

# 1 Lifting and Extensions

## 1.1 Smoothness

**Definition 1.1.1.** We say that a map  $T \rightarrow T'$  is an *order  $n$  infinitesimal thickening (or extension)* if it is a closed immersion whose defining ideal  $\mathcal{I}$  satisfies  $\mathcal{I}^{n+1} = 0$ .

*Remark.* Notice that a zeroth order infinitesimal thickening is an isomorphism. Furthermore, in the affine case, this corresponds to  $A = A'/I$  for an ideal  $I \subset A'$  with  $I^{n+1} = 0$ .

**Definition 1.1.2.** Let  $f : X \rightarrow Y$  be a morphism of schemes. If for any diagram,

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow f \\ T' & \longrightarrow & Y \end{array}$$

with  $T \rightarrow T'$  a first-order infinitesimal thickening of *affine schemes* we say that  $f$  is

- (a) *formally smooth* if there exists at least one dashed arrow
- (b) *formally unramified* if there exists at most one dashed arrow
- (c) *formally étale* if there exists exactly one dashed arrow.

Furthermore, we say that  $f$  is smooth (resp. unramified, resp. étale) if  $f$  is formally smooth (resp. unramified, resp. étale) and locally of finite presentation.

*Remark.* Notice that any order- $n$  infinitesimal thickening  $T \rightarrow T'$  may be factored as,

$$T = T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n = T'$$

where  $T_i$  is the closed subscheme of  $T'$  cut out by  $I^{i+1}$ . Therefore,  $T_i \rightarrow T_{i+1}$  is a closed immersion cut out by  $I^{i+1}/I^{i+2}$  which has zero square and thus is a first-order infinitesimal thickening. Therefore, by repeatedly applying the lifting criteria, we may replace “first-order” in the definition by  $n^{\text{th}}$ -order.

*Remark.* The definition given above appears in the Stacks project. The definition in our text refers to diagrams of (possibly not affine) infinitesimal thickenings,

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow f \\ T' & \longrightarrow & Y \end{array}$$

and asks only about liftings  $T' \rightarrow X$  *Zariski-locally* on  $T'$ . It is clear that Stacks project formal smoothness implies [I] formal smoothness. In fact, both definitions for all three properties agree. Furthermore, if  $f$  is formally étale then the uniqueness of the lift on affines implies gluing so there exists a unique *global* map  $T' \rightarrow X$  for any infinitesimal thickening  $T \rightarrow T'$ . This contrasts the smooth case for which we will construct a global obstruction to the existence of a global lift  $T' \rightarrow X$ .

## 1.2 Extensions

**Definition 1.2.1.** Let  $f : X \rightarrow S$  be an  $S$ -scheme and  $\mathcal{I}$  a quasi-coherent  $\mathcal{O}_X$ -module. A  $S$ -extension of  $X$  by  $\mathcal{I}$  is a  $S$ -morphism  $\iota : X \rightarrow X'$  which is a first-order infinitesimal thickening by an ideal isomorphic to  $\mathcal{I}$  via the data of an  $\mathcal{O}_{X'}$ -module map  $\varphi : \iota_* \mathcal{I} \rightarrow \mathcal{O}_{X'}$ .

*Remark.* If  $\mathcal{I} \subset \mathcal{O}_{X'}$  is the sheaf of ideals corresponding to the thickening  $\iota : X \rightarrow X'$  then  $\mathcal{I}^2 = 0$  so  $\mathcal{I}$  is naturally a  $\mathcal{O}_X = \mathcal{O}_{X'}/\mathcal{I}$ -module.

*Remark.* The situation to have in mind is a  $R$ -algebra  $A$  and an  $A$ -module  $I$ . Then a first-order  $R$ -extension of  $A$  by  $I$  is a map of  $R$ -algebras  $A' \twoheadrightarrow A$  such that,

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

is exact such that the image of  $I \rightarrow A'$  is an ideal with  $I^2 = 0$  in  $A'$ .

*Remark.* We now need a notion of when two extensions are equivalent or more generally the concept of a morphism between them.

