1 Introduction

Theorem 1.0.1 (Hodge-to-de Rham-degeneration)**.** Let X/k be a smooth proper scheme with k a field of characteristic zero and $\Omega^\bullet_{X/k}$ is de Rham complex. Then, the Hodge-to-de Rham spectral sequence,

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \implies H^{p+q}_{\mathrm{dR}}(X)$$

degenerates at the E_1-page.

Remark. This is equivalent to the numerical equality

$$\dim H^n_{\mathrm{dR}}(X) = \sum_{p+q=n} \dim H^q(X, \Omega^p_{X/k})$$

Theorem 1.0.2 (Kodaira-Nakano-vanishing)**.** Let X/k be a smooth proper scheme with k a field of characteristic zero and $\mathcal{L}$ an ample. Then $H^q(X, \Omega^p_X \otimes \mathcal{L}) = 0$ for $p + q > \dim X$.

Theorem 1.0.3 (Deligne-Illusie)**.** Let X/k be a smooth proper scheme of pure dimension n. Let k be a perfect field of characteristic p. Suppose

(a) $p > n$

(b) X lifts to $W_2(k)$

then the Hodge-to-de Rham spectral sequence degenerates at E_1 and Kodaira-Nakano vanishing holds for any ample $\mathcal{L}$.

Corollary 1.0.4. The theorems also hold over k characteristic zero.

Proof. Both are completely numerical statements. For X/k in characteristic zero, we can spread out to a finite type $\mathbb{Z}$-algebra $A \subset k$ and smooth proper morphism $f : \mathscr{X} \to S = \mathrm{Spec}\,(A)$ with a relatively ample $\mathcal{L}$ so that the the dimensions of all the relevant cohomology groups are constant. Hence we just need to prove degeneration and Nakano vanishing for some fiber. Shrinking, we may assume $S \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth. By Chevallay, there is a prime $p > \dim X$ in the image of $S \to \mathrm{Spec}\,(\mathbb{Z})$ so choose $s \mapsto p$ and by smoothness there is a map $\mathrm{Spec}\,(W_2(\kappa(s))) \to S$ hence $\mathscr{X}_s$ satisfies the hypothesis and we apply Deligne-Illusie's results to win. $\square$

1.1 Recall: the Frobenius

Definition 1.1.1. Let X be a scheme of characteristic p (meaning $p\mathcal{O}_X = 0$). Then there is a natural map $\mathrm{Fr} : X \to X$ via id on topological spaces and $\mathcal{O}_X \to \mathcal{O}_X$ via $x \mapsto x^p$. This is natural, in the sense that for any map $f : X \to Y$ there is a commutative diagram,

$$\begin{tikzcd} X \arrow[r, "\mathrm{Fr}_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{Fr}_Y"] & Y \end{tikzcd}$$

Therefore, we can define via pullbacks,

$$\begin{tikzcd} X \arrow[r, dashed, "F_{X/Y}"] \arrow[dr, "f"'] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Y \arrow[r, "\mathrm{Fr}_Y"] & Y \end{tikzcd}$$

giving the relative Frobenius $F_{X/Y} : X \to X^{(p)}$.

Proposition 1.1.2. If Y has characteristic p and $f : X \to Y$ is smooth of relative dimension n then $F_{X/Y} : X \to X^{(p)}$ is finite and flat of degree n. Therefore, $F_* \mathcal{O}_X$ is locally free of rank n as a $\mathcal{O}_{X^{(p)}}$-module.

1.2 The Incredible Trick: Cartier Isomorphisms

Let $F = F_{X/k}$ for X smooth over a perfect field k of characteristic p.

The following is a crucial remark. The differentials $(\Omega_X^\bullet, \mathrm{d})$ are nonlinear so it does not form an element of $D^b(X)$. However, $F_* \Omega_X^\bullet \in D^b(X^{(p)})$ because the differentials are $\mathcal{O}_{X^{(p)}}$-linear! This is because

$$\mathrm{d}s^p = p\, s^{p-1} \mathrm{d}s = 0$$

The incredible observation is that under our hypotheses $F_* \Omega_X^\bullet$ decomposes into its cohomology in the derived category.

