Mathematics GR6261 Commutative Algebra
Assignment # 4

Benjamin Church

March 9, 2019

Problem 1

(a)

Let A be a Noetherian ring. Consider a descending chain of closed subsets V_i of $\mathrm{Spec}\,(A)$,

$$V_1 \supset V_2 \supset V_3 \supset \cdots$$

Any closed set can be written as $V_i = V(I_i)$ for some ideal $I_i \subset A$ which implies that, by Lemma 1.2, we have an ascending chain of radical ideals,

$$\sqrt{I_1} \subset \sqrt{I_2} \subset \sqrt{I_3} \subset \cdots$$

Since A is Noetherian, this chain must stabilize meaning that the sequence $V_i = V(I_i) = V(\sqrt{I_i})$ will also stabilize. Thus, $\mathrm{Spec}\,(A)$ is a Noetherian space.

(b)

Let A be a ring and $F \subset \mathrm{Spec}\,(A)$ a closed set. Then we know $F = V(I)$ for some ideal $I \subset A$. Suppose that $F = V(\mathfrak{p})$ for some $\mathfrak{p} \in F$. First, such a generic point is unique because if $F = V(\mathfrak{p}')$ for some other $\mathfrak{p}' \in F$ then $\mathfrak{p} \in V(\mathfrak{p}')$ and $\mathfrak{p}' \in V(\mathfrak{p})$ so $\mathfrak{p}' \supset \mathfrak{p}$ and $\mathfrak{p} \supset \mathfrak{p}'$ and thus $\mathfrak{p}' = \mathfrak{p}$.

Suppose we can decompose $F = Z_1 \cup Z_2$ into closed proper subsets. Then $Z_1 = V(I_1)$ and $Z_2 = V(I_2)$ and $Z_1 \cup Z_2 = V(I_1 I_2)$. Since $Z_1, Z_2 \subsetneq F$ we must have $\mathfrak{p} \subsetneq \sqrt{I_1}$ and $\mathfrak{p} \subsetneq \sqrt{I_2}$. Furthermore, $\sqrt{I_1 I_2} = \sqrt{I_1} \cap \sqrt{I_2} \supset \mathfrak{p}$. But if $\mathfrak{p} \supset \sqrt{I_1} \cap \sqrt{I_2}$ then $\mathfrak{p}$ must lie above one of them which we know it does not. Thus, $\mathfrak{p} \subsetneq \sqrt{I_1} \cap \sqrt{I_2}$ so $Z_1 \cup Z_2$ must also be a proper subset contradicting the reducibility.

Now suppose that F is irreducible. Since F is closed, we may write $F = V(J) = V(\sqrt{J})$ and take $I = \sqrt{J}$. Suppose that $xy \in I$ then the ideals $I_1 = I + x$ and $I_2 = I + y$ lie above I so $Z_1 = V(I_1)$ and $Z_2 = V(I_2)$ are subsets of $F = V(I)$. Furthermore,

$$Z_1 \cup Z_2 = V(I_1 I_2) = V(I^2 + Ix + Iy + xy) \supset V(I)$$

since $I^2 + Ix + Iy + xy \subset I$. But since $Z_1, Z_2 \subset F$ we have $Z_1 \cup Z_2 = F$ so by irreducibility we must have $Z_1 = F$ or $Z_2 = F$. WLOG take $Z_1 = F$ so $V(I + x) = V(I)$. However, if $x \notin I$ then no power of x lies in J so $S_x = \{1, x, x^2, \dots, \}$ is disjoint from J. By Lemma 1.4, there exists a prime ideal not containing powers of x above J (and thus above $\sqrt{J} = I$) which contradicts $V(I) = V(I + x)$. Thus, $x \in I$ so I is a prime ideal. Therefore, $F = V(\mathfrak{p})$.

