

# 1 Geometry Identities

## 1.1 Interior Derivatives

**Definition:** Let  $\omega$  be a  $k$ -form and  $X$  a vector field  $X$ . Then, we define the interior derivative,

$$(\iota_X \omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1})$$

**Remark 1.** By antisymmetry of forms  $(\iota_X \circ \iota_Y + \iota_Y \circ \iota_X)\omega = 0$  and thus  $\iota_X \circ \iota_X = 0$ .

**Lemma 1.1.**

$$\mathcal{L}_X f = df(X) = X(f)$$

*Proof.* Consider the flow  $\phi_t : M \rightarrow M$  along the vector field  $X$ . Then we define,

$$\begin{aligned} (\mathcal{L}_X f)(x) &= \frac{d}{dt} \Big|_{t=0} (\phi_t^* f) = \frac{d}{dt} \Big|_{t=0} (f \circ \phi_t) \\ &= df \circ d\phi(x) \left( \frac{\partial}{\partial t} \right) = df(X) \end{aligned}$$

because, by definition,

$$\frac{d}{dt} \phi_t(x) = d\phi(x) \left( \frac{\partial}{\partial t} \right) = X_x$$

□

**Theorem 1.2.** For any  $k$ -form  $\omega$  and vector field  $X$  we have,

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$$

*Proof.* We will prove this by induction on  $k$ . For  $k = 0$  we have,

$$\mathcal{L}_X f = df(X)$$

and furthermore,

$$d\iota_X f + \iota_X df = \iota_X df = df(X)$$

Now we can also consider,

$$\mathcal{L}_X(df) = d(\mathcal{L}_X f) = dX(f)$$

Furthermore,

$$[d\iota_X + \iota_X d](df) = d(\iota_X df) = dX(f)$$

Now, since  $\Omega_M^1$  is generated as a  $\mathcal{O}_M$ -module by the forms  $df$  it will suffice to show that both sides are derivations. Then, for  $\alpha$  a  $p$ -form and  $\beta$  a  $q$ -form,

$$\begin{aligned} [d\iota_X + \iota_X d](\alpha \wedge \beta) &= d(\iota_X \alpha \wedge \beta + (-1)^p \alpha \wedge \iota_X \beta) + \iota_X (d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta) \\ &= d\iota_X \alpha \wedge \beta + (-1)^{p-1} \iota_X \alpha \wedge d\beta + (-1)^p d\alpha \wedge \iota_X \beta + \alpha \wedge d\iota_X \beta \\ &\quad + \iota_X d\alpha \wedge \beta + (-1)^{p+1} d\alpha \wedge \iota_X \beta + (-1)^p \iota_X \alpha \wedge d\beta + \alpha \wedge \iota_X d\beta \\ &= [d\iota_X + \iota_X d]\alpha \wedge \beta + \alpha \wedge [d\iota_X + \iota_X d]\beta \end{aligned}$$

so both sides are derivations and thus they must be equal since they agree for a basis of 1-forms. □

## 2 The Hodge Complex

**Definition:** Let  $(M, g)$  be an oriented Riemannian  $n$ -manifold and  $\omega$  the canonical volume form. Then  $g : TM \otimes TM \rightarrow \mathcal{O}_M$  defines a fiberwise nondegenerate inner product which we may view as an isomorphism  $g : TM \rightarrow T^*M$  which, along with its inverse  $g^{-1} : T^*M \rightarrow TM$ , extends to isomorphisms on dual tensor bundles  $T_m^n M \xrightarrow{\sim} T_m^n M$  and thus a nondegenerate pairing  $\langle -, - \rangle : T_m^n M \otimes T_m^n M \rightarrow \mathcal{O}_M$ .

Then we can define a Hilbert space  $L^2(\mathcal{C}^\infty(M, T_m^n M))$  on the tensor bundles  $T_m^n$  via the inner product,

$$\langle \langle \alpha, \beta \rangle \rangle = \int_M \langle \alpha, \beta \rangle \omega$$

Since  $\omega$  is nonvanishing and the functions are smooth (and thus continuous) then,

$$\|\alpha\|^2 = \langle \langle \alpha, \alpha \rangle \rangle = 0 \iff \alpha = 0$$

**Definition:** On an oriented Riemannian  $n$ -manifold with canonical volume form  $\omega$  we define the Hodge dual  $\star : \Omega_M^k \rightarrow \Omega_M^{n-k}$  as the unique map such that,

$$\forall \alpha, \beta \in \Omega_M^k(U) : \alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle \omega$$

Furthermore, we have  $\star \star \eta = (-1)^{n(n-k)} \eta$ .

