Infinitesimal Calculations in Fundamental Groups

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Algebraic Structures in Topology, June 2024
San Juan, Puerto Rico

website

slides

Infinitesimal Calculations in Fundamental Groups

arXiv:2403.20264

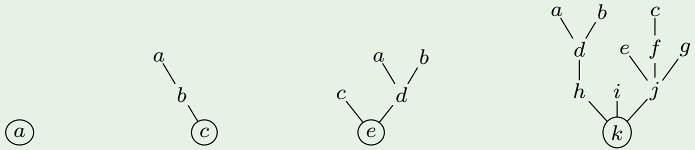
Joint work with:

- Nir Gadish — *University of Michigan → Pennsylvania*
- Aydin Ozbek — *University of Oregon*
- Dev Sinha — *University of Oregon*

Elephant.
(outside Hayden library, MIT)

Finite rooted trees

Examples

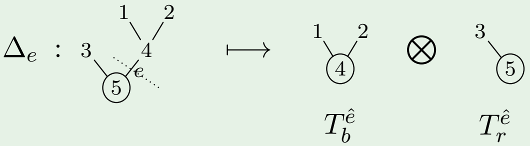


Finite rooted trees: Coproduct and subtrees

Removing an edge e from a tree T divides it into two subtrees:

- Branch subtree $T_b^{\hat{e}}$ (with root given by vertex incident to e)
- Root subtree $T_r^{\hat{e}}$

Example



Finite rooted trees: Cobracket and Lie coalgebra

Consider the vector space spanned by trees.

Definition

- The coproduct of a tree cuts edges one at a time $\Delta T = \sum_{e\in E(T)} \Delta_e T$
- The cobracket of a tree is the anti-commutative coproduct $]T[= \Delta T - \tau\Delta T$

Dual of [Chapoton-Livernet]

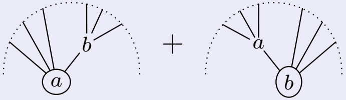
- *Coproduct makes nonplanar trees into a preLie coalgebra.*
- *Cobracket makes nonplanar trees into a Lie coalgebra.*

Finite rooted trees: Free nilpotent Lie coalgebras

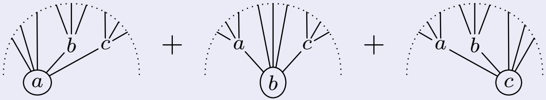
Proposition

The kernel of cobracket is spanned by the "root change" and "root Arnold" expressions

(root change)



(root Arnold)

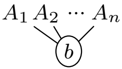


Corollary

Nonplanar trees modulo root change and Arnold ("Eil trees") is isomorphic to tensor algebra modulo shuffles. It models free nilpotent Lie coalgebras.

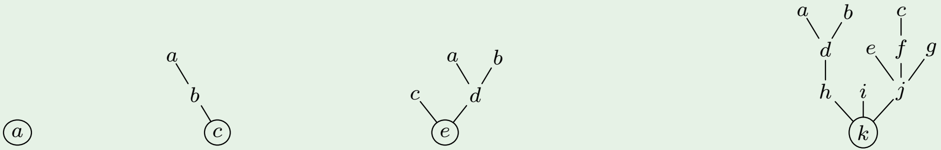
Finite rooted trees $\leftrightarrow$ Symbols

The **symbol** for a tree is defined recursively (on subtrees) as follows:

- A tree with only one vertex $T = (a)$ has symbol a.
- Otherwise $T =$ 

has symbol $\lfloor \alpha_1 \rfloor \lfloor \alpha_2 \rfloor \cdots \lfloor \alpha_n \rfloor\, b$ where $\alpha_1, \ldots, \alpha_n$ are the symbols for the branch subtrees $A_1, \ldots, A_n$ above the root.

