1 Deligne's Paper on Lifting

Here let X_0 be a K3 surface over k an algebraically closed field of cahracteristic $p > 0$. Let $W(k)$ be the ring of Witt vectors.

Proposition 1.0.1. The sepectral sequence

$$E_1^{i,j} = H^j(X_0, \Omega^i_{X_0/k}) \implies H^{i+j}_{\mathrm{dR}}(X_0/k)$$

degenerates at E_1 and the hodge numbers are as expected. Furthermore

(a) $H^i(X_0, \mathcal{T}_{X_0}) = 0$ for $i = 0, 2$ and $h^1(X_0, \mathcal{T}_{X_0}) = 20$

(b) the crystaline cohomology W-modules are free of rank $1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$.

Corollary 1.0.2. The formal versal deformation space of X_0 is a W-algebra artian local with residue field k is universal and is smooth of dimension 20 meaning

$$\mathrm{Def}_{X_0} = S := \mathrm{Spf}\,(W[\![t_1, \dots, t_{20}]\!])$$

From now on, let $\mathscr{X} \to S$ be the universal deformation of X_0.

1.1 Line Bundles

Let L_0 be an invertible sheaf on X_0. Write $\underline{\mathrm{Def}}(X_0, L_0)$ for the functor

$$\mathrm{Art}_k \to \mathrm{Set}$$

which takes A to deformations of the pair (X_0, L_0) over A. There is a forgetful map

$$\underline{\mathrm{Def}}(X_0, L_0) \to \underline{\mathrm{Def}}(X_0)$$

Proposition 1.1.1. $\underline{\mathrm{Def}}(X_0, L_0)$ is pro-representable and the map

$$\underline{\mathrm{Def}}(X_0, L_0) \to \underline{\mathrm{Def}}(X_0)$$

is a closed imersion defined by one equation.

This means there is a closed formal subscheme

$$\Sigma(L_0) \subset S$$

such that L_0 extends over $\underline{X} \times_S \Sigma(L_0)$ and this extension is unique.

Proof. We verify that $F = \underline{\mathrm{Def}}(X_0, L_0)$ satisfies Schlessinger's condition for the existence of a hull.

(H1) let $A_1' \twoheadrightarrow A$ be a small extension of Artin rings and $A_2' \to A$ any map of Artin rings. Consider

$$F(A_1' \times_A A_2') \to F(A_1') \times_{F(A)} F(A_2')$$

Notice that

$$\begin{array}{ccc} \mathrm{Spec}\,(A) & \longrightarrow & \mathrm{Spec}\,(A_1') \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(A_2') & \longrightarrow & \mathrm{Spec}\,(A_1' \times_A A_2') \end{array}$$

is a pushout diagram of schemes. By affinianity we can always glue deformations of schemes to form the pushout in schemes.

Given a line bundles on the reduction to A_i' that are isomorphic over A we need to glue them. This holds by representability of the Picard scheme but we can argue directly. In fact it is true for any coherent sheaves. We show there is a unique way to do it so we can argue locally. Given a fiber product of rings $A \times_R B$ and modules M, N over A, B with an isomorphism $\psi : M \otimes_A R \xrightarrow{\sim} N \otimes_B R$ we can build a glued module

$$M \times_\psi N = \{(m, n) \in M \times N \mid \psi(m \otimes 1) = n \otimes 1\}$$

it is clear how to endow this with an $A \times_R B$-module structure. The claim is that this is unique amoung modules equipped reductions to M, N. Indeed, given any other G we construct an isomorphism as follows. The reduction maps give

$$G \to M \quad G \to N$$

which are compatible with ring projections. Hence we get a map

$$G \to M \times N$$

which is equivariant for the ring map $A \times_R B \to A \times B$. The image must land inside $M \times_\psi N$ because $G \otimes A \xrightarrow{\sim} M$ and $G \otimes B \xrightarrow{\sim} N$ so reducing mod R the isomorphisms are required to be compatible with ψ. The map $G \to M \times_\psi N$ is then an isomorphism because this is true after tensoring with A or B which are scheme-theoretically dense.

(H1) We have shown that the underlying deformation category satisfies (RS) so the functor of isomorphism classes also satsifies (H2)

(H3) finiteness of the tangent space is clear by usual deformation theory and properness

Therefore, we get a formal scheem $S' = \mathrm{Spf}\,(R')$ with R' a local W-algebra that is noetherian and complete with resoude field k and there is a deformation (X', L') over S' hence there is a map

$$\mathrm{Hom}\,(R', A) \to \underline{\mathrm{Def}}(X_0, L_0)(A)$$

which is surjective and bijective over $A = k[\epsilon]$. Let R be the ring of S. Since R pro-represents $\underline{\mathrm{Def}}(X_0)$ there is a composition

$$\mathrm{Hom}\,(R', A) \to \underline{\mathrm{Def}}(X_0, L_0)(A) \to ul\mathrm{Def}(X_0)(A) = \mathrm{Hom}\,(R, A)$$

of natural transformations there is a map $u : R \to R'$ of local W-algebras. To prove closedness, it suffices to show that u is surjective since then the composition and hence the first map are injective and hence the first map is bijective.

According to a well-known lemma [15, 1.1], it suffices to show that if $\mathfrak{m}$ (resp. $\mathfrak{m}'$) is the maximal ideal of R (resp. R') the map

$$\mathfrak{m}/(pR + \mathfrak{m}^2) \to \mathfrak{m}'/(pR' + \mathfrak{m}'^2)$$

is surjective or equivalently (using Schlessinger H2) that

$$\underline{\mathrm{Def}}(X_0, L_0)(k[\epsilon]) \to \underline{\mathrm{Def}}(X_0)(k[\epsilon])$$

is injective. The Atiyah extension

$$0 \to \mathcal{O}_{X_0} \to \mathrm{At}(L_0) \to \mathcal{T}_{X_0} \to 0$$

controlls the deformation theory in the sense that $H^i(X_0, \mathrm{At}(L_0))$ forms an automorphism-tangent-obstruction theory for $\underline{\mathrm{Def}}(X_0, L_0)$. The long exact sequence gives,

$$H^1(X_0, \mathcal{O}_{X_0}) \to H^1(X_0, \mathrm{At}(L_0)) \to H^1(X_0, \mathcal{T}_{X_0})$$

and $H^1(X_0, \mathcal{O}_{X_0}) = 0$ so the map on tangent spaces is injective.

