Mathematics GR6261 Commutative Algebra
Assignment # 3

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Problem 1

Let B be an A-algebra such that,

$$\sum_{f \in \mathrm{Hom}_A(B,A)} f(B) = A$$

where $\mathrm{Hom}_A(B,A)$ are all A-linear maps as A-modules (not necessarily ring maps). Therefore, for each $x \in A$ there exists a finite list of maps $f_i \in \mathrm{Hom}_A(B,A)$ and elements $b_i \in B$ such that $f_1(b_1) + \cdots + f_n(b_n) = x$. Therefore, the evaluation map,

$$\mathrm{ev} : \mathrm{Hom}_A(B,A) \otimes_A B \to A$$

given by $f \otimes b \mapsto f(b)$ is surjective. In particular, there exists some,

$$e = \tilde{f}_1 \otimes \tilde{b}_1 + \cdots + \tilde{f}_n \otimes \tilde{b}_n \in \mathrm{Hom}_A(B,A) \otimes_A B$$

such that $\mathrm{ev}(e) = 1_A$ i.e.

$$\tilde{f}_1(\tilde{b}_1) + \cdots \tilde{f}_n(\tilde{b}_n) = 1_A$$

Furthermore, $\mathrm{Hom}_A(B,A) \otimes_A B$ is a B-module via extension of scalars so the map $q : B \to \mathrm{Hom}_A(B,A) \otimes_A B$ via,

$$b \mapsto b \cdot e = \tilde{f}_1 \otimes (b\tilde{b}_1) + \cdots + \tilde{f}_n \otimes (b\tilde{b}_n)$$

is a B-linear map. Thus, $r = \mathrm{ev} \circ q$ is A-linear and,

$$r(1_B) = \mathrm{ev} \circ q(1_B) = \mathrm{ev}(1_B \cdot e) = \mathrm{ev}(e) = 1_A$$

For $a \in A$ we have $r(a \cdot 1_B) = a \cdot r(1_B) = a \cdot 1_A = a$. Let $\iota : A \to B$ denote the canonical map induced by the A-algebra structure of B given by $a \mapsto a \cdot 1_B$. Therefore we have $r \circ \iota = \mathrm{id}_A$ which is a retraction of $B \to A$. Firstly, this implies that $\iota : A \to B$ is injective so we may view A as a subring of B i.e. we can write $A \subset B$ and thus $r|_A = \mathrm{id}_A$.

Now for an ideal $I \subset A$ consider,

$$r(I \cdot B \cap A) \subset r(I \cdot B) \cap r(A) = I \cdot r(B) \cap A = I \cdot A \cap A = I$$

Thus, if $x \in I \cdot B \cap A$ then $x \in A$ so $r(x) = x$ and therefore $I \cdot B \cap A \subset I$. However, if $x \in I$ then $x \in A$ and $x = x \cdot 1_B \in I \cdot B$ so $x \in I \cdot B \cap A$. Therefore $I \cdot B \cap A = I$ proving property (i). Lemma 1.1 shows that property (i) implies property (ii) i.e. the map $\iota^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Problem 2

Let k be a field and $A = k[t]_{(t)}$. Then consider the ideal $(tX - 1) \subset A[X]$. Since $k[t]_{(t)}$ is a local domain with maximal ideal $tk[t]_{(t)}$ we know that adjoining t^{-1} gives the entire field of fractions $k(t)$. Thus, the map $\mathrm{ev}_{t^{-1}} : A[X] \to k(t)$ given by evaluation at t^{-1} is surjective. Furthermore, the polynomial $p = tX - 1 \in A[X]$ is clearly in the kernel $\ker \mathrm{ev}_{t^{-1}}$ and if $p \in A[X]$ is in the kernel then $p \in k(t)[X]$ with $p(t^{-1}) = 0$ so because $k(t)$ is a field, we can write, $p(X) = (tX - 1)g(X)$ where $g \in k(t)[X]$. But then $p(X) = tXg(X) - g(X) \in A[X]$ so the linear coefficient must be in A i.e. $g_0 \in A$. Suppose that $g_i \in A$ then $p_{i+1} = tg_i - g_{i+1}$ so $g_{i+1} = tg_i - p_{i+1} \in A[X]$ and thus by induction $g \in A[X]$ since each coefficient is in A. Thus, $p \in (tX - 1)A[X]$. Therefore, $\ker \mathrm{ev}_{t^{-1}} = (tX - 1)$ so by the first isomorphism theorem,