Examples



a $\qquad$ $\lfloor \lfloor a \rfloor b \rfloor c$ $\qquad$ $\lfloor c \rfloor \lfloor \lfloor a \rfloor \lfloor b \rfloor d \rfloor e$ $\qquad$ $\lfloor \lfloor \lfloor a \rfloor \lfloor b \rfloor d \rfloor h \rfloor \lfloor i \rfloor \lfloor \lfloor e \rfloor \lfloor \lfloor c \rfloor f \rfloor \lfloor g \rfloor j \rfloor k$

Non-Planar trees $\leftrightarrow$ Commutative symbols

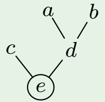
Non-planar finite rooted trees. (Compare to “dendrices” in ∞-operad literature)

- Symbols for non-planar trees are commutative at corollas: $\lfloor a\rfloor\lfloor b\rfloor c = \lfloor b\rfloor\lfloor a\rfloor c$
- We also allow the root vertex to commute inside notation: $\lfloor a\rfloor\lfloor b\rfloor c = \lfloor a\rfloor c\lfloor b\rfloor$

  (Use Koszul signs if graded.) $= c\lfloor a\rfloor\lfloor b\rfloor$

Example

Non-planar tree can be written as any of the following



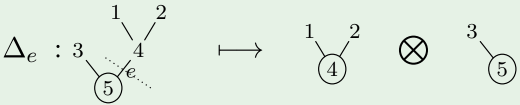
$$\lfloor c\rfloor\big\lfloor\lfloor a\rfloor\lfloor b\rfloor d\big\rfloor e = \big\lfloor\lfloor a\rfloor\lfloor b\rfloor d\big\rfloor\lfloor c\rfloor e = \lfloor c\rfloor e\big\lfloor\lfloor a\rfloor d\lfloor b\rfloor\big\rfloor \quad \textit{(left-to-right)}$$

$$= \big\lfloor\lfloor b\rfloor\lfloor a\rfloor d\big\rfloor\lfloor c\rfloor e = e\lfloor c\rfloor\big\lfloor d\lfloor a\rfloor\lfloor b\rfloor\big\rfloor \quad \textit{(bottom-to-top)}$$

Symbols: Coproduct

Note: Cutting an edge from a tree corresponds to “excising a subsymbol”.

Example



$T_b^{\hat{e}}$ $T_r^{\hat{e}}$

$$\lfloor 3\rfloor \Big\lfloor \underbrace{\lfloor 1\rfloor \lfloor 2\rfloor 4}_{\psi} \Big\rfloor 5 \longmapsto \underbrace{\lfloor 1\rfloor \lfloor 2\rfloor 4}_{\psi} \otimes \underbrace{\lfloor 3\rfloor 5}_{T^{\widehat{\psi}}}$$

Symbols: Root change and root Arnold

Root change and root Arnold for symbols

$$\text{(root change)} \qquad \lfloor\alpha\rfloor\beta \;+\; \alpha\,\lfloor\beta\rfloor,$$

$$\text{(root Arnold)} \qquad \lfloor\alpha\rfloor\,\lfloor\beta\rfloor\,\gamma \;+\; \lfloor\alpha\rfloor\,\beta\,\lfloor\gamma\rfloor \;+\; \alpha\,\lfloor\beta\rfloor\,\lfloor\gamma\rfloor$$

where α, β, γ are expressions for subtrees.

Proposition

Root change and root Arnold span the Liebniz expressions

$$\sum_i \lfloor\alpha_1\rfloor \cdots \lfloor\alpha_{i-1}\rfloor\,\alpha_i\,\lfloor\alpha_{i+1}\rfloor \cdots \lfloor\alpha_n\rfloor$$

Historical Note

Symbols appeared in computations before we noticed connection to trees!

Definition

Tree symbols are nested parenthesization expressions such that each nesting contains exactly one (“free”) element which is not further parenthesized.

- W- and Shiri. The left greedy Lie algebra basis and star graphs. Involve, 9(5):783–795, 2016.
- Monroe and Sinha. Linking of letters and the lower central series of free groups. Comm. Alg., 50(9):3678–3703, 2022.

Note: using the containment partial ordering converts to dendrices.
Q: Faces? Horns?

Discrete functions

Subsumes and generalizes Monroe and Sinha.

Idea: "Knot theory in groups"

Let's tackle the word problem (distinguishing words in groups) like knot theory! Given a word of length n presenting a group element $g \in G$, make functions assigning a number to each position of the word and build (finite type) "word invariants"!

Definition

Write $[n] = \{1, 2, \ldots, n\}$.
A **d-function** is a map $f : [n] \to \mathbb{Q}$.
The **integral** of a d-function is $\displaystyle\int_{[n]} f = \sum_{i=1}^{n} f(i)$.

