1 Sep. 30

1.1 Introduction

Theorem 1.1.1 (Deligne-Illusie)**.** Let X/k be a smooth proper scheme with k a field of characteristic zero and $\Omega^\bullet_{X/k}$ is deRham complex. Then, the Hodge-to-deRham spectral sequence,

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \implies H^{p+q}_{\mathrm{dR}}(X)$$

degenerates at the E_1-page.

Corollary 1.1.2. Then,

$$\dim H^n_{\mathrm{dR}}(X) = \sum_{p+q=n} \dim H^q(X, \Omega^p_{X/k})$$

Remark. For $k = \mathbb{C}$, we can prove the above equality using analytic techniques (i.e. Hodge theory).

Remark. D-I give an purely algebraic proof. The idea is use degeneration in positive characteristic to get degeneration in characteristic zero.

1.2 de Rham Complex

Let $f : X \to Y$ be a morphism of schemes.

Definition 1.2.1. Then $\Omega^1_{X/Y}$ is the sheaf of relative differentials on X/Y. Then,

$$\Omega^1_{X/Y} = \Delta^* \mathcal{C}_{X \times_Y X / X}$$

is the conormal bundle for the diagonal $\Delta_{X/Y} : X \to X \times_Y X$. Then,

$$\Omega^i_{X/Y} = \bigwedge^i \Omega^1_{X/Y}$$

and let $\Omega^0_{X/Y} = \mathcal{O}_X$. Furthermore, there exists a unique family of maps $\mathrm{d}^i : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ such that,

(a) d^i is a Y-antiderivation of the total complex,

$$\Omega_{X/Y} = \bigoplus_{i=0}^{\infty} \Omega^i_{X/Y}$$

meaning that d is $f^{-1}\mathcal{O}_Y$-linear and on local sections it satisfies the graded Leibniz law,

$$\mathrm{d}(a \wedge b) = \mathrm{d}a \wedge b + (-1)^i a \wedge \mathrm{d}b$$

(b) $\mathrm{d}^2 = 0$

(c) $\mathrm{d}a = \mathrm{d}_{X/Y} a$ for $\deg a = 0$.

Then $(\Omega^\bullet_{X/Y}, \mathrm{d})$ is the deRham complex of X/Y,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_{X/Y} \longrightarrow \Omega^2_{X/Y} \longrightarrow \Omega^3_{X/Y} \longrightarrow \cdots$$

Remark. Working over $k = \mathbb{C}$, there is also an analytic deRham complex $(\Omega^\bullet_{X/Y})^{\mathrm{an}}$. Then GAGA tells you that you get the same cohomology in the algebraic and analytic cases. Furthermore, the analytic deRham complex is a (not acyclic!!) resolution of the constant sheaf $\mathbb{C}$.

Definition 1.2.2. $H^n_{\mathrm{dR}}(X) = \mathbb{H}^n(X, \Omega^\bullet_{X/Y})$

Remark. $\mathbb{H}^n(X, \Omega^\bullet_{X/Y}) = R^n\Gamma(\Omega^\bullet_{X/Y})$.

Remark. There exists a hypercohomology spectral sequence,

$$E_1^{p,q} = R^q\Gamma(X, C^p) \implies \mathbb{H}^{p+q}(C^\bullet)$$

Applying this to the deRham complex gives the Hodge-to-deRham spectral sequence,

$$H^q(X, \Omega^p_{X/Y}) \implies H^{p+q}_{\mathrm{dR}}(X)$$

1.3 Frobenius and Cartier Isomorphisms

Definition 1.3.1. Let X be a scheme of characteristic p (meaning $p\mathcal{O}_X = 0$). Then there is a natural map $\mathrm{Fr} : X \to X$ via id on topological spaces and $\mathcal{O}_X \to \mathcal{O}_X$ via $x \mapsto x^p$. This is natural, in the sense that for any map $f : X \to Y$ there is a commutative diagram,

$$\begin{tikzcd} X \arrow[r, "\mathrm{Fr}_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{Fr}_Y"] & Y \end{tikzcd}$$

Therefore, we can define via pullbacks,

$$\begin{tikzcd} X \arrow[r, dashed, "F_{X/Y}"] \arrow[dr, "f"'] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Y \arrow[r, "\mathrm{Fr}_Y"] & Y \end{tikzcd}$$

giving the relative Frobenius $F_{X/Y} : X \to X^{(p)}$.

Proposition 1.3.2. If Y has characteristic p and $f : X \to Y$ is smooth of relative dimension n then $F_{X/Y} : X \to X^{(p)}$ is finite and flat of degree n. Therefore, $F_*\mathcal{O}_X$ is locally free of rank n as a $\mathcal{O}_{X^{(p)}}$-module.

