1 Fourier-Mukai Transforms

Theorem 1.0.1 (Mukai)**.** Let A/k be an abelian variety then there is an equivalence of categories,

$$D^b(A) \xrightarrow{\sim} D^b(A^\vee)$$

Remark. $D^b(A)$ is the category of complexes of quasi-coherent sheaves whose cohomology sheaves are coherent and only finitely many nonzero.

1.1 Derived Categories

Let X be smooth projective over a field k. We make these assumptions because,

(a) smooth: such that all rings are regular hence finite modules have finite projective dimension

(b) projective: every coherent sheaf is a quotient by a vector bundle

therefore every element in $D^b(X)$ is represented by a finite complex of vector bundles.

Fact: there exists $\underline{\mathbb{R}\mathrm{Hom}}$ and $\otimes^{\mathbb{L}}$. For a map $f : X \to Y$ (which is automatically proper because X, Y are projective varities) then for any $E \in D^b(X)$ we get $\mathbb{R}f_* E \in D^b(Y)$ and for $E \in D^b(Y)$ then $\mathbb{L}f^* E \in D^b(X)$.

Furthermore, there is a projection formula,

$$\mathbb{R}f_*(E \otimes^{\mathbb{L}}_X \mathbb{L}f^* F) \cong \mathbb{R}f_* E \otimes^{\mathbb{L}}_Y F$$

Proposition 1.1.1. If we have a Cartesian diagram,

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

and either f or g is flat then for $E \in D^b(X)$ there is a base change isomorphism,

$$\mathbb{L}g^*(\mathbb{R}f_* E) \cong \mathbb{R}f'_*(\mathbb{L}g'^* E)$$

Proposition 1.1.2. If $f : X \to Y$ is smooth,

$$(\mathbb{R}f_* E)^\vee \cong \mathbb{R}f_*(E^\vee \otimes \omega_{X/Y}[\dim X - \dim Y])$$

1.2 Integral functors

Given the following situation,

$$\begin{tikzcd} & X \times_k Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

And fix $\mathcal{P} \in D^b(X \times Y)$.

Definition 1.2.1. The *integral functor with kernel* $\mathcal{P}$ is,

$$\Phi^{\mathcal{P}}_{Y \to X} : D^b(Y) \to D^b(X)$$

given by

$$E \mapsto \mathbb{R}\pi_{X*}(\mathcal{P} \otimes^{\mathbb{L}}_{X \times Y} \mathbb{L}\pi_Y^* E)$$

Proposition 1.2.2. The composition of integral functors is an integral functor.

Proof. To compute $\Phi^{\mathcal{P}}_{Y \to X} \circ \Phi^{\mathcal{Q}}_{Z \to Y}$ draw the diagram,

$$\begin{tikzcd}
 & & X \times Y \times Z \arrow[dl] \arrow[dr] & & \\
 & X \times Y \arrow[dl] \arrow[dr] & & X \times Z \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

then we compute that by base change,

$$\mathbb{L}(\pi_Y^{XY})^* \circ \mathbb{R}(\pi_Y^{YZ})_* \cong \mathbb{R}(\pi_{XY}^{XYZ})_* \circ \mathbb{L}(\pi_{YZ}^{XYZ})^*$$

Therefore,

$$\Phi^{\mathcal{P}}_{Y \to X} \circ \Phi^{\mathcal{Q}}_{Z \to Y} = \mathbb{R}(\pi_X^{XYZ})_*[\mathcal{R} \otimes^{\mathbb{L}}_{XYZ} \mathbb{L}(\pi_Z^{XYZ})^*]$$

where

$$\mathcal{R} = (\pi_{YZ}^{XYZ})^* \mathcal{Q} \otimes^{\mathbb{L}}_{XYZ} (\pi_{XY}^{XYZ})^* \mathcal{P}$$

□

Proposition 1.2.3. All integral functors have both left and right adjoints which are also integral functors.

Proof.

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}_X\left(E, \mathbb{R}\pi_{X*}[\mathcal{P} \otimes^{\mathbb{L}} \mathbb{L}\pi_Y^* F\right) &= \mathbb{R}\mathrm{Hom}_{X \times Y}\left(\pi_X^* E, \mathcal{P} \otimes^{\mathbb{L}} \mathbb{L}\pi_Y^* F\right) \\
&= \mathbb{R}\mathrm{Hom}_{X \times Y}\left(\mathcal{P}^\vee \otimes^{\mathbb{L}} \mathbb{L}\pi_X^* E, \mathbb{L}\pi_Y^* F\right) \\
&= \mathbb{R}\mathrm{Hom}_Y\left(\mathbb{R}\pi_{Y*}(\mathcal{P}^\vee \otimes^{\mathbb{L}} \mathbb{L}\pi_X^* E \otimes^{\mathbb{L}} \mathbb{L}\pi_X^* \omega_X[\dim X], F\right)
\end{aligned}$$

therefore the left adjoint is given by $\Phi^{\mathcal{P}^\vee \otimes^{\mathbb{L}} \mathbb{L}\pi_X^* \omega_X[\dim X]}_{X \to Y}$. Similarly the right adjoint is given by $\Phi^{\mathcal{P}^\vee \otimes^{\mathbb{L}} \mathbb{L}\pi_Y^* \omega_Y[\dim Y]}_{X \to Y}$. □

1.3 Full faithfulness

General fact: if a functor F has a left adjoint G then $\mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(F(X), F(Y)) = \mathrm{Hom}\,(GF(X), Y)$ so $GF = \mathrm{id}$ is enough to show that F is fully faithful.

