THE BRAUER OBSTRUCTION FOR THE FERMAT QUARTIC

Abstract. We study the Brauer obstruction for the Fermat quartic $C\colon x^4+y^4+z^4=0$ over $\mathbb{Q}$. A quadric decomposition over $K=\mathbb{Q}(\sqrt{-3})$ produces a 2-torsion class $\eta \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$, corresponding to an étale double cover $D \to C$ of genus 5. By computing the descent cocycle $\lambda = f\cdot\sigma(f) = -2/3$, which is negative and hence not a norm from K^*, we establish $\delta(\eta) \neq 0$ in $\mathrm{Br}(\mathbb{Q})[2]$: the cover $D \to C$ does not descend to $\mathbb{Q}$. However, the abstract curve D *does* admit a $\mathbb{Q}$-model. The twist $C_2\colon x^4+y^4-z^4=0$, isomorphic to C over $\mathbb{Q}(\zeta_8)$, has a rational point $(1:0:1)$, and the transported class $\varphi(\eta)$ remains in $J[2](C_2)(\mathbb{Q})$. Since a rational point rigidifies the Picard scheme, the Brauer obstruction vanishes on C_2, so the corresponding cover $D' \to C_2$ descends to $\mathbb{Q}$, giving a $\mathbb{Q}$-model of D. All computations were performed in Magma [5].

1. Introduction

Let $C \subset \mathbb{P}^2_{\mathbb{Q}}$ be the Fermat quartic curve defined by

$$C\colon\ x^4+y^4+z^4=0.$$

This is a smooth curve of genus 3 with Jacobian J. A quadric decomposition of the defining equation over the quadratic field $K=\mathbb{Q}(\sqrt{-3})$ produces a 2-torsion class $\eta \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$, where $V_{\mathrm{rat}} \subset J[2](\mathbb{Q})$ is the subgroup arising from rational bitangent lines. The class η corresponds to an étale double cover $D \to C$ of genus 5, defined over K. The cover $D \to C$ does not descend to $\mathbb{Q}$: the Brauer obstruction $\delta(\eta) \neq 0$ prevents it. However, the abstract curve D *does* admit a $\mathbb{Q}$-model, obtained by transporting η to a twist of C that has a rational point.

More precisely, the Hochschild–Serre spectral sequence provides a connecting homomorphism

$$\delta\colon\ \mathrm{Pic}(\overline{C})^{G_{\mathbb{Q}}} \longrightarrow \mathrm{Br}(\mathbb{Q}),$$

whose kernel is $\mathrm{Pic}(C)$, the group of line bundles actually defined over $\mathbb{Q}$. A class $\eta \in J[2](\mathbb{Q}) \subset \mathrm{Pic}^0(\overline{C})^{G_{\mathbb{Q}}}$ with $\delta(\eta)\neq 0$ witnesses a Galois-invariant line bundle that does not descend to $\mathbb{Q}$.

Theorem 1.1. *Let $C\colon x^4+y^4+z^4=0$ and let $J=\mathrm{Jac}(C)$. Then $J[2](\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$, and the obstruction map*

$$\delta\colon\ J[2](\mathbb{Q}) \longrightarrow \mathrm{Br}(\mathbb{Q})[2]$$

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has kernel $V_{\mathrm{rat}} \cong (\mathbb{Z}/2\mathbb{Z})^2$*, the subgroup spanned by differences of rational bitangent contact divisors. In particular,* δ *is nonzero: there exists a Galois-invariant* 2*-torsion line bundle on* $C_{\overline{\mathbb{Q}}}$ *that does not descend to* $\mathbb{Q}$*.*

The proof uses the quadric decomposition method of Bruin [1] over $K = \mathbb{Q}(\sqrt{-3})$, followed by an explicit descent cocycle computation.

2. Background

2.1. The Brauer group and the Hochschild–Serre spectral sequence. Let X be a smooth projective variety over a field k with separable closure $\bar{k}$ and absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$. The *Brauer group* $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ fits into a filtration

$$\mathrm{Br}_0(X) \subset \mathrm{Br}_1(X) \subset \mathrm{Br}(X),$$

where $\mathrm{Br}_0(X) := \mathrm{im}(\mathrm{Br}(k) \to \mathrm{Br}(X))$ and $\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{\bar{k}}))$ is the *algebraic Brauer group* [3].

The Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G_k,\, H^q(X_{\bar{k}},\, \mathbb{G}_m)) \implies H^{p+q}(X,\, \mathbb{G}_m)$$

yields, via the identification $H^1(X_{\bar{k}}, \mathbb{G}_m) = \mathrm{Pic}(X_{\bar{k}})$ and Hilbert's Theorem 90 ($H^1(G_k, \bar{k}^*) = 0$), the exact sequence [7, Theorem 5.5.1]

(1)

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\bar{k}})^{G_k} \xrightarrow{\delta} \mathrm{Br}(k) \to \mathrm{Br}_1(X) \to H^1(G_k,\, \mathrm{Pic}(X_{\bar{k}})) \to H^3(G_k,\, \bar{k}^*).$$

For a smooth projective curve C/k, the group $\mathrm{Br}(C_{\bar{k}})$ vanishes [7, Corollary 6.4.6], so $\mathrm{Br}_1(C) = \mathrm{Br}(C)$.

The connecting homomorphism δ in (1) sends a Galois-invariant line bundle class $[\mathscr{L}] \in \mathrm{Pic}(X_{\bar{k}})^{G_k}$ to the Brauer class measuring the obstruction to descending $\mathscr{L}$ from $\bar{k}$ to k. Its kernel is precisely $\mathrm{Pic}(X)$, the subgroup of classes representable by line bundles defined over k.

2.2. Descent of line bundles over quadratic extensions. For a quadratic extension K/k with $\mathrm{Gal}(K/k) = \{1, \sigma\}$, the obstruction to descending a K-defined line bundle $\mathscr{L}$ to k is computed as follows [9, §5.4]. Suppose $\mathscr{L}$ is Galois-invariant, i.e., $\sigma^*\mathscr{L} \cong \mathscr{L}$. Choose an isomorphism $\psi\colon \sigma^*\mathscr{L} \xrightarrow{\sim} \mathscr{L}$. The *descent cocycle* is

$$\lambda \;:=\; \psi \circ \sigma^*(\psi) \;\in\; \mathrm{Aut}(\mathscr{L}) \;=\; K^*.$$

One checks that $\sigma(\lambda) = \lambda$, so $\lambda \in k^*$. Replacing ψ by $c \cdot \psi$ for $c \in K^*$ changes λ to $\mathrm{N}_{K/k}(c) \cdot \lambda$. Hence the obstruction class

$$[\lambda] \;\in\; k^*/\,\mathrm{N}_{K/k}(K^*) \;\cong\; \mathrm{Br}(K/k) \;\hookrightarrow\; \mathrm{Br}(k)[2]$$

is well-defined. The line bundle $\mathscr{L}$ descends to k if and only if $\lambda \in \mathrm{N}_{K/k}(K^*)$.

Remark 2.1. For $K = \mathbb{Q}(\sqrt{-3})$, the norm form is $\mathrm{N}(a + b\sqrt{-3}) = a^2 + 3b^2$, which is non-negative for all $a, b \in \mathbb{Q}$. Therefore, $\lambda \in \mathbb{Q}^*$ is a norm from K^* only if $\lambda > 0$.

2.3. **Quadric decompositions and 2-torsion on Jacobians.** Let $C \subset \mathbb{P}^2$ be a smooth plane quartic defined by a degree-4 form $F(x, y, z)$. A *quadric decomposition* of F over a field $L \supset k$ is an identity

$$F = Q_1 Q_3 - Q_2^2, \tag{2}$$

where $Q_1, Q_2, Q_3 \in L[x, y, z]$ are homogeneous of degree 2. Such a decomposition determines a 2-torsion divisor class on $J = \mathrm{Jac}(C)$ as follows [1].

Restricting Q_1 to C gives a rational function $q_1 = Q_1|_C \in L(C)^*$. The identity (2) implies $q_1 q_3 = q_2^2$, so $\mathrm{div}(q_1) + \mathrm{div}(q_3) = 2\ \mathrm{div}(q_2)$. In particular, $\mathrm{div}(q_1)$ has all-even multiplicities (since $\mathrm{div}(q_1 q_3)$ does), and the class

$$\eta \ := \ \left[\tfrac{1}{2}\,\mathrm{div}(q_1)\right] \ \in \ \mathrm{Pic}^0(C_{\bar{k}})$$

satisfies $2\eta = [\mathrm{div}(q_1)] = 0$ in Pic^0 (as q_1 is a rational function). Thus $\eta \in J[2]$.

