# NONTRIVIAL BRAUER OBSTRUCTIONS TO DESCENT OF ÉTALE DOUBLE COVERS

Abstract. We study the Brauer obstruction for the Fermat quartic $C\colon x^4+y^4+z^4=0$ over $\mathbb{Q}$. A quadric decomposition over $K=\mathbb{Q}(\sqrt{-3})$ produces a 2-torsion class $\eta \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$, corresponding to an étale double cover $D \to C$ of genus 5. By computing the descent cocycle $\lambda = f \cdot \sigma(f) = -2/3$, which is negative and hence not a norm from $K^*$, we establish $\delta(\eta) \neq 0$ in $\mathrm{Br}(\mathbb{Q})[2]$: the cover $D \to C$ does not descend to $\mathbb{Q}$. However, the abstract curve $D$ *does* admit a $\mathbb{Q}$-model. The twist $C_2\colon x^4+y^4-z^4=0$, isomorphic to $C$ over $\mathbb{Q}(\zeta_8)$, has a rational point $(1:0:1)$, and the transported class $\varphi(\eta)$ remains in $J[2](C_2)(\mathbb{Q})$. Since a rational point rigidifies the Picard scheme, the Brauer obstruction vanishes on $C_2$, so the corresponding cover $D' \to C_2$ descends to $\mathbb{Q}$, giving a $\mathbb{Q}$-model of $D$.

We also exhibit a "generic" smooth plane quartic $C'$ with $\mathrm{Aut}(C'_{\overline{\mathbb{Q}}}) = 1$ possessing phantom 2-torsion: $J[2](\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a class $\eta'$ with Brauer obstruction $\delta(\eta') = (-1,-3)_{\mathbb{Q}}$, ramified at $\infty$ and 3. This shows that phantom 2-torsion is not specific to curves with large automorphism groups. All computations were performed in Magma [6].

## 1. Introduction

Let $C \subset \mathbb{P}^2_{\mathbb{Q}}$ be the Fermat quartic curve defined by

$$C\colon\ x^4+y^4+z^4=0.$$

This is a smooth curve of genus 3 with Jacobian $J$. A quadric decomposition of the defining equation over the quadratic field $K=\mathbb{Q}(\sqrt{-3})$ produces a 2-torsion class $\eta \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$, where $V_{\mathrm{rat}} \subset J[2](\mathbb{Q})$ is the subgroup arising from rational bitangent lines. The class $\eta$ corresponds to an étale double cover $D \to C$ of genus 5, defined over $K$. The cover $D \to C$ does not descend to $\mathbb{Q}$: the Brauer obstruction $\delta(\eta) \neq 0$ prevents it. However, the abstract curve $D$ *does* admit a $\mathbb{Q}$-model, obtained by transporting $\eta$ to a twist of $C$ that has a rational point.

More precisely, the Hochschild–Serre spectral sequence provides a connecting homomorphism

$$\delta\colon\ \mathrm{Pic}(\overline{C})^{G_{\mathbb{Q}}} \longrightarrow \mathrm{Br}(\mathbb{Q}),$$

whose kernel is $\mathrm{Pic}(C)$, the group of line bundles actually defined over $\mathbb{Q}$. A class $\eta \in J[2](\mathbb{Q}) \subset \mathrm{Pic}^0(\overline{C})^{G_{\mathbb{Q}}}$ with $\delta(\eta) \neq 0$ witnesses a Galois-invariant line bundle that does not descend to $\mathbb{Q}$.

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**Theorem 1.1.** *Let $C\colon x^4+y^4+z^4=0$ and let $J=\mathrm{Jac}(C)$. Then $J[2](\mathbb{Q})\cong(\mathbb{Z}/2\mathbb{Z})^3$, and the obstruction map*

$$\delta\colon J[2](\mathbb{Q})\longrightarrow \mathrm{Br}(\mathbb{Q})[2]$$

*has kernel $V_{\mathrm{rat}}\cong(\mathbb{Z}/2\mathbb{Z})^2$, the subgroup spanned by differences of rational bitangent contact divisors. In particular, $\delta$ is nonzero: there exists a Galois-invariant 2-torsion line bundle on $C_{\overline{\mathbb{Q}}}$ that does not descend to $\mathbb{Q}$.*

The proof uses the quadric decomposition method of Bruin [1] over $K=\mathbb{Q}(\sqrt{-3})$, followed by an explicit descent cocycle computation.

## 2. Background

### 2.1. The Brauer group and the Hochschild–Serre spectral sequence.

Let $X$ be a smooth projective variety over a field $k$ with separable closure $\bar{k}$ and absolute Galois group $G_k=\mathrm{Gal}(\bar{k}/k)$. The *Brauer group* $\mathrm{Br}(X):=H^2_{\text{ét}}(X,\mathbb{G}_m)$ fits into a filtration

$$\mathrm{Br}_0(X)\subset\mathrm{Br}_1(X)\subset\mathrm{Br}(X),$$

where $\mathrm{Br}_0(X):=\mathrm{im}(\mathrm{Br}(k)\to\mathrm{Br}(X))$ and $\mathrm{Br}_1(X):=\ker(\mathrm{Br}(X)\to\mathrm{Br}(X_{\bar{k}}))$ is the *algebraic Brauer group* [3].

The Hochschild–Serre spectral sequence

$$E_2^{p,q}=H^p(G_k,\,H^q(X_{\bar{k}},\,\mathbb{G}_m))\implies H^{p+q}(X,\,\mathbb{G}_m)$$

yields, via the identification $H^1(X_{\bar{k}},\mathbb{G}_m)=\mathrm{Pic}(X_{\bar{k}})$ and Hilbert's Theorem 90 ($H^1(G_k,\bar{k}^*)=0$), the exact sequence [8, Theorem 5.5.1]

(1)
$$0\to\mathrm{Pic}(X)\to\mathrm{Pic}(X_{\bar{k}})^{G_k}\xrightarrow{\delta}\mathrm{Br}(k)\to\mathrm{Br}_1(X)\to H^1(G_k,\,\mathrm{Pic}(X_{\bar{k}}))\to H^3(G_k,\,\bar{k}^*).$$

For a smooth projective curve $C/k$, the group $\mathrm{Br}(C_{\bar{k}})$ vanishes [8, Corollary 6.4.6], so $\mathrm{Br}_1(C)=\mathrm{Br}(C)$.

The connecting homomorphism $\delta$ in (1) sends a Galois-invariant line bundle class $[\mathscr{L}]\in\mathrm{Pic}(X_{\bar{k}})^{G_k}$ to the Brauer class measuring the obstruction to descending $\mathscr{L}$ from $\bar{k}$ to $k$. Its kernel is precisely $\mathrm{Pic}(X)$, the subgroup of classes representable by line bundles defined over $k$.

### 2.2. Descent of line bundles over quadratic extensions.

For a quadratic extension $K/k$ with $\mathrm{Gal}(K/k)=\{1,\sigma\}$, the obstruction to descending a $K$-defined line bundle $\mathscr{L}$ to $k$ is computed as follows [10, §5.4]. Suppose $\mathscr{L}$ is Galois-invariant, i.e., $\sigma^*\mathscr{L}\cong\mathscr{L}$. Choose an isomorphism $\psi\colon\sigma^*\mathscr{L}\xrightarrow{\sim}\mathscr{L}$. The *descent cocycle* is

$$\lambda\;:=\;\psi\circ\sigma^*(\psi)\;\in\;\mathrm{Aut}(\mathscr{L})\;=\;K^*.$$

One checks that $\sigma(\lambda)=\lambda$, so $\lambda\in k^*$. Replacing $\psi$ by $c\cdot\psi$ for $c\in K^*$ changes $\lambda$ to $\mathrm{N}_{K/k}(c)\cdot\lambda$. Hence the obstruction class

$$[\lambda]\;\in\;k^*/\,\mathrm{N}_{K/k}(K^*)\;\cong\;\mathrm{Br}(K/k)\;\hookrightarrow\;\mathrm{Br}(k)[2]$$

is well-defined. The line bundle $\mathscr{L}$ descends to $k$ if and only if $\lambda\in\mathrm{N}_{K/k}(K^*)$.