It remains to show that the closed immersion $S' \to S$ is defined by a single equation i.e. the ideal $\ker u = I$ is monogenic. To do this, consider $S'' = \mathrm{Spf}\,(R/\mathfrak{m}I)$ which is a thickening of S' inside S by the square-zero ideal $I/\mathfrak{m}I$. The obstruction to extend the sheaf L' defined over $\mathscr{X} \times_S S''$ is an element

$$\mathrm{ob} \in H^2(X_0, I/\mathfrak{m}I) = H^2(X_0, \mathcal{O}_{X_0}) \otimes I/\mathfrak{m}I$$

which can be regarded as an element of $I/\mathfrak{m}I$ given a choice of generator of $H^2(X_0, \mathcal{O}_{X_0})$. Let $\Sigma = \mathrm{Spf}\,((R/(\mathfrak{m}I + (f)))$ where $f \in I$ lifts a. We then have

$$S' \subset \Sigma \subset S'' \subset S$$

and by construction (and functoriality of the aobstruction) L' lifts to $\mathscr{X} \times_S \Sigma$. By the universal property of S' this means $S' = \Sigma$ menaing $\mathfrak{m}I + (f) = I$. Then by Nakayama's lemma, f generates I. □

Theorem 1.1.2 (A)**.** Let L_0 be a nontrivial sheaf on X_0. Then the formal scheme $\Sigma(L_0)$ pro-represents $\underline{\mathrm{Def}}(X_0, L_0)$ and is flat over W of relative dimension 19.

In other words, if f is the equation defining $\Sigma(L_0)$ over S then p does not divide f. This still means that $\Sigma(L_0)$ is not contained in the reduction S_0 of S mod p hence L_0 does not lift to $\mathscr{X} \times_S S_0$.

We prove this is section 2. We finish the section with some consequences of this theorem.

Corollary 1.1.3. Let L_0 be a nontrivial invertible sheaf on X_0. There exists a trait T finite over W, a deformation of X_0 to a formal scheme X flat over T, and an extension of L_0 to an invertible sheaf L on X.

It suffices to show there is a W-morphism $T \to \Sigma(L_0)$ with T a trait finite over W. Since p is not a zero-divisor in R', there exists elements $x_1, \dots, x_n \in \mathfrak{m} \subset R'$ forming along with p a system of parameters. The quotient $B = R'/(x_1, \dots, x_n)$ is quasi-finite over W, hence finite over W. There exists a local W-homomorphism $B \to C$ to a complete DVR finite over W so the composition $R' \to B \to C$ answers the question.

Applying Grothendieck's algebraization theorem (EGA III, 5.4.5), we deduce from 1.7 the following theorem:

Corollary 1.1.4. Let L_0 be an ample invertible sheaf on X_0. There exists a trait T finite over W and a deformation of X_0 to a proper smooth scheme $X \to T$ and an extension L of L_0 over X.

Remark. We do not know whether any K3 surface over k lifts to a proper smooth scheme over W. Ogus [13] shows that (a) every K3 surface over k lifts over W except perhaps the superspecial case, non elliptic, which actually should not exist if we accept Artin's conjecture [1]. (b) if $p > 2$ every K3 surface over k lifts over $W[\sqrt{p}]$ therefore only the specal case of 1.8 for $p = 2$ and X_0 superspecial is not covered by other results. Let's point out that the other part of Ogus' article has interesting additions on the structure of $\Sigma(L_0)$, cf. also the following exposition in the ordinary case.

Corollary 1.1.5. If k is the algebraic closure of a finite field then on which X_0 is defined, the Frobenius has a semisimple action on $H^2(X_0, \mathbb{Q}_\ell)$ for $\ell \neq p$.

1.2 de Rham Cohomology and Theorem A

Use the same notation as the previous section. Let X_0 be a K3 surface over k, and $\mathscr{X}/S$ the universal W-deformation. The reader familar with de Rham cohomology is invited to skip the first section which develops standard material about the Gauss-Manin connection, the Hodge filtration, the action of Frobenius, and the Chern classes of invertible sheaves.

Let $\Omega^\bullet_{\mathscr{X}/S}$ be the de Rham complex of the formal scheme which is a relative scheme (by definition, it is the projective limit of the de Rham complex for infinitesimal neighbrohoods of $\operatorname{Spec}(k)$).

Proposition 1.2.1. The spectral sequence

$$E_1^{i,j} = H^j(\mathscr{X}, \Omega^i_{\mathscr{X}/S}) \implies H^\bullet_{\mathrm{dR}}(\mathscr{X}/S)$$

degenerates at E_1 and the Hodge cohomology groups are free of finite ype and the canonical maps

$$H^j(\mathscr{X}, \Omega^i_{\mathscr{X}/S}) \otimes k \to H^j(X_0, \Omega^i_{X_0/k})$$

are isomorphisms. The $\mathcal{O}_S$-modules $H^i_{\mathrm{dR}}(\mathscr{X}/S)$ are free of finite type, and the canonical maps

$$H^i_{\mathrm{dR}}(\mathscr{X}/S) \otimes k \to H^i_{\mathrm{dR}}(X_0/k)$$

are isomorphims. The cup-product

$$\smile : H^2_{\mathrm{dR}}(\mathscr{X}/S) \otimes H^2_{\mathrm{dR}}(\mathscr{X}/S) \to H^4_{\mathrm{dR}}(\mathscr{X}/S)$$

is a perfect pairing.

Proof. Indeed, since the Hodge table is "interlaced with zeros" cohomology and base change applies to show these results. The last assertion follows from flatness and Poincare duality over k. $\square$

1.2.1 2.3

Let $\Omega^\bullet_{S/W}$ be the de Rham complex of "formal" differentials of S/W menaing

$$\Omega^i_{S/W} = \bigwedge^i \Omega_{X/S} \quad \Omega_{S/W} = \varprojlim_n \Omega_{S_n/W_n}$$

where Ω_{S_n/W_n} is the module of compelte differentials of S_n/W_n the mod p^n-reduction of $W[\![t_1, \dots, t_n]\!]$. This is free over $\mathcal{O}_S$ with basis $\mathrm{d}t_1, \dots, \mathrm{d}t_{20}$. Then $H^i_{\mathrm{dR}}(\mathscr{X}/S)$ is equipped with a canonical integrable connection, the Gauss-Manin connection

$$\nabla : H^i_{\mathrm{dR}}(\mathscr{X}/S) \to H^i_{\mathrm{dR}}(\mathscr{X}/S) \otimes \Omega^1_{S/W}$$

We can use the fact that $H^i_{\mathrm{dR}}(\mathscr{X}/S)$ is the value over S of a cystal in $\mathcal{O}$-modules on the crystalline site of S_0/W

$$H^i_{\mathrm{dR}}(\mathscr{X}/S) = R^i(f_0)_{\mathrm{crys}*}(\mathcal{O}_{\mathscr{X}_0/W})(S)$$

2 Helene's Paper

Let X be a K3 surface over an algebraically closed field k of characteristic $p > 0$. We assume $p > 3$. Let S be the formal deformation space of X and $\mathrm{Spec}\,(R) \to S$ a morphism from a DVR such that $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(W)$ is dominant. Let $X_R \to \mathrm{Spec}\,(R)$ be a proper model of X. Let $K = \mathrm{Frac}\,(R)$.

There is a specialization homomorphism

$$\iota : \mathrm{Aut}^e(X_{\overline{K}}/\overline{K}) \to \mathrm{Aut}(X/k)$$

where

$$\mathrm{Aut}^e(X_{\overline{K}}/\overline{K}) \subset \mathrm{Aut}(X_{\overline{K}}/\overline{K})$$

is the subgroup of automorphisms that lift to some model. We say that $f \in \mathrm{Aut}(X/k)$ is not geometrically liftable to characteristic 0 if it is not in the image of ι.

