we have $\sigma(z) = z + O(z^2)$. Since θ_1 is an odd function we can expand it about $z=0$ as,

$$\theta_1(z|\tau) = \theta_1'(0|\tau)z + \tfrac{1}{3!}\theta_1'''(0|\tau)z^3 + \cdots$$

Therefore,

$$\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} = z\left(1 + \frac{1}{6}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}z^2 + \cdots\right)$$

Thus, since $e^{\frac{1}{2}\eta_1 z^2} = 1 + O(z^2)$, in the limit $z \to 0$ we have,

$$\log\left(\frac{\sigma(z)}{e^{\frac{1}{2}\eta_1 z^2}\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}}\right) = \log 1 = 0$$

and thus $f(z) \equiv 0$ since it is constant. This implies that,

$$\sigma(z) = e^{\frac{1}{2}\eta_1 z^2}\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}$$

Furthermore,

$$\begin{aligned}
\wp(z) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\sigma(z) &= -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} + \frac{1}{2}\eta_1 z^2\right) \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} - \eta_1
\end{aligned}$$

(e)

Since θ_1 is an odd function we can expand it about $z=0$ as,

$$\theta_1(z|\tau) = \theta_1'(0|\tau)z + \tfrac{1}{3!}\theta_1'''(0|\tau)z^3 + \cdots$$

Therefore,

$$\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} = z\left(1 + \frac{1}{6}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}z^2 + \cdots\right)$$

Taking the logarithm,

$$\log \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} = \log z + \log\left(1 + \frac{1}{6}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} z^2 + \cdots\right) = \log z + \frac{1}{6}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} z^2 + \cdots$$

Meaning that,

$$\wp(z) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} - \eta_1 = \frac{1}{z^2} - \frac{1}{3}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} - \eta_1 + O(z)$$

However,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda^\times}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]\\ &= \frac{1}{z^2} - \sum_{\omega\in\Lambda^\times}\left[\frac{z^2+2\omega z}{\omega^2(z+\omega)^2}\right]\\ &= \frac{1}{z^2} + O(z)\end{aligned}$$

Therefore, we must have,

$$\eta_1 = -\frac{1}{3}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}$$

Problem 3

Let $L \to X$ be a holomorphic line bundle over a compact Riemann surface X.

(a)

A metric h on L is a strictly positive section of $L^{-1} \otimes \bar{L}^{-1}$. It makes sense to call such a section positive because, if $\{t_{\alpha\beta}\}$ are a defining set of transition functions for the line bundle L, then h transforms as,

$$h_\alpha = t_{\alpha\beta}^{-1}\bar{t_{\alpha\beta}}^{-1} h_\beta = |t_{\alpha\beta}|^{-2} h_\beta$$

but $|t_{\alpha\beta}|^{-2}$ is a positive real so the transition functions preserve positivity of components of the section h. Such a metric allows the definition of the Chern unitary connection which acts on a holomorphic section via, $\nabla\varphi = h^{-1}\partial(h\varphi)$ and $\overline{\nabla}\varphi = \bar{\partial}\varphi$. These connections are linear maps,

$$\nabla : \Gamma(X, L) \to \Gamma(X, L \otimes K_X) \quad \text{and} \quad \bar{\nabla} : \Gamma(X, L) \to \Gamma(X, L \otimes \bar{K}_X)$$

These connections define a curvature via the commutator,

$$[\nabla, \overline{\nabla}]\varphi = -F_{z\bar{z}}\varphi$$

Explicitly,

$$[\nabla, \overline{\nabla}]\varphi = \nabla\bar{\partial}\varphi - \bar{\nabla}(h^{-1}\partial(h\varphi))$$

Since $\nabla\varphi \in \Gamma(X, L \otimes K_X)$ it is a section of a holomorphic line bundle so $\bar{\nabla}\nabla\varphi = \bar{\partial}\nabla\varphi$. Furthermore, $\bar{\partial}\varphi \in \Gamma(X, L \otimes \bar{K}_X)$ which is not the space of sections of a holomorphic line bundle. However, $h\bar{\partial} \in \Gamma(X, L \otimes \bar{K}_X \otimes (L^{-1} \otimes \bar{L}^{-1})) = \Gamma(X, \bar{L}^{-1} \otimes \bar{K}_X)$ is a section of an anti-holomorphic line bundle. This implies that $\nabla\bar{\partial}\varphi = h^{-1}\partial(h\bar{\partial}\varphi)$ is covariant because ∂ is a Chern unitary connection on the section $h\bar{\partial}\varphi$ of the anti-holomorphic bundle $\bar{L}^{-1} \otimes \bar{K}_X$. Therefore,

