1 Geometry Identities

1.1 Interior Derivatives

Definition: Let ω be a k-form and X a vector field X. Then, we define the interior derivative,

$$(\iota_X \omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1})$$

Remark 1. By antisymmetry of forms $(\iota_X \circ \iota_Y + \iota_Y \circ \iota_X)\omega = 0$ and thus $\iota_X \circ \iota_X = 0$.

Lemma 1.1.

$$\mathcal{L}_X f = \mathrm{d}f(X) = X(f)$$

Proof. Consider the flow $\phi_t : M \to M$ along the vector field X. Then we define,

$$\begin{aligned} (\mathcal{L}_X f)(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\phi_t^* f) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (f \circ \phi_t) \\ &= \mathrm{d}f \circ \mathrm{d}\phi(x) \left(\frac{\partial}{\partial t} \right) = \mathrm{d}f(X) \end{aligned}$$

because, by definition,

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_t(x) = \mathrm{d}\phi(x) \left(\frac{\partial}{\partial t} \right) = X_x$$

□

Theorem 1.2. For any k-form ω and vector field X we have,

$$\mathcal{L}_X \omega = \mathrm{d}\iota_X \omega + \iota_X \mathrm{d}\omega$$

Proof. We will prove this by induction on k. For $k = 0$ we have,

$$\mathcal{L}_X f = \mathrm{d}f(X)$$

and furthermore,

$$\mathrm{d}\iota_X f + \iota_X \mathrm{d}f = \iota_X \mathrm{d}f = \mathrm{d}f(X)$$

Now we can also consider,

$$\mathcal{L}_X(\mathrm{d}f) = \mathrm{d}(\mathcal{L}_X f) = \mathrm{d}X(f)$$

Furthermore,

$$[\mathrm{d}\iota_X + \iota_X \mathrm{d}](\mathrm{d}f) = \mathrm{d}(\iota_X \mathrm{d}f) = \mathrm{d}X(f)$$

Now, since Ω^1_M is generated as a $\mathcal{O}_M$-module by the forms $\mathrm{d}f$ it will suffice to show that both sides are derivations. Then, for α a p-form and β a q-form,

$$\begin{aligned} [\mathrm{d}\iota_X + \iota_X \mathrm{d}](\alpha \wedge \beta) &= \mathrm{d}(\iota_X \alpha \wedge \beta + (-1)^p \alpha \wedge \iota_X \beta) + \iota_X(\mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta) \\ &= \mathrm{d}\iota_X \alpha \wedge \beta + (-1)^{p-1} \iota_X \alpha \wedge \mathrm{d}\beta + (-1)^p \mathrm{d}\alpha \wedge \iota_X \beta + \alpha \wedge \mathrm{d}\iota_X \beta \\ &+ \iota_X \mathrm{d}\alpha \wedge \beta + (-1)^{p+1} \mathrm{d}\alpha \wedge \iota_X \beta + (-1)^p \iota_X \alpha \wedge \mathrm{d}\beta + \alpha \wedge \iota_X \mathrm{d}\beta \\ &= [\mathrm{d}\iota_X + \iota_X \mathrm{d}]\alpha \wedge \beta + \alpha \wedge [\mathrm{d}\iota_X + \iota_X \mathrm{d}]\beta \end{aligned}$$

so both sides are derivations and thus they must be equal since they agree for a basis of 1-forms. □

2 The Hodge Complex

Definition: Let (M, g) be an oriented Riemannian n-manifold and vol_g the canonical volume form. Then $g : TM \otimes TM \to \mathcal{O}_M$ defines a fiberwise nondegenerate inner product which we may view as an isomorphism $g : TM \to T^*M$ which, along with its inverse $g^{-1} : T^*M \to TM$, extends to isomorphisms on dual tensor bundles $T^n_m M \xrightarrow{\sim} T^m_n M$ and thus a nondegenerate pairing $\langle -, - \rangle : T^n_m M \otimes T^n_m M \to \mathcal{O}_M$.

Then we can define a Hilbert space $L^2(\mathcal{C}^\infty(M, T^n_m M))$ on the tensor bundles T^n_m via the inner product,

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle \,\mathrm{vol}_g$$

Since vol_g is nonvanishing and the functions are smooth (and thus continuous) then,

$$||\alpha||^2 = \langle\langle \alpha, \alpha \rangle\rangle = 0 \iff \alpha = 0$$

Definition: On an oriented Riemannian n-manifold with canonical volume form vol_g we define the Hodge dual $\star : \Omega^k_M \to \Omega^{n-k}_M$ as the unique map such that,

$$\forall \alpha, \beta \in \Omega^k_M(U) : \alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle \,\mathrm{vol}_g$$

Furthermore, we have $\star \star \eta = (-1)^{n(n-k)}\eta$.

