Physics GR6047 Quantum Field Theory I
Assignment # 6

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1 Computing the Fermionic Two-Point Function

Consider the decomposition of the two-point function,

$$\begin{aligned}\langle\Omega|\,\psi_\alpha(x)\bar{\psi}_\beta(y)\,|\Omega\rangle &= \sum_\lambda\sum_s\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{2p^0}\,\langle\Omega|\,\psi_\alpha(x)\,|\lambda_{sp}\rangle\,\langle\lambda_{sp}|\,\bar{\psi}_\beta(y)\,|\Omega\rangle \\ &= \sum_s\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{2p^0}\,\langle\Omega|\,\psi_\alpha(x)\,|\lambda_{sp}\rangle\,\langle\Omega|\,\psi_{\beta'}(x)\,|\lambda_{sp}\rangle^*\,\gamma^0_{\beta'\beta}\end{aligned}$$

Where I have made the sum over λ implicit. For convinience, I will omit this sum throughout the calculation. Therefore, we should investigate the amplitude,

$$\langle\Omega|\,\psi_\alpha(x)\,|\lambda_{sp}\rangle$$

in more detail.

1.1 The Component Amplitude

Returning to the calculation of the amplitude,

$$\langle\Omega|\,\psi(x)\,|\lambda_{sp}\rangle = \langle\Omega|\,e^{iP_\mu x^\mu}\psi(0)e^{-iP_\mu x^\mu}\,|\lambda_{sp}\rangle = \langle\Omega|\,\psi(0)\,|\lambda_{sp}\rangle\,e^{-ip\cdot x}$$

where $p\cdot x = p_\mu x^\mu$. This result follows because $|\Omega\rangle$ is shift invariant and $P_\mu\,|\lambda_{sp}\rangle = p_\mu\,|\lambda_{sp}\rangle$. Now, note that the states $|\lambda_{sp}\rangle$ are Lorentz boosts of zero momentum states,

$$|\lambda_{sp}\rangle = U(\vec{\chi}_p)\,|\lambda_s\rangle$$

Therefore, since the vacuum is Lorentz-invariant,

$$\langle\Omega|\,\psi(0)\,|\lambda_{sp}\rangle = \langle\Omega|\,U(\vec{\chi}_p)^{-1}\psi(0)U(\vec{\chi}_p)\,|\lambda_s\rangle = \langle\Omega|\,e^{\vec{\chi}_p\cdot\vec{S}}\psi(0)\,|\lambda_s\rangle$$

where we have defined the generators of the matrix algebra for the spin-$\frac{1}{2}$ representation as,

$$S^{\mu\nu} = \tfrac{1}{4}[\gamma^\mu,\gamma^\nu]$$

with, in particular, the boost part,

$$(\vec{S})^i = S^{0i} = \tfrac{1}{4}[\gamma^0,\gamma^i] = \tfrac{1}{2}\gamma^0\gamma^i = \frac{1}{2}\begin{pmatrix}-\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

Therefore,

$$(e^{\vec{\chi_p}\cdot\vec{S}})^2 = e^{\gamma^0\vec{\chi_p}\cdot\vec{\gamma}} = \mathbf{1} + \gamma^0\vec{\chi_p}\cdot\vec{\gamma} + \frac{1}{2}(\gamma^0\vec{\chi_p}\cdot\vec{\gamma})^2 + \cdots$$

However,

$$(\gamma^0\vec{\chi_p}\cdot\vec{\gamma})^2 = \gamma^0(\chi_p)_i(\chi_p)_j\gamma^i\gamma^0\gamma^j = -(\chi_p)_i(\chi_p)_j\gamma^i\gamma^j = -\tfrac{1}{2}(\chi_p)_i(\chi_p)_j\{\gamma^i,\gamma^j\} = -(\chi_p)_i(\chi_p)_j\eta^{ij}\mathbf{1} = (\chi_p)^2\mathbf{1}$$

Thus, combining terms in the sequence,

$$(e^{\vec{\chi_p}\cdot\vec{S}})^2 = \mathbf{1}(1 + \tfrac{1}{2}\chi_p^2 + \tfrac{1}{4!}\chi_p^4 + \cdots) + \gamma^0\hat{p}\cdot\vec{\gamma}(\chi_p + \tfrac{1}{3!}\chi_p^3 + \cdots) = \mathbf{1}\cosh\chi_p + \gamma^0\hat{p}\cdot\vec{\gamma}\sinh\chi_p$$

