Mathematics GU4042 Modern Algebra II
Assignment # 1

Benjamin Church

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2 Let A be an abelian group and $\text{End}(A) = \{f : A \to A \mid f \text{ is a homomorphism}\}$.

Now take $f, g \in \text{End}(A)$ then $(f+g)(x+y) = f(x+y) + g(x+y) = f(x) + f(y) + g(x) + g(y) = (f(x)+g(x)) + (f(y)+g(y))$ since A is abelian and f, g are homomorphisms.

Let $0_{End} \in \text{End}(A)$ given by $0_{End}(x) = 0_A$ is a homomorphism because $0_{End}(x+y) = 0_A = 0_A + 0_A = 0_{End}(x) + 0_{End}(y)$ and $(0_{End} + f)(x) = 0_{End}(x) + f(x) = 0_A + f(x) = f(x)$ and $(f + 0_{End})(x) = f(x) + 0_{End}(x) = f(x) + 0_A = f(x)$ (Identity)

$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$ (Commutativity)

Now define $-f$ by $(-f)(x) = -f(x)$, $(-f)(x+y) = -f(x+y) = -(f(x) + f(y)) = -f(x) + (-f(y))$ so $-f \in \text{End}(A)$ also $(-f+f)(x) = -f(x) + f(x) = 0_A$ and $(f + (-f))(x) = f(x) + (-f(x)) = 0_A$. (Inverses)

Let $f, g, h \in \text{End}(A)$ then $((f+g)+h)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g+h)(x) = (f + (g+h))(x)$

$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) = f((g \circ h)(x)) = (f \circ (g \circ h))(x)$ (Associativity)

Let $1_{End} \in \text{End}(A)$ given by $1_{End}(x) = x$ is a homomorphism because $1_{End}(x+y) = x + y = 1_{End}(x) + 1_{End}(y)$ (Identity) also $(1_{End} \circ f)(x) = 1_{End}(f(x)) = f(x)$ and $(f \circ 1_{End})(x) = f(x)$ thus $1_{End} \circ f = f \circ 1_{End} = f$ (Multiplicative Identity)

Let $f, g, h \in \text{End}(A)$ then $(f \circ (g+h))(x) = f((g+h)(x)) = f(g(x) + h(x)) = f(g(x)) + f(h(x)) = (f \circ g)(x) + (f \circ h)(x)$. Also $((f+g) \circ h)(x) = (f+g)(h(x)) = f(h(x)) + g(h(x)) = (f \circ h)(x) + (g \circ h)(x)$. (Distributive Law) Thus $(\text{End}(A), +, \circ)$ is a ring.

3 Let R be a ring. Then $U(R) = \{u \in R \mid \exists v \in R : uv = vu = 1_R\}$.
Now $1_R \cdot 1_R = 1_R$ so $1_R \in U(R)$ (Identity)

If $u, v \in U(R)$ then $\exists u', v' \in R : uu' = u'u = 1_R = vv' = v'v$ Thus $uv \cdot (v'u') = u(vv')u' = uu' = 1_R$ and $(v'u') \cdot uv = v'(u'u)v = 1_R$ so $uv \in R$. (Closure)

If $u \in U(R)$ then $\exists v \in R : uv = vu = 1_R$ so $v \in U(R)$ and $vu = uv = 1_R$ (Inverses)

Furthermore, $U(R)$ is a subset of R and therefore inherents associativity.

4 Let $u \in R$ be a unit then $\exists v \in R : uv = vu = 1_R$ so take $x = y = v$.

Let $\exists x, y \in R : xu = uy = 1_R$ then $x(uy) = x$ but $x(uy) = (xu)y = y$ so $x = y$ thus $ux = xu = 1_R$ so $u \in U(R)$.

7 $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$. Then for $z_i, z_2 \in \mathbb{Z}[i]$ we must check that $z_1 + z_2, z_1 \cdot z_2 \in \mathbb{Z}[i]$ and $-z_1, 1_{\mathbb{Z}[i]}, 0_{\mathbb{Z}[i]} \in \mathbb{Z}[i]$. Associativity (of both addition and multiplication), Distributivity, and Commutativity of addition are inherited from $\mathbb{C}$.

If $z_1, z_2 \in \mathbb{Z}[i]$ then $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ with $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Then $z_1 + z_2 = (a_1 + a_1) + i(b_1 + b_2) \in \mathbb{Z}[i]$ because $a_1 + a_2, b_1 + b_2 \in \mathbb{Z}[i]$ Also, $z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \in \mathbb{Z}[i]$ because $a_1a_2 - b_1b_2 \in \mathbb{Z}$ and $a_1b_2 + a_2b_1 \in \mathbb{Z}$. Therefore, $1 = 1+i0 \in \mathbb{Z}[i]$ takes $1 \cdot z_1 = z_1 \cdot 1 = z_1$ and $0 = 0+i0 \in \mathbb{Z}[i]$ takes $0+z_1 = z_1+0 = z_1$. Also $-z_1 = -a_1 - ib_1 \in \mathbb{Z}[i]$ then $z_1 + (-z_1) = a_1 - a_1 + i(b_1 - b_1) = 0$. By Commutativity, we don't neet to check the other direction.