Definition: We define the codifferential $\delta : \Omega^{k+1}_M \to \Omega^k_M$ via $\delta = (-1)^{k+1} \star^{-1} \mathrm{d}\star$.

Remark 2. This makes a chain complex since,

$$\delta \circ \delta = (-1)^{2k-1}(\star^{-1}\mathrm{d}\star) \circ (\star^{-1}\mathrm{d}\star) = - \star^{-1} \mathrm{d} \circ \mathrm{d}\star = 0$$

Lemma 2.1. For all $\alpha \in \Omega^k_M(U)$ and $\beta \in \Omega^{k+1}_M(U)$ we have,

$$\langle\langle \mathrm{d}\alpha, \beta \rangle\rangle = \langle\langle \alpha, \delta\beta \rangle\rangle$$

Proof. We have $\langle \mathrm{d}\alpha, \beta \rangle \,\mathrm{vol}_g = \mathrm{d}\alpha \wedge (\star\beta)$. Now consider,

$$\begin{aligned} \mathrm{d}(\alpha \wedge (\star\beta)) &= \mathrm{d}\alpha \wedge (\star\beta) + (-1)^k \alpha \wedge \mathrm{d}(\star\beta) \\ &= \mathrm{d}\alpha \wedge (\star\beta) + (-1)^k \alpha \wedge (\star(\star^{-1}\mathrm{d}(\star\beta))) \\ &= \mathrm{d}\alpha \wedge (\star\beta) - \alpha \wedge (\star\delta\beta) \end{aligned}$$

Then, by Stokes' theorem,

$$\int_M \mathrm{d}(\alpha \wedge (\star\beta)) = \int_{\partial M} \alpha \wedge (\star\beta) = 0$$

because M is closed. Therefore,

$$\langle\langle \mathrm{d}\alpha, \beta \rangle\rangle = \int_M \langle \mathrm{d}\alpha, \beta \rangle \,\mathrm{vol}_g = \int_M \mathrm{d}\alpha \wedge (\star\beta) = \int_M \alpha \wedge (\star\delta\beta) = \int_M \langle \alpha, \delta\beta \rangle \,\mathrm{vol}_g = \langle\langle \alpha, \delta\beta \rangle\rangle$$

□

Definition: We define the Laplace-deRham operator,

$$\Delta = \delta \circ \mathrm{d} + \mathrm{d} \circ \delta : \Omega^k_M \to \Omega^k_M$$

We say a k-form ω is *harmonic* if $\Delta\omega = 0$ and we denote the space of harmonic k-forms as $\mathcal{H}^k(M)$.

Remark 3. To motivate this definition, choose local coordinates such that,

$$\omega = g \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

and consider a k-form in local coodinates,

$$\eta = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

Then,

$$\mathrm{d}\eta = \sum_j \sum_{i_1 < \cdots < i_k} \frac{\partial f_{i_1, \dots, i_k}}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

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Lemma 2.2. $\star\Delta = \Delta\star$

Proof. It is clear that $\star\delta = (-1)^k \mathrm{d}\star$ and $\star\mathrm{d} = (-1)^k \delta\star$. Therefore,

$$\begin{aligned}\star\Delta &= \star(\delta\mathrm{d} + \mathrm{d}\delta) = (-1)^k \mathrm{d} \star \mathrm{d} + (-1)^k \delta \star \delta \\ &= \mathrm{d}\delta \star + \delta\mathrm{d}\star = \Delta\star\end{aligned}$$

□

Lemma 2.3. We have $\Delta\omega = 0$ iff $\mathrm{d}\omega = \delta\omega = 0$.

Proof. Clearly if $\mathrm{d}\omega = \delta\omega = 0$ then $\Delta\omega = 0$. Conversely, suppose that,

$$\Delta\omega = [\delta\mathrm{d} + \mathrm{d}\delta]\omega = 0$$

Consider,

$$\langle\langle \Delta\omega, \omega \rangle\rangle = \langle\langle \delta\mathrm{d}\omega, \omega \rangle\rangle + \langle\langle \mathrm{d}\delta\omega, \omega \rangle\rangle = \langle\langle \mathrm{d}\omega, \mathrm{d}\omega \rangle\rangle + \langle\langle \delta\omega, \delta\omega \rangle\rangle = ||\mathrm{d}\omega||^2 + ||\delta\omega||^2$$

