1 Smooth Functions

First we recall the definition of smoothness for domains in $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be open with $\mathbf{p} \in U$.

Definition: We say that $f : U \to \mathbb{R}^m$ is differentiable at $\mathbf{p} \in U$ if there is a linear map $f'_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$\lim_{\mathbf{h} \to 0} \frac{1}{|\mathbf{h}|} \left| f(\mathbf{p} + \mathbf{h}) - f(\mathbf{p}) - f'_{\mathbf{p}}(\mathbf{h}) \right| = 0$$

$f : U \to \mathbb{R}^m$ is differentiable if f is differentiable at each $\mathbf{p} \in U$.

Definition: Denote the vectorspace of continuous functions $U \to \mathbb{R}^m$ by $\mathcal{C}^0$ and for $n > 0$ define,

$$\mathcal{C}^n = \{ f : U \to \mathbb{R}^m \mid \forall p \in U : f'_p \text{ exists and } \forall \mathbf{v} \in \mathbb{R}^n : f'_{\mathbf{v}} \in \mathcal{C}^{n-1} \}$$

where $f'_{\mathbf{v}}$ is the map $\mathbf{p} \mapsto f'_{\mathbf{p}}(\mathbf{v})$. Furthermore, the space of smooth functions is,

$$\mathcal{C}^\infty = \bigcap_k \mathcal{C}^k$$

Proposition 1.1. A function $f : U \to \mathbb{R}^m$ is $\mathcal{C}^k$ if f is differentiable k times and the k^{th}-derivative $f^{(k)} : U \to \mathbb{R}^m$ is continuous. Furthermore f is called $\mathcal{C}^\infty$ or smooth if it is $\mathcal{C}^k$ for all $k \leq 0$.

Proposition 1.2. If $f : U \to \mathbb{R}^m$ is differentiable then f is continuous.

Proof. Becuase f'_p is linear it has limit zero as $\mathbf{h} \to 0$,

$$\lim_{\mathbf{h} \to 0} |f(\mathbf{p} + \mathbf{h}) - f(\mathbf{p})| = \lim_{\mathbf{h} \to 0} \left| f(\mathbf{p} + \mathbf{h}) - f(\mathbf{p}) - f'_p(\mathbf{h}) \right| = 0$$

and the second term has zero limit by differentiability. □

Proposition 1.3. Let $f : U \to V$ and $g : V \to \mathbb{R}^m$ be differentiable. Then the composition has derivative,

$$(g \circ f)'_{\mathbf{p}} = g'_{f(\mathbf{p})} \circ f'_{\mathbf{p}}$$

Proof. □

Proposition 1.4. Let $f, g : U \to \mathbb{R}$ be differentiable functions on $U \subset \mathbb{R}^n$ then,

$$(fg)'_{\mathbf{p}} = f(\mathbf{p}) g'_{\mathbf{p}} + g(\mathbf{p}) f'_{\mathbf{p}}$$

Proof. Consider,

$$\begin{aligned}(fg)(\mathbf{p}+\mathbf{h})-(fg)(\mathbf{p}) &= f(\mathbf{p}+\mathbf{h})g(\mathbf{p}+\mathbf{h})-f(\mathbf{p})g(\mathbf{p})\\ &= f(\mathbf{p}+\mathbf{h})[g(\mathbf{p}+\mathbf{h})-g(\mathbf{p})]+[f(\mathbf{p}+\mathbf{h})-f(\mathbf{p})]g(\mathbf{p})\end{aligned}$$

Therefore,

$$\begin{aligned}Q(\mathbf{h}) &= (fg)(\mathbf{p}+\mathbf{h})-(fg)(\mathbf{p})-[f(\mathbf{p}+\mathbf{h})g'_{\mathbf{p}}(\mathbf{h})+g(\mathbf{p})f'_{\mathbf{p}}(\mathbf{h})]\\ &= f(\mathbf{p}+\mathbf{h})[g(\mathbf{p}+\mathbf{h})-g(\mathbf{p})-g'_{\mathbf{p}}(\mathbf{h})]+[f(\mathbf{p}+\mathbf{h})-f(\mathbf{p})-f'_{\mathbf{p}}(\mathbf{h})]g(\mathbf{p})\end{aligned}$$

which implies that,

$$\lim_{\mathbf{h}\to 0}\frac{1}{|\mathbf{h}|}|Q(\mathbf{h})|=0$$

since both,

$$\lim_{\mathbf{h}\to 0}\frac{1}{|\mathbf{h}|}\left|f(\mathbf{p}+\mathbf{h})-f(\mathbf{p})-f'_{\mathbf{p}}(\mathbf{h})\right|=0$$

and

$$\lim_{\mathbf{h}\to 0}\frac{1}{|\mathbf{h}|}\left|g(\mathbf{p}+\mathbf{h})-g(\mathbf{p})-g'_{\mathbf{p}}(\mathbf{h})\right|=0$$

Finally,

$$D(\mathbf{h})=(fg)(\mathbf{p}+\mathbf{h})-(fg)(\mathbf{p})-[f(\mathbf{p})g'_{\mathbf{p}}(\mathbf{h})+g(\mathbf{p})f'_{\mathbf{p}}(\mathbf{h})]=Q(\mathbf{h})+[f(\mathbf{p}+\mathbf{h})-f(\mathbf{p})]g'_{\mathbf{p}}(\mathbf{h})$$

And thus,

$$\frac{1}{|\mathbf{h}|}|D(\mathbf{h})|=\frac{1}{|\mathbf{h}|}\left|Q(\mathbf{h})+[f(\mathbf{p}+\mathbf{h})-f(\mathbf{p})]g'_{\mathbf{p}}(\mathbf{h})\right|\le\frac{1}{|\mathbf{h}|}|Q(\mathbf{h})|+\frac{|g'_{\mathbf{p}}(\mathbf{h})|}{|\mathbf{h}|}\left|f(\mathbf{p}+\mathbf{h})-f(\mathbf{p})\right|$$

Since $g'_{\mathbf{p}}$ is linear, by Lemma 15.1, we can find some bound M such that,

$$\frac{|g'_{\mathbf{p}}(\mathbf{h})|}{|\mathbf{h}|}\le M$$

for all $\mathbf{h}$. Therefore,

$$\frac{1}{|\mathbf{h}|}|D(\mathbf{h})|\le\frac{1}{|\mathbf{h}|}|Q(\mathbf{h})|+M\left|f(\mathbf{p}+\mathbf{h})-f(\mathbf{p})\right|$$

and both terms have limit zero. Thus,

$$\lim_{bfh\to 0}\frac{1}{|\mathbf{h}|}|D(\mathbf{h})|=0$$

$\square$

2 Manifolds

Definition: A topolgoical space M is an n-manifold if it is Hausdorff, second countable, and locally Euclidean. That is, there exists an open cover by charts (U, φ) with homeomorphisms $\varphi : U \to V$ where $V \subset \mathbb{R}^n$ is open. Such an open cover by charts (U, φ) is called an atlas.

Definition: An atlas is smooth if for each pairs of charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) the transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are $\mathcal{C}^\infty$. Likewise, an atlas is analytic, rational, or holomorphic if the transition maps are.

Definition: A smooth atlas is maximal if whenever (U, φ) is a chart compatible with the atlas then (U, φ) is a member of the atlas.

Proposition 2.1. Every smooth atlas is contained in a unique maximal atlas.

Definition: A topological space M is a smooth n-manifold if it is a manifold with a smooth structure. That is M equiped with a maximal smooth atlas.

Remark 1. Any smooth atlas is contained in a unique maximal atlas and thus any smooth atlas on M defines a unique smooth structure.

Definition: A map $F : M \to N$ between smooth manifolds is smooth if for each point $\mathbf{p} \in M$ there exists a chart (U, φ) of M and (V, ψ) of N such that $\mathbf{p} \in U$ and $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

3 Sheaves

Definition: Let X be a topological space and $\mathcal{C}$ a category. A pre-sheaf on X is a contravariant functor $\mathcal{F} : X^{\mathrm{op}} \to \mathcal{C}$ where X is a directed category on the open sets with inclusion maps. We call $\mathcal{F}(U)$ the sections restricted to U and for $U \subset V$ the maps $\mathrm{res} : \mathcal{F}(V) \to \mathcal{F}(U)$ restriction maps denoted $s|_U = \mathrm{res}_{U,V}(s)$.

Definition: A sheaf on X is a pre-sheaf such that for each open $U \subset X$ and open cover $\{U_\alpha\}$ of U the diagram,

$$\mathcal{F}(U) \xrightarrow{eq} \prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_\alpha \cap U_\beta)$$

is an equalizer of the maps defined by the products $\mathrm{res} : U_\alpha \to U_\alpha \cap U_\beta$ and the products of the maps $\mathrm{res} : U_\beta \to U_\alpha \cap U_\beta$ respectively. This is equivalent to the following conditions: let $\{U_\alpha\}$ be an open cover of $U \subset X$ then,

- If $s, t \in F(U)$ such that $s|_{U_\alpha} = t|_{U_\alpha}$ for each U then $s = t$.

- Suppose we have $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ then there exists a section $s \in F(U)$ such that $s|_\alpha = s_\alpha$ for each α.

Definition: Let $\mathcal{F}$ and G be sheaves on X. Then a morphisms of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. That is a collection of maps $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that,

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\mathrm{res}_{V,U}"] \arrow[d, "\varphi_U"] & \mathcal{F}(V) \arrow[d, "\varphi_V"] \\
\mathcal{G}(U) \arrow[r, "\mathrm{res}_{V,U}"] & \mathcal{G}(V)
\end{tikzcd}$$

whenever $V \subset U$.

Definition: Let $f : X \to Y$ be a continuous map and $\mathcal{F}$ a sheaf on X. Then $f_*\mathcal{F}$ is a sheaf on Y defined by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ for $V \subset Y$ with the restriction maps on the preimages. Furthermore, f_* is a functor from the category of sheaves over X to sheaves over Y by sending a sheaf map $g^\# : \mathcal{F} \to \mathcal{G}$ to the sheaf map $f_*g^\# : f_*\mathcal{F} \to f_*\mathcal{G}$ given by $(f_*g^\#)_U = g^\#_{f^{-1}(U)}$ such that the diagram commutes due to naturality of $g^\#$,

$$\begin{tikzcd}
\mathcal{F}(f^{-1}(U)) \arrow[r, "f_*\mathrm{res}_{V,U}"] \arrow[d, "g^\#_{f^{-1}(U)}"] & \mathcal{F}(f^{-1}(V)) \arrow[d, "g^\#_{f^{-1}(V)}"] \\
\mathcal{G}(f^{-1}(U)) \arrow[r, "f_*\mathrm{res}_{V,U}"] & \mathcal{G}(f^{-1}(V))
\end{tikzcd}$$

Definition: Let $\mathcal{F}$ be a sheaf on X. For $p \in X$ the stalk at p is given by,

$$\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$$

under the directed system given by restricting two neighborhoods to their intersection.

