1 Regular Rings and Schemes

Example 1.1. Consider $X = \mathrm{Spec}\,(k[x]/(x^2))$. Then consider the unique point $p = (x)$ and $\mathcal{O}_{X,p} = (k[x]/(x^2))_{(x)}$. Then $\mathfrak{m}_p = (x)$ and thus we have,

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = \mathfrak{m}_p = kx$$

so $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 = 1$ but $\dim \mathcal{O}_{X,p} = 0$.

2 Normal Crossings Divisors

Definition Let X be a locally Noetherian scheme. A *strict normal crossings divisor* on X is an effective Cartier divisor $D \subset X$ such that for each $p \in D$ the local ring $\mathcal{O}_{X,p}$ is regular and there exists a regular system of parameters $x_1, \dots, x_d \in \mathfrak{m}_p$ and $1 \le r \le d$ sch that D is cut out by $x_1 \cdots x_r \in \mathcal{O}_{X,p}$

Example 2.1. Consider the closed subscheme of $\mathbb{A}^2_k$,

$$X = \mathrm{Spec}\,(k[x,y]/(xy))$$

Then consider the point $p = (x, y)$ so we need to consider the ring,

$$\mathcal{O}_{X,p} = (k[x,y]/(xy))_{(x,y)}$$

with maximal ideal,

$$\mathfrak{m}_p = (x, y)$$

I claim that this is a regular system of parameters and

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = kx \oplus ky$$

However, $\dim \mathcal{O}_{X,p} = 1$ since we have the maximal chain of primes $(y) \subset (x, y)$ so $\mathcal{O}_{X,p}$ is not regular. However, X is a strict normal crossings divisor of $\mathbb{A}^2_k$ since X is cut out by xy.

Example 2.2. Consider the closed subscheme of $\mathbb{A}^2_k$,

$$X = \mathrm{Spec}\,\big(k[x,y]/(y(x^2 - y))\big)$$

Then consider the point $p = (x, y)$ so we need to consider the ring,

$$\mathcal{O}_{X,p} = (k[x,y]/(y(x^2 - y)))_{(x,y)}$$

with maximal ideal,

$$\mathfrak{m}_p = (x, y)$$

I claim that this is a regular system of parameters and

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = kx \oplus ky$$

However, $\dim \mathcal{O}_{X,p} = 1$ since we have the maximal chain of primes $(y) \subset (x, y)$ so $\mathcal{O}_{X,p}$ is not regular. Furthermore, X is a strict normal crossings divisor of $\mathbb{A}^2_k$ is not cut out by the products of the regular parameters.

3 Introdution

Remark. For me a curve is a separated dimension one scheme of finite type over a field k. We will be careful to distinguish between smooth and singular curves.

Proposition 3.1. Proper curves are automatically projective.

Remark. We will generically be in the following situation, let R be a DVR and $K = \mathrm{Frac}\,(R)$ is field of fractions. Then we will be given a curve C over K.

Alternatively, given a field K with a discrete valuation ν then we may take R_ν to be the valuation ring which reduces to the previous situation.

Definition Given a smooth curve C over K a *regular model* for C over R is a regular R-scheme $X \to \mathrm{Spec}\,(R)$ such that the generic fibre $X_K = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}\,(K)$ (the fibre over the generic point $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(R)$) is K-isomorphic to C.

Theorem 3.2 (Existence of Regular Models)**.** If C is a proper smooth curve over K then C admits a regular model over R.

(CHECK THIS)

4 Models of Curves

Remark. R is a DVR with fraction field K and residue field κ and uniformizer π.

Lemma 4.1 (01WS)**.** Let X be a regular model of a smooth curve C over K. Then,

(a). the special fibre X_κ is an effective Cartier divisor on X,

(b). each irreducible component $C - i$ of X_κ is an effective Cartier divisor on X,

(c). as Cartier divisors,
$$X_\kappa = \sum_i m_i C_i$$
where m_i is the multiplicity of C_i in X_κ,

(d). $\mathcal{O}_X(X_\kappa) \cong \mathcal{O}_X$.

Definition Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and X a regular proper model of C. Let $C_1, \dots, C_n$ be the irreducible components of the special fibre X_κ. Then we write,

$$X_\kappa = \sum_i m_i C_i$$

where m_i is the multiplicity of C_i.

4.1 Minmal Models and Uniqueness

Definition Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A *minimal model* is a regular, proper model X of C such that X does not contain an exceptional curve of the first kind.