However, we know that the boost factor is given by $\gamma = \cosh\chi_p$ and $\beta\gamma = \sinh\chi_p$ so $E = m_\lambda\cosh\chi_p$ and $\vec{p} = m_\lambda\hat{p}\sinh\chi_p$. Therefore, since $(\gamma^0)^2 = \mathbf{1}$ we have,

$$(e^{\vec{\chi_p}\cdot\vec{S}})^2 = \frac{1}{m_\lambda}\left(\mathbf{1}E + \gamma^0\vec{p}\cdot\vec{\gamma}\right) = \frac{\gamma^0 p_\mu\gamma^\mu}{m_\lambda} = \frac{1}{m_\lambda}\gamma^0\not{p} = \frac{1}{m_\lambda}\begin{pmatrix} p_\mu\sigma^\mu & 0 \\ 0 & p_\mu\bar{\sigma}^\mu \end{pmatrix}$$

Finally,

$$e^{\vec{\chi_p}\cdot\vec{S}} = \frac{1}{\sqrt{m_\lambda}}\sqrt{\gamma^0\not{p}} = \frac{1}{\sqrt{m_\lambda}}\sqrt{\begin{pmatrix} p_\mu\sigma^\mu & 0 \\ 0 & p_\mu\bar{\sigma}^\mu \end{pmatrix}} = \frac{1}{\sqrt{m_\lambda}}\begin{pmatrix} \sqrt{p_\mu\sigma^\mu} & 0 \\ 0 & \sqrt{p_\mu\bar{\sigma}^\mu} \end{pmatrix}$$

Putting this together, we see that,

$$\langle\Omega|\,\psi(x)\,|\lambda_{sp}\rangle = \sqrt{\frac{\gamma^0\not{p}}{m_\lambda}}\,\langle\Omega|\,\psi(0)\,|\lambda_s\rangle\,e^{-ip\cdot x}$$

1.2 Inner Products of Spinors

Since parity is going to be a symmetry of the theory, we need to have both left and right handed spinors. We will give the chiral components of the spinor field special names,

$$\psi(x) = \begin{pmatrix} L(x) \\ R(x) \end{pmatrix}$$

By the spin-$\frac{1}{2}$ transformation properties,

$$\langle\Omega|\,L_{s'}(0)\,|\lambda_s\rangle = C_{\lambda,L}\delta_{s's} \quad\text{and}\quad \langle\Omega|\,R_{s'}(0)\,|\lambda_s\rangle = C_{\lambda,R}\delta_{s's}$$

However, these matrix element can be related via parity transformation. Since parity commutes with spin operators and, by assumption, with the Hamiltonian, we can choose the rest states $|\lambda_s\rangle$ to be parity eigenstates. This would not work for the states $|\lambda_{sp}\rangle$ because $\mathbf{P}^{-1}P^i\mathbf{P} = -P^i$ and thus we cannot simultaneously diagonalize both operators. Take the state $|\lambda_s\rangle$ to have intrinsic parity η_λ. That is,

$$\mathbf{P}\,|\lambda_s\rangle = \eta_\lambda\,|\lambda_s\rangle$$

Furthermore, the right-handed and left-handed spinors are related by,

$$\mathbf{P}^{-1}L(x)\mathbf{P} = R(\bar{x}) \quad\text{and}\quad \mathbf{P}^{-1}R(x)\mathbf{P} = L(\bar{x})$$

Therefore, since $|\Omega\rangle$ is invariant under parity,

$$\langle\Omega| L_{s'}(0) |\lambda_s\rangle = \langle\Omega| \mathbf{P}^{-1} R_{s'}(0) \mathbf{P} |\lambda_s\rangle = \eta_\lambda \langle\Omega| R_{s'}(0) |\lambda_s\rangle = \eta_\lambda C_{\lambda,R}\delta_{s's} = C_{\lambda,L}\delta_{s's}$$

Likewise,

$$\langle\Omega| R_{s'}(0) |\lambda_s\rangle = \langle\Omega| \mathbf{P}^{-1} L_{s'}(0) \mathbf{P} |\lambda_s\rangle = \eta_\lambda \langle\Omega| L_{s'}(0) |\lambda_s\rangle = \eta_\lambda C_{\lambda,L}\delta_{s's} = C_{\lambda,R}\delta_{s's}$$