$$\begin{aligned}
[\nabla, \overline{\nabla}]\varphi &= \nabla\bar{\partial}\varphi - \bar{\nabla}(h^{-1}\partial(h\varphi)) = h^{-1}\partial(h\bar{\partial}\varphi) - \bar{\partial}(h^{-1}\partial(h\varphi)) \\
&= \partial\bar{\partial}\varphi + h^{-1}(\partial h)\bar{\partial}\varphi - \bar{\partial}\partial\varphi - \bar{\partial}(h^{-1}(\partial h)\varphi) \\
&= h^{-1}(\partial h)\bar{\partial}\varphi - h^{-1}(\partial h)(\bar{\partial}\varphi) - (\bar{\partial}\partial \log h)\varphi \\
&= -(\bar{\partial}\partial \log h)\varphi
\end{aligned}$$

Therefore,

$$F_{\bar{z}z} = -\partial\bar{\partial} \log h$$

(b)

The first Chern class of L is defined by,

$$c_1(L) = \frac{i}{2\pi} \int_X F_{z\bar{z}} \, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

Theorem 0.1. Let L be a line bundle over X and $\varphi \in \Gamma(X, L)$ a meromorphic section of L which is not identically zero. Then,

$$(\#\text{ zeros of } \varphi) - (\#\text{ poles of } \varphi) = \frac{i}{2\pi} \int_X F_{\bar{z}z} \, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

Proof. Consider,

$$-\partial\bar{\partial} \log |\varphi|_h^2 = -\partial\bar{\partial} \log (\varphi\bar{\varphi}h)$$

When we are away from D the set of zeros and poles of φ we can write,

$$-\partial\bar{\partial} \log |\varphi|_h^2 = -\partial\bar{\partial} \left(\log \varphi + \overline{\log \varphi} + \log h\right) = -\partial\bar{\partial} \log h = F_{\bar{z}z}$$

because $\log \varphi$ is holomorphic and $\overline{\log \varphi}$ is anti-holomorphic on $X \setminus D$. Consider the union of disks,

$$D_\epsilon = \bigcup_{p \in D} B_\epsilon(p)$$

where we choose ϵ small enough for $B_\epsilon(p)$ to lie in the image of a single chart so we can identify $B_\epsilon(p)$ in the image of a chart with a disk on X and small enough that only one $p \in D$ lies in each disk. We can always do this because φ is a nonzero meromorphic section and thus has isolated poles and zeros. Then,

$$\int_X F_{\bar{z}z}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \lim_{\epsilon \to 0} \int_{X \setminus D_\epsilon} F_{\bar{z}z}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

However,

$$\mathrm{d}(-\bar{\partial} \log |\varphi|_h^2\, \mathrm{d}\bar{z}) = -\partial\bar{\partial} \log |\varphi|_h^2\, \mathrm{d}z \wedge \mathrm{d}\bar{z} - \bar{\partial}\bar{\partial} \log |\varphi|_h^2\, \mathrm{d}\bar{z} \wedge \mathrm{d}\bar{z} = -\partial\bar{\partial} \log |\varphi|_h^2\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

since $\log |\varphi|_h^2$ is a well-defined scalar function on $X \setminus D$ unlike $\log h$ whose argument transforms as a section of the nontrivial line bundle $L^{-1} \otimes \bar{L}^{-1}$. Therefore, by Stokes' theorem,

$$\begin{aligned} \int_X F_{\bar{z}z}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} &= \lim_{\epsilon \to 0} \int_{X \setminus D_\epsilon} \mathrm{d}(-\bar{\partial} \log |\varphi|_h^2\, \mathrm{d}\bar{z}) = -\lim_{\epsilon \to 0} \int_{\partial(X \setminus D_\epsilon)} \bar{\partial} \log |\varphi|_h^2\, \mathrm{d}\bar{z}) \\ &= \lim_{\epsilon \to 0} \sum_{p \in D} \oint_{\partial B_\epsilon(p)} \bar{\partial} \log |\varphi|_h^2\, \mathrm{d}\bar{z} \end{aligned}$$