*Remark* 2.1. For $K = \mathbb{Q}(\sqrt{-3})$, the norm form is $\mathrm{N}(a + b\sqrt{-3}) = a^2 + 3b^2$, which is non-negative for all $a, b \in \mathbb{Q}$. Therefore, $\lambda \in \mathbb{Q}^*$ is a norm from $K^*$ only if $\lambda > 0$.

2.3. **Quadric decompositions and 2-torsion on Jacobians.** Let $C \subset \mathbb{P}^2$ be a smooth plane quartic defined by a degree-4 form $F(x, y, z)$. A *quadric decomposition* of $F$ over a field $L \supset k$ is an identity

$$F = Q_1 Q_3 - Q_2^2, \tag{2}$$

where $Q_1, Q_2, Q_3 \in L[x, y, z]$ are homogeneous of degree 2. Such a decomposition determines a 2-torsion divisor class on $J = \mathrm{Jac}(C)$ as follows [1].

Restricting $Q_1$ to $C$ gives a rational function $q_1 = Q_1|_C \in L(C)^*$. The identity (2) implies $q_1 q_3 = q_2^2$, so $\mathrm{div}(q_1) + \mathrm{div}(q_3) = 2\,\mathrm{div}(q_2)$. In particular, $\mathrm{div}(q_1)$ has all-even multiplicities (since $\mathrm{div}(q_1 q_3)$ does), and the class

$$\eta \;:=\; \left[\tfrac{1}{2}\,\mathrm{div}(q_1)\right] \;\in\; \mathrm{Pic}^0(C_{\bar{k}})$$

satisfies $2\eta = [\mathrm{div}(q_1)] = 0$ in $\mathrm{Pic}^0$ (as $q_1$ is a rational function). Thus $\eta \in J[2]$.

*Remark* 2.2. The class $\eta$ is the correct formula for the 2-torsion element: one halves *all* multiplicities (both zeros and poles) of $\mathrm{div}(q_1)$. An alternative formula sometimes seen in the literature, $[\frac{1}{2}\,\mathrm{div}_+(q_1) - \frac{1}{2}\,\mathrm{div}_+(q_3)]$ (halving only the positive parts), equals $[\mathrm{div}(q_2/q_3)]$, which is always principal and hence trivial.

## 3. The Fermat quartic: basic properties

Let $C\colon x^4 + y^4 + z^4 = 0$ over $\mathbb{Q}$.

3.1. **The Jacobian and its 2-torsion.** The curve $C$ has genus $g = 3$. Its Jacobian $J$ is isogenous (over $\overline{\mathbb{Q}}$) to $E^3$, where $E\colon y^2 = x^3 - x$ is the elliptic curve with CM by $\mathbb{Z}[i]$ and $j$-invariant 1728 [5]. The full 2-torsion group is $J[2](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/2\mathbb{Z})^6$, with 2-torsion field $\mathbb{Q}(\zeta_8)$ [11]. Over $\mathbb{Q}$, the Galois-invariant subgroup is

$$J[2](\mathbb{Q}) \;\cong\; (\mathbb{Z}/2\mathbb{Z})^3.$$

3.2. **Bitangent lines and $V_{\mathrm{rat}}$.** A smooth plane quartic of genus 3 has exactly 28 bitangent lines over $\bar{k}$, and their pairwise contact divisor differences generate $J[2](\bar{k})$. The Fermat quartic has exactly four rational bitangent lines:

$$x + y + z = 0, \quad x + y - z = 0, \quad x - y + z = 0, \quad x - y - z = 0.$$

The pairwise differences of the half-contact-divisors span a subgroup

$$V_{\mathrm{rat}} \;\cong\; (\mathbb{Z}/2\mathbb{Z})^2 \;\subset\; J[2](\mathbb{Q}).$$

These classes lie in $\ker(\delta)$, since the corresponding line bundles are visibly defined over $\mathbb{Q}$ (they arise from intersecting $C$ with rational lines, giving effective divisors in $\mathrm{div}(C)$).

Since $\dim_{\mathbb{F}_2} J[2](\mathbb{Q}) = 3$ and $\dim_{\mathbb{F}_2} V_{\mathrm{rat}} = 2$, there is a "missing direction" $\eta_0 \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$, and the content of Theorem 1.1 is that $\delta(\eta_0) \neq 0$.

## 4. The quadric decomposition over $\mathbb{Q}(\sqrt{-3})$

4.1. **Nonexistence over $\mathbb{Q}$.** A computational search over $\mathbb{Q}$ (testing all quadratic forms $Q_2$ with integer coefficients in $[-5, 5]$, a total of 885,780 candidates) finds *no* decomposition $F = Q_1Q_3 - Q_2^2$ over $\mathbb{Q}$ that produces a class outside $V_{\mathrm{rat}}$.

4.2. **Rational decompositions and $V_{\mathrm{rat}}$.** While no rational quadric decomposition produces the class $\eta$, rational decompositions *do* exist with a modified scaling. The identity

$$2(x^4 + y^4 + z^4) \;=\; \big((a-b)^2 + c^2\big)\big((a+b)^2 + c^2\big) \;+\; (a^2 + b^2 - c^2)^2 \tag{3}$$

holds for any permutation $\{a, b, c\}$ of $\{x, y, z\}$, giving three decompositions of the form $2F = Q_1Q_3 + Q_2^2$ over $\mathbb{Q}$. The three choices produce exactly the nonzero elements of $V_{\mathrm{rat}}$:

$$\begin{array}{lcl} Q_1 = (x-y)^2 + z^2 & \longrightarrow & v_2, \\ Q_1 = (y-z)^2 + x^2 & \longrightarrow & v_1 + v_2, \\ Q_1 = (x-z)^2 + y^2 & \longrightarrow & v_1. \end{array}$$

On $C$, the relation $2F = Q_1Q_3 + Q_2^2$ becomes $Q_1Q_3 = -Q_2^2$, so $\mathrm{div}(q_1)$ has all-even multiplicities and the half-divisor class $[\frac{1}{2}\,\mathrm{div}(q_1)]$ is well-defined regardless of the scaling factor.

These decompositions account for all of $\ker(\delta) = V_{\mathrm{rat}}$ via quadric methods: the classes $v_1$, $v_2$, and $v_1 + v_2$ are realized by $\mathbb{Q}$-rational quadrics, while $\eta$ and its $V_{\mathrm{rat}}$-translates require an extension of $\mathbb{Q}$.

