Nonvanishing 1-forms on varieties admitting a good minimal model.

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1 Talk

We start with a theorem of Popa and Schnell that I will state via the contrapositive of the ususal fashion:

Theorem 1.0.1 (Popa-Schnell '14)**.** If X is a smooth projective variety carrying a 1-form $\omega \in H^0(X, \Omega_X)$ with no zeros then $\kappa(X) \leq n-1$.

This shows that having a 1-form with no zeros constrains the geometry of X. However, we expect there should be a much more stringent restriction on those varieties with $\kappa(X) \leq n-1$ that actually do carry a 1-form with no zeros.

Question: How do you produce 1-forms, they always arises by pulling back along a map $f : X \to A$ (say to the Albanese). The 1-form will have no zeros if f is smooth. So we might guess that every nonvanishing 1-form arises from the pullback along a smooth map to an abelian variety.

Example 1.0.2. This is not true: let $X = E \times C$ where C is any curve of genus $g \geq 2$ such that E is not an isogeny factor of $\mathrm{Jac}\,(C)$. Then the only smooth map to an abelian variety is $f : X \to E$. However $\pi_1\omega_E + \pi_2\omega_C$ are all nonvanishing for any nonzero ω_E and any ω_C. The only one pulled back from f are of the form ω_E. Therefore, we have to be careful. It seems that having a nonvanishing 1-form ω implied that some smooth map to an abelian variety exists but ω may not be pulled back along it. Indeed, you have to deform ω (by taking $\omega_C \to 0$ in this case) to get it as a pullback from a smooth map.

However, this is still not enough.

Example 1.0.3. Let E_1, E_2 are nonisogenous elliptic curves. Let X be the blowup of $E_1 \times E_2 \times \mathbb{P}^1$ along $E_1 \times \{0\} \times \{0\}$ and $\{0\} \times E_2 \times \{\infty\}$. Then the pullback of $\pi_1\omega_1 + \pi_2\omega_2$ to X has no zeros. However, there is no "diagonal map" to an elliptic curve since E_1, E_2 are not isogenous. Indeed, the only smooth maps to abelian varieties are (up to composition with an isogeny) the projections $X \to E_i$ and both are not smooth since they have reducible fiber along the exceptional.

Therefore the best we could do is the following conjecture of Hao and Schreieder:

Conjecture 1.0.4 (Hao-Schreieder '21, A)**.** Let X be a smooth projective variety and $\omega \in H^0(X, \Omega_X)$ a 1-form with no zeros. Then there is a diagram,

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & A & \end{tikzcd}$$

where $X \dashrightarrow X'$ is a birational modification and $X' \to A$ is a smooth map to an abelian variety.

Furthermore, they conjecture that when $\kappa(X) \geq 0$ we can choose $X' \to A$ to be isotrivial

Remark. When I say "isotrivial" I mean the stronger asusmption than $X \to Y$ is an analytic / étale fiber bundle, I mean it is trivial by a *finite* étale cover $Y' \to Y$. This is always true for constant families of curves of genus $g \geq 1$ over a regular base. But it already fails for smooth conic bundles over a surface (e.g. any nontrivial Brauer class on a K3).

Conjecture 1.0.5 (Hao-Schreieder '21, B)**.** *With the assumptions as above, if moreover, $\kappa(X) \geq 0$ then there is a diagram*

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & A & \end{tikzcd}$$

where $X \dashrightarrow X'$ is a birational modification and $X' \to A$ is a smooth isotrivial map (meaning it is an analytic fiber bundle and moreover is trivialized by an isogeny $A' \to A$).

It turns out our work will also have applications to the case where, instead of asuming there is a nonvanishing 1-form, we assume that we are given a map $f : X \to A$ that is close to smooth.

