Mathematics GU4053 Algebraic Topology
Assignment # 3

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Note. My order of path concatenation follows lectures,

$$\gamma * \delta(x) = \begin{cases} \delta(2x) & x \le \frac{1}{2} \\ \gamma(2x-1) & x \ge \frac{1}{2} \end{cases}$$

Problem 1.

Let $M_0 = \mathbb{R}^n$ for $n \geq 3$. Let $S \subset \mathbb{R}^n$ be finite. Take a finite sequence s_i which exhausts S. Then define $M_i = M_{i-1} \setminus \{s_i\}$. Suppose that M_{i-1} is a simply-connected n-manifold. Then, by Lemma , M_i is simply-connected. However, $M_i = \mathbb{R}^n \setminus A$ for $A \subset S \subset \mathbb{R}^n$ finite so, by Lemma , M_i is a simply-connected n-manifold. Also, $M_0 = \mathbb{R}^n$ is contractable and thus simply-connected. By induction, every M_n is simply-connected. Therefore, $M_N = \mathbb{R}^n \setminus S$ for $N = |S|$ is simply connected.

Problem 2.

Let $X \subset \mathbb{R}^3$ be a union of n lines through the origin. First, consider $\mathbb{R}^3$ minus a single line which deformation retracts onto a hollow cylinder about the line. The next $n-1$ lines each intersect the cylinder in exactly two points. Therefore, $\mathbb{R}^3 \setminus X$ deformation retracts to a cylinder minus $2(n-1)$ points which futher deformation retracts to a plane minus $2n-1$ points (since a cylinder is homeomorphic to a punctured plane under the logarithm map). The plane minus $2n-1$ points deformation retracts onto the wedge of $2n-1$ circles. Thus,

$$\pi_1(\mathbb{R}^3 \setminus X) \cong \pi_1 \left(\bigvee_{i=1}^{2n-1} S^1 \right) \cong \mathbb{Z}^{(2n-1)*}$$

Problem 3.

Let A be an open set containing the first torus and a small section (not containing any nontrivial loops) of the second torus. Similarly, let B be an open set containing the second torus and a small section of the first. Then, $X = A \cup B$. Because we did not contain any nontrivial loops in the other tori, we can deformation retract A and B onto a single torus. Thus, $\pi_1(A, x_0) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(B, x_0) \cong \mathbb{Z} \times \mathbb{Z}$. However, the intersection $A \cap B$ is path-connected and deformation retracts to $S^1 \times \{x_0\}$ which is the circle identified in both tori. Therefore, $\pi_1(X, x_0) \cong (\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z}) / \sim$ where $(n, 0)_1 \sim (n, 0)_2$ since the inclusion maps of $S^1 \times \{x_0\}$ into A and B will take a loop that goes arround n times into the terms $(n,0)_1$ and $(n,0)_2$ which then must be equal. Denote $(n,0)_1$ by a^n and $b^n = (0,n)_1$ and $c^n = (0,n)_2$. Then, $ab = ba$ because,

$$(1,0)_1 * (0,1)_1 = (1,0)_1 + (0,1)_1 = (1,1)_1 = (0,1)_1 * (1,0)_1$$

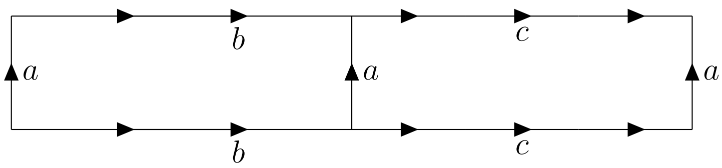
Likewise, $ac = ca$ because,

$$(1,0)_1 * (0,1)_2 = (1,0)_2 * (0,1)_2 = (1,0)_2 + (0,1)_2 = (1,1)_2 = (0,1)_2 * (1,0)_2 = (0,1)_2 * (1,0)_1$$

Therefore,

$$\pi_1(X, x_0) \cong \left\langle a, b, c \mid aba^{-1}b^{-1} = e, aca^{-1}c^{-1} = e \right\rangle$$

This is more easily seen from the identification space,



In which the relations $aba^{-1}b^{-1} = e$ and $aca^{-1}c^{-1} = e$ are obvious by contracting the squares.

