Physics GR6037 Quantum Mechanics I
Assignment # 3

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Problem 8.

(a). Let $\Delta x \Delta p \approx \hbar$ then

$$E = \frac{\Delta p^2}{2m} - \frac{e^2}{\Delta x} = \frac{1}{\Delta x^2}\frac{\hbar^2}{2m} - \frac{e^2}{\Delta x}$$

Now, we minimize the energy with respect to Δx:

$$E' = -\frac{2}{\Delta x^3}\frac{\hbar^2}{2m} + \frac{e^2}{\Delta x^2} = 0 \quad \text{gives} \quad \Delta x = \frac{\hbar^2}{me^2}$$

Now plugging in,

$$E = \frac{m^2 e^4}{\hbar^4}\frac{\hbar^2}{2m} - \frac{me^4}{\hbar^2} = -\frac{me^4}{2\hbar^2}$$

(b). $\Delta E \Delta t \approx \hbar$. Assuming that $\Delta t \approx \tau_{1/2} = 4 \times 10^{-6}\text{s}$ then $\Delta E \approx 2.6 \times 10^{-29}\text{J} = 1.6 \times 10^{-10}\text{eV}$.

(c). Model a ballanced pencil as an inverted physical pendulum with equation of motion

$$\ddot{\theta} = \frac{g}{\ell_{CM}} \frac{m\ell^2}{m\ell^2 + I_{CM}} \sin\theta$$

where l_{CM} is the height of the center of mass, m is the mass, and I_{CM} is the moment of inertia about the center of mass. Define $\omega_0 = \sqrt{\frac{g}{\ell_{CM}}\frac{m\ell^2}{m\ell^2+I_{CM}}}$. Now, applying a small angle approximation, the general solution is given by:

$$\theta(t) = \theta_0 \cosh \omega t + \frac{\dot{\theta}_0}{\omega_0} \sinh \omega_0 t$$

Let $\zeta_0 = \frac{\dot{\theta}_0}{\omega_0}$ then,

$$\theta(t) = \theta_0 \cosh \omega t + \zeta_0 \sinh \omega_0 t = \frac{1}{2}(\theta_0 + \zeta_0)e^{\omega_0 t} + \frac{1}{2}(\theta_0 - \zeta_0)e^{-\omega_0 t}$$

Since we are looking for a maximum time, we assume that the time is much larger than $\frac{1}{\omega_0}$ so that the decaying exponential term can be ignored. Then,

$$\theta(t) = \frac{1}{2}(\theta_0 + \zeta_0)e^{\omega_0 t}$$

Thus,

$$t = \frac{1}{\omega_0} \log\left(\frac{2\theta_{max}}{\theta_0 + \zeta_0}\right)$$

To maximize t, we minimize the denominator $\theta_0 + \zeta_0$. However, from the uncertianty principle, when we try to set $\theta = 0$ the uncertainty in coordinates effectivly puts the true initial conditions at $\theta_0 = \frac{\Delta x}{\ell_{CM}}$ and $\dot{\theta}_0 = \frac{\Delta p}{m\ell}$ so $\theta_0 \dot{\theta}_0 \approx \frac{\hbar}{m\ell^2}$ i.e. $\theta_0 \zeta_0 \approx \frac{\hbar}{m\ell^2 \omega_0}$.

Now set $\frac{\mathrm{d}t}{\mathrm{d}\theta_0} = 0$ which is equivalent to $\frac{\mathrm{d}}{\mathrm{d}\theta_0}(\theta_0 + \zeta_0) = 0$ so $1 - \frac{1}{\theta_0^2}\frac{\hbar}{m\ell^2\omega_0} = 0$ so the maximum time occurs for

$$\theta_0 = \sqrt{\frac{\hbar}{m\ell^2\omega_0}} \text{ giving } \zeta_0 = \sqrt{\frac{\hbar}{m\ell^2\omega_0}}$$

which corresponds to a time,

$$t = \frac{1}{\omega_0} \log\left(\theta_{max}\sqrt{\frac{m\ell^2\omega_0}{\hbar}}\right)$$

For $\ell_{CM} = 6\text{cm}$, $m = 10\text{g}$, $I_{CM} = \frac{1}{3}m\ell^3$, and $\theta_{max} = 15°$, then the maximum ballance time occurs for $\theta_0 = 1.62 \times 10^{-16}$ giving $t = 3.16\,\text{s}$.

