Calculus in Groups: Linkings and Configurations of Letters

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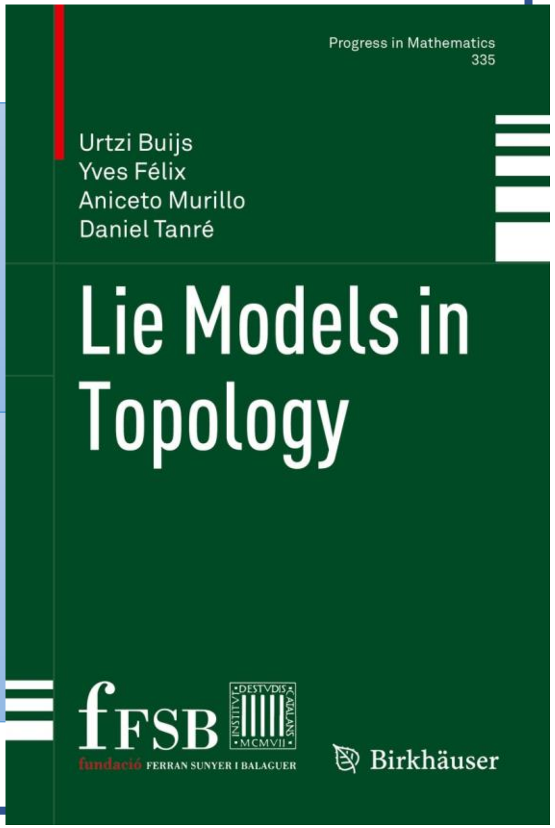
Introduction.

Joint work with N.Gadish, A.Ozbek, D.Sinha

Lie coalgebras and rational homotopy III:
Rational measurement of the fundamental group *(working title)*

D.Sinha and B.Walter
Lie coalgebras and rational homotopy I: Graph coalgebras (2011)
Lie coalgebras and rational homotopy II: Hopf invariants (2013)

J.Monroe and D.Sinha
Linking of letters and the lower central series of free groups (2022)



5 Minute Groups Intro.

"Def": *A **finitely presented group*** $G = \langle S \mid R \rangle$ ("generators S and relations R")
is the collection of "words" in the "alphabet" $S \cup S^{-1}$ modulo the relations R.

Classical example: S = set of actions

wake up → w
get dressed → g
brush teeth → b
etc.

"word" = ordered list of actions

$wgb = \text{wake up, get dressed, brush teeth}$

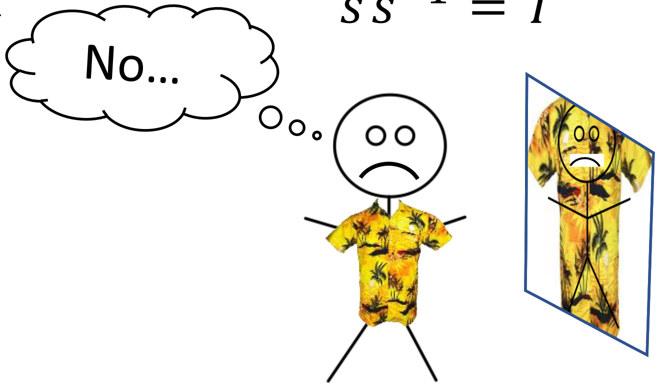
If we think of these as *functions*, then we should probably write in the other order…

Every action has an "inverse" action

put shirt on → s
take shirt off → s^{-1}

$s\,s^{-1} = I$

"empty word" = "identity"



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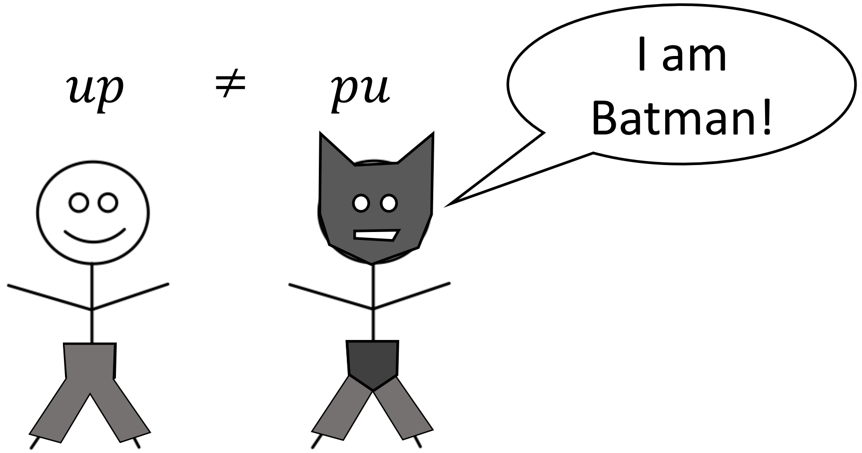
Ordering is important!

