Physics GR6037 Quantum Mechanics I
Assignment # 2

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October 12, 2017

Problem 3.

Let an operator O on a 3 dimensional vector space be given as

$$O = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a). Let $\det(I\lambda - O) = 0$ then $\lambda^3 - 2\lambda = 0$ so $\lambda = 0, \pm\sqrt{2}$

For $\lambda = 0$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a + c \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $a = -c$ and $b = 0$ so

$$|v_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda = \sqrt{2}$,

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\sqrt{2}a + b \\ a - \sqrt{2}b + c \\ b - \sqrt{2}c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $b = a\sqrt{2}$ and $c = a$ so

$$|v_{\sqrt{2}}\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

For $\lambda = -\sqrt{2}$,

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sqrt{2}a + b \\ a + \sqrt{2}b + c \\ b + \sqrt{2}c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $b = -\sqrt{2}a$ and $c = a$ so

$$|v_{-\sqrt{2}}\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

(b). $\mathbf{P}\left(\lambda=0\right)=\left|\langle v_0|\psi\rangle\right|^2=\left(\frac{1}{2}(1+0+0)\right)^2=\frac{1}{4}$

$\mathbf{P}\left(\lambda=\sqrt{2}\right)=\left|\langle v_{\sqrt{2}}|\psi\rangle\right|^2=\left(\frac{1}{2\sqrt{2}}(1+\sqrt{2}+0)\right)^2=\frac{1}{8}(3+2\sqrt{2})$

$\mathbf{P}\left(\lambda=-\sqrt{2}\right)=\left|\langle v_{-\sqrt{2}}|\psi\rangle\right|^2=\left(\frac{1}{2\sqrt{2}}(1-\sqrt{2}+0)\right)^2=\frac{1}{8}(3-2\sqrt{2})$

Problem 4.

(a). For every λ, $E(\lambda)$ is an orthogonal projection so $E(\lambda)^2=E(\lambda)$ and $E(\lambda)^\dagger=E(\lambda)$. Now consider $\langle\psi|\,E(\lambda)\,|\psi\rangle=\langle\psi|\,E(\lambda)E(\lambda)\,|\psi\rangle=\langle E(\lambda)\psi|E(\lambda)\psi\rangle\geq 0$ by Hermiticity and positive definiteness.

Also, let $\lambda_1<\lambda_2$ then

$$\begin{aligned}&\langle\psi|\,E(\lambda_2)\,|\psi\rangle=\langle\psi|\,E(\lambda_2)-E(\lambda_1)\,|\psi\rangle+\langle\psi|\,E(\lambda_1)\,|\psi\rangle \text{ but}\\ &\langle\psi|\,E(\lambda_2)-E(\lambda_1)\,|\psi\rangle=\langle\psi|\,(E(\lambda_2)-E(\lambda_1))^2\,|\psi\rangle=|(E(\lambda_2)-E(\lambda_1))\,|\psi\rangle\,|^2\geq 0\end{aligned}$$

Therefore, $\langle\psi|\,E(\lambda_2)\,|\psi\rangle\geq\langle\psi|\,E(\lambda_1)\,|\psi\rangle$

(b). Let

$$\begin{aligned}F&=\int_{-\infty}^{\infty}\lambda\,\mathrm{d}E_F(\lambda)\text{ then }F^2=\int_{-\infty}^{\infty}\lambda\,\frac{\mathrm{d}E_F(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda\int_{-\infty}^{\infty}\lambda'\,\frac{\mathrm{d}E_F(\lambda')}{\mathrm{d}\lambda'}\mathrm{d}\lambda'\\ &=\iint_{-\infty}^{\infty}\lambda\lambda'\,|\lambda\rangle\,\langle\lambda|\lambda'\rangle\,\langle\lambda'|\,\mathrm{d}\lambda\mathrm{d}\lambda'=\int_{-\infty}^{\infty}\lambda^2\,|\lambda\rangle\langle\lambda|\,\mathrm{d}\lambda=\int_{-\infty}^{\infty}\lambda^2\frac{\mathrm{d}E_F(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda\end{aligned}$$

