1 Theorem of the Base

Theorem 1.0.1. Let X be a projective variety over a field k. Then $\mathrm{NS}\, X$ is finitely generated. Moreover, if X varies in a flat family over a connected Noetherian scheme S then $\mathrm{rank}\, \mathrm{NS}\, X$ and $\#\, \mathrm{NS}\, X_{\mathrm{tors}}$ in the fibers are bounded.

Over $\mathbb{C}$ we can use the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

which gives a long exact sequence

$$H^1(X, \mathcal{O}_X) \to \mathrm{Pic}\,(X) \to H^2(X, \mathbb{Z})$$

Therefore, if we can show that $H^1(X, \mathcal{O}_X)$ map to algebraically trivial cycles then $\mathrm{NS}\, X \hookrightarrow H^2(X, \mathbb{Z})$ and hence it is finitely generated.

1.1 The Picard Scheme

Definition 1.1.1. Let X be a scheme over S then we define the *Picard stack* $\mathscr{Pic}_{X/S}(T)$ is the groupoid of invertible sheaves on X_T.

Definition 1.1.2. Let X be a scheme over S then we define

(a) *Picard presheaf* $p\mathrm{Pic}_{X/S}(T)$ to be the isomorphism classes of $\mathscr{Pic}_{X/S}(T)$ this is usually not a sheaf, for example any line bundle arising from the base T is locally trivial but does not have to be trivial

(b) Let $\sigma : S \to X$ be a section. Then the rigidified Picard sheaf is the sheaf,

$$T \mapsto \{(\mathcal{L}, \alpha) \mid \mathcal{L} \in \mathrm{Pic}\,(X_T) \quad \alpha : \sigma_T^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_T\}$$

Definition 1.1.3. The *Picard scheme* is the coarse moduli space for $\mathscr{Pic}_{X/S}$ and this represents the fppf sheafification of $p\mathrm{Pic}_{X/S}$.

Proposition 1.1.4. Let X be a scheme over S such that $p_* \mathcal{O}_X = \mathcal{O}_S$ holds universaly (i.e. for any $T \to S$ the map $\mathcal{O}_T \xrightarrow{\sim} (p_T)_* \mathcal{O}_{X_T}$ is an isomorphism) then

(a) for any geometric point $\mathrm{Spec}\,(K) \to S$ the geometric points of the functor are

$$p\mathrm{Pic}^{\mathrm{fppf}}_{X_K/K}(K) = \mathrm{Pic}\,(X_K)$$

(b) if X/S has a section σ, then forgetting α induces an isomorphism

$$\mathrm{Pic}\,(X \times_S T) / \mathrm{Pic}\,(T) \xrightarrow{\sim} \mathrm{Pic}_{(X,\sigma)/S}(T)$$

Moreover, pairs $(\mathcal{L}, \sigma)$ have trivial automorphism group.

Proof. For (b) given $\mathcal{L}$ on $X \times_S T$ we alter it to get the rigidifed bundle $\mathcal{L} \otimes p^* \sigma^* \mathcal{L}$. $\square$

1.2 Over a field

Theorem 1.2.1. Let X be a projective variety over a field k. Then

(a) $p\mathrm{Pic}^{\text{ét}}_{X/k}$ is represetnable by a group scheme lfp over k

(b) $\mathrm{Pic}^{\circ}_{X/k}$ makes sense and is of finite type

(c) If X is geometrically normal then $\mathrm{Pic}^{\circ}_{X/k}$ is proper

(d) there is an isomorphism $T_0\mathrm{Pic}_{X/k} \xrightarrow{\sim} H^1(X, \mathcal{O}_X)$ hence $\dim \mathrm{Pic}_{X/k} \le \dim_k H^1(X, \mathcal{O}_X)$ which equality iff $\mathrm{Pic}_{X/k}$ is smooth.

Remark. If k has characteristic zero then $\mathrm{Pic}_{X/k}$ is always smooth by Cartier's theorem. However, if $H^2(X, \mathcal{O}_X) = 0$ then $\mathrm{Pic}_{X/k}$ is smooth.

