*Remark.* Unless otherwise stated, all rings are commutative and unital.

# 1 Definitions

**Definition 1.0.1.** An element $p \in A$ is prime if $(p)$ is a prime ideal. Equivalently $p$ is prime if whenever $p \mid xy$ either $p \mid x$ or $p \mid y$.

**Definition 1.0.2.** An element $r \in A$ which is nonzero and not a unit is irreducible if whenever $r = xy$ either $x \in A^\times$ or $y \in A^\times$.

# 2 Domains

**Definition 2.0.1.** A ring $A$ is a domain if $A$ has no zero divisors i.e. if $ab = 0$ then $a = 0$ or $b = 0$.

**Proposition 2.0.2.** Let $A$ be a domain then any nonzero prime element is irreducible.

*Proof.* Let $p \in A$ be a prime. Now suppose that $p = xy$ for $x, y \in A$. Thus, $p \mid xy$ so (WLOG) we have $p \mid x$ so $x = pz$ and thus $p = pzy$. However, $p$ is nonzero and $A$ is a domain so $zy = 1$ and thus $y \in A^\times$ proving that $p$ is irreducible. $\square$

# 3 Principal Ideal Domains

**Definition 3.0.1.** A principal ideal domain (PID) is a domain $A$ such that every ideal is principal.

**Lemma 3.0.2.** If $A$ is a PID then $A$ is Noetherian.

*Proof.* Every ideal is principal and thus finitely generated. $\square$

**Lemma 3.0.3.** Let $A$ be a PID and $r \in A$ irreducible then $(r)$ is maximal and thus $r$ is prime.

*Proof.* Consider an intermediate ideal $(r) \subset J \subset A$ then since $A$ is a PID we have $J = (a)$ so $r \in (a)$ and thus $r = ac$ so either $a \in A^\times$ in which case $J = A$ or $c \in A^\times$ in which case $J = (r)$ so $(r)$ is maximal and thus a prime ideal. $\square$

**Theorem 3.0.4.** Let $A$ be a PID and not a field then $\dim A = 1$.

*Proof.* Any prime ideal $\mathfrak{p} \subset A$ is principal so $\mathfrak{p} = (p)$ and $p$ is prime. Either $p = 0$ which is prime since $A$ is a domain or $p$ is irreducible and so we have shown $(p)$ is maximal. So every prime ideal is zero or maximal and thus $\dim A \leq 1$. If $\dim A = 0$ then $(0)$ is maximal so $A$ is local and any nonzero element is thus invertible so $A$ is a field. $\square$

**Theorem 3.0.5** (Kaplansky)**.** Let $A$ be Noetherian then $A$ is a principal ideal ring iff every maximal ideal is prime.

**Theorem 3.0.6** (Cohen)**.** A ring $A$ is Noetherian iff every prime ideal is finitely generated.

**Corollary 3.0.7.** A ring $A$ is a principal ideal ring iff every prime ideal is principal.

# 4 Unique Factorization Domains

**Definition 4.0.1.** A domain $A$ is a unique factorization domain (UFD) if every nonzero element has a unique factorization into irreducible elements.

**Definition 4.0.2.** A factorization ring $A$ is a ring such that every nonzero element has a factorization into irreducible elements.

**Lemma 4.0.3.** If $A$ is a Noetherian domain then it is a factorization domain.

*Proof.* Take $a_0 \in A$. If $a$ is irreducible, zero, or a unit then we are done. Then we can write, $a = a_1^{(1)} a_2^{(1)}$ for $a_1, b_1 \notin A^\times$. Continuing in this manner we get,

$$(a) \subsetneq (a_1^{(1)}, a_2^{(1)}) \subsetneq (a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, a_4^{(3)}) \subsetneq \cdots$$

(CHECK THIS) This sequence is proper since if $a = bc$ and $b \in (a)$ then $a = arc$ so $rc = 1$ and thus $c \in A^\times$ contradicting our construction. However, $A$ is Noetherian then the sequence must terminate so at some point the factorization must become irreducible. $\square$

**Theorem 4.0.4.** Let $A$ be a factorization domain. Then $A$ is a UFD iff every irreducible is prime.

*Proof.* If $A$ is a UFD and $p$ an irreducible. Let $x, y \in A$ and $p \mid xy$ then $p$ is in the factorization of $xy$ and thus, by uniqueness must be in the factorization of either $x$ or $y$ so $p \mid x$ or $p \mid y$.

