# Mathematics GU4044 Representations of Finite Groups
## Assignment # 4

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# Problem 1.

(a). Let $V_1$ and $V_2$ be $G$-representations. Consider the vectorspace $(V_1 \oplus V_2)^G$ and take any $(v_1, v_2) \in (V_1 \oplus V_2)^G$. Then,

$$(\rho_1 \oplus \rho_2)(v_1, v_2) = (\rho_1(v_1), \rho_2(v_2)) = (v_1, v_2) \iff \rho_1(v_1) = v_1 \text{ and } \rho_2(v_2) = v_2$$

Thus, $(v_1, v_2) \in (V_1 \oplus V_2)^G \iff (v_1, v_2) \in V_1^G \oplus V_2^G$. Therefore, $(V_1 \oplus V_2)^G = V_1^G \oplus V_2^G$. In particular, since $(\mathrm{Hom}\,(V, W))^G = \mathrm{Hom}^G\,(V, W)$ and $\mathrm{Hom}\,(W, V_1 \oplus V_2) \cong \mathrm{Hom}\,(W, V_1) \oplus \mathrm{Hom}\,(W, V_2)$, we have the relationship,

$$\mathrm{Hom}^G\,(W, V_1 \oplus V_2) \cong \mathrm{Hom}^G\,(W, V_1) \oplus \mathrm{Hom}^G\,(W, V_2)$$

(b). Let $W$ be an irreducible $G$-representation of a finite group $G$ and let $V$ be another $G$-representation which is totally reducible. Thus, we can write, $V \cong V_1 \oplus \cdots V_l$ where each $V_i$ is an irreducible $G$-representation. Applying the lemma above inductively,

$$\mathrm{Hom}^G\,(W, V) \cong \mathrm{Hom}^G\,(W, V_1) \oplus \cdots \oplus \mathrm{Hom}^G\,(W, V_k)$$

and therefore,

$$\dim \mathrm{Hom}^G\,(W, V) = \dim \mathrm{Hom}^G\,(W, V_1) + \cdots + \dim \mathrm{Hom}^G\,(W, V_k)$$

However, by Schur's lemma, since $W$ and $V_i$ are both irreducible $G$-representations, $\dim \mathrm{Hom}^G\,(W, V_i) = 1$ if $W \cong V_i$ and zero otherwise. Therefore,

$$\dim \mathrm{Hom}^G\,(W, V) = \sum_{i=1}^{k} \mathbf{1}(W \cong V_i) = \#\{V_i \text{ isomorphic to } W\}$$

# Problem 2.

(i) Let $G = D_3$ with a two-dimensional representation given by,

$$\rho(r) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus, $\chi_2(e) = 2$ and $\chi_2(r) = -1$ and $\chi_2(s) = 0$. The group $D_3$ has three conjugacy classes, $[e] = \{e\}, [r] = \{r, r^{-1}\}, [s] = \{s, rs, r^2 s\}$. Since the character $\chi_2$ is a class function,

$$\frac{1}{6} \sum_{g \in D_3} |\chi_2(g)|^2 = \frac{1}{6} \left( |\chi_2(e)|^2 + 2|\chi_2(r)|^2 + 3|\chi_2(s)|^2 \right) = \frac{1}{6} \left( 2^2 + 2 \right) = 1$$

(ii) Now, consider the standard representation of $S_3$ on $\mathbb{C}^3$. The group $S_3$ is similarly generated by $\sigma = (1\,2\,3)$ and $\tau = (1\,2)$. These are represented by the matrices.

$$\rho_{st}(\sigma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \rho_{st}(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $\chi_{st}$ is a class function, we need only to compute it on representatives of the three conjugacy classes, $[e] = \{e\}$, $[\sigma] = \{\sigma, \sigma^{-1}\}$, $[\tau] = \{\tau, \sigma\tau, \sigma^2\tau\}$. Now, $\chi_{st}(e) = \dim \mathbb{C}^3 = 3 = \chi_2(e) + 1$. Likewise, $\chi_{st}(\sigma) = 0 = \chi_2(r) + 1$ and $\chi_{st}(\tau) = 1 = \chi_2(s) + 1$. Therefore, because they are both class functions which agree on each conjugacy class, $\chi_{st} = \chi_2 + 1$.

