1 Noether-Lefschetz

1.1 Outline of Grothendieck-Lefschetz

(ADD STACKS PROJECT REFERENCES)

Definition 1.1.1. Let $Y \subset X$ be a closed subscheme. We say that (X, Y) satisfies,

(a) $\text{Lef}(X, Y)$ if for any open $U \subset X$ containing X and any finite locally free $\mathcal{E}$ on U then,

$$\Gamma(U, \mathcal{E}) \to \Gamma(\hat{X}, \hat{\mathcal{E}})$$

is an isomorphism

(b) $\text{Leff}(X, Y)$ if it satisfies $\text{Lef}(X, Y)$ are moreover for any finite locally free $\mathfrak{E}$ on $\hat{X}$ there exists an open $U \subset X$ containing Y an a finite locally free $\mathcal{E}$ on U such that $\hat{\mathcal{E}} = \mathfrak{E}$.

Proposition 1.1.2. If (X, Y) satisfies $\text{Leff}(X, Y)$ then,

$$\text{Pic}(\hat{X}) = \varinjlim_{\substack{U \subset X \\ Y \subset Y}} \text{Pic}(U)$$

Proof. There is a natural map,

$$\varinjlim_{\substack{U \subset X \\ Y \subset Y}} \text{Pic}(U) \to \text{Pic}(\hat{X})$$

via completion. Then $\text{Lef}(X, Y)$ shows that this map is injective since if $\hat{\mathcal{L}}_1 \cong \mathcal{O}_{\hat{X}}$ then the section spreads out to the open U and since it is nonvanishing on Y it is nonvanishing on some smaller open $U' \subset U$ containing Y. Furthermore, the property $\text{Leff}(X, Y)$ says exactly that this map is surjective. $\square$

If we have the $\text{Leff}(X, Y)$ extension property then Lefschetz theorems can be reduced to computing formal liftings.

Theorem 1.1.3 (Grothendieck-Lefschetz (Hartshorne, Ample subvarities, Theorem 3.1))**.** Let X be an S_2 and locally-factorial variety and $Y \subset X$ a closed subscheme. Assume that,

(a) $\text{Lef}(X, Y)$ (resp. $\text{Leff}(X, Y)$)

(b) Y meets every effective divisor on X

(c) $H^i(Y, \mathscr{I}^n/\mathscr{I}^{n+1}) = 0$ for $i = 1$ (resp. for $i = 1, 2$) and all $n \geq 1$ where $\mathscr{I}$ is the ideal cutting out Y.

Then the natural map,

$$\text{Pic}(X) \to \text{Pic}(Y)$$

is injective (resp. an isomorphism).

Proof. Consider the restriction maps,

$$\text{Pic}(X) \to \varinjlim_{\substack{U \subset X \\ Y \subset U}} \text{Pic}(U) \to \text{Pic}(\hat{X}) \to \text{Pic}(Y)$$

we will show these are all injective (isomorphisms).

(a) the first is an isomorphism since by (b) any U containing Y satisfies $\operatorname{codim}(X, X \setminus U) \geq 2$ and X is S_2 and locally factorial which implies that $\operatorname{Pic}(X) \to \operatorname{Pic}(U)$ is an isomorphism

(b) the second is an injection (resp. isomorphism) by $\operatorname{Lef}(X, Y)$ (resp. by $\operatorname{Leff}(X, Y)$)

(c) the third is an injection (resp. isomorphism) via the following lifting computation. Consider the unit exact sequence,

$$0 \longrightarrow \mathscr{I}^n/\mathscr{I}^{n+1} \xrightarrow{x \mapsto 1+x} \mathcal{O}_{Y_{n+1}}^\times \longrightarrow \mathcal{O}_{Y_n}^\times \longrightarrow 0$$

On cohomology this gives,

$$H^1(Y, \mathscr{I}^n/\mathscr{I}^{n+1}) \longrightarrow \operatorname{Pic}(Y_{n+1}) \longrightarrow \operatorname{Pic}(Y_n) \longrightarrow H^2(Y, \mathscr{I}^n/\mathscr{I}^{n+1})$$

Therefore, since the left hand (resp. outer) groups are all zero for $n \geq 1$ there exists at most one lift to $\hat{X}$ (resp. we can lift any line bundle on $Y_1 = Y$ uniquely all the way to $\hat{X}$) giving an injection (resp. isomorphism),

$$\operatorname{Pic}(\hat{X}) = \varprojlim_n \operatorname{Pic}(Y_n) \hookrightarrow \operatorname{Pic}(Y)$$

$\square$

Therefore, it suffices to check when a pair (X, Y) satisfies $\operatorname{Leff}(X, Y)$ or $\operatorname{Leff}(X, Y)$. Luckily Grothendieck did this for us.

Theorem 1.1.4 (SGA2 Expose X, Example 2.2)**.** Let X be a proper k-scheme and $\mathcal{L}$ an ample line bundle. Let $Y = V(s)$ for $s \in \Gamma(X, \mathcal{L})$ a regular section. Then,

(a) if $\operatorname{depth}(\mathcal{O}_{X,x}) \geq 2$ for all closed points $x \in X$ then $\operatorname{Lef}(X, Y)$ holds,

(b) if moreover $\operatorname{depth}(\mathcal{O}_{X,y}) \geq 3$ for all closed points $y \in Y$ then $\operatorname{Leff}(X, Y)$ holds as well.

Corollary 1.1.5. Let X be a CM locally-factorial projective k-variety and $\mathcal{L}$ an ample line bundle. Let $Y = V(s)$ for $s \in \Gamma(X, \mathcal{L})$ a regular section. Then if k has characteristic zero then, the map

$$\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$$

(a) is injective if $\dim X \geq 3$

(b) is an isomorphism if $\dim X \geq 4$

and otherwise the above is true up to a finite p-torsion kernel and cokernel.