Conversely, if $A$ is a factorization domain and every irreducible is prime then given two factorizations of $x$ each irreducible must, by primality, divide an irreducible in the other factorization so they are equal. (DO THIS BETTER) $\square$

**Corollary 4.0.5.** If $A$ is a PID then $A$ is a UFD.

*Proof.* If $A$ is a PID then it is Noetherian and thus a factorization domain. Furthermore, every irreducible is prime so $A$ is a UFD. $\square$

## 4.1 Height One Prime Ideals

**Proposition 4.1.1.** Let $A$ be Noetherian. Then any principal prime ideal has height at most one.

*Proof.* Let $\mathfrak{p} = (p) \subset A$ be a principal prime ideal. Then consider the localization which is $A_{(p)}$ Noetherian and the unique maximal ideal $pA_{(p)}$ is principal. Take $N = \text{nilrad}\left(A_{(p)}\right)$ then,

$$\dim A_{(p)}/N = \dim A_{(p)} = \mathbf{ht}\,(\mathfrak{p})$$

but $A_{(p)}/N$ is a Noetherian domain and the unique maximal ideal $pA_{(p)}$ is principal so $A_{(p)}/N$ is a PID and thus $\dim A_{(p)}/N \leq 1$. $\square$

**Proposition 4.1.2.** If $A$ is a UFD then every prime ideal of height one is principal.

*Proof.* Let $\mathfrak{p} \subset A$ be a prime ideal with $\mathbf{ht}\,(\mathfrak{p}) = 1$. Take any nonzero element $x \in \mathfrak{p}$ and consider its factorization into irreducibles. Since $\mathfrak{p}$ is prime some irreducible factor $p \mid x$ must be in $\mathfrak{p}$ so $(p) \subset \mathfrak{p}$. Since $A$ is a UFD all irreducibles are prime so $(p) \subset \mathfrak{p}$ is prime. However $\mathbf{ht}\,(\mathfrak{p}) = 1$ and $(p) \neq (0)$ so $(p) = \mathfrak{p}$ and thus $\mathfrak{p}$ is principal. $\square$

**Theorem 4.1.3.** Let $A$ be a Noetherian domain. Then $A$ is a UFD iff every height one prime ideal is principal.

*Proof.* We showed one direction above. Conversely, suppose every height one prime ideal is principal. Since $A$ is a Noetherian domain, it suffices to show that each irreducible is prime. Let $r$ be irreducible and consider a minimal prime $\mathfrak{p} \supset (r)$. Then by Krull's Hauptidealsatz, $\mathfrak{p}$ has height one so by our assumption $\mathfrak{p} = (p)$ is principal. However, $(r) \subset (p)$ so $p \mid r$ but $r$ is irreducible so we must have $(r) = (p) = \mathfrak{p}$ and thus $r$ is prime. $\square$

**Theorem 4.1.4** (Krull's Hauptidealsatz)**.** Let $I \subset A$ be an ideal in a Noetherian ring $A$ with $n$ generators then any minimal prime ideal $\mathfrak{p} \supset I$ has height at most $n$.

# 5 Simple Modules

**Definition 5.0.1.** A nonzero $R$-module is *simple* if it has no nontrivial submodules.

**Proposition 5.0.2.** Let $R$ be a ring and $M$ an $R$-module. Then the following are equivalent,

(a) $M$ is simple

(b) $\ell_R(M) = 1$

(c) $M = R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.

*Proof.* The first two are equivalent by definition. Clearly if $\mathfrak{m} \subset R$ is maximal then $R/\mathfrak{m}$ is simple. Now suppose that $M$ is simple and take a nonzero $x \in M$. Then $(x) = M$ by simplicity so consider $I = \ker(R \xrightarrow{x} M) = \mathrm{Ann}_A(x) = \{r \in R \mid rx = 0\}$. Since $M = Rx$ we know that $M \cong R/I$. However, by the lattice isomorphism theorem, submodules of $R/I$ correspond to ideals above $I$ so since $M$ is simple we must have $I$ maximal. $\square$

# 6 Artinian Modules

**Definition 6.0.1.** An $R$-module $M$ is *noetherian/artinian* if it satisfies the ascending/descending chain condition on submodules.

