# ASTR GR6001 Radiative Processes
# Assignment # 4

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October 7, 2019

## 1 Problem 1

Consider an atmosphere of uniform temperature and uniform composition which is emitting, absorbig, and scattering. We write the equation of radiative diffusion,

$$\frac{1}{3}\frac{\mathrm{d}^2 J_\nu}{\mathrm{d}\tau_\nu^2} = \epsilon_\nu (J_\nu - B_\nu)$$

where,

$$\epsilon_\nu = \left( \frac{\alpha_\nu}{\sigma_\nu + \alpha_\nu} \right)$$

### (a)

Consider the difference $D_\nu = B_\nu - J_\nu$. Since we assume that the atmosphere has uniform temperature,

$$\frac{\mathrm{d}^2 D_\nu}{\mathrm{d}\tau_\nu^2} = -\frac{\mathrm{d}^2 J_\nu}{\mathrm{d}\tau_\nu^2} = -3\epsilon_\nu (J_\nu - B_\nu) = 3\epsilon_\nu D_\nu$$

This has solutions,

$$D_\nu = C_1 e^{\sqrt{3\epsilon_\nu}\tau_\nu} + C_2 e^{-\sqrt{3\epsilon_\nu}\tau_\nu}$$

Since the atmosphere is semi-infinite we must have $C_1 = 0$ since otherwise in the limit $\tau_\nu \to \infty$ we would have $J_\nu \to \infty$ which is unphysical. Therefore, we simply need to consider the boundary conditions at $\tau_\nu = 0$. In the two stream approximation, we have an incoming ray $I_0$, reflected ray $I_R$ and a pair of internal rays $I_\nu^\pm$ at angles $\mu = \pm\frac{1}{\sqrt{3}}$. Then, internally, we have,

$$J_\nu = \tfrac{1}{2}(I_\nu^+ + I_\nu^-)$$
$$F_\nu = \frac{2\pi}{\sqrt{3}}(I_\nu^+ - I_\nu^-)$$
$$P_\nu = \frac{2\pi}{3c}(I_\nu^+ + I_\nu^-) = \frac{4\pi}{3c} J_\nu$$

We can solve these equations to give,

$$I_\nu^\pm = J_\nu \pm \frac{\sqrt{3}}{4\pi} F_\nu$$

Furthermore, from the first moment equation,

$$c\frac{\mathrm{d}P_\nu}{\mathrm{d}\tau_\nu} = F_\nu$$

however, in the Eddington approximation,

$$P_\nu = \frac{4\pi}{3c}J_\nu$$

which implies that,

$$F_\nu = \frac{4\pi}{3}\frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}$$

Thus the two-stream rays take the form,

$$I_\nu^\pm = J_\nu \pm \frac{1}{\sqrt{3}}\frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}$$

Now at the surface $\tau_\nu = 0$ we impose continuity of the specific intensity so the outgoing intensities match, $I_R = I_\nu^+(0)$, and the incoming intensities match, $I_\nu^-(0) = 0$, which must be zero since we assume there is no radiation incident on the atmosphere. Thus,

$$I_\nu^-(0) = J_\nu(0) - \frac{1}{\sqrt{3}}\frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\bigg|_{\tau_\nu=0} = 0$$

which implies that,

$$J_\nu(0) = \frac{1}{\sqrt{3}}\frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\bigg|_{\tau_\nu=0}$$

Now we use our solution,

$$J_\nu(\tau_\nu) = C_\nu e^{-\sqrt{3\epsilon_\nu}\tau_\nu} + B_\nu$$

so we must match,

$$J_\nu(0) = C_\nu + B_\nu = \frac{1}{\sqrt{3}}\frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\bigg|_{\tau_\nu=0} = -\sqrt{\epsilon_\nu}C_\nu$$

which implies that,

$$C_\nu = -\frac{B_\nu}{1+\sqrt{\epsilon_\nu}}$$

In particular,

$$D_\nu = B_\nu - J_\nu = -C_\nu e^{-\sqrt{3\epsilon_\nu}\tau_\nu} = \frac{B_\nu}{1+\sqrt{\epsilon_\nu}}e^{-\sqrt{3\epsilon_\nu}}$$

Since $D_\nu > 0$ for all $\tau_\nu$ we have demonstrated that $J_\nu < B_\nu$ at all optical depths.

