

Type III and IV Partial Fractions

Types III-IV Partial Fractions.

$$\frac{P(x)}{(x^2 + a)^2 Q(x)} = \frac{A_1 x + A_2}{x^2 + a} + \dots$$

$$\frac{P(x)}{(x^2 + a)^2 Q(x)} = \left[\frac{A_1 x + A_2}{(x^2 + a)^2} + \frac{A_3 x + A_4}{x^2 + a} \right] + \dots$$

Basic Forms.

$$\int \frac{x}{x^2 + a^2} dx = \frac{1}{2} \ln|x^2 + a^2| + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{x}{(x^2 + a^2)^2} dx = -\frac{1}{2} \frac{1}{x^2 + a^2} + C$$

$$\int \frac{1}{(x^2 + a^2)^2} dx = \frac{x}{2a^2(x^2 + a^2)} + \frac{1}{2a^2} \arctan\left(\frac{x}{a}\right) + C$$

1. Split the following Type III rational functions into sums of fractions.

(a) $\frac{x + 3}{(x^2 + 1)(x + 1)}$

(b) $\frac{3x}{(x^2 + x + 1)(x - 1)}$

2. Write the **form** of partial fractions for the following Type IV rational functions (do not solve).

(a) $\frac{9x}{(x + 1)^2 (x^2 + 1)^2}$

(b) $\frac{6x - 4}{x^2 (x^2 + x + 1)^3}$

3. Split as $\int \frac{du}{u} + \int \frac{a}{u} dx$ and integrate the following basic Type III forms.

(a) $\int \frac{3x - 4}{x^2 + 1} dx$

(b) $\int \frac{3x + 4}{x^2 + 9} dx$

Solving $\int \frac{P(x)}{Q(x)} dx$ via Partial Fractions Integration.

1. Do long division on $\frac{P(x)}{Q(x)}$ so that $\deg(P) < \deg(Q)$.
2. Factor the denominator $Q(x)$.
3. Write form of partial fractions using unknown constants A, B , etc.
(*What Types? How many terms?*)
4. Solve for unknown constants.
 - (a) Multiply by $Q(x)$ to convert to polynomial equation (canceling denominator terms).
 - (b) Plug in roots to get Type I terms and highest order Type II terms.
 - (c) Use “coefficient equations” to get remaining terms (solve system).
(*Top and bottom coefficient equations are easiest!*)
5. Integrate using memorized “basic forms.”