Comparative Risk Attitudes in Stochastic Choice*

Po Hyun Sung[†]
Ben Wincelberg[‡]

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Abstract

Under the CARA and CRRA utility families, Fechnerian models of stochastic choice—a non-parametric class that includes the logit and probit models—are known to suffer from the paradoxical property that a more risk-averse individual is often predicted to choose a riskier lottery more frequently. We show that the paradox persists under broad generalizations: when noise is non-Fechnerian, differs arbitrarily across individuals, and even varies across menus. Paradoxical predictions arise under empirically relevant coefficients of risk aversion and lotteries, revealing that, even theoretically, parameter estimates depend on the observed lottery comparisons. We establish that two utility functions will produce paradoxical reversals whenever the ratio of their second derivatives is unbounded. Using this characterization, we propose a number of parametric utility families that do not suffer from this condition, offering well-behaved alternatives to CARA and CRRA for stochastic choice.

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[†]Caltech. Email: psung@caltech.edu
[‡]Caltech. Email: bwincelb@caltech.edu

1 Introduction

A precise understanding of how humans evaluate risk is central to economics. Since choices are inherently noisy, analysts deploy stochastic choice models to recover individuals' risk preferences and noise from choice data. The most widely used stochastic choice models for this purpose are *Fechnerian models*, a non-parametric class that includes the logit and probit models. In practice, the analyst chooses a functional form for utility and a functional form for noise which maps utility differences to choice probabilities (e.g., Harrison, List, and Towe, 2007; Von Gaudecker, Van Soest, and Wengström, 2011; Holzmeister and Stefan, 2021).

However, predictions from these models under CARA or CRRA—the most commonly used utility functions—suffer from a well-known problem. When two individuals differ in their risk-aversion parameters but face the same noise, the more risk-averse individual exhibits more risk-seeking behavior in some choices (Wilcox, 2008, 2011; Blavatskyy, 2011; Apesteguia and Ballester, 2018). In response to these critiques, subsequent work has argued that non-monotonicity is particularly worrisome under homoskedastic noise and has advocated allowing heteroskedasticity for calibration of noise to utility (Barseghyan, Molinari, O'Donoghue, and Teitelbaum, 2018; O'Donoghue and Somerville, 2024; Keffert and Schweizer, 2024). Moreover, it is common to model heterogeneity in both preferences and noise and to estimate them jointly (e.g., Hey and Orme, 1994; Von Gaudecker, Van Soest, and Wengström, 2011; Meissner, Gassmann, Faure, and Schleich, 2023).

In this paper, we show that the problems noted for Fechnerian models are far more pervasive than previously recognized: they persist even when individuals differ arbitrarily in their Fechnerian noise, when noise varies across menus, and when noise takes a more general, non-Fechnerian form. That is, under CARA and CRRA utilities, none of these generalizations can eliminate the counterintuitive predictions. We thus propose alternative utility functions that allow for more intuitive comparative statics, where a more risk-averse individual always chooses safer options more frequently under various noise specifications. Beyond their intuitive appeal, these comparative statics align with empirical findings (Bruner, 2017).

To build intuition, we begin with the baseline Fechnerian framework, in which each individual is associated with a Bernoulli utility function u and a strictly increasing function F. We refer to F as a Fechnerian noise structure, where

$$F(U(X) - U(Y))$$

is the probability that X is chosen over Y, and $U(X) = \mathbb{E}[u(X)]$. The most popular Fechnerian noise structures are CDFs of normal and logistic distributions, corresponding to probit and logit models. Indeed, any additive random utility models with i.i.d. shocks fit the Fechnerian framework.[1]

Our first result is that, for some pairwise choices between a safe and a risky lottery,[2] a more risk-averse individual chooses the risky option more often than a less risk-averse individual, even when the two differ arbitrarily in their Fechnerian noise structures (Theorem 1). Thus, the problems identified for the homoskedastic noise model cannot be resolved by jointly estimating risk and noise, even when noise is estimated non-parametrically.

For example, consider an analyst who observes Anne and Bob choosing from three distinct menus—each menu consists of a safe lottery—a sure payoff of \$8, \$10, or \$14—and a 50–50 risky lottery paying plus or minus \$4, \$6, or \$8 from the sure amount. The choice probabilities for the safe lotteries are shown in the second and third columns of Table 1. Assuming CRRA expected utilities for their risk preferences and normal distributions for their noise structures, Anne is estimated to be more risk averse and experience less noise than Bob. Indeed, the choice probabilities in Table 1 correspond to Anne having a CRRA coefficient of 0.8 and noise variance of 0.5, compared to Bob's 0.3 and 1, respectively.[3]

[1]Technically, for F to be strictly increasing, the difference in these shocks must have full support.

[2]We say that lottery X is safer than lottery Y if X dominates Y in the concave order. Equivalently, every expected utility maximizer with a concave utility function must prefer X to Y.

[3]I.e., the analyst deploys the probit model for Anne and Bob

$$F_A(U_A(X) - U_A(Y)) \quad \text{and} \quad F_B(U_B(X) - U_B(Y)),$$

respectively, where U_A and U_B are CRRA expected utility functions with relative risk aversions a and b, and F_A and F_B are the CDFs of normal distributions with zero means and respective variances σ_A^2 and σ_B^2.

