1 Small Contractions

We want to study the structure of birational maps $f : X \to Y$. From experience with smooth varities we expect the exceptional locus to be a divisor. First, we define the exceptional locus.

Definition 1.0.1. Let $f : X \to Y$ be a birational map of varieties. Then there exists a largest open $U \subset Y$ such that $f : f^{-1}(U) \to U$ is an isomorphism. Then the *exceptional locus* if the closed subscheme,

$$\mathrm{Ex}\,(f) = X \setminus f^{-1}(U)$$

Proposition 1.0.2 (Kollar-Mori, Cor. 2.63)**.** If $f : X \to Y$ is birational where X is projective and Y is $\mathbb{Q}$-factorial then $\mathrm{Ex}\,(f)$ is pure codimension 1.

Proof. Heuristically, f is projective hence is a blowup at some ideal $\mathscr{I} \subset \mathcal{O}_Y$. Using the methods [Hartshorne, Ex. 7.11] (which only requires that Y has $\mathbb{Q}$-factorial singularities) we modify $\mathscr{I}$ such that it has support equal to $Y \setminus U$ where U is the largest open over which f is an isomorphism. Therefore, $\mathrm{Ex}\,(f)$ is the total transform of $V(\mathscr{I})$ which is a Cartier divisor by the definition of blowing up. □

Example 1.0.3. Let $Y = \mathrm{Spec}\,(k[x, y, z, w]/(xy - zw))$ be the affine cone over a the quadric surface $Q = \mathrm{Proj}\,(k[x, y, z, w]/(xy - zw))$. Thus Y which has an isolated singularity at the origin which is not $\mathbb{Q}$-factorial. Indeed, consider the prime divisor,

$$D = V(x, z)$$

Then I claim that nD is never Cartier for $n \neq 0$. Indeed, the vanishing of each coordinate function x, y, z, w contains components.

Set $\widetilde{X} = \mathrm{Bl}_0 Y$ which has exceptional fiber Q. We can blow down along the two rulings to get two smooth 3-folds,

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & X^+ \arrow[dl] \\ & Y & \end{tikzcd}$$

These can be described as the blowups along $I = (x, z)$ and $I^+ = (x, w)$. Since the exceptional of $\widetilde{X} \to Q$ is codimension 1 then by contracting Q to a curve on X and X^+ we see that these blowups $X \to Y$ and $X^+ \to Y$ has codimension 2 exceptional divisors.

Let's compute this in coordinates. By symmetry, it suffices to consider $X \to Y$ which is the blowup of $I = (x, z)$. Then,

$$\mathrm{Bl}_I(A) = A[u, v]/(uz - vx, uy - vw)$$

Then we get two charts for X,

$$U_0 = \mathrm{Spec}\left(A[\tfrac{u}{v}]/(\tfrac{u}{v}z - x, \tfrac{u}{v}y - w)\right) = \mathrm{Spec}\left(k[y, z, \tfrac{u}{v}]\right)$$
$$U_1 = \mathrm{Spec}\left(A[\tfrac{v}{u}]/(z - \tfrac{v}{u}x, y - \tfrac{v}{u}w)\right) = \mathrm{Spec}\left(k[x, w, \tfrac{v}{u}]\right)$$

so we indeed see that X is smooth (in fact it is locally affine space). The fiber over I is,

$$f^{-1}(V(I)) = \text{Proj}\,(k[y,w][u,v]/(uy - vw))$$

However, this is *not* the exceptional locus since I is invertible on $Y \setminus \{0\}$. Indeed, the exceptional locus is exactly over the origin since these are blowdowns of $\widetilde{X}$. Then the exceptional locus is,

$$E = \text{Proj}\,(\text{Bl}_I(A)/\mathfrak{m}\text{Bl}_I(A)) = \text{Proj}\,(k[u,v])$$

which is a copy of $\mathbb{P}^1$.

Definition 1.0.4. A *small contraction* is a birational map $f : X \to Y$ with $\text{codim}\,(\text{Ex}\,(f)\,, X) \geq 2$.

Remark. We have seen if Y is not $\mathbb{Q}$-Cartier there are often small contractions. However, this does not always happen. For example, if Y is the projective cone over a degree d plane curve, this is normal and projective but not $\mathbb{Q}$-factorial. However, there is no small contraction over Y. Indeed, since $\dim Y = 2$, such a small contraction would have zero dimensional exceptional locus. However, Y is normal so any birational map has connected fibers. Thus, we see that small contractions are a dimension ≥ 3 phenomenon.

