Commutative Algebra Facts for Algebraic Geometry

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Contents

Remark. Unless otherwise stated, all rings are commutative and unital.

1 Definitions

Definition 1.0.1. An element $p \in A$ is prime if (p) is a prime ideal. Equivalently p is prime if whenever $p \mid xy$ either $p \mid x$ or $p \mid y$.

Definition 1.0.2. An element $r \in A$ which is nonzero and not a unit is irreducible if whenever $r = xy$ either $x \in A^\times$ or $y \in A^\times$.

2 Domains

Definition 2.0.1. A ring A is a domain if A has no zero divisors i.e. if $ab = 0$ then $a = 0$ or $b = 0$.

Proposition 2.0.2. Let A be a domain then any nonzero prime element is irreducible.

Proof. Let $p \in A$ be a prime. Now suppose that $p = xy$ for $x, y \in A$. Thus, $p \mid xy$ so (WLOG) we have $p \mid x$ so $x = pz$ and thus $p = pzy$. However, p is nonzero and A is a domain so $zy = 1$ and thus $y \in A^\times$ proving that p is irreducible. □

3 Principal Ideal Domains

Definition 3.0.1. A principal ideal domain (PID) is a domain A such that every ideal is principal.

Lemma 3.0.2. If A is a PID then A is Noetherian.

Proof. Every ideal is principal and thus finitely generated. □

Lemma 3.0.3. Let A be a PID and $r \in A$ irreducible then (r) is maximal and thus r is prime.

Proof. Consider an intermediate ideal $(r) \subset J \subset A$ then since A is a PID we have $J = (a)$ so $r \in (a)$ and thus $r = ac$ so either $a \in A^\times$ in which case $J = A$ or $c \in A^\times$ in which case $J = (r)$ so (r) is maximal and thus a prime ideal. □

Theorem 3.0.4. Let A be a PID and not a field then $\dim A = 1$.

Proof. Any prime ideal $\mathfrak{p} \subset A$ is principal so $\mathfrak{p} = (p)$ and p is prime. Either $p = 0$ which is prime since A is a domain or p is irreducible and so we have shown (p) is maximal. So every prime ideal is zero or maximal and thus $\dim A \le 1$. If $\dim A = 0$ then (0) is maximal so A is local and any nonzero element is thus invertible so A is a field. □

Theorem 3.0.5 (Kaplansky)**.** Let A be Noetherian then A is a principal ideal ring iff every maximal ideal is prime.

Theorem 3.0.6 (Cohen)**.** A ring A is Noetherian iff every prime ideal is finitely generated.

Corollary 3.0.7. A ring A is a principal ideal ring iff every prime ideal is principal.

4 Unique Factorization Domains

Definition 4.0.1. A domain A is a unique factorization domain (UFD) if every nonzero element has a unique factorization into irreducible elements.

Definition 4.0.2. A factorization ring A is a ring such that every nonzero element has a factorization into irreducible elements.

Lemma 4.0.3. If A is a Noetherian domain then it is a factorization domain.

Proof. Take $a_0 \in A$. If a is irreducible, zero, or a unit then we are done. Then we can write, $a = a_1^{(1)} a_2^{(1)}$ for $a_1, b_1 \notin A^\times$. Continuing in this manner we get,

$$(a) \subsetneq (a_1^{(1)}, a_2^{(1)}) \subsetneq (a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, a_4^{(3)}) \subsetneq \cdots$$

(CHECK THIS) This sequence is proper since if $a = bc$ and $b \in (a)$ then $a = arc$ so $rc = 1$ and thus $c \in A^\times$ contradicting our construction. However, A is Noetherian then the sequence must terminate so at some point the factorization must become irreducible. □

Theorem 4.0.4. Let A be a factorization domain. Then A is a UFD iff every irreducible is prime.

Proof. If A is a UFD and p an irreducible. Let $x, y \in A$ and $p \mid xy$ then p is in the factorization of xy and thus, by uniqueness must be in the factorization of either x or y so $p \mid x$ or $p \mid y$.

Conversely, if A is a factorization domain and every irreducible is prime then given two factorizations of x each irreducible must, by primality, divide an irreducible in the other factorization so they are equal. (DO THIS BETTER) □

Corollary 4.0.5. If A is a PID then A is a UFD.

Proof. If A is a PID then it is Noetherian and thus a factorization domain. Furthermore, every irreducible is prime so A is a UFD. □

4.1 Height One Prime Ideals

Proposition 4.1.1. Let A be Noetherian. Then any principal prime ideal has height at most one.

Proof. Let $\mathfrak{p} = (p) \subset A$ be a principal prime ideal. Then consider the localization which is $A_{(p)}$ Noetherian and the unique maximal ideal $pA_{(p)}$ is principal. Take $N = \text{nilrad}\left(A_{(p)}\right)$ then,

$$\dim A_{(p)}/N = \dim A_{(p)} = \mathbf{ht}\left(\mathfrak{p}\right)$$

but $A_{(p)}/N$ is a Noetherian domain and the unique maximal ideal $pA_{(p)}$ is principal so $A_{(p)}/N$ is a PID and thus $\dim A_{(p)}/N \leq 1$. $\square$

Proposition 4.1.2. If A is a UFD then every prime ideal of height one is principal.

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal with $\mathbf{ht}\left(\mathfrak{p}\right) = 1$. Take any nonzero element $x \in \mathfrak{p}$ and consider its factorization into irreducibles. Since $\mathfrak{p}$ is prime some irreducible factor $p \mid x$ must be in $\mathfrak{p}$ so $(p) \subset \mathfrak{p}$. Since A is a UFD all irreducibles are prime so $(p) \subset \mathfrak{p}$ is prime. However $\mathbf{ht}\left(\mathfrak{p}\right) = 1$ and $(p) \neq (0)$ so $(p) = \mathfrak{p}$ and thus $\mathfrak{p}$ is principal. $\square$

Theorem 4.1.3. Let A be a Noetherian domain. Then A is a UFD iff every height one prime ideal is principal.

Proof. We showed one direction above. Conversely, suppose every height one prime ideal is principal. Since A is a Noetherian domain, it suffices to show that each irreducible is prime. Let r be irreducible and consider a minimal prime $\mathfrak{p} \supset (r)$. Then by Krull's Hauptidealsatz, $\mathfrak{p}$ has height one so by our assumption $\mathfrak{p} = (p)$ is principal. However, $(r) \subset (p)$ so $p \mid r$ but r is irreducible so we must have $(r) = (p) = \mathfrak{p}$ and thus r is prime. $\square$

Theorem 4.1.4 (Krull's Hauptidealsatz)**.** Let $I \subset A$ be an ideal in a Noetherian ring A with n generators then any minimal prime ideal $\mathfrak{p} \supset I$ has height at most n.

5 Simple Modules

Definition 5.0.1. A nonzero R-module is *simple* if it has no nontrivial submodules.

Proposition 5.0.2. Let R be a ring and M an R-module. Then the following are equivalent,

(a) M is simple

(b) $\ell_R(M) = 1$

(c) $M = R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.

Proof. The first two are equivalent by definition. Clearly if $\mathfrak{m} \subset R$ is maximal then $R/\mathfrak{m}$ is simple. Now suppose that M is simple and take a nonzero $x \in M$. Then $(x) = M$ by simplicity so consider $I = \ker\left(R \xrightarrow{x} M\right) = \text{Ann}_A\left(x\right) = \{r \in R \mid rx = 0\}$. Since $M = Rx$ we know that $M \cong R/I$. However, by the lattice isomorphism theorem, submodules of R/I correspond to ideals above I so since M is simple we must have I maximal. $\square$

6 Artinian Modules

Definition 6.0.1. An R-module M is *noetherian/artinian* if it satisfies the ascending/descending chain condition on submodules.

Theorem 6.0.2. An R-module M has finite length iff it is both noetherian and artinian.

Proof. If M has finite length then clearly it is noetherian and artinian since chains of submodules are bounded in length. Alternatively, simple modules are noetherian and artinian so given a composition series we see that M is noetherian and artinian by repeated extension. Now, conversely, assume that M is noetherian and artinian. By the artinian property we can take a minimal nonzero submodule $M_1 \subset M$. Then M_1 is simple. Either M/M_1 is simple or we may repeat to get $M_2 \supset M_1$ and M_2/M_1 is simple. Thus we get an ascending chain $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$ with M_{i+1}/M_i simple. Since M is Noetherian, this must terminate at $M_n = M$ so we get a finite length composition series showing that M has finite length. ☐

7 Artinian Rings

Definition 7.0.1. A ring A is *artinian* if it satisfies the descending chain condition on ideals: given a chain of ideals,

$$I_0 \supset I_1 \supset I_2 \supset \cdots$$

the chain stabilizes $I_{n+i} = I_n$.

Remark. A is artinian iff it is artinian as a module over itself.

Proposition 7.0.2. An artinian ring has finitely many maximal ideals.

Proof. Let $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots$ be a list of maximal ideals. Then consider the chain,

$$\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supset \cdots$$

By the artinian condition, we must have $\mathfrak{m}_1 \cdots \mathfrak{m}_n = \mathfrak{m}_1 \cdots \mathfrak{m}_n\mathfrak{m}_{n+1}$ for some n. But then by prime avoidence $\mathfrak{m}_{n+1}$ must be one of $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ since $\mathfrak{m}_{n+1} \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n$ so $\mathfrak{m}_{n+1} \supset \mathfrak{m}_i$ and $\mathfrak{m}_i$ is maximal. ☐

Proposition 7.0.3. Let A be an artinian ring. Then every prime ideal is maximal so $\dim A = 0$.

Proof. Let $\mathfrak{p}$ be prime and $x \notin \mathfrak{p}$. Consider the chain,

$$(x) \supset (x^2) \supset (x^3) \supset \cdots$$

By the artinian condition $(x^n) = (x^{n+1})$ for some n so $x^n = rx^{n+1}$ for some $r \in A$. Thus,

$$x^n(rx - 1) = 0$$

However, $x^n \notin \mathfrak{p}$ so $rx - 1 \in \mathfrak{p}$ and thus $x \in A/\mathfrak{p}$ is invertible so $A/\mathfrak{p}$ is a field and thus $\mathfrak{p}$ is maximal. ☐

Proposition 7.0.4. Let A be artinian. Then $\operatorname{nilrad}(A)$ is a nilpotent ideal.

Proof. Let $I = \text{nilrad}(A)$. Consider the chain of ideals,

$$I \supset I^2 \supset I^3 \supset \cdots$$

By the artinian condition, $I^{n+1} = I^n$ for some n. Consider $J = \{x \in A \mid xI^n = 0\}$. If $J \neq R$ we can choose $J' \supsetneq J$ minimal (using the artinian property). Then take $y \in J'$ so by minimality $J' = J + (y)$. Suppose $J + I(y) = J'$ then, since $J \subset \text{Jac}(A)$ and (y) is finitely generated, by Nakayama, $J' = J + I(y) = J$ which is false so $J \subset J + I(y) \subsetneq J'$ and thus $J = J + I(y)$ by minimality so $I(y) \in J$. Therefore, $y \cdot I^{n+1} = 0$ but $I^{n+1} = I^n$ so $y \cdot I^n = 0$ and thus $y \in J$ contradicting our situation so $J = R$ and thus $I^n = 0$. □

Proposition 7.0.5. Every artinian ring is a product of local artinian rings: $A_{\mathfrak{m}_i} = A/\mathfrak{m}_i^n$.

Proof. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals. Then we know that $\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r} = 0$ for some integers $n_1, \ldots, n_r \in \mathbb{Z}$. Therefore, by the Chinese remainder theorem,

$$A = A/(\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r}) = \prod_{i=1}^{r} A/\mathfrak{m}_i^{n_i}$$

Furthermore, $A/\mathfrak{m}_i^{n_i}$ is local because $\mathfrak{m}_i$ is the only maximal ideal above $\mathfrak{m}_i^{n_i}$. Furthermore,

$$A_{\mathfrak{m}_i} = (A/\mathfrak{m}_i^{n_i})_{\mathfrak{m}_i} = A/\mathfrak{m}_i^{n_i}$$

since $A \setminus \mathfrak{m}_i$ is not contained in any maximal ideal of $A/\mathfrak{m}_i^{n_i}$ and thus is invertible. □

Proposition 7.0.6. A ring A is artinian iff it has finite length as a module over itself.

Proof. If A has finite length as an A-module then it satisfies both the ascending and descending chain conditions on A-submodules i.e. ideals thus A is both noetherian and artinian. Conversely, let A be artinian. Since A is a finite product of local artinian rings we may reduce to the case that A is local artinian with maximal ideal $\mathfrak{m}$. Since $\text{nilrad}(A) = \mathfrak{m}$ then $\mathfrak{m}^n = 0$ for some n so we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

Then $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a $A/\mathfrak{m}$-module and,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

which must be finite since $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is an artinian module and thus must have finite dimension else there would be a nonterminating descending chains. Thus from the series A has finite length. □

Theorem 7.0.7. A ring A is artinian iff A is noetherian and $\dim A = 0$.

Proof. If A is artinian then it has finite length over itself and thus is noetherian. Also every prime is maximal so $\dim A = 0$. Conversely, suppose that A is noetherian and $\dim A = 0$. Then $\text{Spec}(A)$ is a noetherian topological space which has finitely many irreducible componets so A has finitely many minimal primes which are also maximal since $\dim A = 0$. Thus A has finitely many primes all of which are maximal. Since $\dim A = 0$ we have $I = \text{Jac}(A) = \text{nilrad}(A)$ so any $f \in I$ is nilpotent so I is nilpotent because A is noetherian so I is finitely generated. Thus by the Chines remainder theorem A is a finite product of local rings so we reduce to the case that A is local with maximal ideal $\mathfrak{m}$. Then we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

but $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite $A/\mathfrak{m}$-module since A is noetherian so,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is finite and thus $\ell_A(A)$ is finite from the series showing that A is artinian. □

Proposition 7.0.8. Let A be an artinian ring. Then,

$$\ell_A(A) = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i})$$

Proof. We can write, $A = A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r}$ and thus the formula immediately follows. □

Proposition 7.0.9. Any finite dimensional k-algebra is artinian.