put underwear on → u
put pants on → p



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put shirt on → s
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put underwear on → u
put pants on → p

R = collection l actions

- Put on underwear
- Put on pants
- Take off underwear
- Take off pants

- Go to hospital

$ss^{-1} = I$

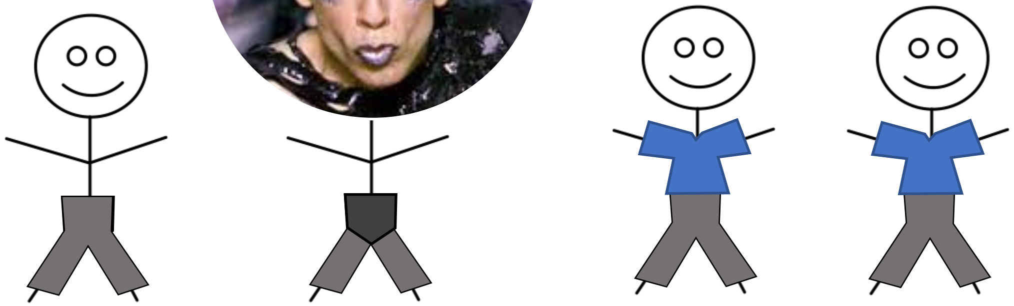
$up \neq pu \iff upu^{-1}p^{-1} \neq I$

$sp = ps \iff sps^{-1}p^{-1} = I$

- Put on shirt
- Put on pants
- e off shirt
- off pants

- Do nothing

up $sp = ps$



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Classical example: S = set of actions

wake up → w
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brush teeth → b
etc.

Every action has an "inverse" action

put shirt on → s
take shirt off → s^{-1}

Ordering is important!

put underwear on → u
put pants on → p

R = collection of equivalent / trivial actions

$$ss^{-1} = I$$

$$up \neq pu \iff upu^{-1}p^{-1} \neq I$$

$$sp = ps \iff sps^{-1}p^{-1} = I$$

$$G = \left\langle u, p, s \mid sps^{-1}p^{-1}, sus^{-1}u^{-1} \right\rangle$$

Now you know about groups!
(And FASHION!)

Questions:

1. When are two groups equivalent?
   (different choice of generators & relations)
2. When are two words equivalent?

"Word Problem"

Plan: Invariants!

Want to tell apart things in here:

G

Dualize!

Commutator
$[a,b] = aba^{-1}b^{-1}$

Define functions on the level of words so that
if $w_1 = w_2$ in G (i.e. modulo relations)
then $f(w_1) = f(w_2)$
These are "**invariants**"

$$\mathrm{Fun}(G;\mathbb{R}) = \{\text{Functions } G \to \mathbb{R}\}$$

$$\mathrm{Fun}(G,[G,G]\,;\mathbb{R}) = \left\{\begin{matrix}\text{Functions } G \to \mathbb{R} \\ \text{vanishing on } [G,G]\end{matrix}\right\}$$

"linear"

$$\mathrm{Fun}\left(G,G^{[3]}\,;\mathbb{R}\right) = \left\{\begin{matrix}\text{Functions } G \to \mathbb{R} \\ \text{vanishing on } [G,[G,G]]\end{matrix}\right\}$$

"quadratic"

$\vdots$

$\vdots$

$$\mathrm{Fun}\left(\hat{G}\,;\mathbb{R}\right) = \left\{\begin{matrix}\text{Functions } G \to \mathbb{R} \\ \text{on } \textit{nilpotent completion}\end{matrix}\right\}$$

"power series"

More general, fancier version uses Baker–Campbell–Hausdorff formula.

Bernoulli numbers appear, possible connection to Riemann zeta function.

