

1 The Kobayashi Pseudodistance

Definition 1.0.1. A *directed pair* (X, V) is a pair of a complex manifold X and a holomorphic subbundle $V \subset T_X$.

Here let Δ be the unit disk in \mathbb{C} and ρ the Poincare metric on Δ .

Definition 1.0.2. Let X be a complex manifold. The *Kobayashi pseudodistance* is the pseduometric defined,

$$d_X(p, q) = \inf_{\alpha} \ell(\alpha)$$

where α is a chain of holomorphic disk $f_i : \Delta \rightarrow X$ and points $p = p_0, p_1, \dots, p_k = q$ of X and pairs $a_1, b_1, \dots, a_k, b_k \in \Delta$ such that,

$$f_i(a_i) = p_{i-1} \quad f_i(b_i) = p_i$$

and the length $\ell(\alpha)$ of the chain is defined as,

$$\ell(\alpha) := \rho(a_1, b_1) + \dots + \rho(a_k, b_k)$$

where ρ is the Poincare metric on Δ .

Example 1.0.3. Let $X = \mathbb{C}$ then $d_X = 0$. Indeed, by choosing larger and larger discs containing p, q their pullback to the unit disk is then closer and closer to the origin and hence have vanishing Poincare distance.

Remark. Recall the Schwartz-Pick lemma says that any holomorphic map $f : \Delta \rightarrow \Delta$ is a contraction for the Poincare metric. Therefore, $d_\Delta = \rho$.

Lemma 1.0.4. Let $f : X \rightarrow Y$ be holomorphic. Then $d_Y(f(x), f(y)) \leq d_X(x, y)$

Proof. Indeed, choosing any chain of disks $g_i : \Delta \rightarrow X$ computing $d_X(x, y)$ we see that $f \circ g_i$ is a chain of disks connecting $f(x)$ and $f(y)$ of the same length. Therefore,

$$d_Y(f(x), f(y)) = \inf_{\alpha} \ell(\alpha) \leq d_X(x, y)$$

□

Corollary 1.0.5. If $f : \mathbb{C} \rightarrow X$ is an entire curve then for $x, y \in f(\mathbb{C})$ we have $d_X(x, y) = 0$ meaning if f is nonconstant then d_X is degenerate along the image of f .

Proof. Indeed, let $z_1, z_2 \in \mathbb{C}$ map to x, y respectively. Then,

$$d_X(x, y) \leq d_{\mathbb{C}}(z_1, z_2) = 0$$

□

Brody's theorem is a converse to this result. We start by considering an infinitesimal analogue of the Kobayashi pseudodistance. Let $v \in T_{X, x_0}$ be a holomorphic tangent vector at $x_0 \in X$ and define,

$$\mathbf{k}_X(v) = \inf\{\lambda > 0 \mid \exists f : \Delta \rightarrow X \text{ such that } f(0) = x_0 \text{ and } \lambda f'(0) = v\}$$

where $f : \Delta \rightarrow X$ is holomorphic. It is easy to check that holomorphic maps contract this pseduometric and for $X = \Delta$ it agrees with the Poincare metric.

Theorem 1.0.6. Let X be a complex manifold. Then,

$$d_X(p, q) = \inf_{\gamma} \int_{\gamma} \mathbf{k}_X(\gamma'(t)) dt$$

where the infimum is taken over all piecewise smooth curves joining p and q .

Definition 1.0.7. A *Brody curve* $f : \mathbb{C} \rightarrow X$ is an entire curve which has bounded derivative (wrt to some/any hermitian metric).

Theorem 1.0.8 (Brody). Let X be a compact complex manifold. If d_X is degenerate then there exists a Brody curve in X .

Remark. Of course, in the case that X is compact any entire curve is automatically Brody.

2 Definitions

Definition 2.0.1. We say a directed pair (X, V) is,

- (a) *Brody hyperbolic* if there does not exist a nonconstant entire map $f : \mathbb{C} \rightarrow X$ tangent to V
- (b) *Kobayashi hyperbolic* if the Kobayashi pseudodistance is nondegenerate (i.e. it is a metric).

Theorem 2.0.2 (Brody). Let X be a compact complex manifold. Then X is Kobayashi hyperbolic if and only if it is Brody hyperbolic.

