Converted wave RTM in anisotropic media

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ABSTRACT

Reverse-time migration(RTM) is a powerful migration method provided that an accurate velocity model can be constructed. Converted wave data, pressure wave energy that has converted into shear wave energy upon reflection, can be used to aid interpretation of pressure wave data. We show how a converted wave RTM can be used to image a simple ramp model, which has a homogenous tilted transverse anisotropic(TTI) overburden. We simplify the propagation of the wavefield necessary for migration by propagating each mode of the elastic wavefield separately by a pseudo-acoustic wave equation. This involves solving the Christoffel equations for the elastic wave equation and creating a pseudo-acoustic wave equation from the derived dispersion relation. Our solutions of the pseudo-acoustics wave equations are based on combining exact dispersion free homogenous solutions.

INTRODUCTION

Multi-component seismic data is a record of all the components of the seismic wavefield. However more expensive and sophisticated processing flows are required to leverage this data for improved hydrocarbon detection. The speed of propagation in anisotropic media is sensitive to the direction of propagation. The shear wave competent is more sensitive to anisotropy than the pressure waves. Tilted Transverse anisotropy (TTI) is a model of anisotropy that reasonably approximates many anisotropy rocks found in the ground. The mathematical simplification of the equations motion provide tractable mathematics for numerical computation while provide a reasonable accurate model of the physics. Converted waves are a pressure wave or shear wave that converts upon reflection into the opposite kind of body wave. TTI Reverse time migration is an accurate method of imaging record seismic data provided an accurate velocity model can be constructed. Converted wave pseudo-acoustic reverse time migration images waves that convert at most once. Normally the source is assumed to be a pressure wave while the record wave field is processed for the shear wave component. The literature contains many examples using converted waves to image the earths subsurface.

Many researches have published results detailing two-way modeling and migration using pseudo-acoustic wavefield propagators. Alkhalifah (2000) derived a pseudo-acoustic wave equation from the coupled dispersion relation for pressure and vertical shear waves. However, it propagates s-wave artifacts and solves a fourth-order wave equation. **?** explained that the shear arises from setting the vertical shear wave to zero. However the horizontal shear wave velocity is non zero as a result of the anisotropy. The shear wave artifact gives rise to instabilities and artifacts. There have been many suggestions to stabilize and reduce the shear wave artifacts. For example, if the vertical shear wave is not set to zero but instead is finite than then instabilities arising from a variable tilt angle can be eliminated. However, this increasing the size of the shear wave artifacts. To reduce the shear wave artifacts the shear wave velocity can be solved for as to reduce refections from P into Sv energy (**?**). **?** suggestions a number of difference waves to use the fourth order dispersion relation to derive a variety of different coupled partial differential equations (PDEs). When the combination of Thomsen parameters (**?**) $\epsilon - \delta$ are either positive or negative a different PDE is necessary. To avoid complications with shear wave artifacts and to avoid solving a couple system of PDEs **?** suggested the pseudo-analytic method that instead of solving a coupled system of PDEs solves a second-order PDE but with spatial derivatives defined by pseudo-differential operators. **?** method interpolates between exact solutions however it has a difficult time handling variable tilt angle of the anisotropy.

DISPERSION RELATIONS FOR P AND S WAVES IN ANISOTROPIC

Waves in the earth can be modeled using the elastic wave equation. This assume the earth is a lossless linear elastic solid where inertia terms are ignored. The elastic stiffness matrix C_{ijlk} contains 21-independent quantities. We may further simplify the physics by assuming that the elastic stiffness matrix has TTI symmetry. This reduces the number of independent quantizes to 7, 2 orientation angles and 5 elastic constants.

Pseudo-acoustic wave equations are simplifications of the elastic wave equation that only propagates a single mode without coupling effects of the different body wave modes. Instead each mode propagates with the same phase velocity as that mode would have as in determined by the depression relation derived from the elastic wave equation.

The equation of motion for lossless linear elastic wave equation is (aki,richards)

$$\rho \frac{\partial^2 U}{\partial t^2} = c_{ijkl} U_{k,lj}, \tag{1}$$

where $U(\vec{x})$ is the paritical displacement and c_{ijkl} is the elastic stiffness tensor. The Einstein index summation convention is used throughout where repeated indices are sum over.

