1 The Ehrhart Polynomial

Theorem 1.0.1 (Ehrhart Polynomial)**.** Let P be an n-dimensionaly lattice polytope in $\mathbb{R}^n$. Then there exists a unique polynomial with rational coefficients $E_P \in \mathbb{Q}[x]$ ssatsfying:

(a) For any integer $\nu \in \mathbb{N}$,
$$E_P(\nu) = \#\left((\nu P) \cap M\right)$$

(b) The leading coefficient of E_P is $\mathrm{Vol}\left(P\right)$ i.e. the volume of P normalized to the lattice cell volume of M.

(c) There is a reciprocity law for positive integers $\nu > 0$,
$$E_P(-\nu) = (-1)^d \#\left(\nu P^\circ \cap M\right)$$

Remark. To prove the power of this theorem, we can easily derive the classical Pick's theorem as a special case.

Theorem 1.0.2 (Pick)**.** Let $n = 2$ and $P \subset \mathbb{R}^2$ be a lattice polygon. Then,

$$\#(P \cap M) = \mathrm{Vol}_M\left(P\right) + \tfrac{1}{2}\#(\partial P \cap M) + 1$$

Proof. Consider the Ehrhart polynomial which takes the form,

$$E_P(x) = \mathrm{Vol}_M\left(P\right) x^2 + Bx + 1$$

Now we can decompose $P = P^\circ \cup \partial P$ which implies that,

$$E_P(1) = \#\left(P \cap M\right) = \#\left(P^\circ \cap M\right) + \#\left(\partial P \cap M\right)$$

Furthermore, by the reciprocity law,

$$E_P(-1) = \#\left(P^\circ \cap M\right)$$

Putting these together, we find,

$$E_P(1) - E_P(-1) = \#\left(\partial P \cap M\right)$$

However, applying the polynomial form,

$$E_P(1) - E_p(-1) = 2B \implies B = \tfrac{1}{2}\#\left(\partial P \cap M\right)$$

Thus the Ehrhart polynomial is,

$$E_P(x) = \mathrm{Vol}_M\left(P\right) x^2 + \tfrac{1}{2}\#\left(\partial P \cap M\right) x + 1$$

Which, for $x = 1$ we find,

$$E_P(1) = \#\left(P \cap M\right) = \mathrm{Vol}_M\left(P\right) + \tfrac{1}{2}\#\left(\partial P \cap M\right) + 1$$

giving Pick's formula. $\square$

2 Construction of the Toric Variety

Let $N = \#(P \cap \mathbb{Z}^n) - 1$. Then consider the map $(\mathbb{C}^\times)^n \hookrightarrow \mathbb{P}^N$ defined by sending,

$$\underline{t} \mapsto [\underline{t}^{\underline{v}_0} : \cdots : \underline{t}^{\underline{v}_N}]$$

for the lattice poitns $\underline{v}_i \in P \cap \mathbb{Z}^n$. Then we define the toric variety X_P to be the closure of the image of this map. The additional points are called the boundary strata and can be shown to also be rational varities of lower dimensions. (Note, if P does not have "enough lattice points" then this construction does not work because the associated divisor is only ample not very ample but this does work replacing P by νP for $\nu \gg 0$ and then mucking around with divisors to return to counting points for the divisor P. Alternatively, one can construct X_P intrinsically from the combinatorial data of P in a way that is manifestly independent of scaling).

Furthermore, notice that the embedding $(\mathbb{C}^\times)^n \hookrightarrow \mathbb{P}^N$ is equivariant for the following action $(\mathbb{C}^\times)^n \circlearrowright \mathbb{P}^N$ given by,

$$\underline{t} \cdot [z_0 : \cdots : z_N] = [\underline{t}^{\underline{v}_0} z_0 : \cdots : \underline{t}^{\underline{v}_N} z_N]$$

Therefore, we get an action $(\mathbb{C}^\times)^n \circlearrowright X_P$ extending the standard left action of the torus $(\mathbb{C}^\times)^n \subset X_P$. Because the embedded torus $(\mathbb{C}^\times)^n \subset X_P$ is a dense open, the function field which is the field of meromorphic functions on X_P, is equal to that of $(\mathbb{C}^\times)^n$ which is,

$$K(X_P) = \mathbb{C}(\chi_1, \dots, \chi_n)$$

These are generated by rational functions $\chi_i : (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ which are the standard characters $\underline{t} \mapsto t_i$. We call functions of the form $\chi^u = \chi_1^{u_1} \cdots \chi_n^{u_n}$ characters because they are exactly the set of group homomorphisms.

