Mathematics GU4044 Representations of Finite Groups
Assignment # 9

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Problem 1.

Let G be a finite group with $N \lhd G$. Suppose that $\rho_V = \pi^*(\psi_V)$ where $\psi_V : G/N \to \text{Aut}(V)$ is a G/N-representation. Then, for $g \in G$ and $x \in N$, consider the character,

$$\chi_V(gx) = \text{Tr}\, \psi_V \circ \pi(gx) = \text{Tr}\, \psi_V(gxN) = \text{Tr}\, \psi_V(gN) = \text{Tr}\, \psi_V \circ \pi(g) = \chi_V(g)$$

Conversely, suppose that $\chi_V(gx) = \chi_V(g)$ for all $g \in G$ and $x \in N$. We know that $\rho_V(e) = \text{id}_V$. And thus, $\chi_V(e) = \text{Tr}\, \text{id}_V = \dim V$. However, by the hypothesis, $\chi_V(x) = \chi_V(e) = \dim V$ and thus $\rho_V(x) = \text{id}_V$ for each $x \in N$. In the last line, I have used the fact that,

$$\chi_V(g) \iff \rho_V(g) = \text{id}_V \iff g \in \ker \rho_V$$

Therefore, $N \subset \ker \rho_V$ so the map ρ_V is constant on N cosets and thus ρ_V factors through the quotient by a map $\psi_V : G/N \to \text{Aut}(V)$ such that $\rho_V = \psi_V \circ \pi = \pi^*(\psi_V)$.

Problem 2.

Let D_n be the diherdral group of order $2n$ which is generated as $D_n = \langle \rho, \tau \mid \rho^n = \tau^2 = e,\ \tau\rho\tau^{-1} = \rho^{-1} \rangle$.

(a). For each $a \in \mathbb{Z}/n\mathbb{Z}$, there exists a one-dimensional representation $W - a$ of $\langle \rho \rangle$ with basis u defined by $\rho_{W_a}(\rho^k) \cdot u = e^{2\pi i ak/n} \cdot u$ and hence a two-dimensional representation $V_a = \text{Ind}_{\langle \rho \rangle}^{D_n} W_a$ with character χ_{V_a} given by,

$$\chi_{V_a}(\rho^k) = e^{2\pi i ak/n} + e^{-2\pi i ak/n} = 2\cos(2\pi ak/n) \quad \text{and} \quad \chi_{V_a}(\tau\rho^k) = 0$$

Consider the inner products of characters,

$$\begin{aligned} \langle \chi_{V_a}, \chi_{V_a} \rangle &= \frac{1}{2n}\left(\sum_{k=0}^{n-1} 4\cos^2(2\pi ak/n) + 0\right) \\ &= \frac{1}{n}\sum_{k=0}^{n-1}(1 + \cos(4\pi ak/n)) = \begin{cases} 1 & 2a \not\equiv 0 \bmod n \\ 2 & 2a \equiv 0 \bmod n \end{cases} \end{aligned}$$

by Lemma 0.1. Thus, Therefore, V_a is irreducible iff $\langle \chi_{V_a}, \chi_{V_a} \rangle = 1$ iff $2a \not\equiv 0 \bmod n$. Furthermore, using the same lemma,

$$\begin{aligned}
\langle \chi_{V_a}, \chi_{V_b} \rangle &= \frac{1}{2n}\left(\sum_{k=0}^{n-1} 4\cos(2\pi ak/n)\cos(2\pi bk/n) + 0\right) \\
&= \frac{1}{n}\sum_{k=0}^{n-1}(\cos(2\pi(a+b)k/n) + \cos(2\pi(a-b)k/n) \\
&= \begin{cases} 2 & a+b \equiv a-b \equiv 0 \bmod n \\ 1 & a+b \equiv 0 \bmod n \\ 1 & a-b \equiv 0 \bmod n \\ 0 & \text{else} \end{cases}
\end{aligned}$$

First suppose that both V_a and V_b are irreducible. By Schur's lemma, $\langle \chi_{V_a}, \chi_{V_b} \rangle > 0 \iff V_a \cong V_b$ then by above $V_a \cong V_b \iff a \equiv \pm b \bmod 0$. Suppose that V_b is irreducible then $2b \equiv 0 \bmod n$ so $b \equiv -b \bmod n$. Therefore, if $a + b \equiv 0 \bmod n \iff a - b \equiv 0 \bmod n$ so if $a \equiv \pm b \bmod n$ then $\langle \chi_{V_a}, \chi_{V_b} \rangle = 2$ and thus $V_a \cong V_b$ since each have multiplicity two since $2a \equiv 2b \equiv 0 \bmod n$.