Remark 2.2. The class η is the correct formula for the 2-torsion element: one halves *all* multiplicities (both zeros and poles) of $\mathrm{div}(q_1)$. An alternative formula sometimes seen in the literature, $[\frac{1}{2}\,\mathrm{div}_+(q_1) - \frac{1}{2}\,\mathrm{div}_+(q_3)]$ (halving only the positive parts), equals $[\mathrm{div}(q_2/q_3)]$, which is always principal and hence trivial.

3. The Fermat quartic: basic properties

Let $C\colon x^4 + y^4 + z^4 = 0$ over $\mathbb{Q}$.

3.1. **The Jacobian and its 2-torsion.** The curve C has genus $g = 3$. Its Jacobian J is isogenous (over $\overline{\mathbb{Q}}$) to E^3, where $E\colon y^2 = x^3 - x$ is the elliptic curve with CM by $\mathbb{Z}[i]$ and j-invariant 1728 [4]. The full 2-torsion group is $J[2](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/2\mathbb{Z})^6$, with 2-torsion field $\mathbb{Q}(\zeta_8)$ [10]. Over $\mathbb{Q}$, the Galois-invariant subgroup is

$$J[2](\mathbb{Q}) \ \cong \ (\mathbb{Z}/2\mathbb{Z})^3.$$

3.2. **Bitangent lines and V_{rat}.** A smooth plane quartic of genus 3 has exactly 28 bitangent lines over $\bar{k}$, and their pairwise contact divisor differences generate $J[2](\bar{k})$. The Fermat quartic has exactly four rational bitangent lines:

$$x + y + z = 0, \quad x + y - z = 0, \quad x - y + z = 0, \quad x - y - z = 0.$$

The pairwise differences of the half-contact-divisors span a subgroup

$$V_{\mathrm{rat}} \ \cong \ (\mathbb{Z}/2\mathbb{Z})^2 \ \subset \ J[2](\mathbb{Q}).$$

These classes lie in $\ker(\delta)$, since the corresponding line bundles are visibly defined over $\mathbb{Q}$ (they arise from intersecting C with rational lines, giving effective divisors in $\mathrm{div}(C)$).

Since $\dim_{\mathbb{F}_2} J[2](\mathbb{Q}) = 3$ and $\dim_{\mathbb{F}_2} V_{\mathrm{rat}} = 2$, there is a "missing direction" $\eta_0 \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$, and the content of Theorem 1.1 is that $\delta(\eta_0) \neq 0$.

4. The quadric decomposition over $\mathbb{Q}(\sqrt{-3})$

4.1. **Nonexistence over $\mathbb{Q}$.** A computational search over $\mathbb{Q}$ (testing all quadratic forms Q_2 with integer coefficients in $[-5, 5]$, a total of 885,780 candidates) finds *no* decomposition $F = Q_1Q_3 - Q_2^2$ over $\mathbb{Q}$. The polynomial $F + Q_2^2$ remains irreducible over $\mathbb{Q}$ for all tested Q_2, and also over $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\zeta_8)$.

4.2. **Decomposition over $K = \mathbb{Q}(\sqrt{-3})$.** Let $K = \mathbb{Q}(\sqrt{-3})$ with $w = \sqrt{-3}$. The identity

(3)

$$x^4+y^4+z^4 = \big(2x^2+(1-w)y^2+(1+w)z^2\big)\big(x^2+\tfrac{1+w}{2}\,y^2+\tfrac{w-1}{2}\,z^2\big) - \big(x^2+y^2+w\,z^2\big)^2$$

gives a quadric decomposition (2) over K with

$$Q_1 = 2x^2 + (1-w)y^2 + (1+w)z^2, \quad Q_2 = x^2 + y^2 + w\,z^2.$$

4.3. **Identification of the 2-torsion class.** To identify the class $\eta = [\frac{1}{2}\operatorname{div}(q_1)] \in J[2]$, we reduce modulo 3. Since $w = \sqrt{-3} \equiv 0 \pmod 3$, the decomposition (3) reduces over $\mathbb{F}_3$ (after a coordinate permutation $(x, y, z) \mapsto (y, z, x)$) to the decomposition with $Q_2 = y^2 + z^2$.

