Physics C2801, Fall 2013 Prof. Brian Cole

Issued: **Sept. 5**	Problem Set # 1	Due: **Sept. 12**

Problem 1. Approximations

a. Evaluate the first three non-zero terms in the Taylor expansion of $\tan\theta$ around $\theta = 0$. Suppose $\theta << 1$ so we can approximate $\tan\theta$ by the first term in the expansion (which we will often do). We can estimate the *relative* error resulting from this approximation by taking the ratio of the second non-zero term in the Taylor expansion to the first. In principle we should include terms of higher power in θ when evaluating the error, but if the approximation is good, the second term will be sufficient. For what value of θ does approximating $\tan\theta$ by the first term in the Taylor expansion result in a 10% error? Express the angle both in radians and degrees. For what angles is the relative error 1% and 25%.

b. Binomial approximation on quadratic:

   In lecture I discussed the "binomial approximation",

   $$(1+\delta)^a \approx 1 + a\delta$$

   where δ is a **dimensionless** number $<< 1$ and a is any real power. This approximation includes the first order in δ correction to the zeroth order result, namely one (1). Apply this approximation to the calculation of the length of the hypotenuse of a right triangle with side lengths d and L where $d << L$. Remember that we can only approximate in terms of a small **dimensionless** parameter so you will need to manipulate the quadratic to put it in the form appropriate for the binomial approximation. *note:* we will use this approximation to the length of the hypotenuse of a right triangle very often in 2801-2802.

c. Show by an appropriate grouping of terms that the Taylor expansion of the sum of two functions, $f(x) + g(x)$ is the same as the sum of the Taylor expansions of each function.

d. Using the results from parts b and c, approximate $\dfrac{1}{a+x} - \dfrac{1}{a-x}$ for $x << a$ to the lowest power in x/a for which the sum is non-zero. You may do the Taylor expansion explicitly or using the binomial approximation.

e. Now do the same for the combination $\dfrac{1}{a+x} + \dfrac{1}{a-x} - \dfrac{2}{a}$. What happens if you use the binomial approximation for each of the fractions here? What is the appropriate step to take to "fix" the binomial approximation?

Problem 2. Oscillating acceleration

Solve the problem in example 1.11 (page 22) for a time-dependent acceleration, $a(t) = \left(\frac{eE_0}{m}\right)\cos\omega t$ with initial condition $x_0 = 0$, $v_0 = 0$. Compare your solution to the result in the book. Explain the difference. *hint: consider the range of velocities that the particle can take on in the two cases.*

Problem 3. Integrating $\dot{x}$ when expressed as a function of x instead of t

As discussed in lecture we will encounter many situations in C2801-C2802 when we need to integrate an expression of the form

$$\dot{x} = dx/dt = f(x).$$

Even in basic mechanics we are more likely to obtain an expression for the velocity of a particle as a function of position than as a function of time (consider conservation of energy with a position-dependent potential energy function). Such an equation can, in principle, be easily integrated. Since the above equation involves only an ordinary first-order derivative, we can manipulate the differentials dx and dt. So, we can write

$$\frac{dx}{f(x)} = dt.$$

Then, all we have to do is integrate both sides with consistent initial conditions which for this problem corresponds to specifying the position at some initial time, t_0. So if $x(t_0) = x_0$ and we want to determine $x(t)$, we integrate the left hand side from x_0 to x and the right-hand side from t_0 to t

$$\int_{x_0}^{x} \frac{dx'}{f(x')} = \int_{t_0}^{t} dt'.$$

The right-hand side is, of course, trivial, so we obtain

$$\int_{x_0}^{x} \frac{dx'}{f(x')} = t - t_0.$$

If the integral on the left-hand side can be performed analytically, then we can, in principle, solve for $x(t)$. Sometimes this procedure yields a transcendental equation that does not have a closed-form solution. Even if the integral cannot be evaluated analytically, it is often possible to make an approximation to the function $f(x)$ that will make the integral tractable.

