Contents

1 Irreducible Spaces

1.1 Irreducibility

Definition 1.1.1. A topological space X is *irreducible* if X is nonempty and whenever $X = Z_1 \cup Z_2$ for closed subsets $Z_1, Z_2 \subset X$ then either $Z_1 = X$ or $Z_2 = X$.

Lemma 1.1.2. Let X be a topological space. The following are equivalent,

(a) X is irreducible

(b) every nomepty open $U \subset X$ is dense

(c) any two nonempty opens $U_1, U_2 \subset X$ have nonempty intersection $U_1 \cap U_2$.

Proof. Let X be irreducible and suppose $U \subset X$ is open. Then $\overline{U} \cup U^C = X$ so either $\overline{U} = X$ or $U^C = X$ because both $\overline{U}, U^C \subset X$ are closed. Thus, if U is nonempty then $\overline{U} = X$.

Conversely, let $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed. Then $Z_1^C \subset Z_2$ so either Z_1^C is empty or dense so $Z_2 = X$ thus either $Z_1 = X$ or $Z_2 = X$ so X is irreducible.

Now (a) and (c) are equivalent because,

$$U_1 \cap U_2 = \varnothing \iff (U_1 \cap U_2)^C = X \iff U_1^C \cup U_2^C = X$$

So,

$$\begin{aligned}[][U_1, U_2 \neq \varnothing \implies U_1 \cap U_2 \neq \varnothing] &\iff [U_1 \cap U_2 = \varnothing \implies U_1 = \varnothing \text{ or } U_2 = \varnothing] \\ &\iff [U_1^C \cup U_2^C = X \implies U_1^C = X \text{ or } U_2^C = X]\end{aligned}$$

□

Lemma 1.1.3. Let $S \subset X$ be a subspace with the subspace topology. Then S is irreducible iff for any closed $Z_1, Z_2 \subset X$ such that $S \subset Z_1 \cup Z_2$ then either $S \subset Z_1$ or $S \subset Z_2$.

Proof. Suppose that S is irreducible. Then $\tilde{Z}_i = Z_i \cap S$ are closed in S and $S = \tilde{Z}_1 \cup \tilde{Z}_2$ so $S = \tilde{Z}_i$ i.e. $S \subset Z_i$ for some i.

Conversely, let $\tilde{Z}_1, \tilde{Z}_2 \subset S$ be closed such that $S = \tilde{Z}_1 \cup \tilde{Z}_2$. Then $\tilde{Z}_i = Z_i \cap S$ for some closed $Z_i \subset X$ because S has the subspace topology. Then $S \subset Z_1 \cup Z_2$ so $S \subset Z_1$ or $S \subset Z_2$ and thus $S = \tilde{Z}_1$ or $S = \tilde{Z}_2$ so S is irreducible. □

Remark. If $S \subset Y \subset X$ with the subspace topologies then,

$$S \text{ is “irreducible in } Y\text{”} \iff S \text{ is “irreducible in } X\text{”}$$

because irreducibility is an absolute property.

Explicitly, if S is “irreducible in Y” and $S \subset Z_1 \cup Z_2$ for $Z_1, Z_2 \subset X$ closed then $Z_1 \cap Y, Z_2 \cap Y \subset Y$ are closed and $S \subset (Z_1 \cap Y) \cup (Z_2 \cap Y)$ so $S \subset Z_1 \cap Y$ or $S \subset Z_2 \cap Y$ so $S \subset Z_1$ or $S \subset Z_2$ menaing S is “irreducible in X”. Conversely, if S is “irreducible in X” then if $S \subset Z_1 \cup Z_2$ for closed $Z_1, Z_2 \subset Y$ then there exist closed $Z_i' \subset X$ such that $Z_i = Z_i' \cap Y$ and $S \subset Z_1' \cup Z_2'$ so $S \subset Z_1'$ or $S \subset Z_2'$ and thus $S \subset Z_1$ or $S \subset Z_2$ showing that S is “irreducible in Y”.

Lemma 1.1.4. Let $U \subset X$ be open and $Z \subset X$ irreducible. Then $Z \cap U$ is irreducible iff $Z \cap U \neq \varnothing$.

Proof. If $Z \cap U = \varnothing$ then it is not irreducible by definition. Otherwise, assume $Z \cap U \neq \varnothing$ and suppose $Z \cap U \subset Z_1 \cup Z_2$ for closed subsets $Z_1, Z_2 \subset X$. Then $Z \subset Z_1 \cup Z_2 \cup U^C$ so $Z \subset Z_1$ or $Z \subset Z_2$ or $Z \subset U^C$ by irreducibility of Z and the previous lemma. However, $Z \not\subset U^C$ because $Z \cap U \neq \varnothing$ so $Z \subset Z_1$ or $Z \subset Z_2$ so by the above lemma $Z \cap U$ is irreducible. $\square$

Lemma 1.1.5. Let $Z \subset X$ be irreducible. Then $\overline{Z} \subset X$ is irreducible.

Proof. Suppose that $\overline{Z} = Z_1 \cup Z_2$ with Z_1 and Z_2 closed. Then $Z \subset Z_1 \cup Z_2$ so either $Z \subset Z_1$ or $Z \subset Z_2$. But since Z_1 and Z_2 are closed, we get $\overline{Z} = Z_1$ or $\overline{Z} = Z_2$. $\square$

1.2 Irreducible Components

Lemma 1.2.1. Increasing unions of irreducible subsets are irreducible.