Proof. In characteristic zero, Kodaira vanishing shows that the obstruction spaces $H^i(Y, \mathcal{L}^{\otimes -n}) = 0$ vanish for $i = 1, 2$. However if $\operatorname{char}(k) = p$ then we replace Kodaira vanishing with asymptotic Serre vanishing and note that the nonzero terms are finite by properness and are p-torsion. HOW DOES FINITENESS WORK OVER A NONFINITE FIELD DO YOU DO SPREADING OUT? $\square$

Corollary 1.1.6. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d with $\dim X \geq 3$. Then,

$$\operatorname{Pic}(X) = \mathbb{Z}H$$

Proof. By the above theorems, it suffices to check that,

$$H^i(X, \mathcal{O}_X(-nd)) = 0$$

for $i = 1, 2$ and all $n \geq 0$. However, we have an exact sequence,

$$H^i(\mathbb{P}^n, \mathcal{O}_X(-nd)) \longrightarrow H^i(X, \mathcal{O}_X(-nd)) \longrightarrow H^{i+1}(\mathbb{P}^n, \mathcal{O}_X(-(n+1)d))$$

and since $n \geq 4$ and $i = 1, 2$ we see that the outside terms are zero. $\square$

Remark. However, the proof fails completely for hypersurfaces $X \subset \mathbb{P}^3$ because the obstruction space $H^2(X, \mathcal{O}_X(-nd))$ is large for $n \gg 0$ by Serre duality. Indeed the conclusion fails for $d < 4$ (smooth quadric surfaces have $\mathrm{Pic}(X) = \mathbb{Z}^2$ and smooth cubic surfaces have $\mathrm{Pic}(X) = \mathbb{Z}^7$). However, when $d \geq 4$ the result is true for the *general* hypersurface.

1.2 Setup for Noether-Lefschetz

Let k be any field. Let X be an integral Cohen-Macaullay locally-factorial projective 3-fold and $Y \subset X$ the vanishing locus of a regular section $s \in V \subset H^0(X, \mathcal{O}_X(d))$ where $\mathcal{O}_X(1)$ is a very ample line bundle on X and $V \subset H^0(X, \mathcal{O}_X(d))$ is a base-point free linear system.

Theorem 1.2.1. With the above notation, suppose that,

(a) K_X is d-regular[1]

(b) $S^n V \twoheadrightarrow H^0(X, \mathcal{O}_X(nd))$ for all $n \geq 0$ (e.g. if V is complete and $\mathcal{O}_X$ is d-regular)

(c) $\mathrm{Pic}^0_{X/k}$ is smooth

then for a very general member $Y \in |V|$ the map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism.

Remark. The condition (a) that K_X is d-regular is a slight strengthening of the condition that $K_X(Y)$ is globally generated considered in [RS].

Remark. The condition (a) exactly corresponds to the following vanishing conditions,

$$\begin{aligned} H^1(X, K_X(d-1)) &= H^2(X, \mathcal{O}_X(1-d))^* = 0 \\ H^2(X, K_X(d-2)) &= H^1(X, \mathcal{O}_X(2-d))^* = 0 \\ H^3(X, K_X(d-3)) &= H^0(X, \mathcal{O}_X(3-d))^* = 0 \end{aligned}$$

In characteristic zero, these are equivalent to $d \geq 4$ by Kodaira vanishing. In positive characteristic these conditions are more stringent. Furthermore condition (b) for complete V is implied by the stronger condition that $\mathcal{O}_X$ is d-regular. This amounts to the vanishing conditions,

$$\begin{aligned} H^1(X, \mathcal{O}_X(d-1)) &= 0 \\ H^2(X, \mathcal{O}_X(d-2)) &= 0 \\ H^3(X, \mathcal{O}_X(d-3)) &= 0 \end{aligned}$$

which are not directly implied by Kodaira vanishing. However, for each $(X, \mathcal{O}_X(1))$ there is garunteed, by amplness, some $d \gg 0$ such that these conditions hold.

[1]The main property we will use is that $H^0(X, \omega_X(d)) \otimes H^0(X, \mathcal{O}_X(nd)) \twoheadrightarrow H^0(X, \omega_X((n+1)d))$ for all $n \geq 0$ as well as some H^1 vanishing properties of $\mathcal{O}_X$.

Remark. Notice that K_X being d-regular implies that it is $(nd+1)$-regular for all $n \geq 1$ and hence,

$$H^2(X, \mathcal{O}_X(-nd)) = H^1(X, K_X(nd))^* = 0$$

for all $n \geq 1$. Likewise, K_X is $(nd+2)$-regular for all $n \geq 1$ and hence,

$$H^1(X, \mathcal{O}_X(-nd)) = H^2(X, K_X(nd))^* = 0$$

for all $n \geq 1$.

Remark. Notice that from the sequence,

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and therefore,

$$H^1(X, \mathcal{O}_X(-d)) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^1(X, \mathcal{O}_X(-d))$$

by the above remarks the outside groups vanish via the regularity assumption on K_X. Therefore,

$$H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^1(Y, \mathcal{O}_Y)$$

which implies that

$$\mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{Y/k}$$

is an isogeny. In particular, $\mathrm{coker}\,(\mathrm{Pic}(X) \to \mathrm{Pic}(Y))$ is a quotient of $\mathrm{NS}\,(Y)$ which is finitely generated. Furthermore, $\mathrm{Pic}_{Y/k}$ is smooth if and only if $\mathrm{Pic}_{X/k}$ is smooth.

Corollary 1.2.2. Let $X \subset \mathbb{P}^3_k$ be a very general hypersurface of degree $d \geq 4$. Then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$.

Remark. This gives a proof over any field that does not require application of characteristic zero techniques.

1.3 Passing Between "very general" and the geometric generic fiber

Here we prove three different perspectives on passing between the very general fiber and the geometric generic fiber which will apply to the case of Picard groups.

1.3.1 Version 1

Remark. Note that if S is qcqs then retrocompact open and quasicompact open coincide. Indeed, if $j : U \to S$ is a quasicompact open immersion then since S is quasicompact we see that $j^{-1}(S) = U$ is also quasicompact. Conversely, if U is quasicomact then for any quasicompact open $V \subset S$ we have $j^{-1}(V) = V \cap U$ is quasicompact since S is quasiseparated.

Remark. Even if S is qcqs then constructibility is a nontrivial condition. For example $S = \mathrm{Spec}\,(k[x_0, x_1, \dots])$ and $U = V(x_0, x_1, \dots)^C$ is not quasicompact and hence not constructible.

Proposition 1.3.1. Let $\mathcal{P}$ be a property of schemes over an algebraically closed field. Suppose,

(a) if K/k is an extension of algebraically closed fields and X is a k-scheme then

$$\mathcal{P}(X_K) \iff \mathcal{P}(X_k)$$

(b) for any $\mathcal{X} \to \mathrm{Spec}\,(R)$ flat and finitely presented we have $\mathcal{P}(\mathcal{X}_{\bar{\eta}}) \implies \mathcal{P}(\mathcal{X}_{\bar{s}})$

Then for any flat finitely presented morphism $f : X \to S$ of schemes the locus,

$$\{s \in S \mid \mathcal{P}(X_{\bar{s}})\}$$

is a countable union of constructible closed subsets (meaning their complements are retrocompact).

