1 Reieder's Theorem and separating Points

Theorem 1.0.1. Let X be a smooth projective surface and L a nef line bundle with $L^2 > 0$. Suppose that for any effective divisor D we have $D \cdot L \geq \alpha$. Then $|K_X + dL|$ separates at least

$$\min\{\alpha(d - \alpha/L^2) - 1, (d/2)^2 L^2\}$$

distinct points.

Remark. This is optimal for $X = \mathbb{P}^2$ and $L = \mathcal{O}_X(1)$ and $\alpha = 1$. Indeed, $|K_X + dL|$ separates $d-2$ points for $d > 2$ and no points for $d \leq 2$. Indeed any smooth plane curve of degree $d \geq 2$ has gonality $d-1$.

Lemma 1.0.2. If $|K_X + L|$ does not separate d-points then there exists a reduced subscheme Z of length d (the union of the bad points) and an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{L} \otimes \mathscr{I}_Z \to 0$$

such that $\mathcal{E}$ is a vector bundle.

Proof. From the sequence

$$0 \to \mathcal{L} \otimes \omega_X \otimes \mathscr{I}_Z \to \mathcal{L} \otimes \omega_X \to \mathcal{L} \otimes \omega_X \otimes \mathcal{O}_Z \to 0$$

we have

$$H^0(X, \mathcal{L} \otimes \omega_X) \to H^0(Z, L \otimes \omega_X|_Z) \to H^1(X, L \otimes \omega_X \otimes \mathscr{I}_Z)$$

Therefore, we get a map via Serre duality

$$\mathrm{Ext}^1_X(\mathcal{L} \otimes \mathscr{I}_Z, \mathcal{O}_X) = H^1(X, \mathcal{L} \otimes \omega_X \otimes \mathscr{I}_Z)^\vee \to H^0(Z, \mathcal{L} \otimes \omega_X|_Z)^\vee$$

□

Proof of Theorem. Suppose it does not separate m points. Then there is a length m subscheme Z and an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to L^{\otimes d} \otimes \mathscr{I}_Z \to 0$$

where $\mathcal{E}$ is a rank 2 vector bundle. We compute

$$\det \mathcal{E} \cong L^{\otimes d} \quad c_2(\mathcal{E}) = m$$

suppose that

$$d^2 L^2 > 4m$$

meaning that $\mathcal{E}$ violates the Bogomolov inequality and hence is unstable. By Bogomolov's theorem, there is a destabilizing sequence

$$0 \to \mathcal{O}_X(A) \to \mathcal{E} \to \mathcal{O}_X(B) \otimes \mathscr{I}_W \to 0$$

for divisors A, B such that

(a) $dL = A + B$

(b) $c_2(\mathcal{E}) = m = A \cdot B + \text{length}\,(W)$

(c) $(A-B)^2 > 0$ and $(A-B)\cdot H > 0$ for all ample H.

Now consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_X(A) \arrow[d] \arrow[dr, "\theta"] & & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{L}\otimes\mathscr{I}_Z \arrow[r] & 0 \\
 & & \mathcal{O}_X(B)\otimes\mathscr{I}_W \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

I claim the map θ is nonzero. Otherwise, there is a nonzero map $\mathcal{O}_X(A) \to \mathcal{O}_X$ meaning $-A$ is effective. By assumption $L\cdot A \le -\alpha < 0$ and

$$dL\cdot A = A^2 + A\cdot B$$

Note that the following hold

(a) $d^2L^2 = A^2 + 2A\cdot B + B^2 \ge 0$ since L is nef

(b) $A^2 - B^2 = (A-B)\cdot(A+B) \ge 0$ since $L = A+B$ is nef and hence the limit is ample divisors

(c) $(A-B)^2 = A^2 - 2A\cdot B + B^2 > 0$

therefore

$$2dL\cdot A = 2(A^2 + A\cdot B) = (A^2 + 2A\cdot B + B^2) + (A^2 - B^2) \ge 0$$

a contradiction. Hence $\theta \neq 0$. This means there is an effective divisor D continaing W such that $D \sim dL - A = B$. By assumption, $L\cdot B \ge \alpha$.

Rewrite what we know,

(a) $(dL-2D)^2 > 0$

(b) $(dL-2D)\cdot L \ge 0$

(c) $(dL-D)\cdot D = m - \text{length}\,(W) \le m$

Therefore, by Hodge index, need a strict inequality

$$(L\cdot D)^2 d \ge (L^2D^2)d > 2(L\cdot D)D^2 \ge 2(L\cdot D)((L\cdot D)d - m)$$

Furthermore, $L\cdot D \ge \alpha$ by assumption. Dividing by $L\cdot D$ we get,

$$\tfrac{d}{2}(D\cdot L) > D^2 \ge d(D\cdot L) - m$$

We set,

$$a = D\cdot L$$
$$b = D^2$$

so we have inequalities

$$a \geq \alpha$$
$$\tfrac{d}{2}\alpha \geq b \geq da - m$$

$\square$

2 Complete Intersections and smooth extensions

Question: when does a smooth complete intersection $X \subset \mathbb{P}^{n+r}$ have an extension X' to a smooth complete intersection of one larger dimension.