Proof. By dimensionality arguments every descending chain stabilizes. □

Proposition 7.0.10. Let $A \to B$ be a local map and M an B-module of finite length. Then,

$$\ell_A(M) = \ell_B(M) \cdot [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

and in particular $\ell_A(M)$ is finite if $\kappa(\mathfrak{m}_B)$ is a finite extension of $\kappa(\mathfrak{m}_A)$.

Proof. Consider a composition series,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

Then M_i/M_{i-1} is a simple B-module so $M_i/M_{i-1} \cong B/\mathfrak{m}_B = \kappa(\mathfrak{m}_B)$ since B is local. Therefore,

$$\ell_A(M) = \sum_{i=1}^{n} \ell_A(M_i/M_{i-1}) = \sum_{i=1}^{n} \ell_A(\kappa(\mathfrak{m}_B)) = n \cdot [\kappa(B_\mathfrak{m}) : \kappa(A_\mathfrak{m})]$$

where $\ell_A(\kappa(\mathfrak{m}_B)) = \ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$ because $A \to B$ is local and,

$$\ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = \dim_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

□

Corollary 7.0.11. If A is a local artinian finite type k-algebra. Then,

$$\dim_k A = \ell_A(A) \cdot \dim_k (A/\mathfrak{m})$$

in particular A is a finite k-module.

Proof. Viewing A as a module over itself we know it has finite length since A is artinian. Furthermore, $A/\mathfrak{m}$ is a field finitely generated over k and thus a finite extension of k by the Nullstellensatz. Then applying the previous result we conclude. □

Corollary 7.0.12. Let A be an artinian finite type k-algebra. Then,

$$\dim_k A = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}) \cdot \dim_k (A/\mathfrak{m}_i)$$

Proof. Since A is artinian we can write,

$$A = \prod_{i=1}^{r} A_{\mathfrak{m}_i}$$

where $A_{\mathfrak{m}_i}$ are the local artinian factors associated to the finitely many prime ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$. The result follows from above by additivity of the dimensions. $\square$

Remark. We can generalize this to the following proposition.

Proposition 7.0.13. Let A be local with maximal ideal $\mathfrak{m}$ and B be semi-local with maximal ideals $\mathfrak{m}_i$. Let $A \to B$ be a homomorphism of rings such that $\mathfrak{m}_i$ lie over $\mathfrak{m}$ and $[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ is finite. Let M be a finite length B-module. Then,

$$\ell_A(M) = \sum_{i=1}^{n} \ell_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \cdot [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$$

8 Weakly Associated Points

8.1 Weakly Associated Primes

Definition 8.1.1. Let A be a ring and M an A-module. Then a prime $\mathfrak{p} \subset A$ is *weakly associated* to M if $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A(m)$ for some $m \in M$. We denote these primes $\mathrm{WAss}_A(M)$.

Lemma 8.1.2. Let M be an A module then the natural map,

$$M \to \prod_{\mathfrak{p} \in \mathrm{WAss}_A(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Suppose that $m \in M$ maps to zero. Then $\mathfrak{p} \not\subset \mathrm{Ann}_A(m)$ for each $\mathfrak{p} \in \mathrm{WAss}_A(M)$ which implies $\mathrm{Ann}_A(m) = A$ since otherwise some associated prime will be minimal over $\mathrm{Ann}_A(m)$. Thus $m = 0$. $\square$

Lemma 8.1.3. Let M be an A-module. Then,

$$M = (0) \iff \mathrm{WAss}_A(M) = \varnothing$$

Proof. If $M = (0)$ then this is clear. Otherwise, by the previous lemma $M \hookrightarrow (0)$ is injective so $M = (0)$. $\square$

Lemma 8.1.4. Let A be a ring and M an A-module. Then,

$$\mathfrak{p} \in \mathrm{WAss}_A(M) \iff \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{WAss}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

Proof. Consider the exact sequence for each $m \in M$,

$$0 \longrightarrow \mathrm{Ann}_A(m) \longrightarrow A \xrightarrow{m} M$$

Since localization is exact, we get an exact sequence,

$$0 \longrightarrow (\mathrm{Ann}_A(m))_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}} \xrightarrow{m} M_{\mathfrak{p}}$$

Therefore, $\mathrm{Ann}_{A_\mathfrak{p}}(m) = (\mathrm{Ann}_A(m))_\mathfrak{p}$. If $\mathfrak{p} \supset \mathrm{Ann}_A(m)$ is minimal then $\mathfrak{p}A_\mathfrak{p} \supset (\mathrm{Ann}_A(m))_\mathfrak{p} = \mathrm{Ann}_{A_\mathfrak{p}}(m)$ is minimal. Conversely, if $\mathfrak{p}A_\mathfrak{p} \supset \mathrm{Ann}_{A_\mathfrak{p}}(m/s)$ is minimal then,

$$\mathrm{Ann}_{A_\mathfrak{p}}(m/s) = \mathrm{Ann}_{A_\mathfrak{p}}(m) = (\mathrm{Ann}_A(m))_\mathfrak{p}$$

which implies that $\mathfrak{p} \supset \mathrm{Ann}_A(m)$ is minimal because if $x \in \mathrm{Ann}_A(m)$ and $x \notin \mathfrak{p}$ then $(\mathrm{Ann}_A(m))_\mathfrak{p} = A_\mathfrak{p}$ and any prime $\mathfrak{q}$ such that $\mathfrak{p} \subset \mathfrak{q} \subset \mathrm{Ann}_A(m)$ implies that $\mathfrak{q}A_\mathfrak{p}$ is intermediate. □

Lemma 8.1.5. Let A be a ring and M an A-module. Then $\mathrm{WAss}_A(M) \subset \mathrm{Supp}_A(M)$ furthermore any minimal element of $\mathrm{Supp}_A(M)$ is an element of $\mathrm{WAss}_A(M)$.

Proof. Since $\mathfrak{p} \supset \mathrm{Ann}_A(m)$ we know $M_\mathfrak{p} \neq 0$ since m is nonzero in $M_\mathfrak{p}$. Furthermore, suppose that $\mathfrak{p} \in \mathrm{Supp}_A(M)$ is minimal. Then $\mathrm{Supp}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{p}A_\mathfrak{p}\}$ and $M_\mathfrak{p} \neq 0$ so $\mathrm{WAss}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \{A_\mathfrak{p}\}$ and thus $\mathfrak{p} \in \mathrm{WAss}_A(M)$. □

Proposition 8.1.6. Let M be finite or A finite-dimensional. Every element of $\mathrm{Supp}_A(M)$ is contained in a minimal element. Likewise for $\mathrm{WAss}_A(M)$ and the sets of minimal elements coincide.

Proof. For Zorn's lemma, we need to show that every downward chain in $\mathrm{Supp}_A(M)$ has a lower bound. If $\dim A < \infty$ then any downward chain of primes stabilizes. Alternatively, assume that M is finite and consider a chain $\{\mathfrak{p}_i\}_{i \in I}$ then I claim that,

$$\mathfrak{q} = \bigcap_{i \in I} \mathfrak{p}_i \in \mathrm{Supp}_A(M)$$

First, $\mathfrak{q}$ is prime because if $xy \in \mathfrak{q}$ then $xy \in \mathfrak{p}_i$ at each stage so either $x \in \mathfrak{p}_i$ or $y \in \mathfrak{p}_i$ but because I is totally ordered either there is a maximal $i \in I$ at which x appears in which case $y \in \mathfrak{q}$ or x lies is $\mathfrak{p}_i$ for arbitrarily large i meaning that $x \in \mathfrak{p}_i$ for all i so $x \in \mathfrak{q}$. Now I claim that $M_\mathfrak{q} \neq 0$. Let $m_1, \dots, m_r \in M$ generate. It suffices to show that $\mathfrak{q} \supset \mathrm{Ann}_A(m_j)$ for some j or equivalently that $\mathfrak{p}_i \supset \mathrm{Ann}_A(m_j)$ for some fixed j and all i. Indeed for each i there is some j so that $\mathfrak{p}_i \supset \mathrm{Ann}_A(m_j)$. Therefore, at least one j must satisfy $\mathfrak{p}_i \supset \mathrm{Ann}_A(m_j)$ for unbounded i and hence $\mathfrak{p}_i \supset \mathrm{Ann}_A(m_j)$ for all i.

Now let $\mathfrak{p} \in \mathrm{WAss}_A(M)$ then $\mathfrak{p} \in \mathrm{Supp}_A(M)$ so choose $\mathfrak{q} \subset \mathfrak{p}$ minimal in $\mathrm{Supp}_A(M)$ then we have shown that $\mathfrak{q} \in \mathrm{WAss}_A(M)$ and is minimal in $\mathrm{WAss}_A(M)$ because $\mathrm{WAss}_A(M) \subset \mathrm{Supp}_A(M)$ and it is minimal in $\mathrm{Supp}_A(M)$. We have shown that any minimal element of $\mathrm{Supp}_A(M)$ is in $\mathrm{WAss}_A(M)$ and hence is minimal in $\mathrm{WAss}_A(M)$. This discussion shows the converse. □

Remark. The condition that M is finite is necessary if A is not finite dimensional (in which case downward chains of primes always stabilize). For example, let $A = k[x_0, x_1, \dots]$ and,

$$M = \bigoplus_{i=0}^{\infty} A/\mathfrak{p}_i \text{ where } \mathfrak{p}_i = (x_i, x_{i+1}, \dots)$$

Then,

$$\mathrm{Supp}_A(M) = \bigcup_{i=0}^{\infty} V(\mathfrak{p}_i)$$

Thus if $\mathfrak{q} \in \mathrm{Supp}_A(M)$ then $\mathfrak{q} \supset \mathfrak{p}_i$ for some i but then $\mathfrak{q} \supset \mathfrak{p}_i \supsetneq \mathfrak{p}_{i+1}$ so $\mathrm{Supp}_A(M)$ has no minimal elements.

Remark. The set $\mathrm{WAss}_A(M)$ need not be a downward set (even when every element is contained in a minimal element) even in the best situations of A a finite-dimensional noetherian ring and M a finite A-module. For example let $A = k[x, y, z]/(x^2, xy, xz)$ and $M = A$ then $\mathrm{WAss}_A(M) = \{(x), (x, y, z)\}$ so the intermediate prime (x, y) is not associated.

Lemma 8.1.7. Let A be a ring and M an A-module and $S \subset A$ a multiplicative subset. Then.

(a) $\mathrm{WAss}_A(S^{-1}M) = \mathrm{WAss}_{S^{-1}A}(S^{-1}M)$

(b) $\mathrm{WAss}_A(M) \cap \mathrm{Spec}(S^{-1}A) = \mathrm{WAss}_A(S^{-1}M)$.

Proof. We have,

$$\mathfrak{p} \in \mathrm{WAss}_A\left(S^{-1}M\right) \iff \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{WAss}_{A_{\mathfrak{p}}}\left(S^{-1}M_{\mathfrak{p}}\right)$$

For $\mathfrak{p} \in \mathrm{Spec}(S^{-1}A)$ (i.e. $S \subset A \setminus \mathfrak{p}$) we have $S^{-1}M_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $(S^{-1}A)_{\mathfrak{p}} = A_{\mathfrak{p}}$ so both equalities hold. Otherwise, $\mathfrak{p}A_{\mathfrak{p}}$ containes an element of S so $\mathfrak{p}A_{\mathfrak{p}}$ has some nonzero divisor on $S^{-1}M_{\mathfrak{p}}$ and thus $\mathfrak{p} \notin \mathrm{WAss}_A(S^{-1}M)$. ☐

Proposition 8.1.8. Let A be a ring M an A-module then $\mathfrak{p} \in \mathrm{Supp}_A(M)$ if and only if there exists $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \in \mathrm{WAss}_A(M)$. Therefore,

$$\bigcap_{\mathfrak{p} \in \mathrm{Supp}_A(M)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{WAss}_A(M)} \mathfrak{p} \quad \text{and} \quad \mathrm{Supp}_A(M) = \bigcup_{\mathfrak{p} \in \mathrm{WAss}_A(M)} V(\mathfrak{p})$$

Proof. Take $\mathfrak{p} \in \mathrm{Supp}_A(M)$ so $M_{\mathfrak{p}} \neq 0$ and then $\mathrm{Ass}_A(M_{\mathfrak{p}}) \neq \varnothing$. Using the previous lemma, there exists $\mathfrak{q} \in \mathrm{Ass}_A(M_{\mathfrak{p}}) = \mathrm{Ass}_A(M) \cap \{\mathfrak{q} \mid \mathfrak{q} \subset \mathfrak{p}\}$. Furthermore, the support is an upward set (if $\mathfrak{q} \subset \mathfrak{p}$ and $M_{\mathfrak{q}} \neq 0$ then $M_{\mathfrak{p}} \neq 0$ since $M_{\mathfrak{p}} \to M_{\mathfrak{q}}$ is localization). Thus, if we have $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \in \mathrm{Ass}_A(M) \subset \mathrm{Supp}_A(M)$ then $\mathfrak{p} \in \mathrm{Supp}_A(M)$. ☐

Lemma 8.1.9. Let $M \hookrightarrow N$ be an injection of A-modules. Then $\mathrm{WAss}_A(M) \subset \mathrm{WAss}_A(N)$.