Now, the eigenvectors of F^2 are $\xi=\lambda^2$. Then $F^2=\int_0^\infty\lambda^2\frac{\mathrm{d}E_F(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda+\int_{-\infty}^0\lambda^2\frac{\mathrm{d}E_F(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda$. In the first integral, reparametrize by $\lambda=\sqrt{\xi}$ and in the second, $\lambda=-\sqrt{\xi}$. Thus,

$$\begin{aligned}F^2&=\int_0^\infty\xi\frac{\mathrm{d}E_F(\sqrt{\xi})}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\xi}\mathrm{d}\xi+\int_\infty^0\xi\frac{\mathrm{d}E_F(-\sqrt{\xi})}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\xi}\mathrm{d}\xi\\ &=\int_0^\infty\xi\frac{\mathrm{d}E_F(\sqrt{\xi})}{\mathrm{d}\xi}\mathrm{d}\xi-\int_0^\infty\xi\frac{\mathrm{d}E_F(-\sqrt{\xi})}{\mathrm{d}\xi}\mathrm{d}\xi\int_0^\infty\xi\left[\frac{\mathrm{d}E_F(\sqrt{\xi})}{\mathrm{d}\xi}-\frac{\mathrm{d}E_F(-\sqrt{\xi})}{\mathrm{d}\xi}\right]\mathrm{d}\xi\\ &=\int_0^\infty\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\left[E(\sqrt{\xi})-E(-\sqrt{\xi})\right]\mathrm{d}\xi\end{aligned}$$

This is a resolution of the identity for F^2 if we let $E_{F^2}(\xi)=E_F(\sqrt{\xi})-E(-\sqrt{\xi})$ for $\xi\geq 0$ and $E_{F^2}=\mathbf{0}$ for $\xi<0$.

Problem 5.

Let both A and B be commuting Hermitian operators with complete spectra:

$$A\,|n_A\rangle=a_n\,|n_A\rangle\text{ and }B\,|n_B\rangle=b_n\,|n_B\rangle$$

(a). Suppose that A has a non-degenerate spectrum. Then $AB|n_A\rangle = BA|n_A\rangle = Ba_n|n_A\rangle$. Thus, $A(B|n_A\rangle) = a_n(B|n_A\rangle)$ so $B|n_A\rangle$ is an eigenvector with of A with eigenvalue a_n. Since there is no degeneracy, $B|a_n\rangle = \omega|n_A\rangle$ and therefore, $|n_A\rangle$ is also an eigenvector for B so the basis $\{|n_A\rangle\}$ consists of eigenvectors of both A and B.

(b). let $V_\lambda^A = \{|v\rangle \in \mathcal{H} \mid A|v\rangle = \lambda|v\rangle\}$. For any $|v\rangle \in V_\lambda^A$ take $AB|v\rangle = BA|v\rangle = B\lambda|v\rangle$. Thus, $A(B|v\rangle = \lambda(B|v\rangle$ so $B|v\rangle \in V_\lambda^A$. Therefore, restricting B to the subspace V_λ^A which by assumption is finite dimensional, we get a linear map $B : V_\lambda^A \to V_\lambda^A$ which is Hermitian on finite dimensional spaces. Thus, by the finite dimensional spectral theorem (problem 6), there exists a basis of V_λ^A consisting of eigenvectors of B namely, $\{|w_1^\lambda\rangle, \dots, |w_{n_\lambda}^\lambda\rangle\}$. Now since $\text{span}\{|w_1^\lambda\rangle, \dots, |w_{n_\lambda}^\lambda\rangle\} = V_\lambda^A$ then since every $|n_A\rangle \in V_{a_n}^A$ then

$$\bigcup_{\lambda \in \{a_n\}} \{|w_1^\lambda\rangle, \dots, |w_{n_\lambda}^\lambda\rangle\}$$

Is a complete set because every $|n_A\rangle$ is contained in its span. However each vector in the set is an eigenvector of B by construction. Also, $|w_i^\lambda\rangle \in V_\lambda^A$ so $A|w_i^\lambda\rangle = \lambda|w_i^\lambda\rangle$ so $|w_i^\lambda\rangle$ is also an eigenvector of A.