4.3. **Decomposition over $K = \mathbb{Q}(\sqrt{-3})$.** Let $K = \mathbb{Q}(\sqrt{-3})$ with $w = \sqrt{-3}$. The identity

$$x^4{+}y^4{+}z^4 \;=\; \big(2x^2{+}(1{-}w)y^2{+}(1{+}w)z^2\big)\big(x^2{+}\tfrac{1+w}{2}\,y^2{+}\tfrac{w-1}{2}\,z^2\big) \;-\; \big(x^2{+}y^2{+}w\,z^2\big)^2 \tag{4}$$

gives a quadric decomposition (2) over $K$ with

$$Q_1 = 2x^2 + (1-w)y^2 + (1+w)z^2, \quad Q_2 = x^2 + y^2 + w\,z^2.$$

4.4. **Alternative decomposition over $\mathbb{Q}(i)$.** Since the Brauer class $\delta(\eta) = (-\frac{2}{3}, -3)_{\mathbb{Q}}$ has local invariants $\frac{1}{2}$ at $v = \infty$ and $v = 2$ only (see §5.5), any quadratic extension that splits both places must kill the obstruction—and hence must support a quadric decomposition producing the class $\eta$. The field $\mathbb{Q}(i)$ has this property: $\mathbb{Q}(i)$ is complex (splitting $\infty$) and 2 ramifies in $\mathbb{Q}(i)$ (splitting the local Brauer class at 2, since any quadratic extension of $\mathbb{Q}_2$ splits the unique nontrivial element of $\mathrm{Br}(\mathbb{Q}_2)[2]$).

A computational search confirms the existence of a decomposition over $\mathbb{Q}(i)$. The simplest example is

$$x^4 + y^4 + z^4 \;=\; (2x^2 + 2iz^2)(x^2 + iy^2) \;-\; (x^2 + iy^2 + iz^2)^2, \tag{5}$$

with $Q_1 = 2x^2 + 2iz^2$, $Q_2 = x^2 + iy^2 + iz^2$, $Q_3 = x^2 + iy^2$, where $i = \sqrt{-1}$.

A computation over $\mathbb{F}_{13}$ and $\mathbb{F}_{37}$ (primes $\equiv 1 \pmod{12}$ where both $\sqrt{-1}$ and $\sqrt{-3}$ exist) verifies that the half-divisor class $[\frac{1}{2}\operatorname{div}(q_1)]$ from (5) is **equal** to the class $\eta$ produced by the $\mathbb{Q}(\sqrt{-3})$ decomposition (4), and is not in $V_{\mathrm{rat}}$.

*Remark* 4.1. The decomposition (5) is in some ways more natural than (4): the factors $Q_1 = 2(x^2 + iz^2)$ and $Q_3 = x^2 + iy^2$ visibly exploit the Gaussian factorization of sums of squares. More generally, a quadric decomposition producing $\eta$ exists over any quadratic extension $\mathbb{Q}(\sqrt{d})$ that splits both ramified places of $\delta(\eta)$. Since $\delta(\eta)$ is ramified at $\infty$ and 2, the extension must be *imaginary* (to split $\infty$) and 2 must *not split* in $\mathbb{Q}(\sqrt{d})$ (equivalently $d \not\equiv 1 \pmod 8$, so that the local degree $[\mathbb{Q}_2(\sqrt{d}) : \mathbb{Q}_2] = 2$ kills the 2-local Brauer class). The imaginary quadratic fields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-3})$ all satisfy these conditions. On the other hand, $\mathbb{Q}(\sqrt{-7})$ does *not*: since $-7 \equiv 1 \pmod 8$, the prime 2 splits in $\mathbb{Q}(\sqrt{-7})$, leaving the local Brauer class at 2 intact.

4.5. **Identification of the 2-torsion class.** To identify the class $\eta = [\frac{1}{2}\operatorname{div}(q_1)] \in J[2]$, we reduce modulo 3. Since $w = \sqrt{-3} \equiv 0 \pmod 3$, the decomposition (4) reduces over $\mathbb{F}_3$ (after a coordinate permutation $(x, y, z) \mapsto (y, z, x)$) to the decomposition with $Q_2 = y^2 + z^2$.

An exhaustive computation of all quadric decompositions over $\mathbb{F}_3$ yields four distinct $J[2]$ classes. Writing $J[2](\mathbb{F}_3) = \langle e_1, e_2, e_3\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$, three of these classes ($e_1$, $e_2$, and $e_1 + e_2$) lie in $V_{\mathrm{rat}} = \langle e_1, e_2\rangle$, and the fourth is

$$\eta \;=\; e_1 + e_2 + e_3 \;\notin\; V_{\mathrm{rat}}.$$

This is the "missing" class.

4.6. **Bitangent decompositions over $\mathbb{Q}(\sqrt{-2})$.** The four rational bitangent lines $L_i$ pair into six products $L_iL_j$, each a reducible conic. Over $\mathbb{Q}(\sqrt{-2})$, each product gives a decomposition $F = L_iL_j \cdot Q_3 - Q_2^2$ with $Q_2 = \sqrt{-2}\,P$ for a rational quadric $P$. Explicitly, with $L_1 = x + y + z$, $L_2 = x + y - z$, $L_3 = x - y + z$, $L_4 = x - y - z$:

$$F = L_1L_2 \cdot \bigl(-(x{+}y)^2 - z^2\bigr) - \bigl(\sqrt{-2}\,(x^2{+}xy{+}y^2)\bigr)^2,$$
$$F = L_3L_4 \cdot \bigl(-(x{-}y)^2 - z^2\bigr) - \bigl(\sqrt{-2}\,(x^2{-}xy{+}y^2)\bigr)^2,$$

and similarly for the other four products (obtained by permuting the roles of $x, y, z$).

The six bitangent products fall into three $S_3$-orbits, each corresponding to a nonzero element of $V_{\mathrm{rat}}$:

$$\{L_1L_2,\, L_3L_4\} \to v_2, \qquad \{L_1L_3,\, L_2L_4\} \to v_1, \qquad \{L_1L_4,\, L_2L_3\} \to v_1 + v_2.$$

Each pair gives the same 2-torsion class as the corresponding rational decomposition from (3).

4.7. **The $\mathrm{GL}_2$ orbit structure.** The equation $F = Q_1Q_3 - Q_2^2$ can be written as $\det M = F$ where $M = \left(\begin{smallmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{smallmatrix}\right)$. The group $\mathrm{GL}_2$ acts by $M \mapsto gMg^\top$, preserving $\det M$ up to $(\det g)^2$; elements with $(\det g)^2 = 1$ preserve the decomposition.

Over $\mathbb{Q}(i)$, all 12 distinct decompositions with $\lambda = 1$ (three $S_3$-basic families and their $\mathrm{GL}_2$ transforms) lie in a **single** $\mathrm{SL}_2(\mathbb{Q}(i))$-orbit. This is the unique orbit producing the class $\eta$; the classes in $V_{\mathrm{rat}}$ require $\sqrt{2} \notin \mathbb{Q}(i)$ and hence are not accessible at this scaling.

The rational decompositions (§4.2) and bitangent decompositions (§4.6) both produce $V_{\mathrm{rat}}$ classes but with different scalings. In the common normalization $-2F = Q_1Q_3 - Q_2^2$ (absorbing the $\sqrt{-2}$ from the bitangent decomposition), both become rational, and the two decompositions for each theta characteristic are $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-2}))$-conjugate but *not* $\mathrm{GL}_2(\mathbb{Q})$-conjugate. For instance, the decomposition with $Q_1 = (x-y)^2 + z^2$ maps to the one with $Q_1 = (x+y)^2 - z^2 = L_1L_2$ via

$$g = \begin{pmatrix} \frac{\sqrt{-2}}{2} & -\frac{\sqrt{-2}}{2} \\ 0 & \sqrt{-2} \end{pmatrix}, \qquad \det g = -1.$$

The $\sqrt{-2}$ obstruction to $\mathbb{Q}$-conjugacy reflects the field of definition of the bitangent decomposition itself.

## 5. The descent cocycle

5.1. **Galois invariance of $\eta$.** Since $\eta$ arises from a decomposition over $K = \mathbb{Q}(\sqrt{-3})$, it is *a priori* an element of $J[2](K)$. To apply descent, we first verify that $\sigma(\eta) = \eta$, where $\sigma$ is the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$ acting by $w \mapsto -w$.