**Definition 1.2.2.** A morphism between two  $S$ -extensions of  $X$  by  $\mathcal{I}$ , namely  $\iota_1 : X \rightarrow X'_1$  and  $\iota_2 : X \rightarrow X'_2$  is an  $X$ -morphism  $g : X'_1 \rightarrow X'_2$  meaning that

$$\begin{array}{ccc} X'_1 & \xrightarrow{g} & X'_2 \\ & \swarrow \iota_1 \quad \searrow \iota_2 & \\ & X & \end{array}$$

commutes and such that,

$$\begin{array}{ccc} \iota_1^{-1} \mathcal{O}_{X'_1} & \xleftarrow{g^\#} & \iota_2^{-1} \mathcal{O}_{X'_2} \\ & \swarrow \varphi_1 \quad \searrow \varphi_2 & \\ & \mathcal{I} & \end{array}$$

commutes as a diagram of  $f^{-1} \mathcal{O}_S$ -modules (notice that  $\iota$  is a homeomorphism so we may apply  $\iota_*$  and  $\iota^{-1}$  freely as inverses to get the sheaves on the correct spaces).

*Remark.* In the affine case, this corresponds exactly to an  $R$ -algebra map  $g : A'_1 \rightarrow A'_2$  giving a morphism of exact sequences,

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A'_1 & \longrightarrow & A \longrightarrow 0 \\ & & \parallel & & \downarrow g & & \parallel \\ 0 & \longrightarrow & I & \longrightarrow & A'_2 & \longrightarrow & A \longrightarrow 0 \end{array}$$

Notice, by the 5-lemma,  $g$  is an isomorphism so a morphism of lifts is always an isomorphism.

**Definition 1.2.3.** We say that an extension  $\iota : X \rightarrow X'$  is *split* if there exists a section  $s : X' \rightarrow X$  such that  $s \circ \iota = \text{id}_X$ . In this case, the exact sequence of  $\iota^{-1} \mathcal{O}_{X'}$ -modules,

$$0 \longrightarrow \mathcal{I} \longrightarrow \iota^{-1} \mathcal{O}_{X'} \xrightarrow{s^\#} \mathcal{O}_X \longrightarrow 0$$

is split meaning that,

$$\iota^{-1}\mathcal{O}_{X'} \cong \mathcal{O}_X \oplus \mathcal{I}$$

with the unique  $\mathcal{O}_X$ -algebra structure such that  $\mathcal{I}^2 = 0$ . Therefore, there is a unique split extension up to isomorphism.

*Remark.* The map  $\iota^{-1}\mathcal{O}_{X'} \rightarrow \mathcal{O}_X$  is surjective because  $\iota$  is a closed immersion and a homeomorphism so  $\iota^{-1}$  and  $\iota_*$  are inverse functors.

*Remark.* The split extension in the affine case is given by  $A' = A \oplus I$  with the unique  $A$ -algebra structure such that  $I^2 = 0$ .

**Definition 1.2.4.** We denote the set of isomorphism classes of  $S$ -extensions of  $X$  by  $\mathcal{S}$  as,

$$\text{Ext}_S(X, \mathcal{S})$$

This is a group under ‘‘Bayer sum’’ with the split extension as the identity as we shall soon see.

**Example 1.2.5.** Let  $X \rightarrow X_1 \rightarrow X \times_S X$  be the first infinitesimal neighborhood of the diagonal i.e. if  $\Delta_{X/S} : X \rightarrow X \times_S X$  is cut out by  $\mathcal{I}$  then  $X_1$  is cut out by  $\mathcal{I}^2$ . Then  $\Delta_1 : X \rightarrow X_1$  is a first-order infinitesimal thickening with ideal  $\mathcal{I}/\mathcal{I}^2 \cong \Omega_{X/S}^1$ . Then the two projections  $p_1, p_2 : X_1 \rightarrow X$  split the extension giving two splittings of,

$$0 \longrightarrow \Omega_{X/S} \longrightarrow \mathcal{P}_{X/S} \begin{array}{c} \xleftarrow{j_1} \\ \xrightarrow{j_2} \end{array} \mathcal{O}_X \longrightarrow 0$$

where  $\mathcal{P}_{X/S} = \iota^{-1}\mathcal{O}_{X_1}$  is the sheaf of first principal parts. The two splittings correspond to two  $\mathcal{O}_X$ -module structures on  $\mathcal{P}$  and  $d_{X/S} = j_2 - j_1$  is the universal derivation  $d_{X/S} : \mathcal{O}_X \rightarrow \Omega_{X/S}$ .

**Proposition 1.2.6.** Let  $\iota : X \rightarrow X'$  be an  $S$ -extension by  $\mathcal{I}$ . Then the automorphisms of  $X'$  as a lift are naturally isomorphic to  $\text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{I})$ .

