1 Pseudo-effective

Definition 1.0.1. A divisor class $D \in N^1(X)_{\mathbb{R}}$ is *pseudo-effective* if it is in the closure of the cone of effective divisors.

Definition 1.0.2. A class $\alpha \in N_1(X)_{\mathbb{R}}$ is *movable* if $\alpha \cdot D \geq 0$ for any effective Cartier divisor D.

Proposition 1.0.3. If D is pseudo-effective if and only if $D \cdot \alpha \geq 0$ for all movable classes α.

Proof. If D is pseudo-effective then by definition,

$$D = \lim_{t \to 0} D_t$$

for D_t effective $\mathbb{R}$-divisors. If α is movable then by definition $D_t \cdot \alpha \geq 0$ for $t > 0$. Since intersection products are continuous (they are really polynomials in the coefficients) we have $D \cdot \alpha \geq 0$. The coverse holds for duals of cones in finite-dimensional vector spaces. Indeed, if D is not pseudo-effective, the separating hyperplane theorem ensures the existence of a numerical curve class α such that $E \cdot \alpha \geq 0$ on all effective divisors, i.e. α is movable, but $D \cdot \alpha < 0$. $\square$

2 Miyaoka's Theorem

3 Relations Between Notions of Semipositiviity

Theorem 3.0.1 (Mehta-Ramanathan)**.** Let X be a normal projective variety of dimension ≥ 2 and H an ample divisor. Let $\mathcal{E}$ be torsion-free sheaf. Then for $m \gg 0$ the restriction of $\mathcal{E}$ to a gneral member $Y \in |mH|$ is $H|_Y$-semistable if and only if $\mathcal{E}$ is H-semistable.

Therefore, we can reduce to sufficently large degree complete intersection curves.

4 The Main Theorem

Proposition 4.0.1. Let X be a smooth projective variety over a field of characteristic $p > 0$. Assume there is a $\mathbb{Q}$-divisor D with $\deg D > 0$ such that when restricted to a generated complete intersection curve $\mathscr{F}(-D)$ ample and $(\mathcal{T}_X/F)(-D)$ negative. Then on the open U where $\mathscr{F} \subset \mathcal{T}_X$ is a subbundle we have that $\mathscr{F}$ is a p-closed foliation.

Proof. The bracket defines an $\mathcal{O}_X$-linear map $\bigwedge^2 \mathscr{F} \to \mathcal{T}_X/\mathscr{F}$. This must be zero because $(\wedge^2 \mathscr{F})(-D)$ is ample but $(\mathcal{T}_X/\mathscr{F})(-2D)$ is negative if restricted to a general curve. Hence $\mathscr{F}$ is a foliation.

The p^{th}-power map induces $F^* \mathscr{F} \to (\mathcal{T}_X/\mathscr{F})$ then $F^* \mathscr{F}(-D)$ is ample on a generic curve but $(\mathcal{T}_X/\mathscr{F})(-D)$ is negative so the map is zero. $\square$

Theorem 4.0.2. Let X be a normal projective variety over an algebraically closed field of characteristic zero. If $\mathcal{T}_X$ is not generically semi-negative then X is uniruled.

Proof. Let $\mathscr{F} \subset \mathcal{T}_X$ be the maximal destabilizer and we assume $\mu(\mathscr{F}) > 0$. Then let $D = cH$ with $\mu(\mathscr{F}) > c > \mu_{\max}(\mathcal{T}_X/\mathscr{F})$ so that $\mathscr{F}(-D)$ is ample and $(\mathcal{T}_X/\mathscr{F})(-D)$ is negative on the generic complete intersection curve. Then applying the previous result we get modulo almost all primes a

p-closed foliation $\mathscr{F} \subset \mathcal{T}_X$. Then we apply the previous theorem so for almost all p the reduction of X is uniruled by rational curves C of degree bounded uniformly by,

$$C \cdot H \le \frac{3H^n}{(\det \mathscr{F}) \cdot K_X}$$

because $\mu(\mathscr{F}) > 0$ so the denominator is nonzero. Therefore, because the Hom scheme is finite type X must be uniruled. □

Is it true that X uniruled implies Ω_X not generically semipositive?