$$A[X]/(tX - 1) \cong k(t)$$

which implies that $(tX - 1)$ is a maximal ideal since $k(t)$ is a field. Suppose the ideal $(tX - 1) \cap A$ is not maximal then it is contained in some maximal ideal $\mathfrak{m} \subset A$. However, if there existed a prime ideal $\mathfrak{P} \supset (tX - 1)A[X]$ such that $\mathfrak{P} \cap A = \mathfrak{m}$ then, since $\mathfrak{m}$ is strictly larger then $(tX - 1) \cap A$, we would have $\mathfrak{P} \supsetneq (tX - 1)A[X]$ contradicting the maximality of $(tX - 1)A[X]$. Therefore, if $(tX - 1) \cap A$ is not maximal then the going-up property fails. However, $(tX - 1) \cap A = (0)$ because every element of $(tX - 1)$ has X-degree at least 1. Furthermore, $k[t]_{(t)}$ has a nonzero maximal ideal $tk[t]_{(t)}$ so $(tX - 1) \cap A = (0)$ is not maximal. Thus, the going-up property fails. However, $A[X]$ is always free over A with a decomposition as a graded A-module,

$$A[X] \cong \bigoplus_{n=0}^{\infty} Ax^n$$

with each factor naturally isomorphic to A.

Problem 3

Let $A \subset B$ be an integral extension of rings such that a unique prime $\mathfrak{P}$ of B lies above the prime $\mathfrak{p} \subset A$. By Lemma 1.4, the extension $A_\mathfrak{p} \subset B_\mathfrak{p} = B \otimes_A A_\mathfrak{p}$ is integral. Notate the map $\iota : A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ then the map $\iota^* : \mathrm{Spec}\,(B \otimes_A A_\mathfrak{p}) \to \mathrm{Spec}\,(A_\mathfrak{p})$ is surjective and has the property that the maximal primes of $B_\mathfrak{p}$ are exactly those above the maximal ideals of $A_\mathfrak{p}$ by Cohen's theorem. However, $A_\mathfrak{p}$ is local with unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$ and the ideal $\mathfrak{P}B_\mathfrak{p}$ lies above $\mathfrak{p}A_\mathfrak{p}$. By Lemma 1.6, the primes of $B_\mathfrak{p}$ are exactly those primes $\mathfrak{P}B_\mathfrak{p} \in \mathrm{Spec}\,(B)$ which are disjoint from $S = A \setminus \mathfrak{p}$ and thus $\mathfrak{P} \cap A \subset \mathfrak{p}$. Therefore, any prime of $B_\mathfrak{p}$ above $\mathfrak{p}A_\mathfrak{p}$ is of the form $\mathfrak{P}B_\mathfrak{p}$ with $\mathfrak{P} \cap A = \mathfrak{p}$ and there is a unique such $\mathfrak{P}$ by assumption. Therefore, $B_\mathfrak{p}$ has a unique maximal ideal $\mathfrak{P}B_\mathfrak{p}$. However, $B_\mathfrak{p} = S^{-1}A$ where $S = A \setminus \mathfrak{p}$ so if $B \to C$ maps all $B \setminus \mathfrak{P}$ to units then it maps $A \cap (B \setminus \mathfrak{P}) = A \setminus \mathfrak{p}$ to units. Thus $B \to C$ factors through $B_\mathfrak{p} \to C$ by the universal property of localization. Furthermore, the map $B \to B_\mathfrak{p}$ sends $B \setminus \mathfrak{P}$ to units because $B_\mathfrak{p}$ is local with unique maximal ideal $\mathfrak{P}B_\mathfrak{p}$. Thus, $B_\mathfrak{p}$ is the initial object in the category of B-algebras with canonical homomorphisms $B \to C$ sending $B \setminus \mathfrak{P}$ to units which is the universal property of the localization $B_{\mathfrak{P}}$. This universal property uniquely fixes $B_{\mathfrak{p}} \cong B_{\mathfrak{P}}$ up to unique B-isomorphism.