(c)

Let X be a Noetherian topological space. Let Σ be the set of all closed subsets of X which are cannot be written as the finite union of closed irreducible sets without any inclusion relations. We assume that $X \in \Sigma$ so that it is nonempty. Any chain of Σ (comprised of closed sets) stabilizes below by the Noetherian property and thus has a minimal element. By Zorn's Lemma, Σ has a minimal element W. Since W is not irreducible we can decompose $W = W_1 \cup W_2$ into proper closed sets W_1 and W_2. By minimality, $W_1, W_2 \notin \Sigma$ but they are closed and thus can be written as the union of finitely many closed irreducible sets with no inclusions,

$$W_1 = Z_1 \cup \cdots \cup Z_n \quad \text{and} \quad W_2 = Y_1 \cup \cdots \cup Y_k$$

Then we have,

$$W = W_1 \cup W_2 = Z_1 \cup \cdots \cup Z_n \cup Y_1 \cup \cdots \cup Y_k$$

which is a finite union of closed irreducible sets. There are no inclusions between different Z_i and different Y_i. Furthermore, if $Z_i \subset Y_i$ then I can remove Z_i from the union. Thus, I can assume that there are no inclusions. This means that $W \notin \Sigma$ which contradicts its definition so the assumption that Σ was nonempty is false. Thus, $X \notin \Sigma$ so X may be decomposed into a finite union of closed irreducible sets without any inclusion relations between them.

Problem 2

(a)

Suppose that Z and Z' are constructible. We can write,

$$Z = \bigcup_{i=1}^{n} O_i \cap F_i \quad \text{and} \quad \bigcup_{i=1}^{n'} O_i' \cap F_i'$$

with O_i and O_i' open and F_i and F_i' closed. Then clearly,

$$Z \cup Z' = \bigcup_{i=1}^{n} O_i \cap F_i \cup \bigcup_{i=1}^{n'} O_i' \cap F_i'$$

is also constructible. Note, by adding sets of the form $O_i = F_i = \varnothing$ which does not change the union we can assume that $n = n'$ in the decompositions of Z and Z'.

Then,

$$Z \cap Z' = \bigcup_{i=1}^{n} [O_i \cap O_i'] \cap [F_i \cap F_i']$$

which is constructible since $O_i \cap O_i'$ is open (finite intersection of open sets) and $F_i \cap F_i'$ is closed (arbitrary intersection of closed sets). Next,

$$Z \setminus Z' = Z \cap (X \setminus Z')$$

so, using the intersection property, it suffices to show that $X \setminus Z'$ is constructible to show that set differences are. However, we can write,

$$Z^C = \bigcap_{i=1}^{n} O_i^C \cup F_i^C$$

Since $O_i^C = X \cap O_i^C$ is closed and $F_i^C = F_i^C \cap X$ is open both are constructible. Since Z^C is produced from finite unions and intersections of constructible sets it is constructible.

(b)

Let X be Noetherian. Suppose that $Z \subset X$ is constructible then we can write,

$$Z = \bigcup_{i=1}^{n} O_i \cap F_i$$

for open O_i an closed F_i. Suppose that X_0 is a closed irreducible subset of X and $X_0 \cap Z$ is dense in X_0. Then,

$$Z \cap X_0 = \bigcup_{i=1}^{n} (O_i \cap X_0) \cap (F_i \cap X_0)$$

Therefore,

$$X_0 = \overline{(Z \cap X_0)} \subset \bigcup_{i=1}^{n} (F_i \cap X_0) = X_0 \cap \bigcup_{i=1}^{n} F_i$$

making the inclusion an equality. Thus,

$$X_0 = \bigcup_{i=1}^{n} (F_i \cap X_0)$$

However, X_0 is irreducible and $F_i \cap X_0$ is closed in X_0 so we must have $X_0 = X_0 \cap F_i$ for some F_i. Therefore, $O_i \cap X_0 = (O_i \cap X_0) \cap (F_i \cap X_0) \subset Z \cap X_0$ so $Z \cap X_0$ contains a nonempty open set.

