Week 8. Confidence Intervals (Concepts)

Definition: An *(unbiased point)* **estimator** for a parameter is a **random variable** *(or statistic)* whose **expected value equals the parameter.**

Examples:

- **Sample mean** $\overline{X} = \frac{X_1+X_2+\cdots+X_n}{n}$ has $E\left[\,\overline{X}\,\right] = \mu_X$ *(by Central Limit Theorem)*
- If X is Binomial(n, p) then **sample proportion** $\hat{P} = \frac{X}{n}$ has $E\left[\,\hat{P}\,\right] = p$ *(because $E[X] = np$)*

Problem: *Estimators only equal parameters in expected value; not in actual value….*

For example, $E\left[\,\overline{X}\,\right] = \mu_X$, so observed values $\overline{x} \approx \mu_X$ but probably $\overline{x} \neq \mu_X$…

Solution: *"Thicken" estimators into **"error estimates"** or **"confidence intervals"***

Error estimate for parameter:

$$\mu_X = \overline{x} \pm \varepsilon$$

$$p = \frac{x}{n} \pm \varepsilon$$

Confidence interval for parameter:

$$\overline{x} - \varepsilon \leq \mu_X \leq \overline{x} + \varepsilon$$

$$\frac{x}{n} - \varepsilon \leq p \leq \frac{x}{n} + \varepsilon$$

ε is the ***error*** at some ***"confidence level"***

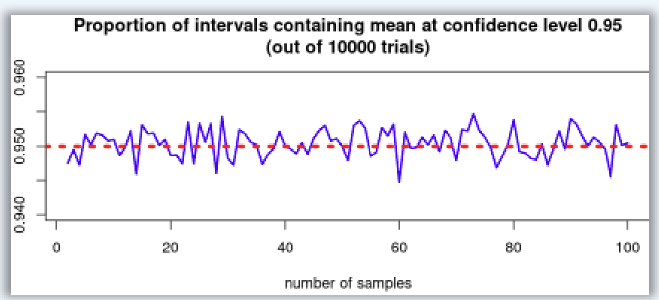
Usually, $\varepsilon = c\ {}^{\sigma}/_{\sqrt{n}}$ *(or something similar)*

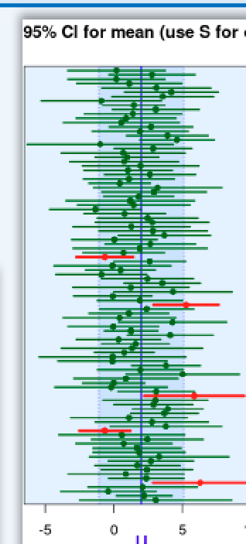
- c is like "***z*-score** for *confidence level*"
- σ is **std. dev.** of underlying population X
- n is **number of samples** used
- *"confidence level"* is the **probability** that the confidence interval contains target

Confidence level is the *expected proportion of intervals* containing the target value.

"confidence level = probability of being correct"

Example: *95% confidence interval for mean will contain the mean 95% of the time.*





Goal: Small confidence intervals with **big** confidence.

Small confidence intervals $\Leftrightarrow$ Small error ε

- More **confidence** (bigger %) → **bigger** CI
- More **data** (bigger n) → **smaller** CI.
- More **variance** (bigger σ) → **bigger** CI.

Week 8. Computing CI for <u>mean</u> with <u>known std.dev σ_X</u>

Recall: z-scores

$$z = \frac{x - \mu}{\sigma}$$

Recall: CLT info on $\overline{X}$

mean $\mu_{\overline{X}} = \mu_X$

std.err. $\sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}}$

Combine:

z-score for $\overline{x}$

$$z = \frac{\overline{x} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} = \frac{\overline{x} - \mu_X}{\frac{\sigma_X}{\sqrt{n}}}$$

Observe sample mean: $\overline{x}$.

Want to infer population mean: μ_X.