Since $||\alpha|| \geq 0$ we see that if $\langle\langle \Delta\omega, \omega \rangle\rangle = 0$ then $||\delta\omega||^2 = 0$ and $||\mathrm{d}\omega||^2 = 0$ and thus $\delta\omega = 0$ and $\mathrm{d}\omega = 0$. □

Remark 4. Using this alternative characterization, we can make an alternative motivation for the definition of Δ. Suppose we wanted to choose the representative $[\alpha] \in H^k_{\mathrm{dR}}(X)$ with minimum norm $||\alpha||$. According to calculus of variation we should perturb alpha slightly by an exact form to give $\alpha t d\eta$ and compute,

$$||\alpha + t\,\mathrm{d}\eta||^2 = ||\alpha||^2 + 2t\,\langle\langle \alpha, \mathrm{d}\eta \rangle\rangle + t^2 ||\mathrm{d}\eta||^2 = ||\alpha||^2 + 2t\,\langle\langle \delta\alpha, \eta \rangle\rangle + O(t^2)$$

Therefore, since we want the norm to be extremal we require it be constant to first order for every test form η. Setting $\eta = \delta\alpha$ forces $\delta\alpha = 0$. Since $[\alpha]$ is a cohomology class, we also have $\mathrm{d}\alpha = 0$ and thus the minimal norm classes are represented by harmonic forms $\Delta\alpha = 0$.

Theorem 2.4 (Hodge)**.** The space $\mathcal{H}^k(M)$ is finite dimensional and there is a canonical decomposition,

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k(M) = \mathrm{d}(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k(M)$$

Proof. The decomposition,

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k(M) = \operatorname{Im}\Delta \oplus \ker\Delta$$

follows immediately from splitting the sequence,

$$0 \longrightarrow \ker\Delta \longrightarrow \Omega^k(M) \longrightarrow \operatorname{Im}\Delta \longrightarrow 0$$

First, suppose that $\eta = \mathrm{d}\alpha = \delta\beta$ then,

$$||\eta||^2 = \langle\langle \eta, \eta \rangle\rangle = \langle\langle \mathrm{d}\alpha, \delta\beta \rangle\rangle = \left\langle\left\langle \mathrm{d}^2\alpha, \beta \right\rangle\right\rangle = 0$$

and thus $\eta = 0$. Thus, $\mathrm{d}(\Omega^{k-1}(M)) \cap \delta(\Omega^{k+1}(M)) = (0)$ so,

$$\mathrm{d}(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \subset \Omega^k(M)$$

Clearly,

$$\Delta(\Omega^k(M)) \subset \mathrm{d}(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))$$

because $\Delta\alpha = \mathrm{d}(\delta\alpha) + \delta(\mathrm{d}\alpha)$. Furthermore, if $\mathrm{d}\alpha \in \mathcal{H}^k(M)$ then $\delta\mathrm{d}\alpha = 0$ but

$$||\mathrm{d}\alpha||^2 = \langle\langle \mathrm{d}\alpha, \mathrm{d}\alpha \rangle\rangle = \langle\langle \alpha, \delta\mathrm{d}\alpha \rangle\rangle = 0$$

so $\mathrm{d}\alpha = 0$ and similarly if $\delta\beta \in \mathcal{H}^k(M)$ then $\mathrm{d}\delta\beta = 0$ but,

$$||\delta\beta||^2 = \langle\langle \delta\beta, \delta\beta \rangle\rangle = \langle\langle \mathrm{d}\delta\beta, \beta \rangle\rangle = 0$$

so $\delta\beta = 0$. Therefore,

$$[\mathrm{d}(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))] \cap \mathcal{H}^k(M) = (0)$$

showing that,

$$\Delta(\Omega^k(M)) = \mathrm{d}(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))$$

The finite dimensionality of $\mathcal{H}^k(M)$ follows from the theory of elliptic operators on compact manifolds. However, we will prove it using the following result plus the following results: de Rham's theorem $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{sing}}(M)$, the fact that singular cohomology is finitely generated for a finite CW complex, and that any compact manifold has the homotopy type of a finite CW complex. $\square$

Theorem 2.5 (Hodge)**.** Let M be compact oriented Riemann manifold. Then every deRham cohomology class on M has a unique harmonic representative and thus the canonical map,

$$\mathcal{H}^k(M) \xrightarrow{\sim} H^k_{\mathrm{dR}}(M)$$

is an isomorphism.