1.3 Over a Base

Theorem 1.3.1. Let $X \to S$ be a flat projective scheme over a locally noetherian scheme S. Then

(a) if X has integral geometric fibers then $p\mathrm{Pic}^{\text{fppf}}_{X/S}$ is representable by a separated group scheme locally of finite type over S

(b) If in addition X has geometrically normal fibers then there exists a closed subscheme $\mathrm{Pic}^{\circ}_{X/S} \hookrightarrow \mathrm{Pic}_{X/S}$ which is fiberwise Pic° and it is proper over S

(c) if $\mathrm{Pic}_{X/S}$ exists then there is an isomorphism $\mathcal{N}_0 \xrightarrow{\sim} R^1 p_* \mathcal{O}_X$

(d) if in addition S is a reduced scheme of characteristic zero then $\mathrm{Pic}_{X/S}$ is smooth or if $s \in S$ such that $H^2(X_s, \mathcal{O}_{X_s}) = 0$ then $\mathrm{Pic}_{X/S}$ is smooth over a neighborhood of s. In both of these cases $\mathrm{Pic}^{\circ}_{X/S}$ is an open group subscheme of $\mathrm{Pic}_{X/S}$.

[EGA IV, vol 3, prop 15.6.8]
and [EGA IV, vol 3, 15.6.4].

1.4 Notions of Equivalence

Definition 1.4.1. Let X be a projective variety over k. Let $\mathcal{L} \in \mathrm{Pic}\,(X)$ say that $\mathcal{L}$ is algebraically trivial if there exists a connected scheme T over k and $x, x' \in T(k)$ and $\Xi \in p\mathrm{Pic}_{X/k}(T)$ such that $\Xi|_x \cong \mathcal{O}_X$ and $\Xi|_{x'} \cong \mathcal{L}$. Furthermore,

(a) $\mathcal{L}$ is algebraically torsion, if $\exists m \neq 0$ such that $\mathcal{L}^{\otimes m}$ is algebraically trivial

(b) numerically trivial if for every curve $C \subset X$ we have $\deg \mathcal{L}|_C = 0$.

Theorem 1.4.2. The following are equivalent

(a) $\mathcal{L}$ is algebraically torsion

(b) $\{\mathcal{L}^m\}$ is bounded (meaning it lies in a quasi-compact open of $\mathrm{Pic}_{X/k}$)

(c) $\mathcal{L}$ is numerically trivial

Proof. $(a) \implies (b) \implies (c)$ are easy. We will spend the next section proving the coverse. $\square$

Definition 1.4.3. Let X be a projective variety over k. Let $\mathscr{F}$ be a coherent sheaf on X. We say that $\mathscr{F}$ is m-regular if for all $i > 0$ we have $H^i(X, \mathscr{F}(m-i)) = 0$.

Remark. Notice that m-regularity is independent of the field so we may assume that $k = \bar{k}$. Then there always exsits a hyperplane section avoiding all the associated points of $\mathscr{F}$ so there always exists a sequence of the form

$$0 \longrightarrow \mathscr{F}(-1) \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}|_H \longrightarrow 0$$

Proposition 1.4.4. Let X be a projective variety over k and $\mathscr{F}$ is an m-regular coherent sheaf then,

(a) $\mathscr{F}|_H$ is m-regular.

(b) $\mathscr{F}$ is $(m+1)$-regular.

(c) the map $H^0(X, \mathcal{O}_X(i)) \otimes H^0(X, \mathscr{F}(m)) \to H^0(X, \mathscr{F}(m+i))$ is surjective

(d) $\mathscr{F}(m)$ is generated by global sections.

Proof. Using the exact sequence,

$$0 \longrightarrow \mathscr{F}(m-1) \longrightarrow \mathscr{F}(m) \longrightarrow \mathscr{F}|_H(m) \longrightarrow 0$$

to get the LES

$$H^i(X, \mathscr{F}(m-i-1)) \longrightarrow H^i(X, \mathscr{F}(m-i)) \longrightarrow H^i(H, \mathscr{F}|_H(m-i)) \longrightarrow H^{i+1}(X, \mathscr{F}(m-i-1))$$

the terms $H^i(X, \mathscr{F}(m-i)) = H^{i+1}(X, \mathscr{F}(m-i-1)) = 0$ by hypothesis and hence $H^i(H, \mathscr{F}|_H(m-i)) = 0$.

For (b) we induct on dimension and the previous sequence with $m \mapsto m+1$. Then

For (c) because $H^1(X, \mathscr{F}(m-1)) = 0$ we can lift sections of $H^0(H, \mathscr{F}|_H(m))$ to $H^0(X, \mathscr{F}(m))$ and therefore we can proceed by induction.