Base Lottery	Probability of S		Probability of $3S$	
(S vs R)	Anne	Bob	Anne	Bob
8 vs $4, 12$	0.60	0.57	0.62	0.65
10 vs $4, 16$	0.65	0.62	0.69	0.74
14 vs $6, 22$	0.64	0.63	0.68	0.77

Table 1: CRRA Choice Probabilities for S with normal errors.

In the last two columns of Table 1, we report the choice probabilities of safe options whose outcomes are tripled from the base lotteries under the estimated parameters. Note that when the stakes are increased, Bob chooses safer options more than Anne from all menus. Thus, despite Anne being estimated as more risk-averse and more precise, the model predicts that she will behave more risk-seeking than Bob once the stakes are scaled up.

This reversal pattern does not hinge on these specific CRRA risk coefficients and noise structures, nor on the specific choice of lotteries. Indeed, given lotteries X and Y with X safer than Y, we show how to construct lotteries X' and Y' offering higher potential rewards, such that X' remains safer than Y' yet leads to a reversal (Proposition 2). In the special case that the noise structures satisfy $F_A(t/\sigma_A) = F_B(t/\sigma_B)$ for some $\sigma_A, \sigma_B > 0$ (e.g., when both are probit, with different variances as the example above), a reversal occurs if both lotteries are scaled sufficiently: for all large enough scales k, Bob chooses kX over kY more frequently than Anne does (Proposition 1). For CARA utilities, reversals occur after shifting the lotteries by the same sufficiently large outcome under arbitrary noise structures for Anne and Bob (Proposition 2). Thus, CARA and CRRA Fechnerian models with arbitrary noise structures make the perverse and systematic prediction that those most inclined toward choosing safer options when stakes are small must become the least inclined when stakes are larger.

Assuming that this reversal pattern does not always bear out in reality—and indeed Bruner (2017) finds that it does not—our results indicate that the estimates yielded by these models will be sensitive to the level of stakes that are analyzed, making them unreliable.[4] Indeed, if the analyst instead observed the choice prob-

[4]This echoes the critique of expected utility in Rabin (2000), which also reveals paradoxes when

abilities of the scaled lotteries in Table 1, the analyst may conclude that Bob is more risk-tolerant than Anne, even if he exhibited more risk-averse behavior. As we illustrate in the tables, the stake increases are often modest, indicating that such paradoxical non-monotonicities can emerge even in low-stakes experimental settings. Moreover, this will lead to counterintuitive, or perhaps counterfactual, out-of-sample predictions.

We next show that the paradoxes of CARA and CRRA Fechnerian models are not artifacts of overly restrictive Fechnerian noise structures. To this end, we explore *noisy expected utility* (NEU) models, which consist of a vNM utility function U and a noise structure $H\colon \mathbb{R}^2 \to [0,1]$ that translates the utilities of two alternatives into choice probabilities.[5] That is, $H(U(X), U(Y))$ is the probability that X is chosen over Y. NEU models generalize Fechnerian models by not requiring choice probabilities to depend only on utility differences. We demonstrate that NEU models based on CARA or CRRA expected utilities are guaranteed to produce paradoxical reversals under a mere smoothness condition on noise structures (Theorem 2).

One might suspect, at this point, that these paradoxes arise because H is applied to vNM utilities, which are just representations of preferences and carry no cardinal significance. A natural alternative would be to calculate, for each lottery X, its certainty equivalent $u^{-1}(\mathbb{E}[u(X)])$, and apply H to these certainty equivalents. It turns out that this does not solve the problem. In fact, we will still have paradoxes, even if we more generally apply H to $f(\mathbb{E}[u(X)])$ for some strictly increasing f, because absorbing f into the noise structure H simply generates a new noise structure (Corollary 1). These results show that the paradoxical reversals originate from the structure of CARA and CRRA utilities themselves, rather than from any particular assumption about noise or utility representation.

We next examine whether allowing noise structures to vary across menus can eliminate these paradoxes. In a *menu-dependent Fechnerian noisy expected-utility* (MNEU) model, each individual is associated with a vNM utility function and a *noise assignment* that specifies, for every pair of lotteries, a menu-specific Fechnerian noise

stakes are increased. While Rabin's result is about the curvature of utility functions in deterministic models, our result is about the interaction between noise structures and CARA/CRRA utilities.

[5]We require choice probabilities to be monotone in utility, meaning that H is increasing in its first argument and decreasing in its second. These models of noise were studied in depth by Tversky and Russo (1969).

structure. An important subclass of menu-dependent models arises in the additive random utility framework, when the random shocks to utility that are associated with each lottery are independent but not identically distributed. For example, lotteries with large stakes may induce increased attention and be associated with less-variable shocks. To rule out pathological noise assignments, we impose a continuity condition requiring that menus containing similar lotteries yield similar noise structures. Our next result shows that MNEU models necessarily yield paradoxical reversals (Theorem 3). Thus, even when allowing for arbitrary noise, varying across individuals and menus, the CARA and CRRA expected utility functional forms lead to untenable predictions.