Proof. I claim that,

$$\ker (\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)) = \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$$

We can write $\eta = \mathrm{d}\alpha + \delta\beta + \varphi$ where φ is harmonic. Suppose that $\mathrm{d}\eta = 0$ then $\mathrm{d}\delta\beta = 0$ which we have shown implies that $\delta\beta = 0$ so $\eta = \mathrm{d}\alpha + \varphi$ and thus,

$$\ker \mathrm{d} \subset \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$$

but it is clear that $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ vanishes on $\mathrm{d}(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$ so,

$$\ker \mathrm{d} = \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$$

Using this we immediately see that the map,

$$\mathcal{H}^k(M) \to H^k_{\mathrm{dR}}(M) \qquad \varphi \mapsto [\varphi]$$

is an isomorphism because,

$$\ker \mathrm{d} / \operatorname{Im} \mathrm{d} = [\mathrm{d}(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)]/\mathrm{d}(\Omega^{k-1}(M)) = \mathcal{H}^k(M)$$

Explicily, if $[\varphi] = 0$ then $\varphi = \mathrm{d}\alpha$ but then $\Delta \mathrm{d}\alpha = \delta \mathrm{d}\alpha = 0$ which implies that $\varphi = \mathrm{d}\alpha = 0$ so $\mathcal{H}^k(M) \to H^k_{\mathrm{dR}}(M)$ is injective. Furthermore, consider a class $[\alpha] \in H^k_{\mathrm{dR}}(N)$ with $\mathrm{d}\alpha = 0$ then, by above, $\alpha \in \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$ so $\alpha = \varphi + \mathrm{d}\beta$ for some harmonic form $\varphi \in \mathcal{H}^k(M)$ and thus,

$$[\alpha] = [\varphi]$$

so the map $\mathcal{H}^k(M) \to H^k_{\mathrm{dR}}(M)$ is surjective. □

Theorem 2.6 (Poincare)**.** Let M be compact oriented Riemann manifold. There is a canonical isomorphism $H^k_{\mathrm{dR}}(M) \xrightarrow{\sim} H^{n-k}_{\mathrm{dR}}(M)^\vee$.

Proof. Consider the bilinear pairing $H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}$ via,

$$B([\omega], [\eta]) = \int_M \omega \wedge \eta$$

This is well-defined since if $\tilde{\omega} = \omega + \mathrm{d}\alpha$ and $\tilde{\eta} = \eta + \mathrm{d}\beta$ then,

$$\begin{aligned} \int_M \tilde{\omega} \wedge \tilde{\eta} &= \int_M (\omega + \mathrm{d}\alpha) \wedge (\eta + \mathrm{d}\beta) \\ &= \int_M \omega \wedge \eta + \int_M \omega \wedge \mathrm{d}\beta + \int_M \mathrm{d}\alpha \wedge (\eta + \mathrm{d}\beta) \end{aligned}$$

However,

$$\mathrm{d}(\omega \wedge \beta) = \mathrm{d}\omega \wedge \beta + (-1)^k \omega \wedge \mathrm{d}\beta$$

But ω is closed so we have,

$$\int_M \omega \wedge \mathrm{d}\beta = (-1)^k \int_M \mathrm{d}(\omega \wedge \beta) = (-1)^k \int_{\partial M} \omega \wedge \beta = 0$$

since M has no boundary. Likewse, since $\eta + \mathrm{d}\beta$ is closed we have,

$$\int_M \mathrm{d}\alpha \wedge (\eta + \mathrm{d}\beta) = \int_{\partial M} \alpha \wedge (\eta + \mathrm{d}\beta) = 0$$

Thus,

$$\int_M \tilde{\omega} \wedge \tilde{\eta} = \int_M \omega \wedge \eta$$

so this bilinear pairing is well-defined.

Now, it suffices to prove that the pairing is non-degenerate. For any class $[\omega]$ we can choose a harmonic representative φ. Furhtermore $\star\varphi$ is harmonic since,

$$\Delta \star \varphi = \star\Delta\varphi = 0$$

so it represents a class $[\star\varphi] \in H^{n-k}_{\mathrm{dR}}$. Then,

$$B([\omega], [\star\varphi]) = B([\varphi], [\star\varphi]) = \int_M \varphi \wedge (\star\varphi) = \int_M \langle \varphi, \varphi \rangle \, \omega = ||\varphi||^2 = 0 \iff \varphi = 0$$

which shows that B is nondegenerate. $\square$

3 Local Systems

Definition: A $\mathcal{A}$-local system is a locally constant sheaf in the category $\mathcal{A}$ i.e. a sheaf $\mathcal{L}$ on X such that for each $x \in X$ there exists some open neighborhood U and an object A such that $\mathcal{L}|_U \cong \underline{A}$.

