PHYSICS C2801 FALL 2013 PROBLEM SET 7

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Problem 1. Kleppner and Kolenkow 7.1

a. To find the instantaneous angular velocity of the wheel, you have to take into account that the wheel has two components of angular velocity, one about the pivot point at the end of the rod and one about the center of the wheel. We have the angular velocity about the pivot point Ω. To find the angular velocity about the wheel, we can consider that the the wheel has speed $v = \Omega R$, where R is the distance from the pivot point to the center of the wheel, since it is rolling without slipping. This lets us determine the angular velocity of the wheel about its center $\omega = v/R = \Omega$. Therefore, the angular velocity is:

$$\begin{aligned}\vec{\omega} &= \Omega\hat{x} - \Omega\hat{z} \\ |\omega| &= \sqrt{2\Omega^2} = \sqrt{2}\Omega \\ \theta &= arctan(-\Omega/\Omega) = -45^\circ\end{aligned}$$

b. To find the angular momentum of the hoop in the inertial frame, we need the moment of inertias along each axis of rotation. There is rotation about the z axis and about the x axis. For the I about the z axis we need to use the parallel axis theorem.

$$\begin{aligned}I_z &= \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2 \\ I_x &= MR^2\end{aligned}$$

The angular momentum in the x and z directions are:

$$\begin{aligned}L_x &= MR^2\Omega \\ L_z &= \frac{3}{2}MR^2\Omega \\ \vec{L} &= MR^2\Omega\hat{x} + \frac{3}{2}MR^2\Omega\hat{z}\end{aligned}$$

To determine if they angular momentum and angular velocity are parallel to each other take the cross product and see if it is 0.

$$\vec{\omega} X \vec{L} = -\hat{y}(MR^2\Omega^2 + \frac{3}{2}\Omega^2)$$

This does not reduce to 0, so they are not parallel to each other.

Problem 2. Kleppner and Kolenkow 7.2

To find the direction of the axle, we need to make the approximation that to first order the rotation perpindicular to the plane of the disk is negligible. Then we can use the equation for torque in terms

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of change in angular momentum. In this equation, $\vec{\Omega}X\vec{L}$, the angular momentum is in the body frame.

$$\begin{aligned} \tau &= \frac{dL}{dt} = \vec{\Omega}X\vec{L} \\ &= L_0\Omega = I_0\omega_0\Omega \\ \tau &= T2lsin(\theta) + T2lsin(\theta) \\ 4Tlsin(\theta) &= I_0\omega_0\Omega \\ sin(\theta) &\approx \theta \\ \theta &= \frac{I_0\omega_0\Omega}{4lT} \end{aligned}$$

Problem 3. Kleppner and Kolenkow 7.3

This problem can be done by using the equation for a gyroscope.

$$\tau = \dot{L} = L_0\Omega$$

The torque is the gravitational torque acting on the object at a distance l from the object. The momentum L_0 is the angular momentum in the body frame $I_0\omega_0$.

$$mgl = I_0\omega_0\Omega$$

Then in order to find the angle β we can look at the forces in the horizontal and vertical direction.

$$\begin{aligned} Tcos(\beta) - mg &= 0 \\ Tsin(\beta) &= ma_c \end{aligned}$$

Where a_c is the centripetal acceleration of the mass as it processes. Therefore, it will be proportional to to procession angular frequency.

$$Tsin(\beta) = ma_c = mv^2/l = m(\Omega l)^2/l = ml\Omega^2$$

Then by assuming that the angle β is small, we can solve for it in terms of known parameters.

$$\begin{aligned} T\beta &\approx ml\Omega^2 \\ T &\approx mg \\ \Omega &= \frac{mgl}{I_0\omega_0} \\ mg\beta &\approx \frac{m^3g^2l^3}{I_0^2\omega_s^2} \\ \beta &\approx \frac{gm^2l^3}{I_0^2\omega_s^2} \end{aligned}$$

Problem 4. Kleppner and Kolenkow 7.6

To find the tilt angle of the coin during rotation, start with the equation for torque. Since we have the velocity of the center of mass you can find the angular velocity from this depending on what circle the object is rotating around. We have one circle at b, which is just rotating about the center of mass of the coin and another circle at R which is the circle of procession of the coin. Thus, if we find the torque (due to gravity acting at the center of mass at a distance $bsin(\theta)$ from the edge) and the angular momentum in the body frame, we can solve for the angle.