Proof. This follows because the set of annihilators of elements of M is a subset of the set of annihilators of elements of N. ☐

Lemma 8.1.10. Consider an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

then,

$$\mathrm{WAss}_A(M_2) \subset \mathrm{WAss}_A(M_1) \cup \mathrm{WAss}_A(M_3)$$

Proof. Let $\mathfrak{p} \in \mathrm{WAss}_A(M_2)$ and $\mathfrak{p} \notin \mathrm{WAss}_A(M_1)$. Using the previous lemma it suffices to consider the case that A is local with maximal ideal $\mathfrak{p}$ (since we may localize the exact sequnence at $\mathfrak{p}$). Then $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A(m)$ for some $m \in M_2$ not in the image of $M_1 \to M_2$ (else $\mathfrak{p} \in \mathrm{WAss}_A(M_1)$). Therefore $\bar{m} \in M_3$ is nonzero and $\mathrm{Ann}_A(\bar{m}) \supset \mathrm{Ann}_A(m)$ but $\mathrm{Ann}_A(\bar{m})$ is proper since $\bar{m}$ is nonzero and thus contained in $\mathfrak{p}$. Since $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A(m)$ it must also be minimal over $\mathrm{Ann}_A(\bar{m})$ and thus we conclude that $\mathfrak{p} \in \mathrm{WAss}_A(M_3)$. ☐

Lemma 8.1.11. Let A be a ring and M and A-module. Then,

$$\bigcup_{\mathfrak{p} \in \mathrm{WAss}_A(M)} = \{\text{zero divisors on } M\}$$

Proof. Let $m \in M$ have zero divisors then there is exists a minimal prime (by Zorn's Lemma) above $\mathrm{Ann}_A(m)$ which must be associated. Conversely, if $f \in \mathfrak{p} \in \mathrm{WAss}_A(M)$ then $\mathfrak{p}$ is minimal over $\mathrm{Ann}_A(m)$ for some $m \in M$. Then $R = (A/\mathrm{Ann}_A(m))_{\mathfrak{p}}$ has a unique minimal prime $\mathfrak{p}$ so $\mathfrak{p} = \mathrm{nilrad}(R)$ and thus $gf^n \in \mathrm{Ann}_A(m)$ for some least $n > 0$ and $g \notin \mathfrak{p}$. Thus $gf^n m = 0$ so $f(gf^{n-1}m) = 0$ but $gf^{n-1}m \neq 0$ because n is minimal so f is a zero divisor. □

Proposition 8.1.12. Let A be reduced then $\mathrm{WAss}_A(A)$ are exactly the minimal primes of A.

Proof. The minimal primes are in $\mathrm{WAss}_A(A)$ by Lemma 8.1.5. Because $\mathfrak{p} \in \mathrm{WAss}_A(A) \iff \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{WAss}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ is suffices to consider the case of a reduced local ring $(R, \mathfrak{m})$ and $\mathfrak{m} \in \mathrm{WAss}_R(R)$. Then $\mathfrak{m}$ is minimal over $\mathrm{Ann}_R(x)$ for some $x \in \mathfrak{m}$ so $\mathfrak{m} = \sqrt{\mathrm{Ann}_R(x)}$. Thus $x^n \in \mathrm{Ann}_R(x)$ so $x^{n+1} = x^n \cdot x = 0$ so $x = 0$ because R is reduced a contradiction unless $\mathfrak{m} = 0$ so R is a field so $\mathfrak{m}$ is minimal showing that $\mathfrak{p}A_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ and thus $\mathfrak{p} \subset A$ are minimal primes and that $A_{\mathfrak{p}}$ is a field. □

Lemma 8.1.13. Let A be a ring and $\mathfrak{p} \subset A$ a prime then $\mathrm{WAss}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

Proof. For nonzero $a \in A/\mathfrak{p}$ (i.e. $a \notin \mathfrak{p}$) the set $\mathrm{Ann}_A(a) = \{r \in A \mid ra \in \mathfrak{p}\} = \mathfrak{p}$ since $\mathfrak{p}$ is prime and therefore therefore $\mathfrak{p}$ is the unique minimal prime over an annihilator. □

Proposition 8.1.14. Let A be a ring and M a Noetherian A-module. Then,

(a) there exists a finite filtration,

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

such that each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_A(M)$

(b) for any such filtration, $\mathrm{WAss}_A(M) \subset \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$

(c) $\mathrm{WAss}_A(M)$ is finite.

Proof. Since $M \neq (0)$ there is some $\mathfrak{p} \in \mathrm{WAss}_A(M)$ so we have an injection $A/\mathfrak{p} \to M$ let $M_1 \subset M$ be the image of this map so $M_1/M_0 \cong A/\mathfrak{p}_1$. Now take M/M_1 and $\mathfrak{p}_2 \in \mathrm{WAss}_A(M/M_1)$ then we have an injection $A/\mathfrak{p}_2 \to M/M_1$ so take $\bar{M}_2$ to be the image inside M/M_1 and M_2 its preimage in M. Then $M_2/M_1 \cong A/\mathfrak{p}_2$ and continuing by induction we construct a sequence,

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$$

with $M_i/M_{i-1} = A/\mathfrak{p}_i$ and

$$\mathfrak{p}_i \in \mathrm{WAss}_A(M/M_{i-1}) \subset \mathrm{Supp}_A(M/M_{i-1}) \subset \mathrm{Supp}_A(M)$$

However, M is Noetherian so this sequence must stabilize but it is striclty increasing when $M_i \subset M$ is proper. Thus, $M_n = M$ for some n.

For any such filtration we get exact sequences,

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow A/\mathfrak{p}_{i+1} \longrightarrow 0$$

Assume for induction that $\mathrm{Ass}_A(M_i) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_i\}$ then, by Lemma 8.1.10,

$$\mathrm{WAss}_A(M_{i+1}) \subset \mathrm{WAss}_A(M_i) \cup \mathrm{WAss}_A(A/\mathfrak{p}_{i+1}) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_{i+1}\}$$

proving (b) by induction. (c) follows directly from (a) and (b). □

8.2 Associated Primes

Definition 8.2.1. Let A be a ring and M an A-module. We say that $\mathfrak{p} \subset A$ is an *associated prime* of M if $\mathfrak{p} = \mathrm{Ann}_A(m)$ for some $m \in M$. We write $\mathrm{Ass}_A(M)$ for the set of associated primes of M.

Remark. Note $\mathfrak{p} = \mathrm{Ann}_A(m) \iff A/\mathfrak{p} \hookrightarrow M$ via $a \mapsto a \cdot m$.

Remark. Clearly $\mathrm{Ass}_A(M) \subset \mathrm{WAss}_A(M)$. We will see equality holds when A is Noetherian.

Lemma 8.2.2. Given an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

we have,

$$\mathrm{Ass}_A(M_2) \subset \mathrm{Ass}_A(M_1) \cup \mathrm{Ass}_A(M_3)$$

Proof. If $\mathfrak{p} \in \mathrm{Ass}_A(M)$ then we have an embedding

$$A/\mathfrak{p} \longhookrightarrow M_2$$

which is injective and $\iota(A/\mathfrak{p}) \cap N_1 = (0)$ then we get an injective map $A/\mathfrak{p} \to M_3$ so $\mathfrak{p} \in \mathrm{Ass}_A(M_3)$. If $\iota(A/\mathfrak{p}) \cap M_1 \neq (0)$ then take nonzero $n \in \iota(A/\mathfrak{p}) \cap M_1$. Then $\mathrm{Ann}_A(n) = \mathrm{Ann}_A(\iota(x))$ for $x \in A/\mathfrak{p}$ nonzero. However, if $a \cdot \iota(x) = 0$ then $\iota(a \cdot x) = 0$ but ι is injective so $a \cdot x = 0$ and thus $\mathrm{Ann}_A(\iota(x)) = \mathrm{Ann}_A(x) = \mathfrak{p}$ because if $a \cdot x \in \mathfrak{p}$ for $x \notin \mathfrak{p}$ then $a \in \mathfrak{p}$. □

Lemma 8.2.3. Let $S_{M,\mathfrak{p}} = \{\mathrm{Ann}_A(m) \subset \mathfrak{p} \mid m \in M \setminus \{0\}\}$ then any maximal element in $S_{M,\mathfrak{p}}$ is a prime ideal.

Proof. Let $\mathfrak{q} \in S_{M,\mathfrak{p}}$ be maximal with $\mathfrak{q} = \mathrm{Ann}_A(m)$ for $m \neq 0$. Suppose $ab \in \mathfrak{q}$ and $a, b \notin \mathfrak{q}$. Then $\mathfrak{q} \subsetneq \mathrm{Ann}_A(am)$ since $b \in \mathrm{Ann}_A(am) \setminus \mathrm{Ann}_A(m)$ so by maximality $\mathrm{Ann}_A(am) \not\subset \mathfrak{p}$. Choose $s \in \mathrm{Ann}_A(am) \setminus \mathfrak{p}$. Then $a \in \mathrm{Ann}_A(sm)$ so $\mathrm{Ann}_A(m) \subsetneq \mathrm{Ann}_A(sm)$ and thus by maximality we can choose $t \in \mathrm{Ann}_A(sm) \setminus \mathfrak{p}$ so $st \in \mathrm{Ann}_A(m) \subset \mathfrak{p}$ but $s, t \notin \mathfrak{p}$ contradicting the primality of $\mathfrak{p}$. Thus $\mathfrak{q}$ is prime. □

Proposition 8.2.4. Let A be Noetherian and M be an A-module. Then,

$$\mathrm{Ass}_A(M) = \mathrm{WAss}_A(M)$$

In particular, $\mathrm{Ass}_A(M) \neq \varnothing$ and all other properties of $\mathrm{WAss}_A(M)$ apply to $\mathrm{Ass}_A(M)$.

Proof. $\mathrm{Ass}_A(M) \subset \mathrm{WAss}_A(M)$ is obvious. If $\mathfrak{p} \in \mathrm{WAss}_A(M)$ then $\mathfrak{p} \supset \mathrm{Ann}_A(m)$ for some $m \in M$ and thus m is nonzero in $M_\mathfrak{p}$ so $\mathfrak{p} \in \mathrm{Supp}_A(M)$. Let A be Noetherian then ascending chains in $S_{M,\mathfrak{p}}$ stabilize and thus by Zorn's Lemma every annhilator $\mathrm{Ann}_A(m) \subset \mathfrak{p}$ is contained in some maximal $\mathrm{Ann}_A(m') \subset \mathfrak{p}$. Thus, if $\mathfrak{p} \in \mathrm{WAss}_A(M)$ then $\mathfrak{p}$ is a minimal prime over some $\mathrm{Ann}_A(m)$ so $\mathfrak{p} = \mathrm{Ann}_A(m')$ since $\mathrm{Ann}_A(m')$ is prime and $\mathrm{Ann}_A(m) \subset \mathrm{Ann}_A(m') \subset \mathfrak{p}$. □

Lemma 8.2.5. Let A be a ring and M an A-module and $S \subset A$ a multiplicative subset. Then.

(a) $\mathrm{Ass}_A(S^{-1}M) = \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$

(b) $\mathrm{Ass}_A(M) \cap \mathrm{Spec}(S^{-1}A) \subset \mathrm{Ass}_A(S^{-1}M)$ with equality when A is Noetherian.

Proof. Tag 05BZ. □

Proposition 8.2.6. Let A be a Noetherian ring and M a finite A-module. Then,

(a) there exists a finite filtration,

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

such that each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_A(M)$

(b) for any such filtration, $\mathrm{Ass}_A(M) \subset \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$

(c) $\mathrm{Ass}_A(M)$ is finite.

Proof. M is a Noetherian module so this applies directly from Prop. 8.1.14. $\square$

Proposition 8.2.7. Let A be a Noetherian ring and $I \subset A$ an ideal and M a finite A-module. Then the following are equivalent,

(a) $I \subset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_A(M)$

(b) $I \subset \{\text{zero divisors on } M\}$

Proof. If $I \subset \mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Ass}_A(M)$ then,

$$I \subset \mathfrak{p} \subset \{\text{zero divisors on } M\}$$

Conversely, if $I \subset \{\text{zero divisors on } M\}$ then,

$$I \subset \{\text{zero divisors on } M\} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(M)} \mathfrak{p}$$

By Proposition 8.2.6, the set $\mathrm{Ass}_A(M)$ is finite so by prime avoidance $I \subset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_A(M)$. $\square$

Corollary 8.2.8. Let $\mathfrak{m} \subset A$ be a maximal ideal with A noetherian and M a finite A-module. Then $\mathfrak{m} \in \mathrm{Ass}_A(M)$ if and only if $\mathfrak{m} \subset \{\text{zero divisors on } M\}$.

Corollary 8.2.9. Let $(A, \mathfrak{m})$ be a noetherian local ring then $\mathfrak{m} \in \mathrm{Ass}_A(A)$ iff $\mathfrak{m} = \{\text{zero divisors}\}$.

Proof. Immediate from the above since zero divisors are not units and thus contained in $\mathfrak{m}$. $\square$

Corollary 8.2.10. Let A be noetherian and M be a finite A-module then for all $\mathfrak{p} \in \mathrm{Spec}(A)$,

$$\mathfrak{p} \in \mathrm{Ass}_A(M) \iff \mathfrak{p}A_{\mathfrak{p}} = \{\text{zero divisors on } M_{\mathfrak{p}}\}$$

8.3 Primary Decomposition

Remark. In this section we let A be a Noetherian ring.

Definition 8.3.1. An A-module M is called coprimary if $\mathrm{Ass}_A(M) = \{\mathfrak{p}\}$ and if $N \subset M$ we say that N is $\mathfrak{p}$-primary if M/N is coprimary with $\mathrm{Ass}_A(M/N) = \{\mathfrak{p}\}$.

Lemma 8.3.2. M is coprimary iff any zero divisor of M is locally nilpotent i.e. if $a \cdot m = 0$ for some $m \in M \setminus \{0\}$ then $\forall m' \in M : a^n \cdot m' = 0$ for some n.

