1 Jan. 30 Tag 0280

(DO EIGHT EXERCISES)

2 Feb. 6

2.1 Tag 028O

(DO FOUR EXERCISES)

2.2 Tag 02A1

Exercise 2.1. Let $n \ge 1$ be an integer. Find a surjective morphism $X \to \mathbb{P}^n_k$ where X is affine.

A "dumb" solution is as follows. Let $U_i = D_+(T_i)$ be the standard open cover of $\mathbb{P}^n_k$. Then there is a surjective map,

$$\coprod_{i=0}^{n} U_i \twoheadrightarrow \mathbb{P}^n_k$$

However, it would be more interesting to give an example where X is an (irreducible) k-variety.

(DO THIS)

3 Feb. 13

3.1 Tag 029Q

Exercise 3.1. Let X be a scheme and $x, x' \in X$. Let $\mathscr{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. If x is a specialization of x' and $\mathscr{F}_{x'} \neq 0$ then show that $\mathscr{F}_x \neq 0$.

The terminology means that $x \in \overline{x'}$. Then for any open $x \in U$ we must have $x' \in U$ else U^C would be a closed containing x' not containing x. Now, choose some affine open $x \in U = \mathrm{Spec}\,(A)$ such that $\mathscr{F}|_U = \widetilde{M}$ and primes $\mathfrak{p} = x$ and $\mathfrak{q} = x'$ so $\mathfrak{p} \supset \mathfrak{q}$. Then $\mathscr{F}_x = M_{\mathfrak{p}}$ and $\mathscr{F}_{x'} = M_{\mathfrak{q}} \neq 0$. However, since $\mathfrak{p} \supset \mathfrak{q}$ then $M_{\mathfrak{q}} = (M_{\mathfrak{p}})_{\mathfrak{q}}$ so $\mathscr{F}_x = M_{\mathfrak{p}} \neq 0$.

3.2 Tag 069T

Exercise 3.2. Show that the composition of affine morphisms is affine.

I think this is trivial. If $f : X \to Y$ and $g : Y \to Z$ are affine and $U \subset Z$ is affine open then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ but $g^{-1}(U)$ is affine open since g is affine and thus $f^{-1}(g^{-1}(U))$ is affine open since f is affine so $g \circ f$ is affine.

3.3 Tag 028Z

Exercise 3.3. Find a morphism of integral schemes $f : X \to Y$ such that $f^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective for all $x \in X$ but f is not a closed immersion.

Consider the map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ which gives $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$. Then, the stalk map at the unique point $(0) \in \mathrm{Spec}(\mathbb{Q})$ is $\mathbb{Z}_{(0)} \to \mathbb{Q}_{(0)}$ which is simply the identity $\mathbb{Q} \to \mathbb{Q}$ and thus surjective. However, $\mathbb{Z} \to \mathbb{Q}$ is not surjective so $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$ is not a closed immersion.

Notice that if $\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y$ were always surjective then the map of sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$ would necessarily be surjective. However, the map in question is $\mathcal{O}_{Y,f(x)} \to (f_* \mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$. Note, in our example, that we have the map of sheaves $\widetilde{\mathbb{Z}} \to \widetilde{\mathbb{Q}}$ which on stalks at $(p) \in \mathrm{Spec}(\mathbb{Z})$ gives $\mathbb{Z}_{(p)} \to \mathbb{Q}$ which is not surjective but $\mathbb{Z}_{(0)} \to \mathbb{Q}$ is at the point in the image.

Consider $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ sending every point to zero, i.e. the map $k[x] \to k[x]$ sending $x \mapsto 0$. Then $f_* \mathcal{O}_X = \widetilde{M}$ where $M = k[x]$ as an abelian group but has multiplication defined by $k[x] \to k[x]$ sending $x \mapsto 0$. Then the map $\mathcal{O}_{Y,(x)} \to M_{(x)}$ is $k[x]_{(x)} \to k[x]$ sending $x \mapsto 0$. Then $\mathcal{O}_{Y,(x)} \to \mathcal{O}_{X,\mathfrak{p}}$ is $k[x]_{(x)} \to k[x]_{\mathfrak{p}}$ sending $x \mapsto 0$. This gives an example where $(f_* \mathcal{O}_X)_{f(x)}$ and $\mathcal{O}_{X,x}$ differ.