Remark. Therefore, we will call manifolds with this property just “hyperbolic” or “analytically hyperbolic” for emphasis.

Definition 2.0.3. If X is a complex projective algebraic variety we say (X, V) is

- (a) *algebraically hyperbolic* if there exists $\epsilon > 0$ such that for every complete integral curve $C \subset X$ we have,

$$2g(C) - 2 \geq \epsilon \deg_H C$$

where $g(C)$ is the geometric genus of C

- (b) *algebraically quasi-hyperbolic* if X contains finitely many genus 0 and genus 1 curves.

Theorem 2.0.4 (Demailly). Let X be a smooth projective variety. Then the following hold,

$$X \text{ is hyperbolic} \implies X \text{ is algebraically hyperbolic}$$

Theorem 2.0.5. If X is algebraically hyperbolic then X admits no nonconstant morphisms from an abelian variety.

Some references:

- (a) [Xi Chen](#)
- (b) [Javanpeykar](#)

2.1 The Green-Griffiths Locus and Jets

Theorem 2.1.1 ([Demilly's Notes](#) Theorem 7.9). Let (X, V) be a direct projective manifold and A an ample line bundle. Then for any entire curve $f : \mathbb{C} \rightarrow X$ tangent to V and any $P \in H^0(X, E_{k,m}^{GG}(V^*) \otimes A^{-1})$ we have $P(f', f'', \dots, f^{(k)}) = 0$ identically.

Therefore, if we fix an ample line bundle we can consider the locus cut out by all these differential equations.

Definition 2.1.2. The *Green-Griffiths locus* $GG_A(X, V)$ is the set $x \in X$ such that for all $k > 0$ there exists a k -jet $\varphi_k : (\mathbb{C}, 0) \rightarrow (X, x)$ tangent to V so that for all $m > 0$ every global jet differential $P \in H^0(X, E_{k,m}^{GG}(V^*) \otimes A^{-1})$ satisfies $P(\varphi_k) = 0$.

Remark. The locus $GG_A(X, V)$ is independent of the choice of ample line bundle (see [Diverio and Rousseau](#) Lemma 2.2. This paper also gives many examples showing that $\text{Exc}(X)$ can be strictly smaller than $GG(X)$. However, it is conjectured that if X is general type then $GG(X) \subsetneq X$.

LOOK AT THE HILBERT MODULAR SURFACES FOR WHICH THE GG LOCUS IS EVERYTHING

3 Conjectures

Conjecture 3.0.1 (Kobayashi). For $n \geq 2$ and $D \subset \mathbb{P}^n$ a very general hypersurface of degree $\deg D \geq 2n + 1$ then,

- (a) D is hyperbolic
- (b) $\mathbb{P}^n \setminus D$ is hyperbolic.

Conjecture 3.0.2 (Green-Griffiths-Lang). Let X be a projective variety of general type. Then there exists a proper algebraic subvariety containing all non-constant entire curves $f : \mathbb{C} \rightarrow X$.

Conjecture 3.0.3 (Demailly). If X is algebraically hyperbolic then X is hyperbolic.

Proposition 3.0.4. The Green-Griffiths-Lang conjecture implies the Demailly conjecture.

WHY?

Proof. Suppose X is algebraically hyperbolic. If X is not of general type then X has a fibration over its canonical model by varieties of Kodaira dimension 0. (I NEED THAT IF NOT GENERAL TYPE THEN NOT ALGEBRAICALLY HYPERBOLIC DOES THIS FOLLOW FROM MMP) \square

4 Theorems

Theorem 4.0.1 (Bogomolov). Let X is a smooth projective surface with $s_2(X) = c_1(X)^2 - c_2(X) > 0$ then X has finitely many genus 0 or genus 1 curves (i.e. it is algebraically quasi-hyperbolic).

Theorem 4.0.2 (McQuillan). Let X is a smooth projective surface with $s_2(X) = c_1(X)^2 - c_2(X) > 0$ and X has *no* genus 0 or genus 1 curves then X is hyperbolic.

5 Bogomolov's Theorem

The notion of stability of a point on a space of linear representations of a reductive group, due to Mumford [10], leads to a notion of stability for fiber bundles over a curve, whose properties were studied in [13] and [19].