An exhaustive computation of all quadric decompositions over $\mathbb{F}_3$ yields four distinct $J[2]$ classes. Writing $J[2](\mathbb{F}_3) = \langle e_1, e_2, e_3\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$, three of these classes (e_1, e_2, and $e_1 + e_2$) lie in $V_{\mathrm{rat}} = \langle e_1, e_2\rangle$, and the fourth is

$$\eta = e_1 + e_2 + e_3 \notin V_{\mathrm{rat}}.$$

This is the "missing" class.

5. The descent cocycle

5.1. **Galois invariance of η.** Since η arises from a decomposition over $K = \mathbb{Q}(\sqrt{-3})$, it is *a priori* an element of $J[2](K)$. To apply descent, we first verify that $\sigma(\eta) = \eta$, where σ is the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$ acting by $w \mapsto -w$.

The conjugate decomposition has $\sigma(Q_1) = 2x^2 + (1 + w)y^2 + (1 - w)z^2$. A direct computation in the class group of the function field of C over $\mathbb{F}_7$ (where $\sqrt{-3} \equiv 2$ and $\sigma(\sqrt{-3}) \equiv 5$) confirms $[\frac{1}{2}\operatorname{div}(q_1)] = [\frac{1}{2}\operatorname{div}(\sigma(q_1))]$ in $J[2](\mathbb{F}_7)$.

Since the reduction map $J[2](\mathbb{Q}) \hookrightarrow J[2](\mathbb{F}_7)$ is injective (as 7 is a prime of good reduction), this implies $\sigma(\eta) = \eta$ globally. Thus $\eta \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$.

5.2. **Setup of the cocycle computation.** Working in the function field $K(C)$ with affine coordinates $t = x/z$, $u = y/z$ satisfying $u^4 + t^4 + 1 = 0$, we set

$$q_1 = 2t^2 + (1-w)u^2 + (1+w),$$
$$\sigma(q_1) = 2t^2 + (1+w)u^2 + (1-w).$$

A direct expansion using $w^2 = -3$ yields the *norm identity*

$$q_1 \cdot \sigma(q_1) \;=\; 4g, \qquad g := t^2u^2 + t^2 - u^2 \;\in\; \mathbb{Q}(C)^*. \tag{4}$$

Geometrically, (4) states that the norm $\mathrm{N}_{K/\mathbb{Q}}(\eta) = \eta + \sigma(\eta) = [\frac{1}{2}\operatorname{div}(g)]$ is a $\mathbb{Q}$-rational divisor class, a necessary condition for descent.

The divisors $D := \frac{1}{2}\operatorname{div}(q_1)$ and $\sigma(D) := \frac{1}{2}\operatorname{div}(\sigma(q_1))$ are well-defined (all multiplicities of $\operatorname{div}(q_1)$ and $\operatorname{div}(\sigma(q_1))$ are even). Since $\eta = \sigma(\eta)$ in $J[2]$, the divisor $D - \sigma(D)$ is linearly equivalent to 0, and there exists $f \in K(C)^*$ with

$$\operatorname{div}(f) = D - \sigma(D). \tag{5}$$

5.3. **Computation of λ.** Using the Riemann–Roch space $L(\sigma(D)-D)$ over $K(C)$, Magma finds the unique (up to scalar) function f satisfying (5):

$$f \;=\; \frac{u^2 + \frac{w}{3}(t^2+1)}{t^2 - \frac{w+1}{2}}. \tag{6}$$

Applying $\sigma\colon w \mapsto -w$ gives

$$\sigma(f) \;=\; \frac{u^2 - \frac{w}{3}(t^2+1)}{t^2 + \frac{w-1}{2}}.$$

The descent cocycle is $\lambda = f \cdot \sigma(f)$. Multiplying the numerators:

$$\begin{aligned}\Big(u^2 + \tfrac{w}{3}(t^2+1)\Big)\Big(u^2 - \tfrac{w}{3}(t^2+1)\Big) &= u^4 - \tfrac{w^2}{9}(t^2+1)^2 \\ &= u^4 + \tfrac{1}{3}(t^2+1)^2.\end{aligned}$$