3 Some Ideas

The deformation theory of pairs (X, ϕ) is controlled by the complex

$$C^\bullet = [\mathcal{T}_X \xrightarrow{\mathrm{d}\phi - \mathrm{id}} \mathcal{T}_X]$$

placed in degrees $0, 1$ in the sense that $\mathbb{H}^i(C^\bullet)$ is a automorphisms-tangent-obstruction theory. Indeed, by definition it fits into an exact triangle

$$C \to \mathcal{T}_X \xrightarrow{\mathrm{d}\phi - \mathrm{id}} \mathcal{T}_X \to +1$$

since $H^i(X, \mathcal{T}_X) = 0$ for $i \neq 1$ we get an exact sequence

$$0 \to H^1(C^\bullet) \to H^1(X, \mathcal{T}_X) \xrightarrow{\mathrm{d}\phi - \mathrm{id}} H^1(X, \mathcal{T}_X) \to H^2(C^\bullet) \to 0$$

Hence the moduli space of deformations (X, ϕ) has virtual dimension zero. It looks like it also has a perfect obstruction theory.

4 Automorphisms of K3 surfaces and lifting

Let X/k be a K3 surface over a field k. Write $\phi : X \to X$ for an automorphism defined over k. Let $G \circlearrowright X$ be a group of automorphisms defined over k acting on X.

4.1 The representation on $H^1(X, \mathcal{T}_X)$

Lemma 4.1.1. $G \circlearrowright H^1(X, \mathcal{T}_X)$ via the representation $H^1(X, \Omega_X)^\vee \otimes H^0(X, \omega_X)^\vee$.

Proof. Consider the pairing

$$\Omega_X \otimes \mathcal{T}_X \xrightarrow{\langle -,- \rangle} \mathcal{O}_X$$

this induces via the Yoneda pairing a morphism

$$H^1(X, \Omega_X) \otimes H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X) \xrightarrow{\sim} H^0(X, \omega_X)^\vee$$

where the last map is Serre duality. These are equivariant maps. I claim that this pairing is perfect meaning that the map

$$H^1(X, \mathcal{T}_X) \to H^1(X, \Omega_X)^\vee \otimes H^0(X, \omega_X)^\vee$$

is an isomorphism. This would exhibit a natural G-equivariant isomorphism as required. Choose a generator $\omega \in H^0(X, \omega_X)$ this gives a commutative diagram

$$\begin{tikzcd} \Omega_X \otimes \mathcal{T}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d, "\omega"] \\ \Omega_X \otimes \Omega_X \arrow[r, "\wedge"] & \omega_X \end{tikzcd}$$

where the downward map is given by

$$\eta \otimes \xi \mapsto \eta \otimes \omega(\xi, -)$$

since the downward maps are isomorphisms this gives a diagram of pairings

$$\begin{tikzcd} H^1(\Omega_X) \otimes H^1(\mathcal{T}_X) \arrow[r] \arrow[d] & H^2(\mathcal{O}_X) \arrow[d, "\omega"] \\ H^1(\Omega_X) \otimes H^1(\Omega_X) \arrow[r, "\wedge"] & H^2(\omega_X) \end{tikzcd}$$

and the bottom is perfect by Poincaré duality [D, 2.3.13]. $\square$

We need one other ingredient:

Lemma 4.1.2. Let X be Shioda supersingular (meaning $\rho = b_2$) and with Artin invariant 1. Then the map

$$c_1^{\mathrm{Hodge}} : \mathrm{NS}(X) \otimes k \to H^1(X, \Omega^1_X)$$

is a surjective map of $k[G]$-modules.

Proof. Theorem 11.10 of van der Geer and Katsura. $\square$

4.2 Computing representations

These facts give us enough information to completely understand the representation $G \circlearrowright H^1(X, \mathcal{T}_X)$ using the information computed by [KS]. Let $X = X(3)$ be the Fermat quartic over $\mathbb{F}_3$ and set $k = \mathbb{F}_9$ over which the automorphism group and Neron-Severi groups are defined. We use the KS description of the automorphism group:

$$\mathrm{Aut}(X) := \langle \tau_1, \tau_2, \mathrm{Aut}(X, h) \rangle$$

where τ_i are involutions associated to generically finite 2-to-1 covers

$$\varphi_i : X \to \mathbb{P}^2$$

Over the locus where φ_i is finite (which is an open set in $\mathbb{P}^2$ whose complement has finitely many points) it induces an isomorphism $\varphi_i^* : H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}) \to H^0(X, \omega_X)^{\tau_i}$ via the trace map (here we use that 2 is coprime to the characteristic) whch extends because it is defined away from codimension 2 on the base. Since $H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}) = 0$ we must have $\tau_i^* \omega = -\omega$ since it is an involution that acts nontrivially on $H^0(X, \omega_X)$. Let $h = \iota^* \mathcal{O}_{\mathbb{P}^3}(1)$ under the standard embedding $\iota : X \hookrightarrow \mathbb{P}^3$. Then, as in [KS], we identify

$$\mathrm{Aut}(X, h) = \mathrm{PGU}_4(\mathbb{F}_9) \hookrightarrow \mathrm{PGL}_4(\mathbb{F}_9)$$

induced by the embedding $\iota : X \hookrightarrow \mathbb{P}^3$.

This group is described as follows. First define,

$$\mathrm{U}_4(\mathbb{F}_9) = \{ A \in \mathrm{GL}_4(\mathbb{F}_9) \mid AA^* = I \}$$

where A^* is the conjugate transpose using the Frobenius σ generating the Galois group of $\mathbb{F}_9/\mathbb{F}_3$.

Remark. The identification $\mathrm{Aut}(X, h) = \mathrm{PGU}_4(\mathbb{F}_9)$ arises from viewing the defining function

$$F(X_0, X_1, X_2, X_3) = X_0^4 + X_1^4 + X_2^4 + X_3^4$$

as the inner product of (X_0, X_1, X_2, X_3) and $(X_0^\sigma, X_1^\sigma, X_2^\sigma, X_3^\sigma)$. Hence the group of automorphisms of $\mathbb{P}^3_{\mathbb{F}_9}$ preserving F is exactly $\mathrm{PGU}_4(\mathbb{F}_9)$. Any automorphism preserving F up to scaling is represented by an element of $\mathrm{PGU}_4(\mathbb{F}_9)$. Indeed, if $F(A\underline{X}) = \lambda F(\underline{X})$ then plugging in $\underline{X} = (1, 0, 0, 0)$ we see that λ is a sum of 4-th powers of elements in $\mathbb{F}_9$ and hence lies in $\mathbb{F}_3^\times$. Either element has a 4-th root ξ in $\mathbb{F}_9$ (since the Norm is surjective) and hence we can modify A to $\xi^{-1}A$ so that $F(\xi^{-1}A\underline{X}) = F(\underline{X})$ hence $A \in \xi \cdot \mathrm{U}_4(\mathbb{F}_9)$ so the image in $\mathrm{PGL}_4(\mathbb{F}_9)$ lies in $\mathrm{PGU}_4(\mathbb{F}_9)$.

