1 The Formal Immersion Step (the new hotness on tiktak)

Theorem 1.0.1. Let N be a prime, either 11 or ≥ 17 (ensuring that $X_0(N)$ has genus > 0) then there are no elliptic curves over $\mathbb{Q}$ with a torsion point of order N.

Kep points,

(a) if E has good reduction at 3 then $E[N](\mathbb{Q}) \hookrightarrow \overline{E}(\mathbb{F}_3)$ which has order at most 9 by Hasse so $N < 9$.

(b) if E has multiplicative reduction we can get crazy polygons so no control on N

(c) if E has additive reduction: what can the special fiber of the minimal regular proper model be? From Kodaia classification, there are a bounded number of components and hence a bound on $\#\overline{E}(\mathbb{F}_3) \leq 12$.

Assume from now on that $N = 11$ or $N > 17$.

Proposition 1.0.2. If (E, C) is a pair of an elliptic curve over $\mathbb{Q}$ and a cyclic subgroup scheme $C \subset E$ of order N. Then E has potentially good reduction away from $2N$.

Remark. This implies you can't have multiplicative reduction because potentially good reduction means the semistable reduction is good but multiplicative reduction is also semistable.

Remark. Recall that,

$$\text{good reduction} \iff T_\ell E \text{ is unramified}$$
$$\text{mult. reduction} \iff I \to \mathrm{GL}(V_\ell E) \text{ is (nontrivial) unipotent}$$

Proposition 1.0.3. Let $\mathcal{A}$ be the Neron model over $\mathbb{Z}[1/2N]$ of the Eisenstein quotient A of $J = \mathrm{Jac}\,(X_0(N))$. Define,

$$X_0(N)_{\mathbb{Q}} \longrightarrow J \longrightarrow A$$

$f : X_0(N) \to \mathcal{A}$ over $\mathbb{Z}[1/2N]$ sends $\infty \mapsto 0$. Then if $p \not| \, 2N$ then $\infty \in X_0(N)(\mathbb{Z}_{(p)})$ is the only $\mathbb{Z}_{(p)}$-point of $X_0(N)$ mapping to $0 \in \mathcal{A}(\mathbb{Z}_{(p)})$ which reduces to $\infty \in X_0(N)(\mathbb{F}_p)$.

Definition 1.0.4. Let $f : Y \to Z$ is lft and Y, Z are locally noetherian. If $y\ inU$ say f is a *formal immersion at* y if $\mathcal{O}^\wedge_{Z,f(y)} \twoheadrightarrow \mathcal{O}^\wedge_{Y,y}$ is surjective.

Definition 1.0.5. Y, Z are ft + sep over a locally noetherian base S. If f is an S-morphism and $y \in Y(S)$ is a section then f is a *formal immersion along* y if,

(a) f is a formal immersion along all points of y

(b) f_s is a formal immersion at y_s for all $s \in S$.

Remark. This is supposed to be equivalent to $\widehat{Y}_y \hookrightarrow \widehat{Z}_{f(y)}$.

Lemma 1.0.6. Let A, B be complete noeth. local rings and $f : A \to B$ is a local map such that $f : A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ and $f : \mathfrak{m}_A/\mathfrak{m}_A^2 \twoheadrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.

Proof. Approximate. $\square$

Proposition 1.0.7. Let Y be separated and $f : Y \to Z$ be a formal immersion at $y \in Y$. Let T be an integral noetherian scheme with $p_1, p_@ \in Y(T)$ are s.t. $y = p_1(t) = p_2(t)$ at some $t \in T$ and $f \circ p_1 = f \circ p_2$ then $p_1 = p_2$.

Lemma 1.0.8. Let A, B be complete noetherian local rings flat over a dvr (R, π) with a map $A \to B$ such that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A$ is an isomorphism. Then $A \to B$ is surjective iff $A/\pi \to B/\pi$ is surjective.

Proof. This follows from the fact that $\mathfrak{m}_A/(\mathfrak{m}_A^2 + \pi A) \twoheadrightarrow \mathfrak{m}_B/(\mathfrak{m}_B^2 + \pi B)$ being surjective implies that it was surjective before moding by π. ☐

Corollary 1.0.9. We can check formal immersions at the special fiber of a DVR.

