Homework 04 - Subgroups

1. Recall that we defined the symmetric group on three letters S_3 as the set of bijections from $\{1, 2, 3\}$ to itself, using function composition as the group law. In class I named two functions,

$$(\text{"shift right"}) \;\; x : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{cases} \qquad \text{and} \qquad (\text{"swap } 1 \leftrightarrow 2\text{"}) \;\; y : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{cases}$$

In class, I showed that $x^3 = e$ and $y^2 = e$, and I computed xy as well as x^2y.

Homework: Compute yx. (Show that $yx = x^2y$.)

2. In class I said that $S_3 = \{e, x, x^2, y, xy, x^2y\}$. All other products can be reduced to one of these six elements using **relations** $x^3 = e$, $y^2 = e$, and $yx = x^2y$. I did an example of this with $x^{-1}y^3x^2y$.

Homework: Apply relations to show that $x^4y^3x^2y^{-1}x^{-1} = x$ in S_3.

3. Recall that a subgroup is a subset of a group which is also a group using the same composition law. A subgroup is **nontrivial** if it is not the trivial group $\{e\}$. It is **proper** if it is not equal to the containing group.

Homework: S_3 has **four** nontrivial proper subgroups. State at **least two** of them.

Bonus: Consider the **order** of the subgroups versus that of S_3. There is a special relationship between these numbers.... try to guess it!

Super Bonus: Find all four nontrivial proper subgroups and prove that there are no others.

4. In class, Kimi-Lee showed that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ with addition as the group law. In fact, these are the **only** subgroups of $\mathbb{Z}$! Will showed that intersections of subgroups are subgroups.

Homework: Which $n\mathbb{Z}$ subgroup is $4\mathbb{Z} \cap 6\mathbb{Z}$? Explain!

5. Intersections of subgroups are subgroups, but unions usually aren't!

Homework: Show why $4\mathbb{Z} \cup 6\mathbb{Z}$ isn't a subgroup of $\mathbb{Z}$.

6. The **centralizer** of an element $a \in G$ is the set of all other elements which commute with it, written $C_G(a)$.

Homework: Prove that the centralizer is a subgroup. *(Hint: see proof that center is a subgroup.)*