Note that for $A \in \mathrm{U}_4(\mathbb{F}_9)$ we have

$$(\det A)\, \sigma(\det A) = 1$$

hence $\det A \in \ker \mathrm{Nm}$ where $\mathrm{Nm}(x) = x\sigma(x)$ is the norm for $\mathbb{F}_9/\mathbb{F}_3$. Likewise, if $\lambda I \in \mathrm{U}_4(\mathbb{F}_9)$ it satisfies the same condition so the central torus satisfies,

$$\mathbb{F}_9^\times \cdot I \cap U_4(\mathbb{F}_9) = \ker \mathrm{Nm} \cdot I$$

Thus we define

$$\mathrm{PGU}_4(\mathbb{F}_9) = \mathrm{U}_4(\mathbb{F}_9) / \ker \mathrm{Nm} \cdot I$$

Crucially the determinant

$$\mathrm{PGU}_4(\mathbb{F}_9) \xrightarrow{\det} \ker \mathrm{Nm} \subset \mathbb{F}_9^\times$$

is well-defined since the kernel of the norm map has order 4. The kernel of this map is $\mathrm{PSL}_4(\mathbb{F}_9)$ by definition. Note that $\mathrm{PSL}_4(\mathbb{F}_9)$ is simple (e.g. by these notes by Keith Conrad) so the determinant map coincides with the abelianization of $\mathrm{PGU}_4(\mathbb{F}_9)$. In particular, the representation

$$\mathrm{PGU}_4(\mathbb{F}_9) \circlearrowright H^0(\omega_X)$$

factors through the determinant.

Proposition 4.2.1. The representation $\mathrm{Aut}(X) \circlearrowright H^0(\omega_X)$ is determined by the following data

(a) τ_i acts via -1

(b) $\mathrm{Aut}(X, h) = \mathrm{PGU}_4(\mathbb{F}_9)$ acts via det

Proof. Because $\mathrm{PGU}_4(\mathbb{F}_9)$ must act factoring through the determinant, it suffices consider the action of diagonal matrices of the form $(1, \lambda, 1, 1)$ since $\det A = \det(1, \det A, 1, 1)$. If

$$F(X_0, \dots, X_n) = 0$$

is the equation for a Calabi-Yau hypersurface in $\mathbb{P}^n$ then the top form can be written

$$\omega = \frac{\mathrm{d}\left(\frac{X_1}{X_0}\right) \wedge \cdots \wedge \mathrm{d}\left(\frac{X_{n-1}}{X_0}\right)}{X_0^{-(n+1)} \partial_{X_n} F(X_0, \dots, X_n)}$$

For us,

$$F(X_0, \dots, X_3) = X_0^4 + \cdots + X_3^4$$

and therefore $\partial_{X_4} F = 4X_3^3$ so the above matrix acts by multiplication by λ hence proving the claim.

We can give an alternative argument that does not rely on the standard form for ω. There is a natural GL_{n+1} linearization on $\mathcal{O}_{\mathbb{P}^n}(1)$ and a natural PGL_{n+1}-linearization on $\omega_{\mathbb{P}^n}^\vee$ as the dual of the usual equivariant structure on the canonical bundle. These are not quite compatible. The discrepancy is exactly

$$\mathcal{O}_{\mathbb{P}^n}(n+1) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes(n+1)} \cong \omega_{\mathbb{P}^n}^\vee \otimes \det$$

where det is the 1-dimensional determinant representation of GL_{n+1} (or equivalently $\mathcal{O}_{\mathbb{P}^n}$ endowed with this nontrivial linearization). To see this, recall that the PGL_{n+1} linearization is defined by taking the induced SL_{n+1}-linearization on $\omega_{\mathbb{P}^n}^\vee \cong \mathcal{O}_{\mathbb{P}^n}(n+1)$ and notincing that it kills μ_{n+1} hence factors through PGL_{n+1}. Since any matrix can be written as λA for $A \in \mathrm{SL}_{n+1}$ (at the level of $\bar{k}$-points) we see that the action on $\mathcal{O}_{\mathbb{P}^n}(n+1)$ is via $\lambda^{n+1} A_*$ but $\lambda^{n+1} = \det(\lambda A)$ which demonstrates the claim. Consider the inclusion of a Calabi-Yau hypersurface

$$X \hookrightarrow \mathbb{P}^n$$

inducing $G = \mathrm{Aut}(X, \mathcal{O}_X(1)) \hookrightarrow \mathrm{GL}_{n+1}$ these are automorphism of the pair $(X, \mathcal{O}_X(1))$ (this is a larger group than those automorphisms preserving the line bundle up to isomorphism, indeed it is an extension by $\mathbb{G}_m$ of $\mathrm{Aut}(X, h)$). The canonical construction of ω_X, which gives it a G-equivariant structure is

$$\omega_X = \mathscr{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \omega_{\mathbb{P}^n})$$

this can be computed via a G-equivariant resolution

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-(n+1)) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$$

however, this is not quite G-equivariant. Since $G \circlearrowright X$ we see that f is preserved up to scaling but the character $s_f : G \to k^\times$ is nontrivial and must enter into the exact sequence. The correct G-equivariant sequence is

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-(n+1)) \otimes s_f \xrightarrow{f} \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$$

where $\mathcal{O}_{\mathbb{P}^{n+1}}(-(n+1))$ is given a G-structure through $G \hookrightarrow \mathrm{GL}_{n+1}$ and the others have the trivial G-structure. Using this resolution,

$$\omega_X \cong (\mathcal{O}_{\mathbb{P}^{n+1}}(n+1) \otimes s_f^\vee \otimes \omega_{\mathbb{P}^n})|_X = \det \otimes s_f^\vee$$

Therefore, $\mathrm{Aut}(X, \mathcal{O}_X(1))$ acts on $H^0(\omega_X)$ via $\det \otimes s_f^\vee$. Notice the action on $H^0(\omega_X)$ factors through

$$\mathrm{Aut}(X, \mathcal{O}_X(1)) \to \mathrm{Aut}(X, h)$$

as it must since elements of the kernel define trivial automorphisms of X. Indeed, this holds because for a scalar matrix $\lambda \cdot \mathrm{id} \in \mathrm{Aut}(X, \mathcal{O}_X(1)) \hookrightarrow \mathrm{GL}_{n+1}$ acts via $\lambda^{n+1} \cdot \lambda^{-(n+1)} = 1$ because it scales f by λ^{n+1}. However, this action is nontrivial on elements of $\mathrm{Aut}(X, h)$. Indeed, in our case of interest, $\mathrm{U}_4(\mathbb{F}_9) \subset \mathrm{Aut}(X, \mathcal{O}_X(1))$ is the kernel of s_f so on $\mathrm{U}_4(\mathbb{F}_9)$ and hence on $\mathrm{PGU}_4(\mathbb{F}_9) = \mathrm{Aut}(X, h)$ the action on $H^0(\omega_X)$ is via det. $\square$

4.3 The representation on $H^1(X, \Omega_X)$

We leverage the following trick to compute $G \circlearrowright H^1(X, \Omega_X)$.