Now conversely, we assume the classification: $Z \cap X_0$ is either not dense in X_0 or

$Z \cap X_0$ contains an open of X_0 for each closed irreducible X_0. Consider the poset Σ of all closed subsets of X_0 of X such that $Z \cap X_0$ is not constructible ordered by inclusion. We assume this poset is nonempty. Since X is Noetherian, each descending chain of Σ (being comprised of closed sets) stabilizes and thus has a minimal element. Therefore, by Zorn's Lemma, Σ has a minimal element $\tilde{X}$. Thus, if $X_0 \subset \tilde{X}$ is any proper closed subset then $Z \cap X_0$ is constructible by minimality. I claim that $\tilde{X}$ is irreducible. Suppose that we decompose $\tilde{X} = X_1 \cup X_2$ into closed proper subsets then $Z \cap \tilde{X} = (Z \cap X_1) \cup (Z \cap X_2)$. However, by minimality, $Z \cap X_1$ and $Z \cap X_2$ are constructible and thus $Z \cap \tilde{X} = (Z \cap X_1) \cup (Z \cap X_2)$ is also constructible, a contradiction of $\tilde{X} \in \Sigma$.

Now suppose that $Z \cap \tilde{X}$ is not dense in $\tilde{X}$. Then, $\overline{(Z \cap \tilde{X})}$ is a proper closed subset of $\tilde{X}$ so $Z \cap \tilde{X} = (Z \cap \tilde{X}) \cap \overline{(Z \cap \tilde{X})}$ is constructible by the minimality of $\tilde{X}$. But this again contradicts $\tilde{X} \in \Sigma$. Therefore, $Z \cap \tilde{X}$ must be dense in $\tilde{X}$. Using the classification, there exists a nonempty open (in $\tilde{X}$) set $U \subset Z \cap \tilde{X}$. Then $X_0 = \tilde{X} \setminus U$ is a proper closed subset of X and thus $Z \cap X_0$ is constructible. However, then $Z \cap \tilde{X} = U \cup (Z \cap X_0)$ since $U \subset Z \cap \tilde{X}$ which implies that $Z \cap \tilde{X}$ is constructible since U is open and $Z \cap X_0$ is constructible. Therefore, $\tilde{X} \notin \Sigma$, a contradiction of its definition as least element. Thus, Σ must be nonempty so $Z \cap Y$ is constructible for any closed Y. In particular, $Z \cap X = Z$ is constructible.

Problem 3

Let A be a Noetherian ring and $Z \subset \mathrm{Spec}\,(A)$ a constructible set. Let $U(f)$ denote the elementary open set $V((f))^C$. By Lemma 1.7, we may write any open set of $\mathrm{Spec}\,(A)$ as a finite union of such open sets. Therefore, when we decompose,

$$Z = \bigcup_{i=1}^{n} O_i \cap F_i$$

where O_i is open and F_i is closed we may assume that each O_i is an elementary open by adding each factor to the union and distributing the intersection with the closed set. Therefore we can write,

$$Z = \bigcup_{i=1}^{n} U(f_i) \cap V(I_i)$$

for some $f_i \in A$ and ideal $I_i \subset A$. Let $S_f = \{1, f, f^2, f^3, \cdots\}$ be the multiplicative set of powers of f. I claim that $B = S_f^{-1}(A/I)$ is an A-algebra whose image under $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is exactly $U(f) \cap V(I)$. First, S_f does not contain zero when reduced to A/I because $f \notin I$ otherwise $U(f) \cap V(I)$ would be empty which we can assume is false. The prime ideals of A/I are exactly those primes of A above I i.e. $V(I)$. Furthermore, we proved previously that the image of the map $\mathrm{Spec}\left(S_f^{-1}(A/I)\right) \to \mathrm{Spec}\,((A/I))$ is exactly those prime ideals of A/I disjoint from S_f. However, $S_f \cap \mathfrak{p} = \varnothing$ if and only if $f \notin \mathfrak{p}$ since $\mathfrak{p}$ is prime. Therefore, the image is exactly $U(f) = V(f)^C$ (with primes ranging over those of A/I which are given by $V(I)$). Thus, the image under the composition of maps,

$$\text{Spec}\left(S_f^{-1}(A/I)\right) \to \text{Spec}\,(A/I) \to \text{Spec}\,(A)$$

gives exactly $U(f) \cap V(I)$.