(c). Since the eigenvectors of A span the entire space, the problem is reduced to diagonalizing B in each eigenspace of A. Then these vectors will be simultaneous eigenvectors of A and B and will space each eigenspace and thus span the entire space. Now, for any $|v\rangle \in V_\lambda^A$ take $AB|v\rangle = BA|v\rangle = B\lambda|v\rangle$. Thus, $A(B|v\rangle = \lambda(B|v\rangle$ so $B|v\rangle \in V_\lambda^A$. Since $B|_{V_\lambda^A}$ is self-adjoint, there is a resolution of the identity,

$$B|_{V_\lambda^A} = \int_{-\infty}^{\infty} \lambda_B \frac{\mathrm{d}E_B(\lambda_B)}{\mathrm{d}\lambda_B} \mathrm{d}\lambda_B$$

With $E_B(\lambda_B)V_\lambda^A \subset V_\lambda^A$. Then

$$\begin{aligned} B|_{V_\lambda^A} \frac{\mathrm{d}E_B(\lambda_B)}{\mathrm{d}\lambda_B} V_\lambda^A &= \int_{-\infty}^{\infty} \lambda_B |\lambda_B'\rangle\langle\lambda_B'|\lambda_B\rangle\langle\lambda_B| \,\mathrm{d}\lambda_B' V_\lambda^A \\ &= \lambda_B |\lambda_B\rangle\langle\lambda_B| \,\mathrm{d}\lambda_B V_\lambda^A = \lambda_B \frac{\mathrm{d}E_B(\lambda_B)}{\mathrm{d}\lambda_B} V_\lambda^A \end{aligned}$$

Thus, $\frac{\mathrm{d}E_B(\lambda_B)}{\mathrm{d}\lambda_B} V_\lambda^A$ is an eignvector of B. Furthermore,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}E_B(\lambda_B)}{\mathrm{d}\lambda_B} V_\lambda^A \mathrm{d}\lambda_B = \int_{-\infty}^{\infty} \mathrm{d}E_B(\lambda_B) V_\lambda^A = (E(\infty) - E(-\infty))V_\lambda^A = V_\lambda^A$$

So these eigenvectors of B span the eigenspace of V_λ^A. Because we are working only in V_λ^A, these vectors are automatically eigenvectors of A as well.

Problem 6.

Let $\dim \mathcal{H} = N$ and $O : \mathcal{H} \to \mathcal{H}$ be hermitian. Then let $S = \{|\psi\rangle \in \mathcal{H} \mid \langle\psi|\psi\rangle = 1\}$ is an N-sphere and thus is compact in $\mathcal{H}$. Since O is hermitian, it has real expectation values so $\langle\psi| O |\psi\rangle : S \to \mathbb{R}$ is a continuous function by the linearity of O. $\langle\psi| O |\psi\rangle$ is a continuous function and S is compact, therefore, Im $\langle\psi| O |\psi\rangle$ is compact in $\mathbb{R}$ so it is closed and bounded and in particular much achive a minumum value $\langle\psi_0| O |\psi_0\rangle \in \mathbb{R}$.

(a). Take normalized $|\delta\psi\rangle \in (\mathrm{span}\{|\psi_0\rangle\})^\perp$ and $\epsilon \in \mathbb{C}$ then define:

$$|\psi_\epsilon\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}} \left(|\psi_0\rangle + \epsilon\, |\delta\psi\rangle\right)$$

Now calculate: $\langle\psi_\epsilon|\psi_\epsilon\rangle =$

$$\frac{1}{1+|\epsilon|^2}\left(\langle\psi_0|\psi_0\rangle + \epsilon\,\langle\psi_0|\delta\psi\rangle + \epsilon^*\,\langle\delta\epsilon|\psi_0\rangle + |\epsilon|^2\,\langle\delta\psi|\delta\psi\rangle\right) = \frac{1}{1+|\epsilon|^2}\left(1+|\epsilon|^2\right) = 1$$

Because $\langle\psi_0|\delta\psi\rangle = 0$ and $\langle\psi|\psi\rangle = \langle\delta\psi|\delta\psi\rangle = 1$.