Substituting a plane wave of the form

$$U_k = d_k \exp(i\omega(n_i x_i / v - t)), \tag{2}$$

where d_k is the polarization direction, ω is the frequency, n_i is the unit outward normal gives,

$$\left(c_{ijkl} n_j n_l - \rho v^2 \delta_{ik}\right) = 0. \tag{3}$$

Let $\Gamma_{kl} = c_{ijkl} n_j n_l$ then equation **??** is

$$\left(\Gamma_{kl} - \rho v^2 \delta_{ik}\right) = 0. \tag{4}$$

This is an eigenvalue problem and since Γ_{kl} is symmetric the eigenvalues ρv^2 are positive and real. The eigenvectors correspond to the directions of displacement of the waves.

For TTI media the elastic constants matrix can be rewritten as In the case of vertical tranverse anisotropy(VTI) medium the eigenvalues or velocities in Thomsen's anisotropic parameters are

$$V_P(\theta)^2 = V_{P_0}^2 \left[1 + \epsilon \sin^2\theta - \frac{f}{2} + \frac{f}{2}\sqrt{1 + \frac{4\sin^2\theta}{f}(2\delta\cos^2\theta - \epsilon\cos 2\theta) + \frac{4\epsilon^2\sin^4\theta}{f^2}}\right] \tag{5}$$

$$V_{SV}(\theta)^2 = V_{P_0}^2 \left[1 + \epsilon \sin^2\theta - \frac{f}{2} - \frac{f}{2}\sqrt{1 + \frac{4\sin^2\theta}{f}(2\delta\cos^2\theta - \epsilon\cos 2\theta) + \frac{4\epsilon^2\sin^4\theta}{f^2}}\right] \tag{6}$$

$$V_{SH}(\theta)^2 = V_{SH_0}^2 \left[1 + 2\gamma\sin^2\theta\right] \tag{7}$$

where $f = 1 - V_{S_0}/V_{P_0}$.

Equations (5) (6), and (7) can be linearized about the small parameters ϵ, δ, and γ to give

$$V_P(\theta)^2 = V_{P_0}^2 \left[1 + (2\delta - 2\epsilon)\sin^2\theta\cos^2\theta + 2\epsilon\sin^2\theta\right] \tag{8}$$

$$V_S(\theta)^2 = V_{S_0}^2 \left[1 + \frac{V_{P_0}^2}{V_{S_0}^2}(2\epsilon - 2\delta)\sin^2\theta\cos^2\theta\right] \tag{9}$$

$$V_S(\theta)^2 = V_{S_0}^2 \left[1 + 2\gamma\sin^2\theta\right]. \tag{10}$$

This linearization is designed NMO correction of seismic data and so is less accurate for velocities near horizontal propagation.

WAVENUMBER REPRESENTATION OF DISPERSION RELATIONS

The dependance of the wavefield on direction of propagation is achieved through expressing the dispersion relation in terms of wavenumbers, $\vec{k}$. In terms of the angles theta and alpha

$$\begin{aligned} k_x &= |\vec{k}|\sin\theta \\ k_z &= |\vec{k}|\cos\theta, \end{aligned} \tag{11}$$

in 2D and

$$\begin{aligned} k_x &= |\vec{k}|\sin\theta\cos\phi \\ k_y &= |\vec{k}|\sin\theta\sin\phi \\ k_z &= |\vec{k}|\cos\theta, \end{aligned} \tag{12}$$

in 3D.

Substituting equation (12) into equation (6) gives a dispersion relation,

$$\omega_{V_s}^2(k_x,k_z)^2 = |\vec{k}|^2 V_{P_0}^2 \left[1+\epsilon\sin^2\theta - \frac{f}{2} - \frac{f}{2}\sqrt{1+\frac{4\sin^2\theta}{f}(2\delta\cos^2\theta - \epsilon\cos 2\theta) + \frac{4\epsilon^2\sin^4\theta}{f^2}}\right]$$

$$\omega_{V_s}^2(k_x,k_z) = V_{P_0}^2 \left[k^2(1-\frac{f}{2}) + \epsilon k_x^2 - \frac{f}{2}\sqrt{k^4 + \frac{4k_x^2}{f}(2\delta k_z^2 - \epsilon(k_z^2-k_x^2)) + \frac{4\epsilon^2 k_x^4}{f^2}}\right] \tag{13}$$

ROTATIONS FOR TTI MEDIA

For TTI media the axis of rotation is not alighted with the coordinate axis. Given a tilt of angle β from vertical, dispersion relation for weak linearly approximation is

$$V_P(\theta)^2 = V_{P_0}^2\left[1+(2\delta-2\epsilon)\sin^2(\theta-\beta)\cos^2(\theta-\beta) + 2\epsilon\sin^2(\theta-\beta)\right] \tag{14}$$

