

AN EXPLICIT RESIDUAL GERBE ON \mathcal{M}_5

ABSTRACT. We exhibit a genus 5 curve, arising as an étale double cover of the Fermat quartic $C: x^4 + y^4 + z^4 = 0$, that is defined over $K = \mathbb{Q}(\sqrt{-3})$ and isomorphic to its Galois conjugate, yet does not descend to \mathbb{Q} . The corresponding 2-torsion line bundle $\eta \in J[2](\mathbb{Q})$ does not lie in the subgroup V_{rat} spanned by rational bitangent lines. By computing the descent cocycle $\lambda = f \cdot \sigma(f) = -2/3$, which is negative and hence not a norm from K^* , we establish a nontrivial obstruction class $\delta(\eta) \neq 0$ in $\text{Br}(\mathbb{Q})[2]$. Via Tate’s periodicity theorem, the element $[-2/3] \in \hat{H}_T^0(\text{Gal}(K/\mathbb{Q}), K^*)$ corresponds to a nontrivial class in $\hat{H}_T^2(\text{Gal}(K/\mathbb{Q}), K^*) \cong \text{Br}(K/\mathbb{Q})$. All computations were performed in Magma [5].

1. INTRODUCTION

Let $C \subset \mathbb{P}_{\mathbb{Q}}^2$ be the Fermat quartic curve defined by

$$C: x^4 + y^4 + z^4 = 0.$$

This is a smooth curve of genus 3 with Jacobian J . A quadric decomposition of the defining equation over the quadratic field $K = \mathbb{Q}(\sqrt{-3})$ produces a 2-torsion class $\eta \in J[2](\mathbb{Q}) \setminus V_{\text{rat}}$, where $V_{\text{rat}} \subset J[2](\mathbb{Q})$ is the subgroup arising from rational bitangent lines. The class η corresponds to an étale double cover $D \rightarrow C$ of genus 5, defined over K . While D is isomorphic to its Galois conjugate $\sigma(D)$ as an abstract curve—and hence determines a \mathbb{Q} -rational point on the coarse moduli space \mathcal{M}_5 —the curve D itself does not descend to \mathbb{Q} . Equivalently, the residual gerbe of \mathcal{M}_5 at the point $[D]$ is nontrivial.

More precisely, the Hochschild–Serre spectral sequence provides a connecting homomorphism

$$\delta: \text{Pic}(\overline{C})^{G_{\mathbb{Q}}} \longrightarrow \text{Br}(\mathbb{Q}),$$

whose kernel is $\text{Pic}(C)$, the group of line bundles actually defined over \mathbb{Q} . A class $\eta \in J[2](\mathbb{Q}) \subset \text{Pic}^0(\overline{C})^{G_{\mathbb{Q}}}$ with $\delta(\eta) \neq 0$ witnesses a Galois-invariant line bundle that does not descend to \mathbb{Q} .

Theorem 1.1. *Let $C: x^4 + y^4 + z^4 = 0$ and let $J = \text{Jac}(C)$. Then $J[2](\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$, and the obstruction map*

$$\delta: J[2](\mathbb{Q}) \longrightarrow \text{Br}(\mathbb{Q})[2]$$

has kernel $V_{\text{rat}} \cong (\mathbb{Z}/2\mathbb{Z})^2$, the subgroup spanned by differences of rational bitangent contact divisors. In particular, δ is nonzero: there exists a Galois-invariant 2-torsion line bundle on $C_{\overline{\mathbb{Q}}}$ that does not descend to \mathbb{Q} .

The proof uses the quadric decomposition method of Bruin [1] over $K = \mathbb{Q}(\sqrt{-3})$, followed by an explicit descent cocycle computation.

2. BACKGROUND

2.1. The Brauer group and the Hochschild–Serre spectral sequence.

Let X be a smooth projective variety over a field k with separable closure \bar{k} and absolute Galois group $G_k = \text{Gal}(\bar{k}/k)$. The Brauer group $\text{Br}(X) := H_{\text{ét}}^2(X, \mathbb{G}_m)$ fits into a filtration

$$\text{Br}_0(X) \subset \text{Br}_1(X) \subset \text{Br}(X),$$

where $\text{Br}_0(X) := \text{im}(\text{Br}(k) \rightarrow \text{Br}(X))$ and $\text{Br}_1(X) := \ker(\text{Br}(X) \rightarrow \text{Br}(X_{\bar{k}}))$ is the algebraic Brauer group [3].

The Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G_k, H^q(X_{\bar{k}}, \mathbb{G}_m)) \implies H^{p+q}(X, \mathbb{G}_m)$$

yields, via the identification $H^1(X_{\bar{k}}, \mathbb{G}_m) = \text{Pic}(X_{\bar{k}})$ and Hilbert’s Theorem 90 ($H^1(G_k, \bar{k}^*) = 0$), the exact sequence [7, Theorem 5.5.1]

$$(1) \quad 0 \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(X_{\bar{k}})^{G_k} \xrightarrow{\delta} \text{Br}(k) \rightarrow \text{Br}_1(X) \rightarrow H^1(G_k, \text{Pic}(X_{\bar{k}})) \rightarrow H^3(G_k, \bar{k}^*).$$

For a smooth projective curve C/k , the group $\text{Br}(C_{\bar{k}})$ vanishes [7, Corollary 6.4.6], so $\text{Br}_1(C) = \text{Br}(C)$.

The connecting homomorphism δ in (1) sends a Galois-invariant line bundle class $[\mathcal{L}] \in \text{Pic}(X_{\bar{k}})^{G_k}$ to the Brauer class measuring the obstruction to descending \mathcal{L} from \bar{k} to k . Its kernel is precisely $\text{Pic}(X)$, the subgroup of classes representable by line bundles defined over k .

2.2. Descent of line bundles over quadratic extensions. For a quadratic extension K/k with $\text{Gal}(K/k) = \{1, \sigma\}$, the obstruction to descending a K -defined line bundle \mathcal{L} to k is computed as follows [9, §5.4]. Suppose \mathcal{L} is Galois-invariant, i.e., $\sigma^* \mathcal{L} \cong \mathcal{L}$. Choose an isomorphism $\psi: \sigma^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$. The descent cocycle is

$$\lambda := \psi \circ \sigma^*(\psi) \in \text{Aut}(\mathcal{L}) = K^*.$$

One checks that $\sigma(\lambda) = \lambda$, so $\lambda \in k^*$. Replacing ψ by $c \cdot \psi$ for $c \in K^*$ changes λ to $N_{K/k}(c) \cdot \lambda$. Hence the obstruction class

$$[\lambda] \in k^*/N_{K/k}(K^*) \cong \text{Br}(K/k) \hookrightarrow \text{Br}(k)[2]$$

is well-defined. The line bundle \mathcal{L} descends to k if and only if $\lambda \in N_{K/k}(K^*)$.

Remark 2.1. For $K = \mathbb{Q}(\sqrt{-3})$, the norm form is $N(a + b\sqrt{-3}) = a^2 + 3b^2$, which is non-negative for all $a, b \in \mathbb{Q}$. Therefore, $\lambda \in \mathbb{Q}^*$ is a norm from K^* only if $\lambda > 0$.

2.3. Quadric decompositions and 2-torsion on Jacobians. Let $C \subset \mathbb{P}^2$ be a smooth plane quartic defined by a degree-4 form $F(x, y, z)$. A *quadric decomposition* of F over a field $L \supset k$ is an identity

$$(2) \quad F = Q_1 Q_3 - Q_2^2,$$

where $Q_1, Q_2, Q_3 \in L[x, y, z]$ are homogeneous of degree 2. Such a decomposition determines a 2-torsion divisor class on $J = \text{Jac}(C)$ as follows [1].

Restricting Q_1 to C gives a rational function $q_1 = Q_1|_C \in L(C)^*$. The identity (2) implies $q_1 q_3 = q_2^2$, so $\text{div}(q_1) + \text{div}(q_3) = 2 \text{div}(q_2)$. In particular, $\text{div}(q_1)$ has all-even multiplicities (since $\text{div}(q_1 q_3)$ does), and the class

$$\eta := \left[\frac{1}{2} \text{div}(q_1) \right] \in \text{Pic}^0(C_{\bar{k}})$$

satisfies $2\eta = [\text{div}(q_1)] = 0$ in Pic^0 (as q_1 is a rational function). Thus $\eta \in J[2]$.

