0.1 Notation

Let $\omega = \alpha H$ and $B = \beta H$ where H is an ample class. Let

$$\mathrm{ch}^B(E) = e^{-B}\mathrm{ch}(E)$$

therefore

$$\begin{aligned}
\mathrm{ch}_0^B(E) &= \mathrm{ch}_0(E)\\
\mathrm{ch}_1^B(E) &= \mathrm{ch}_1(E) - B\\
\mathrm{ch}_2^B(E) &= \mathrm{ch}_2(E) - B\mathrm{ch}_1(E) + \tfrac{1}{2}B^2\mathrm{ch}_0(E)\\
\mathrm{ch}_3^B(E) &= \mathrm{ch}_3(E) - B\mathrm{ch}_2(E) + \tfrac{1}{2}B^2\mathrm{ch}_1(E) - \tfrac{1}{6}B^3\mathrm{ch}_0(E)
\end{aligned}$$

Furthermore, we define the slope

$$\mu_{\omega,B}(E) := \frac{\omega^{n-1}\mathrm{ch}_1^B(E)}{\omega^n\mathrm{ch}_0^B(E)} = \frac{\omega^{n-1}\mathrm{ch}_1(E) - \omega^{n-1}\cdot B\mathrm{ch}_0(E)}{\omega^n\mathrm{ch}_0(E)}$$

when E is a torsion-free sheaf then

$$\mu_{\omega,B}(E) = \frac{\omega^{n-1}c_1(E)}{\operatorname{rank} E} - \omega^{n-1}\cdot B$$

We define the central charge

$$Z_{\omega,B}(E) = \int_X e^{-i\omega - B}\mathrm{ch}(E)$$

Define two subcategories of $\mathfrak{Coh}\,(X)$.

Definition 0.1.1. Let $\mathcal{T}_{\omega,B}$ be the subcategory generated, via extensions, by $\mu_{\omega,B}$-semistable sheaves of slope $\mu_{\omega,B} > 0$ where we set the torsion sheaves to have $\mu_{\omega,B} = \infty$.

This is the category of sheaves E such that E/E_{tors} has all HN-slopes strictly positive.

Definition 0.1.2. Let $\mathcal{F}_{\omega,B}$ be the subcategory generated under extensions by $\mu_{\omega,B}$-semistable sheaves of slope $\mu_{\omega,B} \leq 0$ (hence these are torsion-free).

Definition 0.1.3. $\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B}\rangle \subset D^b(X)$ what we call the *tilt*. We define the slope function on $\mathcal{B}_{\omega,B}$

$$\nu_{\omega,B}(E) = \frac{\mathrm{Im}\,(Z_{\omega,B}(E))}{\omega^2\mathrm{ch}_1^B(E)} = \frac{\omega\mathrm{ch}_2^B(E) - \frac{1}{6}\omega^3\mathrm{ch}_0^B(E)}{\omega^2\mathrm{ch}_1^B(E)}$$

Conjecture 0.1.4. For any tilt-stable object $E \in \mathcal{B}_{\omega,B}$ satisfying $\nu_{\omega,B}(E) = 0$ meaning

$$\tfrac{1}{6}\mathrm{ch}_0^B(E) = \omega\,\mathrm{ch}_2^B(E)$$

we have the inequality

$$\mathrm{ch}_3^B(E) \leq \tfrac{1}{18}\omega^2\,\mathrm{ch}_1^B(E)$$

0.2 A few computations

By Grothendieck-Riemann-Roch

$$\mathrm{ch}(\mathcal{O}_C) = \iota_*(\mathrm{Td}_C\cdot\mathrm{Td}_X^{-1}) = \iota_*(1 - (g-1)[*] + \tfrac{1}{2}K_X\cdot C[*]) = [C] + \tfrac{1}{2}(K_X\cdot C - 2(g-1))[*]$$

0.3 Tangent Bundle

Let X be not uniruled then Miyaoka's theorem shows that $\mu_{\max}(\mathcal{T}_X) \leq 0$. Therefore, as long as $\beta \geq 0$ we have $\mathcal{T}_X \in \mathcal{F}_{\omega,B}$.

(a) $\mathcal{T}_X[1] \in \mathcal{B}_{\omega,B}$ for $\beta \geq 0$

(b) if $\mathcal{T}_X$ satisfies $\mu_{\max}(\mathcal{T}_X) < -\epsilon$ then $\mathcal{T}_X \in \mathcal{B}_{\omega,B}$ for $-H^3\epsilon > \beta$

(c) if $\mu_{\min}(\Omega_X) > \epsilon$ then $\Omega_X \in \mathcal{B}_{\omega,B}$ for $\beta < \omega^3\epsilon$

(d) $\mathscr{I}_C[1] \in \mathcal{B}_{\omega,B}$ for $\beta \geq 0$ and $\mathscr{I}_C \in \mathcal{B}_{\omega,B}$ for $\beta < 0$