Problem 4

Let $A \subset B \subset Q$ be domains where Q is the field of fractions of A and thus also of B. If $A = B$ then clearly B is faithfully-flat over A since $(-) \otimes_A B = (-) \otimes_A A$ is the trivial functor.

Conversely, suppose that B is faithfully flat over A. Consider the exact sequence of A-modules,

$$0 \longrightarrow A \xrightarrow{\ \iota\ } B \xrightarrow{\ \pi\ } B/A \longrightarrow 0$$

and apply the functor $(-)\otimes_A B$ which is exact since B is flat to get the exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & A \otimes_A B \arrow[r, "\iota\otimes\mathrm{id}_B"] & B \otimes_A B \arrow[r, "\pi\otimes\mathrm{id}_B"] \arrow[l, bend left, dashed, "m"] & B/A \otimes_A B \arrow[r] & 0 \end{tikzcd}$$

in which there exists the section $m : B \otimes_B B \to A \otimes_A B$ given by $b \otimes b' \mapsto 1 \otimes bb'$. This map is well-defined by the universal property of the tensor product since the multiplication on tensors is bilinear. Furthermore,

$$m \circ (\iota \otimes \mathrm{id}_B) : a \otimes b \mapsto a \otimes b \mapsto 1 \otimes ab = a \otimes b$$

and thus $m \circ (\iota \otimes \mathrm{id}_B) = \mathrm{id}_{A \otimes_A B}$ so the sequence is left-split. Since $A \otimes_A B \cong B$, the splitting gives an isomorphism,

$$B \otimes_A B \cong B \oplus (B/A \otimes_A B)$$

which implies that there is an embedding,

$$B/A \otimes_A B \hookrightarrow B \otimes_A B$$

However, I claim that $B/A \otimes_A B$ is a torsion A-module. Since $B \subset Q$ we can write an arbitrary element of B as some fraction $\frac{x}{y}$ for $x, y \in A$. Then,

$$y \cdot ([\tfrac{x}{y}] \otimes b) = [y \cdot \tfrac{x}{y}] \otimes b = [x] \otimes b = 0$$

since $x \in A$. Furthermore, we can clear the denominators of a sum of tensors by taking the products of the denominators. Thus, $B/A \otimes_A B$ is entirely A-torsion so it is embedded into the A-torsion submodule of $B \otimes_A B$. However, B is A-flat so $B \otimes_A B$ is also A-flat since the composition of exact functors is exact. Therefore $B \otimes_A B$ is torsion-free by Corollary 1.8 so we have,

$$B/A \otimes_A B \hookrightarrow T_A(B \otimes_A B) = 0$$

and thus $B/A \otimes_A B = 0$. However, B is faithfully-flat over A so this implies that $B/A = 0$ and thus $A = B$.

