1 Lie Groups

Definition 1.0.1. A Lie Group X is a smooth manifold with a smooth group stucture.

Definition 1.0.2. Let G be a Lie group and X a manifold. A smooth action of G on X is a smooth map $A : G \times X \to X$ where we write $g \cdot x = A(g, x)$ such that $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $1 \cdot x = x$. This is equivalent to a smooth map $G \to \mathrm{Diffeo}(X)$.

2 Quotients and Subgroups

2.1 Immersed and Embedded Submanifolds

Remark. First we recall the definition of immersed and embedded submanifolds.

Definition 2.1.1. An *immersed submanifold* of M is an equivalence class of immersions $\iota : N \to M$.

Definition 2.1.2. An *embedded submanifold* of M is an equivalence class of immersive topological embeddings $\iota : N \to M$ i.e. $\mathrm{d}\iota$ is injective and $\iota : N \to \iota(N)$ is a homeomorphism.

Remark. In the previous case, the image $\iota(N)$ (with the subspace topology) has the natural structure of a smooth manifold making $\iota : N \to \iota(N)$ a diffeomorphism.

Lemma 2.1.3. A proper map $f : X \to Y$ of compactly generated hausdorff (CGH) spaces is closed.

Proof. embedded submanifold is closed iff inclusion is proper □

Proposition 2.1.4. An embedded submanifold $\iota : N \to M$ is closed iff ι is proper.

Proof. Because ι is injective, ι closed is equivalent to $\iota(N) \subset M$ being closed. First proper maps between CGH spaces are closed. Conversely, if ι is closed then for any compact $K \subset M$ consider $\iota^{-1}(K)$. Because ι is an embedding $\iota^{-1}(K) \xrightarrow{\sim} K \cap \iota(N)$ which is a closed subspace of a compact space and thus compact so ι is proper. □

Remark. We can easily produce non-closed embedded submanifold simply by deleting points (open subsets of a manifold are embedded submanifolds). However, we will see that morally this is the only thing that can go wrong.

Proposition 2.1.5. Embedded submanifolds are locally closed.

Proof. TO DOO!!! □

Proposition 2.1.6. Let $\iota : N \to M$ be an injective immersion such that one of the following holds,

(a) ι is open

(b) ι is closed

(c) ι is proper

then ι is a clsoed embedding thus $\iota : N \to M$ gives an embedded submanifold.

Proof. Lee Prop. 4.22 □

2.2 Lie Subgroups

Definition 2.2.1. An *immersed Lie subgroup* of G is an equivalence class of immersive homomorphisms $\iota : H \to G$ of Lie groups.

Definition 2.2.2. An *embedded Lie subgroup* of G is an equivalence class of embedding homomorphisms $\iota : H \to G$ of Lie groups.

Remark. For example, consider $\mathbb{R} \to S^1 \times S^1$ mapping injectively through an irrational slope. This is an immersed subgroup but not embedded because the image becomes dense and thus the manifold structure on $\mathbb{R}$ cannot agree with the subspace topology.

Proposition 2.2.3. Any embedded Lie subgroup is closed.

Proof. Assume without loss of generality that $H \subset G$ is a subgroup and H is a Lie group with the supspace topology. $\overline{H} \subset G$ is a subgroup by facts about topological groups. Because $H \subset G$ is an embedded submanifold, it is locally closed meaning for each $h \in H$ there is an open $V \subset G$ such that $H \cap V$ is closed in V. Thus for each $x \in \overline{H}$ we have $xH \cap xV = \overline{H} \cap xV$ so each $xh \in xH$ has an open nieghborhood $\overline{H} \cap xV$ in $\overline{H}$ contained in xH so $xH \subset \overline{H}$ is open. Finally, if $x \in \overline{H}$ then because $x \in xH$ and $xH \subset \overline{H}$ is open we know that $xH \cap H$ is nonempty so $x \in H$. □

Remark. More generally, this shows that locally closed subgroups of topological groups are closed.

Theorem 2.2.4 (Cartan)**.** Let G be a Lie group. Any closed subgroup $H \subset G$ has a unique smooth structure making it an embedded Lie subgroup.

