1 Flag Varieties

Definition Let V be a vectorspace then the Grassmaniann $G_d(V)$ is the space of d-dimensional subspaces of V. For any subspace $S \subset V$ of complementary dimension $n-d$ we define,

$$G_d(V)_S = \{W \in G_d(V) \mid W \cap S = \{0\}\}$$

Thus, writing $V = W_0 \oplus S$ for some fixed W_0 then projection $W \to W_0$ defines an isomorphism,

$$G_d(V)_S \xrightarrow{\sim} \mathrm{Hom}\,(W_0, S)$$

These form an affine open cover of $G_d(V)$ with its variety structure with,

$$G_d(V)_S \cong \mathbb{A}(\mathrm{Hom}\,(W_0, S))$$

Furthermore this shows that $G_d(V)_S$ is smooth with tangent space,

$$T_{W_0} G_d(V) = \mathrm{Hom}\,(W_0, S) = \mathrm{Hom}\,(W_0, V/W_0)$$

Definition We extend the above discussion to chain s of subspaces. Let $\mathbf{d} = (\mathbf{d_1}, \dots, \mathbf{d_r})$ be a sequence of integers with $n > d_1 > d_2 > \cdots > d_r > 0$ and let $G_{\mathbf{d}}(V)$ be the space of flags,

$$F : \ V \supset V^1 \supset \cdots \supset V^r \supset 0$$

with $\dim V^i = d_i$. The map,

$$G_{\mathbf{d}}(V) \xrightarrow{F \mapsto (V^i)} \prod_i G_{d_i}(V) \subset \prod_i \mathbb{P}\left(\bigwedge^{d_i} V \right)$$

gives an embedding of $G_{\mathbf{d}}(V)$ inside $\prod_i G_{d_i}(V)$ showing that $G_{\mathbf{d}}(V)$ is a projective variety.

2 The Hodge Filtration

Given any Hodge structure of weight n there is a filtration,

$$F^\bullet : \ F^{-n} \supset F^1 \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \qquad\qquad F^p = \bigoplus_{r \ge p} V^{r,s} \subset V_{\mathcal{C}}$$

Note that when $p + q = n$ we have,

$$\overline{F^q} = \bigoplus_{s \ge q} \overline{V^{s,r}} = \bigoplus_{S \ge q} V^{r,s} = \bigoplus_{r \le p} V^{r,s}$$

Therefore,

$$V^{p,q} = F^p \cap \overline{F^q}$$

Recall that Hodge structures correspond to representations of the Deligne torus, $h : \S \to \mathbb{G}_m^{\mathbb{R}}$ which is a map of $\mathbb{R}$-algebraic groups. Recall that h acts on $V^{p,q}$ on real points $z \in \S(\mathbb{R}) = \mathcal{C}^\times$ via $h(z) \cdot v = z^{-p} \bar{z}^{-q} \cdot v$ for $v \in V^{p,q}$.

3 Variations of Hodge Structures