3.4 Tag 0293

(DO TWO EXERCISES)

4 Feb. 20

4.1 Tag 02A3

Exercise 4.1. Given examples of R such that on $\mathrm{Proj}(R)$,

(a). $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.

(b). $\mathcal{O}_X(1)$ is invertible but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is not an isomorphism.

Consider the ring $R = k[x_0, x_1]$ graded with x_i in degree 2.

4.2 Tag 02AL

(VERY LONG)

4.3 Tag 0DT4

Exercise 4.2. Give an example of a scheme X and a nontrivial invertible $\mathcal{O}_X$-module $\mathcal{L}$ such that $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are nonzero.

I will give two examples. First, consider the number field $K = \mathbb{Q}(\sqrt{-5})$ with ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. Then the class group $\mathrm{Cl}(\mathcal{O}_K) = \mathrm{Pic}(\mathcal{O}_K) = \mathbb{Z}/2\mathbb{Z}$ with generator $J = (2, 1+\sqrt{-5})$. Thus, $J = J^{\otimes -1}$ and $H^0(\mathrm{Spec}(\mathcal{O}_K), \widetilde{J}) = J$ is nontrivial and its dual is the same module.

(NO, NOT POSSIBLE FOR SMOOTH CURVES PROVE THIS!!!!, FIND EXAMPLE) For the second example, let (X, O) be an elliptic curve over $k = \overline{k}$. Riemann-Roch on X applied to a line bundle $\mathcal{L}$ with $\deg \mathcal{L} = 0$ shows that,

$$\dim_k H^0(X, \mathcal{L}) - \dim_k H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee) = 0$$

but on an elliptic curve $\omega_X = \mathcal{O}_X$ so we find,

$$\dim_k H^0(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L}^{\otimes -1})$$

Therefore, it suffices to find a line bundle on X with $\deg \mathcal{L} = 0$ and $H^0(X, \mathcal{L}) \neq 0$. (FINISH THIS)

The isomorphism $X \to \mathrm{Pic}^0(X)$ shows that, for points, $P, Q \in X$, we have,

$$[P] + [Q] \sim [P + Q] + [O]$$

and thus,

$$[P] + [Q] - [P + Q] - [O] \sim 0$$

4.4 Tag 029U

Exercise 4.3. Let $X = \mathrm{Spec}\,(R)$ be an affine scheme.

(a). Let $f \in R$ and $\mathscr{G}$ be a quasi-coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathscr{G} = \mathscr{F}|_{D(f)}$ for some coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$.

(b). Let $I \subset R$ be an ideal. Let $\iota : Z \hookrightarrow X$ be the closed subscheme of X corresponding to I. Let $\mathscr{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Z$-modules on the closed subscheme Z. Show that $\mathscr{G} = \iota^* \mathscr{F}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$. (Why is this silly?).

(c). Assume that R is Noetherian. Let $f \in R$ and $\mathscr{G}$ be a coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathscr{G} = \mathscr{F}|_{D(f)}$ for some coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$.

For the first, we know $\mathscr{G} = \widetilde{M}$ for some R_f-module M since $D(f) = \mathrm{Spec}\,(R_f)$ is affine. Then consider M as a A-module and take $\mathscr{F} = \widetilde{M}$ then $\mathscr{F}|_{D(f)} = \widetilde{M \otimes_R R_f} = \widetilde{M}$ since M is an R_f-module.

Any quasi-coherent sheaf $\mathscr{G}$ on $Z = \mathrm{Spec}\,(R/I)$ is $\widetilde{M}$ for some A/I-module M. Then consider M as a A-module giving a quasi-coherent sheaf $\mathscr{F} = \widetilde{M}$ on $\mathrm{Spec}\,(A)$. Then $\iota^* \mathscr{F} = \widetilde{M \otimes_A A/I} = \widetilde{M}$ since M is an A/I-module. This is “silly” because in general for a closed immersion $\iota : Z \hookrightarrow X$ we have $\iota^* \iota_* \mathscr{G} = \mathscr{G}$.