Lemma 2.0.1. Let X be smooth and $\mathscr{I}_Z \subset \mathcal{O}_X$ the ideal sheaf of a smooth subvariety $Z \subset X$ with $\dim Z < \frac{1}{2}\dim X$. If $\mathscr{I}_Z \otimes \mathcal{L}$ is globally generated for some $\mathcal{L}$ then the generic section $s \in H^0(X, \mathscr{I} \otimes \mathcal{L})$ defines a smooth hypersurface $V(s)$.

Proof. Let $P = \mathbb{P}(H^0(X, \mathscr{I} \otimes \mathcal{L}))$ and consider the incidence correspondence $\mathscr{X} \subset P \times X$ of (s, x) for $s(x) = 0$. Further, let $\mathscr{Z} \subset \mathscr{X}$ be the locus (s, x) where $s(x) = 0$ and x is a singular point of $V(s)$. Consider the map $S \to X$. The fiber of x consists of the projectivization of the linear space of those s such that

(a) $\bar{s} \in \mathcal{L}/\mathfrak{m}_x\mathcal{L}$ is zero

(b) $\bar{s} \in \mathfrak{m}_x\mathcal{L}/\mathfrak{m}_x^2\mathcal{L}$ is zero

For $x \notin Z$ we see that $H^0(X, \mathscr{I}_Z \otimes \mathcal{L}) \to \mathcal{L}/\mathfrak{m}_x^2\mathcal{L}$ is surjective so the fiber has dimension $(\dim P + 1) - (\dim X + 1) - 1 = \dim P - \dim X - 1$. For $x \in Z$ we get

$$H^0(X, \mathscr{I}_Z \otimes \mathcal{L}) \to \mathscr{I}_x\mathcal{L}/\mathfrak{m}_x^2\mathcal{L}$$

is surjective because $\mathscr{I} \otimes \mathcal{L}$ is globally generated. Since Z is smooth, $\mathscr{I}_x$ is cut out by a regular sequence so this vector space has dimension $\dim X - \dim Z$. Therefore, the fibers over this point has dimension

$$(\dim P + 1) - (\dim X - \dim Z) - 1 = \dim P - \dim X + \dim Z$$

Therefore, by the following lemma

$$\dim \mathscr{X} \leq \max\{\dim P - 1, \dim P - \dim X + 2\dim Z\}$$

Hence, as long as $\dim X - 2\dim Z \geq 1$ we see that $\mathscr{X} \to P$ cannot be dominant. $\square$

Lemma 2.0.2. If $X \to Y$ is a map of finite type k-schemes with Y irreducble. Let $Z \subset Y$ be a closed subscheme. Suppose that $X_y \leq d_1$ for $y \in Z$ and $X_y \leq d_2$ for $y \in Y \backslash Z$. Then

$$\dim X \leq \max\{d_1 + \dim Z, d_2 + \dim Y\}$$

Proof. Write $X = X_1 \cup \cdots \cup X_r$ be the irreducible components. It suffices to prove the claim for each X_i. We know that $(X_i)_y$ also satisfies the dimension bounds. If $X_i \to Y$ factors through Z then a general fiber (over its image) has dimension $\leq d_1$ hence $\dim X_i \leq d_1 + \dim Z$. Otherwise, the general point of the image is not contained in Z so the general fiber has dimension $\leq d_2$ hence $\dim X_i \leq d_2 + \dim Y$. Therefore, we win. $\square$

Now let $X \subset \mathbb{P}^{n+r}$ be a complete intersection of type $(d_1, \dots, d_r)$ with $d_1 \leq d_2 \leq \cdots \leq d_r$. Suppose $2n < n + r$ i.e. $n < r$ then if $\mathscr{I}_X(d)$ is globally, generated, we can put $X \subset X_d$ for a smooth hypersurface X_d. Note that

$$\mathcal{O}(-d_1) \oplus \cdots \oplus \mathcal{O}(-d_r) \twoheadrightarrow \mathscr{I}_X$$

therefore $\mathscr{I}_X(d_r)$ is globally generated.