Therefore, $C_{\lambda,L} = \eta_\lambda C_{\lambda,R}$ and $C_{\lambda,R} = \eta_\lambda C_{\lambda,L}$. Therefore, $\eta_\lambda = \pm 1$. Now, define the field strength,

$$Z_\lambda^{1/2} = \frac{1}{\sqrt{m_\lambda}} C_{\lambda,L} = \frac{\eta_\lambda}{\sqrt{m_\lambda}} C_{\lambda,R}$$

which, because the λ states are relativistically normalized, has the correct units. Using this definition, we see that the entire inner product becomes,

$$\langle\Omega| \psi(0) |\lambda_s\rangle = \langle\Omega| \begin{pmatrix} L(0) \\ R(0) \end{pmatrix} |\lambda_s\rangle = Z_\lambda^{1/2} \sqrt{m_\lambda} \begin{pmatrix} \xi^s \\ \eta_\lambda \xi^s \end{pmatrix}$$

where ξ^s is the two-component spinor defined by,

$$\xi^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Putting it all together,

$$\langle\Omega| \psi(x) |\lambda_{sp}\rangle = Z_\lambda^{1/2} \sqrt{\gamma^0 \not{p}} \begin{pmatrix} \xi^s \\ \eta_\lambda \xi^s \end{pmatrix} e^{-ip\cdot x} = Z_\lambda^{1/2} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \eta_\lambda \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} e^{-ip\cdot x} = Z_\lambda^{1/2} u^s_{\eta_\lambda} e^{-ip\cdot x}$$

where we have found the bispinors,

$$u^s_{\eta_\lambda} = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \eta_\lambda \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}$$

If we restrict this to the cases $\eta_\lambda = \pm 1$ then we find,

$$u^s_+ = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = u^s \quad \text{and} \quad u^s_- = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = v'^s$$

where v'^s is the spinor v^s with the reversed spin. Therefore, the two-point function becomes,

$$\begin{aligned}
\langle\Omega| \psi_\alpha(x)\bar{\psi}_\beta(y) |\Omega\rangle &= \sum_s \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{1}{2p^0} \langle\Omega| \psi_\alpha(x) |\lambda_{sp}\rangle \langle\Omega| \psi_{\beta'}(x) |\lambda_{sp}\rangle^* \gamma^0_{\beta'\beta} \\
&= \sum_s \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{Z_\lambda}{2p^0} (u^s_{\eta_\lambda})_\alpha (u^s_{\eta_\lambda})^*_{\beta'} \gamma^0_{\beta'\beta}\, e^{-ip\cdot(x-y)}
\end{aligned}$$

Where $Z_\lambda \geq 0$ because $Z_\lambda = C_{\lambda,L} C^*_{\lambda,L}$. Now, we need to compute the outer product,

$$\begin{aligned}
\sum_s (u^s_{\eta_\lambda})_\alpha (u^s_{\eta_\lambda})^*_{\beta'} \gamma^0_{\beta'\beta} &= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \eta_\lambda \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \begin{pmatrix} \eta_\lambda (\xi^s)^\dagger \sqrt{p\cdot\bar{\sigma}} & (\xi^s)^\dagger \sqrt{p\cdot\sigma} \end{pmatrix} \\
&= \sum_s \begin{pmatrix} \eta_\lambda \sqrt{p\cdot\sigma}\xi^s(\xi^s)^\dagger \sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\xi^s(\xi^s)^\dagger \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\xi^s(\xi^s)^\dagger \sqrt{p\cdot\bar{\sigma}} & \eta_\lambda \sqrt{p\cdot\bar{\sigma}}\xi^s(\xi^s)^\dagger \sqrt{p\cdot\sigma} \end{pmatrix}
\end{aligned}$$

However,

$$\sum_s \xi^s (\xi^s)^\dagger = \mathbf{1}$$

Therefore,

$$\begin{aligned}
\sum_s (u^s_{\eta_\lambda})_\alpha (u^s_{\eta_\lambda})^*_{\beta'} \gamma^0_{\beta'\beta} &= \begin{pmatrix} \eta_\lambda \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & p\cdot\sigma \\ p\cdot\bar\sigma & \eta_\lambda\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix} \\
&= \begin{pmatrix} \eta_\lambda m_\lambda & p\cdot\sigma \\ p\cdot\bar\sigma & \eta_\lambda m_\lambda \end{pmatrix} = (\not{p} + \eta_\lambda m_\lambda)_{\alpha\beta}
\end{aligned}$$

where I have used the fact that,

$$\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} = \sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)} = \sqrt{p_\mu p_\nu \sigma^\mu \bar\sigma^\nu} = \sqrt{p_\mu p_\nu \eta^{\mu\nu}} = m_\lambda$$

Finally, the two-point function becomes,

$$\langle\Omega|\, \psi_\alpha(x)\bar\psi_\beta(y)\,|\Omega\rangle = \sum_\lambda \int \frac{\mathrm{d}^3\vec p}{(2\pi)^3}\frac{Z_\lambda}{2p^0}(\not{p} + \eta_\lambda m_\lambda)_{\alpha\beta}\, e^{-ip\cdot(x-y)}$$