Definition 5.0.1. Over a smooth proper integral curve, a vector bundle E of rank $r(E)$ and degree $d(E)$ is *stable* (resp. *semistable*) if for every nonzero proper subbundle $F \subsetneq E$ we have,

$$\frac{d(F)}{r(F)} < \frac{d(E)}{r(E)} \quad \left(\text{resp. } \frac{d(F)}{r(F)} \leq \frac{d(E)}{r(E)} \right)$$

A vector bundle is *unstable* if it is not semistable.

Now let X be a smooth proper surface over a field k , and E a vector bundle over rank 2 over X . Then a linear representation $\rho : \mathrm{GL}_2 \rightarrow \mathrm{GL}(V)$ produces an associated bundle $E^{(\rho)} := E \times_{\mathrm{GL}_2} V$ of rank $\dim V$.

Definition 5.0.2. We say a rank 2 vector bundle is *instable* if there exists a representation $\rho : \mathrm{GL}_2 \rightarrow \mathrm{GL}(V)$ with $\det \rho = 1$ such that $E^{(\rho)}$ admits a nonzero section which vanishes at some point.

If the characteristic of k is zero, which we will assume for the remainder, then Bogomolov's instability criterion is simply expressed in terms of devissage of bundles of rank 2 (WHAT?). It is interesting to note that we can here short-circuit the theory and prove directly using these simpler methods.

There are many applications. We quote from memory a proof, elegant and algebraic, of the vanishing theorem of Kodaira-Ramanujan. In the remaining section we prove the following:

Theorem 5.0.3 (0.3). Let X be a proper smooth surface of general type. Then Ω_X is not unstable.

As a consequence, we obtain the inequality $c_1^2 \leq 4c_2$ where c_1, c_2 are the Chern classes of the sheaf Ω_X^1 – improved by Miyaoka [9] which is the best form possible $c_1^2 \leq 3c_2$ – and a geometric result that we will develop here.

The problem is the following: can we show 'bound' the family of curves of bounded geometric genus on a smooth proper surface X ? We construct easily examples where the answer is negative. Bogomolov provides a partial solution in the case that X is a surface of general type. We summarize briefly the method.

Let $\pi : P = \mathbb{P}(\Omega_X^1) \rightarrow X$ be the canonical projection from the projectivization of the canonical bundle. We construct on P a good linear system of divisors allowing it to be mapped to the projective space \mathbb{P}^N . If C is a smooth proper curve and $f : C \rightarrow X$ is a nonconstant morphism there is a lift $t_f : C \rightarrow P$ via the differential defined over points $\alpha \in P$ where f is unramified as $t_f(\alpha) = (f(\alpha), f(v_\alpha))$ where v_α is a nonzero tangent vector to C at α . We apply this to the normalizations of curves embedded in X and study their images in \mathbb{P}^N .

We prove the following result:

Theorem 5.0.4. Let X be a smooth proper surface minimal of general type.

- (a) If $c_1^2 > c_2$ then the curves of bounded geometric genus on X form a bounded family.

- (b) If $c_1^2 \leq c_2$ and $\text{rank NS}(X) \geq 2$ then there exists a nonempty open cone $C \subset \text{NS}(X)_{\mathbb{R}}$ containing the cone $\{z \mid z \in \text{NS}(X)_{\mathbb{R}}, z^2 \leq 0\}$ such that for any closed cone C' contained in C the family of curves of bounded geometric genus on X have image in $\text{NS}(X)_{\mathbb{R}}$ contained in C' forms a bounded family. Moreover, any translate of C parallel to K_X has the same property.

As a corollary, we obtain finiteness of curves with negative self-intersection and bounded geometric genus on surfaces of general type. In particular a solution to Mordell's problem.

Let's point out finally that Bogomolov uses a powerful result of Deidenberg on differential equations [18] but a recent paper of Jouanolou [5] allows us to avoid the use of this sledgehammer.

5.1 Criteria for instability of vector bundles of rank 2 on surfaces

Considering the form of representations of PGL_2 we give a definition equivalent to above.

Definition 5.1.1. A vector bundle E of rank 2 on a surface is *unstable* if and only if there exists $n > 0$ such that $S^{2n}E \otimes (\det E)^{-n}$ has a nonzero section vanishing at some point of X .