Finally, we turn our attention toward the characterization of families of Bernoulli utilities that interact well with noise. Following Wilcox (2011), we say that Anne is *stochastically more risk-averse* than Bob if Anne consistently chooses safer options more frequently than Bob. Our final result is that there exist noise structures for which Anne is stochastically more risk averse than Bob if and only if $ku_A - u_B$ is a concave function for some $k > 0$ (Theorem 4). For twice-differentiable utility functions, this condition is equivalent to the boundedness of

$$u''_B(x)/u''_A(x),$$

providing a simple condition to check.[6] This condition is not satisfied by a class of utility functions that generalizes CARA and CRRA.

We conclude by proposing families of utility functions, where the traditional and stochastic notions of comparative risk agree (Proposition 4). For example, for any $0 \leq r < s$, let crra_r and crra_s be CRRA Bernoulli utility functions with respective relative risk coefficients r and s. Interpolating between crra_r and crra_s yields a family of utility functions parameterized by their weight on the more risk averse crra_s, with a higher weight corresponding to greater risk aversion.[7] These simple families of utility functions offer well-behaved alternatives for empirical analysis.

[6]Indeed, for CARA and CRRA utilities, this condition fails, leading to paradoxical reversals in every model of noise we have considered.

[7]This works as long as the weight on crra_s is positive. Otherwise, we will run into the problem of comparing crra_r and crra_s which leads to reversals.

2 Related Literature

We contribute to a large body of literature on risky choices in the presence of noise (e.g., Becker, Degroot, and Marschak, 1963; Harless and Camerer, 1994; Hey and Orme, 1994; Loomes, Moffatt, and Sugden, 2002; Blavatskyy, 2007). See Wilcox (2021) for a recent survey. In particular, we study Fechnerian noise structures and their generalizations. Axiomatic investigations of these models have been undertaken by Debreu (1958), Tversky and Russo (1969), and Fudenberg, Iijima, and Strzalecki (2015), among others. See Strzalecki (2025) for an extensive review.

The papers most closely related to ours are Wilcox (2011) and Apesteguia and Ballester (2018), which demonstrate that CARA and CRRA utilities coupled with identical Fechnerian noise are problematic: a more risk-averse individual exhibits more risk-seeking behavior in some pairwise choices.[8] Our work builds on their insights in several ways. First, we allow for heterogeneous noise structures and show that the paradoxical reversals persist without any parametric assumptions on how noise differs across individuals (Theorem 1). Second, we show that the reversals occur under more general non-Fechnerian noise (Theorem 2).

We further allow noise to vary across menus, recognizing that some comparisons may involve greater cognitive effort or uncertainty. Menu-dependent noise in risky choice has been extensively studied. Hey (1995), Buschena and Zilberman (2000), and Loomes (2005) model Fechnerian noise that depends on, for example, value differences between lotteries and question difficulty. More recently, He and Natenzon (2024) and Shubatt and Yang (2024) characterize Fechnerian noise that depends on a measure of distance between alternatives. We show that, under very general conditions, paradoxes persist even when we allow noise to vary across menus (Theorem 3).

Finally, we characterize which families of utility functions, beyond CARA and CRRA, suffer from these reversals (Theorem 4) and propose alternative parametric families of utilities that do not suffer from the problem. This is in contrast with the solutions proposed by Wilcox (2011) and Apesteguia and Ballester (2018) who retain CARA and CRRA utilities.

Wilcox (2011) considers a model of menu-dependent noise and shows it is mono-

[8] Wilcox (2011) shows that other parameterizations of CARA and CRRA utilities also lead to reversals. This corresponds to two individuals having noise structures that are related by a scale factor. We study this special case in Proposition 1.

tone for lotteries over three fixed outcomes. Apesteguia and Ballester (2018) show that this model is no longer monotone for more than three outcomes and propose instead the random parameter model, where each individual is associated with a distribution over CARA or CRRA preferences and chooses probabilistically as if they draw a random preference from their distribution. Random parameter models do not lead to paradoxical reversals, but they are based on a different foundation than traditional models of noise. In particular, in random parameter models there is no longer a core preference over alternatives such that an individual usually chooses the more preferred outcome. This is in contrast with our proposal to use alternative vNM utility functions which maintains the idea that individuals choose according to a core preference under noise.

We compare the riskiness of lotteries with the concave order of Rothschild and Stiglitz (1978). In the deterministic setting, Kihlstrom, Romer, and Williams (1981) and Ross (1981) have noted that insurance and risk premia are not increasing in the Arrow-Pratt order. Ross (1981) thus proposes a stronger notion of comparative risk preferences than Arrow-Pratt that leads to monotonicity. Our characterization result (Theorem 4) shows that this stronger notion implies our condition, and thus is sufficient to eliminate paradoxical reversals under some noise structures.

Finally, we contribute to a literature examining failures of expected utility theory in predicting choices across varying ranges of stakes. The most prominent critique in this area is by Rabin (2000), who demonstrates that within the expected utility framework, even modest levels of risk aversion exhibited over small-stakes, imply absurd, counterfactual levels of risk aversion for large stakes. While this result holds for deterministic choice, Propositions 1 and 2 echo Rabin (2000) in the stochastic setting: out of sample predictions made from small-stakes observations are counterintuitive.

