## Math GR6262 Algebraic Geometry Assignment # 9

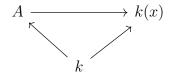
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## 1 Problem 1

Let X be a scheme over a field k and  $x \in X$  have residue field k in the sense that the map  $X \to \operatorname{Spec}(k)$  induces the identity at the stalk  $\mathcal{O}_{\operatorname{Spec}(k),(0)} \to \mathcal{O}_{X,x} \to k(x)$ .

Let  $U \subset X$  be any affine open neighborhood  $U = \operatorname{Spec}(A)$  and  $x \in U$  corresponds to  $\mathfrak{p} \subset A$  then  $k(x) = k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = (A/\mathfrak{p})_{\mathfrak{p}}$ . Furthermore, the map  $\operatorname{Spec}(A) \to \operatorname{Spec}(k)$  makes A a k-algebra compatibly with the isomorphism k(x) = k i.e. the diagram commutes,



We may factor this map via,

$$k \longleftrightarrow A \longrightarrow A/\mathfrak{p} \longleftrightarrow (A/\mathfrak{p})_{\mathfrak{p}} \stackrel{\sim}{\longrightarrow} k(x)$$

which composes the the identity. Because  $A/\mathfrak{p}$  is a domain, the map  $A/\mathfrak{p} \hookrightarrow (A/\mathfrak{p})_{\mathfrak{p}}$  is injective. Therefore, the tower of inclusions collapses showing  $A/\mathfrak{p} = k(x) = k$  which implies that  $\mathfrak{p}$  is maximal since k is a field. Thus  $\mathfrak{p} \in \operatorname{Spec}(A)$  is a closed point. Therefore,  $x \in U$  is closed for each affine open neighborhood. Therefore there exists a closed  $C \subset X$  such that  $C \cap U = \{x\}$  and thus

$$U^{C} \cup \{x\} = (U \setminus \{x\})^{C} = (C^{C} \cap U)^{C} = C \cup U^{C}$$

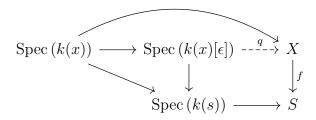
is closed. Now let  $\{U_{\alpha}\}$  be an affine cover of X. If  $x \in U_{\alpha}$  then we have shown that  $U_{\alpha}^{C} \cup \{x\}$  is closed otherwise  $x \in U_{\alpha}^{C}$  so  $U_{\alpha}^{C} \cup \{x\}$  is closed. Therefore, using the fact that  $U_{\alpha}$  cover X, the set

$$\bigcap_{\alpha} U_{\alpha}^{C} \cup \{x\} = \left(\bigcap_{\alpha} U_{\alpha}\right) \cup \{x\} = \emptyset \cup \{x\} = \{x\}$$

is closed.

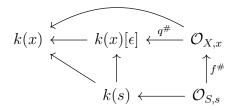
## 2 Tag: 029E

Let  $f: X \to S$  be a morphism of schemes. Let  $x \in X$  be a point and s = f(x). Note that  $\operatorname{Spec}(k(x)[\epsilon]) = \{(\epsilon)\}$  and  $\epsilon^2 = 0$ . Consider the commutative diagram,



where Spec  $(k(x)) \to \text{Spec}(k(x)[\epsilon])$  is induced by the quotient map  $k(x)[\epsilon] \to k(x)[\epsilon]/(\epsilon) = k(x)$  and Spec  $(k(x)[\epsilon]) \to \text{Spec}(k(s))$  is induced by the inclusion  $k(s) \to k(x)[\epsilon]$  and the maps Spec  $(k(x)) \to X$  and Spec  $(k(s)) \to S$  are the canonical maps inducing the identity at the residue field.

Given a morphism  $q: \operatorname{Spec}(k(x)[\epsilon]) \to X$  making the diagram commute we may consider the corresponding maps at stalks,



Consider the restriction  $q^{\#}: \mathfrak{m}_{x} \to (\epsilon) \subset k(x)[\epsilon]$  since this map is local its image lies in  $(\epsilon)$  the maximal ideal of  $k(x)[\epsilon]$ . Then  $q^{\#}(\mathfrak{m}_{x}^{2}) \subset (\epsilon^{2}) = 0$  and thus  $\mathfrak{m}_{x}^{2} \subset \ker q^{\#}$ . Furthermore, by the commutativity of the diagram, the map  $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \xrightarrow{q^{\#}} k(x)[\epsilon]$  factors through k(s) and thus  $q^{\#}(\mathfrak{m}_{s}\mathcal{O}_{X,x}) = 0$  so  $\mathfrak{m}_{s}\mathcal{O}_{X,x} \subset \ker q^{\#}$ . Thus we may factor,

$$\mathfrak{m}_{x} \xrightarrow{q^{\#}} (\epsilon) \cong k(x)$$

$$\underset{\mathfrak{m}_{x}^{2} + \mathfrak{m}_{s} \mathcal{O}_{X,x}}{\underbrace{\mathfrak{m}_{x}^{2} + \mathfrak{m}_{s} \mathcal{O}_{X,x}}}$$

Furthermore,  $\mathcal{O}_{X,x} \to k(x)[\epsilon] \to k(x)$  is the identity so the induced map,

$$\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to (\epsilon)$$

is k-linear.

