Harvard-MIT Algebraic Geometry Seminar

Curves on complete intersections and measures of irrationality.

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Contents

Pretalk

0.0.1 Stable Maps

For a projective variety X equipped with an embedding $X \hookrightarrow \mathbb{P}^N$, we write $\overline{\mathcal{M}}_g(X, b)$ for the Kontsevich moduli space of stable maps from genus g curves which have degree b with respect to the embedding. Likewise, if $(\mathscr{X}, L) \to T$ is a flat projective morphism with L relatively ample we write $\overline{\mathcal{M}}_g(\mathscr{X}/T, b)$ for the relative moduli space of stable maps of degree b with respect to the fixed embedding.

Theorem 0.0.1. Let $\pi : \mathscr{X} \to T$ be a flat projective morphism. Then there exists a proper DM-stack $\overline{\mathcal{M}}_{g,n}(\mathscr{X}/T, b)$ over T representing the moduli problem:

$$(S \to T) \mapsto \{(\pi : \mathcal{C} \to S, \sigma_1, \dots, \sigma_n) \text{ prestable curve with } f : \mathcal{C} \to \mathscr{X}_S \text{ over } S \text{ fiberwise stable}\}$$

such that the fibers of π are curves of genus g and degree b computed against L.

Proof. First notice that the moduli problem for $\mathscr{X}_S \to S$ is the base change along $S \to T$ of the moduli problem for $\mathscr{X} \to T$. Choose an embedding $\mathscr{X} \hookrightarrow \mathbb{P}^N_T$ and consider universal stable map $f : \mathcal{U} \to \mathbb{P}^N \times \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, d)$ over $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^N_T/T, d) = \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, b) \times T$. For $S \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, b) \times T$ we see that f_S factors through $\mathscr{X}_S \hookrightarrow \mathbb{P}^N_S$ if and only if S factors through

$$\overline{\mathcal{M}}_{g,n}(\mathscr{X}/T, b) \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, d) \times T$$

hence it is a closed substack (it is the locus where $f^*\mathscr{I}_{\mathscr{X} \times \overline{\mathcal{M}}} \to \mathcal{O}_{\mathcal{U}}$ is zero, which is represented by a closed substack [FGA explained, Theorem 5.8] since $\mathcal{O}_{\mathcal{U}}$ is flat over $\overline{\mathcal{M}}$).

Therefore, we reduce to the case $\mathscr{X} = \mathbb{P}^N$ and $T = *$. We can build $\overline{\mathcal{M}}$ explicitly from a quotient of the Hilbert scheme. $\square$

Definition 0.0.2. Let $\mathscr{X} \to S$ be a morphism of schemes, $s_0 \in S$ a point, and $\mu_{s_0} : C \to \mathscr{X}_{s_0}$ be a stable map. For any point $s \in S$, we say that μ_{s_0} *deforms to* $\mathscr{X}_s$ if there exists a family of stable maps

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "\mu"] \arrow[d] & \mathscr{X} \arrow[d] \\ (T, t_0) \arrow[r] & (S, s_0) \end{tikzcd}$$

such that T is connected and s is in the image of $T \to S$ and $(\mu_{s_0})_{\kappa(t_0)} \cong \mu_{t_0}$.

Note that when s specializes to s_0, we can always take T to be the spectrum of a DVR.

Definition 0.0.3. A *covering family of curves* on a projective variety X consists of a smooth family

$$\pi \colon \mathcal{C} \to T$$

of projective curves parameterized by an irreducible quasi-projective variety T, together with a dominant morphism (meaning it hits the generic point of each component of X)

$$f \colon \mathcal{C} \to X,$$

such that for general $t \in T$, the map $f_t \colon C_t \to X$ is birational onto its image.

Definition 0.0.4. Let (X, L) be a quasi-projective polarized variety. The *covering degree* of (X, L), denoted by $\operatorname{cov.deg}(X, L)$, is the minimal integer b such that there exists a covering family of curves $\{\pi \colon \mathcal{C} \to T, f \colon \mathcal{C} \to X\}$ with

$$\deg f^*L|_{\mathcal{C}_t} = b.$$

Remark. Alternatively, by standard compactification arguments (cf. [**?**, Prop 2.6]) one can define the *covering degree* $\operatorname{cov.deg}(X, \mathcal{L})$ as the minimal $d \in \mathbb{Z}_{>0}$ such that there exists a family of stable curves $\mathcal{C} \to T$ over an irreducible base scheme T and a *surjective* stable map $f : \mathcal{C} \to X$ such that $\deg f^*L|_{\mathcal{C}_t} = b$. Since X is irreducible, one can reduce to checking the degrees of stable maps where the general curve C_t is smooth and irreducible.