I will remind you that in lecture I made a point to discuss the use of the primed, dummy variable in the integral. That variable is different from the x in the upper limit of the integral which is the position for which we are solving. The x' in the integral runs over the "history" of the system between x_0 and x and so is a different variable. Thus, we give it a different name when we do the integration.

a. One of the most common problems that we will encounter in 2801-2802 involves an observable, x, that has a time (or other variable) derivative that satisfies,

$$\frac{dx}{dt} = -\frac{x}{\tau}, \tag{1}$$

i.e. where the rate of change of the observable is (negatively) proportional to the value of the observable. Show that the solution to this equation for initial condition, $x(t_0) = x_0$, is

$$x(t) = x_0\, e^{-(t-t_0)/\tau}.$$

Consider the dimensions in Eq. 1 and how those dimensions work out in the solution. We will frequently refer to τ for such negative exponential solutions as the "time constant". It controls the rate at which $x(t)$ decreases with time.

b. consider what would happen if there were no minus sign in Eq. 1. Could such an equation of motion be physically valid for arbitrarily large values of x? Justify your answer.

c. Another problem that we will encounter frequently has time derivative,

$$\frac{dx}{dt} = x_0\omega\sqrt{1-\frac{x^2}{x_0^2}}$$

(consider the dimensions of ω and the role of x_0 here). Determine $x(t)$ when $x(0) = x_0$. *hint: use an appropriate trigonometric substitution to simplify the integral.*

Problem 4. Drag forces on a falling body

The effects of drag forces are usually neglected in the analysis of the motion of falling bodies in introductory mechanics. But, we will investigate the consequences of such drag forces while applying some of the techniques covered in lecture 1. We will start by assuming that the drag forces on an object are proportional to the square of the velocity. This assumption is more often valid than that made in lecture 1 that the drag force is proportional to the magnitude of the velocity. Then, **for a falling object** the total vertical acceleration can be written $a = -g + bv^2$. Note that the choice of sign here means that the drag force is acting upwards – appropriate for a falling object. Consider the difference between this expression and the linear drag velocity term we discussed in lecture. There the direction of the acceleration was easily handled with a minus sign. **Before proceeding further, determine for yourself the dimensions of b.** Then, have a look at `http://en.wikipedia.org/wiki/Drag_coefficient` and consider the relationship between b and the drag coefficient. Convince yourself that the role of the other parameters in that relationship make physical sense, i.e. that if you express the drag force using those the drag coefficient and those parameters, the resulting expression makes physical sense.

a. Show that with the form of the acceleration given above, the object will reach a maximum or "terminal" velocity. Find an expression for that terminal velocity, v_t, without explicitly integrating the equations of motion.

b. We will evaluate the time dependence of the velocity **for an object starting at rest** using the method of integration from problem 3. Start by writing

$$\frac{dv}{dt} = -g + bv^2.$$

Then, cast this equation in the form

$$\frac{dv}{f(v)} = dt$$

and integrate both sides applying appropriate limits to both integrals. The velocity integral can be simplified using the method of partial fractions. You should obtain a result from the velocity integral that can be simplified into a single logarithm. **Beware:** it is possible to write the partial fraction expression in a form that yields logs of negative numbers when you integrate. A judicious choice of signs for the partial fractions will prevent this from happening. However, you can also "solve" this problem by combining the logarithms and then appropriately handling the minus signs. In fact, if you always write

$$\int_a^b \frac{dx}{x} = \ln(b/a)$$

(and generalizations of this integral) you will generally avoid such problems with logarithmic integrals.

c. Exponentiate both sides of your result form part b and algebraically solve for v. Show that

$$v = -v_t \left(\frac{1 - e^{-2t/\tau}}{1 + e^{-2t/\tau}} \right),$$

where τ is a characteristic time constant for the problem. What is τ in terms of other parameters of the problem?