The minus sign is canceled by the change in orientation of the integration contours since ∂D and ∂D^C are equal but have reversed orientation. Since φ is meromorphic, near $p \in D$ we can write $\varphi = z^N u(z)$ for $u(z) \neq 0$ on $B_\epsilon(p)$. Therefore, we have,

$$|\varphi|_h^2 = |z|^{2N} |u(z)|^2 h(z)$$

which implies that,

$$\log |\varphi|_h^2 = N \log |z|^2 + \log |u(z)|^2 + \log h(z)$$

and thus,

$$\bar{\partial} \log |\varphi|_h^2 = \frac{N}{\bar{z}} + \bar{\partial} \log |u(z)|^2 + \bar{\partial} \log h(z)$$

About each $p \in D$ we can compute,

$$\lim_{\epsilon \to 0} \oint_{\partial B_\epsilon(p)} \bar{\partial} \log |\varphi|_h^2\, \mathrm{d}\bar{z} = \lim_{\epsilon \to 0} \oint_{|z| = \epsilon} \left[\frac{N}{\bar{z}} + \bar{\partial} \log |u(z)|^2 + \bar{\partial} \log h(z) \right] \mathrm{d}\bar{z}$$

Since both $\bar{\partial} \log |u(z)|^2$ and $\bar{\partial} \log h(z)$ are smooth and have no singularities on $B_\epsilon(p)$ so their integrals go to zero in the limit $\epsilon \to 0$. Therefore,

$$\lim_{\epsilon \to 0} \oint_{\partial B_\epsilon(p)} \bar{\partial} \log |\varphi|_h^2\, \mathrm{d}\bar{z} = \lim_{\epsilon \to 0} \oint_{|z| = \epsilon} \frac{N}{\bar{z}}\, \mathrm{d}\bar{z} = -\lim_{\epsilon \to 0} \overline{\oint_{|z| = \epsilon} \frac{N}{z}\, \mathrm{d}z} = -2\pi i N$$

Thus,

$$\int_X F_{\bar{z}z}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \sum_{p \in D} \lim_{\epsilon \to 0} \oint_{\partial B_\epsilon(p)} \bar{\partial} \log |\varphi|_h^2\, \mathrm{d}z = -\sum_{p \in D} 2\pi i N_p$$

Which gives the theorem since N_p counts the multiplicity of each zero and the negative of the multiplicity of each pole. $\square$

(c)

Suppose that $c_1(L) < 0$. Then any nontrivial meromorphic section of L must satisfy,

$$(\# \text{ zeros of } \varphi) - (\# \text{ poles of } \varphi) = c_1(L) < 0$$

which implies that,

$$(\# \text{ zeros of } \varphi) < (\# \text{ poles of } \varphi)$$

In particular, the number of poles and zeros are positive integers, $(\# \text{ poles of } \varphi) > 0$. However, any nontrivial holomorphic section of L is a meromorphic section with no poles which cannot exist by the above observation. Thus, L admits no nontrivial holomorphic sections.

Problem 4

Let $L \to X$ be a holomorphic line bundle over a compact Riemann surface X. Let h and g be metrics on the bundles L and K_X^{-1} respectively. Then g is a positive section of $K_X \otimes \bar{K}_X$ which we can write as $g = g_{\bar{z}z}\, \mathrm{d}z\ \mathrm{d}\bar{z}$. Let $\bar{\partial}$ be the derivative operator from the space of smooth sections of L to the space of smooth sections of $L \otimes \bar{K}_X$,

$$\bar{\partial} : \Gamma(X, L) \to \Gamma(X, L \otimes \bar{K}_X)$$

which coincides with the Chern unitary connection $\bar{\nabla}$ on L.

(a)

The metrics h and g define L^2 norms on the spaces $\Gamma(X, L)$ and $\Gamma(X, L \otimes \bar{K}_X)$ via,

$$\varphi \in \Gamma(X, L) \implies ||\varphi||^2 = \int_X \varphi\bar{\varphi} h g_{\bar{z}z}$$

$$\psi \in \Gamma(X, L \otimes \bar{K}_X) \implies ||\psi||^2 = \int_X \psi\bar{\psi} h$$

These combinations are covariant because $\varphi\bar{\varphi}h$ is a section of $L \otimes \bar{L} \otimes L^{-1} \otimes \bar{L}^{-1}$ and thus a function on X and g antisymmetrized as $g_{\bar{z}z}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$ is a $(1,1)$-form. Thus, $\varphi\bar{\varphi}hg$ is a well-defined $(1,1)$-form on X which can be covariantly integrated. Similarly, ψ is a section of the line bundle $L \otimes \bar{K}_X$ and thus $\phi\bar{\psi}h$ is a section of the line bundle $(L \otimes \bar{K}_X) \otimes (\bar{L} \otimes K_X) \otimes (L^{-1} \otimes \bar{L}^{-1}) = \bar{K}_X \otimes K_X$ which is the bundle $\Lambda^{1,1}$ of $(1,1)$-forms. Thus, $\phi\bar{\psi}h$ is a well-defined $(1,1)$-form on X which may be covariantly integrated.