*Proof.* Since  $\iota$  is a homeomorphism any  $S$ -automorphism  $g : X' \rightarrow X'$  over  $X$  must topologically be the identity. Therefore, we just need to classify sheaf maps  $g^\# : \varphi : \iota^{-1}\mathcal{O}_{X'} \rightarrow \iota^{-1}\mathcal{O}_{X'}$  over  $\varphi : \mathcal{I} \rightarrow \iota^{-1}\mathcal{O}_{X'}$ . Thus we consider diagrams,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \iota^{-1}\mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \parallel & & \downarrow g^\# & & \parallel \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \iota^{-1}\mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \end{array}$$

Then  $\theta = g^\# - \text{id}$  is a map  $\iota^{-1}\mathcal{O}_{X'} \rightarrow \mathcal{I}$  but  $\mathcal{I}^2 = 0$  so  $\theta$  factors through an  $S$ -linear derivation  $\mathcal{O}_X \rightarrow \mathcal{I}$  giving an element  $\theta \in \text{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{I}) = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{I})$ .

Conversely, any  $S$ -derivation  $\theta : \mathcal{O}_X \rightarrow \mathcal{I}$  produces an automorphism  $\text{id} + \tilde{\theta} : \iota^{-1}\mathcal{O}_{X'} \rightarrow \iota^{-1}\mathcal{O}_{X'}$  where  $\tilde{\theta}$  is the composite map  $\iota^{-1}\mathcal{O}_{X'} \rightarrow \mathcal{O}_X \xrightarrow{\theta} \mathcal{I}$ .  $\square$

*Remark.* For any extension  $(\iota : X \rightarrow X', \varphi : \mathcal{I} \rightarrow \iota^{-1}\mathcal{O}_{X'})$  the map  $\iota : X \rightarrow X'$  is a closed immersion. Therefore, there is an exact sequence of  $\mathcal{O}_X$ -modules,

$$\mathcal{I} \xrightarrow{\varphi} \iota^*\Omega_{X'/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

coming from the second fundamental sequence and the map  $\varphi : \mathcal{I} \rightarrow \iota^{-1}\mathcal{O}_{X'}$  identifying  $\mathcal{I}$  with the defining ideal such that  $\mathcal{I}^2 = 0$ . If  $f : X \rightarrow S$  is smooth then the sequence is short exact.

**Proposition 1.2.7.** If  $f : X \rightarrow S$  is smooth then the map,

$$\mathrm{Ext}_S(X, \mathcal{I}) \rightarrow \mathrm{Ext}_{\mathcal{O}_X}^1(\Omega_{X/S}, \mathcal{I})$$

defined by sending an extension  $(\iota : X \rightarrow X', \varphi : \mathcal{I} \rightarrow \iota^{-1}\mathcal{O}_{X'})$  to the extension class of,

$$0 \longrightarrow \mathcal{I} \xrightarrow{\varphi} \iota^*\Omega_{X'/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

is a bijection sending the split extension to the trivial extension.

### 1.3 Lifting

**Proposition 1.3.1.** Let  $f : X \rightarrow Y$  be smooth and  $\iota : T \rightarrow T'$  an extension of  $T$  by  $\mathcal{I}$ . Then given a diagram,

$$\begin{array}{ccc} T & \xrightarrow{g} & X \\ \iota \downarrow & \nearrow g' & \downarrow f \\ T' & \longrightarrow & Y \end{array}$$

there exists an obstruction class,

$$c(g_0) \in \mathrm{Ext}_{\mathcal{O}_T}^1(g^*\Omega_{X/Y}, \mathcal{I})$$

to the existence of a  $Y$ -morphism  $g : T \rightarrow X$  extending  $g$ . Furthermore, if  $c(g) = 0$  then the set of extensions  $g'$  of  $g$  is a  $\mathrm{Hom}_{\mathcal{O}_T}(g^*\Omega_{X/Y}, \mathcal{I})$ -torsor.