Let $B_i = S_{f_i}^{-1}(A/I_i)$ and $B = B_1 \times \cdots \times B_n$. Now, by Lemma 1.8, the image of the map $\text{Spec}\,(B) \to \text{Spec}\,(A)$ is equal to,

$$\bigcup_{i=1}^{n} \text{Im}(\text{Spec}\,(B_i) \to \text{Spec}\,(A)) = \bigcup_{i=1}^{n} U(f_i) \cap V(I_i) = Z$$

Furthermore, B is a finitely generated A-algebra since there exists a surjective map,

$$A[x_1, \ldots, x_n] \twoheadrightarrow B = S_{f_1}^{-1}(A/I_1) \times \cdots \times S_{f_n}^{-1}(A/I_n)$$

given by sending $x_i \mapsto \bar{f}_i^{\,-1}$ and reducing elements of a modulo I_i in each factor.

Problem 4

Let A be a Noetherian domain and B a finitely generated A-algebra such that $A \to B$ is injective. First, since B is finitely generated as an A-algebra there is a surjection,

$$A[x_1, \ldots, x_n] \twoheadrightarrow B$$

and from now on let $x_1, \ldots, x_n$ denote the images inside B. We may assume that $x_1, \ldots, x_r$ are algebraically independent and therefore form a transcendence basis for the fraction field of B over the fraction field of A. Let $\tilde{A} = A[x_1, \ldots, x_r]$. If $B = \tilde{A}$ then we are done because $\text{Spec}\,(A[x_1, \ldots, x_r]) \to \text{Spec}\,(A)$ is surjective and thus its image contains every elementary open set.

Now, for each x_i with $r < i \leq n$ we must have a polynomial relation over $\tilde{A}$ in B,

$$\sum_{j=0}^{d_j} a_{ij} x_i^{d_j - j} = 0$$

with each $a_{ij} \in \tilde{A}$. Define the element of $\tilde{A}$

$$a = \prod_{i=r+1}^{n} a_{i0}$$

which is a polynomial in $x_1, \ldots, x_r$. Let c denote any one of this polynomial's nonzero coefficients. I claim that the image of the map $\text{Spec}\,(B) \to \text{Spec}\,(A)$ contains $U(c)$. Let $\mathfrak{p} \in U(c)$ that is a prime ideal of A with $c \notin \mathfrak{p}$. Let $\mathfrak{q} = \mathfrak{p}[x_1, \ldots, x_r] \subset \tilde{A}$

be a prime ideal of $\tilde{A}$ above $\mathfrak{p}$. Since c is not in $\mathfrak{p}$ and is a nonzero coefficient of the polynomial a we cannot have $a \in \mathfrak{q}$. Consider the extension of rings $B_{\mathfrak{q}}$ over $\tilde{A}_{\mathfrak{q}}$. Since $a \notin \mathfrak{q}$ we know that a is inevitable in $\tilde{A}_{\mathfrak{q}}$ which implies that each relation of the $x_{r+1}, \cdots, x_n$ over $\tilde{A}_{\mathfrak{q}}$ can be reduced to a monic one since their leading coefficients are units. Therefore, the generators of $B_{\mathfrak{q}}$ are integral over $\tilde{A}_{\mathfrak{q}}$ which implies that $B_{\mathfrak{q}}$ is integral over $\tilde{A}_{\mathfrak{q}}$ since sums and products of integral elements remain integral. By Cohen's theorem, the map $\text{Spec}\,(B_{\mathfrak{q}}) \to \text{Spec}(\tilde{A}_{\mathfrak{q}})$ is surjective so there exists a prime $\mathfrak{P}B_{\mathfrak{q}}$ of $B_{\mathfrak{q}}$ which lies above $\mathfrak{q}\tilde{A}_{\mathfrak{q}}$. Therefore, $\mathfrak{P} \cap \tilde{A} = \mathfrak{q}$ and hence $\mathfrak{P} \cap A = \mathfrak{P} \cap \tilde{A} \cap A = \mathfrak{q} \cap A = \mathfrak{p}$. Therefore, there exists a prime of B above $\mathfrak{p}$ so $U(c)$ is contained in the image of $\text{Spec}\,(B) \to \text{Spec}\,(A)$.

Problem 5

Let A be a Noetherian ring and B a finitely generated A-algebra. Since there exists a surjection,

$$A[x_1, \ldots, x_n] \longrightarrow B$$

we know that B is Noetherian by the Hilbert basis theorem. Thus, both $\text{Spec}\,(A)$ and $\text{Spec}\,(B)$ are Noetherian topological spaces.