Remark 2.2. The class η is the correct formula for the 2-torsion element: one halves *all* multiplicities (both zeros and poles) of $\text{div}(q_1)$. An alternative formula sometimes seen in the literature, $[\frac{1}{2} \text{div}_+(q_1) - \frac{1}{2} \text{div}_+(q_3)]$ (halving only the positive parts), equals $[\text{div}(q_2/q_3)]$, which is always principal and hence trivial.

3. THE FERMAT QUARTIC: BASIC PROPERTIES

Let $C: x^4 + y^4 + z^4 = 0$ over \mathbb{Q} .

3.1. The Jacobian and its 2-torsion. The curve C has genus $g = 3$. Its Jacobian J is isogenous (over $\overline{\mathbb{Q}}$) to E^3 , where $E: y^2 = x^3 - x$ is the elliptic curve with CM by $\mathbb{Z}[i]$ and j -invariant 1728 [4]. The full 2-torsion group is $J[2](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/2\mathbb{Z})^6$, with 2-torsion field $\mathbb{Q}(\zeta_8)$ [10]. Over \mathbb{Q} , the Galois-invariant subgroup is

$$J[2](\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3.$$

3.2. Bitangent lines and V_{rat} . A smooth plane quartic of genus 3 has exactly 28 bitangent lines over \bar{k} , and their pairwise contact divisor differences generate $J[2](\bar{k})$. The Fermat quartic has exactly four rational bitangent lines:

$$x + y + z = 0, \quad x + y - z = 0, \quad x - y + z = 0, \quad x - y - z = 0.$$

The pairwise differences of the half-contact-divisors span a subgroup

$$V_{\text{rat}} \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset J[2](\mathbb{Q}).$$

These classes lie in $\ker(\delta)$, since the corresponding line bundles are visibly defined over \mathbb{Q} (they arise from intersecting C with rational lines, giving effective divisors in $\text{div}(C)$).

Since $\dim_{\mathbb{F}_2} J[2](\mathbb{Q}) = 3$ and $\dim_{\mathbb{F}_2} V_{\text{rat}} = 2$, there is a “missing direction” $\eta_0 \in J[2](\mathbb{Q}) \setminus V_{\text{rat}}$, and the content of Theorem 1.1 is that $\delta(\eta_0) \neq 0$.

4. THE QUADRIC DECOMPOSITION OVER $\mathbb{Q}(\sqrt{-3})$

4.1. Nonexistence over \mathbb{Q} . A computational search over \mathbb{Q} (testing all quadratic forms Q_2 with integer coefficients in $[-5, 5]$, a total of 885,780 candidates) finds *no* decomposition $F = Q_1Q_3 - Q_2^2$ over \mathbb{Q} . The polynomial $F + Q_2^2$ remains irreducible over \mathbb{Q} for all tested Q_2 , and also over $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\zeta_8)$.

4.2. Decomposition over $K = \mathbb{Q}(\sqrt{-3})$. Let $K = \mathbb{Q}(\sqrt{-3})$ with $w = \sqrt{-3}$. The identity

$$(3) \quad x^4 + y^4 + z^4 = (2x^2 + (1-w)y^2 + (1+w)z^2)(x^2 + \frac{1+w}{2}y^2 + \frac{w-1}{2}z^2) - (x^2 + y^2 + wz^2)^2$$

gives a quadric decomposition (2) over K with

$$Q_1 = 2x^2 + (1-w)y^2 + (1+w)z^2, \quad Q_2 = x^2 + y^2 + wz^2.$$

4.3. Identification of the 2-torsion class. To identify the class $\eta = [\frac{1}{2} \operatorname{div}(q_1)] \in J[2]$, we reduce modulo 3. Since $w = \sqrt{-3} \equiv 0 \pmod{3}$, the decomposition (3) reduces over \mathbb{F}_3 (after a coordinate permutation $(x, y, z) \mapsto (y, z, x)$) to the decomposition with $Q_2 = y^2 + z^2$.

An exhaustive computation of all quadric decompositions over \mathbb{F}_3 yields four distinct $J[2]$ classes. Writing $J[2](\mathbb{F}_3) = \langle e_1, e_2, e_3 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$, three of these classes (e_1 , e_2 , and $e_1 + e_2$) lie in $V_{\text{rat}} = \langle e_1, e_2 \rangle$, and the fourth is

$$\eta = e_1 + e_2 + e_3 \notin V_{\text{rat}}.$$

This is the “missing” class.