Proof. When f is étale then $F_{X/Y}$ is actually an isomorphism. Indeed, $F_{X/Y}$ composed with $X^{(p)} \to Y$ is étale and $X^{(p)} \to Y$ is étale by base change so $F_{X/Y}$ is étale but it is also radicial since Fr_X is. Thus $F_{X/Y}$ is a surjective open immerison. In general, this is a local question so we reduce to a standard smooth which factors as the composition of an étale map and a projection from affine space which can be done directly. □

Proposition 1.3.3. Let $\mathrm{d} = \mathrm{d}_{X/Y}$. Let s be a local section of $\mathcal{O}_X$. Then,

$$\mathrm{d}(s^p) = ps^{p-1}\mathrm{d}s = 0$$

since $\mathrm{d}(s^p) = F^*_{X/Y}(\mathrm{d}s) = F^*_{X/Y}(1 \otimes \mathrm{d}s)$. Thus,

(a) $\mathrm{Fr}^*\Omega^i_{X/Y} \to \Omega^i_{X/Y}$ is zero

(b) $F_{X/Y}^* \Omega^i_{X^{(p)}/Y} \to \Omega^i_{X/Y}$ is zero

(c) d on the complex $(F_{X/Y})_* \Omega^\bullet_{X/Y}$ is $\mathcal{O}_{X^p}$-linear.

Theorem 1.3.4 (Cartier)**.** There exists a unique morphism of graded $\mathcal{O}_{X^{(p)}}$-algebras,

$$\gamma : \bigoplus_i \Omega^i_{X^{(p)}/Y} \to \bigoplus_i \mathcal{H}^i((F_{X/Y})_* \Omega^\bullet_{X/Y})$$

such that

(a) for $i = 0$, we have γ is the map $\mathcal{O}_{X^{(p)}} \to (F_{X/Y})_* \mathcal{O}_X$

(b) for $i = 1$, we have $\gamma(1 \otimes \mathrm{d}s) = s^{p-1}\mathrm{d}s$ in $\mathcal{H}^i(F_{X/Y*}\Omega^\bullet_{X/Y})$

Furthermore, if f is smooth then γ is an isomorphism and we call $c = \gamma^{-1}$.

Remark. If $Y = \mathrm{Spec}\,(k)$ and X is smooth then γ is called the absolute Cartier isomorphism.

Remark. The theorem tells us that γ is determined by how it acts in degree 0 and degree 1 because it is a morphism of graded algebras and the deRham complex is generated in degrees 0 and 1. Explicitly,

$$\gamma(\tau \wedge \sigma) = \gamma(\tau) \wedge \gamma(\sigma)$$

1.4 Relationship to the HdDSS

Now let $Y = \mathrm{Spec}\,(k)$ with k a perfect field. D-I realized that the Cartier isomorphism is related to degeneration of the HdDSS,

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \implies H^{p+q}_{\mathrm{dR}}(X/k)$$

Consider the complex,

$$C = \bigoplus_i \Omega^i_{X^{(p)}/Y}[-i]$$

Then $\mathcal{H}^i(C)$ is the graded parts of the domain of the Cartier isomorphism. Furthermore, the codomain is $\mathcal{H}^i(F_* \Omega^\bullet_{X/k})$. Then we might ask if there is a map of complexes,

$$\phi : C \to F_* \Omega^\bullet_{X/k}$$

which induces the Cartier map.

Proposition 1.4.1. If there is such a quasi-isomorphism ϕ, then the HdRSS degenerates at E_1.

Proof. This follows from the chain of isomorphisms,

$$\mathbb{H}^n(X, \Omega^\bullet_X) \cong \mathbb{H}^n(X^{(p)}, F_* \Omega^\bullet_X) \cong \bigoplus_i H^{n-i}(X^{(p)}, \Omega^i_{X^{(p)}}) \cong \bigoplus_i H^{n-i}(X, \Omega^i_X)$$

The first isomorphism comes from the fact that F is finite and thus affine. The second isomorphism is the inverse of the map induced by ϕ on cohomology. Finally,

$$H^j(X^p, \Omega^i_{X^{(p)}}) = H^j(X, \Omega^i_X)$$

becuase $F : X \to X^{(p)}$ is an isomorphism of schemes (not of k-schemes). Therefore the dimensions match whcih implies that the spectral sequence must have degenerated since the dimensions of the terms matches those of the filtered pieces already. ☐

2 Oct. 14

2.1 Degeneration in Characteristic p

First we state the main theorem for today.

Theorem 2.1.1. Let $S \to \mathbb{Z}/p\mathbb{Z}$ be a scheme of characteristic p and a flat lift to $\mathbb{Z}/p^2\mathbb{Z}$,

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] & \widetilde{S} \arrow[d] \\ \mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z}) \arrow[r, hook] & \mathrm{Spec}\,(\mathbb{Z}/p^2\mathbb{Z}) \end{tikzcd}$$

If X/S is smooth and proper and $X^{(p)}$ admits a smooth lift over $\widetilde{S}$ then,

$$\tau^{<p}(F_{X/S})_* \Omega^\bullet_{X/S}$$

is decomposable in $D(X^{(p)})$ meaning it is isomorphic to a complex whose differentials are all zero (i.e. it is isomorphic to its cohomology).