Definition We call the following an *exceptional curve of the first kind*:

Let X be a Noetherian scheme. Let $E \subset X$ be a closed subscheme with the following properties,

(a). E is an effective Cartier divisor on X,

(b). there exists a field k and an isomorphism $\mathbb{P}^1_k \to E$,

(c). the normal sheaf $\mathcal{N}_{E/X}$ pulls back to $\mathcal{O}_{\mathbb{P}^1_k}(-1)$.

Lemma 4.2. In the above situation, the special fibre X_κ is connected.

Lemma 4.3. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. If X is a regular proper model for C, then there exists a sequence of morphisms,

$$X = X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

of proper regular models of C, such that each morphism is a contraction of an exceptional curve of the first kind, and such that X_0 is a minimal model.

Remark. Let $f : X \to Y$ be a morphism of schemes and $D \subset X$ an effective Cariter divisor. Then $f : X \to Y$ is a contraction of D if f is proper such that $f(E) = \{y\}$ for some closed point $y \in Y$ where $\mathcal{O}_{Y,y}$ is regular and $\dim \mathcal{O}_{Y,y} = 2$ and such that $f : X \to Y$ is the blowup of Y at y.

Lemma 4.4 (0C5J)**.** Let X be a Noetherian scheme. Let $E \subset X$ be an exceptional curve of the first kind. If a contraction $f : X \to X'$ of E exists, then it satisfies the following univesal property: for every morphism $\varphi : X \to Y$ such that $\varphi(E)$ is a point, then φ factors uniquely through $f : X \to X'$,

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "f"] & X \arrow[r, "\varphi"] \arrow[d, "f"] & Y \\
\operatorname{Spec}(\kappa(x')) \arrow[r, hook] & X' \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

Corollary 4.5. If it exists, any contraction of $E \subset X$ is unique up to unique isomorphism.

Proposition 4.6. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$. A minimal model X of C over R exists.

Proof. Choose a closed immersion $C \to \mathbb{P}^n_K$ and let X be the scheme theoretic image of $C \to \mathbb{P}^n_K \to \mathbb{P}^n_R$. Then by some lemmas $X \to \operatorname{Spec}(R)$ is a proejctive model of C and there exists a resolution of singularities $X' \to X$ and X' is a model for C. Then $X' \to \operatorname{Spec}(R)$ is proper as a composition of proper morphisms. Then we use the previous result to obtain a minimal model by blowing down. □

Proposition 4.7. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and positive genus. The minimal model X of C over R is unique.

Proof. □

Lemma 4.8. Let C be a smooth projective curve over K with $H^0(C, \mathcal{O}_C) = K$ and positive genus. Let X be the minimal model for C over R. Let Y be a regular proper model for C. Then there is a unique morphism of model $Y \to X$ which is a sequence of contractions of exceptional curves of the first kind.

Remark. If the curve C has genus zero. Then minimal models are generically nonunique.

Remark. The minimal model (proper, regular, no exceptional curves of the first kind, then minimal with respect to these conditions) does not necessarily agree with the minimal regular normal crossings model (proper, regular, strict normal crossings divisiors in the special fibre, minimal with respect to these conditions). This is because the minimal model may require blowing up to get strict normal crossings. However, the minimal regular normal corssings model gives the minimal model via blowing down.

5 Picard Groups of Curves

Lemma 5.1 (0C63)**.** There is an exact sequence,

$$0 \longrightarrow \mathbb{Z}^{\oplus n} \longrightarrow \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(C) \longrightarrow 0$$

sending $1 \mapsto (m_1, \ldots, m_n)$ and $e_i \mapsto \mathcal{O}_X(C_i)$

6 Neron Model

Definition Let R be a Dedekind domain and K its field of fractions such that $\operatorname{Spec}(K) \to \operatorname{Spec}(R)$ is the inclusion of the generic point. Then given a K-scheme X_K we say a *model* of X_K over R is an R-scheme $f : X \to \operatorname{Spec}(R)$ such that the generic fiber $X \times_R K$ of the structure map f is K-isomorphic to X_K.

Definition Let A be a smooth separated scheme of finite type over K A Neron model of A_K over R is a model A over R such that for any smooth separated R-scheme X we have the following extension property, given a K-map $f : X_K \to A_K$ there is a unique extension to $\phi : X \to A_R$ such that $f = \phi \times_R K$.

$$\begin{tikzcd}
X_K \arrow[r] \arrow[d] & A_K \arrow[d] \\
X \arrow[r, dashed, "\exists!"] & A_R
\end{tikzcd}$$

In particular, take $X = \mathrm{Spec}\,(R)$ then $X_K = \mathrm{Spec}\,(K)$ so the K-points of A_K give unique R-points of A_R. Thus $A_K(K) \to A_R(R)$ is an isomorphism since any R-point of A_R base changes to a K-point of $A_K = A_R \times_R K$