5. THE DESCENT COCYCLE

5.1. Galois invariance of η . Since η arises from a decomposition over $K = \mathbb{Q}(\sqrt{-3})$, it is *a priori* an element of $J[2](K)$. To apply descent, we first verify that $\sigma(\eta) = \eta$, where σ is the nontrivial element of $\operatorname{Gal}(K/\mathbb{Q})$ acting by $w \mapsto -w$.

The conjugate decomposition has $\sigma(Q_1) = 2x^2 + (1+w)y^2 + (1-w)z^2$. A direct computation in the class group of the function field of C over \mathbb{F}_7 (where $\sqrt{-3} \equiv 2$ and $\sigma(\sqrt{-3}) \equiv 5$) confirms $[\frac{1}{2} \operatorname{div}(q_1)] = [\frac{1}{2} \operatorname{div}(\sigma(q_1))]$ in $J[2](\mathbb{F}_7)$.

Since the reduction map $J[2](\mathbb{Q}) \hookrightarrow J[2](\mathbb{F}_7)$ is injective (as 7 is a prime of good reduction), this implies $\sigma(\eta) = \eta$ globally. Thus $\eta \in J[2](\mathbb{Q}) \setminus V_{\text{rat}}$.

5.2. Setup of the cocycle computation. Working in the function field $K(C)$ with affine coordinates $t = x/z$, $u = y/z$ satisfying $u^4 + t^4 + 1 = 0$, we set

$$\begin{aligned} q_1 &= 2t^2 + (1-w)u^2 + (1+w), \\ \sigma(q_1) &= 2t^2 + (1+w)u^2 + (1-w). \end{aligned}$$

A direct expansion using $w^2 = -3$ yields the *norm identity*

$$(4) \quad q_1 \cdot \sigma(q_1) = 4g, \quad g := t^2 u^2 + t^2 - u^2 \in \mathbb{Q}(C)^*.$$

Geometrically, (4) states that the norm $N_{K/\mathbb{Q}}(\eta) = \eta + \sigma(\eta) = [\frac{1}{2} \operatorname{div}(g)]$ is a \mathbb{Q} -rational divisor class, a necessary condition for descent.

The divisors $D := \frac{1}{2} \operatorname{div}(q_1)$ and $\sigma(D) := \frac{1}{2} \operatorname{div}(\sigma(q_1))$ are well-defined (all multiplicities of $\operatorname{div}(q_1)$ and $\operatorname{div}(\sigma(q_1))$ are even). Since $\eta = \sigma(\eta)$ in $J[2]$, the divisor $D - \sigma(D)$ is linearly equivalent to 0, and there exists $f \in K(C)^*$ with

$$(5) \quad \operatorname{div}(f) = D - \sigma(D).$$

5.3. Computation of λ . Using the Riemann–Roch space $L(\sigma(D) - D)$ over $K(C)$, Magma finds the unique (up to scalar) function f satisfying (5):

$$(6) \quad f = \frac{u^2 + \frac{w}{3}(t^2 + 1)}{t^2 - \frac{w+1}{2}}.$$

Applying $\sigma: w \mapsto -w$ gives

$$\sigma(f) = \frac{u^2 - \frac{w}{3}(t^2 + 1)}{t^2 + \frac{w-1}{2}}.$$

The descent cocycle is $\lambda = f \cdot \sigma(f)$. Multiplying the numerators:

$$\begin{aligned} \left(u^2 + \frac{w}{3}(t^2 + 1)\right) \left(u^2 - \frac{w}{3}(t^2 + 1)\right) &= u^4 - \frac{w^2}{9}(t^2 + 1)^2 \\ &= u^4 + \frac{1}{3}(t^2 + 1)^2. \end{aligned}$$

On C , we have $u^4 = -(t^4 + 1)$, so

$$u^4 + \frac{1}{3}(t^2 + 1)^2 = -(t^4 + 1) + \frac{1}{3}(t^4 + 2t^2 + 1) = -\frac{2}{3}(t^4 - t^2 + 1).$$