Proof. Consider a chain T of irreducible subsets and consider,

$$U = \bigcup_{S \in T} S$$

Suppose $U = Z_1 \cup Z_2$ for closed subsets Z_1 and Z_2 of U. Then for each $S \in T$ we have $S \subset Z_1$ or $S \subset Z_2$. If for some $S_0 \in T$ we have $S_0 \not\subset Z_2$ (otherwise $Z_2 \supset U$ and we are done) then $S_0 \subset Z_1$ and for any $S \in T$ with $S \supset S_0$ we cannot have $S \subset Z_2$ else $S_0 \subset Z_2$. Therefore, $S \subset Z_1$. For any $S \in T$, since T is totally ordered, either $S \subset S_0$ in which case $S \subset Z_1$ or $S \supset S_0$ in which case $S \subset Z_1$ (as we have just shown). Therefore, $U \subset Z_1$ so U is irreducible. $\square$

Definition 1.2.2. Let X be a topological space then its irreducible components are the maximal irreducible subsets of X.

Remark. The irreducible subsets of X form a poset under inclusion. Furthermore, since chains have a maximum, by Zorn's lemma X always has some irreducible component.

Lemma 1.2.3. Let X be a topological space. The following hold,

(a) irreducible components are closed

(b) every irreducible subset of X is contained in some irreducible component

(c) the irreducible components of X cover X.

Proof. Let $C \subset X$ be an irreducible component. Then $\overline{C}$ is irreducible and $S \subset \overline{C}$ so $\overline{C} = C$ by maximality. Thus, C is closed. For any irreducible set $S \subset X$, Zorn's Lemma gives a maximal element in the irreducible components above S i.e. $S \subset C$ is contained in some irreducible component. In particular, since any point $x \in X$ is irreducible so $x \in C$ is contained in some irreducible component. Thus the irreducible components cover X. $\square$

Lemma 1.2.4. Noetherian spaces have finitly many irreducible components.

Proof. Let S be the poset of closed subspaces with infinitely many components ordered by inclusion. By the Noetherian hypothesis, descending chains in S have minima so, by Zorn's lemma, S has a minimum Z which has infinitely many irreducible components. Clearly, Z cannot be irreducible so we can write $Z = Z_1 \cup Z_2$ with $Z_1, Z_2 \subsetneq Z$ are proper closed subsets. By minimality, $Z_1, Z_2 \notin S$ and thus Z_1, Z_2 have finitely many irreducible components. Thus, $Z = Z_1 \cup Z_2$ has finitely many irreducible components so S is empty. $\square$

2 Quasi-Compactness and Noetherian Spaces

2.1 Noetherian Spaces

Definition 2.1.1. A topological space X is Noetherian if every descending chain of closed sets stabilizes.

Lemma 2.1.2. Subspaces of Noetherian subspaces are Noetherian.

Proof. Let $S \subset X$ with X noetherian. Then the closed sets of S are exactly $S \cap Z$ for $Z \subset X$ closed. Thus descending chains of closed sets in S stabilize. $\square$

Definition 2.1.3. A space is quasi-compact if every open cover has a finite subcover.

Lemma 2.1.4. Noetherian spaces are quasi-compact.

Proof. Let U_α be an open cover of X which is Noetherian. Then consider the poset A under inclusion of finite unions of the U_α all of which are open sets of X. Since X is Noetherian any ascending chain of opens must stabilize so any chain in A has a maximum. Then by Zorn's lemma A has a maximal element which must be X since the U_α form a cover. Therefore there exists a finite subcover. $\square$

Corollary 2.1.5. Every subset of a noetherian topological space is quasi-compact.

Definition 2.1.6. A continuous map $f : X \to Y$ is quasi-compact if for each quasi-compact open $U \subset Y$ then $f^{-1}(U)$ is quasi-compact open.

2.2 The Case for Schemes

Lemma 2.2.1. Affine schemes are quasi-compact.

Proof. Let U_i be an open cover of $\mathrm{Spec}\,(A_i)$. Since $D(f)$ for $f \in A$ forms a basis of the topology on $\mathrm{Spec}\,(A_i)$ we can shrink to the case $U_i = D(f_i)$. Then.

$$X = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D((\{f_i \mid i \in I\}))$$

And thus the ideal $I = (\{f_i \mid i \in I\})$ is not contained in any maximal ideal so $I = (1)$. Therefore, there are $f_1, \dots, f_n$ such that $a_1 f_1 + \cdots a_n f_n = 1$ and thus $(f_1, \dots, f_n) = (1)$ which implies that,

$$X = D((f_1, \dots, f_n)) = \bigcup_{i=1}^{n} D(f_i)$$

so X is quasi-compact. $\square$

Definition 2.2.2. A scheme X is *locally Noetherian* if for every affine open U the ring $\mathcal{O}_X(U)$ is Noetherian. X is *Noetherian* if it is quasi-compact and locally-Noetherian.

Lemma 2.2.3. If $(f_1, \dots, f_n) = A$ and A_{f_i} is Noetherian then A is Noetherian.

Proof. For any ideal $I \subset A$ we know $I_{f_i} \subset A_{f_i}$ is finitely generated. Clearing denominators and collecting the finite union of these finite generators gives a map $A^N \to I$ which is surjective when localized $A^N_{f_i} \twoheadrightarrow I_{f_i}$. Consider the A-module $K = \mathrm{coker}\,(A^N \to I)$ then for any $x \in K$ we have $f_i^{n_i} \cdot x = 0$ for each i but $f_i^{n_i}$ generate the unit ideal (since $D(f_i^{n_i}) = D(f_i)$ which cover $\mathrm{Spec}\,(A)$) so $x = 0$ to $A^N \twoheadrightarrow I$ so I is finitely generated showing that A is Noetherian. $\square$

Lemma 2.2.4. If X has an open affine cover $U_i = \text{Spec}\,(A_i)$ with A_i noetherian then X is locally noetherian. Moreover, if the cover can be made finite then X is noetherian.