Conjecture 1.0.6 (Meng-Popa '21, C)**.** *Let $f : X \to A$ be an algebraic fiber space, with X a smooth projective variety and $\kappa(X) \geq 0$ (equivalently by their work $\kappa(F) \geq 0$ for the general fiber). If f is smooth away from codimension 2 in A then there there is a birational modification*

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & A & \end{tikzcd}$$

so that $X' \to A$ is a smooth isotrivial fiber bundle. Equivalently $X \to A$ is birationally trivialized after an isogeny $A' \to A$.

From the example, we can see that we had to blow up to make a birational modification necessary. Therefore, Nathan, Hao, and I conjectured that:

Conjecture 1.0.7 (Chen-C-Hao '23, D)**.** *If X has a nonvanishing 1-form and moreover X is minimal then there is a smooth isotrivial map $X \to A$.*

Theorem 1.0.8 (C '24)**.** *These conjectures hold under the assumption that X admits a good minimal model (exists $X \dashrightarrow X'$ such that $K_{X'}$ is semiample, in particular we must have $\kappa(X) \geq 0$).*

Corollary 1.0.9. *Suppose $\kappa(X) \geq 0$ if moveover one of*

(a) $\dim X - \kappa(X) \leq 4$

(b) $f : X \to \mathrm{Alb}_X$ has generic fiber (over its image) of dimesnion ≤ 3

then the conjectures hold.

Notice that because we had to assume $\kappa(X) \geq 0$ to get a minimal model, our theorem says nothing about Conjecture A when X is uniruled. Our main technical theorem partially rectifies this issue.

Theorem 1.0.10. Let X be a smooth projective variety equipped with a map $f : X \to A$ to an abelian variety satisfying and there are 1-forms $\omega_1, \ldots, \omega_g \in H^0(A, \Omega_A)$ such that $f^*\omega_1, \ldots, f^*\omega_g$ are independent pointwise. Assume the base Y of the MRC fibration $X \dashrightarrow Y$ admits a good minimal model. Then there exists a quotient with connected kernel $q : A \to B$ to an abelian variety B of dimension $\geq g$ and a birational map $Y \dashrightarrow Z \times^G B'$ making the diagram

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[rr, bend left, "f"] & Y \arrow[r] \arrow[d, dashed] & A \arrow[d, "q"] \\ & Z \times^G B' \arrow[r] & B \end{tikzcd}$$

commute. Here, $B' \to B$ is an isogeny with kernel G, and Z is a smooth projective variety with a G-action.

1.1 Proof of the Main Result

There are two main ingredients in the proof. Here is a sketch of the argument:

(a) the Iitaka fiber $F \to Y \dashrightarrow S$ has image in A an abelian variety B of dimension $\geq g$

(b) choose a good minimal model Y' of Y then the Iitaka fibration

$$\begin{tikzcd} Y' \arrow[r, "f"] \arrow[d] & B \times S \arrow[dl] \\ S & \end{tikzcd}$$

over the locus $U \subset S$ where the fibers are at worst klt is a (weak) Calabi-Yau fibration. In this case prove that: $Y'_U \cong Z \times^G B'$ for an isogeny $B' \to B$ with kernel G where $Z = f^{-1}(0) \cap Y'_U$

(c) choose a G-equivariant smooth compactification $Z \hookrightarrow \overline{Z}$ thus Y'_U is an open set of $\overline{Z} \times^G B'$ which is a smooth variety with an obvious smooth isotrivial map

$$\overline{Z} \times^G B' \to B$$

hence giving our diagram

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[rr, bend left, "f"] & Y \arrow[r] \arrow[d, dashed] & A \arrow[d, "q"] \\ & Z \times^G B' \arrow[r] & B \end{tikzcd}$$