On C, we have $u^4 = -(t^4+1)$, so

$$u^4 + \tfrac{1}{3}(t^2+1)^2 = -(t^4+1) + \tfrac{1}{3}(t^4+2t^2+1) = -\tfrac{2}{3}(t^4-t^2+1).$$

Multiplying the denominators:

$$\Big(t^2 - \tfrac{w+1}{2}\Big)\Big(t^2 + \tfrac{w-1}{2}\Big) = t^4 - \tfrac{(w+1)(1-w)}{4} \cdot (-1) = t^4 - t^2 + 1.$$

Therefore:

$$\boxed{\lambda \;=\; f \cdot \sigma(f) \;=\; \frac{-\frac{2}{3}(t^4-t^2+1)}{t^4-t^2+1} \;=\; -\frac{2}{3}.} \tag{7}$$

5.4. **The norm condition.**

Proposition 5.1. *The element $\lambda = -2/3$ is not in the image of the norm map $\mathrm{N}_{K/\mathbb{Q}}\colon K^* \to \mathbb{Q}^*$ for $K = \mathbb{Q}(\sqrt{-3})$.*

Proof. For $a + b\sqrt{-3} \in K^*$, the norm is $\mathrm{N}(a + b\sqrt{-3}) = a^2 + 3b^2 \geq 0$, with equality only when $a = b = 0$. Since $-2/3 < 0$, it cannot be a norm. □

By the discussion in §2.2, this means the line bundle $\mathscr{L} = \mathscr{O}_C(D)$ on C_K corresponding to η does not descend to $\mathbb{Q}$, i.e., $[\lambda] = [-2/3] \neq 0$ in $\mathbb{Q}^*/\,\mathrm{N}_{K/\mathbb{Q}}(K^*)$.

5.5. **Identification of the Brauer class via Tate cohomology.** The cocycle λ naturally lives in the Tate cohomology group $\widehat{H}^0_T(G,\, K^*)$, and we now relate it to the Brauer group $\widehat{H}^2_T(G,\, K^*) \cong \mathrm{Br}(K/\mathbb{Q})$, where $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Recall that for a cyclic group G of order n with generator σ acting on a G-module M, the Tate cohomology groups are [8, §VIII.4]

$$\widehat{H}^0_T(G,M) = M^G/\,\mathrm{N}(M), \qquad \widehat{H}^{-1}_T(G,M) = \ker(\mathrm{N})/(1-\sigma)M,$$

where $\mathrm{N} = \sum_{g\in G} g$ is the norm map. Tate's periodicity theorem [6, Theorem 6.2.3] states that cup product with the canonical generator $u \in \widehat{H}^2_T(G,\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ induces isomorphisms

$$\widehat{H}^r_T(G,M) \xrightarrow[\sim]{\cup\, u} \widehat{H}^{r+2}_T(G,M) \qquad \text{for all } r \in \mathbb{Z}. \tag{8}$$

Applied to $M = K^*$ with $G = \mathrm{Gal}(K/\mathbb{Q})$:

- $\widehat{H}^0_T(G,K^*) = (K^*)^G/\,\mathrm{N}(K^*) = \mathbb{Q}^*/\,\mathrm{N}_{K/\mathbb{Q}}(K^*)$, where $\lambda = -2/3$ represents a nontrivial class.
- $\widehat{H}^2_T(G,K^*) = H^2(G,K^*) = \mathrm{Br}(K/\mathbb{Q})$, the relative Brauer group.

The periodicity isomorphism (8) identifies $[-2/3] \in \widehat{H}^0_T(G,K^*)$ with a nontrivial element of $\mathrm{Br}(K/\mathbb{Q}) \hookrightarrow \mathrm{Br}(\mathbb{Q})[2]$.

Explicitly, the isomorphism $\mathbb{Q}^*/\,\mathrm{N}_{K/\mathbb{Q}}(K^*) \xrightarrow{\sim} \mathrm{Br}(K/\mathbb{Q})$ sends $[a]$ to the class of the quaternion algebra $(a,d)_{\mathbb{Q}}$ where $K = \mathbb{Q}(\sqrt{d})$ [2, §2.5]. In our case $d = -3$ and $a = -2/3$, so the Brauer class is

$$\delta(\eta) \;=\; (-\tfrac{2}{3},\, -3)_{\mathbb{Q}} \;\in\; \mathrm{Br}(\mathbb{Q})[2].$$

One computes (via the Hilbert symbol) that this quaternion algebra has local invariants $\mathrm{inv}_v = 1/2$ at $v = \infty$ and $v = 2$, and $\mathrm{inv}_v = 0$ at all other places.

