Computing Riemann Sums

Formulas for Sums of Sequences of Powers

$$\sum_{i=1}^{n} 1 = 1 + 1 + 1 + \ldots = n$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots = \frac{n\,(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \ldots = \frac{n\,(n+1)\,(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \ldots = \left[\frac{n\,(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} i^4 = 1 + 16 + 81 + \ldots = \frac{n\,(n+1)\,(2n+1)\,(3n^2+3n-1)}{30}$$

$$\sum_{i=1}^{n} i^5 = 1 + 32 + 271 + \ldots = \frac{n^2\,(n+1)^2\,(2n^2+2n-1)}{12}$$

General formula.

Writing a general formula is somewhat tricky, though Pascal found a clever way to get the next formula by combining all of the previous ones. Applying the binomial theorem, the difference of consecutive kth powers is an expression involving lower powers combined using binomial coefficients.

$$(i+1)^k - i^k = \binom{k}{1} i^{k-1} + \binom{k}{2} i^{k-2} + \ldots + 1$$

We can add these to get an expression relating the sums of various powers.

For example

$$\begin{array}{rcccccccccc}
2^5 - 1^5 & = & 5 \cdot 1^4 & + & 10 \cdot 1^3 & + & 10 \cdot 1^2 & + & 5 \cdot 1 & + & 1 \\
3^5 - 2^5 & = & 5 \cdot 2^4 & + & 10 \cdot 2^3 & + & 10 \cdot 2^2 & + & 5 \cdot 2 & + & 1 \\
4^5 - 3^5 & = & 5 \cdot 3^4 & + & 10 \cdot 3^3 & + & 10 \cdot 3^2 & + & 5 \cdot 3 & + & 1
\end{array}$$

Adding these gives the relation

$$4^5 - 1^5 = 5 \cdot \sum_1^3 i^4 + 10 \cdot \sum_1^3 i^3 + 10 \cdot \sum_1^3 i^2 + 5 \cdot \sum_1^3 i + \sum_1^3 1$$

In general

$$(n+1)^{k+1} - 1^{k+1} = \binom{k+1}{1} \sum_1^n i^k + \binom{k+1}{2} \sum_1^n i^{k-1} + \ldots + \sum_1^n 1$$

Which can be solved for the sum of kth powers:

$$\sum_1^n i^k = \frac{(n+1)^{k+1} - 1 - \binom{k+1}{2} \sum_1^n i^{k-1} - \binom{k+1}{3} \sum_1^n i^{k-2} - \cdots - \binom{k+1}{k} \sum_1^n i - \sum_1^n 1}{k+1}$$