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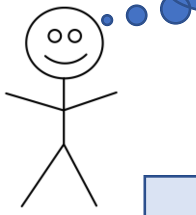
Calculus in Groups

Big picture (for experts)



Do knot theory with words in group

Malcev completion

$\mathrm{Fun}(G, G^{[n]};\ \mathbb{R})$

Functions vanishing on n-fold bracket words

Lower Central Series

Lie algebra structure

$$
\begin{array}{ccccc}
\vdots & & & & \vdots \\
\cup & & & & \\
G^{[4]} = [G,[G,[G,G]]] & \twoheadrightarrow & G^{[4]}/G^{[5]} & = & \Gamma_4 \\
\cup & & & & \\
G^{[3]} = [G,[G,G]] & \twoheadrightarrow & G^{[3]}/G^{[4]} & = & \Gamma_3 \\
\cup & & & & \\
G^{[2]} = [G,G] & \twoheadrightarrow & G^{[2]}/G^{[3]} & = & \Gamma_2 \\
\cup & & & & \\
G & \twoheadrightarrow & G/[G,G] & = & \Gamma_1
\end{array}
$$

Group

Dualize!

Lie coalgebra structure

$$
\begin{array}{ccc}
 & & \mathrm{Fun}(\hat{G};\ \mathbb{R}) \\
 & & \vdots \\
\vdots & & \cup \\
\mathrm{Fun}(\Gamma_4;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{Fun}(G, G^{[5]};\ \mathbb{R}) \\
 & & \cup \\
\mathrm{Fun}(\Gamma_3;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{Fun}(G, G^{[4]};\ \mathbb{R}) \\
 & & \cup \\
\mathrm{Fun}(\Gamma_2;\ \mathbb{R}) & \twoheadleftarrow & \mathrm{Fun}(G, G^{[3]};\ \mathbb{R}) \\
 & & \cup \\
\mathrm{Fun}(\Gamma_1;\ \mathbb{R}) & = & \mathrm{Fun}(G, G^{[2]};\ \mathbb{R})
\end{array}
$$

Invariants

Power Series

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \cup \\
dx^3 & \twoheadleftarrow & a + bx + cx^2 + dx^3 \\
 & & \cup \\
cx^2 & \twoheadleftarrow & a + bx + cx^2 \\
 & & \cup \\
bx & \twoheadleftarrow & a + bx \\
 & & \cup \\
 & & a
\end{array}
$$

Analogy

Finite type (Vassiliev) knot invariants.

Recall: singularity of knot resolves as "over-crossing" − "under-crossing"

$\mathrm{H}^0(\mathcal{K},\mathcal{K}_n;\ \mathbb{R})$
Functions vanishing on knots with n singularities

Conjecture: $\mathrm{H}^0(\mathcal{K};\ \mathbb{R})$

Singular Knots | **Chord Diagrams** | Bi-algebra structure | Bi-algebra structure | **"Symbol"**

$$\begin{tikzcd}
 & \vdots \arrow[d, hook] & & \vdots & & \vdots & & \vdots \arrow[d, hook'] \\
3-\text{singular} = & \mathcal{K}_3 \arrow[r, two heads] \arrow[d, hook] & \mathcal{K}_3/\mathcal{K}_4 \arrow[r, equal] & \Gamma_3 & & \mathrm{H}^0(\Gamma_3;\ \mathbb{R}) & & \mathrm{H}^0(\mathcal{K},\mathcal{K}_4;\ \mathbb{R}) \arrow[ll, two heads] \\
2-\text{singular} = & \mathcal{K}_2 \arrow[r, two heads] \arrow[d, hook] & \mathcal{K}_2/\mathcal{K}_3 \arrow[r, equal] & \Gamma_2 & & \mathrm{H}^0(\Gamma_2;\ \mathbb{R}) & & \mathrm{H}^0(\mathcal{K},\mathcal{K}_3;\ \mathbb{R}) \arrow[ll, two heads] \arrow[u, hook] \\
1-\text{singular} = & \mathcal{K}_1 \arrow[r, two heads] \arrow[d, hook] & \mathcal{K}_1/\mathcal{K}_2 \arrow[r, equal] & \Gamma_1 & & \mathrm{H}^0(\Gamma_1;\ \mathbb{R}) & & \mathrm{H}^0(\mathcal{K},\mathcal{K}_2;\ \mathbb{R}) \arrow[ll, two heads] \arrow[u, hook] \\
\mathbb{Z}\ \text{Links} = & \mathcal{K} \arrow[r, two heads] & \mathcal{K}/\mathcal{K}_1 \arrow[r, equal] & \Gamma_0 & & \mathrm{H}^0(\Gamma_0;\ \mathbb{R}) \arrow[rr, equal] & & \mathrm{H}^0(\mathcal{K},\mathcal{K}_1;\ \mathbb{R}) \arrow[u, hook]
\end{tikzcd}$$

Dualize!