$$\begin{aligned}
\tau &= \dot{L} = \vec{\Omega} X \vec{L} = L_0 \Omega \\
\tau &= mgbsin(\beta) \approx mgb\beta \\
L_0 &= I_0 \omega \\
\omega &= v/b \\
\Omega &= v/R \\
I_0 &= \frac{1}{2}Mb^2 + Mb^2 = \frac{3}{2}Mb^2 \\
mgb\beta &\approx \frac{3}{2}\frac{Mb^2v^2}{bR} \\
\beta &\approx \frac{3v^2}{2gR}
\end{aligned}$$

Problem 5. Kleppner and Kolenkow 7.7

To analyze the problem, let's start with the equation for torque.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\Omega} X \vec{L}$$

Then we need to find the angular momentum in the body frame so we need to find the moment of inertia tensor and angular velocity in the body frame too.

$$\begin{aligned}
\vec{L} &= I\vec{\omega} \\
I &= \begin{bmatrix} \frac{1}{2}MR^2 & 0 & 0 \\ 0 & \frac{1}{2}MR^2 & 0 \\ 0 & 0 & MR^2 \end{bmatrix} \\
\vec{\omega} &= \begin{bmatrix} \omega sin(\beta) \\ 0 \\ \omega cos(\beta) \end{bmatrix} \\
\vec{L} &= \begin{bmatrix} \frac{1}{2}MR^2\omega sin(\beta) \\ 0 \\ MR^2\omega cos(\beta) \end{bmatrix} \approx \begin{bmatrix} \frac{1}{2}MR^2\omega\beta \\ 0 \\ MR^2\omega \end{bmatrix}
\end{aligned}$$

Then we need to find the torque. Let's evaluate the torque in the body frame with respect to the center of mass.This comes from the component of the tension in the string perpindicular to the lasso.

$$\begin{aligned}
\tau &= -TRsin(\pi/2 - \alpha + \beta) = -TRcos(\alpha - \beta) = -TR(cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta) \\
&\approx -TR(cos(\alpha) + sin(\alpha)\beta)
\end{aligned}$$

Then we can look at the the forces in the y direction in the inertial frame to find the tension.

$$\begin{aligned} Tcos(\alpha) &= mg \\ \tau &= -mgR - mgRtan(\alpha)\beta \end{aligned}$$

Then we need to evaluate the cross product in the body frame.

$$\begin{aligned} \tau &= \vec{\omega} X \vec{L} = \begin{bmatrix} \hat{x}' & \hat{y}' & \hat{z}' \\ \omega\beta & 0 & \omega \\ \frac{1}{2}MR^2\omega\beta & 0 & MR^2\omega \end{bmatrix} \\ &= -\hat{y}'(MR^2\omega^2\beta - \frac{1}{2}MR^2\omega^2\beta) = -\frac{1}{2}MR^2\omega^2\beta\hat{y}' \\ MgR + MgRtan(\alpha)\beta &= \frac{1}{2}MR^2\omega^2\beta \\ \beta &\approx \frac{2g}{\omega^2 R}\frac{1}{1 - \frac{2g}{\omega^2 R}tan(\alpha)} \approx \frac{2g}{\omega^2 R} \end{aligned}$$

b. Now we need to find the radius of the small circle traced out by the center of mass. We can look at the equation of motion in the horizontal direction to determine this.

$$\begin{aligned} Tsin(\alpha) &= m\omega^2 R \\ T &= mg/cos(\alpha) \\ R &= gtan(\alpha)/\omega^2 \end{aligned}$$

Problem 6. Rotations and rotation matrixes

a. We need to show that the matrix given is the identiy matrix in 2 dimensions. This can be shown by multiplying it with with a vector and showing it returns that same vector.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

b. Now evaluate the rotation matrix at specifc angles and see how the vector is rotating.

$$\begin{aligned} \theta &= \pi/2 \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -y \\ x \end{bmatrix} \end{aligned}$$

This is an interchange of x and y and an inversion in the x direction, which is a 90 degrees rotation.

$$\begin{aligned} \theta &= \pi \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -x \\ -y \end{bmatrix} \end{aligned}$$

