

1 Cohomology Review

Definition Let X be a smooth complete variety over \mathbb{C} (a smooth proper scheme over \mathbb{C}). There is a corresponding analytic manifold X^{an} whose exact topology depends on the structure map $X \rightarrow \text{Spec}(\mathbb{C})$. This gives us access to topological cohomology denoted $H_B^n(X) = H^n(X^{\text{an}}, \mathbb{Q})$.

Definition For each embedding $\sigma : k_0 \rightarrow \mathbb{C}$ there is a corresponding $X^\sigma = X \times_\sigma \text{Spec}(\mathbb{C})$ and we write $H_\sigma^p(X) = H_B^p(X^\sigma) = H^p((X^\sigma)^{\text{an}}, \mathbb{Q})$.

Remark. In the case that X is projective, a projective embedding $X \rightarrow \mathbb{P}^n$ defines an embedding $X^{\text{an}} \rightarrow \mathbb{CP}^n$ which pulls back the canonical Kahler form on \mathbb{CP}^n to give X a Kahler structure. By Hodge theory, this gives a decomposition,

$$H_B^n(X, \mathbb{C}) = H_{\text{dR}}^n(X^{\text{an}}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X)$ can be identified with a complex form of type (p, q) and also with the sheaf cohomology,

$$H^{p,q}(X) = H^p(X, \Omega^q)$$

Definition The algebraic deRham cohomology is given by the hyper cohomology of the deRham complex,

$$H_{\text{dR}}^n(X/k) = \mathbb{H}^n(X, \Omega^\bullet)$$

Theorem 1.1. There is a Hodge-to-deRham spectral sequence,

$$E_1^{p,q} = H^p(X, \Omega^q) \implies \mathbb{H}^{p+q}(X, \Omega^\bullet) = H_{\text{dR}}^{p+q}(X)$$

which gives a filtration on the algebraic deRham cohomology. Furthermore, the continuous map $X \rightarrow X^{\text{an}}$ induces an isomorphism,

$$H_{\text{dR}}^n(X) \xrightarrow{\sim} H_{\text{dR}}^n(X^{\text{an}})$$

which sends the filtration of the Hodge-to-deRham spectral sequence to the filtration of $H_{\text{dR}}^n(X^{\text{an}})$ given by Hodge theory.

Remark. In general, let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor and $\mathbf{Ch}\mathcal{A}$ its category of complexes. Then there is a spectral sequence computing the hyperderived functor,

$$E_1^{p,q} = R^q F(C^p) \implies \mathbb{R}^{p+q} F(C^\bullet) = \mathbb{H}^{p+q}(C^\bullet)$$

Proposition 1.2. Consider a resolution (exact sequence) in an abelian category \mathcal{A}

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \dots$$

and an additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$. Then, the derived functors of F on A agree with the hyperderived functors of F on C^\bullet ,

$$R^p F(A) = \mathbb{R}^p F(C^\bullet)$$

In particular, in the category of sheaves on X , given any resolution $\mathcal{F} \rightarrow \mathcal{G}^\bullet$ we have,

$$H^p(X, \mathcal{F}) = \mathbb{H}^p(X, \mathcal{G}^\bullet)$$

Proof. We choose a resolution of C^\bullet which is an complex of injectives I^\bullet and a quasi-isomorphism $\alpha : C^\bullet \rightarrow I^\bullet$. Consider the diagram,

$$\begin{array}{ccccccc} & & A & & & & \\ & & \downarrow \varepsilon & & & & \\ 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 \longrightarrow \dots \\ & & \downarrow \alpha_0 & & \downarrow \alpha_1 & & \downarrow \alpha_2 \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 \longrightarrow \dots \end{array}$$

Since $A \xrightarrow{\varepsilon} C^\bullet$ is a resolution, the top row is exact except in degree zero where $\ker(C^0 \rightarrow C^1) = A$. Since $\alpha : C^\bullet \rightarrow I^\bullet$ is a quasi-isomorphism the complex I^\bullet must also be exact in positive degree and at degree zero $\alpha_* : H^0(C^\bullet) \xrightarrow{\sim} H^0(I^\bullet)$ is an isomorphism so $\alpha_0 \circ \varepsilon : A \rightarrow \ker(C^0 \rightarrow C^1) \rightarrow \ker(I^0 \rightarrow I^1)$ is an isomorphism. Thus the complex $0 \rightarrow A \xrightarrow{\alpha_0 \circ \varepsilon} I^0$ is exact so it is an injective resolution of A . Therefore,

$$R^p F(A) = H^p(F(I^\bullet)) = \mathbb{R}^p F(C^\bullet)$$

□

Remark. When the resolution $A \rightarrow C^\bullet$ is acyclic then, applying the spectral sequence,

$$E_1^{p,q} = R^q F(C^p) \implies \mathbb{R}^{p+q} F(C^\bullet)$$

we see that $E_1^{p,0} = F(C^p)$ and all others are zero. Thus, $E_2^{p,0} = H^p(F(C))$ so the spectral sequence converges giving,

$$\mathbb{R}^p F(C^\bullet) = H^p(F(C^\bullet))$$

Together with the previous proposition we conclude,

$$R^p F(A) = H^p(F(C^\bullet))$$

that we can compute derived functors on any acyclic resolution.

Remark. Applying these remarks to the case of a complex manifold X , we consider the resolution of the constant sheaf $\underline{\mathbb{C}}_X$ by the holomorphic differential forms Ω_X^k ,

$$0 \longrightarrow \underline{\mathbb{C}}_X \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^2 \longrightarrow \dots$$

This complex is exact by the Poincare lemma. Thus we have an isomorphism,

$$H_{\text{sing}}^p(X; \mathbb{C}) = H^p(X, \underline{\mathbb{C}}_X) \xrightarrow{\sim} \mathbb{H}^p(X, \Omega_X^\bullet) = H_{\text{dR}}^p(X)$$

Definition When $k = \bar{k}$ we write the Etale cohomology as,

$$H^n(X, \mathbb{A}_{\mathbb{Q}, \text{fin.}}) = \varprojlim H_{\text{et}}^n(X_{\text{et}}, \mathbb{Z}/m\mathbb{Z})$$

Theorem 1.3. For $k = \mathbb{C}$ there is a canonical isomorphism,

$$H_B^n(X) \otimes \mathbb{A}_{\mathbb{Q}, \text{fin.}} \rightarrow H_{\text{et}}^n(X)$$

Therefore $H_B^n(X) \otimes \mathbb{A}_{\mathbb{Q}, \text{fin.}}$ is independent of the choice of structure map $X \rightarrow \text{Spec}(\mathbb{C})$.

Remark. Recall that we have defined an algebraic cycle via the cohomology class of a smooth subvariety $Z \subset X$ of codimension p ,

$$\mathrm{cl}(Z) \in \mathrm{Hdg}^p(X) = H_B^{2p}(X) \cap H^p(X, \Omega^p)$$

We give an alternative definition in terms of Chern classes.

Definition First, we define a Chern class $c_1 : \mathrm{Pic}(X) \rightarrow H_{\mathrm{dR}}^2(X)$ via the following. Consider the map $\mathrm{dlog} : \mathcal{O}_X^\times \rightarrow \Omega_X^1$ which takes $f \mapsto f^{-1}\mathrm{d}f$. Then there is a map of complexes,

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_X^\times & \longrightarrow & 0 \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow \mathrm{dlog} & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{\mathrm{d}} & \Omega_X^1 & \xrightarrow{\mathrm{d}} & \Omega_X^2 \longrightarrow \cdots \end{array}$$

Which gives a map on hypercohomology,

$$H^{n-1}(X, \mathcal{O}_X^\times) = \mathbb{H}^n(X, 0 \rightarrow \mathcal{O}_X^\times \rightarrow 0 \rightarrow \cdots) \rightarrow \mathbb{H}^n(X, \Omega_X^\bullet) = H_{\mathrm{dR}}^n(X)$$

Recall that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ and therefore we have a map,

$$c_1 : \mathrm{Pic}(X) \rightarrow H_{\mathrm{dR}}^2(X)$$

Then, note that we may extend this to $c_p : \mathrm{Pic}(X) \rightarrow H_{\mathrm{dR}}^{2p}(X)$ via splitting.

Definition For any smooth codimension p subvariety $Z \subset X$ we can define,

$$\mathrm{cl}(Z) = \frac{1}{(p-1)!} c_p(\iota_* \mathcal{O}_Z)$$

To make this definition make any sense, we need to note that the Chern class is defined on the Grothendieck group of X which, when X is smooth is equivalent to the Grothendieck group of the category of coherent \mathcal{O}_X -modules. This correspondence defines $c_p(\iota_* \mathcal{O}_Z)$ when $\iota_* \mathcal{O}_Z$ is not a vector bundle only a coherent sheaf.

1.1 Basic Properties of Absolutly Hodge Cycles

Remark. We first need to discuss algebraic connnections on bundles. The setup is k_0 is a field of characteristic zero and S is a smooth k_0 -scheme.

Definition A k_0 -connection on a coherent \mathcal{O}_S -module \mathcal{E} is a morphism of sheaves of k_0 -modules,

$$\nabla : \mathcal{E} \rightarrow \Omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{E}$$

(not as \mathcal{O}_S -modules) which further satisfies the Leibniz rule, for $f \in \mathcal{O}_S(U)$ and $s \in \mathcal{E}(U)$,

$$\nabla(fs) = \mathrm{d}f \otimes e + f\nabla(e)$$

where $\mathrm{d} : \mathcal{O}_S \rightarrow \Omega_S^1$ is the canonical map. We define the subsheaf of horizontal sections, $\mathcal{E}^\nabla = \ker \nabla$

Remark. Any connection may be extended to \mathcal{E} -valued k -forms,

$$\nabla_k : \Omega_S^k \otimes_{\mathcal{O}_S} \mathcal{E} \rightarrow \Omega_S^{k+1} \otimes_{\mathcal{O}_S} \mathcal{E}$$

via,

$$\nabla_k(\omega \otimes e) = d\omega \otimes e + (-1)^k \omega \wedge \nabla e$$

Definition The connection ∇ defines a corresponding curvature form,

$$\omega_\nabla = \nabla_1 \circ \nabla : \mathcal{E} \rightarrow \Omega_S^2 \otimes_{\mathcal{O}_S} \mathcal{E}$$

We say that ∇ is flat or integrable if the curvature vanishes $\omega_\nabla = \nabla_1 \circ \nabla = 0$.

Lemma 1.4. The curvature $\omega_\nabla : \mathcal{E} \rightarrow \Omega_S^2 \otimes_{\mathcal{O}_S} \mathcal{E}$ is a \mathcal{O}_S -module map.

Proof. Consider,

$$\begin{aligned} \omega_\nabla(fs) &= \nabla_1(df \otimes s + f\nabla s) = dd f \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla_1 \circ \nabla \\ &= f\nabla_1 \circ \nabla s = f\omega_\nabla(s) \end{aligned}$$

□

Remark. If we write locally,

$$\nabla e = \sum_i f_i dg_i \otimes s_i$$

then the curvature takes the form,

$$\omega_\nabla(e) = \sum_i (df_i \wedge dg_i \otimes e - f_i dg_i \otimes \nabla s_i)$$

Proposition 1.5. ∇ is flat iff the \mathcal{O}_S -map $Q : \text{Der}(\mathcal{O}_S, \mathcal{O}_S) \rightarrow \text{End}(\mathcal{E})$ given by sending D to,

$$\mathcal{E} \xrightarrow{\nabla} \Omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes \text{id}} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{a \otimes b \mapsto ab} \mathcal{E}$$

is a morphism of Lie algebras.