Knots

Finite-Type Invariants

Power Series

$$\begin{tikzcd}
 & \vdots \\
 & \vdots \arrow[u, hook] \\
\boxed{dx^3} & a + bx + cx^2 + \boxed{dx^3} \arrow[l, two heads] \arrow[u, hook] \\
\boxed{cx^2} & a + bx + \boxed{cx^2} \arrow[l, two heads] \arrow[u, hook] \\
\boxed{bx} & a + \boxed{bx} \arrow[l, two heads] \arrow[u, hook] \\
 & \boxed{a} \arrow[u, hook]
\end{tikzcd}$$

Analogy

Functions on words. (Not invariants yet)

Notation:

- Fix a presentation $G = \langle\, S \mid R \,\rangle$ of the group G.
- A **word** of length k is an element of $(S \coprod S^{-1})^{\times k}$
- The word $w = x_1 x_2 \ldots x_k$ has **letters** $x_1, x_2, \ldots, x_k$
- The **sign** of a letter is $dx_i = \pm 1$ depending on whether $x_i \in S$ or $x_i \in S^{-1}$

Examples:

$a,\ ab,\ aaa^{-1},\ aba^{-1}b^{-1}$

Signs:

$da = 1$

$da^{-1} = -1$

Functions and Forms:

- A **function** on $w = x_1 x_2 \ldots x_k$ is a map $f: \{1,2,\ldots,k\} \to \mathbb{R}$
- The **1-form** dx gives *sign* of associated letter.
- General 1-forms are $\varphi = f\, dx$
- These behave like usual….

Example:

dx	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1
f	1	3	-1	0	0	1	2	-2	3	1	0	0	1	0
$w =$	a	a	b	a	c	b^{-1}	c^{-1}	a^{-1}	a^{-1}	c	b	c^{-1}	a^{-1}	b^{-1}

$= [\, aa, [\, b, ac \,]\,]$

Integrals of functions. (Still not invariants)

Definition: The **integral** (or count) of $f\ dx$ over the word w is

$$\int_w f\ dx\ =\ \sum_i f(i)\ dx_i$$

Notation: We'll say that $f\ dx$ **cobounds** (over w) if

$$\int_w f\ dx\ =\ 0$$

Example:

$$\int f\ dx = 1 - 2 - 1 + 3 = 1$$

$$\begin{array}{cccccc} f & & 1 & -2 & 1 & -3 \\ w & = & a & b & a^{-1} & b^{-1} \end{array}$$

Example:

$$\int f\ dx = 1+3-1+0+0-1-2+2-3+1+0+0-1+0 = -1$$

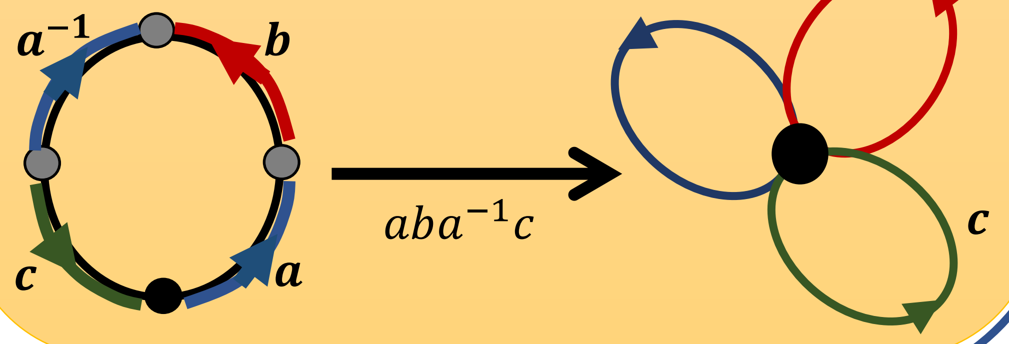
$$\begin{array}{cccccccccccccccc} f & & 1 & 3 & -1 & 0 & 0 & 1 & 2 & -2 & 3 & 1 & 0 & 0 & 1 & 0 \\ w & = & a & a & b & a & c & b^{-1} & c^{-1} & a^{-1} & a^{-1} & c & b & c^{-1} & a^{-1} & b^{-1} \end{array}$$

Basic idea is from algebraic topology.