**Definition:** We define the codifferential  $\delta : \Omega_M^{k+1} \rightarrow \Omega_M^k$  via  $\delta = (-1)^{k+1} \star^{-1} d \star$ .

**Remark 2.** This makes a chain complex since,

$$\delta \circ \delta = (-1)^{2k-1} (\star^{-1} d \star) \circ (\star^{-1} d \star) = - \star^{-1} d \circ d \star = 0$$

**Lemma 2.1.** For all  $\alpha \in \Omega_M^k(U)$  and  $\beta \in \Omega_M^{k+1}(U)$  we have,

$$\langle \langle d\alpha, \beta \rangle \rangle = \langle \langle \alpha, \delta\beta \rangle \rangle$$

*Proof.* We have  $\langle d\alpha, \beta \rangle \omega = d\alpha \wedge (\star \beta)$ . Now consider,

$$\begin{aligned} d(\alpha \wedge (\star \beta)) &= d\alpha \wedge (\star \beta) + (-1)^k \alpha \wedge d(\star \beta) \\ &= d\alpha \wedge (\star \beta) + (-1)^k \alpha \wedge (\star (\star^{-1} d(\star \beta))) \\ &= d\alpha \wedge (\star \beta) - \alpha \wedge (\star \delta \beta) \end{aligned}$$

Then, by Stokes' theorem,

$$\int_M d(\alpha \wedge (\star \beta)) = \int_{\partial M} \alpha \wedge (\star \beta) = 0$$

because  $M$  is closed. Therefore,

$$\langle \langle d\alpha, \beta \rangle \rangle = \int_M \langle d\alpha, \beta \rangle \omega = \int_M d\alpha \wedge (\star \beta) = \int_M \alpha \wedge (\star \delta \beta) = \int_M \langle \alpha, \delta \beta \rangle \omega = \langle \langle \alpha, \delta \beta \rangle \rangle$$

□

**Definition:** We define the Laplace-deRham operator,

$$\Delta = \delta \circ d + d \circ \delta : \Omega_M^k \rightarrow \Omega_M^k$$

We say a  $k$ -form  $\omega$  is *harmonic* if  $\Delta\omega = 0$  and we denote the space of harmonic  $k$ -forms as  $\mathcal{H}^k(M)$ .

**Remark 3.** To motivate this definition, choose local coordinates such that,

$$\omega = g dx_1 \wedge \cdots \wedge dx_n$$

and consider a  $k$ -form in local coordinates,

$$\eta = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} dx_1 \wedge \cdots \wedge dx_{i_k}$$

Then,

$$d\eta = \sum_j \sum_{i_1 < \cdots < i_k} \frac{\partial f_{i_1, \dots, i_k}}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_{i_k}$$

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**Lemma 2.2.**  $\star\Delta = \Delta\star$

*Proof.* It is clear that  $\star\delta = (-1)^k d\star$  and  $\star d = (-1)^k \delta\star$ . Therefore,

$$\begin{aligned} \star\Delta &= \star(\delta d + d\delta) = (-1)^k d\star + (-1)^k \delta\star \\ &= d\delta\star + \delta d\star = \Delta\star \end{aligned}$$

□

**Lemma 2.3.** We have  $\Delta\omega = 0$  iff  $d\omega = \delta\omega = 0$ .

*Proof.* Clearly if  $d\omega = \delta\omega = 0$  then  $\Delta\omega = 0$ . Conversely, suppose that,

$$\Delta\omega = [\delta d + d\delta]\omega = 0$$

Consider,

$$\langle\langle\Delta\omega, \omega\rangle\rangle = \langle\langle\delta d\omega, \omega\rangle\rangle + \langle\langle d\delta\omega, \omega\rangle\rangle = \langle\langle d\omega, d\omega\rangle\rangle + \langle\langle \delta\omega, \delta\omega\rangle\rangle = \|d\omega\|^2 + \|\delta\omega\|^2$$