0.0.2 Matching Condition

Definition 0.0.5. Let R be a DVR, and $s, \eta \in \operatorname{Spec}(R)$ be the closed point and generic point, respectively. An *SNC degeneration of varieties* over R is a flat proper family $f : \mathscr{X} \to \operatorname{Spec}(R)$ such that $\mathscr{X}_\eta$ is a smooth variety and $\mathscr{X}_s$ is reduced with simple normal crossing (SNC) singularities.

Let $f : \mathscr{X} \to \operatorname{Spec}(R)$ be a SNC degeneration of varieties such that $\mathscr{X}_s = X_1 \cup_Z X_2$ is the union of two smooth irreducible varieties along a smooth divisor Z.

Definition 0.0.6. Let $\mu : C \to X_1 \cup_Z X_2$ be a stable map whose target has two smooth components glued along Z. A sub-curve $C' \subseteq C$ is said to be of

(a) *ghost type* if $\mu(C')$ is a point in Z;

(b) *type* Z if it is not of ghost type and $\mu(C') \subseteq Z$;

(c) *type* X_i (for $i = 1$ or 2) if it is neither of ghost type nor of type Z, and $\mu(C') \subseteq X_i$.

Lemma 0.0.7. Let $W \subseteq \mathscr{X}_s$ be the singular locus of the total space. Suppose $\mu_s : C \to \mathscr{X}_s$ is a nonconstant stable map that deforms to $\mathscr{X}_\eta$, and $z \in Z \backslash W$ is a point in the image of μ. Then one of the following holds:

(a) z lies on the image of a component of type Z; or

(b) z lies on the image of a component of type X_1 and also on the image of a component of type X_2.

Proof. The idea is as follows: consider a stable map

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mu"] \arrow[d] & \mathscr{X} \arrow[d] \\ \operatorname{Spec}(A) \arrow[r] & \operatorname{Spec}(R) \end{tikzcd}$$

deforming μ_s. We assume the total space $\mathscr{X}$ is smooth but in general we could either shrink and do a local calculation or resolve singularities not chaning the neighborhood of z and work with a more complicated SNC degeneration. Hence $X_1, X_2 \subset \mathscr{X}$ are Cartier divisors. Consider $L_i = \mu_s^* \mathcal{O}_{\mathscr{X}}(X_i)$ and s_i the canonical sections. By definition, $V(s_i) = \mu_s^{-1}(X_i)$ with the correct scheme-theoretic structure. Finally, note that $\mathcal{O}_{\mathscr{X}}(X_1) \otimes \mathcal{O}_{\mathscr{X}}(X_2) \cong \mathcal{O}_{\mathscr{X}}$ with $s_1 s_2 = \pi$ the uniformizer since this is the entire special fiber. Thus $L_1 \otimes L_2 \cong \mathcal{O}_{\mathcal{C}}$ on this surface. Therefore, on any curve $C' \subset \mathcal{C}_s$

$$\deg_{C'} L_1 + \deg_{C'} L_2 = 0$$

but $L_i|_{\mathcal{C}_\eta}$ are trivial so the total degree on $\mathcal{C}_s$ must be zero. Since there is a component $C' \subset \mathcal{C}_s$ properly meeting X_2 we have $\deg_{C'} L_2 > 0$ (given by the vanishing locus of s_2) so there is some component $C'' \subset \mathcal{C}_s$ with $\deg_{C''} L_2 < 0$ but hence s_2 must vanish identically on C'' so C'' is contained in X_2 and $\deg_{C''} L_1 = -\deg_{C''} L_2 > 0$ so C'' meets X_1. Therefore either C'' is type X_2 and meets X_1 property or is type Z. A more detailed analysis shows that there are two such components meeting at z. □

0.0.3 Covering Gonality and Separating Points

One the main "measures of irrationality" I will discuss is the *covering gonality*

$$\operatorname{cov.gon}(X) \ := \ \min\Big\{c > 0 \mid \exists \text{ a curve of gonality } c \text{ through a general point } x \in X\Big\}.$$

Remark. Note that we define the gonality of a singular curve to be the gonality of its normalization. This is equivalent to the minimal c such that X admits a covering family whose general member has gonality c.