This is an inversion in the of x and y, which is a 180 degrees rotation.

$$\begin{aligned} \theta &= 3\pi/2 \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} y \\ -x \end{bmatrix} \end{aligned}$$

This is an interchange of x and y and an inversion in the y direction, which is a 270 degrees rotation.

$$\begin{aligned} \theta &= 2\pi \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

This ihas no change on x or y, which is a 360 degrees rotation.
c. Compute the matrix that represents 2 rotations done one after the other.

$$\begin{aligned} R'' &= \begin{bmatrix} cos(\theta') & -sin(\theta') \\ sin(\theta') & cos(\theta') \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} cos(\theta')cos(\theta) - sin(\theta')sin(\theta) & -cos(\theta')sin(\theta) - sin(\theta')cos(\theta) \\ sin(\theta')cos(\theta) + cos(\theta')sin(\theta) & -sin(\theta')sin(\theta) + cos(\theta')cos(\theta) \end{bmatrix} \\ R'' &= \begin{bmatrix} cos(\theta'+\theta) & -sin(\theta'+\theta) \\ sin(\theta'+\theta) & cos(\theta'+\theta) \end{bmatrix} = R(\theta'+\theta) \end{aligned}$$

This is just a rotation of the two angles added together.
d. We want to show the $R^{-1}(\theta) = R(-\theta)$, which can be done by showing that if $R(\theta)$ and $R(-\theta)$ are multiplied together, you get the identity. This shows that it is the inverse of R. We can skip some steps by using the results from part c for 2 matrixes multiplied together with different angles of rotation.

$$R(-\theta)R(\theta) = R(-\theta+\theta) = R(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \vec{1}$$

e. We want to show that the transpose of the rotation matrix is the same as the inverse of the rotation matrix.

$$\begin{aligned} R^T &= R^{-1} = R(-\theta) \\ R^T &= \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \\ R(-\theta) &= \begin{bmatrix} cos(-\theta) & -sin(-\theta) \\ sin(\theta) & cos(-\theta) \end{bmatrix} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} = R^T \end{aligned}$$

f. We want to show that the final product in equation 3 in the problem set reproduces what we expect for x' and y'.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} xcos(\theta) - ysin(\theta) & xsin(\theta) + xcos(\theta) \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix}$$

g. Now we want to show that the transformation preserves length. Start with the primed vectors and try to get to the unprimed vectors.

$$
\begin{aligned}
(X')^T X' &= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\
&= \begin{bmatrix} xcos(\theta) - ysin(\theta) & xsin(\theta) + ycos(\theta) \end{bmatrix} \begin{bmatrix} xcos(\theta) - ysin(\theta) \\ xsin(\theta) + ycos(\theta) \end{bmatrix} \\
&= x^2 cos^2(\theta) - xycos(\theta)sin(\theta) - yxsin(\theta)cos(\theta) + y^2 sin^2(\theta) \\
&\quad + x^2 sin^2(\theta) + xysin(\theta)cos(\theta) + y^2 cos^2(\theta) + yxcos(\theta)sin(\theta) \\
&= (x^2 + y^2)(cos^2(\theta) + sin^2(\theta)) - 2xycos(\theta)sin(\theta) + 2xycos(\theta)sin(\theta) \\
&= x^2 + y^2 = X^T X
\end{aligned}
$$

h. The rotations about the y and x axis in 2 dimensions are given below.

$$
\begin{aligned}
R_y(\theta_y) &= \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \\
R_x(\theta_x) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}
\end{aligned}
$$

i. Show that the rotations do not commute about the z and y axis.

$$
\begin{aligned}
R_y(\theta_y)R_z(\theta_z) &= \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} cos(\theta_y)cos(\theta_z) & -cos(\theta_y)sin(\theta_z) & sin(\theta_y) \\ sin(\theta_z) & cos(\theta_z) & 0 \\ -sin(\theta_y)cos(\theta_z) & sin(\theta_y)sin(\theta_z) & cos(\theta_y) \end{bmatrix} \\
R_z(\theta_z)R_y(\theta_y) &= \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} cos(\theta_y)cos(\theta_z) & -sin(\theta_z) & cos(\theta_z)sin(\theta_y) \\ sin(\theta_z)cos(\theta_y) & cos(\theta_z) & sin(\theta_z)sin(\theta_y) \\ -sin(\theta_y) & 0 & cos(\theta_y) \end{bmatrix}
\end{aligned}
$$