Proof. Let $V = \text{Spec}\,(B) \subset X$ be an affine open, Then $V \cap U_i \subset V$ is open so it may be covered by principal opens $D(f_{ij}) \subset V \cap U_i$ for $f_{ij} \in B$. Since V is quasi-compact we may find a finite subcover. We need to show that $B_{f_{ij}}$ is Noetherian then since $D(f_{ij})$ cover V we use the lemma to conclude that B is Noetherian. However, $D(f_{ij}) \subset V \cap U_i$ can be covered by principal opens (of $U_i = \text{Spec}\,(A_i)$) $W_{ijk} \subset D(f_{ij}) \subset U_i = \text{Spec}\,(A_i)$ and each $(A_i)_{f_{ijk}}$ is Noetherian since A_i is, so using the same lemma we find that $B_{f_{ij}}$ is Noetherian.

Now suppose the cover is finite and let V_j be any open cover of X. We need to show X is quasi-compact so we must show that V_i has a finite subcover. Consider $U_i \cap V_j$ which is open in the affine $U_i = \text{Spec}\,(A_i)$ so it may be covered by principal opens $D(f_{ijk}) \subset U_i \cap V_j$. Now,

$$U_i = \bigcup_{j,k} D(f_{ijk})$$

but U_i is affine and thus quasi-compact so we may find an finite subcover which only uses finitely many V_i but the cover U_i of X is also finite so only finitely many V_i are needed to cover X. $\square$

Corollary 2.2.5. $X = \text{Spec}\,(A)$ is Noetherian iff A is a Noetherian ring.

Proof. If X is Noetherian then $\mathcal{O}_X(X) = A$ is a Noetherian ring (X is affine and thus quasi-compact). Conversely $\text{Spec}\,(A)$ is a finite Noetherian affine cover so X is Noetherian. $\square$

Remark. It is not the case that for a Noetherian scheme we must have $\mathcal{O}_X(X)$ a noetherian ring even for varieties. See http://sma.epfl.ch/ ojangure/nichtnoethersch.pdf.

Corollary 2.2.6. A Noetherian ring has finitely many minimal primes.

Proof. Let A be Noetherian then primes $\mathfrak{p} \in \text{Spec}\,(A)$ correspond to irreducible closed subsets $V(\mathfrak{p})$ and thus minimal primes correspond to irreducible components of $\text{Spec}\,(A)$. Therefore, since $\text{Spec}\,(A)$ is Noetherian, we see that $\text{Spec}\,(A)$ has finitely many irreducible components and thus finitely many minimal primes. $\square$

Lemma 2.2.7. If A is Noetherian then $\text{Spec}\,(A)$ is a Noetherian topological space.

Proof. Every descending chain of subsets is of the form $V(I_1) \supsetneq V(I_2) \supsetneq V(I_3) \supsetneq \cdots$ but the ideals,

$$\sqrt{I_1} \subsetneq \sqrt{I_2} \subsetneq \sqrt{I_3} \subsetneq \cdots$$

satbilize since A is Noetherian and thus so does the chain of closed subsets. $\square$

Lemma 2.2.8. If X is a Noetherian scheme then its underlying topological space is Noetherian.

Proof. Choose a finite covering $U_i = \text{Spec}\,(A_i)$ by Noetherian rings. Then for any descending chain of closed subsets $Z_1 \supsetneq Z_2 \supsetneq Z_3 \supsetneq \cdots$ we know $Z \cap U_i$ stabilizes at n_i since $\text{Spec}\,(A_i)$ is a Noetherian space. Thus, Z satibilizes at $\max n_i$ which exists since the cover is finite. $\square$

Remark. The converses of the above are false and so is X Noetherian. Let R be a non-Noetherian valuation ring for example.

Lemma 2.2.9. If X is locally Noetherian then any immersion $\iota : Z \hookrightarrow X$ is quasi-compact.

Proof. Closed immersions are affine and thus quasi-compact so it suffices okay to show that open immersions are quasi-compact. Let $j : U \to X$ be an open immersion. It suffices to check that $j^{-1}(U_i)$ is quasi-compact on an affine open cover $U_i = \text{Spec}(A_i)$ with A_i Noetherian. But $j : j^{-1}(U_i) \to U_i \cap U$ is a homeomorphism and $\text{Spec}(A_i)$ is a Noetherian topological space so every subset is quasi-compact and, in particular, $U_i \cap U$ is quasi-compact so $j^{-1}(U_i)$ is also. ☐

Remark. When X is Noetherian then it is a Noetherian space so any inclusion map $\iota : Z \hookrightarrow X$ for *any* subset $Z \subset X$ is quasi-compact since every subset is quasi-compact. In particular, every subset of X is retrocompact.

2.3 Quasi-Compact Morphisms

Lemma 2.3.1. A morphism $f : X \to Y$ is quasi-compact iff Y has a cover by affine opens V_i such that $f^{-1}(V_i)$ is quasi-compact.

Proof. Clearly if f is quasi-compact then any affine open cover V_i of Y satisfies $f^{-1}(V_i)$ is quasi-compact since V_i is a quasi-compact open by virtue of being affine open.