d. The result from part c can be simplified using hyperbolic trigonometric functions. Show that

$$v = -v_t \tanh{(t/\tau)}.$$

If you are not familiar with hyperbolic trigonometric functions, do not be alarmed – they are simple combinations of positive and negative exponential ($e^{\pm x}$) functions. Consult a reference book or a good calculus book for the definitions. *Hint: you can recast your result into the tanh form by multiplying the terms on the top and bottom of the ratio by the (same) appropriate factor.*

e. The result from part d is compact, but you may not have good intuition for what the tanh function looks like. First, show that you obtain expected results for the velocity at $t = 0$ and $t = \infty$. What fraction (numerically) of the terminal velocity does the object reach at $t = \tau$, $t = \tau/2$ and $t = 2\tau$?

f. Now, using either hand graphing, excel (or some other spreadsheet program), Mathematica (or some other plotting program) plot the velocity as a function of time.

> **note:** You don't need to write this up in your problem set, but consider the behavior of the numerator and denominator in the result from part c. Which is more important in determining the time dependence of the velocity?

Problem 5. Drag forces and horizontal motion

An object moving (approximately) horizontally at high velocity will also experience drag forces but because of the lack of constant acceleration, the resulting time dependence of the horizontal velocity will be different from the result in problem 4. Assume initial conditions, $x = 0$ and $v_x = v_0$ at $t = 0$.

a. Using the same *drag* acceleration as in problem 4 and a procedure similar to that used in problem 4, find an expression for the time dependence of the horizontal velocity, $v_x(t)$. At what time (symbolically) does the velocity reach half its initial value? Consider how b (or better $1/b$) sets the distance scale over which the velocity decreases.

b. A baseball has $b \approx 2 \times 10^{-3}$ m^{-1}. Using your result from part a, evaluate the fractional decrease in the velocity of a baseball initially thrown at 45 m/s from the pitching mound to home plate 16 m away. The straightforward solution to this problem requires evaluating the time that it takes the baseball to go the 16 m. It turns out that we can evaluate that time exactly, but we will often be faced with problems where an exact solution isn't possible and we have to make appropriate approximations. So we will explore the application of such an approximation in this problem and then compare to the exact solution.

It turns out that a reasonable estimate for the time it takes the baseball to go the 16 m can be obtained from the ideal distance vs time relationship (ideal here means neglecting the drag). We will call t_{ideal} that time estimate. Calculate t_{ideal} and then evaluate the fractional decrease in the baseball's velocity using the value you obtained. Can you see why our approximation may be reasonable?

c. Now integrate the result from part a again to obtain an expression for the horizontal position as a function of time, $x(t)$. Express your result symbolically.

d. Using a Taylor expansion of your result in part c around $t = 0$, keeping only the **first non-zero term** in the expansion, show that for small times you obtain the "ideal" result, $x(t) \approx v_0 t$. We will sometimes refer to the first non-zero term in a Taylor expansion as the "zeroth" order result.

e. Calculate the **second non-zero term** in the Taylor expansion of your results from part c which we will call the "first-order" correction to the ideal, zeroth order result that you calculated in part d.

f. Evaluate symbolically the ratio of first order correction term from part e to the ideal, zeroth order, result from part d as a function of time. This ratio is a *rough* estimate of the fractional error that we would make in approximating the complete result from part c by the zeroth order result in part d. Numerically evaluate this fractional error using t_{ideal}.

g. Now, re-evaluate symbolically the travel time of the baseball to the plate including the second term in the Taylor expansion (part e). We will call this new time estimate t_{1st}. Evaluate numerically the fractional change in the estimated travel time from your result in part b,

$$\frac{t_{1st} - t_{ideal}}{t_{ideal}},$$

using the parameters in part b.

h. The result of part c is sufficiently simple that we can directly solve $x(t) = 16$ m for t, yielding the exact travel time to home plate. Do so. Compare this time to t_{ideal} and to t_{1st}.

Problem 6. Kleppner and Kolenkow problem 1.21