Now let R be Noetherian. Any coherent $\mathcal{O}_{D(f)}$-module is of the form $\mathscr{G} = \widetilde{M}$ for a finitely generated R_f-module M. We need a coherent sheaf $\mathscr{F} = \widetilde{N}$ for some finitely generated R-module N such that $N \otimes_R R_f = M$.

4.5 Tag 0D8V

Exercise 4.4. Let A be a ring and $\mathbb{P}^n_A = \text{Proj}\,(A[T_0, \dots, T_n])$ be projective space over A. Let $\mathbb{A}^{n+1}_A = \text{Spec}\,(A[T_0, \dots, T_n])$ and let,

$$U = \bigcup_{i=0}^{n} D(T_i) \subset \mathbb{A}^{n+1}_A$$

be the complement of the closed point $(T_0, \dots, T_n) \in \mathbb{A}^{n+1}_A$. Construct an affine surjective morphism,

$$f : U \to \mathbb{P}^n_A$$

such that $f_* \mathcal{O}_U = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n_A}(d)$. More generally, show that for any graded $A[T_0, \dots, T_n]$-module M we have,

$$f_*(\widetilde{M}|_U) = \bigoplus_{d \in \mathbb{Z}} \widetilde{M(d)}$$

where on the left we have the quasi-coherent sheaf $\widetilde{M}$ associated to M on $\mathbb{A}^{n+1}_A$ and on the right we have the quasi-coherent sheaves $\widetilde{M(d)}$ associated to the graded module $M(d)$.

Consider the line bundle $\mathcal{O}_U$ on U and choose sections $s_i = T_i$ which generate $\mathcal{O}_U$ globally since these do not all vanish on U. These sections define the map $U \to \mathbb{P}^n_A$ which satisfies $f^{-1}(D_+(T_i)) = D(T_i)$ and thus is affine. On the affine opens $D_+(T_i)$ this is given by the ring map,

$$A[T_0/T_i, \dots, T_n/T_i] \to A[T_0, \dots, T_n, T_i^{-1}]$$

which is not surjective because this is not a closed immersion but it is injective so the associated map,

$$\text{Spec}\left(A[T_0, \dots, T_n, T_i^{-1}]\right) \to \text{Spec}\,(A[T_0/T_i, \dots, T_n/T_i])$$

is dominant and, in fact, surjective because (SHOW)

Now, suppose that M is an $A[T_0, \dots, T_n]$-module. We can check the equality of sheaves on the affine cover $D_+(T_i)$. The map $f : D(T_i) \to D_+(T_i)$ takes $\widetilde{M}|_{D(T_i)} = \widetilde{M_{T_i}} \mapsto \widetilde{M_{T_i}}$ viewing M_{T_i} as a $A[T_0/T_i, \dots, T_n/T_i]$-module. Since M is a graded $A[T_0, \dots, T_n]$-module we can wrtie,

$$M_{T_i} = \bigoplus_{d \in \mathbb{Z}} (M_{T_i})_d = \bigoplus_{d \in \mathbb{Z}} (M(d))_{(T_i)}$$

since the degree d part of M_{T_i} is the degree zero part of $(M(d))_{T_i}$ Therefore,

$$f_*(\widetilde{M}|_{D(T_i)}) = \bigoplus_{d \in \mathbb{Z}} \widetilde{M(d)}|_{D_+(T_i)}$$

and thus, globally,

$$f_*(\widetilde{M}|_U) = \bigoplus_{d \in \mathbb{Z}} \widetilde{M(d)}$$

Alternatively, we can use the fact that for any quasi-coherent $\mathcal{O}_{\mathbb{P}^n_A}$-module $\mathscr{F}$ we have $\widetilde{\Gamma_*(\mathscr{F})} \xrightarrow{\sim} \mathscr{F}$ and since $f : U \to \mathbb{P}^n_A$ is quasi-compact and quasi-separated then $f_*(\widetilde{M}|_U)$ is quasi-coherent so,