Theorem 1.2.1. If X lifts to $W_2(k)$ then there is a quasi-isomorphism in $D^b(X^{(p)})$

$$\varphi : \tau^{<p} F_* \Omega_X^\bullet \xrightarrow{\sim} \bigoplus_{i<p} \Omega^i_{X^{(p)}}[-i]$$

In particular, if $p > \dim X$ then $\tau^{<p} F_* \Omega_X^\bullet = F_* \Omega_X^\bullet$ decomposes.

Let's assume this and prove the main theorem.

1.3 Hodge-to-de Rham degeneration

Let X/k be satisfying the hypotheses so φ exists and $p < \dim X$. Then

$$\mathbb{H}^n(X, \Omega_X^\bullet) = \mathbb{H}^n(X^{(p)}, F_* \Omega_X^\bullet) \xrightarrow{\sim} \bigoplus_i H^{n-i}(X^{(p)}, \Omega^i_{X^{(p)}}) = \bigoplus_i \mathrm{Fr}_k^* H^{n-i}(X, \Omega^i_X)$$

The first map is because F is a homeomorphism[1], the second is φ on cohomology the last one uses flat base change for the pullback diagram

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,(k) \arrow[r, "\mathrm{Fr}_k"] & \mathrm{Spec}\,(k) \end{tikzcd}$$

Since F_k^* does not change the dimension of a vector space (only the k-action) we conclude using that E_1-degeneration is equivalent to the numerical equality given by taking dimensions above.

[1]Since it is not a complex of $\mathcal{O}_X$-modules, because the maps are nonlinear, affine is not enough. However, the Leray spectral sequence is completely general so quasi-finite is enough because then the higher derived pushforwards vanish when the fibers are zero dimensional.

1.4 Kodaira Vanishing

We will use φ to prove the following inductive step.

Definition 1.4.1. We say that $\mathcal{M} \in \text{Pic}\,(X)$ satisfies (NV) if

$$H^q(X, \Omega_X^p \otimes \mathcal{M}) = 0 \quad \text{for all } p + q > \dim X$$

We will prove

$$(*) \qquad \mathcal{M}^{\otimes p} \text{ satisfies (NV)} \implies \mathcal{M} \text{ satisfies (NV)}$$

Why does $(*)$ suffice. By downward induction, we just need to show that if $\mathcal{L} \in \text{Pic}\,(X)$ is ample then $\mathcal{L}^{\otimes p^k}$ satisfies (NV) for $k \gg 0$. But this is clear: large enough powers of $\mathcal{L}$ kill *all* higher cohomology of anything by Serre vanishing.

Lemma 1.4.2. For any invertible module $\mathcal{M}$,

$$F_X^* \mathcal{M} \xrightarrow{\sim} \mathcal{M}^{\otimes p}$$

Proof. The map is defined by adjunction of $\mathcal{M} \to (F_X)_* \mathcal{M}^{\otimes p}$ via $m \mapsto m^{\otimes p}$ which is linear because,

$$am \mapsto (am)^p = a^p m^p = a \cdot m^p$$

We check $F_X^* \mathcal{M} \xrightarrow{\sim} \mathcal{M}^{\otimes p}$ locally. □

Corollary 1.4.3. Let $\mathcal{M}'$ be the pullback of $\mathcal{M}$ under $X^{(p)} \to X$. Then $F^* \mathcal{M}' = \mathcal{M}^{\otimes p}$.

Remark. The point of this is that the p-th power of any line bundle is pulled back from a line bundle on $X^{(p)}$.

Proof of induction. Assume (NV) holds for $\mathcal{M}^{\otimes p}$. By the projection formula,

$$F_*(\mathcal{M}^{\otimes p} \otimes \Omega_X^i) \cong F_*(F^* \mathcal{M}' \otimes \Omega^i) \cong \mathcal{M}' \otimes F_* \Omega_X^i$$

Consider the hypercohomology spectral sequence computing the cohomology of $\mathcal{M}' \otimes F_* \Omega_X^\bullet$,

$$E_1^{i,j} = H^j(X^{(p)}, \mathcal{M}' \otimes F_* \Omega_X^i) \implies \mathbb{H}^{i+j}(X^{(p)}, \mathcal{M}' \otimes F_* \Omega_X^\bullet)$$