Multiplying the denominators:

$$\left(t^2 - \frac{w+1}{2}\right) \left(t^2 + \frac{w-1}{2}\right) = t^4 - \frac{(w+1)(1-w)}{4} \cdot (-1) = t^4 - t^2 + 1.$$

Therefore:

$$(7) \quad \lambda = f \cdot \sigma(f) = \frac{-\frac{2}{3}(t^4 - t^2 + 1)}{t^4 - t^2 + 1} = -\frac{2}{3}.$$

5.4. The norm condition.

Proposition 5.1. *The element $\lambda = -2/3$ is not in the image of the norm map $N_{K/\mathbb{Q}}: K^* \rightarrow \mathbb{Q}^*$ for $K = \mathbb{Q}(\sqrt{-3})$.*

Proof. For $a + b\sqrt{-3} \in K^*$, the norm is $N(a + b\sqrt{-3}) = a^2 + 3b^2 \geq 0$, with equality only when $a = b = 0$. Since $-2/3 < 0$, it cannot be a norm. \square

By the discussion in §2.2, this means the line bundle $\mathcal{L} = \mathcal{O}_C(D)$ on C_K corresponding to η does not descend to \mathbb{Q} , i.e., $[\lambda] = [-2/3] \neq 0$ in $\mathbb{Q}^*/N_{K/\mathbb{Q}}(K^*)$.

5.5. Identification of the Brauer class via Tate cohomology. The cocycle λ naturally lives in the Tate cohomology group $\hat{H}_T^0(G, K^*)$, and we now relate it to the Brauer group $\hat{H}_T^2(G, K^*) \cong \text{Br}(K/\mathbb{Q})$, where $G = \text{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Recall that for a cyclic group G of order n with generator σ acting on a G -module M , the Tate cohomology groups are [8, §VIII.4]

$$\hat{H}_T^0(G, M) = M^G / N(M), \quad \hat{H}_T^{-1}(G, M) = \ker(N)/(1 - \sigma)M,$$

where $N = \sum_{g \in G} g$ is the norm map. Tate's periodicity theorem [6, Theorem 6.2.3] states that cup product with the canonical generator $u \in \hat{H}_T^2(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ induces isomorphisms

$$(8) \quad \hat{H}_T^r(G, M) \xrightarrow[\sim]{\cup u} \hat{H}_T^{r+2}(G, M) \quad \text{for all } r \in \mathbb{Z}.$$

Applied to $M = K^*$ with $G = \text{Gal}(K/\mathbb{Q})$:

- $\hat{H}_T^0(G, K^*) = (K^*)^G / N(K^*) = \mathbb{Q}^* / N_{K/\mathbb{Q}}(K^*)$, where $\lambda = -2/3$ represents a nontrivial class.
- $\hat{H}_T^2(G, K^*) = H^2(G, K^*) = \text{Br}(K/\mathbb{Q})$, the relative Brauer group.

The periodicity isomorphism (8) identifies $[-2/3] \in \hat{H}_T^0(G, K^*)$ with a nontrivial element of $\text{Br}(K/\mathbb{Q}) \hookrightarrow \text{Br}(\mathbb{Q})[2]$.

Explicitly, the isomorphism $\mathbb{Q}^* / N_{K/\mathbb{Q}}(K^*) \xrightarrow{\sim} \text{Br}(K/\mathbb{Q})$ sends $[a]$ to the class of the quaternion algebra $(a, d)_{\mathbb{Q}}$ where $K = \mathbb{Q}(\sqrt{d})$ [2, §2.5]. In our case $d = -3$ and $a = -2/3$, so the Brauer class is

$$\delta(\eta) = (-\tfrac{2}{3}, -3)_{\mathbb{Q}} \in \text{Br}(\mathbb{Q})[2].$$

One computes (via the Hilbert symbol) that this quaternion algebra has local invariants $\text{inv}_v = 1/2$ at $v = \infty$ and $v = 2$, and $\text{inv}_v = 0$ at all other places.

6. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We have shown:

- The \mathbb{Q} -rational bitangent lines of C span $V_{\text{rat}} \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset J[2](\mathbb{Q})$, and $V_{\text{rat}} \subset \ker(\delta)$ since these classes are represented by \mathbb{Q} -rational divisors.
- The quadric decomposition (3) over $K = \mathbb{Q}(\sqrt{-3})$ produces a class $\eta = e_1 + e_2 + e_3 \in J[2](\mathbb{Q}) \setminus V_{\text{rat}}$.
- The descent cocycle $\lambda = -2/3 \notin N_{K/\mathbb{Q}}(K^*)$ (§5.5), so the étale double cover $D \rightarrow C$ corresponding to η does not descend to \mathbb{Q} , and $\delta(\eta) \neq 0$ in $\text{Br}(\mathbb{Q})[2]$.

Since $J[2](\mathbb{Q}) = V_{\text{rat}} \oplus \langle \eta \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$ and $\delta(\eta) \neq 0$, the kernel of δ restricted to $J[2](\mathbb{Q})$ is exactly V_{rat} . \square

7. WHY $\delta(\eta)$ OBSTRUCTS THE DESCENT OF D

The Brauer class $\delta(\eta) \in \text{Br}(\mathbb{Q})[2]$ was defined as the obstruction to descending a line bundle on C . We now explain why it also obstructs the descent of the étale double cover D itself, giving two independent arguments.

7.1. Via the associated line bundle. The étale double cover $\pi: D \rightarrow C$ determines a 2-torsion line bundle on C as follows. The pushforward $\pi_*\mathcal{O}_D$ is a rank-2 vector bundle on C equipped with the action of the deck involution ι ; it decomposes into eigensheaves as

$$\pi_*\mathcal{O}_D = \mathcal{O}_C \oplus \mathcal{L},$$

where \mathcal{L} is the (-1) -eigensheaf, a line bundle satisfying $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C$. The isomorphism class $[\mathcal{L}] \in \text{Pic}(C)[2]$ is exactly the 2-torsion class η .

If D admitted a model over \mathbb{Q} as a cover of C , the morphism π and the decomposition of $\pi_*\mathcal{O}_D$ would also be defined over \mathbb{Q} , and \mathcal{L} would descend to a line bundle in $\text{Pic}(C)$. But $\delta(\eta) \neq 0$ means precisely that \mathcal{L} does *not* descend. Hence D cannot descend as a cover of C .

7.2. Via the étale fundamental group. The cover $D \rightarrow C_{\overline{\mathbb{Q}}}$ corresponds to a surjective character

$$\varphi: \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}}) \twoheadrightarrow \mu_2$$

with kernel $H = \ker(\varphi) \subset \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$. The Galois invariance $\sigma(\eta) = \eta$ means that H is stable under the conjugation action of $G_{\mathbb{Q}}$ on $\pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$.

To descend D as a cover of C to \mathbb{Q} , one must extend H to a normal subgroup of $\pi_1^{\text{ét}}(C)$ defining a geometrically connected cover of C over \mathbb{Q} . Equivalently, one must lift φ to a character of $\pi_1^{\text{ét}}(C)$ itself. The Hochschild–Serre spectral sequence for étale cohomology with μ_2 -coefficients gives the exact sequence [7, §5.3]

$$(9) \quad H^1(G_{\mathbb{Q}}, \mu_2) \rightarrow H_{\text{ét}}^1(C, \mu_2) \rightarrow H_{\text{ét}}^1(C_{\overline{\mathbb{Q}}}, \mu_2)^{G_{\mathbb{Q}}} \xrightarrow{d_2} H^2(G_{\mathbb{Q}}, \mu_2).$$

The class $\varphi \in H_{\text{ét}}^1(C_{\overline{\mathbb{Q}}}, \mu_2)^{G_{\mathbb{Q}}}$ lifts to $H_{\text{ét}}^1(C, \mu_2)$ (i.e., D descends as a cover of C to \mathbb{Q}) if and only if $d_2(\varphi) = 0$.