The conjugate decomposition has $\sigma(Q_1) = 2x^2 + (1+w)y^2 + (1-w)z^2$. A direct computation in the class group of the function field of $C$ over $\mathbb{F}_7$ (where $\sqrt{-3} \equiv 2$ and $\sigma(\sqrt{-3}) \equiv 5$) confirms $[\frac{1}{2}\operatorname{div}(q_1)] = [\frac{1}{2}\operatorname{div}(\sigma(q_1))]$ in $J[2](\mathbb{F}_7)$.

Since the reduction map $J[2](\mathbb{Q}) \hookrightarrow J[2](\mathbb{F}_7)$ is injective (as 7 is a prime of good reduction), this implies $\sigma(\eta) = \eta$ globally. Thus $\eta \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$.

5.2. **Setup of the cocycle computation.** Working in the function field $K(C)$ with affine coordinates $t = x/z$, $u = y/z$ satisfying $u^4 + t^4 + 1 = 0$, we set

$$q_1 = 2t^2 + (1-w)u^2 + (1+w),$$
$$\sigma(q_1) = 2t^2 + (1+w)u^2 + (1-w).$$

A direct expansion using $w^2 = -3$ yields the *norm identity*

$$q_1 \cdot \sigma(q_1) \;=\; 4g, \qquad g := t^2u^2 + t^2 - u^2 \;\in\; \mathbb{Q}(C)^*. \tag{6}$$

Geometrically, (6) states that the norm $\mathrm{N}_{K/\mathbb{Q}}(\eta) = \eta + \sigma(\eta) = [\frac{1}{2}\operatorname{div}(g)]$ is a $\mathbb{Q}$-rational divisor class, a necessary condition for descent.

The divisors $D := \frac{1}{2}\operatorname{div}(q_1)$ and $\sigma(D) := \frac{1}{2}\operatorname{div}(\sigma(q_1))$ are well-defined (all multiplicities of $\operatorname{div}(q_1)$ and $\operatorname{div}(\sigma(q_1))$ are even). Since $\eta = \sigma(\eta)$ in $J[2]$, the divisor $D - \sigma(D)$ is linearly equivalent to 0, and there exists $f \in K(C)^*$ with

$$\operatorname{div}(f) = D - \sigma(D). \tag{7}$$

5.3. **Computation of $\lambda$.** Using the Riemann–Roch space $L(\sigma(D) - D)$ over $K(C)$, Magma finds the unique (up to scalar) function $f$ satisfying (7):

$$f \;=\; \frac{u^2 + \frac{w}{3}(t^2+1)}{t^2 - \frac{w+1}{2}}. \tag{8}$$

Applying $\sigma\colon w \mapsto -w$ gives

$$\sigma(f) \;=\; \frac{u^2 - \frac{w}{3}(t^2+1)}{t^2 + \frac{w-1}{2}}.$$

The descent cocycle is $\lambda = f \cdot \sigma(f)$. Multiplying the numerators:

$$\begin{aligned}\left(u^2 + \tfrac{w}{3}(t^2+1)\right)\left(u^2 - \tfrac{w}{3}(t^2+1)\right) &= u^4 - \tfrac{w^2}{9}(t^2+1)^2 \\ &= u^4 + \tfrac{1}{3}(t^2+1)^2.\end{aligned}$$

On $C$, we have $u^4 = -(t^4+1)$, so

$$u^4 + \tfrac{1}{3}(t^2+1)^2 = -(t^4+1) + \tfrac{1}{3}(t^4+2t^2+1) = -\tfrac{2}{3}(t^4 - t^2 + 1).$$

Multiplying the denominators:

$$\left(t^2 - \tfrac{w+1}{2}\right)\left(t^2 + \tfrac{w-1}{2}\right) = t^4 + \tfrac{(w-1)-(w+1)}{2}\, t^2 + \tfrac{1-w^2}{4} = t^4 - t^2 + 1.$$

Therefore:

$$\boxed{\lambda \;=\; f \cdot \sigma(f) \;=\; \frac{-\frac{2}{3}(t^4 - t^2 + 1)}{t^4 - t^2 + 1} \;=\; -\frac{2}{3}.} \tag{9}$$

5.4. **The norm condition.**

**Proposition 5.1.** *The element $\lambda = -2/3$ is not in the image of the norm map $\mathrm{N}_{K/\mathbb{Q}}\colon K^* \to \mathbb{Q}^*$ for $K = \mathbb{Q}(\sqrt{-3})$.*

*Proof.* For $a + b\sqrt{-3} \in K^*$, the norm is $\mathrm{N}(a + b\sqrt{-3}) = a^2 + 3b^2 \geq 0$, with equality only when $a = b = 0$. Since $-2/3 < 0$, it cannot be a norm. $\square$

By the discussion in §2.2, this means the line bundle $\mathscr{L} = \mathscr{O}_C(D)$ on $C_K$ corresponding to $\eta$ does not descend to $\mathbb{Q}$, i.e., $[\lambda] = [-2/3] \neq 0$ in $\mathbb{Q}^*/\,\mathrm{N}_{K/\mathbb{Q}}(K^*)$.

5.5. **Identification of the Brauer class via Tate cohomology.** The cocycle $\lambda$ naturally lives in the Tate cohomology group $\widehat{H}^0_T(G, K^*)$, and we now relate it to the Brauer group $\widehat{H}^2_T(G, K^*) \cong \mathrm{Br}(K/\mathbb{Q})$, where $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Recall that for a cyclic group $G$ of order $n$ with generator $\sigma$ acting on a $G$-module $M$, the Tate cohomology groups are [9, §VIII.4]

$$\widehat{H}^0_T(G, M) = M^G/\,\mathrm{N}(M), \qquad \widehat{H}^{-1}_T(G, M) = \ker(\mathrm{N})/(1-\sigma)M,$$

where $\mathrm{N} = \sum_{g\in G} g$ is the norm map. Tate's periodicity theorem [7, Theorem 6.2.3] states that cup product with the canonical generator $u \in \widehat{H}^2_T(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ induces isomorphisms

$$\widehat{H}^r_T(G, M) \xrightarrow[\sim]{\cup u} \widehat{H}^{r+2}_T(G, M) \qquad \text{for all } r \in \mathbb{Z}. \tag{10}$$

Applied to $M = K^*$ with $G = \mathrm{Gal}(K/\mathbb{Q})$:

- $\widehat{H}^0_T(G, K^*) = (K^*)^G/\,\mathrm{N}(K^*) = \mathbb{Q}^*/\,\mathrm{N}_{K/\mathbb{Q}}(K^*)$, where $\lambda = -2/3$ represents a nontrivial class.
- $\widehat{H}^2_T(G, K^*) = H^2(G, K^*) = \mathrm{Br}(K/\mathbb{Q})$, the relative Brauer group.

The periodicity isomorphism (10) identifies $[-2/3] \in \widehat{H}^0_T(G, K^*)$ with a nontrivial element of $\mathrm{Br}(K/\mathbb{Q}) \hookrightarrow \mathrm{Br}(\mathbb{Q})[2]$.

Explicitly, the isomorphism $\mathbb{Q}^*/\,\mathrm{N}_{K/\mathbb{Q}}(K^*) \xrightarrow{\sim} \mathrm{Br}(K/\mathbb{Q})$ sends $[a]$ to the class of the quaternion algebra $(a, d)_{\mathbb{Q}}$ where $K = \mathbb{Q}(\sqrt{d})$ [2, §2.5]. In our case $d = -3$ and $a = -2/3$, so the Brauer class is

$$\delta(\eta) \;=\; (-\tfrac{2}{3},\, -3)_{\mathbb{Q}} \;\in\; \mathrm{Br}(\mathbb{Q})[2].$$

One computes (via the Hilbert symbol) that this quaternion algebra has local invariants $\mathrm{inv}_v = 1/2$ at $v = \infty$ and $v = 2$, and $\mathrm{inv}_v = 0$ at all other places.

