1 Sheaves Stuff

Question: does there exists an fpqc torsor for a reasonable group not representable by an algebraic space?

Lemma 1.0.1. Descent holds along a τ-cover for sheaves in the τ-topology. Explicitly, let $\mathcal{C}_\tau$ be a site and consider the natural map

$$\mathrm{Shv}_S(\mathcal{C}_\tau) \to \mathrm{DD}_{S'/S}(\mathrm{Shv}_{S'}(\mathcal{C}_\tau))$$

is an equivaence of categories.

Remark. Note that $\mathrm{Shv}_S(\mathcal{C}_\tau)$, the slice category of sheaces on $\mathcal{C}_\tau$ over the representable h^S (in presheaves if h^S is not a τ-sheaf), is equivalent to $\mathrm{Shv}(\mathcal{C}_{\tau/S})$ the sheaves on the slice category over S. Indeed, the map $\varphi : F \to S$ gives a map $F(T) \to \mathrm{Hom}\,(T, S)$ so it lives over the slice category already. Conversely, given a sheaf G on the slice category we define F via

$$T \mapsto \{(\alpha, \beta) \mid \alpha : T \to S \text{ and } \beta \in G(\alpha : T \to S)\}$$

Proof. This is just unwinding definitions. For full faithfulness, we need to show that

$$\mathrm{Hom}_S\,(F, G) \to \mathrm{Hom}_{S'}\,(F_{S'}, G_{S'}) \rightrightarrows \mathrm{Hom}_{S'\times_S S'}\,(F_{S'\times_S S'}, G_{S'\times_S S'})$$

is an equalizer. This is exactly the sheaf condition for $\mathrm{Hom}\,(F, G)$. Indeed, let's prove it. Let $\varphi, \psi : F \to G$ be S-morphisms that become equal upon pulling back to S'. For any $T \to S$ consider the cover $T_{S'} \to T$ then $\varphi_{T_{S'}} = \psi_{T_{S'}}$ so by local uniqueness: $\varphi_T = \psi_T$. Now supppose that $\varphi' : F_{S'} \to G_{S'}$ is equalized. Let φ be defined as follows: $\varphi_T(x) \in G(T)$ is obtained by gluing $\varphi_{T_{S'}}(x|_{T_{S'}})$ along $T_{S'} \to T$ which exists because of the overlap condition on φ_T.

Now we prove essential surjectivity. Let (G, α) be a descent datum. We produce a sheaf F as follows. Base changing along $T \to S$ we can replace S by any T so it suffices to produce $F(S)$. Define $F(S)$ as the limit (equalizer) of the diagram

$$\begin{tikzcd}
 & & G(\pi_1 : S' \times_S S' \to S') \arrow[dd, "\alpha"] \\
F(S) \arrow[r] & G(S') \arrow[ur] \arrow[dr] & \\
 & & G(\pi_2 : S' \times_S S' \to S')
\end{tikzcd}$$

□

2 Accessible Categories

Lurie works only with ∞-categories that are sets basically by definition since an ∞-category is a simplicial set. To differentiate between "small" and "large" he fixes a regular cardinal κ (meaning it is not a limit over less than κ smaller cardinals, eg. an inaccessible limit cardinal) and lets the the "small" simplicial sets be those in the corresponding Grothendieck universe of sets of rank $\leq \kappa$ in the Von Neumann hierarchy.

Definition 2.0.1. An ∞-category $\mathcal{C}$ is κ-accessible if it is closed under κ-filtered colimits and there exists a κ-small subcategory $\mathcal{C}^0 \subset \mathcal{C}$ such that the natural map

$$\mathrm{Ind}_\kappa(\mathcal{C}^0) \to \mathcal{C}$$

is an equivalence.

Usually people say "$\mathcal{C}$ is accessible if it is generated by $\mathcal{C}^0$ under κ-filtered colimits" which is true but confusing since it is really a stronger property than "everything is a colimit". The natural map being an equivalence says that $\mathcal{C}$ really is the category of Ind-objects not just a quotient of it. For example, the category of free R-modules is not accessible. It is obviously generated under colimits by the trivial module R but it is not isomorphic to the ind-objects since filtered colimits produce all flat R-modules. These is a filtered colimit of frees that gives a non-free finite projective and we are required to have this as well. The definition is equivalent to:

Lemma 2.0.2. An ∞-category $\mathcal{C}$ is κ-accessible if and only if it is

(a) locally small

(b) closed under κ-filtered colimits

(c) the full subcategory $\mathcal{C}^\kappa \subset \mathcal{C}$ of κ-compact objects is essential small

(d) $\mathcal{C}^\kappa$ generates $\mathcal{C}$ under small, κ-filtered colimits

3 Stable Motivic Homotopy Theory

Stable category: natural home for compatible sequences of spaces. Natural source for cohomology theories.