The polarization identities can be used to construct L^2 inner products on on the spaces $\Gamma(X, L)$ and $\Gamma(X, L \otimes \bar{K}_X)$. Explicitly,

$$\varphi, \psi \in \Gamma(X, L) \implies \langle \varphi, \psi \rangle_L = \int_X \varphi\bar{\psi} h g_{\bar{z}z}$$

$$\varphi, \psi \in \Gamma(X, L \otimes \bar{K}_X) \implies \langle \varphi, \psi \rangle_{L \otimes \bar{K}_X} = \int_X \varphi\bar{\psi} h$$

These inner products are similarly covariant.

(b)

Then we define the formal adjoint via,

$$\left\langle \bar{\partial}\varphi, \psi \right\rangle_{L \otimes \bar{K}_X} = \left\langle \varphi, \bar{\partial}^\dagger \psi \right\rangle_L$$

Specifically, we require that, for $\varphi \in \mathcal{C}^\infty(X, L)$ and $\varphi \in \mathcal{C}^\infty(X, L \otimes \bar{K}_X)$ we have[1],

$$\left\langle \bar{\partial}\varphi, \psi \right\rangle_{L \otimes \bar{K}_X} = \left\langle \varphi, \bar{\partial}^\dagger \psi \right\rangle_L$$

Therefore,

$$\int_X h(\bar{\partial}\varphi)\bar{\psi} = \int_X \varphi \overline{(\bar{\partial}^\dagger \psi)} h g_{\bar{z}z}$$

However, $g_{\bar{z}z}$ is a metric on $\bar{K}_X^{-1}$ and therefore a positive section of $K_X \otimes \bar{K}_X$ which is a $(1,1)$-form on X. We can integrate the first expression by parts,

$$\begin{aligned}
\int_X h(\bar{\partial}\varphi)\bar{\psi} &= -\int_X \varphi \bar{\partial}\left(h\bar{\psi}\right) \\
&= -\int_X \varphi \overline{\partial\left(h\psi\right)} = -\int_X \varphi h \overline{h^{-1}\partial(h\psi)} g^{\bar{z}z} g_{\bar{z}z} = -\int_X \varphi \overline{h^{-1}\partial(h\psi) g^{\bar{z}z}} h g_{\bar{z}z}
\end{aligned}$$

Therefore,

$$\int_X \varphi \overline{(\partial^\dagger \psi)} h g_{\bar{z}z} = -\int_X \varphi \overline{h^{-1}\bar{\partial}(h\psi) g^{\bar{z}z}} h g_{\bar{z}z}$$

Since this must hold for all possible φ and ψ we must have,

$$\bar{\partial}^\dagger \psi = -g^{\bar{z}z}\left(h^{-1}\partial(h\psi)\right) = -g^{\bar{z}z}\nabla\psi$$

Define the maps,

$$\begin{aligned}
\Delta_+ &= \bar{\partial}^\dagger \bar{\partial} : \Gamma(X, L) \to \Gamma(X, L) \\
\Delta_- &= \bar{\partial}\bar{\partial}^\dagger : \Gamma(X, L \otimes \bar{K}_X) \to \Gamma(X, L \otimes \bar{K}_X)
\end{aligned}$$

If $\varphi \in \ker \Delta_+$ then $\bar{\partial}^\dagger \partial \varphi = 0$ which implies that,

$$||\bar{\partial}\varphi||^2 = \left\langle \bar{\partial}\varphi, \bar{\partial}\varphi \right\rangle = \left\langle \varphi, \bar{\partial}^\dagger \bar{\partial}\varphi \right\rangle = 0$$

which implies that $\bar{\partial}\varphi = 0$. Since $H^0(X, L)$ is the space of holomorphic sections which is exactly a smooth section satisfying the Cauchy-Riemann equation $\bar{\partial}\varphi = 0$, we have,

$$\varphi \in \ker \Delta_+ \iff \bar{\partial}\varphi = 0 \iff \varphi \in H^0(X, L)$$

implying that $\ker \Delta_+ = \ker \bar{\partial} = H^0(X, L)$.