5.6. **Alternative cocycle via $\mathbb{Q}(i)$.** The descent cocycle computation simplifies considerably when performed over $K' = \mathbb{Q}(i)$ using the decomposition (5). Let $\sigma'\colon i \mapsto -i$ denote the nontrivial element of $\mathrm{Gal}(K'/\mathbb{Q})$.

Set $q_1 = 2t^2 + 2i$ and $\sigma'(q_1) = 2t^2 - 2i$ in $K'(C)$. Their product is

$$q_1 \cdot \sigma'(q_1) = (2t^2 + 2i)(2t^2 - 2i) = 4t^4 + 4 = 4(t^4 + 1) = -4u^4,$$

using $t^4 + u^4 + 1 = 0$ on $C$. Hence $D + \sigma'(D) = 2\,\mathrm{div}(u)$ (where $D = \frac{1}{2}\,\mathrm{div}(q_1)$), and therefore

$$D - \sigma'(D) \;=\; \mathrm{div}(q_1) - 2\,\mathrm{div}(u) \;=\; \mathrm{div}\Big(\frac{q_1}{u^2}\Big), \tag{11}$$

so the function $f = q_1/u^2 = (2t^2 + 2i)/u^2$ satisfies $\mathrm{div}(f) = D - \sigma'(D)$. The descent cocycle is

$$\boxed{\lambda' \;=\; f \cdot \sigma'(f) \;=\; \frac{q_1 \cdot \sigma'(q_1)}{u^4} \;=\; \frac{-4u^4}{u^4} \;=\; -4.} \tag{12}$$

Since $\mathrm{N}_{K'/\mathbb{Q}}(a+bi) = a^2+b^2 \geq 0$ and $-4 < 0$, the cocycle $\lambda'$ is not a norm, confirming $\delta(\eta) \neq 0$ via this second splitting field. The resulting quaternion algebra is

$$(-4,\, -1)_{\mathbb{Q}} \;=\; (-1,\, -1)_{\mathbb{Q}},$$

the Hamilton quaternions (since $-4 = -1 \cdot 2^2$ and $\mathrm{N}(2) = 4$). This is the unique quaternion algebra over $\mathbb{Q}$ ramified at $\{\infty, 2\}$, consistent with the earlier computation $(-\frac{2}{3},\, -3)_{\mathbb{Q}}$.

*Remark* 5.2. The $\mathbb{Q}(i)$ computation avoids the Riemann–Roch step entirely: the function $f = q_1/u^2$ is obtained by inspection from the identity $q_1 \cdot \sigma'(q_1) = -4u^4$, and the cocycle $\lambda' = -4$ follows by a one-line calculation. In contrast, the $\mathbb{Q}(\sqrt{-3})$ descent requires finding $f$ via a Riemann–Roch space computation (equation (8)) and a more involved cancellation to reach $\lambda = -2/3$ (equation (9)).

## 6. Proof of Theorem 1.1

*Proof of Theorem 1.1.* We have shown:

- (i) The $\mathbb{Q}$-rational bitangent lines of $C$ span $V_{\mathrm{rat}} \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset J[2](\mathbb{Q})$, and $V_{\mathrm{rat}} \subset \ker(\delta)$ since these classes are represented by $\mathbb{Q}$-rational divisors.
- (ii) The quadric decomposition (4) over $K = \mathbb{Q}(\sqrt{-3})$ (or equivalently (5) over $K' = \mathbb{Q}(i)$) produces a class $\eta = e_1 + e_2 + e_3 \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$.
- (iii) The descent cocycle over $K'$ gives $\lambda' = -4 \notin \mathrm{N}_{K'/\mathbb{Q}}(K'^*)$ (§5.6), so the étale double cover $D \to C$ corresponding to $\eta$ does not descend to $\mathbb{Q}$, and $\delta(\eta) \neq 0$ in $\mathrm{Br}(\mathbb{Q})[2]$.

Since $J[2](\mathbb{Q}) = V_{\mathrm{rat}} \oplus \langle \eta \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$ and $\delta(\eta) \neq 0$, the kernel of $\delta$ restricted to $J[2](\mathbb{Q})$ is exactly $V_{\mathrm{rat}}$. □

## 7. Why $\delta(\eta)$ obstructs the descent of $D$

The Brauer class $\delta(\eta) \in \mathrm{Br}(\mathbb{Q})[2]$ was defined as the obstruction to descending a line bundle on $C$. We now explain why it also obstructs the descent of the étale double cover $D$ itself, giving two independent arguments.

### 7.1. Via the associated line bundle.

The étale double cover $\pi\colon D \to C$ determines a 2-torsion line bundle on $C$ as follows. The pushforward $\pi_*\mathscr{O}_D$ is a rank-2 vector bundle on $C$ equipped with the action of the deck involution $\iota$; it decomposes into eigensheaves as

$$\pi_*\mathscr{O}_D \;=\; \mathscr{O}_C \;\oplus\; \mathscr{L},$$

where $\mathscr{L}$ is the $(-1)$-eigensheaf, a line bundle satisfying $\mathscr{L}^{\otimes 2} \cong \mathscr{O}_C$. The isomorphism class $[\mathscr{L}] \in \mathrm{Pic}(C)[2]$ is exactly the 2-torsion class $\eta$.

If $D$ admitted a model over $\mathbb{Q}$ *as a cover of* $C$, the morphism $\pi$ and the decomposition of $\pi_*\mathscr{O}_D$ would also be defined over $\mathbb{Q}$, and $\mathscr{L}$ would descend to a line bundle in $\mathrm{Pic}(C)$. But $\delta(\eta) \neq 0$ means precisely that $\mathscr{L}$ does *not* descend. Hence $D$ cannot descend as a cover of $C$.

7.2. **Via the étale fundamental group.** The cover $D \to C_{\overline{\mathbb{Q}}}$ corresponds to a surjective character

$$\varphi\colon \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}}) \twoheadrightarrow \mu_2$$

with kernel $H = \ker(\varphi) \subset \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$. The Galois invariance $\sigma(\eta) = \eta$ means that $H$ is stable under the conjugation action of $G_{\mathbb{Q}}$ on $\pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$.

To descend $D$ as a cover of $C$ to $\mathbb{Q}$, one must extend $H$ to a normal subgroup of $\pi_1^{\text{ét}}(C)$ defining a geometrically connected cover of $C$ over $\mathbb{Q}$. Equivalently, one must lift $\varphi$ to a character of $\pi_1^{\text{ét}}(C)$ itself. The Hochschild–Serre spectral sequence for étale cohomology with $\mu_2$-coefficients gives the exact sequence [8, §5.3]

$$H^1(G_{\mathbb{Q}}, \mu_2) \;\to\; H^1_{\text{ét}}(C, \mu_2) \;\to\; H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}}, \mu_2)^{G_{\mathbb{Q}}} \xrightarrow{\;d_2\;} H^2(G_{\mathbb{Q}}, \mu_2). \tag{13}$$

The class $\varphi \in H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}}, \mu_2)^{G_{\mathbb{Q}}}$ lifts to $H^1_{\text{ét}}(C, \mu_2)$ (i.e., $D$ descends as a cover of $C$ to $\mathbb{Q}$) if and only if $d_2(\varphi) = 0$.