Problem 4.

Let $X = \mathbb{R}^2 \backslash \mathbb{Q}^2$. Take your favorite irrational number $x_0 \in \mathbb{R}\backslash\mathbb{Q}$. Take any $x \in \mathbb{R}\backslash\mathbb{Q}$ then consider the square defined by the coordinates, $(x_0, x_0), (x_0, x), (x, x_0), (x, x)$. The boundary of this square has no points in $\mathbb{Q}^2$ because $x, x_0 \notin \mathbb{Q}$ so traversing this boundard defines a loop $\gamma_x : I \to X$ at (x_0, x_0). I claim that if $x \neq y$ for $x, y \in \mathbb{R}\backslash\mathbb{Q}$ then γ_x and γ_y are not homotopic. Without loss of generality, take $x < y$ so there exists $q \in [x, y] \cap \mathbb{Q}$. Because $X \subset \mathbb{R}^2 \backslash \{(q,q)\}$ if the paths γ_x and γ_y are homotopic in X then they must also be homotopic in $\mathbb{R}^2\backslash\{(q,q)\}$. However, (q,q) is in the interior of γ_y but not of γ_x so γ_x is homotopic to the constant path but γ_y cannot be because it must generate the fundamental group. However, the punctured plane retracts onto the circle so it has a nontrivial fundamental group. Therefore, γ_x and γ_y are not homotopic in $\mathbb{R}^2\backslash\{(q,q)\}$ and thus not homotopic in X thus proving the claim. Therefore, there is an injection from $\mathbb{R}$ into homotopy classes of loops at (x_0, x_0) in X. Thus, $\pi_1(\mathbb{R}^2\backslash\mathbb{Q}^2, (x_0, x_0))$ is uncountable.

Problem 5.

Are the following categories?

(a). Objects are finite sets, morphisms are injective maps of sets:

Yes because the identity is always injective and the composition of injective maps is always injective.

(b). Objects are sets, morphisms are surjective maps of sets:

Yes because the identity is always surjective and the composition of surjective maps is always surjective.

(c). Objects are abelian groups, morphisms are isomorphisms of abelian groups:

Yes because the identity is always an isomorphisms and the composition of isomorphisms is an isomorphism.

(d). Objects are sets, morphisms are maps of sets which are not surjective:

No! because the identity is always surjective so this category cannot contain identity maps.

(e). Objects are topological spaces, morphisms are homeomorphisms:

Yes because the identity is always a homeomorphisms and the composition of homeomorphisms is always a homeomorphism.

Problem 6.

(a). Define a category $\mathcal{C}$ with a single object $\mathbb{Z}/5\mathbb{Z}$ with morphisms that are automorphisms of $\mathbb{Z}/5\mathbb{Z}$. We know that this group has exactly four automorphisms including one identity and the composition of two automorphisms is an automorphism.

(b). Let $\mathcal{C}$ be the category with objects given by the groups $\mathbb{Z}/2\mathbb{Z}$ and the trivial group $\{0\}$ with morphisms given by all homomorphisms on these groups. There are exactly two homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to itself, namely the identity and the zero map. There is exactly one homomorphism from $\{0\}$ to itself, namely the identity (which equals the zero map). Furthermore, there is exactly one homomorphism (the zero map) between the groups in each direction. Thus, there are a total of 5 morphisms.

Problem 7.

(a). Let $F : \mathbf{0} \to \mathcal{C}$ be an empty diagram in $\mathcal{C}$. Then, the colimit X is an object of $\mathcal{C}$ such that given any object A (which vacuously has a natural transormation $\eta : F \to \underline{A}$ since there are no objects in the domain of F and thus η is the empty set of maps) there is a unique map $f : X \to A$ such that f perserves the empty natural transformation which is true of any map. This is equivalent to the condition that for any $A \in \mathrm{Ob}(\mathcal{C})$ there exists a unique map $f : X \to A$.