Naive way: sequential spectra: a sequence of spaces $\{X_n\}_{n\geq 0}$ and bonding maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Definition 3.0.1. If $\mathcal{C}$ is a category and $F : \mathcal{C} \to \mathcal{C}$ is a functor then define

$$\mathrm{Sp}^{\mathbb{N}}(\mathcal{C}, F) := \mathrm{colim}\,(\mathcal{C} \xrightarrow{F} \mathcal{C} \xrightarrow{F} \mathcal{C} \to \cdots)$$

The problem is it is hard to preserve nice properties of $\mathcal{C}$ under this construction. Nice properties:

(a) presentable

(b) symmetric monoidal structure

Issue with presentability: $\mathrm{Pr}^L \subset \mathrm{Cat}_\infty$ is not closed under colimits. However, there is a hacky trick.

Proposition 3.0.2. If $\mathcal{C}$ is presentable and G is a right adjoint to F then

$$\mathrm{Sp}^{\mathbb{N}\mathbb{N}}(\mathscr{G}, F) \xrightarrow{\sim} \lim\,(\cdots \to \mathcal{C} \xrightarrow{G} \mathcal{C} \xrightarrow{G} \mathcal{C})$$

in particular it is presentable since $\mathrm{Pr}^R \subset \mathrm{Cat}_\infty$ is limit-closed.

Example 3.0.3. Say we want to invert Σ on Spaces. Instead of the colimit of iterating Σ we use the right adjoint Ω to form spectra via a limit.

More generally if $\mathcal{C}$ is pointed and has limits then there is an endofunctor

$$\Omega : \mathcal{C} \to \mathcal{C}$$

given by taking the limit of the diagram

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

Then

$$\mathrm{Sp}(\mathcal{C}) = \lim\left(\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}\right)$$

Definition 3.0.4. An object $X \in \mathcal{C}$ in a symmetric monoidal category is *symmetric* if for some $n \geq 2$ the n-cycle

$$(1\,2\,\ldots\,n) : X^{\otimes n} \to X^{\otimes n}$$

is homotopic to the identity.

Theorem 3.0.5. If $\mathcal{C}$ is presentably symmetric monoid, and $X \in \mathcal{C}$, then there is a natural functor

$$\mathrm{Sp}^{\mathbb{N}}(\mathcal{C}, X \otimes -) \to \mathcal{C}[X^{-1}]$$

is an equivalence if X is symmetric.

Corollary 3.0.6. The category of spectra, as a presentably symmetric monoidal category, can be modeled in three equivalent ways:

(a) $\operatorname{colim}\left(S_* \xrightarrow{\Sigma} S_* \xrightarrow{\Sigma} S_* \to \cdots\right)$

(b) $\lim\left(\cdots \to S_* \xrightarrow{\Omega} S_* \xrightarrow{\Omega} S_*\right)$

(c) $S_*[(S^1)^{-1}]$

Proof. The first two are by adjunction. For the last, we need check that S^1 is symmetric. Indeed,

$$(1\,2\,3) : S^1 \wedge S^1 \wedge S^1 \to S^1 \wedge S^1 \wedge S^3$$

is homotopic to the identity as a self-map of S^3. $\square$

We get a natural adjunction:

$$\Sigma^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C}) : \Omega^\infty$$

3.1 Motivic Spectra

Could take $\mathrm{PSh}(\mathrm{Sm}_k)$ and stabilize it, we would get $\mathrm{Fun}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{Sp})$. Could look at the presheaves that are Nisnevich sheaves of spectra. Denote this by

$$\mathrm{Sp}(k) = \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp}) = \mathrm{Sp}(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k))$$

Example 3.1.1. For any sheaf of abelian groups A, get a reoresenting object $HA \in \mathrm{Sp}(k)$, defined as $\{K(A,n)\}_{n\geq 0}$ along with the maps $K(A,n) \xrightarrow{\sim} \Omega K(A, n+1)$.