(a)

Suppose that $Z \subset \text{Spec}\,(B)$ is a constructible set. First, assume that $Z = \text{Spec}\,(B)$. We want to show that its image Z' in $\text{Spec}\,(A)$ is constructible. Let $X_0 \subset \text{Spec}\,(A)$ be a closed irreducible set. We have shown that such a set must have a generic point $X_0 = V(\mathfrak{p})$. Suppose that $X_0 \cap Z'$ is dense in X_0, we need to show that $X_0 \cap Z'$ contains an open set of X_0 to prove that Z' is constructible.

Consider the quotients $A/\mathfrak{p}$ and $B/\mathfrak{p}B$. Since $A/\mathfrak{p}$ is a Noetherian domain since A is Noetherian and $\mathfrak{p}$ is prime and $B/\mathfrak{p}B$ is a finitely generated $A/\mathfrak{p}$-algebra since B is a finitely generated A-algebra then problem 4 applies. Therefore, there exists $c \in A/\mathfrak{p}$ such that $\text{Spec}\,(B/\mathfrak{p}B) \to \text{Spec}\,(A/\mathfrak{p})$ contains the open set $U(c)$ in its image.

$$\begin{array}{ccc}
\text{Spec}\,(B/\mathfrak{p}B) & \longrightarrow & \text{Spec}\,(A/\mathfrak{p}) \\
\downarrow & & \downarrow \\
\text{Spec}\,(B) & \longrightarrow & \text{Spec}\,(A)
\end{array}$$

However, the image of $\text{Spec}\,(A/\mathfrak{p})$ in $\text{Spec}\,(A)$ is $V(\mathfrak{p}) = X_0$ and thus $U(c) \subset X_0 \cap Z'$ since $U(c)$ is contained in the image of $\text{Spec}\,(B)$ since it is contained in the image of $\text{Spec}\,(B/\mathfrak{p}B)$. Therefore, Z' is constructible.

For the general case, let $Z \subset \text{Spec}\,(B)$ be constructible. Since B is a finitely generated A-algebra over a Noetherian ring, it is Noetherian itself and thus we can apply problem 3. There exists a finitely-generated B-algebra B' over B such that the image of the induced map $\text{Spec}\,(B') \to \text{Spec}\,(B)$ is exactly Z. Then consider,

$$\begin{tikzcd} \mathrm{Spec}\,(B') \arrow[r] \arrow[dr] & \mathrm{Spec}\,(B) \arrow[d] \\ & \mathrm{Spec}\,(A) \end{tikzcd}$$

The induced map $\mathrm{Spec}\,(B') \to \mathrm{Spec}\,(A)$ is the composition of $\mathrm{Spec}\,(B') \to \mathrm{Spec}\,(B)$ and $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$. Therefore, the image of $\mathrm{Spec}\,(B')$ in $\mathrm{Spec}\,(A)$ is exactly Z' the image of Z (which is the image of $\mathrm{Spec}\,(B') \to \mathrm{Spec}\,(B)$) under $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$. Therefore, Z' is constructible since it is the image of $\mathrm{Spec}\,(B')$ and B' is a finitely-generated A-algebra.

(b)

Suppose that the going-down property holds for $A \to B$. Let $U \subset \mathrm{Spec}\,(B)$ be open and thus constructible. By the previous proposition, its image U' in $\mathrm{Spec}\,(A)$ is also constructible. Let $Z = \mathrm{Spec}\,(A) \setminus U'$ which is also constructible and let W be an irreducible component of $\overline{Z}$. Thus, W is a closed irreducible subset of $\mathrm{Spec}\,(A)$ and Z is constructible and $Z \cap W$ is dense in W (since W is a component of $\overline{Z}$) so there exists O a nonempty open subset of W contained in $Z \cap W$. Since W is closed and irreducible in $\mathrm{Spec}\,(A)$ it must have a generic point $W = V(\mathfrak{p})$. Since O is a nonempty open subset of W we must have $\mathfrak{p} \in O$ otherwise $\mathfrak{p} \in O^C$ but $\mathfrak{p}$ is a generic point so its closure is W contradicting the fact that O^C is a proper subset of W. Thus $\mathfrak{p} \in Z$. Furthermore, for any prime $\mathfrak{p} \subset \mathfrak{p}'$ if $\mathfrak{p}'$ is in the image of $\mathrm{Spec}\,(B)$ then by the going-down property we must also have $\mathfrak{p}$ in the image. Therefore, if $\mathfrak{p} \subset \mathfrak{p}'$ since $\mathfrak{p} \in Z$ we also have $\mathfrak{p}' \in Z$ so $W = V(\mathfrak{p}) \subset Z$. Since each irreducible component of $\overline{Z}$ lies in Z we must have $\overline{Z} = Z$ and thus Z is closed so $U = Z^C$ is open proving that $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is an open map.