Definition: Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. Then a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Lemma 3.1. If $f^\# : \mathcal{F} \to \mathcal{G}$ is a map of sheaves on X then $f^\#$ induces a map on stalks $f^\# : \mathcal{F}_p \to \mathcal{G}_p$ for any $p \in X$.

Proof. Since $f^\#$ is a map of sheaves, each inclusion of $f^\#_U$ into $\mathcal{G}_p$ is compatible with the restriction maps and thus lists to a map $f^\# : \mathcal{F}_p \to \mathcal{G}_p$. Furthermore, $\square$

Definition: Given a continous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X there is a natural inclusion $(f_*\mathcal{F})_{f(p)} \to \mathcal{F}_p$.

Proof. By definition, the inclusion maps $\iota : \mathcal{F}(f^{-1}(V)) \to \mathcal{F}_p$ for $f(p) \subset V$ are compatible with restrictions. Therefore, we get a map $(f_*\mathcal{F})_{f(p)} \to \mathcal{F}_p$. $\square$

Definition: A space $(X, \mathcal{O}_X)$ is locally ringed if $\mathcal{O}_{X,p}$ is a local ring for each $p \in X$. A morphism of locally ringed spaces is a morphism of ringed spaces such that the induced map $f^\# : \mathcal{O}_{Y,f(p)} \to (f_*\mathcal{O}_X)_{f(p)} \to \mathcal{O}_{X,p}$ is a local map. That is, considering the unique maximal ideals $\mathfrak{m}_{Y,f(p)} \subset \mathcal{O}_{Y,f(p)}$ and $\mathfrak{m}_{X,p} \subset \mathcal{O}_{X,p}$ then $f^\#(\mathfrak{m}_{Y,f(p)}) \subset \mathfrak{m}_{X,p}$.

3.1 The Sheaf of Smooth Functions

Definition: Let M be a smooth manifold and let $\mathcal{C}^k_M$ be the sheaf of $\mathcal{C}^k$ functions on M. Define $\mathcal{O}_M = \mathcal{C}^\infty_M$ to be the sheaf of smooth functions on M.

Now we can redefine the basics of smooth manifolds in terms of structure sheaves.

Definition:

Definition: A smooth manifold is a locally ringed second countable Hausdroff space $(M, \mathcal{O}_M)$ with a covering by open sets $(U, \mathcal{O}_U)$ which are isomorphic as ringed spaces to $(V, \mathcal{O}_M)$ for some open $V \subset \mathbb{R}^n$ where $\mathcal{O}_V$ is the sheaf of smooth functions on V.

Definition: Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be smooth manifolds. A smooth map from M to N is a morphism of locally ringed spaces $(F, F^\#) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$.

Theorem 3.2. These definitions coincide with the classical definitions.

4 The Tangent Space

Definition: Let $(M, \mathcal{O})$ be a smooth-manifold. The cotangent space at $p \in M$ is the quotient $T^*_p M = \mathfrak{m}_p / \mathfrak{m}_p^2$ where $\mathfrak{m}_p$ is the maximal ideal of the stalk $\mathcal{O}_p$ given by germs of functions vanishing at p. The tangent space is the dual $T_pM = (T^*_pM)^*$.

Remark 2. Since T_pM is defined from the stalk of $\mathcal{O}$ then T_pU is identical to T_pM for any open U containing p.

Proposition 4.1. Given the manifold $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ we have $T^*_{\mathbf{p}}\mathbb{R}^n \cong (\mathbb{R}^n)^*$.

Proof. Let $f \in \mathcal{C}^\infty_{\mathbb{R}^n}$ be smooth. Then we have,

$$f(\mathbf{x}) = f(\mathbf{p}) + f'_{\mathbf{p}}(\mathbf{x} - \mathbf{p}) + E(\mathbf{x})$$

where $E(\mathbf{p}) = E'_{\mathbf{p}} = 0$. Then we have,

$$\mathfrak{m}_{\mathbf{p}} = \{[f] \mid f \in \mathcal{C}^\infty_U \text{ such that } f(\mathbf{p}) = 0\}$$

If $[g] = [h_1 h_2]$ so on some neighborhood U we have $g = ab$ with $a(\mathbf{p}) = b(\mathbf{p}) = 0$. Thus,

$$g'_{\mathbf{p}}(\mathbf{x}) = a(\mathbf{p}) \cdot b'_{\mathbf{p}}(\mathbf{x}) + a'_{\mathbf{p}}(\mathbf{x}) \cdot b(\mathbf{p}) = 0$$

so if $[g] \in \mathfrak{m}_{\mathbf{p}}^2$ then $g'_{\mathbf{p}} = 0$. Furthermore, if $E(\mathbf{p}) = 0$ and $E'_{\mathbf{p}} = 0$ then I claim that $[E] \in \mathfrak{m}_{\mathbf{p}}^2$. Consider the smooth functions,

$$f_i = \frac{x_i - p_i}{(\mathbf{x} - \mathbf{p})^2} E$$

These are smooth because

$$\lim_{|\mathbf{h}| \to 0} |f_i(\mathbf{p} + \mathbf{h})| = \lim_{|\mathbf{h}| \to 0} \left| \frac{E(\mathbf{p} + \mathbf{h})}{|\mathbf{h}|} \right| \cdot \frac{|h_i|}{|\mathbf{h}|} = 0$$

which is zero because E has zero derivative and value at $\mathbf{p}$. Let $g_i(\mathbf{x}) = (x_i - p_i)$ then

$$E = \sum_{i=1}^{n} f_i(\mathbf{x}) g_i(\mathbf{x})$$

where $f_i(\mathbf{p}) = g_i(\mathbf{p}) = 0$ and thus $E \in \mathfrak{m}_{\mathbf{p}}^2$. Define the map, $\Phi : \mathfrak{m}_{\mathbf{p}} \to (\mathbb{R}^n)^*$ by $\Phi([f]) = f'_{\mathbf{p}}$ which is a linear functional. We have shown that $\ker \Phi = \mathfrak{m}_{\mathbf{p}}^2$. Furthermore, Φ is surjective because $[g_i] \mapsto \hat{e}_i$. By the first isomorphism theorem,

$$T_{\mathbf{p}}^* \mathbb{R}^n = \mathfrak{m}_{\mathbf{p}} / \mathfrak{m}_{\mathbf{p}}^2 \cong (\mathbb{R}^n)^*$$

$\square$

Definition: A smooth $f : M \to N$ defines a linear map $f_p^* : T_{f(p)}^* N \to T_p^* M$ given by sending $[g] \in \mathfrak{m}_{f(p)}^N$ to $[g \circ f] \in \mathfrak{m}_p^M$. The dual map defines the differential,

$$\mathrm{d}f_{\ p} = (f_*)_p = (f_p^*)^* : T_p M \to T_{f(p)} M$$

Remark 3. The map $f_p^* : T_{f(p)}^* N \to T_p^* M$ is well-defined by Lemma 15.2 because the induced map $f_p^* : \mathfrak{m}_{f(p)}^N \to \mathfrak{m}_p^M$ is an algebra homomorphism.

Proposition 4.2. The tangent T_p is a covariant and the cotangent T_p^* is a contravariant functor from the category of smooth manifolds to the category of $\mathbb{R}$-vectorspaces.

Proof. For $f : M \to N$ and $g : N \to R$ smooth maps then,

$$(g \circ f)_p^*([h]) = [h \circ g \circ f] = f_p^*[h \circ g] = f_p^*(g_{f(p)}^*([h]))$$

and $\mathrm{id}_p^*([h]) = [h \circ \mathrm{id}] = [h]$. Thus, taking the dual map of vectorspaces,

$$\mathrm{d}(g \circ f)_{\ p} = ((g \circ f)_p^*)^* = (f_p^* \circ g_{f(p)}^*)^* = \mathrm{d}g_{\ f(p)} \circ \mathrm{d}f_{\ p}$$

$\square$

Corollary 4.3. Let M be a smooth n-manifold. Then $T_p M \cong \mathbb{R}^n$ at each $p \in M$.

Proof. For $p \in M$ there is some chart $\varphi : U \xrightarrow{\sim} V$ for open $V \subset \mathbb{R}^n$. with $p \in U$. Thus, since φ has a smooth inverse, the map $\mathrm{d}\varphi_{\ p} : T_p M \xrightarrow{\sim} T_{\varphi(p)} \mathbb{R}^n \cong \mathbb{R}^n$ is an isomorphism. $\square$

Definition: A linear functional $X : \mathcal{O}_p \to \mathbb{R}$ is called a derivation if is satisfies the Leibniz rule,

$$X(fg) = X(f)g(p) + f(p)X(g)$$

Proposition 4.4. T_pM is canonically isomorphic to the space of derivations at p.

Proof. Take $X \in T_pM = (T_p^*M)^*$ then $X : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}$. Given any germ $f \in \mathcal{O}_p$ we can extend X to act on f by $X(f) = X(\tilde{f})$ where $\tilde{f} = f - f(p) \in \mathfrak{m}_p/\mathfrak{m}_p^2$. Then take $f, g \in \mathcal{O}_p$ and consider,

$$\begin{aligned} X(fg) &= X(fg - f(p)g(p)) = X(\tilde{f}\tilde{g} + \tilde{f}g(p) + f(p)\tilde{g}) \\ &= X(\tilde{f}\tilde{g}) + X(\tilde{f})g(p) + f(p)X(\tilde{g}) = X(f)g(p) + f(p)X(g) \end{aligned}$$

because $\tilde{f}\tilde{g} \in \mathfrak{m}_p^2$ so $X(\tilde{f}\tilde{g}) = 0$. Thus, X is a derivation at p. Furthermore, any derivation is automatically a linear map $\mathfrak{m}_p \to \mathbb{R}$ so we only need to show that it descends to the quotient. Take $f, g \in \mathfrak{m}_p$ then we have,

$$X(fg) = X(f)g(p) + f(p)X(g) = 0$$

because $f(p) = g(p) = 0$. Thus, X is zero on $\mathfrak{m}_p^2$ so it factors through the quotient as a linear map $\tilde{X} : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}$ which an element of the dual space $(\mathfrak{m}_p/\mathfrak{m}_p^2)^* = T_pM$. Therefore, T_pM is canonically identified with the space of derivations. □

4.1 The Tangent Bundle

4.2 Product Manifolds

Proposition 4.5. Let M and N be smooth manifolds of dimensions m and n then $M \times N$ natually has a smooth stucture making it a smooth $m+n$-manifold such that the projection maps are smooth.

Proof. □

Proposition 4.6. For $(p, q) \in M \times N$ we have, $T_{p,q}(M \times N) \cong T_pM \oplus T_qN$.

