The Mean Value Theorem

Bounding Function Values

If $\quad m \quad \leq f'(x) \leq \quad M, \quad$ then

$$f(a) + m\,\Delta x \;\leq\; f(b) \;\leq\; f(a) + M\,\Delta x$$

where $\Delta x = b - a$.

Counting # of Roots

$\#\big(f(x) \text{ Zeros}\big) \;\leq\; 1 \;+\; \#\big(\boldsymbol{f'(x)} \text{ Zeros}\big)$

If $f'(x)$ doesn't change sign, then $f(x)$ has at most one root!

1. Bounding function values.

A. If $-2 < f'(x) < 3$ for $1 < x < 4$, then what values are possible for $f(4) - f(1)$?

B. If $f'(x) > 2$ for all x and $f(0) = 1$, then what is the smallest possible value for $f(5)$?

C. If $f(0) = 2$ and $-1 < f'(x) < 2$ for $0 < x < 3$, then what values are possible for $f(3)$?

2. Counting roots.

A. Show that $x^3 - 15x + c = 0$ cannot have more than one solution for $-2 \leq x \leq 2$.

B. Show that $x^4 + 4x + c = 0$ cannot have more than two solutions.

3. Identities.

A. Show that $\ln(2x) = \ln(2) + \ln(x)$ by computing derivatives.

B. Show that $\sin^2 x + \cos^2 x = 1$ by computing derivatives.

4. Verifying the mean value theorem.

A. Compute the average change of $f(x) = 2x^2 - 4x + 5$ on $[-1, 3]$.
Then find c so that $f'(c) =$ avg. change.

B. Compute the average change of $f(x) = x^3 - 2x + 2$ on $[-2, 2]$.
Then find c so that $f'(c) =$ avg. change.