**Theorem 6.0.2.** An $R$-module $M$ has finite length iff it is both noetherian and artinian.

*Proof.* If $M$ has finite length then clearly it is noetherian and artinian since chains of submodules are bounded in length. Alternatively, simple modules are noetherian and artinian so given a composition series we see that $M$ is noetherian and artinian by repeated extension. Now, conversely, assume that $M$ is noetherian and artinian. By the artinian property we can take a minimal nonzero submodule $M_1 \subset M$. Then $M_1$ is simple. Either $M/M_1$ is simple or we may repeat to get $M_2 \supset M_1$ and $M_2/M_1$ is simple. Thus we get an ascending chain $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$ with $M_{i+1}/M_i$ simple. Since $M$ is Noetherian, this must terminate at $M_n = M$ so we get a finite length composition series showing that $M$ has finite length. $\square$

# 7 Artinian Rings

**Definition 7.0.1.** A ring $A$ is *artinian* if it satisfies the descending chain condition on ideals: given a chain of ideals,

$$I_0 \supset I_1 \supset I_2 \supset \cdots$$

the chain stabilizes $I_{n+i} = I_n$.

*Remark.* $A$ is artinian iff it is artinian as a module over itself.

**Proposition 7.0.2.** An artinian ring has finitely many maximal ideals.

*Proof.* Let $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots$ be a list of maximal ideals. Then consider the chain,

$$\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supset \cdots$$

By the artinian condition, we must have $\mathfrak{m}_1 \cdots \mathfrak{m}_n = \mathfrak{m}_1 \cdots \mathfrak{m}_n \mathfrak{m}_{n+1}$ for some $n$. But then by prime avoidence $\mathfrak{m}_{n+1}$ must be one of $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ since $\mathfrak{m}_{n+1} \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n$ so $\mathfrak{m}_{n+1} \supset \mathfrak{m}_i$ and $\mathfrak{m}_i$ is maximal. □

**Proposition 7.0.3.** Let $A$ be an artinian ring. Then every prime ideal is maximal so $\dim A = 0$.

*Proof.* Let $\mathfrak{p}$ be prime and $x \notin \mathfrak{p}$. Consider the chain,

$$(x) \supset (x^2) \supset (x^3) \supset \cdots$$

By the artinian condition $(x^n) = (x^{n+1})$ for some $n$ so $x^n = rx^{n+1}$ for some $r \in A$. Thus $x^n(rx-1) = 0$. However, $x^n \notin \mathfrak{p}$ so $rx - 1 \in \mathfrak{p}$ and thus $x \in A/\mathfrak{p}$ is invertible so $A/\mathfrak{p}$ is a field and thus $\mathfrak{p}$ is maximal. □

**Proposition 7.0.4.** Let $A$ be artinian. Then $\text{nilrad}\,(A)$ is a nilpotent ideal.

*Proof.* Let $I = \text{nilrad}\,(A)$. Consider the chain of ideals,

$$I \supset I^2 \supset I^3 \supset \cdots$$

By the artinian condition, $I^{n+1} = I^n$ for some $n$.

Consider $J = \{x \in A \mid xI^n = 0\}$. If $J \neq R$ we can choose $J' \supsetneq J$ minimal (using the artinian property). Then take $y \in J'$ so by minimality $J' = J + (y)$. Suppose $J + I(y) = J'$ then, since $J \subset \text{Jac}\,(A)$ and $(y)$ is finitely generated, by Nakayama, $J' = J + I(y) = J$ which is false so $J \subset J + I(y) \subsetneq J'$ and thus $J = J + I(y)$ by minimality so $I(y) \in J$. Therefore, $y \cdot I^{n+1} = 0$ but $I^{n+1} = I^n$ so $y \cdot I^n = 0$ and thus $y \in J$ contradicting our situation so $J = R$ and thus $I^n = 0$. □

**Proposition 7.0.5.** Every artinian ring is a product of local artinian rings: $A_{\mathfrak{m}_i} = A/\mathfrak{m}_i^n$.

*Proof.* Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals. Then we know that $\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r} = 0$ for some integers $n_1, \ldots, n_r \in \mathbb{Z}$. Therefore, by the Chinese remainder theorem,

$$A = A/(\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r}) = \prod_{i=1}^{r} A/\mathfrak{m}_i^{n_i}$$

Furthermore, $A/\mathfrak{m}_i^{n_i}$ is local because $\mathfrak{m}_i$ is the only maximal ideal above $\mathfrak{m}_i^{n_i}$. Furthermore,

$$A_{\mathfrak{m}_i} = (A/\mathfrak{m}_i^{n_i})_{\mathfrak{m}_i} = A/\mathfrak{m}_i^{n_i}$$

since $A \setminus \mathfrak{m}_i$ is not contained in any maximal ideal of $A/\mathfrak{m}_i^{n_i}$ and thus is invertible. □