$$f_*(\widetilde{M}|_U) = \widetilde{\Gamma_*(f_*(\widetilde{M}|_U))} = \bigoplus_{d \in \mathbb{Z}} \widetilde{\Gamma(\mathbb{P}^n_A, f_*(\widetilde{M}|_U)(d))} = \bigoplus_{d \in \mathbb{Z}} \widetilde{M(d)}$$

5 Feb. 27

5.1 Tag 0D8Q

Exercise 5.1. Let $X = \mathbb{R}$ with the usual Euclidean topology. Using only formal δ-functor properties of cohomology show that there exists a sheaf $\mathscr{F}$ on X with nonzero $H^1(X, \mathscr{F})$.

Let $p, q \in \mathbb{R}$ be two distinct points and consider the inclusions $\iota_x : \{x\} \to \mathbb{R}$ at the point x. Then let $\mathscr{F} = \underline{\mathbb{Z}}$ be the constant sheaf $\mathbb{Z}$ over $\mathbb{R}$ and $\mathscr{G} = (\iota_p)_*(\underline{\mathbb{Z}}) \oplus (\iota_p)_*(\underline{\mathbb{Z}})$ be the sum of skyscraper sheaves at p and at q. Then there is a surjection of sheaves $\mathscr{F} \to \mathscr{G}$ determined on the open cover $U = (-\infty, \frac{1}{2}(p+q))$ and $V = (\frac{1}{2}(p+q), \infty)$ by the identity map $\mathbb{Z} \to \mathbb{Z}$ on both since each open set contains exactly one of p and q. This map is a surjection because $f_p : \mathscr{F}_p \to \mathscr{G}_p$ and $f_q : \mathscr{F}_q \to \mathscr{G}_q$ are both the identity maps $\mathbb{Z} \to \mathbb{Z}$ and $f_x : \mathscr{F}_x \to \mathscr{G}_x$ is the zero map for $x \neq p, q$. Taking the kernel of this map gives an exact sequence of sheaves,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

which gives a long exact sequence of cohomology,

$$0 \longrightarrow \Gamma(\mathbb{R}, \mathcal{K}) \longrightarrow \Gamma(\mathbb{R}, \mathscr{F}) \longrightarrow \Gamma(\mathbb{R}, \mathscr{G}) \longrightarrow H^1(\mathbb{R}, \mathcal{K}) \longrightarrow \cdots$$

However, $\Gamma(\mathbb{R}, \mathscr{F}) = \mathbb{Z}$ and $\Gamma(\mathbb{R}, \mathscr{G}) = \Gamma(\mathbb{R}, (\iota_p)_*(\underline{\mathbb{Z}})) \oplus \Gamma(\mathbb{R}, (\iota_q)_*(\underline{\mathbb{Z}})) = \mathbb{Z} \oplus \mathbb{Z}$. In particular, there is an exact sequence,

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H^1(\mathbb{R}, \mathcal{K})$$

Since there does not exist a surjection $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ the map $\Gamma(\mathbb{R}, \mathscr{G}) \to H^1(X, \mathcal{K})$ cannot be the zero map so $H^1(\mathbb{R}, \mathcal{K})$ must be nontrivial.

5.2 Tag 0D8R

(See Assigment 6)

5.3 Tag 0D8S

Exercise 5.2. Show that if X has two or fewer points then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$ and any $\mathscr{F}$ on X. What about if X has three points.

(DO THIS!!!!)

5.4 Tag 0D8T

Exercise 5.3. Let $X = \mathrm{Spec}\,(R)$ for a local ring R. Then show that $H^i(X, \mathscr{F}) = 0$ for all $i > 0$ and any abelian sheaf $\mathscr{F}$.