Now assume that such a cover exists. Let $U \subset Y$ be a quasi-compact open. Then U is covered by finitely may $V_1, \dots, V_n$. Then $U \cap V_i$ is open in V_i which is affine so it is covered by standard opens W_{ij}. Since U is quasi-compact then we can choose finitely many W_{ij}. Now $f^{-1}(V_i)$ is quasi-compact by assumption so it has a finite cover by affine opens,

$$f^{-1}(V_i) = \bigcup_{j=1}^{N} \tilde{V}_{ij}$$

Then $f : \tilde{V}_{ik} \to V_i$ is a morphism of affine schemes so $f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$ is a principal affine. Therefore,

$$f^{-1}(U) = \bigcup_{i=1}^{n} f^{-1}(V_i \cap U) = \bigcup_{i,j} f^{-1}(W_{ij}) = \bigcup_{i,j,k} f^{-1}(W_{ij}) \cap \tilde{V}_{ik}$$

is a finite union of principal affines each of which is quasi-compact so $f^{-1}(U)$ is quasi-compact. ☐

Proposition 2.3.2. *X is quasi-compact iff any morphism $X \to T$ for some affine scheme T is quasi-compact.*

Proof. If X is quasi-compact then $f : X \to T$ is quasi-compact since T is an affine open cover of itself and $f^{-1}(T)$ is quasi-compact. Conversely, if $f : X \to T$ is quasi-compact with T affine then T is quasi-compact open in T so $X = f^{-1}(T)$ is quasi-compact. ☐

Lemma 2.3.3. The base change of a quasi-compact morphism is quasi-compact.

Proof. (DO THIS) ☐

2.4 Affine Morphisms

Definition 2.4.1. A morphism $f : X \to Y$ is *affine* if the preimage of every affine open is affine.

Lemma 2.4.2. Every morphism of affine schemes is affine and thus quasi-compact.

Proof. Let $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ and $f : X \to Y$ be a morphism of affine schemes given by a ring map $\varphi : B \to A$. Then, any affine open $\mathrm{Spec}\,(C) = V \subset Y$ can be covered by principal opens $D(f_i)$ for $f_i \in B$. Note that under $\psi : B \to C$ we see that $D(f_i) = D(\psi(f_i))$ since $D(f_i) \subset \mathrm{Spec}\,(C)$. Since $D(\psi(f_i))$ cover $\mathrm{Spec}\,(C)$ then $\psi(f_i) \in C$ generate the unit ideal. Then we have $f^{-1}(D(f_i)) = D(\varphi(f_i))$ which is affine and $\varphi(f_i)$ generate the unit ideal of $\Gamma(f^{-1}(V), \mathcal{O}_X)$ so f^{-1} is affine. $\square$

Remark. An alternative proof goes as follows. Consider the pullback diagram,

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ X \arrow[r] & Y \end{tikzcd}$$

then open immersions are stable under base change so $f^{-1}(U) = U \times_Y X = \mathrm{Spec}\,(C \otimes_B A)$ if affine.

Remark. In fact, by Tag 01S8, a morphism $f : X \to S$ is affine iff X is relatively affine over S meaning $X = \mathbf{Spec}_S\,(\mathcal{A})$ for some quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$.

Lemma 2.4.3. Let $f : X \to Y$ be a morphism and W_i an affine open cover of Y such that $f^{-1}(W_i)$ is affine. Then f is affine.

Proof. Let $\mathrm{Spec}\,(A) = V \subset Y$ be affine open. Then $V_i = V \cap W_i$ is open in the affine open $V = \mathrm{Spec}\,(A)$ so it can be covered by principal opens $D(f_{ij}) \subset V \cap W_i$ for $f_{ij} \in A$. Since $f : f^{-1}(W_i) \to W_i$ is a morphism of affine schemes, the preimage of the affine open $D(f_{ij}) \subset V \cap W_i$ is affine $f^{-1}(D(f_{ij}))$ (note that $D(f_{ij}) \subset V \cap W_i$ is not necessarily a prinicpal affine open of W_i). But since $D(f_{ij})$ cover $\mathrm{Spec}\,(A)$ the $f_{ij} \in A$ generate the unit ideal and thus $f^{\#}(f_{ij}) \in \Gamma(f^{-1}(V), \mathcal{O}_X)$ generate the unit ideal and $(f^{-1}(V))_{f_{ij}} = f^{-1}(D(f_{ij}))$ is affine so $f^{-1}(V)$ is affine. $\square$

Lemma 2.4.4. The base change of an affine morphism is affine.

Proof. (DO THIS) $\square$

Lemma 2.4.5. Affine morphisms are quasi-compact.

Proof. If $f : X \to Y$ is affine then any affine open cover V_i of Y gives $f^{-1}(V_i)$ is affine and thus quasi-compact so f is quasi-compact. $\square$

2.5 Separatedness

Definition 2.5.1. A morphism $f : X \to Y$ with diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is,

(a) *separated* if the diagonal $\Delta_{X/Y}$ is a closed immersion

(b) *affine-separated* if the diagonal $\Delta_{X/Y}$ is affine

(c) *quasi-separated* if the diagonal $\Delta_{X/Y}$ is quasi-compact

Lemma 2.5.2. Any morphism of affine schemes is separated. Furthermore, affine morphisms are separated.

Proof. For a map $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(B)$ the diagonal is $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(A \otimes_B A)$ given by $A \otimes_B A \to A$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ which is surjective so the diagonal is a closed immersion. The second fact is Tag 01S7. $\square$

Lemma 2.5.3. The composition of (quasi/affine)-separated morphisms are (quasi/affine)-separated.

Proof. (DO THIS) ☐

Lemma 2.5.4. For any morphism $f : X \to Y$ the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is an immersion.

Proof. Let V_i be an affine cover of Y then choose an affine open cover U_{ij} of X with $f(U_{ij}) \subset V_i$. Then the diagonal of the affine map $U_{ij} \to V_j$ is $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ which is a closed immersion since it corresponds to $A_{ij} \otimes_{B_i} A_{ij} \to A_{ij}$ via $a_1 \otimes a_2 \mapsto a_1 a_2$ is surjective. Therefore $f : X \to Y$ is locally on X a closed immersion and thus an immersion. ☐

Remark. Therefore, to show that $f : X \to Y$ is separated, it suffices to show that the diagonal is closed (here equivalently meaning that the map or its image is closed).