I want to describe a general method for proving lower bounds on the covering gonality.

Notice that there is a trivial bound $\operatorname{cov.gon}(X) \leq \deg X$ given by taking linear slices and noting that any curve C in projective space can be projected to $\mathbb{P}^1$ with a map of degree $\deg C$.

Proposition 0.0.8. If $H^0(X, \omega_X)$ separates m distinct general points then $\operatorname{cov.gon}(X) \geq m + 1$.

Proof. Consider a family of gonality $\leq c$ curves covering X,

$$\begin{tikzcd} \mathbb{P}^1 \times S \arrow[dr] & \mathcal{C} \arrow[l, "c"'] \arrow[r, "f"] \arrow[d, "\pi"] & X \\ & S & \end{tikzcd}$$

We can choose $\mathcal{C} \to X$ so that the general curve $\mathcal{C}_s$ is birational onto its image. Suppose that $H^0(X, \Omega_X)$ can separate c general points on X. For some $s \in S$ so that $\mathcal{C}_s \to X$ is birational onto its image, we can chose these points to be exactly a fiber (contained in the immersed locus of $\mathcal{C}_s \to X$) of the gonality map $\mathcal{C}_s \to \mathbb{P}^1$ over a point $y \in \mathbb{P}^1$ where the map is étale. Thus there exists a top-form $\omega \in H^0(X, \omega_X)$ zero at all but one of these points. Hence applying the trace map along $\mathcal{C} \to \mathbb{P}^1 \times S$ to $f^*\omega$ we get a form that is nonzero on $\mathbb{P}^1 \times S$ over the point (y, s). Since $\mathbb{P}^1 \times S$ has no global top forms this is a contradiction. $\square$

0.0.4 Multiplier Ideals

Definition 0.0.9. Let (X, D) projective log pair such that D is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, and fix a log resolution $\mu : X' \to (X, D)$. Then the *multiplier ideal sheaf*

$$\mathcal{J}(D) = \mathcal{J}(X, D) \subseteq \mathcal{O}_X$$

associated to D is defined to be

$$\mathcal{J}(D) = \mu_* \mathcal{O}_{X'}(K_{X'} - \lfloor \mu^*(K_X + D) \rfloor)$$

where $\lfloor \mu^* D \rfloor$ denotes the round-down. Let $Z(X, D)$ (or $Z(D)$ if there is no confusion about the ambient space) denote the scheme defined by the multiplier ideal $\mathcal{J}(D)$.

The most important fact obout multiplier ideas is their vanishing theory. In particular, the Nadel vanishing theorem:

Theorem 0.0.10. (Nadel vanishing theorem) Let X be a $\mathbb{Q}$-Gorenstein normal projective variety, and L be a line bundle on X. Let D be an effective $\mathbb{Q}$-divisor such that $L - D$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0, \quad \text{for} \quad i > 0.$$

Definition 0.0.11. Let (X, D) be a log pair. A prime divisor E over X is called a *log canonical place* or *lc place* of (X, D) if there exists a birational model (or equivalently, a log resolution) $\pi : X' \to X$ of (X, D) such that

$$\operatorname{ord}_E \Big(\pi^*(K_X + D) - K_{X'}\Big) \;\geq\; 1.$$

The closed subset $\pi(E)$ of X is called an *lc center* of (X, D). The *non-klt locus* of (X, D), denoted by $\mathrm{NKLT}(X, D)$, is the union of all lc centers of (X, D). Equivalently it is the complement of the locus of (X, D) having klt singularities.

Remark. $\mathrm{NKLT}(X, D) = \operatorname{Supp}(\mathcal{O}_X/\mathcal{J}(D))$ so it is just the reduced structure on the subscheme defined by the muliplier ideal.

