Mathematics GU4053 Algebraic Topology
Assignment # 1

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Note. My order of path concatenation follows Hatcher,

$$\gamma * \delta(x) = \begin{cases} \gamma(2x) & x \le \frac{1}{2} \\ \delta(2x-1) & \ge \frac{1}{2} \end{cases}$$

Problem 1.

Let X be a contractible space. Then, there exists a homotopy $H : X \times I \to X$ between id_X and constant map $f : X \to \{x_0\} \subset X$. For any $x \in X$ consider the path $\gamma : I \to X$ given by $\gamma(t) = H(x, t)$ which satisfies $\gamma(0) = H(x, 0) = \mathrm{id}_X(x) = x$ and $\gamma(1) = H(x, 1) = x_0$. Therefore, any x is path connected to x_0. However, because path connection is an equivalence relation on points, any $x, y \in X$ are path connected by transitivity.

Problem 2.

Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be pairs of homotopic maps. Then, there exist homotopies, $F : X \times I \to Y$ and $G : Y \times I \to Z$ between these maps. Consider the function $H : X \times I \to Z$ given by $H(x, t) = G(F(x, t), t)$ which is continuous by composition of continuous maps. Now, $H(x, 0) = G(F(x, 0), 0) = G(f(x), 0) = g \circ f(x)$ and $H(x, 1) = G(F(x, 1), 1) = G(f'(x), 1) = g' \circ f'(x)$. Therefore, H is a homotopy between $g \circ f$ and $g' \circ f'$.

Problem 3.

(a). Let $f : X \to Y$ and $g : Y \to Z$ be homotopy equivalences with homotopy "inverses" such that the compositions are homotopy equivalent to identity maps, $f' : Y \to X$ and $g' : Z \to Y$. Consider the maps $g \circ f$ and $f' \circ g'$. Now, using the result of problem 2,

$$(g \circ f) \circ (f' \circ g') = g \circ ((f \circ f') \circ g') \simeq g \circ (\mathrm{id}_Y \circ g') = g \circ g' \simeq \mathrm{id}_Y$$

and similarly,

$$(f' \circ g') \circ (g \circ f) = f' \circ ((g' \circ g) \circ f) \simeq f' \circ (\mathrm{id}_Y \circ f) = f' \circ f \simeq \mathrm{id}_X$$

therefore $g \circ f$ is a homotopy equivalence. Therefore, $\simeq$ is an equivalence relation on topological spaces because $X \simeq X$ under the identity map. If $X \simeq Y$ then there are maps $f : X \to Y$ and $g : Y \to X$ which are homotopy "inverses" and thus $Y \simeq X$ by swapping f and g. And finally, if $X \simeq Y$ and $Y \simeq Z$ then by above the composition of homotopy equivalences gives a homotopy equivalence $X \simeq Z$ so the relation is transitive.

(b). Consider the maps from X to Y under homotopy. Clearly, $f \simeq f$ under the homotopy $H(x,t) = f(x)$. If $f \simeq g$ then there exists a homotopy $H : X \times I \to Y$ then consider the map $H'(x,t) = H(x, 1-t)$. Now, $H'(x,0) = H(x,1) = g(x)$ and $H'(x,1) = H(x,0) = f(x)$ so $g \simeq f$. Finally, let $f \simeq g$ and $g \simeq h$. Then, we have homotopies $F, G : X \times I \to Y$ between f and g and between g and h respectively. Define the map $H : X \times I \to Y$ given by,

$$H(x,t) = \begin{cases} F(x, 2t) & t \in [0, \frac{1}{2}] \\ G(x, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

At $t = \frac{1}{2}$ the maps $F(x,1) = g(x) = G(x,0)$ so the map H is continous by the gluing lemma. Furthermore, $H(x,0) = F(x,0) = f(x)$ and $H(x,1) = G(x,1) = h(x)$ so H is a homotopy between f and h. Thus, $f \simeq h$ so $\simeq$ is transitive and then an equivalence relation on maps with common domains and codomains.

(c). Let $f : X \to Y$ be a homotopy equivalence with homotopy "inverse" $g : Y \to X$ and let $h \simeq f$. Then, by problem 2, $h \circ g \simeq f \circ g \simeq \mathrm{id}_Y$ so by transitivity, $h \circ g \simeq \mathrm{id}_Y$. Similarly, $g \circ h \simeq g \circ f \simeq \mathrm{id}_X$ so $g \circ h = \mathrm{id}_X$. Therefore, h is a homotopy equivalence with homotopy "inverse" g.