Lemma 4.3.1. $G \circlearrowright \mathrm{NS}(X) \otimes \mathbb{F}_9$ has a unique rank 2 submodule

Proof. In fact, a MAGMA computation shows that this is already true about the action $\langle \tau_1, \tau_2, \tau \rangle$. Using the explicit integer matrices for the action on $\mathrm{NS}(X)$ we find a submodule lattice:

```
Submodule Lattice of GModule M of dimension 22 over GF(3^2)

Partially ordered set of submodule classes
------------------------------------------


---
[7]  Dim 22  Maximal submodules: 5 6
---
[6]  Dim 21  Maximal submodules: 4
[5]  Dim 3   Maximal submodules: 4
---
[4]  Dim 2   Maximal submodules: 2 3
---
[3]  Dim 1   Maximal submodules: 1
[2]  Dim 1   Maximal submodules: 1
---
[1]  Dim 0   Maximal submodules:
```

so we see that [4] is the unique submodule of rank 2 and it consists of two invariant lines. Call this submodule $U_2 \subset \mathrm{NS}(X) \otimes \mathbb{F}_9$. $\square$

Since $\ker c_1^{\mathrm{Hodge}}$ is also rank 2 because c_1^{Hodge} is surjective onto a rank 20 space and is G-invariant, we conclude that

$$U_2 = \ker c_1^{\mathrm{Hodge}}$$

Therefore, the action $G \circlearrowright H^1(X, \Omega_X)$ is isomorphic to the action

$$G \circlearrowright (\mathrm{NS}(X)_{\mathbb{F}_9}/U_2)$$

which is easily computed in MAGMA.

4.4 Computations

[KS] computed explicit matrices for τ_i and $\mathrm{PGU}_4(\mathbb{F}_9)$ acting on $\mathrm{NS}(X)$.

Remark. In [KS] the matrices act *on the right* meaning $\tau_*(x) = \vec{x}T_\tau$ where $\vec{x}$ in row vector corresponding to $x \in \mathrm{NS}(X)$ the line basis specified by [KS]. This is consistent with MAGMA conventions for right actions. The reader should be aware that we will adopt this convention throughout so $G \circlearrowright X$ is a right action and $\phi_1 \circ \phi_2$ is the automorphism given by first applying ϕ_1 and then ϕ_2. This will not change anything substantial about the calculation (even if the wrong convention were employed) because transpose does not change the rank or spectrum.

We let ϕ_i be the i^{th}-automorphism in $\mathrm{PGU}_4(\mathbb{F}_9)$ according to the indexing of [KS, data files]. Also let τ be the special automorphism of order 28 considered in [KS12, Ex. 3.4]. and in [EO]. Explicitly,

$$\tau := \begin{pmatrix} i & 0 & i & -1+i \\ 1 & 1-i & -1 & 0 \\ 1 & i & i & -i \\ 1 & -1 & -i & -1 \end{pmatrix} \tag{1}$$

where $i \in \mathbb{F}_9$ is a choice of square root of -1. Note that i generates $\ker \mathrm{Nm}$. We also record two other elements of $\mathrm{PGU}_4(\mathbb{F}_9)$

$$\phi_2 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

$$\phi_5 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

Notice that:

(a) $\det \tau = 1$

(b) $\det \phi_2 = -1$

(c) $\det \phi_5 = -1$.

Now we define

$$\phi := \tau_1 \circ \tau \circ \phi_2 \circ \tau_2 \circ \tau \circ \phi_5 \circ \tau_1 \circ \tau \in \mathrm{Aut}(X)$$

Note that $\phi \circlearrowright H^0(\omega_X)$ by -1 since each τ_i acts by -1 and ϕ_2, ϕ_5 act through their determinant by -1 and τ acts through its determinant by 1.

Theorem 4.4.1. Let $\phi \circlearrowright X$ be the automorphism as above. Then

(a) $\phi \circlearrowright H^1(X, \mathcal{T}_X)$ has no 1-eigenspace meaning that $\phi_* - \mathrm{id}$ is an isomorphism

(b) $\phi \circlearrowright \mathrm{NS}(X)$ has characteristic polynomial

$$\begin{aligned} p_\phi(x) = x^{22} &- 30x^{21} + 15x^{20} - 14x^{19} + 16x^{18} + 7x^{17} - 19x^{16} \\ &+ 4x^{15} - 14x^{14} + 15x^{13} - 4x^{12} + 10x^{11} - 4x^{10} + 15x^9 - 14x^8 \\ &+ 4x^7 - 19x^6 + 7x^5 + 16x^4 - 14x^3 + 15x^2 - 30x + 1 \end{aligned}$$

which is irreducible

(c) ϕ has positive entropy $h(\phi)$ equal to the logarithm of a Salem number a of degree 22 which is numerically

$$a = 29.5071\cdots, \quad h(g) = 3.38463\cdots$$

(d) ϕ does not lift to any projective model $X_R \to \mathrm{Spec}\,(R)$

(e) ϕ lifts to a canonically-defined formal scheme $\mathscr{X} \to \mathrm{Spf}\,(W(k))$ lift of X.

Proof. Let T be the matrix representing the action $\phi \circlearrowright H^1(X, \Omega_X)$. In the previous section we found a method to compute this. From the G-isomorphism

$$H^1(X, \mathcal{T}_X) \xrightarrow{\sim} H^1(X, \Omega_X)^\vee \otimes H^0(\omega_X)^\vee$$

we see that $\phi \circlearrowright H^1(X, \mathcal{T}_X)$ via the matrix $-T^\top$ (using that $\phi \circlearrowright H^0(\omega_X)$ by -1). MAGMA computes the factorization of the characteristic polynomial of this action on the 20-dimension $\mathbb{F}_9$-vectorspace $H^1(X, \mathcal{T}_X)$ to be

```
[
    <t + 1, 2>,
    <t + F.1^2, 1>,
    <t + F.1^6, 1>,
    <t^8 + 2*t^7 + F.1^2*t^6 + F.1^7*t^5 + F.1^6*t^4 + F.1^7*t^3 + F.1^2*t^2 +
        2*t + 1, 1>,
    <t^8 + 2*t^7 + F.1^6*t^6 + F.1^5*t^5 + F.1^2*t^4 + F.1^5*t^3 + F.1^6*t^2 +
        2*t + 1, 1>
]
```

where $F.1$ is a generator of $\mathbb{F}_9^\times$ in this case $1+i$. Since $F.1^2 = -i$ and $F.1^6 = i$ this means that $\phi \circlearrowright H^1(X, \mathcal{T}_X)$ does not have any 1-eigenspace (or generalized eigenspace). This proves (a). For (b) MAGMA performs the calculation using the explicit matrices defined in the data associated to [KS]. Then (c) follows immediately. (d) follows from the results of [EO] showing that no automorphism with entropy the log of a Salem number of degree 22 can lift to a projective K3 surface over characteristic zero. (e) then follows from the subsequent discussion. $\square$

4.5 Deformations of Automorphism

Lemma 4.5.1. Let X be a smooth proper k-scheme. Let $f : X \to X$ be an endomorphism. The complex

$$C^\bullet = [\mathcal{T}_X \xrightarrow{f^* - \mathrm{d}f} f^*\mathcal{T}_X]$$

supported in degrees $[0,1]$ controls the deformation theory of the pair (X, f). Explicitly, for any small extension of Artin rings $A' \twoheadrightarrow A$ with residue field A and given a deformation (X_A, f_A) of (X, f) over A there is

(a) an obstruction class $\mathrm{ob} \in \mathbb{H}^2(C^\bullet)$

(b) if $\mathrm{ob} = 0$ then the deformations of (X_A, ϕ_A) to A' form a torsor over $\mathbb{H}^1(C^\bullet)$

(c) the automorphisms of any deformation to A' is isomorphic to $\mathbb{H}^0(C^\bullet)$.