Since  $\|\alpha\|^2 \geq 0$  we see that if  $\langle\langle\Delta\omega, \omega\rangle\rangle = 0$  then  $\|\delta\omega\|^2 = 0$  and  $\|d\omega\|^2 = 0$  and thus  $\delta\omega = 0$  and  $d\omega = 0$ . □

**Theorem 2.4** (Hodge). The space  $\mathcal{H}^k(M)$  is finite dimensional and there is a canonical decomposition,

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k(M) = d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k(M)$$

*Proof.* The decomposition,

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k(M) = \text{Im } \Delta \oplus \ker \Delta$$

follows immediately from splitting the sequence,

$$0 \longrightarrow \ker \Delta \longrightarrow \Omega^k(M) \longrightarrow \text{Im } \Delta \longrightarrow 0$$

First, suppose that  $\eta = d\alpha = \delta\beta$  then,

$$\|\eta\| = \langle \langle \eta, \eta \rangle \rangle = \langle \langle d\alpha, \delta\beta \rangle \rangle = \langle \langle d^2\alpha, \beta \rangle \rangle = 0$$

and thus  $\eta = 0$ . Thus,  $d(\Omega^{k-1}(M)) \cap \delta(\Omega^{k+1}(M)) = (0)$  so,

$$d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \subset \Omega^k(M)$$

Clearly,

$$\Delta(\Omega^k(M)) \subset d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))$$

because  $\Delta\alpha = d(\delta\alpha) + \delta(d\alpha)$ . Furthermore, if  $d\alpha \in \mathcal{H}^k(M)$  then  $\delta d\alpha = 0$  but

$$\|d\alpha\| = \langle \langle d\alpha, d\alpha \rangle \rangle = \langle \langle \alpha, \delta d\alpha \rangle \rangle = 0$$

so  $d\alpha = 0$  and similarly if  $\delta\beta \in \mathcal{H}^k(M)$  then  $d\delta\beta = 0$  but,

$$\|\delta\beta\| = \langle \langle \delta\beta, \delta\beta \rangle \rangle = \langle \langle d\delta\beta, \beta \rangle \rangle = 0$$

so  $\delta\beta = 0$ . Therefore,

$$[d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))] \cap \mathcal{H}^k(M) = (0)$$

showing that,

$$\Delta(\Omega^k(M)) = d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))$$

The finite dimensionality of  $\mathcal{H}^k(M)$  follows from the theory of elliptic operators on compact manifolds. However, we will prove it using the following result plus the following results: de Rham's theorem  $H_{\text{dR}}^k(M) \cong H_{\text{sing}}^k(M)$ , the fact that singular cohomology is finitely generated for a finite CW complex, and that any compact manifold has the homotopy type of a finite CW complex.  $\square$

**Theorem 2.5** (Hodge). Let  $M$  be compact oriented Riemann manifold. Then every deRham cohomology class on  $M$  has a unique harmonic representative and thus the canonical map,

$$\mathcal{H}^k(M) \xrightarrow{\sim} H_{\text{dR}}^k(M)$$

is an isomorphism.

*Proof.* I claim that,

$$\ker(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)) = d(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$$

We can write  $\eta = d\alpha + \delta\beta + \varphi$  where  $\varphi$  is harmonic. Suppose that  $d\eta = 0$  then  $d\delta\beta = 0$  which we have shown implies that  $\delta\beta = 0$  so  $\eta = d\alpha + \varphi$  and thus,

$$\ker d \subset d(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$$

but it is clear that  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  vanishes on  $d(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$  so,

$$\ker d = d(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$$

Using this we immediately see that the map,

$$\mathcal{H}^k(M) \rightarrow H_{\text{dR}}^k(M) \quad \varphi \mapsto [\varphi]$$

is an isomorphism because,

$$\ker d / \text{Im } d = [d(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)] / d(\Omega^{k-1}(M)) = \mathcal{H}^k(M)$$

