Phase-shift time stepping for wavefield propagation

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SUMMARY

We consider time-marching algorithms that numerically solve the acoustic wave equation. Unlike finite-difference solvers, the methods described have reduced numerical dispersion and can obtain larger timesteps and coarser grid spacings. We first describe methods that combines analytic solutions for constant velocity with discontinuous partitions of unity. We then compare these time-marching solutions qualitatively to Lax-Wendroff pseudospectral methods.

INTRODUCTION

Reverse-time migration (RTM) (Baysal et al., 1983; McMechan, 1983) and forward modeling by differencing the two-way acoustic wave equation are computationally expensive. However with an accurate velocity model they are very effective methods for migration and modeling. We consider a number of wavefield propagators to solve the acoustic wave equation in the wavenumber domain.

One-way in time marching algorithms (Zhang et al., 2005; Zhang and Zhang, 2009) have been used for wavefield propagation in RTM. They have an advantage over two-way marching algorithms in that they have no CFL-type timestep limitation. They incorrectly propagate amplitudes. A similar two-way in time marching equation (Etgen, 1989; Dablain, 1986) correctly propagates kinetics and amplitudes and various methods have been used to calculate it (Tal-Ezer, 1986; Soubaras and Zhang, 2008; Wards et al., 2008; Etgen and Brandsberg-Dahl, 2009). Solving the wave equation as an evolution equation (Chen, 2006) allows a much larger timestep to be used. However each timestep is more costly to compute because the wavefield and its derivative are required. We consider a number of methods to solve the wave equation using constant velocity extrapolators and a partition of unity.

PSEUDOSPECTRAL AND PHASE SHIFT TIME STEPPING METHODS

Pseudospectral methods are numerically efficient methods to solve the full two-way acoustic wave equation. They compute the spacial Laplacian exactly by using a Fourier transform and as a result they allow larger spatial sampling rates. However, they use a finite difference approximation for the time derivative and so are still dispersive. Additionally, as compared to finite differencing, the dispersive errors caused by calculating the spacial derivatives cannot be canceled with errors due to calculating the time derivative.

The symbols $\mathscr{F}_{\vec{x}\to\vec{k}}$, and $\mathscr{F}^{-1}_{\vec{k}\to\vec{x}}$ are used to denote the forward-like and inverse Fourier-like transforms with respect the spacial and wavenumber variables, respectively. The acoustic constant-density variable-velocity wave equation is

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = v^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}\right) \\ U(0,\vec{x}) = U^0, U(-\delta t,\vec{x}) = U^1 \end{cases}, \tag{1}$$

where $U(t,\vec{x})$ is the amplitude of the wave at the point $(t,U)\in(\mathbb{R}^+,\mathbb{R}^2)$, x is the lateral coordinate, z is the depth coordinate, t is the time coordinate. The pseudospectral time-marching algorithm is

$$U^{n+1} = 2U^n - U^{n-1} - \delta t^2 v^2 \mathscr{F}^{-1}_{\vec{k}\to\vec{x}}\left[(2\pi|\vec{k}|)^2\mathscr{F}_{\vec{x}\to\vec{k}}[U]\right], \tag{2}$$

where the superscripts n refers to the approximation at time $n\delta t$. Alternatively, the modified equation approach (Cohen, 2001) can be used to calculate the time derivative more precisely. Substituting the Taylor series expansion of the second-order time derivative into the scalar wave equation and using the approximation $\partial^{2n}U/\partial t^{2n}\simeq v^{2n}\Delta^n U$ gives formally an infinite order in time solution (Etgen, 1989; Dablain, 1986),

$$U^{n+1} = -U^{n-1} + 2\sum_{m=0}^{\infty}\frac{(\delta t v)^{2m}}{(2m)!}\left(\Delta^m U\right)^n \tag{3}$$

where ΔU refers to the Laplacian of the function U and $\Delta^2 U$, for example, is the biharmonic or the Laplacian applied twice to U. Taking the Fourier transform of both sides of equation (3) with respect to the spatial coordinates gives

$$\begin{aligned} \widehat{U}^{n+1} &= -\widehat{U}^{n-1} + 2\sum_{m=0}^{\infty}\frac{(\delta t v)^{2m}}{(2m)!}\left((-2\pi|\vec{k}|)^{2m}\widehat{U}\right)^n \\ &= -\widehat{U}^{n-1} + 2\cos\left(2\pi v|k|\delta t\right)\widehat{U}^n. \end{aligned} \tag{4}$$