However,

$$H^j(X^{(p)}, \mathcal{M}' \otimes F_* \Omega_X^i) = H^j(X^{(p)}, F_*(\mathcal{M}^{\otimes p} \otimes \Omega_X^i)) = H^j(X, \mathcal{M}^{\otimes p} \otimes \Omega_X^i) = 0$$

for $i + j > \dim X$ by the induction hypothesis. Therefore, we conclude from the spectral sequence,

$$\mathbb{H}^k(X^{(p)}, \mathcal{M}' \otimes F_* \Omega_X^\bullet) = 0$$

for $k > \dim X$. Now we use the decomposition

$$\mathcal{M}' \otimes F_* \Omega_X^\bullet \xrightarrow{\sim} \bigoplus_i \mathcal{M}' \otimes \Omega_X^i[-i]$$

so the hypercohomology is given by,

$$\mathbb{H}^k(X^{(p)}, \mathcal{M}' \otimes F_* \Omega_X^\bullet) = \bigoplus_{i+j=k} H^j(X^{(p)}, \mathcal{M}' \otimes \Omega_{X^{(p)}}^i) = \bigoplus_{i+j=k} \text{Fr}_k^* H^j(X, \mathcal{M} \otimes \Omega_X^i)$$

and thus by vanihsing of the hypercohomology for $n > \dim X$ we get vanishing,

$$H^j(X, \mathcal{M} \otimes \Omega_X^i) = 0$$

for $i + j > \dim X$ proving (NV) for $\mathcal{M}$ thus completing the induction. □

1.5 The Cartier Operator

We need to construct φ. The first step is to understand the Cartier operator. There is a graded isomorphism,

$$C^{-1} : \bigoplus_i \Omega^i_{X^{(p)}} \xrightarrow{\sim} \bigoplus_i \mathcal{H}^i(F_* \Omega^\bullet_X)$$

such that,

(a) in $i = 0$ the map $\mathcal{O}_{X^{(p)}} \to F_* \mathcal{O}_X$ is exactly $F^\#$

(b) in $i = 1$,

$$C^{-1}(1 \otimes \mathrm{d}s) = s^{p-1} \mathrm{d}s \in \mathcal{H}^1(F_* \Omega^\bullet_X)$$

think of this as "$\frac{F^*(\mathrm{d}s)}{p}$".

To prove the theorem, we will exhibit a quasi-isomorphism

$$\varphi : \bigoplus_{i<p} \Omega^i_{X^{(p)}}[-i] \to F_* \Omega^\bullet_X$$

that induces C^{-1} on cohomology for $i < p$ (and thus is a quasi-isomorphism to the trucation). We want to reduce to constructing φ^1 where φ^i are the components of the map from the direct sum. For φ^0 we just define,

$$\varphi^0 : \mathcal{O}_{X^{(p)}} \xrightarrow{C^{-1}} F_* \mathcal{O}_X = \mathcal{H}^0(F_* \Omega^\bullet_X) \hookrightarrow F_* \Omega^\bullet_X$$

Now assume we have constructed,

$$\varphi^1 : \Omega^1_{X^{(p)}}[-1] \to \widetilde{F_*} \Omega^\bullet_X$$

inducing C^{-1} on $\mathcal{H}^1$. Then there exists,

$$\left(\Omega^1_{X^{(p)}}\right)^{\otimes i} \to \Omega^i_{X^{(p)}}$$

by sending,

$$w_1 \otimes \cdots \otimes w_i \mapsto w_1 \wedge \cdots \wedge w_i$$

If $i < p$ (or in characteristic zero) then there exists a section to this map,

$$a(w_1 \wedge \cdots \wedge w_i) = \frac{1}{i!} \sum_{\sigma \in S_i} \mathrm{sign}(i) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(i)}$$

Therefore we get,

$$\begin{tikzcd}
(\Omega^1_{X^{(p)}})^{\otimes i} \arrow[r, "\varphi_1^{\otimes i}"] & (F_* \Omega^\bullet_X)^{\otimes^{\mathbb{L}} i} \arrow[d] \\
\Omega^i_{X^{(p)}} \arrow[u] \arrow[r, dashed, "\varphi^i"] & F_* \Omega^\bullet_X
\end{tikzcd}$$

Because this construction agrees with the product structure and the Cartier isomorphism is determined (using the product structure) by its values in degree 1 this means that φ^i must induce C^{-1} in degree i.

1.6 Construction of φ^1

First we consider the case when F admits a global lift over $W_2(k)$. This means there is a diagram,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "F"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{X} \arrow[d, "\widetilde{F}"] \\ X^{(p)} \arrow[r] & \widetilde{X}^{(p)} \end{tikzcd}$$

where $\widetilde{X} \to \mathrm{Spec}\,(W_2(k))$ and $\widetilde{X^{(p)}} \to \mathrm{Spec}\,(W_2(k))$ are smooth lifts of X and $X^{(p)}$ over $W_2(k)$.