Proof. By absolute Noetherian approximation write $S = \varprojlim_\lambda S_\lambda$ where each S_λ is a finite type $\mathbb{Z}$-scheme. By spreading out we get a fiber product diagram for some λ,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_\lambda \arrow[d] \\ S \arrow[r] & S_\lambda \end{tikzcd}$$

with $X_\lambda \to S_\lambda$ flat and finitely presented. Now we show that that locus,

$$J = \{s \in S_\lambda \mid \mathcal{P}((X_\lambda)_{\bar{s}})\} = \bigcup_{n \in \mathbb{N}} C_n$$

where C_n is a constructible closed subset. Then we conclude since the locus $\{s \in S \mid \mathcal{P}(X_{\bar{s}})\}$ is the preimage of J under $S \to S_\lambda$ by propety (a). Property (b) shows that J is closed under specialization so J is the union over the closures of its points. However, S_λ is a finite type $\mathbb{Z}$-scheme and hence its underlying space is countable. Hence J is a countable union. $\square$

1.3.2 Version 2

Proposition 1.3.2. Let $f : X \to S$ be a finitely presented morphism of schemes with S qcqs and irreducible. Then there exists a countable union of proper constructible closed subsets $C \subset S$ such that for each $t \in S \setminus C$ there exists an algebraically closed field K containing both $\kappa(t)$ and $\kappa(\eta)$ such that $(X_t)_K \cong (X_\eta)_K$ as K-schemes where $\eta \in S$ is the generic point.

Proof. By absolute Noetherian approximation write $S = \varprojlim_\lambda S_\lambda$ where each S_λ is a finite type $\mathbb{Z}$-scheme. By spreading out we get a fiber product diagram for some λ,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_\lambda \arrow[d] \\ S \arrow[r] & S_\lambda \end{tikzcd}$$

Let C be the preimage of the union of all proper subschemes of S_λ which is a countable union since S_λ is finite type over $\mathbb{Z}$. Then for each $t \in S \backslash C$ we have $t \mapsto \eta_\lambda$ where η_λ is the generic point of S_λ and hence X_t and X_η are both base changes of the same scheme $(X_\lambda)_{\eta_\lambda}$ by the fields $\kappa(t)$ and $\kappa(\eta)$ and hence the claim follows immediately taking any algebraic closed K containing both $\kappa(t)$ and $\kappa(\eta)$. $\square$

Proposition 1.3.3. Let $\mathcal{P}$ be a property of schemes over an algebraically closed field. Suppose,

(a) if K/k is an extension of algebraically closed fields and X is a k-scheme then

$$\mathcal{P}(X_K) \iff \mathcal{P}(X_k)$$

Then for any finitely presented morphism $f : X \to S$ of schemes with S qcqs and irreducible such that $\mathcal{P}$ does not hold for the generic fiber $X_{\bar{\eta}}$ then $\{s \in S \mid \mathcal{P}(X_{\bar{s}})\}$ is contained in a countable union of proper construcible closed subsets.

Proof. Take C as in the previous proposition. By (a) and the conclusion of the above proposition we see that if $s \in S \setminus C$ then $\neg\mathcal{P}(X_{\bar{s}})$ so,

$$\{s \in S \mid \mathcal{P}(X_{\bar{s}})\} \subset C$$

which proves the claim. □

1.3.3 Checking the conditions for Noether-Lefschetz Property

CHECK PROPERTIES (a) AND (b) FOR "Noether-Lefschetz fails"

1.4 Smoothness of the Picard Scheme

Instead of passing between Noether-Lefschetz for the geometric generic fiber and for the very general fiber we can use properties of the Picard scheme directly to make the same sort of reduction.

Lemma 1.4.1. Let $f : X \to S$ be a smooth proper morphism of schemes with geometrically connected fibers. Assume S is quasi-compact and quasi-separated. Then the relative picard functor $\mathrm{Pic}_{X/S}$ is represented by an algebraic space over S. Furthermore, if $Z \subset \mathrm{Pic}_{X/S}$ denotes the (closed) complement of the smooth locus of $\mathrm{Pic}_{X/S} \to S$, then the image of Z in S is a countable union of closed, constructible subsets of S.

Proof. The representability of $\mathrm{Pic}_{X/S}$ follows from [**?**, Tag 0D2C], since X/S having geometrically connected fibers implies $\mathcal{O}_T \to f_{T,*}\mathcal{O}_X$ is an isomorphism for every S-scheme T. By a limit argument, there are a scheme S_0 of finite type over $\mathbf{Z}$, a smooth proper morphism $f_0 : X_0 \to S_0$ with geometrically connected fibers, an affine morphism $S \to S_0$, and an isomorphism $X_0 \times_{S_0} S = X$. Then $\mathrm{Pic}_{X/S} = \mathrm{Pic}_{X_0/S_0} \times_{S_0} S$. Since the smooth locus of the base change of a morphism is the pullback of the smooth locus of the morphism, we see that we may replace S with S_0 and thus assume S is of finite type over $\mathbf{Z}$. But then the underlying topological space of S is Noetherian and countable, so we only have to show the image of Z in S is closed under specialization. This follows since X/S is smooth so $\mathrm{Pic}_{X/S}$ satisfies the existence part of the valuative criterion of properness. □

How much can we weaken smoothness?

1.5 The Universal Family

We want to show that the lifting problem for $\mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(Y)$ is unobstructed for the very general member $Y \in |V|$. However, the natural obstruction spaces $H^2(Y, \mathcal{O}_Y(-nd))$ never vanish by Serre duality. The main idea is to compare the obstruction for lifting along the formal neighborhoods of $Y \subset X$to the lifting problem for the formal neighborhood of Y viewed as a fiber in the universal family. This latter lifting problem will then be unobstructed for the general member by a spreading out argument.

