Homework 05 - Cyclic Groups

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1. Write the elements of the cyclic subgroup $\langle 18 \rangle < \mathbb{Z}_{30} = \{0, 1, 2, \ldots, 29\}$.

2. Consider the set $S = \big\{\text{strings of alternating } a \text{ and } b\big\}$ (including the empty string).

Homework: Show that this is a group if we set $a^2 = e$ and $b^2 = e$ and use string concatenation as the group operation (for example $aba \circ ab = abaab = abb = a$).
(Note: you will need to figure out what e is and what inverses are for this group.)

> This group is interesting because despite a and b both having order 2, the product ab has **infinite** order!

3. Recall that $|x| = n$ if and only if n is the smallest number with $x^n = e$.

Homework: For any $a, b \in G$ (not necessarily cyclic or commutative), prove that $|ab| = |ba|$.

4. Recall that $|x| = n$ if and only if n is the smallest number with $x^n = e$.

Homework: Prove that conjugation preserves order, $\big|aba^{-1}\big| = \big|b\big|$.

5. Recall that $\langle x^k \rangle = \langle x \rangle$ if and only if k and $|x|$ are relatively prime.

Homework: Prove that if b has odd order and $ab^2 = b^2a$, then $\langle aba^{-1} \rangle = \langle b \rangle$.

6. Note that if $\langle aba^{-1} \rangle = \langle b \rangle$, then $aba^{-1} = b^k$ for some $k < |b|$. Continuing from 5...

Homework: Prove that if b has odd order and $ab^2 = b^2a$, then $ab = ba$

Other interesting problems (which are not assigned as homework).

- Prove that if $a \in G$ and G is cyclic of order n then $a^n = e$.

  (We will eventually show this even for non-cyclic G!)

- The $\mathbf{n}^{\text{th}}$ **roots of unity** are the n solutions to $z^n = 1$ in $\mathbb{C}$.
  Prove that the n^{th} roots of unity form a cyclic group.

  Recall that roots of unity come in conjugate pairs.
  What is conjugation doing in terms of the cyclic group structure?

- Prove that product of all elements in a cyclic group $\langle g \rangle$ is
  - ... e if $|g|$ is odd.
  - ... $g^{n/2}$ if $|g| = n$ is even.

  What does this mean in the context of roots of unity?