The Kummer sequence $1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{(\cdot)^2} \mathbb{G}_m \to 1$ on $\mathrm{Spec}(\mathbb{Q})$ yields the identification

$$H^2(G_{\mathbb{Q}}, \mu_2) \;\cong\; \mathrm{Br}(\mathbb{Q})[2], \tag{14}$$

and the differential $d_2$ in (13) is identified with the restriction of the connecting homomorphism $\delta$ from (1) to 2-torsion classes. Concretely, under the natural map $H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}}, \mu_2) \to \mathrm{Pic}(C_{\overline{\mathbb{Q}}})[2] = J[2](\overline{\mathbb{Q}})$, the character $\varphi$ maps to $\eta$, and

$$d_2(\varphi) \;=\; \delta(\eta) \;=\; (-\tfrac{2}{3}, -3)_{\mathbb{Q}} \;\neq\; 0 \;\in\; \mathrm{Br}(\mathbb{Q})[2].$$

Thus the obstruction to extending $H \subset \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$ to a normal subgroup of $\pi_1^{\text{ét}}(C)$ defining a geometrically connected cover over $\mathbb{Q}$ is precisely the Brauer class $\delta(\eta)$.

7.3. **The automorphism group of $D$.** We now determine the full automorphism group of $D$ and its field of definition.

The class $\eta$ is the *unique* $\mathrm{Aut}(C)$-invariant element of $J[2](\overline{\mathbb{Q}}) \setminus \{0\}$: for every $\sigma \in \mathrm{Aut}(C)$, the ratio $\sigma^*(q_1)/q_1$ has all-even valuations and principal half-divisor, so $\sigma^*(\eta) = \eta$. Thus $\mathrm{Stab}_{\mathrm{Aut}(C)}(\eta) = \mathrm{Aut}(C)$, the full group of order 96, and the exact sequence

$$1 \;\to\; \langle\iota\rangle \;\to\; \mathrm{Aut}(D_{\overline{\mathbb{Q}}}) \;\to\; \mathrm{Aut}(C_{\overline{\mathbb{Q}}}) \;\to\; 1$$

gives $|\mathrm{Aut}(D_{\overline{\mathbb{Q}}})| = 2 \times 96 = 192$. The group is identified as

$$\mathrm{Aut}(D_{\overline{\mathbb{Q}}}) \;\cong\; C_2^3.S_4 \quad \big(\mathrm{SmallGroup}(192, 181)\big),$$

confirmed by computing $|\mathrm{Aut}(D/\mathbb{F}_q)| = 192$ at $q = 9, 49, 97, 193, 241$.

To determine the field of definition, we analyse the *lifting constant*. For each $\sigma \in \mathrm{Aut}(C)$ fixing $\eta$, write $\sigma^*(q_1)/q_1 = c_\sigma \cdot h_\sigma^2$ with $c_\sigma \in k^*/(k^*)^2$; then

$\sigma$ lifts to $\mathrm{Aut}(D)$ over $k$ if and only if $c_\sigma$ is a square. Computations over $\mathbb{F}_p$ for $p = 7, 13, 19, 37, 43$ show:

- Even permutations (id and 3-cycles in the $S_3$ factor of $\mathrm{Aut}(C) = (\mathbb{Z}/4\mathbb{Z})^2 \rtimes S_3$) have $c_\sigma = 1$.
- Odd permutations (transpositions) have $c_\sigma \equiv -2 \pmod{(\mathbb{Q}(i)^*)^2}$.

The cover $D \to C$ and its deck involution are defined over $\mathbb{Q}(i)$, since the Brauer obstruction $\delta(\eta)$ is ramified only at $\infty$ and 2, both of which split in $\mathbb{Q}(i)$ (see §4.4). Over $\mathbb{Q}(i)$, the even permutations lift but the transpositions do not (as $-2$ is not a square in $\mathbb{Q}(i)^*$). Since $\sqrt{-2} = i\sqrt{2} \in \mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(\zeta_8)$, we obtain

$$|\,\mathrm{Aut}(D/\mathbb{Q}(i))| = 96, \qquad |\,\mathrm{Aut}(D/\mathbb{Q}(\zeta_8))| = 192.$$

This is confirmed by the LMFDB [4]: the family `5.192-181.0.2-3-8` lists exactly 4 refined passports for genus-5 curves with automorphism group $[192, 181]$ and signature $(0; 2, 3, 8)$. These correspond to 4 geometric points of the scheme $\mathscr{P}$ parametrising marked pairs $(X, G \hookrightarrow \mathrm{Aut}(X))$. The outer automorphism of $G$ pairs the four into two orbits of size 2, giving two isomorphism classes of unmarked curves, so $\mathscr{P}$ is an étale $\mathbb{Q}$-scheme of degree 4. Since $[\mathbb{Q}(\zeta_8) : \mathbb{Q}] = 4$ and $\mathscr{P}$ acquires a rational point over $\mathbb{Q}(\zeta_8)$—namely the curve $D$ with its full automorphism group—but not over any proper subfield (the intermediate fields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-2})$ each lack either $\sqrt{-1}$ or $\sqrt{-2}$), we conclude

$$\mathscr{P} \;\cong\; \mathrm{Spec}\; \mathbb{Q}(\zeta_8).$$

*Remark* 7.1 (Inflation and the descent cocycle)*.* The extension

$$1 \;\to\; \langle\iota\rangle \;\to\; \mathrm{Aut}(D_{\overline{\mathbb{Q}}}) \;\to\; \mathrm{Aut}(C_{\overline{\mathbb{Q}}}) \;\to\; 1 \tag{15}$$

is *non-split*: a computation in SmallGroup(192, 181) confirms that no element $g \in \mathrm{Aut}(D_{\overline{\mathbb{Q}}})$ satisfies $g^2 = \iota$, and the centre $Z(\mathrm{Aut}(D_{\overline{\mathbb{Q}}})) = \langle\iota\rangle$ admits no complement.

The cover $D \to C$ is defined over $\mathbb{Q}(i)$ (see §4.4), and the descent cocycle from $\mathbb{Q}(i)$ to $\mathbb{Q}$ is $c(\sigma) = \iota$, where $\sigma\colon i \mapsto -i$ generates $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$. Trivialising this cocycle requires $\alpha \in \mathrm{Aut}(D_{\overline{\mathbb{Q}}})$ with $\alpha^{-1}\,\sigma(\alpha) = \iota$. Over $\mathbb{Q}(i)$ alone, only 96 of the 192 geometric automorphisms are available (the lifts of even permutations), and among these no such $\alpha$ exists—indeed $\iota$ is central and no element squares to $\iota$.

The lifts of odd permutations (transpositions) provide the needed $\alpha$: their lifting constant $c_\sigma = -2$ requires $\sqrt{-2} = i\sqrt{2} \in \mathbb{Q}(\zeta_8) \setminus \mathbb{Q}(i)$, and conjugation by $\sigma$ introduces precisely the sign flip $\sigma(\sqrt{-2}) = -\sqrt{-2}$ that produces $\alpha^{-1}\,\sigma(\alpha) = \iota$. This is the mechanism behind the descent of $D$ to $\mathbb{Q}$ constructed in §7.4 below: the isomorphism $\varphi\colon C \xrightarrow{\sim} C_2$ given by $(x{:}y{:}z) \mapsto (x{:}y{:}\zeta_8\, z)$ involves an odd permutation of the 4th-root scalings (since $\zeta_8^4 = -1$), and its lift to $D$ absorbs the cocycle $\iota$.

7.4. **Descent of $D$ as an abstract curve.** The arguments above show that $D$ does not descend to $\mathbb{Q}$ *as a cover of* $C$. We now show that, nevertheless, $D$ *does* descend to $\mathbb{Q}$ as an abstract curve.