Proposition 3.1.2 (representability of cohomology)**.** For $X \in \mathrm{Sm}_k$ and $n \geq 0$

$$H^n_{\mathrm{Nis}}(X, A) = [\Sigma^{-n}\Sigma^\infty_+ X, HA]_{\mathrm{Sp}(k)}$$

Proof.

$$\begin{aligned}
[\Sigma^\infty_+ \Sigma^n X, HA]_{\mathrm{Sp}(k)} &\cong [\Sigma^n X_+, \Omega^\infty HA]_{\mathrm{Shv}} \\
&\cong [\Sigma^n X_+, K(A, 0)] \\
&\cong [X_+, \Omega^n K(A, 0)]_{\mathrm{Shv}_*} \\
&\cong [X_+, K(A, n)]_{\mathrm{Shv}_*} \\
&= [X, K(A, n)]_{\mathrm{Shv}} \\
&= H^n(X, A)
\end{aligned}$$

□

Definition 3.1.3. For $E \in \mathrm{Sp}(k)$, can define $\pi_n(E)$ to be the sheafifcation of the presheaf

$$U \mapsto [\Sigma^\infty_+ \Sigma^n U, E]_{\mathrm{Sp}(k)}$$

Example 3.1.4.

$$\pi_n HA = \begin{cases} 0 & n \neq 0 \\ A & n = 0 \end{cases}$$

This induces a t-structure such that

$$\mathrm{Sp}(k) = \mathbf{Ab}(\mathrm{Shv}(\mathrm{Sm}_k)_{\leq 0})$$

Notation: Denote by $\mathrm{Sp}_{\mathbb{A}^1}(k) \subset \mathrm{Sp}(k)$ the full subcategory of $\mathbb{A}^1$-invariant sheaves of spectra, i.e. those $E \in \mathrm{Sp}(k)$ for which $X \times \mathbb{A}^1 \to X$ induces an equivalence

$$E(X) \xrightarrow{\sim} E(X \times \mathbb{A}^1)$$

for every X.

Remark. In the literature ${}^{S^1}(k)$ or ${}_{S^1}(k)$ or ${}^{s}_{S^1}(k)$ for $\mathrm{Sp}_{\mathbb{A}^1}(k)$ called motivic S^1-spectra. Here only S^1 has been inverted not $\mathbb{G}_m$.

Remark. $\mathrm{Sp}_{\mathbb{A}^1}(k) = HI(k)$ the strongly invariant sheaves.

Want: invert all motivic spheres not just S^1.

Proposition 3.1.5. $\mathbb{P}^1$ is symmetric

Proof. I can identify $\mathbb{P}^1 \wedge \mathbb{P}^1 \wedge \mathbb{P}^1 \cong \mathbb{A}^3/(\mathbb{A}^3 \backslash 0)$ and the cycle $(1\,2\,3)$ becomes the map

$$\mathbb{A}^3/(\mathbb{A}^3 \backslash 0) \to \mathbb{A}^3/(\mathbb{A}^3 \backslash 0) \quad (x, y, z) \mapsto (y, z, x)$$

hence given by the matrix

$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ this is a product of elementary matrices and any elementary matrix is homotopic to the identity through invertible maps. □

Definition 3.1.6. The *stable motivic category* is

$$(k) = (k)_*[(\mathbb{P}^1)^{-1}]$$

This is presentably symmetric monoidal because $\mathbb{P}^1$ is symmetric.

3.2 Eilenberg-Maclane Spaces

How to build HA. If we have some $A = K(A,0)$ we need a delooping of it

$$\Omega_{\mathbb{P}^1} K(\mathbb{A}^1, 1) = K(A,0)$$

but this is

$$\Omega^{1,1}\Omega^{1,0}K(A',1) = \Omega^{1,1}K(A',0) = K((A')_{-1},0)$$

Therefore, we want A' should be the "decontraction". Hence we need A to be an infinite contraction.

In other words, need A_* to be a homotopy module.

Proposition 3.2.1. Any homotopy module gives rise to an eilenberg-Maclane spectrum in (k) and in fact

$$(k) \cong (k)$$

Now for any A_* and k, n

$$H^n_{\text{Nis}}(X, A_{-n}) = [\Sigma^\infty X, \Sigma^{n+k,k} HA]_{(k)}$$

Definition 3.2.2. For $a, b \in \mathbb{Z}$ can define homotopy groups of $E \in (k)$ to be the sheafifcation of

$$U \mapsto [\Sigma^\infty \Sigma^{a,b} U, E]_{(k)}$$

3.3 Representability of K-Theory

Goal: show algebraic K-theory is represented by $\in (k)$.

What is group completetion? Given a monoid M then its group completetion M is the initial group with a map from M.

Example 3.3.1. $\mathbb{N} = \mathbb{Z}$.