Remark. Note that $Q(D)$ is in fact a \mathcal{O}_S -morphism using the universal property,

$$\text{Der}(\mathcal{O}_S, \mathcal{O}_S) \cong \text{Hom}_{\mathcal{O}_S}(\Omega_S^1, \mathcal{O}_S)$$

Proof. We need to check that $Q[D_1, D_2] = [Q(D_1), Q(D_2)]$ is equivalent to $\nabla_1 \circ \nabla = 0$. Now,

$$[D_1, D_2] \in \text{Hom}_{\mathcal{O}_S}(\Omega_S^1, \mathcal{O}_S)$$

is the unique \mathcal{O}_S -map such that,

$$[D_1, D_2] \circ d = D_1 \circ d \circ D_2 \circ d - D_2 \circ d \circ D_1 \circ d$$

Now consider this action locally,

$$[D_1, D_2] \otimes \text{id} \circ \nabla = \sum_i f_i (D_1 \circ d \circ D_2 \circ d - D_2 \circ d \circ D_1 \circ d)(g_i) \cdot s_i$$

Furthermore,

$$[Q(D_1), Q(D_2)] = (D_1 \otimes \text{id}) \circ \nabla \circ (D_2 \otimes \text{id}) \circ \nabla - (D_2 \otimes \text{id}) \circ \nabla \circ (D_1 \otimes \text{id}) \circ \nabla$$

Again consider its local action,

$$\begin{aligned} Q(D_1) \circ Q(D_2)(e) &= (D_1 \otimes \text{id}) \circ \nabla \left(\sum_i f_i D_2(\text{d}g_i) \cdot s_i \right) \\ &= \sum_i \left([D_2(\text{d}g_i) D_1(\text{d}f_i) + f_i D_1(\text{d}(D_2(\text{d}g_i)))] \cdot s_i + f_i D_2(\text{d}g_i) D_1(\nabla s_i) \right) \end{aligned}$$

Now consider,

$$\begin{aligned} &\left[Q(D_1) \circ Q(D_2) - Q(D_2) \circ Q(D_1) \right] - Q([D_1, D_2])(e) \\ &= \sum_i \left(D_1(\text{d}f_i) D_2(\text{d}g_i) - D_2(\text{d}f_i) D_1(\text{d}g_i) \right) \cdot s_i \\ &\quad + \sum_i f_i \left(D_1(\text{d}(D_2(\text{d}g_i))) - D_2(\text{d}(D_1(\text{d}g_i))) \right) \cdot s_i \\ &\quad + \sum_i \left(f_i D_2(\text{d}g_i) D_1(\nabla s_i) - g_i D_1(\text{d}g_i) D_2(\nabla s_i) \right) \\ &\quad - \sum_i f_i (D_1 \circ \text{d} \circ D_2 \circ \text{d} - D_2 \circ \text{d} \circ D_1 \circ \text{d})(g_i) \cdot s_i \\ &= \sum_i \left(D_1(\text{d}f_i) D_2(\text{d}g_i) - D_2(\text{d}f_i) D_1(\text{d}g_i) \right) \cdot s_i \\ &\quad + \sum_i \left(f_i D_2(\text{d}g_i) D_1(\nabla s_i) - g_i D_1(\text{d}g_i) D_2(\nabla s_i) \right) \\ &= (D_1 \otimes D_2 - D_2 \otimes D_1) \otimes \text{id}_{\mathcal{E}} \circ \omega_{\nabla} \end{aligned}$$

which is defined on $(\Omega_S^1)^{\otimes 2} \otimes_{\mathcal{O}_S} \mathcal{E}$ but descends to $\Omega_S^2 \otimes_{\mathcal{O}_S} \mathcal{E}$ since it sends the ideal $\omega \otimes \omega \mapsto 0$. Therefore, we see that Q is a Lie algebra map iff

$$\forall D_1, D_2 \in \text{Hom}_{\mathcal{O}_S}(\Omega_S^1, \mathcal{O}_S) : (D_1 \otimes D_2 - D_2 \otimes D_1) \otimes \text{id}_{\mathcal{E}} \circ \omega_{\nabla} = 0$$

in particular when $\omega_{\nabla} = 0$. Furthermore when Q is a Lie algebra map then we must have $\omega_{\nabla} = 0$ since, for any fixed form, there exists sections of Ω_S^1 which do not kill it. \square

Example 1.6. For $\mathcal{E} = \mathcal{O}_S$ we have the universal connection $\text{d} : \mathcal{O}_S \rightarrow \Omega_S^1$. Then the statement that d is flat is equivalent to $\text{d}^2 = 0$.

Remark. Recall that given $f : X \rightarrow S$ there is an exact sequence of \mathcal{O}_X -modules,

$$f^* \Omega_S^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/S}^1 \longrightarrow 0$$

We may define,

$$\Omega_{X/S}^k = \bigwedge^k \Omega_{X/S}^1$$

to give $\Omega_{X/S}^\bullet$, the relative deRham complex of X over S ,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_{X/S}^1 \xrightarrow{d} \Omega_{X/S}^2 \longrightarrow \dots$$

Definition Now consider a proper smooth morphism $\pi : X \rightarrow S$ of smooth varieties. We define its sheaf of relative deRham cohomology by the hyperderived functors applied to the relative de Rham complex,

$$\mathcal{H}_{\text{dR}}^n(X/S) = \mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet)$$

Remark. Note that for the structure map $\pi : X \rightarrow \text{Spec}(k_0)$ map we have $\pi_* \mathcal{F} = \Gamma(X, \mathcal{F})$ and thus its hyperderived functors are simply hypercohomology of sheaves so,

$$\mathcal{H}_{\text{dR}}^n(X/k_0) = \mathbb{H}^n(\Omega_{S/k_0}^\bullet) = H_{\text{dR}}^n(X/k_0)$$

recovering algebraic de Rham cohomology.

Definition Let S and $\pi : X \rightarrow S$ be smooth. Then there is a decreasing filtration,

$$F^p \Omega_X^q = \bigoplus_{p \geq p'} \text{Im} \left((\pi^* \Omega_S^{p'} \otimes_{\mathcal{O}_X} \Omega_X^{q-p'} \rightarrow \Omega_X^q) \right)$$

There is always an exact sequence of sheaves of k_0 -modules,

$$0 \longrightarrow F^1/F^2 \longrightarrow F^0/F^2 \longrightarrow F^0/F^1 \longrightarrow 0$$

which, in this case, gives an exact sequence of complexes,

$$0 \longrightarrow \Omega_{X/S}^{\bullet-1} \otimes_{\mathcal{O}_X} \pi^* \Omega_S^1 \longrightarrow \Omega_X^\bullet / F^2 \Omega_X^\bullet \longrightarrow \Omega_{X/S}^\bullet \longrightarrow 0$$

The associated long exact sequence of hypercohomology,

$$\begin{array}{ccccc} \mathbb{R}^n \pi_*(\Omega_{X/S}^{\bullet-1}) \otimes_{\mathcal{O}_S} \Omega_S^1 & \longrightarrow & \mathbb{R}^n \pi_*(\Omega_X^\bullet / F^2 \Omega_X^\bullet) & \longrightarrow & \mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet) \xrightarrow{\nabla} \mathbb{R}^{n+1} \pi_*(\Omega_{X/S}^{\bullet-1}) \otimes_{\mathcal{O}_S} \Omega_S^1 \\ \parallel & & & & \parallel \\ \mathbb{R}^{n-1} \pi_*(\Omega_{X/S}^\bullet) \otimes_{\mathcal{O}_S} \Omega_S^1 & & & & \mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet) \otimes_{\mathcal{O}_S} \Omega_S^1 \end{array}$$

In partiucular, the connecting map $\nabla : \mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet) \rightarrow \mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet) \otimes_{\mathcal{O}_S} \Omega_S^1$ is a flat connection on the relative deRham sheaf, $\mathcal{H}_{\text{dR}}^n(X/S) = \mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet)$. We call this connection the Gauss-Manin connection.

Remark. For example, if $f : X \rightarrow S$ is etale then we know that $f^* \Omega_S^1 \rightarrow \Omega_X^1$ is an isomorphism and thus $\Omega_{X/S}^1 = 0$. Therefore, the sheaf of relative deRham cohomology is,

$$\mathbb{R}^n \pi_*(\Omega_{X/S}^\bullet) = \mathbb{R}^n \pi_*(0 \rightarrow \mathcal{O}_X \rightarrow 0 \rightarrow \dots) = R^n \pi_*(\mathcal{O}_X)$$

Then the connecting map $\nabla : R^n \pi_*(\mathcal{O}_X) \rightarrow R^n \pi_*(\mathcal{O}_X) \otimes_{\mathcal{O}_S} \Omega_S^1$ is simply induced by the exterior derivative,

$$\nabla = R^n \pi_*(d : \mathcal{O}_X \rightarrow \Omega_X^1)$$

where $\pi_*(\Omega_X^1) = \pi_* \mathcal{O}_X \otimes_{\mathcal{O}_S} \Omega_S^1$.

Remark. If we take $k_0 = \mathbb{C}$ then GAGA implies that,

$$\mathcal{H}_{\mathrm{dR}}^n(X/S)^{\mathrm{an}} \cong \mathcal{H}_{\mathrm{dR}}^n(X^{\mathrm{an}}/S^{\mathrm{an}})$$

and ∇^{an} is a flat connection on $\mathcal{H}_{\mathrm{dR}}^n(X^{\mathrm{an}}/S^{\mathrm{an}})$ so there is a relative deRham complex,

$$0 \longrightarrow \mathcal{O}_X^{\mathrm{an}} \xrightarrow{\mathrm{d}} (\Omega_{X/S}^1)^{\mathrm{an}} \xrightarrow{\mathrm{d}} (\Omega_{X/S}^2)^{\mathrm{an}} \longrightarrow \dots$$

However, by Ehresmann's lemma, locally above $s \in S$ we may write $\pi^{-1}(U) = U \times X_s$ and choose U to be contractible. Then, locally, $\Omega_{X^{\mathrm{an}}/S^{\mathrm{an}}}^{\bullet} = \underline{\mathbb{C}}_X \otimes (\Omega_{X_s}^{\bullet})^{\mathrm{an}}$ which, using the projection formula, gives,

$$\mathcal{H}_{\mathrm{dR}}^n(X^{\mathrm{an}}/S^{\mathrm{an}}) = \mathbb{R}^n \pi_*(\underline{\mathbb{C}}_X \otimes (\Omega_{X_s}^{\bullet})^{\mathrm{an}}) = R^n \pi_*(\underline{\mathbb{C}}_X) \otimes \mathcal{O}_S^{\mathrm{an}}$$

In particular, there is a natural connection on this analytic sheaf,

$$\begin{aligned} \nabla^{\mathrm{an}} : R^n \pi_*(\underline{\mathbb{C}}_X) \otimes_{\underline{\mathbb{C}}_S} \mathcal{O}_S &\rightarrow (R^n \pi_*(\underline{\mathbb{C}}_X) \otimes_{\underline{\mathbb{C}}_S} \mathcal{O}_S) \otimes_{\mathcal{O}_S} \Omega_S^1 = R^n \pi_*(\underline{\mathbb{C}}_X) \otimes_{\underline{\mathbb{C}}_S} \Omega_S^1 \\ \nabla^{\mathrm{an}} : (\alpha \otimes f) &\mapsto \alpha \otimes \mathrm{d}f \end{aligned}$$

Clearly this connection satisfies $\mathcal{H}_{\mathrm{dR}}^n(X^{\mathrm{an}}/S^{\mathrm{an}})^{\nabla^{\mathrm{an}}} \cong R^n \pi_*(\underline{\mathbb{C}}_X)$. In fact, there is a unique connection satisfying this property which is the GAGA equivalent analytic connection to the algebraic Gauss-Manin connection.