5.1.1 Remark: devissage of vector bundles of rank 2

Let E be a vector bundle of rank 2 and L an invertible sheaf and $s : L \rightarrow E$ a nonzero map. The bidual M of E/L is reflexive and hence invertible (since X is a smooth surface), and the kernel L_1 of the homomorphism $E \rightarrow M$ is a larger invertible subsheaf of E containing L . We say that it is a saturated line bundle of E . The cokernel E/L_1 is torsion-free in rank 1, and hence of the form $I_Z \otimes M$ for M an invertible sheaf and I_Z a sheaf of ideals defining a closed subscheme Z of dimension 0 outside of which L' is a subbundle of E . We have a diagram of exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & E/L & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & E & \longrightarrow & I_Z \otimes M & \longrightarrow & 0 \end{array}$$

We will say that the second line is a devissage of E . We can deduce the Chern classes of E ,

$$c_1(E) = c_1(L_1) + c_1(M) \quad c_2(E) = c_1(L_1) \cdot c_1(M) + \deg Z$$

Theorem 5.1.2 (Bogomolov-Mumford). A vector bundle E of rank 2 over a surface X is unstable if and only if there exists a devissage,

$$0 \rightarrow L \rightarrow E \rightarrow I_Z \otimes M \rightarrow 0$$

such that if $L' = L \otimes M^{-1} = L^2 \otimes (\det E)^{-1}$ then either,

- (a) L' is in the cone $C_+ \subset \text{NS}(X)_{\mathbb{Q}}$ generated by positive divisors (IS THIS NEF?)
- (b) or $L' = \mathcal{O}_X$ and Z is nonempty

Moreover the devissage is unique.

We will prove this using only Mumford's theory of instability.

Let $P = \mathbb{P}(E)$ and $p : P \rightarrow X$ the projection and $\mathcal{O}_P(1)$ the canonical relatively ample bundle on P . A nonzero section $s \in H^0(X, S^{2n}E \otimes (\det E)^{-n})$ corresponds to a nonzero section $t \in H^0(P, \mathcal{O}_P(2n) \otimes p^*(\det E)^{-n})$. Let $\xi \in X$ be the generic point and $K = \kappa(\xi)$. If we chose a basis of E_K then $s(\xi)$ corresponds to a homogeneous polynomial F of degree $2n$ in two variables. Since s is zero at some point of X , we know $s(\xi)$ is unstable for the action of PGL_2 on $S^{2n}E_K \otimes (\det E_K)^{-n}$ (WHY?). We deduce from the stability criterion using 1-parameter subgroups [11] that F has a root of order greater than n in the algebraic closure of K , so also in K (WHAT? WHY?), that's to say there exists an integer $r \geq 1$ and two polynomials G, H homogeneous of degrees 1 and $n - r$ respectively such that $F = G^{n+r}H$. Let D be the divisor of t and Δ the closure of the divisor defined over a generic point by G . We can write $D = (n+r)\Delta + \Delta'$ where Δ has degree 1 and Δ' has degree $n - r$ on P . Therefore, there exist invertible modules L, L' on X such that,

$$\mathcal{O}_P(\Delta) = \mathcal{O}_P(1) \otimes p^*L \quad \mathcal{O}_P(\Delta') = \mathcal{O}_P(n-r) \otimes p^*L'$$

and hence,

$$(\det E)^{-n} = L^{n+r} \otimes L'$$

The divisor Δ corresponds to a section of $E \otimes L$ and thus an injection $L^{-1} \hookrightarrow E$ which by construction is saturated in E . We verify that it provides the desired devissage.

5.2 Operations on unstable bundles

Instability is preserved by passage to the dual and tensor product with a line bundle.

- (a) Let $f : Y \rightarrow X$ be a surjective morphism of surfaces, E a vector bundle of rank 2 over X . Then E is unstable if and only if f^*E is.
- (b) Let $f : Y \rightarrow X$ be a finite faithfully flat morphism of surfaces, F a fiber bundle of rank 2 on Y . Then if F is unstable so is f_*F .