Problem 4.

If every map $f : X \to Y$ for any Y is nullhomotopic then in particular, $\mathrm{id}_X : X \to X$ is nullhomotopic so X is contractable. Conversely, if X is contractable then $\mathrm{id}_X : X \to X$ is homotopic to some constant map $g : X \to X$. For any map, $f : X \to Y$ we have $f = f \circ \mathrm{id}_X \simeq f \circ g$ which is a constant map because g is constant. Thus, f is nullhomotopic.

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Problem 5.

Suppose there exist map $f : X \to Y$ and $g, h : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $h \circ f \simeq \mathrm{id}_X$. Then consider the composition,

$$h = h \circ \mathrm{id}_Y \simeq h \circ (f \circ g) = (h \circ f) \circ g \simeq \mathrm{id}_X = g$$

Therefore, by transitivity, $h \simeq g$. Thus, $g \circ f \simeq h \circ f \simeq \mathrm{id}_X$. However, $f \circ g \simeq \mathrm{id}_Y$ so f is a homotopy equivalence.

Let $f \circ g$ and $h \circ f$ be homotopy equivalences with homotopy "inverses" $a : Y \to X$ and $b : X \to Y$ respectively. Therefore, $f \circ (g \circ a) = (f \circ g) \circ a \simeq \mathrm{id}_Y$ and $(b \circ h) \circ f = b \circ (h \circ f) \simeq \mathrm{id}_X$. Therefore, by the above argument, f is a homotopy equivalence.

Problem 6.

Let X be path-connected. Suppose that $\pi_1(X)$ is abelian and thus $\pi_1(X, x)$ is abelian at any point $x \in X$ because these groups are isomorphic on path-connected points. Now, let $h, h' : I \to X$ be paths with equal endpoints $x_0, x_1 \in X$ and let β_h and $\beta_{h'}$ be the respective basepoint change isomorphisms. Take any loop $[\gamma] \in \pi_1(X, x_1)$. The maps $\bar{h} * h'$ and $\bar{h'} * h$ are loops at x_1 satisfying,

$$(\bar{h} * h') * (\bar{h'} * h) = \bar{h} * ((h' * \bar{h'}) * h) \simeq \bar{h'} * h' \simeq e_{x_1}$$

Therefore, $[\gamma] = [(\bar{h} * h') * (\bar{h'} * h) * \gamma] = [(\bar{h} * h') * \gamma * (\bar{h'} * h)]$ using the commutativity of $\pi_1(X, x_1)$. Then,

$$\beta_h([\gamma]) = \beta_h([(\bar{h} * h') * \gamma * (\bar{h'} * h)]) = [h * (\bar{h} * h') * \gamma * (\bar{h'} * h) * \bar{h}] = [h' * \gamma * \bar{h'}] = \beta_{h'}([\gamma])$$

and therefore, $\beta_h = \beta_{h'}$.

Conversely, suppose that for any two paths with equal endpoints h and h' the change of basepoint maps are equal i.e. $\beta_h = \beta_{h'}$. In particular, take $x_0 \in X$ and let h be any loop at x_0. Also, set $h' = e_{x_0}$ the constant loop at x_0. Then, for any loop $[\gamma] \in \pi_1(X, x_0)$ we know that,

$$\beta_h([\gamma]) = [h * \gamma * \bar{h}] = [h][\gamma][h]^{-1} = \beta_{h'}([\gamma]) = [e_{x_0} * \gamma * \bar{e_{x_0}}] = [\gamma]$$

because $e_{x_0} * \gamma * \bar{e_{x_0}} \simeq \gamma$. Therefore, conjugation by $[h] \in \pi_1(X, x_0)$ is trivial for any h so the group is abelian.

Problem 7.

To show that the three conditions are equivalent, I will show that $(a) \implies (b) \implies (c) \implies (a)$.