Explicitly, if  $[\varphi] = 0$  then  $\varphi = d\alpha$  but then  $\Delta d\alpha = \delta d\alpha = 0$  which implies that  $\varphi = d\alpha = 0$  so  $\mathcal{H}^k(M) \rightarrow H_{\text{dR}}^k(M)$  is injective. Furthermore, consider a class  $[\alpha] \in H_{\text{dR}}^k(N)$  with  $d\alpha = 0$  then, by above,  $\alpha \in d(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$  so  $\alpha = \varphi + d\beta$  for some harmonic form  $\varphi \in \mathcal{H}^k(M)$  and thus,

$$[\alpha] = [\varphi]$$

so the map  $\mathcal{H}^k(M) \rightarrow H_{\text{dR}}^k(M)$  is surjective.  $\square$

**Theorem 2.6** (Poincare). Let  $M$  be compact oriented Riemann manifold. There is a canonical isomorphism  $H_{\text{dR}}^k(M) \xrightarrow{\sim} H_{\text{dR}}^{n-k}(M)^\vee$ .

*Proof.* Consider the bilinear pairing  $H_{\text{dR}}^k(M) \times H_{\text{dR}}^{n-k}(M) \rightarrow \mathbb{R}$  via,

$$B([\omega], [\eta]) = \int_M \omega \wedge \eta$$

This is well-defined since if  $\tilde{\omega} = \omega + d\alpha$  and  $\tilde{\eta} = \eta + d\beta$  then,

$$\begin{aligned} \int_M \tilde{\omega} \wedge \tilde{\eta} &= \int_M (\omega + d\alpha) \wedge (\eta + d\beta) \\ &= \int_M \omega \wedge \eta + \int_M \omega \wedge d\beta + \int_M d\alpha \wedge (\eta + d\beta) \end{aligned}$$

However,

$$d(\omega \wedge \beta) = d\omega \wedge \beta + (-1)^k \omega \wedge d\beta$$

But  $\omega$  is closed so we have,

$$\int_M \omega \wedge d\beta = (-1)^k \int_M d(\omega \wedge \beta) = (-1)^k \int_{\partial M} \omega \wedge \beta = 0$$

since  $M$  has no boundary. Likewise, since  $\eta + d\beta$  is closed we have,

$$\int_M d\alpha \wedge (\eta + d\beta) = \int_{\partial M} \alpha \wedge (\eta + d\beta) = 0$$

Thus,

$$\int_M \tilde{\omega} \wedge \tilde{\eta} = \int_M \omega \wedge \eta$$

so this bilinear pairing is well-defined.

Now, it suffices to prove that the pairing is non-degenerate. For any class  $[\omega]$  we can choose a harmonic representative  $\varphi$ . Furthermore  $\star\varphi$  is harmonic since,

$$\Delta \star \varphi = \star \Delta \varphi = 0$$

so it represents a class  $[\star\varphi] \in H_{\text{dR}}^{n-k}$ . Then,

$$B([\omega], [\star\varphi]) = B([\varphi], [\star\varphi]) = \int_M \varphi \wedge (\star\varphi) = \int_M \langle \varphi, \varphi \rangle \omega = \|\varphi\|^2 = 0 \iff \varphi = 0$$

which shows that  $B$  is nondegenerate.  $\square$

### 3 Local Systems

**Definition:** A  $\mathcal{A}$ -local system is a locally constant sheaf in the category  $\mathcal{A}$  i.e. a sheaf  $\mathcal{L}$  on  $X$  such that for each  $x \in X$  there exists some open neighborhood  $U$  and an object  $A$  such that  $\mathcal{L}|_U \cong \underline{A}$ .

**Lemma 3.1.** If  $X$  is connected then any local system has constant fibers and thus we may take its constant objects on the trivializing neighborhoods to be equal.

*Proof.* For some fixed  $p \in X$  let  $D_p = \{x \in X \mid \mathcal{L}_x \cong \mathcal{L}_p\}$ . Since  $\mathcal{L}$  is a local system, for any  $x \in X$  we have an open  $U$  s.t.  $\mathcal{L}|_U = \underline{A_x}$ . If  $x \in D_p$  then  $\mathcal{L}_x \cong A_x \cong \mathcal{L}_p$ . But then for any  $y \in U$  we have,

$$\mathcal{L}_y \cong A_x \cong \mathcal{L}_x \cong \mathcal{L}_p$$

so  $x \in U \subset D_p$  and thus  $D_p$  is open. Therefore,

$$X = \bigcup_{p \in X} D_p$$

is an open partition which implies that,

$$D_p^C = \bigcup_{x \neq p} D_p$$

is open so  $D_p$  is clopen. Since  $X$  is connected and  $p \in D_p$  we have  $D_p = X$ .  $\square$

**Proposition 3.2.** Let  $X$  be locally connected and  $\mathcal{L}$  be a  $\mathcal{A}$ -local system. Then there is a canonical functor  $A : \Pi_1(X) \rightarrow \mathcal{A}$ .