## (b)

The outgoing intensity at the surface is given by,

$$I_\nu^+(0) = J_\nu(0) + \frac{1}{\sqrt{3}} \frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\bigg|_{\tau_\nu = 0}$$

Our previous solution gives,

$$J_\nu(\tau_\nu) = C_\nu e^{-\sqrt{3\epsilon_\nu}\tau_\nu} + B_\nu = B_\nu \left[1 - \frac{e^{-\sqrt{3\epsilon_\nu}\tau_\nu}}{1 + \sqrt{\epsilon_\nu}}\right]$$

Thus, using the boundary conditions,

$$J_\nu(0) = \frac{1}{\sqrt{3}} \frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\bigg|_{\tau_\nu = 0} = B_\nu \left[\frac{\sqrt{\epsilon_\nu}}{1 + \sqrt{\epsilon_\nu}}\right]$$

$$I_\nu^+(0) = B_\nu \left[\frac{2\sqrt{\epsilon_\nu}}{1 + \sqrt{\epsilon_\nu}}\right]$$

Note that in the limit of weak scattering $\sigma_\nu \to 0$ thus $\epsilon_\nu \to 0$ and then ,

$$I_\nu^+(0) \to B_\nu$$

so the atmosphere becomes a blackbody in the limit of no scattering. Furthermore, for strong scattering, $\sigma_\nu \gg \alpha_\nu$ then $\epsilon_\nu \ll 1$ which implies that,

$$I_\nu^+(0) \approx 2\sqrt{\epsilon_\nu} B_\nu$$

is highly suppressed from an ideal blackbody.

## (c)

We have derived a formula for the total flux in the Eddington approximation,

$$F_\nu = \frac{4\pi}{3} \frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}$$

However, at the boundary $\tau_\nu = 0$, in our two-stream approximation, we have shown that the incoming ray vanishes so the entirety of the flux at the boundary is from the outgoing ray and thus is emergent so,

$$F_\nu^+ = F_\nu(0) = \frac{4\pi}{3} \frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\bigg|_{\tau_\nu = 0} = \frac{4\pi}{\sqrt{3}} \cdot \left[\frac{\sqrt{\epsilon_\nu}}{1 + \sqrt{\epsilon_\nu}}\right] B_\nu$$

In the strong scattering limit we have,

$$F_\nu^+ = \frac{4\pi}{\sqrt{3}} \sqrt{\epsilon_\nu} B_\nu$$

which is much less than the blackbody flux $F_\nu^B = \pi B_\nu$. However, in the weak scattering limit we have $\epsilon_\nu \to 1$ and so we find,

$$F_\nu^+ \to \frac{2\pi}{\sqrt{3}} B_\nu$$

which slightly exceeds the blackbody flux $F_\nu^B = \pi B_\nu$. This cannot be correct since we know that a blackbody emitts the greatest possible intensity in each part of the spectrum of any body in thermal equilibirum at a given temperature. Since this does not seem right I may have made a mistake or it may be related to the two-stream approximation since the outgoing radiation is fixed with angle $\mu = \frac{1}{\sqrt{3}}$ unlike that of a blackbody with isotropic emergent radiation giving the $F_\nu^B = \pi B_\nu$ relation. In fact, the factor $\frac{2\pi}{\sqrt{3}}$ is exactly the integration factor computing the flux from a cone of rays at fixed angle $\mu = \frac{1}{\sqrt{3}}$ while $\pi$ is the integration factor for isotropic emerging radiation given that both have the same (constant) specific intensity (over angles where it is nonzero).

# 2 Problem 2

Consider a plane EM wave propagating in the $z$-direction of the form,

$$\vec{E}(z,t) = (\varepsilon_1 e^{i\phi_1} \hat{x} + \varepsilon_2 e^{i\phi_2} \hat{y}) e^{i(kz - \omega t)}$$

Now, evaluating near an observer at $z = 0$ the electric field is,

$$\vec{E}(0,t) = (\varepsilon_1 e^{i\phi_1} \hat{x} + \varepsilon_2 e^{i\phi_2} \hat{y}) e^{-i\omega t}$$

Now we define the following Stokes parameters,

$$\begin{aligned} I &= \varepsilon_1^2 + \varepsilon_2^2 \\ Q &= \varepsilon_1^2 - \varepsilon_2^2 \\ U &= 2\varepsilon_1\varepsilon_2 \cos(\phi_1 - \phi_2) \\ V &= 2\varepsilon_1\varepsilon_2 \sin(\phi_1 - \phi_2) \end{aligned}$$