Proof. Let X be uniruled by $f : \mathbb{P}^1 \times B \dashrightarrow X$ and Ω_X be generically semipositive. Consider a generic complete intersection curve $C \subset X$ and its preimage $C' \subset \mathbb{P}^1 \times B$. Then $g : C' \to C$ is finite. Since $\Omega_X|_C$ is semipositive $g^*\Omega_X|_C$ is semipositive so $\Omega_X|_C \to \Omega_{\mathbb{P}^1 \times B}|_{C'}$ which is generically injective and of the same rank means that $\Omega_{\mathbb{P}^1 \times B}|_{C'}$ must also be semipositive. However, $\Omega_{\mathbb{P}^1 \times B} = \Omega_{\mathbb{P}^1} \boxtimes \Omega_B$ and C' is a generic complete intersection curve so $\Omega_{\mathbb{P}^1}|_{C'}$ is negative giving a contradiciton. □

5 Supplementary Lemmas

Proposition 5.0.1. Let C be a smooth projective curve over an algebraically closed field of characteristic zero. Let $\mathcal{E}$ be a locally free sheaf of rank r and $\pi : \mathbb{P}(\mathcal{E}) \to C$ the projective bundle. Let $M = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - (1/\operatorname{rank} \mathcal{E})\pi^* c_1(\mathcal{E})$. Then the following are equivalent,

(a) for any finite $f : C' \to C$ then $f^*\mathcal{E}$ is μ-semistable

(b) M is nef

(c) $\operatorname{Nef}(X) = \mathbb{R}_+ M + \mathbb{R}_+ \pi^* P$ for $P \in N^1(C)$ a generator

(d) $\overline{\operatorname{NE}}(X) = \mathbb{R}_+ M^{r-1} + \mathbb{R}_+ M^{r-2}\pi^* P$

(e) $\overline{\operatorname{Eff}}(X) = \operatorname{Nef}(X)$

(f) $\overline{\operatorname{Eff}}(X) \subset \operatorname{Nef}(X)$

(g) $M - \pi^* D$ is not pseduo-effective for any $\mathbb{Q}$-divisor D with $\deg D > 0$

(h) $M + \pi^* D$ is ample for some $\mathbb{Q}$-divisor D with $0 < \deg D < 1/r!$

(i) $M - \pi^* D$ is not pseudo-effective, where D is some $\mathbb{Q}$-divisor with $0 < \deg D < 1/r!$

(j) $\mathcal{E}$ is μ-semistable.

Proof. Let $r = \operatorname{rank} \mathcal{E}$ and $X = \mathbb{P}(\mathcal{E})$. By the canonical bundle formula, setting $\xi := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ we get

$$\xi^r = \xi^{r-1}\pi^* c_1(\mathcal{E})$$

Therefore,

$$M^r = (\xi - 1/r\, \pi^* c_1(\mathcal{E}))^r = \xi^r - \xi^{r-1}\pi^* c_1(\mathcal{E}) = 0$$

since $(\pi^* c_1(\mathcal{E}))^i = 0$ for $i > 1$. This implies that,

$$M^{r-2} \cdot (M + \pi^* D) \cdot (M - \pi^* D) = M^r - M^{r-2}(\pi^* D)^2 = 0$$

since the square of any pullback divisor is zero.

Note that $\overline{\mathrm{NE}}(X) \subset N_1(X)$ is the dual cone of $\mathrm{Nef}(X) \subset N^1(X)$ basically by definition. Let $P \in N^1(C)$ be a generator. We know that $N^1(X)$ has a basis M and P.

Suppose $D = aM + b\pi^* P$ is nef. Since $\pi^* P \cdot M^{r-2}$ is a line in a fiber which is an effective curve we see $a = D \cdot (\pi^* P) \cdot M^{r-2} \geq 0$. Furthermore, $D^r = a^{r-1} b \geq 0$ so for $a > 0$ this implies $b \geq 0$ (which is also clear for $a = 0$). Since $\pi^* P$ is nef we see $(b) \iff (c)$.