[1] To define the true adjoint on a Hilbert space we must also impose the defining adjoint on the limits of smooth functions with may no longer be smooth. What we have defined is called the "formal adjoint."

Similarly, we know that $\ker \Delta_- = \ker \bar{\partial}^\dagger$ because if $\Delta_- \varphi = 0$ then,

$$\langle \bar{\partial}^\dagger \varphi, \bar{\partial}^\dagger \varphi \rangle = \langle \bar{\partial} \bar{\partial}^\dagger \varphi, \varphi \rangle = 0$$

which implies that $\bar{\partial}^\dagger \varphi = 0$. However,

$$\bar{\partial}^\dagger \psi = 0 \iff \partial(h\psi) = 0$$

since $g^{\bar{z}z} h^{-1}$ is non-vanishing. h is a non-vanishing section of $L^{-1} \otimes \bar{L}^{-1}$ so,

$$\psi \in \Gamma(X, L \otimes \bar{K}_X) \iff \bar{\Psi} = h\psi \in \Gamma(X, \bar{L}^{-1} \otimes \bar{K}_X) \iff \Psi = h\bar{\psi} \in \Gamma(X, L^{-1} \otimes K_X)$$

Therefore, $\psi \in \ker \bar{\partial}^\dagger \iff \Psi \in H^0(X, L^{-1} \otimes K_X)$ since,

$$\bar{\partial}^\dagger \psi = 0 \iff \partial(h\psi) = 0 \iff \bar{\partial}(h\bar{\psi}) = 0 \iff \bar{\partial}\Psi = 0$$

Thus, $\dim \ker \bar{\partial}^\dagger = \dim H^0(X, L^{-1} \otimes K_X)$ since there is a correspondence between their elements. We have shown,

$$\begin{aligned} \dim \ker \Delta_+ &= \dim \ker \bar{\partial} = \dim H^0(X, L) \\ \dim \ker \Delta_- &= \dim \ker \bar{\partial}^\dagger = \dim H^0(X, L^{-1} \otimes K_X) \end{aligned}$$

(c)

Suppose that Δ_+ and Δ_- are completely diagonalizable with discrete spectra. There is a one-to-one correspondence between the eigenfunctions of Δ_+ and Δ_- with nonzero eigenvalues. Let $V_\lambda^\pm$ be the space of eigenfunctions of $\Delta_\pm$ with eigenvalue λ. For $\lambda \neq 0$, I claim that,

$$\lambda^{-\frac{1}{2}} \bar{\partial} : V_\lambda^+ \to V_\lambda^- \quad \text{and} \quad \lambda^{-\frac{1}{2}} \bar{\partial}^\dagger : V_\lambda^- \to V_\lambda^+$$

are a pair of inverse linear isomorphisms. If $\varphi \in V_\lambda^+$ then,

$$\Delta_- \bar{\partial} \varphi = \bar{\partial} \bar{\partial}^\dagger \bar{\partial} = \bar{\partial} \Delta_+ \varphi = \lambda \bar{\partial} \varphi$$

Thus, if $\lambda \neq 0$ then $\bar{\partial}\varphi \neq 0$ (otherwise $\Delta_+ \varphi = \bar{\partial}^\dagger \bar{\partial} \varphi = 0$) and thus $\bar{\partial}\varphi \in V_\lambda^-$. Similarly, if $\psi \in V_\lambda^-$ then,

$$\Delta_+ \bar{\partial}^\dagger \psi = \bar{\partial}^\dagger \bar{\partial} \bar{\partial}^\dagger \psi = \bar{\partial}^\dagger \Delta_- \psi = \lambda \bar{\partial}^\dagger \psi$$

Thus, if $\lambda \neq 0$ then $\bar{\partial}^\dagger \psi \neq 0$ (since $\ker \Delta_- = \ker \bar{\partial}^\dagger$) and thus $\bar{\partial}^\dagger \psi \in V_\lambda^+$.