Proof. Consider the map $\Phi : T_p^*M \oplus T_q^*N \to T_{p,q}^*(M \times N)$ defined by $\Phi = \pi_1^* + \pi_2^*$ acting on $f \in \mathfrak{m}_p$ and $g \in \mathfrak{m}_q$ via $[f] \oplus [g] \mapsto [f \circ \pi_1 + g \circ \pi_2]$. Suppose that $\Phi([f] \oplus [g]) = 0$ then let $F = f \circ \pi_1 + g \circ \pi_2 \in \mathfrak{m}_{p,q}$ on some neighborhood of p, q. Thus we can write,

$$F = \sum_{i=1}^{n} a_i b_i$$

with $a_i, b_i \in \mathfrak{m}_{p,q}$. Then,

$$F(x, q) = \sum_{i=1}^{n} a_i(x, q) b_i(x, q)$$

and $a_i(-,q), b_i(-,q) \in \mathfrak{m}_p$ thus $F(-,q) \in \mathfrak{m}_p^2$. However, $F(x,p) = f(x) + g(q) = f(x)$ so $f \in \mathfrak{m}_p^2$. Similarly, $g \in \mathfrak{m}_q$. Thus Φ is injective. Clearly, Φ is linear. Furthermore,

$$\dim (T_p^* M \oplus T_q^* N) = m + n = \dim T_{p,q}^*(M \times N)$$

Thus, Φ must be surjective by rank-nullty. Thus, Φ is an isomorphism. Taking the dual map we get an isomorphism,

$$\Phi^* : T_{p,q}(M \times N) \xrightarrow{\sim} T_p M \oplus T_q N$$

given explictly by $\Phi^* = (\pi_1)_* \oplus (\pi_2)_*$ since,

$$\begin{aligned}\Phi^*(X)([f] \oplus [g]) &= X(\Phi([f] \oplus [g])) = X([f \circ \pi_1 + g \circ \pi_2]) \\ &= X([f \circ \pi_1]) + X([g \circ \pi_2]) = (\pi_1)_* X([f]) + (\pi_2)_* X([g]) \\ &= \Big((\pi_1)_* X \oplus (\pi_2)_* X \Big)([f] \oplus [g])\end{aligned}$$

□

Proposition 4.7. Let $f : P \to M \times N$ a smooth map of smooth manifolds then

$$\mathrm{d}f_{\,p} = \mathrm{d}(\,\pi_M \circ f)_p \oplus \mathrm{d}(\pi_N \circ f)_{\,p} : T_p P \to T_{\pi_1(f(p))} M \oplus T_{\pi_2(f(p))} N$$

Proof. Let $X \in T_p P$ be a derivation (or linear functional on $T_p^* P = \mathfrak{m}_p / \mathfrak{m}_p^2$). Then consider,

$$\Phi^* \circ \mathrm{d}f_{\,p}(X) = (\pi_1)_* \circ \mathrm{d}f_{\,p} X \oplus (\pi_2)_* \circ \mathrm{d}f_{\,p} X = \Big(\mathrm{d}(\,\pi_M \circ f)_p \oplus \mathrm{d}(\pi_N \circ f)_{\,p} \Big)(X)$$

□

Proposition 4.8. Let $f : M \times N \to P$ a smooth map of smooth manifolds then

$$\mathrm{d}f_{\,p,q} = \mathrm{d}(\,f \circ \iota_M^q)_p + \mathrm{d}(f \circ \iota_N^p)_{\,q}$$

where $\iota_M^q : M \to M \times N$ is the inclusion $x \mapsto (x,q)$.

Proof. Let $X_1 \oplus X_2 \in T_p M \oplus T_q N$ be derivations corresponding to $X \in T_{p,q}(M \times N)$
I claim that $(\Phi^*)^{-1}(X_1 \oplus X_1) = (\iota_M^q)_* X + (\iota_N^p)_* X$. It is easy to check that,

$$\begin{aligned}\Phi^* \circ (\Phi^*)^{-1}(X_1 \oplus X_1) &= \Phi^*((\iota_M^q)_* X_1 + (\iota_N^p)_* X_2) = (\pi_1)_*(\iota_M^q)_* X_1 \oplus (\pi_2)_*(\iota_N^p)_* X_2 \\ &= (\pi_1 \circ \iota_M^q)_* X \oplus (\pi_2 \circ \iota_N^p)_* X\end{aligned}$$

where the cross terms vanish because $\pi_1 \circ \iota_N^p$ is a constant map which has zero differential. Furthermore, $\pi_1 \circ \iota_M^q = \mathrm{id}_M$ so $(\pi_1 \circ \iota_M^q)_* = \mathrm{id}_{T_p M}$. Thus,

$$\Phi^* \circ (\Phi^*)^{-1}(X_1 \oplus X_1) = (\pi_1)_*(\iota_M^q)_* X_1 \oplus (\pi_2)_*(\iota_N^p)_* X_2 = X_1 \oplus X_2$$

Since Φ^* is invertible, this must be its two-sided inverse. Now consider,

$$\begin{aligned}\mathrm{d}f_{\,p,q} \circ (\Phi^*)^{-1}(X_1 \oplus X_2) &= \mathrm{d}f_{\,p,q}((\iota_M^q)_* X_1 + (\iota_N^p)_* X_2) \\ &= \big(\mathrm{d}f_{\,p,q} \circ (\iota_M^q)_*\big) X_1 + \big(\mathrm{d}f_{\,p,q} \circ (\iota_N^p)_*\big) X_2 \\ &= \mathrm{d}(f \circ \iota_M^q)\, X_1 + \mathrm{d}(f \circ \iota_N^p)\, X_2\end{aligned}$$

□

4.3 Vector Fields

4.4 Tensors and Tensor Fields

5 Oct. 2

Lemma 5.1. Let X be a smooth vector field on a smooth manifold M such that $\text{Supp}\,(X) = \overline{\{p \in M \mid X_p \neq 0\}}$ is compact. Then there exists a smooth $\phi : \mathbb{R} \times M \to M$ such that $\mathrm{d}\phi_p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = X(\phi(t,p))$ and $\phi(0,p) = p$.

Proof. Let $K = \text{Supp}\,(X)$ which is compact in M. For $p \notin K$ then $X(p) = 0$ so $\phi(t,p) = \phi(p)$. For $p \in K$ let U_p be an open neighborhood of p then a function satisfying the properties $\phi : (-\epsilon_p, \epsilon_p) \times U_p \to M$ is defined. By the compactess of K we need only a finite number of U_p to cover K so we can take the minimum of the ϵ_p. Then ϕ is defined on K. □

Definition: Take $X \in \mathcal{C}^\infty(M, TM)$. The Lie derivative $\mathcal{L}_X : \mathcal{C}^\infty M \to \mathcal{C}^\infty M$ given by $\mathcal{L}_X(f)(p) = X_p(f)$ is $\mathbb{R}$-linear and satisfies the product rule.

Definition: For $X, Y \in \mathcal{C}^\infty(M, TM)$ then the Lie derivative on vector fields is a map $\mathcal{L}_X : \mathcal{C}^\infty(M, TM) \to \mathcal{C}^\infty(M, TM)$ given by $\mathcal{L}_X Y = [X, Y]$ and is $\mathbb{R}$-linear and satisfies the product rule,

$$\mathcal{L}_X(fY) = (\mathcal{L}_X f)Y + f\mathcal{L}_X Y$$

Definition: Let $F; M \to N$ be a diffeomorphism and $X \in \mathcal{C}^\infty(M, TM)$ define F_*X by $(F_*X)_q = \mathrm{d}F_{\,F^{-1}(q)}\left(X_{F^{-1}(q)}\right)$ and for $Y \in \mathcal{C}^\infty(N, TN)$ then $F^*Y = (F_{-1})_*Y$.

Proposition 5.2. Let $X \in \mathcal{C}^\infty(M, TM)$ and ϕ the local flow of X then,

1. For $f \in \mathcal{C}^\infty(M)$ we have,

$$X_p(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi_t^* f)(p) = \lim_{t \to 0} \frac{f \circ \phi_t(p) - f(p)}{t}$$

2. For $Y \in \mathcal{C}^\infty(M, TM)$ we have,

$$[X, Y]_p = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} ([\phi_t]_* Y)_p = \lim_{t \to 0} \frac{Y_p - ([\phi_t]_* Y)_p}{t}$$

Proof. Let,

$$T = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}$$

be the standard derivation at zero on $\mathbb{R}$. Then $T(f \circ \phi_t(p)) = \mathrm{d}\phi_p(T)(f) = X_p(f)$. □

6 Vector Bundles

Definition: $\pi_E : E \to M$ is a $\mathcal{C}^\infty$-vector bundle over M of rank r if there exists an open cover $\{U_\alpha\}$ of M such that π_E is locally trivialized via,

$$\begin{tikzcd} \pi_E^{-1}(U_\alpha) \arrow[r, "\sim"] \arrow[d, "\pi_E"'] & U_\alpha \times \mathbb{R}^r \arrow[dl, "\pi_1"] \\ U_\alpha & \end{tikzcd}$$

for each U_α.

Definition: The (r, s)-tensor bundle on M is the bundle,

$$T^r_s M = (TM)^{\otimes r} \otimes (T^* M)^{\otimes s}$$

Furthermore, the bundle of k-forms is given by the k^{th} exterior power,

$$\bigwedge^k T^* M \subset (T^* M)^{\otimes k}$$

Thus every k-form is a $(0, k)$-tensor.

7 Differential Forms

Definition: The space of differential k-forms on M is,

$$\Omega^k(M) = \mathcal{C}^\infty \left(M, \bigwedge^k T^* M \right)$$

which are smooth sections of the bundle of k-forms.

Definition: The exterior derivative is an $\mathbb{R}$-linear map,

$$\mathrm{d} : \Omega^s(M) \to \Omega^{s+1}(M)$$

satisfying,

1. For $f \in \Omega^0(M) = \mathcal{C}^\infty(M)$ the 1-form $\mathrm{d}f$ is the differential.
2. For $f \in \Omega^0(M)$ we have $\mathrm{dd}f = 0$.
3. For $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^s(M)$ then $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta$.
4. We have $\mathrm{d} \circ \mathrm{d} = 0$ i.e. $\forall \omega \in \Omega^s(M)$ we have $\mathrm{dd}\omega = 0$.
5. Let $\phi : M \to N$ be smooth and $\omega \in \Omega^s(N)$ then $\phi^* \mathrm{d}\omega = \mathrm{d}(\phi^* \omega)$ i.e. $\mathrm{d} \circ \phi^* = \phi^* \circ \mathrm{d}$.