(b). Suppose that $n = 2m + 1$ then for $1 \leq a, b \leq m$ we cannot have $2a \equiv 0 \bmod n$ since $0 < 2a, 2b \leq 2m < n$ and cannot have $a \pm b \equiv 0 \bmod n$ unless $a = b$. Therefore, we have at least m irreducible V_a two-dimensional representations. Let $c_1 \geq 1$ be the number of one-dimensional irreducible representations of D_{2n} (greater than one due to the trivial representation) and c_2 the number of two-dimensional irreducible representations of D_{2n} and c' the sum of the squared dimensions of higher dimension irreducible D_{2n}-representations (if any exist). Then,

$$2n = 4m + 2 = \sum_{i=1}^{h} d_i^2 = c_1 + 4c_2 + c' \geq c_1 + 4m + c'$$

However, $c_1 \geq 1$ and $c' \geq 9$ (if it is nonzero since the smallest rep with $d_i > 2$ has $d_i^2 = 9$) so we must have $c_1 = 2$ and $c_2 = m$ and $c' = 0$.

(c). Suppose that $n = 2m$ then for all $1 \leq a, b \leq m-1$ we cannot have $2a \equiv 0 \mathrm{mod} n$ or $a \equiv \ \mathrm{mod} n$ since $0 < 2a, 2b \leq 2m - 2 < n$ so there are at least $m - 1$ distinct V_a. Therefore, we have $c_2 \geq m - 1$ and $c_1 \geq 1$. Using the same notation as above,

$$2n = 4m = \sum_{i=1}^{h} d_i^2 = c_1 + 4c_2 + c' \geq c_1 + 4(m-1) + c'$$

Therefore, $c_1 + c' \leq 4$ but $c' \geq 9$ if $c' \neq 0$ so $c' = 0$. However, due to the trivial representation, $c_1 \geq 1$ so we cannot have $c_2 \geq m - 1$ because $4m = c_1 + 4c_2$ and thus $4(m - c_2) \geq 1$. Therefore $c_2 = m - 1$ and thus $c_1 = 4$. Therefore, D_{4m} has $m - 1$ two-dimensional irreducible representations and 4 irreducible one-dimensional representations.

Problem 3.

Let H be a subgroup of G and let $\mathbb{C}[G/H]$ be the permutation representation of G on the cosets G/H. The restriction representation $\mathrm{Res}_H^G \mathbb{C}[G/H]$ is the direct sum of trivial representation if the representation acts trivially for each $h \in H$. Suppose that H is normal in G. Then we know that for $h \in H$ we have $\rho_{\mathbb{C}[G/H]}(h) \cdot gH = hgH = hHg = Hg = gH$ so $\rho_V(h)$ fixes the basis of $\mathbb{C}[G/H]$ and thus $\rho_{\mathbb{C}[G/H]}(h) = \mathrm{id}_{\mathbb{C}[G/H]}$ so $\mathbb{C}[G/H]$ is a direct sum of the trivial representation. Convsersely, suppose that $\mathrm{Res}^G_H \mathbb{C}[G/H]$ is the direct sum of trivial representation then we know that $\rho_V(h) = \mathrm{id}_{\mathbb{C}[G/H]}$ for each $h \in H$. Then for any $h \in H$ and any $g \in G$ we know that,

$$\rho_V(h) \cdot gH = hgH = gH \implies g^{-1}hgH = H \implies g^{-1}hg \in H$$

Thus, $H \lhd G$.

Problem 4.

Suppose that $H \lhd G$. Let W be an H-representation and let $V = \mathrm{Ind}^G_H W$. Let $x_1, \dots, x_k$ be coset representatives of G/H. Using the character formula,

$$\chi_V(g) = \sum_{x_i^{-1} g x_i \in H} \chi_W(x_i^{-1} g x_i)$$

Since H is normal, $x_i^{-1} g x_i \in H \iff g \in x_i H x_i^{-1} = H$. Therefore,

$$\chi_V(g) = \begin{cases} \sum_{i=1}^{k} \chi_W(x_i^{-1} g x_i) & g \in H \\ 0 & g \notin H \end{cases}$$

Lemmas

Lemma 0.1.

$$\frac{1}{n} \sum_{k=0}^{n-1} \cos(2\pi ak/n) = \begin{cases} 1 & a \equiv 0 \bmod n \\ 0 & a \not\equiv 0 \bmod n \end{cases}$$

Proof.

$$\frac{1}{n} \sum_{k=0}^{n-1} \cos(2\pi ak/n) = \frac{1}{2n} \sum_{k=0}^{n-1} (\zeta_n^{ak} + \zeta_n^{-ak}) = \frac{1}{2n} \left[\frac{1 - \zeta_n^{an}}{1 - \zeta_n^{a}} + \frac{1 - \zeta_n^{-an}}{1 - \zeta_n^{-a}} \right]$$

Therefore, if $a \not\equiv 0 \bmod n$ then $\zeta_n^a \neq 1$ and since $\zeta_n^{an} = 1$ we have that,

$$\sum_{k=0}^{n-1} \cos(2\pi ak/n) = 0$$

However, if $a \not\equiv 0 \bmod n$ then $\zeta_n^{ak} = 1$ so,

$$\frac{1}{n} \sum_{k=0}^{n-1} \cos(2\pi ak/n) = \frac{1}{2n} \sum_{k=0}^{n-1} (\zeta_n^{ak} + \zeta_n^{-ak}) = 1$$

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