Equation (4) is a dispersion-free method for solving the acoustic wave equation, and for constant velocity is an exact solution. However, it is expensive because a fast Fourier transform cannot be used to calculated the inverse Fourier-like transform. Taking a partial sum of equation (3) or equivalently taking the Taylor series expansion of equation (4) gives the higher-order in time (Lax-Wendroff) pseudospectral methods. The power series expansion about the velocity v_0 with the variation $\delta v = v(x) - v_0$ for the function $\cos(2\pi v(\vec{x})|\vec{k}|\Delta t)$ is

$$\begin{aligned} \cos(2\pi v(\vec{x})|\vec{k}|\Delta t) &= \cos(2\pi v_{ref}|\vec{k}|\delta t) \\ &- \sin(2\pi v_{ref}|\vec{k}|\delta t)\delta v(\vec{x})2\pi|\vec{k}|\Delta t \\ &- \frac{1}{2}\cos(2\pi v_{ref}|\vec{k}|\delta t)\left[\delta v(\vec{x})2\pi|\vec{k}|\Delta t\right]^2 \\ &+ \ldots \end{aligned} \tag{5}$$

Substituting the Taylor series expansion (5) into equation (4) gives the second-order split-step correction. For large velocity variations δv, it is necessary to take a higher order split-step correction or smaller timestep to ensure stability. The timestep must satisfy $\delta t < \delta x/\sqrt{2}V_{max}$ due to aliasing considerations.

Alternatively, a discontinuous partition of unity can used to approximate equation (4). Suppose that a set of reference velocities have been chosen so that $v_1 < \ldots < v_j < \ldots < v_J$. Suppose $\sum_j \Omega_j(\vec{x}) v_j^2 = v^2(\vec{x})$ and $\sum_j \Omega_j(\vec{x}) = 1$ with $\Omega_j(\vec{x}) = 0$ if $v(x) > v_{j+1}$ or $v(x) < v_{j-1}$. The Ω_j's are windowing functions that correspond will corresponding to propagation with a constant velocity. Then (Etgen and Brandsberg-Dahl, 2009),

$$\Omega_j(\vec{x}) = \begin{cases} \frac{v^2(\vec{x}) - v_{j-1}^2}{v_j^2 - v_{j-1}^2} & v_{j-1} < v(x) < v_j \\ \frac{v_{j+1}^2 - v^2(\vec{x})}{v_{j+1}^2 - v_j^2} \quad \text{if} & v_j < v(x) < v_{j+1} \\ 0 & otherwise \end{cases} . \quad (6)$$

To approximate equation (4), we propagate the wavefield with a collection of constant velocities and then recombine the result with the partition of unity,

$$U^{n+1} = \sum_{j=1}^{J} \Omega_j(\vec{x}) \mathscr{F}^{-1}_{\vec{k}\to\vec{x}} \left[2\cos(2\pi v_j \delta t |\vec{k}|) \mathscr{F}_{\vec{x}\to\vec{k}}[U^n] \right] - U^{n-1}. \quad (7)$$

Expanding the cosines in equation (7) and (4) with the power series expansion about $v = 0$ shows that equation (7) is at least as accurate as a second-order pseudospectral method.