Therefore, the maps in both directions are well-defined. The map, $\lambda^{-1} \bar{\partial} \bar{\partial}^\dagger : V_\lambda^- \to V_\lambda^-$ acts on ψ via,

$$\lambda^{-1} \bar{\partial} \bar{\partial}^\dagger \psi = \lambda^{-1} \Delta_- \psi = \psi$$

and similarly, $\lambda^{-1} \bar{\partial}^\dagger \bar{\partial} : V_\lambda^+ \to V_\lambda^+$ acts on φ via,

$$\lambda^{-1} \bar{\partial}^\dagger \bar{\partial} \varphi = \lambda^{-1} \Delta_+ \varphi = \varphi$$

Therefore, these maps are well-defined inverses and thus isomorphisms. Therefore, there is an exact correspondence between the spaces of eigenfunctions of Δ_+ and Δ_- with corresponding nonzero eigenvalue.

The operator $e^{-t\Delta_+}$ is defined by its action on an arbitrary section $\varphi \in \Gamma(X, L)$ which may be written in a basis as,

$$\varphi = \sum_j c_j \varphi_j$$

which converges in the sense,

$$\left|\left| \varphi - \sum_{j=1}^{n} c_j \varphi_j \right|\right| \to 0$$

Now we define,

$$e^{-t\Delta_+}\varphi = \sum_j e^{-t\lambda_j} c_j \varphi_j$$

This converges in the Hilbert space since if the sequence $\{d_j = e^{-t\lambda_j} c_j\}$ is square summable if we take $t \geq 0$ since

$$\sum_j |d_j|^2 = \sum_j e^{-2t\lambda_j} |c_j|^2 \leq \sum_j |c_j|^2 = ||\varphi||$$

since $\lambda_i \geq 0$ and thus $t\lambda_j \geq 0$.

The eigenfunctions of $e^{-t\Delta_+}$ are clearly the eigenfunctions $\{\varphi_j\}$ of Δ_+ (which we assumed form a complete set) with eigenvalues $e^{-t\lambda_j}$. For any positive $t > 0$, the trace is given by summing over the eigenvalues counting each multiplicity $\dim V_\lambda^+$,

$$\mathrm{Tr}\left(e^{-t\Delta_+}\right) = \sum_j e^{-t\lambda_j} = \sum_{\lambda \in \mathrm{Spec}\Delta_+} e^{-t\lambda} \dim V_\lambda^+$$

This sum converges for $t > 0$ because the λ_j grow as a power-law and have finite multiplicity. The same construction applies for Δ_-. However, Δ_+ and Δ_- have the same spectra (up to the multiplicity of $\lambda = 0$) because $V_\lambda^+ \cong V_\lambda^-$ so these spaces contain nontrivial eigenfunctions for exactly the same values of λ so we can compute,

$$\mathrm{Tr}\left(e^{-t\Delta_+}\right) - \mathrm{Tr}\left(e^{-t\Delta_-}\right) = \sum_{\lambda \in \mathrm{Spec}\Delta_\pm} e^{-t\lambda}\left(\dim V_\lambda^+ - \dim V_\lambda^-\right) = \dim V_0^+ - \dim V_0^+$$

However, the space of zero eigenvalues is exactly the kernel since

$$\varphi \in V_0^\pm \iff \Delta_\pm \varphi = 0 \iff \varphi \in \ker \Delta_\pm$$

and therefore we arrive at the trace formula,

$$\mathrm{Tr}\left(e^{-t\Delta_+}\right) - \mathrm{Tr}\left(e^{-t\Delta_-}\right) = \dim\ker\Delta_+ - \dim\ker\Delta_- = \dim H^0(X, L) - \dim H^0(X, L^{-1} \otimes K_X)$$

Problem 5

Let X be a compact Riemann surface of genus $g \geq 1$, and form the cut polygonal surface X_{cut} with $4g$ sides and boundary $\prod_{I=1}^{g} A_I B_I A_I^{-1} B_I^{-1}$ where A_I and B_I are a fixed Canonical homology basis. Fix a reference point $p_0 \in X_{\text{cut}}$.

(a)

Let ω be a closed 1-form on X i.e. $\mathrm{d}\omega = 0$ and define the Abelian integral of ω by,

$$f_\omega(z) = \int_{p_0}^{z} \omega$$

for $z \in X_{\text{cut}}$. The function $f_\omega(z)$ is well-defined on X_{cut} because, after the cutting, X_{cut} becomes contractible (even though X has nontrivial homology) and thus all paths with the same start and end points are homotopic. Thus, suppose that γ and γ' are both paths in X_{cut} from p_0 to z. Because γ and γ' are homotopic, they bound a disk in X_{cut}. Therefore, applying Stokes' theorem,

$$\int_{\gamma'} \omega - \int_{\gamma} \omega = \int_{\gamma' * \gamma^{-1}} \omega = \int_{\partial D^2} \omega = \int_{D} \mathrm{d}\omega = 0$$

since $\mathrm{d}\omega = 0$. Therefore, the Abelian integral $f_\omega(z)$ is path-independent and thus well-defined.