Remark. The de Rham complex is not an element of the derived category of $\mathcal{O}_X$-modules because the transition maps are not $\mathcal{O}_X$-linear. However, the useful fact about $(F_{X/S})_* \Omega^\bullet_{X/S}$ is that the transition maps are $\mathcal{O}_{X^{(p)}}$-linear because for any $f \in \mathcal{O}_{X^{(p)}}(U)$ and $\omega \in \Omega_{X/S}(F_{X/S}^{-1}(U))$ we have,

$$\mathrm{d}(f \cdot \omega) = \mathrm{d}(F^\#_{X/S}(f)\omega) = \mathrm{d}(F^\#_{X/S}(f)) \wedge \omega + F^\#_{X/S}(f)\mathrm{d}\omega = f \cdot \mathrm{d}\omega$$

because $\mathrm{d}(F^\#_{X/S}(f)) = 0$ since this is d relative to S and $F_{X/S}$ acts via $x \mapsto x^p$ "relative to S".

Corollary 2.1.2. If k is a perfect field and X/k is smooth, proper, and $\dim X < p$ and X lifts over $W_2(k)$ then the Hodge-to-de Rham spectral sequence degenerates at E_1.

Proof. We apply this to the case $S = \mathrm{Spec}\,(k)$ and $\widetilde{S} = \mathrm{Spec}\,(W_2(k))$. By above, we have that $(F_{X/S})_* \Omega^\bullet_{X/S}$ is decomposable and the hyperderived spectral sequence of any decomposable complex degenerates at E_1 just because the differentials of the spectral sequence are formed from the transition maps on the complex which are zero up to quasi-isomorphism. Therefore,

$$\dim \mathbb{H}^n(X, \Omega^\bullet_{X/k}) = \dim \mathbb{H}^n(X^{(p)}, (F_{X/k})_* \Omega^\bullet_{X/k}) = \sum_{p+q=n} h^q(X^{(p)}, (F_{X/k})_* \Omega^p_{X/k}) = \sum_{p+q=n} h^q(X, \Omega^p_{X/k})$$

because the Frobenius is affine and therefore the dimensions add up for the Hodge-to-de Rham spectral sequence already at the E_1 page proving that the differentials must already be zero. $\square$

2.2 Recall the Cartier Isomorphism

Let X/S be a smooth scheme with S characteristic p. Then there is a graded isomorphism,

$$C^{-1} : \bigoplus_i \Omega^i_{X^{(p)}/S} \xrightarrow{\sim} \bigoplus_i \mathcal{H}^i((F_{X/S})_* \Omega^\bullet_{X/S})$$

such that,

(a) in $i = 0$ the map $\mathcal{O}_{X^{(p)}} \to (F_{X/S})_* \mathcal{O}_X$ is $F^\#_{X/S}$

(b) in $i=1$,

$$C^{-1}(1\otimes \mathrm{d}s) = s^{p-1}\mathrm{d}s \in \mathcal{H}^1((F_{X/S})_*\Omega^\bullet_{X/S})$$

think of this as like "$\frac{F^*(\mathrm{d}s)}{p}$".

To prove the main theorem for today, we will exhibit a quasi-isomorphism

$$\varphi : \bigoplus_{i<p} \Omega^i_{X^{(p)}/S}[-i] \to \tau^{<p}(F_{X/S})_*\Omega^\bullet_{X/S}$$

that induces C^{-1} on cohomology for $i<p$ (and thus is a quasi-isomorphism).

Remark. Note that when S is perfect (meaning Fr_S is an isomorphism) we also get an "absolute" version of the theorem since $(\mathrm{Fr}_S)_{X*}\Omega^i_{X^{(p)}/S} = \Omega^i_{X/S}$ because $(\mathrm{Fr}_S)_X : X^{(p)} \to X$ is also an isomorphism. Therefore, pushing forward φ gives a quasi-isomorphism

$$(\mathrm{Fr}_S)_{X*}\varphi : \bigoplus_{i<p} \Omega^i_{X^{(p)}/S} \xrightarrow{\sim} (\mathrm{Fr}_X)_*\Omega^\bullet_{X/S}$$

We want to reduce to constructing φ^1 where φ^i are the components of the map from the direct sum. For φ^0 we just define,

$$\varphi^0 : \mathcal{O}_{X^{(p)}} \xrightarrow{C^{-1}} \mathcal{H}^0((F_{X/S})_*\Omega^\bullet_{X/S}) \hookrightarrow (F_{X/S})_*\Omega^\bullet_{X/S}$$

Now assume we have constructed,

$$\varphi^1 : \Omega^1_{X^{(p)}/S}[-1] \to (F_{X/S})_*\Omega^\bullet_{X/S}$$

inducing C^{-1} on $\mathcal{H}^1$. Then there exists,

$$\left(\Omega^1_{X^{(p)}/S}\right)^{\otimes i} \to \Omega^i_{X^{(p)}/S}$$

by sending,

$$w_1 \otimes \cdots \otimes w_i \mapsto w_1 \wedge \cdots \wedge w_i$$

If $i<p$ (or in characteristic zero) then there exists a section to this map,

$$a(w_1\wedge\cdots\wedge w_i) = \frac{1}{i!}\sum_{\sigma\in S_i} \mathrm{sign}(i) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(i)}$$

Therefore we get,

$$\begin{tikzcd}
(\Omega^1_{X^{(p)}/S})^{\otimes i} \arrow[r, "\varphi_1^{\otimes i}"] & \left((F_{X/S})_*\Omega^\bullet_{X/S}\right)^{\otimes^{\mathbb{L}} i} \arrow[d] \\
\Omega^i_{X^{(p)}/S} \arrow[u] \arrow[r, dashed, "\varphi^i"] & (F_{X/S})_*\Omega^\bullet_{X/S}
\end{tikzcd}$$

Because this construction agrees with the product structure and the Cartier isomorphism is determined (using the product structure) by its values in degree 1 this means that φ^i must induce C^{-1} in degree i.