$(a) \implies (b)$

Suppose that every map $f : S^1 \to X$ is homotopic to a constant map $g : S^1 \to \{p\}$. Then, there exists a homotopy $F : S^1 \times I \to X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = p$. Now, identify all the points $S^1 \times 1$ in the cylinder $S^1 \times I$. Under this identification of gluing together one end of the cylinder, the quotient space is the disk D^2. Now, $F(x, 1) = p$ so F is constant on $S^1 \times \{1\}$ and thus constant on all equivalence classes.

$$\begin{tikzcd}
S^1 \arrow[r, "\iota"] \arrow[rr, bend left, "f"] \arrow[dr, "\tilde{\iota}"'] & S^1 \times I \arrow[r, "F"] \arrow[d, "\pi"] & X \\
& D^2 \cong S^1 \times I/\sim \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

Therefore, F descends to the quotient space giving a map $\tilde{F} : D^2 \to X$ such that,

$$\tilde{F}|_{S^1 \times \{0\}} = \tilde{F} \circ \tilde{\iota} = \tilde{F} \circ \pi \circ \iota = F \circ \iota = f$$

where $\iota : S^1 \to S^1 \times I$ is the inclusion onto $S^1 \times \{0\}$ on which $F(x, 0) = f(x)$ and $\pi : S^1 \times I \to D^2$ is the projection onto the quotient.

$(b) \implies (c)$

Suppose that every map $f : S^1 \to X$ extends to a map $F : D^2 \to X$. For any loop $\gamma : I \to X$ based at x_0, because $\gamma(0) = \gamma(1)$ the map $\gamma : I \to X$ descends to a map $f : I/\sim \to X$ on quotient space under the identification $0 \sim 1$. However, $I/\{0,1\} \cong S^1$ so $f : S^1 \to X$ maps the generator of the fundamental group of S^1 to γ. Now, let $\iota : S^1 \to D^2$ be the inclusion onto the boundary of D^2. Then, $F \circ \iota(x) = f(x)$ because F is an extension of f. The functor π_1 takes this diagram in Top to the analogous diagram in Grp,

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "\iota"] & X \\ D^2 \arrow[ur, "F"] & \end{tikzcd}$$

(a) Top

$$\begin{tikzcd} \pi_1(S^1, s_0) \arrow[r, "f_*"] \arrow[d, "\iota_*"] & \pi_1(X, x_0) \\ \pi_1(D^2, \iota(s_0)) \arrow[ur, "F_*"] & \end{tikzcd}$$

(b) Grp

However, D^2 is homeomorphic to a convex subset of $\mathbb{R}^2$ and is thus contractable. Therefore, $\pi_1(D^2) = 0$ and thus $i_*(\pi_1(S^1, s_0)) \subset \pi_1(D^2, \iota(s_0)) = 0$ so $i_*(\pi_1(S^1, s_0)) = 0$. Therefore, $f_*(\pi_1(S^1, s_0)) = F_* \circ \iota_*(\pi_1(S^1, s_0)) = 0$. However, letting $[1]$ generate $\pi_1(S^1, s_0) \cong \mathbb{Z}$, we have $f_*([1]) = [\gamma]$ so $[\gamma] = [e_{x_0}]$ because f_* is the zero map. Therefore, $[\gamma]$ is trivial so $\pi_1(X, x_0) = 0$.

$(c) \implies (a)$

Suppose that $\pi_1(X, x_0) = 0$ for any $x_0 \in X$. Given any map $f : S^1 \to X$, take the map $\pi : I \to S^1$ given by the quotient map under the identification $0 \sim 1$. Then, $f \circ \pi$ is a loop in X at some basepont $f \circ \pi(0) = x_0 = f \circ \pi(1)$. Because X is simply connected, this loop is path-homotopic to the constant loop at x_0 under a homotopy $H : I \times I \to X$. Because $H(0,t) = H(1,t) = x_0$ the map descends to a map $\tilde{H} : S^1 \times I \to X$ on the quotient space under the same identification. $\tilde{H}$ is a homotopy between f and a constant map, $\tilde{H}(x, 1) = x_0$. Thus, every map $f : S^1 \to X$ is homotopic to a constant map.

Therefore,

$$(a) \iff (b) \iff (c)$$

simply connected $\iff$ all maps $S^1 \to X$ are homotopic:

If all maps $f : S^1 \to X$ are homotopic then, in particular, every map $f : S^1 \to X$ is homotopic to a constant map. Using $(a) \implies (c)$ we conclude that $\pi(X, x_0) = 0$ at any basepoint. Furthermore, all constant maps from S^1 are homotopic which implies that X is path-connected. Thus, X is simply connected.