1.2 Step (a)

Suppose that Y itself admits g poinwise independent 1-forms. Consider the following diagram,

$$\begin{tikzcd} \widetilde{Y} \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d, dashed] & \mathrm{Alb}_X \arrow[r] \arrow[d] & B \\ & S \arrow[r, dashed] & Q & \end{tikzcd}$$

where $Q = \mathrm{coker}\,(\mathrm{Alb}_F \to A)$ (recall that since $\kappa(F) = 0$ Kawamata proves that $F \twoheadrightarrow \mathrm{Alb}_F$) since $\widetilde{Y} \to B$ contracts the general fiber of $\widetilde{Y} \to S$ by definition, rigidity shows that there is a rational map $S \dashrightarrow Q$. Hence the map $Y \to Q$ factors birationally through the Iitaka fibration. Now we use the full power of Popa-Schnell

Theorem 1.2.1 (PS '14)**.** Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. If $H^0(X, \omega_X^{\otimes n} \otimes f^*\mathcal{L}^{-1}) \neq 0$ for some ample $\mathcal{L} \in \mathrm{Pic}\,(A)$ and some $n > 0$ then every $Z(f^*\omega) \neq \varnothing$ for all $\omega \in H^0(A, \Omega_A)$.

Let $W \subset H^0(A, \Omega_A)$ be spanned by the $\omega_1, \dots, \omega_g$. Then the above theoem shows that $W \cap H^0(Q, \Omega_Q) = \{0\}$ so $\dim Q + g \le \dim A$ proving the claim.

But since $X \dashrightarrow Y$ is a rational map, its' not actually clear that $\omega_1, \dots, \omega_g$ are independent everywhere on Y. We need a slight improvement of PS14.

Theorem 1.2.2. Let $f : X \to A$ be in $\mathbf{Var}_A$. Consider the sheaf of k-forms killed by $- \wedge f^*\omega$ for all $\omega \in H^0(A, \Omega_A)$

$$P\Omega_X^k := \ker\,(\Omega_X^k \to \Omega_X^{k+1} \otimes H^0(A, \Omega_A^\vee))$$

Suppose there is a line bundle $\mathcal{N} \hookrightarrow P\Omega_X^k$ and an ample $\mathcal{L} \in \mathrm{Pic}\,(()\,A)$ so that $H^0(X, \mathcal{N}^{\otimes d} \otimes f^*\mathcal{L}^{-1}) \neq 0$ for some $d \ge 1$. Then f does not satisfy $(*)_1$ i.e. every $\omega \in H^0(A, \Omega_A)$ has nonempty $Z(f^*\omega) \neq \varnothing$.

I claim that $X \to Q$ will satisfy the assumption of this result. Indeed, $m^*\omega_Y \hookrightarrow P\Omega_X^{\dim Y}$ and is positive for $Y \to Q$ because this factors through Iitaka.

1.3 Step (b)

Theorem 1.3.1. Let $g : (X, \Delta) \to S$ be a flat projective family of pairs over a locally noetherian reduced base scheme S of pure characteristic zero whose fibers satisfy

(a) (X_s, Δ_s) are klt pairs (in particular the fibers are integral with $K_{X_s} + \Delta_s$ a $\mathbb{Q}$-Cartier divisor)

(b) $K_{X_s} + \Delta_{X_s} \equiv_{\mathrm{num}} 0$

equipped with a surjective S-morphism $g : X \to \mathcal{A}$ where $\mathcal{A} \to S$ is a polarized abelian scheme. Let $Z = f^{-1}(0_A)$. Then there is an isogeny $\pi : \mathcal{B} \to \mathcal{A}$ such that in the diagram

$$\begin{tikzcd} Z \times_S \mathcal{B} \arrow[dr, dashed, "\tilde{\sigma}"] \arrow[drr, bend left] \arrow[ddr, bend right, "\sigma"'] & & \\ & X \times_{\mathcal{A}} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "\pi"] \\ & X \arrow[r, "f"] & \mathcal{A} \end{tikzcd}$$

the unique map $\tilde{\sigma} : Z \times \mathcal{B} \xrightarrow{\sim} X \times_{\mathcal{A}} \mathcal{B}$ induced by the action is an isomorphism. Hence there is an S-isomorphism $X \cong Z \times^G_S \mathcal{B}$ where $G = \ker(\mathcal{B} \to \mathcal{A})$.