A word w defines a map

$$w : S^1 \to \vee_S\, S^1$$

Forms on $\vee_S\, S^1$ can be *pulled back* and integrated: $\int_{S^1} w^*\ f\ dx$



Indicator functions. (Weight 0 Invariants)

Definition: The **indicator function** for a letter $a \in S$ is

$$\delta_a(i) = \begin{cases} 1, & \text{if } x_i = a \text{ or } a^{-1} \\ 0, & \text{otherwise} \end{cases}$$

Notation:

- Use lowercase letters for generators: $a, b, c \in S$
- Use uppercase letters for corresponding indicator forms: $A = \delta_a\,dx,\;\; B = \delta_b\,dx,$ etc..

Examples:

δ_a: 1 0 1 0

$w = a\;\; b\;\; a^{-1}\;\; b^{-1}$

$\delta_a + 2\delta_b$: 1 2 1 2

$w = a\;\; b\;\; a^{-1}\;\; b^{-1}$

Convention: omit 0's

Bigger Example:

$\delta_a + 2\delta_b$: 1 1 2 1 0 2 0 1 1 0 2 0 1 2

$w = a\;\; a\;\; b\;\; a\;\; c\;\; b^{-1} c^{-1} a^{-1} a^{-1} c\;\; b\;\; c^{-1} a^{-1} b^{-1}$

Indicator forms are pull-backs of bump forms

$$w : S^1 \to \vee_S\; S^1$$

They tell *whether* w went around some circle and *when*.

Note: The integral of an indicator form **counts** (with sign) the number of occurrences of the corresponding letter.

$$\int_w A = \#\{a \text{ in } w\} - \#\{a^{-1} \text{ in } w\}$$

If **all** indicators cobound then the word **vanishes** in the abelianization $G^{ab} = G\,/\,[G, G]$

Anti-derivatives of functions. (Cobounding)

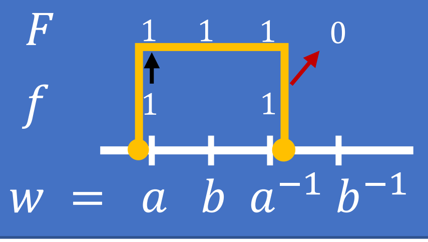
Definition: The **anti-derivative** (or cobounding function) of $\varphi = f\,dx$ is the function

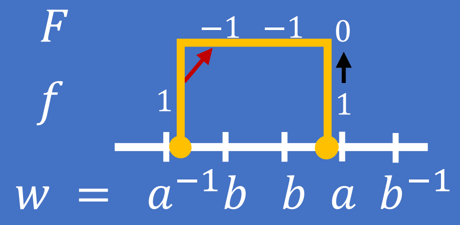
$$d^{-1}(\varphi) = F(i) = \sum_{j<i} f(j)\,dx_j + f(i)\,\epsilon_i$$

where $\epsilon_i = 1$ if $dx_i = 1$ and 0 otherwise.

This is necessary to make *something* be a group homomorphism…

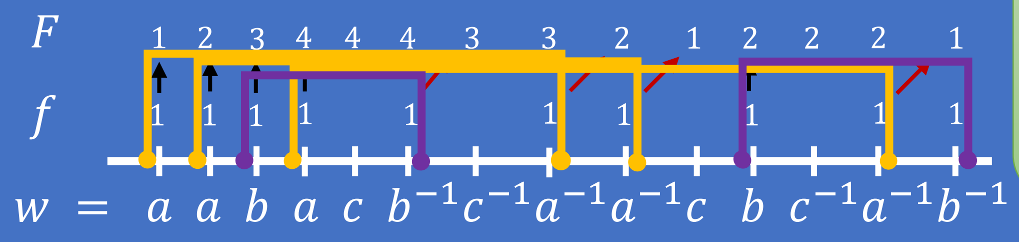
Basic Examples:





dx **'nudges'** value of $f(x)$ to the **left** or **right** of letter.