(e) $\mathscr{I}_C^\vee[1] \in \mathcal{B}_{\omega,B}$ for $\beta \geq 0$

We want to compute

$$\operatorname{Ext}^i_X\left(\mathcal{O}_X, \mathscr{I}_C \otimes \mathcal{T}_X\right) = \operatorname{Ext}^i_X\left(\mathscr{I}_C^\vee, \mathcal{T}_X\right) = \operatorname{Ext}^i_X\left(\Omega_X, \mathscr{I}_C\right)$$

0.4 Case (a)

We consider X such that $\mu(\Omega_X) = \epsilon > 0$ and Ω_X is μ-semistable. We need $0 < \beta H^3 < \epsilon$.

$$\operatorname{Ext}^2_X\left(\Omega_X, \mathscr{I}_C\right) = \operatorname{Hom}_{D^b(X)}\left(\Omega_X, \mathscr{I}_C[2]\right)$$

We consider a nonzero element ξ then we consider the extension defined by (ξ, ξ, ξ)

$$\mathscr{I}_C[1]^{\oplus 3} \to E_\xi \to \Omega_X$$

Now we compute

$$\operatorname{ch}^B(E_\xi) = 3\operatorname{ch}^B(\mathscr{I}_C[1]) + \operatorname{ch}^B(\Omega_X)$$

likewise

$$\begin{aligned}\operatorname{ch}^B(\mathscr{I}_C[1]) &= e^{-B}\operatorname{ch}(\mathscr{I}_C[1]) = -e^{-B}(\operatorname{ch}(\mathcal{O}_X) - \operatorname{ch}(\mathcal{O}_C)) = -e^{-B}(1 - [C] + \tfrac{1}{2}(K_X \cdot C - 2(g-1))[*]) \\ &= \Big(-1, B, -\tfrac{1}{2}B^2 + [C], \tfrac{1}{6}B^3 - \tfrac{1}{2}(K_X \cdot C - 2(g-1))\Big)\end{aligned}$$

Therefore

$$\operatorname{ch}(E_\xi) = \Big(0, c_1, \tfrac{1}{2}c_1^2 - c_2 + 3[C], \tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) - \tfrac{3}{2}(c_1 \cdot C - 2(g-1))\Big)$$

hence

$$\begin{aligned}\operatorname{ch}^B(E_\xi) = \Big(&0, c_1, -c_1 \cdot B + \tfrac{1}{2}c_1^2 - c_2 + 3[C], \\ &\tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) - \tfrac{3}{2}(c_1 \cdot C - 2(g-1)) + \tfrac{1}{2}B^2 \cdot c_1 - B \cdot (\tfrac{1}{2}c_1^2 - c_2 + 3[C])\Big)\end{aligned}$$

We want

$$H \cdot \Big(\tfrac{1}{2}c_1^2 + c_1 \cdot B - c_2 + 3[C]\Big) = 0$$

i.e.

$$\beta = \frac{H \cdot (c_2 - \frac{1}{2}c_1^2) - 3\deg_H C}{c_1 \cdot H^2}$$

but this is a series condition on the curves we can consider. Now we consider the conjectural inequality:

$$\operatorname{ch}_3^B(E_\xi) \le \tfrac{1}{18}\omega^2 \operatorname{ch}_1^B(E_\xi)$$

This says

$$\tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) - \tfrac{3}{2}(c_1 \cdot C - 2(g-1)) + \tfrac{1}{2}B^2 \cdot c_1 - B \cdot (\tfrac{1}{2}c_1^2 - c_2 + 3[C]) \le \tfrac{1}{18}\omega^2 \operatorname{ch}_1^B(E_\xi) = \tfrac{1}{18}\alpha^2 H^2 \cdot c_1$$

We are going to take the limit as $\alpha \to 0$. Therefore, E_ξ is destabilized if the RHS is positive. Therefore

$$B \cdot (\tfrac{1}{2}c_1^2 - c_2 + 3[C]) = \beta H \cdot (\tfrac{1}{2}c_1^2 - c_2 + 3[C]) = -\beta H \cdot B \cdot c_1 = -B^2 \cdot c_1$$

Therefore we get

$$\tfrac{1}{9}(c_1^3 - 3c_1c_2 + 3c_3) - (c_1 \cdot C - 2(g-1)) + B^2 \cdot c_1 \le 0$$

0.5 (b)

Alternatively, lets look at

$$\operatorname{Ext}_X^1(\mathcal{O}_X, \mathscr{I}_C \otimes \mathcal{T}_X) = \operatorname{Ext}_X^1(\mathscr{I}_C^\vee, \mathcal{T}_X) = \operatorname{Ext}_X^1(\mathscr{I}_C^\vee[1], \mathcal{T}_X[1])$$

then we get an extension

$$\mathcal{T}_X[1] \to E_\xi \to \mathscr{I}_C^\vee[1]$$