2 Local Systems

Proposition 2.1. Let \mathcal{E} be a vector bundle on X with a flat connection

$$\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$$

Then $\mathcal{E}^{\nabla} = \ker \nabla$ is a local system.

Proof. Since \mathcal{E} is locally free, we can find a cover of trivializing neighborhoods U such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$. Then $\nabla : \mathcal{O}_U^{\oplus n} \rightarrow (\Omega_U^1)^{\oplus n}$ is a connection. Define,

$$\nabla e_j = \sum_{i=1}^n \omega_{ij} \otimes e_i$$

where $\omega_{ij} \in \Omega_X^1(U)$ is a form. This uniquely defines the connection since,

$$\begin{aligned} \nabla(f_1, \dots, f_n) &= \nabla \left(\sum_{i=1}^n f_i e_i \right) = \sum_{i=1}^n (f_i \nabla e_i + \mathrm{d}f_i \otimes e_i) \\ &= \sum_{i,j=1}^n \omega_{ij} \otimes f_j e_i + (\mathrm{d}f_1, \dots, \mathrm{d}f_n) \end{aligned}$$

Therefore, \mathcal{E}^{∇} is given locally by (f_1, \dots, f_n) solving the linear system of differential equations,

$$\mathrm{d}f_i + \sum_{j=1}^n \omega_{ij} f_j = 0$$

The condition of flatness is that,

$$\nabla_1 \circ \nabla = 0$$

which locally is,

$$\begin{aligned} \nabla_1 \circ \nabla(f_1, \dots, f_n) &= \nabla_1 \left(\sum_{i,j=1}^n \omega_{ij} \otimes f_j e_i + \sum_{j=1}^n df_j \otimes e_j \right) \\ &= \sum_{i,j=1}^n [d\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \nabla(f_j e_i)] + \sum_{i=1}^n [ddf_i \otimes e_i - df_j \wedge \nabla e_j] \\ &= \sum_{i,j=1}^n \left[d\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \left(df_j \otimes e_i + f_j \sum_{k=1}^n \omega_{ki} \otimes e_k \right) \right] - \sum_{i,j=1}^n [df_j \wedge \omega_{ij} \otimes e_i] \\ &= \sum_{i,j=1}^n \left[d\omega_{ij} \otimes e_i - \sum_{k=1}^n \omega_{ij} \wedge \omega_{ki} \otimes e_k \right] f_j \\ &= \sum_{i,j=1}^n \left[d\omega_{ij} + \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} \right] \otimes f_j e_i \end{aligned}$$

So the curvature ω_∇ is given by coefficients,

$$\Theta_{ij} = d\omega_{ij} + \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj}$$

This vanishing is exactly the criterion in Frobenius' theorem for integrability. \square

3 Principle B

Proposition 3.1. Let $k_0 \subset \mathbb{C}$ have finite transcendence degree over \mathbb{Q} and X be a complete smooth variety over a field k that is finitely generated over k_0 . Let ∇ be the Gauss-Manin connection on $\mathcal{H}_{\text{dR}}^n(X)$ relative to $X \rightarrow \text{Spec}(k) \rightarrow \text{Spec}(k_0)$.

If $t \in H_{\text{dR}}^n(X)$ is rational relative to all embeddings $k \hookrightarrow \mathbb{C}$ then $\nabla t = 0$.

Proof. Let A be a finite-type k_0 -algebra and $\pi : X_A \rightarrow \text{Spec}(A)$ a smooth proper map with generic fibre $X_{(0)} = X \rightarrow \text{Spec}(k)$ and such that t extends to $\Gamma(\text{Spec}(A), \mathcal{H}_{\text{dR}}^n(X/\text{Spec}(A)))$. After base change via $k_0 \hookrightarrow \mathbb{C}$ to $S = \text{Spec}(A_{\mathbb{C}})$ there are maps,

$$X_S \rightarrow S \rightarrow \text{Spec}(\mathbb{C})$$

and a global section $t' = t \otimes 1$ of $\mathcal{H}_{\text{dR}}^n(X_S^{\text{an}}/S^{\text{an}})$. We need to show that $(\nabla \otimes 1)t' = 0$. However, if we recall that,

$$\mathcal{H}_{\text{dR}}^n(X_S^{\text{an}}/S^{\text{an}}) = \mathbb{R}^n \pi_*^{\text{an}}(\Omega_{X_S^{\text{an}}/S^{\text{an}}}^\bullet) = (R^n \pi_* \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S^{\text{an}}} = H^n(X_S^{\text{an}}, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S^{\text{an}}}$$

and that the Gauss-Manin connection kills exactly those sections purely in,

$$\mathcal{H}^n(X_S^{\text{an}}, \underline{\mathbb{C}}_X) = R^n \pi_*(\underline{\mathbb{C}}_X)$$

An embedding $\sigma : k \hookrightarrow \mathbb{C}$ gives a point $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(A)$ of s . Since t is rational,

$$t(s) \in H^n(X_s^{\text{an}}, \mathbb{Q}) \subset H_{\text{dR}}^n(X_s^{\text{an}})$$

Then locally on S we have $\mathcal{H}_{\text{dR}}^n(X^{\text{an}}/S^{\text{an}}) = R^n \pi_* (\underline{\mathbb{C}}_X) \otimes \mathcal{O}_S^{\text{an}}$ which is locally free and $\mathcal{H}^n(X^{\text{an}}, \underline{\mathbb{C}}_X)$ gives its sheaf of locally constant sections. However, t takes rational vatonal values on the closed points which are dense so it must be locally constant and thus $t \in \mathcal{H}^n(X^{\text{an}}, \underline{\mathbb{C}}_X)$ so $\nabla t = 0$. \square

Definition Let $\pi : X \rightarrow S$ e a proper smooth map of smooth varieties / \mathbb{C} with S connected. Then,

$$\mathcal{H}_{\text{ét}}^n(X/S)(m) = \varprojlim_r (R^n \pi_{\text{ét}}^* \mu_r^{\otimes m}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and

$$\mathcal{H}_{\mathbb{A}}^n(X/S)(m) = \mathcal{H}_{\text{dR}}^n(X/S)(m) \times \mathcal{H}_{\text{ét}}^n(X/S)(m)$$

and

$$\mathcal{H}_B^{2p}(X/S)(p) = R^{2p} \pi_*^{\text{an}} \mathbb{Q}(p)$$

Remark. By Ehresmann's lemma we can locally write $\pi^{-1}(U) = U \times X_s$ with U contractible. Therefore, by Kunnet,eth,

$$H_B^{2p}(X/S)(p)(U) = H^{2p}(\pi^{-1}(U), \mathbb{Q}(p)|_U) = H_B^{2p}(X_s, \mathbb{Q}(p)) \otimes_{\mathbb{Q}} H^0(U, \mathbb{Q}(p)) = H_B^{2p}(X_s, \mathbb{Q}(p))$$

since U is contractible. This is a constant sheaf so $H_B^{2p}(X/S)(p)$ is a local system. A similar arument holds for the other sheaves.

Theorem 3.2 (Principle B). Let $t \in \Gamma(\mathcal{H}_{\mathbb{A}}^{2p}(X/S)(p))$ such that $\nabla t_{\text{dR}} = 0$. If $(t_{\text{dR}})_s \in F^0 H_{\text{dR}}^{2p}(X_s)(p)$ for each $s \in S$ and t_s is an absolute Hodge cyclce in $H_{\mathbb{A}}^{2p}(X_s)(p)$ for some s then it is an absolute Hodge cycle for every s .

Proof. We suppose that t_s is an absolute Hodge cycle for some some $s \in S$. For any $s' \in S$ we need to show that $t_{s'}$ is absolutly Hodge meaning that it is rational relative to every isomorphism $\sigma : \mathbb{C} \rightarrow \mathbb{C}$. However, such an isomorphism gives a morphism $\sigma\pi : \sigma X \rightarrow \sigma S$ and a section $\sigma(t)$ of $\mathcal{H}_{\mathbb{A}}^n(\sigma X/\sigma S)(p)$. We know that $\sigma(t)_{\sigma s}$ is rationl and we must show that $\sigma(t)_{\sigma s'}$ is rational. It suffices to prove this for $\sigma = \text{id}$ given that there is some σ for which this global rationality holds.

First, consider the component t_{dR} of t (relative the the construction of $\mathcal{H}_{\mathbb{A}}^n(\sigma X/\sigma S)(p)$ as a product. By assumption $\nabla t_{\text{dR}} = 0$ so t_{dR} is a global section of $\mathcal{H}^{2p}(X^{\text{an}}, \underline{\mathbb{C}}_X)$ which we have shown is the vanishing of the analytic Gauss-Manin connection. Since t_{dR} is rational at one point, it must be rational at every point since $\mathcal{H}^{2p}(X^{\text{an}}, \underline{\mathbb{C}}_X)$ is locally constant and X^{an} is connected.

Thus, it suffices to prove the rationality of the other factor $t_{\text{ét}}$. Since the relative cohomology sheaves defined above are local systems, for any point s we have a monodromy action of $\pi_1(S, s)$ on their stalks at s whose fixed points are those germs which extend globally. In particular, this induces isomorphism,

$$\begin{aligned} \Gamma(S, \mathcal{H}_B^{2p}(X/S)(p)) &\cong H_B^{2p}(X_s)^{\pi_1(S, s)} \\ \Gamma(S, \mathcal{H}_{\text{ét}}^{2p}(X/S)(p)) &\cong H_{\text{ét}}^{2p}(X_s)^{\pi_1(S, s)} \end{aligned}$$

Then consider the diagram,

$$\begin{array}{ccccc}
\Gamma(S, \mathcal{H}_B^{2p}(X/S)(p)) & \hookrightarrow & \Gamma(S, \mathcal{H}_B^{2p}(S/X)(p)) \otimes \mathbb{A}_{\text{fin}} & \xrightarrow{\sim} & \Gamma(S, \mathcal{H}_{\text{ét}}^{2p}(X/S)(p)) \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
H_B^{2p}(X_s)(p)^{\pi_1(S,s)} & \hookrightarrow & H_B^{2p}(X_s)(p)^{\pi_1(S,s)} \otimes \mathbb{A}_{\text{fin}} & \xrightarrow{\sim} & H_{\text{ét}}^{2p}(X_s)(p)^{\pi_1(S,s)} \\
\downarrow & & \downarrow & & \downarrow \\
H_B^{2p}(X_s) & \hookrightarrow & H_B^{2p}(X_s) \otimes A_{\text{fin}} & \xrightarrow{\sim} & H_{\text{ét}}^{2p}(X_s)(p)
\end{array}$$

We have $t_{\text{ét}} \in \Gamma(S, \mathcal{H}_{\text{ét}}^{2p}(X/S)(p))$ which is rational at s so its image in $H_{\text{ét}}^{2p}(X_s)(p)$ lies in $H_B^{2p}(X_s)(p)$. Now we need the following lemma which allows us to conclude that $t_{\text{ét}} \in \Gamma(S, \mathcal{H}_B^{2p}(X/S)(p))$ and thus $(t_{\text{ét}})_{s'} \in H_B^{2p}(X_s)(p) \subset H_{\text{ét}}^{2p}(X_s)(p)$ for all s' completing the theorem. \square

Lemma 3.3. Let $W \hookrightarrow V$ be an inclusion of vector spaces. Let Z be a third vector space and take nonzero $z \in Z$. Embed V in $V \otimes Z$ via $v \mapsto v \otimes z$. Then, in $V \otimes Z$,

$$(W \otimes V) \cap (V \otimes z) = W \otimes z$$

Proof. This is clear if we choose a basis e_i for W which extends to a basis of V . Then any $x \in V \otimes Z$ has a unique expansion,

$$x = \sum e_i \otimes z_i$$

If $x \in W \otimes Z$ then $z_i = 0$ for each e_i not in W and if $x \in V$ then $z_i = z$ for each nonzero z_i . \square

Remark. The proof of principle B concludes taking $Z = \mathbb{A}_{\text{fin}}$ and $z = 1$ over the inclusion $H_B^{2p}(X_s)^{\pi_1(S,x)(p)} \rightarrow H_B^{2p}(X_s)(p)$. The lemma then implies that, in $H_{\text{ét}}^{2p}(X_s)(p)$,

$$\begin{aligned}
\Gamma(S, \mathcal{H}_B^{2p}(X/S)(p)) \cap H_B^{2p}(X_s)(p) &= [H_B^{2p}(X_s)(p)^{\pi_1(S,s)} \otimes \mathbb{A}_{\text{fin}}] \cap H_B^{2p}(X_s)(p) \\
&= H_B^{2p}(X_s)(p)^{\pi_1(S,s)} = \Gamma(S, \mathcal{H}_B^{2p}(X/S)(p))
\end{aligned}$$

so we get a global rational section.

