Convergence Rules for Improper Integrals

Type I (Integrate to $\pm\infty$)

$\int_1^\infty \frac{1}{x^p}\,dx$ **Divergent** iff $p \le 1$

Examples.

$\int_1^\infty \frac{1}{x^2}\,dx$ **convergent**

$\int_1^\infty \frac{1}{x}\,dx$ **divergent**

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ **divergent**

Type II (Integrate to **asymptote**)

$\int_0^1 \frac{1}{x^p}\,dx$ **Divergent** iff $p \ge 1$

Examples.

$\int_0^1 \frac{1}{x^2}\,dx$ **divergent**

$\int_0^1 \frac{1}{x}\,dx$ **divergent**

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ **convergent**

Comparison Theorem

If $0 \le f \le g$ and

- $\int f\,dx$ is **divergent**, then $\int g\,dx$ is **divergent**
- $\int g\,dx$ is **convergent**, then $\int f\,dx$ is **convergent**

Example. $\int_{10}^\infty \frac{1}{x^3+2x+1}\,dx$

Compare to $\frac{1}{x^3}$.

Both are positive for $x \ge 10$.

Comparing to **bigger** function $\frac{1}{x^3+2x+1} < \frac{1}{x^3}$

Which has $\int_{10}^\infty \frac{1}{x^3}\,dx$ **convergent** $(p = 3 > 1)$.

So $\int_{10}^\infty \frac{1}{x^3+2x+1}\,dx$ is **convergent** also.

Example. $\int_2^5 \frac{x}{(x-2)^3}\,dx$

Compare to $\frac{1}{(x-2)^3}$.

Both are positive for $2 < x < 5$.

Comparing to **smaller** function $\frac{1}{(x-2)^3} < \frac{x}{(x-2)^3}$

Which has $\int_2^5 \frac{1}{(x-2)^3}\,dx$ **divergent** $(p = 3 \ge 1)$.

So $\int_2^5 \frac{x}{(x-2)^3}\,dx$ is **divergent** also.

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1. Decide "Convergent" or "Divergent" for the following Type I Improper Integrals. (DO NOT INTEGRATE)

(a) $\int_8^\infty \frac{5}{\sqrt[4]{x}}\,dx$

(b) $\int_4^\infty \frac{2x+1}{x^2-x-3}\,dx$

(c) $\int_{-\infty}^{-2} \frac{7}{x^3}\,dx$

(d) $\int_{-\infty}^{-7} \frac{2x^2-1}{x^4+1}\,dx$

2. Decide "Convergent" or "Divergent" for the following Type II Improper Integrals. (DO NOT INTEGRATE)

(a) $\displaystyle\int_0^7 \frac{5}{\sqrt[4]{x}}\,dx$

(b) $\displaystyle\int_1^3 \frac{x+5}{(x-1)(x+4)}\,dx$

(c) $\displaystyle\int_{-5}^0 \frac{7}{x^3}\,dx$

(d) $\displaystyle\int_2^{12} \frac{x-1}{(x+1)(x-2)^2}\,dx$