Lets show that M^{r-1} and $M^{r-2}\pi^* P$ form a basis of $N_1(X)$. Indeed, against the basis $M, \pi^* P \in N^1(X)$ the intersection pairing is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is nondegenerate. Therefore $(c) \iff (d)$ using the intersection pairing.

If M is nef then $M + \epsilon \pi^* D$ by (c) is in the interior of the nef cone hence is ample. If $D = aM + b\pi^* P$ is pseduo-effective then $D \cdot (M + \epsilon \pi^* D)^{r-2} \in \overline{\mathrm{NE}}(X)$ and so is its limit $\epsilon \to 0$ so $D \cdot M^{r-2} = aM^{r-1} + bM^{r-2}\pi^* P \in \overline{\mathrm{NE}}(X)$ hence $a, b \geq 0$ by (d). If $a, b > 0$ then D is ample and hence effective so we conclude (e).

$\square$

Lemma 5.0.2. Let $f : C' \to C$ be a separable surjective k-map of smooth complete curves. Let $\mathcal{E}$ be a bundle on C. Then the Harder-Narishiman filtration of $f^*\mathcal{E}$ is the pullback of the Harder-Narishiman filtration of $\mathcal{E}$.

Proof. Note that $\deg f^* \mathcal{E} = \deg f^* \det \mathcal{E} = (\deg f) \cdot (\deg \mathcal{E})$. By factoring the morphism it suffices to consider the case where f is Galois with galois group G. We need to show that if

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$$

is the Harder-Narisimhan filatration then $f^* \mathcal{E}_i$ is the Harder-Narisimhan filatration of $\mathcal{E}$. Since the slopes of the graded parts are still strictly decreasing after applying f^*, it suffices to show that if $\mathcal{E}$ is semistable then $f^*\mathcal{E}$ is semistable and then we apply this to the graded parts (here we use flatness of f to ensure that f^* is exact). Let $\mathscr{F} \subset f^*\mathcal{E}$ be the maximal destabilizer. Consider the G-action on $f^*\mathcal{E}$ then $\sigma_g : f^*\mathcal{E} \to f^*\mathcal{E}$ must preserve $\mathscr{F}$ since it is canonical (there is a unique maximal subbundle containing all subbundles of maximal slope) and hence $\mathscr{F}$ descends to $\mathscr{F}_0 \subset \mathcal{E}$ but $\mu(\mathscr{F}_0) = \deg f \mu(\mathscr{F})$ so since $\mu(\mathscr{F}_0) \leq \mu(\mathcal{E})$ we must have $\mu(\mathscr{F}) = \mu(f^*\mathcal{E})$ and hence $f^*\mathcal{E} = \mathscr{F}$. $\square$

IS THE FOLLOWING TRUE

Proposition 5.0.3. Let C be a smooth projective curve over an algebraically closed field of characteristic zero. Let $\mathcal{E}$ be a locally free sheaf of rank r and $\pi : \mathbb{P}(\mathcal{E}) \to C$ the projective bundle. Let $M = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Then the following are equivalent,

(a) for any finite $f : C' \to C$ then $f^*\mathcal{E}$ is semipositive

(b) M is nef

(c) $M - \pi^* D$ is not pseduo-effective for any $\mathbb{Q}$-divisor D with $\deg D > 0$

(d) $M + \pi^* D$ is ample for some $\mathbb{Q}$-divisor D with $0 < \deg D < 1/r!$

(e) $M - \pi^* D$ is not pseudo-effective, where D is some $\mathbb{Q}$-divisor with $0 < \deg D < 1/r!$

(f) $\mathcal{E}$ is semipositive.

Proof. Notice that $M^2 = c_1(\mathcal{E})$ □

Corollary 5.0.4. Let (X, H) be a normal, projective, polarized scheme over a ring R of characteristic zero, finitely generated over $\mathbb{Z}$. Let $\mathcal{E}$ be a torsion free sheaf on X. Let $K = \overline{\mathrm{Frac}\,(R)}$. If $\mathcal{E}_K$ is H-semistable on X_K then $\mathcal{E}$ is H-semistalbe on reduction mod p for almost all p.