5.3 Proof of Theorem 0.3

Suppose that Ω_X^1 is unstable. Then there exists a devissage:

$$0 \rightarrow L \rightarrow \Omega_X^1 \rightarrow I_Z \otimes M \rightarrow 0$$

and an integer $n > 0$ such that there is an injection $\mathcal{O}_X \hookrightarrow (L \otimes M^{-1})^{\otimes n}$. Note that $L \otimes M^{-1} = L^2 \otimes (\det \Omega_X^1)^{-1} = L^2 \otimes (\Omega_X^2)^{\otimes -1}$. Also, for $m \gg 0$,

$$h^0(L^{2m}) = h^0((L \otimes M^{-1})^{\otimes m} \otimes (\Omega_X^2)^{\otimes m}) \geq h^0((\Omega_X^2)^{\otimes m}) \in O(m^2)$$

Therefore, the theorem is a consequence of the following.

Theorem 5.3.1 (Bogomolov). Let X be a smooth proper surface and $L \hookrightarrow \Omega_X^1$ an invertible subsheaf. Then $h^0(L^n) \in O(n)$.

First recall the pretty result of Castelnuovo and of Franchis which we will need for the proof.

Lemma 5.3.2 (4, 12). Let ω_1, ω_2 be two holomorphic 1-forms on X which are linearly independent over k such that $\omega_1 \wedge \omega_2 = 0$. Then there exists a curve C which is proper and smooth over k of genus $g \geq 2$ and two holomorphic 1-forms θ_1, θ_2 on C and a morphism $u : X \rightarrow C$ such that $\omega_i = u^*\theta_i$ for $i = 1, 2$.

There exists a meromorphic function $f : X \dashrightarrow \mathbb{P}^1$ such that $\omega_2 = f\omega_1$. This defines a morphism $f : X' \rightarrow \mathbb{P}^1$ where X' is a blowup of X . Let $u : X' \rightarrow C \rightarrow \mathbb{P}^1$ be the Stein factorization. We have an exact sequence of modules of differentials,

$$0 \rightarrow u^*\Omega_C^1 \rightarrow \Omega_{X'}^1 \rightarrow \Omega_{X'/C}^1 \rightarrow 0$$

We know $\omega_2 = f\omega_1$ and $0 = d\omega_2 = df \wedge \omega_1$ (since ω_i are global holomorphic forms they are closed by Hodge theory).

WHY DOES IT WORK ON AN OPEN

But df is pulled back from an open of U so ω_1 is also as it is parallel to df hence also $\omega_2 = f\omega_1$. So above an open $U \subset C$ the forms ω_1, ω_2 are in the image of,

$$H^0(u^{-1}(U), u^*\Omega_C^1) = H^0(U, \Omega_C^1) \rightarrow H^0(u^{-1}(U), \Omega_{X'}^1)$$

so we choose θ_1, θ_2 holomorphic forms on U which pull back to ω_1, ω_2 . However, $u_*\mathcal{O}_{X'} = \mathcal{O}_C$ so θ_1, θ_2 extend to global sections of Ω_C because ω_1, ω_2 are global sections of $\Omega_{X'}$. Indeed, **(WHY DOES IT EXTEND??) THIS SEEMS WRONG**

Since ω_1, ω_2 are k -independent so are θ_1, θ_2 . Hence $g(C) \geq 2$ and therefore the map $u : X' \rightarrow C$ contracts all rational curves and hence factors through $X' \rightarrow X$ giving the required map.

5.3.1 Interlude: regularizing meromorphic 1-forms via covers

Definition 5.3.3.

Proposition 5.3.4. Let $f : X \rightarrow Y$ be a proper dominant morphism of locally noetherian integral S -schemes that are smooth over S at the generic points of all divisors. If f is tame and $\omega \in (\Omega_{Y/S})_\eta$ is a meromorphic differential such that $f^*\omega \in H^0(X, \Omega_{X/S}^{\vee\vee})$ is a global reflexive differential then $\omega \in H^0(Y, \Omega_{Y/S}^{\vee\vee})$ is a global reflexive differential.

Proof. Since Y is regular in codimension 1 it suffices to show that for each $\xi \in Y$ of height 1 that $\omega_\xi \in (\Omega_Y)_\xi$. Since f is proper and dominant it is surjective so we may choose $\xi' \in X$ mapping to ξ . ξ' in the regular locus hence $\mathcal{O}_{X, \xi'}$ is a noetherian local domain by [Hartshorne, Ex.4.11] there exists a DVR $R \subset \text{Frac}(\mathcal{O}_{X, \xi'})$ dominating $\mathcal{O}_{X, \xi'}$. \square

Remark. For example, this holds for any tame dominant map of normal proper varieties over a perfect field.