Proof. DOD THIS!!!! □

Remark. The previous results imply a correspondence between closed subgroups and embedded Lie subgroups. From now on, we will simply refer to such objects as *closed subgroups* or *embedded subgroups* since there is no ambiguity about the smooth structure, embedding vs immersion, or closure of such a group.

2.3 Quotients

Definition 2.3.1. A continuous action of a topological group G on a topological space X is *proper* if the map $\pi : G \times X \to X \times X$ given by $(g, x) \mapsto (g \cdot x, x)$ is a proper map. In particular, if

$$\mathrm{Stab}(x) \times \{x\} = \pi^{-1}(\{(x, x)\})$$

is compact.

Lemma 2.3.2. If G is compact then any action of G on X is proper.

Proof. Let $D \subset X \times X$ be compact and thus closed because D is compact in a Hausdorff manifold. Thus, $\pi^{-1}(D) = \{(g, x) \mid (g \cdot x, x) \in D\}$ is closed in $G \times X$ and thus closed in $G \times \pi_2(D)$ which is compact. Therefore $\pi^{-1}(D)$ is compact. Notice that $\pi_2(D)$ is compact because D is compact and thus $G \times \pi_2(D) \subset G \times X$ is closed because it is compact in a Hausdorff space. □

Proposition 2.3.3. Let G be a topological group and $H \subset G$ a closed subgroup. The left and right actions of H on G are proper.

Proof. Let $D \subset G \times G$ be compact and consider,

$$\pi^{-1}(D) = \{(h,g) \in H \times G \mid (hg, g) \in D\} = \{(g'g^{-1}, g) \mid (g', g) \in D\} \cap H \times G$$

However, $\{(g'g^{-1}, g) \mid (g', g) \in D\}$ is homeomorphic to D via $(x, y) \mapsto (xg^{-1}, y)$ and is thus compact so its intersection with the closed subspace $H \times G$ is compact. The same argument works for a right action. $\square$

Proposition 2.3.4. Let G be a topological group. The adjoint action $G \circlearrowright G$ via $g \cdot x = gxg^{-1}$ is proper if and only if G is compact.

Proof. If the action is proper then $\mathrm{Stab}(1) = G$ must be compact but if G is compact then every action is proper. $\square$

Theorem 2.3.5. Let G be a Lie group and X a smooth manifold with a smooth action $\rho : G \circlearrowright X$ which is,

(a) proper: $\pi : G \times X \to X \times X$ is a proper map

(b) free: $\forall x : G_x = \{\mathrm{id}_G\}$ i.e. if $g \cdot x = x$ then $g = \mathrm{id}$

then X/G has a unique smooth structure such that $\pi : X \to X/G$ is a smooth submersion. In fact, this smooth surjection locally admits sections and thus $\pi : X \to X/G$ is a principal G-bundle with the given action $G \circlearrowright X$.

Proof. See a proof here and also look here. $\square$

Corollary 2.3.6. If $H \subset G$ is a closed subgroup then G/H is a manifold and $\pi : G \to G/H$ is a principal G-bundle.

Proof. The action of G on H is free because if $g \cdot h = gh = h$ then $g = e$. Furthermore, the action of H on G is proper by Proposition 2.3.3. Furthermore, there is an obvious action of G on G/H which is free and transitive on the fibers. It suffices to show that $\pi : G \to G/H$ is locally trivial which is equivalent to finding local sections. Here is a reference showing how to do it. $\square$

Remark. We really do need $H \subset G$ to be closed otherwise G/H is not even Hausdorff!