4 The Main Theorem

Theorem 4.1 (Deligne). Let X be an abelian variety over an algebraically closed field k and $t \in H_{\mathbb{A}}^{2p}(X)(p)$. If t is a Hodge cycle relative to some embedding $\sigma : k \hookrightarrow \mathbb{C}$ then it is a Hodge cycle with respect to every embedding. That is, every Hodge cycle is absolutely Hodge.

5 Hodge Structures and Mumford-Tate Groups

5.1 The Deligne Torus

Remark. Let $T \rightarrow S$ be a morphism of schemes. Given an S -scheme X and a T -scheme Y ,

$$\text{Hom}_T(Y, X \times_S T) = \text{Hom}_S(Y, X)$$

where,

$$\begin{array}{ccccc}
Y & & & & \\
& \searrow & & \searrow & \\
& X \times_S T & \longrightarrow & T & \\
& \downarrow & & \downarrow & \\
& X & \longrightarrow & S &
\end{array}$$

(Note: In the original image, there is a curved arrow from Y to T and a dashed curved arrow from Y to X.)

Definition Let $T \rightarrow S$ be a morphism of schemes. Given an T -scheme X we define the restriction of scalars functor $\mathcal{R}_{T/S}(X) : \mathbf{Sch}_S^{\text{op}} \rightarrow \mathbf{Set}$ via,

$$Y \mapsto X(Y \times_S T) = \text{Hom}_T(Y \times_S T, X)$$

When the functor $\mathcal{R}_{T/S}(X)$ is representable in \mathbf{Sch}_S then we call the (unique up to unique isomorphism) S -scheme representing it $X' = \text{Res}_{T/S}(X_T)$ such that,

$$\mathcal{R}_{T/S}(X) = \text{Hom}_S(-, \text{Res}_{T/S}(X))$$

In this case, we have an isomorphism of functors,

$$\text{Hom}_T(- \times_S T, X) = \text{Hom}_S(-, \text{Res}_{T/S}(X))$$

which makes $\text{Res}_{T/S}(X)$ be right-adjoint to extension of scalars functor,

$$Y_S \mapsto Y_S \times_S T$$

Remark. Starting with $\mathbb{G}_m^A = \text{Spec}(A[z, z^{-1}])$ we define some algebraic groups as follows.

Definition The Deligne torus \mathbb{S} is an algebraic group over \mathbb{R} defined as,

$$\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m^{\mathbb{C}}$$

where $\text{Res}_{\mathbb{C}/\mathbb{R}}$ is restriction of scalars from \mathbb{C} to \mathbb{R} .

Remark. We may characterize $\text{Res}_{\mathbb{C}/\mathbb{R}}$ as the right-adjoint to base change so the S -points are,

$$\begin{aligned}
\mathbb{S}(S) &= \text{Hom}_{\mathbb{R}}(S, \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m^{\mathbb{C}}) = \mathbb{G}_m^{\mathbb{C}}(S \times_{\mathbb{R}} \mathbb{C}) = \text{Hom}_{\mathbb{C}}(S \times_{\mathbb{R}} \mathbb{C}, \mathbb{G}_m^{\mathbb{C}}) \\
&= \text{Hom}_{\mathbb{C}}(\mathbb{C}[z, z^{-1}], \Gamma(S \times_{\mathbb{R}} \mathbb{C})) = \Gamma(S \times_{\mathbb{R}} \mathbb{C})^{\times}
\end{aligned}$$

In particular, the \mathbb{R} -points of \mathbb{S} are,

$$\mathbb{S}(\mathbb{R}) = \mathbb{G}_m^{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^{\times}$$

Furthermore, the \mathbb{C} -points of \mathbb{S} are,

$$\mathbb{S}(\mathbb{C}) = \mathbb{G}_m^{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \text{Hom}_{\mathbb{C}}(\mathbb{C}[z, z^{-1}], \mathbb{C} \oplus i\mathbb{C}) = \mathbb{C}^{\times} \times i\mathbb{C}^{\times}$$

Definition We define a set of characters and cocharacters of \mathbb{S} . First we define the character,

$$\mathrm{Nm} : \mathbb{S} \rightarrow \mathbb{G}_m^{\mathbb{R}}$$

on \mathbb{R} -points $\mathbb{S}(\mathbb{R}) \rightarrow \mathbb{G}_m^{\mathbb{R}}(\mathbb{R})$ as $\mathbb{C}^\times \rightarrow \mathbb{R}^\times$ via $z \mapsto z\bar{z}$.

Furthermore, we define the cocharacter,

$$w : \mathbb{G}_m^{\mathbb{R}} \rightarrow \mathbb{S}$$

on \mathbb{R} -points $\mathbb{G}_m^{\mathbb{R}}(\mathbb{R}) \rightarrow \mathbb{S}(\mathbb{R})$ by the natural inclusion $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$.

Lastly, we define a \mathbb{C} -cocharacter,

$$\mu : \mathbb{G}_m^{\mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}}$$

on \mathbb{C} -points via $\mathbb{G}_m^{\mathbb{C}}(\mathbb{C}) \rightarrow \mathbb{S}_{\mathbb{C}}(\mathbb{C})$ as $\mu(z) = (z, i)$ where,

$$\mathbb{S}_{\mathbb{C}}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{S} \times_{\mathbb{R}} \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{S}) = \mathbb{S}(\mathbb{C}) = \mathbb{C} \oplus i\mathbb{C}$$

5.2 Hodge Structures

Definition Let V be a finite-dimensional \mathbb{Q} -vectorspace. A \mathbb{Q} -rational Hodge structure of weight n on V is a decomposition,

$$V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $V^{q,p} = \overline{V^{p,q}}$.

Definition A Hodge structure defines a cocharacter,

$$\mu : \mathbb{G}_m^{\mathbb{C}} \rightarrow \mathrm{GL}(V_{\mathbb{C}})$$

via $\mu(z)v^{p,q} = z^{-p}v^{p,q}$ for $v^{p,q} \in V^{p,q}$.

Furthermore, $\overline{\mu(z)} \cdot v^{p,q} = \bar{z}^{-q}v^{p,q}$ commutes with the action of $\mu(z)$. Therefore, we may take their product to give a map of real algebraic groups,

$$h : \mathbb{S} \rightarrow \mathrm{GL}(V_{\mathbb{R}})$$

via $h(z)v^{p,q} = z^{-p}\bar{z}^{-q}v^{p,q}$. where \mathbb{C}^\times is the algebraic group,

$$\mathrm{Spec}(\mathbb{C}[x, x^{-1}]) \rightarrow \mathrm{Spec}(\mathbb{R})$$

Remark. Conversely, any homomorphism of \mathbb{R} -algebraic groups $h : \mathbb{S} \rightarrow \mathrm{GL}(V_{\mathbb{R}})$ which, on \mathbb{R} , restricts to $r \mapsto r^{-n}\mathrm{id}_V$ defines a Hodge structure of weight n on V by taking $V^{p,q}$ to be the eigenspace of eigenvalue $z^{-p}\bar{z}^{-q}$ for $h(z)$ i.e.,

$$V^{p,q} = \{v \in V_{\mathbb{C}} \mid \forall z \in \mathbb{S}(\mathbb{R}) : h(z) \cdot v = z^{-p}\bar{z}^{-q}v\}$$

Definition The Weil operator $C \in \mathrm{GL}(V_{\mathbb{R}})$ of a Hodge structure (V, h) is $C = h(i)$.

Proposition 5.1. Given a Hodge structure on V there is a decreasing filtration of $V_{\mathbb{C}}$ via,

$$F^p V = \bigoplus_{p' \geq p} V^{p', n-p'}$$

(ASK RAYMOND ABOUT TATE TWISTS AND THIS HODE STRUCTURE)

Example 5.2. For any m we define a Hodge structure of weight $-2m$ denoted $\mathbb{Q}(m)$ via taking $\mathbb{Q}(m)_{\mathbb{C}} = \mathbb{Q}(m)^{-m, -m}$

5.3 Mumford-Tate Groups

Definition The Mumford-Tate group $M(V)$ associated to Hodge structure (V, h) is the smallest \mathbb{Q} -algebraic subgroup of $\mathrm{GL}(V)$ such that,

$$\mathrm{Im}(h)(\mathbb{R}) \subset M(V)(\mathbb{R})$$

Example 5.3. For $\mathbb{Q}(m)$ as a Hodge structure the map $h : \mathbb{C}^\times \rightarrow \mathrm{GL}_1(\mathbb{R})$ is given by $h(z) = |z|^{-m}$ which is surjective for $m \neq 0$. Thus, for $n \neq 0$ we have,

$$M_h = \mathbb{G}_m^{\mathbb{Q}}$$

and for $n = 0$ it is $\mathrm{Spec}(\mathbb{Q})$ the trivial \mathbb{Q} -group scheme.

(BADDD)

Proposition 5.4. Let V be a \mathbb{Q} -vectorspace with Hodge structure h of weight n . The tensor space,

$$T = V^{\otimes m_1} \otimes V^{\vee \otimes m_2} \otimes \mathbb{Q}(1)^{\otimes m_3}$$

has a Hodge structure of weight $(m_1 - m_2)n - 2m_3$. Then the Mumford-Tate group G of (V, h) is the subgroup of $\mathrm{GL}_n(V) \times \mathbb{G}_m$ fixing all rational tensors of type $(0, 0)$ in T .

Proof. For any $t \in T$ the element t is of type $(0,)$ iff it is fixed by $\mu(\mathbb{G}_m)$ so $M_h = H'$. We will now prove that characters of H lift and thus $H = H'$. \square

5.4 DO IT RIGHT

Remark. Let (V, h) be a Hodge structure of weight d . Then the tensor space,

$$T^{m,n}(V) = \bigoplus_{j=1}^n V^{\otimes m_j} \otimes (V^\vee)^{\otimes n_j}$$

is a Hodge structure of weight,

$$N = \sum_{j=1}^n (m_j - n_j)d$$

Furthermore, let $M(V)$ be the Mumford-Tate group of (V, h) i.e. the intersection of all \mathbb{Q} -algebraic subgroups of $\mathrm{GL}(V)$ whose \mathbb{R} -points contain $\mathrm{Im}(h)$.