Example 1.0.5. Consider the 1-forms,

$$\begin{array}{lll}
\omega = \mathrm{d}x & U_0 : \frac{u}{v}\mathrm{d}z + z\mathrm{d}\frac{u}{v} & U_1 : \mathrm{d}z \\
\omega = \mathrm{d}y & U_0 : \mathrm{d}y & U_1 : \frac{v}{u}\mathrm{d}w + w\mathrm{d}\frac{v}{u} \\
\omega = \mathrm{d}z & U_0 : \mathrm{d}z & U_1 : \frac{v}{u}\mathrm{d}x + x\mathrm{d}\frac{v}{u} \\
\omega = \mathrm{d}w & U_0 : \frac{u}{v}\mathrm{d}y + y\mathrm{d}\frac{u}{v} & U_1 : \mathrm{d}w
\end{array}$$

So we see that for each of these 1-forms, one vanishes on exactly one but not both of the flops. Moreover, the form $\omega = \mathrm{d}\frac{u}{v}$ flops to the rational form $-\left(\frac{v}{u}\right)^{-2}\mathrm{d}\frac{v}{u}$.

2 Main Results of BCHM

Definition 2.0.1. (See Definition 3.1.1 of BCHM for all definitions) Let $\pi : X \to U$ be a projective morphism of quasi-projective varities and D a $\mathbb{R}$-Cartier divisor on X. We say that

(a) D is π-big if $D|_F$ is big on a general fiber F of π (equivalently $D \sim_{\mathbb{R},U} A + B$ for A ample over U and $B \geq 0$)

(b) D is π-nef if $D \cdot C \geq 0$ for all curves C contained in a fiber

(c) D is π-pseudo-effective if $D|_F$ is pseudo-effective for the generic fiber

Remark. If $\pi : X' \to X$ is projective and birational then any proper divisor $D \subset X'$ (if X' is irreducible, otherwise we need that D does not contain any component) is π-big and π-pseudo-effective (since it is zero on the generic fiber) but usually not π-nef.

Theorem 2.0.2. Let (X, Δ) be a klt pair, where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi : X \to U$ be a projective morphism of quasi-projective varieties. If either Δ is π-big and Δ is π-pseudo-effective or $K_X + \Delta$ is π-big, then

(a) $K_X + \Delta$ has a log terminal model over U,

(b) if $K_X + \Delta$ is π-big then $K_X + \Delta$ has a log canonical model over U, and

(c) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then the $\mathcal{O}_U$-algebra

$$\mathcal{R}(\pi, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}\mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely fenerated.

3 MMP Learning Seminar Week 6

Definition 3.0.1. $\varphi : X \to W$ is a *flipping contraction* if (X, Δ) is klt $\mathbb{Q}$-factorial $\rho(X/W) = 1$ and φ is a small birational contraction and $-(K_X + \Delta)$ is ample over W.

Remark. W is never $\mathbb{Q}$-factoral and K_W is not $\mathbb{Q}$-Cartier!

Definition 3.0.2. Let $\varphi : X \to W$ be a flipping contraction. We say that $\pi : X \dashrightarrow X'$ is a *flip* if it is a small birational map $K_{X^+} + \Delta^+$ is $\mathbb{Q}$-Cartier where $\Delta^+ = \pi_* \Delta$. There is a projective morphism $\varphi^+ : X^+ \to W$ so that $K_{X^+} + \Delta^+$ is ample over W.

Lemma 3.0.3. Let $f : X \dashrightarrow Y$ a small birational map between normal varieties. D a Weil divisor then

$$H^0(X, \mathcal{O}_X(D)) = H^0(Y, \mathcal{O}_Y(f_* D))$$

Proof. Indeed, by Harthog and isomorphism in codimension 1. $\square$

Lemma 3.0.4. Let $\varphi : X \to W$ be a flipping contraction of (X, Δ) and $\pi : X \dashrightarrow X'$ a flip. Then $\rho(X) = \rho(X^+)$ and X^+ is $\mathbb{Q}$-factorial.

Remark. When we work with log pairs (X, Δ) we assume $-(K_X + \Delta)$ is φ-ample for a flipping contraction.

Proof. Consider D^+ on X^+ and corresponding D on X by isomorphism in codim 1. Find r such that $R \cdot (D + r(K_X + \Delta)) = 0$ here R is the extremal ray defining the flipping contraction. We know X is $\mathbb{Q}$-factorial hence $m(D + r(K_X + \Delta))$ is Cariter for m big so we can descent it to D_W a Cartier divisor on W. Then

$$mD^+ = m\pi_* D \sim (\varphi^+)^* D_W - (mr)(K_{X^+} + \Delta^+)$$

which is Cartier. For equality of ρ, we use that π is an isomorphism in codimension 1 so it is injective and surjective on divisors. $\square$

Lemma 3.0.5. Let $f : X \to Y$ is a projective contraction between normal varieties with $\rho(X/Y) = 1$. Assume $\operatorname{Exc} \varphi$ contains a divisor. Then φ s the contraction of a unique irreducible divisor.