ONE-WAY IN TIME WAVE EQUATIONS

The one-way in time wave equation,

$$\partial U / \partial t = v\sqrt{-\Delta} U \quad (8)$$

subject to the initial condition $U(0) = U^0$ is a Fourier integral operator. It has the approximate solution

$$U^{n+1} = \mathscr{F}^{-1}_{\vec{k}\to\vec{x}} [\exp(2\pi i v(\vec{x}) \delta t |\vec{k}|) \mathscr{F}_{\vec{x}\to\vec{k}}[U^n]]. \quad (9)$$

Expanding the exponential with a Taylor series expansion at zero or any reference velocity provides a numerical implementation. Alternatively, a discontinuous partition of unity can used to approximate equation (9) similar to what was done to derive equation (7). In this case it is more appropriate to choose $\sum_j \Omega_j(\vec{x}) v_j = v(\vec{x})$ and so,

$$\Omega_j(\vec{x}) = \begin{cases} \frac{v(\vec{x}) - v_{j-1}}{v_j - v_{j-1}} & v_{j-1} < v(x) < v_j \\ \frac{v_{+1} - v(\vec{x})}{v_{j+1} - v_i} \quad \text{if} & v_j < v(x) < v_{j+1} \\ 0 & otherwise \end{cases} . \quad (10)$$

To approximate equation (9), we propagate the wavefield with a collection of constant velocities and recombine the result with the POU,

$$U^{n+1} = \sum_{j=1}^{J} \Omega_j(\vec{x}) \mathscr{F}^{-1}_{\vec{k}\to\vec{x}} [\exp(2\pi v_j \delta t |\vec{k}|) \mathscr{F}_{\vec{x}\to\vec{k}}[U^n]]. \quad (11)$$

Expanding the exponential functions in equation (11) and (9) with a second-order Taylor series shows that equation (11) is as accurate as second-order pseudospectral method. However it does show significantly less dispersion.

EVOLUTIONARY WAVE EQUATIONS

Transforming the acoustic wave equation into a system of first-order linear partial differential equaions (Chen, 2006; Tal-Ezer, 1986) gives

$$\frac{\partial}{\partial t} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ v^2(x,z)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}, \quad (12)$$

subject to the initial condition

$$\begin{bmatrix} U(0,\vec{x}) \\ V(0,\vec{x}) \end{bmatrix} = \begin{bmatrix} U^0(\vec{x}) \\ V^0(\vec{x}) \end{bmatrix}., \quad (13)$$

where $V(t) = \frac{\partial U}{\partial t}(t)$. The solution to equation (12) for constant velocity is,

$$\begin{bmatrix} \hat{U}(t+\Delta t,\vec{k}) \\ \hat{V}(t+\Delta t,\vec{k}) \end{bmatrix} = K(\Delta t, \vec{k}, v) \begin{bmatrix} \hat{U}(0,\vec{k}) \\ \hat{V}(0,\vec{k}) \end{bmatrix}, \quad (14)$$

where

$$K(\Delta t, \vec{k}, v) = \begin{bmatrix} \cos(2\pi|\vec{k}|\Delta t v) & \frac{\sin(2\pi|\vec{k}|\Delta t v)\Delta t)}{(2\pi|\vec{k}|v)} \\ -2\pi|\vec{k}|v\sin(2\pi|\vec{k}|\Delta t v) & \cos(2\pi|\vec{k}|\Delta t v) \end{bmatrix}. \quad (15)$$

Replacing the constant velocity v in equation (14) with a variable velocity gives an approximate solution for a small timestep. A further approximate timestepper for the wave equation is

$$\begin{bmatrix} U^{n+1} \\ V^{n+1} \end{bmatrix} = \left[\sum_{j=1}^{J} \Omega_j \mathscr{F}^{-1}_{\vec{k}\to\vec{x}} K(t, |\vec{k}|, v_j) \mathscr{F}_{\vec{x}\to\vec{k}} \begin{bmatrix} U^n \\ V^n \end{bmatrix} \right]. \quad (16)$$

This solution does not have an aliasing limit but the approximation for variable velocity does suffer errors if the timestep is taken too large or their are not enough reference velocities. Alternatively, taking a Taylor series approximation of sines and cosines in equation (15) derives a higher order pseudospectral method for the evolution equation. This solution has a much more restrictive condition on the timestep because of the numerical error introduced in the Taylor series expansion.