6. Let X be a smooth vector field then $\mathrm{d} \circ \mathcal{L}_X = \mathcal{L}_X \circ \mathrm{d}$.
7. For $\alpha \in \Omega^s(M)$ and $X_0, \ldots, X_s$ are smooth vector fields then,

$$\mathrm{d}\alpha(X) = \sum_{i=0}^{s}(-1)^i X_i\left(\alpha(X|_{\text{not } i})\right) + \sum_{1 \le i < j \le s}(-1)^{i+j}\alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_s)$$

Definition: The space of all differential forms is,

$$\Omega^*(M) = \bigoplus_{k=1}^{\infty} \Omega^k(M)$$

Definition: Consider the cochain complex,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}^0} \Omega^1(M) \xrightarrow{\mathrm{d}^1} \Omega^2(M) \xrightarrow{\mathrm{d}^2} \Omega^3(M) \xrightarrow{\mathrm{d}^3} \Omega^4(M) \longrightarrow \cdots$$

The de Rham cohomology is the cohomology of this complex,

$$H^k_{\mathrm{dR}}(M, \mathbb{R}) = \ker d^k / \mathrm{Im}(d^{k-1})$$

Definition: The interior derivative, for a smooth vector field $X \in \mathcal{C}^\infty(M, TM) = \mathscr{X}(M)$, is an $\mathbb{R}$-linear map on forms,

$$\iota_X : \Omega^s(M) \to \Omega^{s-1}(M)$$

defined by,

$$\iota_X(\alpha)(Y_1, \ldots, Y_{s-1}) = \alpha(X, Y_1, \ldots, Y_{s-1})$$

for any smooth vector fields $Y_1, \ldots, Y_{s-1} \in \mathscr{X}(M)$.

8 Remannian Manifolds

Definition: A Remannian metric g on a smooth manifold M is a smooth $(0,2)$-tensor such that $g_p : T_pM \times T_pM \to \mathbb{R}$ is an inner product on T_pM.

Definition: A Remannian manifold (M, g) is a smooth manifold M with a Remannian metric g on M.

8.1 Isometric Immersions

Let $f : M \to N$ be a smooth map and (N, g) a Riemannian manifold. Then f^*g is a symmetric $(0,2)$-tensor on M. When is f^*g a Riemannian manifold on M? We have that,

$$(f^*g)_p(v, v) = g(\mathrm{d}f_{\,p}(v), \mathrm{d}f_{\,p}(v))$$

Thus we must have that $\mathrm{d}f_{\,p}(v) = 0$ implies $v = 0$ in order that f^*g be nondegenerate.

Proposition 8.1. If $f : M \to N$ is a smooth immersion and g is a Riemannian metric on N then f^*g is a Riemannian metric on M.

Proof. We know that f^*g is a symmetric $(0,2)$-tensor. Furthermore, we know that,

$$(f^*g)_p(v,v) = g(\mathrm{d}f_{\,p}(v), \mathrm{d}f_{\,p}(v)) = 0 \implies \mathrm{d}f_{\,p}(v) = 0$$

but f is an immersion so $\mathrm{d}f_{\,p}(v) = 0$ implies that $v = 0$. Thus, f^*g is nondegenerate. Then, since g is positive-defininite, so is f^*g. $\square$

Definition: A map $f : (M, g_M) \to (N, g_N)$ is,

1. an isometry if f is a diffeomorphism and $f^*g_N = g_M$.
2. an isometric immersion if f is a smooth immersion and $f^*g_N = g_M$.
3. an isometric embedding if f is a smooth embedding and $f^*g_N = g_M$.
4. a local isometry if f is a local diffeomorphism and $f^*g_N = g_M$.

Proposition 8.2. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, then their product $(M_1 \times M_2, g)$ is canonically a Riemannian manifold with $g = \pi_1^* g_1 + \pi_2^* g_2$.

Proof. We must check that g is non-degenerate. Suppose,

$$g((v,u),(v,u)) = g_1(v) + g_2(u) = 0$$

then since g_1 and g_2 are positive-definite we must have $v = u = 0$. $\square$

8.2 Distance

Definition: Let (M, g) be a connected Remannian manifold. For $x, y \in M$ we have $d(x,y) = \inf \operatorname{len}(\gamma)$ for all picewise smooth paths $\gamma : I \to M$ from x to y.

Proposition 8.3. For all $x, y, z \in M$ we have,

$$d(x,z) + d(z,y) \geq d(x,y)$$

Proof. Let $\gamma_1, \gamma_2 : I \to M$ be picewise smooth paths from x to z and z to y respectively. Then $\gamma_2 * \gamma_1$ is a piecewise smooth path from x to y and the lengths add. $\square$

8.3 Volume Forms

Definition: Let M be a smooth n-manifold then a *volume form* on M is a nonvanishing smooth n-form $\omega \in \Omega^n(M)$.

Definition: An orientation on a smooth manifold M is an atlas on M such that each transition map has positive jacobian i.e. its differential has positive determinant.

Lemma 8.4. Let (M, g) be an oriented Riemannian n-manifold then there exists a unique volume form $\omega \in \Omega^n(M)$ such that at each point $p \in M$ there exists an ordered basis of (T_pM, g_p) compatible with the orientation.

Proof. For any $p \in M$ define $\omega(p) = e_1^* \wedge \cdots \wedge e_n^*$ where $(e_1^*, \dots, e_n^*)$ is an ordered dual basis of T_p^*M orthonormal with respect to g_p and compatible with the orientation. If we choose a different set $(\tilde{e}_1, \dots, \tilde{e}_n)$ then we can write,

$$\tilde{e}_i^* = \sum_{j=1}^n A_{ji} e_j^*$$

with $A \in \mathrm{O}\,(n)$ because it must preserve the metric g. Furthermore, since e_i' is also compatible with the orientation we must have $\det A > 0$ so $A \in \mathrm{SO}\,(n)$ and thus $\det A = 1$. Furthermore,

$$\omega'(p) = \tilde{e}_1^* \wedge \cdots \wedge \tilde{e}_n^* = e_1^* \wedge \cdots \wedge e_n^* \det A = e_1^* \wedge \cdots \wedge e_n^* = \omega(p)$$

$\square$

Lemma 8.5. Let M be a smooth n-manifold. There exists a volume form on M if and only if $\bigwedge^n T^*M$ is trivial.

Proof. There exists a volume form on M if and only if there exists a nonvanishing smooth section of $\bigwedge^n T^*M$ if and only if there there exists a smooth global frame of $\bigwedge^n T^*M$ if and only if $\bigwedge^n T^*M$ is a trivial vector bundle of rank 1 over M. $\square$

Lemma 8.6. Let M be a smooth n-manifold then every volume form is in correspononce to a choice of orientation. Thus, M admits a volume form if and only if M is orientible.

9 Connections

Definition: An affine connection ∇ on M is a bilinear map,

$$\mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$$

Definition: Let M be a $\mathcal{C}^\infty$ manifold. An affine connection ∇ on M is symmetric if,

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for any $X, Y \in \mathscr{X}(M)$.

Definition: The torsion of ∇ is defined as,

$$T_\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$$

where,

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Then T_∇ is bilinear and antisymmetric so $T_\nabla \in \mathcal{C}^\infty(M, \Omega^2(M, TM) \otimes TM)$.

Remark 4. An affine connection ∇ is symmetric or torsion-free iff $T_\nabla = 0$.

Proposition 9.1. The space of all affine connections on M is an affine space with associated vectorspace $\mathcal{C}^\infty(M, T_2^1 M)$. The space of all symmetric affine connections is also an affine space with associate vector space $\mathcal{C}^\infty(M, \mathrm{Sym}^2(T^*M) \otimes TM)$.

Definition: Let (M, g) be a Riemannian manifold. An affine connection ∇ on M is compatible with the Riemannian structure g if,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any $X, Y, Z \in \mathscr{X}(M)$.

Remark 5. If g is a symmetric T_2 field then $\nabla_X g \in \mathcal{C}^\infty(M, (T^*M)^{\otimes 2})$ defined by,

$$(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

Therefore, ∇ is compatible with $g \iff \nabla_X g = 0 \quad \forall X \in \mathscr{X}(M)$.

Theorem 9.2 (Levi-Civita)**.** If (M, g) is a Riemannian manifold then there exists a unique symmetric affine connection ∇ on M which is compatible by the Riemannian structure.

Proof. Suppose that ∇ is an affine connection on M which is symmetric and compatible with g. Then, by compatibility,

$$\begin{aligned} X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \\ Y(g(X, Z)) - g(\nabla_Y X, Z) - g(X, \nabla_Y Z) = 0 \\ Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = 0 \end{aligned}$$

Then take (1) + (2) - (3), and use symmetry,

$$X(g(X, Z)) + Y(g(Z, X)) - Z(g(X, Y)) = g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, [X, Y]) + 2g(Z, \nabla_Y X)$$

Therefore,

$$g(Z, \nabla_Y X) = \tfrac{1}{2}\left[X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [X, Y])\right]$$

Which implies that ∇ is uniquely determined by the metric g. To show existence, define $\nabla_X Y$ by the above equation for any $Z \in \mathscr{X}(M)$. $\square$

Definition: Let (M, g) be a Riemannian manifold and $p \in M$ then there exists an open neighborhood V of p in M and $\epsilon > 0$ such that $\phi(t, q, w)$ and $\gamma(t, q, w)$ are defined for $|t| < 2$ and $q \in V$ and $|w| \le \epsilon$ then $\exp : U_{(v,\epsilon)} \to M$ is defined by $\exp(q, w) = \gamma(1, q, w)$.

10 Curvature

Definition: Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection determined by g. Then given $X, Y \in \mathscr{X}(M)$ the Riemann map,

$$R(X,Y) : \mathscr{X}(M) \to \mathscr{X}(M)$$

is defined by,

$$R(X,Y)(Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Proposition 10.1. Viewing the Riemann map as,

$$R : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$$

we have,

1. R is antisymmetric in the first two arguments, $R(X,Y,Z) = -R(Y,X,Z)$.
2. R is $\mathcal{C}^\infty(M)$-linear viewing $\mathscr{X}(M)$ as a $\mathcal{C}^\infty(M)$-module.
3. R is equivalent to an element of $\mathcal{C}^\infty(M, (\Lambda^2 T^* M) \otimes T^* M \otimes TM) = \Omega^2(M, \mathrm{End}\,(TM))$ so R is an $\mathrm{End}\,(TM)$-valued 2-form.

Remark 6. Let $\pi : E \to M$ be a $\mathcal{C}^\infty$ vector bundle over M with connection $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ then we may define,

$$R_\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \times \Omega^0(M, E) \to \Omega^0(M, E)$$

by,

$$R_\nabla(X,Y)(S) = \nabla_X \nabla_Y S - \nabla_Y \nabla_X S - \nabla_{[X,Y]} S$$

Theorem 10.2 (Bianchi)**.**

$$R(X,Y)(Z) + R(Y,Z)(X) + R(Z,X)(Y) = 0$$

Proof. This property follows from the symmetry of the Levi-Civita connection and the Jacobi identity. We have,

$$\begin{aligned} Q = R(X,Y)(Z) + R(Y,Z)(X) + R(Z,X)(Y) &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \\ &+ \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X - \nabla_{[Z,Y]} X \\ &+ \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y - \nabla_{[X,Z]} Y \end{aligned}$$

Using symmetry,

$$\begin{aligned} Q &= \nabla_Y [X,Z] - \nabla_{[Y,X]} Z + \nabla_X [Z,Y] - \nabla_{[Z,Y]} X + \nabla_Z [Y,X] - \nabla_{[X,Z]} Y \\ &= [Y,[X,Z]] + [X,[Z,Y]] + [Z[Y,X]] = 0 \end{aligned}$$

Which is zero by the Jacobi identity. $\square$

Definition: For $X, Y, Z, W \in \mathscr{X}(M)$, the Riemman tensor is defined by,

$$\mathcal{R}(X, Y, Z, W) = g(R(X, Y)(Z), W) \in \mathcal{C}^\infty(M)$$

Thus, $\mathcal{R}$ is a smooth $(0, 4)$-tensor field on M.