Lemma 5.5. There are morphism of \mathbb{R} -algebraic subgroups,

$$\mathbb{S} \xrightarrow{\psi} M(V)_{\mathbb{R}} \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$$

Conversely, given any \mathbb{Q} -vectorspace H with an algebraic representation,

$$\rho : M(V) \rightarrow \mathrm{GL}(H)$$

gives H a Hodge structure via,

$$\mathbb{S} \xrightarrow{\psi} M(V)_{\mathbb{R}} \xrightarrow{\rho} \mathrm{GL}(H_{\mathbb{R}})$$

Proposition 5.6. Let $H \subset T^{m,n}(V)$ be any rational subspace. Then H is a Hodge substructure iff H is stable under $M(V)$. Furthermore, a rational vector $t \in T^{m,n}(V)$ is of type $(0,0)$ iff it is fixed by $M(V)$.

Proof. If H is stable under the action of the Mumford-Tate group then it becomes a representation $\rho : M(V) \rightarrow \mathrm{GL}(H)$ since it is rational this gives a Hodge structure on H .

Conversely, suppose that $V \subset T^{m,n}(V)$ is a substructure then consider its stabilizer $G_H \subset \mathrm{GL}(V)$ which is a \mathbb{Q} -algebraic subgroup since H is rational. Moreover, $(G_H)_{\mathbb{R}}$ contains $\mathrm{Im}(h)$ because as a Hodge structure it splits into eigenspaces of h so is preserved by its image. Thus $M(V) \subset G_H$ by definition so $M(V)$ preserves V .

Likewise, it is clear that t is fixed by the action of $\mathbb{S}(\mathbb{R})$ iff t is of Hodge type $(0,0)$. Thus it suffices to prove that t is fixed by $\mathbb{S}(\mathbb{R})$ iff it is fixed by $M(V)$. A similar argument will show this.

First, if t is fixed by $M(V)$ then it is fixed by $M(V)(\mathbb{R})$ which contains $\mathrm{Im}(h)$ and thus t is fixed by $\mathbb{S}(\mathbb{R})$.

Conversely, if t is fixed by $\mathbb{S}(\mathbb{R})$ then its stabilizer $G_t \subset \mathrm{GL}(V)$ is a \mathbb{Q} -algebraic subgroup since t is rational. Furthermore, by assumption, $\mathrm{Im}(h) \subset (G_t)(\mathbb{R})$ and thus $M(V) \subset G_t$ by definition showing that $M(V)$ fixes t . \square

Corollary 5.7. The space $\mathrm{End}(V)$ is an algebraic $M(V)$ -rep and therefore a Hodge structure. Furthermore, the type- $(0,0)$ Hodge classes are exactly morphisms of Hodge structures since they must commute with the action of \mathbb{S} . Therefore,

$$\mathrm{Hom}_{\mathrm{HS}}(V, V) = \mathrm{End}(V)^{M(V)}$$

5.5 Polarization

Definition A polarization ψ of (V, h) is a morphism of Hodge structures,

$$\psi : V \times V \rightarrow \mathbb{Q}(-n)$$

such that $\psi(x, Cy)$ on $V_{\mathbb{R}}$ is an inner product where $C = h(i)$ is the Weil operator.

Remark. Under the canonical isomorphism,

$$\mathrm{Hom}(V \otimes V, \mathbb{Q}(-n)) \cong V^{\vee} \otimes V^{\vee}(-n)$$

a polarization is a tensor of bidegree $(0,0)$ because it is a morphism of Hodge structures and thus is fixed by the Mumford-Tate group G ,

$$\forall v, v' \in V : \forall (g_1, g_2) \in G(\mathbb{Q}) : \psi(g_1 v, g_2 v') = g_2^n \psi(v, v')$$

Remark. Let $C = h(i)$ be a Weil operator. For $v^{p,q} \in V^{p,q}$ we have $Cv^{p,q} = i^{-p+q}v^{p,q}$ and thus C^2 acts as $(-1)^n$ on all of V where $n = p + q$ is the weight of V .

Definition Let H be a real algebraic group with an involution σ of $H_{\mathbb{C}}$. Then a real-form of H is a real algebraic group H_{σ} and an isomorphism $H_{\mathbb{C}} \rightarrow (H_{\sigma})_{\mathbb{C}}$ sending complex conjugation to the action of σ on complex conjugation on $H(\mathbb{C})$.

Theorem 5.8. The Mumford-Tate group $M(V)$ is connected and if (V, h) is polarizable then $M(V)$ is reductive.

Proof. $M(V)$ is clearly connected else its connected component of the identity would be a smaller \mathbb{Q} -algebraic subgroup also satisfying the property that its \mathbb{R} -points contain $\text{Im}(h)$ (because \mathbb{S} is connected the image must lie in this connected component). Now, we use the fact that a connected algebraic group is reductive if it has a faithful semisimple representation. We will show that the tautological representation $M(V) \hookrightarrow \text{GL}(V)$ which is clearly faithful is also semisimple when V is polarizable. \square

Proposition 5.9. If V is polarizable then $M(V) \subset \text{GL}(V)$ is semisimple.

Proof. We will prove that a real algebraic group H is semisimple if it has a *compact* real-form. It suffices to show that H_σ is semisimple. By the unitarian trick, any finite-dimensional H -rep has an H_σ -invariant positive definite symmetric form via,

$$\langle u, v \rangle_0 = \int_{H_\sigma} \langle h \cdot u, h \cdot v \rangle$$

to conclude that every finite-dimensional H_σ -rep is semisimple. This implies that H_σ is reductive.

Thus, it suffices to prove that the Mumford-Tate group has a *compact* real-form (the compactness here is the magic ingredient). Consider the special Mumford-Tate group of (V, h) ,

$$G^0 = \ker(G \rightarrow \mathbb{G}_m)$$

and G^1 be the smallest \mathbb{Q} -rational subgroup of $\text{GL}(V) \times \mathbb{G}_m$ (WHY THIS GROUP) such that $G^1_{\mathbb{R}}$ contains $h(U^1)$ where U^1 is the \mathbb{R} -algebraic groups whose \mathbb{R} -points are $S^1 \subset \mathbb{C}^\times$. Then, $G^1 \subset G^0 \subset G$ since,

$$G^1_{\mathbb{R}} \cdot h(C^\times) = G_{\mathbb{R}} \text{ and } h(U^1) = \ker(h(C^\times) \rightarrow \mathbb{G}_m)$$

so $G^0 = G^1$ and thus G^0 is connected since G^1 is.

Since $C = h(i)$ acts trivially on $\mathbb{Q}(1)$ we know $C \in G^0(\mathbb{R})$. Furthermore C^2 acts as $(-1)^n$ on V and thus is in the center of $G^0(\mathbb{R})$. The inner automorphism $a_C : g \mapsto C^{-1}gC$ of $G_{\mathbb{R}}$ is therefore an involution since its square satisfies,

$$a_C^2(g) = C^{-2}gC^2 = g$$

because C^2 is in the center.

Now let ψ be a polarization of V . For $u, v \in V_{\mathbb{C}}$ and $g \in G^0(\mathbb{C})$ we have,

$$\psi(u, C\bar{v}) = \psi(gu, gC\bar{v}) = \psi(g, CC^{-1}gC\bar{v}) = \psi(gu, C\overline{a_C(\bar{g})}v)$$

Thus, the positive-definite bilinear form $\phi(u, v) = \psi(u, C\bar{v})$ on $V_{\mathbb{R}}$ is invariant under the G^0 -real-form $G^0_{a_C}$ since the action of \bar{g} is sent to $a_C(\bar{g})$ under the isomorphism $G^0_{\mathbb{C}} \rightarrow (G^0_{a_C})_{\mathbb{C}}$. Since $G^0_{a_C}$ has an invariant inner-product on V it must be compact. (ASK HARRIS ABOUT THAT) \square

5.6 Characterizing Subgroups

Here let G be a reductive algebraic group over a field k of characteristic zero and let V_α be a faithful family of finite-dimensional representations of G over k such that $G \rightarrow \prod \mathrm{GL}(V_\alpha)$ is injective. We may define a tensor algebra,

$$T^{m,n} = \bigotimes_{\alpha} V_{\alpha}^{\otimes m(\alpha)} \otimes \bigotimes_{\alpha} (V_{\alpha}^{\vee})^{\otimes n(\alpha)}$$

which is also a finite G -rep.

Definition Then for any algebraic subgroup $H \subset G$ we write H' for the subgroup fixing all tensors appearing in some T fixed by H . That is, H' is the largest subgroup $H \subset H'$ which fixes every tensor fixed by H .

Definition Given an algebraic group G over k we define its character group,

$$X_k(G) = \mathrm{Hom}_k(G, \mathbb{G}_m^k)$$

Theorem 5.10. We have the following,

- (a). Every finite G -rep is contained in a sum of $T^{m,n}$
- (b). Every subgroup $H \subset G$ is the stabilizer of a line D in some finite G -rep.
- (c). If $H \subset G$ is reductive or $X_k(G) \rightarrow X_k(H)$ is surjective then $H = H'$.

Proof. Let W be a finite G -rep and W_0 be the trivial rep on the underlying space of W . There is a morphism of G -reps, $W \rightarrow W_0 \otimes_k k[G] \cong k[G]^{\dim W}$ so it suffices to prove that the regular representation can be expressed in terms of tensors.

There must be a finite sum $V = \bigoplus_{\alpha} V_{\alpha}$ such that the action $G \rightarrow \mathrm{GL}(V)$ is faithful then embed,

$$\mathrm{GL}(V) \rightarrow \mathrm{End}(V) \times \mathrm{End}(V^{\vee})$$

identifying $\mathrm{GL}(V)$ with a closed subvariety of $\mathrm{End}(V) \times \mathrm{End}(V^{\vee})$ (FIX)

Let $I \subset \Gamma(G, \mathcal{O}_G)$ be the ideal of global functions on G whose value is zero on H . Consider the regular G -representation $k[G]$ (FIX)

The subgroup H is the stabilizer of a line D in some G -representation V which, by (a), we may take to be a direct sum of tensor representations $T^{m,n}$. Now suppose that H is reductive then V must be a semisimple H -representation so we can write $V = W \oplus D$ for some H -representation W . Furthermore, dualizing $V^{\vee} = W^{\vee} \oplus D^{\vee}$. Since H is the stabilizer of D \square

(WHAT IS THE POINT)

Lemma 5.11. Every \mathbb{Q} -character of H (above) extends to $\mathrm{GL}(V) \times \mathbb{G}_m$

Proof. Any \mathbb{Q} -character restricted to \mathbb{G}_m is $\mathbb{Q}(n)$ for some n . After tensoring with $\mathbb{Q}(-n)$ we find that the character is trivial on $\mu(\mathbb{G}_m)$. But H as the minimal subgroup must act trivially then we use the fact that trivial characters extend. \square

(OF THIS)

Theorem 5.12. Let $G \subset \mathrm{GL}(V)$ be the subgroup of all elements which fix every $(0,0)$ -hodge class in every tensor space $T^{m,n}(V)$. Then $M(V) = G$.

Proof. We have shown that $M(V) \subset G$. Furthermore, $M(V)' = G$ since $(0,0)$ -tensors are exactly the tensors fixed by the Mumford-Tate group and thus G is the group of all elements fixing all tensors fixed by $M(V)$. Now we use the general fact about reductive groups that if G is reductive and $H \subset G$ is a reductive subgroup then $H' = H$. \square

5.7 Back to Principle B

Remark. We need a slightly stronger version of Principle B proved as a corellary.

Theorem 5.13. Let $\pi : X \rightarrow S$ be a smooth proper map of smooth varieties over \mathbb{C} with S connected and let V be a local subsyttem of $R^{2p}\pi_*\mathbb{Q}(p)$ such that V_s consists purely of $(0,0)$ -cycles for all s and consists of absolute Hodge cycles at at least one $s \in S$. Then V_s consists of absolute Hodge cycles for all $s \in S$.