Conversely, if X is simply connected then $\pi_1(X, x_0) = 0$ for any basepoint $x_0 \in X$. From the result, $(c) \implies (a)$ we have that every map $f : S^1 \to X$ is homotopic to some constant map $f_c : S^1 \to \{c\} \subset X$. However, since X is path connected, all constant maps are homotopic. Therefore, given two maps $f_1, f_2 : S^1 \to X$, we know that $f_1 \simeq f_{c_1}$ and $f_2 \simeq f_{c_2}$ and $f_{c_1} \simeq f_2$ because both are constant maps. Thus, $f_1 \simeq f_{c_1} \simeq f_{c_2} \simeq f_{c_2}$ because homotopy is an equivalence relation on maps. Therefore, any two maps $f : S^1 \to X$ are homotopic.

At last, we have shown that X is simply-connected iff all maps $f : S^1 \to X$ are homotopic.

Problem 8.

Let $\gamma : I \to X$ be a loop at x_0 and $\delta : I \to Y$ be a loop at y_0. Then, consider the map, $H : I \times I \to X \times Y$ given by,

$$H(x,t) = \begin{cases} (\gamma(3xt), y_0) & x \leq \frac{1}{3} \\ (\gamma(t), \delta(3x-1)) & x \in [\frac{1}{3}, \frac{2}{3}] \\ (\gamma((3x-2)(1-t)+t), y_0) & x \geq \frac{2}{3} \end{cases}$$

First, consider the overlaps. At $x = \frac{1}{3}$, we have, $(\gamma(3xt), y_0) = (\gamma(t), y_0)$ and $(\gamma(t), \delta(0)) = (\gamma(t), y_0)$. At $x = \frac{2}{3}$, we have, $(\gamma(t), \delta(1)) = (\gamma(t), y_0)$ and $(\gamma((2-2)(1-t)+t), y_0) = (\gamma(t), y_0)$ so by the glueing lemma, H is a continuous map. Futhermore, $H(0,t) = (\gamma(0), y_0) = (x_0, y_0)$ and $H(1,t) = (\gamma(1), y_0) = (x_0, y_0)$. Also,

$$H(x,0) = \begin{cases} (x_0, y_0) & x \leq \frac{1}{3} \\ (x_0, \delta(3x-1)) & x \in [\frac{1}{3}, \frac{2}{3}] \\ (\gamma(3x-2), y_0) & x \geq \frac{2}{3} \end{cases}$$

which is the path (using a triple concatenation with time divided into 1/3 intervals) $e_{(x_0,y_0)} * (\{x_0\} \times \delta) * (\gamma \times \{y_0\}) \simeq (\{x_0\} \times \delta) * (\gamma \times \{y_0\})$. Likewise,

$$H(x,1) = \begin{cases} (\gamma(3x), y_0) & x \leq \frac{1}{3} \\ (x_0, \delta(3x-1)) & x \in [\frac{1}{3}, \frac{2}{3}] \\ (x_0, y_0) & x \geq \frac{2}{3} \end{cases}$$

which is the path $(\gamma \times \{y_0\}) * (\{x_0\} \times \delta) * e_{(x_0,y_0)} \simeq (\gamma \times \{y_0\}) * (\{x_0\} \times \delta)$. Thus, H is a path-homotopy from $e_{(x_0,y_0)} * (\{x_0\} \times \delta) * (\gamma \times \{y_0\})$ to $e_{(x_0,y_0)} * (\{x_0\} \times \delta) * (\gamma \times \{y_0\})$

These paths are themselves easily equivalent via reparametrization to $(\{x_0\} \times \delta) * (\gamma \times \{y_0\})$ and $(\{x_0\} \times \delta) * (\gamma \times \{y_0\})$.

Problem 9.

Let $p : \tilde{X} \to X$ be a covering map and $A \subset X$ have the subspace topology. Then, consider $\tilde{A} = p^{-1}(A)$ and $p' = p|_{\tilde{A}} : \tilde{A} \to A$. For each $x \in X$ there is an evenly covered neighborhood U such that $p^{-1}(U)$ is a disjoint union of sets W_λ each of which is homeomorphic to U under p. Now, for $x \in A$ consider $p|_{\tilde{A}}^{-1}(U \cap A) = p|_{\tilde{A}}^{-1}(U) \cap p|_{\tilde{A}}^{-1}(A) = p^{-1}(U) \cap \tilde{A} = \bigsqcup_{\lambda \in \Lambda} W_\lambda \cap \tilde{A}$. The sets $W_\lambda \cap \tilde{A}$ are disjoint because W_λ are. Also, p is a homeomorphism on W_λ to U and thus $p|_{\tilde{A}}$ is a homeomorphism restricted to $W_\lambda \cap \tilde{A}$ to its image $p(W_\lambda \cap \tilde{A}) = p(W_\lambda) \cap p(\tilde{A}) = U \cap A$ by properties of a bijection. Thus, $X \cap A$ is evenly covered by $p|_{\tilde{A}}$. Thus, $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering map.