Proof. Assume that M is coprimary, $\mathrm{Ass}_A(M) = \{\mathfrak{p}\}$. If $x \in M$ is nonzero then Ax is a nonzero submodule of M so $\mathrm{Ass}_A(Ax) = \{\mathfrak{p}\}$ since it is nonempty. Therefore, $\mathfrak{p}$ is a minimal element in $\mathrm{Supp}_A(Ax) = V(\mathrm{Ann}_A(x))$ because $Ax \cong A/\mathrm{Ann}_A(x)$. Thus, $\sqrt{\mathrm{Ann}_A(x)} = \mathfrak{p}$. If a is a zero divisor of M then $a \in \mathfrak{p}$ so $a^n \in \mathrm{Ann}_A(x)$ so a is locally nilpotent. Converely, assume that the set of zero divisors equals the set of locally nilpotent elements. Take $\mathfrak{p}$ to be the ideal of all locally nilpotents. Take $\mathfrak{q} \in \mathrm{Ass}_A(M)$ then $\mathfrak{q} = \mathrm{Ann}_A(x)$ for some x. If $a \in \mathfrak{p}$ then $a^n \cdot x = 0$ for some n implies that $a^n \in \mathfrak{q}$ so $a \in \mathfrak{q}$. so $\mathfrak{p} \subset \mathfrak{q}$. Furthermore,

$$\bigcup_{\mathfrak{q} \in \mathrm{Ass}_A(M)} \mathfrak{q} = \{\text{zero divisors}\} = \mathfrak{p}$$

so for any $\mathfrak{q} \in \mathrm{Ass}_A(M)$ we have $\mathfrak{q} \subset \mathfrak{p}$. Thus, $\mathfrak{p} = \mathfrak{q}$ so $\mathrm{Ass}_A(M)$ constains a unique prime. $\square$

Corollary 8.3.3. If $I \subset A$ is an ideal then $\mathrm{Ass}_A(A/I) = \{\mathfrak{p}\}$ if and only if I is a primary ideal and in that case $\sqrt{I} = \mathfrak{p}$.

Proof. Consider $I \subset A$ and A/I is coprimary then take $x, y \in A$ such that $y \notin I$ and $\bar{x} \cdot \bar{y} = 0$ in A/I. Then $\bar{x}$ is a zero divisor of A/I so it is locally nilpotent by the above. Thus, $\bar{x}^n \cdot 1 = 0$ for some n so $x^n \in I$ so $x \in \sqrt{I}$ and thus I is primary. Furthermore,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Supp}_A(A/I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(A/I)} \mathfrak{p} = \mathfrak{p}$$

since $\mathrm{Ass}_A(M)$ is the set of minimal primes of $\mathrm{Supp}_A(M)$ and $\mathrm{Ass}_A(A/I) = \mathfrak{p}$. $\square$

Definition 8.3.4. Let M be an A-module and $N \subset M$. We say N has a primary decomposition if,

$$N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

where each Q_i is primary. Moreover, we say that this decomposition is irredundant if

(a) if $i \neq j$ then $\mathrm{Ass}_A(M/Q_i) \neq \mathrm{Ass}_A(M/Q_j)$

(b) we cannot remove any Q_j from the intersection.

Lemma 8.3.5. Let M be an A-module then,

(a) If $Q_1, Q_2 \subset M$ are $\mathfrak{p}$-primary then $Q_1 \cap Q_2$ is $\mathfrak{p}$-primary.

(b) If $N = Q_1 \cap \cdots \cap Q_n$ is a irredundant primary decomposition and for each i, Q_i is $\mathfrak{p}_i$-primary then,
$$\mathrm{Ass}_A(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$$

Proof. Consider the injection,

$$0 \longrightarrow M/Q_1 \cap Q_2 \hookrightarrow M/Q_1 \oplus M/Q_2$$

which implies that,

$$\mathrm{Ass}_A(M/Q_1 \cap Q_2) \subset \mathrm{Ass}_A(M/Q_1 \oplus M/Q_2) = \mathrm{Ass}_A(M/Q_1) \cup \mathrm{Ass}_A(M/Q_2) = \{\mathfrak{p}\}$$

proving the first. For the second, consider the injection,

$$M/N \hookrightarrow M/Q_1 \oplus \cdots \oplus M/Q_n$$

which implies that,

$$\mathrm{Ass}_A(M/N) \subset \mathrm{Ass}_A(M/Q_1) \cup \cdots \cup \mathrm{Ass}_A(M/Q_n) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

We need to show that $\mathfrak{p}_i \in \mathrm{Ass}_A(M/N)$ for each i. We have the exact sequence,

$$0 \longrightarrow N \longrightarrow Q_2 \cap \cdots \cap Q_n \longrightarrow M/Q_1$$

and therefore,

$$(Q_2 \cap \cdots \cap Q_n)/N \hookrightarrow M/Q_1$$

which implies that,

$$\mathrm{Ass}_A((Q_2 \cap \cdots \cap Q_n)/N) \subset \mathrm{Ass}_A(M/Q_1) = \{\mathfrak{p}_1\}$$

so since it is nonempy we have,

$$\{\mathfrak{p}_1\} = \mathrm{Ass}_A((Q_2 \cap \cdots \cap Q_n)/N) \subset \mathrm{Ass}_A(M/N)$$

where the inclusion holds via the exact sequence,

$$0 \longrightarrow N \longrightarrow Q_2 \cap \cdots \cap Q_n \longrightarrow M/N$$

The same argument holds for each i. $\square$

Theorem 8.3.6. Let M be Noetherian. For each $\mathfrak{p} \in \mathrm{Ass}_A(M)$, there exist $Q_{\mathfrak{p}} \subset M$ which are $\mathfrak{p}$-primary such that,

$$\bigcap_{\mathfrak{p} \subset \mathrm{Ass}_A(M)} Q_{\mathfrak{p}} = 0$$

Proof. Fix $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and consider the set $S_{\mathfrak{p}} = \{Q \subset M \mid \mathfrak{p} \notin \mathrm{Ass}_A(Q)\} \neq \varnothing$ since the zero module is contained in this set. Since M is Noetherian ascending chains stabilize so by Zorn's lemma there exists a maximal element $Q_{\mathfrak{p}} \in S_{\mathfrak{p}}$. We know,

$$\mathrm{Ann}_A(M/Q_{\mathfrak{p}}) \neq \varnothing$$

since we have $M/Q_{\mathfrak{p}} \neq (0)$. Otherwise, $M = Q_{\mathfrak{p}}$ which implies $\mathfrak{p} \in \mathrm{Ass}_A(Q_{\mathfrak{p}})$ but $Q_{\mathfrak{p}} \in S_{\mathfrak{p}}$. Let $\mathfrak{p}' \in \mathrm{Ass}_A(M/Q_{\mathfrak{p}})$ and suppose that $\mathfrak{p}' \neq \mathfrak{p}$ then we have,

$$A/\mathfrak{p}' \hookrightarrow M/Q_{\mathfrak{p}}$$

The image of this embedding is a submodule, $Q_{\mathfrak{p}} \subsetneq Q' \subset M$ such that $Q'/Q_{\mathfrak{p}} \cong A/\mathfrak{p}'$ implying that,

$$\mathrm{Ass}_A(Q'/Q_{\mathfrak{p}}) = \{\mathfrak{p}'\}$$

Thus we have an exact sequence,

$$0 \longrightarrow Q_{\mathfrak{p}} \longrightarrow Q' \longrightarrow A/\mathfrak{p} \longrightarrow 0$$

which implies that $\mathrm{Ass}_A(Q') \subset \mathrm{Ass}_A(Q_\mathfrak{p}) \cup \mathrm{Ass}_A(A/\mathfrak{p}') = \mathrm{Ass}_A(Q_\mathfrak{p}) \cup \{\mathfrak{p}'\}$. However, this contradicts the fact that $Q_\mathfrak{p}$ is maximal in $S_\mathfrak{p}$ since $Q' \in S_\mathfrak{p}$ as long as $\mathfrak{p}' \neq \mathfrak{p}$. Therefore, $\mathfrak{p}' = \mathfrak{p}$ so $\mathrm{Ass}_A(A/Q_\mathfrak{p}) = \{\mathfrak{p}\}$. Now consider,

$$\mathrm{Ass}_A\left(\bigcap_{\mathfrak{p}\in\mathrm{Ass}_A(M)} Q_\mathfrak{p}\right) \subset \bigcap_{\mathfrak{p}\in\mathrm{Ass}_A(M)} \mathrm{Ass}_A(Q_\mathfrak{p}) = \varnothing$$

because for any $\mathfrak{p}$ we know $\mathfrak{p} \notin \mathrm{Ass}_A(Q_\mathfrak{p})$. Therefore,

$$\bigcap_{\mathfrak{p}\in\mathrm{Ass}_A(M)} Q_\mathfrak{p} = (0)$$

since it has no associated primes. □

Corollary 8.3.7. If M is a finite A-module then any submodule has a primary decomposition.

Proof. Let $N \subset M$ be a submodule. Apply the theorem to $\bar{M} = M/N$ which has finite type so $\mathrm{Ass}_A(M/N)$ is finite. Write, $\mathrm{Ass}_A(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Therefore, there exist primary ideals Q_i such that,

$$Q_{\mathfrak{p}_1} \cap \cdots \cap Q_{\mathfrak{p}_r} = (0)$$

in M/N. Take Q_i to be the preimage of $Q_{\mathfrak{p}_i}$. Thus,

$$Q_1 \cap \cdots \cap Q_r = N$$

and

$$M/Q_i \cong \bar{M}/Q_{\mathfrak{p}_i} \implies \mathrm{Ass}_A(M/Q_i) = \{\mathfrak{p}_i\}$$

□

8.4 Weakly Associated Points

Definition 8.4.1. Let X be a scheme and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then we define,

(a) $x \in X$ is *weakly associated* to $\mathscr{F}$ if $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is weakly associated to $\mathscr{F}_x$

(b) $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ is the set of weakly associated points of $\mathscr{F}$

(c) the (weakly) associated points of X are $\mathrm{WAss}_{\mathcal{O}_X}(\mathcal{O}_X)$.

Proposition 8.4.2. Let $X = \mathrm{Spec}(A)$ and $\mathscr{F} = \widetilde{M}$ be a quasi-coherent $\mathcal{O}_X$-module then we have,

$$\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F}) = \mathrm{WAss}_A(M)$$

Proof. Immediate consequence of Lemma 8.1.4. □

Proposition 8.4.3. Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then,

$$\mathscr{F} = 0 \iff \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F}) = 0$$

Proof. Choose an affine open cover $U_i = \mathrm{Spec}(A_i)$ such that $\mathscr{F}|_{U_i} = \widetilde{M_i}$. Then $\mathrm{WAss}_A(M_i) = \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F}) \cap U_i = \varnothing$ so $M_i = 0$ and thus $\mathscr{F} = 0$. □

Proposition 8.4.4. Let X be a scheme and $\mathscr{F} \to \mathscr{G}$ a morphism of quasi-coherent $\mathcal{O}_X$-modules. If $\mathscr{F}_x \to \mathscr{G}_x$ is injective for each $x \in \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ then $\mathscr{F} \to \mathscr{G}$ is injective.

Proof. Consider the sequence,

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}$$

Since $\mathscr{F}_x \to \mathscr{G}_x$ is an injection $\mathscr{K}_x = 0$ for each $x \in \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$. Furthermore, $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{K}) \subset \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ and thus $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{K}) = \varnothing$ so $\mathscr{K} = 0$. □

8.5 Associated Points: the Noetherian Case

Remark. By analogy, we might define an *associated point* of $\mathscr{F}$ on X to be a point $x \in X$ such that $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is an associated prime of $\mathscr{F}_x$. However, this definition is problematic because, in general, associated primes do not play nicely with localization. In particular $\mathfrak{p} \in \mathrm{Ass}_A(M) \implies \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ but the converse may not hold. Therefore, we may have a scheme X and a quasi-coherent sheaf $\mathscr{F}$ such that on an affine open $U = \mathrm{Spec}(A)$ with $\mathscr{F}|_U = \widetilde{M}$ we have $\mathfrak{p} \in \mathrm{Ass}_A(M)$ but $\mathfrak{p} = x \in X$ is not as associated point of $\mathscr{F}$ on X. To recify this pathology, we only consider associated points on locally noetherian schemes in which case there is no difference between weakly associated points and associated points.

Definition 8.5.1. Let X be a locally noetherian scheme and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. We say $x \in X$ is an *associated point* of $\mathscr{F}$ if x is a *weakly associated point*. Likewise we write,

$$\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F}) = \mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$$

Remark. Notice this definition is purely notational. In the locally noetherian case we simply will write $\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F})$ for $\mathrm{WAss}_{\mathcal{O}_X}(\mathscr{F})$ as a reminder that these sets behave as expected for associated points in the case of Noetherian rings.

Proposition 8.5.2. Let X be noetherian and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module. Then $\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F})$ is finite.

Proof. Since X is quasi-compact we may choose a finite open cover $U_i = \mathrm{Spec}(A_i)$ with A_i Noetherian on which $\mathscr{F}|_{U_i} = \widetilde{M_i}$ for finite A_i-modules. Then $\mathrm{Ass}_{\mathcal{O}_X}(\mathscr{F}) \cap U = \mathrm{Ass}_{A_i}(M_i)$ each of which is finite since M_i is a Noetherian module. $\square$

9 Depth

9.1 Definitions

Definition 9.1.1. Let A be a ring $I \subset A$ an ideal and M a finite A-module. Then $x_1, \dots, x_r \in I$ are an *M-regular sequence in I* if

(a) x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$ for each $i \in \{1, \dots, r\}$

(b) $M/(x_1, \dots, x_r)M$ is nonzero.

We say that $\mathrm{depth}_I(M)$ is the supremum of the lengths of M-regular sequence in I unless $IM = M$ in which case $\mathrm{depth}_I(M) = \infty$.

Remark. If $IM \subsetneq M$ then $\mathrm{depth}_I(M) = 0$ iff $I \subset \{\text{zero divisors on } M\}$.

Remark. If $(A, \mathfrak{m})$ is a local ring then we define $\mathrm{depth}(M) := \mathrm{depth}_{\mathfrak{m}}(M)$.

9.2 The Cohomological Criterion

Lemma 9.2.1. Let A be a Noetherian ring, $I \subset R$ an ideal, and M a finite A-module with $IM \neq M$. Then the following are equivalent,

(a) $\mathrm{Ext}^i_A(N, M) = 0$ for all $i < n$ and all finite A-modules N with $\mathrm{Supp}_A(N) \subset V(I)$

(b) $\mathrm{Ext}^i_A(A/I, M) = 0$ for all $i < n$

(c) there exists a finite A-module N with $\mathrm{Supp}_A(N) = V(I)$ and $\mathrm{Ext}^i_A(N, M) = 0$ for all $i < n$

(d) there exists an M-regular sequence $x_1, \dots, x_n \in I$ of length n

and therefore $\mathrm{depth}_I(M) = \inf\{n \in \mathbb{Z} \mid \mathrm{Ext}^i_A(A/I, M) \neq 0\}$.