Lemma 2.5.5. If X is Noetherian then every morphism $f : X \to S$ is quasi-compact and quasi-seperated.

Proof. Every subset of X is quasi-compact since X is (topologically) Noetherian. Then apply the first part to the diagonal $\Delta_{X/S} : X \to X \times_S X$ which is then quasi-compact and thus $f : X \to S$ is quasi-separated. ☐

Lemma 2.5.6. Let $f : X \to S$ be affine-separated/quasi-separated with $S = \mathrm{Spec}\,(A)$ affine. Then for any two affine opens $U, V \subset X$ the intersection $U \cap V$ is affine/quasi-compact.

Proof. Consider the pullback diagram,

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d, hook] & U \times_S V \arrow[d, hook] \\ X \arrow[r, "\Delta_{X/S}"] & X \times_S X \end{tikzcd}$$

where $U \cap V = \Delta_{X/S}(U \times_S V)$ using the basechange of an open immersion is an open immersion. Then since S is affine, $U \times_S V$ is affine and thus quasi-compact open of $X \times_S X$. Then if f is affine-separated then $\Delta_{X/S}$ is affine so $U \cap V = \Delta_{X/S}(U \times_S V)$ is affine. If f is quasi-separated then $\Delta_{X/S}$ is quasi-compact so $U \cap V = \Delta_{X/S}(U \times_S V)$ is quasi-compact. ☐

Remark. In the separated case, we see that $U \cap V$ is affine and $\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.

Remark. Tag 01KO gives a generalization of this lemma. For the separated case see Tag 01KP.

Lemma 2.5.7. Let $f : X \to Y$ be quasi-compact and quasi-separated and $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module then $f_* \mathscr{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

Proof. Sinsce this is local on Y we can restrict to the case that Y is affine. Then $X = f^{-1}(Y)$ is quasi-compact (when Y is not affine $f^{-1}(V)$ will be quasi-compact) so take a finite affine open cover U_i and since $f : X \to Y$ is quasi-seperated over an affine then by the above lemma $U_i \cap U_j$ is quasi-compact so it has a finite affine open cover U_{ijk}. Then, by the sheaf property, there is an exact sequence of sheaves on Y

$$0 \longrightarrow f_* \mathscr{F} \longrightarrow \bigoplus_i f_*(\mathscr{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} f_*(\mathscr{F}|_{U_{ijk}})$$

which works because these are finite sums. However, $f : U_{ijk} \to Y$ is a morphism of affine schemes and since $\mathscr{F}$ is quasi-coherent we have $\mathscr{F}|_{U_{ijk}} = \widetilde{M_{ijk}}$ so $f_*(\mathscr{F}|_{U_{ijk}}) = \widetilde{M_{ijk}}$ as an $\mathcal{O}_Y(Y)$-module. Thus, $f_*\mathscr{F}$ is a kernel of quasi-coherent $\mathcal{O}_Y$-modules and thus is quasi-coherent. ☐

Remark. If X is Noetherian then $f : X \to Y$ is automatically quasi-compact and quasi-separated so there is no issue in the above lemma.

3 Sober Spaces

Definition 3.0.1. A topological space is T_0 if for each pair of distinct points there is a neighborhood of one that does not contain the other.

Proposition 3.0.2. All schemes are T_0.

Proof. Let X be a scheme and $x, y \in X$ distinct points. If x and y lie in different affine opens then this is an open seperation. If x, y lie in the same affine open $U = \text{Spec}\,(A)$ then they correspond to distinct prime ideals $\mathfrak{p}, \mathfrak{q} \subset A$. Since $\mathfrak{p} \neq \mathfrak{q}$ there exists some element of one that is not in the other. Without loss of generality suppose that there is some $f \in \mathfrak{p}$ with $f \notin \mathfrak{q}$. Thus, $\mathfrak{q} \in D(f)$ and $\mathfrak{p} \notin D(f)$ so x and y are seperated by some open $D(f) \subset U \subset X$. ☐

Definition 3.0.3. A *generic point* $\xi \in Z$ of a closed irreducible set Z is such that $\overline{\{\xi\}} = Z$.

Proposition 3.0.4. Let X be a toplogical space and $\xi \in X$ then $\overline{\{\xi\}}$ is a closed irreducible set with generic point ξ.

Proof. Clearly, $\overline{\{\xi\}}$ is closed. Suppose that $\overline{\{\xi\}} \subset Z_1 \cup Z_2$ then $\xi \subset Z_1$ or $\xi \subset Z_2$ and thus $\overline{\{\xi\}} \subset Z_1$ or $\overline{\{\xi\}} \subset Z_2$ so $\overline{\{\xi\}}$ is irreducible. Clearly, ξ is a generic point of $\overline{\{\xi\}}$. ☐

Definition 3.0.5. A topological space is *sober* if every irreducible closed set has a unique generic point.

Proposition 3.0.6. Any Hausdorff space is sober.