Consider the maximal ideal $\mathfrak{m} \in \mathrm{Spec}\,(R)$. Note that if $f \notin \mathfrak{m}$ then $f \in R^\times$ so the only open set containing $\mathfrak{m}$ is X. Thus,

$$\mathscr{F}_{\mathfrak{m}} = \varinjlim_{\mathfrak{m} \in U} \mathscr{F}(U) = \mathscr{F}(X)$$

This means that given an exact sequence of abelian sheaves,

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

it must be exact on the stalks. In particular,

$$0 \longrightarrow \mathscr{F}_{\mathfrak{m}} \longrightarrow \mathscr{G}_{\mathfrak{m}} \longrightarrow \mathscr{H}_{\mathfrak{m}} \longrightarrow 0$$

is exact but this is just,

$$0 \longrightarrow \mathscr{F}(X) \longrightarrow \mathscr{G}(X) \longrightarrow \mathscr{H}(X) \longrightarrow 0$$

showing that $\Gamma(X, -)$ is an exact functor so $H^i(X, -) = 0$ for $i > 0$.

5.5 Tag 0DAK

Exercise 5.4. Let A be a ring and $I = (f_1, \dots, f_t)$ a finitely generated ideal of A. Let $U \subset \mathrm{Spec}\,(A)$ be $V(I)^C$. For any A-module M find a complex of A-modules (in terms of $A, f_1, \dots, f_n, M$) whose cohomology groups compute $H^n(U, \widetilde{M})$.

The affine opens $D(f_i) \subset \mathrm{Spec}\,(A)$ cover U since,

$$\bigcup D(f_i) = D((f_1, \dots, f_n)) = D(I)$$

Furthermore, $D(f_i) \cap D(f_j) = D(f_i f_j)$ is affine so this gives a cover whose Cech cohomology computes the cohomology of quasi-coherent modules, in particular $\widetilde{M}$. Then, the ordered Cech complex gives,

$$0 \longrightarrow \prod_{i_0} M_{f_{i_0}} \longrightarrow \prod_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \longrightarrow \prod_{i_0 < i_1 < i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_t} \longrightarrow 0$$

The cohomology of this complex computes $\check{H}^i(\mathfrak{U}, \widetilde{M}) = H^i(U, \widetilde{M})$.

6 March 5

6.1 Tag 0DB9

(KUNNETH)

6.2 Tag 0DBA

(TWISTS ON $\mathbb{P}^1 \times \mathbb{P}^1$)

6.3 Tag 0DCF

Let $P_{\mathscr{F}}(t) = \chi(X, \mathscr{F}(t))$ be the Hilbert polynomial for $X = \mathbb{P}^n_k$ and $\mathscr{F}(t) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)$.

(a). For $\mathscr{F}(-d)$ we have,

$$P_{\mathscr{F}(-d)} = \chi(X, \mathscr{F}(t-d)) = P_{\mathscr{F}}(t-d)$$

(b). If $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2$,

$$P_{\mathscr{F}}(t) = \chi(X, \mathscr{F}(t)) = \chi(X, \mathscr{F}_1(t) \oplus \mathscr{F}_2(t)) = \chi(X, \mathscr{F}_1(t)) + \chi(X, \mathscr{F}_2(t)) = P_{\mathscr{F}_1}(t) + P_{\mathscr{F}_2}(t)$$

(c). Let,

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

be an exact sequence of coherent sheaves. Then twising by $\mathcal{O}_X(t)$ gives an exact sequence,

$$0 \longrightarrow \mathscr{F}_1(t) \longrightarrow \mathscr{F}_2(t) \longrightarrow \mathscr{F}_3(t) \longrightarrow 0$$