Proof. Let $C \sim mH^{n-1}$ be a general complete intersection curve on X of large degree. Then we may assume that $\mathcal{E}|_C$ is μ-semistable on C_K hence using the above notation $M + c\pi^* H$ is ample on $\mathbb{P}(\mathcal{E}_C)_K$ but ampleness is an open condition for projective morphisms so this is satisfied for $\mathcal{E}|_C$ modulo almost every p, which implies H-semistability modulo almost every prime. □

Lemma 5.0.5. Let C be a smooth curve and $\mathcal{E}$ a vector bundle. Then $\mathcal{E}$ is μ-semistable if and only if $\mathcal{E}(-\mu)$ is semipositive.

Proof. This is almost immediate from the definition. Semistable means that for any $\mathcal{E} \twoheadrightarrow \mathcal{L}$ we have $\mu(\mathcal{L}) \geq \mu(\mathcal{E})$ and semipositive means $\mu(\mathcal{L}) \geq 0$ so shifting by $-\mu(\mathcal{E})$ these are the same condition. □

Corollary 5.0.6. Over a field of characteristic zero, if $\mathcal{E}$ is H-semistable then $\mathcal{E}^{\otimes n}$ is H-semistable. Hence the direct summands $S^m\mathcal{E}$ and $\bigwedge^m \mathcal{E}$ are H-semistable. More generally, if $\mathcal{E}_1, \mathcal{E}_2$ are H-semistable then $\mathcal{E}_1 \otimes \mathcal{E}_2$ are H-semistable.

Proof. We can reduce to a complete intersection curve of sufficiently divisible degree. Suppose $\mathcal{E}_1, \mathcal{E}_2$ are μ-semistable this means that $\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)$ are nef for $i = 1, 2$. □

Corollary 5.0.7.

Proof. We can reduce to a complete intersection curve of sufficiently divisible degree. Then we just need to show that if $\mathcal{E}_1, \mathcal{E}_2$ are semipositive then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is semipositive. Consider $\mathcal{E}_1 \otimes \mathcal{E}_2 \twoheadrightarrow \mathscr{F}$ where $\mathscr{F}$ is a vector bundle. I ONLY SEE HOW TO DO THIS IF ONE IS GLOBALLY GENERATED? □

Definition 5.0.8. Let X be a projective variety and $\mathscr{F}$ a torsion-free coherent sheaf. We say that $\mathscr{F}$ is *generically H-semipositive* if $\mu_{\min}(\mathscr{F}) \geq 0$.

Remark. This is equivalent to "generically nef". WHY?

6 Talk

7 Harder-Narasimhan Filtration

Let X be a smooth projective variety of dimension n with an ample divisor H. Then for any torsion-free coherent sheaf $\mathcal{E}$ define,

$$\mu(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\operatorname{rank} \mathcal{E}}$$

Definition 7.0.1. We say that a torsion-free sheaf $\mathcal{E}$ on X is *H-stable (resp. semistable)* if every subsheaf $0 \subsetneq \mathscr{F} \subsetneq \mathcal{E}$ satisfies,

$$\mu(\mathscr{F}) < \text{(resp. } \le \text{)}\, \mu(\mathcal{E})$$

Remark. Note that it is not true that a nonzero map $\varphi : \mathcal{E} \to \mathscr{F}$ of vector bundles implies that $c_1(\mathcal{E}) \cdot H^{n-1} \le c_1(\mathscr{F}) \cdot H^{n-1}$ unless both have the same rank. For example, consider on $\mathbb{P}^1$ the map $\mathcal{O}_X(1) \to \mathcal{O}_X(1) \oplus \mathcal{O}_X(-1)$. However, if X is smooth $\varphi : \mathcal{E} \to \mathscr{F}$ is a nonzero map of torsion-free sheaves of the same rank r then there is a map $\det \varphi : \det \mathcal{E} \to \det \mathscr{F}$ and hence we get that $c_1(\mathscr{F}) - c_1(\mathcal{E}) = c_1(\det \mathscr{F}) - c_1(\det \mathcal{E})$ is effective.