3 Preliminaries

We denote by $\mathcal{L}$ the set of all bounded real random variables defined over a non-atomic probability space $(\Omega, \Sigma, \mathbb{P})$. We use the term *lotteries* to refer to elements of $\mathcal{L}$. A vNM *utility function* is a map $U\colon \mathcal{L} \to \mathbb{R}$ given by $U(X) = \mathbb{E}[u(X)]$ for some strictly increasing, concave, and continuous Bernoulli utility $u\colon \mathbb{R} \to \mathbb{R}$.

We consider a decision maker that evaluates lotteries according to such a utility function, but chooses stochastically: there is a function $H\colon \mathbb{R}^2 \to [0,1]$ such that they choose lottery X over lottery Y with probability

$$H(U(X), U(Y)),$$

where H is increasing in the first argument and decreasing in the second argument. To ensure that the sum of choice probabilities is 1, we require that $H(s,t) + H(t,s) = 1$. We refer to H as a *noise structure*.

When H is differentiable, we denote by H_i the partial derivative of H with respect to the i^{th} argument. We denote by $\mathcal{H}$ the set of all continuously differentiable noise structures H such that $H_1 > 0$ and $H_2 < 0$.[9] We refer to a model of the form (U, H) where U is a vNM utility and $H \in \mathcal{H}$, as a *noisy expected utility* (NEU) model.[10]

We say that X dominates Y in the *concave order*, denoted $X \geq_{\mathrm{c}} Y$ if $\mathbb{E}[g(X)] \geq \mathbb{E}[g(Y)]$ for all concave functions $g\colon \mathbb{R} \to \mathbb{R}$. We denote the strict part of $\geq_{\mathrm{c}}$ by $>_{\mathrm{c}}$. Recall that $X \geq_{\mathrm{c}} Y$ if and only if Y is a mean-preserving spread of X. If $X \geq_{\mathrm{c}} Y$, then $U(X) \geq U(Y)$ for all vNM utilities U.[11] We will often use the variables S and R when referring to lotteries where $S >_{\mathrm{c}} R$, in order to highlight that S is *safer* while R is *riskier*.

3.1 Stochastic Comparative Risk Aversion

We are interested in comparing the choices of agent A with those of agent B. Each agent is associated with a stochastic choice model (U, H) consisting of a utility function and noise structure. We will often refer to agents and their associated stochastic choice models interchangeably.

We say A is *more risk-averse than* B if $U_A(X) \geq U_A(y)$ implies $U_B(X) \geq U_B(y)$

[9]This implies that H is strictly increasing in the first argument and strictly decreasing in the second argument, and furthermore that these partial derivatives do not vanish.

[10]Notably, choice probabilities resulting from an NEU model (U, H) are monotone in the utility U. I.e., if $U(X) \geq U(Y)$, then X is chosen over any alternative Z more frequently than Y is chosen over Z. Using a strengthening of this property, Tversky and Russo (1969) characterize a more general model where the choice probability of X over Y is given by some noise structure H and some arbitrary utility function V. In Appendix H, we discuss the axiomatic foundations of these models and characterize when V is a vNM utility.

[11]Since we defined vNM utilities as expectations of concave Bernoulli utilities, they are increasing in $\geq_{\mathrm{c}}$.

for any $X \in \mathcal{L}$ and degenerate lottery y.[12] Suppose these agents have noise structures $H_A, H_B \in \mathcal{H}$.

Definition 1. *We say that $S, R \in \mathcal{L}$ are* paradoxical *for A and B, if $S >_{\mathrm{c}} R$, yet*

$$H_A(U_A(S), U_A(R)) < H_B(U_B(S), U_B(R)).$$

This means that A, the more risk-averse individual, paradoxically exhibits more risk-tolerant behavior by choosing the safer asset S less frequently than B.

A's behavior can also be seen as more risk-tolerant in the following sense: Let s_A and s_B denote A's and B's respective choice probabilities of S. The ex-ante lottery that A receives is the compound lottery resulting in S with probability s_A and in R with probability $1 - s_A$, and likewise for B. Since $s_A < s_B$, the compound lottery that A receives is riskier than the one B receives. Since these individuals' choice probabilities yield these compound lotteries, A evidently makes riskier choices than B.

We show that NEU models yield paradoxical lotteries under the Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) utility specifications, which are the most commonly used parametric families of utility functions in both theory and practice. CARA and CRRA utilities are defined by

$$\mathrm{CARA}_a(X) = \begin{cases} \mathbb{E}[X] & a = 0 \\ \mathbb{E}\left[\frac{1-\mathrm{e}^{-aX}}{a}\right] & a > 0 \end{cases} \tag{1}$$

$$\mathrm{CRRA}_a(X) = \begin{cases} \mathbb{E}\left[\frac{X^{1-a}-1}{1-a}\right] & a \neq 1 \\ \mathbb{E}[\ln(X)] & a = 1, \end{cases} \tag{2}$$

where $a \geq 0$ is the coefficient of absolute/relative risk aversion. The coefficients reflect varying levels of risk-aversion, with higher coefficients corresponding to greater aversion to risk.[13] For $\mathrm{CRRA}_a(X)$ to be well defined, X must be non-negative, which

[12]This is Yaari (1969)'s notion of comparative risk, which coincides with the Arrow-Pratt notion of comparative risk since U_A and U_B are vNM utility functions. I.e., U_A being more risk-averse than U_B is equivalent to the concavity of $u_A \circ {u_B}^{-1}$, when u_A and u_B are the corresponding Bernoulli utility functions to U_A and U_B. Indeed, CARA and CRRA utilities are totally ordered by this order.