Discrete indicator functions on words

Setup: Free group $F = \langle S \rangle$ and word $w = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ ($s_i \in S$ and $\varepsilon_i = \pm 1$).

Definition

The letter $a \in S$ has **indicator function on** w given by $A(i) = \begin{cases} \varepsilon_i & \text{if } s_i = a \\ 0 & \text{otherwise.} \end{cases}$

A **counting function** is a linear combination of indicator functions.

Example

Suppose $w = aba^{-1}b^{-1}$.

A maps $\begin{array}{cccc} a & b & a^{-1} & b^{-1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ +1 & 0 & -1 & 0 \end{array}$ and $(2A - B)$ maps $\begin{array}{cccc} a & b & a^{-1} & b^{-1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ +2 & -1 & -2 & +1 \end{array}$

Cobounding discrete functions on letters in words

Setup: Free group $F = \langle S \rangle$, word $w = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$, and d-function $f : [n] \to \mathbb{Q}$

Definition

The **cobounding of** f **on** w is the d-function $\lfloor f \rfloor(i) = \begin{cases} \int_{[i-1]} f & \text{if } \varepsilon_i = 1 \\ \int_{[i]} f & \text{if } \varepsilon_i = -1 \end{cases}$

- If $\int_{[n]} f = 0$ then we say $\lfloor f \rfloor$ is a **closed cobounding**.
- Otherwise $\lfloor f \rfloor$ is an **open cobounding**.

Remark

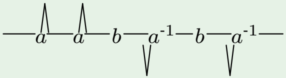
This is the **discrete anti-derivative** with a small twist accounting for whether letters are generators or inverses.

Visual intuition for cobounding

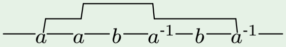
Example

We visualize the **closed cobounding** $\lfloor A \rfloor$ on $aaba^{-1}ba^{-1}$ as follows.

Graph of A



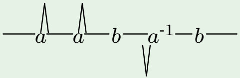
Graph of $\lfloor A \rfloor$



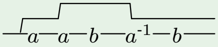
Example

We visualize the **open cobounding** $\lfloor A \rfloor$ on $aaba^{-1}b$ as follows.

Graph of A



Graph of $\lfloor A \rfloor$



(Combinatorial) Braiding product and linking product

Definition

The (combinatorial) **letter braiding product** of f and g is the pointwise product $\lfloor f \rfloor g$. If $\lfloor f \rfloor$ is a closed cobounding, we say this is a **linking product**.

Example

w	b	c	a^{-1}	b^{-1}	c^{-1}	b	b
$2A+B$	1	0	-2	-1	0	1	1
$\lfloor 2A+B \rfloor$	$^\wedge 0$	$^\wedge 1$	$-1\,^\wedge$	$-2\,^\wedge$	$-2\,^\wedge$	$^\wedge{-2}$	$^\wedge{-1}$
$-C$	0	-1	0	0	1	0	0
$\lfloor 2A+B \rfloor(-C)$	0	-1	0	0	-2	0	0

Letter configurations and letter linking

Definition

The value of a d-function f on a word w of length n is $f(w) = \int_{[n]} f$.

Example

- $A(w)$ counts (with sign) occurrences of the letter a in w.
- $\lfloor A \rfloor B(w)$ counts (with sign) pairs a followed by b in w.
  If linking product, this equals a count of b between a-a^{-1} pairs.
- $\lfloor \lfloor A \rfloor B \rfloor C(w)$ counts triplets a then b then c in w.
  If linking product, this is nesting c between b-b^{-1} pairs each between a-a^{-1} pairs.
- More generally if τ is a tree of indicator functions, then $\tau(w)$ counts "τ-configurations" of letters in w. (Convert tree $\to$ partial order w/ root maximal)

Theorems

Definition

A **letter braiding function** is a nested braiding product (tree) of counting functions.

Proposition

Letter braiding functions are well-defined invariants on free groups.

Theorem

Letter braiding functions whose cobrackets vanish on relations are well-defined on $\langle\, S \mid R \,\rangle$.

Theorem

Length filtration of braiding functions is dual to lower central series of group.