Proof. Let's say $\operatorname{Exc} \varphi$ has two divisors E_1, E_2. Then we can find C_i covering E_i with $C_i \cdot E_i < 0$ (what does this have to do with Picard rank?). Furthermore, E_1, E_2 are numerically dependent over Y so $E_1 + aE_2 \equiv_Y 0$. Assume $C_1 \cdot E_2$ then

$$C_1 \cdot (E_1 + aE_2) = C_1 \cdot E_1 < 0$$

but C_1 is contracted so this is impossible. Thus choosing C_1 general we get $C_1 \cdot E_2 > 0$. Thus

$$a = -\frac{C_1 \cdot E_1}{C_1 \cdot E_2} > 0$$

Thus $E = E_1 + aE_2$ is an effective divisor which is contracted so it must be covered by E-negative curves contradicting that it is numerically trivial. Therefore, there is at most 1 irreducible divisor in $\operatorname{Exc}\varphi$. Suppose $\operatorname{Exc}\varphi$ contains another component W. We can find a curve $C \subset W$ intersecting E properly (since the fibers are connected) but then $E \cdot C > 0$ contradicting the fact that all contracted curves are numerically equivalent since we also have negative curves on E since it is contracted. $\square$

This is called a divisorial contraction, it contracts an irreducible divisor to a higher-codimension locus.

Proposition 3.0.6. Let $\varphi : X \to W$ be a flipping contraction for (X, Δ) klt. The flip exists iff

$$R = \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(m(K_X + \Delta))$$

is a fg $\mathcal{O}_W$-algebra. If this is the case then

$$X^+ = \mathbf{Proj}_W \left(\bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(m(K_X + \Delta)) \right)$$

Proof. Assume the flip

$$\begin{tikzcd} X \arrow[rr, dashed, "\pi"] \arrow[dr] & & X^+ \arrow[dl] \\ & W & \end{tikzcd}$$

exists. Then π is small so

$$R = \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(m(K_X + \Delta)) = \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_{X^+}(m(K_{X^+} + \Delta^+))$$

Moreover, $K_{X^+} + \Delta^+$ is ample over W hence R is finitely generated and hence the Proj equals X^+ over W.

Assume R is finitely generated and define

$$X^+ = \mathbf{Proj}_W (R)$$

The natural map

$$\pi : X \dashrightarrow X^+$$

is an isomorphism in codimension 1 on X because away from the flipping locus $K_X + \Delta$ is ample and hence the map is an isomorphism. We need to show the same is true of the inverse. It could happen that there exists $E \subset X^+$ contracted to X. Then we would have E contracted under $\varphi^+ : E \to W$ so

$$\varphi^+_* \mathcal{O}_{X^+}(1) = \varphi_* \mathcal{O}_X(m(K_X + \Delta)) = \mathcal{O}_W(m(K_W + \varphi_* \Delta))$$

so

$$\mathcal{O}_W(tm(K_W + \varphi_* \Delta) = \varphi^+_* \mathcal{O}_{X^+}(t) \subsetneq \varphi^+_* \mathcal{O}_{X^+}(t)(E)$$

however we have a natural inclusion

$$\varphi^+_* \mathcal{O}_{X^+}(t)(E) \hookrightarrow \mathcal{O}_W(tm(K_W + \varphi_* \Delta))$$

because it is contracted. Thus φ^+ is small. Then by Lemma 2 they have the same Picard rank. The same argument shows that $\rho(X/W) = \rho(X^+/W)$. $\square$

4 MMP Learning Seminar Week 9

Let X be a $\mathbb{Q}$-factorial terminal projective 3-fold. Suppose we have a flipping contraction $f : X \to W$ then we want

$$\begin{tikzcd} X \arrow[rr, dashed, "\pi"] \arrow[dr] & & X^+ \arrow[dl] \\ & W & \end{tikzcd}$$

we know

(a) $\rho(X/W) = 1$

(b) $-K_X$ is f-ample

(c) X is smooth in codim 2

What we want is a birational modification π which is an isomorphism in codim 1 such that

(a) $\rho(X^+/W) = 1$

(b) K_{X^+} is ample over W

(c) X^+ has $\mathbb{Q}$-factorial terminal singularities

Lemma 4.0.1. Such X^+ is unique and equals

$$\mathbf{Proj}_W \left(\bigoplus_{n \geq 0} f_* \mathcal{O}_X(nK_X) \right)$$

provided that this is a fg $\mathcal{O}_W$-algebra.

Proposition 4.0.2.

$$R(X) = \bigoplus_{n \geq 0} f_8 \mathcal{O}_X(nK_X)$$

is fg as a $\mathcal{O}_W$-algebra iff it is fg locally (even analytically) over W.

Because X is terminal, it has isolated singularities.

Mori 1988: proved that these curves can be contracted one by one in the analytic sense (you can't algebraically since they are numerically equivalent).

Terminal 3-fold flipping contractions implies we should study extremal neighborhoods.