*Proof.* Consider the sheaf of sets  $\mathcal{F}$  on  $T$  of local lifts,

$$U \mapsto \{g' \in \mathrm{Hom}_Y(U', X) \mid g \circ \iota = g_0\}$$

Since  $\iota$  is a homeomorphism, opens of  $T$  and  $T'$  agree but they have different scheme structures. Let  $\mathcal{G} = \mathcal{H}om_{\mathcal{O}_T}(g^*\Omega_{X/Y}, \mathcal{I})$ . Given  $s \in \mathcal{G}(U)$  and  $g' \in \mathcal{F}(U)$  we can form  $g' + s \in \mathcal{F}(U)$  as follows. For simplicity of notation we take  $U = T$  since the construction works functorially for all  $T$  with a fixed  $\mathcal{I}$ -extension. Since  $\iota$  is a homeomorphism, maps  $g'$  are determined by sheaf maps  $g^{-1}\mathcal{O}_X \rightarrow \iota^{-1}\mathcal{O}_{T'}$ . Thus consider,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \iota^{-1}\mathcal{O}_{T'} & \longrightarrow & \mathcal{O}_T \longrightarrow 0 \\ & & & & \nwarrow & & \uparrow \\ & & & & & & g^{-1}\mathcal{O}_X \end{array}$$

For two dashed maps  $g_1, g_2$  the difference  $D = g_2 - g_1 : g^{-1}\mathcal{O}_X \rightarrow \mathcal{I}$  is a  $g^{-1}f^{-1}\mathcal{O}_Y$ -derivation because  $\mathcal{I}^2 = 0$  and likewise to any  $g$  we may add an  $\mathcal{I}$ -valued derivation and retain an algebra morphism. Therefore  $\mathcal{F}$  is a pseudo-torsor (possibly non-split) over,

$$\begin{aligned} \mathcal{D}er_{g^{-1}f^{-1}\mathcal{O}_Y}(g^{-1}\mathcal{O}_X, \mathcal{I}) &= \mathcal{H}om_{g^{-1}\mathcal{O}_X}(\Omega_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_Y}, \mathcal{I}) \\ &= \mathcal{H}om_{g^{-1}\mathcal{O}_X}(g^{-1}\Omega_{X/Y}, \mathcal{I}) = \mathcal{H}om_{\mathcal{O}_T}(g^*\Omega_{X/Y}, \mathcal{I}) = \mathcal{G} \end{aligned}$$

(See [Tag 08RR](#) if this makes you uncomfortable). Since  $f$  is smooth, lifts exist Zariski locally so  $\mathcal{F}$  is a  $\mathcal{G}$ -torsor and thus it corresponds to a class,

$$c(g) \in H^1(T, \mathcal{G}) = H^1(T, \mathcal{H}om_{\mathcal{O}_T}(g^*\Omega_{X/Y}, \mathcal{I})) = \text{Ext}_{\mathcal{O}_T}^1(g^*\Omega_{X/Y}, \mathcal{I})$$

which is zero iff  $\mathcal{F}$  is trivial iff  $\mathcal{F}$  has a global section. Furthermore, if  $\mathcal{F}$  is trivial then  $\Gamma(T, \mathcal{F})$  is an affine space over  $\Gamma(X, \mathcal{G}) = \text{Hom}_{\mathcal{O}_T}(g^*\Omega_{X/Y}, \mathcal{I})$ .  $\square$

**Definition 1.3.2.** Let  $\iota : Y \rightarrow Y'$  be a first-order infinitesimal thickening and  $f : X \rightarrow Y$  a  $Y$ -scheme. A *lift* of  $X$  over  $Y'$  is a  $Y'$ -scheme  $f' : X' \rightarrow Y'$  and an isomorphism  $\varphi : X' \times_{Y'} Y \xrightarrow{\sim} X$ . A morphism between lifts  $f'_1 : X'_1 \rightarrow Y'$  and  $f'_2 : X'_2 \rightarrow Y'$  is a  $Y'$ -morphism  $g : X'_1 \rightarrow X'_2$  such that,

$$\begin{array}{ccc} X'_1 \times_{Y'} Y & \xrightarrow{g \times \text{id}} & X'_2 \times_{Y'} Y \\ & \swarrow \varphi_1 & \searrow \varphi_2 \\ & X & \end{array}$$

commutes.