Which are not necessarily equal.
j. In this small angle approximation to first order, $sin(\theta) \approx \theta$ and $cos(\theta) \approx 1$ in both matrixes.

$$\begin{aligned}
R_z &= \begin{bmatrix} 1 & -\theta_z & 0 \\ \theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
R_y &= \begin{bmatrix} 1 & 0 & \theta_y \\ 0 & 1 & 0 \\ -\theta_y & 0 & 1 \end{bmatrix} \\
R_z R_y &= \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & \theta_y\theta_z \\ -\theta_y & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & 0 \\ -\theta_y & 0 & 1 \end{bmatrix} \\
R_y R_z &= \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & 0 \\ -\theta_y & \theta_y\theta_z & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & 0 \\ -\theta_y & 0 & 1 \end{bmatrix}
\end{aligned}$$

Which are equal.
k. We need to find all 3 infinitesimal matrixes multiplied together. Do this for the first order approximation in part j. Therefore, we already have the y and z rotations multiplied together.

$$\begin{aligned}
R_y(d\theta_y)R_z(d\theta_z) &= \begin{bmatrix} 1 & -d\theta_z & 1 \\ d\theta_z & 1 & 1 \\ -d\theta_y & 0 & 1 \end{bmatrix} \\
R_x(d\theta_x)R_y(d\theta_y)R_z(d\theta_z) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\theta_x \\ 0 & d\theta_x & 1 \end{bmatrix} \begin{bmatrix} 1 & -d\theta_z & d\theta_y \\ d\theta_z & 1 & 0 \\ -d\theta_y & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -d\theta_z & d\theta_y \\ d\theta_z + d\theta_x d\theta_y & 1 & -d\theta_x \\ d\theta_x d\theta_z - d\theta_y & d\theta_x & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -d\theta_z & d\theta_y \\ d\theta_z & 1 & -d\theta_x \\ -d\theta_y & d\theta_x & 1 \end{bmatrix}
\end{aligned}$$

We need to show that this motivatives $d\vec{r} = d\vec{\theta} X \vec{r}$.

$$d\vec{\theta} X \vec{r} = \begin{bmatrix} zd\theta_y - yd\theta_z \\ xd\theta_z - zd\theta_z \\ yd\theta_x - xd\theta_y \end{bmatrix}$$

Let's multiply the 3 rotations by the vector r.

$$\begin{aligned}
R\vec{r} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} zd\theta_y - yd\theta_z \\ xd\theta_z - zd\theta_z \\ yd\theta_x - xd\theta_y \end{bmatrix} \\
&= \vec{r} + d\vec{\theta} X \vec{r}
\end{aligned}$$

Then we can say that $d\vec{r} = \vec{r}(R(d\theta) - \vec{1})$ so therefore the rotation matrix shows the relation $d\vec{r} = d\vec{\theta} X \vec{r}$.

Problem 7. Skew rod with non-point masses
a. To find the angular momentum of the rod, use the equations for angular momentum from chapter 6.

$$\begin{aligned}
\vec{L} &= 2I\omega\hat{k} + \vec{r}_1 X \vec{p}_1 + \vec{r}_2 X \vec{p}_2 \\
\vec{r}_1 &= -lcos(\alpha)\hat{\imath} + lsin(\alpha)\hat{k} \\
\vec{r}_2 &= lcos(\alpha)\hat{\imath} - lsin(\alpha)\hat{k} \\
\vec{p}_1 &= -ml\omega cos(\alpha)\hat{\jmath} \\
\vec{p}_2 &= +ml\omega cos(\alpha)\hat{\jmath} \\
\vec{L} &= 2I\omega\hat{k} + (-lcos(\alpha)\hat{\imath} + lsin(\alpha)\hat{k})X(-ml\omega cos(\alpha)\hat{\jmath}) + (lcos(\alpha)\hat{\imath} - lsin(\alpha)\hat{k})X(ml\omega cos(\alpha)\hat{\jmath}) \\
&= (2I + 2ml^2)\omega\hat{k} + ml^2\omega(cos^2(\alpha)\hat{k} + sin(\alpha)cos(\alpha)\hat{\imath}) + ml^2\omega(cos^2(\alpha)\hat{k} + cos(\alpha)sin(\alpha)\hat{\imath}) \\
\vec{L} &= \omega(2I + 2ml^2cos^2(\alpha))\hat{k} + 2ml^2cos(\alpha)sin(\alpha)\omega\hat{\imath} \\
tan(\theta) &= L_x/L_z = \frac{2ml^2cos(\alpha)sin(\alpha)}{2ml^2cos^2(\alpha) + 2I}
\end{aligned}$$