# Problem 3.

Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group with $\rho : Q \to \mathrm{GL}(2, \mathbb{C})$ given by,

$$\rho(\pm 1) = \pm \mathrm{id} \quad \rho(\pm i) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \rho(\pm j) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \rho(\pm k) = \pm \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Therefore, $\chi(\pm 1) = \pm 2$ and $\chi(\pm i) = \chi(\pm j) = \chi(\pm k) = 0$. Then,

$$\frac{1}{8} \sum_{g \in Q} \chi(g)\overline{\chi(g)} = \frac{1}{8}\left(|\chi(1)|^2 + |\chi(2)|^2 + 0\right) = \frac{1}{8}\left(2^2 + 2^2\right) = 1$$

# Problem 4.

(i) Let $V_1$ and $V_2$ be two one-dimensional represnetations of $G$ correspondsing to homomorphisms $\lambda_1, \lambda_2 : G \to \mathbb{C}^\times$. Clearly, if $\lambda_1 = \lambda_2$ then $V_1 \cong V_2$ because they correspond to equal representations. Conversely, if $V_1 \cong V_2$ then there exists a $G$-isomorphism $F : V_1 \to V_2$ such that, $F \circ \lambda_1(g) = \lambda_2(g) \circ F$ for all $g \in G$. Then, because $F$ is linear,

$$F \circ \lambda_1(g)(v) = F(\lambda_1(g)v) = \lambda_1(g)F(v) = \lambda_2(g) \circ F(v)$$

for all $g \in G$. Since $F$ is an isomorphism and the representations are nontrivial, there must be a nonzero vector in the image of $F$. Therefore, $\lambda_1(g) = \lambda_2(g)$ for all $g \in G$.

(ii) Let $V$ be a one-dimensional $G$-representation with corresponding homomorphism $\lambda : G \to \mathbb{C}^\times$. Consider the dual representation $(V^*, \rho_{V^*})$ such that $\rho_{V^*}(g) \cdot f = f \circ \rho_V(g)^{-1} = f \circ \rho_V(g^{-1})$. Consider a linear functional $f \in V^*$ and a vector $v \in V$ then, $\rho_{V^*}(g) \cdot f(v) = f \circ \rho_V(g^{-1})(v) = f(\lambda(g^{-1}v) = \lambda(g^{-1})f(v)$. Therefore, $\rho_{V^*}$ acts on $V^*$ by multiplication by $\lambda(g^{-1})$. Thus, the $G$-representation $\rho_{V^*}$ corresponds to the homomorphism $\lambda^{-1} : G \to \mathbb{C}^\times$ where $\lambda^{-1}(g) = \lambda(g^{-1})$.

(iii) Let $V_1$ denote the one-dimensional representatiob of $\mathbb{Z}/n\mathbb{Z}$ corresponding to the homomorphism $\lambda_1(k) = e^{2\pi i k/n}$. Suppose that $n > 2$ then, the corresponding representation on the dual space $(V_1)^*$ is given by the homomorphism $\lambda_1^{-1}(k) = e^{2\pi i(-k)/n} = e^{-2\pi i k/n}$. For $n > 2$ we have that $\lambda_1^{-1}(1) = e^{-2\pi/n} \neq e^{2\pi i/n}$ else $4\pi/n \in 2\pi\mathbb{Z}$ which it cannot be for $n > 2$. Therefore, by part (i), we have that $V_1 \not\cong (V_1)^*$ because the corresponding complex homomorphisms are not equal. For the case $n = 2$, inversion is the identity automorphism on $\mathbb{Z}/2\mathbb{Z}$ so $\lambda_1 = \lambda_1^{-1}$ and thus $V_1 \cong (V_1)^*$ because they correspond to equal homomorphisms into $\mathbb{C}^\times$.

