1 The pro-étale Topology

1.1 Problems with the étale Topology

Perhaps the most obviously disquieting fact about the étale topology is that,

$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell) = \varprojlim_n H^i(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

cannot be defined as a derived functor (sheaf cohomology) of a natural representing object $\mathbb{Q}_\ell$ in the étale topos. This is the main problem we wish to fix.

Another problem arises as follows. The site of finite étale covers is isomorphic to the site of finite continous $\pi_1^{\text{ét}}(X)$-sets and thus sheaves on the finite étale site are equivalent to continuous $\pi_1^{\text{ét}}(X)$-modules. Since lcc sheaves are exactly those represented by finite étale covers we conclude that these are equivalent to continuous $\pi_1^{\text{ét}}(X)$-modules. This transfers over to the case of lisse $\mathbb{Z}_\ell$-sheaves which are limits of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules with a $\pi_1^{\text{ét}}(X)$-action and thus $\mathbb{Z}_\ell$-modules with a $\pi_1^{\text{ét}}(X)$-action. However, the problem arrises when we pass to $\mathbb{Q}_\ell$-sheaves. The correspondence between $\mathbb{Q}_\ell$-local systems and continuous $\mathbb{Q}_\ell$-representations of $\pi_1^{\text{ét}}(X)$ fails for non-normal schemes X. For example, let X be $\mathbb{P}^1$ with 0 and ∞ glued and take the $\mathbb{Q}_\ell$-local system $\mathscr{F}$ on X given by identifying the fibers over 0 and ∞ by the automorphism $\ell \in \mathbb{Q}_\ell^\times = \mathrm{GL}_1(\mathbb{Q}_\ell)$ giving $\mathscr{F}_0 \to \mathscr{F}_\infty$ by $\mathbb{Q}_\ell \xrightarrow{\times \ell} \mathbb{Q}_\ell$. However, this cannot corespond to any continuous representation $\pi_1^{\text{ét}}(X) \to \mathrm{GL}_1(\mathbb{Q}_\ell)$ sending $1 \in \pi_1^{\text{ét}}(X) = \hat{\mathbb{Z}}$ to ℓ because then the image of $\hat{\mathbb{Z}} \to \mathrm{GL}_1(\mathbb{Q}_\ell)$ is invariant under multiplication by ℓ^{-1} and thus not bounded but $\hat{\mathbb{Z}}$ is compact so its image is compact giving a contradiciton. Thus suggests that the "real" $\pi_1(X)$ should be $\mathbb{Z}$ such that we do indeed get a representation $\mathbb{Z} \to \mathrm{GL}_1(\mathbb{Q}_\ell)$.

1.2 Preliminaries

Definition 1.2.1. A map $f : X \to Y$ of schemes is *weakly étale* if f is flat and $\Delta_f : X \to X \times_Y X$ is flat.

Proposition 1.2.2. An étale map (in fact any flat and unramified morphism) is weakly-étale.

Proof. An étale map $f : X \to Y$ is flat and unramified so $\Delta_f : X \to X \times_Y X$ is an open immersion and thus flat. ☐

Definition 1.2.3. The pro-étale site $X_{\text{proét}}$ is the site of weakly-étale X-schemes, with covers given by fpqc covers.

Theorem 1.2.4. Let $f : A \to B$ be a map of rings.

(a) f is étale if and only if f is weakly-étale and finitely presented.

(b) If f is ind-étale (a filtered colimit of étale A-algebras) then f is weakly étale.

(c) If f is weakly-étale, then there exists a faithfully flat ind-étale $f : B \to C$ such that $g \circ f$ is ind-étale.

Proof. The diagonal $\delta : B \otimes_A B \to B$ is always surjective (a closed immersion) so if δ is flat then it is open and thus an open immersion so $A \to B$ is unramified but if $A \to B$ is unramified then δ is an open immersion and thus flat. Furthermore, étale is equivalent to flat an unramified and finitely presented so we conclude the first part.

Because tensor products commute with colimits an ind-limit of flat A-algebras is flat. ☐

Remark. This shows that for a ring A, the ind-étale A-algebras are cofinal for the weakly-étale A-algebras and thus we can work with ind-étale covers. Therefore $\mathrm{Spec}\,(A)_{\text{proét}}$ has a basis of ind-étale covers which is why its called the pro-étale topology.

1.3 The pro-étale Fundamental Group

Theorem 1.3.1. Let X be a connected scheme whose underlying topological space is locally noetherian. Then the following categories are equivalent,

(a) The category Loc_X of locally constant sheaves on $X_{\text{proét}}$.

(b) The category Cov_X of étale X-schemes satisfying the aluative criterion o properness.

Definition 1.3.2. A topological group G is call a Noohi group if G is complete, and admits a basis of open neighborhoods of 1 given by open subgroups.

Definition 1.3.3. Let x be a geometric point of X. Equiping Cov_X (or equivalently Loc_X) with the fiber functor F_x gives a Galois theory giving rise to a Noohi group $\pi_1^{\text{proét}}(X, x) = \mathrm{Aut}\,(F_x)$.

Proposition 1.3.4. There is an equivalence of categories between $\mathbb{Q}_\ell$-local systems on X and finite dimensional continuous representations of $\pi_1^{\text{proét}}$.

Remark. The above is also true with $\mathbb{Q}_\ell$ replaced by $\overline{\mathbb{Q}}_\ell$.

Proposition 1.3.5. The pro-finite completion of $\pi_1^{\text{proét}}(X, x)$ is $\pi_1^{\text{ét}}(X, x)$ and the pro-discrete completion of $\pi_1^{\text{proét}}(X, x)$ is $\pi_1^{\text{SGA3}}(X, x)$.