2.3 Construction of φ^1

First we consider the case when $F_{X/S}$ admits a global lift. Given,

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \tilde{S} \arrow[d] \\ \mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z}) \arrow[r, hook] & \mathrm{Spec}\,(\mathbb{Z}/p^2\mathbb{Z}) \end{tikzcd}$$

and X/S is smooth and proper. We want there to be a digram,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "F_{X/S}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X} \arrow[d, "\widetilde{F_{X/S}}"] \\ X^{(p)} \arrow[r] & \widetilde{X}^{(p)} \end{tikzcd}$$

where $\widetilde{X} \to \widetilde{S}$ and $\widetilde{X}^{(p)} \to S$ are smooth (flat implies this) lifts of $X \to S$ and $X^{(p)} \to S$. Note that we assumed the existence of the smooth lift $\widetilde{X} \to \tilde{S}$ in the hypothesis of the thorem but we did not assume the existence of a lift of $F_{X/S}$. However, a lift of $F_{X/S}$ exists locally so we will assume a lift exists and then use uniqueness to patch together the results obtained for each local lift.

Remark. We will only apply this for $S = \mathrm{Spec}\,(k)$ with k a perfect field and $\widetilde{S} = \mathrm{Spec}\,(W_2(k))$. Note that $W_2(k)$ is the *unique* flat lift of k along

$$\mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z}) \hookrightarrow \mathrm{Spec}\,\left(\mathbb{Z}/p^2\mathbb{Z}\right)$$

This is what people mean when they say $W(k)$ lifting is an "unramified" lift, it is unramified over $\mathbb{Z}_p$. Indeed, another characteristic zero lift say over a ring R will have $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(\mathbb{Z}_p)$ ramified (the fiber over (p) is a nonreduced structure on $\mathrm{Spec}\,(k)$) so it does not induce a flat deformation of $\mathrm{Spec}\,(k)$ over $\mathbb{Z}/p^2\mathbb{Z}$ only of the nonreduced scheme.

Remark. Note that if $S = \mathrm{Spec}\,(k)$ for k a perfect field, if X lifts to $W_2(k)$ then so does $X^{(p)}$. Indeed, absolute frobenius of k lifts to $W_2(k)$ so we can pullback a lift of X along this. Also Fr_k is an automorphism so it is directly clear that X lifts if and only if $X^{(p)}$ lifts.

Remark. Because of flatness, multiplication by p induces an isomorphism $p : \mathcal{O}_S \xrightarrow{\sim} p\mathcal{O}_{\widetilde{S}}$. Inded, from the exact sequence,

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\ p\ } \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

we see that $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} p\mathbb{Z}/p^2\mathbb{Z}$ meaning that this is an extension by the module $\mathbb{Z}/p\mathbb{Z}$. Then by the flatness of $\widetilde{S} \to \mathrm{Spec}\,(\mathbb{Z}/p^2\mathbb{Z})$ the exact sequence,

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\ p\ } \mathcal{O}_{\widetilde{S}} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

so the extension is by the ideal $p \cdot \mathcal{O}_{\widetilde{S}}$ which is isomorphic to $\mathcal{O}_S$. The exact same argument for $X \hookrightarrow \widetilde{X}$ which is also a flat lift over $\mathrm{Spec}\,(\mathbb{Z}/p\mathbb{Z}) \to \mathrm{Spec}\,(\mathbb{Z}/p^2\mathbb{Z})$ shows that $\widetilde{X}$ is an extension of X by $\mathcal{O}_X \xrightarrow{\sim} p\mathcal{O}_{\widetilde{X}}$. Therefore, by local freeness, we get a similar isomorphism,

$$p : \Omega^1_{X/S} \xrightarrow{\sim} p \cdot \Omega^1_{\tilde{X}/\tilde{S}}$$

Now to perform the construction notice that,

$$\operatorname{im}(\widetilde{F_{X/S}}^* : \Omega^1_{\widetilde{X^{(p)}}/\widetilde{S}} \to (\widetilde{F_{X/S}})_* \Omega^1_{\widetilde{X}/\widetilde{S}}) \subset p \cdot (\widetilde{F_{X/S}})_* \Omega^1_{\widetilde{X}/\widetilde{S}}$$

because pulling back differentials by Frobenius introduces a factor of p. Therefore, we get a diagram,