[13]Note that there are many possible ways of parameterizing CARA and CRRA preferences, since applying a positive affine transformation to each Bernoulli utility does not change the underlying preference/coefficient. Nevertheless, we will show that our results hold for every possible parameterization of these families.

is an assumption we maintain whenever referring to CRRA.

In Sections 5 and 6, we show that noisy CARA and CRRA models and their generalizations yield paradoxical lotteries. To build intuition, we first consider the special case of Fechnerian models.

4 Fechnerian Models

An important subclass of NEU models yields choice probabilities

$$H(U(X), U(Y)) = F(U(X) - U(Y)),$$

where $F\colon \mathbb{R} \to [0,1]$ is a strictly increasing function that is continuously differentiable with non-vanishing derivative. Moreover, $F(t) + F(-t) = 1$. We denote by $\mathcal{F}$ the set of all such functions and refer to a model of the form (U, F), where U is a vNM utility and $F \in \mathcal{F}$, as a *Fechnerian noisy expected utility* (FNEU) model.[14] In these models, the larger the difference between $U(X)$ and $U(Y)$, the higher the probability that X is chosen.

FNEU models are among the most widely used stochastic choice models (Becker, Degroot, and Marschak, 1963; Loomes, Moffatt, and Sugden, 2002). An important subclass of FNEU models yields choice probabilities

$$\mathbb{P}(U(X) + \varepsilon \geq U(Y) + \varepsilon') = F(U(X) - U(Y)), \tag{3}$$

where ε and ε' are i.i.d. continuous random variables and F is the CDF of $\varepsilon - \varepsilon'$. This subclass includes many widely used discrete choice models such as the logit and probit models, where the random shocks are Gumbel and Gaussian, respectively.

In this section, we focus on FNEU models and show that the CARA and CRRA specifications display paradoxes: they predict that a more risk-averse individual will sometimes choose a riskier asset more frequently than a more risk-tolerant individual.

Theorem 1. *For any $F_A, F_B \in \mathcal{F}$ and distinct* CARA (CRRA) *utilities U_A, U_B, there exist paradoxical lotteries for (U_A, F_A) and (U_B, F_B).*

[14] The assumptions that F must be continuously differentiable with positive derivative can be relaxed without affecting our results, provided these assumptions hold at 0. Note that we do not assume that $\lim_{t\to\infty} F(t) = 1$, nor that $\lim_{t\to-\infty} F(t) = 0$, although this will be the case for the examples we study.

Theorem 1 is an impossibility result for the commonly used CARA/CRRA-expected-utility Fechnerian models as paradoxical lotteries arise under arbitrary pairs of noise structures and risk coefficients. In the case of $F_A = F_B$, i.e., when individuals face the same noise structures, it was shown by Apesteguia and Ballester (2018) that the probability of choosing the safer option from a pair is non-monotone in the CARA/CRRA coefficients.[15] In practice, however, it is common to model heterogeneity in both preferences and noise and to estimate them jointly (Hey and Orme, 1994; Von Gaudecker et al., 2011). Indeed, subsequent work has argued that non-monotonicity is particularly worrisome under homoskedastic noise and has advocated allowing heteroskedasticity for calibration of noise to utility (Barseghyan et al., 2018; O'Donoghue and Somerville, 2024; Keffert and Schweizer, 2024).

Yet, perhaps surprisingly, Theorem 1 demonstrates that non-monotonicity persists even when noise structures vary arbitrarily across individuals. Specifically, if one estimates the model (CARA_a, F_A) for one individual and (CARA_b, F_B) for another with $a > b$, the out-of-sample predictions are for the individual with coefficient a to choose some safer options less frequently than the individual with coefficient b, regardless of how F_A and F_B are fit in-sample. If these out-of-sample predictions are to be taken seriously, it is unclear that the parameter a reflects greater risk aversion.

Note that choice probabilities are not invariant to reparameterizations of CARA and CRRA utilities. For example, consider the model (U, F), where U is a CARA or CRRA utility and $F \in \mathcal{F}$. Let $V(X) = \frac{1}{c}U(X) + d$ for some $c > 0$. While V represents the same preference as U, replacing U with V has the same effect on choice probabilities as changing the noise structure from F to G which is defined by $G(t) = F(t/c)$. However, since G is also a member of $\mathcal{F}$, it follows from Theorem 1 that reparameterizations of the CARA or CRRA utility family cannot resolve the non-monotonicity.

4.1 Scale-Family Heteroskedasticity

To develop intuition for Theorem 1, we first consider individuals with distinct CARA or CRRA utilities U_A and U_B with respective coefficients a and b, where $a > b$ and whose noise structures are related by a scale factor. That is, $F_A(t) = F(t/\sigma_A)$ and

[15] When $F_A = F_B$, Theorem 1 follows from Corollary 1 and Proposition 3 of Apesteguia and Ballester (2018).