Guess: μ_X is a value which makes $\overline{x}$ have "common" z-score (like $|z| \leq 2$)

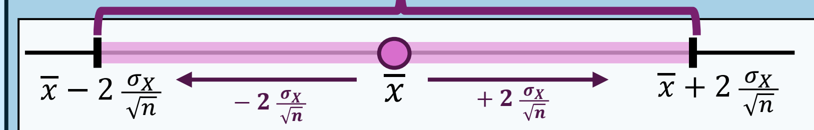
What μ values work??? Solve for μ_X:

$$z = \frac{\overline{x} - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} \longrightarrow \mu_X = \overline{x} - z\,\frac{\sigma_X}{\sqrt{n}}$$

If $-2 \leq z \leq 2$ (*"common" z-score*) then

$$\overline{x} - 2\,\frac{\sigma_X}{\sqrt{n}} \leq \mu_X \leq \overline{x} + 2\,\frac{\sigma_X}{\sqrt{n}}$$

μ_X in here



Confidence Interval for μ_X with known σ_X:

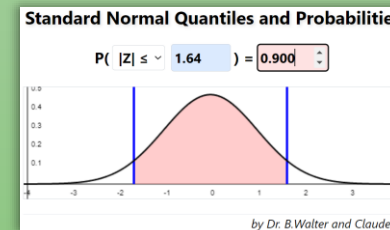
$$\boxed{\overline{x} - z\,\frac{\sigma_X}{\sqrt{n}} \leq \mu_X \leq \overline{x} + z\,\frac{\sigma_X}{\sqrt{n}}}$$

Value of z depends on "confidence level"

90% → $z = 1.64 \approx 1.6$

95% → $z = 1.96 \approx 2.0$

99% → $z = 2.58 \approx 2.6$



As an *"error estimate"*:

$$\mu_X = \overline{x} \pm \boxed{z}\,\frac{\sigma_X}{\sqrt{n}}$$

Common notation:

- z_α is value with *right-tailed* probability α
  $P(Z \geq z_\alpha) = \alpha$
- Then z for confidence level $(1-\alpha)$ is $z_{\alpha/2}$

Often, we shorten this to "*error estimate*" form:

$$\mu_X = \boxed{\overline{x}} \pm \boxed{2\,\frac{\sigma_X}{\sqrt{n}}}$$

estimate "$\hat{\mu}$"

error "ε"

This is a **warm-up** example, since usually we **won't know σ.**

Week 8. Confidence Intervals (Usual Applications)

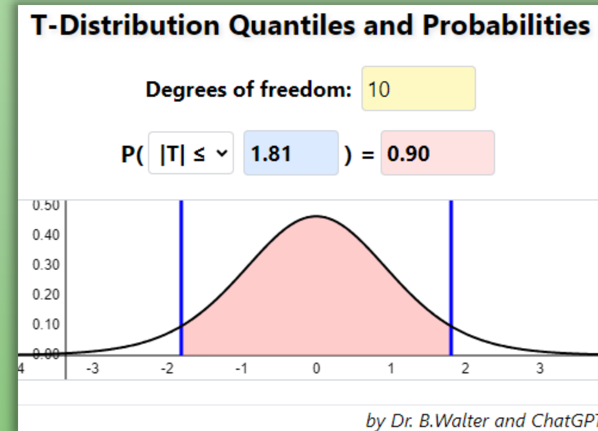
Confidence Interval for <u>mean</u> μ_X with unknown σ_X:

$$\boxed{\overline{x} - t_\nu \frac{s}{\sqrt{n}} \;\le\; \mu_X \;\le\; \overline{x} + t_\nu \frac{s}{\sqrt{n}}}$$

- *Use sample std. dev. s instead of population std.dev σ_X*
- *Use t-distribution with $\nu = (n-1)$ "degrees of freedom"*

As an *"error estimate"*:

$$\mu_X = \overline{x} \pm t_\nu \frac{s}{\sqrt{n}}$$



Confidence Interval for <u>population proportion</u> p:

$$\boxed{\frac{x}{n} - z\sqrt{\frac{x(n-x)}{n^3}} \;\le\; p \;\le\; \frac{x}{n} + z\sqrt{\frac{x(n-x)}{n^3}}}$$

If X is a sum of Bernoulli trials, then $\frac{X}{n}$ is their sample mean. Bernoulli probability is $\hat{P} \approx \frac{X}{n}$

- *Better to use "+4 rule" (unless n is REALLY big)*
  - *Replace n by $(n+4)$*
  - *Replace x by $(x+2)$*

Value of z depends on "confidence level"

$90\% \rightarrow z = 1.64 \approx 1.6$

$95\% \rightarrow z = 1.96 \approx 2.0$

$99\% \rightarrow z = 2.58 \approx 2.6$

As an *"error estimate"*:

$$p = \frac{x}{n} \pm z\sqrt{\frac{x(n-x)}{n^3}}$$

There are other, fancier formulas for this... but none are perfect.

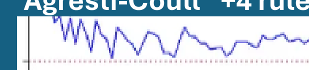
Wald

Clopper-Pearson "exact"

Wilson "score"

Corrected Wilson "score"

Agresti-Coull "+4 rule"



Bayes HPD