Then we get,

$$\chi(X, \mathscr{F}_2(t)) = \chi(X, \mathscr{F}_1(t)) + \chi(X, \mathscr{F}_3(t))$$

and thus,

$$P_{\mathscr{F}_2} = P_{\mathscr{F}_1} + P_{\mathscr{F}_2}$$

and in particular,

$$P_{\mathscr{F}_1} = P_{\mathscr{F}_2} - P_{\mathscr{F}_3}$$

6.4 Tag 0DCG

For $X = \mathbb{P}^n_k$ and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module. Then we define the Hilbert polynomial,

$$P_{\mathscr{F}}(t) = \chi(X, \mathscr{F}(t))$$

Note that,

$$\chi(X, \mathcal{O}_X(t)) = \binom{n+t}{n} + (-1)^n \binom{-t-1}{n}$$

In particular, in the case $n = 1$ we have,

$$\forall t \in \mathbb{Z} : \chi(X, \mathcal{O}_X(t)) = t + 1$$

Now we can take, $X = \mathbb{P}^1_k$ then take $\mathscr{F} = \mathcal{O}_X(-102)$ which has,

$$P_{\mathscr{F}}(t) = \chi(X, \mathscr{F}(t)) = t - 101$$

In general, on $X = \mathbb{P}^n_k$ the pushforward of this sheaf on any line will have $P(t) = t - 101$.

6.5 Tag 0DCH

Let $X = \mathbb{P}^2_k$ and consider,

$$\mathscr{F} = \mathcal{O}_X(-3) \oplus \mathcal{O}_H(-2)^{\oplus 2}$$

for some hyperplane $H \subset X$. Then we get,

$$\begin{aligned} P_{\mathscr{F}}(t) &= \chi(X, \mathscr{F}(t)) = \chi(X, \mathcal{O}_X(t-2)) = 2\chi(X, \mathcal{O}_H(t-2)) \\ &= \binom{t-2}{2} + 2\binom{t-1}{1} = \tfrac{1}{2}(t-1)(t-2) = \tfrac{1}{2}t^2 - \tfrac{3}{2}t + 1 + 2(t-1) \\ &= \tfrac{1}{2}t^2 + \tfrac{1}{2}t - 1 \end{aligned}$$

7 March 12

7.1 Tag 0DD1

Exercise 7.1. Let k be a field and $X = \mathbb{P}^3_k$. Let $L \subset X$ be a line and $P \subset X$ a plane defined as closed subschemes cut out by 1, resp., 2 linear equations. Compute,

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_L, \mathcal{O}_P)$$

7.2 Tag 0EEN

Let k be a field and $X = \mathbb{P}^1_k$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module and $h^i(X, \mathcal{E}(d)) = \dim_k H^i(X, \mathcal{E}(d))$ where $\mathcal{E}(d) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$. Then for $\mathbb{P}^1_k$ we can always decompose,

$$\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_X(q_i)$$

and then,

$$H^p(X, \mathcal{E}(d)) = \bigoplus_i H^p(X, \mathcal{O}_X(q_i + d))$$

Therefore,

$$h^0(X, \mathcal{E}(d)) = \sum_{i=1}^n \dim_k H^0(X, \mathcal{O}_X(q_i + d)) = \sum_{i=1}^n \begin{cases} q_i + d + 1 & q_i \ge -d \\ 0 & q_i < -d \end{cases}$$

and, by Serre duality,

$$h^1(X, \mathcal{E}(d)) = \sum_{i=1}^n \dim_k H^1(X, \mathcal{O}_X(q_i + d)) = \dim_k H^0(X, \mathcal{O}_X(-q_i - d - 2)) = \sum_{i=1}^n \begin{cases} -q_i - d - 1 & q_i \le -d - 2 \\ 0 & q_i > -d - 2 \end{cases}$$

Thus, we get some explicit formulae,

$$h^0(X, \mathcal{E}) = \sum_{q_i \ge 0} (q_i + 1)$$

$$h^0(X, \mathcal{E}(1)) = \sum_{q_i \ge -1} (q_i + 2)$$

Notice that $h^0(X, \mathcal{E}) \le h^0(X, \mathcal{E}(1))$ showing that:

(1) there does not exist $\mathcal{E}$ with $h^0(X, \mathcal{E}) = 5$ and $h^0(X, \mathcal{E}(1)) = 4$.