Proof. Let $\{U_i\}$ be an affine cover of X. Consider the Čech complex computing $\mathbb{H}^*(C^\bullet)$:

$$\check{C}^{p,q} = \bigoplus_{i_0 < \cdots < i_p} \Gamma(U_{i_0 \ldots i_p}, C^q)$$

with differentials $d_1 = \check{d}$ (the Čech differential) and $d_2 = f^* - \mathrm{d}f$. This is a double complex whose total complex computes $\mathbb{H}^*(C^\bullet)$.

Given a deformation (X_A, f_A) over A and a small extension $A' \twoheadrightarrow A$ with kernel I, let $\widetilde{U_i}$ be local lifts of U_i and $\widetilde{f_i} : \widetilde{U_i} \to \widetilde{U_i}$ be local lifts of f. For compatibility, we need diagrams

$$\begin{tikzcd}
\widetilde{U_i}|_{U_{ij}} \arrow[r, "\widetilde{f_i}+\eta_i"] \arrow[d, "\varphi_{ij}+\xi_{ij}"] & \widetilde{U_i}|_{U_{ij}} \arrow[d, "\varphi_{ij}+\xi_{ij}"] \\
\widetilde{U_j}|_{U_{ij}} \arrow[r, "f_i+\eta_i"] & \widetilde{U_j}|_{U_{ij}}
\end{tikzcd}$$

for some choice of isomorphism $\varphi_{ij} : \widetilde{U_i}|_{U_{ij}} \xrightarrow{\sim} \widetilde{U_j}|_{U_{ij}}$ such that $\varphi_{ij} = \mathrm{id} \mod I = \ker(A' \to A)$. This exists since there is a unique isomorphism class of deformation of a smooth affine scheme. Here $\xi_{ij} \in \check{C}^{1,0} \otimes I$ and $\eta_i \in \check{C}^{0,1} \otimes I$ and we can write any such deformation of f_i in this form any any gluing as $\varphi_{ij} + \xi_{ij}$ defining a deformation of X. The key compatibility equation is derives from commutativity of the above diagram and reads:

$$(\widetilde{f_i} + \eta_i) \circ (\varphi_{ij} + \xi_{ij})^\# = (\varphi_{ij} + \xi_{ij})^\# \circ (\widetilde{f_j} + \eta_j)^\#$$

which expands to

$$\widetilde{f_i}^\# \varphi_{ij}^\# + \eta_i + f_i^\# \xi_{ij}^\# = \varphi_{ij}^\# \widetilde{f_j}^\# + \xi_{ij} f_j^\# + \eta_j$$

using that $I^2 = 0$ and that ξ or η land in I so multiplying by either has the effect of reduction of the other term mod I. This means we need the equation

$$\widetilde{f_i}^\# \varphi_{ij}^\# - \varphi_{ij}^\# \widetilde{f_j}^\# = f_j^\# + \eta_j - \eta_i + \xi_{ij} f_j^\# - f_i^\# \xi_{ij}^\#$$

The RHS is the total differential of (η, ξ) in the total complex projected to $\check{C}^{1,1}$. Therefore, the space of solutions (if one exists) forms a torsor over $\mathbb{H}^1(C^\bullet)$. We need to simulteneously be able to solve this equation along with the cocycle that says

$$d(\xi_{ij}) = \varphi_{ij}^\# \circ (\varphi_{ik}^\#)^{-1} \circ \varphi_{jk}^\#$$

This defines an element

$$(\varphi_{ij}^{\#} \circ (\varphi_{ik}^{\#})^{-1} \circ \varphi_{jk}^{\#}, \widetilde{f}_i^{\#} \varphi_{ij}^{\#} - \varphi_{ij}^{\#} \widetilde{f}_j^{\#}) \in \check{C}^{2,0} \oplus \check{C}^{1,1}$$

which is easily checked to be a cocycle and we are asking if it is a coboundary hence giving an obstruction

$$\mathrm{ob} \in \mathbb{H}^2(C^\bullet)$$

□

Remark. If we assume Def_X is unobstructed so we can choose φ_{ij} to satisfy the cocycle condition then we obtain The obstruction class lies in

$$\mathrm{coker}(H^1(\mathcal{T}_X) \xrightarrow{f^* - \mathrm{d}f} H^1(f^* \mathcal{T}_X))$$

since we are asking if a class in $H^1(f^*\mathcal{T}_X)$ lies in the image of $f^* - \mathrm{d}f$ up to a boundary $\{\eta_i\}$. When this vanishes, the choices of compatible deformations form a torsor over $\mathbb{H}^1(C^\bullet)$ as claimed.

Remark. When $\phi : X \to X$ is an automorphism then $\phi^* : \mathcal{T}_X \to \phi^*\mathcal{T}_X$ is an isomorphism of sheaves so we can form $\phi_* : \mathcal{T}_X \to \mathcal{T}_X$ as $(\phi^*)^{-1} \circ \mathrm{d}\phi$ which is the pushforward of vector fields in the sense used in differential geometry. It is clear that $C^\bullet$ is isomorphic to the complex

$$[\mathcal{T}_X \xrightarrow{\mathrm{id} - \phi_*} \mathcal{T}_X]$$

supported in degrees $[0, 1]$.

Lemma 4.5.2. If X is a smooth proper k-variety with $H^0(X, \mathcal{T}_X) = H^2(X, \mathcal{T}_X) = 0$ then there is an exact sequence

$$0 \to \mathbb{H}^1(C^\bullet) \to H^1(X, \mathcal{T}_X) \xrightarrow{\phi_* - \mathrm{id}} H^1(X, \mathcal{T}_X) \to \mathbb{H}^2(C^\bullet) \to 0$$

Corollary 4.5.3. If moreover, $\phi_* - \mathrm{id}$ is an isomorphism on $H^1(X, \mathcal{T}_X)$ then (X, ϕ) is unobstructed and has a trivial tangent space.