1 Lemmata

Lemma 1.1. For any ideal $I \subset A$,

$$V(I) = V(\sqrt{I})$$

Proof. Suppose that $\mathfrak{p} \in V(I)$ and then $\mathfrak{p} \supset I$ so clearly,

$$\mathfrak{p} \supset \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I}$$

and thus $\mathfrak{p} \in V(\sqrt{I})$. Conversely, since,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} \supset I$$

we know that $V(\sqrt{I}) \subset V(I)$. Therefore, $V(I) = V(\sqrt{I})$. $\square$

Lemma 1.2. $V(I) \subset V(J)$ if and only if $\sqrt{I} \supset \sqrt{J}$.

Proof. If $V(I) \subset V(J)$ then for each prime $\mathfrak{p} \supset I$ we have $\mathfrak{p} \supset J$. Therefore,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} \supset \bigcap_{\mathfrak{p} \supset J} \mathfrak{p} = \sqrt{J}$$

since the left is an intersection over a subset. Furthermore, if $\sqrt{I} \supset \sqrt{J}$ then clearly $V(\sqrt{I}) \subset V(\sqrt{J})$ since $\mathfrak{p} \supset \sqrt{I} \supset \sqrt{J}$ implies $\mathfrak{p} \supset \sqrt{J}$. However, $V(I) = V(\sqrt{I})$ for any ideal. Together, this implies that $V(I) \subset V(J)$. □

Corollary 1.3. If $V(I) = V(J)$ then $\sqrt{I} = \sqrt{J}$.

Lemma 1.4. Let A be a ring and $S \subset A$ a multiplicative set. There exists a prime ideal $\mathfrak{p}$ disjoint to S above any ideal disjoint to I.

Proof. Let Σ be the poset under inclusion of ideals of A disjoint to S. Σ is nonempty since it contains (0). Suppose that $\mathcal{I} \subset \Sigma$ is a chain and consider,

$$U = \bigcup_{I \in \mathcal{I}} I$$

This is an ideal because if $x, y \in U$ then $x \in I$ and $y \in I'$ for $I, I' \in \mathcal{I}$ but $\mathcal{I}$ is totally ordered so either $I \subset I'$ or $I' \subset I$ and thus the larger contains both x and y and therefore the sum and the multiples are in U. Furthermore,

$$S \cap U = \bigcup_{I \in \mathcal{I}} S \cap I = \varnothing$$

so $U \in \Sigma$. However, $\forall I \in \mathcal{I} : I \subset U$. Since every chain has a maximum, by Zorn's Lemma there exist maximal elements of Σ above every ideal $I \in \Sigma$. Suppose $\mathfrak{m} \in \Sigma$ is maximal and $x, y \notin \mathfrak{m}$. Then, by maximality, $(x) + \mathfrak{m} \notin \Sigma$ and $(y) + \mathfrak{m} \notin \Sigma$ so they must contain elements of S say $s_x \in (x) + \mathfrak{m}$ and $s_y \in (y) + \mathfrak{m}$ then $s_x s_y \in [(x) + \mathfrak{m}][(y) + \mathfrak{m}] \subset (xy) + \mathfrak{m}$ but $s_x s_y \notin \mathfrak{m}$ since $s_x s_y \in S$ but $\mathfrak{m} \in \Sigma$. Therefore, $xy \notin \mathfrak{m}$ otherwise $s_x s_y \in (xy) + \mathfrak{m} = \mathfrak{m}$. Thus, $\mathfrak{m}$ is a prime disjoint to S above a particular $I \in \Sigma$. □

Lemma 1.5. Every irreducible component is closed.

