# Nonvanishing 1-forms on varieties admitting a good minimal model.

#### December 5, 2024

#### 1 Talk

We start with a theorem of Popa and Schnell that I will state via the contrapositive of the ususal fashion:

**Theorem 1.0.1** (Popa-Schnell '14). If X is a smooth projective variety carrying a 1-form  $\omega \in H^0(X, \Omega_X)$  with no zeros then  $\kappa(X) \leq n-1$ .

This shows that having a 1-form with no zeros constrains the geometry of X. However, we expect there should be a much more stringent restriction on those varieties with  $\kappa(X) \leq n-1$  that actually do carry a 1-form with no zeros.

**Question:** How do you produce 1-forms, they always arises by pulling back along a map  $f: X \to A$  (say to the Albanese). The 1-form will have no zeros if f is smooth. So we might guess that every nonvanishing 1-form arises from the pullback along a smooth map to an abelian variety.

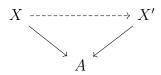
**Example 1.0.2.** This is not true: let  $X = E \times C$  where C is any curve of genus  $g \geq 2$  such that E is not an isogeny factor of  $\operatorname{Jac}(C)$ . Then the only smooth map to an abelian variety is  $f: X \to E$ . However  $\pi_1 \omega_E + \pi_2 \omega_C$  are all nonvanishing for any nonzero  $\omega_E$  and any  $\omega_C$ . The only one pulled back from f are of the form  $\omega_E$ . Therefore, we have to be careful. It seems that having a nonvanishing 1-form  $\omega$  implied that some smooth map to an abelian variety exists but  $\omega$  may not be pulled back along it. Indeed, you have to deform  $\omega$  (by taking  $\omega_C \to 0$  in this case) to get it as a pullback from a smooth map.

However, this is still not enough.

**Example 1.0.3.** Let  $E_1, E_2$  are nonisogenous elliptic curves. Let X be the blowup of  $E_1 \times E_2 \times \mathbb{P}^1$  along  $E_1 \times \{0\} \times \{0\}$  and  $\{0\} \times E_2 \times \{\infty\}$ . Then the pullback of  $\pi_1 \omega_1 + \pi_2 \omega_2$  to X has no zeros. However, there is no "diagonal map" to an elliptic curve since  $E_1, E_2$  are not isogenous. Indeed, the only smooth maps to abelian varieties are (up to composition with an isogeny) the projections  $X \to E_i$  and both are not smooth since they have reducible fiber along the exceptional.

Therefore the best we could do is the following conjecture of Hao and Schreieder:

Conjecture 1.0.4 (Hao-Schreieder '21, A). Let X be a smooth projective variety and  $\omega \in H^0(X, \Omega_X)$  a 1-form with no zeros. Then there is a diagram,

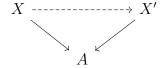


where  $X \xrightarrow{--} X'$  is a birational modification and  $X' \to A$  is a smooth map to an abelian variety.

Furthermore, they conjecture that when  $\kappa(X) \geq 0$  we can choose  $X' \to A$  to be isotrivial

Remark. When I say "isotrivial" I mean the stronger assumption than  $X \to Y$  is an analytic / étale fiber bundle, I mean it is trivial by a *finite* étale cover  $Y' \to Y$ . This is always true for constant families of curves of genus  $g \ge 1$  over a regular base. But it already fails for smooth conic bundles over a surface (e.g. any nontrivial Brauer class on a K3).

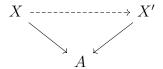
Conjecture 1.0.5 (Hao-Schreieder '21, B). With the assumptions as above, if moreover,  $\kappa(X) \geq 0$  then there is a diagram



where X oup X' is a birational modification and  $X' \to A$  is a smooth isotrivial map (meaning it is an analytic fiber bundle and moreover is trivialized by an isogeny  $A' \to A$ ).