Definition 0.0.12. Let (X, D) be a pair such that D is $\mathbb{Q}$-Cartier. For an effective $\mathbb{Q}$-divisor D and a point $x \in Y$, the *LC-locus* of D locally at x, denoted by $\mathrm{LC}_x(X, D)$, is the union of all irreducible components of $Z(c \cdot D)$ that pass through x, with a reduced scheme structure, where

$$c = \operatorname{lct}_x(X; D) := \inf\{c' > 0 \mid \mathcal{J}(c' \cdot D) \text{ is nontrivial at } x\}$$

is the local log canonical threshold at x.

Remark. In the above definition of LC-locus, the pair (X, D) does not have to be log canonical. While this may not be compatible with some other conventions in the literature, it is convenient for our proofs. If (X, D) is log canonical, we call it the non-klt locus (compare with Definition 0.0.11). In general, for any non-lc pair (X, D), by definition we have that

$$\mathrm{LC}(X, D) \;=\; \mathrm{NKLT}(X, cD),$$

where $c = \operatorname{lct}(X, D)$. The local version is also true: for any point $p \in X$, we have

$$\mathrm{LC}_p(X, D) \;=\; \mathrm{NKLT}_p(X, c_p D),$$

where $c_p = \operatorname{lct}_p(X, D)$.

1 Introduction

The main question: given a projective variety X what is the geometry of the curves on X. More precisely suppose $X \subset \mathbb{P}^N$ has a fixed embedding in projective space. We would like to ask:

(a) what possible values for the numerical invariants of curves on X can appear e.g.

   (a) degree (computed against $\mathcal{O}_X(1)$)

   (b) genus

   (c) gonality

(b) we also have a natural source of curves on X arsing from the embedding: taking a linear space Λ of dimension $N - \dim X + 1$ we get linear slice curves $C_\Lambda := X \cap \Lambda$ that cover X. How close are the "simplest" (in terms of the above numbers) curves to the linear slices

When $X \subset \mathbb{P}^{n+r}$ is a general complete intersection cut out by homogeneous polynomials of degrees $d_1, \ldots, d_r$, we write X is CI of type $(d_1, \ldots, d_r)$ the following result gives a first step towards these questions:

Theorem A (Chen-C-Zhao, '24). *Let $X \subseteq \mathbb{P}^{n+r}$ be a general complete intersection variety of dimension $n \geq 1$ cut out by polynomials of degrees $d_1, \ldots, d_r \geq 2n$. Then any curve $C \subseteq X$ satisfies*

$$\deg(C) \;\geq\; (d_1 - 2n + 1) \cdots (d_r - 2n + 1).$$

Moreover, there exists $N := N(n, r)$ such that if $d_1, \ldots, d_r \geq N$, then

$$\deg(C) \;\geq\; d_1 \cdots d_r.$$

This result is in the same spirit as famous conjectures of Griffiths and Harris originally stated for complete intersection 3-folds. They gave a series of conjectures the weakest of which is:

Conjecture 1.0.1 (Griffiths-Harris, '85). Let $X_d \subset \mathbb{P}^4$ be a (very) general hypersurface of degree $d \geq 6$. Then every curve $C \subset X$ has degree divisible by d.

In partcular, this says that curves are numerically equivalent to complete intersection curves. They further conjecture that all curves are actually *linearly* equivalent to complete intersection curves but I won't be able to say anything about that.

Given the divisibility, we might ask: is every curve a complete intersection. Moreover, for surfaces, the corresponding statement ($d \geq 4$) is a consequence of the Noether-Lefschetz theorem which further says that all curves are complete intersections. However, if $\dim X \geq 3$, Voisin showed that for all d and general X_d there are non-complete intersection curves on X_d so the situtation is quite subtle indeed. However, there is a partial result of Wu

Theorem 1.0.2 (Wu, '90). Let $X_d \subset \mathbb{P}^4$ be a very general hypersurface of degree $d \geq 6$ and $C \subset X$ a curve of degree ℓ. If $\ell < 2d - 1$ then C is a complete intersection.

In particular, Wu's theorem implies our result for hypersurface 3-folds but to my knowledge the methods do not extend past this case.

Regarding divisibility, recent work of Paulsen building on work of Kollár has proved Griffiths and Harris' conjecture for a positive density set of d. We use this result in the proof of Thm. A.