**Proposition 1.3.3.** Assume that  $f : X \rightarrow Y$  is smooth and  $\iota : Y \hookrightarrow Y'$  has ideal  $\mathcal{I}$ . Then,

(a) there exists an obstruction,

$$\omega(f) \in \text{Ext}_{\mathcal{O}_X}^2(\Omega_{X/Y}, f^*\mathcal{I})$$

to the existence of a smooth lift of  $X$  over  $Y'$

(b) If  $\omega(f) = 0$  then the set of isomorphism classes of smooth lifts is a torsor over,

$$\text{Ext}_{\mathcal{O}_X}^1(\Omega_{X/Y}, f^*\mathcal{I})$$

(c) If  $f' : X' \rightarrow Y'$  is a smooth lift of  $X$  then the group of automorphisms of  $f'$  is naturally,

$$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, f^*\mathcal{I})$$

*Remark.* Since  $f : X \rightarrow Y$  is smooth, then  $\Omega_{X/Y}$  is locally free and therefore,

$$\text{Ext}_{\mathcal{O}_X}^i(\Omega_{X/Y}, f^*\mathcal{I}) = H^i(X, \mathcal{T}_{X/Y} \otimes_{\mathcal{O}_X} f^*\mathcal{I})$$

*Proof.* First, we consider shrinking  $X$  until it is affine and it maps to affines so we have  $Y = \text{Spec}(B)$  and  $X = \text{Spec}(A)$  and  $Y' = \text{Spec}(B')$  where  $B = B'/I$  and  $I^2 = 0$ . We need to show that there exists a unique lift over  $A$  and that the group of automorphisms of this lift is  $\text{Hom}_A(\Omega_{A/B}, I \otimes_B A)$ .

Suppose that  $A'$  is a smooth lift meaning  $A' \otimes_{B'} B = A$ . Applying  $-\otimes_{B'} A'$  to the sequence,

$$0 \longrightarrow I \longrightarrow B' \longrightarrow B \longrightarrow 0$$

we get an exact sequence because  $A'$  is flat over  $B'$ ,

$$0 \longrightarrow I \otimes_{B'} A' \longrightarrow A' \longrightarrow A' \otimes_{B'} B \longrightarrow 0$$

then using that  $A' \otimes_{B'} B = A$  and that  $I$  is naturally a  $B$ -module because  $I^2 = 0$  we get,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \otimes_B A & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\
& & \uparrow & & \uparrow f' & & \uparrow f & & \\
0 & \longrightarrow & I & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

identifying  $I_A = \ker(A' \rightarrow A) = I \otimes_B A$  which is great because it is fixed by the given data. The only unknown is the  $A'$  that fill in the diagram.

First we consider automorphisms of  $A'$  preserving the diagram. If  $\varphi : A' \rightarrow A'$  is a ring automorphism then  $\varphi - \text{id} : A' \rightarrow I_A$  is a  $B'$ -module map because  $\varphi - \text{id}$  projects to zero in  $A$ . Because  $I^2 = 0$  it is easy to check that  $\tilde{D} = \varphi - \text{id}$  is a derivation. Moreover, any  $B'$ -derivation kills  $I_A$  because  $I_A = IA'$  and  $D(ia) = iD(a) \in I^2 A' = 0$  so it factors through a  $B$ -derivation  $D : A \rightarrow I_A$  (since  $A'/I_A = A$ ). Conversely, given a  $B$ -derivation  $D : A \rightarrow I_A$  we produce a  $B'$ -map,

$$\tilde{D} : A' \rightarrow A \xrightarrow{D} I_A \rightarrow A'$$

and a direct calculation shows that  $\varphi = \text{id} + \tilde{D}$  is a  $B$ -algebra automorphism making the diagram commute (because  $D$  lands in  $I_A = \ker(A' \rightarrow A)$ ). Therefore,

$$\text{Aut}_B(A'/A) = \text{Der}_B(A, I_A) = \text{Hom}_A(\Omega_{A/B}, I \otimes_B A)$$

Next we show the uniqueness of lifts. Suppose that  $A'_1$  and  $A'_2$  are two smooth lifts of  $A$ . Then,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \otimes_B A & \longrightarrow & A'_2 & \longrightarrow & A & \longrightarrow & 0 \\
& & \nearrow & \uparrow & \nearrow & & \uparrow & & \\
0 & \longrightarrow & I \otimes_B A & \longrightarrow & A'_1 & \longrightarrow & A & \longrightarrow & 0 \\
& & \uparrow & \downarrow & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\
& & \uparrow & \nearrow & \nearrow & & \uparrow & & \\
0 & \longrightarrow & I & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

commutes giving a commutative square,

$$\begin{array}{ccc}
A & \longleftarrow & A'_2 \\
\uparrow & \nwarrow & \uparrow \\
A'_1 & \longleftarrow & B'
\end{array}$$

where the lift  $A'_1 \rightarrow A'_2$  exists because  $B' \rightarrow A'_2$  is smooth and  $A'_1 \rightarrow A$  is an infinitesimal extension by  $I'$ . Applying the 5-lemma to the top of the preceding diagram we see that  $A'_1 \rightarrow A'_2$  is an isomorphism proving uniqueness of the lift.