*Proof.* Consider a path  $\gamma : I \rightarrow X$  from  $x$  to  $y$ . Then, since  $\text{Im } \gamma$  is compact, we can choose a finite cover of  $\text{Im } \gamma$  by connected trivializing neighborhoods  $U_i$  s.t.  $U_i \cap U_{i+1} \neq \emptyset$  and  $x \in U_0$  and  $y \in U_n$ . Then on each we have  $\mathcal{L}|_{U_i} \cong F$ . Now we construct a map  $[\gamma] : \mathcal{F}_x \rightarrow \mathcal{F}_y$  as follows. For a germ  $f \in \mathcal{L}_x$  we lift to a section  $f \in \mathcal{L}(U_0)$  since  $f$  is constant on  $U_0$ . Now, suppose we have a section  $f_i \in \mathcal{L}(U_i)$ , choose a connected open  $V \subset U_i \cap U_{i+1}$  then  $f_i|_V \in \mathcal{L}(V)$ . Since  $\mathcal{L}|_{U_{i+1}}$  is constant then the restriction map,

$$\text{res}_{V, U_{i+1}} : \mathcal{L}(U_{i+1}) \rightarrow \mathcal{L}(V)$$

is an isomorphism and thus we get a section  $f_{i+1} = \text{res}_{V, U_{i+1}}^{-1}(f_i|_V)$ . Then we choose  $\alpha_\gamma f = f_n$  which is the germ of  $f_n \in \mathcal{L}(U_n)$ . It is clear that this is a morphism and invariant under homotopy giving a well-defined map  $\Pi_1(X, x, y) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{L}_x, \mathcal{L}_y)$ .  $\square$

**Proposition 3.3.** Let  $X$  be path-connected and locally connected and  $\mathcal{L}$  be a local system with fiber  $\mathcal{L}_p \cong F$ . Then there is a canonical action  $\pi_1(X, x_0) \rightarrow \text{Aut}(\mathcal{L}_{x_0})$  and  $\Gamma(X, \mathcal{F}) = \mathcal{L}_{x_0}^{\pi_1(X, x_0)}$ .

*Proof.* Consider the case  $x_0 = x = y$  then we have a map  $\pi_1(X, x_0) \rightarrow \text{Aut}(\mathcal{L}_{x_0})$ . Now, consider the restriction map  $\Gamma(X, \mathcal{L}) \rightarrow \mathcal{L}_{x_0}$ . Since restrictions compose we have  $\alpha_\gamma f|_{x_0} = f|_{x_0}$  since  $f_i = f|_{U_i}$  and  $(f|_{U_n})_{x_0} = f_{x_0}$  so the image lies in  $\mathcal{L}_{x_0}^{\pi_1(X, x_0)}$ . Conversely, consider  $f \in \mathcal{L}_{x_0}^{\pi_1(X, x_0)}$  such that  $[\gamma] \cdot f = f$  for any loop  $\gamma : I \rightarrow X$ . Now, taking  $x \in X$  we can define  $f_x = [\gamma] \cdot f$  where  $\gamma$  is a path from  $x_0$  to  $x$ . This is well-defined because if  $\gamma, \delta : I \rightarrow X$  are two paths from  $x_0$  to  $x$  then  $\delta^{-1} * \gamma$  is a loop at  $x_0$  and  $\alpha_{\delta^{-1} * \gamma} = \alpha_\delta^{-1} \circ \alpha_\gamma$  but by assumption  $\alpha_{\delta^{-1} * \gamma} = \text{id}$  so  $\alpha_\gamma = \alpha_\delta$ . Furthermore, each  $f_x$  lifts to  $f_x \in \mathcal{L}(U_x)$  for some trivializing neighborhood and these sections glue to a global section by the construction of the morphisms. This construction gives an inverse map  $\mathcal{L}_{x_0}^{\pi_1(X, x_0)} \rightarrow \Gamma(X, \mathcal{L})$  showing the given isomorphism.  $\square$