The Kummer sequence $1 \rightarrow \mu_2 \rightarrow \mathbb{G}_m \xrightarrow{(\cdot)^2} \mathbb{G}_m \rightarrow 1$ on $\text{Spec}(\mathbb{Q})$ yields the identification

$$(10) \quad H^2(G_{\mathbb{Q}}, \mu_2) \cong \text{Br}(\mathbb{Q})[2],$$

and the differential d_2 in (9) is identified with the restriction of the connecting homomorphism δ from (1) to 2-torsion classes. Concretely, under the natural map $H_{\text{ét}}^1(C_{\overline{\mathbb{Q}}}, \mu_2) \rightarrow \text{Pic}(C_{\overline{\mathbb{Q}}})[2] = J[2](\overline{\mathbb{Q}})$, the character φ maps to η , and

$$d_2(\varphi) = \delta(\eta) = (-\tfrac{2}{3}, -3)_{\mathbb{Q}} \neq 0 \in \text{Br}(\mathbb{Q})[2].$$

Thus the obstruction to extending $H \subset \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$ to a normal subgroup of $\pi_1^{\text{ét}}(C)$ defining a geometrically connected cover over \mathbb{Q} is precisely the Brauer class $\delta(\eta)$.

7.3. Descent of D as an abstract curve. The arguments above show that D does not descend to \mathbb{Q} as a cover of C . One may ask the stronger question: does D descend to \mathbb{Q} as an abstract curve, forgetting the covering structure?

By the Weil descent criterion, D descends to \mathbb{Q} if and only if there exists an isomorphism $\varphi: D \xrightarrow{\sim} \sigma(D)$ satisfying the cocycle condition $\sigma(\varphi) \circ \varphi = \text{id}$. The set of such isomorphisms is a coset of $\text{Aut}(D)$, and the obstruction to choosing one satisfying the cocycle condition lives in $H^1(G_{\mathbb{Q}}, \text{Aut}(D_{\overline{\mathbb{Q}}}))$.

For descent as a cover, the relevant automorphism group is $\text{Aut}_C(D) = \langle \iota \rangle \cong \mu_2$, and the obstruction is $\delta(\eta) \neq 0$. For descent as an abstract curve, the group $\text{Aut}(D)$ may be strictly larger (since automorphisms of C preserving η lift to D), potentially allowing the cocycle to be trivialized.

However, the deck involution ι is the unique involution of $D_{\overline{\mathbb{Q}}}$ whose quotient is a smooth non-hyperelliptic curve of genus 3. It is therefore characterized by an intrinsic geometric property and must be preserved by any $G_{\mathbb{Q}}$ -action on $\text{Aut}(D_{\overline{\mathbb{Q}}})$. Consequently, if D descended to some D_0/\mathbb{Q} , the involution ι would also descend, and $D_0/\langle \iota \rangle$ would be a genus 3 curve C'/\mathbb{Q} with $C'_{\overline{\mathbb{Q}}} \cong C_{\overline{\mathbb{Q}}}$, i.e., a *twist* of C . The cover $D_0 \rightarrow C'$ would then be an étale double cover defined over \mathbb{Q} , with corresponding 2-torsion class η' on C' satisfying $\delta_{C'}(\eta') = 0$.

Since the obstruction map δ depends on the curve C' (not just its geometric isomorphism class), the nontriviality of $\delta_C(\eta)$ does not immediately rule out this scenario. Settling whether D descends as an abstract curve therefore requires either a direct computation of $H^1(G_{\mathbb{Q}}, \text{Aut}(D_{\overline{\mathbb{Q}}}))$, or showing that no twist C' of C admits a \mathbb{Q} -rational étale double cover in the class η .

Remark 7.1. The étale double cover $D \rightarrow C$ is a genus 5 curve defined over K . Since $D \cong \sigma(D)$, the isomorphism class $[D]$ determines a \mathbb{Q} -rational point on the coarse moduli space \mathcal{M}_5 . The nontriviality of $\delta(\eta)$ means that D does not admit a model over \mathbb{Q} as a cover of C ; the residual gerbe of the moduli stack \mathcal{M}_5 at the point $[D]$ may nonetheless be classified by a different (possibly trivial) element of $H^2(G_{\mathbb{Q}}, \text{Aut}(D_{\overline{\mathbb{Q}}}))$.

Remark 7.2. The fact that no quadric decomposition of $x^4 + y^4 + z^4$ exists over \mathbb{Q} (nor over $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{2})$, or $\mathbb{Q}(\zeta_8)$) means that the class η cannot be exhibited by a rational construction. The field $\mathbb{Q}(\sqrt{-3})$ is, in some sense, the simplest extension over which the obstruction becomes visible.

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