$F_B(t) = F(t/\sigma_B)$ for some common $F \in \mathcal{F}$.[16] The choice probabilities of S over R are then given by

$$F\left(\frac{U_A(S) - U_A(R)}{\sigma_A}\right) \tag{4}$$

$$F\left(\frac{U_B(S) - U_B(R)}{\sigma_B}\right). \tag{5}$$

Since F is strictly increasing, the pair (S, R) is paradoxical (i.e., (5) is strictly larger than (4)) if and only if

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} > \frac{\sigma_B}{\sigma_A}.$$

Note that $\frac{\sigma_B}{\sigma_A}$, which is the relative noise level, does not depend on the lotteries. Given this, the next lemma immediately implies that to construct paradoxical S and R, we can start with any $S >_{\mathrm{c}} R$, and get a paradoxical pair by adding to both a large enough constant x (in the CARA case) or multiplying both by a large enough constant (in the CRRA case).

Lemma 1. *For any $S >_{\mathrm{c}} R$, $a, b > 0$, and all $x \in \mathbb{R}$ and $k > 0$,*

1. $$\frac{\mathrm{CARA}_b(S + x) - \mathrm{CARA}_b(R + x)}{\mathrm{CARA}_a(S + x) - \mathrm{CARA}_a(R + x)} = C_1 \cdot \mathrm{e}^{(a-b)x}$$
2. $$\frac{\mathrm{CRRA}_b(k \cdot S) - \mathrm{CRRA}_b(k \cdot R)}{\mathrm{CRRA}_a(k \cdot S) - \mathrm{CRRA}_a(k \cdot R)} = C_2 \cdot k^{a-b},$$

for some positive constants C_1 and C_2.

Note that when $a > b$, the ratios are strictly increasing in x and α and tend to infinity. Moreover, these properties hold for any $S >_{\mathrm{c}} R$. We thus have the following stronger result when noise structures are related by a scale factor.

Proposition 1. *Let $\sigma_A, \sigma_B > 0$ and $F \in \mathcal{F}$. Let $F_A(t) = F(\frac{t}{\sigma_A})$, $F_B(t) = F(\frac{t}{\sigma_B})$, and $S >_{\mathrm{c}} R$. Let U_A, U_B be distinct CARA utilities and let V_A, V_B be distinct CRRA utilities. Then there exist unique x_0 and k_0 such that*

1. *$S + x$ and $R + x$ are paradoxical for (U_A, F_A) and (U_B, F_B) if and only if $x > x_0$;*
2. *kS and kR are paradoxical for (V_A, F_A) and (V_B, F_B) if and only if $k > k_0$.*

[16]This is the case, for example, when F_A and F_B are both normal or both logistic CDFs.

Proposition 1 highlights two problems with the CARA and CRRA Fechnerian models. First, there is a disconnect between the standard Arrow-Pratt notion of comparative risk-aversion developed in the deterministic framework with the resulting stochastic choice predictions. Namely, more risk-averse individuals are not more likely to make risk-averse choices. This counterintuitive prediction is contradicted by empirical evidence that risk aversion is negatively correlated with decision noise, as measured by the frequency with which individuals choose the concave-order dominated option (Bruner, 2017).

In terms of estimation, this means that even if Bob chooses safer lotteries more frequently than Anne, we may conclude that Bob is more risk-tolerant than Anne.[17] This is the case, for example, if we only observe the choice probabilities of the scaled lotteries in Table 1.

Second, these models make the systematic prediction that those most inclined to choose safer options when stakes are small must become the least inclined once the stakes grow modestly larger. Even without a specific model of risk aversion in mind, this prediction is implausible.

4.2 Arbitrary Noise Structures

So far we have considered the special case that F_A and F_B are related by a scale factor. To prove the more general case considered in Theorem 1, we establish the following lemma, which shows that the properties of CARA and CRRA utilities given in Lemma 1 give rise to non-monotonicity for any $F_A, F_B \in \mathcal{F}$.

Lemma 2. *Let $F_A, F_B \in \mathcal{F}$. Let U_A and U_B be vNM utility functions with U_A more risk averse. Suppose that for each $M \in \mathbb{R}$ there exist lotteries $S >_{\mathrm{c}} R$ such that*

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M. \tag{6}$$

Then there exist paradoxical lotteries for (U_A, F_A) and (U_B, F_B).

Lemma 2 identifies a sufficient condition on the vNM utility functions to yield the negativity result in Theorem 1. By Lemma 1, CARA and CRRA utilities satisfy this condition, and thus Theorem 1 follows immediately from these lemmas.

[17] This holds under any consistent estimator.

Let $S >_c R$ satisfying (12) for some $M \in \mathbb{R}$. The key step of the proof of Lemma 2 is to construct, $S' >_c R'$ such that

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} = \frac{U_B(S') - U_B(R')}{U_A(S') - U_A(R')},$$

while making the utility differences arbitrarily small. To this end, we consider the menu $(S, R_\lambda S)$, where $\lambda \in (0,1)$ and $R_\lambda S$ is distributed as a compound lottery that yields R with probability λ and S with probability $1-\lambda$. As $\lambda \to 0$, the utility difference between S and $R_\lambda S$ vanishes. Lemma 2 exploits the linearity of vNM utilities, which ensures that the ratio of utility differences is independent of λ. It also relies on the differentiability of F_A and F_B to approximate them around 0 by affine functions. Lemma 2 is proved in Appendix A.