$$\begin{tikzcd} \Omega^1_{\widetilde{X^{(p)}}/\widetilde{S}} \arrow[r, "\widetilde{F_{X/S}}"] \arrow[d, two heads] & p \cdot (\widetilde{F_{X/S}})_* \Omega^1_{\widetilde{X}/\widetilde{S}} \\ \Omega^1_{X^{(p)}/S} \arrow[r, dashed, "\varphi^1"] & (F_{X/S})_* \Omega^1_{X/S} \arrow[u, "p\cdot(-)"'] \end{tikzcd}$$

which exists because the right upward map is an isomorphism and the kernel of the left downward map is the multiples of p which are sent to zero. I claim that

$$\operatorname{im} \varphi^1 \subset Z^1((F_{X/S})_* \Omega^\bullet_{X/S})$$

and φ^1 induces C^{-1} in degree 1. For local section $a' \in \Gamma(U^{(p)}, \mathcal{O}_{\widetilde{X^{(p)}}})$ pulled back from $a \in \Gamma(U, \mathcal{O}_X)$, the differential $\mathrm{d}a$ is acted on via

$$\widetilde{F_{X/S}}^*(\mathrm{d}a') = \mathrm{d}\, \widetilde{F_{X/S}}^\# a' = p a^{p-1} \mathrm{d}a + p\, \mathrm{d}b$$

where $\widetilde{F_{X/S}}^\# a' = a^p + p\, b$ where $p\, b$ is the error term. Hence

$$\varphi^1(\mathrm{d}a') = a^{p-1}\mathrm{d}a + \mathrm{d}b$$

which is clearly an exact form (lies in Z^1). But notice that the second term is exact and therefore dies in the quotient

$$Z^1((F_{X/S})_* \Omega^\bullet_{X/S}) \to \mathcal{H}^1((F_{X/S})_* \Omega^\bullet_{X/S})$$

so the induced map is exactly given by the Cartier isomorphism in degree 1.

2.4 What about if F doesn't lift?

From smoothness, we know that lifts exist locally. We need to compare the outputs of different lifts.

Lemma 2.4.1. Given flat lifts $\widetilde{X}_i$ of X and $G_i : \widetilde{X} \to \widetilde{X^{(p)}}$ of $F_{X/S}$ over $\widetilde{S}$ there is a canonical element,

$$h(G_1, G_2) : \Omega^1_{X^{(p)}/S} \to (F_{X/S})_* \mathcal{O}_X$$

such that,

$$\varphi^1_{G_2} - \varphi^1_{G_1} = \mathrm{d}h(G_1, G_2)$$

and if $G_3 : \widetilde{X}_3 \to \widetilde{X^{(p)}}$ is a third lifting then

$$h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3)$$

Proof. Choose an isomorphism $u : \widetilde{X}_1 \xrightarrow{\sim} \widetilde{X}_2$ of lifts (which may only exist locally) then

$$u^*G_2 - G_1 : \mathcal{O}_{X^{(p)}} \to (F_{X/S})_*\mathcal{O}_X$$

is a derivation which does not depend on the choice of isomorphism u. Indeed, given $u' : \widetilde{X}_1 \xrightarrow{\sim} \widetilde{X}_2$ the difference is a derivation or equivalently a map

$$\delta : \Omega^1_{X/S} \to \mathcal{O}_X$$

Then u^*G_2 and u'^*G_2 differ by the composition of δ with the pullback $F^*_{X/S}\Omega^1_{X^{(p)}/S} \to \Omega^1_{X/S}$ which is zero. Hence $u^*G_2 = u'^*G_2$. Therefore, working locally on X so that an isomorphism u exists, we get a well-defined derivation

$$h(G_1, G_2) : \Omega^1_{X^{(p)}/S} \to (\widetilde{F_{X/S}})_*\mathcal{O}_X$$

via the difference above. Then

$$\varphi^1_{G_2} - \varphi^1_{G_1} = \mathrm{d}h(G_1, G_2)$$

from the formula for φ^1 since $G_2^\#(a') - G_1^\#(a') = b_2 - b_1$ in $(F_{X/S})_*\mathcal{O}_X = p \cdot (\widetilde{F_{X/S}})_*\mathcal{O}_{\widetilde{X}}$ then

$$\varphi^1_{G_2}(a') - \varphi^1_{G_1}(a') = \mathrm{d}(b_2 - b_1)$$

□

2.5 Proof of the Theorem

Now fix the lifting $\widetilde{X^{(p)}}$ of $X^{(p)}$ over $\widetilde{S}$. Choose an open covering $\mathrm{U} = (U_i)_{i \in I}$ of X so that for each i there is a lifting $\widetilde{U}_i$ of U_i over $\widetilde{S}$ and a lifting $G_i : \widetilde{U}_i \to \widetilde{X^{(p)}}$ of $F|_{U_i}$. We have built, for each i a map of complexes

$$f_i = \varphi^1_{G_i} : \Omega^1_{X^{(p)}/S}[-1] \to F_*\Omega^\bullet_{X/S}|_{U_i}$$