Proposition 10.3. The Riemann tensor $\mathcal{R}$ satisfies,

1. $\mathcal{R}(X, Y, Z, W) + \mathcal{R}(Y, Z, X, W) + \mathcal{R}(Z, X, Y, W) = 0$.
2. $\mathcal{R} \in \mathcal{C}^\infty(M, \mathrm{Sym}^2(\Lambda^2 T^* M))$ or equivalently,
   (a) $\mathcal{R}(X, Y, Z, W) = -\mathcal{R}(Y, X, Z, W)$
   (b) $\mathcal{R}(X, Y, Z, W) = -\mathcal{R}(X, Y, W, Z)$
   (c) $\mathcal{R}(X, Y, Z, W) = \mathcal{R}(Z, T, X, Y)$

11 Covariant Derivatives of Tensors

Let M be a smooth n-manifold with an affine connection ∇ such that for $X \in \mathscr{X}(M)$ we have an $\mathbb{R}$-linear map, $\nabla_X : \mathscr{X}(M) \to \mathscr{X}(M)$. We will extend this to a derivative on all tensors inductively by imposing the Leibniz rule.

$$\begin{array}{lll} (0,0) & f \in \mathcal{C}^\infty(M) & \nabla_X f = X(f) \\ (1,0) & Y \in \mathscr{X}(M) & \nabla_X Y \\ (0,1) & \omega \in \Omega^1(M) & (\nabla_X \omega)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y) \end{array}$$

Suppose that T_1 is an (r_1, s_1)-tensor and T_2 is an (r_2, s_2)-tensor. Then we want,

$$\nabla_X(T_1 \otimes T_2) = \nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2$$

This extends ∇_X to a map $\nabla_X : \mathcal{C}^\infty(M, T^r_s M) \to \mathcal{C}^\infty(M, T^r_s M)$ Given T a (r, s)-tensor we can write T as a map,

$$T : \mathscr{X}(M)^{\otimes s} \to \mathscr{X}(M)^{\otimes r}$$

which is $C^\infty(M)$-linear. Now, define,

$$\nabla T : \mathscr{X}(M)^{\otimes(s+1)} \to \mathscr{X}(M)^r$$

given by,

$$(\nabla T)(X_1, \dots, X_{s+1}) = (\nabla_{X_{s+1}} T)(X_1, \dots, X_s)$$

Thus, ∇T is a $(r, s+1)$-tensor.

Proposition 11.1. Let ∇ be an affine connection on a Riemannian manifold (M, g).

1. ∇ is symmetric $\iff$ $\mathrm{d}\alpha\,(X, Y) = \nabla\alpha(Y, X) - \nabla\alpha(X, Y)$ for any $\alpha \in \Omega^1(M)$ and $X, Y \in \mathscr{X}(M)$
2. ∇ is compatible with g $\iff$ $\nabla g = 0$.

Proof. □

11.1 Covariant Derivatives In Local Coordinates

Take the local coordinates $(x_1, \dots, x_n)$ on U. We can write,

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}$$

for some $\Gamma_{ij}^k \in \mathcal{C}^\infty(M)$ since this is a $\mathcal{C}^\infty(M)$-basis of vector fields. Now,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_i}} \mathrm{d}x_j &= \sum_{k=1}^{n} \left(\nabla_{\frac{\partial}{\partial x_i}} \mathrm{d}x_j \right) \left(\frac{\partial}{\partial x_k} \right) \mathrm{d}x_k = \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_i} \left(\mathrm{d}x_j \left(\frac{\partial}{\partial x_k} \right) \right) - \mathrm{d}x_j \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} \right) \right) \mathrm{d}x_k \\ &= \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_i} \delta_{jk} - \mathrm{d}x_j \left(\Gamma_{ik}^{\ell} \frac{\partial}{\partial x_\ell} \right) \right) \mathrm{d}x_k = - \sum_{k=1}^{n} \Gamma_{ik}^{j} \mathrm{d}x_k \end{aligned}$$

Finally, we can compute the covariant derivative of an arbitrary tensor in local coordinates.

12 Jacobi Fields

Let (M, g) be a Riemannian manifold and $\gamma : [0, a] \to M$ a geodesic. Then a Jacobi field may arise as follows. Let $f_s : [0, a] \to M$ with $s \in (-\epsilon, \epsilon)$ be a smooth family of geodesics. That is,

$$f : (-\epsilon, \epsilon) \times [0, a] \to M$$

such that f_s is a geodesic and $f_0 = \gamma$. Then set $J(t) = \frac{\partial f}{\partial s}(0, t)$.

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Let $f : (M, g) \to (\bar{M}, \bar{g})$ be an isometric immersion. Then let ∇ and $\bar{\nabla}$ be the Levi-Civita connections and $\nabla = f^* \bar{\nabla}$ on $f^* T\bar{M}$.

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Theorem 14.1. Let (M, g) be a connected Riemannian manifold then (M, d) is a metric space with d induced by g.

Proof. □

Theorem 14.2 (Hopf-Rinow)**.** Let (M, g) be a connected Riemannian manifold so (M, d) is a metric space with d induced by g. Then for any $p \in M$ TFAE,

1. $\exp_p$ is defined on T_pM
2. closed bounded subsets of M are compact

3. (M, d) is complete

4. (M, g) is geodesically complete

5. there exists compact sets $K_n \subset M$ covering M such that $q_n \notin K_n \implies d(p, q_n) \to \infty$ as $n \to \infty$.

6. $\forall q \in M$ there exists minimizing geodesic from p to q.

Proof. □

Corollary 14.3. If M is a compact smooth manifold ten (M, g) is a geodesically complete Riemannian manifold for any Riemannian metric g.

Definition: A connected Riemannian manifold (M, g) is *expandible* if there exists a connected Riemannian manifold (M', g') and an isometric proper open embedding $\iota : (M, g) \to (M', g')$. Otherwise (M, g) is nonextendible.

Proposition 14.4. If (M, g) is complete then (M, g) is extendible.

Proof. If there exists $\iota : (M, g) \to (M', g')$ an isometric proper open embedding then $\iota(M)$ must be geodesically incomplete because it is a proper open subset of M'. Thus M is geodesically incomplete because ι is an isometric embedding and thus an isometry onto its image. □

Corollary 14.5. Any (induced) connected Riemannian submanifold of a complete connected Riemannian manifold is complete.

Proof. Let $(N, \iota^* g) \xrightarrow{\iota} (M, g)$ be a Riemannian submanifold. Then $\forall p, q \in N$ we have $d_N(p, q) \geq d_M(p, q)$ so any Cauchy sequence in N is also Cauchy on M and thus converges in M. However, $N \subset M$ is closed and thus contains all its limit points. Thus all Cauchy sequences converge in N. □

Definition: Let (M, g) be a connected complete Riemannian manifold then $\forall p \in M$ we have $\exp_p : T_p : M \to M$. We say that p is a pole if $\exp_p : T_p M \to M$ is a local diffeomorphism i.e. $\forall p \in T_p M$ the map,

$$\mathrm{d}(\exp_p)_v \ : T_v(T_p M) \to T_{\exp_p(v)} M$$

is a linear isomorphism.

Lemma 14.6. Let (Mg) be a connected complete Riemannian manifold such that $\forall p \in M$ and any 2-plane $\sigma \subset T_p M$ then $K(p, \sigma) \leq 0$ then $\forall p \in M$ the point p is a pole.

Proof. □

Lemma 14.7. Let (M, g) and (N, h) be complete Riemannian manifolds with smooth surjective map $f : (M, g) \to (N, h)$ which is a local diffeomorphism satisfying $\forall p \in M, v \in T_p M$ then $|\mathrm{d}f_p(v)\,|_{f(p)} \geq |v|_p$ then $f : (M, g) \to (N, h)$ is a covering map.

Lemma 14.8. Let (M, g) be a complete connected Riemannian manifold and $p \in M$ is a pole then $\exp_p : T_pM \to M$ is a covering map.

Corollary 14.9. If (M, g) is a complete, connected, simply-connected Riemannian manifold with a pole p then $\exp_p : T_pM \to M$ is a diffeomorphism.

Theorem 14.10 (Cartan-Hadamard)**.** Let (M, g) be a connected complete Riemannian manifold with $K(p, \sigma) \leq 0$ forall $p \in M$ and $\sigma \in \mathrm{Gr}(2, T_pM)$ then $\forall p \in M$ the exponential map $\exp_p : T_pM \to M$ is a covering map. In particular, if M is simply-connected then $M \cong \mathbb{R}^{\dim M}$.

15 General Lemmata

Lemma 15.1. Let $T : V \to W$ be a linear map of finite-dimensional real normed spaces. Then there exists $M \in \mathbb{R}$ such that for all $v \in V$,

$$||T(v)|| \leq M||v||$$

Proof. For $v = 0$, the inequality is trivial. Suppose $v \neq 0$ then we can always scale,

$$T(v) = ||v||T\left(\frac{v}{||v||}\right)$$

Let $\dim V = n$ and thus $V \cong \mathbb{R}^n$ as normed spaces. Thus, $\{v \in V \mid ||v|| = 1\} \cong S^{n-1}$. Furthermore, $T : V \to W$ is linear and thus continuous. Therefore, the restriction $T : S^{n-1} \to \mathbb{R}^n$ is also continuous. Since S^{n-1} is compact its image $T(S^{n-1}) \subset W$ is compact and is thus bounded by Heine-Borel. Therefore, there exists $M \in \mathbb{R}$ such that whever $||v|| = 1$ then $||T(v)|| \leq M$. Finally, for any $v \in V$,

$$T(v) = ||v||T\left(\frac{v}{||v||}\right) \leq M||v||$$

becaues $v/||v||$ has unit norm. □

Lemma 15.2. Let $f : A \to B$ be an K-algebra homomorphism. The quotient A/A^k is a K-vector space and $f : A/A^k \to B/B^k$ is a well-defined map of K-vectorspaces.

Proof. Clearly, $A^k \subset A$ is a subvector space so the quotuent is a K-vectorspace. Furthermore, consider the map,

$$A \xrightarrow{f} B \xrightarrow{\pi} B/B^k$$

However, $f(A^k) \subset B^k$ since f is an algebra homomorphism and thus $\pi \circ f(A^k) = (0)$ so $A^k \subset \ker \pi \circ f$ and thus $\pi \circ f$ factors through the quotient A/A^k. □

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Theorem 16.1 (Hadamard Theorem)**.**

Theorem 16.2 (Cartan)**.**

Lemma 16.3. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a surjective local diffeomorphism and $\forall p \in M : |\mathrm{d}f_p(v)| \geq |v|$ and (M_1, g_1) is complete then f is a covering map.

