

Homework 03 - Introduction to Groups

0. Write down the requirements for a set to be a group.
1. Verify the requirements to show that the following are groups.
 - Positive rational numbers $\mathbb{Q}^{>0} = \{q \in \mathbb{Q} \mid q > 0\}$ with binary operation $a \diamond b = \frac{ab}{2}$.
 - Integers \mathbb{Z} with binary operation $a \oplus b = a + b + 2$.
 - Upper triangular matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ using matrix multiplication.
2. When p is prime, $\mathbb{Z}_p^\times = \{1, 2, \dots, (p-1)\}$ forms a group using multiplication modulo p .
Compute the inverses in \mathbb{Z}_5^\times of the values 1, 2, 3, 4.
3. When n is not prime, \mathbb{Z}_n^\times does **not** form a group using multiplication mod n .
Which group property is not satisfied? Give an example in \mathbb{Z}_4^\times .
4. Prove that $aba^{-1}b^{-1} = e$ if and only if $ab = ba$.
5. Prove that $(ab)^2 = a^2 b^2$ if and only if $ab = ba$.
6. Prove that $(aba^{-1})^n = ab^n a^{-1}$.