Problem 9.

(a). At $t = 0$, let

$$\psi_0(x) = \frac{1}{(2\pi)^{1/4} d^{1/2}} \exp\left[-\frac{x^2}{4d^2} + i\frac{p_0}{\hbar}x\right]$$

Applying the Fourier transform to the Schrodinger equation,

$$\mathcal{F}_x\left[\hbar\frac{\partial}{\partial t}\psi(x,t)\right] = \mathcal{F}_x\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)\right]$$

$$\frac{\partial}{\partial t}\tilde{\psi}(k,t) = -i\frac{\hbar k^2}{2m}\tilde{\psi}(k,t)$$

Thus,

$$\tilde{\psi}(k,t) = \tilde{\psi}_0(k)\exp\left(-i\frac{\hbar k^2}{2m}t\right)$$

Now,

$$\tilde{\psi}_0(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi)^{1/4}d^{1/2}}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{4d^2} + i\frac{p_0}{\hbar}x - ikx\right]\mathrm{d}x$$

Now expanding the exponent with $q = \frac{p_0}{\hbar}$:

$$-\frac{x^2}{4d^2} + iqx - ikx = -\frac{1}{4d^2}[x + 2id^2(k-q)]^2 - d^2(k-q)^2$$

Therefore,

$$\begin{aligned}
\tilde{\psi}_0(k) &= \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi)^{1/4}d^{1/2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{4d^2}[x+2id^2(k-q)]^2 - d^2(k-q)^2\right]\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi)^{1/4}d^{1/2}}\exp\left[-d^2(k-q)^2\right]\int_{-\infty}^{\infty}\exp\left[-\frac{1}{4d^2}u^2\right]\mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}}\frac{2d\sqrt{\pi}}{(2\pi)^{1/4}d^{1/2}}\exp\left[-d^2(k-q)^2\right] = \frac{1}{(2\pi)^{1/4}(2/d)^{1/2}}\exp\left[-d^2(k-q)^2\right]
\end{aligned}$$

So,

$$\tilde{\psi}(k,t) = \frac{(2d)^{1/2}}{(2\pi)^{1/4}}\exp\left[-d^2(k-q)^2 - i\frac{\hbar k^2}{2m}t\right]$$

Applying the inverse transform,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\frac{(2d)^{1/2}}{(2\pi)^{1/4}}\int_{-\infty}^{\infty}\exp\left[-d^2(k-q)^2 - i\frac{\hbar k^2}{2m}t + ikx\right]\mathrm{d}k$$

Expanding the exponent,

$$\begin{aligned}
-d^2(k-q)^2 - i\frac{\hbar k^2}{2m}t + ikx &= -d^2(k-q)^2 - \frac{i\hbar t}{2m}(k-q)^2 - \frac{i\hbar t}{m}kp + \frac{i\hbar t}{2m}q^2 + ikx \\
&= -\left[d^2 + \frac{i\hbar t}{2m}\right](k-q)^2 + i\left(x - \frac{p}{m}t\right)(k-q) + iqx - \frac{i\hbar t}{2m}q^2 \\
&= -\left[d^2 + \frac{i\hbar t}{2m}\right]\left[k - q - \frac{i}{2\tilde{d}^2}\left(x - \frac{p}{m}t\right)\right]^2 - \frac{1}{4\tilde{d}^2}\left(x - \frac{p}{m}t\right)^2 + iqx - \frac{i\hbar t}{2m}q^2
\end{aligned}$$