Besides intrinsic interest, our motivation is a conjecture of Bastianelli–De Poi–Ein–Lazarself–Ullery [BDELU17] on the measures of irrationality of complete intersections.

2 Measures of Irrationality

These are quantitative measures of "how far from being rational" a variety.

For a projective variety X of dimension n, the *degree of irrationality* and the *covering gonality* are defined as follows:

$$\mathrm{irr}(X) \ := \ \min\Big\{\delta > 0 \mid \exists \text{ dominant rational map } X \dashrightarrow \mathbb{P}^n \text{ of degree } \delta\Big\};$$

$$\mathrm{cov.\,gon}(X) \ := \ \min\Big\{c > 0 \mid \exists \text{ a curve of gonality } c \text{ through a general point } x \in X\Big\}.$$

From their descriptions, we see that the degree of irrationality is a measure of how far X is from being rational, while the covering gonality is a measure of how far X is from being uniruled.

These are related by:

$$\mathrm{irr}(X) \geq \mathrm{cov.\,gon}(X)$$

For me, irr is the more fundamental measure. However, in practice cov. gon is much easier to study. Since we are interested in lower bounds, it suffices to bound cov. gon

In their landmark paper BDELU prove

Theorem 2.0.1 (BDELU, '17)**.** Let $X_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq n+2$. Then $\mathrm{cov.\,gon}(X_d) \geq d-n$. If X_d is very general then $\mathrm{irr}(X_d) = d-1$.

Geoff Smith also established similar bounds in characteristic p.

Both of their method can prove if $X_{d_1,\ldots,d_r} \subset \mathbb{P}^{n+r}$ is a general complete intersection then $\mathrm{cov.\,gon}(X_{d_1,\ldots,d_r}) \gtrapprox d_1 + \cdots + d_r$ giving an *additive* bound in the degrees. Basically because it comes from positivity of $K_X = (d_1 + \cdots + d_r - n + r)H$.

BDELU ask: are there *multiplicative bounds* of the form

$$\mathrm{cov.\,gon}(X_{d_1,\ldots,d_r}) \geq C d_1 \cdots d_r$$

hence likewise for $\mathrm{irr}(X_{d_1,\ldots,d_r})$.

2.1 History

(a) First evidence: Lazarsfeld '97 proves the case of complete intersection curves

(b) in his thesis: Stapleton '17 gives supperadditive bounds (of the form $e\sqrt[n+1]{d}$ for type (e,d)) for $X_{e,d} \subset \mathbb{P}^{n+2}$ CI of codimension 2

(c) in his thesis: Chen '21 proved a multiplicative bound for dimension 2 and for codimesnion 2 but with a constant $C \ll_n 1$

(d) Levinson-Stapleton-Ullery '23 establish sharp multiplicative bounds for the degree of irrationality of complete intersections whose degrees are sufficiently spread out (i.e. $d_1 \gg d_2 \gg \cdots \gg d_r \gg 0$).

We prove this conjecture and give the sharpest possible constant $C = 1 - \epsilon$.

Theorem B (Chen-C-Zhao, '24)**.** *Let $X \subseteq \mathbb{P}^{n+r}$ be a general CI of type $(d_1, \ldots, d_r)$. For any $0 < \epsilon \ll 1$, there exists an integer $N_\epsilon = N(\epsilon, n, r) > 0$ such that if $d_1, \ldots, d_r \geq N_\epsilon$, then*

$$\mathrm{cov.\,gon}(X) \ \geq \ (1-\epsilon) \cdot d_1 \cdots d_r.$$

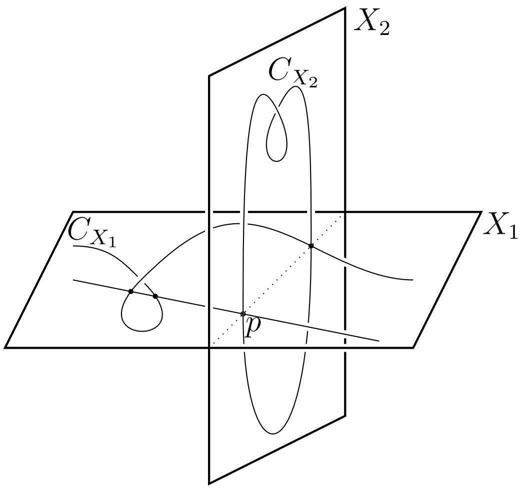

Figure 1: Case (a)