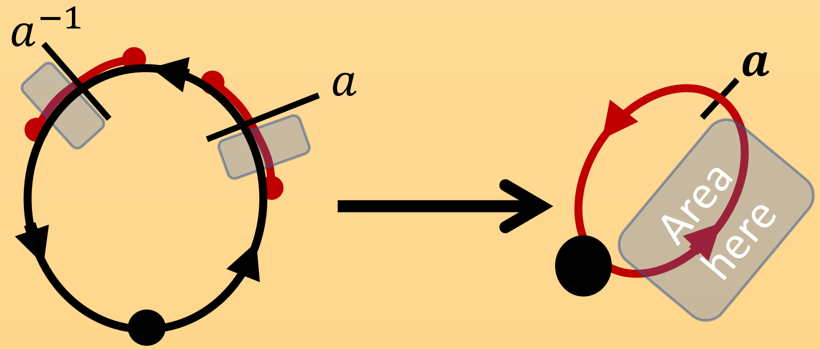
Bigger Example:



Anti-derivatives of **indicators** are **square waves**

Why ϵ_i ????

Before discretizing, we 'push' area to one side of loop.



- When we do loop forwards, we've already seen area by discretization position.
- When we do loop backwards, we don't see area until **after** the discretization position.

Linking Products and Symbols.

Definition: The **linking product** of $\varphi = f\,dx$ and $\psi = g\,dx$ is the form

$$(d^{-1}\,\varphi)\,\psi \;=\; F\,g\,dx$$

We're mostly interested in this when either φ or ψ cobound.

Notation: For indicator forms, we will drop the "d^{-1}"

$$(d^{-1}A)\,B \;=\; (A)B$$
$$(d^{-1}A)(d^{-1}B)C \;=\; (A)(B)C$$
$$\left(d^{-1}((d^{-1}A)B)\right)C \;=\; ((A)B)C$$

Definition: A **symbol** Λ is a formal expression of iterated operations of

- linear combination and
- linking product of

indicator forms.

These will give our invariants!

Examples:

A	B	$A+2B$
$(A+B)C$	$(A)\{B-C\}$	$(A)B-A$
$\big((A)B-C\big)A$	$\big((A)\{B-C\}\big)A$	$(A)(B-C)B$

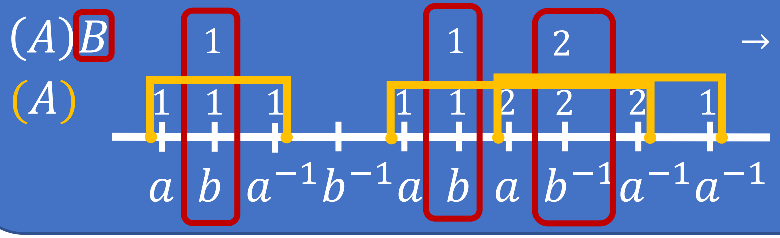
Definition: The **weight** of symbol is # (linking products) + 1

Invariants.

Definition: The **value of a symbol on a word** is given by integrating the symbol over the word.

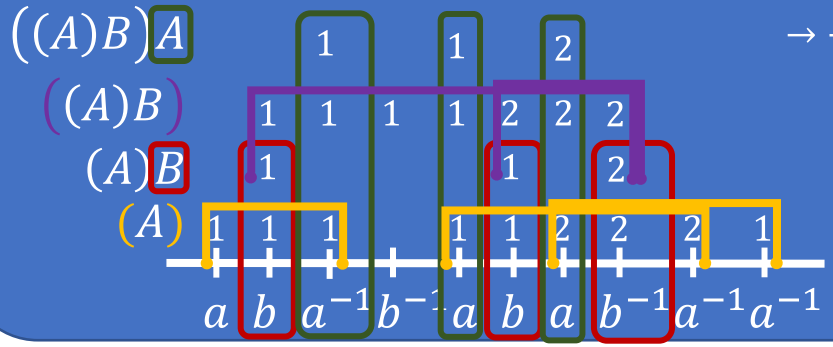
$$\Lambda(w) = \int_w \Lambda$$

Example: $(A)B$ on $w = [[a, b], a]$



$\to 1 + 1 - 2 = 0$

Example: $((A)B)A$ on $w = [[a, b], a]$



$\to -1 + 1 + 2 = 2$

Theorem: If G is a free group, then a word is an n-fold commutator $w \in G^{[n]}$ if and only if all symbols of weight $< n$ vanish on the word.