b. The expression for $\hat{\imath}$ and $\hat{\jmath}$ are found by looking at the angle swept out by the rotating rod with respect to time $\theta = \omega t$.

$$\begin{aligned}
\hat{\imath}'(t) &= cos(\omega t)\hat{\imath} + sin(\omega t)\hat{\jmath} \\
\hat{\jmath}'(t) &= -sin(\omega t)\hat{\imath} + cos(\omega t)\hat{\jmath}
\end{aligned}$$

c. Now we want to find the time derivative of the momentum in both frames.

$$\begin{aligned}
\frac{d\vec{L}}{dt} &= \omega(2ml^2cos^2(\alpha) + 2I)\omega\dot{\hat{k}}' + 2ml^2cos(\alpha)sin(\alpha)\omega\dot{\hat{\imath}}' \\
\dot{\hat{k}}' &= \dot{\hat{k}} = 0 \\
\dot{\hat{\imath}}' &= -\omega sin(\omega t)\hat{\imath} + w cos(\omega t)\hat{\jmath} = \omega\hat{\jmath}' \\
\dot{\vec{L}} &= 2ml^2cos(\alpha)sin(\alpha)\omega^2\hat{\jmath}' \\
&= 2ml^2cos(\alpha)sin(\alpha)(-sin(\omega t)\hat{\imath} + cos(\omega t)\omega^2\hat{\jmath})
\end{aligned}$$

d. Nowe went want to evalulate the time derivative of the momentum from the general expression:

$$\begin{aligned}
\dot{\vec{L}} &= \vec{\Omega} X \vec{L}_{bc} + \dot{\vec{L}}_{bc} \\
\dot{\vec{L}}_{bc} &= 0 \\
\vec{L}_{bc} &= \omega(2I + 2ml^2cos^2(\alpha))\hat{k}' + 2ml^2cos(\alpha)sin(\alpha)\hat{\imath}' \\
\vec{\Omega} &= \omega\hat{k} \\
\vec{\Omega} X \vec{L}_{bc} &= 2ml^2cos(\alpha)sin(\alpha)\omega^2\hat{k}X\hat{\imath}' \\
\hat{k}X\hat{\imath}' &= cos(\omega t)\hat{\jmath} - sin(\omega t)\hat{\imath} = \hat{\jmath}' \\
\dot{\vec{L}} &= 2ml^2cos(\alpha)sin(\alpha)\omega^2\hat{\jmath}'
\end{aligned}$$

d. Let's try to evaluate the moment of inertial tensor in the body-centered frame.

$$\begin{aligned}
\vec{L}_{bc} &= \begin{bmatrix} 2ml^2cos(\alpha)sin(\alpha)\omega \\ 0 \\ (2ml^2cos^2 + 2I)\omega \end{bmatrix} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \\
I_{x'z'} &= 2ml^2cos(\alpha)sin(\alpha) = I_{z'x'} \\
I_{y'z'} &= 0 = I_{z'y'} \\
I_{z'z'} &= 2ml^2cos^2 + 2I
\end{aligned}$$

Thus, we can not find $I_{x'x'}$, $I_{y'y'}$, and $I_{x'y'} = I_{y'x'}$.
Now we want to find the unprimed moment of inertia tensor.