Proof. Let Z be irreducible and closed. Suppose that Z has more than one point. Take distinct $x, y \in Z$ and, using the Hausdorff property, open sets $x \in U$ and $y \in V$ such that $U \cap V = \varnothing$. Now consider $Z_1 = Z \cap U^C$ and $Z_2 = Z \cap V^C$ which are closed in Z proper because $x \notin Z_1$ and $y \notin Z_2$. Furthermore, $Z_1 \cup Z_2 = Z \cap (U^C \cup V^C) = Z \cap (U \cap V)^C = Z$ so Z cannot be irreducble. Thus, the only irreducible sets are points which clearly have a unique generic point because all points in a T_2 space are closed. ☐

Lemma 3.0.7. Any prime $\mathfrak{p} \in \text{Spec}\,(A)$ in an affine scheme satisfies $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

Proof. Any closed set in $\text{Spec}\,(A)$ is of the form $V(I)$ for some ideal $I \subset A$. Consider the closed sets $\mathfrak{p} \in V(I)$ containing $\mathfrak{p}$ which correspond to $\mathfrak{p} \supset I$. Clearly, $\mathfrak{p} \in V(\mathfrak{p})$ and if $\mathfrak{p} \in V(I)$ then $V(\mathfrak{p}) \subset V(I)$ since $\mathfrak{p} \supset I$. Therefore $V(\mathfrak{p})$ is the closure of $\mathfrak{p}$. ☐

Lemma 3.0.8. Every closed irreducible set of an affine scheme $\text{Spec}\,(A)$ is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p} \subset A$.

Proof. First, all closed subsets of $\text{Spec}\,(A)$ are of the form $V(I)$. First, if $I = \mathfrak{p}$ is prime and $V(\mathfrak{p}) \subset V(I_1) \cup V(I_2) = V(I_1 I_2)$ then $\mathfrak{p} \supset I_1 I_2$. However, since $\mathfrak{p}$ is prime we have either $\mathfrak{p} \supset I_1$ or $\mathfrak{p} \subset I_2$ so $V(\mathfrak{p}) \subset V(I_1)$ or $V(\mathfrak{p}) \subset V(I_2)$ proving that $V(\mathfrak{p})$ is irreducible. Conversely, if $V(I)$ is irreducible then take $x, y \in A$ such that $xy \in \sqrt{I}$ and thus,

$$\sqrt{(xy)} \subset \sqrt{I} \implies V(I) \subset V((xy)) = V((x)) \cup V((y))$$

Since $V(I)$ is irreducible we must have either $V(I) \subset V((x))$ or $V((y)) \subset V(I)$ which implies that $\sqrt{(x)} \subset \sqrt{I}$ or $\sqrt{(y)} \subset \sqrt{I}$. Therefore, $x \in \sqrt{I}$ or $y \in \sqrt{I}$ so $\sqrt{I}$ is prime and $V(I) = V(\sqrt{I})$. $\square$

Proposition 3.0.9. Any scheme is sober.

Proof. First consider the affine case $X = \text{Spec}\,(A)$. Any irreducible closed set in X is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p} \subset A$. Thus $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ is the unique generic point. Now let X be any scheme and $Z \subset X$ a closed irreducible subset. X has a cover by affine opens so take some affine open U which intersects Z. Since U is an affine scheme and $U \cap Z$ is a closed irreducible subet of U there exists a unique generic point $\xi \in U \cap Z$. Because Z is closed in X we then have $Z \cap U \subset \overline{\{\xi\}} \subset Z$. However, $Z \cap U$ is open in Z and $\overline{\{\xi\}}$ is closed in Z, an irreducible, which implies that either $U \cap Z$ is empty (which is false by assumption) or $\overline{\{\xi\}} = Z$. Thus Z has a generic point ξ. Suppose that $\xi, \xi' \in Z$ were both generic points then both must be limit points of each other and thus have exactly the same open neighborhoods contradicting the fact that $Z \subset X$ is T_0. $\square$

3.1 Specialization

Definition 3.1.1. Let X be a topological space and $\xi_1, \xi_2 \in X$. We write $\xi_1 \rightsquigarrow \xi_2$ if $\xi_2 \in \overline{\{\xi_1\}}$ i.e if ξ_2 is a limit point of ξ_1. We say ξ_1 is a *generalization* of ξ_2 and ξ_2 is a *specialization* of ξ_1.

4 Dimension Theory

4.1 Introduction

Definition 4.1.1. Let X be a topological space. The *Krull dimension* or *combinatorial dimension* of X is the maximal length of chains of irreducible closed subsets,

$$\dim(X) = \max\{n \in \mathbb{Z} \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ is a chain of closed irreducible subsets } Z_i \subset X\}$$

and $\dim X = \infty$ if there is no maximum and $\dim X = -\infty$ if X is empty.

Definition 4.1.2. For $x \in X$ we define the dimension at x as,

$$\dim_x(X) = \inf_{x \in U} \dim\,(U)$$

taken over open neighborhoods U of x.

Remark. For any subset $S \subset X$, if $Z \subset S$ is closed irreducible then $\overline{Z} \subset X$ is closed irreducible so we get an inclusion of chains in S to chains in X. Thus,

$$\dim S \leq \dim X$$

Definition 4.1.3. Let $Z \subset X$ be a closed irreducible subset. Then,

$$\text{codim}\,(Z, X) = \sup\{n \in \mathbb{Z} \mid Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ is a chain of closed irreducible subsets } Z_i \subset X\}$$

and for any closed subspace $Y \subset X$ we define,

$$\text{codim}\,(Y, X) = \inf_{Z \subset Y} \text{codim}\,(Z, X)$$

over $Z \subset Y \subset X$ closed irreducible subsets in X. Furthermore, for any subspace $S \subset X$ we may define,

$$\text{codim}\,(S, X) = \text{codim}\left(\overline{S}, X\right)$$

Proposition 4.1.4. For any subspace $Y \subset X$,

$$\dim(X) \geq \text{codim}\,(Y, X) + \dim(Y)$$

Proof. Let $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ be a maximal chain of closed irreducible subset of Y realizing $\dim(Y)$. Then taking closures gives a chain of irreducible closed subsets of X conainted in $\overline{Y}$. Then choose a maximal chain $\tilde{Z}_i$ realizing $\text{codim}\left(\overline{Z_n}, X\right)$ to give a chain,

$$\overline{Z_0} \subsetneq \cdots \subsetneq \overline{Z_n} = \tilde{Z}_0 \subsetneq \tilde{Z}_1 \subsetneq \cdots \subsetneq \tilde{Z}_k$$

Therefore, $n + k \leq \dim(X)$. However, $n = \dim(Y)$ and because $\overline{Z_n} \subset \overline{Y}$ we have,

$$k = \text{codim}\left(\overline{Z_n}, X\right) \geq \text{codim}\,(Y, X)$$

and thus,

$$\dim(X) \geq n + k \geq \text{codim}\,(Y, X) + \dim(Y)$$

$\square$

Lemma 4.1.5. If $Z \subset X$ is irreducible and U is open and $U \cap Z \neq \varnothing$ then $Z \cap U$ is irreducible. Furthermore, if $Z \subset X$ is irreducible then $\overline{Z}$ is irreducible.