2 Computing the Reversed Two-Point Function

Consider the decomposition of the two-point function,

$$\begin{aligned}
\langle\Omega|\, \bar\psi_\beta(y)\psi_\alpha(x)\,|\Omega\rangle &= \sum_{\bar\lambda}\sum_s \int \frac{\mathrm{d}^3\vec p}{(2\pi)^3}\frac{1}{2p^0}\langle\Omega|\, \bar\psi_\beta(y)\,\big|\bar\lambda_{sp}\big\rangle \big\langle\bar\lambda_{sp}\big|\, \psi_\alpha(x)\,|\Omega\rangle \\
&= \sum_s \int \frac{\mathrm{d}^3\vec p}{(2\pi)^3}\frac{1}{2p^0}\langle\Omega|\, \psi^*_{\beta'}(y)\,\big|\bar\lambda_{sp}\big\rangle \langle\Omega|\, \psi^*_\alpha(x)\,\big|\bar\lambda_{sp}\big\rangle^* \gamma^0_{\beta'\beta}
\end{aligned}$$

As before, I will make the sum over $\bar\lambda$ implicit. Therefore, we should investigate the amplitude,

$$\langle\Omega|\, \psi^*_\alpha(x)\,\big|\bar\lambda_{sp}\big\rangle$$

in more detail.

2.1 The Component Amplitude

Returning to the calculation of the amplitude,

$$\langle\Omega|\, \psi^*(x)\,\big|\bar\lambda_{sp}\big\rangle = \langle\Omega|\, e^{iP_\mu x^\mu}\psi^*(0)e^{-iP_\mu x^\mu}\,\big|\bar\lambda_{ps}\big\rangle = \langle\Omega|\, \psi^*(0)\,\big|\bar\lambda_{sp}\big\rangle\, e^{-ip\cdot x}$$

where $p\cdot x = p_\mu x^\mu$. This result follows because $|\Omega\rangle$ is shift invariant and $P_\mu\,\big|\bar\lambda_{ps}\big\rangle = p_\mu\,\big|\bar\lambda_{ps}\big\rangle$. Now, note that the states $\big|\bar\lambda_{sp}\big\rangle$ are Lorentz boosts of zero momentum states,

$$\big|\bar\lambda_{sp}\big\rangle = U(\vec\chi_p)\,\big|\bar\lambda_s\big\rangle$$

Therefore, since the vacuum is Lorentz-invariant,

$$\langle\Omega|\, \psi^*(0)\,\big|\bar\lambda_{sp}\big\rangle = \langle\Omega|\, U(\vec\chi_p)^{-1}\psi^*(0)U(\vec\chi_p)\,\big|\bar\lambda_{sp}\big\rangle = \langle\Omega|\, e^{\vec\chi_p\cdot\vec S^*}\psi^*(0)\,\big|\bar\lambda_s\big\rangle$$

As before, we have that,

$$e^{\vec{\chi}_p\cdot\vec{S}^*} = \frac{1}{\sqrt{m_{\bar{\lambda}}}}\sqrt{\gamma^0 \not{p}^*} = \frac{1}{\sqrt{m_{\bar{\lambda}}}}\sqrt{\begin{pmatrix} p_\mu\sigma^{*\mu} & 0 \\ 0 & p_\mu\bar{\sigma}^{*\mu}\end{pmatrix}} = \frac{1}{\sqrt{m_{\bar{\lambda}}}}\begin{pmatrix}\sqrt{p_\mu\sigma^{*\mu}} & 0 \\ 0 & \sqrt{p_\mu\bar{\sigma}^{*\mu}}\end{pmatrix}$$

Putting this together, we see that,

$$\langle\Omega|\,\psi^*(x)\left|\bar{\lambda}_{sp}\right\rangle = \sqrt{\frac{\gamma^0\not{p}^*}{m_{\bar{\lambda}}}}\,\langle\Omega|\,\psi^*(0)\left|\bar{\lambda}_s\right\rangle e^{-ip\cdot x}$$

2.2 Inner Products of Conjugate Spinors

Consider the chiral components of the conjugated spinor,

$$\psi^*(x) = \begin{pmatrix} L^*(x) \\ R^*(x)\end{pmatrix}$$

We cannot directly use the spin-$\frac{1}{2}$ transformation properties because L^* is not a spinor. However, ϵL^* is a right handed spinor. This can be checked by considering its transformation properties under rotations,

$$\epsilon L^* \mapsto \epsilon\left(e^{-\frac{i}{2}\vec{\phi}\cdot\vec{\sigma}}L\right)^* = \epsilon e^{\frac{i}{2}\vec{\phi}\cdot\vec{\sigma}^*}L^* = e^{-\frac{i}{2}\vec{\phi}\cdot\vec{\sigma}}\epsilon L^*$$

and under boosts,

$$\epsilon L^* \mapsto \epsilon\left(e^{-\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}}L\right)^* = \epsilon e^{-\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}^*}L^* = e^{\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}}\epsilon L^*$$