Proof. □

Theorem 16.4. Let $(\tilde{M}, \tilde{g})$ be a simply connectd, complete manifold with constant sectional curvature K then $(tildeM, \tilde{g})$ is isometric to

1. (H^n, g_{can}) if $K = -1$
2. (R^n, g_{can}) if $K = 0$
3. (S^n, g_{can}) if $K = 1$

Proof. Let Δ be the required space in the cases $K = -1$ and $K = 0$. Take $\tilde{p} \in \tilde{M}$ and $p \in \Delta$ be any point. Let $\iota : T_{\tilde{p}}\tilde{M} \to T_p\Delta$ be any linear isometry. By Hadamard's theorem, the exponetial maps are diffeomorphism. Therefore, $f = \exp_p \circ \circ \iota \circ \exp_{\tilde{p}}^{-1}$ is an isometry $\tilde{M} \to \Delta$.

Now conisder the case $K = 1$. Let $p \in S^n$ and $\tilde{p} \in \tilde{M}$ be some maps and $\iota : T_pS^n \to T_{\tilde{p}}\tilde{M}$ any linear isometry. Again, define the map $f = \exp_{\tilde{p}} \circ \iota \circ \exp_p^{-1}$ taking the open neighborhood $S^n \setminus \{-p\}$ to $\tilde{M}$. By Cartan, f is a local isometry. Now, choose any other $p' \in S^n$ besides p and $-p$ and $\tilde{p}' = f(p')$. Then we may construct the map $f' : S^n \setminus \{-p'\} \to \tilde{M}$ via $f = \exp_{p'} \circ \iota' \circ \exp_{\tilde{p}'}$ via the linear isometry defined as $\iota' = \mathrm{d}f_{p'} : T_{p'}S^n \to T_{\tilde{p}'}\tilde{M}$. Therefore, $f(p') = f'(p')$ and $\mathrm{d}f_{p'} = \iota = \mathrm{d}f'_{p'}$ and thus, on the overlap $f = f'$. Therefore the two functions glue to form a local isometry $h : S^n \to \tilde{M}$. By the lemma, h is a covering map but $\tilde{M}$ is simply connected so h is a diffeomorphism and thus a global isometry. □

Corollary 16.5. Let (M^n, g) be a space form (a complete Riemannian manifold with constant sectional curvature K) then (M^n, g) is isometric to $(\tilde{M}/\Gamma, \hat{g})$ where,

$$(\tilde{M}, \hat{g}) = \begin{cases} (S^n, \lambda^{-1}g_{\text{can}}) & K = \lambda \\ (\mathbb{R}^n, g_{\text{can}}) & K = 0 \\ (H^n, \lambda^{-1}g_{\text{can}}) & K = -\lambda \end{cases}$$

where Γ is a discrete subgroup of isometries of $\tilde{M}$ which acts freely and properly discontinuously. Furthermore the map $(\tilde{M}, \tilde{g}) \to (\tilde{M}/\Gamma, \hat{g})$ is a covering map and local isometry.

Proof. Let $\tilde{M}$ be the universal cover of M. Then equip $\tilde{M}$ with the unique smooth structure such that $\pi : \tilde{M} \to M$ is a local diffeomorphism. Let $\tilde{g} = \pi^*(g)$ then $\tilde{g}$ is a Riemannian metric on $\tilde{M}$. Let $\Gamma = D(\pi)$ be the group of deck transformations of the covering map $\pi : \tilde{M} \to M$. Since $\tilde{M}$ is simply-connected $D(\pi) \cong \pi_1(M)$. We have that Γ facts isometrically on $(\tilde{M}, \tilde{g})$ since it commutes with π and $\tilde{g} = \pi^*(g)$. Furthermore, Γ acts freely and properly discontinuously since it is the deck transformations $\tilde{M} \to M$ is a covering map. $\square$

17 Feb. 28

Proposition 17.1. Let (M^n, g) ne a complete Riemannian manifold with constant sectional curvature $K = +1$ and $n = 2m$ even then $M^n = S^n/\Gamma$ for $\Gamma \subset O(n+1)$ and thus (M^n, g) is isometric to either (S^n, g_{can}) or $(\mathbb{RP}^n, \hat{g})$. In particular, if M^n is orientable then $M^n \cong S^n$.

Proof. We have $M^n \cong S^{2m}/\Gamma$ with $\Gamma \subset O(n+1)$. Then Γ acts freely and properly discontinuously on S^{2m}. All $O(n+1)$ maps are normal and thus diagonalizable with an odd number of eigenvalues each of the from $e^{i\theta}$. If $\gamma \in \Gamma$ has a $+1$ eigenvalue then γ has a fixed point on S^{2m} but the action is free so $\gamma = \text{id}$. For any $\gamma \in \Gamma$, if $\det \gamma = +1$ then γ has $+1$ as an eigenvalue so $\gamma = \text{id}$. Otherwise for $\gamma \in O(n+1)$ we must have $\det \gamma = -1$ so $\det \gamma^2 = 1$ and therefore $\gamma^2 = \text{id}$. This implies that all the eigenvalues of γ are ± 1. If $\gamma \neq \text{id}$ then all its eigenvalues must be -1 and thus $\gamma = -\text{id}$. Therefore either $\Gamma = \{\text{id}\}$ or $\Gamma = \{\text{id}, -\text{id}\}$. $\square$

17.1 Conformal Maps

Definition: Let V and W be n-dimensional inner product spaces. Then $T : V \to W$ is a linear conformal map if T is a linear isomorphism and,

$$\cos\theta(T(v), T(u)) = \frac{\langle T(v), T(u)\rangle}{|T(u)| \cdot |T(v)|} = \frac{\langle v, u\rangle}{|v| \cdot |u|} = \cos\theta(v, u)$$

Lemma 17.2. Let V and W be inner product spaces of dimension n and $T : V \to W$ a linear map. Then T is a linear conformal map iff there exists $\lambda > 0$ such that $|L(v)|_W = \lambda |v|_V$ for all $v \in V$ iff $\langle T(v), T(u)\rangle_W = \lambda^2 \langle v, u\rangle_V$ for all $v, u \in V$.

Definition: Let (M, g) and (N, h) be Riemannian manifolds then a smooth map $f : M \to N$ is *conformal* if $\forall p \in M$ the differential $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is a linear conformal map.

Remark 7. linear conformal map $\implies$ linear isomorphism $\implies$ $\dim M = \dim N$ and f is a local diffeomorphism. By the lemma, f is confroal map if f is a clocal diffomorphism and $f^*h = \lambda^2 g$ for some smooth function $\lambda : M \to \mathbb{R}^+$. In particular, if f is a local isometry then it is conformal wit $\lambda = 1$. Local isometry $\implies$ conformal $\implies$ local diffeomorphism but neither arrow is reversable.

Example 17.3. Take $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(\vec{x}) = \lambda\vec{x}$ for $\lambda \in \mathbb{R}^+$. Then,

$$f_\lambda^* g = f_\lambda^*(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2) = \lambda^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2) = \lambda^2 g$$

Then $\forall \vec{x} : \mathrm{d}f_{\lambda\ \vec{x}} : T_{\vec{x}}\mathbb{R}^n \to T_{\vec{x}}\mathbb{R}^n$. Also $\det \mathrm{d}f_\lambda = \lambda^n > 0$ so f_λ is a conformal orientation preserving map $(\mathbb{R}^n, g_0) \to (\mathbb{R}^n, g_0)$.

Example 17.4. For $\vec{x}_0 \in \mathbb{R}^n$ take $\iota_{\vec{x}_0} : \mathbb{R}^n \setminus \{\vec{x}_0\} \to \mathbb{R}^n \setminus \{\vec{x}_0\}$ which inverts the point across the unit sphere arround $\vec{x}_0$ such that,

$$|\iota_{\vec{x}_0}(\vec{x}) - \vec{x}_0| = \frac{1}{|\vec{x} - \vec{x}_0|}$$

We define,

$$\iota_{\vec{x}_0}(\vec{x}) = \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|^2} + \vec{x}_0$$

Then the differential is,

$$\mathrm{d}\iota_{\vec{x}_0\ \vec{x}}(\vec{v}) = \frac{1}{|\vec{x} - \vec{x}_0|^2}\left(\vec{v} - 2\frac{\langle \vec{x} - \vec{x}_0, \vec{v}\rangle}{|\vec{x} - \vec{x}_0|^2}(\vec{x} - \vec{x}_0)\right)$$

Therefore, the differential simply scales by $|\vec{x} - \vec{x}_0|^{-2}$ and reverses the component of $\vec{v}$ perpendicular to the unit sphere which leaves the length of vectors invariant up to the overall scaling. Therefore $\iota_{\vec{x}_0}$ is conformal.

Theorem 17.5 (Liouville)**.** Let $U \subset \mathbb{R}^n$ is connected open and $f : U \to \mathbb{R}^n$ is conformal with respect to g_0. If $n \geq 3$ then f is the restriction to U of a composition of isometries, dilations (f_λ), and inversions $\iota_{\vec{x}_0}$ at most one of each.

17.2 Möbius Transformation

Consider the map,

$$f(z) = \frac{az + b}{cz + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}\,(2, \mathbb{C})$$

Theorem 17.6. For $n \geq 2$, the isometries of H^n are restrictions to $H^n \subset \mathbb{R}^n$ of the conformal transformations of $\mathbb{R}^n$ that map $H^n \to H^n$.

Proof. □

18 March 4

Definition: Let $c : [0, a] \to M$ be a piecewise smooth curve in a smooth manifold M. A *variation* of c is a continuous map $f : (-\epsilon, \epsilon) \times [0, a] \to M$ denoted as $(s, t) \mapsto f_s(t)$ such that,

1. $f_0(t) = c(t)$

2. $\exists 0 = t_0 < t_1 < \cdots < t_k < k_{k+1} = a$ such that $f|_{(-\epsilon,\epsilon)\times[t_k,t_{k+1}]}$ is smooth.

Given a variation we have the following situations,

1. For fixed $s \in (-\epsilon, \epsilon)$ the map $f_s : [0, a] \to M$ is called the *curve of variation* of f.
2. For fixed $t \in (0, a]$ the function $g_t : (-\epsilon, \epsilon) \to M$ given by $g_t(s) = f_s(t)$ is called a *transverse curve in the variation* f.
3. The vector $V(t) = \frac{\partial f}{\partial s}(s, t)$ for $t \in [0, a]$ is called the *variational field* of f.
4. We say that f is *proper* if $\forall s \in (-\epsilon, \epsilon)$ we have $f_s(0) = c(0)$ and $f_s(a) = c(a)$ i.e. $V(0) = V(a) = 0$.