5.3.2 Completion of the Theorem

Either, for all $n > 0$ we have $h^0(X, L^{\otimes n}) \leq 1$ or there exists $n > 0$ such that $h^0(X, L^{\otimes n}) \geq 2$. In the second case, there is a standard method of extracting an n^{th} -root **(WHAT THE HELL DOES THIS MEAN)** to get $h^0(X, L) \geq 2$. In this case, there are two forms $\omega_1, \omega_2 \in H^0(X, \Omega_X^1)$ such that $\omega_1 \wedge \omega_2 = 0$ since they arise from the same subsheaf of rank 1. Therefore, by the lemma, there exists a curve C and a morphism $u : X \rightarrow C$ and an invertible sheaf L_0 on C such that,

$$L \subset u^*(L_0)$$

AGAIN WHY? so we can conclude that,

$$h^0(X, L^n) \leq h^0(C, L_0^n) \in O(n)$$

Corollary 5.3.5. If c_1 and c_2 are the Chern classes of Ω_X^1 then $c_1^2 \leq 4c_2$.

5.3.3 Curves of bounded genus on a minimal surface of general type

We provide a few examples showing that X being general type plays an essential role, and that in the contrary case, there can be unbounded families of curves of fixed geometric genus.

Example 5.3.6. Let $X = \mathbb{P}^2$ then $\text{NS}(X) = \mathbb{Z}$. There exist in the projective plane curves of bounded geometric genus but arbitrarily large degree.

Example 5.3.7. Let E be an elliptic curve without complex multiplication and let $X = E \times E$. Then $\text{NS}(X) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \mathbb{Z}\Delta$ where f_i are the fiber classes and Δ is the diagonal. For every pair of integers (m, n) the image in X of the morphism $f_{m,n} : E \rightarrow X$ given by $f_{m,n}(\alpha) = (m\alpha, n\alpha)$ is a curve of class $m^2f_1 + n^2f_2 + (m - n)^2\Delta$ and genus 1.

Example 5.3.8. Let B be a smooth proper curve and $\pi : X \rightarrow B$ a nonisotrivial (that is to say it does not become trivial after some finite base change $B' \rightarrow B$) minimal elliptic fibration admitting a section $\sigma : B \rightarrow X$ of infinite order. Let ω be the conormal bundle of σ . Then there exist global sections $g_2 \in H^0(X, \omega^4)$ and $g_3 \in H^0(X, \omega^6)$ such that X is the minimal resolution of the surface $Y \subset \mathbb{P}_B(\omega^2 \oplus \omega^3 \oplus \mathcal{O}_B)$ defined by the Weierstrass equation,

$$y^2z = x^3 - g_2xz^2 - g_3z^3$$

Moreover, ω is independent of the section σ as has degree $-\sigma(B)^2$. If the degree is zero, then g_2 and g_3 are constant and the fibration π is isotrivial. There exist infinitely many sections of negative self-intersection and the classes are algebraically distinct.

Notation: we write K_X for a canonical divisor of X and T_X the tangent bundle and $\pi : \mathbb{P}(\Omega_X) \rightarrow X$ the canonical projection and $L = \mathcal{O}_P(1)$ the relatively ample bundle for π .

Let F be an invertible bundle on X . We note that F is a divisor of some linear system (DOES HE MEAN F IS THE ZERO LOCUS OF SOME SECTION). Moreover, for any rational number $\ell \in \mathbb{Q}$, we allow ourselves to form the sheaf ℓF , extending that we consider the tensor powers $(\ell F)^{\otimes m}$ for which m is such that $m\ell$ is an integer.

5.4 COnstruction of a good linear system of divisors on P

Proposition 5.4.1. Let F be an invertible sheaf on X and ℓ a rational positive number such that,

- (a) $K \cdot F \geq 0$
- (b) $(K + 2\ell F)^2 > 0$
- (c) $c_1^2(\Omega_X \otimes \ell F) - c_2(\Omega_X \otimes \ell F) > 0$

Then for $m \gg 0$ the linear system $(L \otimes \pi^*(\ell F))^m$ defines a rational map $u_F : \mathbb{P}(\Omega_X^1) \dashrightarrow \mathbb{P}^N$ birational onto its image.