We need to relate these quantities to the intensities which pass through various polarizers. First note that we measure the energy flux via the Poynting vector,

$$S = c(E \times B) = c\hat{z}E^2$$

so we simply need to consider the time-averaged square of the field to get,

$$\langle |S| \rangle = c \left\langle E^2 \right\rangle$$

However, we need to be somewhat careful to use the *real* fields in this calculation since taking real parts does not commute with complex multiplication. Therefore, consider,

$$\begin{aligned} \left\langle E^2 \right\rangle &= \tfrac{1}{4} \left\langle (E_{\mathbb{C}} + \bar{E}_{\mathbb{C}}) \cdot (E_{\mathbb{C}} + \bar{E}_{\mathbb{C}}) \right\rangle \\ &= \tfrac{1}{4} \left\langle E_{\mathbb{C}}^2 \right\rangle + \tfrac{1}{2} \left\langle E_{\mathbb{C}} \cdot \bar{E}_{\mathbb{C}} \right\rangle + \tfrac{1}{4} \left\langle \bar{E}_{\mathbb{C}}^2 \right\rangle \end{aligned}$$

where $E_{\mathbb{C}}$ is the complex field given above. However, the first and last terms will have pure phase time dependence and therefore time-average to zero. Only the middle term remains,

$$\begin{aligned}
\left\langle E^2 \right\rangle &= \tfrac{1}{2} \left\langle E_{\mathbb{C}} \cdot \bar{E}_{\mathbb{C}} \right\rangle \\
&= \tfrac{1}{2}(\varepsilon_1 e^{i\phi_1} \hat{x} + \varepsilon_2 e^{i\phi_2} \hat{y}) \cdot (\varepsilon_1 e^{-i\phi_1} \hat{x} + \varepsilon_2 e^{-i\phi_2} \hat{y}) \\
&= \tfrac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2)
\end{aligned}$$

and thus,

$$I = 2 \left\langle E^2 \right\rangle$$

is proportional to the total intensity.

Now we consider various polarizing filters $L_\theta$ which only permit radiation linearly polarized at $\theta$ pass through and $C_\pm$ which only permit circularly polarized light of a certain handedness to pass. To accomplish such a filtering we simply need to expand the electric field in the corresponding polarization basis. For linear polarization, this corresponds to finding the component of $\vec{E}$ at an angle $\theta$ in the $x$-$y$ plane. Thus,

$$I_\theta \vec{E} = \vec{E} \cdot \hat{n}_\theta = (\varepsilon_1 e^{i\phi_1} \cos\theta + \varepsilon_2 e^{i\phi_2} \sin\theta) e^{-i\omega t}$$

Thus, the corresponding intensity is proportional to,

$$\begin{aligned}
P_\theta = \left\langle (L_\theta \vec{E})^2 \right\rangle &= \tfrac{1}{2} \left\langle (L_\theta E_{\mathbb{C}})(L_\theta \bar{E}_{\mathbb{C}}) \right\rangle \\
&= \tfrac{1}{2}(\varepsilon_1 e^{i\phi_1} \cos\theta + \varepsilon_2 e^{i\phi_2} \sin\theta)(\varepsilon_1 e^{-i\phi_1} \cos\theta + \varepsilon_2 e^{-i\phi_2} \sin\theta) \\
&= \tfrac{1}{2} \left[ \varepsilon_1^2 \cos^2\theta + \varepsilon_2^2 \sin^2\theta + 2\varepsilon_1 \varepsilon_2 \sin\theta \cos\theta \cos(\phi_1 - \phi_2) \right]
\end{aligned}$$

Consider now,

$$2(I_0 - I_{\frac{\pi}{2}}) = \varepsilon_1^2 - \varepsilon_2^2 = Q$$

so $Q$ is the difference in intensity between vertical and horizontal polarization. Furthermore, consider,

$$2(I_{\frac{\pi}{4}} - I_{\frac{3\pi}{4}}) = 2\varepsilon_1 \varepsilon_2 \cos(\phi_1 - \phi_2) = U$$

so $U$ is the difference in intensity between 45° and 135° polarization. Finally, we need to consider circular polarizers. We expand in the basis,

$$\hat{e}_\pm = \tfrac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$$

which are pure circular polarization states. We can write,

$$\hat{x} = \tfrac{1}{\sqrt{2}}(\hat{e}_+ + \hat{e}_-) \qquad \hat{y} = \tfrac{1}{i\sqrt{2}}(\hat{e}_+ - \hat{e}_-)$$

and therefore,

$$\vec{E} = \left[ \tfrac{1}{\sqrt{2}}(\varepsilon_1 e^{i\phi_1} - i\varepsilon_2 e^{i\phi_2})\hat{e}_+ + \tfrac{1}{\sqrt{2}}(\varepsilon_1 e^{i\phi_1} + i\varepsilon_2 e^{i\phi_2})\hat{e}_- \right] e^{-i\omega t}$$