Corollary 2.3.7. For any closed subgroup $H \subset G$ there is a fibration,

$$H \hookrightarrow G \twoheadrightarrow G/H$$

and therefore there is a long exact sequence of homotopy groups,

$$\cdots \longrightarrow \pi_n(H) \longrightarrow \pi_n(G) \longrightarrow \pi_n(G/H) \longrightarrow \pi_{n-1}(H) \longrightarrow \cdots$$

Proof. It follows from the fact that fiber bundles over paracompact bases are fibrations and manifolds are paracompact. $\square$

2.4 Orbits and Stabilizers

Remark. In general for a smooth action $\rho : G \circlearrowright X$ we can topologize the orbits $G \cdot x$ by declaring that the G-equivariant bijection $G/G_x \to G \cdot x$ from the orbit-stabilizer theorem is a homeomorphism. Since $G_x = \rho_x^{-1}(\{x\})$ (where $\rho_x : G \to X$ is $g \mapsto g \cdot x$) is a closed subgroup so G/G_x, and thus $G \cdot x$, is a smooth manifold via the homeomorphism $G/G_x \xrightarrow{\sim} G \cdot x$.

Proposition 2.4.1. The map $G/G_x \xrightarrow{\sim} G \cdot x \to X$ makes $G \cdot x \hookrightarrow X$ an injective immersed submanifold.

Proof. Clearly $f_x : G/G_x \to X$ is injective since it is bijective onto $G \cdot x \to X$ which is injective. Since $f_x : G/G_x \to X$ is G-equivariant for the left G-action on G/G_x which is transitive, f_x has constant rank. By the constant rank theorem, locally we can replace f_x by $\mathrm{d}f_x$ which thus must be injective because f_x is proving that f_x is an immersion. □

Lemma 2.4.2. Let G be a Lie group G acting properly on a smooth manifold X. Then each orbit map $\rho_x : G \to X$ is proper.

Proof. Indeed, the map $\phi : G \times X \to X \times X$ restricted to $G \times \{x\}$ sends $g \mapsto (g \cdot x, x) = (\rho_x(g), x)$ and thus if $K \subset X$ is compact then $\phi^{-1}(K \times \{x\}) = \rho_x^{-1}(K) \times \{x\}$ is compact because the action is proper so $\rho_x^{-1}(K)$ is compact. □

Lemma 2.4.3. Let $f : X \to Y$ be a map and $q : \tilde{X} \to X$ surjective such that $f \circ q$ is proper. then f is proper.

Proof. Let $K \subset Y$ be compact. Then $q^{-1}(f^{-1}(K))$ is compact but because q is surjective $f^{-1}(K) = q(q^{-1}(f^{-1}(K)))$ and thus $f^{-1}(K)$ is compact because it is the continuous image of a compact space. □

Proposition 2.4.4. Let G be a Lie group G acting properly on a smooth manifold X. Then the orbits of $G \circlearrowright X$ are embedded closed submanifolds of X and $G/G_x \xrightarrow{\sim} G \cdot x$ is a diffeomorphism.

Proof. When the action is proper, each $\rho_x : G \to X$ is proper. Therefore, by the lemma, $G/G_x \to X$ is proper. Since it is proper and also an injective immersion it is a closed embedding. Therefore, $G/G_x \xrightarrow{\sim} G \cdot x \hookrightarrow X$ is a closed embedding so, in particular, $G \cdot x \hookrightarrow X$ is a closed embedding. □

Proposition 2.4.5. For a smooth action $G \circlearrowright X$ and a point $x \in X$ the following are equivalent:

(a) the map $G/G_x \xrightarrow{\sim} G \cdot x \hookrightarrow X$ is a smooth embedding

(b) $G \cdot x \subset X$ with the subspace topology is an embedded submanifold

(c) $G \cdot x \subset X$ is closed in the subspace topology

(d) $\rho_x : G \to X$ is proper.

Proof. Here is a partial reference. □

Proposition 2.4.6. If $G \circlearrowright X$ properly then for any point $x \in X$ the orbit $G \cdot x \subset X$ is a closed embedded submanifold and $G \to G/G_x \xrightarrow{\sim} G \cdot x$ is a principal G_x-bundle giving a fibration,

$$G_x \hookrightarrow G \twoheadrightarrow G/G_x \cong G \cdot x$$

Proof. This is simply a combination of Corollary 2.3.6 and Proposition 2.4.4 and noting that fiber bundles over paracompact bases are fibrations. □

3 Lie Algebras

Definition 3.0.1. A Lie Algebra $\mathfrak{g}$ over a field K is a algebra over K with multiplication written $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying,

(a) $[x, y] = -[y, x]$

(b) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$

Definition 3.0.2. Let G be a Lie group. There is a canonical Lie group structure on T_1G.

