Theorem 0.0.1. all vector bundles on $\mathbb{P}^1$ split into line bundles.

Definition 0.0.2. $E(r,d)$ is the set of isomorphism classes of rank r vector bundles of degree d.

Definition 0.0.3. For $\mathscr{F}$ a coherent sheaf on X,

$$\chi(X,\mathscr{F}) = \sum_{i=0}^{\infty}(-1)^i \dim H^i(X,\mathscr{F})$$

Definition 0.0.4. For X proper E coherent then,

$$\deg E = \chi(X,E) - \operatorname{rank}(E)\cdot\chi(X,\mathcal{O}_X)$$

If E is a vector bundle then,

$$\deg E = \deg\det E$$

Remark. If X is an elliptic curve $g=1$ and,

$$\chi(X,\mathcal{O}_X) = 0$$

therefore,

$$\deg E = \chi(X,E)$$

Proposition 0.0.5. Degree is additive on short exact sequences and,

$$\chi(X,\mathscr{F}\otimes\mathscr{G}) = \operatorname{rank}(\mathscr{F})\cdot\chi(X,\mathscr{G}) + \operatorname{rank}(\mathscr{G})\cdot\chi(X,\mathscr{F})$$

Theorem 0.0.6 (Devissage)**.** If P is a property of coherent sheaves on a locally noetherian scheme X such that,

(a) If there is a short exact sequence of coherent sheaves,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

if $\mathscr{F}_1$ and $\mathscr{F}_2$ have P then $\mathscr{F}_3$ has P.

(b) for all $Z\subset X$ closed integral with generic point $\xi\in Z$ there is some coherent sheaf $\mathscr{G}$ on X such that $\operatorname{Supp}(\mathscr{G}) = Z$ and $\dim_{\kappa(\xi)}\mathscr{G}_\xi = 1$ and $\mathscr{G}$ has P

Then P holds for all coherent sheaves on X.

Remark. We can use this to prove the previous tensor formula for curves since the closed irreducible sets are just a point.

First operation in "Euclidean Algorithm",

$$E(r,d)\to E(r,d+rk)$$

for any $k\in\mathbb{Z}$ fix a line bundle $\mathcal{L}$ of degree 1 and send,

$$\mathcal{E}\mapsto\mathcal{E}\otimes\mathcal{L}^k$$

so by the formula,

$$\deg(\mathcal{E}\otimes\mathcal{L}^k) = d + rk$$

Therefore, we can always reduce to $0 \leq d < r$ via choosing k. Now we want to reduce r,

$$E(r+d, d) \xrightarrow{\sim} E(r, d)$$

Idea: given $\mathcal{E} \in E(r,d)$ find a trivial subbundle of rank d with indecomposable quotient.

Main tool,

$$\{\text{extensions of } \mathcal{E}' \text{ by } \mathcal{O}_X^s\} \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}', \mathcal{O}_X^s) = H^1(X, \mathcal{E}'^\vee \otimes \mathcal{O}_X^s) = \mathrm{Hom}\left(\Gamma(\mathcal{O}_X^s)^\vee, H^1(X, \mathcal{E}'^\vee)\right)$$

Fact: $\mathcal{E}$ is indecomposable iff the class δ in Hom is injective. For now, assume $\deg \mathcal{E} = d > 0$ then $s = \dim \Gamma(\mathcal{E}) > 0$ because,

$$\deg \mathcal{E} = \chi(\mathcal{E}) = h^0(\mathcal{E}) - h^1(\mathcal{E})$$

since $\chi(X, \mathcal{O}_X) = 0$.

Lemma 0.0.7. If $r > d$ then any maximal degree sublinebundle in $\mathcal{E}$ must have degree 0.

Proof. Find a maximal degree line bundle $\mathcal{L} \subset \mathcal{E}$ then we proceed to $\mathcal{E}/\mathcal{L}$ to get a filtration,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

where $\mathcal{E}_{i+1}/\mathcal{E}_i \cong \mathcal{L}_i$ with $\mathcal{L}_0 = \mathcal{L}$. Then we can show that,

$$\deg \mathcal{L}_{i+1} \geq \deg \mathcal{L}_i$$

and that,

$$\deg \mathcal{E} = \sum_{i=0}^{r} \deg \mathcal{L}_i$$

Therefore, if $\deg \mathcal{L}_0 > 0$ then the above sum is at least r proving that $\deg \mathcal{E} \geq r$ contradicting the assumption. $\square$

By Lemma, any $\varphi \in \Gamma(\mathcal{E})$ generates a line bundle of degree 0 (it cannot have negative degree because its generated by the section φ).

Example 0.0.8. If $\mathcal{L}$ is a line bundle on an elliptic curve, $\deg \mathcal{L} = 0$ and $h^0(\mathcal{L}) > 0$ then $\mathcal{L}$ is trivial. (DONT NEED ELLIPTIC CURVE HERE)

If all φ generate trivial line bundles then $\Gamma(\mathcal{E}) \to \mathcal{E}_x$ is injective. Therefore $\Gamma(\mathcal{E})$ generate $\mathcal{O}_X^s \subset \mathcal{E}$.

Now we want to show that $d = s$. By indcution on r: if $r = 1$, $d > 0$, then $h^0(\mathcal{E}) = d$ by Riemann-Roch. Assuming $\mathcal{E}' = \mathcal{E}/\mathcal{O}_X^s$ is indecomposable, then $\dim \Gamma(\mathcal{E}') = d$ by induction. Assuming that $\mathcal{E}, \mathcal{E}'$ are indecomposable, claim if $\mathcal{E}' = \mathcal{E}/\mathcal{O}_X^s$ then,

$$\dim \Gamma(\mathcal{E}') = \dim \Gamma(\mathcal{E})$$

then from,

$$0 \longrightarrow \mathcal{E}'^\vee \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X^s \longrightarrow 0$$

we see that $\deg \mathcal{E} = \deg \mathcal{E}'$ and also this gives a long exact sequence,

$$0 \longrightarrow \Gamma(X, \mathcal{E}'^{\vee}) \longrightarrow \Gamma(X, \mathcal{E}^{\vee}) \xrightarrow{\delta} H^1(X, \mathcal{E}'^{\vee})$$

given that $\mathcal{E}$ is indecomposble iff δ is injective this implies that $\Gamma(\mathcal{E}'^{\vee}) = \Gamma(\mathcal{E}^{\vee})$. Then by Serre duality (recalling that $\omega_X = \mathcal{O}_X$),

$$h^1(X, \mathcal{E}') = h^1(X, \mathcal{E})$$

and also $\deg \mathcal{E} = \deg \mathcal{E}'$ so $h^0(X, \mathcal{E}) = h^0(X, \mathcal{E}')$.

Given $\mathcal{E}' \in E(r, d)$ and $d > 0$ there is a unique $\mathcal{E} \in E(r + d, d)$ which is an extension of $\mathcal{E}'$ by $\mathcal{O}_X^d$. As for degree 0 case, for $\mathcal{E} \in E(r, 0)$ there exists a unique $\mathscr{F}_r \in E(r, 0)$ with $\Gamma(\mathscr{F}_r) \neq 0$, and for all $\mathcal{E} \in E(r, 0)$ there exists a unique $\mathcal{L} \in E(1, 0)$ such that,

$$\mathcal{E} = \mathscr{F}_r \otimes \mathcal{L}$$