Proof. If we have closed $Z_1, Z_2 \subset X$ with $Z_1 \cup Z_2 \supset Z \cap U$ then $Z_1 \cup Z_2 \cup U^C \supset Z$ so one must cover Z since it is irreducible but $Z \not\subset U^C$ so either $Z_1 \supset Z \cap U$ or $Z_2 \supset Z \cap U$.

Likewise, for closed $Z_1, Z_2 \subset X$ with $Z_1 \cup Z_2 \supset \overline{Z} \supset Z$ then by irreducibility $Z_1 \supset Z$ or $Z_1 \supset Z$ but these are closed so $Z_1 \supset \overline{Z}$ or $Z_2 \supset \overline{Z}$. $\square$

Lemma 4.1.6. Consider a closed subset $Y \subset X$ and an open $U \subset X$ with $U \cap Z \neq \varnothing$ for each irreducible component $Z \subset Y$. Then $\text{codim}\,(Y, X) = \text{codim}\,(Y \cap U, U)$.

Proof. Consider a chain of irreducibles $Z_i \supsetneq Z_{i+1}$ with $Z_0 \subset Y$. I claim that $Z_i \mapsto Z_i \cap U$ and $Z_i \mapsto \overline{Z_i}$ are inverse functions giving a bijection between closed irreducible chains in X with final terms containined in Y and closed irreducible chains in U with final term contained in $Y \cap U$. Note, if $Z_i \subset Y \cap U$ then $\overline{Z_i} \subset Y$ since Y is closed in X. Furthermore, $Z_i \mapsto Z_i \cap U$ remains irreducible if it is nonempty. The chain Z_i realizing $\text{codim}\,(Y, X)$ must begin an irreducible component of Y so we have indeed that $Z_i \cap U \neq \varnothing$.

First, $\overline{Z_i \cap U} \subset Z_i$ and is closed in X. Then $\overline{Z_i \cap U} \cup U^C \supset Z_i$ so because Z_i is irreducible $\overline{Z_i \cap U} = Z_i$ since by assumption $Z_i \not\subset U^C$. Conversely, if $Z_i \subset U$ is a closed irreducible subset then $\overline{Z_i}$ is closed and irreducible in X and $Z_i \subset \overline{Z_i} \cap U$ but $Z_i = C \cap U$ for closed $C \subset X$ so $Z_i \subset C$ and thus $\overline{Z_i} \subset C$ so $\overline{Z_i} \cap U \subset C \cap U = Z_i$ meaning $Z_i = \overline{Z_i} \cap U$. Thus we have shown these operations are inverse to eachother.

Finally, if $Z_i \cap U = Z_{i+1} \cap U$ then $\overline{Z_i \cap U} = \overline{Z_i \cap U}$ so $Z_i = Z_{i+1}$ so the chain does not degenerate. Likewise, if $\overline{Z_i} = \overline{Z_{i+1}}$ then $\overline{Z_i} \cap U = \overline{Z_{i+1}} \cap U$ so $Z_i = Z_{i+1}$. Therefore, we get a length-preserving bijection between the chains defining $\text{codim}\,(Y, X)$ and $\text{codim}\,(Y \cap U, U)$. $\square$

4.2 Equidimensionality

Proposition 4.2.1. Let X be a topological space and Z_i its irreducible components. Then,

$$\dim(X) = \sup_{i \in I} \dim(Z_i)$$

Proof. Clearly, $\dim(X) \geq \dim(Z_i)$. Furthermore, choose a maximal chain of closed irreducible subsets of X,

$$W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n$$

Since W_n is irreducible, we must have $W_n \subset Z_i$ for some $i \in I$ so this is a chain in Z_i showing that,

$$\dim(Z_i) \geq \dim(X)$$

□

Definition 4.2.2. We say that X is *equidimensional* if $\dim(Z) = \dim(X)$ for any irreducible component $Z \subset X$.

Remark. Equidimensionality is equivalent to: all irreducible components have the same dimension.

Proposition 4.2.3. Let X be a topological space. Then,

$$\dim(X) = \sup_{x \in X} \dim_x(X)$$

Proof. Clearly $\dim(X) \geq \dim_x(X)$. Furthermore, choose a maximal chain of closed irreducibles,

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

and choose a point $x \in Z_0$. Then for any open neighborhood $x \in U$ we see that,

$$Z_0 \cap U \subsetneq Z_1 \cap U \subsetneq \cdots \subsetneq Z_n \cap U$$

is a chain of closed irreducible subsets of U (since all are nonempty because they contain x). Thus $\dim_x(X) \geq \dim(X)$. □

Definition 4.2.4. A space X is *equicodimensional* if $\operatorname{codim}(x, X) = \dim(X)$ for every point $x \in X$.

Definition 4.2.5. A space X is *biequidimensional* if every maximal chain of closed irreducible subsets has length $\dim(X)$.