### 3.1 Connections

**Definition:** Let  $\mathcal{E}$  be a coherent sheaf on  $X$ . Then a *connection* on  $\mathcal{E}$  is a morphism  $\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$  of *abelian* sheaves (not  $\mathcal{O}_X$ -modules) which satisfies the Leibniz rule,

$$\nabla(fs) = df \otimes s + f \nabla s$$

**Proposition 3.4.** Given a connection  $\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$  it naturally extends to a connection  $\nabla_k : \Omega_X^k \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \Omega_X^{k+1} \otimes_{\mathcal{O}_X} \mathcal{E}$  via,

$$\nabla_k(\omega \otimes e) = d\omega \otimes e + (-1)^k \omega \wedge \nabla e$$

**Definition:** The connection  $\nabla$  defines a corresponding curvature form,

$$\omega_\nabla = \nabla_1 \circ \nabla : \mathcal{E} \rightarrow \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$$

We say that  $\nabla$  is flat or integrable if the curvature vanishes  $\omega_\nabla = \nabla_1 \circ \nabla = 0$ .

**Proposition 3.5.** When  $\nabla$  is flat we have  $\nabla_{k+1} \circ \nabla_k = 0$  for all  $k$ . In this case we have the  $\mathcal{E}$ -valued deRham complex,

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\nabla_1} \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \dots$$

whose hypercohomology gives the deRham cohomology with coefficients in  $\mathcal{E}$ ,

$$H_{\text{dR}}^k(X, \mathcal{E}) = \mathbb{H}^k(X, \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$$

**Definition:** A connection  $\nabla$  on  $\mathcal{E}$  defines a subsheaf  $\mathcal{E}^\nabla = \ker \nabla \subset \mathcal{E}$  of *horizontal* or *flat* sections.

**Lemma 3.6.** The curvature  $\omega_\nabla : \mathcal{E} \rightarrow \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$  is a  $\mathcal{O}_X$ -module map.

*Proof.* Consider,

$$\begin{aligned} \omega_\nabla(fs) &= \nabla_1(df \otimes s + f\nabla s) = ddf \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla_1 \circ \nabla \\ &= f\nabla_1 \circ \nabla s = f \omega_\nabla(s) \end{aligned}$$

□

**Remark 4.** If we write locally,

$$\nabla e = \sum_i f_i dg_i \otimes s_i$$

then the curvature takes the form,

$$\omega_\nabla(e) = \sum_i (df_i \wedge dg_i \otimes e - f_i dg_i \otimes \nabla s_i)$$

**Proposition 3.7.**  $\nabla$  is flat iff the  $\mathcal{O}_X$ -map  $Q : \mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X) \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$  given by sending  $D$  to,

$$\mathcal{E} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{D \otimes \text{id}} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{a \otimes b \mapsto ab} \mathcal{E}$$

is a morphism of sheaves of Lie algebras.

**Remark 5.** In the definition of  $Q(D)$  we have used  $D$  as an  $\mathcal{O}_X$ -module morphism  $\Omega_X^1 \rightarrow \mathcal{O}_X$  via the universal property of  $\Omega_X^1$ ,

$$\mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) = \mathcal{T}_X$$

which identifies  $\mathcal{D}er(\mathcal{O}_X, \mathcal{O}_X)$  with the tangent sheaf  $\mathcal{T}_X$ .

*Proof.* We need to check that  $Q[D_1, D_2] = [Q(D_1), Q(D_2)]$  is equivalent to  $\nabla_1 \circ \nabla = 0$ . Now,

$$[D_1, D_2] \in \text{Hom}_{\mathcal{O}_U}(\Omega_U^1, \mathcal{O}_U)$$

is the unique  $\mathcal{O}_X$ -map such that,

$$[D_1, D_2] \circ d = D_1 \circ d \circ D_2 \circ d - D_2 \circ d \circ D_1 \circ d$$



Now consider this action locally,

$$[D_1, D_2] \otimes \text{id} \circ \nabla = \sum_i f_i (D_1 \circ d \circ D_2 \circ d - D_2 \circ d \circ D_1 \circ d)(g_i) \cdot s_i$$