Proof of Proposition. $A = \{x \in T \mid p_1(x) = p_2(x)\}$ then Y is separated implies $A \subset T$ closed and T is integral so suffices to show $\mathrm{Spec}\,(\mathcal{O}_{T,t}) \to T$ factors through $A \hookrightarrow T$. So assume T is local with closed point t. Can assume Y is local with closed point y.

$$\begin{tikzcd}
\mathcal{O}_{T,t} \arrow[r, hook] & \widehat{\mathcal{O}}_{T,t} & \\
\mathcal{O}_{Y,y} \arrow[u, shift left] \arrow[u, shift right] \arrow[r] & \widehat{\mathcal{O}}_{Y,y} \arrow[u, shift left] \arrow[u, shift right] & \widehat{\mathcal{O}}_{Z,f(y)} \arrow[l]
\end{tikzcd}$$

thus the maps must agree on the local rings since they agree after composing with the surjection. ☐

Goal show that if $T_{\mathbb{Q}} \twoheadrightarrow A$ is any surjection of abelian varities with connected kernel (what we call an optimal quotient) then $X_0(N) \to J \to \mathcal{A}$ over $\mathbb{Z}[1/2N]$ is a formal immerison.

Setup N is prime > 2 and $S = \mathrm{Spec}\,(\mathbb{Z}[1/2N])$ and $X = X_0(N)$ then $J = J_0(N)$ and $\mathbb{T} \hookrightarrow \mathrm{End}\,(J)$ the Hecke algebra.

Remark. all optimal quotients of J are of the form J/IJ where $I \subset \mathbb{T}$ is a *saturated* ideal ($\mathbb{T}/I$ is torsion-free). Then $J_{\mathbb{Q}} = J_0(N)^{\text{new}}_{\mathbb{Q}}$ so everything in Daniel's talk applies. In particular,

$$J_{\mathbb{Q}} \sim \prod_{f \in C} A_f$$

with C Galois orbits of cusp forms. Also,

$$\mathrm{End}_{\mathbb{Q}}\,(A_f) = K_f = \mathrm{im}\,\mathbb{T}$$

with $[K_f : \mathbb{Q}] = \dim A_f$. Then any optimal quotient of $J_{\mathbb{Q}}$ is $\prod_{g \in C'} A_g$ with $C' \subset C$.

Theorem 1.0.10. The tangent space $T_0(F)$ is a free $\mathcal{T}_{\mathbb{Z}[1/2N]}$-module of rank 1 generated by $\frac{\mathrm{d}}{\mathrm{d}q}|_0$.

Remark. This is saying,

$$S_2(N)_R \cong H^0(J_R, \Omega^1_{J_R/R}) = T_0^*(J_R)$$

for any ring R. This is because level N cusp 2-forms are exactly given by forms on $X_0(N)$ and these are the same as forms on its Jacobian.

Corollary 1.0.11. If A is an optimal quotient of J then $X \to \mathcal{A}$ sending $\infty \mapsto 0$ is a formal immersion over S.

Proof. It suffices to show that $T_\infty X \hookrightarrow T_0\mathcal{A}$ over each prime. Then in the a sequence,

$$0 \longrightarrow B \longrightarrow J \longrightarrow A \longrightarrow 0$$

since J and A have good reduction so does B by Neron-Ogg-Shafarevich. Then Raynaud's theorem gives an exact sequence,

$$0 \longrightarrow T_0(\mathcal{B}) \longrightarrow T_0(J) \longrightarrow T_0(\mathcal{A}) \longrightarrow 0$$

(HMMM) $\square$

Reduction, $M' = T_0(T)/(\mathbb{T}_{\mathbb{Z}[1/2N]}\frac{\mathrm{d}}{\mathrm{d}q})$. But $T_0(J)$ is finite over $\mathbb{Z}[1/2N]$ hence also $\mathbb{T}_{\mathbb{Z}[1/2N]}$. Suffices to show that $M'/\mathfrak{m}M' = 0$ for all $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$ i.e. $\frac{\mathrm{d}}{\mathrm{d}q}$ generated $T_0(T)/\mathfrak{m}T_0(J)$.

Lemma 1.0.12. $S_2(N)^{\mathrm{new}}_{\mathbb{Q}}$ is a free $\mathbb{T}_{\mathbb{Q}}$-module of rank 1 generated by $\frac{\mathrm{d}}{\mathrm{d}q}|_0$.

Lemma 1.0.13. For $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$ and $T_0(J)/\mathfrak{m}T_0(J) = 0$.