Note that by Lemma 1, (12) is satisfied for all concave-ordered lotteries under sufficient shifting/scaling. Moreover, for CARA utilities, the utility difference $U(S+x) - U(R+x)$ tends to 0 as x tends to infinity (Lemma 4). For CRRA utilities, on the other hand, $U(kS) - U(kR)$ increases with k under CRRA coefficients less than unity. We therefore apply an additional transformation of scaling down the probability of receiving the risky prospect so that the utility difference is arbitrarily small. Based on these observations, the following proposition extends the non-monotonicity result of Proposition 1 to all $F_A, F_B \in \mathcal{F}$.

Proposition 2. *Let $F_A, F_B \in \mathcal{F}$ and $S >_c R$. Let U_A and U_B be distinct CARA utilities and V_A and V_B be distinct CRRA utilities. Then there exist x_0, $k_0 > 0$ such that*

1. *$S + x$ and $R + x$ are paradoxical for (U_A, F_A) and (U_B, F_B) for $x > x_0$.*
2. *$k \cdot S$ and $k \cdot (R_\lambda S)$ are paradoxical for (V_A, F_A) and (V_B, F_B) for $k > k_0$ and λ small enough.*

Proposition 2 is proved in Appendix B. While CARA preferences are invariant to changes in background wealth, the above proposition shows that non-monotonicity occurs in FNEU models under CARA utilities for any concave-ordered lotteries under sufficient background wealth. Likewise, while CRRA preferences are invariant to the scaling of stakes and to mixing with the safe lottery, these transformations can always generate a non-monotonicity.

Proposition 2 demonstrates that non-monotonicity is a pervasive problem. It holds for all pairs of risk parameters, for all pairs of Fechnerian noise specifications, and for all concave-ordered lotteries after a transformation that preserves the concave ordering as well as the family of preferences being measured.

5 Noisy Expected-Utility Models

Recall that an NEU model (U, H) consists of a vNM utility U and noise structure $H \in \mathcal{H}$, where $H(U(X), U(Y))$ is the probability of choosing lottery X over Y. The central property of NEU models is that choice probabilities are monotone in the utility. I.e., if $U(X) \geq U(Y)$, then, since H is increasing in its first argument, for all lotteries Z,

$$H(U(X), U(Z)) \geq H(U(Y), U(Z)).$$

This means that X is chosen over Z more frequently than Y is chosen over Z.[18]

We think of H as a measure of preference intensities between alternatives that depends on their utilities. When this measure is simply the utility difference, this reduces to FNEU. NEU models are much more flexible, allowing for any measure of intensities that is monotone in the utilities.

The next result shows that the paradoxes of FNEU models with CARA or CRRA utilities are not artifacts of overly restrictive Fechnerian noise structures.

Theorem 2. *For any $H_A, H_B \in \mathcal{H}$ and any distinct* CARA (CRRA) *utilities U_A and U_B, there exist paradoxical lotteries for (U_A, H_A) and (U_B, H_B).*

One may guess, at this point, that paradoxes arise in NEU models under CARA or CRRA utilities because H is applied to vNM utility U, which is merely a representation of the preference and carries no cardinal significance. A natural alternative would be to calculate, for each lottery X, its certainty equivalent $u_A^{-1}(\mathbb{E}[u_A(X)])$,

[18] In Appendix H, we show that in any stochastic choice model with this property, the choice probability of X over Y takes the form

$$H(U(X), U(Y)),$$

for some H that is increasing in the first argument and satisfying $H(s,t) + H(t,s) = 1$. This is weaker than the requirement that $H \in \mathcal{H}$, which additionally requires continuous non-vanishing partial derivatives.

and apply H to the certainty equivalents. The next result shows that this does not solve the problem. In fact, we will still have paradoxes, even if we more generally apply H to $f \circ U$ where f is any continuously differentiable function with positive derivative. This is because H_f defined by $H_f(s,t) = H(f(s), f(t))$ is also a member of $\mathcal{H}$.

Corollary 1. *Let f_A and f_B be continuously differentiable functions with positive derivatives. Then, for any $H_A, H_B \in \mathcal{H}$ and any distinct* CARA (CRRA) *utilities U_A and U_B, there exist paradoxical lotteries for $(f_A \circ U_A, H_A)$ and $(f_B \circ U_B, H_B)$.*

To see what drives non-monotonicity in these models, note that when a noise structure H is applied to a utility function, it measures the strength of preference between alternatives in a way that depends on the particular utility representation. CARA and CRRA expected utilities exhibit pathological comparative statics in their risk coefficients, preventing any hope of monotonicity. When these utilities are transformed to, say, certainty equivalents, the continuous differentiability of the monotone transformation f ensures that the resulting utilities change locally like expected utilities. It turns out that this remnant of CARA/CRRA expected utility is enough to preclude monotonicity.

Theorem 2 is proved in Appendix E. The proof is based on the observation that the class of NEU models is contained within the broader class of *menu-dependent Fechnerian noisy expected utility models*, which also suffer from non-monotonicity. We discuss this class of models and show how they generalize NEU in the upcoming section.