Corollary 4.5.4. Suppose that X is a smooth proper k-variety with an endomorphism $\phi : X \to X$. Let k be a perfect field of characteristic $p > 0$. Suppose that

(a) $H^0(X, \mathcal{T}_X) = 0$

(b) $H^2(X, \mathcal{T}_X) = 2$

(c) $\phi_* - \mathrm{id} : H^1(X, \mathcal{T}_X) \to H^1(X, \mathcal{T}_X)$ is an isomorphism

then there exists a canonical lift of (X, ϕ) to a formal scheme $\mathscr{X} \to \mathrm{Spf}\,(W(k))$.

Proof. Indeed, the above shows that the deformation space of (X, ϕ) is smooth over $W(k)$ of relative dimension 0. Since $\mathrm{Def}_{(X,\phi)} \to \mathrm{Spf}\,(W)$ is an isomorphism over k it is an isomorphism. □

5 Entropy

5.1 Spectral Radius Entropies

Recall that for an action $\varphi : V \to V$ on a complex vectorspace the *spectral radius* is

$$\rho(\varphi) := \sup_{\lambda} |\lambda| = \sup_{v \in V \setminus \{0\}} \frac{||\varphi v||}{||v||}$$

which holds for any choice of norm on V. The *entropy* of φ is $h(\varphi) := \log \rho(\varphi)$.

Definition 5.1.1. Let X be a finite CW complex (we just need the total ring $H^\bullet(X, \mathbb{C})$ finite dimensional over $\mathbb{C}$) and $\varphi : X \to X$ an automorphism. We define the *topological entropy* $h(\varphi)$ as the log of the spectral radius of $\varphi \circlearrowright H^\bullet(X, \mathbb{C})$.

Definition 5.1.2 (Esnault-Srinivas)**.** Let X be a smooth proper variety over a field k and $\varphi : X \to X$ an endomorphism over k. Then we define

(a) for a prime ℓ invertible on k, the characteristic polynomial of $\varphi \circlearrowright H^\bullet_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ is independent of ℓ and has integer coefficients and algebraic integer roots so we made defien the spectral radius of the action as a real number and its logarithm as the *topological entropy* $h(\varphi)$

(b) Let $\mathrm{CH}^\bullet_{\text{num}}(X_{\bar{k}})$ be the Chow ring modulo numerical equivalence. The underlying abelian groups is a finite free $\mathbb{Z}$-module hence we can define the *algebraic entropy* of φ as the log of the spectral radius of $\varphi \circlearrowright \mathrm{CH}^\bullet_{\text{num}}(X_{\bar{k}})$.

Remark. Note, if $k \hookrightarrow \mathbb{C}$ then by Artin comparison $h_{\mathrm{T}}(\varphi)$ can also be computed as the spectral radius on singular cohomology.

Theorem 5.1.3. If X is an algebraic surface then the algebraic entropy and topological entropy coincide.

5.2 Dinh and Sibony

Let X be a compact Kähler manifold of dimension n and $[\omega] \in H^2(X, \mathbb{R})$ a Kähler class. Choose a norm $|| \bullet ||$ on $H^{p,p}(X, \mathbb{R}) := H^{p,p}(X) \cap H^{2p}(X, \mathbb{R})$. For any endomorphism $f : X \to X$ we define

$$d_{p,n} := ||(f^n)^*[\omega^p]||$$

and

$$d_p := \limsup_{n \to \infty} d_{p,n}^{1/n} \quad H(f) := \sup_{0 \le p \le n} \log d_p$$

Note that $d_{p,n}$ does not depend on the choice of Kähler class. Furthermore, d_p and $H(f)$ do not depend on the the choice of form ω or on the choice of norm $||\bullet||$. Moreover, d_p is the spectral radius of f^* on $H^{p,p}(X)$ WHY?. We say that d_p is the dynamical degree of order p of f. In particular, we can compute d_p from the formula

$$d_p := \lim_{n \to \infty} \left(\int_X (f^n)^* \omega^p \wedge \omega^{k-p} \right)^{1/n}$$

5.2.1 Topological Entropy

Let dist denote the metric distance on (X, ω) induced by the associated Kähler metric. Then we denote

$$\mathrm{dist}_n(x, y) := \max_{0 \le i \le n-1} \mathrm{dist}(f^i(x), f^i(y))$$

Let $s_n(\epsilon)$ be the largest number of balls of radius $\epsilon/2$ defined in the metric dist_n that can fit disjointly in X. The *topological entropy* of f is defined by the formula

$$h(f) := \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log s_n(\epsilon)}{n}$$

Theorem 5.2.1 (Yomdin-Gromov)**.** Let f be a holomorphic endomorphism of a compact Kähler manifold X. Then $h(f) = H(f)$.

6 Salem Numbers

Definition 6.0.1. A *Salem* number is an algebraic integer $\lambda \in \mathbb{C}$ such that

(a) $\lambda \in \mathrm{R}_{>1}$

(b) every conjugate of λ has $|\lambda| \le 1$

(c) at least one conjugate of λ lies on the unit circle: $|\lambda| = 1$.

Proposition 6.0.2. The minimal polynomial of an algebraic integer on the unit circle (besides 1) is reciprocal meaning its coefficients are a palindrome.

Proof. Indeed, let $z \in \mathbb{C}$ have $|z|$ and $p(x)$ the minimal polynomial. Let $n = \deg p$ then since $z\bar{z} = 1$,

$$z^n \overline{p(1/\bar{z})} = z^n \overline{p(z)} = 0$$

so z is a root of $z^n \overline{p(1/\bar{z})}$ which is a nonzero polynomial of degree $\le n$ so we must have, by minimality,

$$p^\dagger = cp$$

where $p^\dagger(z) = z^n \overline{p(1/\bar{z})}$ is the conjugate reciprocal. Note that because $p \in \mathbb{Z}[x]$ we have that $p^\dagger$ is the reciprocal, the polynomial whose coefficients are reversed. Thus it suffices to show $c = 1$. The above formula says that if

$$p(x) = \sum_i a_i x^i$$

then $ca_i = \overline{a_{n-i}} = a_{n-i}$ since the coefficients are integers. Taking the sum,

$$c(a_0 + \cdots + a_n) = a_n + \cdots + a_0$$

and the sum is nonzero because $p(1) \ne 0$ because it is irreducible and $z \ne 0$ so we conclude $c = 1$. $\square$

Corollary 6.0.3. The minimal polynomial of a Salem number is reciprocal meaning its coefficients are a palindrome.

Lemma 6.0.4. Let p be a conjugate-reciprocal polynomial. Then $p(\alpha) = 0$ iff $p(\alpha^{-1}) = 0$.

Proof. This is obvious from $p^\dagger(x) = x^n\overline{p(1/\bar{x})}$ and $p = p^\dagger$. $\square$

Corollary 6.0.5. Let λ be a Salem number. Then the conjugates of λ are exactly

$$\lambda, \lambda^{-1}, \alpha_1, \ldots, \alpha_{n-2}$$

where $|\alpha_i| \in \mathbb{C}$. Since λ^{-1} is also an algebraic integer, λ is a unit in the ring of algebraic integers.