Proof. Finiteness of $T_0(J)$ and NAK and $T_0(J) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$. $\square$

Lemma 1.0.14. For $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$ then $\frac{\mathrm{d}}{\mathrm{d}q}$ has nonzero image in $T_0(J)/\mathfrak{m}T_0(J)$.

Proof. If $f \in S_2(N)_{\overline{\mathbb{F}}_\ell}$ has a q-expansion,

$$f = \sum_{n=1}^{\infty} a_n q^n$$

then $\frac{\mathrm{d}}{\mathrm{d}q}(f) = a_1$ and we win by showing that if f is an eigenform with $a_1 = 0$ then $f = 0$. This is because $\frac{\mathrm{d}}{\mathrm{d}q}(T_n f) = a_n$ so if $T_n f = \lambda f$ for $\lambda \neq 0$ then we also have all $a_n = 0$.

Let's do this in more detail. Let ℓ be the characteristic of $F = (\mathbb{T} \otimes \mathbb{Z}[1/2N])/\mathfrak{m}$ and $R = (\mathbb{T} \otimes \mathbb{Z}[1/2N]) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell$. And let $M = T_0(J) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell$. Then there is an exact sequence,

$$\begin{tikzcd}
 & \mathfrak{m} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell \arrow[r] & R \arrow[r] & F \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_\ell \arrow[r] \arrow[d, equal] & 0 & \\
 & & & \prod_{i \in I} \overline{\mathbb{F}}_\ell \arrow[d, two heads] & & \\
0 \arrow[r] & \mathfrak{m}R \arrow[r] & R \arrow[r] & R \arrow[r] & \overline{\mathbb{F}}_\ell \arrow[r] & 0 \\
 & & & & F \arrow[u, hook] &
\end{tikzcd}$$

by tensoring the inclusion $F \hookrightarrow \overline{\mathbb{F}}_\ell$ we get $T_0(J)/\mathfrak{m}T_0(J) \hookrightarrow M/\mathfrak{m}M$. As R-modules,

$$(M/\mathfrak{m}M)^\vee \cong M^\vee[\mathfrak{m}] \cong H^0(X_{\overline{\mathbb{F}}_\ell}, \Omega^1_{X/\overline{\mathbb{F}}_\ell})[\mathfrak{m}]$$

$\square$

Theorem 1.0.15. if $f : X \to S$ is a smooth proper relative curve then $R^i f_* \Omega_{X/S}$ commutes with all base change.

Proof. If S is reduced this comes from Grauert. Otherwise use cohomology and base change. $\square$

In particular: if $f \in S_2(N)_{\overline{\mathbb{F}}_\ell}[\mathfrak{m}]$ is nonzero can lift to char 0 and then $\frac{\mathrm{d}}{\mathrm{d}q}(T_n f) = a_n(f)$ follows from analysis.

Lemma 1.0.16. For every $\mathfrak{m} \subset \mathbb{T}_{\mathbb{Z}[1/2N]}$. Then $T_0(J)/\mathfrak{m}T_0(J)$ is free over $\mathbb{T}_{\mathbb{Z}[1/2N]}/\mathfrak{m}$ generated by $\frac{\mathrm{d}}{\mathrm{d}q}$.

Proof. $\dim_F T_0(J)/\mathfrak{m}T_0(J) = \dim_{\overline{\mathbb{F}}_\ell} M^\vee[\mathfrak{m}]$ then let a_n be the image of T_n in $R/\mathfrak{m} = \overline{\mathbb{F}}_\ell$ then if $f \in S_2(N)_{\overline{\mathbb{F}}_\ell}[\mathfrak{m}]$ and $T_n(f) = a_n(F)$ so f is a multiple of $q + a_2 q^2 + \cdots$. ☐

2 Final Talk

2.1 Kodaira Classification of Special Fibers

Definition 2.1.1. An *elliptic surface* is a regular connected 2-dimensional scheme X equipped with a map proper map $\pi : X \to C$ to a regular connected 1-dimensional scheme C (e.g. a Dedekind scheme) whose generic fiber is a smooth geometrically-connected curve of genus 1.

Remark. The map $\pi : X \to C$ is dominant (since the generic fiber is nonempty) and X and C are integral so we get an injection $K(C) \hookrightarrow K(X)$ hence $\mathcal{O}_{X,x}$ are torsion-free $\mathcal{O}_{C,\pi(x)}$-modules and hence flat since the base is a DVR. Thus π is flat. Since X is irreducible the fibers must be all pure dimension 1. Then π is also proper so since C is normal and the generic fiber is geometrically-connected $\pi_* \mathcal{O}_X = \mathcal{O}_C$ so π has connected fibers. Thus every fiber of π is a connected genus 1 curve (not necessarily reduced) and π is smooth iff these are smooth genus 1 curves.