≈ [Monroe-Sinha]

Theorem: If $G = \langle\, S \mid R\, \rangle$, then a word is an n-fold commutator $w \in G^{[n]}$ if and only if all symbols of weight $< n$ *in the coideal generated by symbols vanishing on the relations* vanish on the word.

[GOSW]

Integration by parts. (Anti-commutativity and Arnold identities)

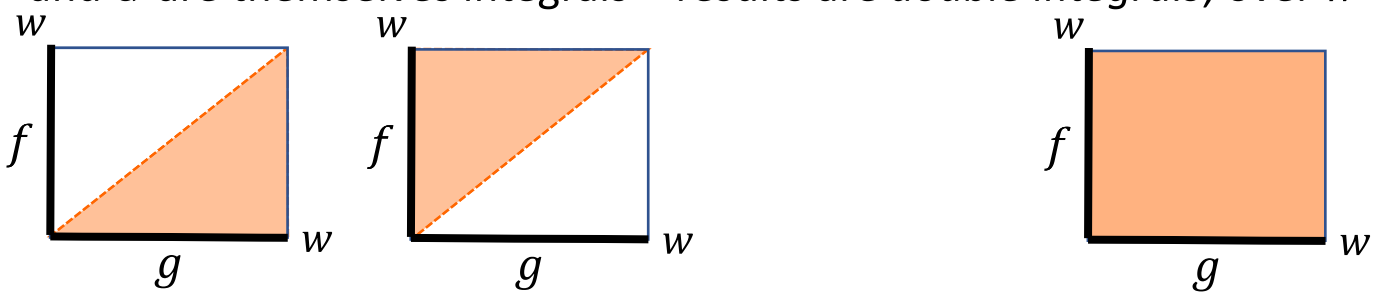
Lemma: Integrals of linking products satisfy

$$\int_w F\,g\,dx + \int_w G\,f\,dx - \int_w f g\,dx = \int_w f\,dx \int_w g\,dx$$

Notation:
The **total intersection** of $f\ dx$ and $g\ dx$ is

$$\int_w f g\,dx$$

Proof (sketch): F and G are themselves integrals – results are *double integrals*, over $w \times w$



On diagonal, $f(i)g(i)$ appears **twice** if $dx_i = 1$ and **not at all** if $dx_i = -1$

On diagonal, all $f(i)g(i)$ appear with sign $dx_i^2 = 1$

If $f\ dx$ and $g\ dx$ cobound and have total intersection 0 then linking product is anti-commutative!

Integration by parts. (Anti-commutativity and Arnold identities)

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Lemma: Integrals of linking products satisfy

$$\int_w FG\,h\,dx + \int_w GH\,f\,dx + \int_w HF\,g\,dx$$

$$-\int_w F\,gh\,dx - \int_w G\,hf\,dx - \int_w H\,fg\,dx + \int_w fgh\,dx$$

$$= \int_w f\,dx \int_w g\,dx$$

If no pairwise total intersection and cobounding then this is the **Arnold Identity**.

Integration by parts. (Anti-commutativity and Arnold identities)

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$$\int_w F\, g\, dx + \int_w G\, f\, dx - \int_w f g\, dx = \int_w f\, dx \int_w g\, dx$$

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The **total intersection** of $f\ dx$ and $g\ dx$ is

$$\int_w fg\ dx$$

If $f\ dx$ and $g\ dx$ cobound and have total intersection 0 then linking product is anti-commutative!

tal

this

ntity.

Proposition: If these forms are **symbols** and at least one of them **cobounds**, then all total intersections vanish!

As functions on words, the following relations hold:

Anticommutativity $\quad (M)N = -(N)M$

Arnold Identity $\quad (L)(M)N + (M)(N)L + (N)(L)M = 0$

Linking of Letters

Switch to camera.

Configuration of Letters.

Switch to camera.

From Symbols to Graphs!

Switch to camera.

The End