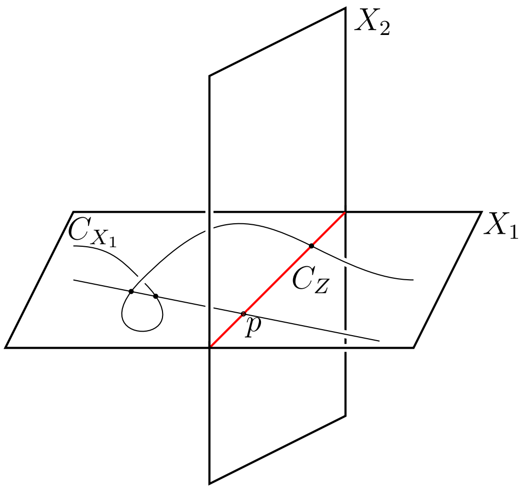

Figure 2: Case (b)

Remark. Note that $\operatorname{cov.gon}(X) \leq \deg X$ by taking linear slices so this is the optimal constant.

Remark. Note that we prove this for X *general* not *very general* as one might expect for gonality results. This means in pariturlar it holds for almost all CIs defined over $\mathbb{Q}$.

It turns out that our proof Theorem B depends on Theorem A.

3 Proof of Thm A

The idea is very simple, we break our complete intersection $X \rightsquigarrow X_1 \cup_Z X_2$ into two complete intersections with $\deg X_1 + \deg X_2 = \deg X$. Crucially we do this so that, Z is also a complete intersection of one higher codimension: say by degenerating one of the equations cutting out X into a product of two lower degree equations. Start with some curve $C \subset X$ and degenerate it to a curve $C' \subset X_1 \cup_Z X_2$.

By wishful thinking: suppose that when we break X the curve C' has a component on each side then

$$\operatorname{min.deg}(X) \geq \operatorname{min.deg}(X_1) + \operatorname{min.deg}(X_2)$$

and immediately win by induction. Unfortunately, this is not true. Consider the degeneration of the lines on a quadric suface to the union of two planes. *DRAW PICTURE*

However, we can fix this strategy by using an idea deloped in log Gromov-Witten theory for finding degeration formulas for GW-invariants in a family specializing as above. The crutial observation is due to Jun Li:

if $\mathscr{X}$ is the total space of the degeneration and we have a nodal curve $C' \to \mathscr{X}_0 = X_1 \cup_Z X_2$ which deforms to the generic fiber. Suppose $p \in C'$ is a point of the curve meeting Z then p must be a node meeting two components C_1, C_2 where $C_i \subset X_i$ and they meet Z at p with the same multiplicity *provided* the following are satisfied:

(1) no component of C' meeting p is contained in Z

(2) $p \in \mathscr{X}$ is a smooth point of the *total space*.

But there is still a problem: what if the curve specializes to C' meeting Z only inside the singular points of $\mathscr{X}$. Indeed this happens for the quadric! The next trick is to consider only those curves that move in a covering family. Then we can ensure that C deforms to a curve that passes through a general point of Z hence outside of $\mathscr{X}^{\text{sing}}$. for the quadric this is given by the line specializing to the intersction of the two planes, ** DRAW **

Hence for covering families, the matching condition applies at some point $z \in Z \setminus \mathscr{X}^{\text{sing}}$ so we can conclude:

$$\text{cov.deg}(X) \geq \min\{\text{cov.deg}(X_1) + \text{cov.deg}(X_2), \text{cov.deg}(Z)\}$$

Because X_1, X_2, Z are also complete intersections, this alows us to do a complicated induction on degrees, codimension, and dimension simultaneously to prove the bound,

$$\text{min.}\deg(X) \geq (d_1 - 2n + 1) \cdots (d_r - 2n + 1)$$

Then a trick of Reidl-Yang reduces the problem of computing the minimal degree of any curve to computing the degrees of curves that cover X. Finally, the same induction coupled with the divisibility results of Paulsen given

$$\deg C \geq d_1 \dots d_r$$

for any curve as long as d_i are very large.