Background, I

Definition

Harrison cohomology $H^*_{\mathcal{E}}(X)$ is homology of Harrison complex $\mathcal{E}(C^*X)$, where C^*X is a commutative model for cochains on X (such as A_{PL}).

$\mathcal{E}$ is the "bar complex" for commutative algebras. We can make it with trees!

Proposition (classical)

For $n \geq 1$, there is a canonical isomorphism $H^{n-1}_{\mathcal{E}}(S^n) \cong H^n(S^n) \xrightarrow[\int_{S^n}]{\cong} \mathbb{Q}$

Definition

The Hopf pairing $H^{n-1}_{\mathcal{E}}(X) \times \pi_n(X) \to \mathbb{Q}$ is $\langle \gamma, [f] \rangle = \displaystyle\int_{S^n} f^*\gamma$

Background, II

- Sinha and W-. Lie coalgebras and rational homotopy theory II: Hopf invariants. Trans. Am. Math. Soc., 365(2):861–883, 2013.

Theorem [SW 13]

For rational simply connected spaces, the Hopf pairing is well defined, perfect, and respects Lie algebra / coalgebra structures.

To compute $\langle \gamma, [f] \rangle$ directly

- Use $f : S^n \to X$ to pull back $\gamma \in H_{\mathcal{E}}^{n-1}(X)$ to $\gamma^* f \in H_{\mathcal{E}}^{n-1}(S^n)$
- Find a cohomologous form of weight 0 (“weight reduce”)
- Evaluate on the fundamental class of S^n

Infinitesimal calculations in fundamental groups

Theorem [GOSW 24]

The Hopf pairing $H^0_{\mathcal{E}}(X) \times \pi_1(X) \to \mathbb{Q}$ is well defined and identifies $H^0_{\mathcal{E}}(X)$ as the "universal Lie dual" of the group $\pi_1(X)$.

Corollary

- *The Hopf pairing realizes the rational duality between $H^0_{\mathcal{E}}(X)$ and the Malcev Lie algebra of $\pi_1(X)$.*
- *Cobracket in $H^0_{\mathcal{E}}(X)$ is compatible with BCH product in $\pi_1(X)$.*

Generalizes Magnus expansion, Fox derivatives, Chen iterated integrals.
Quasi-isomorphism invariant. Computational. Algorithmic.
Compare to "Lie Models I & II" [BFMT] and [Rivera-Zeinalian].

Letter linking is Hopf pairing!

Approximately:

- Given $G = \langle S \mid R \rangle$, make $X \simeq \bigvee_S S^1 \coprod_R \{D^2\}_R$ So $\pi_1(X) \cong G$
- $\mathcal{E}(C^*X)$ is trees of (linear comb. of) bump forms on the "generator circles"
- Cycles $\gamma \in \mathcal{E}(C^*X)$ are equivalent to trees of counting functions from S
- Words $w \in G$ are maps $w : S^1 \to \bigvee_S S^1 \subset X$, yielding $[w] \in \pi_1(X)$
- Pullbacks $w^*\gamma$ are trees of bump forms on S^1 with positive mass at generators s and negative mass at inverses s^{-1} (recall "visual intuition" slide!)
- Cobounding operation performs weight reduction in $\mathcal{E}(C^*X)$
- Summing all values performs evaluation on S^1

Many thanks!

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Calculus in Groups

Finite type (Vassiliev) knot invariants.

Recall: singularity of knot resolves as "over-crossing" – "under-crossing"

$\mathbf{H^0}(\mathcal{K}, \mathcal{K}_n;\ \mathbb{R})$

Functions vanishing on knots with n singularities

Conjecture: $\mathrm{H}^0(\mathcal{K}\,;\ \mathbb{R})$

Knots

Singular Knots

Chord Diagrams

Bi-algebra structure

$$\begin{array}{rcccl}
 & \vdots & & & \vdots \\
 & \downarrow & & & \\
3-\text{singular} = & \mathcal{K}_3 & \twoheadrightarrow \mathcal{K}_3/\mathcal{K}_4 & = & \Gamma_3 \\
 & \downarrow & & & \\
2-\text{singular} = & \mathcal{K}_2 & \twoheadrightarrow \mathcal{K}_2/\mathcal{K}_3 & = & \Gamma_2 \\
 & \downarrow & & & \\
1-\text{singular} = & \mathcal{K}_1 & \twoheadrightarrow \mathcal{K}_1/\mathcal{K}_2 & = & \Gamma_1 \\
 & \downarrow & & & \\
\mathbb{Z}\ \text{Links} = & \mathcal{K} & \twoheadrightarrow \mathcal{K}/\mathcal{K}_1 & = & \Gamma_0
\end{array}$$

Dualize!