Furthermore, for $z \in B_I^{-1}$, consider $f_\omega(z + A_I)$. Since we can choose any path from p_0 to $z + A_I$ along which to compute this integral,

$$f_\omega(z + A_I) = \int_{p_0}^{z + A_I} \omega = \int_{p_0}^{z} \omega + \oint_{A_I} \omega = f_\omega(z) + \oint_{A_I} \omega$$

Similarly,

$$f_\omega(z + B_I) = \int_{p_0}^{z + B_I} \omega = \int_{p_0}^{z} \omega + \oint_{B_I} \omega = f_\omega(z) + \oint_{B_I} \omega$$

(b)

Let ω and η be closed 1-forms on X. Consider,

$$\mathrm{d}f_\omega \eta = \mathrm{d}f_\omega \wedge \eta + f_\omega \,\mathrm{d}\eta = \omega \wedge \eta$$

because $\mathrm{d}\eta = 0$ and $\mathrm{d}f_\omega = \omega$. Therefore, by Stokes's theorem,

$$\int_X \omega \wedge \eta = \int_{X_{\text{cut}}} \mathrm{d}f_\omega \eta = \int_{\partial X_{\text{cut}}} f_\omega \eta$$

Now expanding this integral along the boundary, we get,

$$\oint_{\partial X_{\text{cut}}} f_\omega \eta = \sum_{I=1}^{g} \left[\oint_{A_I} (f_\omega(z)\eta(z) - f_\omega(z + B_I)\eta(z + B_I)) + \oint_{B_I} (f(z + A_I)\eta(z + A_I) - f(z)\eta(z)) \right]$$

However, η is a form on X and therefore must be periodic on X_{cut} under all homology cycles. Therefore,

$$\begin{aligned}
\oint_{\partial X_{\text{cut}}} f_\omega \eta &= \sum_{I=1}^{g} \left[\oint_{A_I} \left(f_\omega(z) - f_\omega(z + B_I) \right) \eta(z) + \oint_{B_I} \left(f(z + A_I) - f(z) \right) \eta(z) \right] \\
&= \sum_{I=1}^{g} \left[\oint_{A_I} \left(- \oint_{B_I} \omega \right) \eta + \oint_{B_I} \left(\oint_{A_I} \omega \right) \eta \right] \\
&= \sum_{I=1}^{g} \left[\left(\oint_{A_I} \omega \right) \left(\oint_{B_I} \eta \right) - \left(\oint_{B_I} \omega \right) \left(\oint_{A_I} \eta \right) \right]
\end{aligned}$$

(c)

We know that $\dim H^0(X, K_X) = g$ from Riemann-Roch meaning we can choose a basis ω_I of holomorphic $(1,0)$-forms on X with $I = 1, \dots, g$. Furthermore, we may choose this basis to diagonalize,

$$\oint_{A_J} \omega_I = \delta_{IJ}$$

and then set,

$$\oint_{B_J} \omega_I = \Omega_{IJ}$$

First, we need to check that all ω_I and $\overline{\omega}_I$ are closed 1-forms. In local coordinates we can write,

$$\omega_I = f_I(z) \, \mathrm{d}z \quad \text{and} \quad \overline{\omega}_I = \bar{f}_I(z) \, \mathrm{d}\bar{z}$$

Therefore,

$$\mathrm{d}\omega_I = \partial f_I(z) \, \mathrm{d}z \wedge \mathrm{d}z + \bar{\partial} f_I(z) \, \mathrm{d}\bar{z} \wedge \mathrm{d}z = 0$$

