Substitution

Linear Substitution.

These are so simple that it isn't worth explicitly substituting.

- Form: $$\int \boldsymbol{f}(mx+b)\,dx \;=\; \boldsymbol{F}(mx+b)\cdot\frac{1}{m} \;+\; C$$

- Examples: $$\int (3x+5)^{\mathbf{7}}\,dx \;=\; \tfrac{\mathbf{1}}{\mathbf{8}}\,(3x+5)^{\mathbf{8}}\cdot\frac{1}{3} \;+\; C$$
$$\int \mathbf{sin}(5x-2)\,dx \;=\; \mathbf{-\,cos}(5x-2)\cdot\frac{1}{5} \;+\; C$$

Nonlinear Substitution.

These are complicated enough that you should probably include the substitution as a step.

- Form: $$\int \boldsymbol{f}(u)\,u'\,dx \;=\; \int \boldsymbol{f}(u)\,du \;=\; \boldsymbol{F}(u) \;+\; C \qquad\qquad \textit{(Note: } u'\,dx \;=\; du\textit{)}$$

- Example: $$\int x\,(3x^2+5)^{\mathbf{7}}\,dx \underset{\substack{3x^2+5\,=\,u\\ 6x\,dx\,=\,du\\ x\,dx\,=\,\frac{1}{6}\,du}}{=} \int (u)^{\mathbf{7}}\,\tfrac{1}{6}\,du \;=\; \tfrac{\mathbf{1}}{\mathbf{8}}(u)^{\mathbf{8}}\,\tfrac{1}{6} \;+\; C \underset{u\,=\,3x^2+5}{=} \tfrac{1}{48}(3x^2+5)^{\mathbf{8}} \;+\; C$$

More complicated problems may be easier to substitute via two steps as follows.

$$\int x^3\,(x^2+1)^{\mathbf{7}}\,dx \underset{\substack{x^2+1\,=\,u\\ 2x\,dx\,=\,du\\ x\,dx\,=\,\frac{1}{2}du}}{=} \int x^2\,(u)^{\mathbf{7}}\,\tfrac{1}{2}\,du \underset{\substack{x^2+1\,=\,u\\ x^2\,=\,u-1}}{=} \int (u-1)\,(u)^{\mathbf{7}}\;\tfrac{1}{2}\,du \;=\; \tfrac{1}{2}\int u^8-u^7\,du \;=\; \cdots$$

Definite Integral Substitution.

For definite integrals you should also substitute into the limits!
To be safe, it is best to write "$x =$" and "$u =$" in your limits while substituting.

$$\int_{\mathbf{x=0}}^{\mathbf{x=1}} \frac{x^3}{4-x^2}\,dx \underset{\substack{4-x^2\,=\,u\\ -2x\,dx\,=\,du\\ x\,dx\,=\,-\frac{1}{2}\,du}}{=} \int_{\mathbf{u=4-(0)^2}}^{\mathbf{u=4-(1)^2}} \frac{x^2}{u}\left(-\tfrac{1}{2}\right)du \underset{\substack{4-x^2\,=\,u\\ x^2\,=\,4-u}}{=} -\tfrac{1}{2}\int_{\mathbf{u=4}}^{\mathbf{u=3}} \frac{4-u}{u}\,du \;=\; -\tfrac{1}{2}\int_4^3 \frac{4}{u}-1\,du \;=\; \cdots$$

As a general rule, you should **NEVER, EVER** solve problems with some bits in x and some bits in u. If you do this regularly, then you will eventually make a terrible mistake. And it will cause trouble in multivariable integration (math 342).

Indefinite Integral Practice.

1. $\displaystyle\int \sqrt{x}\, dx$

$\displaystyle\int \sqrt{2x+3}\, dx$

$\displaystyle\int x\sqrt{x^2+3}\, dx$

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2. $\displaystyle\int \frac{1}{x}\, dx$

$\displaystyle\int \frac{1}{2x+3}\, dx$

$\displaystyle\int \frac{x}{x^2+3}\, dx$

Definite Integral Practice.

3. $\displaystyle\int_0^2 e^x\, dx$

$\displaystyle\int_0^2 e^{2x+3}\, dx$

$\displaystyle\int_0^2 x\, e^{x^2+3}\, dx$