Conversely, suppose that k(x) = k(s). Given the diagram, the doted morphism is uniquely determined on the underlying topological spaces since it must send the unique point of Spec  $(k(x)[\epsilon])$  to x. Therefore it suffices to show that a local stalk map  $q^{\#}: \mathcal{O}_{X,x} \to k(x)[\epsilon]$  is uniquely determined by a k(x)-linear map,

$$z: \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to k(x)$$

First, note that since  $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$  is local we have maps,

$$\mathcal{O}_{S,s}/\mathfrak{m}_s \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x$$

whose composition gives the natural map  $k(s) \to k(x)$  which we assume to be an isomorphism. Denote k(s) = k(x) = k then the above maps give  $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$  a natural k-algebra structure. The projection map (defined since  $\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x} \subset \mathfrak{m}_x$ ),

$$\frac{(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x})}{\mathfrak{m}_x^2} \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k$$

has kernel  $\mathfrak{m}_x/(\mathfrak{m}_x^2+\mathfrak{m}_s\mathcal{O}_{X,x})$  giving a canonical decomposition as k-modules,

$$\frac{(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x})}{\mathfrak{m}_x^2} = \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}} = k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}}$$

Therefore, we get a map  $q: \mathcal{O}_{X,x} \to k(X)[\epsilon]$  via,

$$\mathcal{O}_{X,x} \to \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \to k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} \xrightarrow{\mathrm{id} \oplus \epsilon z} k(x)[\epsilon]$$

where the last map sends  $(a, m) \mapsto a + z(m)\epsilon$ . I claim that this map makes the diagram commute and is unique. First, it is clear that restructing q to  $\mathfrak{m}_x$  recovers the map z with image embdded as  $k(x)\epsilon \subset k(x)[\epsilon]$ . Next, the diagram commutes because the map sends  $\mathcal{O}_{X,x} \to k(x)$  under projection to the first factor exactly by the quotient  $\pi : \mathcal{O}_{X,x} \to k(x)$  since,

$$a \mapsto [a] \mapsto \pi(a) \oplus [a'] \mapsto \pi(a)$$

for some  $a' \in \ker (\mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x} \to k)$ . Furthermore,  $\mathcal{O}_{S,s} \to k(s) \to k(x)[\epsilon]$  is exactly given by  $\mathcal{O}_{S,s} \to \mathcal{O}_{S,s}/\mathfrak{m}_s \to \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x) \subset k(x)[\epsilon]$  which is just  $\pi \circ f^{\#}$ . Since the diagram commutes, it suffices to show that such a construction will recover the original map  $q^{\#} : \mathcal{O}_{X,x} \to k[\epsilon]$ . The difference  $\tilde{q} = q - q^{\#}$  is a map  $\mathcal{O}_{X,s} \to k[\epsilon]$  which factors through,

$$\frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}} = k \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x}}$$

but is zero on each factor because q and  $q^{\#}$  agree on  $\mathcal{O}_{X,x} \to k$  and on  $\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s \mathcal{O}_{X,x})$  by construction. Thus  $\tilde{q} = 0$  since it factors though the zero map on each factor of the quoitent. Therefore,  $q = q^{\#}$  proving the result.

## 3 Tag: 029G

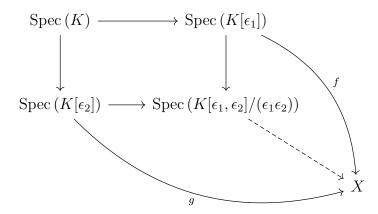
Let K be a field then consider the diagram of schemes,

$$\operatorname{Spec}(K) \longrightarrow \operatorname{Spec}(K[\epsilon_1])$$

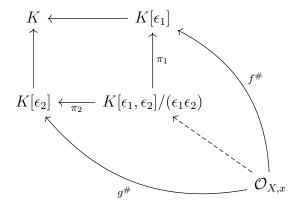
$$\downarrow \qquad \qquad \downarrow$$

$$\operatorname{Spec}(K[\epsilon_2]) \longrightarrow \operatorname{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1 \epsilon_2))$$

we are asked to show that this diagram is a pushout in the category of schemes. Let X be any scheme and consider a commutative diagram,



Each affine scheme has one point so a map Spec  $(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \to X$  is given by choosing a point  $x \in X$  and map  $\mathcal{O}_{X,x} \to K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)$ . We chose the point  $x \in X$  as the image of f which equals the image of f. The sheaf maps (which on a one point space are equivalent to the maps on the stalk) must satisfy the diagram,



However,  $K[\epsilon_1, \epsilon_2]/(\epsilon_1 \epsilon_2) = K[\epsilon_1] \times_K K[\epsilon_2]$  is the pullback in the category of rings and thus there exists a unique map  $\mathcal{O}_{X,x} \to K[\epsilon_1, \epsilon_2]/(\epsilon_1 \epsilon_2)$  making the diagram commute. Since the topological part is fixed this is equivalent to a giving a unque morphism of schemes Spec  $(K[\epsilon_1, \epsilon_2]/(\epsilon_1 \epsilon_2)) \to X$  such that the first diagram commutes. This proof works because  $K[\epsilon_1] \times_K K[\epsilon_2]$  is the pullback in the category of rings making (by the antiequivalence of the Spec functor) the origional diagram a pushout in the category of affine schemes. However, any morphism Spec  $(k[\epsilon_i]) \to X$  factors through an open immersion of some affine patch because the image is a single point which must lie in some affine open. Therefore, this pushout diagram in the category of affine schemes is a pushout in the category of schemes.