Finally, we need to consider existence. It turns out it will be easiest to consider what seems like a more complication problem: lifting closed subschemes inside an ambient space that is endowed with a fixed lift. In our case, the ambient space will be affine space over  $B$  (since  $A$  is a finitely presented  $B$ -algebra) which is easy to lift (since  $B'[x_1, \dots, x_n]$  is obviously a lift of  $B[x_1, \dots, x_n]$ ) so we only need to show that we can lift smooth subschemes of affine space. Hartshorne's deformation theory chapter 2 considers this problem in detail and proves existence in much more generality. I will sketch Hartshorne's argument [H, Thm. 9.2] which applies for local complete intersections. (LOOK AT SEAN'S REFERENCE HERE!!)

Let  $P \twoheadrightarrow A$  be the ambient embedding (in our case  $P = B[x_1, \dots, x_n]$ ) and  $P'$  a fixed lift of  $P$  over  $B'$  (in our case  $P' = B'[x_1, \dots, x_n]$ ). Then we need to find  $A'$  that fit into a diagram,

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I \otimes_B J & \longrightarrow & J' & \longrightarrow & J \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I \otimes_B P & \longrightarrow & P' & \longrightarrow & P \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I \otimes_B A & \longrightarrow & A' & \longrightarrow & A \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where the bottom two rows are exact because  $A'$  and  $P'$  are flat over  $B'$  and the leftmost column is exact because  $A$  is flat over  $B$  and the other two columns are exact by definition so the top row is also exact by the 9-lemma (c.f. [H, Thm. 6.2]). Let  $J = (f_1, \dots, f_r)$  and we define  $J'$  (and thus  $A'$ ) by lifting  $f'_i \in P'$  to give an ideal  $J' = (f'_1, \dots, f'_r)$ . Because  $A$  is a local complete intersection in  $P$  the Koszul complex  $K_\bullet(P; f_1, \dots, f_r)$  is exact and forms a free resolution of  $A$ . Since  $P'$  is flat over  $B'$ , tensoring the complex of free  $P'$ -modules  $K_\bullet(P'; f'_1, \dots, f'_r)$  over the sequence,

$$0 \longrightarrow I \longrightarrow B' \longrightarrow B \longrightarrow 0$$

we get an exact sequences of complexes,

$$0 \longrightarrow I \otimes_B K_\bullet(P; f_1, \dots, f_r) \longrightarrow K_\bullet(P'; f'_1, \dots, f'_r) \longrightarrow K_\bullet(P; f_1, \dots, f_r) \longrightarrow 0$$

again using that  $I$  is a  $B$ -module. Furthermore,  $K_\bullet(P; f_1, \dots, f_r) \otimes_B I$  remains exact because  $P$  is flat over  $B$ . Therefore, taking the long exact sequence on cohomology shows that  $K_\bullet(P'; f'_1, \dots, f'_r)$  is exact in positive degrees and that their quotients form an exact sequence,

$$0 \longrightarrow I \otimes_B A \longrightarrow A' \longrightarrow A \longrightarrow 0$$

so  $A'$  fits into the above diagram and moreover is flat over  $B'$  (use [H Prop. 2.2]) and therefore smooth because its fibers over  $B'$  are equal to the fibers of  $A$  over  $B$  (the ideal  $I$  is nilpotent so  $B = B'/I$  has the same points and residue fields) which are smooth. Therefore, we have produced a lift of  $A$  inside the lifted ambient space  $P'$  giving our desired lift of  $A$ .