6. Proof of Theorem 1.1

Proof of Theorem 1.1. We have shown:

(i) The $\mathbb{Q}$-rational bitangent lines of C span $V_{\mathrm{rat}} \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset J[2](\mathbb{Q})$, and $V_{\mathrm{rat}} \subset \ker(\delta)$ since these classes are represented by $\mathbb{Q}$-rational divisors.

(ii) The quadric decomposition (3) over $K = \mathbb{Q}(\sqrt{-3})$ produces a class $\eta = e_1 + e_2 + e_3 \in J[2](\mathbb{Q}) \setminus V_{\mathrm{rat}}$.

(iii) The descent cocycle $\lambda = -2/3 \notin \mathrm{N}_{K/\mathbb{Q}}(K^*)$ (§5.5), so the étale double cover $D \to C$ corresponding to η does not descend to $\mathbb{Q}$, and $\delta(\eta) \neq 0$ in $\mathrm{Br}(\mathbb{Q})[2]$.

Since $J[2](\mathbb{Q}) = V_{\mathrm{rat}} \oplus \langle\eta\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$ and $\delta(\eta) \neq 0$, the kernel of δ restricted to $J[2](\mathbb{Q})$ is exactly V_{rat}. $\square$

7. Why $\delta(\eta)$ obstructs the descent of D

The Brauer class $\delta(\eta) \in \mathrm{Br}(\mathbb{Q})[2]$ was defined as the obstruction to descending a line bundle on C. We now explain why it also obstructs the descent of the étale double cover D itself, giving two independent arguments.

7.1. **Via the associated line bundle.** The étale double cover $\pi\colon D \to C$ determines a 2-torsion line bundle on C as follows. The pushforward $\pi_*\mathscr{O}_D$ is a rank-2 vector bundle on C equipped with the action of the deck involution ι; it decomposes into eigensheaves as

$$\pi_*\mathscr{O}_D \;=\; \mathscr{O}_C \;\oplus\; \mathscr{L},$$

where $\mathscr{L}$ is the (-1)-eigensheaf, a line bundle satisfying $\mathscr{L}^{\otimes 2} \cong \mathscr{O}_C$. The isomorphism class $[\mathscr{L}] \in \mathrm{Pic}(C)[2]$ is exactly the 2-torsion class η.

If D admitted a model over $\mathbb{Q}$ *as a cover of* C, the morphism π and the decomposition of $\pi_*\mathscr{O}_D$ would also be defined over $\mathbb{Q}$, and $\mathscr{L}$ would descend to a line bundle in $\mathrm{Pic}(C)$. But $\delta(\eta) \neq 0$ means precisely that $\mathscr{L}$ does *not* descend. Hence D cannot descend as a cover of C.

7.2. **Via the étale fundamental group.** The cover $D \to C_{\overline{\mathbb{Q}}}$ corresponds to a surjective character

$$\varphi\colon \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}}) \twoheadrightarrow \mu_2$$

with kernel $H = \ker(\varphi) \subset \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$. The Galois invariance $\sigma(\eta) = \eta$ means that H is stable under the conjugation action of $G_{\mathbb{Q}}$ on $\pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$.

To descend D as a cover of C to $\mathbb{Q}$, one must extend H to a normal subgroup of $\pi_1^{\text{ét}}(C)$ defining a geometrically connected cover of C over $\mathbb{Q}$. Equivalently, one must lift φ to a character of $\pi_1^{\text{ét}}(C)$ itself. The Hochschild–Serre spectral sequence for étale cohomology with μ_2-coefficients gives the exact sequence [7, §5.3]

$$H^1(G_{\mathbb{Q}}, \mu_2) \;\to\; H^1_{\text{ét}}(C, \mu_2) \;\to\; H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}}, \mu_2)^{G_{\mathbb{Q}}} \;\xrightarrow{d_2}\; H^2(G_{\mathbb{Q}}, \mu_2). \tag{9}$$

The class $\varphi \in H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}}, \mu_2)^{G_{\mathbb{Q}}}$ lifts to $H^1_{\text{ét}}(C, \mu_2)$ (i.e., D descends as a cover of C to $\mathbb{Q}$) if and only if $d_2(\varphi) = 0$.