Problem 10.

Let X and Y be path-connected and locally path-connected and let $\tilde{X}$ and $\tilde{Y}$ be simply-connected covering spaces with covering maps $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$. Also let $f : X \to Y$ be a homotopy equivalence with homotopy "inverse" $g : Y \to X$. Now, by Lemma , the covering spaces, $\tilde{X}$ and $\tilde{Y}$ are locally path-connected. Since they are also simply-connected, all maps from $\tilde{X}$ or $\tilde{Y}$ to X or Y satisfy the lifting criterion. This is because $f_*(\pi_1(\tilde{X}, \tilde{x_0})) = 0$ which is trivially a subgroup of any group.

Now, consider lifts of the maps $f \circ p : \tilde{X} \to Y$ and $g \circ q : \tilde{Y} \to X$, namely, $\widetilde{f \circ p} : \tilde{X} \to \tilde{Y}$ and $\widetilde{g \circ q} : \tilde{X} \to \tilde{Y}$ which satisfy

$$p \circ \widetilde{g \circ q} = g \circ q \qquad q \circ \widetilde{f \circ p} = f \circ p$$

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, bend left, "\widetilde{f\circ p}"] \arrow[d, "p"] & \tilde{Y} \arrow[l, dashed, bend left, "\widetilde{g\circ q}"] \arrow[d, "q"] \\
X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"]
\end{tikzcd}$$

Now, consider the composition,

$$p \circ (\widetilde{g \circ q} \circ \widetilde{f \circ p}) = g \circ q \circ \widetilde{f \circ p} = g \circ f \circ p = (g \circ f) \circ p \simeq \mathrm{id}_X \circ p = p$$

Therefore, by homotopy lifting, $(\widetilde{g \circ q} \circ \widetilde{f \circ p})$ is homotopic to some lift of p, namely, $r_p : \tilde{X} \to \tilde{X}$. Because r_p is a lift of p, we must have that $p \circ r_p = p$ so r is a deck transformation. However, the deck transformations form a group so if $(\widetilde{g \circ q} \circ \widetilde{f \circ p}) \simeq r_p$ then $(r_p^{-1} \circ \widetilde{g \circ q}) \circ \widetilde{f \circ p} \simeq \mathrm{id}_{\tilde{X}}$.

Similarly,

$$q \circ (\widetilde{f \circ p} \circ \widetilde{g \circ q}) = f \circ p \circ \widetilde{g \circ q} = f \circ g \circ q = (f \circ g) \circ q \simeq \mathrm{id}_Y \circ q = q$$

Therefore, by homotopy lifting, $(\widetilde{f \circ p} \circ \widetilde{g \circ q})$ is homotopic to some lift of q, namely, $r_q : \tilde{Y} \to \tilde{Y}$. Because r_q is a lift of q, we must have that $q \circ r_q = q$ so r is a deck transformation. However, the deck transformations form a group so if $(\widetilde{f \circ p} \circ \widetilde{g \circ q}) \simeq r_q$ then $\widetilde{f \circ p} \circ (\widetilde{g \circ q} \circ r_q^{-1}) \simeq \mathrm{id}_{\tilde{Y}}$.

Therefore, by problem 5, we know that $\widetilde{f \circ p}$ is a homotopy equivalence.

Problem 11.

(a). Let $p : \tilde{X} \to X$ be a covering map and let X be path-connected, locally path-connected, and semi-locally simply-connected. Since X is locally path-connected, the path-components and components correspond. Let $x \sim y$ iff there is a path connecting x and y in X. Take $\tilde{x} \in p^{-1}(x_0)$ and consider the orbit $\mathrm{Orb}(\tilde{x})$ under the action of $\pi_1(X, x_0)$ via $[\gamma] \cdot \tilde{x} = \widetilde{\gamma}(1)$ where $\widetilde{\gamma}$ is the unique lift of γ with initial point $\tilde{x}$. Now, assosicate, $\mathrm{Orb}(()\tilde{x})$ with $[\tilde{x}]$ under $\sim$. We need to show that this assoication is well-defined and one-to-one.