4 Thm A implies Thm B

Precisely, we consider $X = X_{d_1} \cap Y$ as a divisor in $|d_1 H|$ on a complete intersection Y of type $(d_2, \dots, d_r)$. The covering gonality of X is controlled by the number of points separated by canonical sections $H^0(X, \omega_X)$. This means we need to study the number of points separated by the linear series $|K_Y + d_1 H|$. Angehrn and Siu's work towards the Fujita conjecture shows that lower bounds on the degrees of subvarities implies separation of points for adjoint linear series. However, the original theorem of Angehrn and Siu gives insufficient numerics to answer the conjecture of BDELU given only our knowledge of degrees of curves. By optimizing the multiplier ideal construction for large numbers of points using only information on degrees of curves (rather than input from higher-dimensional subvarities) we obtain

Theorem C. *Let (X, H) be a polarized smooth variety. Suppose for $\alpha > 0$ any positive dimensional subvariety $W \subseteq X$ satisfies $\deg_H W \geq \alpha$. Then for any $\epsilon > 0$, there exists an integer $d_0 := d_0(\dim(X), \alpha, \epsilon)$ such that for all $d \geq d_0$, the linear series $|K_X + dH|$ separates at least $(1-\epsilon) \cdot d \cdot \alpha$ distinct points on X.*

Note that the degree bound on curves gives a linear degree bound on all subvarities.

Remark. Theorem C gives the optimal asymtotics for the number of points $|K_X + dH|$ can separate when X is a CI.

4.1 Method

The desiderata: for any choice of m distinct points $p_1, ..., p_m \in U$, we will construct a $\mathbb{Q}$-divisor D which satisfies the following conditions:

- $D \sim_{\mathbb{Q}} c \cdot H$, where $0 < c < d$ is a rational number,
- the support of $\mathcal{O}_X / \mathcal{J}(X, D)$ contains $p_1, ..., p_m$, and
- some point p_j is an isolated point of $\text{NKLT}(D) := \text{Supp}\,(\mathcal{O}_X / \mathcal{J}(X, D))$.

We claim this suffices. Nadel vanishing gives

$$H^1\Big(X, \mathcal{O}_X(K_X + dH) \otimes \mathcal{J}(X, D)\Big) = 0,$$

and hence the restriction map

$$H^0\Big(X, \mathcal{O}_X(K_X + dH)\Big) \longrightarrow H^0\Big(X, \mathcal{O}_X(K_X + dH) \otimes \mathcal{O}_X/\mathcal{J}(X, D)\Big)$$

is surjective hence there is a section $s_j \in H^0(X, \mathcal{O}_X(K_X+dH))$ nonzero at p_j but zero at the rest. Removing the point p_j from $\{p_1, \ldots, p_m\}$ and repeating the same construction for $\{p_1, \ldots, \hat{p}_j, \ldots, p_m\}$, one will end up with m sections $s_1, \ldots, s_m$ separating $p_1, ..., p_m$ since their evaluations at $p_1, \ldots, p_m$ form a triangular matrix with nonzero elements on the diagonal (so the map is surjective).

To produce such a D we perform cutting down operation on $\mathrm{NKLT}(X, D)$ which reduces the local dimension around a point p_j while ensuring that all $p_1, \ldots, p_m$ are still contained.

5 In Case Someone Asks

The correct breaking lemma: let R be a DVR and $s, \eta \in \mathrm{Spec}\,(R)$ be the closed point and generic point, respectively. Let $f : \mathscr{X} \to \mathrm{Spec}\,(R)$ be a SNC degeneration of varieties such that $\mathscr{X}_s = X_1 \cup_Z X_2$ is the union of two smooth irreducible varieties along a smooth divisor Z.

Lemma 5.0.1. Let $W \subseteq \mathscr{X}_s$ be the singular locus of the total space. Suppose $\mu_s : C \to \mathscr{X}_s$ is a nonconstant stable map that deforms to $\mathscr{X}_\eta$, and $z \in Z \backslash W$ is a point in the image of μ. Then one of the following holds:

(a) z lies on the image of a component of type Z; or

(b) z lies on the image of a component of type X_1 and also on the image of a component of type X_2.