Proof. Clearly (a) $\implies$ (b) $\implies$ (c). Now we show that (c) $\implies$ (d).

Finally, we need to show that (d) $\implies$ (a). (DOOOOOOOOOOOOOOOOOOOOOO!! OR SPLIT UP THIS PROOF!!) □

Remark. From here on, let A be a Noetherian ring and $I \subset A$ an ideal and M a finite A-module with $IM \neq M$.

Lemma 9.2.2. Consider an exact sequence of finite A-modules such that $IM_i \neq M_i$,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

Then the following hold,

(a) $\mathrm{depth}_I(M_2) \geq \min\{\mathrm{depth}_I(M_1), \mathrm{depth}_I(M_3)\}$

(b) $\mathrm{depth}_I(M_1) \geq \min\{\mathrm{depth}_I(M_2), \mathrm{depth}_I(M_3) + 1\}$

(c) $\mathrm{depth}_I(M_3) \geq \min\{\mathrm{depth}_I(M_1) - 1, \mathrm{depth}_I(M_2)\}$

Proof. Apply the functor $\mathrm{Hom}_A(A/I, -)$ to give the long exact sequence,

$$\mathrm{Ext}^i_A(A/I, M_1) \longrightarrow \mathrm{Ext}^i_A(A/I, M_2) \longrightarrow \mathrm{Ext}^i_A(A/I, M_3) \longrightarrow \mathrm{Ext}^{i+1}_A(A/I, M_1)$$

If $i < n = \min\{\mathrm{depth}_I(M_1), \mathrm{depth}_I(M_3)\}$ then $\mathrm{Ext}^i_A(A/I, M_2) = 0$ applying the cohomological criterion and the exact sequence so $\mathrm{depth}_I(M_3) \geq n$. The other parts follow similarly. □

Lemma 9.2.3. Let x be a nonzerodivisor on M then $\mathrm{depth}_I(M/xM) = \mathrm{depth}_I(M) - 1$.

Proof. Applying the previous Lemma to the exact sequence,

$$0 \longrightarrow M \xrightarrow{\times x} M \longrightarrow M/xM \longrightarrow 0$$

gives $\mathrm{depth}_I(M/xM) \geq \mathrm{depth}_I(M) - 1$. However, for any M/xM-regular sequence $x_1, \dots, x_n \in I$ we get a M-regular sequence $x, x_1, \dots, x_n \in I$ and thus $\mathrm{depth}_I(M) \geq \mathrm{depth}_I(M/xM) + 1$. □

Corollary 9.2.4. Any M-regular sequence $x_1, \dots, x_r \in I$ can be extended to a regular sequence of length $\mathrm{depth}_I(M)$ and thus all maximal regular sequences have the same length.

Proof. Given an M-regular sequence $x_1, \dots, x_r \in I$ we apply the previous Lemma to show that,

$$\mathrm{depth}_I(M/(x_1, \dots, x_r)M) = \mathrm{depth}_I(M) - r$$

and thus there exists a regular sequence $x_{r+1}, \dots, x_d \in I$ for $M/(x_1, \dots, x_r)M$ meaning that $x_1, \dots, x_r, \cdots, x_d \in$ gives a M-regular sequence of length $\mathrm{depth}_I(M)$ extending $x_1, \dots, x_r$. □

9.3 Vanishing Criteria on Ext

(GRADE AND (Ischebeck))

9.4 Locality of Depth

Proposition 9.4.1. Let A be a noetherian ring, $I \subset A$ an ideal, and M a finite A-module. Then,

$$\operatorname{depth}_I(M) = \inf\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I)\}$$

Proof. DOOOOOOOOOOO!!!! □

9.5 Additional Lemmas

Proposition 9.5.1. Let A be Noetherian ring, $I \subset A$ an ideal, and M a finite A-module. Then there exists an exact sequence of finite A-modules,

$$0 \longrightarrow K \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_i are finite free A-modules and $r = \operatorname{depth}(A) - \operatorname{depth}(M)$. Furthermore, given any such sequence, $\operatorname{depth}(K) = \operatorname{depth}(A)$.

Proof. There always exists a surjection $F_0 \twoheadrightarrow M$ from a finite free module F_0 because M is finite. Extending to an exact sequence,

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

gives $\operatorname{depth}_I(K) \geq \min\{\operatorname{depth}_I(A), \operatorname{depth}_I(M) + 1\}$ because F_0 is free so clearly $\operatorname{depth}_I(F_0) = \operatorname{depth}_I(A)$ by the cohomological criterion. Thus either $\operatorname{depth}_I(K) \geq \operatorname{depth}_I(A)$ already or $\operatorname{depth}_I(K) \geq \operatorname{depth}_I(M)+1$. Therefore, repeating this process r times we see that $\operatorname{depth}_I(K_r) \geq \operatorname{depth}_I(M)$ □

9.6 Cohen-Macaulay Rings

(IS THIS CORRECT AS STATED!!)

Proposition 9.6.1. Let A be a ring, $I \subset A$ an ideal, and M a finite A-module. Then,

$$\operatorname{depth}_I(M) \leq \min_{\mathfrak{p} \in \mathrm{WAss}_A(M)} \dim A/\mathfrak{p} \leq \dim \operatorname{Supp}_A(M)$$

Definition 9.6.2. Let A be a Noetherian local ring. A finite A-module M is *Cohen-Macaulay* if,

$$\operatorname{depth}(M) = \dim \operatorname{Supp}_A(M)$$

We say that A is Cohen-Macaulay if it is Cohen-Macaulay as an A-module i.e. if $\operatorname{depth}(A) = \dim A$.

Lemma 9.6.3. If A is a Cohen-Macaualy Noetherian local ring then for any prime $\mathfrak{p} \in \operatorname{Spec}(A)$ the local ring $A_{\mathfrak{p}}$ is Cohen-Macaulay.

Proof. Tag 0AAG □

Remark. This Lemma allows for the following definition.

Definition 9.6.4. A ring A is Cohen-Macaulay if A is Noetherian and $A_{\mathfrak{p}}$ is Cohen-Macaualy for each $\mathfrak{p} \in \operatorname{Spec}(A)$.

(UNIVERSALLY CATENARY ETC..)
(FIX THIS STATEMENT!!)

Proposition 9.6.5. Let R be a regular local ring and M a finite A-module. Then any exact sequence of finite A-modules

9.7 Dimension

Proposition 9.7.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. Then,

$$\dim A/(f) \geq \dim A - 1$$

with equality iff f is a nonzero divisor.

Proof. https://math.stackexchange.com/questions/2085779/the-dimension-modulo-a-principal-ideal-in-a-noetherian-local-ring □

9.8 Properties

Proposition 9.8.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$ a nonzero divisor. Then A is Cohen-Macaulay iff $A/(f)$ is Cohen-Macaulay.

Proof. We have $\text{depth}\,(A/(f)) = \text{depth}\,(A) - 1$ and $\dim A/(f) = \dim A - 1$. □

10 Finite Projective Modules over Local Rings

Remark. It is well know that if $\phi : M \to M$ is an endomorphism of Noetherian R-modules which is surjective then it is injective. However, we can remove the Noetherian hypothesis and only require M to be finitely generated (which does not imply Noetherian unless R is Noetherian).

Remark. The following proposition crucially only holds for *commutative* rings.

Theorem 10.0.1. Let M be a finite R-module and $\phi : M \to M$ a surjective endomorphism then ϕ is injective.

Proof. We consider M as a $R[X]$-module with $X \cdot m = f(m)$. Let $I = (X) \subset R[X]$ then $I \cdot M = M$ since f is surjective. Thus, by Nakayama, $\exists P(X) \in I$ such that $(1 - P(X)) \cdot M = 0$. Thus, for all $m \in M$ we have $P(X) \cdot m = m$ i.e. $m = P(f)(m)$ so if $f(m) = 0$ then $m = 0$ since $P(X) \in I$ and thus has no constant terms. □

Lemma 10.0.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring and M a finite R-module with $M \otimes_R \kappa = 0$. Then $M = 0$.

Proof. If $M \otimes_R \kappa = M/\mathfrak{m}M = 0$ then $\mathfrak{m}M = M$. However, since R is local $\mathfrak{m} = \text{Jac}\,(R)$ and M is finite so by Nakayama, $M = 0$. □

Lemma 10.0.3. Let $(R, \mathfrak{m}, \kappa)$ be a local ring and $\phi : M \to N$ a map of R modules with N finite such that $\phi \otimes \text{id}_\kappa : M \otimes_R \kappa \to N \otimes_R \kappa$ is surjective. Then ϕ is surjective.

Proof. Consider the exact sequence,

$$M \xrightarrow{\phi} N \longrightarrow \text{coker}\,\phi \longrightarrow 0$$

Since $- \otimes_R \kappa$ is right-exact, we get an exact sequence,

$$M \otimes_R \kappa \xrightarrow{\phi \otimes \text{id}_\kappa} N \otimes_R \kappa \longrightarrow \text{coker}\,\phi \otimes_R \kappa \longrightarrow 0$$

However, $\phi \otimes \text{id}_\kappa$ is surjective so by exactness $\text{coker}\,\phi \otimes_R \kappa = 0$. However, since N is finite so is $\text{coker}\,\phi$ and thus $\text{coker}\,\phi = 0$ by the lemma showing that ϕ is surjective. □

Lemma 10.0.4. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Suppose that M is a finite R-module with an endomorphism $\phi : M \to M$ such that $\phi \otimes \mathrm{id} : M \otimes_R \kappa \to M \otimes_R \kappa$ is an isomorphism then ϕ is an isomorphism.

Proof. Consider the exact sequence,

$$M \xrightarrow{\phi} M \longrightarrow \operatorname{coker} \phi \longrightarrow 0$$

and apply the right-exact functor $(-) \otimes_R \kappa$ to get,

$$M \otimes_R \kappa \xrightarrow{\phi \otimes \mathrm{id}} M \otimes_R \kappa \longrightarrow (\operatorname{coker} \phi) \otimes_R \kappa \longrightarrow 0$$

But $\phi \otimes \mathrm{id}$ is an isomorphism and the sequence is exact so $(\operatorname{coker} \phi) \otimes_R \kappa = 0$ and thus, by the previous lemma, $\operatorname{coker} \phi = 0$ so ϕ is surjective. Now we apply the previous theorem to get that ϕ is an isomorphism. $\square$

Lemma 10.0.5. Let M be a finite module over R a local ring then bases of $M \otimes_R \kappa$ lift to generating sets $R^n \twoheadrightarrow M$ giving,

$$\operatorname{rank}(M) = \dim_\kappa (M \otimes_R \kappa)$$

Proof. If M is generated by $m_1, \dots, m_n$ then $M \otimes_R \kappa = M/\mathfrak{m}M$ is generated by $\bar{m}_1, \dots, \bar{m}_n$ over $\kappa = R/\mathfrak{m}R$ since surjectivity of $R^n \to M$ is preserved after applying $(-) \otimes_R \kappa$. Thus,

$$\operatorname{rank}(M) = \dim_\kappa M \otimes_R \kappa \le n$$

Now suppose that $v_1, \dots, v_n$ is a κ-basis of $M \otimes_R \kappa = M/\mathfrak{m}M$ then choose lifts $m_1, \dots, m_n \in M$. I claim that $m_1, \dots, m_n$ generate M as an R-module. Let $N \subset M$ be the R-submodule generated by the $m_1, \dots, m_n$ and let $K = M/N$. Then I claim that $\mathfrak{m}K = K$. To see this it suffices to show that $K \subset \mathfrak{m}K$. For any $m \in M$ we know that its image $\bar{m} \in M/\mathfrak{m}M$ is in the span of the basis $v_1, \dots, v_n$ so,

$$\bar{m} = r_1 v_1 + \cdots r_n v_n$$

for $r_i \in R$. Thus,

$$m - (r_1 m_1 + \cdots r_n m_n) \in \mathfrak{m}M$$

This implies that in K we have $m \in \mathfrak{m}K$ so $K = \mathfrak{m}K$. Then since $\operatorname{Jac}(R) = \mathfrak{m}$ (because R is local) by Nakayama $K = 0$ so M is generated by $m_1, \dots, m_n$. $\square$

Theorem 10.0.6. Every finite projective module over a local ring is free.

Proof. Let P be a finite projective R-module where $(R, \mathfrak{m}, \kappa)$ is a local ring. Then there is a surjection $R^n \to P$ which we may assume gives a basis $\kappa^n \xrightarrow{\sim} P \otimes_R \kappa$. We extend to a short exact sequence,

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

but P is projective so the sequence splits giving $R^n \cong K \oplus P$ and a surjection $R^n \to K$ making K finitely generated. Since split exact sequences are preserved under additive functors,

$$0 \longrightarrow K \otimes_R \kappa \longrightarrow \kappa^n \longrightarrow P \otimes_R \kappa \longrightarrow 0$$

but the second map is an isomorphism so $K \otimes_R \kappa = 0$ and K is finite so by the lemma $K = 0$. Thus $R^n \xrightarrow{\sim} P$ is an isomorphism so P is free. $\square$

Lemma 10.0.7. Let P be a projective R-module and $S \subset R$ a multiplicative subset. Then $S^{-1}P$ is a projective $S^{-1}R$-module.

Proof. Let M, N be $S^{-1}R$-modules and consider a diagram in the category of R-modules,

$$\begin{tikzcd}
& & M \arrow[d, two heads] \\
P \arrow[r] \arrow[urr, dashed, "\phi"] & S^{-1}P \arrow[r] \arrow[ur, dashed, "\tilde{\phi}"'] & N
\end{tikzcd}$$

then $P \to N$ lifts to $\phi : P \to M$ since P is projective. Now we define $\tilde{\phi} : S^{-1}P \to M$ via $\tilde{\phi}(x \otimes r/s) = (r/s) \cdot \phi(x)$ using the decomposition $S^{-1}P = P \otimes_R S^{-1}R$. This makes the diagram commute. $\square$

Remark. We can also use the fact that (See Tag 05G3),

$$\mathrm{Hom}_{S^{-1}R}\left(S^{-1}P, -\right) = \mathrm{Hom}_{S^{-1}R}\left(P \otimes_R S^{-1}R, -\right) = \mathrm{Hom}_R\left(P, \mathrm{Res}_R^{S^{-1}R}(-)\right)$$

and that projectiveity of P is equivalent to $\mathrm{Hom}_R\left(P, \mathrm{Res}_R^{S^{-1}R}(-)\right)$ being exact showing that $S^{-1}P$ is $S^{-1}R$-projective.