and for each pair (i, j), a homomorphism

$$h_{ij} = h(G_i|_{U_{ij}}, G_j|_{U_{ij}}) : \Omega^1_{X^{(p)}/S}|_{U_{ij}} \to F_*\Omega^\bullet_{X/S}|_{U_{ij}}$$

where $U_{ij} = U_i \cap U_j$. These datum are related via

$$f_j - f_i = \mathrm{d}h_{ij} \text{ over } U_{ij}$$

$$h_{ij} + h_{jk} = h_{ik} \text{ over } U_{ijk} = U_i \cap U_j \cap U_k$$

These make it possible to define a homomorphism of complexes of $\mathcal{O}_{X^{(p)}}$-modules

$$\varphi^1_{\widetilde{X^{(p)}},(U_i,G_i)} : \Omega^1_{X^{(p)}/S}[-1] \to \check{C}(\mathrm{U}, F_*\Omega^\bullet_{X/S})$$

where the target is the total complex associated to the Čech bicomplex of the cover U with values in the complex $F_*\Omega^\bullet_{X/S}$. Explicitly, this the complex

$$\check{C}(\mathrm{U}, F_*\Omega^\bullet_{X/S})^n = \bigoplus_{i+j=n} \check{C}^j(\mathrm{U}, F_*\Omega^i_{X/S})$$

with differential $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$ where d_1 is the de Rham differential and d_2 is, in bidegree (i, j), equal to $(-1)^i \sum(-1)^i \partial^i$ for the Čech differential. In particular,

$$\check{C}(\mathrm{U}, F_*\Omega^\bullet_{X/S})^1 = \check{C}(\mathrm{U}, F_*\mathcal{O}_X) \oplus \check{C}^0(\mathrm{U}, F_*\Omega^1_{X/S})$$

The morphism $\varphi^1_{\widetilde{X^{(p)}},(U_i,G_i)}$ is defined as having for components (φ_1, φ_2) in degree 1, with

$$(\varphi_1\omega)(i, j) = h_{ij}(\omega)|_{U_{ij}} \quad (\varphi_2\omega)(i) = f_i(\omega)|_{U_i}$$

Using the fact that the f_i are morphisms of complexes, together with the above forumlas relating the f_i and h_{ij}, it follows that $\varphi^1_{X^{(p)},(U_i,G_i)}$ is thus a well-defined morphism of complexes. We also has at our disposal the natural augmentation

$$\epsilon : F_*\Omega^\bullet_{X/S} \to \check{C}(\mathrm{U}, F_*\Omega^\bullet_{X/S})$$

which is a quasi-isomorphism. Because for each i, the complex $\check{C}(\mathrm{U}, F_*\Omega^i_{X/S})$ is a resolution of $F_*\Omega^i_{X/S}$. We then define φ^1_Z by inverting ϵ. Comparing two coverings we can show that φ^1 does not depend on the choices.

3 Passage to Characteristic Zero

Remark. Today again all schemes are noetherian.

Proposition 3.0.1 (Nullstellensatz)**.** If K is a finite type k-algebra and K is a field then K/k is finite.

Proof. Suppose not. Then there is an injection $k[t] \hookrightarrow K$ because K cannot be algebraic. Then $\mathrm{Spec}\,(K) \to \mathbb{A}^1_k$ so by Chevalley the image is constructible. But the image the generic point which is not constructible giving a contradiction. ☐

Corollary 3.0.2. Every nonempty constructible subset of a finite type k-scheme has a closed point.

Proof. Let $C \subset X$ be locally closed and affine let $C = \mathrm{Spec}\,(A)$. Then $A/\mathfrak{m}$ is a field finite type over k so it is finite. Then consider $\overline{\{\mathfrak{m}\}} \subset X$ is closed. However, the generic point of $\overline{\{\mathfrak{m}\}}$ has transcendence degree zero. ☐

Definition 3.0.3. X is *Jacobson* if every nonempty constructible subset has a closed (in X) point.

Remark. This is equivalent to every closed set is the closure of its closed points.

Example 3.0.4. Some (non) examples of Jacobson schemes,

(a) finte type k-schemes are Jacobson

(b) $\mathrm{Spec}\,(\mathbb{Z})$ is Jacobson

(c) if R is a local ring of $\dim R \geq 1$ then not Jacobson

(d) $X = \mathrm{Spec}\,(R) \setminus \{\mathfrak{m}_R\}$ is Jacobson.

Proposition 3.0.5. Let S be Jacobson and $f : X \to S$ is finite type.

(a) If $x \in X$ is a closed point then $f(x)$ is closed.

(b) X is Jacobson.

Proof. For (a) let $x \in X$ be a closed point then Chevalley's theorem implies that $\{f(x)\}$ is constructible so $\{f(x)\}$ is closed because S is Jacobson. For (b) let $C \subset X$ be constructible. Then Chevalley's theorem implies that $f(C) \subset S$ is constructible so there is a closed point $s \in f(C)$. Then $X_s \to \kappa(s)$ is finite type so X_s is Jacobson. Then $X_s \cap C \subset X_s$ is nonempty constructible so it has a closed point $x \in C \cap X_s$ and X_s is closed (because $s \in S$ is closed) so x is a closed point. □

Corollary 3.0.6. Finite type $\mathbb{Z}$-schemes are Jacobson and have finite residue fields at closed points.