Given a monoid M in a category $\mathcal{C}$, then it has some data $M \times M \to M$ and a unit $1 \to M$ and there is associativity relations.

Let be the category of finite sets, and () the category with

(a) objects: finite sets

(b) morphisms are roofs $X \leftarrow Z \to Y$ maps of finite sets

to form compositions we take fiber products.

In a monoid M, can

(a) add $x + y$

(b) perform iterated addition $x + \cdots + x = n \cdot x$

hence can build, evaluate, and compose systems of linear mulivariate polynomials.

We can encode these operations in spans and composition of spans. We use a set with n elements to mean M^n and use repedition along the first map and grouping along the second map to represent addition.

Definition 3.3.2. If $\mathcal{C}$ is an ∞-category, the category of commutative monoids

$$(\mathcal{C}) = \text{Fun}^\times((), \mathcal{C})$$

is the product-preserving functors $() \to \mathcal{C}$.

Example 3.3.3. The span $\{x,y\} \leftarrow \{x,y\} \to \{f\}$ maps to the multiplication map $M^2 \to M$.

4 Nov. 21 - Monoids

Definition 4.0.1. We define the full subcategory

$$\mathbf{Ab}(\mathcal{C}) \subset (\mathcal{C})$$

of "abelian group objects" as those for which the distinguished span $z \leftarrow z \rightarrow z$ is an equivalence.

Proposition 4.0.2. If $\mathcal{C}$ is presentable then

(a) $(\mathcal{C})$ and $\mathbf{Ab}(\mathcal{C})$ are presentable

(b) the inclusion $\mathbf{Ab}(\mathcal{C}) \subset (\mathcal{C})$ preserves limits and filtered colimits, and admits a right adjoint

$$(-) : (\mathcal{C}) \to \mathbf{Ab}(\mathcal{C})$$

Example 4.0.3. For $\mathcal{C} =$ then this is classical group completion.

Example 4.0.4. If $F : \mathcal{C} \to \mathcal{D}$ preserves finite products get a diagram

$$\begin{array}{ccc} \mathbf{Ab}(\mathcal{C}) & \longrightarrow & (\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathbf{Ab}(\mathcal{D}) & \longrightarrow & (\mathcal{D}) \end{array}$$

Main case: category S of spaces. In this case a commutative monoid is a commutative group iff it is a loop space.

Example 4.0.5. $B\mathbb{N} = B\mathbb{Z}$.

Given $M \in (\mathcal{C})$, a candidate for its delooping is BM and we consider

$$M \to \Omega BM$$

this is a super interesting map, we can study it at the level of homology (McDull-Segal).

Definition 4.0.6. Take a collection of generators for $\pi_0 M$ and denote by M the colimit of multiplying by these generators infinitely many times on M. More precisely,

$$\langle I \rangle = \pi_0 M$$

then for any $S \subset I$ finite we produce

$$M_\infty = \operatorname*{colim}_{S \subset I} \operatorname{colim} (M \xrightarrow{\prod S} M \xrightarrow{\prod S} M \to \cdots)$$

In this setting

$$M_\infty = M[(\pi_0 M)^{-1}]$$

The map

$$M \to \Omega BM$$

has target a group and therefore we get a factorization

$$M_\infty \to \Omega BM$$

Theorem 4.0.7. The map $M_\infty \to \Omega BM$ is a plus construction.

Remark. The plus construction

(a) abelianizes π_1 (since if it is a group it must be abelian)

(b) fixes homology to agree with the input space

(c) totally messes up $\pi_\bullet$.

Example 4.0.8. $(B\mathrm{GL}_\infty(R) \times \mathbb{Z})^+ = K(R)$.

There is a natural map

$$B\Sigma_n \to (M^{\times n})_{h\Sigma_n} \to M$$

inducing a homomorphism

$$\Sigma_n = \pi_1(B\Sigma_n) \to \pi_1(M)$$

Theorem 4.0.9. The following are equivalent:

(a) the natural map $M_\infty \to B\Omega M$ is an equivalence

(b) the cyclic permutation $(1\,2\,3)$ is in the kernel of

$$\Sigma_3 \to \pi_1(M) \to \pi_1(M_\infty)$$

Definition 4.0.10. For a ring R, let (R) be the groupoid of finitely generated projective R-modules. Then *algebraic K-theory* is the group completion in S

$$K(R) = (R)$$

Note: $(-)$ as a functor can be extended to

$$(-) : \mathrm{Sm}_k^{\mathrm{op}} \to\hookrightarrow S$$

is a fppf sheaf.

