Fundamental Theorem of Calculus

Summary.

The Fundamental Theorem of Calculus says that derivatives and integrals are “inverses”:

$$\frac{d}{dx}\int_a^x f(t)\;dt \;=\; f(x) \qquad\qquad \int_a^x \frac{d}{dt}\,f(t)\;dt \;=\; f(x)-f(a)$$

- Combining the first statement with the chain rule yields a fancier statement:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\;dt \;=\; f\big(b(x)\big)\,b'(x) \;-\; f\big(a(x)\big)\,a'(x)$$

Example: $\displaystyle \frac{d}{dx}\int_{x^3}^{x^7} \sin(t)\;dt \;=\; \sin(x^7)\,7x^6 \;-\; \sin(x^3)\,3x^2$

- The second statement is usually written:

$$\int_a^b f(x)\;dx \;=\; F(x)\,\Big|_a^b \;=\; F\big(b\big) \;-\; F\big(a\big) \qquad \text{for } F(x) \text{ ANY anti-derivative of } f(x)$$

  Integrals missing start/stop values are **indefinite integrals**.

$$\int f(x)\;dx \;=\; F(x) \;+\; C$$

  So far we have one anti-derivative function to work with:

$$f(x) = x^n \qquad \Longrightarrow \qquad F(x) = \frac{1}{n+1}\,x^{n+1}$$

Example: $$\int_3^6 x^{2/5}\;dx \;=\; \frac{5}{7}\,x^{7/5}\,\Big|_3^6 \;=\; \frac{5}{7}\left(6^{7/5}-3^{7/5}\right)$$

Example: $$\int 4x^2-7x+3\;dx \;=\; 4\Big(\frac{1}{3}\,x^3\Big) \;-\; 7\Big(\frac{1}{2}\,x^2\Big) \;+\; 3(x) \;+\; C \;=\; \frac{4}{3}\,x^3 \;-\; \frac{7}{2}\,x^2 \;+\; 3\,x \;+\; C$$

Practice Work.

1. Combine $\int_0^3 f(x)\,dx = 6$ and $\int_2^3 f(x)\,dx = 2$ and $\int_3^4 f(x)\,dx = 3$ to answer the following.

   - $\int_0^2 f(x)\,dx =$

   - $\int_0^4 f(x)\,dx =$

2. Use the Fundamental Theorem of Calculus to compute the following derivatives:

   - If $f(x) = \int_2^x e^{t^2}\,dt$, then $f'(x) =$

   - If $f(x) = \int_2^{x^3} e^{t^2}\,dt$, then $f'(x) =$

   - If $f(x) = \int_{2x}^{x^3} e^{t^2}\,dt$, then $f'(x) =$

3. Evaluate the following definite integrals:

   - $\int_1^3 x^2 + 5x + 1\,dx$

   - $\int_1^3 \sqrt{x}\,dx$

   - $\int_1^3 \sqrt[3]{x^2}\,dx$