Lemma 3.1. If X is connected then any local system has constant fibers and thus we may take its constant objects on the trivializing neighborhoods to be equal.

Proof. For some fixed $p \in X$ let $D_p = \{x \in X \mid \mathcal{L}_x \cong \mathcal{L}_p\}$. Since $\mathcal{L}$ is a local system, for any $x \in X$ we have an open U s.t. $\mathcal{L}|_U = \underline{A_x}$. If $x \in D_p$ then $\mathcal{L}_x \cong A_x \cong \mathcal{L}_p$. But then for any $y \in U$ we have,

$$\mathcal{L}_y \cong A_x \cong \mathcal{L}_x \cong \mathcal{L}_p$$

so $x \in U \subset D_p$ and thus D_p is open. Therefore,

$$X = \bigcup_{p \in X} D_p$$

is an open partition which implies that,

$$D_p^C = \bigcup_{x \neq p} D_p$$

is open so D_p is clopen. Since X is connected and $p \in D_p$ we have $D_p = X$. $\square$

Proposition 3.2. Let X be locally connected and $\mathcal{L}$ be a $\mathcal{A}$-local system. Then there is a canonical functor $A : \Pi_1(X) \to \mathcal{A}$.

Proof. Consider a path $\gamma : I \to X$ from x to y. Then, since $\operatorname{Im}\gamma$ is compact, we can choose a finite conver of $\operatorname{Im}\gamma$ by connected trivializing neighborhoods U_i s.t. $U_i \cap U_{i+1} \neq \varnothing$ and $x \in U_0$ and $y \in U_n$. Then on each we have $\mathcal{L}|_{U_i} \cong F$. Now we construct a map $[\gamma] : \mathcal{F}_x \to \mathcal{F}_y$ as follows. For a germ $f \in \mathcal{L}_x$ we lift to a section $f \in \mathcal{L}(U_0)$ since is constant on U_0. Now, suppose we have a section $f_i \in \mathcal{L}(U_i)$, choose a connected open $V \subset U_i \cap U_{i+1}$ then $f_i|_V \in \mathcal{L}(V)$. Since $\mathcal{L}|_{U_{i+1}}$ is constant then the restriction map,

$$\mathrm{res}_{V,U_{i+1}} : \mathcal{L}(U_{i+1}) \to \mathcal{L}(V)$$

is an isomorphism and thus we get a section $f_{i+1} = \mathrm{res}^{-1}_{V,U_{i+1}}(f_i|_V)$. Then we choose $\alpha_\gamma f = f_n$ which is the germ of $f_n \in \mathcal{L}(U_n)$. It is clear that this is a morphism and invariant under homotopy giving a well-defined map $\Pi_1(X, x, y) \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{L}_x, \mathcal{L}_y)$. □

Proposition 3.3. Let X be path-connected and locally connected and $\mathcal{L}$ be a local system with fiber $\mathcal{L}_p \cong F$. Then there is a canonical action $\pi_1(X, x_0) \to \operatorname{Aut}(\mathcal{L}_{x_0})$ and $\Gamma(X, \mathcal{F}) = \mathcal{L}_{x_0}^{\pi_1(X,x_0)}$.

Proof. Consider the case $x_0 = x = y$ then we have a map $\pi_1(X, x_0) \to \operatorname{Aut}(\mathcal{L}_{X_0})$. Now, consider the restriction map $\Gamma(X, \mathcal{L}) \to \mathcal{L}_{x_0}$. Since restrictions compose we have $\alpha_\gamma f|_{x_0} = f|_{x_0}$ since $f_i = f|_{U_i}$ and $(f|_{U_n})_{x_0} = f_{x_0}$ so the image lies in $\mathcal{L}_{x_0}^{\pi_1(X,x_0)}$. Conversely, consider $f \in \mathcal{L}_{x_0}^{\pi_1(X,x_0)}$ such that $[\gamma] \cdot f = f$ for any loop $\gamma : I \to X$. Now, taking $x \in X$ we can define $f_x = [\gamma] \cdot f$ where γ is a path from x_0 to x. This is well-defined because if $\gamma, \delta : I \to X$ are two paths from x_0 to x then $\delta^{-1} * \gamma$ is a loop at x_0 and $\alpha_{\delta^{-1}*\gamma} = \alpha_\delta^{-1} \circ \alpha_\gamma$ but by assumption $\alpha_{\delta^{-1}*\gamma} = \mathrm{id}$ so $\alpha_\gamma = \alpha_\delta$. Furthermore, each f_x lifts to $f_x \in \mathcal{L}(U_x)$ for some trivializing neighborhood and these sections glue to a global section by the construction of the morphisms. This construction gives an inverse map $\mathcal{L}_{x_0}^{\pi_1(X,x_0)} \to \Gamma(X, \mathcal{L})$ showing the given isomorphism. □