# Problem 5.

Let $V = \mathbb{C}^n$ be the standard representation of $S_n$ with a homomorphism $\rho_{st} : S_n \to \mathrm{Aut}(\mathbb{C}^n)$. We have $S_n$-invariant projection maps, $p : V \to \mathbb{C}$ and $p' : V \to V^G = \mathrm{span}\{e_1 + \cdots + e_n\}$ given by,

$$p(t_1, \cdots, t_n) = \frac{1}{n}\sum_{i=1}^{n} t_i \quad \text{and} \quad p'(v) = \frac{1}{\#(S_n)} \sum_{\sigma \in S_n} \rho_{st}(v) = \frac{1}{n!} \sum_{\sigma \in S_n} \rho_{st}(v)$$

These maps are, in fact, equal under the identification $\mathbb{C} \cong \mathrm{span}\{e_1 + \cdots + e_n\}$. This holds because there are exactly $(n-1)!$ permutations in $S_n$ taking $e_i$ to $e_j$ for any $1 \leq i, j \leq n$. Now, write $v = t_1 e_1 + \cdots + t_n e_n$ and consider,

$$\begin{aligned} p'(v) &= \frac{1}{n!} \sum_{\sigma \in S_n} \rho_{st}(t_1 e_1 + \cdots + t_n e_n) = \frac{1}{n!} \sum_{i=1}^{n} \sum_{\sigma \in S_n} t_i \, \rho_{st}(e_i) \\ &= \frac{1}{n!} \sum_{i=1}^{n} t_i \sum_{j=1}^{n} (n-1)! \, e_j = \frac{1}{n} \sum_{i=1}^{n} t_i \, (e_1 + \cdots + e_n) = p(t_1, \cdots, t_n)(e_1 + \cdots + e_n) \end{aligned}$$

# Problem 6.

Let $V$ be a representation of a finite group $G$. Let $\mathbb{C}[G] \cdot v = \mathrm{span}\{\rho(g) \cdot v \mid g \in G\}$

(a). Take $w \in \mathbb{C}[G] \cdot v$ then $w = \sum\limits_{g \in G} t_g \rho(g) v$ with coeficients $t_g \in \mathbb{C}$. Then, for any $h \in G$ consider,

$$\rho(h) w = \sum_{g \in G} \rho(h)(t_g \, \rho(g) v) = \sum_{g \in G} t_g \, \rho(hg) v) = \sum_{g' \in G} t_{h^{-1} g'} \, \rho(g') v) \in \mathbb{C}[G] \cdot v$$

Therefore $\mathbb{C}[G] \cdot v$ is a $G$-invariant subspace of $V$.

(b). Let $V$ be irreducible and take $v \neq 0$. Then, $v \in \mathbb{C}[G] \cdot v$ so $\mathbb{C}[G] \cdot v$ is a nonempty $G$-invariant subspace of $V$. However, since $V$ is irreducible, there is exactly one such subspace, namely $\mathbb{C}[G] \cdot v = V$.

(c). Since $\mathbb{C}[G] \cdot v = \mathrm{span}\{\rho(g) \cdot v \mid g \in G\} = V$ the set $\{\rho(g) \cdot v \mid g \in G\} = G \cdot v$ spans $V$ and therefore must be at least a large as the dimension of the space. Thus, $\#(G \cdot v) \geq \dim V$.

(d). Suppose that $V$ is irreducible and $H \leq G$ is an abelian subgroup. Since $H$ is abelian, the restricted representation has a common eigenvector $v$ for all $h \in H$. Suppose that $g_1, g_2 \in G$ lie in the same coset $G/H$ then $g_2 - 1 g_1 = h$ so $\rho_{g_2^{-1} g_1} v = \rho_h v = \lambda(h) v$ and thus $\rho_{g_1} v = \lambda(h) \rho_{g_2} v$. Thus, $\mathrm{span}\{\rho_{g_1} v\} = \mathrm{span}\{\rho_{g_2} v\}$. Thus, each coset produces a one-dimensional span. Since for purposes of calculating spans, each coset can be replaced by a single element, $\mathbb{C}[G] \cdot v = \mathrm{span}\{\rho(g) v \mid g \in G\} = \mathrm{span}\{\rho(h) v \mid hH \in G/H\}$ and thus, $\dim \mathbb{C}[G] \cdot v \leq \#(G/H)$. However, $\mathbb{C}[G] \cdot v = V$ so $\dim V \leq \#(G/H)$.