First we fix some notaiton. Consider the universal hypersurface $\mathcal{Y}$ which is the incidence correspondence,

$$\mathcal{Y} \subset X \times S$$

where $S = \mathbb{P}(H^0(X, \mathcal{O}_X(d))$. Thus $\mathcal{Y}$ is equipped with projection maps $p : \mathcal{Y} \to X$ and $q : \mathcal{Y} \to S$ where q is flat and proper and p is smooth and proper, in fact it is a projective bundle. Consider the sequence,

$$0 \longrightarrow \mathcal{V} \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(d) \longrightarrow 0$$

where $\mathcal{V}$ is the bundle of sections in V which vanishing at a given point of x. Hence we see that $\mathcal{Y} = \mathbb{P}(\mathcal{V}) := \mathbf{Proj}_X\left(\mathrm{Sym}(\mathcal{V}^\bullet)\right)$ compatible with $p : \mathcal{Y} \to X$. Now let $\hat{X}$ be the formal completion of X along Y and viewing $Y \subset \mathcal{Y}$ as the fiber over $s \in S$ let $c\hat{Y}$ be the formal completion of $\mathcal{Y}$ along Y. Since we have a morphism of schemes $p : \mathcal{Y} \to X$ mapping the fiber $\mathcal{Y}_s = Y$ into (scheme-theoretically) the closed subscheme $Y \subset X$ (in fact it is an isomorphism over Y) we get a morphism of formal schemes,

$$\hat{p} : \hat{\mathcal{Y}} \to \hat{X}$$

Then let $\mathcal{I}$ be the ideal cutting out $Y \subset X$. Since the map p sends $\mathcal{Y}_s \to Y$ we have $p^* \mathcal{I} \subset \mathfrak{m}_s \mathcal{O}_{\mathcal{Y}}$ or we say that $\mathcal{I}$ maps into $\mathfrak{m}_s \mathcal{O}_{\mathcal{Y}}$ under the sheaf map $p^\#$. This also implies that $\mathcal{I}^n$ maps into $\mathfrak{m}_s^n \mathcal{O}_{\mathcal{Y}}$ under $p^\#$ which is what induces the map of formal schemes $\hat{p} : \hat{\mathcal{Y}} \to \hat{X}$.

1.6 Formal Deformations and Formal Noether-Lefschetz

Definition 1.6.1. We say that (X, Y) satisfies *formal Noether-Lefschetz* (FNL) if,

$$\mathrm{im}\left(\mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(Y)\right) = \mathrm{im}\left(\mathrm{Pic}(\hat{\mathcal{Y}}) \to \mathrm{Pic}(Y)\right)$$

We say that $(X, |V|)$ satisfies FNL if there is a nonempty open $U \subset S$ so that for each $s \in U$ the pair $(X, \mathcal{Y}_s)$ satisfies FNL.

First we will show how FNL implies the Noether-Lefschetz property using that obstructions for $Y \subset \mathcal{Y}$ are unobstructed for the generic member $Y \in |V|$.

Theorem 1.6.2. Suppose that $(X, |V|)$ satisfies FNL and $H^1(X, \mathcal{O}_X) = 0$. Let $K = k(S)$. Then,

$$\mathrm{Pic}(X_{\overline{K}}) \to \mathrm{Pic}(\mathcal{Y}_{\overline{K}})$$

is surjective.

In the proof I assumed that k is algebraically closed. Does it work regardless?

Proof. Choose $\alpha \in \mathrm{Pic}(\mathcal{Y}_K)$. First, note that $H^1(\mathcal{Y}_K, \mathcal{O}_{\mathcal{Y}_K}) = 0$ by the vanishing conditions so $\mathrm{Pic}_{\mathcal{Y}_K/K}$ is an étale K-group. Hence every point appears over K^{sep} so we may assume that $\alpha \in \mathrm{Pic}(\mathcal{Y}_{K^{\mathrm{sep}}})$ Then by spreading out there is some finite separable extension L/K such that α is the pullback of $\alpha_L \in \mathrm{Pic}(\mathcal{Y}_L)$. Spreading out gives an étale map $U \to S$ inducing L/K on the generic point and a class $\alpha_U \in \mathrm{Pic}(\mathcal{Y}_U)$. Choose some $t \in U$ mapping to a closed point $s \in S$ in the locus where FNL holds. Since $U \to S$ is étale it induces an isomorphism of formal neighborhoods. Hence, the completion of $\mathcal{Y}_U$ along $(\mathcal{Y}_U)_t$ is isomorphic to $\hat{\mathcal{Y}}$ since k is algebraically closed so $\kappa(t) = \kappa(s) = k$. Therefore we get a class $\hat{\alpha} \in \mathrm{Pic}((\hat{\mathcal{Y}})_{\kappa(t)})$ so by FNL we get a lift to $\alpha' \in \mathrm{Pic}(\hat{X})$ and hence to $\mathrm{Pic}(X)$ by Grothendieck-Lefschetz. Then $\alpha_U - q^* \alpha'$ is trivial on a fiber of p. Since $H^1(Y, \mathcal{O}_Y) = 0$ for all $Y \in |V|$ then $\mathrm{Pic}_{\mathcal{Y}/S}$ is generically étale so if we do the above construction chosing s in the étale locus then $\alpha_U - q^* \alpha'$ and the zero section agree at the generic point. Hence we have shown that α arises as the pullback of $\alpha' \in \mathrm{Pic}(X)$. $\square$

Now we need to prove the formal Noether-Lefschetz property. We want to compare obstructions for lifting line bundles from Y to $\hat{\mathcal{Y}}$ to those same obstructions for lifting Y. Choose $s \in U$ and $Y = \mathcal{Y}_s$. Since p is flat, p^* is exact and therefore there is a diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{I}^n/\mathcal{I}^{n+1} & \longrightarrow & p^*\mathcal{O}_{Y_{n+1}} & \longrightarrow & p^*\mathcal{O}_{Y_n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{J}^n/\mathcal{J}^{n+1} & \longrightarrow & \mathcal{O}_{\mathcal{Y}_{n+1}} & \longrightarrow & \mathcal{O}_{\mathcal{Y}_n} & \longrightarrow & 0 \end{array}$$

where $\mathcal{Y}_n = (\mathcal{Y}_s)_n$ is the n^{th}-formal neighborhood of $\mathcal{Y}_s \subset \mathcal{Y}$ cut out by the ideal $J^n = \mathfrak{m}_s^n \mathcal{O}_{\mathcal{Y}}$. Since p is a projective bundle $Rp_*\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_X$ and therefore by the projection formula p^* preserves cohomology. Therefore, applying the long exact sequences of cohomology gives a diagram,

$$\begin{array}{ccccccc} H^1(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) & \longrightarrow & \mathrm{Pic}(Y_{n+1}) & \longrightarrow & \mathrm{Pic}(Y_n) & \longrightarrow & H^2(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma_n} \\ H^1(Y, \mathcal{J}^n/\mathcal{J}^{n+1}) & \longrightarrow & \mathrm{Pic}(\mathcal{Y}_{n+1}) & \longrightarrow & \mathrm{Pic}(\mathcal{Y}_n) & \longrightarrow & H^2(Y, \mathcal{J}^n/\mathcal{J}^{n+1}) \end{array}$$