NUMERICAL EXAMPLES

We compare a number of methods for wavefield propagation by looking at some snapshots of a forward propagated zero phase wavelet. The wavelet is injected at the center of a salt-dome model which is a portion of the BP dataset in Figure 3(a). The wavelet is bandlimit from $5Hz$ to $50Hz$. All of the comparisons were done on the same model with a grid spacing of $12.5m$. The BP data set contains a rugose saltdome embedded in a background sediment whose velocity smoothly increases with depth. Figure 1(a), and (b) are snapshots propagated using second-order, and fourth-order pseudospectral method derived in equation (2), respectively. The lower-order method is computationally efficient but contains some dispersion while their is little dispersion in the higher order implementation. Figure 1(c) is the first order split-step correction derived in equation (5). The snapshot is not dispersive but their are large kinematic errors due to the low order of the approximation.

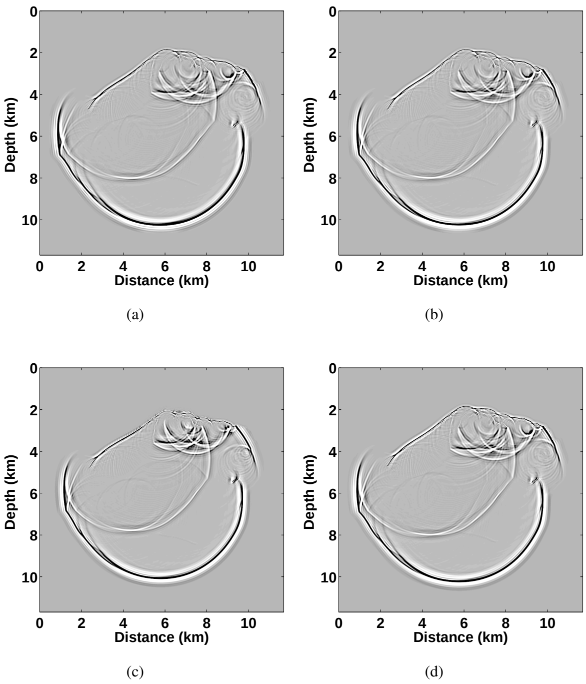


Figure 1: (a) A snapshot of a wavefield propagated with second-order pseudo-spectral method using equation (7). (b) A snapshot of a wavefield propagated with forth-order pseudo-spectral method using equation (7). (c) A snapshot of a wavefield propagated with first-order split-step correction with one correction term in equation (5). (d) A snapshot of a wavefield propagated with second-order split-step correction with two correction terms in equation (5).

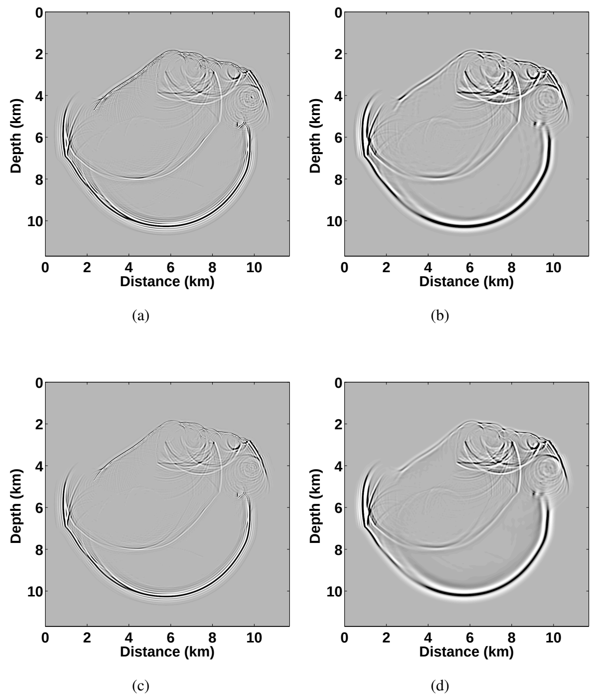
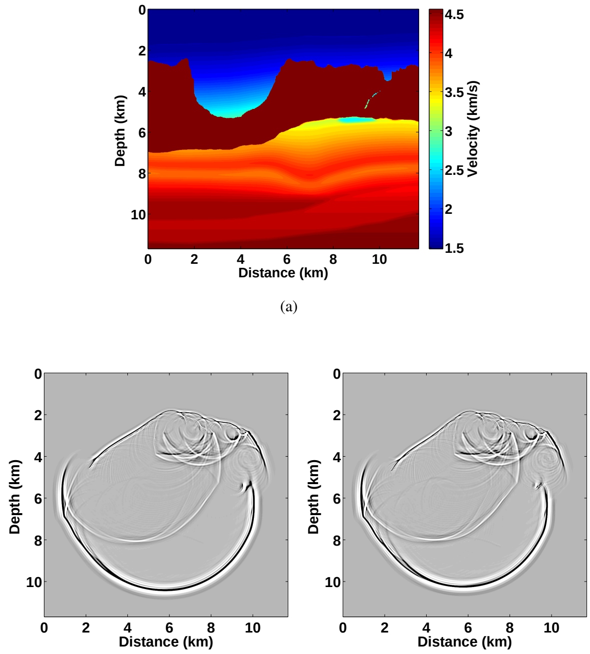