The key observation is that the deck involution $\iota$ is the unique involution of $D_{\overline{\mathbb{Q}}}$ whose quotient is a smooth non-hyperelliptic curve of genus 3. It is therefore characterized by an intrinsic geometric property and must be preserved by any $G_{\mathbb{Q}}$-action on $\mathrm{Aut}(D_{\overline{\mathbb{Q}}})$. Consequently, if $D$ descends to some $D_0/\mathbb{Q}$, the involution $\iota$ also descends, and $D_0/\langle\iota\rangle$ is a twist $C'$ of $C$ over $\mathbb{Q}$, with $D_0 \to C'$ an étale double cover defined over $\mathbb{Q}$.

Consider the twist $C_2\colon x^4 + y^4 - z^4 = 0$, which is isomorphic to $C$ via $\varphi\colon (x : y : z) \mapsto (x : y : \zeta_8 z)$ over $\mathbb{Q}(\zeta_8)$, where $\zeta_8^4 = -1$. The isomorphism $\varphi$ induces a map $\varphi_*\colon J[2](C) \to J[2](C_2)$, and a computation over $\mathbb{F}_{49}$ (where both $\sqrt{-3}$ and $\zeta_8$ exist) verifies that

$$\varphi_*(\eta) \in J[2](C_2)(\mathbb{Q}).$$

Specifically, over $\mathbb{F}_{49}$ the class $\eta$ has coordinates $(1, 0, 0, 0, 1, 0)$ in the $J[2]$ basis, and $\varphi_*(\eta) = (0, 0, 1, 0, 0, 0)$, which is fixed by $\mathrm{Frob}_7$. Since the 2-rank of $J(C_2)$ at $p = 7$ equals $3 = \dim J[2](C_2)(\mathbb{Q})$, the Frobenius-fixed subspace equals $J[2](C_2)(\mathbb{Q})$, confirming rationality.

**Proposition 7.2.** *The abstract curve $D$ is defined over $\mathbb{Q}$.*

*Proof.* The curve $C_2$ has the rational point $(1 : 0 : 1)$, so its Picard scheme $\mathrm{Pic}_{C_2/\mathbb{Q}}$ admits a section. This rigidifies the Picard scheme and implies that every Galois-invariant line bundle on $(C_2)_{\overline{\mathbb{Q}}}$ descends to $\mathbb{Q}$ [8, §5.5]. In particular, $\delta_{C_2}$ is identically zero on $J[2](C_2)(\mathbb{Q})$.

Since $\varphi_*(\eta) \in J[2](C_2)(\mathbb{Q})$ and $\delta_{C_2}(\varphi_*(\eta)) = 0$, the étale double cover $D' \to C_2$ corresponding to $\varphi_*(\eta)$ descends to $\mathbb{Q}$—both the cover map and the total space $D'$. Over $\overline{\mathbb{Q}}$, $D' \cong D$ (via $\varphi$), so $D$ has a $\mathbb{Q}$-model. □

*Remark* 7.3. This shows that the residual gerbe of the moduli stack $\mathscr{M}_5$ at the point $[D]$ is *trivial*: $D$ does admit a $\mathbb{Q}$-model. What $\delta(\eta) \neq 0$ obstructs is only the descent of the *cover* $D \to C$, not the descent of $D$ as a curve.

*Remark* 7.4. The $J[2](\mathbb{Q})$ subspaces of $C$ and $C_2$ are *not* equal under $\varphi_*$: the intersection $\varphi_*(J[2](C)(\mathbb{Q})) \cap J[2](C_2)(\mathbb{Q})$ has dimension 2 over $\mathbb{F}_2$. The specific class $\eta$ survives (i.e., $\varphi_*(\eta) \in J[2](C_2)(\mathbb{Q})$), but some elements of $V_{\mathrm{rat}}(C)$ do *not* remain rational on $C_2$.

*Remark* 7.5. No quadric decomposition of $x^4 + y^4 + z^4$ producing a class outside $V_{\mathrm{rat}}$ exists over $\mathbb{Q}$ itself. However, decompositions producing $\eta$ exist over *every* imaginary quadratic field in which 2 does not split (see Remark 4.1), reflecting the fact that each such field kills the Brauer class $\delta(\eta)$.

## 8. A generic quartic with phantom 2-torsion

The Fermat quartic has a large automorphism group ($|\mathrm{Aut}(C_{\overline{\mathbb{Q}}})| = 96$), and one may ask whether the phenomenon of phantom 2-torsion—i.e., a class

$\eta \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$ with $\delta(\eta) \neq 0$—is special to curves with extra symmetry. We now exhibit a smooth plane quartic $C'$ with $\mathrm{Aut}(C'_{\overline{\mathbb{Q}}}) = 1$ (no geometric automorphisms) that possesses phantom 2-torsion with a *different* Brauer class from the Fermat quartic.

8.1. **The construction.** The key observation is that the form $f = A^2 + 3B^2 + 3D^2$, for quadratic forms $A, B, D \in \mathbb{Q}[x, y, z]$, automatically carries a nontrivial Brauer obstruction when the curve $C'\colon f = 0$ is smooth.

Over $K = \mathbb{Q}(\sqrt{-3})$ with $w = \sqrt{-3}$, set

$$Q_1 = A + wB, \quad Q_3 = A - wB, \quad Q_2 = wD.$$

Then $Q_1Q_3 = A^2 + 3B^2$ and $Q_2^2 = -3D^2$, so $f = Q_1Q_3 - Q_2^2$ gives a quadric decomposition over $K$.

The descent cocycle is computed as follows. Set $h = q_1/q_2$ in $K(C')$. Since $\sigma(Q_2) = -wD = -Q_2$, we have $\sigma(h) = \sigma(q_1)/\sigma(q_2) = q_3/(-q_2)$, and

$$\lambda = h \cdot \sigma(h) = \frac{q_1}{q_2} \cdot \frac{-q_3}{q_2} = -\frac{q_1q_3}{q_2^2} = -1, \tag{16}$$

where the last step uses $q_1q_3 = q_2^2$ on $C'$.

Since $\lambda = -1 < 0$ and the norm form $\mathrm{N}_{K/\mathbb{Q}}(a + bw) = a^2 + 3b^2 \geq 0$, the cocycle is *not* a norm. The resulting Brauer class is

$$\delta(\eta) = (-1, -3)_{\mathbb{Q}},$$

which has local invariants $\mathrm{inv}_v = 1/2$ at $v = \infty$ and $v = 3$, and $\mathrm{inv}_v = 0$ at all other places.

*Remark* 8.1. The construction $f = A^2 + 3B^2 - Q_2^2$ with $Q_2 \in \mathbb{Q}[x, y, z]$ (i.e., $Q_2$ defined over $\mathbb{Q}$) always yields $\lambda = +1$: since $\sigma(Q_2) = Q_2$, one has $\lambda = q_1q_3/q_2^2 = 1$ on $C'$. The nontrivial cocycle requires $Q_2 = wD$ with $D$ rational, so that $\sigma(Q_2) = -Q_2$.

8.2. **An explicit example.** Taking

$$A = x^2 - xy - xz + y^2 - yz + z^2, \quad B = xy, \quad D = x^2 - z^2,$$

the quartic

$$f = A^2+3B^2+3D^2 = 4x^4-2x^3y-2x^3z+6x^2y^2-3x^2z^2-2xy^3-2xz^3+y^4-2y^3z+3y^2z^2-2yz^3+4z^4 \tag{17}$$

defines a smooth plane quartic $C'$ of genus 3 with the following properties:

(i) *Positive definite*: the minimum of $f$ on the unit sphere is approximately $0.117 > 0$, so $C'(\mathbb{R}) = \varnothing$.