Lemma 10.0.8. Let M be a finitely-presented R-module such that $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module at each prime $\mathfrak{p} \in \mathrm{Spec}\,(R)$. Then M is a localy free R-module.

Proof. Take a prime $\mathfrak{p} \in \mathrm{Spec}\,(R)$ then $M_\mathfrak{p}$ is a finite free $R_\mathfrak{p}$-module say $M_\mathfrak{p} \cong R_\mathfrak{p}^n$. Lift the basis to give a map $R^n \to M$ and an exact sequence,

$$0 \longrightarrow C \longrightarrow R^n \longrightarrow M \longrightarrow K \longrightarrow 0$$

Since M is finitely-presented, both K and C are finitely generated. Futhermore, localizing at $\mathfrak{p}$ gives,

$$0 \longrightarrow C_\mathfrak{p} \longrightarrow R_\mathfrak{p}^n \longrightarrow M_\mathfrak{p} \longrightarrow K_\mathfrak{p} \longrightarrow 0$$

but $R_\mathfrak{p}^n \to M_\mathfrak{p}$ is an isomorphism so $C_\mathfrak{p} = 0$ and $K_\mathfrak{p} = 0$. Since they are finitely generated, there is an element $f \notin \mathfrak{p}$ killing both generating sets and thus $C_f = 0$ and $K_f = 0$. Therefore,

$$0 \longrightarrow C_f \longrightarrow R_f^n \longrightarrow M_f \longrightarrow K_f \longrightarrow 0$$

is exact so $R_f^n \xrightarrow{\sim} M_f$ is an isomorphism so M is free on $D(f) \subset \mathrm{Spec}\,(R)$ for $\mathfrak{p} \in D(f)$ so M is locally free. $\square$

Theorem 10.0.9. Let R be a ring. Then finite projective R-modules are exactly the finite locally free R-modules.

Proof. If P is finite projective then $P_\mathfrak{p}$ is finite projective over $R_\mathfrak{p}$ and thus free. Furthermore, P is finitely presented because there is an exact sequence,

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

which splits $R^n \cong K \oplus P$ since P is projective giving a surjection $R^n \to K$ thus showing that K is finite and giving a finite presentation,

$$R^n \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

Therefore, by the previous lemma, P is locally free.

Conversely, if P is locally free so there exists a finite (Spec (R) is quasi-compact) open cover $D(f_i)$ of Spec (R) such that $P_{f_i} \cong R_{f_i}^n$. Then we need to show that $\mathrm{Hom}_R(P, -)$ is exact. We use that $\mathrm{Hom}_R(P, -)_{f_i} = \mathrm{Hom}_{R_{f_i}}(P_{f_i}, (-)_{f_i})$ which is exact since P_{f_i} is free and localization $(-)_{f_i}$ is an exact functor. Then $\mathrm{Hom}_R(P, -)$ is exact since we can check exactness of the hom sequence locally. ☐

Remark. Look at Tag 00NV for more detailed version.

11 Integral and Finite Extensions

Definition 11.0.1. Let $\varphi : A \to B$ be a map of rings. We say that an element $x \in B$ is *integral* over A if it satisfies a monic polynomial,

$$x^n + \varphi(a_{n-1})x^{n-1} + \cdots + \varphi(a_0) = 0$$

for $a_i \in A$. We say that φ is *integral* if every element $x \in B$ is integral over A.

(DO THIS STUFF).

12 Normal Domains

Definition 12.0.1. Let R be a domain. We say that R is *normal* if R is integrally closed in Frac (R).

Lemma 12.0.2. Let R be a domain. The following are equivalent,

(a) R is a normal domain

(b) for each multiplicative subset $S \subset R$, the localization $S^{-1}R$ is a normal domain

(c) for each prime $\mathfrak{p} \subset R$ the localization $R_{\mathfrak{p}}$ is a normal domain

(d) for each maximal ideal $\mathfrak{p} \subset R$ the localization $R_{\mathfrak{m}}$ is a normal domain.

Proof. Let R be a normal domain and $x \in K = \mathrm{Frac}(R)$ satisfying the monic polynomial,

$$x^n + \frac{r_{n-1}}{s_{n-1}}x^{n-1} + \cdots + \frac{r_0}{s_0}$$

for $\frac{r_i}{s_i} \in S^{-1}R$. Then let $s = s_{n-1} \cdots s_0$ and,

$$(sx)^n + s_0 \cdots s_{n-2} r_{n-1} (sx)^{n-1} + \cdots + s^{n-1} s_1 \cdots s_{n-1} r_0 = 0$$

and therefore $sx \in K$ is integral over R so $sx \in R$ and thus $x \in S^{-1}R$ showing that $S^{-1}R$ is integrally closed.

Clearly, (b) $\implies$ (c) $\implies$ (d). Finally, suppose that each $R_{\mathfrak{m}}$ is integrally closed. Then,

$$R = \bigcap R_{\mathfrak{m}}$$

inside K. Suppose that $x \in K$ is integral over R then x is integral over each $R_{\mathfrak{m}}$ and thus $x \in R_{\mathfrak{m}}$ for each $\mathfrak{m}$ by integral closure so $x \in R$ proving that R is an integrally closed domain. ☐

12.1 Normalization

Lemma 12.1.1. Let $\varphi : A \to B$ be a ring map. Then,

$$B' = \{b \in B \mid b \text{ is integral over } A\}$$

is an integrally closed A-subalgebra of B called the integral closure of A in B.

Proof. (DO THIS!!!) □

Proposition 12.1.2. Let A be a noetherian normal domain with $K = \mathrm{Frac}\,(A)$ and L/K a finite seperable extension. Let A' be the normalization of A in L. Then $A \subset A'$ is a finite extension of rank $n = [L : K]$.

Proof. Consider the trace pairing,

$$L \times L \to K \quad (x, y) \mapsto \langle x, y \rangle := \mathrm{Tr}_{L/K}(xy)$$

Since L/K is separable this is nondegenerate (see algebra review). Furthermore, if $x \in L$ is integral over A then $\mathrm{Tr}_{L/K}(x) \in K$ is integral over A so because A is normal $\mathrm{Tr}_{L/K}(x) \in A$. Therefore, choosing an integral K-basis $x_1, \dots, x_n \in L$ (which we can always do by clearing denominators since L/K is algebraic) then $A' \subset L$ is contained in,

$$M = \{\alpha \in L \mid \langle \alpha, x_i \rangle \in A \text{ for all } i\}$$

which is an A-module because $\langle -, x_i \rangle$ is linear. However, $M \cong A^{\oplus n}$ via choosing the dual basis of $x_1, \dots, x_n$. Thus $A' \subset A^{\oplus n}$ so A' is a finite A-module since A is noetherian. Furthermore,

$$N = Ax_1 \oplus \cdots \oplus Ax_n \subset A'$$

by definition because each $x_i \in L$ is integral. Therefore, $A^{\oplus n} \subset A' \subset A^{\oplus n}$ so by tensoring with K we see that $\mathrm{rank}\,(A') = n$. □

13 Projective and Global Dimension

13.1 Projective Dimension

Definition 13.1.1. Let M be an A-module. Then the projective dimension $\mathrm{pd}_A\,(M)$ is the minimal length r of a projective resolution of M,

$$0 \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and $\mathrm{pd}_A\,(M) = \infty$ if there does not exist a finite-length projective resolution of M.

Lemma 13.1.2 (Schanuel's lemma)**.** Let A be a ring and M an A-module. Let,

$$0 \longrightarrow K \xrightarrow{c_1} P_1 \xrightarrow{p_1} M \longrightarrow 0 \qquad\qquad 0 \longrightarrow L \xrightarrow{c_2} P_2 \xrightarrow{p_2} M \longrightarrow 0$$

be two short exact sequences of A-module where P_i are projective. Then there exists an isomorphism of short exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "(c_1\,\mathrm{id})"] \arrow[d] & P_1 \oplus P_2 \arrow[r, "(p_1\,0)"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "(\mathrm{id}\,c_2)"'] & P_1 \oplus P_2 \arrow[r, "(p_2\,0)"'] & M \arrow[r] & 0
\end{tikzcd}$$

Proof. Using projectivity of P_1 and P_2 we get maps $a : P_1 \to P_2$ and $P_2 \to P_1$ over M meaning that $p_2 \circ a = p_1$ and $p_1 \circ b = p_2$. Therefore, we get a diagram,

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "(c_1\,\mathrm{id})"] & P_1 \oplus P_2 \arrow[r, "(p_1\,0)"] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u, dashed] \arrow[d, dashed] & P_1 \oplus P_2 \arrow[r, "(p_1\,p_2)"] \arrow[u, "t"'] \arrow[d, "s"] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "(\mathrm{id}\,c_2)"'] & P_1 \oplus P_2 \arrow[r, "(p_2\,0)"'] & M \arrow[r] & 0
\end{tikzcd}$$

where $t(x, y) = (x + b(y), y)$ and $s(x, y) = (x, y + a(x))$ such that,

$$(p_1, 0) \circ t = p_1 \circ (\mathrm{id} + b) = p_1 + p_2 \quad \text{and} \quad (0, p_2) \circ s = p_2 \circ (\mathrm{id} + a) = p_1 + p_2$$

so the diagram commutes inducing maps $N \to K \oplus P_2$ and $N \to P_1 \oplus L$ where $N = \ker(P_1 \oplus P_2 \to M)$. It is clear that t and s are isomorphisms and thus the induced maps are also isomorphisms proving the claim. $\square$

Lemma 13.1.3. Let A be a ring and M an A-module with finite projective dimension. Then for any projective resolution,

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

the module $\ker(P_k \to P_{k-1})$ is projective for $k \geq \mathrm{pd}_A(M) - 1$.

Proof. We proceed by induction on $\mathrm{pd}_A(M)$. For the case $\mathrm{pd}_A(M) = 0$ then M is projective so the exact sequence,

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

splits so $P_0 = K \oplus M$ proving that K is also projective giving the case $k = 0$. Replacing M by $K = \ker(P_0 \to M)$ we prove $\ker(P_k \to P_{k-1})$ is projective for all k.

Now for induction suppose $\mathrm{pd}_A(M) = d + 1$ and let,

$$0 \longrightarrow \tilde{P}_{d+1} \longrightarrow \cdots \longrightarrow \tilde{P}_1 \longrightarrow \tilde{P}_0 \longrightarrow M \longrightarrow 0$$

be a minimal length projective resolution. By Schanuel's lemma,

$$\tilde{P}_0 \oplus \ker(P_0 \to M) \cong P_0 \oplus \ker(\tilde{P}_0 \to M)$$

If $\mathrm{pd}_A(M) = 1$ and $k = 0$ then $\ker(\tilde{P}_0 \to M)$ is projective meaning that $\ker(P_0 \to M)$ is projective as well. Otherwise let $k > 0$ and consider the projective resolutions,

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow \ker(P_0 \to M) \longrightarrow 0$$

$$0 \longrightarrow \tilde{P}_{d+1} \longrightarrow \cdots \longrightarrow \tilde{P}_1 \longrightarrow \ker(\tilde{P}_0 \to M) \longrightarrow 0$$

We cannot directly apply induction because these are not resolutions of the same module. However, applying $- \oplus \tilde{P}_0$ to the first sequence and $- \oplus P_0$ to the second we get projective resolutions of $M' = \tilde{P}_0 \oplus \ker(P_0 \to M) \cong P_0 \oplus \ker(\tilde{P}_0 \to M)$

$$\cdots \longrightarrow P_3 \oplus \tilde{P}_0 \longrightarrow P_2 \oplus \tilde{P}_0 \longrightarrow P_1 \oplus \tilde{P}_0 \longrightarrow M' \longrightarrow 0$$

$$0 \longrightarrow \tilde{P}_{d+1} \oplus P_0 \longrightarrow \cdots \longrightarrow \tilde{P}_1 \oplus P_0 \longrightarrow M' \longrightarrow 0$$

because direct sum is exact and preserves projectives. From the second sequence $\mathrm{pd}_A(M') \le d$ so we may apply induction and find that $\ker(P_k \oplus \tilde{P}_0 \to P_{k-1} \oplus \tilde{P}_0) = \ker(P_{k+1} \to P_k) \oplus \tilde{P}_0$ is projective for $k \ge d-1$ and thus $\ker(P_k \to P_{k-1})$ is projective for $k \ge d$ completing the proof. ☐

Lemma 13.1.4. Let A be a Noetherian ring and M a finite A-module. Then the following are equivalent,

(a) $\mathrm{pd}_A(M) \le d$

(b) there exists a resolution of M by finite modules F_i and P_d,

$$0 \longrightarrow P_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the F_i are finite free and P_d is finite projective.

Proof. Clearly the second implies the first since F_i are projective. Given $\mathrm{pd}_A(M) \le d$ we know $d-1 \ge \mathrm{pd}_A(M) - 1$. Since A is Noetherian and M is finite we can build a finite free resolution,

$$0 \longrightarrow P_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

by taking a generating set for M and the kernel $\ker(F_k \to F_{k-1})$ is again a finite A-module by the Noetherian property. Then let $P_d = \ker(F_{d-1} \to F_{d-2})$. Since the F_k are projective, by the previous lemma P_d is projective and finite as a submodule of a finite module. ☐

Lemma 13.1.5. Let A be a Noetherian local ring and M a finite A-module. Then the following are equivalent,

(a) $\mathrm{pd}_A(M) \le d$

(b) there exists a resolution of M by finite free modules F_i,

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Proof. This follows from above noting that finite projective A-modules are free because A is local. ☐

Proposition 13.1.6. Let A be a ring and M an A-module. Then the following are equivalent,

(a) $\mathrm{pd}_A(M) \le n$

(b) $\mathrm{Ext}^i_A(N, M) = 0$ for all A-modules A and all $i \ge n+1$

(c) $\mathrm{Ext}^{n+1}_A(N, M) = 0$ for all A-modules.

