1 Introduction

Study the rep theory of $\pi_1(X(\mathbb{C}))$. How ordinary Hodge theory of H^1 is the $r = 1$ case of this.

2 Higgs Bundles

In this section we work on a smooth variety X.

Definition 2.0.1. A *Higgs* bundle is a pair $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a vector bundle and ϕ is a $\mathcal{O}_X$-linear map,

$$\phi : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$$

such that $\phi \wedge \phi = 0$.

Remark. We should define the notation $\phi \wedge \phi$. Such a linear map is equivalent to a section of,

$$\phi \in \Gamma(X, \mathrm{End}\,(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_X$$

This is a sheaf of $\mathcal{O}_X$-algebras under the following operation,

$$(\varphi_1 \otimes \omega_1, \varphi_2 \otimes \omega_2) \mapsto (\varphi_1 \circ \varphi_2) \otimes (\omega_1 \wedge \omega_2)$$

then extended $\mathcal{O}_X$-linearly. This operation is denoted $\wedge$. However, do not let this mislead you into thinking that $\wedge$ is antisymmetric since if $\mathrm{rank}\,\mathcal{E} > 1$ then the composition in $\mathrm{End}\,(\mathcal{E})$ is noncommutative. Hence $\phi \wedge \phi = 0$ is a nontrivial condition when $\mathcal{E}$ has rank at least 2.

Remark. We refer to $\phi \wedge \phi = 0$ as the *integrability* condition. This is because we call a flat connection integrable. We will now spell out the relationship of Higgs bundles to flat connections.

There are a number of ways to motivate the definition of a Higgs bundle. My favorite is to think of them as degenerations of a flat connection where we send the nonlinear part to zero. In order to make this precise we introduce the notion of a t-connection.

Definition 2.0.2. Let X be an S-scheme. Let $\mathcal{E}$ be a coherent sheaf on X. A *t-connection* on $\mathcal{E}$ over X/S is a triple $(t, \mathcal{E}, \nabla)$ where $t : X \to \mathbb{A}^1_S$ is a global function and ∇ is a S-linear map,

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$$

satisfying the t-scalled Liebniz law,

$$\nabla(fs) = t\mathrm{d}f \otimes s + f\nabla s$$

Remark. Notice that if $t = 0$ then ∇ is $\mathcal{O}_X$-linear.

Definition 2.0.3. There is a natural extension of ∇ to,

$$\nabla_p : \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^p_X \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{p+1}_X$$

defined on pure tensors as follows

$$\nabla_p(s \otimes \omega) = ts \otimes \mathrm{d}\omega + (-1)^p \nabla s \wedge \omega$$

Then we define the curvature of ∇,

$$\omega_\nabla = \nabla_1 \circ \nabla$$

A straightforward calculation shows that,

$$\omega_\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^2$$

is $\mathcal{O}_X$-linear. We say that ∇ is *flat* or *integrable* if $\omega_\nabla = 0$. In this case ∇ is a differential meaning the de Rham complex,

$$0 \to \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_X^1 \xrightarrow{\nabla_1} \mathcal{E} \otimes \Omega_X^2 \xrightarrow{\nabla_2} \mathcal{E} \otimes \Omega_X^3 \to \cdots$$

is actually a complex.

Remark. Notice in the case that $t = 0$ we saw ∇ is $\mathcal{O}_X$-linear. Call $\phi := \nabla$. Then notice,

$$\omega_\nabla(s) = \nabla_1 \circ \nabla(s) = \nabla_1 \left(\sum_i s_i \otimes \omega_i \right) = -\phi(s_i) \wedge \omega_i = -(\phi \wedge \phi)(s)$$

where,

$$\phi(s) = \sum_i s_i \otimes \omega_i$$

Therefore, $\phi \wedge \phi = 0$ if and only if the t-connection ∇ is flat.

The previous calculation shows that a t-connection is a gadget that interpolates between a flat connection on X_1 over $t = 1$ and a Higgs bundle on X_0 over $t = 0$. If we take the constant t-scheme $X \times \mathbb{A}^1 \to \mathbb{A}^1$ and a constant coherent sheaf $\pi_1^* \mathcal{E}$ then a t-connection is litterally just linearly interpolating between a connection on $\mathcal{E}$ and a Higgs bundle structure on $\mathcal{E}$. This picture is completely functorial so we get a universal interpretation,

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{Dol}}(X) \arrow[r, hook] \arrow[d] & \mathcal{M}_{\mathrm{Hod}}(X) \arrow[d] & \mathcal{M}_{\mathrm{dR}}(X) \arrow[l, hook'] \arrow[d] \\
\{t = 0\} \arrow[r, hook] & \mathbb{A}^1 & \{t = 1\} \arrow[l, hook']
\end{tikzcd}$$

so we get a moduli space $\mathcal{M}_{\mathrm{Hod}}(X)$ of flat t-connections $(t, \mathcal{E}, \nabla)$ with a $\mathbb{G}_m$-equivariant map,

$$\mathcal{M}_{\mathrm{Hod}}(X) \to \mathbb{A}^1 \qquad (t, \mathcal{E}, \nabla) \mapsto t$$

where the $\mathbb{G}_m$-acts via,

$$\lambda \cdot (t, \mathcal{E}, \nabla) = (\lambda t, \mathcal{E}, \lambda \nabla)$$