Therefore,

$$\vec{E}_\pm = \tfrac{1}{\sqrt{2}}(\varepsilon_1 e^{i\phi_1} \mp i\epsilon_2 e^{i\phi_2})\hat{e}_\pm e^{-i\omega t}$$

so the power in each circularly polarized component is,

$$
\begin{aligned}
2I_\pm &= 2\left\langle E_\pm^2\right\rangle = \left\langle E_\pm \cdot \bar{E}_\pm\right\rangle \\
&= \tfrac{1}{2}(\varepsilon_1 e^{i\phi_1} \mp i\varepsilon_2 e^{i\phi_2})(\varepsilon_1 e^{-i\phi_1} \pm i\varepsilon_2 e^{-i\phi_2})\hat{e}_\pm \cdot \bar{\hat{e}}_\pm
\end{aligned}
$$

Now,

$$\hat{e}_\pm \cdot \bar{\hat{e}}_\pm = \tfrac{1}{2}(\hat{x} \pm i\hat{y}) \cdot (\hat{x} \mp i\hat{y}) = \tfrac{1}{2}(1+1) = 1$$

Therefore,

$$
\begin{aligned}
2I_\pm &= \tfrac{1}{2}\left(\varepsilon_1^2 + \varepsilon_2^2 \mp i\varepsilon_1\varepsilon_2 e^{i(\phi_2-\phi_1)} \pm i\varepsilon_1\varepsilon_2 e^{i(\phi_1-\phi_2)}\right) \\
&= \tfrac{1}{2}\left(\varepsilon_1^2 + \varepsilon_2^2 \mp 2\varepsilon_1\varepsilon_2 \sin(\phi_1 - \phi_2)\right)
\end{aligned}
$$

Therefore,

$$2(I_- - I_+) = 2\varepsilon_1\varepsilon_2 \sin(\phi_1 - \phi_2) = V$$

so $V$ measures the difference in intensity between the two circular polarizations.

# 3 Problem 3

The differential Thomson cross section for unpolarized light is,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = r_0^2 \cdot \frac{1+\cos^2\phi}{2}$$

where $r_0$ is the effective electron radius,

$$r_0 = \frac{e^2}{mc^2}$$

We can compute the total cross section integrating in spherical coordinates alinged along the incident directon,

$$
\begin{aligned}
\sigma_T &= r_0^2 \int \frac{1+\cos^2\phi}{2}\mathrm{d}\Omega \\
&= r_0^2 \int_0^\pi \int_0^{2\pi} \frac{1+\cos^2\phi}{2} \sin\phi \, \mathrm{d}\gamma \, \mathrm{d}\phi \\
&= \pi r_0^2 \int_0^\pi (1+\cos^2\phi)\sin\phi \, \mathrm{d}\phi \\
&= \pi r_0^2 \int_{-1}^1 (1+\mu^2)\, \mathrm{d}\mu \; = \frac{8\pi r_0^2}{3}
\end{aligned}
$$

For totally linearly polarized light with polarization vector $\hat{E}$, the differential cross section for Thomson scattering is,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = r_0^2(1 - (\hat{r}\cdot\hat{E})^2)$$

where $\hat{r}$ is the unit vector in the direction of scattering. We can compute the total scattering cross section by integrating in spherical coordinates aligned along the direction of polarization,

$$\begin{aligned}
\sigma_T &= r_0^2 \int (1 - (\hat{r} \cdot \hat{E})^2) \, \mathrm{d}\Omega \\
&= r_0^2 \int_0^\pi \int_0^{2\pi} (1 - \cos^2\theta) \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= 2\pi r_0^2 \int_{-1}^{1} (1 - \mu^2) \, \mathrm{d}\mu \\
&= \frac{8\pi r_0^2}{3}
\end{aligned}$$

which agrees with the earlier computation of the total cross section.