Proof. For $\xi, \eta \in T_1G$ we will define a bracket $[\xi, \eta]$. Consider the map $f_g : G \to G$ given by $x \mapsto gxg^{-1}$ then $\mathrm{d}f_g : T_1G \to T_1G$. Suppose we have a path, $\gamma : I \to G$ such that the unit tangent vector is mapped to $d\gamma(e_1) = \xi$. Then we write,

$$[\xi, \eta] = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{d}f_{\gamma(t)}(\eta)\Big)\Big|_{t=0}$$

$\square$

Proposition 3.0.3. Let $f : G \to H$ be a Lie group homomorphism. Then $\mathrm{d}f : \mathfrak{g} \to \mathfrak{h}$ [1] is a morphism of Lie algebras i.e. $f([\xi, \eta]_G) = [f(\xi), f(\eta)]_H$.

Corollary 3.0.4. Let $H \subset G$ be a Lie subgroup then there is a natural embedding of the Lie algebras $\mathfrak{h} \subset \mathfrak{g}$.

Definition 3.0.5. A Lie Group representation of G on V is a Lie Group homomorphism $G \to \mathrm{Aut}\,(V)$.

Definition 3.0.6. Let $\rho_V : G \to \mathrm{Aut}\,(V)$ be a Lie Group representation. Then we can construct the *dual* representation $\rho_V^* : G \to \mathrm{Aut}\,(V)$ via,

$$\rho_V^*(g) = (\rho_V(g^{-1}))^*$$

which is a representation because,

$$\rho_V^*(gh) = \big(\rho_V(h^{-1}g^{-1})\big)^* = \big(\rho_V(h^{-1})\rho_V(g^{-1})\big)^* = \rho_V(g^{-1})^*\rho_V(h^{-1})^* = \rho_V^*(g)\rho_V^*(h)$$

Definition 3.0.7. The adjoint action $a : G \to \mathrm{Aut}\,(G)$ is given by $g \mapsto a_g : G \to G$ which acts via $x \mapsto gxg^{-1}$. Then, the differential gives, $\mathrm{Ad}(g) = \mathrm{d}a_g : \mathfrak{g} \to \mathfrak{g}$ and the map $\mathrm{Ad} : G \to \mathrm{Aut}\,(\mathfrak{g})$ is a G-representation. Then the differential gives a Lie algebra representation,

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}\,(\mathfrak{g})$$

where $\mathrm{ad}_\xi = \mathrm{d}(\mathrm{Ad})_\xi$.

Theorem 3.0.8. For any $\xi, \in \mathfrak{g}$ and $X \in \mathfrak{g}$ we have,

$$\mathrm{ad}_\xi(X) = [\xi, X]$$

[1] All differentials in this section will be applied at the identity of the group unless explicilty stated otherwise.

Proof. (DO THIS) We may check that $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is in fact a Lie algebra representation by using the Jacobi identity. Recall that,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

which we may reagrange as,

$$[x, [y, z]] - [y, [x, z]] = [[x, y], z]$$

and then rewrite as,

$$(\text{ad}_x \circ \text{ad}_y - \text{ad}_y \circ \text{ad}_x)(z) = \text{ad}_{[x,y]}(z)$$

where the left hand side is the bracket for $\mathfrak{gl}(\mathfrak{g})$ impling that,

$$[\text{ad}_x, \text{ad}_y] = \text{ad}_{[x,y]}$$

so the map $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra representation. □

Theorem 3.0.9 (Lie)**.** For any Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ there exists a unique simply-connected real or complex Lie group G with $\text{Lie}(G) = \mathfrak{g}$.