Proof. Since the minimal polynomial p is reciprocal we see that λ^{-1} is also a root. Furthermore, for any other root α, by assumption $|\alpha| \leq 1$ but since p is reciprocal α^{-1} is also a root so either $\alpha^{-1} = \lambda$ hence $\alpha = \lambda^{-1}$ or $|\alpha^{-1}| \leq 1$ so we conclude $|\alpha| = 1$. $\square$

6.1 Automorphisms of K3 surfaces over $\mathbb{C}$

Let X be a complex (possibly non-algebraic) K3 surface. Then $H^2(X, \mathbb{Z})$ is an even unimodular lattice of singnature $(3, 19)$. By the Torelli theorem $\mathrm{Aut}(X) \circlearrowright H^2(X, \mathbb{Z})$ is faithful. For $f \in \mathrm{Aut}(X)$, define two invariants:

(a) $\lambda(f)$ – the spectral radius of $f \circlearrowright H^2(X)$

(b) $\delta(f)$ – the eigenvalue of f on the line $H^{2,0}(X) = H^0(X, \omega_X) = \mathbb{C} \cdot \omega$ inside $H^2(X, \mathbb{C})$

The topological entropy of f is

$$h(f) = \log \lambda(f) \geq 0$$

Remark. Since f preserves the volume $\int_X \omega \wedge \overline{\omega}$ we see that $|\delta(f)| = 1$. Indeed, f acts on $H^4(X, \mathbb{Z})$ by the degree but $\deg f = 1$ so it must preserve this integral which is proportional to $[X]$ cap the generator of $H^4(X, \mathbb{Z})$. We refer to $\delta(f)$ as the *determinant* of f, since

$$\det Df_p = \delta(f)$$

for any fixed-point $p \in X$.

Proposition 6.1.1. If $h(f) > 0$ then $\lambda(f)$ is the unique eigenvalue outside the unit circle and hence is a Salem number. Otherwise the eigenvalues of f are all roots of unity.

Proof. If $h(f) = 0$ this means $\lambda(f) = 1$ so all eigenvalues are within the unit disk. Since they are non-zero algebraic integers the product of all of them, the constant term of p the characteristic polynomial of $f \circlearrowright H^2(X)$, is a nonzero integer ≤ 1 so it must equal 1 meaning that all $|\lambda_i| = 1$ since their product equals 1 and $|\lambda_i| \leq 1$. If all conjugates of an algebraic integer lie on the unit circle then it is a root of unity so in this case we conclude.

Otherwise, $\lambda(f) > 1$ and let λ be an eigenvalue with $|\lambda| > 1$. Since f^* stabilizes $H^{1,1}(X) \subset H^2(X)$ which has signature $(1, 19)$ it is conjugate to an element $T \in O(2, 0) \times O(1, 19)$. Use the following lemma to see that T has a unique eigenvalue outside the unit disk. Since λ is unique, it must be real.

Since f^* preserves the Kähler cone, the $O(1, 19)$ part does not interchange the sheets of the light-cone in $H^{1,1}_{\mathbb{R}}$, and thus $\lambda > 1$. Hence λ is a Salem number since it is the unique root of p outside the unit disk. Therefore p is a product of at most one Salem polynomial and some number of cyclotomic polynomials. $\square$

Lemma 6.1.2. A transformation $T \in O(p, q)$ has at most $\min(p, q)$ eigenvalues outside the unit circle, counted with multiplicities.

Proof. Consider the subspace

$$S = \bigoplus_{|\lambda|>1} E(\lambda)$$

which is isotropic and defined over $\mathbb{R}$. Thus

$$\dim S \leq \min(p, q)$$

and $\dim S$ is at least as large as the number of eigenvalues outside S^1. $\square$

Proposition 6.1.3. If X is projective then $\delta(f)$ is a root of unity so there cannot be a Siegel disk.

Proof. Since X is projective, there is a $D \in \mathrm{Pic}\,(()\,X) \subset H^2(X,\mathbb{Z})$ with $D^2 > 0$. The supsace $H^{1,1}(X) \cap D^{\perp}$ is negative-definite, with signature $(0,19)$ by the Hodge index theorem and contains $\mathrm{Pic}\,(X) \cap D^{\perp}$. The intersection form on $\mathrm{Pic}\,(X) \otimes \mathbb{R}$ hence has signature $(1,n)$ or some $0 \leq n \leq 19$. Consequence, the rational f^*-invariant subspace

$$S = \mathrm{Pic}\,(X)^{\perp} \supset H^{2,0}(X) \oplus H^{0,2}(X)$$

has signature $(2, 19-n)$. Now $f^*|_S$ preserves the signature $(2,0)$-subspace on the right, so it is conjugate to an element of $O(2) \times O(19-n)$. Thus all eigenvalues of $f^*|_S$, inclusing $\delta(f)$ lie on the unit circle (since it is conjugate to an orthogonal matrix). But $f^*|_S$ also preserves the lattice $S \cap H^2(X,\mathbb{Z})$, so its characteristic polynomial lies in $\mathbb{Z}[t]$, and therefore the eigenvalues are all roots of unity. $\square$

Theorem 6.1.4. Up to isomorphism, there are only countably many pairs (X, f) where $\delta(f)$ is not a root of unity and these all have algebraic periods.

Proof. Assume $\delta = \delta(f)$ is not a root of unity. Since the characteristic polynomial p of $f \circlearrowright H^2(X,\mathbb{Z})$ has a unique Salem factor and the other factors are cyclotomic, δ is a root of the Salem factor and hence has multiplicity 1 (since the Salem factor is irreducible). Therefore, $H^{2,0}(X)$ is an eigenspace for f^*, and therefore f^* determines the Hodge structure on $H^2(X)$ up to finitely many choices (the choice of root of the Salem factor not on the unit circle). By the Torelli theorem, the Hodge structure on $H^2(X)$ together with $f^* \circlearrowright H^2(X,\mathbb{Z})$ determines (X, f) up to isomorphism. There are only countably many because there are only countably many f acting on the lattice $H^2(X,\mathbb{Z})$. $\square$

7 Eratum

I think [KS] has the wrong matrix for τ. The automorphism of X whose corresponding action on $\mathrm{NS}(X)$ corresponds to the action T is given in the data set at line 3370320 (there is a shift by one where line n in FQprojautS corresponds to line $n+1$ in FQprojaut because there is an extra header line. If we look in the data file:

```
sed -n '3370321p' FQprojaut.m
[1, 2, 6, 2, 2, 5, 5, 5, 2, 6, 8, 0, 3, 0, 5, 2],
```

unwinding, we get the matrix

$$\tau = \begin{pmatrix} i & 0 & i & -1+i \\ 1 & 1-i & -1 & 0 \\ 1 & i & i & -1 \\ 1 & -1 & -i & -1 \end{pmatrix}$$

which is different from the matrix given in [KS]. Indeed, that matrix was in $\mathrm{PSL}_4(\mathbb{F}_9)$ but this one has $\det \tau = i$ whcih agrees with my calculations arising from T. Therefore, Helene's choice of automorphism actually will give a canonical lift.