Introduction to Set Theory

Naive Definition. A **set** is a collection of elements.
We write $a \in A$ to say that *"a is an element of the set A"*

Example. Sets can be defined by naming all elements or giving a condition for membership[1]

- "set consisting of $1, 2, 3$" $= \{1, 2, 3\}$
- "set of even integers" $= \{n \mid n \text{ is an even integer}\}$
  $= \{2k \mid k \in \mathbb{Z}\}$

Equality of sets means they have the same elements.

Definition. $A = B$ means

- for each element $a \in A$, have $a \in B$ also; and
- for each element $b \in B$, have $b \in A$ also.

Subset if all elements of one set are also elements of another.

Definition. $A \subset B$ means for all $a \in A$, have $a \in B$ also.

Lemma. *$A = B$ if and only if $A \subset B$ and $B \subset A$.*

Notes:

- We sometimes say *"A is **(contained) in** B"* to mean $A \subset B$. Pay attention to context!
- Containment gives a *partial ordering* on sets. If $A \subset B$, then A is "less than or equal to" B.
- The empty set $\emptyset = \{\}$ is a subset of every set, and every set is a subset of itself.

Union of sets combines elements.

Definition. $A \cup B = \{x \mid x \in A \textbf{ or } x \in B\}$

Universal Property. A and B are both subsets of $A \cup B$. In fact $A \cup B$ is the *smallest set* with this property.
Proposition. *If X is a set with $A \subset X$ and $B \subset X$, then $A \cup B \subset X$*

Intersection of sets restricts to common elements.

Definition. $A \cap B = \{x \mid x \in A \textbf{ and } x \in B\}$

Universal Property. $A \cap B$ is a subset of both A and B. In fact $A \cap B$ is the *largest set* with this property.
Proposition. *If X is a set with $X \subset A$ and $X \subset B$, then $X \subset A \cap B$*

Disjoint sets have no common elements.

Definition. A and B are **disjoint** if and only if $A \cap B = \emptyset$

Theorem. *Rules for triple operations:*

$$\textbf{(Associative)} \quad \begin{aligned} A \cup (B \cup C) = (A \cup B) \cup C &= A \cup B \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C &= A \cap B \cap C \end{aligned}$$

[1] Slightly tricky – see Russell's paradox and Zarmelo-Fraenkel set theory

(Distributive) $A \cup \big(B \cap C\big) = \big(A \cup B\big) \cap \big(A \cup C\big)$
$A \cap \big(B \cup C\big) = \big(A \cap B\big) \cap \big(A \cup C\big)$

Difference of sets removes elements.

Definition. $A \setminus B = \{x \mid x \in A \textbf{ but } x \notin B\}$

This is pronounced "A **not** B" or "A **minus** B". Some people may write $A - B$ or $A \neg B$. One of the biggest applications will be to remove single elements, for example $\mathbb{R} \setminus \{0\}$.

Universal Property. $A \setminus B$ is the *largest subset* of A that is disjoint from B.
Proposition. *If X is a set with $X \subset A$ and $X \cap B = \emptyset$, then $X \subset A \setminus B$.*