IT SEEMS WRONG THAT ℓF IS INSIDE THE S^m THIS GIVES $(\ell F)^{2m}$ NOT $(\ell F)^m$ AS SHOULD BE FROM PROJECTION FORMULA

Proof. By the theorem of Iitaka [20], it suffices to show that for $m \gg 0$,

$$h^0(P, (L \otimes \pi^*(\ell F))^m) = h^0(X, S^m(\Omega_X \otimes \ell F)) \geq O(m^3)$$

The Riemann-Roch formula for E shows that,

$$\chi(S^m E) = \frac{m(m+1)(m+2)}{24}(c_1^2(E) - 4c_2(E)) + \frac{m+1}{2} \left[\frac{m^2}{4}c_1^2(E) - \frac{m}{2}K_X \cdot c_1(E) \right] + (m+1)\chi(\mathcal{O}_X)$$

and hence for $m \gg 0$,

$$h^0(S^m(\Omega_X^1 \otimes \ell F)) + h^2(S^m(\Omega_X^1 \otimes \ell F)) \sim h^1(S^m(\Omega_X^1 \otimes \ell F)) + \frac{m^3}{6} [c_1^2(\Omega_X \otimes \ell F) - c_2(\Omega_X^1 \otimes \ell F)] \geq O(m^3)$$

By Serre duality, **HOW DO I FIX THE DUAL AND S^m IN POSITIVE CHAR**

$$h^2(S^m(\Omega_X \otimes \ell F)) = h^0(K \otimes S^m(T_X \otimes -\ell F))$$

Chosing some divisors D and D' ample and smooth such that,

$$\mathcal{O}_X(-D') \subset K \subset \mathcal{O}_X(D)$$

we find that,

$$\left| h^0(K \otimes S^m(T_X \otimes -\ell F)) - h^0(S^m(T_X \otimes -\ell F)) \right| \in O(m^2)$$

Therefore, we conclude by appealing to the following lemma. □

Lemma 5.4.2. For any $m > 0$ we have $H^0(S^m(T_X \otimes -\ell F)) = 0$.

THE m VS $2m$ DOESNT MAKE SENSE

Proof. We showed that $T_X \otimes -\ell F$ is not unstable. Hence, the only sections of $H^0(S^{2m}(T_X \otimes -\ell F) \otimes (\det(T_X \otimes -\ell F))^{-m})$ are nowhere vanishing. If we show for $m \gg 0$ that $H^0(\det(T_X \otimes -\ell F)^{-m})$ has a nonzero section with a zero at some point $x \in X$ then its product with a section $H^0(S^{2m}(T_X \otimes -\ell F))$ will give a contradiction. Thus, the result will follow from the definition,

$$\det(T_X \otimes -\ell F)^{-m} = m(K + 2\ell F)$$

and Riemann-Roch,

$$\chi(m(K + 2\ell F)) \sim \frac{m^2}{2}(K + 2\ell F)^2 \in O(m^2)$$

and therefore,

$$h^0(m(K + 2\ell F)) + h^2(m(K + 2\ell F)) \geq O(m^2)$$

by Serre duality,

$$h^0(m(K + 2\ell F)) = h^0(K - m(K + 2\ell F))$$

Since $K \cdot (K - m(K + 2\ell F)) = K^2 - mK \cdot (K + 2\ell F) < 0$ and K is nef (we assumed that X is minimal) $h^0(K - m(K + 2\ell F)) = 0$ for $m \gg 0$ giving the result. □

Our any bundle F verifying the conditions of the properosition, we fix, once and for all, m and ℓ and let Z_F be the closed subset of $\mathbb{P}(\Omega_X)$ outside of which u_F is defined.

Definition 5.4.3. Let C be a curve embedded in C and $f : \widetilde{C} \rightarrow C$ its normalization. If $t_f(\widetilde{C})$ is not contained in (resp. is contained in) Z_F , we say that C is F -regular (resp. F -irregular).

5.5 Proof of Theorem 0.4