# Example. Symmetric Groups

---

**Definition.** The $n^{\text{th}}$ **symmetric group** is the set of bijections $S_n = \left\{ \{1,\dots,n\} \xrightarrow{\cong} \{1,\dots,n\} \right\}$ with group operation given by composition.

We call $S_n$ the **"symmetric group on $n$ *letters*"**, because it concerns rearrangements of $n$ element sets.

Symmetric group elements correspond to permutations of the numbers $1,\dots,n$. There are $n!$ of these.

- $S_1$ is the trivial group, whose only element is the identity $S_1 = \{e : 1 \mapsto 1\}$
- $S_2$ has the identity and one nontrivial element, swapping 1 and 2 $S_2 = \{e, \tau\}$ where $\tau : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \end{cases}$
- $S_3$ has 6 elements corresponding to the $3! = 6$ permutations of three numbers. We can write these 6 elements as $S_3 = \{e, x, x^2, y, xy, x^2y\}$ where $x$ increments by 1 and $y$ swaps $1 \leftrightarrow 2$

$$x : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{cases} \qquad \text{and} \qquad y : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{cases}$$

We showed directly that $x^3 = e$ and $y^2 = e$.

$$x^3 : \begin{cases} 1 \overset{x}{\mapsto} 2 \overset{x}{\mapsto} 3 \overset{x}{\mapsto} 1 \\ 2 \overset{x}{\mapsto} 3 \overset{x}{\mapsto} 1 \overset{x}{\mapsto} 2 \\ 3 \overset{x}{\mapsto} 1 \overset{x}{\mapsto} 2 \overset{x}{\mapsto} 3 \end{cases} \qquad \text{and} \qquad y^2 : \begin{cases} 1 \overset{y}{\mapsto} 2 \overset{y}{\mapsto} 1 \\ 2 \overset{y}{\mapsto} 1 \overset{y}{\mapsto} 2 \\ 3 \overset{y}{\mapsto} 3 \overset{y}{\mapsto} 3 \end{cases}$$

We also showed that $xy$ swaps $1 \leftrightarrow 3$ and $x^2y$ swaps $2 \leftrightarrow 3$.

$$xy : \begin{cases} 1 \overset{y}{\mapsto} 2 \overset{x}{\mapsto} 3 \\ 2 \overset{y}{\mapsto} 1 \overset{x}{\mapsto} 2 \\ 3 \overset{y}{\mapsto} 3 \overset{x}{\mapsto} 1 \end{cases} \qquad \text{and} \qquad x^2y : \begin{cases} 1 \overset{y}{\mapsto} 2 \overset{x}{\mapsto} 3 \overset{x}{\mapsto} 1 \\ 2 \overset{y}{\mapsto} 1 \overset{x}{\mapsto} 2 \overset{x}{\mapsto} 3 \\ 3 \overset{y}{\mapsto} 3 \overset{x}{\mapsto} 1 \overset{x}{\mapsto} 2 \end{cases}$$

$S_3$ is the first simple example of a nonabelian (non-commutative) group. Direct computation shows that $xy \neq yx$. In fact $yx = x^2y$ (Homework!).

$S_3$ is also a great example of a group where we can compute using **relations**. Working modulo the relations $x^3 = e$, $y^2 = e$, and $yx = x^2y$ we can simplify any long string of products in $S_3$ to one of the basic elements $e, x, x^2, y, xy, x^2y$. We can use the first two relations to lower the powers of $x$ and $y$ and the third relation to push occurences of $y$ to the right.

**Example.** In $S_3$ the expression $x^{-1}y^3x^2y$ simplifies as follows:

$$\begin{aligned} & x^{-1}y^3x^2y \\ &= x^2 \; y \; x^2y \quad (x^{-1} = x^2, \text{ and } y^3 = y^2y = ey = y) \\ &= x^2x^2x^2y\,y \quad (yx^2 = yxx = x^2yx = x^2x^2y) \\ &= \; x^6 \quad y^2 \\ &= \; e \quad\; e \quad = e \end{aligned}$$

**Comments.** The symmetric groups are the *universal* finite groups. All of the other finite groups sit inside of them. Mantra: "If we understand symmetric groups well, then we understand everything".

Note that we can see each symmetric group inside of the next bigger one $S_n \subset S_{n+1}$ by identifying it with bijections which "fix" the last element (i.e. don't move $(n+1)$). Oh! What does $S_\infty$ look like???

There is also a natural way to write symmetric groups as $n \times n$ *permutation matrices*. They have determinant $= \pm 1$. Restricting to the determinant $= 1$ matrices yields the **alternating groups** $A_n$. Wow!!! There's a lot of excitement in our future!