Suppose that $\alpha \in \mathrm{im}\,(\mathrm{Pic}(\hat{\mathcal{Y}}) \to \mathrm{Pic}(Y))$ then α lifts to some comptabile sequence of $\alpha_n \in \mathrm{Pic}(\mathcal{Y}_n)$. We want to find a sequence $\alpha'_n \in \mathrm{Pic}(Y_n)$ which pulls back to α_n. Indeed, set $\alpha'_1 = \alpha_1$ since $\mathcal{Y}_1 = Y_1 = Y$. Suppose we have built α'_n. If γ_n is injective then the obstruction class of α'_n dies since it maps to α_n which is unobstructed by assumption. Hence there is some lift α'_{n+1} but it is not clear that $\alpha'_{n+1} \mapsto \alpha_{n+1}$. The difference is controlled by some class in $H^1(Y, \mathcal{J}^n/\mathcal{J}^{n+1})$ which may be nonzero. However,

$$H^1(Y, \mathcal{J}^n/\mathcal{J}^{n+1}) = \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes H^1(Y, \mathcal{O}_Y)$$

so if $H^1(Y, \mathcal{O}_Y) = 0$ then there is unique lifting along $\mathcal{Y}_{n+1} \to \mathcal{Y}_n$. Therefore, we have proven:

Lemma 1.6.3. Suppose that for all $Y = Y_s$ in some nonempty open $U \subset S$ we have for all $n \geq 1$,

(a) γ_n is injective

(b) $H^1(Y, \mathcal{O}_Y) = 0$

Then $(X, |V|)$ satisfies FNL.

1.7 Version Using the Picard Scheme

Here we define a variant that uses properties of the Picard scheme rather than the unpalatable condition $H^1(X, \mathcal{O}_X) = 0$.

Proposition 1.7.1. Suppose that for some $s \in S$ setting $Y = Y_s$ we have,

(a) γ_n is injective for all $n \geq 1$

(b) every point of $\mathrm{Pic}_{\mathcal{Y}/S}$ lying over s is a smooth point of the morphism $\mathrm{Pic}_{\mathcal{Y}/S} \to S$

then $\mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(Y)$ is surjective.

Proof. Consider the long exact cohomology sequence as previously, Therefore, applying the long exact sequences of cohomology gives a diagram,

$$
\begin{tikzcd}
H^1(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \arrow[r] \arrow[d] & \mathrm{Pic}(Y_{n+1}) \arrow[r] \arrow[d] & \mathrm{Pic}(Y_n) \arrow[r] \arrow[d] & H^2(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \arrow[d, "\gamma_n"] \\
H^1(Y, \mathcal{J}^n/\mathcal{J}^{n+1}) \arrow[r] & \mathrm{Pic}(\mathcal{Y}_{n+1}) \arrow[r] & \mathrm{Pic}(\mathcal{Y}_n) \arrow[r] & H^2(Y, \mathcal{J}^n/\mathcal{J}^{n+1})
\end{tikzcd}
$$

However, since every point of $\mathrm{Pic}_{\mathcal{Y}/S}$ over s is a smooth point of the morphism it implies the formal lifting criterion which we apply for the extension of Artin local rings $\mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1} \twoheadrightarrow \mathcal{O}_{S,s}/\mathfrak{m}_s^n$. This shows that any $\mathcal{L} \in \mathrm{Pic}(\mathcal{Y}_n)$ is unobstructed to lifting to $\mathrm{Pic}(\mathcal{Y}_{n+1})$. Hence the injectivity of γ_n shows that any $\mathcal{L} \in \mathrm{Pic}(Y_n)$ is unobstructed to lift to $\mathrm{Pic}(Y_{n+1})$ hence proving the claim. $\square$

The same argument shows the following result.

Proposition 1.7.2. If $\mathrm{Spec}\,(L) \to \mathrm{Pic}_{\mathcal{Y}/S}$ corresponds to a smooth point $(s, \mathcal{L}) \in \mathrm{Pic}_{\mathcal{Y}/S}$ of the family and if all $\{\gamma_n\}_{n \ge 1}$ for the pair $(X_L, \mathcal{Y}_L)$ are injective then $\mathcal{L} \in \mathrm{Pic}(\mathcal{Y}_L)$ is in the image of $\mathrm{Pic}(X_L) \to \mathrm{Pic}({}_{)}\mathcal{Y}_L$.

1.8 Injectivity of the maps γ_n

Proposition 1.8.1. Suppose that,

(a) K_X is d-regular

(b) $S^n V \twoheadrightarrow H^0(X, \mathcal{O}_X(nd))$ for all $n \ge 0$ (e.g. if V is complete and $\mathcal{O}_X$ is d-regular)

then the maps γ_n are injective for $n \ge 1$ and any $Y \in |V|$ equidimensional and Cohen-Macaulay

Proof. Let Y be such that it is Cohen-Macaulay so that $K_Y = K_X(d)|_Y$ is a dualizing sheaf. We need to show that the maps,

$$\gamma_n : H^2(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \to H^2(Y, \mathcal{J}^n/\mathcal{J}^{n+1}) = H^2(Y, \mathcal{O}_Y) \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1} = H^2(Y, \mathcal{O}_Y) \otimes S^n(V/s)^*$$

are injective for $n \ge 1$. Applying Serre duality - and using that $K_Y = K_X(d)$ - these are dual to,

$$\gamma_n^\vee : H^0(Y, K_X(d)) \otimes S^n W \to H^0(Y, K_X((n+1)d))$$

so it suffices to show that $\gamma_n^\vee$ are surjective for all $n \ge 1$. Consider the diagram,

$$
\begin{tikzcd}
& H^0(Y, K_X(d)) \otimes H^0(Y, \mathcal{O}_Y(nd)) \arrow[dr] & \\
H^0(Y, K_X(d)) \otimes S^n(V/s) \arrow[ur] \arrow[rr, "\gamma_n^\vee"] & & H^0(Y, K_X((n+1)d)) \\
H^0(X, K_X(d)) \otimes S^n V \arrow[u] \arrow[rr] \arrow[dr, "\alpha"] & & H^0(X, K_X((n+1)d)) \arrow[u, "\gamma"] \\
& H^0(X, K_X(d)) \otimes H^0(X, \mathcal{O}_X(nd)) \arrow[ur, "\beta"] \arrow[uuu, crossing over] &
\end{tikzcd}
$$

From commutativity, it suffices to show that each α, β, γ are surjective. First α is surjective by assumption (b) . Then β is surjective because K_X is d-regular. Finally, for γ we consider the sequence,

$$0 \longrightarrow K_X(nd) \longrightarrow K_X((n+1)d) \longrightarrow K_X((n+1)d)|_Y \longrightarrow 0$$

to get,

$$H^0(X, K_X((n+1)d)) \xrightarrow{\gamma} H^0(Y, K_X((n+1)d)) \longrightarrow H^1(X, K_X(nd))$$

but K_X is 1-regular and hence $(nd+1)$-regular for all $n \geq 1$ and hence $H^1(X, K_X(nd)) = 0$ proving that γ is surjective. □

1.9 Proof of the Main Theorem

Theorem 1.9.1 (Version 1)**.** Suppose that,

(a) K_X is d-regular[2]

(b) $S^n V \twoheadrightarrow H^0(X, \mathcal{O}_X(nd))$ for all $n \geq 0$ (e.g. if V is complete and $\mathcal{O}_X$ is d-regular)

(c) $H^1(X, \mathcal{O}_X) = 0$

then for a very general member $Y \in |V|$ the map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism.