3.1 Connections

Definition: Let $\mathcal{E}$ be a coherent sheaf on X. Then a *connection* on $\mathcal{E}$ is a morphism $\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ of *abelian* sheaves (not $\mathcal{O}_X$-modules) which satisfies the Leibniz rule,

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

Proposition 3.4. Given a connection $\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ it naturally extends to a connection $\nabla_k : \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{E}$ via,

$$\nabla_k(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^k \omega \wedge \nabla e$$

Definition: The connection ∇ defines a corresponding curvature form,

$$\omega_\nabla = \nabla_1 \circ \nabla : \mathcal{E} \to \Omega^2_X \otimes_{\mathcal{O}_S} \mathcal{E}$$

We say that ∇ is flat or integrable if the curvature vanishes $\omega_\nabla = \nabla_1 \circ \nabla = 0$.

Proposition 3.5. When ∇ is flat we have $\nabla_{k+1} \circ \nabla_k = 0$ for all k. In this case we have the $\mathcal{E}$-valued deRham complex,

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\nabla_1} \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \cdots$$

whose hypercholomogy gives the deRham cohomology with coefficients in $\mathcal{E}$,

$$H^k_{\mathrm{dR}}(X, \mathcal{E}) = \mathbb{H}^k(X, \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{E})$$

Definition: A connection ∇ on $\mathcal{E}$ defines a subsheaf $\mathcal{E}^\nabla = \ker \nabla \subset \mathcal{E}$ of *horizontal* or *flat* sections.

Lemma 3.6. The curvature $\omega_\nabla : \mathcal{E} \to \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{E}$ is a $\mathcal{O}_X$-module map.

Proof. Consider,

$$\begin{aligned} \omega_\nabla(fs) &= \nabla_1(\mathrm{d}f \otimes s + f\nabla s) = \mathrm{dd}f \otimes s - \mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + f\nabla_1 \circ \nabla \\ &= f\nabla_1 \circ \nabla s = f\, \omega_\nabla(s) \end{aligned}$$

□

Remark 5. If we write locally,

$$\nabla e = \sum_i f_i \mathrm{d}g_i \otimes s_i$$

then the curvature takes the form,

$$\omega_\nabla(e) = \sum_i (\mathrm{d}f_i \wedge \mathrm{d}g_i \otimes e - f_i \mathrm{d}g_i \otimes \nabla s_i)$$

Proposition 3.7. ∇ is flat iff the $\mathcal{O}_X$-map $Q : \mathscr{D}er(\mathcal{O}_X, \mathcal{O}_X) \to \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{E})$ given by sending D to,

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{D \otimes \mathrm{id}} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{a \otimes b \mapsto ab} \mathcal{E}$$

is a morphism of sheaves of Lie algebras.

Remark 6. In the definition of $Q(D)$ we have used D as an $\mathcal{O}_X$-module morphism $\Omega^1_X \to \mathcal{O}_X$ via the universal property of Ω^1_X,

$$\mathscr{D}er(\mathcal{O}_X, \mathcal{O}_X) \cong \mathscr{H}om_{\mathcal{O}_X}\left(\Omega^1_X, \mathcal{O}_X\right) = \mathscr{T}_X$$

which identifies $\mathscr{D}er(\mathcal{O}_X, \mathcal{O}_X)$ with the tangent sheaf $\mathscr{T}_X$.

Proof. We need to check that $Q[D_1, D_2] = [Q(D_1), Q(D_2)]$ is equivalent to $\nabla_1 \circ \nabla = 0$. Now,

$$[D_1, D_2] \in \mathrm{Hom}_{\mathcal{O}_U}\left(\Omega^1_U, \mathcal{O}_U\right)$$

is the unique $\mathcal{O}_X$-map such that,

$$[D_1, D_2] \circ \mathrm{d} = D_1 \circ \mathrm{d} \circ D_2 \circ \mathrm{d} - D_2 \circ \mathrm{d} \circ D_1 \circ \mathrm{d}$$

Now consider this action locally,

$$[D_1, D_2] \otimes \mathrm{id} \circ \nabla = \sum_i f_i(D_1 \circ \mathrm{d} \circ D_2 \circ \mathrm{d} - D_2 \circ \mathrm{d} \circ D_1 \circ \mathrm{d})(g_i) \cdot s_i$$