Problem 5

Let A be a ring and B a Noetherian A-algebra such that the going-up property holds for the canonical map $\iota : A \to B$. Consider the map $\iota^* : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$. An arbitrary closed set of $\mathrm{Spec}\,(B)$ takes the form $V(I)$ where $I \subset B$ is an ideal. Since B is Noetherian, I admits a primary decomposition,

$$I = Q_1 \cap \cdots \cap Q_n$$

with associated primes $P_i = \sqrt{Q_i}$. Therefore,

$$V(I) = V\left(\bigcap_{i=1}^{n} Q_i\right) = \bigcup_{i=1}^{n} V(Q_i)$$

Furthermore, by Lemma 1.9, $V(Q_i) = V(P_i)$ and thus,

$$V(I) = \bigcup_{i=1}^{n} V(P_i)$$

Since finite unions of closed sets are closed, it suffices to prove that $\iota^*(V(\mathfrak{P}))$ is closed for $\mathfrak{P} \subset B$ prime. If $\mathfrak{P}' \supset \mathfrak{P}$ then $\iota^*(\mathfrak{P}') \supset \iota^*(\mathfrak{P})$ so $\iota^*(\mathfrak{P}') \in V(\iota^*(\mathfrak{P}))$. Therefore, $\iota^*(V(\mathfrak{P})) \subset V(\iota^*(\mathfrak{P}))$. Furthermore, suppose we have $\mathfrak{p} \in V(\iota^*(\mathfrak{P}))$ i.e. we have $\mathfrak{p} \supset \iota^*(\mathfrak{P})$. By the going-up property there exists a prime $\mathfrak{P}' \supset \mathfrak{P}$ such that $\iota^*(\mathfrak{P}') = \mathfrak{p}$ and thus $\mathfrak{p} \in \iota^*(V(\mathfrak{P}))$. Therefore, $\iota^*(V(\mathfrak{P})) = V(\iota^*(\mathfrak{P}))$ and is closed.

1 Lemmata

Lemma 1.1. Let B be an A-algebra where $\iota^{-1}(I \cdot B) = I$ for any ideal $I \subset A$. Then, the canonical map $\iota : A \to B$ induces a surjection $\iota^* : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$.

Proof. Consider the subset of ideals,

$$\Sigma = \{ I \subset B \mid \iota^{-1}(I) = \mathfrak{p} \}$$

which is a poset under inclusion. Furthermore, $\mathfrak{p} \cdot B \in \Sigma$ since, by hypothesis, $\iota^{-1}(\mathfrak{p} \cdot B) = \mathfrak{p}$. Thus Σ is nonempty. Suppose that $\mathcal{I} \subset \Sigma$ is a chain and consider,

$$U = \bigcup_{I \in \mathcal{I}} I$$

This is an ideal because if $x, y \in U$ then $x \in I$ and $y \in I'$ for $I, I' \in \mathcal{I}$ but $\mathcal{I}$ is totally ordered so either $I \subset I'$ or $I' \subset I$ and thus the larger contains both x and y and therefore the sum and the multiples are in U. Furthermore,

$$\iota^{-1}(I) = \iota^{-1}\left(\bigcup_{I \in \mathcal{I}} I\right) = \bigcup_{I \in \mathcal{I}} \iota^{-1}(I) = \bigcup_{I \in \mathcal{I}} \mathfrak{p} = \mathfrak{p}$$

so $U \in \Sigma$. However, $\forall I \in \mathcal{I} : I \subset U$. Since every chain has a maximum, by Zorn's Lemma there exist maximal elements of Σ above every ideal $I \in \Sigma$. Suppose $\mathfrak{m} \in \Sigma$ is maximal and $x, y \notin \mathfrak{m}$. Then, by maximality, $(x)+\mathfrak{m} \notin \Sigma$ and $(y)+\mathfrak{m} \notin \Sigma$. Therefore, $\iota^{-1}(x+\mathfrak{m}) \supsetneq \mathfrak{p}$ and $\iota^{-1}(y+\mathfrak{m}) \supsetneq \mathfrak{p}$ since $\iota^{-1}(\mathfrak{m}) = \mathfrak{p}$ and otherwise these ideals would be elements of Σ contradicting the maximality of $\mathfrak{m}$. Thus, there exists $x', y' \in A \setminus \mathfrak{p}$ such that $\iota(x') \in (x) + \mathfrak{m}$ and $\iota(y') \in (y) + \mathfrak{m}$. Since $\mathfrak{p}$ is prime, $x'y' \in A \setminus \mathfrak{p}$ and furthermore $\iota(x'y') \in [(x) + \mathfrak{m}][(y) + \mathfrak{m}] \subset (xy) + \mathfrak{m}$ so $\iota^{-1}(xy + \mathfrak{m}) \supsetneq \mathfrak{p}$. Therefore $xy \notin \mathfrak{m}$ so $\mathfrak{m}$ is prime. Thus, there exists a prime $\mathfrak{m} \in \mathrm{Spec}\,(B)$ such that $\iota^{-1}(\mathfrak{m}) = \mathfrak{p}$ for each prime $\mathfrak{p} \in \mathrm{Spec}\,(A)$ so the map $\iota^* : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is surjective. □