We now consider the structure of the boundary strata and how these characters behave at the boundary. Let's consider a polynomial map $\lambda : \mathbb{C}^\times \to (\mathbb{C}^\times)^n$. Let e_i be the maximum exponent of λ of $\chi_i \circ \lambda$. Then we see that under the embedding,

$$\lim_{t \to \infty} \lambda(t) = \lim_{t \to \infty} [c_1 t^{\underline{e} \cdot \underline{v}_1} : \cdots : c_N t^{\underline{e} \cdot \underline{v}_N}]$$

Therefore, after rescaling, the only remaining terms are the maximum values of $\underline{e} \cdot \underline{v}_i$. The indices that show up as maximum values in the direction $\underline{e}$ are the extreme shapes of P. Therefore, we get a correspondence between the strata and the faces of P. Furthermore, consider,

$$\lim_{t \to \infty} \chi^u \circ \lambda(t) = t^{u \cdot \underline{e}}$$

so the order of the pole of χ^u on the boundary strata D_e defined by e is $u \cdot \underline{e}$. In particular, the character χ^u has a pole on the boundary strata defined by a direction $\underline{e}$ if and only if $u \cdot \underline{e} > 0$ and has a zero if and only if $u \cdot \underline{e} < 0$. Note, this really only makes sense for top-dimensional boundary strata (corresponding to facets: top dimensional faces of P) because a zero or pole of a rational function only makes sense on a codimension 1 subset. For example, consider $f(x, y) = \frac{x}{y}$ on $\mathbb{C}^2$. This has a pole on the x-axis and a zero on the y-axis so what is its value at the origin?? Indeed, it has a different limit depending on if you approach the origin along the x or y axis.

Now notice the following. If we write decompose our polytope into half-spaces defined by the facets,

$$P = \bigcap_{F \subset P} H^+(u_F, a_F) \quad \text{where} \quad H^+(u, a) = \{x \in \mathbb{R}^n \mid x \cdot u \leq a\}$$

Then notice that $u \in P \cap \mathbb{Z}^n$ if and only if χ^u has a pole of no worse than order a. Therefore, we should define $\mathcal{L}_P$ to be the line bundle of functions with poles along the strata D_F no worse than order a_F. Because D_F are torus-invariant and the torus acts on the function field we see that $\mathcal{L}_P$ is equivariant for the torus and therefore its space of sections is spanned by eigensections for the torus action which are exactly characters. Therefore, we conclude that,

$$H^0(X_P, \mathcal{L}_P) = \bigoplus_{u \in P \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^u$$

Notice that we can describe any other polytope with the same facet normals as P in a similar way just by chaning the pole behavior on the boundary strata. For a tuple of integers $q = (q_F)$ let,

$$P(q) = \bigcap_{F \subset P} H^+(u_F, q_F)$$

and we associate a line bundle $\mathcal{L}_q = \mathcal{L}_{P(q)}$ which we also write in divisor notation as,

$$\mathcal{L}_q = \mathcal{O}_{X_P}(\sum_{F \subset P} q_F D_F)$$

Then the same argument shows that,

$$H^0(X_P, \mathcal{L}_q) = \bigoplus_{u \in P(q) \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^u$$

Theorem 2.0.1 (Demazure)**.** For $i > 0$ the cohomology,

$$H^i(X_P, \mathcal{L}_{P(q)}) = 0$$

and therefore,

$$\chi(X_P, \mathcal{L}_{P(q)}) = \#(P \cap \mathbb{Z}^n)$$

3 The Proof

Proof. Given the lattice polytope P we have constructed a toric variety X_P with a divisor D_P such that,

$$\chi(X_P, \mathcal{O}_{X_P}(D_P)) = \#(P \cap \mathbb{Z}^n)$$

By the Hirzbruch-Riemann-Roch theorem we have,

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \int_{X_P} \mathrm{ch}(\mathcal{O}_{X_P}(\nu D_P)) \, \mathrm{Td}(\mathcal{T}_{X_P})$$

Recall that the Chern character is,

$$\mathrm{ch}(\mathcal{O}_{X_P}(\nu D_P)) = \exp\left(c_1(\mathcal{O}_{X_P}(\nu D_P))\right) = \sum_{m=0}^{d} \frac{c_1(\mathcal{O}_{X_P}(\nu D_P))^m}{m!}$$

where the sum terminates at $d = \dim X_P$ since higher intersections vanish. Recall that the Chern class c_1 is a homomorphism $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$. Thus, since $\mathcal{O}_{X_P}(\nu D_P) = \mathcal{O}_{X_P}(D_P)^{\otimes \nu}$,

$$\mathrm{ch}(\mathcal{O}_{X_P}(\nu D_P)) = \sum_{m=0}^{d} \frac{c_1(\mathcal{O}_{X_P}(D_P)^{\otimes \nu})^m}{m!} = \sum_{m=0}^{d} c_1(\mathcal{O}_{X_P}(D_P))^m \frac{\nu^m}{m!}$$

