Homework 1

CS314

1 Problem — Three simple rewrite systems

Remember our “rewrite game” in the second lecture. We represent arithmetic values > 0 as sequences of “|” symbols. For example, `|` represents value 1, and `|||||` represents value 5. The input to your rewrite system is either a single value representation, or two value representations surrounded by a begin ($) and end (#) marker, and separated by a & marker. For example, the single input value 3 is represented by $|||#, and the input pair 2,5 is represented by $||&|||||#. The normal forms produced by the rewrite systems do not contain any markers. Give rules of rewrite systems that implement different arithmetic operations on our chosen representation. A rewrite system consists of a set of rewrite rules of the form X ⇒ Y as discussed in class. You do not have to worry about incorrect input.

1. **Successor function**: $f(x) = x + 1, x > 0$
   Example: `$|||#` will be rewritten to `||||`
   Show the rewrite sequence of your rewrite system for the example input.

Rule 1: `$|` $\Rightarrow$ `|$`
Rule 2: `$#` $\Rightarrow$ `|`

`$|||#` $\Rightarrow_1$ `|$||#`
`|$||#` $\Rightarrow_1$ `||$|#`
`||$|#` $\Rightarrow_1$ `|||$#`
`|||$#` $\Rightarrow_2$ `||||`

2. **Triple function**: $f(x) = 3 * x, x > 0$
   Example: `$|||#` will be rewritten to `|||||||||` Show the rewrite sequence of your rewrite
   system for the example input.

Rule 1: `$|` $\Rightarrow$ `|||$`
Rule 2: `$#` $\Rightarrow \epsilon$

`$|||#` $\Rightarrow_1$ `|||$||#`
`|||$||#` $\Rightarrow_1$ `||||||$|#`
`||||||$|#` $\Rightarrow_1$ `|||||||||$#`
`|||||||||$#` $\Rightarrow_2$ `|||||||||`

3. **Subtraction function**: $f(x, y) = x - y$, $x > 0$, $y > 0$, and $x > y$
   Example: `$|||&||#` will be rewritten to `|` Show the rewrite sequence of your rewrite
   system for the example input.

Rule 1: `$|` $\Rightarrow$ `|`
Rule 2: `|&|` $\Rightarrow$ `&`
Rule 3: `&#` $\Rightarrow$ `|`

`$|||&||#` $\Rightarrow_1$ `|||&||#`
`|||&||#` $\Rightarrow_2$ `||&|#`
`||&|#` $\Rightarrow_2$ `|&#`
`|&#` $\Rightarrow_3$ `|`

2 Problem — A rewrite system for modulo 3 addition

An interpreter for a language L maps programs written in L to their answers. Remember that a language is a set of words. Let us define our language $L_{add-mod3}$ inductively as follows:

1. The words 0, 1, and 2 are in $L_{add-mod3}$.
2. Assume that both A and B stand for words in the language $L_{add-mod3}$. Then
   (a) $(A + B)$ are also in $Ladd - mod3$.

Examples of add-mod3 expressions are: $((1 + 2) + 0)$ and $(1 + (2 + 2))$. However, $1 + 1$ is not in the language (parenthesis are missing).

Give a rewrite system that "evaluates" or "computes" the value of expressions in $L_{add-mod3}$. The operators $+$ corresponds to the standard modulo 3 addition functions given below:

x	y	x+mod3_y
0	0	0
0	1	1
1	0	1
0	2	2
2	0	2
1	1	2
1	2	0
2	1	0
2	2	1

1. Define a rewrite system for modulo 3 expressions in $L_{add-mod3}$ that produces the final value of the expression. A final value is represented by either 0, 1 or 2. Your rewrite system is basically an interpeter for $L_{add-mod3}$. For example, our two expressions $((1+2)+0)$ and $(1+(2+2))$ should be rewritten to 0 and 2, respectively. You can assume that your rewrite system will only be presented with correct $L_{add-mod3}$ expressions, so don't worry about error messages.