Therefore, we apply the spin-$\frac{1}{2}$ selection rules to the spinors ϵL^* and ϵR^*,

$$\langle\Omega|\,\epsilon_{s's''}L^*_{s''}(0)\left|\bar{\lambda}_s\right\rangle = C_{\bar{\lambda},L}\delta_{s's} \quad\text{and}\quad \langle\Omega|\,\epsilon_{s's''}R^*_{s''}(0)\left|\bar{\lambda}_s\right\rangle = C_{\bar{\lambda},R}\delta_{s's}$$

Therefore,

$$\langle\Omega|\,L^*_{s'}(0)\left|\bar{\lambda}_s\right\rangle = C_{\bar{\lambda},L}\epsilon_{ss'} \quad\text{and}\quad \langle\Omega|\,R^*_{s'}(0)\left|\bar{\lambda}_s\right\rangle = C_{\bar{\lambda},R}\epsilon_{ss'}$$

These matrix element can be related via parity transformation. Since parity commutes with spin operators and, by assumption, with the Hamiltonian, we can choose the rest states $\left|\bar{\lambda}_s\right\rangle$ to be parity eigenstates. Take the state $\left|\bar{\lambda}_s\right\rangle$ to have intrinsic parity $\eta_{\bar{\lambda}}$. That is,

$$\mathbf{P}\left|\bar{\lambda}_s\right\rangle = \eta_{\bar{\lambda}}\left|\bar{\lambda}_s\right\rangle$$

Furthermore, the right-handed and left-handed conjugate spinors are related by,

$$\mathbf{P}^{-1}\epsilon L^*(x)\mathbf{P} = \epsilon R^*(\bar{x}) \quad\text{and}\quad \mathbf{P}^{-1}\epsilon R^*(x)\mathbf{P} = \epsilon L^*(\bar{x})$$

Therefore, since $|\Omega\rangle$ is invariant under parity,

$$\langle\Omega|\,(\epsilon L^*)_{s'}\left|\bar{\lambda}_s\right\rangle = \langle\Omega|\,\mathbf{P}^{-1}(\epsilon R^*)_{s'}\mathbf{P}\left|\bar{\lambda}_s\right\rangle = \eta_{\bar{\lambda}}\,\langle\Omega|\,(\epsilon R^*)_{s'}\left|\bar{\lambda}_s\right\rangle = \eta_{\bar{\lambda}}C_{\bar{\lambda},R}\delta_{s's} = C_{\bar{\lambda},L}\delta_{s's}$$

Likewise,

$$\langle\Omega|\,(\epsilon R^*)_{s'}\left|\bar{\lambda}_s\right\rangle = \langle\Omega|\,\mathbf{P}^{-1}(\epsilon L^*)_{s'}\mathbf{P}\left|\bar{\lambda}_s\right\rangle = \eta_{\bar{\lambda}}\,\langle\Omega|\,(\epsilon L^*)_{s'}(0)\left|\bar{\lambda}_s\right\rangle = \eta_{\bar{\lambda}}C_{\bar{\lambda},L}\delta_{s's} = C_{\bar{\lambda},R}\delta_{s's}$$

Therefore, $C_{\bar\lambda,L} = \eta_{\bar\lambda} C_{\bar\lambda,R}$ and $C_{\bar\lambda,R} = \eta_{\bar\lambda} C_{\bar\lambda,L}$. As before, our choices require, $\eta_{\bar\lambda} = \pm 1$. Now, define the field strength,

$$Z_{\bar\lambda}^{1/2} = \frac{1}{\sqrt{m_{\bar\lambda}}} C_{\bar\lambda,L} = \frac{\eta_{\bar\lambda}}{\sqrt{m_{\bar\lambda}}} C_{\bar\lambda,R}$$

which, because the $\bar\lambda$ states are relativistically normalized, has the correct units. Using this definition, we see that the entire inner product becomes,

$$\langle\Omega|\, \psi^*(0) \left|\bar\lambda_s\right\rangle = \langle\Omega| \begin{pmatrix} L^*(0) \\ R^*(0) \end{pmatrix} \left|\bar\lambda_s\right\rangle = Z_{\bar\lambda}^{1/2} \sqrt{m_{\bar\lambda}} \begin{pmatrix} \epsilon\xi^s \\ \eta_{\bar\lambda}\epsilon\xi^s \end{pmatrix}$$