Proposition 7.0.2. Fix a torsion-free sheaf of rank r on the projective polarized variety (X, H). Then the set of slopes $\{\mu(\mathscr{F}) \mid 0 \neq \mathscr{F} \subset \mathcal{E}\} \subset \frac{1}{r!}\mathbb{Z}$ is bounded above. Let μ_1 be the maximum then $\{\mathscr{F} \subset \mathcal{E} \mid \mu(\mathscr{F}) = \mu_1\}$ contains the largest element with respect to the inclusion relation (the maximal destabilizer).

Proof. Because $\mathcal{E}$ is torsion-free there are injections,

$$\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee} \hookrightarrow \mathcal{O}_X(mH)^N$$

for some integers m, N. Therefore, it suffices to show that slopes of subsheaves of $\mathcal{O}_X(mH)^N$ are bounded. Let $\mathscr{F} \subset \mathcal{E}$ be a rank s subsheaf. At the generic point the matrix corresponding to $\mathscr{F} \hookrightarrow \mathcal{O}_X(mH)^N$ has s independent columns (because it is full rank) and hence we can choose $\mathscr{F} \hookrightarrow \mathcal{O}_X(mH)^N \to \mathcal{O}_X(mH)^s$ such that the composition is injective. Then taking determinants we get $\deg \mathscr{F} \le smH^n$ and hence $\mu(\mathscr{F}) \le mH^n$ proving a uniform bound.

Now suppose that $\mathscr{F}_1, \mathscr{F}_2 \subset \mathcal{E}$ are two subsheaves with $\mu(\mathscr{F}_1) = \mu(\mathscr{F}_2) = \mu_1$. It suffices to show that $\mu(\mathscr{F}_1 + \mathscr{F}_2) = \mu_1$. Consider the exact sequence,

$$0 \longrightarrow \mathscr{F}_1 \cap \mathscr{F}_2 \to \longrightarrow \mathscr{F}_1 \oplus \mathscr{F}_2 \longrightarrow \mathscr{F}_1 + \mathscr{F}_2 \longrightarrow 0$$

and the additivity of Chern classes,

$$r\mu(\mathscr{F}_1 + \mathscr{F}_2) = r_1\mu(\mathscr{F}_1) + r_2\mu(\mathscr{F}_2) - r'\mu(\mathscr{F}_1 \cap \mathscr{F}_2)$$

where $r = \operatorname{rank}(\mathscr{F}_1 + \mathscr{F}_2)$ and $r_i = \operatorname{rank} \mathscr{F}_i$ and $r' = \operatorname{rank}(\mathscr{F}_1 \cap \mathscr{F}_2)$. By definition of μ_1 we have $\mu(\mathscr{F}_1 \cap \mathscr{F}_2) \le \mu_1$ and thus,

$$r\mu(\mathscr{F}_1 + \mathscr{F}_2) \ge (r_1 + r_2 - r')\mu_1$$

and thus $\mu(\mathscr{F}_1 + \mathscr{F}_2) \ge \mu_1$ but trivially $\mu(\mathscr{F}_1 + \mathscr{F}_2) \le \mu_1$ so we win. $\square$

Definition 7.0.3. By the above result, setting $\mu_{\max}(\mathcal{E}) = \mu_1$ is a well-defined invariant of $(X, H, \mathcal{E})$ and so is the maximal destabilizer. By maximality, the maximal destabilizer is saturated and H-semistable.

Lemma 7.0.4. Let $\mathcal{E}$ be torsion-free and $\mathscr{F} \subset \mathcal{E}$ the maximal destabilizer. Then $\mathcal{E}$ is H-semistable iff $\mathscr{F} = \mathcal{E}$ iff $\mu(\mathcal{E}) = \mu_{\max}(\mathscr{F})$. If $\mathcal{E}$ is not H-semistable then $\mu_{\max}(\mathcal{E}/\mathscr{F}) < \mu_{\max}(\mathcal{E}) = \mu(\mathscr{F})$.

Proof. Indeed, $\mathcal{E}$ is H-semistable iff $\mu_{\max}(\mathcal{E}) = \mu(\mathcal{E})$ since this exactly means that every subsheaf has slope at most $\mu(\mathcal{E})$ but this is equivalent to $\mathscr{F} = \mathcal{E}$ since $\mathscr{F}$ is maximal amoung subsheaves with $\mu(\mathscr{F}) = \mu_{\max}(\mathcal{E})$.