Rule 1: `(0+0)` $\Rightarrow$ `0`
Rule 2: `(0+1)` $\Rightarrow$ `1`
Rule 3: `(1+0)` $\Rightarrow$ `1`
Rule 4: `(0+2)` $\Rightarrow$ `2`
Rule 5: `(2+0)` $\Rightarrow$ `2`
Rule 6: `(1+1)` $\Rightarrow$ `2`
Rule 7: `(1+2)` $\Rightarrow$ `0`
Rule 8: `(2+1)` $\Rightarrow$ `0`
Rule 9: `(2+2)` $\Rightarrow$ `1`

2. Show your rewrite system steps that are performed for our two example expressions given above. For each step clearly show the left-hand side of the rule in the current expression that you are rewriting.

Example 1: $((1+2)+0)=0$
$((1+2)+0)$, Rule 7 $(1+2) \Rightarrow 0$
$(0+0)$, Rule 1 $(0+0) \Rightarrow 0$

Example 2: $(1+(2+2))=2$
$(1+(2+2))$, Rule 9 $(2+2) \Rightarrow 1$
$(1+1)$, Rule 6 $(1+1) \Rightarrow 2$

3. Is the choice of your next rewrite rule and its left-hand side always unique in your rewrite system? If not, show an example.

No, it is not.
Example: $((1+2)+(1+1))$

We may use rewrite rules 6 or 7 to get the same output.
Rule 7: $((1+2)+(1+1)) \Rightarrow (0+(1+1))$
Rule 6: $(0+(1+1)) \Rightarrow (0+2)$
Rule 7: $(0+2) \Rightarrow 2$

or, we can do
Rule 6: $((1+2)+(1+1)) \Rightarrow ((1+2)+2)$
Rule 7: $((1+2)+2) \Rightarrow (0+2)$
Rule 7: $(0+2) \Rightarrow 2$

3 Problem — Regular expressions

Describe the formal languages denoted by the following regular expressions using the English language

1. $((\epsilon|1)0^*)^*$
   All strings of "1"s and "0"s
2. $0(0|1)^*1(0|1)1$
   All strings of "1"s and "0"s that have the prefix 0 and the suffix 101 or 111.

4 Problem — Regular expressions

Write a regular expression for the following language. **You must use the regular language definition introduced in class (see lecture 3)** Make the expression as compact as possible.

For the following solutions, take $A?$ to be $(A \mid \epsilon)$, where A is an arbitrary regular expression. This rule has a similar amount of priority as the $+$ and $*$ rules. (**This rule will not be used in the solutions, it is only shorthand for me to use in my work.**)

1. All strings of "a"s, "b"s, and "c"s that contain exactly 2 "a"s
   Solution: `(b|c)*a(b|c)*a(b|c)*`

2. All strings of "a"s, "b"s, and "c"s that contain at least 1 "b" or at least 3 "c"s
   `[abc]*b([abc]*c){3,}` $\Rightarrow$ A string of "a"s, "b"s, "c"s that contains at least 1 "b" and at least 3 "c"s to the right of the b.
   `(a|b|c)*b(a|b|c)*c(a|b|c)*c(a|b|c)*c(a|b|c)*`

For these notes, I use the standard Regular Expression language, where {n,} means n or more occurrences. This is not used in the solution.

`([abc]*c){3,}[abc]*b[abc]*` $\Rightarrow$ A string of "a"s, "b"s, "c"s that contains at least 1 "b" and at least 3 "c"s to the left of the b.
`(a|b|c)*c(a|b|c)*c(a|b|c)*c(a|b|c)*b(a|b|c)*`

`([abc]*c){2,}[abc]*b([abc]*c)+[abc]*` $\Rightarrow$ A string of "a"s, "b"s, "c"s that contains at least 1 "b" and at least 2 "c"s to the left of the b and 1 "c" to the right.
`(a|b|c)*c(a|b|c)*c(a|b|c)*b(a|b|c)*c(a|b|c)*`

`([abc]*c)+[abc]*b([abc]*c){2,}[abc]*` $\Rightarrow$ A string of "a"s, "b"s, "c"s that contains at least 1 "b" and at least 1 "c" to the left of the b and 2 "c"s to the right.