Putting it all together,

$$\begin{aligned}\langle\Omega|\, \psi^*(x) \left|\bar\lambda_{sp}\right\rangle &= Z_{\bar\lambda}^{1/2} \sqrt{\gamma^0 \not{p}^*} \begin{pmatrix} \epsilon\xi^s \\ \eta_{\bar\lambda}\epsilon\xi^s \end{pmatrix} e^{-ip\cdot x} = Z_{\bar\lambda}^{1/2} \begin{pmatrix} \sqrt{p\cdot\sigma^*}\,\epsilon\xi^s \\ \eta_{\bar\lambda}\sqrt{p\cdot\bar\sigma^*}\,\epsilon\xi^s \end{pmatrix} e^{-ip\cdot x} \\ &= Z_{\bar\lambda}^{1/2} \begin{pmatrix} \epsilon\sqrt{p\cdot\bar\sigma}\,\xi^s \\ \eta_{\bar\lambda}\epsilon\sqrt{p\cdot\sigma}\,\xi^s \end{pmatrix} e^{-ip\cdot x} = Z_{\bar\lambda}^{1/2} (v^s_{\eta_{\bar\lambda}})^* e^{-ip\cdot x}\end{aligned}$$

where we have found bispinors,

$$v^s_{\eta_{\bar\lambda}} = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\epsilon\xi^s \\ \eta_{\bar\lambda}\sqrt{p\cdot\bar\sigma}\,\epsilon\xi^s \end{pmatrix} = (u^s_{-\eta_{\bar\lambda}})^{(c)} = \mathcal{C}\cdot(u^s_{-\eta_{\bar\lambda}})^*$$

If we restrict this to the cases $\eta_{\bar\lambda} = \pm 1$ then we find,

$$v^s_+ = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\epsilon\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\epsilon\xi^s \end{pmatrix} = u'^s \quad \text{and} \quad v^s_- = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\epsilon\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\epsilon\xi^s \end{pmatrix} = v^s$$

where u'^s is the spinor u^s with the reversed spin. Therefore, the two-point function becomes,

$$\begin{aligned}\langle\Omega|\, \bar\psi_\beta(y)\psi_\alpha(x) \left|\Omega\right\rangle &= \sum_s \int \frac{\mathrm{d}^3\vec p}{(2\pi)^3} \frac{1}{2p^0} \langle\Omega|\, \psi^*_{\beta'}(y) \left|\bar\lambda_{sp}\right\rangle \langle\Omega|\, \psi^*_\alpha(x) \left|\bar\lambda_{sp}\right\rangle^* \gamma^0_{\beta'\beta} \\ &= \sum_s \int \frac{\mathrm{d}^3\vec p}{(2\pi)^3} \frac{Z_{\bar\lambda}}{2p^0} (v^s_{\eta_{\bar\lambda}})_\alpha (v^s_{\eta_{\bar\lambda}})^*_\beta \gamma^0_{\beta'\beta}\, e^{ip\cdot(x-y)}\end{aligned}$$

Where $Z_{\bar\lambda} \geq 0$ because $Z_{\bar\lambda} = C_{\bar\lambda,L} C^*_{\bar\lambda,L}$. Now, we need to compute the outer product,

$$\begin{aligned}\sum_s (v^s_{\eta_{\bar\lambda}})_\alpha (v^s_{\eta_{\bar\lambda}})^*_{\beta'} \gamma^0_{\beta'\beta} &= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\epsilon\xi^s \\ \eta_{\bar\lambda}\sqrt{p\cdot\bar\sigma}\,\epsilon\xi^s \end{pmatrix} \begin{pmatrix} \eta_{\bar\lambda}(\epsilon\xi^s)^\dagger\sqrt{p\cdot\bar\sigma} & (\epsilon\xi^s)^\dagger\sqrt{p\cdot\sigma} \end{pmatrix} \\ &= \sum_s \begin{pmatrix} \eta_{\bar\lambda}\sqrt{p\cdot\sigma}\,\epsilon\xi^s(\epsilon\xi^s)^\dagger\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\,\epsilon\xi^s(\epsilon\xi^s)^\dagger\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\,\epsilon\xi^s(\epsilon\xi^s)^\dagger\sqrt{p\cdot\bar\sigma} & \eta_{\bar\lambda}\sqrt{p\cdot\bar\sigma}\,\epsilon\xi^s(\epsilon\xi^s)^\dagger\sqrt{p\cdot\sigma} \end{pmatrix}\end{aligned}$$