Proposition 18.1. Let $c : [0, a] \to M$ be a piecewise smooth curve. For any piecewise smooth vector-field $V(t)$ along $c(t)$, there exists a variation $f : (-\epsilon, \epsilon) \times [0, a]$ of c for some $\epsilon > 0$ such that, V is the variational field of f. Moreover, if $V(0) = V(a) = 0$ then f can be chosen to be proper.

Energy Function Given a variation $f : (-\epsilon, \epsilon) \times [0, a] \to M$ of some piecewise smooth curve $c : [0, a] \to M$ on a Riemannian manifold M we define the energy function,

$$E_f(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s, t) \right|^2 \mathrm{d}t$$

which is a (piecewise) smooth map $E_f : (-\epsilon, \epsilon) \to \mathbb{R}$ i.e. a map $E(f_s)$. Let V be the variational field then,

$$\mathrm{d}E_c\,(V) = E'(0)$$

Theorem 18.2 (Formula For the First Variation of Energy)**.** Let (M, g) be a Riemannian manifold and $c : [0, a] \to M$ a piecewise smooth curve and $f : (-\epsilon, \epsilon)\times[0, a] \to M$ the variation of c with variational field V. Let $E : (-\epsilon, \epsilon) \to \mathbb{R}$ be the energy function of f. Then we may compute,

$$\begin{aligned} \tfrac{1}{2}E'(0) = &- \int_0^a \left\langle V(t), \frac{D}{\mathrm{d}t}\frac{\mathrm{d}c}{\mathrm{d}t} \right\rangle \mathrm{d}t \; + \sum_{i=1}^{k} \left\langle V(t_i), \frac{\mathrm{d}c}{\mathrm{d}t}(t_i^+) - \frac{\mathrm{d}c}{\mathrm{d}t}(t_i^-) \right\rangle \\ &- \left\langle V(0), \frac{\mathrm{d}c}{\mathrm{d}t}(0) \right\rangle + \left\langle V(a), \frac{\mathrm{d}c}{\mathrm{d}t}(0) \right\rangle \end{aligned}$$

Proof. Consider the Energy function,

$$E(s) = \int_0^a \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle \mathrm{d}t \; = \sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle \mathrm{d}t$$

Consider, a single term,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_{t_i}^{t_{i+1}}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \mathrm{d}t &= \int_{t_i}^{t_{i+1}}\left\langle \frac{D}{\mathrm{d}s}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \mathrm{d}t \\
&= \int_{t_i}^{t_{i+1}}\left\langle \frac{D}{\mathrm{d}t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle \mathrm{d}t \\
&= \int_{t_i}^{t_{i+1}}\left(\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle - \left\langle \frac{\partial f}{\partial s}, \frac{D}{\mathrm{d}t}\frac{\partial f}{\partial t}\right\rangle\right) \mathrm{d}t \\
&= \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right\rangle \Bigg|_{t=t_i^+}^{t=t_{i+1}^-} - \int_{t_i}^{t_{i+1}}\left\langle \frac{\partial f}{\partial s}, \frac{D}{\mathrm{d}t}\frac{\partial f}{\partial t}\right\rangle \mathrm{d}t
\end{aligned}$$

Therefore,

$$\tfrac{1}{2}E'(s) = -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{\mathrm{d}t}\frac{\partial f}{\partial t}\right\rangle \mathrm{d}t \;+ \sum_{i=1}^{k}\left\langle \frac{\partial f}{\partial s}, \frac{\mathrm{d}f}{\mathrm{d}t}\right\rangle \Bigg|_{t=t_i^+}^{t=t_{i+1}^-}$$

However, at $s = 0$ we have $\frac{\partial f}{\partial s}(0,t) = V(t)$ and $\frac{\partial f}{\partial t}(0,t) = c'(t)$. Therefore,

$$\begin{aligned}
\tfrac{1}{2}E'(0) = -\int_0^a \left\langle V(t), \frac{D}{\mathrm{d}t}\frac{\mathrm{d}c}{\mathrm{d}t}\right\rangle \mathrm{d}t \;&+ \sum_{i=1}^{k}\left\langle V(t_i), \frac{\mathrm{d}c}{\mathrm{d}t}(t_i^+) - \frac{\mathrm{d}c}{\mathrm{d}t}(t_i^-)\right\rangle \\
-\left\langle V(0), \frac{\mathrm{d}c}{\mathrm{d}t}(0)\right\rangle &+ \left\langle V(a), \frac{\mathrm{d}c}{\mathrm{d}t}(0)\right\rangle
\end{aligned}$$

□

Proposition 18.3. Consider the critical points of $E : \Omega_{pq} \to \mathbb{R}$ where $c : [0,a] \to M$ is a piecewise smooth cure. Then for any proper variation f of c, the energy function satisfies $E'(0) = 0$ if and only if c is a geodesic.

Proof. If c is a geodesic then,

$$\frac{D}{\mathrm{d}t}\frac{\mathrm{d}c}{\mathrm{d}t} = 0$$

and because c is smooth we have $c(t_i^+) = c(t_i^-)$. Therefore, by the above formula $E'(0) = 0$.

Conversely, consider two particular variations of c. □

19 Principal Bundles

Definition: We say that $\pi : P \to M$ is a principal G-bundle for a Lie group G acting freely on the right on P such that $\pi : P \to M$ is the quotient map $P \to P/G$ if there are local trivializations (U_α, ψ_α) such that,

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr] & & U_\alpha \times G \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

commutes and ψ_α is G-equivariant i.e. $\psi_\alpha(p \cdot g) = \psi_\alpha(p) \cdot g$ where the action of G on $U_\alpha \times G$ is $(x, h) \cdot g = (x, hg)$.

Definition: Let $\pi : P \to M$ be a pricipal G-bundle. Let F be a smooth manifold equiped with a left G-action. Then G actions on $P \times F$ freely on the right via $(p, \xi) \cdot g = (p \cdot g, g^{-1} \cdot \xi)$.

Definition: Let $P \times_G F$ denote the fibre product given by $(P \times F)/G$. Then the map $\tilde{\pi} : P \times_G F \to M$ given by $[p, \xi] \mapsto \pi(p)$ is a fibre bundle with base M and fiber F.

Definition: Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to \mathrm{Aut}\,(V)$ a representation. Then we may take $P \times_\rho V = P \times_G V$, the associate vector bundle.

Example 19.1. Let $\pi_E : E \to M$ be a vector bundle of rank r over M and take the frame bundle $\mathrm{Aut}\,(E) \to M$ which is a principle $\mathrm{Aut}\,(\mathbb{R}^r)$-bundle. Then we may consider the associated vector bundle $\mathrm{Aut}\,(E) \times_\rho \mathbb{R}^r = E$. However, we may also consider the vector bundle associated to the dual representation, $\mathrm{Aut}\,(E) \times_{\rho^*} \mathbb{R}^r = E^*$. More generally, we may take the representation,

$$\rho^{\otimes s} \otimes (\rho^*)^{\otimes t} : \mathrm{Aut}\,(\mathbb{R}^r) \to \mathrm{Aut}\left(\mathbb{R}^{r^{s+t}}\right)$$

then we find the associated vector bundle,

$$\mathrm{Aut}\,(E) \times_{\rho^{\otimes s} \otimes (\rho^*)^{\otimes t}} \mathbb{R}^{r^{s+t}} = E^{\otimes s} \otimes (E^*)^{\otimes t}$$

In particular,

$$\mathrm{Aut}\,(TM) \times_{\rho^{\otimes s} \otimes (\rho^*)^{\otimes t}} \mathbb{R}^{n^{s+t}} = T^s_t M$$

Example 19.2. Let h be an inner product on a real vector bundle E. Then consider the orthonormal frame bundle $\mathcal{O}(E, h) \to M$ which is a principal $\mathcal{O}(r)$-bundle where r is the rank of E. There is a representation $\rho : \mathcal{O}(n) \to \mathrm{Aut}\,(\mathbb{R}^r)$ and then $\mathcal{O}(E, h) \times_\rho \mathbb{R}^r = E$ and $\mathcal{O}(E, h) \times_{\rho^*} \mathbb{R}^r = E^*$ but these are isomorphic because $\rho = \rho^*$ since it is the orthonormal representation.

Now let h be a hermitian metric on a complex vector bundle $E \to M$ of rank r and consider $\mathrm{U}(E, h) \to M$ the unitary frame bundle. Then there are representations $\rho : \mathrm{U}(r) \to \mathrm{Aut}\,(\mathrm{C}^r)$ and $\rho^* : \mathrm{U}(r) \to \mathrm{Aut}\,(\mathrm{C}^r)$ which give $\mathrm{U}(E, h) \times_\rho \mathrm{C}^r = E$ and $\mathrm{U}(E, h) \times_{\rho^*} \mathrm{C}^r = E^*$. However, $\rho^* = \overline{\rho}$ since it is the representation of the unitary group. Thus, $\mathrm{U}(E, h) \times_{\rho^*} \mathrm{C}^r = \overline{E}$.

20 March 27

20.1 Cross Section

Definition: A *cross section* of a fiber bunle $\pi : E \to$ with fiber F is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$.

Lemma 20.1. Let $\pi : E \to M$ be a trivial fiber bundle with fiber F then cross sections correspond exactly to smooth maps $M \to F$.

Proof. Consider,

$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & M \times F \arrow[dl] \\ & M \arrow[ur, bend right, "s"'] & \end{tikzcd}$$

$s : M \to M \times F$ is a section of $M \times F \to M$ and thus is a map $M \to F$. $\square$

Lemma 20.2. Let $\pi : P \to M$ be a principal G-bundle then $\pi : P \to M$ is trivial iff it admits a cross section.

Proof. If $\pi : P \to M$ is trivial then the above lemma gives a section. Conversely, if $\sigma : M \to P$ is a corss section then define $\phi : M \times G \to P$ via $\phi(x, g) = \sigma(x) \cdot g$. Then ϕ is a G-equivariant diffeomorphism since

$$\phi((x, a) \cdot g)) = \phi(x, ag) = \sigma(x) \cdot (ag) = (\sigma(x) \cdot a) \cdot g = \phi(x, a) \cdot g$$

and furthermore the following diagram commutes,

$$\begin{tikzcd} M \times G \arrow[rr, "\phi"] \arrow[dr] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

because $\pi \circ \phi(x, g) = \pi(\sigma(x) \cdot g) = \pi(\sigma(x)) = x$. $\square$

20.2 Vertical Spaces

Definition: Let $\pi : E \to M$ be a fibre bundle with fibre F. For any $u \in E$ let $x = \pi(u) \in M$ then $\iota_x : \pi^{-1}(x) \hookrightarrow E$ be the inclusion of the firbre $\pi^{-1}(x) = E_x \cong F$. The vertical space $V_u = \mathrm{Im}((\mathrm{d}\iota_x\ )_u) \subset T_uE$. Then $\dim V_u = \dim F = N$. Then $\{V_u \subset T_uE\}$ is a smooth distribution. Equivalently $V \to E$ is a smooth subbundle of $TE \to E$ of rank N. In particular if E is a vector bundle then $V \cong \pi^*E$.