Recall: finitely generated modules of rank r are classified by maps into $_r$ in the sheaf topos. There is an equivalence of categories of groupoids

$$(0) \to \sqcup_{r\geq 0 r}$$

in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$. Idea: do the group completion operation as above

$$\begin{tikzcd} {} \arrow[rr] \arrow[dr] & & {= K} \\ & {}_\infty \arrow[ur] & \end{tikzcd}$$

Notation:

$$= \operatorname*{colim}_{r\to\infty} {}_r$$

along the stabilization maps $E \mapsto E \oplus \mathcal{O}$. Each $_r$ is connected so

$$\pi_0 =_0 \sqcup_{r\geq 0 r} = \mathbb{N}$$

The $+1$ map is given by the stabilization maps inducing "shifts"

$$\sqcup_{r\geq 0 r} \to \sqcup_{r\geq 1 r} \subset \sqcup_{r\geq 0 r}$$

Proposition 4.0.11. $_\infty = \times\mathbb{Z}$

Proof. Can pull disjoint union out of the colimit of shift maps

$$\mathrm{colim}\,(\sqcup_{r\geq 0 r} \to \sqcup_{r\geq 0 r} \to \cdots) = \sqcup_{n\in\mathbb{Z}}\,\mathrm{colim}\,(_n \to _n \to \cdots) = \sqcup_{n\in\mathbb{Z}} = \times\mathbb{Z}$$

$\square$

Since $K =$ the factorization

$$\to_\infty \to$$

gives

$$\to \times\mathbb{Z} \to K$$

Theorem 4.0.12. $\times\mathbb{Z} \to K$ is a motivic equivalence.

Proof. Since L prserves finite product it also preserves commutative monoids and abelian group objects. Also L is a left adjoint so it commutes with limits and therefore it commues with $(-)_\infty$ and as a left adjoint preserving monoids and groups so it commutes with $(-)$. We're trying to show that

$$L(_\infty \to)$$

is an equivalence. This is the same as showing that

$$(L)_\infty \to (L)$$

is an equivalence. We apply the theorem to L. This means we need to show that the permutation $(1\,2\,3)$ on the bundle $\mathcal{O}^{\oplus 3}$ is homotopic to the identity. This is true because the associated matrix is a product of elementary matrices. $\square$

4.1 Algebraic K-theory is a Nisnevich sheaf

Theorem 4.1.1 (Thomason-Trobaugh)**.** Algebraic K-theory i a Nisnevich sheaf of spectra.

Sketch: given a Nisnevich distinguished square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

such that p is étale and $p^{-1}(X\backslash U) \xrightarrow{\simeq} X\backslash U$ is an isomorphism. We want to show that

$$\begin{tikzcd} K(W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & K(V) \\ K(U) \arrow[r] & K(X) \end{tikzcd}$$

is a pullback square of spectra. Taking the category of perfect complexes

$$: \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$$

is an fppfsheaf. Then there is a diagram

$$\begin{tikzcd}
{}_Z(V) \arrow[r] \arrow[d, equal] & (W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (V) \arrow[d] \\
{}_Z(X) \arrow[r] & (U) \arrow[r] & (X)
\end{tikzcd}$$

where the kernels are complexes "supported on Z" and the equivalence comes from the square being a pullback. In order to show K-theory is a Nisnevich sheaf, we have to argue it "preserves fiber sequences"

$$K : \mathrm{Cat}_\infty \to \mathrm{Sp}$$

this follows from K being a *localizing invariant*.

4.2 Algebraic K-theory is $\mathbb{A}^1$-invariant

Fundamental theorem of algebraic K-theory

Theorem 4.2.1 (Quillen)**.** If R is a regular Noetherian ring, then

$$K(R) \to K(R[t])$$

is an equivalence.

Sketch: G-theory ($\mathbf{Mod}_{fg(-)}$) is $\mathbb{A}^1$-invariant on Noetherian rings, and exploit a devisage argument and commutative algebra to show $K(R) = G(R)$ for R regular noetherian.

Theorem 4.2.2. If X is a regular noetherian scheme then

$$K(X) \to K(X \times \mathbb{A}^1)$$

is an equivalence.

Corollary 4.2.3. $K : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Sp}$ is $\mathbb{A}^1$-invariant.

4.3 Projective Bundle Formula

$K(\mathbb{P}^n_R) = K(R)[x]/(x^{n+1})$ can use this to get $\mathbb{P}^1$-bonding maps of $\times\mathbb{Z}$ to itself to get a $\mathbb{P}^1$-spectrum $\in (k)$