$$\begin{aligned}
I_{xx} &= \hat{i}\cdot\mathbf{I}\cdot\hat{i} = \begin{bmatrix} cos(\omega t) & -sin(\omega t) & 0 \end{bmatrix} \begin{bmatrix} I_{x'x'} & I_{x'y'} & 2ml^2 cos(\alpha)sin(\alpha) \\ I_{x'y'} & I_{y'y'} & 0 \\ 2ml^2 cos(\alpha)sin(\alpha) & 0 & 2ml^2cos^2+2I \end{bmatrix} \begin{bmatrix} cos(\omega t) \\ -sin(\omega t) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} cos(\omega t) & -sin(\omega t) & 0 \end{bmatrix} \begin{bmatrix} I_{x'x'}cos(\omega t) - I_{x'y'}sin(\omega t) \\ I_{x'y'}cos(\omega t) - I_{y'y'}sin(\omega t) \\ 2ml^2cos(\alpha)sin(\alpha)cos(\omega t) \end{bmatrix} \\
&= I_{x'x'}cos^2(\omega t) - I_{x'y'}cos(\omega t)sin(\omega t) - I_{x'y'}cos(\omega t)sin(\omega t) + I_{y'y'}sin^2(\omega t) \\
&= I_{x'x'}cos^2(\omega t) + I_{y'y'}sin^2(\omega t) - 2I_{x'y'}cos(\omega t)sin(\omega t) \\
I_{yy} &= \hat{j}\cdot\mathbf{I}\cdot\hat{j} = \begin{bmatrix} sin(\omega t) & cos(\omega t) & 0 \end{bmatrix} \begin{bmatrix} I_{x'x'} & I_{x'y'} & 2ml^2 cos(\alpha)sin(\alpha) \\ I_{x'y'} & I_{y'y'} & 0 \\ 2ml^2 cos(\alpha)sin(\alpha) & 0 & 2ml^2cos^2+2I \end{bmatrix} \begin{bmatrix} sin(\omega t) \\ cos(\omega t) \\ 0 \end{bmatrix} \\
&= I_{x'x'}sin^2(\omega t) + I_{y'y'}cos^2(\omega t) + 2I_{x'y'}cos(\omega t)sin(\omega t) \\
I_{xy} &= I_{yx} = I_{x'x'}cos(\omega t)sin(\omega t) - I_{y'y'}cos(\omega t)sin(\omega t) \\
I_{xz} &= I_{zx} = 2ml^2cos(\alpha)sin(\alpha)cos(\omega t) \\
I_{yz} &= I_{zy} = 2ml^2cos(\alpha)sin(\alpha)sin(\omega t) \\
I_{zz} &= 2ml^2cos^2+2I \\
\vec{L} &= \mathbf{I}\vec{\omega} = \begin{bmatrix} I_{xx} & I_{xy} & 2ml^2cos(\alpha)sin(\alpha)cos(\omega t) \\ I_{xy} & I_{yy} & 2ml^2cos(\alpha)sin(\alpha)sin(\omega t) \\ 2ml^2cos(\alpha)sin(\alpha)cos(\omega t) & 2ml^2cos(\alpha)sin(\alpha)sin(\omega t) & 2ml^2cos^2+2I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \\
&= \begin{bmatrix} 2ml^2cos(\alpha)sin(\alpha)cos(\omega t)\omega \\ 2ml^2cos(\alpha)sin(\alpha)sin(\omega t)\omega \\ (2ml^2cos^2+2I)\omega \end{bmatrix} = 2ml^2cos(\alpha)sin(\alpha)\omega(cos(\omega t)\hat{i} + sin(\omega t)\hat{j}) + (2ml^2cos^2+2I)\omega\hat{k} \\
\dot{\vec{L}} &= \dot{I}\vec{\omega} = \begin{bmatrix} 0 & 0 & -2ml^2cos(\alpha)sin(\alpha)sin(\omega t)\omega \\ 0 & 0 & 2ml^2cos(\alpha)sin(\alpha)cos(\omega t)\omega \\ -2ml^2cos(\alpha)sin(\alpha)sin(\omega t)\omega & 2ml^2cos(\alpha)sin(\alpha)cos(\omega t)\omega & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \\
&= \begin{bmatrix} -2ml^2cos(\alpha)sin(\alpha)sin(\omega t)\omega^2 \\ 2ml^2cos(\alpha)sin(\alpha)cos(\omega t)\omega^2 \\ 0 \end{bmatrix} = 2ml^2cos(\alpha)sin(\alpha)\omega^2(-sin(\omega t)\hat{i} + cos(\omega t)\hat{j})
\end{aligned}$$

e. Now we can define a new axis and recalculate the moment of inertia tensor. In the z and y direction we can use the parallel axis theorem on the spheres.