Lemma 2.0.3. Let $X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection of multidegrees $d_1 \leq d_2 \leq \cdots \leq d_r$. Then there exists an extension $X \subset X' \subset \mathbb{P}^{n+r}$ where X' is a smooth complete intersection of type $(d_i, \dots, d_r)$ such that X is cut out by the same equations $(d_1, \dots, d_{i-1})$ in $(X', \mathcal{O}_{X'}(1))$ as long as $i > \dim X$

Proof. Let $(X, \mathcal{O}_X(1))$ be smooth projective with $\mathcal{O}_X(1)$ very ample such that $H^i(X, \mathcal{O}_X(d)) = 0$ for all $0 < i < \dim X$ and all d. Let $Z \subset X$ a smooth complete intersection of multidegrees $d_1 \leq d_2 \leq \cdots \leq d_r$. If $2r > \dim X$ then there exists a smooth hypersurface $X_{d_r} \subset X$ containing Z such that Z is a complete intersection of type $(d_1, \dots, d_{r-1})$ in $(X_d, \mathcal{O}_{X_d}(1))$. Note:

$$2 \dim Z = 2(\dim X - r) < \dim X \iff 2r > \dim X$$

Indeed, because

$$\mathcal{O}_X(-d_1) \oplus \cdots \oplus \mathcal{O}_X(-d_r) \twoheadrightarrow \mathscr{I}_Z$$

we see that $\mathscr{I}_Z(d_r)$ is globally generated so we can apply the lemma. Therefore, there exists $Z \subset X_d$ with X_d smooth. We just need to show that Z is a complete intersection in X_d. Let $f_i \in H^0(X, \mathcal{O}_X(d_i))$ be the sections cutting out Z and $f' \in H^0(X, \mathcal{O}_X(d_r))$ the section defining X_d. Consider

$$\mathcal{O}_{X_d}(-d_1) \oplus \cdots \oplus \mathcal{O}_{X_d}(-d_{r-1}) \to \mathscr{I}_{Z|X_d}$$

we need to show this is surjective. Consider the Kozul resolution $\mathcal{E}^\bullet$ of Z

$$0 \to \mathcal{O}_X(-(d_1 + \cdots + d_r)) \to \cdots \to \mathcal{O}_X(-d_1) \oplus \cdots \oplus \mathcal{O}_X(-d_r) \to \mathscr{I}_Z \to 0$$

Then we get a spectral sequence

$$E_1^{p,q} = H^q(X, \mathcal{E}^p(d)) \implies H^{p+q}(X, \mathscr{I}_Z(d))$$

Then by the vanishing property, $E_1^{p,q} = 0$ for $q \neq 0, n$ where $n = \dim X$. Since $\mathcal{E}^\bullet$ is supported in degrees $[-(r-1), 0]$ and $r \leq \dim X$ so if $p + q = 0$ then $E_1^{p,q} = 0$ except for $E_1^{0,0}$. Furthermore, the differentials

$$\mathrm{d}_r : E_r^{0,0} \to E_r^{r,1-r}$$

are zero because $r > 0$ so we see there is a surjection

$$E_1^{0,0} \to H^0(X, \mathscr{I}_Z(d))$$

Hence, for $d = d_r$, we can write

$$f' = \lambda_1 f_1 + \cdots + \lambda_r f_r$$

where $\lambda_i \in H^0(X, \mathcal{O}_X(d_r - d_i))$ and $\lambda_r \in \mathbb{C}$. Since the generic element is smooth, we can choose X_d so that $\lambda_r \neq 0$ therefore in $\mathcal{O}_{X_d}$ we see that f_r is in the image of the above map. Hence it is surjective because $\mathscr{I}_{Z|X_d}$ is generated by $f_1, \dots, f_r$.

Now we run induction. We can run it as long as $2\dim Z < \dim X$ therefore we run until we get $Z \subset X' \subset X$ such that $\dim X' = 2\dim Z$ meaning $Z \subset X'$ is type $(d_1, \ldots, d_i)$ and $X' \subset X$ is type $(d_{i+1}, \ldots, d_r)$ in X. Hence

$$\dim X' = \dim X - (r - i) \quad \dim Z = \dim X - r$$

so we must have

$$2(\dim X - r) = \dim X - (r - i)$$

so $i = \dim X - r$. □

For example, if $r = \dim X - 1$ (the case $\dim Z = 1$) then $i = 1$ and we can extend to a smooth surface.

Remark. Consider $f_1 = x, f_2 = y, f_3 = z$ and $f' = x + y$ in $\mathbb{P}^4$ then $Z = V(f_1, f_2, f_3) \subset V(f')$ but obviously Z is not cut out by f_1, f_2 inside $X_1 = V(f')$.

Question: if we don't assume the vanishing, is this false?

3 TODO

(a) Work out the Ryidl-Yang proof.

(b) Reieder for 3-folds and elliptic curves on Calabi-Yau

(c) numerics