

1 Affine Curves with No Immersions

Exercise 2. Find a smooth affine curve C with no immersion $C \hookrightarrow \mathbb{P}_k^2$. Unlike in plane case, $K_X \neq 0$ is not an obstruction ($g = 3$ example).

Exercise 3. An easier version, find a curve with no immersion $C \hookrightarrow \mathbb{A}_k^2$. My guess is that here $K_X \neq 0$ is an obstruction but I can't prove it.

4 Curves with Intersection at One Point in Ambient Space

Exercise 5. First question: given a curve C and a closed point $P \in C$ when is there a function $f \in \Gamma(C, \mathcal{O}_C)$ such that $V(f) = \{P\}$ (topologically it may be nonreduced that is fine).

For affine curves: the previous question is asking: given a one-dimensional Noetherian domain A and a maximal ideal $\mathfrak{m}_0 \in \text{Spec}(A)$ when is there $f \in A$ such that $\sqrt{(f)} = \mathfrak{m}_0$.

Suppose that A is a Dedekind domain. Then by unique factorization, $\sqrt{(f)} = \mathfrak{m}_0$ iff $(f) = \mathfrak{m}_0^n$ meaning that this is possible iff \mathfrak{m}_0 is torsion in the ideal class group. Choosing any nontorsion point of an elliptic curve is then a counter example.

Is the class group not being torsion the only obstruction?

Exercise 6. Given a closed immersion $\iota : C \hookrightarrow S$ with S a surface, given a point $P \in C$ when does there exist a closed curve $Z \subset S$ such that $Z \cap C = \{P\}$?

First, an example. Consider the surface $S = C \times_k \mathbb{P}_k^1$ with $C \hookrightarrow S$ via the closed point $0 \in \mathbb{P}_k^1$. Then for any closed point $P \in C$ we can consider the curve $\{P\} \times_k \mathbb{P}_k^1 \subset C \times_k \mathbb{P}_k^1$ and $\{P\} \times_k \mathbb{P}_k^1 \cap C = \{P\}$. Therefore, this is certainly possible for an arbitrary curve. Perhaps we need to fix

The affine curve $Z \subset X$ is defined by some sheaf of ideals $\mathcal{J} \subset \mathcal{O}_S$. Then $\iota^{-1}\mathcal{J} \cdot \mathcal{O}_C$ gives the sheaf of ideals for $\iota^{-1}(Z)$.

First, consider $S = \mathbb{A}_k^2$ then we have $C = V(f)$ for some $f \in k[x, y]$ and $Z = V(g)$ for some $g \in k[x, y]$ we can assume these are irreducible so C and Z are integral curves. Then, under the map $k[x, y] \rightarrow k[x, y]/(f)$ we get $\iota^{-1}\mathcal{J} \cdot \mathcal{O}_C = g \cdot k[x, y]/(f) = (g)$ and thus we reduce exactly to the previous problem for the affine curve $C = \text{Spec}(k[x, y]/(f))$.

Likewise, consider $S = \mathbb{P}_k^2$ then we have $Z = V(I)$ for a height one homogeneous prime ideal I (CHECK THIS). I am guessing that we can take $I = (f)$ since $k[X_0, X_1, X_2]$ is a UFD so height one primes are principal (MAKE SURE f is HOMOGENEOUS). Then $Z = \text{Proj}(k[X_0, X_1, X_2]/(g))$ and I think $\iota^{-1}(Z) = \text{Proj}(k[X_0, X_1, X_2]/(f, g))$ or equivalently $V(g)$ in $C = \text{Proj}(k[X_0, X_1, X_2]/(f))$. So now reduce to when $V(g)$ on a projective curve have support at only one point.

7 Weakly but not Strongly Toric Curve

Find a curve which is toric (i.e. weakly Δ -nondegenerate for some Δ) but is never nondegenerate i.e. (never Δ -nondegenerate for any Δ).