Definition 2.1.2. We say that an elliptic surface X is *pointed* if it furthermore equipped with a section $\sigma : C \to X$ of π. We say that π is *relatively minimal* if the fibers contain no (-1)-curves. In this case we say that X is a *minimal elliptic surface.*

Remark. A minimal elliptic surface $\pi : X \to C$ is exactly the data of compatible minimal regular models of its generic fiber E over each DVR $\mathcal{O}_{C,p}$. Therefore, to classify the fibers of minimal elliptic surfaces, it suffices to classify the special fibers of minimal regular models of genus 1 curves and, in the equicharacteristic case, then exhibit these fibers in complete families with regular total spaces. (IT IS OBVIOUS THAT THIS CAN ALWAYS BE DONE??)

Remark. DO I NEED TO ASSUME S IS EXCELLENT FOR EVERYTHING I WANT TO BE TRUE.

2.2 General Properties of Regular Models

Remark. Liu's book covers this topic well.

Let $X \to S = \mathrm{Spec}\,(R)$ be a regular proper model with special fiber X_s. Let Γ_i be the irreducible components of X_s appearing with multiplicity d_i. These are (possibly singular) proper curves over $\kappa = R/\mathfrak{m}$.

We want to define an intersection pairing on X. For any nonzero horizontal divisor D we should have $C \cdot X_s > 0$. However, $\mathcal{O}_X(X_s) = \mathcal{O}_X$ because it is cut out by a global function $\pi \in \Gamma(X, \mathcal{O}_X)$. Thus we cannot have an intersection pairing invariant under linear equivalence. This is because S is not "complete" so we can deform X_s to the "boundary" where it vanishes. Arakalov theory solves this but instead we will just ask that $i(-, D)$ is invariant under linear equivalence in its first factor. However, there is still an issue if D contains a horizontal divisor then $X_s \cdot D > 0$ because definition of the intersection pairing is symmetric. To fix this we restrict the second coordinate to only vertical divisors.

Lemma 2.2.1. There is a bilinear intersection pairing,

$$i_s : \mathrm{Div}(X) \times \mathrm{Div}_s(X) \to \mathbb{Z}$$

which satisfies the following properties,

(a) when D and E share no components by,

$$(D, E) \mapsto \sum_{x \in D \cap E} i_x(D, E)[\kappa(x) : \kappa]$$

(b) if $D \sim D'$ then $i_s(D, E) = i_s(D', S)$

(c) if FINDI

When $E \subset X_s$ is an effective divisor in the special fiber this is equivalent to,

$$i_s(D, E) = \deg_\kappa \mathcal{O}_E(D)$$

This is only defined for the second divisor $\mathrm{Div}_s(X)$ supported in the special fiber because the base is "not complete". The following example shows the pathologies of this intersection pairing and why we can't define it for arbitrary divisors.

Remark. $i_s(-, E)$ is invariant under linear equivalence. However $i_s(D, -)$ is *not* invariant unless $D \in \mathrm{Div}_s(X)$. For example, $\mathcal{O}_X(X_s) = \mathcal{O}_X$ because it is cut out by a global function $\pi \in \Gamma(X, \mathcal{O}_X)$. However, we will see that

$$K_{X/S} \cdot X_s = 2g(X_\eta) - 2$$

which is nonzero. However, when by $D, E \in \mathrm{Div}_s(X)$ then i_s is symmetric and hence invariant under linear equivalence in both components. This implies for example that $X_s^2 = 0$.

Remark. The above example shows why we cannot define an intersection theory at all for arbitrary divisors. Indeed, suppose we had,

$$i : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$$

and we just wanted $i(-, D)$ invariant under linear equivalence in the first coordinate. However, if $D = H + V$ where H is a horizontal divisor and V is a vertical divisor we have seen that the usual intersection product means that,

$$i(X_s, H + V) = X_s \cdot H = H \cdot X_s > 0$$

but $X_s \sim 0$.