Proof. By passing to the geometric generic fiber we need to show that $\mathrm{Pic}(X_K) \to \mathrm{Pic}(\mathcal{Y}_K)$ is an isomorphism. Therefore by Theorem 1.6.2 it suffices to show that FNL holds. This is what Lemma 1.6.3 and Proposition 1.8.1 show. □

Theorem 1.9.2 (Version 2)**.** Suppose that,

(a) K_X is d-regular[3]

(b) $S^n V \twoheadrightarrow H^0(X, \mathcal{O}_X(nd))$ for all $n \geq 0$ (e.g. if V is complete and $\mathcal{O}_X$ is d-regular)

(c) $\mathrm{Pic}^0_{X/k}$ is smooth

then for a very general member $Y \in |V|$ the map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism.

Proof. consider the maps,

$$\mathrm{Pic}(X) \to \mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(Y)$$

Grothendieck-Lefschetz shows that the first map is an isomorphism and the second is injective. Hence it suffices to prove surjectivity of the second. Let $K = k(S)$. Since $\mathrm{Pic}^0_{X_K/K} \to \mathrm{Pic}^0_{\mathcal{Y}_K/K}$ is an isogeny and $\mathrm{Pic}^0_{X_K/K} = (\mathrm{Pic}^0_{X/k})_K$ is smooth we conclude that $\mathrm{Pic}_{\mathcal{Y}/S} \to S$ is generically smooth and therefore the image of the nonsmooth locus $C \subset S$ is a countable union of proper closed subvarieties. For any $s \in S \setminus C$ the hypotheses of Proposition 1.7.1 hold by Proposition 1.8.1 so we conclude. □

[2] The main property we will use is that $H^0(X, \omega_X(d)) \otimes H^0(X, \mathcal{O}_X(nd)) \twoheadrightarrow H^0(X, \omega_X((n+1)d))$ for all $n \geq 0$ as well as some H^1 vanishing properties of $\mathcal{O}_X$.

[3] The main property we will use is that $H^0(X, \omega_X(d)) \otimes H^0(X, \mathcal{O}_X(nd)) \twoheadrightarrow H^0(X, \omega_X((n+1)d))$ for all $n \geq 0$ as well as some H^1 vanishing properties of $\mathcal{O}_X$.

2 Talk: Ben, Laure, Noah, Sean, Shreya, Supravat

Throughout we fix the following notation. Let k be any field. Let X be a smooth projective 3-fold and $Y \subset X$ the vanishing locus of a regular section $s \in V \subset H^0(X, \mathcal{O}_X(d))$ where $\mathcal{O}_X(1)$ is a very ample line bundle on X and $V \subset H^0(X, \mathcal{O}_X(d))$ is a base-point free linear system.

2.1 Outline of Grothendieck-Lefschetz

Definition 2.1.1. Let $Y \subset X$ be a closed subscheme. We say that (X, Y) satisfies,

(a) $\mathrm{Lef}(X, Y)$ if for any open $U \subset X$ containing X and any finite locally free $\mathcal{E}$ on U then,

$$\Gamma(U, \mathcal{E}) \to \Gamma(\hat{X}, \hat{\mathcal{E}})$$

is an isomorphism

(b) $\mathrm{Leff}(X, Y)$ if it satisfies $\mathrm{Lef}(X, Y)$ are moreover for any finite locally free $\mathfrak{E}$ on $\hat{X}$ there exists an open $U \subset X$ containing Y an a finite locally free $\mathcal{E}$ on U such that $\hat{\mathcal{E}} = \mathfrak{E}$.

If we have the $\mathrm{Leff}(X, Y)$ extension property then Lefschetz theorems can be reduced to computing formal liftings.

Theorem 2.1.2 (Grothendieck-Lefschetz (Hartshorne, Ample subvarities, Theorem 3.1))**.** Let X be an S_2 and locally-factorial variety and $Y \subset X$ a closed subscheme. Assume that,

(a) $\mathrm{Lef}(X, Y)$ (resp. $\mathrm{Leff}(X, Y)$)

(b) Y meets every effective divisor on X

(c) $H^i(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for $i = 1$ (resp. for $i = 1, 2$) and all $n \geq 1$ where $\mathcal{I}$ is the ideal cutting out Y.

Then the natural map,

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is injective (resp. an isomorphism).

Proof. Consider the restriction maps,

$$\mathrm{Pic}(X) \to \varinjlim_{\substack{U \subset X \\ Y \subset U}} \mathrm{Pic}(U) \to \mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(Y)$$

we will show these are all isomorphisms.

(a) the first is an isomorphism since by (b) any U containing Y satisfies $\mathrm{codim}\,(X, X \backslash U) \geq 2$ and X is S_2 and locally factorial which implies that $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is an isomorphism

(b) the second is an isomorphism by $\mathrm{Leff}(X, Y)$

(c) the third is an isomorphism via the following lifting computation. Consider the unit exact sequence,

$$0 \longrightarrow \mathcal{I}^n/\mathcal{I}^{n+1} \xrightarrow{x \mapsto 1+x} \mathcal{O}_{Y_{n+1}}^\times \longrightarrow \mathcal{O}_{Y_n}^\times \longrightarrow 0$$

On cohomology this gives,

$$H^1(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \longrightarrow \mathrm{Pic}(Y_{n+1}) \longrightarrow \mathrm{Pic}(Y_n) \longrightarrow H^2(Y, \mathcal{I}^n/\mathcal{I}^{n+1})$$

Therefore, since the outside groups are all zero for $n \geq 1$ we can lift any line bundle on $Y_1 = Y$ uniquely all the way to $\hat{X}$ giving isomorphism,

$$\mathrm{Pic}(\hat{X}) = \varprojlim_n \mathrm{Pic}(Y_n) = \mathrm{Pic}(Y)$$

$\square$

Therefore, it suffices to check when a pair (X, Y) satisfies $\mathrm{Leff}(X, Y)$. Luckily Grothendieck did this for us.