(ii) *Trivial automorphism group*: $|\mathrm{Aut}(C'_{\mathbb{F}_p})| = 1$ for all 14 good primes $p \in \{7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59\}$. Since $C'$ is a non-hyperelliptic genus-3 curve, every geometric automorphism acts on the canonical space $H^0(C', \Omega^1) \cong \overline{\mathbb{Q}}^3$ with eigenvalues that are $d$-th roots of unity (for $d$ the order), hence is defined over a subfield of $\mathbb{Q}(\zeta_d)$ and is $\mathbb{F}_p$-rational whenever $p \equiv 1 \pmod{d}$. By the Wiman bound $d \leq 4g+2 = 14$, and for each $d \in \{2, \ldots, 14\}$ our list contains a prime $p \equiv 1 \pmod{d}$ (e.g., 7 for $d \mid 6$; 13 for $d \mid 12$; 17 for $d \mid 8$; 11 for $d \mid 10$; 29 for $d \mid 14$; 23 for $d = 11$; 53 for $d = 13$). Therefore $\mathrm{Aut}(C'_{\overline{\mathbb{Q}}}) = 1$.

(iii) *Nontrivial* $J[2](\mathbb{Q})$: the 2-adic valuation $v_2(\#J(C'/\mathbb{F}_p))$ is at least 1 for all good primes $p \leq 53$, with $v_2(\#J(C'/\mathbb{F}_7)) = 1$. Hence $J[2](\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$, generated by $\eta$.

(iv) *Nontrivial* $\eta$: the divisor $\mathrm{div}(q_1)$ has all-even multiplicities, and $\dim L(\frac{1}{2}\,\mathrm{div}(q_1)) = 0$ in the function field of $C'$ over $K$, confirming $\eta \neq 0$ in $J$.

(v) *Phantom*: $\lambda = -1$ by (16), so $\delta(\eta) \neq 0$.

(vi) *No rational bitangent lines*: an exhaustive search with integer coefficients bounded by 5 finds none, so $V_{\mathrm{rat}} = 0$.

(vii) *Bad primes*: $p = 2$ and $p = 3$ only.

**Proposition 8.2.** *Let* $C'\colon f = 0$ *be the quartic* (17). *Then* $\mathrm{Aut}(C'_{\overline{\mathbb{Q}}}) = 1$, $C'(\mathbb{R}) = \varnothing$, $V_{\mathrm{rat}} = 0$, *and* $J[2](\mathbb{Q}) = \langle \eta \rangle \cong \mathbb{Z}/2\mathbb{Z}$ *with* $\delta(\eta) = (-1, -3)_{\mathbb{Q}} \neq 0$.

In particular, the étale double cover $D' \to C'$ corresponding to $\eta$ does not descend to $\mathbb{Q}$. Moreover, $D'$ does not admit *any* $\mathbb{Q}$-model, even as an abstract curve: since $\mathrm{Aut}(C'_{\overline{\mathbb{Q}}}) = 1$, the geometric automorphism group $\mathrm{Aut}(D'_{\overline{\mathbb{Q}}}) = \langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$ (the deck involution alone), and the obstruction to descending $D'$ as an abstract curve lies in $H^2(G_{\mathbb{Q}}, \mathrm{Aut}(D'_{\overline{\mathbb{Q}}})) \cong \mathrm{Br}(\mathbb{Q})[2]$. The class $\delta(\eta) \neq 0$ in this group is the *complete* obstruction—unlike the Fermat quartic, where $|\mathrm{Aut}(D_{\overline{\mathbb{Q}}})| = 192$ provided enough room to absorb the descent cocycle via a twist (Remark 7.1).

8.3. **Comparison of Brauer classes.** The Brauer class of the phantom quartic differs from that of the Fermat quartic:

| | Brauer class | $\mathrm{inv}_\infty$ | $\mathrm{inv}_2$ | $\mathrm{inv}_3$ |
|---|---|---|---|---|
| Fermat: | $(-\frac{2}{3}, -3)_{\mathbb{Q}} = (-1, -1)_{\mathbb{Q}}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| Phantom: | $(-1, -3)_{\mathbb{Q}}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |

Both classes are nontrivial elements of $\mathrm{Br}(\mathbb{Q})[2]$, but they are ramified at different finite primes (2 vs. 3).

Both classes are split by $\mathbb{Q}(i)$, but for different reasons:

- *Fermat*: the class $(-1, -1)_{\mathbb{Q}}$ has $\mathrm{inv}_2 = \frac{1}{2}$. Since 2 ramifies in $\mathbb{Q}(i)$, the local extension $\mathbb{Q}_2(i)/\mathbb{Q}_2$ has degree 2 and kills $\mathrm{Br}(\mathbb{Q}_2)[2]$.
- *Phantom*: the class $(-1, -3)_{\mathbb{Q}}$ has $\mathrm{inv}_3 = \frac{1}{2}$. Since $-1$ is a quadratic non-residue mod 3, the prime 3 is inert in $\mathbb{Q}(i)$, so $\mathbb{Q}_3(i)/\mathbb{Q}_3$ is the unramified quadratic extension and kills $\mathrm{Br}(\mathbb{Q}_3)[2]$.

In both cases, $\mathbb{Q}(i)$ is imaginary (killing $\mathrm{inv}_\infty$) and the relevant finite prime does not split (killing the finite invariant). In particular, $\eta$ is representable over $\mathbb{Q}(i)$ for both curves.

*Remark* 8.3. For the Fermat quartic, a $\mathbb{Q}(i)$-rational quadric decomposition producing $\eta$ exists explicitly (equation (5)). For the phantom quartic (17), however, a $\mathbb{Q}(i)$-rational quadric decomposition $f = (P + iR)(P - iR) - S^2$ with $P, R, S \in \mathbb{Q}[x, y, z]_2$ would require $f + S^2 = P^2 + R^2$, a representation as a sum of two rational squares. A computational search over $\mathbb{F}_5$ (exhaustive, $5^6$ candidates for $S$) and $\mathbb{F}_{13}$ (structured search) finds no such decomposition.

The class $\eta$ is nevertheless representable over $\mathbb{Q}(i)$: the vanishing of the Brauer obstruction guarantees the existence of a $\mathbb{Q}(i)$-rational line bundle, even without an explicit quadric decomposition.

*Remark* 8.4. The Brauer class $(-1, -3)_{\mathbb{Q}}$ is split by a quadratic extension $\mathbb{Q}(\sqrt{d})$ if and only if $d$ kills both ramified places:

- $\mathrm{inv}_\infty = \frac{1}{2}$: the extension must be imaginary ($d < 0$).
- $\mathrm{inv}_3 = \frac{1}{2}$: the prime 3 must not split in $\mathbb{Q}(\sqrt{d})$, i.e., $d \not\equiv 1 \pmod 3$.

Examples: $\mathbb{Q}(i)$ ($d = -1 \equiv 2 \pmod 3$), $\mathbb{Q}(\sqrt{-2})$ ($d = -2 \equiv 1 \pmod 3$—does *not* work), $\mathbb{Q}(\sqrt{-3})$ ($d = -3$, 3 ramifies). Thus $\mathbb{Q}(\sqrt{-3})$ splits the class (as expected, since the decomposition is defined over $\mathbb{Q}(\sqrt{-3})$), and $\mathbb{Q}(i)$ splits it (since $-1$ is a square), but $\mathbb{Q}(\sqrt{-2})$ does not ($-2 \equiv 1 \pmod 3$ means 3 splits in $\mathbb{Q}(\sqrt{-2})$).

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