Theorem 1.3.1. The functor $F := \Phi^{\mathcal{P}}_{Y \to X}$ is fully faithful if and only if

$$\mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_Y\,(\mathcal{O}_{y_1}, \mathcal{O}_{y_1}) \to \mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_Y\,(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2})$$

is an isomorphism in $D^b(k)$ for all closed points $y_1, y_2 \in Y$. Equivalently, if $y_1 \neq y_2$ then $\mathbb{R}\mathrm{Hom}_Y\,(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0$ and if $y_1 = y_2$ then,

$$\mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_Y\,(\mathcal{O}_y, \mathcal{O}_y) \xrightarrow{\sim} \mathbb{R}\Gamma\mathbb{R}\mathrm{Hom}_Y\,(F\mathcal{O}_y, F\mathcal{O}_Y)$$

Proof. It is clear this is necessary by considering shifts. Therefore, we need to prove sufficience. Let G be the left adjoint and write $GF = \Phi^{\mathcal{Q}}_{Y\to Y}$. The counit map $GF \to \mathrm{id}$ is realized by the map of sheaves $\mathcal{Q} \to \Delta_Y$ inside $D^b(Y \times Y)$. Let K be the fiber of this map. By assumption,

$$\mathbb{R}\mathrm{Hom}_Y\left(\mathcal{O}_{y_1}, \mathcal{O}_{y_1}\right) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_Y\left(GF\mathcal{O}_{y_1}, \mathcal{O}_{y_2}\right)$$

this is given by precomposition with $\mathcal{Q} \to \Delta_Y$ meaning there is an exact triangle

$$K_{y_1} \to \mathcal{Q}_{y_1} \to \mathcal{O}_{y_1} \to K_{y_1}[+1]$$

But the assumption exactly implies that $\mathbb{R}\mathrm{Hom}_Y\left(K_{y_1}, \mathcal{O}_{y_2}\right) = 0$. Then $K_{y_1} = 0$ because the resolution of K_{y_1} must be exact at each y_2. Therefore $K = 0$ because it is zero on all fibers. □

1.4 Fourier-Mukai

Let A/k be an abelian variety and $A^\vee/k$ is dual Pic^0_A. We let $\mathcal{P} \in D^b(A \times A^\vee)$ the Poincare bundle. The claim is that:

$$\Phi^{\mathcal{P}}_{A^\vee \to A} : D^b(A) \to D^b(A^\vee)$$

is an equivalence.

First we show it is fully faithful. To do this set $F = \Phi^{\mathcal{P}}_{A^\vee \to A}$ then

(a) for $y_1 \neq y_2$ we need $\mathbb{R}\mathrm{Hom}\left(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}\right) = 0$ but $F\mathcal{O}_{y_i} = \mathcal{L}_{y_i}$ is the line bundle corresponding to $y_i \in A^\vee$ so we need to show that $\mathbb{R}\mathrm{Hom}_A\left(\mathcal{L}_{y_1}, \mathcal{L}_{y_2}\right) = 0$

(b) for $y = y$ we need to show that $\mathbb{R}\mathrm{Hom}_{A^\vee}\left(\mathcal{O}_y, \mathcal{O}_y\right) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_A\left(\mathcal{L}_y, \mathcal{L}_y\right)$.

For part 1 we just need to show that $H^i(\mathcal{L}_y) = 0$ for all i and $y \neq 0$ by setting $y = y_2 - y_1$.

Lemma 1.4.1 (Mumford)**.** For any $\mathcal{L} \in A^\vee$ nonzero $H^i(\mathcal{L}) = 0$ for all i.

Proof. Consider $m : A \times A \to A$ addition. Then these satisfy the theorem of the square,

$$m^*\mathcal{L}_y \cong \pi_1^*\mathcal{L}_y \otimes \pi_2^*\mathcal{L}_y$$

Then by Kunneth:

$$H^k(A \times A, m^*\mathcal{L}_y) = \bigoplus_{i+j=k} H^i(A, \mathcal{L}_y) \otimes H^j(A, \mathcal{L}_y)$$

Note that if $A \hookrightarrow A \times A \to A$ is the identiy where $A \hookrightarrow A \times A$ is $\mathrm{id} \times 0$. Therefore, the pullback map m^* is injective so,

$$\dim H^k(A, \mathcal{L}_y) \leq H^k(A \times A, m^*\mathcal{L}_y) = \sum_{i+j=k} (\dim H^i(A, \mathcal{L}_y)(\dim H^j(A, \mathcal{L}_y)$$

But we know that $H^0(A, \mathcal{L}_y) = 0$ since $y \neq 0$ so we get that all $H^i(A, \mathcal{L}_y) = 0$ by an inductive argument. □

For part 2 we can use the Kozul resolution to show that,

$$\dim \operatorname{Ext}^i_{A^\vee}(\mathcal{O}_y, \mathcal{O}_y) = \binom{g}{i}$$

and by Hodge theory we also know that,

$$\dim H^i(A, \mathcal{O}_A) = \binom{g}{i}$$

Both sides have the same dimension.

Now we show it is essentially surjective. However, we know its left adjoint is the integral functor for $\mathcal{P}^\vee \otimes \pi_X^* \omega_X[\dim X] = \mathcal{P}^\vee[\dim X]$. Therefore, the same computation works for $\mathcal{P}^\vee$ so we see that the left adjoint is also fully faithful.