Furthermore,

$$[Q(D_1), Q(D_2)] = (D_1 \otimes \mathrm{id}) \circ \nabla \circ (D_2 \otimes \mathrm{id}) \circ \nabla - (D_2 \otimes \mathrm{id}) \circ \nabla \circ (D_1 \otimes \mathrm{id}) \circ \nabla$$

Again consider its local action,

$$\begin{aligned} Q(D_1) \circ Q(D_2)(e) &= (D_1 \otimes \mathrm{id}) \circ \nabla \left(\sum_i f_i D_2(\mathrm{d}g_i) \cdot s_i \right) \\ &= \sum_i \Big([D_2(\mathrm{d}g_i) D_1(\mathrm{d}f_i) + f_i D_1(\mathrm{d}(D_2(\mathrm{d}g_i)))] \cdot s_i + f_i D_2(\mathrm{d}g_i) D_1(\nabla s_i) \Big) \end{aligned}$$

Now consider,

$$\begin{aligned} &\Big[Q(D_1) \circ Q(D_2) - Q(D_2) \circ Q(D_1)] - Q([D_1, D_2]) \Big](e) \\ &= \sum_i \Big(D_1(\mathrm{d}f_i) D_2(\mathrm{d}g_i) - D_2(\mathrm{d}f_i) D_1(\mathrm{d}g_i) \Big) \cdot s_i \\ &+ \sum_i f_i \Big(D_1(\mathrm{d}(D_2(\mathrm{d}g_i))) - D_2(\mathrm{d}(D_1(\mathrm{d}g_i))) \Big) \cdot s_i \\ &+ \sum_i \Big(f_i D_2(\mathrm{d}g_i) D_1(\nabla s_i) - f_i D_1(\mathrm{d}g_i) D_2(\nabla s_i) \Big) \\ &- \sum_i f_i (D_1 \circ \mathrm{d} \circ D_2 \circ \mathrm{d} - D_2 \circ \mathrm{d} \circ D_1 \circ \mathrm{d})(g_i) \cdot s_i \\ &= \sum_i \Big(D_1(\mathrm{d}f_i) D_2(\mathrm{d}g_i) - D_2(\mathrm{d}f_i) D_1(\mathrm{d}g_i) \Big) \cdot s_i \\ &+ \sum_i \Big(f_i D_2(\mathrm{d}g_i) D_1(\nabla s_i) - f_i D_1(\mathrm{d}g_i) D_2(\nabla s_i) \Big) \\ &= (D_1 \otimes D_2 - D_2 \otimes D_1) \otimes \mathrm{id}_{\mathcal{E}} \circ \omega_\nabla \end{aligned}$$

which is defined on $(\Omega^1_X)^{\otimes 2} \otimes_{\mathcal{O}_X} \mathcal{E}$ but descends to $\Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{E}$ since it sends the ideal $\omega \otimes \omega \mapsto 0$. Therefore, we see that Q is a Lie algebra map iff

$$\forall D_1, D_2 \in \mathrm{Hom}_{\mathcal{O}_X}\left(\Omega^1_X, \mathcal{O}_X\right) : (D_1 \wedge D_2) \otimes \mathrm{id}_{\mathcal{E}} \circ \omega_\nabla = 0$$

in particular when $\omega_\nabla = 0$. Furthermore when Q is a Lie algebra map then we must have $\omega_\nabla = 0$ since, for any fixed form, there exists sections of Ω^1_X which do not kill it. $\square$

Example 3.8. For $\mathcal{E} = \mathcal{O}_X$ we have the universal connection $\mathrm{d} : \mathcal{O}_X \to \Omega^1_X$. Then the statment that d is flat is equivalent to $\mathrm{d}^2 = 0$ leading to the deRham complex. Furthermore this means that d induces a Lie algebra map,

$$\mathscr{T}_X \to \mathscr{E}\!nd_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X$$

sending a vector field v to the map $f \mapsto \langle v, \mathrm{d}f \rangle$ proving the identity, $\langle [v,u], \mathrm{d}f \rangle = 0$ since $\mathcal{O}_X$ has trivial Lie algebra structure.

Example 3.9. A connection on a scheme or manifold X is a connection on the cotangent (or equivalently tangent) bundle $\nabla : \Omega^1_X \to (\Omega^1_X)^{\otimes 2}$. Such a connection is equivalent to a choice of global section $g \in \Gamma(X, \mathrm{Sym}^2(\Omega^1_X))$ i.e. a metric. We say that (X, g) is flat if this connection ∇ is flat. In this case we have an augmented deRham complex $(\Omega^\bullet_X \otimes_{\mathcal{O}_X} \Omega^1_X, \nabla)$.