(b). By the minimum property, $\langle\psi_\epsilon|\, O\,|\psi_\epsilon\rangle \geq \langle\psi_0|\, O\,|\psi_0\rangle$ therefore,

$$\frac{1}{1+|\epsilon|^2}\left(\langle\psi_0)|\, O\,|\psi_0)\rangle + \epsilon\,\langle\psi_0|\, O\,|\delta\psi\rangle + \epsilon^*\,\langle\delta\psi|\, O\,|\psi_0\rangle + |\epsilon|^2\,\langle\delta\psi|\, O\,|\delta\psi\rangle\right) \geq \langle\psi_0|\, O\,|\psi_0\rangle$$

To first order in ϵ,

$$2\mathfrak{Re}\left[\epsilon^*\,\langle\delta\psi|\, O\,|\psi_0\rangle\right] \geq 0$$

Thus take $\epsilon = -\varepsilon\,\langle\delta\psi|\, O\,|\psi_0\rangle$ for $\varepsilon \in \mathbb{R}^+$. Therefore,

$$2\mathfrak{Re}\left[-\varepsilon\,|\langle\delta\psi|\, O\,|\psi_0\rangle|^2\right] = -\varepsilon\,|\langle\delta\psi|\, O\,|\psi_0\rangle|^2 \geq 0$$

Which is a contradiction unless $\langle\delta\psi|\, O\,|\psi_0\rangle = 0$.

(c). Because $\mathcal{H}$ is finite dimensional, $\mathcal{H} = W \bigoplus W^\perp$ with $W = (\mathrm{span}\{|\psi_0\rangle\})^\perp$ and also $W^{\perp\perp} = W$ but $\forall\, |\delta\psi\rangle \in W : \langle\delta\psi|\, O\,|\psi_0\rangle = 0$ therefore, $O\,|\psi_0\rangle \in W^\perp = \mathrm{span}\{|\psi_0\rangle\}$. But if $O\,|\psi_0\rangle \in \mathrm{span}\{|\psi_0\rangle\}$ then $O\,|\psi_0\rangle = \lambda\,|\psi_0\rangle$.

(d). Take $W = (\mathrm{span}\{|\psi_0\rangle\})^\perp$ then for $|\psi\rangle \in W$, $\langle\psi_0|O\psi\rangle = \langle O\psi_0|\psi\rangle = \lambda^*\,\langle\psi_0|\psi\rangle = 0$. Thus, $\mathrm{Im}\, O|_W \subset W$. Therefore, $O|_W$ is a well defined operator on W which inherits Hermiticity. We can apply the above argument to W since $\dim W = N-1$ is finite and produce a new eigenvector $|\psi_1\rangle \in W$ which is perpendicular to the span of $|\psi_0\rangle$.

(e). We can therefore prove the finite dimensional spectral theorem by induction on the dimension of $\mathcal{H}$. If $\dim\mathcal{H} = 1$ then $O\,|v\rangle \in \mathrm{span}\{|v\rangle\}$ trivially. Suppose the theorem holds on every space with $\dim\mathcal{H} = N$. Let $\dim\mathcal{H} = N+1$. Then since $\dim W = N$, W admits a orthonormal basis of eigenvectors of $O|_W$ namely $\{|v_1\rangle, |v_2\rangle, \ldots, |v_N\rangle\}$. Then since the eigenvector found above $|\psi_0\rangle \in W^\perp$ then the set $\{|v_1\rangle, \ldots, |v_N\rangle, |\psi_0\rangle\}$ is an orthonormal set of eigenvectors which are therefore independent.

Problem 7.

Let $|\psi\rangle = |A\rangle + \alpha\,|B\rangle$ where $\alpha \in \mathbb{C}$ then $\langle\psi|\psi\rangle \geq 0$ therefore,

$$\langle A|A\rangle + \alpha\,\langle A|B\rangle + \alpha^*\,\langle B|A\rangle + \alpha^2\,\langle B|B\rangle = |B|^2|\alpha|^2 + 2\mathfrak{Re}\left[\langle A|B\rangle\,\alpha\right] + |A|^2 \geq 0$$

Let $\alpha = \langle B|A\rangle\, r$ for $r \in \mathbb{R}$ then because $\langle A|B\rangle \langle B|A\rangle \in \mathbb{R}$

$$|B|^2\ |\langle A|B\rangle|^2\, r^2 + 2\,\langle A|B\rangle \langle B|A\rangle\, r + |A|^2 \geq 0$$

The innequality must hold for every r therefore, the discriminant of the quadratic form must be non-positive. Therefore, $4\,|\langle A|B\rangle|^4 - 4|A|^2|B|^2\ |\langle A|B\rangle|^2 \geq 0$ Thus,

$$|A||B| \geq |\langle A|B\rangle|$$