Suppose that $\mu_{\max}(\mathcal{E}) > \mu(\mathcal{E})$. Then if $0 \neq \mathscr{F}' \subset (\mathcal{E}/\mathscr{F})$ is the maximal destabilizer then its preimage $\mathscr{F}'' \subset \mathcal{E}$ must satisfy $\mu(\mathscr{F}'') < \mu_{\max}(\mathcal{E})$ because $\mathscr{F}''$ strictly contains $\mathscr{F}$ then consider,

$$0 \to \mathscr{F} \to \mathscr{F}'' \to \mathscr{F}' \to 0$$

we have,

$$r\mu(\mathscr{F}) + r'\mu(\mathscr{F}') = r''\mu(\mathscr{F}'') < r''\mu(\mathscr{F})$$

and therefore,

$$r'\mu(\mathscr{F}') < (r'' - r)\mu(\mathscr{F})$$

but $r' = r'' - r$ so we conclude. $\square$

Corollary 7.0.5. There exists a filtration,

$$0 = \mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \cdots \subsetneq \mathscr{F}_s = \mathcal{E}$$

where $\mathscr{F}_{i+1}$ is the preimage in $\mathcal{E}$ of the maximal destabilizer of $\mathcal{E}/\mathscr{F}_i$. Therefore, $\mathscr{F}_{i+1}/\mathscr{F}_i$ is H-semistable and the slopes satisfy,

$$\mu_{\max}(\mathcal{E}) = \mu(\mathscr{F}_1/\mathscr{F}_0) > \mu(\mathscr{F}_2/\mathscr{F}_1) > \cdots > \mu(\mathscr{F}_s/\mathscr{F}_{s-1}) = \mu_{\min}(\mathcal{E})$$

Furthermore, $\mu_{\min}(\mathcal{E}) = -\mu_{\max}(\mathcal{E}^\vee)$ is the minimal slope of a torsion-free quotient of $\mathcal{E}$.

Proof. We need to show that the slopes of $\mathcal{E}^\vee$ are negative of the slopes of $\mathcal{E}$. Consider the filtration $\mathscr{F}'_r := \ker \mathcal{E}^\vee \to \mathscr{F}_r^\vee$. Note that c_1 is insensitive to sheaves of support in codimension ≥ 2 and therefore we may dualize torsion-free bundles without worry. Thus we get $\deg(\mathscr{F}'_r) = -\deg(\mathcal{E}^\vee) + \deg(\mathscr{F}_r)$ and there is a correspondence between saturated subsehaves $\mathscr{F} \subset \mathcal{E}$ and torsion-free quotient sheaves $\mathcal{E}^\vee \twoheadrightarrow \mathrm{im} \subset \mathscr{F}^\vee$ under which the slopes are inverted. $\square$

Lemma 7.0.6. Let $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ be a morphism of torsion-free sheaves with $\mu_{\min}(\mathcal{E}_1) > \mu_{\max}(\mathcal{E}_2)$. Then $\phi = 0$.

Proof. Assume $\phi \neq 0$ so $\mathscr{I} = \mathrm{im}\,\phi$ is nonzero. Consider the exact sequences

$$0 \to \mathscr{K} \to \mathcal{E}_1 \to \mathscr{I} \to 0$$

and

$$0 \to \mathscr{I} \to \mathcal{E}_2 \to \mathcal{Q} \to 0$$

We know that $\mu(\mathscr{I}) \geq \mu_{\min}(\mathcal{E}_1)$ since $\mathscr{I}$ is a torsion-free quotient. But also $\mu(\mathscr{I}) \leq \mu_{\max}(\mathcal{E}_2)$ and hence

$$\mu_{\min}(\mathcal{E}_1) \leq \mu(\mathscr{I}) \leq \mu_{\max}(\mathcal{E}_2)$$

$\square$

Now we need two results about semistability that I don't have time to prove.

Lemma 7.0.7. If k has characteristic zero and $\mathcal{E}, \mathcal{E}'$ are semistable then $\mathcal{E} \otimes \mathcal{E}'$ is semistable. Hence

(a) $\mathcal{E}^{\otimes n}$

(b) $\mathrm{Sym}^n(\mathcal{E})$

(c) $\wedge^n \mathcal{E}$

are all semistable.