If $\mathrm{Orb}(\tilde{x}) = \mathrm{Orb}(\tilde{x}')$ then there must exist a path γ in X such that $[\gamma] \cdot \tilde{x} = \tilde{x}'$ because they lie in the same orbit. Thus, $\tilde{\gamma}(0) = \tilde{x}$ and $\tilde{\gamma}(1) = \tilde{x}'$ so the lift is a path between $\tilde{x}$ and $\tilde{x}'$. Thus, $\tilde{x} \sim \tilde{x}'$ and equivalently $[\tilde{x}] = [\tilde{x}']$. Conversely, if $[\tilde{x}] = [\tilde{x}']$ then these points must be equivalent under path-connection i.e. there exists a path $\delta : I \to \tilde{X}$ taking $\tilde{x}$ to $\tilde{x}'$. Consider, $p \circ \delta$ which is a loop in at x_0 in X because $\delta(0) = \tilde{x} \in p^{-1}(x_0)$ and $\delta(1) = \tilde{x}' \in p^{-1}(x_0)$ so $p \circ \delta(0) = p \circ \delta(1) = x_0$. However, $[p \circ \delta] \cdot \tilde{x} = \tilde{x}'$ because δ is already the unique lift of $p \circ \delta$ at $\tilde{x}$ and thus $\mathrm{Orb}(\tilde{x}) = \mathrm{Orb}(\tilde{x}')$.

(b). Take $Z \subset \tilde{X}$ to be the component containing $\tilde{x}_0$. Under the Galois correspondence, Z corresponds to $p_*(\pi_1(Z, \tilde{x}_0))$. Now, take $[\gamma] \in p_*(\pi_1(Z, \tilde{x}_0))$ then $[\gamma] = [p \circ \delta]$ for some loop $[\delta] \in \pi_1(Z, \tilde{x}_0)$. Consider, $[\gamma] \cdot \tilde{x}_0 = \tilde{\gamma}(1)$. However, δ is already the unique lift at $\tilde{x}_0$ because $\gamma = p \circ \delta$ and δ is based at $\tilde{x}_0$. Thus, $\tilde{\gamma} = \delta$ and δ is a loop at $\tilde{x}_0$ so $\tilde{\gamma}(1) = \tilde{x}_0$. Therefore, $[\gamma] \in \mathrm{Stab}(\tilde{x}_0)$.

Conversely, if $[\gamma] \in \mathrm{Stab}(\tilde{x}_0)$ then $[\gamma] \cdot \tilde{x}_0 = \tilde{x}_0$ so the lift $\tilde{\gamma}$ at $\tilde{x}_0$ is a loop at $\tilde{x}_0$ because $\tilde{\gamma}(1) = [\gamma] \cdot \tilde{x}_0 = \tilde{x}_0$. Furthermore, $\tilde{\gamma}$ must be resticted to Z because the image of any path must be contained in a single path component. Therefore, $[\tilde{\gamma}] \in \pi_1(Z, \tilde{x}_0)$ and thus, $[p \circ \tilde{\gamma}] \in p_*(\pi_1(Z, \tilde{x}_0))$ but $p \circ \tilde{\gamma} = \gamma$ so $[\gamma] \in p_*(\pi_1(Z, \tilde{x}_0))$. Therefore,

$$p_*(\pi_1(Z, \tilde{x}_0)) = \mathrm{Stab}(\tilde{x}_0)$$

Lemmas

Lemma 0.1. *If $p : \tilde{X} \to X$ is a covering map and X is locally path-conected then $\tilde{X}$ is locally path connected.*

Proof. Take $\tilde{x} \in \tilde{X}$ and an open $\tilde{x} \in A \subset \tilde{X}$. Now, consider $x = p(\tilde{x}) \in X$ which has an evenly covered neighborhood $x \in U$. Furthermore, because X is locally path-connected, there is a path-connected neighborhood V of x such that, $x \in V \subset U \cap p(A)$ because $p(A)$ is open since every covering map is an open map. However, $p^{-1}(U)$ is a disjoint union of W_α on each of which p restricts to a homeomorphism. Therefore, since $\tilde{x} \in p^{-1}(U \cap p(A))$ take W_λ to be the slice containing $\tilde{x}$. Then, p restricted to W_λ is a homeomorphism and therefore must take the path connected neighborhood V of x to a path connected neighborhood $\tilde{x} \in p|_{W_\lambda}^{-1}(V) \subset A$ where the final inclusion follows because $V \subset p(A)$ and p is a homeomorphism on W_λ. $\square$