WAVEFIELD PROPAGATION IN ISOTROPIC HOMOGENOUS MEDIA

The following conventions are used for the forward and inverse Fourier transform of the function $\varphi:\mathbb{R}^3\to\mathbb{C}$,

$$\hat{\varphi}(\vec{k}) = \mathcal{F}_{\vec{x}\to\vec{k}}(\varphi) = \int_{\mathbb{R}^3} e^{2\pi i\vec{x}\cdot\vec{k}}\varphi(\vec{x})dxdydz, \tag{15}$$

and

$$\varphi(\vec{x}) = \mathcal{F}^{-1}_{\vec{k}\to\vec{x}}(\hat{\varphi}) = \int_{\mathbb{R}^3} e^{-2\pi i\vec{x}\cdot\vec{k}}\hat{\varphi}(\vec{k})dk_x dk_y dk_z, \tag{16}$$

where $\mathbb{R}$ is the real line, $i=\sqrt{-1}$, $\vec{x}=(x,y,z)\in\mathbb{R}^3$, $\vec{k}=(k_x,k_y,k_z)\in\mathbb{R}^3$ is the Fourier domain coordinate conjugate to $\vec{x}$, and the hat denotes a Fourier transformed function of the spatial coordinates. The symbols $\mathcal{F}_{\vec{x}\to\vec{k}}$, and $\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}$ are used to denote the forward and inverse Fourier transforms as abstract operators, respectively. Later, the symbols $\mathcal{F}_{\vec{x}\to\vec{k}}$, and $\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}$ are also used to denote the Fourier-like integrals when φ or $\hat{\varphi}$ depending upon both $\vec{k}$ and $\vec{x}$ explicitly. In this case, a single FFT cannot be used to calculate the Fourier transform.

Common wave-equation depth migration methods recursively extrapolate the recorded wavefield downward in depth. In contrast, RTM, like forward modeling, recursively propagates the recorded wavefield in time. This is typically done by finite-differencing the two-way wave equation. As an alternative to finite-differencing the wave equation, our time-stepping equation is formulated by phase-shifting the Fourier transform of the wavefield with a cosine operator. The time-stepping equation is based on an exact solution of the constant-velocity wave equation,

$$\frac{\partial^2 U}{\partial t^2} = v^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right), \tag{17}$$

where $U(t, \vec{x})$ is the amplitude of the wave at the point $(t, \vec{x} = (x, y, z))$, x, y are the lateral coordinate, z is the depth coordinate, t is the time coordinate, $\partial^2 U/\partial t^2$ is, for example, the second-order partial derivative of the wavefield with respect to the time coordinate, and c, a constant, is the speed of propagation.

Assume $\vec{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Applying the Fourier transform over the spatial dimensions $\vec{x} = (x, y, z)$ to both sides of equation (17) reduces it to a collection of ordinary differential equations,

$$\frac{\partial^2 \hat{U}}{\partial t^2} = -(2\pi)^2 v^2 \left(k_x^2 + k_y^2 + k_z^2 \right) \hat{U}. \tag{18}$$

Which for the initial conditions

$$\begin{cases} U(0, \vec{x}) = f(\vec{x}) \\ U(-\delta t, \vec{x}) = g(\vec{x}) \end{cases}, \tag{19}$$

where δt denotes a timestep. The resulting exact solution at time $t = \delta t$ of constant velocity wave equation is

$$U(\delta t, \vec{x}) = -\,U(-\delta t, \vec{x}) + 2\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}[\cos(2\pi v|\vec{k}|\delta t)\mathcal{F}_{\vec{x}\to\vec{k}}[U(0, \vec{x})]]. \tag{20}$$

The solution can be calculated by iterating equation (21). The calculation at time step n corresponding to time $t = n\delta t$ is

$$U^{n+1}(\vec{x}) = -\,U^{n-1}(\vec{x}) + 2\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}[\cos(2\pi v|\vec{k}|\delta t)\mathcal{F}_{\vec{x}\to\vec{k}}[U^n(\vec{x})]]. \tag{21}$$

The fast Fourier transform can be employed to compute equation (21) because the kernel of the Fourier integral is independent of the spatial coordinate $\vec{x}$.

To demonstrate the effectiveness of recursively using equation (21) for wavefield propagation, a minimum phase wavelet is stepped forward in time. The minimum phase wavelet is injected at the center point of a constant velocity two-dimensional model at the start of propagation. Figure **??** is the propagation of a minimum phase wavelet using the phase-shift time-stepper and also with conventional second-order finite differencing. In spite of a much smaller timestep, the finite-difference solution shows unacceptable distortion due to numerical grid dispersion.