Proof. (DO THIS!!!) ☐

Lemma 13.1.7. Consider an exact sequence of A-modules,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

(a) if $\mathrm{pd}_A(M_2) \le n$ then $\mathrm{pd}_A(M_1) \le n$ and $\mathrm{pd}_A(M_3) \le n+1$

(b) if $\mathrm{pd}_A(M_1) \le n$ and $\mathrm{pd}_A(M_3) \le n$ then $\mathrm{pd}_A(M) \le n$

(c) if $\mathrm{pd}_A(M_1) \le n$ and $\mathrm{pd}_A(M) \le n+1$ then $\mathrm{pd}_A(M_3) \le n+1$.

Proof. Combine the long exact sequence of Ext groups and the previous result. ☐

13.2 Global Dimension

Definition 13.2.1. Let A be a ring. The global dimension $\mathrm{gldim}(A)$ is the supremum of $\mathrm{pd}_A(M)$ over all A-modules M.

Theorem 13.2.2. Let A be a ring. The following are equivalent,

(a) $\mathrm{gldim}(A) \le n$

(b) $\mathrm{pd}_A(M) \le n$ for all A-modules M

(c) $\mathrm{pd}_A(M) \le n$ for all finite A-modules M

(d) $\mathrm{pd}_A(M) \le n$ for all cyclic A-modules M.

Proof. Tag 065T. ☐

Lemma 13.2.3. Let A be a ring, M an A-module, and $S \subset A$ a multiplicative subset then,

(a) $\mathrm{pd}_{S^{-1}A}(S^{-1}M) \le \mathrm{pd}_A(M)$

(b) $\mathrm{gldim}(S^{-1}A) \le \mathrm{gldim}(A)$

Proof. The functor $S^{-1}(-) : \mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}$ is exact and preserves projectives because it is left-adjoint to restriction which is also exact. Therefore, if M has a projective A-resolution of length n then $S^{-1}M$ has a projective $S^{-1}A$-resolution of length at most n so $\mathrm{pd}_{S^{-1}A}(S^{-1}M) \le \mathrm{pd}_A(M)$. Notice that for any $S^{-1}A$-module M, we have $M = S^{-1}M_A$ viewing M_A as an A-module under the restriction function. Thus, applying the first part

$$\begin{aligned} \mathrm{gldim}\left(S^{-1}A\right) &= \sup\{\mathrm{pd}_{S^{-1}A}(M) \mid M \in \mathbf{Mod}_{S^{-1}A}\} \le \sup\{\mathrm{pd}_A(M_A) \mid M \in \mathbf{Mod}_{S^{-1}A}\} \\ &\le \sup\{\mathrm{pd}_A(M) \mid M \in \mathbf{Mod}_A\} = \mathrm{gldim}(A) \end{aligned}$$

☐

Proposition 13.2.4. Let R be a Noetherian ring. Then,

$$\mathrm{gldim}(R) = \sup\{\mathrm{gldim}(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} = \sup\{\mathrm{gldim}(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \mathrm{mSpec}(R)\}$$

Proof. DOO!!!!!!!!! ☐

13.3 Auslander-Buchsbaum

(MOST GENERAL VERSION!!)

13.4 Regular Rings

Remark. Throughout let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring.

Lemma 13.4.1. We always have,

$$\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$$

Proof. By Nakayma, $n = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ is the minimal number of generators of $\mathfrak{m}$. Then by Krull's ideal theorem, $\dim R = \mathbf{ht}\,(\mathfrak{m}) \leq n$. □

Corollary 13.4.2. When R is a Noetherian local ring, $\dim R$ is finite.

Proof. $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ is finite because $\mathfrak{m}$ is finitely generated since R is Noetherian. □

Definition 13.4.3. We say that R is a *regular local ring* if $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Proposition 13.4.4. Let R be a regular local ring. Then $\text{gldim}\,(R) \leq \dim R$.

Proof. DO!!!! □

Proposition 13.4.5. Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring then $\text{pd}_R\,(\kappa) \geq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

Proof. Tag 00OA. □

Proposition 13.4.6. If $\text{pd}_R\,(\kappa) < \infty$ then $\dim R \geq \text{pd}_R\,(\kappa)$.

Proof. Tag 00OB. □

Proposition 13.4.7. Let R be a Noetherian local ring. If $\text{pd}_R\,(\kappa) < \infty$ then R is a regular local ring.

Proof. The above propositions give $\dim R \geq \text{pd}_R\,(\kappa) \geq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ but $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$. □

Proposition 13.4.8. Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring. Then $\text{gldim}\,(R) < \infty$ if and only if R is a regular local ring in which case $\text{gldim}\,(R) = \dim R$.

Proof. If R is regular local then $\text{gldim}\,(R) \leq \dim R$. Conversely, if $\text{gldim}\,(R)$ is finite then $\text{pd}_R\,(\kappa) < \infty$ so R is reglar local. In this case, $\text{pd}_R\,(\kappa) = \dim R$ and $\text{gldim}\,(R) \leq \dim R$ so $\text{gldim}\,(R) = \dim R$. □

Lemma 13.4.9. If R is reglar local then $R_\mathfrak{p}$ is regular local for each prime $\mathfrak{p} \in \text{Spec}\,(R)$.

Proof. If R is regular local then $\text{gldim}\,(R) < \infty$ and thus $\text{gldim}\,(R_\mathfrak{p}) \leq \text{gldim}\,(R) < \infty$. Since $R_\mathfrak{p}$ is local and noetherian, $R_\mathfrak{p}$ is regular local as well. □

Definition 13.4.10. A noetherian ring R is *regular* if $R_\mathfrak{p}$ is regular local for each $\mathfrak{p} \in \text{Spec}\,(R)$.

Remark. The preceeding Lemma says that a regular local ring is regular.

Remark. It suffices to check regularity at $R_\mathfrak{m}$ for maximal ideals $\mathfrak{m} \in \text{mSpec}\,(R)$ since $R_\mathfrak{p}$ is a localization of some $R_\mathfrak{m}$ and we have shown that localization preserves being regular local.

Proposition 13.4.11. Let R be a Noetherian ring. The following are equivalent for each $n \in \mathbb{N}$,

(a) $\operatorname{gldim}(R) \le n$

(b) for each $\mathfrak{m} \in \operatorname{mSpec}(R)$ the ring $R_\mathfrak{m}$ is regular and $\dim R_\mathfrak{m} \le n$

(c) for each $\mathfrak{p} \in \operatorname{mSpec}(R)$ the ring $R_\mathfrak{p}$ is regular and $\dim R_\mathfrak{p} \le n$.

Therefore, if $\operatorname{gldim}(R) < \infty$ then R is regular and if R is regular then

$$\operatorname{gldim}(R) = \sup\{\dim R_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{mSpec}(R)\} = \sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$$

Proof. This follows from,

$$\operatorname{gldim}(R) = \sup\{\operatorname{gldim}(R_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mSpec}(R)\}$$

and the fact that $\operatorname{gldim}(R_\mathfrak{m}) < \infty$ is equivalent to regularity of $R_\mathfrak{m}$ in which case $\operatorname{gldim}(R_\mathfrak{m}) = \dim R_\mathfrak{m}$. □

Remark. Notice that even when R is regular $\operatorname{gldim}(R)$ may be infinite simply because the dimensions of $R_\mathfrak{m}$ for $\mathfrak{m} \in \operatorname{mSpec}(R)$ may be unbounded even when R is Noetherian. In this case, $\dim R = \infty$ so if $\dim R$ is finite then $\operatorname{gldim}(R)$ is finite iff R is regular.

14 Pseudomorphisms

Lemma 14.0.1. Let $f : X \to Y$ be a morphism of schemes such that for each weakly associated point $y \in Y$ there exists a point $x \in X$ such that $f(x) = y$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. Then the map on sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective.

Proof. To show that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective, it suffices to show that $\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$ is injective on each weakly associated point $y \in Y$. Furthermore, we know there exists $x \in X$ with $f(x) = y$ and the composition $\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$ is injective and thus $\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$ is injective. □

Remark. In particular, if $f : X \to Y$ is a dominant map of integral schemes then $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective.

Example 14.0.2. Consider the map $\operatorname{Spec}(k[x]) \to \operatorname{Spec}(k[x,y]/(xy, y^2))$. Although this map hits the generic point (y), it does not hit the embedded associated point (x, y^2) at the origin and thus $k[x,y]/(xy,y^2) \to k[x]$ is not injective since $y \mapsto 0$.

Definition 14.0.3. We say an immersion $\iota : Y \hookrightarrow X$ is *scheme theoretically dense* if the scheme theoretic image is X.

Lemma 14.0.4. An open immersion $\iota : U \to X$ is scheme theoretically dense iff U contained all weakly associated points of X.

Proof. □

When can we ensure that the coker of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is supported in codimension one.

14.1 Annhiliators

Remark. Here we let X be a scheme. Warning: many of these results do not hold for arbitrary locally ringed spaces. In particular, the kernel of quasi-coherent sheaves need not be quasi-coherent on an arbitrary locally ringed space. However, this holds locally on schemes because kernels and cokerns of sheaves associated to modules are associated to modules.

Definition 14.1.1. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then we define the sheaf of annihilators:

$$\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F}) = \ker\left(\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})\right)$$

Lemma 14.1.2. Let $\mathscr{F}, \mathscr{G}$ be quasi-coherent $\mathcal{O}_X$-modules with $\mathscr{F}$ finitely presented. Then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is quasi-coherent.

Proof. Locally on $U \subset X$ we have a presentation,

$$\bigoplus_{i=1}^m \mathcal{O}_U \longrightarrow \bigoplus_{j=1}^n \mathcal{O}_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Applying the functor $\mathscr{H}om_{\mathcal{O}_U}(-, \mathscr{G})$ gives,

$$0 \longrightarrow \mathscr{H}om_{\mathcal{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U) \longrightarrow \bigoplus_{j=1}^n \mathscr{G}|_U \longrightarrow \bigoplus_{i=1}^m \mathscr{G}|_U$$

Since $\mathscr{G}$ is quasi-coherent and finite sums and kernels of quasi-coherent sheaves are quasi-coherent we see that $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is locally quasi-coherent and thus quasi-coherent. □

Lemma 14.1.3. If $\mathscr{F}$ is finitely presented then $\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ is quasi-coherent.

Proof. From the previous lemma, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})$ is quasi-coherent. Therefore, the kernel,

$$\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F}) = \ker\left(\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})\right)$$

is quasi-coherent. □

Proposition 14.1.4. Let $\mathscr{F}$ be finitely presented. Then $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F})$ is closed and the quasi-coherent sheaf of ideals $\mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ gives a scheme structure on $\mathrm{Supp}_{\mathcal{O}_X}(\mathscr{F})$. Furthermore, $\mathscr{F}$ is naturally a $\mathcal{O}_X / \mathscr{A}nn_{\mathcal{O}_X}(\mathscr{F})$ - module.

Lemma 14.1.5. Let $f : X \to Y$ be a morphism of schemes. Assume that $\mathcal{O}_Y$ and $f_*\mathcal{O}_X$ are coherent on Y. Furthermore, for each generic point of an irreducible component $\xi \in Y$ assume that there exists some $x \in X$ with $f(x) = \xi$ and $\mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,x}$ surjective. Then $\mathscr{C} = \mathrm{coker}\left(\mathcal{O}_Y \to f_*\mathcal{O}_X\right)$ has $Z = \mathrm{Supp}_{\mathcal{O}_Y}(\mathscr{C})$ in positive codimension.

15 Singularities of Curves

Definition 15.0.1. NORMALIZATION

Proposition 15.0.2. Normalization of a curve exists and is regular.

(CAN WE GET $H^0(O_X)$ is the same?)

16 Jacobson Rings

Definition 16.0.1. A ring A is *weakly-Jacobson* if $\text{nilrad}(A) = \text{Jac}(A)$.

Definition 16.0.2. A topological space X is *weakly-Jacobson* if the closed points of X are dense.

Proposition 16.0.3. A ring A is weakly-Jacobson iff $\text{Spec}(A)$ is weakly-Jacobinson.

Proof. The closed points of $\text{Spec}(A)$ are maximal ideals $\mathfrak{m}$ which are dense iff there exists a maximal ideal in each nonempty principal open $\mathfrak{m} \in D(f)$ where $f \notin \text{nilrad}(A)$ since $D(f)$ is nonempty. Thus, $\text{Spec}(A)$ is weakly-Jacobinson iff

$$f \notin \text{nilrad}(A) \implies \exists \mathfrak{m} \in \text{mSpec}(A) : f \notin \mathfrak{m}$$

This is equivalent to $f \in \text{Jac}(A) \implies f \in \text{nilrad}(A)$ so $\text{Spec}(A)$ is weakly-Jacobson iff $\text{Jac}(A) \subset \text{nilrad}(A)$ however $\text{Jac}(A) \supset \text{nilrad}(A)$ by definition so this is equivalent to $\text{nilrad}(A) = \text{Jac}(A)$. □

Definition 16.0.4. A ring A is *Jacobson* if for any ideal $I \subset A$,

$$\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$$

Proposition 16.0.5. A ring is Jacobson iff every quotient is weakly Jacobinson.

Proof. For any ideal $I \subset A$ then consider A/I. We know $\text{nilrad}(A/I) = \text{Jac}(A/I)$ iff

$$\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$$

since $\text{Spec}(A/I) = V(I)$ so the result follows. □

Definition 16.0.6. A topological space X with closed points X_0 is Jacobson if for every closed subspace $Z \subset X$ we have $\overline{Z \cap X_0} = Z$ i.e. the closed points are dense in Z.

Remark. Clearly Jacobson rings and spaces are weakly Jacobson.

Proposition 16.0.7. A ring A is Jacobson iff its spectrum $\text{Spec}(A)$ is Jacobson.