Proof. The first part is immediate. Then if $x \in X$ is a closed point then it lies over some $p \in \mathrm{Spec}\,(\mathbb{Z})$ nonzero (because x is closed) so $x \in X_p$ and X_p is finite type over $\kappa(p) = \mathbb{F}_p$. Then it follows from the Nullstellensatz. □

Proposition 3.0.7. If $X \to \mathrm{Spec}\,(\mathbb{Z})$ is finite tpye and X is reduced then there is a dense open such that $U \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth.

Proof. This follows from two facts:

(a) if k is a perfect field and X is a finite type reduced k-scheme then it is generically smooth.

(b) if $f : X \to S$ is finite type then the smooth locus is open.

We can assume that X is integral then $K(x)/k$ is finitely generated. Since k is perfect there is a separating transcendence basis $t_1, \dots, t_n \in K(X)$ such that $K(X)/k(t_1, \dots, t_n)$ is finite separable. Then $K(X) = k(t_1, \dots, t_n)[T]/(G(T))$ by the primitive element theorem. By localizing on X we get an open affine $U \subset X$ with $U \hookrightarrow \mathbb{A}^{n+1}_k$ defined by G. Then $U \setminus V(G)$ is smooth and $V(G)$ does not contain the generic point so this is a dense open.

To see the second part, locally embedd $X \hookrightarrow \mathbb{A}^N_S$ by $f_1, \dots, f_m$ then smoothness is characterized by the nonvanihsing og some minors of the jacobian of $f_1, \dots, f_m$ which is a closed condition. □

Theorem 3.0.8. If $\pi : X \to S$ is proper, $\mathscr{F}$ is coherent over X then $R^i\pi_*\mathscr{F}$ is also coherent.

Proof. The proof is long but,

(a) first deal with the projective case by showing $H^i(\mathbb{P}^n_A, \mathcal{O}(m))$ is finite over A for all i, m, n.

(b) if $\mathscr{F}$ is coherent on $\mathbb{P}^n_A$ then there exists a surjection $\mathcal{O}(-N)^M \twoheadrightarrow \mathscr{F}$ then we use descending induction to show that $H^i(\mathbb{P}^n_A, \mathscr{F})$ is finite for all i.

(c) X is projective then use $\iota : X \hookrightarrow \mathbb{P}^n_A$ and exactness of affine pushforward to reduce to the case of projective space.

(d) In general, Chow's lemma gives $f : \tilde{X} \to X$ over S such that $\tilde{X}$ is projective over S and f is projective and surjective. Use Leray spectral sequence argument [EGA III, 3.1-2].

□

Remark. The same coherence statement also holds if $\mathscr{F}$ is a bounded complex of coherent sheaves. This follows from the spectral sequence,

$$E_1^{i,j} = R^j f_* K^i \implies R^{i+j} f_* K^\bullet$$

which is just the first spectral sequence for hypercohomology.

Theorem 3.0.9 (flat base change)**.** Consider a Cartesian diagram,

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

where g is flat and f is finite type and separated. Let $\mathscr{F}$ be quasi-coherent on X then the natural base change map,

$$g^* R f_* \mathscr{F} \to R f'_* g'^* \mathscr{F}$$

is an isomorphism. By adjunction this is the same as a map,

$$R f_* \mathscr{F} \to R g_* R f'_* g'^* \mathscr{F} = R f_* R g'_* g'^* \mathscr{F}$$

which we have by applying $R f_*$ to $\mathscr{F} \to R g'_* g'^* \mathscr{F}$.

Theorem 3.0.10 (Cohomology and Base Change)**.** Let $f : X \to S$ be proper and $\mathscr{F}$ is coherent on X and flat over S. Suppose that $R^i f_* \mathscr{F}$ is finite locally free for all i. Then given any diagram,

$$g^* R^i f_* \mathscr{F} \to R^i f'_* g'^* \mathscr{F}$$

is an isomorphism for all n for all maps g.

Remark. The same holds if $\mathscr{F}$ is replaced with a bounded complex of coherent sheaves with flat cohomology sheaves over S such that $R^i f_* K^\bullet$ is finite locally free for all n.

Theorem 3.0.11. If $f : X \to S$ is finite type, the function,

$$x \mapsto \dim_x X_{f(x)}$$

is upper semi-continuous. If f is closed then the function,

$$s \mapsto \dim X_s$$

is also semi-continuous.

Proof. The second follows from the first because,

$$f(\{x \in X \mid \dim_x X_{f(x)} \geq n\})$$

is closed. □

3.1 Completing the Proof

Remark. Previously, we proved the following.

Theorem 3.1.1. Let k be perfect of characteristic $p > 0$ and X is smooth and proper over k and $\dim X < p$ and X admits a lift to $W_2(k)$ then,

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H_{\text{dR}}^{p+q}(X)$$

degenerates at E_1.

Remark. We now use this to deduce the main theorem.