It turns out our work will also have applications to the case where, instead of assuming there is a nonvanishing 1-form, we assume that we are given a map  $f: X \to A$  that is close to smooth.

**Conjecture 1.0.6** (Meng-Popa '21, C). Let  $f: X \to A$  be an algebraic fiber space, with X a smooth projective variety and  $\kappa(X) \ge 0$  (equivalently by their work  $\kappa(F) \ge 0$  for the general fiber). If f is smooth away from codimension 2 in A then there there is a birational modification



so that  $X' \to A$  is a smooth isotrivial fiber bundle. Equivalently  $X \to A$  is birationally trivialized after an isogeny  $A' \to A$ .

From the example, we can see that we had to blow up to make a birational modification necessary. Therefore, Nathan, Hao, and I conjectured that:

Conjecture 1.0.7 (Chen-C-Hao '23, D). If X has a nonvanishing 1-form and moreover X is minimal then there is a smooth isotrivial map  $X \to A$ .

**Theorem 1.0.8** (C '24). These conjectures hold under the assumption that X admits a good minimal model (exists  $X \dashrightarrow X'$  such that  $K_{X'}$  is semiample, in particular we must have  $\kappa(X) \ge 0$ ).

Corollary 1.0.9. Suppose  $\kappa(X) \geq 0$  if moveover one of

- (a)  $\dim X \kappa(X) \le 4$
- (b)  $f: X \to Alb_X$  has generic fiber (over its image) of dimesnion  $\leq 3$

then the conjectures hold.

Notice that because we had to assume  $\kappa(X) \geq 0$  to get a minimal model, our theorem says nothing about Conjecture A when X is uniruled. Our main technical theorem partially rectifies this issue.

**Theorem 1.0.10.** Let X be a smooth projective variety equipped with a map  $f: X \to A$  to an abelian variety satisfying and there are 1-forms  $\omega_1, \ldots, \omega_g \in H^0(A, \Omega_A)$  such that  $f^*\omega_1, \ldots, f^*\omega_g$  are independent pointwise. Assume the base Y of the MRC fibration  $X \dashrightarrow Y$  admits a good minimal model. Then there exists a quotient with connected kernel  $q: A \to B$  to an abelian variety B of dimension  $\geq g$  and a birational map  $Y \dashrightarrow Z \times^G B'$  making the diagram

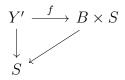
$$X \xrightarrow{f} X \xrightarrow{f} A \qquad \downarrow^{q} \qquad \downarrow^{q} \qquad Z \times^{G} B' \longrightarrow B$$

commute. Here,  $B' \to B$  is an isogeny with kernel G, and Z is a smooth projective variety with a G-action.

#### 1.1 Proof of the Main Result

There are two main ingredients in the proof. Here is a sketch of the argument:

- (a) the Iitaka fiber  $F \to Y \dashrightarrow S$  has image in A an abelian variety B of dimension  $\geq g$
- (b) choose a good minimal model Y' of Y then the Iitaka fibration

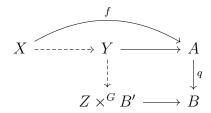


over the locus  $U \subset S$  where the fibers are at worst klt is a (weak) Calabi-Yau fibration. In this case prove that:  $Y'_U \cong Z \times^G B'$  for an isogeny  $B' \to B$  with kernel G where  $Z = f^{-1}(0) \cap Y'_U$ 

(c) choose a G-equivariant smooth compactification  $Z \hookrightarrow \overline{Z}$  thus  $Y'_U$  is an open set of  $\overline{Z} \times^G B'$  which is a smooth variety with an obvious smooth isotrivial map