Therefore,

$$\begin{aligned}\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) &= \int_{X_P} \left(\sum_{m=0}^{d} c_1(\mathcal{O}_{X_P}(D_P))^m \frac{\nu^m}{m!} \right) \mathrm{Td}(\mathcal{T}_{X_P}) \\ &= \sum_{m=0}^{d} \frac{\nu^m}{m!} \left(\int_{X_P} c_1(\mathcal{O}_{X_P}(D_P))^m \, \mathrm{Td}(\mathcal{T}_{X_P}) \right) = h(\nu)\end{aligned}$$

is a degree at most d polynomial in ν. This implies that for $\nu \in \mathbb{N}$ we have proven there is a polynomial,

$$E_P(\nu) = h(\nu) = \chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \dim_{\mathbb{C}} H^0(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \#\left(\nu P \cap \mathbb{Z}^n\right)$$

Furthermore, since D_P is big and $E_P(m)$ counts sections of $\mathcal{O}_{X_P}(mD_P)$, we know that the leading term must be m^d so $\deg E_P = d$. Writing,

$$E_P(x) = a_n x^n + \cdots + a_0$$

we may isolate the leading coefficient as follows,

$$a_n = \lim_{\nu \to \infty} \frac{E_P(\nu)}{\nu^d} = \lim_{\nu \to \infty} \frac{\#\left(\nu P \cap M\right)}{\nu^d} = \mathrm{Vol}_M\left(P\right)$$

Lastly, to prove the duality property, we apply Serre duality. On X_P, the dualizing sheaf is equal to the canonical sheaf,

$$\omega_{X_P} = \mathcal{O}_{X_P}(-\sum_F D_F)$$

where D_F is the divisor $V(\sigma_F)$ for each facet $F \subset P$. Since X_P is a projective Cohen–Macaulay variety (and thus irreducible over k), Serre duality sates that, for any locally free sheaf $\mathscr{F}$ on X_P,

$$H^i(X_P, \mathscr{F}^\vee) = H^{d-i}(X_P, \mathscr{F} \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})^\vee$$

which, by computing dimensions and reordering, implies that,

$$\chi(X_P, \mathscr{F}^\vee) = (-1)^d \chi(X_P, \mathscr{F} \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})$$

In particular, for $\mathscr{F} = \mathcal{O}_{X_P}(\nu D_P)$ we have,

$$E_P(-\nu) = \chi(X_P, \mathcal{O}_{X_P}(-\nu D_P)) = (-1)^d \chi(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})$$

By the Kodaria vanishing theorem, since νD_P is ample for $\nu > 0$,

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P}) = \dim_{\mathcal{C}} H^0(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P})$$

Now we consider the invertible sheaf,

$$\mathcal{O}_{X_P}(\nu D_P) \otimes_{\mathcal{O}_{X_P}} \omega_{X_P} = \mathcal{O}_{X_P}(\nu D_P - \sum_F D_F) = \mathcal{O}_{X_P}(\sum_F (\nu a_F - 1) D_F)$$

which means we should consider the divisor,

$$D' = \sum_F (\nu a_F - 1) D_F$$

which corresponds to the support function $\psi_{D'}$ satisfying $\psi_{D'}(n_F) = -(\nu a_F - 1)$ (recall that cones $\rho \in \Sigma_P(1)$ correspond to facets $F \subset P$). Therefore, the polytope for the divisor D' is,

$$P_{D'} = \bigcap_{\substack{F \subset P \\ \text{a facet}}} H^+(n_F, \psi_{D'}(n_F)) = \bigcap_{\substack{F \subset P \\ \text{a facet}}} H^+(n_F, 1 - \nu a_F)$$

Recall that,

$$\nu P = \bigcap_{\substack{F \subset P \\ \text{a facet}}} H^+(n_F, -a_F) = \bigcap_{\substack{F \subset P \\ \text{a facet}}} \{x \in M_{\mathbb{R}} \mid \forall F : \langle x, n_F \rangle \geq -\nu a_F\}$$

Therefore, the interior is,

$$\nu P^\circ = \bigcap_{\substack{F \subset P \\ \text{a facet}}} \{x \in M_{\mathbb{R}} \mid \forall F : \langle x, n_F \rangle > -\nu a_F\}$$

Therefore, intersecting with the lattice,

$$\nu P^\circ \cap M = \bigcap_{\substack{F \subset P \\ \text{a facet}}} \{m \in M \mid \forall F : \langle m, n_F \rangle \geq -\nu a_F + 1\} = P_{D'} \cap M$$

because the inner product is integer valued on the lattice so,

$$\langle m, n_F \rangle > -\nu a_F \iff \langle n, n_F \rangle \geq -\nu a_F + 1$$

Thus,

$$E_P(-\nu) = (-1)^d \dim_{\mathcal{C}} H^0(X_P, \mathcal{O}_{X_P}(D')) = (-1)^d \# (P_{D'} \cap M) = (-1)^d \# (\nu P^\circ \cap M)$$

$\square$

Remark. Note that $E_P(0) = \#((0 \cdot P) \cap M) = 1$ so the constant term is 1. Furthermore, in the limit $\nu \to \infty$ if $\dim P = d$ then $E_P(\nu) \in O(\nu^d)$ so $\deg E_P = d$.