with $\tilde{d}^2 = \left[d^2 + \frac{i\hbar t}{2m}\right]$ then,

$$\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2\pi}}\frac{(2d)^{1/2}}{(2\pi)^{1/4}}\int_{-\infty}^{\infty}\exp\left[-\tilde{d}^2u^2 - \frac{1}{4\tilde{d}^2}\left(x - \frac{p}{m}t\right)^2 + iqx - \frac{i\hbar t}{2m}q^2\right]\mathrm{d}u \\
&= \frac{1}{\tilde{d}}\frac{\sqrt{\pi}}{\sqrt{2\pi}}\frac{(2d)^{1/2}}{(2\pi)^{1/4}}\exp\left[-\frac{1}{4\tilde{d}^2}\left(x - \frac{p}{m}t\right)^2 + iqx - \frac{i\hbar t}{2m}q^2\right] \\
&= \frac{1}{(2\pi)^{1/4}d^{1/2}\sqrt{1 + \frac{i\hbar t}{2md^2}}}\exp\left[-\frac{\left(x - \frac{p}{m}t\right)^2}{4d^2\left(1 + \frac{i\hbar t}{2md^2}\right)} + i\frac{p_0}{\hbar}x - \frac{ip_0^2 t}{2m\hbar}\right]
\end{aligned}$$

(b). Let $G(x-x',t) = \langle x| e^{-i\hat{H}t/\hbar}|x'\rangle$. Then

$$i\hbar\frac{\partial}{\partial t}G(x-x',t) = i\hbar\langle x|\frac{1}{i\hbar}\hat{H}e^{-i\hat{H}t/\hbar}|x'\rangle = \langle x|\hat{H}e^{-i\hat{H}t/\hbar}|x'\rangle$$

But, $\hat{H} = \frac{1}{2m}\hat{p}^2$ so $\langle x|\hat{H}|\psi\rangle = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\langle x|\psi\rangle$. Thus,

$$i\hbar\frac{\partial}{\partial t}G(x-x',t) = \langle x|\hat{H}e^{-i\hat{H}t/\hbar}|x'\rangle = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\langle x|e^{-i\hat{H}t/\hbar}|x'\rangle = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x-x',t)$$

Futhermore, at $t=0$, $G(x-x',0) = \langle x|\mathbf{1}|x'\rangle = \langle x|x'\rangle = \delta(x-x')$.

(c).

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x| e^{-i\hat{H}t/\hbar}|\psi_0\rangle = \int_{-\infty}^{\infty} \langle x| e^{-i\hat{H}t/\hbar}|x'\rangle\langle x'|\psi_0\rangle \, \mathrm{d}x' = \int_{-\infty}^{\infty} G(x-x',t)\psi_0(x')\mathrm{d}x'$$

(d). Taking the Fouier Transform of,

$$i\hbar\frac{\partial}{\partial t}G(x-x',t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x-x',t)$$

we obtain:

$$i\hbar\frac{\partial}{\partial t}\tilde{G}(k,t) = \frac{\hbar^2 k^2}{2m}\tilde{G}(k,t)$$

Therefore,

$$\tilde{G}(k,t) = \exp\left(-i\frac{\hbar k^2}{2m}\right)\tilde{G}_0(k) = \frac{1}{\sqrt{2\pi}}\exp\left(-i\frac{\hbar k^2}{2m}t\right)$$

because $G_0(x-x') = \delta(x-x')$ so $\tilde{G}(k) = \frac{1}{\sqrt{2\pi}}$. Then applying the inverse Fourier transform,

$$G(x-x',t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\exp\left(-i\frac{\hbar k^2}{2m}t\right)\mathrm{d}k$$