(b). Suppose a category $\mathcal{C}$ has two initial objects X and Y. Because X and Y are initial, there exist unique maps $f : X \to Y$ and $g : Y \to X$. Then, $f \circ g : Y \to Y$ and $g \circ f : X \to X$ are maps from initial objects to themselves. However, an initial object has a unique map from it to any other object including itself and any object has an idenitiy map. Therefore, $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Since f and g were unique, there is a unique isomorphism $X \cong Y$.

(c).
- **Set** has an initial object, namely the empty set $\varnothing$. The trivial map is the unique map from $\varnothing$ to any set.
- **Grp** has an initial object, namely the trivial group $\{0\}$. There is a unique map from $\{0\}$ to any group which sends 0 to the idenity.

- **Top** has an initial object, namely the empty set with the trivial (only possible) topology $\varnothing$. The trivial map is the unique continuous (vacuously) map from $\varnothing$ to any topological space.
- **Top$_\bullet$** has an initial object, namely any one point set $(\{x_0\}, x_0)$ with the trivial (only possible) topology. Given any (Y, y_0) there is a unique map $f : x_0 \mapsto y_0$ from $(\{x_0\}, x_0)$ to (Y, y_0).
- The category of fields with field homomorphisms does not have an initial object. The fields $\mathbb{Q}$ and $\mathbb{F}_2$ both have no subfields and any field homomorphism is an embedding. Therefore, any initial object must be embedded in both $\mathbb{Q}$ and $\mathbb{F}_2$ which implies that it equals both $\mathbb{Q}$ and $\mathbb{F}_2$ which is obviously false.
- The category of infinite-dimensional vectorspaces over a given field with linear maps does not have an initial object because any infinite-dimensional vectorspace has a nontrivial automorphism so it does not have a unique map to itself.
- The category **Cat** of small categories has an initial object, namely **0** the empty category which has a unique functor from itself to any category as defined in the problem statement.

(d). A terminal object X is such that for any $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique map $f : Y \to X$. Consider the following categories,

- **Set** has a terminal object, namely the set with one element. There is exactly one map, the map sending everything to the same place, from any set to a one element set.
- **Grp** has a terminal object, namely the trivial group $\{0\}$. There is a unique map to $\{0\}$ from any group which sends everything to $\{0\}$.
- **Top** has a terminal object, namely a singleton set with the trivial (only possible) topology. There is exactly one (continuous) map from any set to a one element set which is the constant map.
- **Top$_\bullet$** has a terminal object, namely any singleton set $(\{x_0\}, x_0)$ with the trivial (only possible) topology. Given any (Y, y_0) there is a unique map from (Y, y_0) to (X, x_0) which sends everything to x_0.
- The category of fields with field homomorphisms does not have a terminal object. Every field homomorphism is an embedding so a terminal object must have every field embedded inside it. In particular, it must contain a copy of $\mathbb{Q}$ and of $\mathbb{F}_2$. This is clearly impossible as it would have to simultaneously have characteristic zero and 2.
- The category of infinite-dimensional vectorspaces over a given field with linear maps does not have a terminal object because any infinite-dimensional vectorspace has a nontrivial automorphism so it does not have a unique map to itself.
- The category **Cat** of small categories has a terminal object, namely the category with one object and one morphism. From any small category, there is a unique functor sending all objects and all maps to the single object and idenity map respectively.

Problem 8.

Let X be a path-connected topological space and let $\Pi(X)$ and $\pi_1(X, x_0)$ denote the fundamental groupoid and fundamental group at $x_0 \in X$ of X respectively. There is a natural inclusion functor,

$$J : \pi_1(X, x_0) \to \Pi(X)$$

given by $J(x_0) = x_0$ and $J(\gamma) = \gamma$ for any loop at x_0. Now, since x is path-connected, at each point $x \in X$ we can choose a path from x_0 to x called γ_x. For convenience, choose γ_{x_0} to be the trivial loop. The inverse functor is determined by the choice of these paths. Now, define the functor, $K : \Pi(X) \to \pi_1(X, x_0)$ by $K(x) = x_0$ for any point $x \in X$ and if γ is a path from x to y then $K(\gamma) = \hat{\gamma}_y \circ \gamma \circ \gamma_x$ where $\circ$ denotes composition in the category $\Pi(X)$ which is path composition $*$ and $\hat{\gamma}$ denotes the inverse path which always exists because $\Pi(X)$ is a groupoid. First, it must be shown that K is a covariant functor. Consider the diagram in $\Pi(X)$,