4 The Exponential Map

Definition 4.0.1. The multiplication map $m : G \times G \to G$ is smooth. Thus, $m(-, g)$ and $m(g, -)$ are smooth diffeomorphism $G \to G$. Thus, denote the action of $\mathrm{d}m(g, -) : T_eG \to T_gG$ on $\xi \in \mathfrak{g}$ by $g \cdot \xi = \mathrm{d}m(g, -)(\xi) \in T_gG$ and, likewise, the action of $\mathrm{d}m(-, g) : T_eG \to T_gG$ on $\xi \in \mathfrak{g}$ by $\xi \cdot g = \mathrm{d}m(-, g)(\xi) \in T_gG$.

Definition 4.0.2. The exponetial map $\exp : \mathfrak{g} \to G$ is defined as follows. For $\xi \in \mathfrak{g}$ we can define a smooth vector field $X^\xi \in \mathscr{X}(G)$ by $X^\xi_g = \xi \cdot g$. Let $\gamma : I \to G$ be an integral curve of X such that $I(0) = e$. Then the exponential map is defined as $\exp \xi = \gamma(1)$.

Proposition 4.0.3. Let $f : G \to H$ be a Lie group homomorphism. Then the exponential diagram,

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

commutes where $f_* = \mathrm{d}f_e$.

Proof. Let γ be the interval curve of X^ξ. That is,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X^\xi(\gamma(t)) = \xi \cdot \gamma(t)$$

Then consider the smooth path $f \circ \gamma : I \to H$ and its derivative,

$$\frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t} = \mathrm{d}(f \circ \gamma)_t \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \mathrm{d}f_{\gamma(t)} \circ \mathrm{d}\gamma_t \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \mathrm{d}f_{\gamma(t)} \left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) = \mathrm{d}f_{\gamma(t)}(\xi \cdot \gamma(t))$$

We can reqrite this result using $\xi \cdot g = \mathrm{d}m(-,g)(\xi)$,

$$\frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t} = \mathrm{d}f_{\gamma(t)}\mathrm{d}m(-,g)(\xi) = \mathrm{d}(f \circ m(-,g))(\xi)$$

However, $f \circ m(-,g)(x) = f(xg) = f(x)f(g) = m(-,f(g)) \circ f(x)$ and thus $f \circ m(-,g) = m(-,f(g)) \circ f$. Therefore,

$$\mathrm{d}f_g \circ \mathrm{d}m(-,g) = \mathrm{d}(f \circ m(-,g)) = \mathrm{d}(m(-,f(g)) \circ f) = \mathrm{d}m(-,f(g))_e \circ \mathrm{d}f_e$$

Let $g = \gamma(t)$ then,

$$\frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t} = \mathrm{d}(f \circ m(-,\gamma))(\xi) = \mathrm{d}m(-,f(\gamma)) \circ f_*(\xi) = f_*(\xi) \cdot (f \circ \gamma)(t)$$

Thus, $f \circ \gamma$ is the integral curve starting at $f \circ \gamma(0) = f(e) = e$ of the vector field $X^{f_*(\xi)}$ given by $h \mapsto f_*(\xi) \cdot h$. Therefore,

$$\exp(f_*(\xi)) = (f \circ \gamma)(1) = f(\gamma(1)) = f(\exp(\xi))$$

□

Lemma 4.0.4. Let G be a Lie group and let $f_1 : M \to G$ and $f_2 : M \to G$ be smooth maps. Then, $F = f_1 \cdot f_2 = m \circ (f_1, f_2)$ is a smooth map with,

$$\mathrm{d}F(\xi) = \mathrm{d}f_1(\xi) \cdot f_2 + f_1 \cdot \mathrm{d}f_2(\xi)$$

Proof. We have,

$$\mathrm{d}F_p = \mathrm{d}m_{f_1(p),f_2(p)} \circ \mathrm{d}(f_1, f_2) = \mathrm{d}m_{f_1(p),f_2(p)} \circ ((\mathrm{d}f_1)_p \oplus (\mathrm{d}f_2)_p)$$