WAVEFIELD PROPAGATION IN ISOTROPIC INHOMOGENEOUS MEDIA

The variable-velocity acoustic wave equation is

$$\frac{\partial^2 U}{\partial t^2} = v^2(x, y, z) \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right), \tag{22}$$

where $v(x, y, z)$ is the spatially dependent velocity. We now adapt equation (21) which propagates an acoustic wavefield exactly in a constant velocity medium to propagate approximately in a variable velocity medium. Equation (22) has a local causality property which means that the wavefield $U(t, \vec{x})$ only depends on the wavefield locally. As a result, the right hand side of equation (22), $v^2(\vec{x})\,(U_{xx} + U_{yy} + U_{zz})$, can, for small enough δt, be approximated locally near $\vec{x}_0$ by the solution to the frozen term $v^2(\vec{x}_0)\,(U_{xx} + U_{yy} + U_{zz})$. This means that by replacing the constant velocity appearing in the dispersion relation in equation (21) by the variable velocity (i.e. unfreezing the velocity), we have an approximate solution

$$U^{n+1}(\vec{x}) = -U^{n-1}(\vec{x}) + 2\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}\left[\cos\left(2\pi v(\vec{x})|\vec{k}|\delta t\right)\mathcal{F}_{\vec{x}\to\vec{k}}\left[U^n(\vec{x})\right]\right]. \tag{23}$$

Or explicitly,

$$\begin{aligned} &U^{n+1}(\vec{x}_1) = -U^{n-1}(\vec{x}_1) \\ &+2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} U^n(\vec{x})\cos\left(2\pi v(\vec{x}_1)|\vec{k}|\delta t\right) e^{2\pi i(\vec{x}-\vec{x}_1)\cdot\vec{k}} \end{aligned} \tag{24}$$

This is the freezing-unfreezing argument that appears in the literature in the context of hyperbolic and elliptic partial differential equations e.g., (p. 230-231, **?**). Such solutions are often called locally homogeneous approximations (e.g., **?**) and they approximate the solution to the variable velocity wave equation by the solution locally from the constant velocity wave equation. In fact the local homogenous approximation is used for both finite-difference and pseudospectral solvers.

Equation (23), for variable velocity, is too numerically complex to be used directly for wavefield propagation because the integrations cannot be accomplished with fast Fourier transforms (FFT's). We use a Gabor windowing scheme to approximate it and so the resulting operator is a Gabor multiplier. The Gabor Transform is a windowed Fourier transform. Equation (23) is a Fourier integral operator (**?**). Numerical computation of these operators is an active area of research (**?**). Please refer to Appendix A for various definitions of terms that occur in this section.

A partition of unity is used to partition the velocity model into N regions with approximate constant velocity, $v_1 < \ldots < v_j < \ldots < v_J$, called reference velocities. In our method these regions can be spatially very complex when the velocity varies rapidly. The wavefield in each region is then propagated with the corresponding reference velocity. Figure **??** is an example velocity partition, or window, used to migrate the Marmousi data set. The POU is used to window the wavefield into regions at each timestep and the combination of windowing and Fourier transformation results in the Gabor approximation to equation (23) given by

$$U^{n+1}(\vec{x}) = -U^{n-1}(\vec{x}) + \sum_{j=1}^{N}\Omega_j(\vec{x})2\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}\left[\cos\left(2\pi v_j|\vec{k}|\delta t\right)\mathcal{F}_{\vec{x}\to\vec{k}}\left[U^n(\vec{x})\right]\right], \tag{25}$$

where v_j is the reference velocity used for propagation in the jth window $\Omega_j(\vec{x})$ and the integrations are now all

The computational burden of the PSTS equation depends linearly on the number of reference velocities used to approximate the the cosine operator. Constructing accurate approximations with a minimal number of reference velocities allows numerically efficient phase-shift time-stepping algorithms. Methods for choosing reference velocities are found in, for example, **?** or **?**. A simple method is to take the reference velocities equally spaced between the lowest and greatest velocity. A spatially discontinuous partition of unity can used to approximate equation (23). Suppose that a set of reference velocities have been chosen so that $v_1 < \ldots < v_j < \ldots < v_J$ where v_1 and v_N are the minimum velocity and maximum velocity of $v(\vec{x})$, respectively. Suppose $\sum_j \Omega_j(\vec{x})v_j^2 = v^2(\vec{x})$ and $\sum_j \Omega_j(\vec{x}) = 1$ with $0 \leq \Omega_j(\vec{x}) \leq 1$. The auxiliary condition that $\Omega_j(\vec{x}) = 0$ if $v(x) > v_{j+1}$ or $v(x) < v_{j-1}$ so that the operator is interpolated between the closest reference velocities. The Ω_j's window and interpolate the wavefield propagated with constant velocities and form a partition of unity. Then (**?**),

