Homework 08 - Normal Subgroups and Quotients

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1. Recall that if $\left|G\right| = n$ then $g^n = e$ for all $g \in G$.

Prove. If $H \lhd G$ with $\left|G/H\right| = m$ then $g^m \in H$ for all $g \in G$.

2. Recall that $[G, G] = \{aba^{-1}b^{-1} \mid a, b \in G\}$.
In class we showed $[G, G] \lhd G$.
Previously we showed $[G, G] = \{e\}$ if and only if G is commutative.

You will prove that $[G, G]$ is the **smallest** subgroup so that G/H is commutative.

(i) **Prove.** $G/[G, G]$ is commutative.

(ii) **Prove.** G/H is commutative if and only if $[G, G] < H$.

3. The core of a subgroup $H < G$ is $\operatorname{core}(H) = \bigcap_{g \in G} gHg^{-1}$.

You will prove that the core of H is the **largest** normal subgroup of G contained in H.

(i) **Prove.** $\operatorname{core}(H) < H$ and $\operatorname{core}(H) \lhd G$.

(ii) **Prove.** If $K < H$ with $K \lhd G$ then $K < \operatorname{core}(H)$.

4. **Prove.** If $\left|G\right| = 24$ and G has more than one subgroup of order 3, then none are normal!

Later we will show this using the Sylow theorems, but this one needs only the product formula!