Furthermore,

$$[Q(D_1), Q(D_2)] = (D_1 \otimes \text{id}) \circ \nabla \circ (D_2 \otimes \text{id}) \circ \nabla - (D_2 \otimes \text{id}) \circ \nabla \circ (D_1 \otimes \text{id}) \circ \nabla$$

Again consider its local action,

$$\begin{aligned} Q(D_1) \circ Q(D_2)(e) &= (D_1 \otimes \text{id}) \circ \nabla \left( \sum_i f_i D_2(\text{d}g_i) \cdot s_i \right) \\ &= \sum_i \left( [D_2(\text{d}g_i) D_1(\text{d}f_i) + f_i D_1(\text{d}(D_2(\text{d}g_i)))] \cdot s_i + f_i D_2(\text{d}g_i) D_1(\nabla s_i) \right) \end{aligned}$$

Now consider,

$$\begin{aligned} &\left[ Q(D_1) \circ Q(D_2) - Q(D_2) \circ Q(D_1) \right] - Q([D_1, D_2])(e) \\ &= \sum_i \left( D_1(\text{d}f_i) D_2(\text{d}g_i) - D_2(\text{d}f_i) D_1(\text{d}g_i) \right) \cdot s_i \\ &\quad + \sum_i f_i \left( D_1(\text{d}(D_2(\text{d}g_i))) - D_2(\text{d}(D_1(\text{d}g_i))) \right) \cdot s_i \\ &\quad + \sum_i \left( f_i D_2(\text{d}g_i) D_1(\nabla s_i) - g_i D_1(\text{d}g_i) D_2(\nabla s_i) \right) \\ &\quad - \sum_i f_i (D_1 \circ d \circ D_2 \circ d - D_2 \circ d \circ D_1 \circ d)(g_i) \cdot s_i \\ &= \sum_i \left( D_1(\text{d}f_i) D_2(\text{d}g_i) - D_2(\text{d}f_i) D_1(\text{d}g_i) \right) \cdot s_i \\ &\quad + \sum_i \left( f_i D_2(\text{d}g_i) D_1(\nabla s_i) - g_i D_1(\text{d}g_i) D_2(\nabla s_i) \right) \\ &= (D_1 \otimes D_2 - D_2 \otimes D_1) \otimes \text{id}_{\mathcal{E}} \circ \omega_{\nabla} \end{aligned}$$

which is defined on  $(\Omega_X^1)^{\otimes 2} \otimes_{\mathcal{O}_X} \mathcal{E}$  but descends to  $\Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}$  since it sends the ideal  $\omega \otimes \omega \mapsto 0$ . Therefore, we see that  $Q$  is a Lie algebra map iff

$$\forall D_1, D_2 \in \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) : (D_1 \otimes D_2 - D_2 \otimes D_1) \otimes \text{id}_{\mathcal{E}} \circ \omega_{\nabla} = 0$$

in particular when  $\omega_{\nabla} = 0$ . Furthermore when  $Q$  is a Lie algebra map then we must have  $\omega_{\nabla} = 0$  since, for any fixed form, there exists sections of  $\Omega_X^1$  which do not kill it.  $\square$

**Example 3.8.** For  $\mathcal{E} = \mathcal{O}_X$  we have the universal connection  $d : \mathcal{O}_X \rightarrow \Omega_X^1$ . Then the statment that  $d$  is flat is equivalent to  $d^2 = 0$  leading to the deRham complex. Furthermore this means that  $d$  induces a Lie algebra map,

$$\mathcal{T}_X \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X$$

sending a vector field  $v$  to the map  $f \mapsto \langle v, df \rangle$  proving the identity,  $\langle [v, u], df \rangle = 0$  since  $\mathcal{O}_X$  has trivial Lie algebra structure.

**Example 3.9.** A connection on a scheme or manifold  $X$  is a connection on the cotangent (or equivalently tangent) bundle  $\nabla : \Omega_X^1 \rightarrow (\Omega_X^1)^{\otimes 2}$ . Such a connection is equivalent to a choice of global section  $g \in \Gamma(X, \text{Sym}^2(\Omega_X^1))$  i.e. a metric. We say that  $(X, g)$  is flat if this connection  $\nabla$  is flat. In this case we have an augmented deRham complex  $(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \Omega_X^1, \nabla)$ .