Now to perform the construction notice that,

$$\mathrm{im}\,(\widetilde{F}^* : \Omega^1_{\widetilde{X^{(p)}}/\widetilde{S}} \to \widetilde{F}_* \Omega^1_{\widetilde{X}/\widetilde{S}}) \subset p \cdot \widetilde{F}_* \Omega^1_{\widetilde{X}/\widetilde{S}}$$

because pulling back differentials by Frobenius introduces a factor of p. Therefore, we get a diagram,

$$\begin{tikzcd} \Omega^1_{\widetilde{X^{(p)}}/\widetilde{S}} \arrow[r, "\widetilde{F}"] \arrow[d, two heads] & p \cdot \widetilde{F}_* \Omega^1_{\widetilde{X}/\widetilde{S}} \\ \Omega^1_{X^{(p)}} \arrow[r, dashed, "\varphi^1"] & F_* \Omega^1_X \arrow[u, "p\cdot(-)"'] \end{tikzcd}$$

which exists because the right upward map is an isomorphism and the kernel of the left downward map is the multiples of p which are sent to zero. I claim that

$$\mathrm{im}\,\varphi^1 \subset Z^1(F_* \Omega^\bullet_X)$$

and φ^1 induces C^{-1} in degree 1. For local section $a' \in \Gamma(U^{(p)}, \mathcal{O}_{\widetilde{X^{(p)}}})$ pulled back from $a \in \Gamma(U, \mathcal{O}_X)$, the differential $\mathrm{d}a$ is acted on via

$$\widetilde{F}^*(\mathrm{d}a') = \mathrm{d}\,\widetilde{F}^\# a' = p a^{p-1} \mathrm{d}a + p\,\mathrm{d}b$$

where $\widetilde{F}^\# a' = a^p + p\,b$ where $p\,b$ is the error term. Hence

$$\varphi^1(\mathrm{d}a') = a^{p-1}\mathrm{d}a + \mathrm{d}b$$

which is clearly an exact form (lies in Z^1). But notice that the second term is exact and therefore dies in the quotient

$$Z^1(F_*\Omega^\bullet_X) \to \mathcal{H}^1(F_*\Omega^\bullet_X)$$

so the induced map is exactly given by the Cartier isomorphism in degree 1.

1.7 What about if F doesn't lift?

From smoothness, we know that lifts exist locally. We need to compare the outputs of different lifts.

Lemma 1.7.1. Given flat lifts $\widetilde{X}_i$ of X and $G_i : \widetilde{X} \to \widetilde{X^{(p)}}$ of F over $\widetilde{S}$ there is a canonical element,

$$h(G_1, G_2) : \Omega^1_{X^{(p)}} \to F_* \mathcal{O}_X$$

such that,

$$\varphi^1_{G_2} - \varphi^1_{G_1} = \mathrm{d}h(G_1, G_2)$$

and if $G_3 : \widetilde{X}_3 \to \widetilde{X^{(p)}}$ is a third lifting then

$$h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3)$$

Proof. Choose an isomorphism $u : \widetilde{X}_1 \xrightarrow{\sim} \widetilde{X}_2$ of lifts (which may only exist locally) then

$$u^* G_2 - G_1 : \mathcal{O}_{X^{(p)}} \to F_* \mathcal{O}_X$$

is a derivation which does not depend on the choice of isomorphism u. Indeed, given $u' : \widetilde{X}_1 \xrightarrow{\sim} \widetilde{X}_2$ the difference is a derivation or equivalently a map

$$\delta : \Omega^1_X \to \mathcal{O}_X$$

Then $u^* G_2$ and $u'^* G_2$ differ by the composition of δ with the pullback $F^* \Omega^1_{X^{(p)}} \to \Omega^1_X$ which is zero. Hence $u^* G_2 = u'^* G_2$. Therefore, working locally on X so that an isomorphism u exists, we get a well-defined derivation

