

# Homework 03 - Introduction to Groups

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0. Write down the requirements for a set to be a group.
1. Verify the requirements to show that the following are groups.
  - Positive rational numbers  $\mathbb{Q}^{>0} = \{q \in \mathbb{Q} \mid q > 0\}$  with binary operation  $a \diamond b = \frac{ab}{2}$ .
  - Integers  $\mathbb{Z}$  with binary operation  $a \oplus b = a + b + 2$ .
  - Upper triangular matrices of the form  $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$  using matrix multiplication.
2. When  $p$  is prime,  $\mathbb{Z}_p^\times = \{1, 2, \dots, (p-1)\}$  forms a group using multiplication modulo  $p$ . Compute the inverses in  $\mathbb{Z}_5^\times$  of the values 1, 2, 3, 4.
3. When  $n$  is not prime,  $\mathbb{Z}_n^\times$  does **not** form a group using multiplication mod  $n$ . Which group property is not satisfied? Give an example in  $\mathbb{Z}_4^\times$ .
4. Prove that  $aba^{-1}b^{-1} = e$  if and only if  $ab = ba$ .
5. Prove that  $(ab)^2 = a^2b^2$  if and only if  $ab = ba$ .
6. Prove that  $(aba^{-1})^n = ab^na^{-1}$ .