Set Maps and Equivalence

Definition. A **map of sets** $f : A \to B$ is a rule[1] which for every element $a \in A$ assigns an element $f(a) \in B$.
We call A the **source** or **domain** of f and B the **target** or **codomain** of f.
The **image** of f is $f(A) = \{f(a) \mid a \in A\}$.

This is pronounced "f maps A to B". Sometimes we write it as $A \xrightarrow{f} B$.

Definition. A map $f : A \to B$ is an **injection** if it has the following property.
If $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$.

Oftentimes it is more convenient to work with the equivalent contrapositive statement.
If $f(a_1) = f(a_2)$ then $a_1 = a_2$.

Other names for injections are "one-to-one" or "into" maps and "monomorphisms".
People who say "injection" will use hook-arrow notation $f : A \hookrightarrow B$.
People who say "monomorphism" will use tail-arrow notation $f : A \rightarrowtail B$.

Intuition: *Injections "pick up A and set it **inside** of B"*

Definition. A map $f : A \to B$ is an **surjection** if it has the following property.
For each $b \in B$ there is at least one $a \in A$ with $f(a) = b$.

Being a surjection is equivalent to having image equal to codomain $f(A) = B$.

Other names for surjections are "onto" maps and "epimorphisms".
Everyone uses double head arrow notation for surjections (epimorphisms) $f : A \twoheadrightarrow B$.

Intuition: *Surjections "pick up A and use it to **cover over** all of B"*

Definition. A map $f : A \to B$ is a **bijection** if it is both an injection and a surjection.
In this case we say A and B are **isomorphic** sets, written $A \cong B$.

Bijections are sometimes called "isomorphisms". Maps are bijections iff they have (left and right) inverses.
People usually denote bijections using an isomorphism decoration $f : A \xrightarrow{\cong} B$

Definition. Maps are **composable** if the target of one equals the source of another.
$f : A \to B$ and $g : B \to C$ have **composition** $g \circ f : A \xrightarrow{f} B \xrightarrow{g} C$ given by $\big(g \circ f\big)(a) = g\big(f(a)\big)$.

Compositions interact nicely with injections and surjections.

Proposition. *If a composition $g \circ f : A \xrightarrow{f} B \xrightarrow{g} C$ is a injection, then the first map $f : A \to B$ must be an injection.*

Proposition. *If a composition $g \circ f : A \xrightarrow{f} B \xrightarrow{g} C$ is a surjection, then the last map $g : B \to C$ must be a surjection.*

Note: The composition $\{\bullet\} \to \{\bullet, \circ\} \to \{\bullet\}$ is an injection even though the last map is not; and is a surjection even though the first map is not. But if you arrange to avoid this example, then you can make a statement about the "wrong" maps.

[1] We could use a more formal definition via products (defining functions as *graphs* $\{(a, f(a))\}$), but this is fine for us.

Proposition. *If a composition $g \circ f : A \xrightarrow{f} B \xrightarrow{g} C$ is a injection and the first map $f : A \to B$ is a surjection, then the last map $g : B \to C$ must be an injection.*

Proposition. *If a composition $g \circ f : A \xrightarrow{f} B \xrightarrow{g} C$ is a surjection and the last map $g : B \to C$ is an injection, then the first map $f : A \to B$ must be a surjection.*

Injections and surjections satisfy universal properties involving compositions.

Universal Property. Injections distinguish maps into their source $X \rightrightarrows A \xrightarrow{f} B$.
Proposition. *$A \xrightarrow{f} B$ is an injection if and only if it has the following property.*
For every pair of maps $g, h : X \rightrightarrows A$ if $f \circ g = f \circ h$ then $g = h$.

Proof that property implies injection:
Suppose f has the stated property.
If $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$ then let $X = \{\bullet\}$ with $g(\bullet) = a_1$ and $h(\bullet) = a_2$.
Then $f\big(g(\bullet)\big) = f(a_1) = f(a_2) = f\big(h(\bullet)\big)$ so $f \circ g = f \circ h$.
According to the property, this implies $g = h$.
Thus $a_1 = g(\bullet) = h(\bullet) = a_2$. $\square$

Universal Property. Surjections distinguish maps from their target $A \xrightarrow{f} B \rightrightarrows X$.
Proposition. *$A \xrightarrow{f} B$ is an surjection if and only if it has the following property.*
For every pair of maps $g, h : B \rightrightarrows X$ if $g \circ f = h \circ f$ then $g = h$.

Proof that property implies surjection:
Suppose f has the stated property.
Given $b \in B$ use $X = \{\bullet, \circ\}$ with $g(\beta) = \bullet$ and $h(\beta) = \begin{cases} \bullet & \text{if } \beta \neq b \\ \circ & \text{if } \beta = b \end{cases}$
Since $g \neq h$, according to the property $g \circ f \neq h \circ f$.
So there must be $a \in A$ with $g\big(f(a)\big) \neq h\big(f(a)\big)$.
This is only possible if $f(a) = b$, since g and h are identical elsewhere. $\square$

Definition. The **product** of two sets is $A \times B = \{(a, b) \mid a \in A, \text{ and } b \in B\}$.
Products have **projection maps** onto their coordinates. (People use π or p for projections.)

$p_A : A \times B \to A$ by $p_A(a, b) = a$

$p_B : A \times B \to B$ by $p_B(a, b) = b$

Products satisfy a universal property involving projection maps: they are "closest to A and B from the left".
Universal Property. If X is any other set with maps $f_A : X \to A$ and $f_B : X \to B$ then there is a unique map $i : X \to A \times B$ and the maps f_A, f_B factor through i and the projection. Note that $i(x) = \big(f_A(x), f_B(x)\big)$.

$$\begin{tikzcd}
& & A \\
X \arrow[r, dashed, "\exists\, i"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] & A \times B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
& & B
\end{tikzcd}$$

Proposition. *If P is any other set with the property above, then $P \cong A \times B$.*

Definition. An **equivalence relation** on a set X is $\square \sim \square$ satisfying the following three properties.

(Reflexive) $x \sim x$ for all $x \in X$.

(Symmetric) If $x \sim y$, then $y \sim x$.

(Transitive) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition. An **equivalence class** is a subset of all elements of X equivalent to a given element.
$[x] = \{k \in X \mid k \sim x\}$

The transitive property implies that different equivalence classes are disjoint.

Lemma. *If* $[x] \neq [y]$ *then* $[x] \cap [y] = \emptyset$

When a collection of disjoint subsets covers a set, we say that it **partitions** the set.

Definition. The set of unique equivalence classes of an equivalence relation is written $X/\sim = \big\{[x] \bigm| x \in X\big\}$.
This is called the "quotient set X modulo the equivalence".

Note: There is always a surjection $q : X \twoheadrightarrow X/\sim$ given by $q(x) = [x]$. This is called a **quotient map**.