Proof. If V is constant i.e. if the map $\Gamma(S, V) \rightarrow V_s$ is bijective then this follows immedietly from the above argument. However, we may reduce the general case to this as follows.

By Hodge theory on S^{an} , at each point $s \in S$ the stalk $(R^{2p}\pi_*\mathbb{Q}(p))_s$ has a Hodge structure and a polarization which, since $R^{2p}\pi_*\mathbb{Q}(p)$ is a local system, glue to give a form,

$$\psi : R^{2p}\pi_*\mathbb{Q}(p) \times R^{2p}\pi_*\mathbb{Q}(p) \rightarrow \mathbb{Q}(-p)$$

which at each point is a polarization on the Hodge structure $(R^{2p}\pi_*\mathbb{Q}(p))_s$. On the rational $(0,0)$ -subspace,

$$(R^{2p}\pi_*\mathbb{Q}(p))_S \cap (R^{2p}\pi_*\mathbb{C}(p))_s^{0,0}$$

the form is symmetric, bilinear, rational and positive definite. Since V_s everywhere consists of $(0,0)$ -cycles this is a form defined on V_s . Since monodromy preserves the form, the image of $\pi_1(S, s_0)$ in $\mathrm{Aut}(V_{s_0})$ is finite because it is discrete and lies inside the compact group preserving the form. Therefore, after passing to a finite covering we can ensure that $\pi_1(S, s_0)$ acts trivially on V_{s_0} implying that V is globally constant. \square

6 Principle A

Definition Let X_α be a family of complete smooth varieties over k . We define tensor spaces,

$$\begin{aligned} T_{\mathrm{dR}} &= \left(\bigotimes_{\alpha} H_{\mathrm{dR}}^{m(\alpha)}(X_{\alpha}) \right) \otimes \left(\bigotimes_{\alpha} H_{\mathrm{dR}}^{n(\alpha)}(X_{\alpha})^{\vee} \right) (m) \\ T_{\mathrm{dR}} &= \left(\bigotimes_{\alpha} H_{\mathrm{\acute{e}t}}^{m(\alpha)}(X_{\alpha}) \right) \otimes \left(\bigotimes_{\alpha} H_{\mathrm{\acute{e}t}}^{n(\alpha)}(X_{\alpha})^{\vee} \right) (m) \\ T_{\mathbb{A}} &= T_{\mathrm{dR}} \times T_{\mathrm{\acute{e}t}} \end{aligned}$$

Finally, given an inclusion $k \hookrightarrow \mathbb{C}$ we get a Betti tensor space,

$$T_\sigma = \left(\bigotimes_{\alpha} H_{\sigma}^{m(\alpha)}(X_{\alpha}) \right) \otimes \left(\bigotimes_{\alpha} H_{\sigma}^{n(\alpha)}(X_{\alpha})^{\vee} \right) (m)$$

We say that an element $t \in T_{\mathbb{A}}$ is,

- (a). rational relative to σ if its image in $T_{\mathbb{A}} \otimes_{k \times \mathbb{A}_{\text{fin}}} (\mathbb{C} \times \mathbb{A}_{\text{fin}})$ lies in the subspace T_{σ}
- (b). is a Hodge cycle relative to σ if it is rational relative to σ and its first component lies in F^0 meaning it lies in the subspace generated by,

$$F^0 H_{\text{dR}}^{2p}(X)(p) = H_{\text{dR}}^{p,p}(X) \subset H_{\text{dR}}^{2p}(X)(p) \times H_{\text{ét}}^{2p}(X)(m)$$

- (c). is absolutely Hodge if it is a Hodge cycle relative to each $\sigma : k \hookrightarrow \mathbb{C}$.

Theorem 6.1 (Principle A). Let X_{α} be a family of varieties over \mathbb{C} and,

$$T = \bigotimes_{\alpha} H_B^{n_{\alpha}}(X_{\alpha}) \otimes H_B^{n_{\alpha}}(X_{\alpha})^{\vee} \otimes \mathbb{Q}(1)$$

Let $t_i \in T_i$ be absolute Hodge cycles and let G be the subgroup of,

$$\prod_{\alpha, n_{\alpha}} \text{GL}(H_B^{n_{\alpha}}(X_{\alpha})) \times \mathbb{G}_m$$

fixing all t_i . If $t \in T$ and is fixed by G then it is an absolute Hodge cycle.

Remark. We first need a lemma.

(FIX THIS SECTION ON TORSORS)

Lemma 6.2. Let G be an algebraic group over \mathbb{Q} and P be a G -torsor of isomorphism $H_{\sigma}^{\alpha} \rightarrow H_{\tau}^{\alpha}$ where these are families of \mathbb{Q} -rational G -reps. Let T_{σ} and T_{τ} be tensor spaces of H_{σ} and H_{τ} . Then P defines a map $T_{\sigma}^G \rightarrow T_{\tau}$.

Proof. Locally, for the étale topology on $\text{Spec}(\mathbb{Q})$, (MEANING WE CAN CHOSE AN ÉTALE COVERING SUCH THAT THIS IS THE CASE?) points of P give isomorphisms $T_{\sigma} \rightarrow T_{\tau}$. Furthermore, the restriction to T_{σ}^G is independent of the point since P is a G -torsor. Therefore, this map descends to $T_{\sigma}^G \rightarrow T_{\tau}$. \square

Proof. We define our groups over k with an isomorphism $\sigma : k \hookrightarrow \mathbb{C}$. Let $\tau : k \hookrightarrow \mathbb{C}$ be any other isomorphism. We may assume that t and t_i belong to the same tensor space T then because the t_i are absolute Hodge cycles, they lie in T_{σ} for each σ . Then there are inclusions of cohomology,

$$\begin{array}{ccc} H_{\sigma}(X_{\alpha}) & & H_{\tau}(X_{\alpha}) \\ & \searrow & \swarrow \\ & H_{\sigma}(X_{\alpha}) \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}}) & \end{array}$$

defined by these isomorphisms. These inclusions follow from the identification of $H_{\sigma}(X_{\alpha}) \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}})$ with the étale cohomology which is independent of the choice of embedding $k \hookrightarrow \mathbb{C}$. These induce maps on the tensors,

$$\begin{array}{ccc}
T_\sigma & & T_\tau \\
& \searrow & \swarrow \\
& T \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}}) &
\end{array}$$

Now, define a functor,

$$P(R) = \{p : H_\sigma \times R \xrightarrow{\sim} H_\tau \otimes R \mid p : p \text{ preserves each absolute Hodge cycles}\}$$

Recall that, by definition, an absolute Hodge cycle corresponds to another absolute Hodge cycle for each embedding $k \hookrightarrow \mathbb{C}$ so the condition above make sense, p should identify $t_i \in T_\sigma$ with its corresponding absolute Hodge cycle in T_τ .

The inclusions demonstrate that $P(\mathbb{C} \times \mathbb{A}_{\mathbb{Q}, \text{fin}})$ is nonempty and since $H_\sigma \otimes R$ and $H_\tau \otimes \mathbb{R}$ are G -representations we get a G -action on $P(R)$. Since G is the group fixing exactly the absolute Hodge cycles, we can see that P is a G -torsor.

If we apply the previous lemma we obtain a map $T_\sigma^G \rightarrow T_\tau$ making the following diagram commute,

$$\begin{array}{ccc}
T_\sigma^G & \longrightarrow & T_\tau \\
\downarrow & & \downarrow \\
T_\sigma & \hookrightarrow & T \otimes (\mathbb{C} \times \mathbb{A}_{\text{fin}})
\end{array}$$

Therefore, the map $T_\sigma^G \rightarrow T_\tau$ is injective we must have $t \in T_\tau$ since it lies in T_σ^G by hypothesis. Thus t is rational relative to all σ .

It remains to show that the component t_{dR} of $T \otimes \mathbb{C} = T_{\text{dR}}$ lies in the filtration $F^0 T_{\text{dR}}$. For a rational $s \in T_{\text{dR}}$,

$$s \in F^0 T_{\text{dR}} \iff s \text{ is fixed by } \mu(\mathbb{C}^\times)$$

where $\mu(\mathbb{C}^\times)$ corresponds to the real action defining the Mumford-Tate group. Since, by hypothesis, $(t_i)_{\text{dR}} \in F^0$ we know that $G \supset \mu(\mathbb{C}^\times)$ since $\mu(\mathbb{C}^\times)$ must fix all of them. Clearly then if t is fixed by G we must have t fixed by $\mu(\mathbb{C}^\times)$ and thus $t_{\text{dR}} \in F^0 T_{\text{dR}}$. \square

7 Construction of Some Absolute Hodge Cycles

7.1 Hermitian Forms

Remark. Recall that a number field E is a CM-field if for each embedding $E \hookrightarrow \mathbb{C}$ complex conjugation induces a nontrivial automorphism on E independently on the embedding. The fixed field is then a totally real field F and E/F has degree 2.

Definition If E is a CM-field and V is a K -vectorspace then a sesquilinear form $\phi : V \times V \rightarrow \mathcal{E}$ is Hermitian if $\phi(v, w) = \overline{\phi(w, v)}$.

Remark. For any embedding $\tau : F \hookrightarrow \mathbb{R}$ we obtain a Hermitian form ϕ_τ on $V_\tau = V \otimes_\tau \mathbb{R}$. Let a_τ and b_τ be the dimensions of the maximal subspaces of V_τ on which ϕ_τ is positive definite and negative definite respectively.

Furthermore, ϕ defines a Hermitian form on the top forms $\Lambda^{\dim V} V \cong E$ which must be an E -Hermitian form on E and thus is given by an element $f \in F$ defined up to $\text{Nm}_{E/F} E^\times$. We call this the discriminant.

Remark. Let (v_1, \dots, v_d) be an orthogonal basis for ϕ and $\phi(v_i, v_i) = c_i$. Then a_τ is the number of i s.t. $\tau c_i > 0$ and b_τ is the number of i s.t. $\tau c_i < 0$ and $f = c_1 \cdots c_n$. If ϕ is nondegenerate, then $f \in F^\times / \text{Nm}_{E/F} E^\times$ and,

$$a_\tau + b_\tau = \dim V \quad \text{sign}(\tau f) = (-1)^{b_\tau}$$

Proposition 7.1. Given, for each embedding $\tau : F \hookrightarrow \mathbb{C}$, a tripple (a_τ, b_τ) and $f \in F^\times / \text{Nm}_{E/F} E^\times$ satisfying the above. Then there exists a unique pair (V, ϕ) a non-degenerate Hermitian form ϕ on an E -vectorspace V with invariants (a_τ, b_τ) with respect to $\tau : F \hookrightarrow \mathbb{R}$ and f .

Definition A Hermitian space (V, ϕ) of dimension d is *split* if it satisfies the equivalent conditions,

- (a). $a_\tau = b_\tau$ for all τ and $f = (-1)^{d/2}$
- (b). there is a totally isotropic subspace of V of dimnsion $d/2$ (for each $v \in W : \phi(v, v) = 0$).

Lemma 7.2. Let k be a field, k' an etale k -algebra (a finite product of finite separable extensions of k) and V a f.g. free k' -module. Then,

- (a). The map,

$$f \mapsto \text{Tr}_{k'/k} \circ f : \text{Hom}_{k'}(V, k') \rightarrow \text{Hom}_k(V, k)$$

is an isomorphism of k -vectorspaces.

- (b). $\bigwedge_{k'}^n V$ is a direct summand of $\bigwedge_k^n V$ naturally.