8 Affines in the Plane

Exercise 9. Given an example of an immersed curve $C \hookrightarrow \mathbb{A}_k^2$ such that C is not (closed) embedded in any affine open of \mathbb{A}_k^2 . Likewise, give an example of an immersed curve $C \hookrightarrow \mathbb{P}_k^2$ such that C is not (closed) embedded in any affine open of \mathbb{P}_k^2 .

10 Why is this so hard

Proposition 10.0.1. There exists a smooth affine curve C over k with no immersion $C \hookrightarrow \mathbb{A}_k^2$ and, in particular, no immersion $C \hookrightarrow \mathbb{G}_m k^2$. Thus, there are smooth affine curves which are not affine plane curves.

Proof. First, we show that if $j : C \hookrightarrow \mathbb{A}_k^2$ is an immersion then $\Omega_{C/k} \cong \mathcal{O}_C$ is trivial. We can factor j as $C \hookrightarrow \bar{C} \hookrightarrow \mathbb{A}_k^2$ into an open immersion followed by a closed immersion [Stacks, Tag 03DQ]. Then \bar{C} □

Take an algebraically closed field k . Show the following,

- (a) if $C \hookrightarrow \mathbb{P}_k^2$ is an immersion then there is a plane curve $\bar{C} \subset \mathbb{A}_k^2$ (closed immersion) and an open immersion $C \hookrightarrow \bar{C}$.
- (b) for any closed curve $\bar{C} \subset \mathbb{A}_k^2$ we have $\Omega_{\bar{C}/k} = \mathcal{O}_{\bar{C}}$
- (c) thus, since $C \hookrightarrow \bar{C}$ is étale we have $\Omega_{C/k} = \mathcal{O}_C$ so it suffices to construct a smooth affine curve with nontrivial canonical bundle $\Omega_{C/k}$.
- (d) Choose a curve C with genus $g(C) \geq 2$ then $\deg \Omega_{C/k} \geq 2$ and choose a point $P \in C$ such that $K_X \not\sim (2g-2)[P]$ for any $k \in \mathbb{Z}$.
- (e) Show that $U = C \setminus \{P\}$ is affine,
- (f) Then $U \hookrightarrow C$ is étale so $\Omega_{U/k} = f^* \Omega_{C/k}$ so $K_U \sim [P]$.
- (g) Show that this is nontrivial using the exact sequence,

$$\mathbb{Z} \longrightarrow \text{Cl}(C) \longrightarrow \text{Cl}(U) \longrightarrow 0$$

the first map sending $1 \mapsto [P]$ so we need to show that $K_X \not\sim (2g-2)[P]$ for any k .

11 Supersingular Stuff Here

12 TODO

Milne's Notes on Etale Cohomology

Write notes on GAGA and analytification of a Scheme

Finish K-book

Finish Milne's Notes on Shimura Varieties

13 Questions For Johan

Exercise 14. Which hypersurfaces $X \subset \mathbb{P}_k^n$ are rational. I managed to prove that only hyperplanes and conics $X \subset \mathbb{P}_k^2$ are isomorphic to \mathbb{P}_k^{n-1} but which are actually rational?

Exercise 15. Which hypersurfaces $X \subset \mathbb{P}_k^n$ are of the form $\mathbb{P}^{a_1} \times_k \cdots \times_k \mathbb{P}^{a_r}$?

This might be trivial since in projective space all lines intersect but this is not true of these schemes.

Exercise 16. Is there a splitting principal in algebraic geometry?
How difficult is it to compute the Atyiah class for a vector bundle? What tools are there to do this computation?

Exercise 17. What is the “correct” definition of the Chern classes in algebraic geometry?

Exercise 18. Ravi says: Having made a spirited case for smoothness, we should be clear that regularity is still very useful. For example, it is the only concept which makes sense in mixed characteristic. In particular, Z is regular at its points (it is a regular ring), and more generally, discrete valuation rings are incredibly useful examples of regular rings.

Why does smoothness not make sense in mixed characteristic?

Exercise 19. what is the difference between the neron model and the minimal model?

20 Questions For Stack Exchange

Exercise 21. The thing about a function which only cuts out one point.

Exercise 22. What does supersingularity actually mean.

Exercise 23. Soft question. Non mathematical questions with mathematical answers.