Theorem 7.0.8. Semistability is open in flat families. In particular, let (X, H) be a normal, projective, polarized scheme over a ring R of characteristic zero, finitely generated over $\mathbb{Z}$. Let $\mathcal{E}$ be a torsion free sheaf on X. Let $K = \overline{\mathrm{Frac}\,(R)}$. If $\mathcal{E}_K$ is H-semistable on X_K then $\mathcal{E}$ is H-semistalbe on reduction mod p for almost all p.

7.1 Foliations and Bend and Break

Definition 7.1.1. A subsheaf $\mathscr{F} \subset \mathcal{T}_X$ is a *foliation* if

(a) $\mathscr{F}$ is closed under $[-,-]$

(b) $\mathscr{F}$ is saturated meanign $\mathcal{T}_X / \mathscr{F}$ is torsion-free

If X has characteristic p then we say $\mathscr{F}$ is p-closed if it is closed under the map $\partial \mapsto \partial^p$.

Theorem 7.1.2 (Ekedahl)**.** Let X be a smooth variety over a perfect field k. There is a 1-1 correspondence,

$$\{p\text{-closed foliations } \mathscr{F} \subset \mathcal{T}_X\} \iff \{X \to Y \text{ purely inseperable of height 1 with } Y \text{ normal}\}$$

Given by[1],

$$\mathscr{F} \mapsto \mathcal{O}_Y := \mathrm{Ann}_{\mathcal{O}_X}(\mathscr{F}) = \{x \in \mathcal{O}_X \mid \forall \partial \in \mathscr{F} : \partial x = 0\}$$

and

$$Y \mapsto \mathscr{F} := \ker\left(\mathcal{T}_X \to \mathcal{T}_Y\right)$$

Furthermore, $\mathscr{F} \subset \mathcal{T}_X$ is a subbundle if and only if $X \to Y$ is flat (hence Y is also smooth).

Lemma 7.1.3. Let $X \to Y$ correspond to a foliation $\mathscr{F} \subset \mathcal{T}_X$. Then there is an exact sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_Y \longrightarrow \mathrm{Frob}^* \mathscr{F} \longrightarrow 0$$

hence $\omega_X = \omega_Y \otimes (\det \mathscr{F})^{p-1}$.

Theorem 7.1.4 (Miyaoka-Mori)**.** Let k be an algebraically closed field of any characteristic and (X, H) a normal projective polarized variety defined over k. Assume that there exists an irreducible curve $C \subset X$ contained in the smooth locus of X such that $C \cdot K_X < 0$. Then for any closed point $x \in C$ there exists a rational curve $\Gamma \subset X$ through x such that

$$\Gamma \cdot H \leq \frac{2C \cdot H}{-C \cdot K_X}$$

[1]Notice that $X \to Y$ is a homeomorphism so we just need to specify $\mathcal{O}_Y$ as a sheaf of rings on X.

7.2 Main Theorem

Theorem 7.2.1. Let (X, H) be a smooth, polarized projective variety over a field of characteristic $p > 0$. Assume that there is a p-closed foliation $\mathscr{F} \subset \mathcal{T}_X$ such that,

$$(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1} > 0$$

Then X contains a rational curve C through a general point of X such that,

$$C \cdot H \leq \frac{2pH^n}{(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1}}$$

Proof. Let $\pi : X \to Y$ be the quotient by $\mathscr{F}$. Let $H^{(1)}$ be an ample divisor on $X^{(1)}$ such that $\varphi^* H^{(1)} = pH$. Let $mH^{(1)}$ be very ample and $\Gamma^{(1)} \subset X^{(1)}$ be a general complete intersection curve cut out by $mH^{(1)}$ and $\Gamma^* \subset Y$ and $\Gamma \subset X$ its inversel image with reduced structure. The natural projection $\Gamma \to \Gamma^{(1)}$ is Frobenius and Γ is numerically equivalent to $m^{n-1}H^{n-1}$ as a 1-cycle on X. Let d be the degree of $\pi : \Gamma \to \Gamma^*$ which is either 1 or p. Then we have,

$$d(\Gamma^* \cdot (-K_Y)) = \Gamma \cdot (-\pi^* K_Y) = \Gamma \cdot (-K_X + (p-1)\det \mathscr{F}) = m^{n-1}H^{n-1} \cdot (-K_X + (p-1)\det \mathscr{F})$$