Remark 7. Note that a connection $\nabla : \mathcal{O}_X \to \Omega^1_X$ does NOT induce a connection on Ω^1_X. Such a connection induces a connection,

$$\nabla_1 : \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \Omega^1_X \to \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \Omega^2_X = \bigwedge^2 \Omega^1_X$$

but it is only well-defined in the exterior algebra not on the tensor algebra $\Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X$. There is always a canonical derivation i.e. connection $\mathrm{d} : \Omega_X \to \Omega^1_X$ but there is not generically a map $\Omega^1_X \to (\Omega^1_X)^{\otimes 2}$.

3.2 Vector Bundles

Proposition 3.10. Let $\mathcal{E}$ be a vector bundle on X with a flat connection

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

Then $\mathcal{E}^\nabla = \ker \nabla$ is a local system.

Proof. Since $\mathcal{E}$ is locally free, we can find a cover of trivializing neighbrohoods U such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$. Then $\nabla : \mathcal{O}_U^{\oplus n} \to (\Omega^1_U)^{\oplus n}$ is a connection. Define,

$$\nabla e_j = \sum_{i=1}^n \omega_{ij} \otimes e_i$$

where $\omega_{ij} \in \Omega^1_X(U)$ is a form. This uniquely defines the connection since,

$$\begin{aligned} \nabla(f_1, \ldots, f_n) &= \nabla \left(\sum_{i=1}^n f_i e_i \right) = \sum_{i=1}^n (f_i \nabla e_i + \mathrm{d}f_i \otimes e_i) \\ &= \sum_{i,j=1}^n \omega_{ij} \otimes f_j e_i + (\mathrm{d}f_1, \ldots, \mathrm{d}f_n) \end{aligned}$$

Therefore, $\mathcal{E}^\nabla$ is given locally by $(f_1, \ldots, f_n)$ solving the linear system of differential equations,

$$\mathrm{d}f_i + \sum_{j=1}^n \omega_{ij} f_j = 0$$

The condition of flatness is that,

$$\nabla_1 \circ \nabla = 0$$

which locally is,

$$
\begin{aligned}
\nabla_1 \circ \nabla(f_1, \ldots, f_n) &= \nabla_1 \left(\sum_{i,j=1}^{n} \omega_{ij} \otimes f_j e_i + \sum_{j=1}^{n} \mathrm{d} f_j \otimes e_j \right) \\
&= \sum_{i,j=1}^{n} \left[\mathrm{d}\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \nabla(f_j e_i) \right] + \sum_{i=1}^{n} \left[\mathrm{d}\mathrm{d} f_i \otimes e_i - \mathrm{d} f_j \wedge \nabla e_j \right] \\
&= \sum_{i,j=1}^{n} \left[\mathrm{d}\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \left(\mathrm{d} f_j \otimes e_i + f_j \sum_{k=1}^{n} \omega_{ki} \otimes e_k \right) \right] - \sum_{i,j=1}^{n} \left[\mathrm{d} f_j \wedge \omega_{ij} \otimes e_i \right] \\
&= \sum_{i,j=1}^{n} \left[\mathrm{d}\omega_{ij} \otimes e_i - \sum_{k=1}^{n} \omega_{ij} \wedge \omega_{ki} \otimes e_k \right] f_j \\
&= \sum_{i,j=1}^{n} \left[\mathrm{d}\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} \right] \otimes f_j e_i
\end{aligned}
$$

So the curvature ω_∇ is given by coefficients,

$$
\Theta_{ij} = \mathrm{d}\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj}
$$

Now I claim that if ε^∇ as a full set of solutions then $\omega_\Delta = 0$. To show this, consider,

$$
\mathrm{d}\left(\mathrm{d} f_i + \sum_{j=1}^{n} \omega_{ij} f_j \right) = 0
$$

This implies,

$$
\sum_{j=1}^{n} \left(\mathrm{d}\omega_{ij} f_j - \omega_{ij} \wedge \mathrm{d} f_j \right) = 0
$$

However, using the relation,

$$
\sum_{j=1}^{n} \left(\mathrm{d}\omega_{ik} + \omega_{ij} \wedge \omega_{jk} \right) f_k = 0
$$

and thus,

$$
\sum_{j=1}^{n} \Theta_{ij} f_j = 0
$$

If we assume that f_i can be choosen to span then we must have $\Theta_{ij} = 0$ which implies $\omega_\nabla = 0$. This is also sufficient for integrability. $\square$