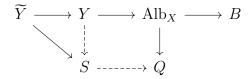
$$\overline{Z} \times^G B' \to B$$

hence giving our diagram



### 1.2 Step (a)

Suppose that Y itself admits g poinwise independent 1-forms. Consider the following diagram,



where  $Q = \operatorname{coker}(\operatorname{Alb}_F \to A)$  (recall that since  $\kappa(F) = 0$  Kawamata proves that  $F \to \operatorname{Alb}_F$ ) since  $\widetilde{Y} \to B$  contracts the general fiber of  $\widetilde{Y} \to S$  by definition, rigidity shows that there is a rational map  $S \dashrightarrow Q$ . Hence the map  $Y \to Q$  factors birationally through the litaka fibration. Now we use the full power of Popa-Schnell

**Theorem 1.2.1** (PS '14). Let  $f: X \to A$  be a morphism from a smooth projective variety to an abelian variety. If  $H^0(X, \omega_X^{\otimes n} \otimes f^* \mathcal{L}^{-1}) \neq 0$  for some ample  $\mathcal{L} \in \text{Pic }(A)$  and some n > 0 then every  $Z(f^*\omega) \neq \emptyset$  for all  $\omega \in H^0(A, \Omega_A)$ .

Let  $W \subset H^0(A, \Omega_A)$  be spanned by the  $\omega_1, \ldots, \omega_g$ . Then the above theoem shows that  $W \cap H^0(Q, \Omega_Q) = \{0\}$  so dim  $Q + g \leq \dim A$  proving the claim.

But since X oup Y is a rational map, its' not actually clear that  $\omega_1, \ldots, \omega_g$  are independent everywhere on Y. We need a slight improvement of PS14.

**Theorem 1.2.2.** Let  $f: X \to A$  be in  $\mathbf{Var}_A$ . Consider the sheaf of k-forms killed by  $- \wedge f^*\omega$  for all  $\omega \in H^0(A, \Omega_A)$ 

$$P\Omega_X^k := \ker (\Omega_X^k \to \Omega_X^{k+1} \otimes H^0(A, \Omega_A^{\vee}))$$

Suppose there is a line bundle  $\mathcal{N} \hookrightarrow P\Omega_X^k$  and an ample  $\mathcal{L} \in \text{Pic}(()A)$  so that  $H^0(X, \mathcal{N}^{\otimes d} \otimes f^*\mathcal{L}^{-1}) \neq 0$  for some  $d \geq 1$ . Then f does not satisfy  $(*)_1$  i.e. every  $\omega \in H^0(A, \Omega_A)$  has nonempty  $Z(f^*\omega) \neq \emptyset$ .

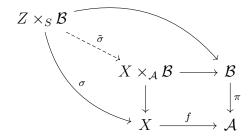
I claim that  $X \to Q$  will satisfy the assumption of this result. Indeed,  $m^*\omega_Y \hookrightarrow P\Omega_X^{\dim Y}$  and is positive for  $Y \to Q$  because this factors through Iitaka.

## 1.3 Step (b)

**Theorem 1.3.1.** Let  $g:(X,\Delta)\to S$  be a flat projective family of pairs over a locally noetherian reduced base scheme S of pure characteristic zero whose fibers satisfy

- (a)  $(X_s, \Delta_s)$  are klt pairs (in particular the fibers are integral with  $K_{X_s} + \Delta_s$  a  $\mathbb{Q}$ -Cartier divisor)
- (b)  $K_{X_s} + \Delta_{X_s} \equiv_{\text{num}} 0$

equipped with a surjective S-morphism  $g: X \to \mathcal{A}$  where  $\mathcal{A} \to S$  is a polarized abelian scheme. Let  $Z = f^{-1}(0_A)$ . Then there is an isogeny  $\pi: \mathcal{B} \to \mathcal{A}$  such that in the diagram



the unique map  $\tilde{\sigma}: Z \times \mathcal{B} \xrightarrow{\sim} X \times_{\mathcal{A}} \mathcal{B}$  induced by the action is an isomorphism. Hence there is an S-isomorphism  $X \cong Z \times_S^G \mathcal{B}$  where  $G = \ker (\mathcal{B} \to \mathcal{A})$ .