If $z \in U(\mathbb{Z}[i])$ then $zz' = 1$ so $|z|^2|z'|^2 = 1$ i.e. $(a^2 + b^2)(a'^2 + b'^2) = 1$ so $a^2 + b^2 \mid 1$ and thus $a^2 + b^2 = 1$ since both are in $\mathbb{N}$. If $|a| > 1$ then $b^2 < 0$ so $a = 0, \pm 1$ and $b = \pm 1, 0$ so the units are $1, -1, i, -i$.

p.112 10 Let A, B be ideals in R. Then $AB = (\{ab \mid a \in A \text{ and } b \in B\}) =$

$$\left\{ \sum_{i=1}^{n} x_i(a_ib_i)y_i \;\middle|\; a_i \in A \text{ and } b_i \in B \text{ and } x_i, y_i \in R \right\}$$

If $r \in AB$ then $r = \sum_{i=1}^{n} x_i(a_ib_i)y_i$ but $x_i(a_ib_i)y_i = (x_ia_i)(b_iy_i)$ and since A and B are ideals then $(x_ia_i) = a_i' \in A$ and $(b_iy_i) = b_i' \in B$. Thus, $r = \sum_{i=1}^{n} a_i'b_i'$.

Also for $r = \sum_{i=1}^{n} a_ib_i$ take $x_i = y_i = 1_R$ then $r = \sum_{i=1}^{n} x_i(a_ib_i)y_i$ so

$$AB = \left\{ \sum_{i=1}^{n} a_ib_i \;\middle|\; a_i \in A \text{ and } b_i \in B \right\}$$

11 Let $x \in (AB)C$ then $\sum_{i=1}^{n} g_ic_i$ s.t. $g_i \in AB$ and $c_i \in C$ with $g_i = \sum_{j=1}^{k_i} a_{ij}b_{ij}$ so $x = \sum_{i=1}^{n}\sum_{j=1}^{k_i} a_{ij}b_{ij}c_i$ distributing and reparametrizing, $x = \sum_{k=1}^{r} a_kb_kc_k$ so $x \in \{\sum_{i=1}^{r} a_ib_ic_i \mid a_i \in A, b_i \in B, c_i \in C\} = S$. Also if $x \in S$ then $x = \sum_{i=1}^{r}(a_ib_i)c_i$ but $a_ib_i \in AB$ so $x \in (AB)C$ thus $(AB)C = S$.

Let $x \in A(BC)$ then $\sum_{i=1}^{n} a_ig_i$ s.t. $g_i \in BC$ and $a_i \in A$ with $g_i = \sum_{j=1}^{k_i} b_{ij}c_{ij}$ so $x = \sum_{i=1}^{n}\sum_{j=1}^{k_i} a_ib_{ij}c_{ij}$ distributing and reparametrizing, $x = \sum_{k=1}^{r} a_kb_kc_k$ so $x \in S$. Also if $x \in S$ then $x = \sum_{i=1}^{r} a_i(b_ic_i)$ but $b_ic_i \in BC$ so $x \in A(BC)$ thus $A(BC) = S = (AB)C$.

12 Let A, B, and C be ideals in R and $x \in A(B + C)$ then $x = \sum_{i=1}^{n} a_i(b_i + c_i) = \sum_{i=1}^{n}(a_ib_i + a_ic_i) = \sum_{i=1}^{n} a_ib_i + \sum_{i=1}^{n} a_ic_i$ therefore, $x \in AB + AC$.

Now let $x \in AB + AC$ then $x = \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n'} a'_i c'_i$

Define:

$$\tilde{a}_i = \begin{cases} a_i & 1 \leq i \leq n \\ a'_{i-n} & n < i \leq n' \end{cases} \quad \tilde{b}_i = \begin{cases} b_i & 1 \leq i \leq n \\ 0_R & n < i \leq n' \end{cases} \quad \tilde{c}_i = \begin{cases} 0_R & 1 \leq i \leq n \\ c'_{i-n} & n < i \leq n' \end{cases}$$

then $\sum_{i_1}^{n+n'} \tilde{a}_i(\tilde{b}_i + \tilde{c}_i) = \sum_{i=1}^{n} a_i(b_i + 0_R) + \sum_{i=n+1}^{n+n'} a'_{i-n}(0 + c'_{i-n}) =$
$\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n'} a'_i c'_i = x$ thus $x \in A(B + C)$

Therefore $A(B + C) = AB + AC$

Take $x \in (A+B)C$ then $x = \sum_{i=1}^{n}(a_i + b_i)c_i = \sum_{i=1}^{n}(a_i c_i + b_i c_i) = \sum_{i=1}^{n} a_i c_i + \sum_{i=1}^{n} b_i c_i$ therefore, $x \in AC + BC$.

Now let $x \in AC + BC$ then $x = \sum_{i=1}^{n} a_i c_i + \sum_{i=1}^{n'} b'_i c'_i$.

Define:

$$\tilde{c}_i = \begin{cases} c_i & 1 \leq i \leq n \\ c'_{i-n} & n < i \leq n' \end{cases} \quad \tilde{a}_i = \begin{cases} a_i & 1 \leq i \leq n \\ 0_R & n < i \leq n' \end{cases} \quad \tilde{b}_i = \begin{cases} 0_R & 1 \leq i \leq n \\ b'_{i-n} & n < i \leq n' \end{cases}$$

then $\sum_{i_1}^{n+n'}(\tilde{a}_i + \tilde{b}_i)\tilde{c}_i = \sum_{i=1}^{n}(a_i + 0_R)c_i + \sum_{i=n+1}^{n+n'}(0 + b'_{i-n})c'_{i-n} =$
$\sum_{i=1}^{n} a_i c_i + \sum_{i=1}^{n'} b'_i c'_i = x$ thus $x \in (A + B)C$

Therefore $(A + B)C = AC + BC$