Solution: `(a|b|c)*((b(a|b|c)*c(a|b|c)*c(a|b|c)*c)|(c(a|b|c)*c(a|b|c)*c(a|b|c)*b)|(c(a|b|c)*c(a|b|c)*b(a|b|c)*c)|(c(a|b|c)*b(a|b|c)*c(a|b|c)*c))(a|b|c)*`

3. All strings of "a"s, "b"s, and "c"s that do not contain more than 1 "b" and no more than 3 "c"s

`a*c?a*c?a*c?a*` ⇒ All strings of "a"s and "c"s that contain 3 or less "c"s
`a*b?a*c?a*c?a*c?a*` ⇒ All strings of "a"s, "b"s, and "c"s that contain at most 1 "b" and at most 3 "c"s to the right
`a*c?a*b?a*a*c?a*c?a*` ⇒ All strings of "a"s, "b"s, "c"s that contain at most 1 "b" with at most 1 "c" to the left and 2 to the right
`a*c?a*c?a*b?a*c?a*` ⇒ All strings of "a"s, "b"s, and "c"s that contain at most 1 "b" with at most 2 "c"s to the left and 1 to the right
`a*c?a*c?a*c?a*b?a*` ⇒ All strings of "a"s, "b"s, and "c"s that contain at most one "b" with at most 3 "c" to the right

Combined: `a*((c?a*c?a*c?)|(c?a*c?a*c?a*b?)|(c?a*c?a*b?a*c?)|(c?a*b?a*c?a*c?)|(b?a*c?a*c?a*c?))a*`

However, this regular expression is far from minimal and therefore we must reduce it.

Solution: `a*((c|ε)a*((c|ε)a*((c|ε)a*((a*(b|ε))|ε)|(b|ε)a*c)|(b|ε)a*(c|ε)a*(c|ε))|(b|ε)a*(c|ε)a*(c|ε)a*(c|ε))a*`

5 Problem — Regular expressions and finite state machines

You are designing a new language with fixed-point numbers. Every fixed-point number should have a unique representation. This means, no leading or trailing 0's are allowed, and every number must have a "point". Examples:

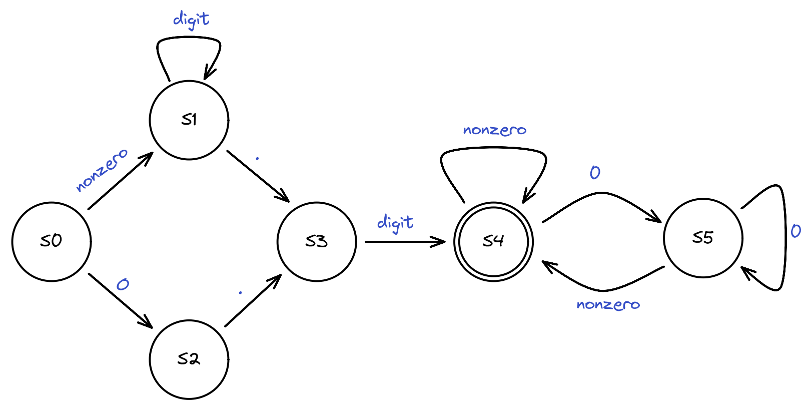
- Allowed: $0.0, 10.0, 45000.007, 0.888$
- Not allowed: $0, 10, 10., 10.00, 045000.007, .888$

1. Write a regular expression for fixed-point numbers for your language.

Let $digit$ be `(0|1|2|3|4|5|6|7|8|9)` (digits)
Let $nonzero$ be `(1|2|3|4|5|6|7|8|9)` (non-zero digits)
Let $ellipse$ be the character `.` for the sake of clarity (specifically for the DFA)

$(0 \mid (nonzero)(digit)^*).\,((digit)^*(nonzero)|0)$

2. Give a DFA in the form of a state transition graph that recognizes your language. Note: No need to introduce error states; your DFA can reject the input if it gets "stuck". Keep your DFA as small as possible.



6 Problem — Regular expressions and finite state machines

Use the discussed "translation" strategy for constructing an ϵ-NFA from a regular expression as discussed in lecture 3 for the regular expression.

```
letter ( letter | digit )*
```

Show the ϵ-NFA for the above regular expression.

where $letter$ is `a|b|c|d|e|f|` ... `w|x|y|z`, a regular expression containing all the letters in the english alphabet
and $digit$ is `(0|1|2|3|4|5|6|7|8|9)`