# REU Notes

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## 1 Introduction

Suppose $f \in \mathbb{Z}[X_0, \cdots, X_n]$ is an integral polynomial in $n$-variables. We want to find relations between the complex solutions to $f = 0$ and the solutions of $f$ over $\mathbb{Z}, \mathbb{Q}, \mathbb{F}_{p^k}$. We want to find out how many solutions to the system,

$$f_1(x_0, \cdots, x_n) = 0$$
$$\vdots$$
$$f_r(x_0, \cdots, x_n) = 0$$

over $\mathbb{F}_q[x_0, \cdots, x_n]$. How many solutions does this system have over $\mathbb{F}_{q^k}$.

**Definition:** $X(\mathbb{F}_{q^k})$ is the set of solutions to $\forall i : f_i = 0$ in $(\mathbb{F}_{q^k})^{n+1}$.

Weil defined the zeta function,

**Definition:**

$$\zeta_X(t) = \exp\left[\sum_{r=1}^{\infty} \frac{\#|X(\mathbb{F}_{q^r})|}{r} t^r\right]$$

and made the following conjectures which were later proven,

**Theorem 1.1.** *The following hold about $\zeta_X$ for a variety $X$,*

1. *$\zeta_X(t)$ is a rational function.*
2. *If $f_i$ are homogenous polynomials define the projective solutions,*

$$X'(\mathbb{F}_{q^k}) = \frac{X(\mathbb{F}_{q^k}) \backslash \{\vec{0}\}}{\mathbb{F}_{q^k}^{\times}}$$

*and $\zeta_{X'}(t)$ satisfied a function equation if $X'$ is smooth.*

3. *All roots and poles $\zeta_X$ in $\mathbb{C}$ are $q$-Weil numbers e.g. if $\alpha$ is a root or pole then $|\alpha| = q^{i/2}$ for some $i \in \mathbb{Z}$.*
4. *If $X$ is smooth and projective then*

$$\zeta_X(t) = \prod_{i=0}^{d} p^i(t)^{(-1)^{i+1}}$$

*where $d = \dim X$ and $p_i \in \mathbb{Z}[t]$. For almost all primes $\deg p_i = b_i(X(\mathbb{C}))$ and the roots of $p_i$ have absolute value $q^{i/2}$ where $b_i$ is the $i^{\text{th}}$ Betti number.*

***Definition:*** *Suppose $f(x_0, x_n)$ is a homogeneous polynomial in $\mathbb{F}_q[x_0, \cdots, x_n]$ then,*

$$X(\mathbb{F}_{q^k} = \frac{\{\vec{x} \in (\mathbb{F}_{q^k})^{n+1} \mid f(\vec{x}) = 0\} \backslash \{\vec{0}\}}{\mathbb{F}_{q^k}^{\times}}$$

***Definition:*** *A variety $X$ is supersigular if $\zeta_X$ has the property that every root and pole is of the form $\omega q^{i/2}$ where $\omega$ is a root of unity.*

*There are may conjectures about supersigular varieties.*

***Definition:*** *A 2-dimensional hypersurface is unirational if there exist $x_i(t_1, t_2) \in \overline{\mathbb{F}_q}(t_1, t_2)$ such that $f(x_0(t_1, t_2), \cdots, x_3(t_1, t_2)) = 0$.*

**Conjecture 1.2** (Shioda)**.** *A supersingular 2-dimensional hypersurface is unirational*

**Conjecture 1.3** (Tate)**.** *Supersingular surjaces have a lot of curves in them!*