Proof. The trace map $\text{Tr}_{k'/k} : k' \times k' \rightarrow k$ is nondegenerate (HOW IS THIS A PAIRING). The map $f \mapsto \text{Tr}_{k'/k} \circ f$ is injective and then onto because the spaces are of the same dimension.

There are obvious maps,

$$\begin{aligned} \bigwedge_k^n V &\rightarrow \bigwedge_{k'}^n V \\ \bigwedge_k^n V^\vee &\rightarrow \bigwedge_{k'}^n V^\vee \end{aligned}$$

where here we deine the dual of k' -modules as,

$$V^\vee = \text{Hom}_{k'}(V, k') = \text{Hom}_k(V, k)$$

(WHAT?) □

7.2 Conditions to Consist of Absolute Hodge Cycles

Remark. In this section we will be in the following situation.

Definition Let A be an abelian variety over \mathbb{C} and E a CM field with a homomorphism $\nu : E \rightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $d = \dim_E H_1(A, \mathbb{Q})$ which has an E -vectorspace structure via ν . Thus, $2 \dim A = d[E : \mathbb{Q}]$.

Proposition 7.3. The analytic space A^{an} is a compact complex Lie group which is a complex torus. Let \mathfrak{g} be the lie Algebra then there is an \mathbb{R} -linear map $\mathfrak{g} \rightarrow H_1(A^{\text{an}}, \mathbb{R})$ sending a tangent vector to the homology class defined by its geodesic (ASK HARRIS ABOUT THIS). Now \mathfrak{g} is a complex vectorspace so $H_1(A^{\text{an}}, \mathbb{R})$ inherents a complex structure given by an \mathbb{R} -linear endomorphism $J : H_1(A^{\text{an}}, \mathbb{R}) \rightarrow H_1(A^{\text{an}}, \mathbb{R})$.

Proposition 7.4. Hoge theory gives a hodge structure on $H^1(A^{\text{an}}, \mathbb{R})$ which is determined by a map $h : \mathbb{S} \rightarrow \text{GL}(H^1(A, \mathbb{R}))$.

Now, on a complex torus of $\dim_{\mathbb{R}}(A^{\text{an}}) = 2g$ there are isomorphisms,

$$H^1(A^{\text{an}}, \mathbb{R})^{\vee} \xrightarrow{\sim} \bigwedge^{2g-1} H^1(A^{\text{an}}, \mathbb{R}) \xrightarrow{\sim} H^{2g-1}(A^{\text{an}}, \mathbb{R}) \xrightarrow{\sim} H_1(X, \mathbb{R})$$

This identification gives an isomorphism,

$$\text{GL}(H^1(A^{\text{an}}, \mathbb{R})) \cong \text{GL}(H_1(A, \mathbb{R}))$$

under which $h(i) \mapsto J$.

Proposition 7.5. Consider the decomposition,

$$\begin{aligned} E \otimes_{\mathbb{Q}} \mathbb{C} &\xrightarrow{\sim} \prod_{\sigma \in \text{Hom}(E, \mathbb{C})} \mathbb{C} \\ e \otimes z &\mapsto (\sigma \mapsto \sigma(e) \cdot z) \end{aligned}$$

Tensoring by $H_B^1(A) = H^1(A^{\text{an}}, \mathbb{Q})$ we find,

$$H_B^1(A) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \text{Hom}(E, \mathbb{C})} H_B^1(A) \otimes_{\sigma} \mathbb{C}$$

where,

$$H_B^1(A) = H^1(A^{\text{an}}, \mathbb{Q})$$

is an E -vectorspace and $e \in E$ acts on $H_B^1(A) \otimes_{\sigma} \mathbb{C}$ via $\sigma(e)$. Since E respects the Hodge structure on $H_B^1(A)$ each $H_{E,\sigma}^1(A) = H^1(A^{\text{an}}, \mathbb{Q}) \otimes_{\sigma} \mathbb{C}$ acquires a Hodge structure,

$$H_{E,\sigma}^1(A) = H_{E,\sigma}^{1,0}(A) \oplus H_{E,\sigma}^{0,1}(A)$$

Define,

$$a_{\sigma} = \dim_{\mathbb{C}} H_{E,\sigma}^{1,0}(A) \quad \text{and} \quad b_{\sigma} = \dim_{\mathbb{C}} H_{E,\sigma}^{0,1}(A) \quad \text{thus} \quad a_{\sigma} + b_{\sigma} = d$$

Proposition 7.6. The subspace,

$$\bigwedge_E^d H_B^1(A) \subset H^d(A^{\text{an}}, \mathbb{Q})$$

has pure bidegree $(\frac{d}{2}, \frac{d}{2})$ iff $a_{\sigma} = b_{\sigma}$ for each $\sigma \in \text{Hom}(E, \mathbb{C})$.

Proof. For a complex torus, we have,

$$H^d(A^{\text{an}}, \mathbb{Q}) \cong \bigwedge_{\mathbb{Q}}^d H^1(A^{\text{an}}, \mathbb{Q})$$

so a previous lemma identifies,

$$\bigwedge_E^d H^1(A^{\text{an}}, \mathbb{Q}) \subset \bigwedge_{\mathbb{Q}}^d H^1(A^{\text{an}}, \mathbb{Q})$$

as a direct summand. Then consider,

$$\begin{aligned}
\left(\bigwedge_E^d H_B^1(A)\right) \otimes_{\mathbb{Q}} \mathbb{C} &\cong \bigwedge_{E \otimes_{\mathbb{Q}} \mathbb{C}}^d (H_B^1(A) \otimes_{\mathbb{Q}} \mathbb{C}) \\
&\cong \bigoplus_{\sigma \in \text{Hom}(E, \mathbb{C})} \bigwedge_{\mathbb{C}}^d (H^1(A^{\text{an}}, \mathbb{Q}) \otimes_{\sigma} \mathbb{C}) \\
&\cong \bigoplus_{\sigma \in \text{Hom}(E, \mathbb{C})} \bigwedge_{\mathbb{C}}^d (H_{E, \sigma}^{1,0}(A) \oplus H_{E, \sigma}^{0,1}(A)) \\
&\cong \bigoplus_{\sigma \in \text{Hom}(E, \mathbb{C})} \bigwedge_{\mathbb{C}}^{a_{\sigma}} H_{E, \sigma}^{1,0}(A) \oplus \bigwedge_{\mathbb{C}}^{b_{\sigma}} H_{E, \sigma}^{0,1}(A)
\end{aligned}$$

Thus, we have decomposed this subspace into a sum of pure bidegree $(a_{\sigma}, 0)$ and $(0, b_{\sigma})$ proving the proposition. \square

Remark. In the case $a_{\sigma} = b_{\sigma}$ then,

$$\left(\bigwedge_E^d H_B^1(A)\right) \left(\frac{d}{2}\right)$$

(ASK HARRIS WHY TATE TWIST HERE?) consists of Hodge cycles. We want to know when this consists of absolute Hodge cycles.

Lemma 7.7. If $A = A_0 \otimes_{\mathbb{Q}} E$ for some abelian variety A_0 of dimension $\frac{d}{2}$ then,

$$\left(\bigwedge_E^d H_B^1(A)\right) \left(\frac{d}{2}\right) \subset H^d(A^{\text{an}}, \mathbb{Q}) \left(\frac{d}{2}\right)$$

consists of absolute Hodge cycles.

Proof. Consier the diagram,

$$\begin{array}{ccc}
H_B^d(A_0) \left(\frac{d}{2}\right) \otimes_{\mathbb{Q}} E & \longrightarrow & H_B^d(A_0) \left(\frac{d}{2}\right) \otimes_{\mathbb{Q}} E \\
\downarrow \sim & & \downarrow \sim \\
\left(\bigwedge_E^d H_B^1(A_0 \otimes_{\mathbb{Q}} E)\right) \left(\frac{d}{2}\right) & \longrightarrow & \left(\bigwedge_{E \otimes_{\mathbb{Q}} \mathbb{A}}^d H_{\mathbb{A}}^1(a_0 \otimes_{\mathbb{Q}} E)\right) \left(\frac{d}{2}\right) \hookrightarrow H_{\mathbb{A}}^d(A_0 \otimes E) \left(\frac{d}{2}\right)
\end{array}$$

The vertical maps are induced by the isomorphism $H_B^d(A_0) \otimes_{\mathbb{Q}} E \xrightarrow{\sim} H_B^d(A_0 \otimes_{\mathbb{Q}} E)$. There is a similar diagram for each embedding $\sigma : E \hookrightarrow \mathbb{C}$ and thus the image of the bottom map must be stable with respect to a choice of $\sigma : E \hookrightarrow \mathbb{C}$. Therefore, the Hodge cycles,

$$\left(\bigwedge_E^d H_B^1(A_0 \otimes_{\mathbb{Q}} E)\right) \left(\frac{d}{2}\right) \subset H_B^d(A_0 \otimes_{\mathbb{Q}} E) \left(\frac{d}{2}\right)$$

are indeed absolutly Hodge. (ASK HARRIS ABOUT THIS PROOF)? I don't understand it. \square

7.3 Riemann Forms

Definition A Hermitian form H on a complex vectorspace V is a complex bilinear form $H : \bar{V} \times V \rightarrow \mathbb{C}$ (sesquilinear on H) which satisfies,

$$H(u, v) = \overline{H(v, u)}$$

Lemma 7.8. Let V be a complex vectorspace. There is a one-to-one correspondence between Hermitian forms H on V and real-valued skew-symmetric forms E on V .

Proof. The correspondence is given by,

$$\begin{aligned} H &\mapsto E_H & E_H(u, v) &= \operatorname{Im}(H(u, v)) \\ E &\mapsto H_E & H_E(u, v) &= E(iu, v) + iE(u, v) \end{aligned}$$

□

Definition A Riemann form $E : V \times V \rightarrow \mathbb{R}$ on a complex vectorspace V is an antisymmetric \mathbb{R} -bilinear form such that,

- (a). $E(iu, iv) = E(u, v)$
- (b). the corresponding Hermitian form H_E is positive definite.

Definition A complex torus $X = V/\Lambda$ is *polarizable* if there exists an antisymmetric form $E : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ such that $E_{\mathbb{R}} : V \times V \rightarrow \mathbb{R}$ (using that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$) is a Riemann form.

(IS THIS EQUIVALENT TO THE POLARIZATION OF THE HODGE STRUCTURE $H_1(X, \mathbb{Q})$)

Theorem 7.9. A complex torus $X = V/\Lambda$ is of the form A^{an} for some abelian variety A iff X is polarizable.

(DOES THIS IMPLY THAT ALL ABELIAN VARIETIES ARE POLARIZABLE IN THE FOLLOWING SENSE)

Definition A polarization of an abelian variety A is an isogeny $\lambda : A \rightarrow A^{\vee}$ such that

Remark. We can identify, $A^{\vee} = \operatorname{Pic}^0(A)$.

Proposition 7.10. For each line bundle \mathcal{L} on A/k there is an associated morphism $\phi_{\mathcal{L}} : A \rightarrow A^{\vee}$ which is an isogeny if \mathcal{L} is ample.