The first term vanishes because $\mathrm{d}z \wedge \mathrm{d}z = 0$. The second term vanishes because $f_I(z)$ is a holomorphic function and thus $\bar{\partial} f_I(z) = 0$. Similarly,

$$\mathrm{d}\overline{\omega}_I = \partial \bar{f}_I(z) \, \mathrm{d}z \wedge \mathrm{d}\bar{z} + \bar{\partial} \bar{f}_I(z) \, \mathrm{d}\bar{z} \wedge \mathrm{d}\bar{z} = 0$$

where $\partial \bar{f}_I(z) = 0$ since $\bar{f}_I$ is anti-holomorphic. Therefore we may apply the previous problem to the $(1,0)$-forms ω_I and the $(0,1)$-forms $\overline{\omega}_I$. We consider,

$$\begin{aligned}
\int_X \omega_I \wedge \omega_J &= \sum_{K=1}^{g} \left[\left(\oint_{A_K} \omega_I \right) \left(\oint_{B_K} \omega_J \right) - \left(\oint_{B_K} \omega_I \right) \left(\oint_{A_K} \omega_J \right) \right] \\
&= \sum_{K=1}^{g} \left[\delta_{IK} \Omega_{JK} - \Omega_{IK} \delta_{JK} \right] = \Omega_{JI} - \Omega_{IJ}
\end{aligned}$$

However, since ω_I and ω_J are holomorphic $(1,0)$-forms, in local coordinates we may write,

$$\omega_I = f_I(z) \, \mathrm{d}z \implies \omega_I \wedge \omega_J = (f_I(z) \, \mathrm{d}z) \wedge (f_J(z) \, \mathrm{d}z) = f_I(z) f_J(z) \, \mathrm{d}z \wedge \mathrm{d}z = 0$$

Thus, $\omega_I \wedge \omega_J = 0$ meaning that,

$$\Omega_{JI} - \Omega_{IJ} = \int_X \omega_I \wedge \omega_J = 0$$

Therefore, the period matrix is symmetric, $\Omega_{IJ} = \Omega_{JI}$.

Furthermore, consider,

$$\begin{aligned} \int_X \omega_I \wedge \overline{\omega}_J &= \sum_{K=1}^{g} \left[\left(\oint_{A_K} \omega_I \right) \left(\oint_{B_K} \overline{\omega}_J \right) - \left(\oint_{B_K} \omega_I \right) \left(\oint_{A_K} \overline{\omega}_J \right) \right] \\ &= \sum_{K=1}^{g} \left[\delta_{IK} \overline{\Omega}_{JK} - \Omega_{IK} \overline{\delta}_{JK} \right] = \overline{\Omega}_{JI} - \Omega_{IJ} = -2i \mathrm{Im}(\Omega_{IJ}) \end{aligned}$$

However, in local coordinates,

$$\omega_I \wedge \overline{\omega}_J = (f_I(z)\, \mathrm{d}z\,) \wedge (\bar{f}_J(z)\, \mathrm{d}\bar{z}\,) = f_I(z) \bar{f}_J(z)\, \mathrm{d}z \,\wedge\, \mathrm{d}\bar{z} \, = -2i f_I(z) \bar{f}_J(z)\, \mathrm{d}x \,\wedge\, \mathrm{d}y$$

Therefore,

$$\mathrm{Im}(\Omega_{IJ}) = \int_X f_I(z) \bar{f}_J(z)\, \mathrm{d}x \,\wedge\, \mathrm{d}y$$

We want to show that this matrix is positive-definite. Take any $X^I \in \mathbb{R}^g$ then consider,

$$X^I \mathrm{Im}(\Omega_{IJ}) X^J = \int_X X^I f_I(z) \bar{f}_J(z) X^J\, \mathrm{d}x \,\wedge\, \mathrm{d}y \, = \int_X \left| X^I f_I(z) \right|^2 \mathrm{d}x \,\wedge\, \mathrm{d}y$$

Since $\mathrm{d}x \,\wedge\, \mathrm{d}y$ is the canonical positive area element and $|X^I f_I(z)|^2$ is a nonactive function,

$$\int_X \left| X^I f_I(z) \right|^2 \mathrm{d}x \,\wedge\, \mathrm{d}y \, \geq 0$$

and, since $\left| X^I f_I(z) \right|^2$ is smooth and nonnegative, we have equality exactly when $\left| X^I f_I(z) \right|^2 \equiv 0$ identically on X. However, $\left| X^I f_I(z) \right|^2 \equiv 0 \iff X^I f_I(z) \equiv 0$. Furthermore,

$$X^I f_I(z) \equiv 0 \iff X^I f_I(z)\, \mathrm{d}z \, = X^I \omega_I = 0$$

But since the ω_I form a basis of $H^0(X, K_X)$ they are independent and thus the only solution is $X^I = 0$. Therefore, we have shown that, $X^I \mathrm{Im}(\Omega_{IJ}) X^J \geq 0$ with equality exactly when $X^I = 0$ proving that $\mathrm{Im}(\Omega_{IJ})$ is a positive definite matrix.