Now we do the general case. As before, we would like to consider a “sheaf of lifts” over open of  $X$  but this does not make sense because lifts have automorphisms. Indeed, we actually have a stack of lifts  $\mathcal{X}$  over  $X_{\text{Zar}}$ . Explicitly, the objects of  $\mathcal{X}$  are smooth lifts  $U'$  of an open  $U \subset X$  over  $Y$  and morphisms are morphisms of  $Y$ -schemes  $\varphi : U' \rightarrow V'$  such that  $U' \times_{Y'} Y \rightarrow V' \times_{Y'} Y$  is identified with the open inclusion  $U \hookrightarrow V$ . This is a fibered category over  $X_{\text{Zar}}$ . An affine local argument shows that every map in  $\mathcal{X}$  is an open immersion and that maps over  $\text{id} : U \rightarrow U$  are isomorphisms so  $\mathcal{X}$  is fibered in groupoids. Morphisms glue because an open cover of  $U$  will pull back a lift  $U'$  of  $U$  to an open cover of  $U'$ . Furthermore, descent is effective because descent data for a cover  $\{U_i \rightarrow U\}$  is exactly gluing data on opens for lifts  $U'_i$  over each  $U_i$  that thus glue to a lift  $U'$  over  $U$ . Let  $\mathcal{G} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, f^* \mathcal{I})$ . A global version of the automorphisms argument shows that  $\mathcal{G}$  acts on the objects of  $\mathcal{X}$  precisely giving an action  $B\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}$ . What we showed is that on affine

opens  $U \subset X$ , the stack  $\mathcal{X}|_U \cong B\mathcal{G}$  because it is connected with automorphism group  $\mathcal{G}$  and on affines  $\mathcal{G}$ -torsors are all trivial (since  $\mathcal{G}$  is quasi-coherent).

Therefore  $\mathcal{X}$  is a  $\mathcal{G}$ -gerbe which corresponds to a class  $[\mathcal{X}] \in H^2(X, \mathcal{G})$  which is zero if and only if  $\mathcal{X}$  is the trivial gerbe if and only if  $\mathcal{X}$  “admits a global section” meaning  $\mathcal{X}(X)$  is nonempty i.e. there is a global lift. Furthermore, if  $\mathcal{X}$  is trivial then  $\mathcal{X} \cong B\mathcal{G}$  so the groupoid  $\mathcal{X}(X)$  is exactly the category of  $\mathcal{G}$ -torsors with isomorphisms meaning that  $H^1(X, B)$  classifies global lifts<sup>1</sup> and the automorphism group of any global lift is  $H^0(X, \mathcal{G})$  as noted earlier.

This can be described more prosaically if  $Y$  is separated. Given an open affine cover  $\{U_i\}$  of  $X$  and lifts  $U'_i$  of each  $U_i$  then pulling back to the double intersections  $U'_{ij} = U'_i|_{U_{ij}}$  (which are affine by separatedness) are lifts over  $U_{ij}$ . We showed that any two lifts over an affine scheme are isomorphic so we can fix isomorphism  $\varphi_{ij} : U'_{ij} \rightarrow U'_{ji}$ . Then on triple overlaps  $U_{ijk}$  (which are affine by separatedness) there is a cocycle automorphism,

$$u_{ijk} = \varphi_{ik}^{-1} \circ \varphi_{jk} \circ \varphi_{ij}$$

of  $U'_i|_{U_{ijk}}$ . Then, as previously,

$$c_{ijk} = u_{ijk} - \text{id} \in \mathcal{G}(U_{ijk})$$

and  $(c_{ijk})$  defines a 2-cocycle for  $\mathcal{G}$  and therefore defines an obstruction class  $[c] \in H^2(X, \mathcal{G})$  to the lifts  $\{U'_i\}$  gluing to a global lift  $X'$ . Of course, this must be checked to be a 2-cocycle independent of our choices up to the addition of a 2-coboundary and that the isomorphisms  $\varphi_{ij}$  can be modified, possibly on a refinement of the cover  $\{U_i\}$ , such that the cocycle vanishes and gluing goes through if and only if  $(c_{ijk})$  is a coboundary but we will leave the discussion here.  $\square$

*Remark.* When affine locally flat lifts of subschemes exist inside a lifted ambient space there is an obstruction to global lifting given by an  $H^1$ -class of a twisted normal bundle. It seems strange that obstructions to lifting subschemes live in  $H^1$  where as obstructions to lifting schemes without an ambient space live in  $H^2$  until we remember that automorphisms played a central part in the above argument. Notice that subschemes have no automorphisms (as subschemes) and therefore the lifts of subschemes actually form a sheaf (rather than a stack) which is a torsor (rather than a gerbe) over some twisted normal bundle and thus the obstruction to a global lift (corresponding to a global section of the torsor which trivializes it) is an  $H^1$ -class corresponding to the torsor.