Theorem 2.1.3 (SGA2 Expose X, Example 2.2)**.** Let X be a proper k-scheme and $\mathcal{L}$ an ample line bundle. Let $Y = V(s)$ for $s \in \Gamma(X, \mathcal{L})$ a regular section. Then,

(a) if depth $(\mathcal{O}_{X,x}) \geq 2$ for all closed points $x \in X$ then $\mathrm{Lef}(X, Y)$ holds,

(b) if moreover depth $(\mathcal{O}_{X,y}) \geq 3$ for all closed points $y \in Y$ then $\mathrm{Leff}(X, Y)$ holds as well.

Corollary 2.1.4. Let X be a regular projective k-variety with $\dim X \geq 4$ and $\mathcal{L}$ an ample line bundle. Let $Y = V(s)$ for $S \in \Gamma(X, \mathcal{L})$ a regular section. Then the map,

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism if k has characteristic zero.

Proof. In characteristic zero, Kodaira vanishing shows that the obstruction spaces $H^i(Y, \mathcal{L}^{\otimes -n}) = 0$ vanish for $i = 1, 2$. $\square$

Corollary 2.1.5. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d with $\dim X \geq 3$. Then,

$$\mathrm{Pic}(X) = \mathbb{Z}H$$

Remark. However, the proof fails completely for hypersurfaces $X \subset \mathbb{P}^3$ because the obstruction space $H^2(X, \mathcal{O}_X(-nd))$ is large for $n \gg 0$ by Serre duality. Indeed the conclusion fails for $d < 4$ (smooth quadric surfaces have $\mathrm{Pic}(X) = \mathbb{Z}^2$ and smooth cubic surfaces have $\mathrm{Pic}(X) = \mathbb{Z}^7$). However, when $d \geq 4$ the result is true for the *general* hypersurface.

2.2 Setup for Noether-Lefschetz

Theorem 2.2.1. Suppose that,

(a) K_X is d-regular[4]

(b) $S^n V \twoheadrightarrow H^0(X, \mathcal{O}_X(nd))$ for all $n \geq 0$ (e.g. if V is complete and $\mathcal{O}_X$ is d-regular)

(c) $H^1(X, \mathcal{O}_X) = 0$

[4]The main property we will use is that $H^0(X, \omega_X(d)) \otimes H^0(X, \mathcal{O}_X(nd)) \twoheadrightarrow H^0(X, \omega_X((n+1)d))$ for all $n \geq 0$ as well as some H^1 vanishing properties of $\mathcal{O}_X$.

then for a very general member $Y \in |V|$ the map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism.

Remark. The condition (a) that K_X is d-regular is a slight strengthening of the condition that $K_X(Y)$ is globally generated considered in [RS].

Remark. Notice that from the sequence,

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and therefore,

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^1(X, \mathcal{O}_X(-d))$$

by (c) the first vanishes and by the above remarks the second does as well from the regularity assumption on K_X. Therefore $H^1(Y, \mathcal{O}_Y) = 0$.

Corollary 2.2.2. Let $X \subset \mathbb{P}^3$ be a very general hypersurface of degree $d \geq 4$. Then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$.

2.3 The Universal Family

We want to show that the lifting problem for $\mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(Y)$ is unobstructed for the very general member $Y \in |V|$. However, the natural obstruction spaces $H^2(Y, \mathcal{O}_Y(-nd))$ never vanish by Serre duality. The main idea is to compare the obstruction for lifting along the formal neighborhoods of $Y \subset X$to the lifting problem for the formal neighborhood of Y viewed as a fiber in the universal family. This latter lifting problem will then be unobstructed for the general member by a spreading out argument.

First we fix some notaiton. Consider the universal hypersurface $\mathcal{Y}$ which is the incidence correspondence,

$$\mathcal{Y} \subset X \times S$$

where $S = \mathbb{P}(H^0(X, \mathcal{O}_X(d))$. Thus $\mathcal{Y}$ is equipped with projection maps $p : \mathcal{Y} \to X$ and $q : \mathcal{Y} \to S$ where q is flat and proper and p is smooth and proper, in fact it is a projective bundle. Consider the sequence,

$$0 \longrightarrow \mathcal{V} \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(d) \longrightarrow 0$$

where $\mathcal{V}$ is the bundle of sections in V which vanishing at a given point of x. Hence we see that $\mathcal{Y} = \mathbb{P}(\mathcal{V}) := \mathbf{Proj}_X\left(\mathrm{Sym}(\mathcal{V}^\bullet)\right)$ compatible with $p : \mathcal{Y} \to X$. Now let $\hat{X}$ be the formal completion of X along Y and viewing $Y \subset \mathcal{Y}$ as the fiber over $s \in S$ let $\hat{cY}$ be the formal completion of $\mathcal{Y}$ along Y. Since we have a morphism of schemes $p : \mathcal{Y} \to X$ mapping the fiber $\mathcal{Y}_s = Y$ into (scheme-theoretically) the closed subscheme $Y \subset X$ (in fact it is an isomorphism over Y) we get a morphism of formal schemes,

$$\hat{p} : \hat{\mathcal{Y}} \to \hat{X}$$

Then let $\mathcal{I}$ be the ideal cutting out $Y \subset X$. Since the map p sends $\mathcal{Y}_s \to Y$ we have $p^*\mathcal{I} \subset \mathfrak{m}_s \mathcal{O}_{\mathcal{Y}}$ or we say that $\mathcal{I}$ maps into $\mathfrak{m}_s \mathcal{O}_{\mathcal{Y}}$ under the sheaf map $p^\#$. This also implies that $\mathcal{I}^n$ maps into $\mathfrak{m}_s^n \mathcal{O}_{\mathcal{Y}}$ under $p^\#$ which is what induces the map of formal schemes $\hat{p} : \hat{\mathcal{Y}} \to \hat{X}$.

2.4 Formal Deformations and Formal Lefschetz

Definition 2.4.1. We say that (X, Y) satisfies *formal Noether-Lefschetz* (FNL) if,

$$\mathrm{im}\,(\mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(Y)) = \mathrm{im}\,(\mathrm{Pic}(\hat{\mathcal{Y}}) \to \mathrm{Pic}(Y))$$

We say that $(X, |V|)$ satisfies FNL if there is a nonempty open $U \subset S$ so that for each $s \in U$ the pair $(X, \mathcal{Y}_s)$ satisfies FNL.