Remark. If X is biequidimensional this clearly implies X is equidimensional, equicodimensional, and catenary but the converse is false in general. However, the converse holds if X is finite dimensional and irreducible [Emerton and Gee, Lem. 2.32] (https://arxiv.org/pdf/1704.07654v2.pdf).

Lemma 4.2.6. If X is biequidimensional then for any closed subset $Y \subset X$,

$$\dim(X) = \operatorname{codim}(Y, X) + \dim(Y)$$

Proof. Choose a chain of closed irreducibles achieving $\operatorname{codim}(Y, X)$ and thus terminating at some $Z \subset Y$. Then this chain may be extended to a maximal chain by adding irreducible closed subsets of Y (since closed subsets of Y are closed in X since Y is closed). By biequidimensionality, all such maximal chains have length $\dim(X)$ and thus,

$$\dim(X) \leq \operatorname{codim}(Y, X) + \dim(Y)$$

which along with the reverse innequality (which holds generally) proves the claim. □

4.3 Catenary Spaces

Definition 4.3.1. A topological space X is *catenary* if for every pair $Z \subset Z'$ of closed irreducible subsets,

(a) $\text{codim}\,(Z, Z') < \infty$

(b) every maximal chain $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = Z'$ has the same length.

Lemma 4.3.2. Let X be a topological space. Then the following are equivalent,

(a) X is catenary

(b) for any triple of irreducible closed subsets $Z_1 \subset Z_2 \subset Z_3$,

$$\text{codim}\,(Z_1, Z_3) = \text{codim}\,(Z_1, Z_2) + \text{codim}\,(Z_2, Z_3)$$

and $\text{codim}\,(Z_1, Z_3)$ is finite.

4.4 Catenary Rings

Definition 4.4.1. We say a ring A is *catenary* if $\text{Spec}\,(A)$ is catenary as a a topological space. Explicitly, A is catenary if for all pairs of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ all chains of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}'$$

can be extended to a maximal chain and all maximal chains have the same length.

Definition 4.4.2. A Noetherian ring A is *universally catenary* if every finite type A-algebra is catenary.

Proposition 4.4.3. If A is one of the following,

(a) a field

(b) a Dedekind domain

(c) a localization of a univerally catenary ring

then A is universally catenary.

Example 4.4.4. There exist Noetherian rings of dimension two which are not universally catenary and thus there exist non catenary Noetherian rings. For an example see Tag 02JE.

4.5 Dimension Theory of Schemes

Lemma 4.5.1. Let $Z \subset X$ be a closed irreducible subset with generic point $\xi \in Z$. Then,

$$\text{codim}\,(Z, X) = \dim \mathcal{O}_{X,\xi}$$

Proof. Take affine open neighborhood $\xi \in U = \text{Spec}\,(A) \subset X$. Then for $\mathfrak{p} \in \text{Spec}\,(A)$ corresponding to ξ we get $A_{\mathfrak{p}} = \mathcal{O}_{X,\xi}$. However, $\text{codim}\,(Z, X) = \text{codim}\,(Z \cap U, U)$ and $Z \cap U = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Therefore,

$$\text{codim}\,(Z, X) = \text{codim}\,(Z \cap U, U) = \mathbf{ht}\,(\mathfrak{p}) = \dim A_{\mathfrak{p}} = \dim \mathcal{O}_{X,\xi}$$

□

4.6 Dimension Theory for Finite Type k-Schemes

5 General Easy Facts

Remark. In a T_0 space, generic points, if they exist, are unique.

Proposition 5.0.1. Let X be a topological space and $Z \subset X$ a closed irreducible subset. Then $\text{codim}\,(Z, X)$ if and only if Z is an irreducible component.

Proof. By definition, $\text{codim}\,(Z, X) = 0$ if and only if Z is a maximal irreducible closed subset. The irreducible components are the maximal irreducible sets and are closed and thus exactly the maximal closed irreducible subsets. □

Proposition 5.0.2. Let X be an irreducible T_0 space with $\dim X = 0$. Then $X = \{x\}$.

Proof. By definition X is nonempty. Then for each $x \in X$ consider $\overline{\{x\}} \subset X$ but $\overline{\{x\}}$ is closed and irreducible so because $\dim X = 0$ we have $X = \overline{\{x\}}$ for each $x \in X$. Since generic points are unique in a T_0 space we have $X = \{x\}$. □

Proposition 5.0.3. Let X be a T_0 space with $\dim X$ finite. Let $Z \subset X$ a closed irreducible subset with $\text{codim}\,(Z, X) = \dim X$ then $Z = \{x\}$.

Proof. Notice that,

$$\dim X \geq \text{codim}\,(Z, X) + \dim Z$$

and therefore $\dim Z = 0$. Therefore Z is a minimal closed irreducible subset. Suppose that $x \in Z$ then $Z' = \overline{\{x\}} \subset Z$ because Z is closed and Z' is also closed and irreducible so $Z' = Z$ by minimality. Since generic points are unique we see that Z contains a unique point which is thus closed. □

Lemma 5.0.4. Let X be a sober space and $Y \subset X$ a closed subspace. The following are equivalent,

(a) $\text{codim}\,(Y, X) = 0$

(b) Y contains the generic point of some irreducible component of X

(c) Y contains some irreducible component of X.

Proof. Suppose Y contains $\xi \in Z$ the generic point of an irreducible component. Then because Y is closed $Z \subset Y$ and the converse is obvious. In this case,

$$\text{codim}\,(Y, X) \leq \text{codim}\,(Z, X) = 0$$

because Z is maximal and thus $\text{codim}\,(Y, Z) = 0$. Conversely, suppose that $\text{codim}\,(Y, X) = 0$ then there is some closed irreducible $Z \subset Y$ such that $\text{codim}\,(Z, X) = 0$ menaing that Z is an irreducible component. □