For (d) we know that $\mathscr{F}(n)$ is generated by global sections for $n \gg 0$. However, we can write,

$$H^0(X, \mathcal{O}_X(n-m)) \otimes H^0(X, \mathscr{F}(m)) \twoheadrightarrow H^0(X, \mathscr{F}(n))$$

and therefore the map $\mathcal{O}_X(n-m) \otimes H^0(X, \mathscr{F}(m)) \to \mathscr{F}(n)$ is surjective hence the map $\mathcal{O}_X \otimes H^0(X, \mathscr{F}(m)) \to \mathscr{F}(n)$ is surjective proving the claim. $\square$

Lemma 1.4.5. Let X be a projective variety over k. Let $\mathscr{F}$ be a coherent sheaf with $\dim \mathrm{Supp}\,(\mathscr{F}) \leq r$ then there exists a constant $A(\mathscr{F})$ such that for all $\mathcal{L} \sim_{\mathrm{num}} 0$ then $h^0(\mathscr{F} \otimes \mathcal{L}(n)) \leq A(\mathscr{F})\binom{n+r}{r}$.

Proof. For sequences

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

then for $\mathscr{G}$ we get the constant $A(\mathscr{F}) + A(\mathscr{H})$. Hence we reduce to sheaves of the form $\mathscr{F} = \mathcal{O}_Z$ for $Z \subset X$ an irreducible closed subscheme. Consider a hyperplane section $H \hookrightarrow Z$ and consider the sequence

$$0 \longrightarrow \mathcal{L}(j-1) \longrightarrow \mathcal{L}(j) \longrightarrow \mathcal{L}(j)|_H \longrightarrow 0$$

therefore

$$h^0(Z, \mathcal{L}(j)) \leq h^0(Z, \mathcal{L}(j-1)) + A(\mathcal{O}_H)\binom{j+r-1}{r-1}$$

and hence taking the sum

$$h^0(Z, \mathcal{L}(n)) \leq A(\mathcal{O}_H)\binom{n+r}{r} + h^0(Z, \mathcal{L})$$

but $\mathcal{L}$ is numerically trivial so $h^0(Z, \mathcal{L}) \leq 1$. □

Proposition 1.4.6. Let X be a projective variety then there exists $m(X)$ such that for all $L \sim_{\text{num}} 0$ then $\mathcal{L}$ is $m(X)$-regular. Also $\chi(X, \mathcal{L}(n)) = \chi(X, \mathcal{O}(n))$.

Proof. Let $\mathcal{L}$ be numerically trivial and $\mathcal{L}(d)$ very ample. Let F be an effective divisor in the class of $\mathcal{O}_X(d)$ and G in the class of $\mathcal{L}(d)$. Consider the sequences

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_F \longrightarrow 0$$

$$0 \longrightarrow \mathcal{L}^{-1}(-d) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_G \longrightarrow 0$$

and apply $- \otimes \mathcal{L}^p(n+d)$ and take Euler characteristics and subtract,

$$\chi(X, \mathcal{L}^p(n)) - \chi(X, \mathcal{L}^{p-1}(n)) = \chi(G, \mathcal{L}^p(n+d)) - \chi(F, \mathcal{L}^p(n+d)) + \chi(X, \mathcal{L}^p(n))$$

For induction we assume that both are known on F and G and therefore the RHS is a polynomial independent of $\mathcal{L}$ so by summation we get

$$\chi(X, \mathcal{L}^p(n)) = \varphi_1(n)p + \varphi_0(n)$$

for some polynomials φ_1, φ_0 independent of $\mathcal{L}$. Consider the sequence

$$0 \longrightarrow \mathcal{L}^p(n) \longrightarrow \mathcal{L}^p(n+d) \longrightarrow \mathcal{L}^p|_F(n+d) \longrightarrow 0$$

For all $n \geq m$ we know $H^i(F, \mathcal{L}^p(n-i)) = 0$ so $H^i(X, \mathcal{L}^p(n)) = H^i(X, \mathcal{L}^p(n+d))$ for all $i \geq 2$ and $n \geq m, p$. But by Serre vanishing this must be zero since we can take the twist very large. Then

$$h^0(\mathcal{L}^p(n)) - h^1(\mathcal{L}^p(n)) = \varphi_1(n)p + \varphi_0(n)$$

and therefore $h^0(\mathcal{L}^p(n)) \geq p_1(n)p + p_0(n)$ taking some limit. But this contradicts what we previously proved so it must be p-independent.

