Lemma 0.1. *For any integers $n > 1$ and $d \geq n^2$ let $q := d - n\sqrt{d} \geq 0$. Then the inequality*

$$\sum_{k=1}^{n} k\sqrt[k]{q} + c \;<\; d$$

is satisfied for some $c := c(n,d) > 0$ strictly positive function.

Proof. The desired inequality is equivalent to showing

$$\sum_{k=2}^{k} k\sqrt[k]{q} < n\sqrt{d}$$

and then we set

$$c := \frac{1}{2}\left[n\sqrt{d} - \sum_{k=2}^{k} k\sqrt[k]{q}\right]$$

which is then positive. For $n = 2$ the desired inequality is obvious. For $n > 2$ write the inequality as

$$\sum_{k=3}^{n} k\sqrt[k]{q} < (n-2)\sqrt{d} + 2(\sqrt{d} - \sqrt{q})$$

Since there are $n - 2$ terms in this sum, it suffices to show for all $3 \leq k \leq n$ that

$$k\sqrt[k]{q} < \sqrt{d}$$

In fact, this is not quite true. It is true for $3 \leq k \leq n - 6$. To see this, let $y = \sqrt{d}$ then the inequality becomes

$$y^{k-1} > k^k(y - n)$$

which is minimized (there is at most one minimum for $y \geq n$) when

$$y = \left(\frac{k^k}{k-1}\right)^{\frac{1}{k-2}}$$

Now it is easily seen that, when $k \geq 4$, this value occurs between k and $k + 6$. Thus the inequality is satisfied unless $k \geq n - 6$. We handle the case $k = 3$ separately (the only subtitly is when $n = 3$ but then $k = 3$ is covered in the final 6 values). Taking into account the remaining term on the right-hand side, it suffices to check that

$$\sum_{k=n-6}^{n} k\sqrt[k]{q} < 6\sqrt{d} + 2(\sqrt{d} - \sqrt{q})$$

□

Proposition 0.2. *Hence in theorem C we can take*

$$d_0(\dim X, \alpha, \epsilon) := \left\lceil \frac{(\dim X)^{\frac{\dim X}{\dim X - 1}}(1 + \ln\frac{\dim X}{2})^2}{\epsilon^2} \right\rceil$$

Proposition 0.3. *Combining this with Theorem A, we get an explicit version of theorem B: general* $X \subset \mathbb{P}^{n+r}$ *of type* $(d_1, \ldots, d_r)$ *has*

$$\operatorname{covgon}(X) \geq (1-\epsilon)d_1 \cdots d_r$$

if (up to reordering the degrees)

$$d_1 \geq \left\lceil \frac{4(n+1)^{1+\frac{1}{n}}(1+\ln\frac{n+1}{2})^2}{\epsilon^2} \right\rceil$$

and

$$d_2, \ldots, d_r \geq \left\lceil \frac{2(n-2)(r-1)}{\epsilon} \right\rceil$$

Proof. Indeed, we just need that $Y := X_{d_2,\ldots,d_r}$ which has dimension $n+1$ satisfies

$$\operatorname{covdeg}(Y) \geq (1-\epsilon/2)d_2 \cdots d_r$$

which by the bound

$$\operatorname{covdeg}(Y) \geq (d_2 - n + 2) \cdots (d_r - n + 2)$$

holds as long as each

$$\sum_{i=2}^{r} \left(1 - \frac{n-2}{d_i}\right) \leq 1 - \epsilon/2$$

which holds if each term is less that $1 - \epsilon/2(r-1)$ hence if

$$d_i \geq \frac{2(n-2)(r-1)}{\epsilon}$$

Hence we apply Theorem C with $\alpha = (1-\epsilon/2)d_2 \ldots d_r$ so we evaluate the constant $d_0(n+1, \alpha, \epsilon/2)$ to get the desired bound on d_1. $\square$

In the proof of theorem A the set S_n should be replaced by S_{n+r-1}. Let $n' = n+r-1$ from now on. Should mention that we need $n' \geq 3$ ofc the cases where this is not true (inside ^{3} are completely understood).

Proposition 0.4. *In theorem A we can take* $N(n,r) := 2^{2r^2(n+r)^3}$.

Proof. The proof says we need to take $k = 2nr$ and find an increasing sequence of primes $p_1, \ldots, p_\ell$ so that $p_1 > 2^{2n'}$ with $p_{i+1} \leq 2p_i$ (possible by Bertrand's postulate) so that

$$\left(\binom{n'}{2} - 1\right) p_\ell^{n'} + \left(n'! - \binom{n'}{2}\right) p_\ell^{n'-1} + (2^{n'}+1) \cdot n'! \leq p_1 \cdots p_\ell$$

since $p_1 > 2^{2n}$ and $n' \geq 3$

$$\left(n'! - \binom{n'}{2}\right)[p_\ell^n - p_\ell^{n-1}] > (2^n + 1) \cdot n!$$

so we can take

$$C_{n'} = n'!$$

and the condition is satisfied as long as

$$C_{n'} p_\ell^{n'} \leq p_1 \cdots p_\ell$$

which is satisfied if

$$2^{n'\ell} C_n p_1^{n'} \leq p_1^\ell$$

since $p_{i+1} < 2p_i$. Therefore, we need

$$\left(\frac{p_1}{2^{n'}}\right)^{\ell - n'} \geq C_{n'} 2^{n'^2}$$

since $p_1 > 2^{2n'}$ this is satisfied as long as

$$\ell \geq 2n' + \frac{\log n'!}{n' \log 2}$$

Hence we can set

$$\ell := 3n'$$

Now we form the requisite numbers as follows. Let g_1 be the product of $p_1, \ldots, p_\ell$ and g_2 be the product of the next ℓ primes and so on. We need to do this $2r$ times to win and we need $d \geq g_r \cdot g_{2r}$ and $k \geq g_i$ for all i. The second condition is clearly satisfied because $nr < 2^{n+r}$ for $n, r > 1$. Therefore N can be taken as the product of the first $2r\ell$ primes larger than $2^{2n'}$. Agan by Bertrand's postulate, this is upper bounded by

$$\prod_{i=1}^{2r\ell} 2^{2n'} 2^i \leq 2^{2n'(r\ell+3r)^2} = 2^{2n'r^2(n+r)^2} \leq 2^{2r^2(n+r)^3}$$

$\square$