However,

$$\sum_s \epsilon\xi^s(\epsilon\xi^s)^\dagger = \epsilon\left(\sum_s \xi^s(\xi^s)^\dagger\right)\epsilon^{-1} = \mathbf{1}$$

Therefore,

$$\begin{aligned}\sum_s (v^s_{\eta_{\bar\lambda}})_\alpha (v^s_{\eta_{\bar\lambda}})^*_{\beta'} \gamma^0_{\beta'\beta} &= \begin{pmatrix} \eta_{\bar\lambda}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & p\cdot\sigma \\ p\cdot\bar\sigma & \eta_{\bar\lambda}\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix} \\ &= \begin{pmatrix} \eta_{\bar\lambda} m_{\bar\lambda} & p\cdot\sigma \\ p\cdot\bar\sigma & \eta_{\bar\lambda} m_{\bar\lambda} \end{pmatrix} = (\not{p} + \eta_{\bar\lambda} m_{\bar\lambda})_{\alpha\beta}\end{aligned}$$

Finally, the reversed two-point function becomes,

$$\langle\Omega|\,\bar{\psi}_\beta(y)\psi_\alpha(x)\,|\Omega\rangle = \sum_{\bar{\lambda}}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{Z_{\bar{\lambda}}}{2p^0}(\not{p}+\eta_{\bar{\lambda}}m_{\bar{\lambda}})_{\alpha\beta}\,e^{ip\cdot(x-y)}$$

3 The Anti-Commutation Relations

Knowing that observable fields must either commute or anti-commute at space-like speration, we want to find out whether we are forced to impose commutation or anti-commutation relations on our spinor fields. Suppose x and y are space-like separated. First, assume that x and y are at equal times. Consider the the reversed two-point function,

$$\begin{aligned}\langle\Omega|\,\bar{\psi}_\beta(y)\psi_\alpha(x)\,|\Omega\rangle &= \sum_{\bar{\lambda}}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{Z_{\bar{\lambda}}}{2p^0}(\not{p}+\eta_{\bar{\lambda}}m_{\bar{\lambda}})_{\alpha\beta}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\\ &= \sum_{\bar{\lambda}}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{Z_{\bar{\lambda}}}{2p^0}(\gamma^0p^0-\vec{p}\cdot\vec{\gamma}+\eta_{\bar{\lambda}}m_{\bar{\lambda}})_{\alpha\beta}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\end{aligned}$$

Flipping the direction of the integrated out variable $\vec{p}$ we get,

$$\begin{aligned}\langle\Omega|\,\bar{\psi}_\beta(y)\psi_\alpha(x)\,|\Omega\rangle &= \sum_{\bar{\lambda}}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{Z_{\bar{\lambda}}}{2p^0}(2\gamma^0p^0-\gamma^0p^0+\vec{p}\cdot\vec{\gamma}+\eta_{\bar{\lambda}}m_{\bar{\lambda}})_{\alpha\beta}\,e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\\ &= \sum_{\bar{\lambda}}\left[\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}Z_{\bar{\lambda}}\gamma^0_{\alpha\beta}\,e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}-\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{Z_{\bar{\lambda}}}{2p^0}(\not{p}-\eta_{\bar{\lambda}}m_{\bar{\lambda}})_{\alpha\beta}\,e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right]\\ &= \sum_{\bar{\lambda}}Z_{\bar{\lambda}}\gamma^0_{\alpha\beta}\delta^3(\vec{x}-\vec{y})-\sum_{\bar{\lambda}}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{Z_{\bar{\lambda}}}{2p^0}(\not{p}-\eta_{\bar{\lambda}}m_{\bar{\lambda}})_{\alpha\beta}\,e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\end{aligned}$$

Therefore, since $Z_\lambda, Z_{\bar{\lambda}}\geq 0$ then we must have equal time anti-commutation relations. Furthermore, compare the above to,

$$\langle\Omega|\,\psi_\alpha(x)\bar{\psi}_\beta(y)\,|\Omega\rangle = \sum_{\bar{\lambda}}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{Z_\lambda}{2p^0}(\not{p}+\eta_\lambda m_\lambda)_{\alpha\beta}\,e^{-ip\cdot(x-y)}$$

If these terms are going to actually be minus eachother, there must be a spectral symmetry, $\lambda\leftrightarrow\bar{\lambda}$ which makes $Z_\lambda = Z_{\bar{\lambda}}$ and $m_\lambda = m_{\bar{\lambda}}$ and finally $\eta_{\bar{\lambda}} = -\eta_\lambda$. In that case,

$$\langle\Omega|\,\psi_\alpha(x)\bar{\psi}_\beta(y)\,|\Omega\rangle + \langle\Omega|\,\bar{\psi}_\beta(y)\psi_\alpha(x)\,|\Omega\rangle = \sum_\lambda Z_\lambda\gamma^0_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$

which fixes the vacuum expectation of equal-time anti-commutation relations,

$$\langle\Omega|\,\{\psi_\alpha(x),\bar{\psi}_\beta(y)\}\,|\Omega\rangle = \sum_\lambda Z_\lambda\gamma^0_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$

By swapping which we call the particle and which the anti-particle, we are free to choose $\eta_\lambda = 1$ and therefore $\eta_{\bar{\lambda}} = -1$.