The exponent is $-\frac{i\hbar t}{2m}\left(k^2 - \frac{2m}{\hbar t}k(x-x')\right) = -\frac{i\hbar t}{2m}\left(k^2 - \frac{m}{\hbar t}(x-x')\right)^2 + \frac{im}{2\hbar t}(x-x')^2$. Thus,

$$\begin{aligned}
G(x-x',t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{i\hbar t}{2m}\left(k^2 - \frac{m}{\hbar t}(x-x')\right)^2 + \frac{im}{2\hbar t}(x-x')^2\right]\mathrm{d}k \\
&= \frac{1}{2\pi}\exp\left[\frac{im}{2\hbar t}(x-x')^2\right]\int_{-\infty}^{\infty}\sqrt{\frac{2m}{i\hbar t}}\exp\left(-u^2\right)\mathrm{d}u = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{im}{2\hbar t}(x-x')^2\right]
\end{aligned}$$

We verify the normalization of this Green's function by applying it to the gaussian wave packet of part (a),

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{\infty} G(x-x',t)\psi_0(x')\mathrm{d}x' \\
&= \int_{-\infty}^{\infty}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{im}{2\hbar t}(x-x')^2\right]\frac{1}{(2\pi)^{1/4}d^{1/2}}\exp\left[-\frac{x^2}{4d^2} + i\frac{p_0}{\hbar}x\right]\mathrm{d}x' \\
&= \sqrt{\frac{m}{2\pi i\hbar t}}\frac{1}{(2\pi)^{1/4}d^{1/2}}\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar t}(x-x')^2 - \frac{x'^2}{4d^2} + i\frac{p_0}{\hbar}x'\right]\mathrm{d}x' \\
&= \sqrt{\frac{m}{2\pi i\hbar t}}\frac{1}{(2\pi)^{1/4}d^{1/2}}\int_{-\infty}^{\infty}\exp\left[\left(\frac{im}{2\hbar t} - \frac{1}{4d^2}\right)x'^2 + \left(i\frac{p_0}{\hbar} - \frac{im}{\hbar t}x\right)x' + \frac{im}{2\hbar t}x^2\right]\mathrm{d}x'
\end{aligned}$$

Let $a = \left(\frac{im}{2\hbar t} - \frac{1}{4d^2}\right)$, $b = \left(i\frac{p_0}{\hbar} - \frac{im}{\hbar t}x\right)$, $c = \frac{im}{2\hbar t}x^2$, and $N = \sqrt{\frac{m}{2\pi i\hbar t}}\frac{1}{(2\pi)^{1/4}d^{1/2}}$
Then,

$$\begin{aligned}
\psi(x,t) &= N\int_{-\infty}^{\infty}\exp\left[ax'^2 + bx' + c\right]\mathrm{d}x' = N\int_{-\infty}^{\infty}\exp\left[a\left(x' + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c\right]\mathrm{d}x' \\
&= N\exp\left[-\frac{b^2}{4a} + c\right]\int_{-\infty}^{\infty}\exp\left[a\left(x' + \frac{b}{2a}\right)^2\right]\mathrm{d}x' = N\sqrt{\frac{\pi}{-a}}\exp\left[-\frac{b^2}{4a} + c\right]
\end{aligned}$$

So pluggin in,

$$\psi(x,t) = \sqrt{\frac{m}{2\pi i\hbar t}}\frac{\sqrt{\pi}}{(2\pi)^{1/4}d^{1/2}}\frac{1}{\sqrt{\frac{m}{2i\hbar t}+\frac{1}{4d^2}}}\exp\left[\frac{\left(i\frac{p_0}{\hbar}-\frac{im}{\hbar t}x\right)^2}{4\left(\frac{m}{2i\hbar t}+\frac{1}{4d^2}\right)}+\frac{im}{2\hbar t}x^2\right]$$

$$= \frac{1}{(2\pi)^{1/4}d^{1/2}\left(1+\frac{i\hbar t}{2md^2}\right)}\exp\left[-\frac{\left(x-\frac{p_0}{m}t\right)^2}{4d^2\left(1+\frac{i\hbar t}{2md^2}\right)}+i\frac{p_0}{\hbar}x-\frac{ip_0^2 t}{2m\hbar}\right]$$

Problem 10.