To complete the induction we need to also prove the first part. Let H be a hyperplane section then consider

$$0 \longrightarrow \mathcal{L}(n) \longrightarrow \mathcal{L}(n+1) \longrightarrow \mathcal{L}|_H(n+1) \longrightarrow 0$$

therefore

$$\chi(H, \mathcal{L}(n+1)) = \chi(X, \mathcal{L}(n+1)) - \chi(X, \mathcal{L}(n)) = \chi(X, \mathcal{O}_X(n+1)) - \chi(X, \mathcal{O}_X(n))$$

by what we just proved. Then there exists $m(H)$ such that $\mathcal{L}|_H$ is $m(H)$-regular and for $n\, gem(H)-2$ then from the long exact sequence we get vanishing $H^i(X, \mathcal{L}(n-i)) = 0$ for $i \geq 2$. For $i = 1$ we need a different argument. Claim: $h^1(\mathcal{L}(n))$ is strictly decreasing. Indeed, otherwise there is some n such that $h^1(\mathcal{L}(n)) = h^1(\mathcal{L}(n+1))$ but then

$$H^0(H, \mathcal{L}(n+1)) \otimes H^0(\mathcal{O}(1)) \twoheadrightarrow H^0(H, \mathcal{L}(n+2))$$

is surjecive so $h^1(\mathcal{L}(n)) = h^2(\mathcal{L}+2)$ and so on then we see that $h^1(\mathcal{L}(n))$ never decreases again but it must go to zero by Serre vanishing. Therefore, we know that

$$h^1(\mathcal{L}(m)) \leq h^1(\mathcal{L}(m-1)) = h^0(\mathcal{L}(m-1)) - \chi(\mathcal{L}(m-1)) \leq A(H)\binom{n+m}{r-1}$$

$\square$

This proves the main theorem because if $\mathcal{L} \sim_{\text{num}} 0$ is a quotient of $\mathcal{O}_X(-m)^{\oplus A(X)\binom{m+r}{r}}$ with hilbert polynomial $\chi(X, \mathcal{L}(n)) = \chi(X, \mathcal{O}_X(n))$ independent of $\mathcal{L}$. Therefore, all $\mathcal{L}$ live in a qc component of the Quot scheme.

2 Talk 2

Theorem 2.0.1. Let X be projective variety ove k then $\mathcal{L} \in \text{Pic}\,(X)$ the following are equivalent

(a) $\mathcal{L}$ is algebraically torsion

(b) $\mathcal{L}$ is numerically trivial

(c) $\{\mathcal{L}^n\}$ is bounded

(d) $\chi(X, \mathcal{L}^p(n)) = \chi(X, \mathcal{O}_X)$ for a zariski dense set of integers $(p, n) \in \mathbb{Z}^2$

Theorem 2.0.2. The family of all numerically trivial line bundles on X is bounded.

Corollary 2.0.3. $\text{NS}(X)_{\text{tors}}$ is finite.

Proof. $\text{NS}(X)_{\text{tors}}$ is numerically trivial mod algebraically trivial line bundles which is the group of connected components of a quasi-compact group scheme which is hence finite. $\square$

2.1 Alterations

Definition 2.1.1. Let X be a noetheiran scheme. An alteration $Y \to X$ is a proper generically finite map with Y regular.

Proposition 2.1.2. If X is a variety over a field k then there exists a finite extension k'/k purely inseparable such that $X_{k'}$ has an alteration by a smooth k'-variety.

For X/k we have $\text{NS}(X) \to \text{NS}(X_{k'}) \to \text{NS}(X)$ is multiplication by $[k' : k]$.

Definition 2.1.3. Let $\text{Num}(X) = \text{Pic}\,(X) / \sim_{\text{num}}$.

Proposition 2.1.4. If $f : X \to Y$ is a surjective map of projective varities then $f^* : \text{Num}(Y) \to \text{Num}(X)$ is well-defined injective map.

Proof. Let $\mathcal{L}$ be a line bundle on Y such that $f^*\mathcal{L}$ is numerically trivial we need to show that $\mathcal{L}$ is numerically trivial. Given $C \subset X$ then $\deg_C f^*\mathcal{L} = \deg_{f(C)} \mathcal{L} = 0$ but also every curve in Y is of the form $f(C)$ because we can pull back and take an irreducible component. ☐

2.2 étale Cohomology

Recall that for smooth X there is a Chern class $c_1 : \text{Pic}\,(X) \to H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$. This preserves algebraic equivalence.

Lemma 2.2.1. c_1 is well-defined modulo numerical equivalence.