Definition: Let G be a Lie group, Y a smooth manifold and G acts on Y on the right. Given any $\xi \in \mathfrak{g}$ we define the fundamental vector field $X^Y_\xi \in \mathscr{X}(Y)$ via,

$$X^Y_\xi(y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} y \cdot \exp(t\xi)$$

In particular if $Y = G$ and G acts on G by right multiplication then,

$$X_\xi^G = X_\xi^L$$

is the unique left invariant vector field on G with $X_\xi^L(e) = \xi$. If $Y = P$ is a princiapl G-bundle over M then $X_\xi^P(u) \in V_u$ because the curve $t \mapsto y \cdot \exp(t\xi)$ is contained in the fiber E_y. Thus, $X_\xi^P \in \mathcal{C}^\infty(P, V)$.

Lemma 20.3. Let $\pi : P \to M$ be a principal G-buundle. Then the vertical bundle is given by $V \cong P \times \mathfrak{g}$ where $\mathfrak{g} = \mathrm{Lie}(G)$.

Proof. Define $\phi : P \times \mathfrak{g} \to V$ via $\phi(u, \xi) = X_\xi^P(u) \in V_u$. This is an isomorphism of vector bundles over P. □

Remark 8. In particular, if $P = G$ is a principal G-bundle over a point then $V = TG \cong G \times \mathfrak{g}$.

20.3 Connections on Principal Bundles

Remark 9. A connection on a principal bundle $\pi : P \to M$ is of one,

1. horizontal spaces
2. connection 1-form
3. parallel transport

Definition: A *connection* on a principal G-bundle $\pi : P \to M$ s an assignment of hoizontal spaces $\{H_u \subset T_u P \mid u \in P\}$ which is a smooth distribution of n-planes on P where $n = \dim M$ such that,

1. $\forall u \in P : T_u P = Vu \oplus H_u$ and thus $TP = V \oplus H$
2. $\forall u \in P : \forall a \in G : H_{u \cdot a} = (\mathrm{d}R_a\,)_u(H_u)$ with $R_a : P \to P$ via $u \mapsto u \cdot a$.

Definition: A *connection 1-form* on a principal G-bundle $\pi : P \to M$ is a smooth $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ (i.e. for each $X \in \mathscr{X}(P)$ we have a smooth map $\omega(X) : P \to \mathfrak{g}$) such that

1. $\forall \xi \in \mathfrak{g} : \omega(X_\xi^P) = (u \mapsto \xi)$
2. $\forall a \in G : R_a^* \omega = \mathrm{Ad}(a^{-1})\omega$ i.e. pointwise,

$$\forall u \in P : \forall a \in G : \forall y \in T_u P : \omega(u \cdot a)((\mathrm{d}R_a\,)_u(y)) = \mathrm{Ad}(a^{-1})(\omega(u)(y))$$

Lemma 20.4. $(R_a)_* X_\xi^P = X_{\mathrm{Ad}(a^{-1})\xi}^P$

Proof. At $u \in P$ we have,

$$[(R_a)_* X^P_\xi](u) = (\mathrm{d}R_a\,)_{u\cdot a^{-1}}(X^P_\xi(u \cdot a^{-1}) = (\mathrm{d}R\,{}_a)_{u\cdot a^{-1}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} u \cdot a^{-1}\exp(t\xi)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} u \cdot a^{-1} \cdot \exp(t\xi) \cdot a = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} u \cdot \exp(t\mathrm{Ad}(a^{-1})\xi) = X^P_{\mathrm{Ad}(a^{-1})\xi}(u)$$

$\square$

Lemma 20.5. Horizontal spaces and connection 1-forms are in correspondence.

Proof. Given $\{H_u \subset T_uP\}$ satisfying the conditions, define, $\omega \in \Omega^1(P, V)$ as follows. $\forall u \in P : \forall u \in T_uP = H_u \oplus V_u$ then write $y = y^H + y^V$. Since $V \cong P \times \mathfrak{g}$ via the fundamental vector fields then $y^V = X^P_\xi(u)$ for a unique ξ. Define $\omega(u)(y) = \xi$. Clearly, $\omega(X^P_\xi) = \xi$.

Now, for $u \in P, a \in G, y \in T_uP$ then in the case $y \in H_u$ we have $(\mathrm{d}R_a\,)_u(y) \in H_{u\cdot a}$ by assumption. Therefore, by construction,

$$\omega(u \cdot a)((\mathrm{d}R_a\,)_u(y)) = 0 = \mathrm{Ad}(a^{-1})(\omega(u)(y))$$

For the case $y \in V_u$ we have $y = X^P_\xi(u)$ for some $\xi \in \mathfrak{g}$ and thus,

$$\omega(u{\cdot}a)((\mathrm{d}R_a)_u(y)) = \omega(u{\cdot}a)((R_a)_* X^P_\xi(u{\cdot}a)) = \omega(u{\cdot}a)(X^P_{\mathrm{Ad}a^{-1}\xi}) = \omega(X^P_{\mathrm{Ad}(a^{-1})\xi})(u{\cdot}a) = \mathrm{Ad}(a^{-1})\xi$$

Furthermore,

$$\mathrm{Ad}(a^{-1})\omega(u)(y) = \mathrm{Ad}(a^{-1})\omega(u)(X^P_\xi(u)) = \mathrm{Ad}(a^{-1})\xi$$

and the general case follows by linearity. $\square$

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Consider $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying the connection 1-form conditions. Then let $H_u = \ker(\omega(u) : T_uP \to \mathfrak{g})$. The first property gives $\omega(u)|_{V_u} : V_u \to \mathfrak{g}$ is a linear isomorphism so $T_uP = V_u \oplus H_u$. The second property is equivalent to,

$$\begin{array}{ccc} T_uP & \xrightarrow{\omega(u)} & \mathfrak{g} \\ \downarrow{\scriptstyle (\mathrm{d}R_a\,)_u} & & \downarrow{\scriptstyle \mathrm{Ad}(a^{-1})} \\ T_{(ua)}P & \xrightarrow{\omega(u\cdot a)} & \mathfrak{g} \end{array}$$

$$v \in H_u = \ker\omega(u) \iff (\mathrm{d}R_a\,)_u(v) \in \ker\omega(u \cdot a) = H_{ua}$$

21.1 ?

We now fix a principal G-bundle $\pi : P \to M$ of the base M and local trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ which are G-equivariant diffeomorphisms and cross sections $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ of $\pi^{-1}(U_\alpha) \to U_\alpha$ given by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$ for $e \in G$. We want to consider all possible connections on this bundle.

Definition: Let G be a Lie group. Then Mauser-Cartan form is the unqiue $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(G, \mathfrak{g})$ on G which is left invariant and $\theta(e) : T_eG \to \mathfrak{g}$ is the identity i.e. $\theta(X_\xi^L = \xi$ for any $\xi \in \mathfrak{g}$. Equivalently,

$$\forall g \in G : \theta(g) = (\mathrm{d}L_g\ )_{g^{-1}} : T_gP \to T_eG$$

We use the notation $\theta = g^{-1}\mathrm{d}g\ $ for a general group G. Then,

$$R_a^*\theta = R_a^* L_{a^{-1}}^*\theta = \mathrm{Ad}(a^{-1}\theta$$

Example 21.1. Let $G = \mathrm{Aut}\,(\mathbb{R}^r)$ then $TG = G \times \mathfrak{g}$

Definition: Given any connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ on a principal bundle $\pi : P \to G$ with given trivializations and sections. Define $\omega_\alpha = \sigma_\alpha^*\omega \in \Omega^1(U_\alpha, \mathfrak{g})$ then $\phi_\alpha^{-1}(\omega) \in \Omega^1(U_\alpha \times G, \mathfrak{g})$

Lemma 21.2. Then,

$$[(\psi_\alpha^{-1})^*\omega](x, g) = \mathrm{Ad}(g^{-1})\omega_\alpha(x) + \theta(g)$$

via,

$$\begin{array}{cc} ((\psi_\alpha^{-1})^*\omega)(x,g) : & T_{(x,g)}(U_\alpha \times G) \longrightarrow \mathfrak{g} \\ & \Big\| \qquad\qquad\qquad \Big\| \\ \mathrm{Ad}(g^{-1})\omega_\alpha(x) + \theta(g) : & T_xU_\alpha \oplus T_gG \longrightarrow \mathfrak{g} \end{array}$$

Proof. First,

$$((\psi_\alpha^{-1})^*\omega)(x, g)\Big|_{T_gG} = \theta(g)$$

and

$$(\psi_\alpha^{-1})^*\omega(x, e)\Big|_{T_xU_\alpha} = (\sigma_\alpha^*\omega)(x) = \omega_\alpha(x)$$

and,

$$(\psi_\alpha^{-1})^*\omega(x, g)\Big|_{T_xU_\alpha} = ((R_g \circ \sigma_\alpha)^*\omega)(x) = [\sigma_\alpha^* R_g^*\omega](x) = [\sigma_\alpha^*(\mathrm{Ad}(g^{-1}))\omega](x) = \mathrm{Ad}(g^{-1})\omega_\alpha(x)$$

□

Lemma 21.3. On $U_\alpha \cap U_\beta$ with $\omega_\alpha = \sigma_\alpha^*\omega$ and $\omega_\beta = \sigma_\beta^*\omega$. Then,

$$\omega_\alpha = \mathrm{Ad}(\psi_{\alpha\beta}^{-1})\omega_\beta + (\psi_{\alpha\beta}^{-1})\theta$$

Proof. □

Example 21.4. Let $\pi : P \to M$ be a pricipal $\mathrm{GL}\left(()\, r, F\right)$-bundle. Consider the fundamental representation ρ given by acting on F^r. Then $E = P \times_\rho F^r$ is a vector bundle of rank r over M. Then $P = \mathrm{Aut}\,(E)$. Given a connection on $P = \mathrm{Aut}\,(E)$ we define a conection $\nabla : \Omega^0(E) \to \Omega^1(E)$ on E as follows. LEt $\{U_\alpha : \alpha \in I\}$ be an open cover of M. Let $\sigma_\alpha(x) = (e_{\alpha,d}(x), \ldots, e_{\alpha,1}(x))$ frame of $E|_{U_\alpha} \to U_\alpha$ with $\pi : P \to M$. On $E|_{U_\alpha}$ we define,

$$\nabla e_{\alpha,i} = \sum_{j=1}^{n} e_{\alpha,j} \otimes \theta_{ji} \qquad \omega_\alpha = \sigma_\alpha^*\omega = (\theta_{ij}) \in \Omega^1(U_\alpha, \mathfrak{g})$$

Theorem 21.5. Given a connection of a principal bundle $\pi : P \to M$ then there is a connection on the associate vector bundle.