Mathematics GU4042 Modern Algebra II
Assignment # 10

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Problem 1.

Let $K = \mathbb{Q}$ and $E = \mathbb{Q}(\sqrt[3]{2})$ and $F = \mathbb{Q}(\zeta_3, \sqrt[3]{2})$ the splitting field of $X^3 - 2$ over $\mathbb{Q}$. Thus, $K \subset E \subset F$ and F/K us normal because F is a splitting field over K. However, E/K is not normal because $\sqrt[3]{2} \in E = \mathbb{Q}(\sqrt[3]{2})$ but $\mathrm{Min}(\sqrt[3]{2}; \mathbb{Q}) = X^3 - 2$ does not split in E because it has complex roots in F which are not contained in E and any root in E must also be a root in F. Therefore, there are not enough roots in E so E/K is not normal.

Problem 4.

Let $f(X) = X^3 - X - 1$ then $\mathrm{Disc}(f) = \Delta = -4p^3 - 27q^2$ where $p = -1$ and $q = -1$ so $\Delta = -23$. Because -23 is not a square in $\mathbb{Q}$ the Galois group of f over $\mathbb{Q}$ is S_3. We will now consider an arbitrary irreducible monic cubic polynomial f with a non-square discriminant over $\mathbb{Q}$. Enumerate the roots of f in $\mathbb{C}$ as $\alpha_1, \alpha_2, \alpha_3$. The nontrivial subgroups of $S_3 = \langle \sigma, \tau \mid \sigma^3 = e, \tau^2 = e, \sigma\tau = \tau\sigma^2 \rangle$ are, $R = \langle \sigma \rangle$, $F_0 = \langle \tau \rangle$, $F_1 = \langle \tau\sigma \rangle$, and $F_2 = \langle \tau\sigma^2 \rangle$. These must correspond to the nontrivial intermediate fields of $E/\mathbb{Q}$, the splitting field of f. The action of these elements must be faithful. One choice of representation is $\sigma = (1\,2\,3)$ and $\tau = (1\,2)$. These fixed fields are,

$$E^R = \mathbb{Q}(\sqrt{\mathrm{Disc}(f)}) = \mathbb{Q}(\sqrt{\Delta})$$

which holds because $\sqrt{\mathrm{Disc}(f)} = \pm \prod_{i<j} (\alpha_i - \alpha_j) \in E = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ so $\mathbb{Q}(\sqrt{\Delta}) \subset E$ however, the degree, $[E : \mathbb{Q}(\sqrt{\Delta})] = 2$ because $X^2 - \Delta$ is irreducible if Δ is not a square in $\mathbb{Q}$. Therefore, $\mathbb{Q}(\sqrt{\Delta})$ corresponds to a subgroup of S_3 with index 2. However, there is only one such subgroup, namely, R. By the Galois correspondence, $E^R = \mathbb{Q}(\sqrt{\Delta})$. Next,

$$E^{F_0} = \mathbb{Q}(\alpha_3)$$

because r_3 is fixed under the permutation $(1\,2)$ and no other root. Also, $[\mathbb{Q}(r_3) : \mathbb{Q}] = \deg f = 3$ which is the minimal polynomial because f is irreducible and monic. However, by the Galois correspondence,

$$[E^{F_0} : \mathbb{Q}] = [S_3 : F_0] = 6/2 = 3$$

Thus, $\mathbb{Q}(\alpha_3) \subset E^{F_0}$ has the degree over $\mathbb{Q}$ of the entire field E^{F_0} and thus must equal E^{F_0}. Similarly,

$$E^{F_1} = \mathbb{Q}(\alpha_2)$$

because $(1\,2\,3)(1\,2)$ fixes α_2 and the previous argument ensures that $\mathbb{Q}(r_2)$ is the entire fixed field. Likewise,

$$E^{F_2} = \mathbb{Q}(\alpha_1)$$

since $(1\,2\,3)^2(1\,2)$ fixes α_1.

For the polynomial $f(X) = X^3 - X - 1$, which does in fact satisfy the necessary conditions, we have the four intermediate fields of its splitting field, $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\alpha_1)$ and $\mathbb{Q}(\alpha_2)$ and $\mathbb{Q}(\alpha_3)$. These roots can be found via Cardano's formula but are sufficiently terrible to not warrant space on this paper.

Problem 5.

Let $f(X) = X^3 - 2$ which is irreducible, monic and has $\Delta = -27 \cdot 2^2 = -108$ which is not a square in $\mathbb{Q}$. Therefore, the earlier discussion applies. Letting E be the splitting field of f, we may immediately conclude that $Gal(E/\mathbb{Q}) \cong S_3$ and that the only intermediate fields are, $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{-108}) = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\alpha_1) = \mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\alpha_2) = \mathbb{Q}(\zeta_3\sqrt[3]{2})$ and $\mathbb{Q}(\alpha_1) = \mathbb{Q}(\zeta_3^2\sqrt[3]{2})$.

Problem 8.