Proof. We define a map $\phi_{\mathcal{L}} : A(\bar{k}) \rightarrow \operatorname{Pic}(A)$ via $\phi_{\mathcal{L}}(x) = t_x^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}$. First, via the Theorem of the Square, for $x, y \in A(\bar{k})$,

$$t_x^* \mathcal{L} \otimes_{\mathcal{O}_A} t_y^* \mathcal{L} = t_{x+y}^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}$$

Therefore,

$$(t_x^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}) \otimes_{\mathcal{O}_A} (t_y^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}) = t_{x+y}^* \mathcal{L} \otimes_{\mathcal{O}_A} \mathcal{L}^{-1}$$

so ϕ is a group homomorphism. Furthermore, $\deg t_x^* \mathcal{L} = \deg \mathcal{L}$ since the map $t_x : A \rightarrow A$ is an isomorphism. (IS THIS TRUE?) Therefore, $\deg \phi_{\mathcal{L}}(x) = 0$ so the image is contained in $\operatorname{Pic}^0(A) = A^{\vee}(\bar{k})$. □

Definition A polarization of A is an isogeny $\phi : A \rightarrow A^{\vee}$ such that $\phi_{\bar{k}} : A_{\bar{k}} \rightarrow A_{\bar{k}}^{\vee}$ is of the form $\phi_{\mathcal{L}}$ for some ample line bundle \mathcal{L} on $A_{\bar{k}}$. Deriving from a line bundle gives symmetry $\phi = \phi^{\vee}$ and ampleness is a positivity condition.

Definition Let A be an abelian variety with a polarization $\phi : A \rightarrow A^\vee$. Since ϕ is an isogeny, it has an “inverse element” in the algebra $\phi^{-1} \in \text{Hom}(A^\vee, A) \otimes \mathbb{Q}$. (This follows from inverting the multiplication by n maps). Then we define the Rosati involution of the endomorphism algebra $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ via,

$$\alpha^\dagger = \phi^{-1} \circ \alpha^\vee \circ \phi \quad \text{for } \alpha \in \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Remark. The Rosati involution depends on the choice of polarization.

Theorem 7.11. A polarization θ on A is determined by a Riemann form ϕ on $H_1(A^{\text{an}}, \mathbb{Q})$. Two forms ϕ, ϕ' determine the same polarization iff $\exists a \in \mathbb{Q}^\times : \phi' = a\phi$. In this case, the Rosati involution is determined by,

$$\forall u, v \in H_1(A^{\text{an}}, \mathbb{Q}) : \phi(\alpha(u), v) = \phi(u, \alpha^\dagger(v)) \quad \alpha \in \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Proof. (HOW DOES ONE PROVE THIS?) □

Theorem 7.12. Let A be an abelian variety over \mathbb{C} and $\nu : E \rightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ the inclusion of a CM-field with $d = \dim_E H^1(A^{\text{an}}, \mathbb{Q})$. Suppose there exists a polarization θ for A such that,

- (a). the Rosati involution of θ induces complex conjugation on $E \subset \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$
- (b). there exists a split E -Hermitian form ϕ on $H_1(A^{\text{an}}, \mathbb{Q})$ and $f \in E^\times$ with $\bar{f} = -f$ such that $\phi(x, y) = \text{Tr}_{E/\mathbb{Q}}(f\phi(x, y))$ is a Riemann form for θ .

Then the subspace,

$$\left(\bigwedge_E^d H_B^1(A) \right) \left(\frac{d}{2} \right) \subset H^d(A^{\text{an}}, \mathbb{Q}) \left(\frac{d}{2} \right)$$

consists of absolute Hodge cycles.

7.4 Shimura Varieties

8 The Proof for Abelian Varieties of CM Type

Definition The Mumford-Tate group $M(A)$ of an abelian variety A is the Mumford Tate group of the rational Hodge structure $H_1(A, \mathbb{Q})$.

Definition An abelian variety is of CM-type if $M(A)$ is abelian.

Remark. Any abelian variety A is isogenous to a product of simple abelian varieties A_α and A is CM-type iff each A_σ is CM-type since the Mumford-Tate group of the product $M(A)$ is contained in the product of $M(A_\alpha)$ and projects fully onto each. Therefore, it will suffice to study simple abelian varieties of CM-type.

Lemma 8.1. Let A be an abelian variety. Then $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the subalgebra of elements in $\text{End}(H_1(A^{\text{an}}, \mathbb{Q}))$ preserving the Hodge structure. Furthermore, preserving the Hodge structure is equivalent to commuting with the image of $\mu : \mathbb{G}_m \rightarrow \text{GL}(H_1(A^{\text{an}}, \mathbb{C}))$.

Proof. (PROVE THIS) □

Proposition 8.2. A simple abelian variety over \mathbb{C} is of CM-type iff $E = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a commutative field over which $H_1(A, \mathbb{Q})$ has dimension 1. In this case, E is a CM-field and the Rosati involution on E for any polarization of A is complex conjugation on $E \subset \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. Let A be an abelian variety with $E \subset \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that,

$$\dim_E H_1(A, \mathbb{Q}) = 1$$

Then $\mu(\mathbb{G}_m)$ commutes with $E \otimes \mathbb{R}$ in $\text{End}(H_1(A^{\text{an}}, \mathbb{R}))$ because the Hodge structure is compatible with the E -vectorspace structure. (WHY THOUGH) The subspace $(E \otimes_{\mathbb{Q}} \mathbb{R}) \subset \text{GL}(H_1(A^{\text{an}}, \mathbb{R}))$ is all diagonal matrices (since $H_1(A^{\text{an}}, \mathbb{R})$ is dimension one over E) and since anything that commutes with all diagonal matrices must itself be diagonal, we have $h(\mathbb{S}) \subset (E \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$ which implies that $M(A) \subset \mathbb{G}_{E^{\times}}$ where $\mathbb{G}_{E^{\times}} \subset \text{GL}(H_1(A^{\text{an}}, \mathbb{Q}))$ is the commutative \mathbb{Q} -algebraic subgroup defined by $\mathbb{G}_{E^{\times}}(F) = (E \otimes_{\mathbb{Q}} F)^{\times}$ and thus whose \mathbb{R} -points are $(E \otimes_{\mathbb{Q}} \mathbb{R})$ containing $h(\mathbb{S})$. Therefore $M(A) \subset \mathbb{G}_{E^{\times}}$ is abelian since $\mathbb{G}_{E^{\times}}$ is a commutative group scheme. (I believe that $\mathbb{G}_{E^{\times}} = \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m^E)$ IS THIS CORRECT?)

Conversely, let A be simple and of CM-type and $\mu : \mathbb{G}_m \rightarrow \text{GL}(H_1(A^{\text{an}}, \mathbb{C}))$ define the Hodge structure on $H_1(A^{\text{an}}, \mathbb{C})$. Since A is simple, $E = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division ring of degree $\leq \dim_{\mathbb{Q}} H_1(A^{\text{an}}, \mathbb{Q})$ over \mathbb{Q} . (COMPLETE THIS PROOF!?) \square

8.1 The Proof For CM Case

Let A_{α} be a finite family of abelian varieties of CM-type. We need to show that every Hodge cycle in,

$$T_{\mathbb{A}} = \left(\bigotimes_{\alpha} H_{\mathbb{A}}^1(X_{\alpha})^{\otimes m_{\alpha}} \right) \otimes \left(\bigotimes_{\alpha} H_{\mathbb{A}}^1(X_{\alpha})^{\vee \otimes n_{\alpha}} \right) (m)$$

is an absolute Hodge cycle. According to Principal A the group G^{AH} fixing all absolute Hodge cycles fixes exactly the absolute Hodge cycles. Thus it suffices to prove that the subgroup $G^H \subset G^{AH}$ fixing all Hodge cycles is equal to G^{AH} .

9 Proof of the Main Theorem

Let A be an abelian variety over \mathbb{C} and t_{α} for $\alpha \in I$ be Hodge cycles on A . We need to show that these are absolute Hodge cycles. Since we know the result in the case that A is CM-type it suffices to prove the following.

Proposition 9.1. There exists a connected smooth algebraic variety S/\mathbb{C} and an abelian scheme $\pi : Y \rightarrow S$ such that,

- (a). for some $s_0 \in S$ the fibre $Y_{s_0} = A$
- (b). for some $s_1 \in S$ the fibre Y_{s_1} is of CM-type
- (c). the cycles t_{α} extend to rational cycles of bidegree $(0, 0)$ on Y . Explicitly, suppose that,

$$t_{\alpha} \in H_B^1(A)^{\otimes m(\alpha)} \otimes H_B^1(A)^{\vee \otimes n(\alpha)}$$

then there is a section t of,

$$(R^1 \pi_* \underline{\mathbb{Q}})^{\otimes m(\alpha)} \otimes (R^1 \pi_* \underline{\mathbb{Q}})^{\otimes n(\alpha)}$$

over a finite cover $\tilde{S} \rightarrow S$ such that for some \bar{s}_0 over s_0 we have $t_{\bar{s}_0} = t_\alpha$ and for all $\tilde{s} \in \tilde{S}$ we have,

$$t_{\tilde{s}} \in H_B^1(Y_{\tilde{s}})^{\otimes m(\alpha)} \otimes H_B^1(Y_{\tilde{s}})^{\vee \otimes n(\alpha)}$$

is a Hodge cycle.

Proof. S will be a Shimura Variety. Extend the set of AH cycles such that some t_α is a polarization of A and let $H = H_1(A, \mathbb{Q})$. Now we consider $G \subset \mathrm{GL}_H(\times) \mathbb{G}_m$ fixing t_α . Since the hodge character must act trivially on t_α then it defines a character $h_0 : \mathbb{C}^\times \rightarrow G(\mathbb{R})$.

Define,

$$X = \{h : \mathbb{C}^\times \rightarrow G(\mathbb{R}) \mid h \text{ is conjugate to } h_0 \in G(\mathbb{R})\}$$

For each $h \in X$ we get a new Hodge structure of H relative to which t_α has bidegree $(0,0)$ since h fixes it. Let $F^0(h) = H^{0,-1} \subset H \otimes \mathbb{C}$ in this new Hodge structure. Sending $h \mapsto F^0(h)$ is a map $X \rightarrow \mathrm{Gr}_k(H \otimes \mathbb{C})$ as real manifolds. The map is injective because the filtration completely determines a hodge structure. Consider the centralizer K_∞ of h_0 . Then,

$$\begin{array}{ccc} T_{h_0}(X) & \xlongequal{\quad} & \mathrm{Lie}(G_{\mathbb{R}})/\mathrm{Lie}(K_\infty) \hookrightarrow \mathrm{End}(H \otimes \mathbb{C})/F^0\mathrm{End}(H \otimes \mathbb{C}) \xlongequal{\quad} T_{\phi(h_0)}\mathrm{Gr}_k(H \otimes \mathbb{C}) \\ & & \parallel \nearrow \\ & & \mathrm{Lie}(G_{\mathbb{C}})/F^0\mathrm{Lie}(G_{\mathbb{C}}) \end{array}$$

where the Filtration on $\mathrm{End}(H \otimes \mathbb{C})$ is given by the Hodge structure h_0 on H . Then, X is a complex manifold.

To each $h \in X$ we attach a complex torus given by the double cosets $F^0(h) \setminus H \otimes \mathbb{C}/H(\mathbb{Z})$ where $H(\mathbb{Z})$ is a fixed lattice inside H . In particular, at h_0 we get,

$$F^0(h_0) \setminus H \otimes \mathbb{C}/H(\mathbb{Z}) = T_0(A)/H(\mathbb{Z})$$

These tori form a family $B \rightarrow X$. Then define the group,

$$\Gamma_n = \{g \in G(\mathbb{Q}) \mid (g - q)H(\mathbb{Z}) \subset nH(\mathbb{Z})\}$$

for some. For sufficiently large n Baily and Borel show that $S = X/\Gamma$ is an algebraic variety, in particular a Shimura variety. \square

10 Ideal for Next Semester

That paper on Slopes of powers of Frobenius on crystalline cohomology.

Course on crystalline cohomology.

Course on Shimura varieties.

Study supersingular curves or K3 surfaces.