Furthermore,

$$\mathrm{d}m = \mathrm{d}(m \circ \iota_1^{f_2(p)}) + \mathrm{d}(m \circ \iota_1^{f_1(p)}) = \mathrm{d}m(-,f_2(p)) + \mathrm{d}m(f_1(p),-)$$

and thus,

$$\mathrm{d}F_p = \mathrm{d}m(-,f_2(p)) \circ (\mathrm{d}f_1)_p + \mathrm{d}m(f_1(p),-) \circ (\mathrm{d}f_2)_p$$

Therefore, for $\xi \in T_pM$ we have,

$$\begin{aligned}\mathrm{d}F_p(\xi) &= \mathrm{d}m(-,f_2(p)) \circ (\mathrm{d}f_1)_p(\xi) + \mathrm{d}m(f_1(p),-) \circ (\mathrm{d}f_2)_p(\xi) \\ &= (\mathrm{d}f_1)_p(\xi) \cdot f_2(p) + f_1(p) \cdot (\mathrm{d}f_2)_p(\xi)\end{aligned}$$

□

Corollary 4.0.5. For any $\xi \in \mathfrak{g}$ we have $\mathrm{Ad}(\exp \xi) = \exp \circ (\mathrm{ad}_\xi)$. Therefore, on the lie algebra, for any $X \in \mathfrak{g}$ we have,

$$(\exp \xi) \cdot X \cdot (\exp \xi)^{-1} = \mathrm{Ad}(\exp \xi) \cdot X = (\exp(\mathrm{ad}_\xi))(X) = (\exp [\xi, -]) \cdot X$$

Proposition 4.0.6. The left and right-invariant vector fields, $X_L^\xi, X_R^\xi \in \mathscr{X}(G)$ associated with $\xi \in \mathfrak{g}$ i.e. $X_L^\xi(g) = g \cdot \xi$ and $X_R^\xi(g) = \xi \cdot g$ have the same integral curves at the identiy. Thus, either can be used to define the exponential map.

Proof. Let $\gamma_1, \gamma_2 : I \to G$ be smooth curves satisfying,

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} = X_L^{\xi}(\gamma_1(t)) = \gamma_1(t) \cdot \xi \quad \text{and} \quad \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} = X_R^{\xi}(\gamma_2(t)) = \xi \cdot \gamma_2(t)$$

First consider,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma \cdot \gamma^{-1}\right) = \frac{\mathrm{d}\gamma}{\mathrm{d}t} \cdot \gamma^{-1} + \gamma \cdot \frac{\mathrm{d}\gamma^{-1}}{\mathrm{d}t}$$

But $\gamma \cdot \gamma^{-1} = e$ so the differential is zero. Thus,

$$\frac{\mathrm{d}\gamma^{-1}}{\mathrm{d}t} = -\gamma^{-1} \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}t} \cdot \gamma^{-1}$$

Therefore, consider,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_1 \cdot \gamma_2^{-1}\right) &= \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \cdot \gamma_2^{-1} + \gamma_1 \cdot \frac{\mathrm{d}\gamma_2^{-1}}{\mathrm{d}t} = \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \cdot \gamma_2^{-1} - \gamma_1\gamma_2^{-1} \cdot \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \cdot \gamma_2^{-1} \\ &= \gamma_1 \cdot \xi \cdot \gamma_2^{-1} - \gamma_1\gamma_2^{-1} \cdot \xi \cdot \gamma_2^{-1}\gamma_2 = \gamma_1 \cdot \xi \cdot \gamma_2^{-1} - \gamma_1\gamma_2^{-1} \cdot \xi \end{aligned}$$

At $t = 0$ we have $\gamma_1(0) = \gamma_2(0) = e$ and thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_1 \cdot \gamma_2^{-1}\right)\bigg|_{t=0} = \xi - \xi = 0$$

Therefore, $\gamma_1 \cdot \gamma_2^{-1} = e$ is constant and thus $\gamma_1 = \gamma_2$. $\square$

5 Lie Algebras

Definition 5.0.1. A Lie Algebra $\mathfrak{g}$ over a commutative ring R is an R-module with a bilinear bracket $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which satisfies,