Since this is positive, by Bend-and-Break through a general point of Y there exists a rational curve C' such that,

$$C' \cdot \pi_* H \leq 2\frac{\Gamma^* \cdot \pi_* H}{\Gamma^* \cdot (-K_Y)}$$

Then its image under $Y^{(-1)} \to X$ produces a rational curve C through a general point of X of degree at most,

$$C \cdot H \leq \frac{2d(\Gamma \cdot H)}{\Gamma \cdot (-\pi^* K_Y)} = \frac{2pm^{n-1}H^n}{m^{n-1}(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1}} = \frac{2pH^n}{(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1}}$$

□

Theorem 7.2.2. Let X be a normal projective variety over an algebraically closed field of characteristic zero. If X is not uniruled then all HS-slopes of Ω_X are non-negative i.e. $\mu_{\min}(\Omega_X) \geq 0$.

This is saying that to have any positivity in $\mathcal{T}_X$ at all requires that X is covering by rational curves.

Proof. Towards contradiction we can assume that $\mu_{\max}(\mathcal{T}_X) > 0$. Let $\mathscr{F} \subset \mathcal{T}_X$ be the maximal destabilizer and we assume $\mu(\mathscr{F}) > 0$. By maximality, $\mathscr{F}$ is saturated. We claim it is a foliation. Indeed, $[-,-]$ defines a map $\wedge^2 \mathscr{F} \to \mathcal{T}_X / \mathscr{F}$ but $\mathscr{F}$ is semistable so $\wedge^2 \mathscr{F}$ is also semistable (using characteristic zero here)

$$\mu(\wedge^2 \mathscr{F}) = \frac{(r-1)H^{n-1} \cdot c_1(\mathscr{F})}{\binom{r}{2}} = \frac{H^{n-1} \cdot c_1(\mathscr{F})}{r/2} = 2\mu(\mathscr{F}) > \mu(\mathcal{T}_X / \mathscr{F})$$

because $\mu(\mathscr{F}) > 0$ so $2\mu(\mathscr{F}) > \mu(\mathscr{F})$. Therefore the map is zero so $\mathscr{F}$ is a foliation.

Now we spread out $(X, H, \mathscr{F})$ to $(X_A, H_A, \mathscr{F}_A)$ over A finite type over $\mathbb{Z}$ and consider the reduction in characteristic $p \gg 0$ (abusing notation to replace $(X, H, \mathscr{F})$ by $(X_\mathfrak{p}, H_\mathfrak{p}, \mathscr{F}_\mathfrak{p})$ for $\mathfrak{p} \subset A$ with $A/\mathfrak{p}$ a fintie field of characteristic p). The claim is that $\mathscr{F}$ is p-closed. Indeed, the p-curature map

$$\psi_p : \mathrm{Frob}_p^* \mathscr{F} \to \mathcal{T}_X / \mathscr{F}$$

is linear and $\mu(\mathrm{Frob}_p^* \mathscr{F}) = p\mu(\mathscr{F}) > \mu(\mathscr{F})$ and $\mu(\mathscr{F}) > \mu(\mathcal{T}_X/\mathscr{F})$ so again since $\mathscr{F}$ is semistable for $p \gg 0$ the map is zero. Then we apply the previous theorem so for almost all p the reduction of X is uniruled by rational curves C of degree bounded asymtotically by,

$$C \cdot H \le \frac{2pH^n}{(-K_X + (p-1)\det \mathscr{F}) \cdot H^{n-1}} \sim \frac{2H^n}{(\det \mathscr{F}) \cdot H^{n-1}}$$

because $\mu(\mathscr{F}) > 0$ so the denominator is positive. The hom scheme $\mathrm{Hom}_A(\mathbb{P}^1_A, X_A)$ of maps of bounded H-degree is finite type over A. Therefore, X must be uniruled. $\square$