Proof. Indeed, if $\mathcal{L} \sim_{\text{num}} 0$ then $\mathcal{L}^n \sim_{\text{alg}} 0$ for some n so $nc_1(\mathcal{L}) = 0$ but $H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$ is a vector space over $\mathbb{Q}_\ell$ which has characteristic zero so $c_1(\mathcal{L}) = 0$. ☐

Hence we have a map $c_1 : \text{Num}(X) \to H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$ and if $c_1(\mathcal{L}) = 0$ then $\deg \mathcal{L}|_C = c_1(\mathcal{L}) \frown [C] = 0$ so $\mathcal{L}$ is numerically trivial so c_1 is injective.

Theorem 2.2.2. $\text{Num}(X)$ is finitely generated.

Proof. We know that $c_1 : \text{Num}(X) \to H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$ is injective and $H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$ is a finite $\mathbb{Q}_\ell$-vectorspace. Let $[C_1], \dots, [C_r]$ be the largest independent set of fundamental classes of curves in $H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$. Consider $\lambda : \text{Num}(X) \to \mathbb{Z}^n$ given by intersection against these curves. Then λ is injective becase if $\lambda(\mathcal{L}) = 0$ then for all $\deg_C \mathcal{L} = 0$ because in cohomology we can write

$$[C] = \sum \alpha_i [C_i]$$

and $\deg_{C_i} \mathcal{L} = 0$ so

$$\deg_C \mathcal{L} = \sum_i \alpha_i c_1(\mathcal{L}) \frown [C_i] = 0$$

Therefore we win. ☐

2.3 Families

Proposition 2.3.1. Let X be a flat proper scheme over a locally noetherian S such that $\text{Pic}_{X/S}$ exists as a scheme. Then the subfunctor $\text{Pic}^\tau_{X/S}$ consisting of points which are algebraically torsion in their fiber is open group subscheme.

Proof. Given $T \to \text{Pic}_{X/S}$ corresponding to a line bundle $\mathcal{L}$ we need to show that the locus of points $t \in T$ such that $\mathcal{L}_t \in \text{Pic}^\tau_{X_t}$ is open. Note that $t \mapsto \chi(X, \mathcal{L}_t^p(n))$ is constant on each connected component of T by flatness. Therefore, we apply the last part of the theorem to say that the locus is a union of connected components of T. ☐

Proposition 2.3.2. Let X be a projective scheme with geometrically integral fibers over a noetherian scheme S then $\#\text{NS}(X_{\bar{s}})_{\text{tors}}$ is bounded.

Proof. By noetherian induction I may assume $X \to S$ is flat. Then $\mathrm{Pic}_{X/S}$ exists as a scheme and $\mathrm{Pic}^{\tau}_{X/S}$ is finite type and therefore the number of geometric connected components of its fibers is bounded by Hilbert scheme arguments. $\square$

Example 2.3.3. Rank of $\operatorname{rank} \mathrm{NS}(X_s)$ is not constructible. Let E_t be a family of elliptic curves then consider $E_t \times E_t$. Then,

$$\operatorname{rank}(E_t \times E_t) = \begin{cases} 3 & E_t \text{ not CM} \\ 4 & E_t \text{ CM} \end{cases}$$

Indeed, if $C \subset E_t \times E_t$ is a cycle then it induces a correspondence and hence a map $\mathrm{Pic}(E_t) \to \mathrm{Pic}(E_t)$. If E_t does not have CM then this is just $[n]$. Then

Theorem 2.3.4. Let X be a projective scheme over a noetherian scheme with geometrically integral fibers. Then $\operatorname{rank} \mathrm{NS}(X_{\bar{s}})$ is bounded. Over a field $\operatorname{rank} \mathrm{NS}(X) \le \dim H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$ and therefore

$$\operatorname{rank} \mathrm{NS}(X_{\bar{S}}) \le \dim H^2_{\text{ét}}(X_{\bar{s}}, \mathbb{Q}_\ell(1)) = \operatorname{rank}(R^2 p_* \mathbb{Q}_\ell)_{\bar{s}}$$

Proof. This is true because $R^2 p_* \mathbb{Q}_\ell$ is an étale local system for X smooth. We may assume S is integral with generic point η. After finite inseperable extension X_η has an alteration $X'_\eta \to X_\eta$ we can spread out X'_η to X'/U' for $U' \to U \subset S$ pure inseparable map. For every $s \in S$ the map $\mathrm{Num}(X_{\bar{s}}) \to \mathrm{Num}(X'_{\bar{s}})$ injective so we are done. $\square$

Theorem 2.3.5 (Generic Representability)**.** Let X/S is a proper scheme over S locally noetherian then $\exists U \subset S$ dense open such that $\mathrm{Pic}_{X_U/U}$ exists.

Corollary 2.3.6.