Let $\hat{X} = \hat{x} - \langle x\rangle$ and $\hat{P} = \hat{p} - \langle p\rangle$. Take $|\psi_\alpha\rangle = (\hat{X} - i\alpha\hat{P})\,|\psi\rangle$. Then expanding $\langle\psi_\alpha|\psi_\alpha\rangle \geq 0$ gives the uncertainty relation. If ψ saturates the uncertianty relation then the discriminant of $\langle\psi_\alpha|\psi_\alpha\rangle$ is zero therefore for $\alpha = -\frac{\hbar}{2\Delta p^2} = -\frac{\Delta x}{\Delta p} = -\frac{2\Delta x^2}{\hbar}$ the quadratic form $\langle\phi_\alpha|\psi_\alpha\rangle = 0$. Thus, $(\hat{X} - i\alpha\hat{P})\,|\psi\rangle = 0$ so take,

$$\langle x|\,(\hat{X} - i\alpha\hat{P})\,|\psi\rangle = (x - \langle x\rangle)\psi(x) - i\alpha\left(\frac{\hbar}{i}\frac{\partial}{\partial x} - \langle p\rangle\right)\psi(x) = 0$$

Letting $y = x - \langle x\rangle$ and $p_0 = \langle p\rangle$ then,

$$\frac{\partial}{\partial y}\psi(y) = \left(\frac{y}{\hbar\alpha} + i\frac{p_0}{\hbar}\right)\psi(y) \text{ so } \psi(x) = N\exp\left[\frac{y^2}{2\hbar\alpha} + i\frac{p_0}{\hbar}y\right] = N\exp\left[-\frac{y^2}{4\Delta x^2} + i\frac{p_0}{\hbar}y\right]$$

Thus,

$$\psi(x) = \frac{1}{(2\pi)^{\frac{1}{4}}(\Delta x)^{\frac{1}{2}}}\exp\left[-\frac{(x-x_0)^2}{4\Delta x^2} + i\frac{p_0}{\hbar}(x-x_0)\right]$$

Problem 11.

Consider $F(\alpha) = e^{\alpha A}e^{\alpha B}$ then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F(\alpha) = Ae^{\alpha A}e^{\alpha B} + e^{\alpha A}Be^{\alpha B} = Ae^{\alpha A}e^{\alpha B} + e^{\alpha A}Be^{-\alpha A}e^{\alpha A}e^{\alpha B} = (A + e^{\alpha A}Be^{-\alpha A})F(\alpha)$$

However,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}e^{\alpha A}Be^{-\alpha A} = e^{\alpha A}ABe^{-\alpha A} - e^{\alpha A}BAe^{-\alpha A} = e^{\alpha A}[A,B]e^{-\alpha A}$$

Now since $[A,B]$ commutes with both A and B we have,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}e^{\alpha A}Be^{-\alpha A} = [A,B]e^{\alpha A}e^{-\alpha A} = [A,B]$$

Therefore, because $[A,B]$ is independent of α,

$$e^{\alpha A}Be^{-\alpha A} = e^{\alpha A}Be^{-\alpha A}\Big|_{\alpha=0} + \alpha[A,B] = B + \alpha[A,B]$$

Applying this result above,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F(\alpha) = (A + B + \alpha[A,B])F(\alpha)$$

Thus,

$$F(\alpha) = F(0)e^{\alpha A + \alpha B + \frac{\alpha^2}{2}[A,B]} = e^{\alpha A + \alpha B + \frac{\alpha^2}{2}[A,B]}$$

Evaluating at $\alpha = 1$,

$$F(1) = e^A e^B = e^{A + B + \frac{1}{2}[A,B]}$$