Since X is regular and flat over a regular base, the fibers are Gorenstein (Tag 0BJJ). Therefore, there exists a relative dualizing line bundle $\omega_{X/S}$ (Tag 0E6R)

Lemma 2.2.2. Let X be a regular surface flat and proper over $\mathrm{Spec}\,(R)$.

(a) X is minimal iff $K_{X/S}$ is numerically effective

(b)

Lemma 2.2.3. Let X

(a) $K_{X/S} \cdot X_s = 2g(X_\eta) - 2$

(b)

2.3 The Relation to Neron Models

Proposition 2.3.1. Let E be an elliptic curve over K with $K = \text{Frac}(R)$ and R a DVR. Then let X be the minimal regular model of E over R. By properness, X is a pointed elliptic surface with $\sigma_K = 0 \in E(K) = X(K)$. Let $\mathcal{E} \subset X$ be the open subscheme obtained by removing the points of X which are singularities of the special fiber. Then $\mathcal{E}$ is the Neron model of E.

Proof. We verify the Neron mapping property in two steps. First, if $\text{Spec}(R') \to \text{Spec}(R)$ is a finite étale cover then by properness,

$$X(R') = E(K')$$

so it suffices to show that maps $\text{Spec}(R') \to X$ land in $\mathcal{E}$. Indeed, let $x \in X$ be the image of a closed point $\mathfrak{m} \subset R'$ then we get a diagram,

$$\begin{tikzcd} R'_{\mathfrak{m}} & \mathcal{O}_{X,x} \arrow[l] \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$

and $R \to R'_{\mathfrak{p}}$ is étale so the uniformizer lands in $\mathfrak{m} \backslash \mathfrak{m}^2$. Thus by commutativity $\pi \in \mathfrak{m}_x \backslash \mathfrak{m}_x^2$. Since $\mathcal{O}_{X,x}$ is regular this implies that $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\pi$ is regular so $x \in \mathcal{E}$. Thus,

$$\mathcal{E}(R') \to X(R') \to E(K')$$

are all isomorphisms. This can be used to show that the group law on E extends uniquely to $\mathcal{E}$. Now, if $T \to \text{Spec}(R)$ is a smooth R-scheme and $f_K : T_K \to E$ is a map then by properness f_K extends to a rational map $f : T \dashrightarrow X$ defined away from codimension 2. However, $T \to \text{Spec}(R)$ has sections through any point after some étale extension $\text{Spec}(R') \to \text{Spec}(R)$. Since we know maps $\text{Spec}(R') \to X$ land in $\mathcal{E}$ we conclude that f factors through $\mathcal{E} \hookrightarrow X$. Finally, since $\mathcal{E}$ is a group by a translation argument $f : T \dashrightarrow \mathcal{E}$ is everywhere defined. Then,

$$\text{Hom}_R(T, \mathcal{E}) \to \text{Hom}_K(T_K, E)$$

is surjective and since $\mathcal{E} \to \text{Spec}(R)$ is separated it is injective. $\square$

Lemma 2.3.2. Suppose that E is an elliptic curve over $\mathbb{Q}$ with additive reduction at p and $\mathcal{E}$ its Neron model over $\mathbb{Z}_{(p)}$. Then $\#\pi_0(\mathcal{E}_{\mathbb{F}_p}) \le 4$.

Proof. Additive reduction means that the special fiber of $\mathcal{E}^0$ is $\mathbb{G}_a$ (since the residue field is perfect). Therefore, all the geometric components (IS IT POSSIBLE FOR $\pi_0(\mathcal{E}_{\mathbb{F}_p})$ NONSPLIT??) of $\mathcal{E}^0$ are isomorphic to $\mathbb{G}_a$. From our construction of the Neron model this is equivalent to, in the special fiber of the minimal regular model, each genus 1 component having a cusp and each genus 0 component intersecting the other components in exactly one point. By inspection of the possible types (II, III, IV, HOW MANY OTHERS, there can be a maximum of 4 geometric components under these restrictions. $\square$

2.4 Completing the Proof

We first need a lemma. (HAS THIS BEEN PROVEN PREVIOUSLY??)

Proposition 2.4.1. Let $p \neq 2$ and $f : H \to G$ be a morphism of finite flat group schemes over a DVR R with mixed characteristic 0 and p. Let $K = \text{Frac}(R)$. If $f_K : H_K \to G_K$ is a closed immersion then f is a closed immersion.