$$\Omega_j(\vec{x}) = \begin{cases} \frac{v^2(\vec{x})-v_{j-1}^2}{v_j^2-v_{j-1}^2} & v_{j-1} < v(x) < v_j \\ \frac{v_{j+1}^2-v^2(\vec{x})}{v_{j+1}^2-v_j^2} \quad \text{if} & v_j < v(x) < v_{j+1} \\ 0 & otherwise \end{cases} . \tag{26}$$

To approximate equation (23), we propagate the wavefield with a collection of constant velocities and then recombine the result with the partition of unity,

$$U^{n+1} = \sum_{j=1}^{J} \Omega_j(\vec{x})\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}\left[2\cos(2\pi v_j|\vec{k}|)\mathcal{F}_{\vec{x}\to\vec{k}}[U^n]\right] - U^{n-1}. \tag{27}$$

Expanding the cosines in equation (27) and (23) with the power series expansion about $v = 0$ shows that equation (27) is at least as accurate as a second-order pseudospectral method. Figure **??** is the image of Ω_j windows corresponding to four reference velocities of the Marmousi dataset (**?**). Choosing the reference velocities and windows as above makes the windowing scheme effectively an interpolation between reference velocities.

WAVEFIELD PROPAGATION IN ANISOTROPIC HOMOGENOUS MEDIA

Given a dispersion relation $\omega(\vec{k}, \vec{x})$ we may define a wave equation where the phase velocity is the same as the phase velocity derived from the elastic wave equation for any particular bode wave mode. We define a homogenous anisotropic wave equation with phase velocity $\omega(\vec{k}, \vec{x})$ as

$$\frac{\partial^2 U}{\partial t^2} = \mathcal{F}^{-1}_{\vec{k}\to\vec{x}}\left(\omega^2(\vec{k})\widehat{U}(\vec{k})\right). \tag{28}$$

This equation is hyperbolic provided $\omega(\vec{k}) > 0$. When ω is independent of $\vec{x}$ equation (28) has the solution

$$U^{n+1}(\vec{x}) = -U^{n-1}(\vec{x}) + 2\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}\left[\cos\left(2\omega(\vec{k})\delta t\right)\mathcal{F}_{\vec{x}\to\vec{k}}\left[U^n(\vec{x})\right]\right], \tag{29}$$

for the initial conditions $U(0,\vec{x}) = U^0, U(\delta t,\vec{x}) = U^1$. In the case of variable velocity, Equation (28) has an approximate solution

$$U^{n+1}(\vec{x}) = -U^{n-1}(\vec{x}) + 2\mathcal{F}^{-1}_{\vec{k}\to\vec{x}}\left[\cos\left(2\omega(\vec{k},\vec{x})\delta t\right)\mathcal{F}_{\vec{x}\to\vec{k}}\left[U^n(\vec{x})\right]\right], \tag{30}$$

WAVEFIELD PROPAGATION IN ANISOTROPIC HETEROGENOUS MEDIA

CONVERTED WAVE IMAGING CONDITIONS

Frequency content for crosscorrelation.

Flipping polarity

THE FRP RAMP MODEL

The ramp model contains three velocities layers contain on anisotropic overburden.

REFERENCES

Alkhalifah, T., 2000, An acoustic wave equation foranisotropic media: Geophysics, **65**, No. 4, 1239–1250.

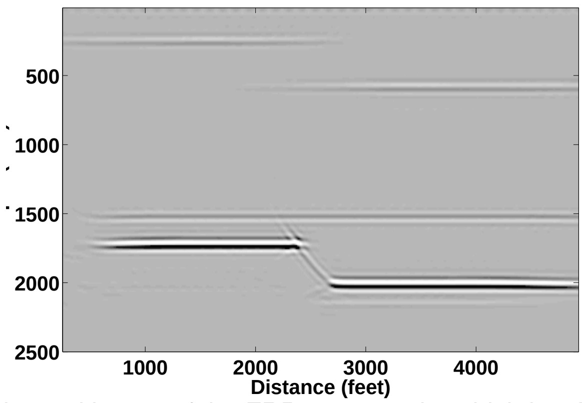

FIG. 1. A PP migrated image of the FRP ramp model which has TTI overburden.