First we will show how FNL implies the Noether-Lefschetz property using that obstructions for $Y \subset \mathcal{Y}$ are unobstructed for the generic member $Y \in |V|$.

Theorem 2.4.2. Let K be the separable closure of the function field of S. Suppose that $(X, |V|)$ satisfies FNL. Then,

$$\mathrm{Pic}(X_K) \to \mathrm{Pic}(\mathcal{Y}_K)$$

is an isomorphism.

Proof. Since $\mathrm{Pic}_{\mathcal{Y}/S}$ is representable and $H^1(Y, \mathcal{O}_Y) = 0$ there is a countable intersection of nonempty opens $C \subset S$ such that $\mathrm{Pic}_{\mathcal{Y}_s}$ is smooth for any $s \in C$. For any $\mathcal{L} \in \mathrm{Pic}(Y)$ such that the map $\mathrm{Spec}\,(L) \to \mathrm{Pic}_{\mathcal{Y}/S}$ lands in the smooth locus, by the formal lifting property, it spreads out to line bundle $\hat{\mathcal{L}} \in \mathrm{Pic}(\hat{\mathcal{Y}})$ and therefore by formal Lefschetz we get $\hat{\mathcal{L}}' \in \mathrm{Pic}(\hat{X}_L)$ which restricts to $\mathcal{L} \in \mathrm{Pic}(X)$. Therefore by Grothendieck-Lefschetz, $\mathcal{L}$ is in the image of $\mathrm{Pic}(X_L) \to \mathrm{Pic}(Y)$. $\square$

Now we need to prove the formal Noether-Lefschetz property. We want to compare obstructions for lifting line bundles from Y to $\hat{\mathcal{Y}}$ to those same obstructions for lifting Y. Choose $s \in U$ and $Y = \mathcal{Y}_s$. Since p is flat, p^* is exact and therefore there is a diagram,

$$\begin{tikzcd}
0 \arrow[r] & p^*\mathcal{I}^n/\mathcal{I}^{n+1} \arrow[r] \arrow[d] & p^*\mathcal{O}_{Y_{n+1}} \arrow[r] \arrow[d] & p^*\mathcal{O}_{Y_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}^n/\mathcal{J}^{n+1} \arrow[r] & \mathcal{O}_{\mathcal{Y}_{n+1}} \arrow[r] & \mathcal{O}_{\mathcal{Y}_n} \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{Y}_n = (\mathcal{Y}_s)_n$ is the n^{th}-formal neighborhood of $\mathcal{Y}_s \subset \mathcal{Y}$ cut out by the ideal $J^n = \mathfrak{m}_s^n \mathcal{O}_{\mathcal{Y}}$. Since p is a projective bundle $Rp_* \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_X$ and therefore by the projection formula p^* preserves cohomology. Therefore, applying the long exact sequences of cohomology gives a diagram,

$$\begin{tikzcd}
H^1(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \arrow[r] \arrow[d] & \mathrm{Pic}(Y_{n+1}) \arrow[r] \arrow[d] & \mathrm{Pic}(Y_n) \arrow[r] \arrow[d] & H^2(Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \arrow[d, "\gamma_n"] \\
H^1(Y, \mathcal{J}^n/\mathcal{J}^{n+1}) \arrow[r] & \mathrm{Pic}(\mathcal{Y}_{n+1}) \arrow[r] & \mathrm{Pic}(\mathcal{Y}_n) \arrow[r] & H^2(Y, \mathcal{J}^n/\mathcal{J}^{n+1})
\end{tikzcd}$$

Suppose that $\alpha \in \mathrm{im}\,(\mathrm{Pic}(\hat{\mathcal{Y}}) \to \mathrm{Pic}(Y))$ then α lifts to some comptabile sequence of $\alpha_n \in \mathrm{Pic}(\mathcal{Y}_n)$. We want to find a sequence $\alpha'_n \in \mathrm{Pic}(Y_n)$ which pulls back to α_n. Indeed, set $\alpha'_1 = \alpha_1$ since $\mathcal{Y}_1 = Y_1 = Y$. Suppose we have built α'_n. If γ_n is injective then the obstruction class of α'_n dies since it maps to α_n which is unobstructed by assumption. Hence there is some lift α'_{n+1} but it is not clear that $\alpha'_{n+1} \mapsto \alpha_{n+1}$. The difference is controlled by some class in $H^1(Y, \mathcal{J}^n/\mathcal{J}^{n+1})$ which may be nonzero. However,

$$H^1(Y, \mathcal{J}^n/\mathcal{J}^{n+1}) = \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes H^1(Y, \mathcal{O}_Y)$$

so if $H^1(Y, \mathcal{O}_Y) = 0$ then there is unique lifting along $\mathcal{Y}_{n+1} \to \mathcal{Y}_n$. Therefore, we have proven:

Lemma 2.4.3. Suppsoe that for all $Y = Y_s$ in some nonempty open $U \subset S$ we have for all $n \geq 1$,

(a) γ_n is injective

(b) $H^1(Y, \mathcal{O}_Y) = 0$

Then $(X, |V|)$ satisfies FNL.

3 Proposal for Stacks Project writeup

Let k be any field. Let X be an integral Cohen-Macaullay locally-factorial projective 3-fold and $Y \subset X$ the vanishing locus of a regular section $s \in V \subset H^0(X, \mathcal{O}_X(d))$ where $\mathcal{O}_X(1)$ is a very ample line bundle on X and $V \subset H^0(X, \mathcal{O}_X(d))$ is a base-point free linear system.

Theorem 3.0.1. With the above notation, suppose that,

(a) K_X is d-regular[5]

(b) $S^n V \twoheadrightarrow H^0(X, \mathcal{O}_X(nd))$ for all $n \geq 0$ (e.g. if V is complete and $\mathcal{O}_X$ is d-regular)

(c) $\mathrm{Pic}^0_{X/k}$ is smooth

then for a very general member $Y \in |V|$ the map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

is an isomorphism.

To prove this we need the following results:

(a) Lemmas on Castelnuovo-Mumford regularity (not yet in stacks project)

(b) Grothendeck-Lefschetz (in the stacks project)

(c) Lemma 1.4.1 (enough is in the stacks project to prove this)

(d) Proposition 1.7.1

(e) Proposition 1.8.1

[5]The main property we will use is that $H^0(X, \omega_X(d)) \otimes H^0(X, \mathcal{O}_X(nd)) \twoheadrightarrow H^0(X, \omega_X((n+1)d))$ for all $n \geq 0$ as well as some H^1 vanishing properties of $\mathcal{O}_X$.