Proof. DO THIS!! □

Theorem 2.4.2. Let $(\mathcal{E}_{\mathbb{Q}}, C)$ be a pair of an elliptic curve over $\mathbb{Q}$ and a cyclic subgroup C of order N with N an odd prime. Then E has potentially good reduction at all odd primes $p \neq N$.

The proof of this theorem relies on the following result from last time.

Proposition 2.4.3. Let $\mathcal{A}$ be the Neron model over $\mathbb{Z}[1/2N]$ of any nonzero optimal quotient A of J. Define $X_0(N)_{\mathbb{Q}} \to J \to A$ by sending the cusp ∞ to 0 and let f denote the morphism extensing this over $\mathrm{Spec}\,(\mathbb{Z}[1/2N])$. Then $\infty \in X_0(N)(\mathbb{Z}_{(p)})$ is the only point reducing to $\infty \in X_0(N)(\mathbb{F}_p)$ that also maps to 0 in $\mathcal{A}(\mathbb{Z}_{(p)})$ under f.

Proof of Theorem. Let $A = \widetilde{J}$ DO THIS PROFO □

Remark. Let $\mathcal{E}$ be the neron model of an elliptic curve E over a DVR R. Then $\mathcal{E}[N]$ is finite flat over R because $\mathcal{E} \xrightarrow{N} \mathcal{E}$ is finite flat so its base change along the zero section is also finite flat. To show $\mathcal{E} \to \mathcal{E}$ is finite flat we use miracle flatness check quasi-finiteness explicitly then show properness because (HOW TO DO!!)

Now we are ready to prove the main theorem.

Theorem 2.4.4 (Mazur)**.** Let N be prime and $N \geq 11$ and not 13. Then there are no elliptic curves over $\mathbb{Q}$ with a torsion subgroup of order divisible by N.

Proof. Suppose $E(\mathbb{Q})[N]$ is nonempty. We proved that E has potentially good reduction at 3. We will now show that $N \leq 7$. First, suppose that E has good reduction at 3 so its Neron model $\mathcal{E}$ over $\mathrm{Spec}\left(\mathbb{Z}_{(3)}\right)$ is proper. Then since $\mathcal{E}[N]$ is finite over $\mathrm{Spec}\left(\mathbb{Z}_{(3)}\right)$ we have,

$$\mathbb{Z}/N\mathbb{Z} \hookrightarrow E(\mathbb{Q})_{\mathrm{tors}} \to \mathcal{E}(\mathbb{F}_3)$$

is injective. But $\mathcal{E}_{\mathbb{F}_3}$ is an elliptic curve so by the Hasse-Weil bound,

$$\#\mathcal{E}_{\mathbb{F}_3} \leq \lfloor 4 + 2 \cdot \sqrt{3} \rfloor = 7$$

Otherwise, E has additive reduction at 3 and let $\mathcal{E}$ be its Neron model over $\mathrm{Spec}\left(\mathbb{Z}_{(3)}\right)$. Consider the exact sequence,

$$0 \longrightarrow \mathcal{E}^0_{\mathbb{F}_3} \longrightarrow \mathcal{E}_{\mathbb{F}_3} \longrightarrow \pi_0(\mathcal{E}_{\mathbb{F}_3}) \longrightarrow 0$$

Since we are in the case of additive reduction, $\mathcal{E}^0_{\mathbb{F}_3} = \mathbb{G}_a$. Consider the map of finite flat group schemes,

$$\underline{\mathbb{Z}/N\mathbb{Z}} \to \mathcal{E}[N]$$

over $R = \mathbb{Z}_{(3)}$ arising from the inclusion $\mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N](\mathbb{Q})$ which spreads out by the Neron mapping property since $\underline{\mathbb{Z}/N\mathbb{Z}}$ is smooth over R. This map is, by construction, a closed immersion on the generic fiber so by Prop 3.3 this map is a closed immerison. In particular,

$$\mathbb{Z}/N\mathbb{Z} \hookrightarrow \mathcal{E}[N]_{\mathbb{F}_3}$$

By the above lemma, $\#\pi_0(\mathcal{E}_{\mathbb{F}_3}) \leq 4$ so if $N > 4$ then $\mathbb{Z}/N\mathbb{Z}$ lands in $\mathcal{E}^0_{\mathbb{F}_3}$ but $\#\mathcal{E}^0_{\mathbb{F}_3}(\mathbb{F}_3) = \#\mathbb{G}_a(\mathbb{F}_3) = 3$ so this is impossible. Therefore we conclude that $N \leq 4$. □