Lemma 1.2. Let $A \subset B$ be an integral extension and $S \subset A$ be a multiplicative subset then $S^{-1}A \subset S^{-1}B$ is integral.

Proof. For any $(b, s) \in S^{-1}B$ we have $b \in B$ so, because B is integral over A, b satisfies a monic polynomial $p \in A[x]$ given by,

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

Then consider the monic polynomial $p' \in S^{-1}A[x]$ given by,

$$p'(x) = x^n + \frac{a_{n-1}}{s}x^{n-1} + \cdots + \frac{a_0}{s^n}$$

Then we have,

$$p'\left(\frac{b}{s}\right) = \left(\frac{b}{s}\right)^n + \frac{a_{n-1}b}{s^n} + \cdots + \frac{a_0}{s^n} = \frac{p(b)}{s^n} = 0$$

Thus $S^{-1}B$ is integral over $S^{-1}A$. □

Lemma 1.3. Take an extension of rings $A \subset B$. Let $\mathfrak{p} \subset A$ be a prime and $S = A \setminus \mathfrak{p}$. Then $B \otimes_A A_\mathfrak{p} = B_\mathfrak{p} = S^{-1}B$ as A-algebras where $A_\mathfrak{p} = S^{-1}A$.

Proof. Consider the map $B \otimes_A A_\mathfrak{p} \to B_\mathfrak{p}$ given by $b \otimes (a, s) \mapsto (ba, s)$. This map is clearly A-linear and a well-defined ring map since acting by $r \in A$ on either factor maps to $r \cdot (ba, s) = (rba, s)$. Furthermore, the map is injective since if $(ba, s) = 0$ then there exists $t \in S$ such that $tba = 0$ in which case $b \otimes (a, s) = b \otimes (at, st) = bat \otimes (1, st) = 0$ since these are A-modules and $at \in A$. Finally the map is surjective because $b \otimes (1, s) = (b, s)$ which is an arbitrary element of $B_\mathfrak{p}$. □

Corollary 1.4. If B is integral over A then $B_\mathfrak{p} = B \otimes_A A_\mathfrak{p}$ is integral over $A_\mathfrak{p}$.

Proof. Let $S = A \setminus \mathfrak{p}$. Since $A \subset B$ is an integral extension then $A_\mathfrak{p} = S^{-1}A \subset S^{-1}B = B_\mathfrak{p} = B \otimes_A A_\mathfrak{p}$ is also an integral extension by the above lemmas. □

Lemma 1.5. Let $S \subset A$ be multiplicative. The proper ideals $I \subset S^{-1}A$ are exactly $I = S^{-1}J$ with $J = \iota^{-1}(I)$ where $J \cap S = \varnothing$ and $\iota : A \to S^{-1}A$ is the canonical map.