$$\begin{aligned}
I_{x''x''} &= 2I \\
I_{z''z''} &= 2(I+ml^2) \\
I_{y''y''} &= 2(I+ml^2)
\end{aligned}$$

f. For the sphere, the off diagonal elements will be 0 because of the symmetry of the sphere. At each point there will be an equal and oppositve contribution on the other side of the sphere that cancels out the contribution in the integral.
Now we want to calculate the off diagonal elements for the skew rod, where the spheres are off centered at some distance l. We can use the parallel axis theorem for off-diagonal elements to evaluate these terms.

$$\begin{aligned}
I_{xy} &= I_{yx} = 2(I^0_{xy} - mxy) = 2(0 - m(L)(0)) = 0 \\
I_{xz} &= I_{zx} = 2(I^0_{xz} - mxz) = 2(0 - m(L)(0)) = 0 \\
I_{yz} &= I_{zy} = 2(I^0_{yz} - myz) = 2(0 - m(0)) = 0
\end{aligned}$$

Therefore, we have diagonalized the moment of inertia tensor in this choice of coordinates.
g. Now we want to evaluate the angular momentum in this choice of coordinates. First, we need to represent $\vec{\omega}$ in the coordinate frame.

$$\begin{aligned}
\vec{\omega} &= \begin{bmatrix} -\omega sin(\alpha) \\ 0 \\ \omega cos(\alpha) \end{bmatrix} \\
\vec{L} &= \mathbf{I}''\vec{\omega} = \begin{bmatrix} 2I & 0 & 0 \\ 0 & 2I + 2ml^2 & 0 \\ 0 & 0 & 2I + ml^2 \end{bmatrix} \begin{bmatrix} -\omega sin(\alpha) \\ 0 \\ \omega cos(\alpha) \end{bmatrix} \\
&= -2l\omega sin(\alpha)\hat{i}'' + 2(I + ml^2)\omega cos(\alpha)\hat{k}''
\end{aligned}$$

Now we want to represent the result from part a in this new coordinate system.

$$\begin{aligned}
\hat{k} &= -sin(\alpha)\hat{i}'' + cos(\alpha)\hat{k}'' \\
\hat{i} &= cos(\alpha)\hat{i}'' + sin(\alpha)\hat{k}'' \\
\vec{L} &= \omega(2I + 2ml^2 cos^2(\alpha))(-sin(\alpha)\hat{i}'' + cos(\alpha)\hat{k}'') + 2ml^2 cos(\alpha)sin(\alpha)\omega(cos(\alpha)\hat{i}'' + sin(\alpha)\hat{k}'') \\
&= \hat{k}''\omega(2Icos(\alpha) + 2ml^2 cos^3(\alpha) + 2ml^2 cos(\alpha)sin^2(\alpha)) \\
&\quad + \hat{i}''\omega(-2Isin(\alpha) - 2ml^2 cos^2(\alpha)sin(\alpha) + 2ml^2 sin(\alpha)cos^2(\alpha)) \\
&= -2Isin(\alpha)\omega\hat{i}'' + (2I + 2ml^2)cos(\alpha)\omega\hat{k}''
\end{aligned}$$

h. Now we want to evaluate the time derivative of the angular momentum.

$$\begin{aligned}
\dot{\vec{L}} &= \vec{\omega} X \vec{L}_{bc} + \dot{\vec{L}}_{bc} \\
&= \begin{bmatrix} \hat{i}'' & \hat{j}'' & \hat{k}'' \\ -\omega sin(\alpha) & 0 & \omega cos(\alpha) \\ -2Isin(\alpha)\omega & 0 & (2I + 2ml^2)cos(\alpha)\omega \end{bmatrix} \\
&= -\hat{j}''((-(2I + 2ml^2)cos(\alpha)\omega)\omega sin(\alpha) + 2Isin(\alpha)\omega^2 cos(\alpha)) \\
&= -2ml^2\omega^2 cos(\alpha)sin(\alpha)\hat{j}'' = -2ml^2\omega^2 cos(\alpha)sin(\alpha)\hat{j}
\end{aligned}$$