Let $f(X) = X^3 - X - 1$ so as before $\Delta = -23$. However, Δ is a square in $\mathbb{Q}(\sqrt{-23})$ since $\sqrt{-23} \in \mathbb{Q}$. Since f is irredcible over $\mathbb{Q}$, by Lemma 0.1, f is an irreducible seperable (because $\mathbb{Q}$ is perfect) cubic with square discriminant over $\mathbb{Q}(\sqrt{-23})$ so $Gal(E/\mathbb{Q}(\sqrt{-23})) \cong A_3 \cong \mathbb{Z}/3\mathbb{Z}$. However, $\mathbb{Z}/3\mathbb{Z}$ has no nontrivial subgroups and thus the extension $E/\mathbb{Q}(\sqrt{-23})$ has no nontrivial intermediate fields.

Problem 9.

The discriminant of $f = X^3 - 10$ is $\Delta = -27 \cdot 10^2 = -2700$ which is not a square in $\mathbb{Q}(\sqrt{2})$ because $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ and therefore does not contain the root of any negtive number. Since f is irredcible over $\mathbb{Q}$, by Lemma 0.1, f is an irreducible seperable (because $\mathbb{Q}$ is perfect) cubic with non-square discriminant over $\mathbb{Q}(\sqrt{2})$ so $Gal(E/\mathbb{Q}(\sqrt{2})) \cong S_3$ where E is the splitting field of f over $\mathbb{Q}(\sqrt{2})$.

Problem 10.

Again, the discriminant of $f = X^3 - 10$ is $\Delta = -27 \cdot 10^2 = -2700$ which, this time, is a square in $\mathbb{Q}(\sqrt{-3})$ because $-2700 = (30\sqrt{-3})^2$. Since f is irredcible over $\mathbb{Q}$, by Lemma 0.1, f is an irreducible seperable (because $\mathbb{Q}$ is perfect) cubic with square discriminant over $\mathbb{Q}(\sqrt{-3})$ so $Gal(E/\mathbb{Q}(\sqrt{-3})) \cong A_3$ where E is the splitting field of f over $\mathbb{Q}(\sqrt{2})$.

Additional Problem.

Let $f \in \mathbb{Q}[Y_1, Y_2, Y_3]$ be $f(Y_1, Y_2, Y_3) = Y_1^3 + Y_2^3 + Y_3^3 = u_1^3 - 3u_1u_2 + 3u_3$. This is most easily checked by direct computation,

$$\begin{aligned}
u_1^3 - 3u_1u_2 + 3u_3 &= (Y_1 + Y_2 + Y_3)^3 - 3(Y_1 + Y_2 + Y_3)(Y_1Y_2 + Y_2Y_3 + Y_1Y_3) + 3Y_1Y_2Y_3 \\
&= Y_1^3 + Y_2^3 + Y_3^3 + 3Y_1^2Y_2 + 3Y_1Y_2^2 + 3Y_2^2Y_3 + 3Y_2Y_3^2 + 3Y_1^2Y_3 + 3Y_1Y_3^2 + 6Y_1Y_2Y_3 \\
&- 3(Y_1^2Y_2 + Y_1Y_2^2 + Y_2^2Y_3 + Y_2Y_3^2 + Y_1^2Y_3 + Y_1Y_3^2 + 3Y_1Y_2Y_3) + 3Y_1Y_2Y_3 \\
&= Y_1^3 + Y_2^3 + Y_3^3
\end{aligned}$$

Thus, given a cubic $g(X) = X^3 + aX^2 + bX + c$ with roots $\alpha_1, \alpha_2, \alpha_3$, by Vieta,

$$\begin{aligned}
-a &= \alpha_1 + \alpha_2 + \alpha_3 = u_1(\alpha_1, \alpha_2, \alpha_3) \\
b &= \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = u_2(\alpha_1, \alpha_2, \alpha_3) \\
-c &= \alpha_1\alpha_2\alpha_3 = u_3(\alpha_1, \alpha_2, \alpha_3)
\end{aligned}$$

Since $f = u_1^3 - 3u_1u_2 + 3u_3$, plugging in the roots, we get,

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = f(\alpha_1, \alpha_2, \alpha_3) = (-a)^3 - 3(-a)b + 3(-c) = -a^3 + 3ab - 3c$$

Therefore, for $g(X) = X^3 - 2$ we have $a = b = 0$ and $c = -2$ so $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = (-3) \cdot (-2) = 6$.

Lemmas

Lemma 0.1. *An irreducible cubic over $\mathbb{Q}$ is irreducible over any quadratic extension $\mathbb{Q}(\sqrt{d})$.*

Proof. Suppose that $\mathbb{Q}(\sqrt{d})$ contained α, a root of f then $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\sqrt{d})$ but since f is irreducible over $\mathbb{Q}$, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ but $[\mathbb{Q}(\sqrt{d}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{d}) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] \geq 3$. This contradicts the fact that $X^2 + d$ is the minimal polynomial of $\sqrt{d}$ and thus $[\mathbb{Q}(\sqrt{d}) : \mathbb{Q}] = 2$. Therefore, f has no roots in $\mathbb{Q}(\sqrt{d})$ so f is irreducible over $\mathbb{Q}(\sqrt{d})$ because otherwise it would split into a linear and a quadratic factor which would imply the existence of a root. $\square$