The Kummer sequence $1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{(\cdot)^2} \mathbb{G}_m \to 1$ on $\mathrm{Spec}(\mathbb{Q})$ yields the identification

$$H^2(G_{\mathbb{Q}}, \mu_2) \;\cong\; \mathrm{Br}(\mathbb{Q})[2], \tag{10}$$

and the differential d_2 in (9) is identified with the restriction of the connecting homomorphism δ from (1) to 2-torsion classes. Concretely, under the natural map $H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}}, \mu_2) \to \mathrm{Pic}(C_{\overline{\mathbb{Q}}})[2] = J[2](\overline{\mathbb{Q}})$, the character φ maps to η, and

$$d_2(\varphi) \;=\; \delta(\eta) \;=\; (-\tfrac{2}{3}, -3)_{\mathbb{Q}} \;\neq\; 0 \;\in\; \mathrm{Br}(\mathbb{Q})[2].$$

Thus the obstruction to extending $H \subset \pi_1^{\text{ét}}(C_{\overline{\mathbb{Q}}})$ to a normal subgroup of $\pi_1^{\text{ét}}(C)$ defining a geometrically connected cover over $\mathbb{Q}$ is precisely the Brauer class $\delta(\eta)$.

7.3. **Obstruction over $K = \mathbb{Q}(\sqrt{-3})$ via $\mathrm{Aut}(D_K)$.** The preceding arguments show that D does not descend to $\mathbb{Q}$ as a cover of C. One may ask whether the Brauer class $\delta(\eta)$ is killed by the larger automorphism group of D itself—that is, whether the image of $\delta(\eta)$ under the map $H^2(\mathrm{Gal}(K/\mathbb{Q}), K^*) \to H^2(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{Aut}(D_K))$ induced by the inclusion $\langle\iota\rangle \hookrightarrow \mathrm{Aut}(D_K)$ is trivial. If so, D_K would descend from K to $\mathbb{Q}$ (through this particular extension).

A Magma computation over $\mathbb{F}_{13}$ (where $\sqrt{-3}$ and $\sqrt{-1}$ both exist) determines the full automorphism group:

$$\mathrm{Aut}(D_{\overline{\mathbb{Q}}}) \;\cong\; C_2^2.\,\mathrm{SL}(2,3) \quad (\mathrm{SmallGroup}(96,3)),$$

with geometric order 96, confirmed at multiple primes $p = 13, 37$. Over K alone (where $\sqrt{-1}$ is not available), only 48 automorphisms are visible: $\mathrm{Aut}(D_K) \cong C_2 \times S_4$ (confirmed at primes $p = 19, 43$ where $\sqrt{-1} \notin \mathbb{F}_p$). The full automorphism group is defined over $\mathbb{Q}(\sqrt{-3}, i) = \mathbb{Q}(\zeta_{12})$.

The deck involution ι is **central** in $\mathrm{Aut}(D_K)$: it generates $Z(\mathrm{Aut}(D_K)) \cong \mathbb{Z}/2\mathbb{Z}$. The Galois group $\mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle$ acts *trivially* on $\mathrm{Aut}(D_K)$, since σ fixes both $Z(\mathrm{Aut}(D_K))$ (the deck involution is canonical) and $\mathrm{Aut}(D_K)/Z(\mathrm{Aut}(D_K)) \cong S_4$ (which is the stabiliser of η in $\mathrm{Aut}(C)$, defined over $\mathbb{Q}$). Furthermore, $\mathrm{Hom}(S_4, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ but the unique lift of the sign character does not split the extension, as one checks that no element $g \in \mathrm{Aut}(D_K)$ satisfies $g^2 = \iota$: all 24 elements of order 4 square to non-central involutions.

Since σ acts trivially on $\mathrm{Aut}(D_K)$, the cocycle $[\sigma \mapsto \iota] \in H^1(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{Aut}(D_K))$ is a homomorphism, and it is nontrivial because ι is central (hence not conjugate to the identity). Therefore the class $\delta(\eta)$ is *not killed* in $H^2(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{Aut}(D_K))$, and D_K does not descend from K to $\mathbb{Q}$ through this extension.