(a) $\forall x \in \mathfrak{g} : [x, x] = 0$

(b) $\forall x, y, z \in \mathfrak{g} : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Definition 5.0.2. The *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ over a ring R is the unital associative R-algebra,

$$U\mathfrak{g} = T_R(\mathfrak{g})/I$$

where I is the ideal generated by $\{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\}$. Note that,

$$x \otimes y - y \otimes x - [x, y] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \subset T(\mathfrak{g})$$

The universal enveloping algebra defines a functor $U : \mathbf{LieAlg}_R \to \mathbf{Mod}_R$

Definition 5.0.3. A representation of a Lie Algebra $\mathfrak{g}$ over R is an R-module M and a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(M)$. That is a linear map $\rho : \mathfrak{g} \to \mathrm{End}(V)$ which preserves the bracket i.e.

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

Proposition 5.0.4. The category of representations of a Lie algebra $\mathfrak{g}$ is equivalent to the category of $U\mathfrak{g}$-modules.

Proof. Any Lie algebra representation $\rho : \mathfrak{g} \to \mathfrak{gl}(M)$ may be extended to a ring map $U\mathfrak{g} \to \mathrm{End}(M)$ by sending $\rho(m) = m \cdot \mathrm{id}$ and $\rho(x \otimes y) = \rho(x)\rho(y)$. Then we have,

$$\rho(x \otimes y - y \otimes x) = \rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x, y])$$

so this extension is well-defined on the quotient. Likewise any map $U\mathfrak{g} \to \mathrm{End}(M)$ restricts to $\mathfrak{g} \to \mathrm{End}(M)$ and sends the bracket to the commutator thus giving a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(M)$. □

Lemma 5.0.5. Let R be a ring and M, N be simple R-modules. Then any R-module morphism $f : M \to N$ is zero or an isomorphism.

Proof. Let $f : V \to W$ be A-linear (i.e. a morphism of A-representations). Then $\ker f \subset V$ is a submodule so $\ker f = 0$ or $\ker f = V$ by simplicity. Thus either $f = 0$ or injective. Furthermore, $\mathrm{Im}(f) \subset W$ is a submodule so either $\mathrm{Im}(f) = 0$ or $\mathrm{Im}(f) = W$ thus either $f = 0$ or surjective. Therefore, either $f = 0$ or f is an isomorphism. □

Lemma 5.0.6 (Schur)**.** Let A be a unital associative K-algebra over an algebraically closed field K and V and W simple A-modules. Then,

$$\mathrm{Hom}_A(V, W) = \begin{cases} K & V \cong W \\ 0 & V \not\cong W \end{cases}$$

Proof. By above, any nonzero map is an isomorphism. In the case, $V \cong W$, fix an isomorphism $f : V \to W$. Consider any $g : V \to W$ then $f^{-1} \circ g : V \to V$ is an endomorphism over vectorspaces over an algebraically closed field so $f^{-1} \circ g$ has an eigenvector $v \in V$ with eigenvalue λ. Thus $f^{-1} \circ g - \lambda \cdot \mathrm{id}_V$ is not injective but is a morphism of representations so, by above, $f^{-1} \circ g - \lambda \cdot \mathrm{id}_V = 0$. Thus, $g = \lambda \cdot f$. □

Remark. For the case $A = \mathbb{C}[G]$ for some group G a simple $\mathbb{C}[G]$-module is the same as irreducible complex G-representation giving the standard form of the lemma.

Corollary 5.0.7. Let A be a unital associative K-algebra over an algebraically closed field and V a semisimple A-modules. Then there is a canoical isomorphism, s

$$\bigoplus_X \mathrm{Hom}_A(X, V) \otimes_A X \xrightarrow{\sim} V$$

where X runs over the simple A-modules.