$$h(G_1, G_2) : \Omega^1_{X^{(p)}} \to \widetilde{F}_* \mathcal{O}_X$$

via the difference above. Then

$$\varphi^1_{G_2} - \varphi^1_{G_1} = \mathrm{d} h(G_1, G_2)$$

from the formula for φ^1 since $G_2^\#(a') - G_1^\#(a') = b_2 - b_1$ in $F_* \mathcal{O}_X = p \cdot \widetilde{F}_* \mathcal{O}_{\widetilde{X}}$ then

$$\varphi^1_{G_2}(a') - \varphi^1_{G_1}(a') = \mathrm{d}(b_2 - b_1)$$

$\square$

This is exactly enough data to modify the local lifts in such a way that the φ^1_G glue to

$$\varphi^1 : \Omega^1_{X^{(p)}} \to Z^1(F_* \Omega^\bullet_X)$$

using the Cech description.

2 Talk

Let $X/\mathbb{C}$ be smooth and proper. Then

$$H^n(X/\mathbb{C}, \mathbb{C}) = \bigoplus_{i+j=n} H^j(X, \Omega^i_X)$$

by the Hodge decomposition. This is because by Grothendieck

$$H^n(X, \mathbb{C}) = H^n_{\mathrm{dR}}(X/\mathbb{C}) = \mathbb{H}^n_{\mathrm{Zar}}(X, \Omega^\bullet_X)$$

we can interpret the Hodge decomposition by saying that we can replace the differentials by 0 and the cohomology does not change.

We also have Kodaira-Akizuki-Nakano vanishing: if $\mathcal{L}$ is ample then

$$H^i(X, \mathcal{L} \otimes \Omega^j_X) = 0$$

for all $i + j > \dim X$.

2.1 The positive characteristic scenario

Let p be a prime and $X/\mathbb{F}_p$ be a smooth projective variety. Is there a natural decomposition?

$$H^n_{\mathrm{dR}}(X/\mathbb{F}_p) = \bigoplus_{i+j=n} H^j(X, \Omega^i_X)$$

but we cannot hope to have a natural such decomposition. Indeed, if $F : X \to X$ is the Frobenius

$$f \in \mathcal{O}_X \quad F^*(f) = f^p$$

Example 2.1.1. Let $E/\mathbb{F}_p$ be a supersingular elliptic curve then $F^* = 0$ on $H^1(E, \mathcal{O}_E)$. Then

$$F^* \circlearrowright H^1_{\mathrm{dR}}(E/\mathbb{F}_p) = \mathbb{F}_p^{\oplus 2}$$

acts via

$$F^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and hence cannot respect any decomposition.

Example 2.1.2. Mumford gave an example of a surface such that

$$\mathrm{d} : H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X)$$

is nonzero and thus

$$\dim H^1_{\mathrm{dR}} < \dim H^0(X, \Omega^1_X) + \dim H^1(X, \mathcal{O})$$

Theorem 2.1.3 (Deligne-Illusie)**.** Let $X/\mathbb{F}_p$ be a smooth variety endowed with a lift $\widetilde{X}$ over $\mathbb{Z}/p^2$ then

$$H^n_{\mathrm{dR}}(X) = \bigoplus_{i+j=n} H^j(X, \Omega^i)$$

for $n < p$ which is natural in $\widetilde{X}$ (so not completely canonical only functorial for morphisms endowed with a lift).

This fixes the issues. In the first example, F cannot be lifted, in the second the variety does not lift at all.

Theorem 2.1.4. If X is smooth projective and $\dim X \leq p$ then

$$H^i(X, \Omega^j \otimes \mathcal{L}) = 0$$

for $i + j > \dim X$ if X admits a lift.

Theorem 2.1.5 (P)**.** There exist smooth projective $X/\mathbb{F}_p$ liftable to $\mathbb{Z}_p$ such that

$$\dim H^p_{\mathrm{dR}}(X/\mathbb{F}_p) < \dim \bigoplus_{i+j=p} H^j(X, \Omega^i_X)$$

So the first case possible fails.