Furthermore, we have explicit formulae,

$$h^1(X, \mathcal{E}) = -\sum_{q_i \le -2} (q_i + 1)$$

$$h^1(X, \mathcal{E}(1)) = -\sum_{q_i \le -3} (q_i + 2)$$

Notice that $h^1(X, \mathcal{E}) \ge h^1(X, \mathcal{E}(1))$ showing that:

(1) there does not exist $\mathcal{E}$ with $h^1(X, \mathcal{E}) = 4$ and $h^1(X, \mathcal{E}(1)) = 5$.

Now we consider $\mathcal{E}$ satisfying the following conditions,

$$\begin{array}{cc} h^0(X, \mathcal{E}) = 1 & h^1(X, \mathcal{E}) = 1 \\ h^0(X, \mathcal{E}(1)) = 2 & h^1(X, \mathcal{E}(1)) = 0 \\ h^0(X, \mathcal{E}(2)) = 4 & h^1(X, \mathcal{E}(2)) = a \end{array}$$

Choose an ordering $q_0 \geq q_1 \geq q_2 \geq \cdots \geq q_n$. The condition $h^0(X, \mathcal{E}) = 1$ implies that $q_0 = 0$ and $q_1 < 0$. Then $h^1(X, \mathcal{E}) = 1$ implies that $q_n = -2$ and $q_{n-1} > -2$. Therefore, we have,

$$\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-1)^{\oplus s}$$

Therefore, $h^0(X, \mathcal{E}(1)) = 2 + s$ and $h^1(X, \mathcal{E}(1)) = 0$ but since $h^0(X, \mathcal{E}(1)) = 2$ so we find $s = 0$ and thus,

$$\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X(-2)$$

Thus we get $h^0(X, \mathcal{E}(2)) = 3 + 1 = 4$ compatible with our requirement. Finally, we compute,

$$h^1(X, \mathcal{E}(2)) = h^1(X, \mathcal{O}_X(2) \oplus \mathcal{O}_X) = 0$$

so we must have $a = 0$. Note that we could have more easily done this by noting that,

$$h^1(X, \mathcal{E}(2)) \leq h^1(X, \mathcal{E}(1))$$

in general but we require $h^1(X, \mathcal{E}(1)) = 0$ so $h^1(X, \mathcal{E}(2)) = 0$ giving $a = 0$.

7.3 Tag 0DT3

Exercise 7.2. Let k be an algebraically closed field and X a reduced, projective scheme over k all of whose irreducible components all have dimension 1. Let $\omega_{X/k}$ be the relative dualizing module. Show that if $\dim_k H^1(X, \omega_{X/k}) > 1$ then X is disconnected.

(DO THIS!!)

7.4 Tag 0EEP

Exercise 7.3. Let X be a topological space which is the union $X = Y \cup Z$ of two closed subsets Y and Z whose intersection is $W = Y \cap Z$. Denote $i : Y \to X$ and $j : Z \to X$ and $k : W \to X$ the inclusion maps.

(a). Show that there is a short exact sequence of sheaves,

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow i_* \underline{\mathbb{Z}}_Y \oplus j_* \underline{\mathbb{Z}}_Z \longrightarrow k_* \underline{\mathbb{Z}}_W \longrightarrow 0$$

(b). What can you conclude about the cohomology of X, Y, Z, W with $\mathbb{Z}$-coefficients.

(DO THIS)

7.5 Tag 0EEQ

Exercise 7.4. Let k be a field and $A = k[x_1, x_2, x_3, \dots]$ and $\mathfrak{m} = (x_1, x_2, x_3, \dots)$. Let $X = \operatorname{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Compute $H^i(U, \mathcal{O}_U)$.

Consider the Cech complex for the cover $U_i = D(x_i)$,

$$0 \longrightarrow \prod_{i_0} A_{x_i} \longrightarrow \prod_{i_0 < i_1} A_{x_{i_0} x_{i_1}} \longrightarrow \prod_{i_0 < i_1 < i_2} A_{x_{i_0} x_{i_1} x_{i_2}} \longrightarrow \cdots$$

(FINSIH THIS!!)