Figure 2: (a) A snapshot of a wavefield propagated with a one-way in time pseudospectral method. Five terms in the power series expansion of equation (9) with respect to $v = 0$ were used. The timestep size is 5ms. (b) A snapshot of a wavefield propagated with a one-way in time wave equation approximated with a POU with three windows with a timestep of 9ms derived in equation (11). (c) A snapshot of a wavefield propagated with a second order pseudospectral method applied to equation (14). (d) A three window POU applied to the evolutionary equation with a timestep of 9ms using equation (16).



(a) (b) (c)

Figure 3: (a) A section of the BP data set showing the high velocity salt dome. (b) A snapshot of a wavefield propagated with equation (7) using a POU with two windows. (c) Similar to (b) but using a 4 window POU.

Figure	Time (s)	Timestep (ms)	Number of FFTs FFTs
1(a)	90	1.2	2
1(b)	113	1.5	3
1(c)	103	1.5	3
1(d)	130	1.5	4
2(a)	70	5	6
2(b)	38	9	4
2(c)	215	1.5	6
2(d)	88	9	8
3(b)	104	1.9	3
3(c)	188	1.9	5

Table 1: Computation time and timestep size used to make Figure 1, 2, and 3. The grid size was kept constant at 12.5m.

Figure 1(d) is the second order split-step correction. The kinematics are much better than in Figure 1(c).

Figure 2(a) is the snapshot propagated with a one-way time algorithm using a Law-Wendroff pseudospectral implementation of equation (9). The amplitudes are incorrect. Figure 2(b) is the snapshot propagated with a one-way time algorithm using equation (11) and has much better amplitudes. Figure 2(c) is the snapshot using a Law-Wendroff pseudospectral implementation of equation (14). The amplitudes are incorrect. Figure 2(d) is a snapshot of the wavefield using equation (16). The amplitudes are much better and because their is less numerical dispersion a much larger timestep was used.

Figure 3(b) is a snapshot of the wavefield propagated with equation (7) using 2 reference velocities. It has velocities errors when compared to the snapshot in Figure 3 computed using 4 reference velocities. Both have little dispersion. Table 1 contains the computational times and timestep sizes used to calculate the the snapshots in the figures. The one-way in time algorithm was the fastest because it can be computed with a large timestep.

Due to their slow convergence, power series expansions of the one-way in time solution and the evolutionary wave equation solution used to derive Lax-Wendroff pseudospectral methods are inaccurate. A partition of unity and propagating with a constant velocity provide less dispersive solutions. The one-way equation had large amplitude errors. Using a partition of unity will provide a solution that has little dispersion but enough reference velocities must be chosen to minimize velocity errors. A power series expansion of the cosine equation is more accurate because of better convergence properties.

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