Proof. Suppose that $D \subset X$ is irreducible. Then suppose we can decompose $\overline{D} = Z_1 \cup Z_2$ as a union of closed proper sets. Then, $Z_1 \cap D$ and $Z_2 \cap D$ are closed in D and cover D so by irreducibility WLOG $Z_1 \cap D = D$. Thus, Z_1 is a closed set containing D so $Z_1 \supset \overline{D}$ proving that $\overline{D}$ is irreducible. Therefore, if D is an irreducible component then it must be a maximal irreducible set so $D = \overline{D}$ otherwise $\overline{D}$ would be an irreducible set properly containing it. □

Lemma 1.6. Any topological space can be written as the union of its irreducible components.

Proof. Let Σ be the set of irreducible subsets of X which is a poset under inclusion. Let $\mathcal{C} \subset \Sigma$ be a chain and consider,

$$U = \bigcup_{D \in \mathcal{C}} C$$

I claim that U is irreducible. If we could decompose $U = Z_1 \cup Z_2$ into proper closed sets then, for each $D \in \mathcal{C}$, the sets $Z_1 \cap D$ and $Z_2 \cap D$ would be closed in D and cover D. Since Z_1 and Z_2 are not proper there must exist D_1 and D_2 with $D_1 \not\subset Z_1$ and $D_2 \not\subset Z_2$ but WLOG $D_1 \subset D_2$ since they are in a chain. Thus, $Z_1 \cap D_2$ and $Z_2 \cap D_2$ are proper closed sets covering D_2 contradicting its irreducibility. Thus, U is irreducible so $U \in \Sigma$. Since each chain has a maximum element, by Zorn's Lemma, there exists a maximal element of Σ above any $D \in \Sigma$. Specifically, if $x \in X$ is any point, then $\overline{\{x\}}$ is irreducible because if $\overline{\{x\}} = Z_1 \cup Z_2$ then WLOG $x \in Z_1$ so Z_1 is a closed set containing x which implies that $\overline{\{x\}} \subset Z_1$. Thus, each point of x is contained in a maximal irreducible set proving the lemma. □

Lemma 1.7. Let A be a Noetherian ring then any open set of $\mathrm{Spec}\,(A)$ can be written as a finite union of elementary open sets $U(f) = V((f))^C$.

Proof. Any open set $U = V(I)^C$ for some ideal $I \subset A$. Since A is Noetherian, the ideal I is finitely-generated so we may write,

$$I = Af_1 + \cdots Af_n$$

and thus,

$$V(I) = V((f_1)) \cap \cdots \cap V((f_n))$$

Therefore,

$$U = V((f_1))^c \cup \cdots \cup V((f_n))^C = U(f_1) \cup \cdots \cup U(f_n)$$

□

Lemma 1.8. Let B_1 and B_2 be A-algebras. Denote the natural maps $\iota_1 : A \to B_1$ and $\iota_2 : A \to B_2$. Then the diagonal gives a map,

$$\mathrm{Spec}\,(B_1 \times B_2) \to \mathrm{Spec}\,(A)$$

whose image in $\mathrm{Spec}\,(A)$ is equal to,

$$\iota_1^*(\mathrm{Spec}\,(B_1)) \cup \iota_2^*(\mathrm{Spec}\,(B_2))$$

Proof. Every prime ideal of $B_1 \times B_2$ is of the form $\mathfrak{P}_1 \times B_2$ or $B_1 \times \mathfrak{P}_2$ where $\mathfrak{P}_1$ is a prime ideal of B_1 and $\mathfrak{P}_2$ is a prime ideal of B_2. Therefore, under the map $\iota_1 \times \iota_2 : A \to B_1 \times B_2$, the preimage of the prime $\mathfrak{P}_1 \times B_1$ is the prime $\iota_1^{-1}(\mathfrak{P}_1)$ and the preimage of the prime $B_1 \times \mathfrak{P}_2$ is $\iota_2^{-1}(\mathfrak{P}_2)$. Therefore, the image of this map over all primes of $B_1 \times B_2$ gives exactly the required union. □