Finite-Type Invariants

Bi-algebra structure

"Symbol"

$$\begin{array}{ccc}
\vdots & & \vdots \\
 & & \uparrow \\
\mathrm{H}^0(\Gamma_3;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{H}^0(\mathcal{K},\mathcal{K}_4;\ \mathbb{R}) \\
 & & \uparrow \\
\mathrm{H}^0(\Gamma_2;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{H}^0(\mathcal{K},\mathcal{K}_3;\ \mathbb{R}) \\
 & & \uparrow \\
\mathrm{H}^0(\Gamma_1;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{H}^0(\mathcal{K},\mathcal{K}_2;\ \mathbb{R}) \\
 & & \uparrow \\
\mathrm{H}^0(\Gamma_0;\ \mathbb{R}) & = & \mathrm{H}^0(\mathcal{K},\mathcal{K}_1;\ \mathbb{R})
\end{array}$$

Analogy

Power Series

$$\begin{array}{ccc}
 & & \vdots \\
 & & \uparrow \\
\boxed{dx^3} & \twoheadleftarrow & a + bx + cx^2 + \boxed{dx^3} \\
 & & \uparrow \\
\boxed{cx^2} & \twoheadleftarrow & a + bx + \boxed{cx^2} \\
 & & \uparrow \\
\boxed{bx} & \twoheadleftarrow & a + \boxed{bx} \\
 & & \uparrow \\
 & & \boxed{a}
\end{array}$$

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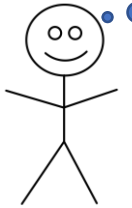
Calculus in Groups

Big picture (for experts)



Do knot theory with words in group

$\mathrm{Fun}(G, G^{[n]};\ \mathbb{R})$

Functions vanishing on n-fold bracket words

Malcev completion

Group

Lower Central Series

Lie algebra structure

$$
\begin{array}{ccccccc}
 & & \vdots & & & & \vdots \\
 & & \cup & & & & \\
G^{[4]} & = & [G,[G,[G,G]]] & \twoheadrightarrow & G^{[4]}/G^{[5]} & = & \Gamma_4 \\
 & & \cup & & & & \\
G^{[3]} & = & [G,[G,G]] & \twoheadrightarrow & G^{[3]}/G^{[4]} & = & \Gamma_3 \\
 & & \cup & & & & \\
G^{[2]} & = & [G,G] & \twoheadrightarrow & G^{[2]}/G^{[3]} & = & \Gamma_2 \\
 & & \cup & & & & \\
 & & G & \twoheadrightarrow & G/[G,G] & = & \Gamma_1
\end{array}
$$

Dualize!

Invariants

Lie coalgebra structure

$$
\begin{array}{ccc}
 & & \mathrm{Fun}(\hat{G};\ \mathbb{R}) \\
\vdots & & \vdots \\
 & & \cup \\
\mathrm{Fun}(\Gamma_4;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{Fun}(G, G^{[5]};\ \mathbb{R}) \\
 & & \cup \\
\mathrm{Fun}(\Gamma_3;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{Fun}(G, G^{[4]};\ \mathbb{R}) \\
 & & \cup \\
\mathrm{Fun}(\Gamma_2;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{Fun}(G, G^{[3]};\ \mathbb{R}) \\
 & & \cup \\
\mathrm{Fun}(\Gamma_1;\ \mathbb{R}) & = & \mathrm{Fun}(G, G^{[2]};\ \mathbb{R})
\end{array}
$$

Analogy

Power Series

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \cup \\
dx^3 & \twoheadleftarrow & a + bx + cx^2 + dx^3 \\
 & & \cup \\
cx^2 & \twoheadleftarrow & a + bx + cx^2 \\
 & & \cup \\
bx & \twoheadleftarrow & a + bx \\
 & & \cup \\
 & & a
\end{array}
$$