Proof. Let $I \subset S^{-1}A$ be an ideal. Consider the ideal $J = \iota^{-1}(I)$. If $(a, s) \in I$ then $(as, s) = (a, 1) \in I$ so $a \in J$. Thus, $(a, s) \in S^{-1}J$. Furthermore, $S^{-1}J \subset I$ since $\iota(J) \subset I$ and I is an ideal. Therefore, $I = S^{-1}J$. However, if $s \in J \cap S$ then $\iota(s) = (s, 1) \in I$ but then $(1, s) \cdot (s, 1) = (s, s) = (1, 1) \in I$ so $I = S^{-1}A$. Thus if $I \subset S^{-1}A$ is proper then $J \cap S = \varnothing$. $\square$

Lemma 1.6. Let $S \subset A$ be multiplicative. Then, the canonical map $\iota : A \to S^{-1}A$ induces an injection $\iota^* : \mathrm{Spec}\,(S^{-1}A) \to \mathrm{Spec}\,(A)$ which which is a homeomorphism onto its image,

$$\iota^*(\mathrm{Spec}\left(S^{-1}A\right)) = \mathrm{Spec}\,(A)\,|_{S^c} = \{\mathfrak{p} \in \mathrm{Spec}\,(A) \mid \mathfrak{p} \cap S = \varnothing\}$$

Proof. Since any prime $\mathfrak{P} \subset S^{-1}A$ is, by definition, proper, by Lemma 1.5, its image under ι^* is disjoint from S. Thus, we may restrict ι^* to a map,

$$\iota^* : \mathrm{Spec}\left(S^{-1}A\right) \to \mathrm{Spec}\,(A)\,|_{S^c}$$

I claim that $\iota_* : \mathrm{Spec}\,(A)\,|_{S^c} \to \mathrm{Spec}\,(S^{-1}A)$ given by $\iota_*(\mathfrak{p}) = \iota(\mathfrak{p})S^{-1}A = S^{-1}\iota(\mathfrak{p})$ is a well-defined continuous inverse of ι^*. First, if $\mathfrak{p} \in \mathrm{Spec}\,(A)\,|_{S^c}$ then suppose that $(a_1, s_1), (a_2, s_2) \in S^{-1}A$ such that $(a_1a_2, s_1s_2) \in \iota_*(\mathfrak{p})$. This implies that $(a_1a_2, s_1s_2) = (x, s)$ for $x \in \mathfrak{p}$ and thus $\exists u \in S$ such that,

$$u\,(xs_1s_2 - a_1a_2s) = 0 \implies a_1s_2su = xs_1s_2u \implies a_1a_2su \in \mathfrak{p}$$

However, $\mathfrak{p} \cap S = \varnothing$ so $su \notin \mathfrak{p}$. Since $\mathfrak{p}$ is prime we have $a_1a_2 \in \mathfrak{p}$ which implies that $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$. Thus, $(a_1, s_1) \in S^{-1}\mathfrak{p} = \iota_*(\mathfrak{p})$ or $(a_2, s_2) \in \iota_*(\mathfrak{p})$. Therefore, $\iota_*(\mathfrak{p}) \in \mathrm{Spec}\,(S^{-1}A)$.

Furthermore, any closed set of $\mathrm{Spec}\,(B)$ is of the form $V(I)$ where $I \subset B$ is an ideal. Consider $(\iota_*)^{-1}(V(I))$. By Lemma 1.5, we have $I = S^{-1}J$ with $J = \iota^{-1}(I)$. If $\mathfrak{p} \in V(J)$ i.e. $\mathfrak{p} \supset J$ then $\iota_*(\mathfrak{p}) = S^{-1}\mathfrak{p} \supset S^{-1}J = I$ so $\mathfrak{p} \in (\iota_*)^{-1}(V(I))$. Conversely, if $\iota_*(\mathfrak{p}) \in V(I)$ then $S^{-1}\mathfrak{p} \supset S^{-1}J$ which implies that $\iota^{-1}(S^{-1}\mathfrak{p}) \supset \iota^{-1}(S^{-1}J) \supset J$. However, if $\iota(a) \in S^{-1}\mathfrak{p}$ then $\iota(a) = (a, 1)$ so $(a, 1) = (x, s)$ with $x \in \mathfrak{p}$. Thus, there exists $u \in S$ such that,