Theorem 3.1.2. Let K be a field of char zero and X is smooth and proper over K. Then,

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H_{\text{dR}}^{p+q}(X)$$

degenerates at E_1.

Proof. Spread out X to some smooth and proper $\mathfrak{X} \to \text{Spec}\,(A)$ for $A \subset K$ finite type over $\mathbb{Z}$. This is because $K = \varinjlim A$ for finite type $\mathbb{Z}$-subalgebras of K then we spread out to schemes over each A and smooth and proper spreads out. Thus we get a Cartesian diagram,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{X} \arrow[d] \\ \text{Spec}\,(K) \arrow[r] & \text{Spec}\,(A) \end{tikzcd}$$

Now by base change we can assume that $K = \bar{K}$ and X is connected of dimension d. By upper-semi continuity we can assume that all fibers of $\mathfrak{X} \to S = \text{Spec}\,(A)$ are of dimension d by shrinking A. Furthermore, we can shirnk A such that $\text{Spec}\,(A) \to \text{Spec}\,(\mathbb{Z})$ is smooth. This is because $A_{\mathbb{Q}}$ is reduced and thus $\text{Spec}\,(A_{\mathbb{Q}}) \to \text{Spec}\,(\mathbb{Q})$ is smooth on a dense open and therefore the smooth locus of $\text{Spec}\,(A) \to \text{Spec}\,(\mathbb{Z})$ contains the generic point and thus is a nonempty open so we can shrink to that open.

Now $R^n f_* \Omega^i_{\mathfrak{X}/S}$ and $R^n f_* \Omega^\bullet_{\mathfrak{X}/S}$ are coherent. Therefore, by shrinking S we can assume that all of them are finite locally free (this works because there are finitely many since it vanishes when $i > d$ and $n > d$) because they are generically free. Let $h^{i,j} = \dim H^j(X, \Omega^i_{X/K})$ and $h^n = \dim_K H^n_{\text{dR}}(X)$. It suffices to show that,

$$h^n = \sum_{i+j=n} h^{i,j}$$

Because all pushforwards in sight are finite locally free and therefore these pushforwards commute with arbitrary base change. In particular if $s \in S$ is any point then,

$$h^{i,j} = \dim_{\kappa(s)} H^j(\mathfrak{X}_s, \Omega^i_{\mathfrak{X}_s/\kappa(s)}) \quad \text{and} \quad h^n = \dim_{\kappa(s)} H^n_{\text{dR}}(\mathfrak{X}_s)$$

We want to find an s such that $\mathfrak{X}_s \to \text{Spec}\,(\kappa(s))$ satisfies our previous conditions for degeneration of Hodge-to-deRham. Thus we want,

(a) $\dim \mathfrak{X}_s < \text{char}(\kappa(s))$

(b) $\mathfrak{X}_s$ lifts to $W_2(\kappa(s))$

If we can do this then,

$$E_1^{i,j} = H^j(\mathfrak{X}_s, \Omega^i_{\mathfrak{X}_s/\kappa(s)}) \implies H^{i+j}_{\mathrm{dR}}(\mathfrak{X}_s)$$

degenerates at E_1 and therefore,

$$h^n = \dim_{\kappa(s)} H^n_{\mathrm{dR}}(\mathfrak{X}_s) = \sum_{i+j=n} \dim_{\kappa(s)} H^j(\mathfrak{X}_s, \Omega^i_{\mathfrak{X}_s/\kappa(s)}) = \sum_{i+j=n} h^{i,j}$$

which is what we wanted to show.

Set,

$$N = \prod_{\substack{p \le d \\ p \text{ prime}}} p$$

Replace A by $A[1/N]$ so no residue field of A can have characteristic $\le d$. Then A is finite over $\mathbb{Z}$ so it has a closed point $s \in \mathrm{Spec}\,(A)$ and thus $\mathrm{char}(\kappa(s)) > d$ and $d = \dim \mathfrak{X}_s$. Choose this point $s \in \mathrm{Spec}\,(A)$.

Now, we have a diagram,

$$\begin{tikzcd}
\mathrm{Spec}\,(\kappa(s)) \arrow[r] \arrow[d, "\text{nilpotent thickening}"'] & S \arrow[d, "\text{smooth}"] \\
\mathrm{Spec}\,(W_2(\kappa(s))) \arrow[r] \arrow[ur, dashed] \arrow[d] & \mathrm{Spec}\,(\mathbb{Z}) \\
\mathrm{Spec}\,(\mathbb{Z}_p) \arrow[ur] &
\end{tikzcd}$$

there exists a lift because $S \to \mathrm{Spec}\,(\mathbb{Z})$ is smooth. Therefore, by pulling back along this lift gives a lift of $\mathfrak{X}_s$,

$$\begin{tikzcd}
\widetilde{\mathfrak{X}_s} \arrow[r] \arrow[d] & \mathfrak{X}_s \arrow[d] \\
\mathrm{Spec}\,(W_2(\kappa(s))) \arrow[r] & S
\end{tikzcd}$$

therefore $\mathfrak{X}_s$ lifts over $W_2(\kappa(s))$ so we are done. $\square$