**Example 1.3.4.** If  $X$  is affine then,

$$H^i(X, \mathcal{T}_{X/Y} \otimes_{\mathcal{O}_X} f^* \mathcal{I}) = 0$$

for all  $i > 0$  and thus smooth lifts always exist and are unique as we demonstrated in the proof.

**Example 1.3.5.** If  $f : X \rightarrow Y$  is étale then  $\Omega_{X/Y} = 0$ . Thus for any first-order thickening  $\iota : Y \rightarrow Y'$  there is a unique smooth lift  $X'$  of  $X$  over  $Y'$  and  $X'$  has no nontrivial automorphisms.

**Example 1.3.6.** If  $f : X \rightarrow Y$  is a family of smooth curves over a zero dimensional base then,

$$H^2(X, \mathcal{T}_{X/Y} \otimes_{\mathcal{O}_X} f^* \mathcal{I}) = 0$$

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<sup>1</sup>Lifts form an affine space over  $H^1(X, \mathcal{G})$  meaning it only classifies lifts up to choosing a base point or equivalently up to choosing an isomorphism  $\mathcal{X} \cong B\mathcal{G}$  which is why the identification is not canonical unlike the case for torsors where 0 corresponds to the trivial torsor (there is no corresponding trivial lift).



because  $\dim X = 1$  so curves over infinitesimal schemes always admit liftings. In the case,

$$Y = \operatorname{Spec}(k) \rightarrow \operatorname{Spec}(R) = Y'$$

is an infinitesimal extension for some Artin local ring  $R$  with residue field  $k$  and maximal ideal  $\mathfrak{m} \subset R$  with  $\mathfrak{m}^2 = 0$  (e.g.  $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z}) \rightarrow \operatorname{Spec}(\mathbb{Z}/p^2\mathbb{Z})$  or  $\operatorname{Spec}(k) \rightarrow \operatorname{Spec}(k[\epsilon])$ ) then we see that  $\mathfrak{m}$  is a  $k$ -vectorspace of dimension  $r$  so  $\mathcal{I} \cong k^{\oplus r}$  and thus,

$$H^1(X, \mathcal{T}_{X/k} \otimes_{\mathcal{O}_X} f^* \mathcal{I}) = H^1(X, \mathcal{T}_{X/Y}^{\oplus r}) = H^0(X, (\omega_{X/Y}^{\otimes 2})^{\oplus r}) \cong \begin{cases} 0 & g = 0 \\ k^{\oplus r} & g = 1 \\ k^{3(g-1)r} & g \geq 1 \end{cases}$$

In particular, for  $r = 1$ , this gives the expected dimension of the moduli space  $\mathcal{M}_g$  of smooth genus  $g$  curves:  $3(g-1)$ . This makes sense because the tangent space  $T_{[C]} \mathcal{M}_g$  should correspond to smooth infinitesimal deformations of a curve  $C/k$  (i.e. smooth lifts of  $C/k$  over  $k[\epsilon]$ ) by the moduli functor description since  $\ker(\mathcal{M}_g(k[\epsilon]) \rightarrow \mathcal{M}_g(k))$  should classify smooth curves over  $k[\epsilon]$  with a fixed pullback to a smooth curve over  $k$  on the closed point.

**Example 1.3.7.** When the extension  $\iota : Y \rightarrow Y'$  splits then the obstruction class always vanishes  $\omega(f) = 0$  because we can form a lift by pulling back along the section  $s : Y' \rightarrow Y$ . This happens, for example, with  $D = k[\epsilon]$  and the split extension,

$$0 \longrightarrow k \xrightarrow{\epsilon} D \longrightarrow k \longrightarrow 0$$

Therefore, lifts over  $D$  always exist and are classified (in the case that  $X \rightarrow \operatorname{Spec}(k)$  is smooth) by,

$$\operatorname{Ext}_{\mathcal{O}_X}^1(\Omega_{X/k}, f^* \underline{k}) = H^1(X, \mathcal{T}_{X/Y})$$

Furthermore, the automorphism group over any lift of  $X$  over  $D$  is naturally isomorphic to,

$$\operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, f^* \underline{k}) = H^0(X, \mathcal{T}_{X/k})$$

which identifies tangent fields with “infinitesimal automorphisms of  $X$ ” meaning  $\operatorname{Aut}_X(X \times_k D)$ .