7.4. **Descent of D as an abstract curve.** The arguments above show that D does not descend to $\mathbb{Q}$ *as a cover of* C. We now show that, nevertheless, D *does* descend to $\mathbb{Q}$ as an abstract curve.

The key observation is that the deck involution ι is the unique involution of $D_{\overline{\mathbb{Q}}}$ whose quotient is a smooth non-hyperelliptic curve of genus 3. It is therefore characterized by an intrinsic geometric property and must be preserved by any $G_{\mathbb{Q}}$-action on $\mathrm{Aut}(D_{\overline{\mathbb{Q}}})$. Consequently, if D descends to some $D_0/\mathbb{Q}$, the involution ι also descends, and $D_0/\langle\iota\rangle$ is a twist C' of C over $\mathbb{Q}$, with $D_0 \to C'$ an étale double cover defined over $\mathbb{Q}$.

Consider the twist $C_2\colon x^4 + y^4 - z^4 = 0$, which is isomorphic to C via $\varphi\colon (x : y : z) \mapsto (x : y : \zeta_8 z)$ over $\mathbb{Q}(\zeta_8)$, where $\zeta_8^4 = -1$. The isomorphism φ induces a map $\varphi_*\colon J[2](C) \to J[2](C_2)$, and a computation over $\mathbb{F}_{49}$ (where both $\sqrt{-3}$ and ζ_8 exist) verifies that

$$\varphi_*(\eta) \;\in\; J[2](C_2)(\mathbb{Q}).$$

Specifically, over $\mathbb{F}_{49}$ the class η has coordinates $(1,0,0,0,1,0)$ in the $J[2]$ basis, and $\varphi_*(\eta) = (0,0,1,0,0,0)$, which is fixed by Frob_7. Since the 2-rank of $J(C_2)$ at $p = 7$ equals $3 = \dim J[2](C_2)(\mathbb{Q})$, the Frobenius-fixed subspace equals $J[2](C_2)(\mathbb{Q})$, confirming rationality.

Proposition 7.1. *The abstract curve D is defined over $\mathbb{Q}$.*

Proof. The curve C_2 has the rational point $(1:0:1)$, so its Picard scheme $\mathrm{Pic}_{C_2/\mathbb{Q}}$ admits a section. This rigidifies the Picard scheme and implies that every Galois-invariant line bundle on $(C_2)_{\overline{\mathbb{Q}}}$ descends to $\mathbb{Q}$ [7, §5.5]. In particular, δ_{C_2} is identically zero on $J[2](C_2)(\mathbb{Q})$.

Since $\varphi_*(\eta) \in J[2](C_2)(\mathbb{Q})$ and $\delta_{C_2}(\varphi_*(\eta)) = 0$, the étale double cover $D' \to C_2$ corresponding to $\varphi_*(\eta)$ descends to $\mathbb{Q}$—both the cover map and the total space D'. Over $\overline{\mathbb{Q}}$, $D' \cong D$ (via φ), so D has a $\mathbb{Q}$-model. □

Remark 7.2. This shows that the residual gerbe of the moduli stack $\mathscr{M}_5$ at the point $[D]$ is *trivial*: D does admit a $\mathbb{Q}$-model. What $\delta(\eta) \neq 0$ obstructs is only the descent of the *cover* $D \to C$, not the descent of D as a curve.

Remark 7.3. The $J[2](\mathbb{Q})$ subspaces of C and C_2 are *not* equal under φ_*: the intersection $\varphi_*(J[2](C)(\mathbb{Q})) \cap J[2](C_2)(\mathbb{Q})$ has dimension 2 over $\mathbb{F}_2$. The specific class η survives (i.e., $\varphi_*(\eta) \in J[2](C_2)(\mathbb{Q})$), but some elements of $V_{\mathrm{rat}}(C)$ do *not* remain rational on C_2.

Remark 7.4. The fact that no quadric decomposition of $x^4 + y^4 + z^4$ exists over $\mathbb{Q}$ (nor over $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{2})$, or $\mathbb{Q}(\zeta_8)$) means that the class η cannot be exhibited by a rational construction on C itself. The field $\mathbb{Q}(\sqrt{-3})$ is, in some sense, the simplest extension over which the obstruction becomes visible.

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