Proof. Every closed subset $Z \subset \text{Spec}(A)$ is of the form $Z = V(I) = \text{Spec}(A/I)$ for some ideal $I \subset A$. Any closed point of X is closed in Z so if $Z \cap X_0$ is dense then Z is weakly-Jacobson so $\text{Spec}(A/I)$ is weakly-Jacobson for each ideal $I \subset A$ and thus A is Jacobson. Conversely, if A is Jacobson and $Z = V(I) \subset \text{Spec}(A)$ is closed. Any nonempty open of Z is of the form $U \cap Z \subset Z$ for an open $U \subset \text{Spec}(A)$ which contains some principal affine open $D(f) \subset U$ such that $Z \cap D(f)$ is nonempty. Then, $f \notin \sqrt{I}$ but A is Jacobson so,

$$\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$$

and thus $f \notin \mathfrak{m}$ for some $\mathfrak{m} \supset I$ so $\mathfrak{m} \in D(f)$ and $\mathfrak{m} \supset I$ i.e. $\mathfrak{m} \in V(I) \cap D(f) \subset Z \cap U$ so closed points are dense in Z so X is Jacobson. □

Theorem 16.0.8. Finitely generated k-algebras are Jacobson.

Proof. Let B be a finitely generated k-algebra. Then for any ideal $I \subset B$ we need to show that $\text{nilrad}(B/I) = \text{Jac}(B/I)$ which is equivalent to $\text{Jac}\left(B/\sqrt{I}\right) = (0)$ since $\text{nilrad}\left(B/\sqrt{I}\right) = (0)$. Since $B/\sqrt{I}$ is reduced and a finitely generated k-alegbra, it suffices to prove that if A is a reduced finitely generated k-algebra then $\text{Jac}(A) = 0$.

Let $f \in A$ be nonzero, we wish to show that $f \notin \text{Jac}(A)$ i.e. there is a maximal ideal in $D(f)$. Since A is reduced and f is nonzero f is not nilpotent so $A_f \neq 0$ meaning there exists a maximal ideal $\mathfrak{m} \in \text{mSpec}(A_f)$. Let $\mathfrak{p} \subset A$ be the preimage of $\mathfrak{m}$ under $A \to A_f$. Then there are inclusions,

$$k \hookrightarrow A/\mathfrak{p} \hookrightarrow A_f/\mathfrak{m}$$

Since $\mathfrak{m} \subset A_f$ is maximal $A_f/\mathfrak{m}$ is a field and $A_f = A[f^{-1}]$ is a finite type k-algebra hence $A_f/\mathfrak{m}$ is also so by the Nullstellensatz $A_f/\mathfrak{m}$ is a finite extension of k. Thus, since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is a domain and $A/\mathfrak{p} \hookrightarrow A_f/\mathfrak{m}$ so $A/\mathfrak{p}$ is a finite dimensional domain and thus a field so $\mathfrak{p}$ is maximal with $f \notin \mathfrak{p}$ so $f \notin \text{Jac}(A)$ and thus $\text{Jac}(A) = (0)$. ☐

17 Versions of Hilbert's Nullstellensatz

Remark. Versions of the Nullstellensatz are loosely divided into "weak" and "strong" depending on if they apply to showing an ideal is proper or to the structure of all radical ideals. Zariski's lemma is also sometimes called a Nullstellensatz because it features critically in many of its proofs.

Lemma 17.0.1 (Zariski, Weak Nullstellensatz I)**.** Let E be a finitely generated k-algebra and a field then E/k is a finite extension of fields.

Proof. This follows directly from Noetherian normalization. ☐

Theorem 17.0.2 (Weak Nullstellensatz II)**.** Let k be algebraically closed. Then the maximal ideals of $k[x_1, \dots, x_n]$ are $\mathfrak{m}_\lambda = (x_1 - \lambda_1, \dots, x_n - \lambda_n)$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in k^n$.

Proof. Let $A = k[x_1, \dots, x_n]$ and $\mathfrak{m} \subset A$ be a maximal ideal. Then $A/\mathfrak{m}$ is a finitely generated k-algebra and a field so $A/\mathfrak{m}$ is a finite extension of k but k is algebraically closed so $A/\mathfrak{m} = k$. The map $A/\mathfrak{m} \to k$ must take $x_i \mapsto \lambda_i \in k$ and thus the kernel of this map is $\mathfrak{m} = (x_1 - \lambda_1, \dots, x_n - \lambda_n)$. ☐

Theorem 17.0.3 (Strong Nullstellensatz I)**.** Finitely generated k-algebras are Jacobson.

Proof. By taking the quotient, it suffices to prove that $\text{nilrad}(A) = \text{Jac}(A)$ for finitely generated k-algebras. Let $a \notin \text{nilrad}(A)$ then A_a is nonzero because a is not nilpotent. Choose a maximal ideal $\mathfrak{m}_0 \subset A_a$. Under the map $\varphi : A \to A_a$ consider $\mathfrak{m} = \varphi^{-1}(\mathfrak{m}_0)$ which is a maximal ideal because $A/\mathfrak{m} \hookrightarrow A_a/\mathfrak{m}_0$ and $A_a/\mathfrak{m}_0$ is a finitely generated k-algebra and a field and thus a finite field extension of k proving that $A/\mathfrak{m} \subset A_a/\mathfrak{m}_0$ is a finite dimensional domain and thus a field so $\mathfrak{m}$ is maximal. However, if $a \in \mathfrak{m}$ then $a \in \mathfrak{m}_0$ but $a \in A_a$ is a unit and thus it cannot lie in any maximal ideal. Thus $a \notin \mathfrak{m}$ so $a \notin \text{Jac}(A)$. Therefore, $\text{Jac}(A) \subset \text{nilrad}(A)$ so $\text{Jac}(A) = \text{nilrad}(A)$. ☐

Definition 17.0.4. Let $I \subset k[x_1, \dots, x_n]$ be an ideal then $V(I) \subset \bar{k}^n$ is the common vanishing set,

$$V(I) = \{z \in k^n \mid \forall f \in I : f(z) = 0\}$$

For a subset $Z \subset \bar{k}^n$ define the ideal $I(Z) \subset k[x_1, \dots, x_n]$ of polynomials vanishing on Z,

$$I(Z) = \{f \in k[x_1, \dots, x_n] \mid \forall z \in Z : f(z) = 0\}$$

Remark. For general k, the set $V(I) \subset k^n$ corresponds exactly to the closed points of $V(I) = \text{Spec}\,(k[x_1, \dots, x_n]/I) \subset \mathbb{A}^n_k = \text{Spec}\,(k[x_1, \dots, x_n])$.

Proposition 17.0.5. Let $Z \subset \bar{k}^n$ be a subset. Then, in the Zariski topology,

$$V(I(Z)) = \overline{Z}$$

Proof. The vanishing of functions is closed and the intersection of such sets is closed so $V(I)$ is always closed. Futhermore, it is clear that $V(I(Z)) \supset Z$ so $V(I(Z)) \supset \overline{Z}$. However, if $Y \supset Z$ is Zariski closed then $Y = V(J)$ for some ideal $J \subset k[x_1, \dots, x_n]$. However, $V(J) \supset Z$ so J must vanish on Z and thus $J \subset I(Z)$ so $V(J) \supset V(I(Z))$. Therefore, $V(I(Z)) = \overline{Z}$. □

Theorem 17.0.6 (Weak Nullstellensatz III)**.** Let k be algebraically closed and $J \subset k[x_1, \dots, x_n]$ be an ideal. Then, $V(J) = \varnothing \iff J = (1)$.

Proof. A point $\lambda \in V(J)$ is equivalent to $\mathfrak{m}_\lambda \supset J$. If J is proper then it is contained in some maximal ideal $\mathfrak{m}$. When k is algebraically closed every maximal ideal is of the form $\mathfrak{m}_\lambda$ so $\mathfrak{m}_\lambda \supset J$ and thus $\lambda \in V(J)$. Therefore, if $J \neq (1)$ then $V(J) \neq \varnothing$. If $J = (1)$ then clearly $V(J) = \varnothing$ since 1 does not vanish anywhere. □

Theorem 17.0.7 (Strong Nullstellensatz II)**.** Let k be algebraically closed and $J \subset k[x_1, \dots, x_n]$ be an ideal. Then, $I(V(J)) = \sqrt{J}$.

Proof. Since $k[x_1, \dots, x_n]$ is Jacobson we have,

$$\sqrt{J} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$$

but since k is algebraically closed every maximal ideal is of the form $\mathfrak{m}_\lambda$ so,

$$\sqrt{J} = \bigcap_{\mathfrak{m}_\lambda \supset J} \mathfrak{m}_\lambda = \bigcap_{\lambda \in V(J)} I(\lambda) = I(V(J))$$

□

Remark. These imply that $I(V(I(Z))) = I(Z)$ and $V(I(V(J))) = V(J)$.

Remark. The weak Nullstellensatz directly implies the strong version using the Rabinowitsch trick. Indeed, let $J = (f_1, \dots, f_m)$ and $f \in I(Z(J))$. Then the polynomials $f_1, \dots, f_m, 1 - x_0 f$ in $k[x_0, x_1, \dots, x_n]$ have no common solution so by the weak form there are $g_i \in k[x_0, x_1, \dots, x_n]$ such that,

$$g_1 f_1 + \cdots + g_m f_m + g_{m+1}(1 - x_0 f) = 1$$

This holds for all x_0 hence for $x_0 = f^{-1}$ in $K(x_1, \dots, x_n)$ so we get,

$$\sum_{i=1}^{r} g_i(f(x_1, \dots, x_n)^{-1}, x_1, \dots, x_n) f_i(x_1, \dots, x_n) = 1$$

The only denominators are f so clearing denominators,

$$f^r = \sum_{i=1}^{n} h_i f_i$$

where $h_i(x_1, \dots, x_n) = f(x_1, \dots, x_n)^r g_i(f(x_1, \dots, x_n)^{-1}, x_1, \dots, x_n)$ proving that $f \in \sqrt{J}$.

This is secretly the same proof that goes from Weak Nullstellensatz I to Strong Nullstellensatz I by localizing $A \to A_f$ which is what the Rabinowitsch trick accomplishes. The condition $f \in Z(I(J))$ becomes that J generates the unit ideal of A_f since it does not lie in any maximal ideal by the weak form. Thus $f \in \sqrt{J}$ (this is the converse of choosing a maximal ideal assuming $f \notin \sqrt{J}$).

Remark. In the context of affine schemes, for a subspace $Z \subset \mathrm{Spec}\,(A)$, we define,

$$I(Z) = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$$

it is clear $V(I(Z)) = \overline{Z}$ and $I(V(J)) = \sqrt{J}$ by definition giving the bijection between Zariski closed subsets and radical ideals. This makes the statements corresponding to classical versions of the Nullstellensatz for schemes definitions rather than theorems.

18 Jacobson Schemes

Remark. Here we use Chevalley's theorem to reprove the versions of the Nullstellensatz.

Theorem 18.0.1 (Chevalley)**.** Let $f : X \to Y$ be finite presentation morphism of schemes and $C \subset X$ locally construcible. Then $f(C)$ is locally constructible.

Remark. In quasi-compact schemes, locally construcible and constructible coincide.

Theorem 18.0.2 (Nullstellensatz)**.** Let K be a finite type k-algebra and a field then K/k is finite.

Proof. Suppose not. Then there is an injection $k[t] \hookrightarrow K$ because K cannot be algebraic. Then $\mathrm{Spec}\,(K) \to \mathbb{A}^1_k$ so by Chevalley the image is constructible. But the image the generic point which is not constructible giving a contradiction. ☐

Definition 18.0.3. X is *Jacobson* if every nonempty constructible subset has a closed (in X) point.

Remark. This is equivalent to the closed points being dense in every closed set.

Corollary 18.0.4. k-schemes locally of finite type are Jacobson.

Proof. Let $C \subset X$ be construtible then there is a closed $Z \subset X$ such that $Z \cap C$ contains a nonempty affine open $U = \mathrm{Spec}\,(A)$ of Z. Since Z is closed it suffices to show that U contains a closed point of Z. Since A is nonzero there is some maximal ideal $\mathfrak{m} \subset A$ and A is a finite-type k-algebra so $A/\mathfrak{m}$ is a finite extension of k. Hence the corresponding point $x \in X$ is closed. ☐

Example 18.0.5. Some (non) examples of Jacobson schemes,

(a) finte type k-schemes are Jacobson

(b) $\mathrm{Spec}\,(\mathbb{Z})$ is Jacobson

(c) if R is a local ring of $\dim R \geq 1$ then not Jacobson

(d) $X = \mathrm{Spec}\,(R) \setminus \{\mathfrak{m}_R\}$ is Jacobson.

Proposition 18.0.6. Let S be Jacobson and $f : X \to S$ is finite type.

(a) If $x \in X$ is a closed point then $f(x)$ is closed.

(b) X is Jacobson.

Proof. For (a) let $x \in X$ be a closed point then Chevalley's theorem implies that $\{f(x)\}$ is constructible so $\{f(x)\}$ is closed because S is Jacobson. For (b) let $C \subset X$ be constructible. Then Chevalley's theorem implies that $f(C) \subset S$ is constructible so there is a closed point $s \in f(C)$. Then $X_s \to \kappa(s)$ is finite type so X_s is Jacobson. Then $X_s \cap C \subset X_s$ is nonempty constructible so it has a closed point $x \in C \cap X_s$ and X_s is closed (because $s \in S$ is closed) so x is a closed point. $\square$

Corollary 18.0.7. Let X be a finite type $\mathbb{Z}$-scheme. Then X is Jacobson and for each closed $x \in X$ the residue field $\kappa(x)$ is finite.

Proof. The first part is immediate from the fact that $\mathrm{Spec}\,(\mathbb{Z})$ is Jacobson. If $x \in X$ is a closed point then it lies over some $p \in \mathrm{Spec}\,(\mathbb{Z})$ nonzero (because x is closed) so $x \in X_p$ and X_p is finite type over $\kappa(p) = \mathbb{F}_p$ so $\kappa(x)/\mathbb{F}_p$ is finite by the Nullstellensatz. $\square$

Proposition 18.0.8. If $X \to \mathrm{Spec}\,(\mathbb{Z})$ is finite type and X is reduced then there is a dense open such that $U \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth.