1 Kodaira Vanishing

Theorem 1.0.1. Let k be a field of characteristic 0 and X is smooth projective of pure dimension d over k. Let $\mathcal{L}$ be an ample line bundle. Then,

(a) $H^j(X, \mathcal{L} \otimes \Omega_X^i) = 0$ if $i + j > d$

(b) $H^j(X, \mathcal{L}^\vee \otimes \Omega^i) = 0$ if $i + j < d$.

Remark. These two statements are Serre dual. Indeed, there is a perfect pairing

$$\bigwedge^i \Omega \times \bigwedge^{d-i} \Omega \to \bigwedge^d \Omega = \omega_X$$

and therefore,

$$H^j(X, \mathcal{L} \otimes \Omega_X^i) = H^{d-j}(X, \mathcal{L}^\vee \otimes (\Omega_X^i)^\vee \otimes \omega)^\vee = H^{d-j}(X, \mathcal{L}^\vee \otimes \Omega_X^{d-i})^\vee$$

and $(d - j) + (d - i) = 2d - (i + j) < d \iff i + j > d$.

Remark. In order to prove this theorem, we will deduce it from a positive characteristic version.

Theorem 1.0.2. Suppose that k has $\operatorname{char} k = p$. If X is smooth and projective over k pure of dimension d with $d < p$ and X lifts (smoothly) over $W_2(k)$ then,

(a) $H^j(X, \mathcal{L} \otimes \Omega_X^i) = 0$ if $i + j > d$

(b) $H^j(X, \mathcal{L}^\vee \otimes \Omega_X^i) = 0$ if $i + j < d$.

Remark. Because these are equivalent by Serre duality, it suffices to prove the second statement.

Remark. Our first case comes from the following classic result of Serre.

Theorem 1.0.3. If X is projective over k and $\mathcal{L}$ is ample for any coherent sheaf $\mathcal{E}$ there exists n_0 such that for $n \geq n_0$ then,

$$H^i(X, \mathcal{E} \otimes \mathcal{L}^{\otimes n}) = 0$$

for all $i > 0$.

Remark. We apply this to $\mathcal{E} = \Omega_X^{d-i}$ then for $n \geq n_0$ we have,

$$H^j(X, \mathcal{L}^{\otimes -n} \otimes \Omega_X^i) = H^{d-j}(X, \mathcal{L}^{\otimes n} \otimes \Omega_X^{d-i})^\vee = 0$$

for all $j < d$.

Proof of Thm. 1.0.2. In particular, we can choose some power $n = p^m$ such that $n \geq n_0$ and thus,

$$H^j(X, (\mathcal{L}^\vee)^{\otimes p^m} \otimes \Omega_X^i) = 0$$

for all $j < d$ and thus also whenever $i + j < d$. Therefore, we can apply descending induction to prove that,

$$H^j(X, \mathcal{L}^\vee \otimes \Omega_X^i) = 0$$

for all $i + j < d$ by applying the following results. $\square$

1.1 The Induction

Proposition 1.1.1. Let $\mathcal{M}$ be any invertible sheaf. Suppose that,

$$H^j(X, \mathcal{M}^{\otimes p} \otimes \Omega^i_X) = 0$$

for all $i + j < d$ then,

$$H^j(X, \mathcal{M} \otimes \Omega^i_X) = 0$$

for all $i + j < d$.

Remark. Let F_X denote the absolute Frobenius $F_X : X \to X$ and $F : X \to X^{(p)}$ the relative Frobenius.

Lemma 1.1.2. For any invertible module $\mathcal{M}$,

$$F_X^* \mathcal{M} \xrightarrow{\sim} \mathcal{M}^{\otimes p}$$

Proof. Consider the map $\mathcal{M} \to (F_X)_* \mathcal{M}^{\otimes p}$ via $m \mapsto m^{\otimes p}$ which is linear because,

$$am \mapsto (am)^p = a^p m^p = a \cdot m^p$$

because this is $(F_X)_* \mathcal{M}^{\otimes p}$. Then by adjunction, we get a map $F_X^* \mathcal{M} \to \mathcal{M}^{\otimes p}$ via $m \otimes r \mapsto m^{\otimes p} r$ which is well-defined because,

$$(am) \otimes r = m \otimes a^p r \mapsto m^{\otimes p} a^p r = (am)^{\otimes p} r$$

Then it suffices to check for the case $\mathcal{M} = \mathcal{O}_X$ in which case we get $\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X$ by $1 \otimes r \mapsto r$. $\square$

Corollary 1.1.3. Let $\mathcal{M}'$ be the pullback of $\mathcal{M}$ under $\pi : X^{(p)} \to X$. Then $F^* \mathcal{M}' = \mathcal{M}^{\otimes p}$.

Proof of induction (Prop. 1.1.1). By the projection formula,

$$F_*(\mathcal{M}^{\otimes p} \otimes \Omega^i_X) \cong F_*(F^* \mathcal{M}' \otimes \Omega^i) \cong \mathcal{M}' \otimes F_* \Omega^i_X$$

Now we apply the hypercohomlogy spectral sequence,

$$E_1^{ij} = R^j T(K^i) \implies R^{i+j} T(K^\bullet)$$

Then we apply this to the above complex with $T = \Gamma(X^{(p)}, -)$ giving,

$$E_1^{ij} = H^j(X^{(p)}, \mathcal{M}' \otimes F_* \Omega^i_X) \implies \mathbb{H}^{i+j}(X^{(p)}, \mathcal{M}' \otimes F_* \Omega^\bullet_X)$$

However,

$$H^j(X^{(p)}, \mathcal{M}' \otimes F_* \Omega^i_X) = H^j(X^{(p)}, F_*(\mathcal{M}^{\otimes p} \otimes \Omega^i_X)) = H^j(X, \mathcal{M}^{\otimes p} \otimes \Omega^i_X) = 0$$

for $i + j < d$ by the induction hypothesis. Therefore, we conclude from the spectral sequence,

$$\mathbb{H}^n(X^{(p)}, \mathcal{M}' \otimes F_* \Omega^\bullet_X) = 0$$

for $n < d$. Now we need to recall the Cartier isomorphism and decomposability in positive characteristic to complete the proof. $\square$

Proposition 1.1.4. The complex $F_* \Omega_X^\bullet$ is decomposable meaning there is a quasi-isomorphism,

$$F_* \Omega_X^\bullet \xrightarrow{\sim} \bigoplus_i \mathcal{H}^i(F_* \Omega_X^\bullet)[-i]$$

Then from the Cartier isomorphism,

$$\gamma : \mathcal{H}^i(F_* \Omega_X^\bullet) \to \Omega_{X^{(p)}}^\bullet$$

we get a quasi quasi-isomorphism,

$$F_* \Omega_X^\bullet \xrightarrow{\sim} \bigoplus_i \Omega_{X^{(p)}}^i[-i]$$

Completing the proof of induction (Prop. 1.1.1). Then the hypercohomlogy of,

$$\mathcal{M}' \otimes F_* \Omega_X^\bullet \xrightarrow{\sim} \bigoplus_i \mathcal{M}' \otimes \Omega_X^i[-i]$$

is given by,

$$\mathbb{H}^n(X^{(p)}, \mathcal{M}' \otimes F_* \Omega_X^\bullet) = \bigoplus_{i+j=n} H^j(X^{(p)}, \mathcal{M}' \otimes \Omega_{X^{(p)}}^i)$$

and thus by vanihsing of the hypercohomology for $n < d$ we get vanishing,

$$H^j(X^{(p)}, \mathcal{M}' \otimes \Omega_{X^{(p)}}^i) = 0$$

for $i + j < d$. However, in general, for a Cartesian diagram,

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & Y' \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

we get a natural isomorphism $g'^* \Omega_{X/Y} = \Omega_{X'/Y'}$. Applying this to $\pi : X^{(p)} \to X$ over $F_S :$ $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k)$ we get $\pi^* \Omega_{X/k} = \Omega_{X^{(p)}/k}$ (where this is $X^{(p)} \to \mathrm{Spec}\,(k)$ is the structure map not composed with F_S i.e. meaning that π is *not* k-linear). Then we have

$$\mathcal{M}' \otimes \Omega_{X^{(p)}}^i = \pi^*(\mathcal{M} \otimes \Omega_X^i).$$

However, F_S is flat because k is a field so π is also flat by preservation under base change. Applying flat base change,

$$F_S^* H^j(X, \mathcal{M} \otimes \Omega_X^i) = H^j(X^{(p)}, \pi^*(\mathcal{M} \otimes \Omega_X^i)) = H^j(X^{(p)}, \mathcal{M}' \otimes \Omega_{X^{(p)}}^i) = 0$$

for $i + j < d$ thus completing the induction. ☐

1.2 Spreading Out

Proposition 1.2.1. Ampleness spreads out. Meaning given L ample on X/k then we can spread out to $\mathfrak{X}/S = \mathrm{Spec}\,(A)$ then we can spread out to $\mathcal{L}$ ample on $\mathfrak{X}/S$.

Proof. WLOG can assume that $\mathcal{L}$ is very ample. Then spread out the closed embedding to a closed embedding into projective space. ☐

Remark. Now we finally prove the main theorem.

Theorem 1.2.2. Let K be a field of characteristic 0 and X is smooth projective of pure dimension d over K. Let L be an ample line bundle. Then,

(a) $H^j(X, L \otimes \Omega^i_X) = 0$ if $i + j > d$

(b) $H^j(X, L^\vee \otimes \Omega^i) = 0$ if $i + j < d$.

Proof. Recall that by Serre duality we need only prove the second statement.

We consider,

$$K = \varinjlim A$$

where A runs over finite-type $\mathbb{Z}$-algerbas. Thus we can spread out over a smooth $S = \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(\mathbb{Z})$ to give a smooth, projective, finite type $f : \mathfrak{X} \to S$ pure of relative dimension d and $\mathcal{L}$ is an ample invertible sheaf on $\mathfrak{X}$. Then by restricting S we can assume that $R^j f_*(\mathcal{L} \otimes \Omega^i_{\mathfrak{X}/S})$ are all free of constant rank (via semicontinuity and cohomlogy and base change). Then we choose a point $s_0 : \mathrm{Spec}\,(k) \to S$ such that $d < \mathrm{char}(k)$ and now by smoothness of S over $\mathbb{Z}$ the point s_0 lifts to $g : \mathrm{Spec}\,(W_2(k)) \to S$. Then $g^* \mathfrak{X}$ gives a lift of $\mathfrak{X}_{s_0}$ over $W_2(k)$ and therefore we have proved that,

$$H^i(\mathfrak{X}_{s_0}, \mathcal{L}^\vee_{s_0} \otimes \Omega^i_{\mathfrak{X}_{s_0}/k}) = 0$$

for all $i + j < d$. However, by cohomology and base change, for any point $s \in S$,

$$H^j(X, \mathcal{L}^\vee_s \otimes \Omega^i_{\mathfrak{X}_s/\kappa(s)}) = (R^j f_*(\mathcal{L}^\vee \otimes \Omega^i_{\mathfrak{X}/S}))_s \otimes \kappa(s)$$

However, because the pushforwards $R^j f_*(\mathcal{L}^\vee \otimes \Omega^i_{\mathfrak{X}/S})$ are locally free of constant rank and thus the cohomology has constant dimension in s. Taking $s = s_0$ we see that this dimension is zero so,

$$R^j f_*(\mathcal{L}^\vee \otimes \Omega^i_{\mathfrak{X}/S}) = 0$$

In particular, taking the fiber over the point $\xi : \mathrm{Spec}\,(K) \to S$ we find that,

$$H^j(X, \mathcal{L}^\vee_s \otimes \Omega^i_X) = (R^j f_*(\mathcal{L}^\vee \otimes \Omega^i_{\mathfrak{X}/S}))_\xi \otimes K = 0$$

for $i + j < d$. ∎

1.3 Counterexamples

Theorem 1.3.1 (Raynaud)**.** Kodaira vanishing can fail in characteristic p when no lifting to $W_2(k)$ exists.

Theorem 1.3.2 (Serre)**.** There exists X in characteristic p *not* lifting to characteristic 0.

1.3.1 The Proof

Let k be of characteristic p and k either infinite or "large" (we will see what this means in a bit).

Proposition 1.3.3 (Godeaux)**.** Suppose we have an action $r_0 : G \to \mathrm{PGL}_n(K)$ then there exists a smooth closed subvariety Y_0 of $\mathbb{P}^{n-1}_K$ a complete intersection such that $G \circlearrowright Y_0$ without fixed-points.

Proposition 1.3.4 (Serre)**.** Suppose $\forall g \neq 1$ the fixed scheme in $\mathbb{P}_K^{n-1}$ has codimension ≥ 4 then can take $\dim Y_0 \geq 3$ and if $X_0 = Y_0/G$ lifts to some A complete noetherian local ring of $\mathrm{char}(A) = 0$ then r_0 lifts to a map $r : G \to \mathrm{PGL}_n(A)$.

Remark. Therefore, it suffices to produce a group action with these properties that does not lift to characteristic zero.

Consider the standard order 5 nilpotent matrix $N \in M_5(k)$. Let $G = \mathbb{G}_a$ or $G = \mathbb{F}_p^5 \subset k$. Then take $G \to \mathrm{PGL}_n$. Then we consider the map $g \mapsto \exp(gN)$. It is not hard to show that there is a unique fixed point in $\mathbb{P}^4(k)$ so it has codimension 4. If we can lift $G \to \mathrm{PGL}_n(A)$ we may assume that A is a domain (because A has characteristic 0 so p is not nilpotent so we can quoitent by a prime not containing p) then we get $G \to \mathrm{PGL}_n(L)$ with $\mathrm{char}(L) = 0$. Then we would get $\mathbb{F}_p^5 \subset \mathrm{PGL}_5(L)$ but this is abelian so we can simultaneously diagonalize but this is not possible because these would have to be diagonal matrces of which we can have at most $\mathbb{F}_p^4$.