**Proposition 7.0.6.** A ring $A$ is artinian iff it has finite length as a module over itself.

*Proof.* If $A$ has finite length as an $A$-module then it satisfies both the ascending and descending chain conditions on $A$-submodules i.e. ideals thus $A$ is both noetherian and artinian. Conversely, let $A$ be artinian. Since $A$ is a finite product of local artinian rings we may reduce to the case that $A$ is local artinian with maximal ideal $\mathfrak{m}$. Since $\text{nilrad}(A) = \mathfrak{m}$ then $\mathfrak{m}^n = 0$ for some $n$ so we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

Then $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a $A/\mathfrak{m}$-module and,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

which must be finite since $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is an artinian module and thus must have finite dimension else there would be a nonterminating descending chains. Thus from the series $A$ has finite length. □

**Theorem 7.0.7.** A ring $A$ is artinian iff $A$ is noetherian and $\dim A = 0$.

*Proof.* If $A$ is artinian then it has finite length over itself and thus is noetherian. Also every prime is maximal so $\dim A = 0$. Conversely, suppose that $A$ is noetherian and $\dim A = 0$. Then $\text{Spec}(A)$ is a noetherian topological space which has finitely many irreducible componets so $A$ has finitely many minimal primes which are also maximal since $\dim A = 0$. Thus $A$ has finitely many primes all of which are maximal. Since $\dim A = 0$ we have $I = \text{Jac}(A) = \text{nilrad}(A)$ so any $f \in I$ is nilpotent so $I$ is nilpotent because $A$ is noetherian so $I$ is finitely generated. Thus by the Chines remainder theorem $A$ is a finite product of local rings so we reduce to the case that $A$ is local with maximal ideal $\mathfrak{m}$. Then we get a series,

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset A$$

but $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite $A/\mathfrak{m}$-module since $A$ is noetherian so,

$$\ell_A(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \ell_{A/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_{A/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is finite and thus $\ell_A(A)$ is finite from the series showing that $A$ is artinian. □

**Proposition 7.0.8.** Let $A$ be an artinian ring. Then,

$$\ell_A(A) = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i})$$

*Proof.* We can write, $A = A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r}$ and thus the formula immediately follows. □

**Proposition 7.0.9.** Any finite dimensional $k$-algebra is artinian.

*Proof.* By dimensionality arguments every descending chain stabilizes. □

**Proposition 7.0.10.** Let $A \to B$ be a local map and $M$ an $B$-module of finite length. Then,

$$\ell_A(M) = \ell_B(M) \cdot [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

and in particular $\ell_A(M)$ is finite if $\kappa(\mathfrak{m}_B)$ is a finite extension of $\kappa(\mathfrak{m}_A)$.

*Proof.* Consider a composition series,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

Then $M_i/M_{i-1}$ is a simple $A$-module so $M_i/M_{i-1} \cong B/\mathfrak{m}_B = \kappa(\mathfrak{m}_B)$ since $B$ is local. Therefore,

$$\ell_A M = \sum_{i=1}^{n} \ell_A M_i/M_{i-1} = \sum_{i=1}^{n} \ell_A(\kappa(\mathfrak{m}_B)) = n \cdot [\kappa(B_\mathfrak{m}) : \kappa(A_\mathfrak{m})]$$

where $\ell_A(\kappa(\mathfrak{m}_B)) = \ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$ because $A \to B$ is local and,

$$\ell_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = \dim_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = [\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

□

**Corollary 7.0.11.** If $A$ is a local artinian finite type $k$-algebra. Then,

$$\dim_k A = \ell_A(A) \cdot \dim_k (A/\mathfrak{m})$$

in particular $A$ is a finite $k$-module.

*Proof.* Viewing $A$ as a module over itself we know it has finite length since $A$ is artinian. Furthermore, $A/\mathfrak{m}$ is a field finitely generated over $k$ and thus a finite extension of $k$ by the Nullstellensatz. Then applying the previous result we conclude. □

**Corollary 7.0.12.** Let $A$ be an artinian finite type $k$-algebra. Then,

$$\dim_k A = \sum_{i=1}^{r} \ell_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}) \cdot \dim_k (A/\mathfrak{m}_i)$$

*Proof.* Since $A$ is artinian we can write,

$$A = \prod_{i=1}^{r} A_{\mathfrak{m}_i}$$

where $A_{\mathfrak{m}_i}$ are the local artinian factors associated to the finitely many prime ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. The result follows from above by additivity of the dimensions. □