2.2 Method of Deligne-Illusie

For $X/\mathbb{F}_p$ smooth the complex

$$F_*\Omega_X^\bullet := [F_*\mathcal{O}_X \xrightarrow{F_*\mathrm{d}} F_*\Omega_X^1 \xrightarrow{F_*\mathrm{d}} F_*\Omega_X^2 \to \cdots]$$

is a *linear* complex of $\mathcal{O}_X$-modules. Indeed,

$$\mathrm{d}f^p\omega = pf^{p-1}\mathrm{d}f \wedge \omega + f^p\omega = f^p\omega$$

But note that

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X) = \mathrm{R}\Gamma_{\mathrm{Zar}}(X, F_*\Omega_X^\bullet)$$

Cartier isomorphism

$$\mathcal{H}^i(F_*\Omega_X^\bullet) \xrightarrow{\sim} \Omega_X^i$$

The key question is whether this lifts to the derived category? This means is $F_*\Omega_X^\bullet$ quasi-isomorphic to a complex with zero differentials? Question: is there a quasi-isomorphism

$$F_*\Omega_X^\bullet = \xrightarrow{\sim} \bigoplus_{i\geq 0} \Omega_X^i[-i]$$

If yes, we get a decomposition

$$H_{\mathrm{dR}}^n(X) \cong \bigoplus_{i+j=n} H^j(X, \Omega_X^i)$$

for all n and we get Kodaira-Akizuki-Nakano vanishing.

Consider

$$\tau^{\leq 1}F_*\Omega_X \xrightarrow{\sim} [\mathcal{O}_X \xrightarrow{0} \Omega_X^1]$$

exists iff X lifts to $\mathbb{Z}/p^2$. Moreover if you are precise about the equivalence of quasi-isomorphisms they are in bijection with isomorphism classes of lifts. It turns out that such a quasi-isomorphism implies

$$\tau^{<p}F_*\Omega_X^\bullet \xrightarrow{\sim} \bigoplus_{i=0}^{p-1} \Omega_X^i[-i]$$

We want to figure out some classes of varieties for which this quasi-isomorphism exists.

2.3 Frobenius Splitting

Definition 2.3.1 (Mehta-Ramanathan)**.** Let $X/\mathbb{F}_p$ be any scheme. We say it is *Frobenius split* if there exists

$$\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$$

such that the composition

$$\mathcal{O}_X \to F_*X \xrightarrow{\sigma} \mathcal{O}_X$$

is the identity.

Note that $F_*\mathcal{O}_X$ is, for X smooth, a vector bundle of rank $p^{\dim X}$.

Remark. In characteristic zero, if $f : X \to Y$ is a finite flat morphism then $\mathcal{O}_Y \to f_*\mathcal{O}_X$ always has a splitting. Indeed, if $\deg f$ is invertible on Y then there is a splitting by taking the trace map which gives multiplication by p. Thus failure of splitting is a strictly characteristic zero phenomenon.

Proposition 2.3.2. If X is any projective variety over $\mathbb{F}_p$ and is F-split then if $\mathcal{L}$ is an ample line bundle then $H^i(X, \mathcal{L}) = 0$ for $i > 0$.

Proof. For any line bundle $\mathcal{L}$ we see that if $H^i(X, \mathcal{L}^{\otimes p}) = 0$ then $H^i(X, \mathcal{L}^{\otimes p}) = 0$. Indeed,

$$H^i(X, \mathcal{L}^{\otimes p}) = H^i(X, F^*\mathcal{L}) = H^i(X, F_*F^*\mathcal{L}) = H^i(X, \mathcal{L} \otimes F_*\mathcal{O}_X)$$

there is always a map $\mathcal{L} \to F_*F^*\mathcal{L}$ but we have a splitting so

$$H^i(X, \mathcal{L}) \to H^i(X, \mathcal{L} \otimes F_*\mathcal{O}_X)$$

is a direct summand. Hence $H^i(X, \mathcal{L}^{\otimes p}) = 0$ implies $H^i(X, \mathcal{L}) = 0$. Since $\mathcal{L}$ is ample, some power will kill higher cohomology so we win by downward induction. □

Analogously we can prove

$$H^i(X, \mathcal{L} \otimes \omega_X) = 0$$

for $i > 0$ if F-split and smooth projective.

Example 2.3.3. The following are F-split

(a) ordinary elliptic curves

(b) flag varities G/P for $P \subset G$ parabolic in a reductive group

(c) toric varieties?

(d) BG for any reductive group G

non F-split varieties:

(a) curve of genus $g > 1$

(b) most varieties

Theorem 2.3.4 (P)**.** If X is smooth and F-split then

$$F_*\Omega_X^\bullet = \bigoplus_{i \geq 0} \Omega_X^i[-i]$$

so there is a decomposition in all degrees. Moreover, if X is smooth projective then

$$H^i(X, \mathcal{L} \otimes \Omega_X^j) = 0$$

for $i + j > \dim X$.