$$\begin{tikzcd}
x_0 \arrow[r, "K(\mu)"] \arrow[d, "\gamma_x"] & x_0 \arrow[r, "K(\delta)"] \arrow[d, "\gamma_y"] & x_0 \arrow[d, "\gamma_z"] \\
x \arrow[r, "\mu"] & y \arrow[r, "\delta"] & z
\end{tikzcd}$$

by definition, both small squares commute. Furthermore,

$$K(\delta \circ \mu) = \hat{\gamma}_z \circ \delta \circ \mu \circ \gamma_x = \hat{\gamma}_z \circ \delta \circ \hat{\gamma}_y \circ \gamma_y \circ \mu \circ \gamma_x = K(\delta) \circ K(\mu)$$

Thus, the large square commutes so the entire diagram commutes. Likewise,

$$K(\mathrm{id}_x) = \hat{\gamma}_x \circ \mathrm{id}_x \circ \gamma_x = \hat{\gamma}_x \circ \gamma_x = \mathrm{id}_x$$

so K is a functor. It suffices to show that $J \circ K$ and $K \circ J$ are naturally equivalent to $\mathrm{id}_{\Pi(X)}$ and $\mathrm{id}_{\pi_1(X,x_0)}$ respectively.

First, the above diagram gives a natural equivalence between $J \circ K$ and $\mathrm{id}_{\Pi(X)}$ with $\eta_x = \gamma_x$ because γ_x is an isomorphism, $J \circ K(x) = x_0$, $J \circ K(\delta) = K(\delta)$, and the squares all commute.

Likewise, $K \circ J(x_0) = x_0$ and $K \circ J(\delta) = \delta$ since we took γ_{x_0} to be trivial i.e. the identity map on x_0. Thus, $K \circ J = \mathrm{id}_{\pi_1(X,x_0)}$ exactly which is obviously naturally equivalent to $\mathrm{id}_{\pi_1(X,x_0)}$ by $\eta_x = \mathrm{id}_x$. Therefore, J and K are equivalences of categories.

Lemma 0.1. Let M be a path-connected manifold of dimension $n \geq 3$ then for any $p \in M$,

$$\pi_1(M) \cong \pi_1(M \backslash \{p\})$$

Proof. Since M is a manifold, there exists an open neighborhood U of p such that $U \cong \mathbb{R}^n$. Take, $V = M \backslash \{p\}$ which is also open. Since M is path-connected, V is path-connected and since $U \cong \mathbb{R}^n$ we have that U is path-connected. Furthermore, $U \cap V = U \backslash \{p\}$ is also path-connected. Clearly, $M = U \cup V$. However $\pi_1(U) \cong \pi_1(\mathbb{R}^n) \cong \{e\}$ and $\pi_1(U \cap V) = \pi_1(U \backslash \{p\}) \cong \pi_1(\mathbb{R}^n \backslash \{p'\}) \cong \{e\}$ because for $n \geq 3$ we know that $\mathbb{R}^n$ minus a point is simply connected. Thus, applying Van-Kampen,

$$\pi_1(M) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \{e\} *_{\{e\}} \pi_1(M \backslash \{p\}) \cong \pi_1(M \backslash \{p\})$$

□

Lemma 0.2. $\mathbb{R}^n$ minus a finite set S is a connected n-manifold.

Proof. For any $x \in \mathbb{R}^n \backslash S$ let $r = \min_{s \in S} |x - s|$ which is positive because $x \notin S$ and S is finite. Take $U = B_x(r) \cong \mathbb{R}^n$ and $U \subset \mathbb{R}^n \backslash S$ because if $y \in U$ then $|x - y| < |x - s|$ for all $s \in S$ so $y \notin S$. Thus, $\mathbb{R}^n \backslash S$ is locally euclidean. The Hausdorff and second-countable properties are inherited by the subspace topology. $\square$