4 The Feynman Propagator

I claim that the Feynman propagator can be written in the form,

$$\Delta_F(x-y)_{\alpha\beta} = \sum_\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon} Z_\lambda e^{-ip\cdot(x-y)}$$

Throughout the derivation, I will drop the sum over λ for notational convenience. I will prove this formula by integrating over p^0,

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon} Z_\lambda e^{-ip\cdot(x-y)} = \int \frac{i\,\mathrm{d}p^0}{2\pi} e^{-ip^0(x^0-y^0)} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} Z_\lambda \frac{(\not{p}+m)_{\alpha\beta}}{(p^0)^2-E_p^2+i\epsilon} e^{\vec{p}\cdot(\vec{x}-\vec{y})}$$

When $x^0 > y^0$ then we can close the contour below such that $-ip^0 < 0$ and thus $e^{-p^0(x^0-y^0)}$ is exponentially small. However, we can write the term,

$$\frac{\not{p}+m}{(p^0)^2-E_p^2+i\epsilon} = \frac{\not{p}+m}{(p^0-E_p+i\epsilon)(p^0+E_p-i\epsilon)}$$

Therefore, the residue at $p^0 = E_p - i\epsilon$ is,

$$\frac{\not{p}+m}{2E_p} e^{-iE_p(x^0-y^0)}$$

where $\not{p}$ is on shell. Therefore, by the residue theorem (remembering to use the factor $-2\pi i$ for a clockwise contour),

$$\begin{aligned}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon} Z_\lambda e^{-ip\cdot(x-y)} &\xrightarrow{x^0>y^0} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} Z_\lambda \frac{(\not{p}+m)_{\alpha\beta}}{2E_p} e^{-iE_p(x^0-y^0)} e^{\vec{p}\cdot(\vec{x}-\vec{y})} \\ &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{Z_\lambda}{2E_p} (\not{p}+m)_{\alpha\beta}\, e^{-ip\cdot(x-y)} = \langle\Omega|\, \psi_\alpha(x)\bar{\psi}_\beta(y)\, |\Omega\rangle\end{aligned}$$

Likewise, for $x^0 < y^0$ we close the contour above such that $-ip^0 > 0$ and thus $e^{-ip^0(x^0-y^0)}$ is exponentially small. The residue at $p^0 = -E_p + i\epsilon$ is given by,

$$\frac{-E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m}{-2E_p} e^{+iE_p(x^0-y^0)}$$

Therefore, by the residue theorem,

$$\begin{aligned}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon} Z_\lambda e^{-ip\cdot(x-y)} &\xrightarrow{x^0<y^0} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} Z_\lambda \frac{(-E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)_{\alpha\beta}}{2E_p} e^{+iE_p(x^0-y^0)} e^{\vec{p}\cdot(\vec{x}-\vec{y})} \\ &= -\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} Z_\lambda \frac{(E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} - m)_{\alpha\beta}}{2E_p} e^{iE_p(x^0-y^0)} e^{-\vec{p}\cdot(\vec{x}-\vec{y})} \\ &= -\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{Z_\lambda}{2E_p} (\not{p}-m)_{\alpha\beta}\, e^{ip\cdot(x-y)} = -\langle\Omega|\, \bar{\psi}_\beta(y)\psi_\alpha(x)\, |\Omega\rangle\end{aligned}$$

where I have changed variables $\vec{p} \to -\vec{p}$. Putting these results together and reintroducing the sum over states λ we find,

$$\sum_\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon} Z_\lambda e^{-ip\cdot(x-y)} = \langle\Omega|\, \mathbf{T}[\psi_\alpha(x)\bar{\psi}_\beta(y)]\, |\Omega\rangle = \begin{cases} \langle\Omega|\, \psi_\alpha(x)\bar{\psi}_\beta(y)\, |\Omega\rangle & x^0 > y^0 \\ -\langle\Omega|\, \bar{\psi}_\beta(y)\psi_\alpha(x)\, |\Omega\rangle & x^0 < y^0 \end{cases}$$

which is the Feynman propagator.