Proposition 2.3.5 (Vologodsky, Bhatt)**.** If $X/\mathbb{F}_p$ is smooth then

$$F^*F_*\Omega_X^\bullet \cong \bigoplus_{i \geq 0} F^*\Omega_X^\bullet[-i]$$

Any possible extension thus die under F^*.

Then the theorem follows quickly. We get

$$F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} F_*\Omega_X^\bullet \cong \bigoplus_{i \geq 0} F_*\mathcal{O}_X \otimes \Omega_X^i[-i]$$

there is always a map

$$F_*\Omega_X^\bullet \to F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} F_*\Omega_X^\bullet$$

but when you are F-split there is a section

$$F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} F_*\Omega_X^\bullet \to F_*\Omega_X^\bullet$$

so $F_*\Omega_X^\bullet$ is a direct summand of a decomposition complex.

2.4 The de Rham stack (Simpson, Drinfeld, Bhatt-Lurie, Ogus-Vologodsky

The proof of these claims uses the de Rham stack. Let $X/\mathbb{F}_p$ be a smooth variety. There is a stack $X^{\mathrm{dR}}/\mathbb{F}_p$ which is a étale stack in groupoids such that

(a) $\mathrm{R}\Gamma(X^{\mathrm{dR}}, \mathcal{O}_{X^{\mathrm{dR}}}) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbb{F}_p)$

(b) $\mathfrak{QCoh}\left(()\, X^{\mathrm{dR}}\right) \cong D_X$-modules with locally nilpotentent p-curvature. These are pairs $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a vector bundle with ∇ a flat connection with locally nilpotent p-curvature.

There is a map $s : X \to X^{\mathrm{dR}}$ such that $s^*(\mathcal{E}, \nabla) = \mathcal{E}$. In characteristic p there is a map $\pi : X^{\mathrm{dR}} \to X$ such that $\pi^*\mathcal{E} = (F^*\mathcal{E}, \nabla^{\mathrm{can}})$ where $\nabla^{\mathrm{can}}(f \otimes s) = \mathrm{d}f \otimes s$ which is well-defined because p-th powers have zero differential. The composition $\pi \circ s = F$ because $s^*\pi^*\mathcal{E} = F^*\mathcal{E}$.

Proposition 2.4.1. $F_*\Omega^\bullet_X = \mathrm{R}\pi_*\mathcal{O}_{X^{\mathrm{dR}}}$

Key property: $\pi : X^{\mathrm{dR}} \to X$ is a gerbe for a group scheme on X.

Lemma 2.4.2 (Ogus-Vologodsky)**.** Assume X has a lift $\widetilde{X}$ over $\mathbb{Z}/p^2$ equipped with a lift of Frobenius $\widetilde{F} : \widetilde{X} \to \widetilde{X}$. Then the category of quasi-coherent sheaves on X with flat locally nilpotent p-curvature connections is equivalent to the category of nilpotent Higgs sheaves $(\mathcal{E}, \theta)$ on X.

In particular, when we have F-lifts, $X^{\mathrm{dR}} \cong B_X T_X^{\#}$ for a certain group scheme $T_X^{\#}$. This is induced by the above equivalence of categories.

Proof. Let $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ be a Higgs sheaf. We want to produce $(\mathcal{E}, \nabla)$. The lift $\widetilde{F}$ produces

$$F^*\Omega^1_X \xrightarrow{\frac{1}{p}\mathrm{d}wtF} \Omega^1_X$$

given by

$$\omega \otimes 1 \mapsto \frac{\mathrm{d}\widetilde{F}(\widetilde{\omega})}{p}$$

for some lift $\widetilde{\omega}$ of ω. Then I can consider $(F^*\mathcal{E}, \nabla^{\mathrm{can}} +$
$frac\mathrm{d}\widetilde{F}p \cdot F^*\theta)$ where we need this curious map or else $F^*\theta$ has target in $F^*\Omega^1_X$. This map of categories works in general, it is an equivalence if we impose nilpotency conditions. ☐

Corollary 2.4.3. Given a lift

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[dr] & X^{\mathrm{dR}} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

then there is a quasi-isomorphism

$$\widetilde{f}^* F_*\Omega^\bullet_X \bigoplus_{i \geq 0} \Omega^i_X[-i]$$

In particular, we can apply this to Frobenius given a lifting.