**Remark 6.** Note that a connection  $\nabla : \mathcal{O}_X \rightarrow \Omega_X^1$  does NOT induce a connection on  $\Omega_X^1$ . Such a connection induces a connection,

$$\nabla_1 : \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X = \Omega_X^1 \rightarrow \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{O}_X = \Omega_X^2 = \bigwedge^2 \Omega_X^1$$

but it is only well-defined in the exterior algebra not on the tensor algebra  $\Omega_X^1 \otimes_{\mathcal{O}_X} \Omega_X^1$ . There is always a canonical derivation i.e. connection  $d : \Omega_X \rightarrow \Omega_X^1$  but there is not generically a map  $\Omega_X^1 \rightarrow (\Omega_X^1)^{\otimes 2}$ .

## 3.2 Vector Bundles

**Proposition 3.10.** Let  $\mathcal{E}$  be a vector bundle on  $X$  with a flat connection

$$\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$$

Then  $\mathcal{E}^\nabla = \ker \nabla$  is a local system.

*Proof.* Since  $\mathcal{E}$  is locally free, we can find a cover of trivializing neighborhoods  $U$  such that  $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$ . Then  $\nabla : \mathcal{O}_U^{\oplus n} \rightarrow (\Omega_U^1)^{\oplus n}$  is a connection. Define,

$$\nabla e_j = \sum_{i=1}^n \omega_{ij} \otimes e_i$$

where  $\omega_{ij} \in \Omega_X^1(U)$  is a form. This uniquely defines the connection since,

$$\begin{aligned} \nabla(f_1, \dots, f_n) &= \nabla \left( \sum_{i=1}^n f_i e_i \right) = \sum_{i=1}^n (f_i \nabla e_i + df_i \otimes e_i) \\ &= \sum_{i,j=1}^n \omega_{ij} \otimes f_j e_i + (df_1, \dots, df_n) \end{aligned}$$

Therefore,  $\mathcal{E}^\nabla$  is given locally by  $(f_1, \dots, f_n)$  solving the linear system of differential equations,

$$df_i + \sum_{j=1}^n \omega_{ij} f_j = 0$$

The condition of flatness is that,

$$\nabla_1 \circ \nabla = 0$$

which locally is,

$$\begin{aligned}
\nabla_1 \circ \nabla(f_1, \dots, f_n) &= \nabla_1 \left( \sum_{i,j=1}^n \omega_{ij} \otimes f_j e_i + \sum_{j=1}^n df_j \otimes e_j \right) \\
&= \sum_{i,j=1}^n [d\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \nabla(f_j e_i)] + \sum_{i=1}^n [ddf_i \otimes e_i - df_i \wedge \nabla e_i] \\
&= \sum_{i,j=1}^n \left[ d\omega_{ij} \otimes f_j e_i - \omega_{ij} \wedge \left( df_j \otimes e_i + f_j \sum_{k=1}^n \omega_{ki} \otimes e_k \right) \right] - \sum_{i,j=1}^n [df_j \wedge \omega_{ij} \otimes e_i] \\
&= \sum_{i,j=1}^n \left[ d\omega_{ij} \otimes e_i - \sum_{k=1}^n \omega_{ij} \wedge \omega_{ki} \otimes e_k \right] f_j \\
&= \sum_{i,j=1}^n \left[ d\omega_{ij} + \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} \right] \otimes f_j e_i
\end{aligned}$$

So the curvature  $\omega_\nabla$  is given by coefficients,

$$\Theta_{ij} = d\omega_{ij} + \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj}$$

Now I claim that if  $\varepsilon^\nabla$  as a full set of solutions then  $\omega_\Delta = 0$ . To show this, consider,

$$d \left( df_i + \sum_{j=1}^n \omega_{ij} f_j \right) = 0$$

This implies,

$$\sum_{j=1}^n (d\omega_{ij} f_j - \omega_{ij} \wedge df_j) = 0$$

However, using the relation,

$$\sum_{j=1}^n (d\omega_{ik} + \omega_{ij} \wedge \omega_{jk}) f_k = 0$$

and thus,

$$\sum_{j=1}^n \Theta_{ij} f_j = 0$$

If we assume that  $f_i$  can be chosen to span then we must have  $\Theta_{ij} = 0$  which implies  $\omega_\nabla = 0$ . This is also sufficient for integrability.  $\square$