Proof. The canonical map sends $f \otimes x \mapsto f(x)$. We need to show that this map is an isomorphism. Decompose,

$$V = \bigoplus_X X^{\oplus n_X}$$

Then, by Schur,

$$\mathrm{Hom}_A(X, V) \cong A^{\oplus n_X}$$

which gives,

$$\bigoplus_X \mathrm{Hom}_A(X, V) \otimes_A X = \bigoplus_X A^{\oplus n_X} \otimes_A X = \bigoplus_X X^{\oplus n_X} = V$$

by the evauluation map. □

Definition 5.0.8. A Casimir element of a Lie algebra $\mathfrak{g}$ is an element of $Z(U\mathfrak{g})$ i.e. an element of $U\mathfrak{g}$ commuting with everything in $\mathfrak{g}$ and thus all of $U\mathfrak{g}$.

Proposition 5.0.9. Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field K and $\omega \in U\mathfrak{g}$ a Casimir. Suppose that $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducble $\mathfrak{g}$-representation then $\rho(\omega) = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in K$ where $\rho : U\mathfrak{g} \to \mathrm{End}(V)$ is the induced map.

Proof. Let ω be a Casimir. I claim that $\rho(\omega)$ is a $\mathfrak{g}$-morphism $V \to V$. This is because $\forall x \in U\mathfrak{g}$: $x \otimes \omega = \omega \otimes x$ in $U\mathfrak{g}$ meaning that $\rho(x) \circ \rho(\omega) = \rho(\omega) \circ \rho(x)$. Thus the map $\rho(\omega)$ is $U\mathfrak{g}$-linear. Since V is irreducible and K is algebraically closed, by Schur's lemma, $\rho(\omega) = \lambda \cdot \mathrm{id}_V$. □

Remark. In the previous case, we call λ the Casimir invariant of the irreducible representation V associated to the Casimir element ω.

6 Misc

Theorem 6.0.1 (Poincare-Hopf)**.** Let M be a compact smooth manifold and X a smooth vector field on M with isolated zeros. Then,

$$\sum_{x \in X} \mathrm{index}_x(X) = \chi(M)$$

Proposition 6.0.2. A vector bundle of rank r is trivial iff it admits r pointwise linearly independent sections.

Proof. r-pointwise linear independent sections define a global frame i.e. an isomorphism of vector bundles $M \times \mathbb{R}^r \xrightarrow{\sim} V$. □

Theorem 6.0.3. Let G be a Lie group, then $TG \cong G \times \mathfrak{g}$ i.e. the tangent bundle is trivial.

Proof. □

Theorem 6.0.4. Let G be a compact Lie group (of positive dimension) then $\chi(G) = 0$.

Proof. Since $\pi : TG \to G$ is a trivial bundle it admits $n = \dim G$ pointwise linearly independent sections (i.e. vector fields) which thus must be nonvanising everywhere (since $n > 0$). Thus, by Poincare-Hopf, $\chi(G) = 0$. □

Theorem 6.0.5. For n even, S^n admits no nonvanishing vector fields.

Proof. Such a vector field would give a homotopy $\mathrm{id} \simeq -\mathrm{id}$ and thus the degrees of these maps must be equal i.e. $(-1)^{n+1} = 1$ so n must be odd. Alternativly, $\chi(S^n) = 1 + (-1)^n$ and therefore, in the case n is even $\chi(S^n) = 2$. In that case, a nonvanishing vector field would contradict the Poincare-Hopf theorem. □

Theorem 6.0.6. Let G be a compact Lie group then $\pi_2(G) = 0$. If G is nonabelian then $\pi_3(G) \neq 0$.

Corollary 6.0.7. S^n admits a Lie group structure exactly when $n = 0, 1, 3$.

Proof. The case S^0 is a zero-dimensional Lie group is clear. Assume $n \geq 1$ so S^n is connected. If G is an abelian Lie group then its Lie algebra is trivial. By the Lie group Lie algebra correspondence, its universal cover must be $\mathbb{R}^n$. However, S^n is simply connected for $n > 1$ so S^1 is the only abelian sphere group. If G is nonabelian then $\pi_3(G) \neq 0$ but $\pi_3(S^n) = 0$ for $n > 3$. Thus we have shown that $n \leq 3$. The case $n = 2$ is excluded by noting that even dimensional spheres have nontrivial tangent bundles and thus cannot be Lie groups. □