$$u\,(x - as) = 0 \implies asu = ux \in \mathfrak{p}$$

but $\mathfrak{p} \cap S = \varnothing$ and $\mathfrak{p}$ is prime so $a \in \mathfrak{p}$. Furthermore if $a \in \mathfrak{p}$ then clearly $\iota(a) = (a, 1) \in S^{-1}\mathfrak{p}$. Therefore, $\iota^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ which, returning to the above, implies that $\mathfrak{p} \supset J$ and thus $\mathfrak{p} \in V(I)$. Therefore,

$$(\iota_*)^{-1}(V(I)) = V(J) = V(\iota^{-1}(I))$$

which is closed. Thus the map $\iota_* : \mathrm{Spec}\,(A)\,|_{S^c} \to \mathrm{Spec}\,(S^{-1}A)$ is continuous.

Consider $\iota_* \circ \iota^*(\mathfrak{P}) = S^{-1}\iota^{-1}(\mathfrak{P}) = \mathfrak{P}$ by Lemma 1.5. Also, $\iota^* \circ \iota_*(\mathfrak{p}) = \iota^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ as shown above. Therefore, ι_* and ι^* are continuous inverses. $\square$

Lemma 1.7. Let M be an A-module and $a \in A$ then the a-torsion is related to the Tor functor via,

$$T_a(M) = \mathrm{Tor}_1^A(A/(a), M)$$

where the a-torsion submodule is defined as,

$$T_a(M) = \{m \in M \mid a \cdot m = 0\}$$

and for $i > 1$ the Tor vanishes i.e.

$$\mathrm{Tor}_i^A(A/(a), M) = 0$$

Proof. Consider the exact sequence,

$$0 \longrightarrow A \xrightarrow{\times a} A \longrightarrow A/(a) \longrightarrow 0$$

Applying the long exact sequence of the derived functor $\mathrm{Tor}_*^A(-, M)$ we get,

$$\begin{tikzcd}
\mathrm{Tor}_1^A(A, M) \arrow[r] \arrow[d] & \mathrm{Tor}_1^A(A/(a), M) \arrow[r] \arrow[d] & A \otimes_A M \arrow[r, "(\times a)\otimes \mathrm{id}_M"] \arrow[d] & A \otimes_A M \arrow[d] \\
0 \arrow[r] & \mathrm{Tor}_1^A(A/(a), M) \arrow[r] & M \arrow[r, "a\cdot"] & M
\end{tikzcd}$$

where the vertical arrows are isomorphisms since $A \otimes_A M \cong M$ and $\mathrm{Tor}_1^A(A, M) = 0$ since A is A-flat. Since $(\otimes a) \otimes \mathrm{id}_M$ sends $a' \otimes m \mapsto aa' \otimes m$ and thus we get the map $a'm \mapsto aa'm$ from $M \to M$ under this isomorphism. Therefore, $\mathrm{Tor}_1^A(A/(a), M)$ is the kernel of multiplication by a since the rows are exact. ☐

Corollary 1.8. If M is a flat A-module then M is A-torsion-free.

Proof. This follows immediately from Lemma 1.7 using the fact that M is flat so $\mathrm{Tor}_1^A(M, -) = 0$ and thus $T_a(M) = \mathrm{Tor}_1^A(M, A/(a)) = 0$ for each $a \in A$. ☐

Lemma 1.9. For any ideal $I \subset A$,

$$V(I) = V(\sqrt{I})$$

Proof. Suppose that $\mathfrak{p} \in V(I)$ and then $\mathfrak{p} \supset I$ so clearly,

$$\mathfrak{p} \supset \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I}$$

and thus $\mathfrak{p} \in V(\sqrt{I})$. Conversely, since,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} \supset I$$

we know that $V(\sqrt{I}) \subset V(I)$. Therefore, $V(I) = V(\sqrt{I})$. ☐