6 Menu-Dependent Models

In the previous sections, we studied models in which noise structures vary across individuals but not across menus for the same individual. In this section, we study a more flexible model that incorporates menu-dependent noise structures. For example, consider CARA agents with probit noise whose variance depends on the sum of the variances of the lotteries in the menu. This captures an intuitive idea that lotteries involving larger sums may generate more noise, allowing for mistake probabilities to stay high even when stakes are high.

In a *menu-dependent Fechnerian noisy expected-utility* (MNEU) model, each individual is associated with a vNM utility function U and menu-dependent Fechnerian noise structures, i.e., a map $\Phi\colon \mathcal{L} \times \mathcal{L} \to \mathcal{F}$, so that

$$\Phi_{X,Y}(U(X) - U(Y))$$

is interpreted as the probability that they choose X over Y, where $\Phi_{X,Y}$ is short for $\Phi(X,Y)$. We refer to Φ as a *noise assignment*, and require that $\Phi_{X,Y}(t) + \Phi_{Y,X}(-t) = 1$.[19]

This is a very flexible class models that generalizes NEU. To see this, let $H \in \mathcal{H}$ and let U be a vNM utility. Define

$$\Phi_{X,Y}(t) = H\left(\bar{U} + \frac{t}{2}, \bar{U} - \frac{t}{2}\right),$$

where $\bar{U} = \frac{U(X)+U(Y)}{2}$. Then

$$\Phi_{X,Y}(U(X) - U(Y)) = H(U(X), U(Y)).$$

We show in Appendix E that our assumptions on H imply that $\Phi_{X,Y} \in \mathcal{F}$ for all $X, Y \in \mathcal{L}$, so Φ is a valid noise assignment.

One way in which MNEU generalizes NEU is that, unlike in NEU where choice probabilities can only depend on utilities, choice probabilities from MNEU models may depend on the distributions of the lotteries. In fact, they may even depend on the joint distribution of the lotteries in a menu, since Φ is a function of the random variables in $\mathcal{L}$. For example, X may be chosen over Y more frequently if X dominates Y state-wise rather than in terms of first-order stochastic dominance.[20]

An important subclass of menu-dependent models arises from replacing the identicality assumption on Equation (3) with symmetry of the shock terms. I.e., the choice probability of X over Y is

$$\mathbb{P}(U(X) + \varepsilon_X \geq U(Y) + \varepsilon_Y) = \Phi_{X,Y}(U(X) - U(Y)),$$

where ε_X and ε_Y are independent and symmetric about zero (but not necessarily identical) random variables and $\Phi_{X,Y}$ is the CDF of $\varepsilon_X - \varepsilon_Y$.[21]

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[19] Since $\Phi_{X,Y}, \Phi_{Y,X} \in \mathcal{F}$, this requirement implies $\Phi_{X,Y} = \Phi_{Y,X}$.

[20] This distinction can only be determined from the joint distribution of (X, Y), not from their marginals.

[21] The symmetry assumption on the shocks ensures that $\Phi_{X,Y} \in \mathcal{F}$. We can relax this assumption by only requiring that the difference is symmetric.

We note that without any restrictions on the noise assignment Φ, the only implication of menu-dependent FNEU models is that individuals will choose their preferred lottery more than half of the time. This requirement is weaker than the central requirement of NEU, which was that choice probabilities must be monotone in the utility.

In order to rule out pathological noise assignments, we require that menus with similar lotteries are associated with similar noise structures. Formally, we define topologies on $\mathcal{L}$ and on $\mathcal{F}$ and require that the noise assignment be continuous. We say that a sequence of lotteries $X_1, X_2, \dots$ converges to a lottery X if for all increasing and continuous functions u it holds that

$$\mathbb{E}[|u(X_n) - u(X)|] \to 0.$$

In other words, $X_1, X_2, \dots$ converges to X if the sequence of random utilities $u(X_1), u(X_2), \dots$ converges to $u(X)$ in L^1 for every increasing and continuous Bernoulli u. An important property of this topology is that vNM utilities are continuous. Moreover, noise assignments are not required to depend only on the marginal distributions of the lotteries in the menu; they may, for example, depend on their joint distribution.

We associate each noise structure in $\mathcal{F}$ with its derivative and say that Φ is *continuous* if whenever $X_n \to X$ and $Y_n \to Y$, it holds that the derivatives of Φ_{X_n,Y_n} converge compactly to that of $\Phi_{X,Y}$.[22]

It turns out that many noise assignments are continuous. In particular, our topology on $\mathcal{L}$ is very fine,[23] making many functions on $\mathcal{L}$ continuous. Moreover, compact convergence is not too strict a requirement in many natural applications. For example, for normal and logistic distributions, parameter convergence implies compact convergence of densities. E.g., a sequence of densities ϕ_n of normal distribution with means $\mu_n \to \mu$ and variances $\sigma_n \to \sigma$ compactly converges to ϕ with corresponding mean μ and variance σ. In fact, parameter convergence is sufficient for the location-scale family of distributions with continuous densities (see Appendix C).

[22]That is, on every compact set $K \subset \mathbb{R}$,

$$\lim_n \Big(\sup_{t \in K} \big| \Phi'_{X_n,Y_n}(t) - \Phi'_{X,Y}(t) \big| \Big) = 0.$$

[23]Indeed, it is finer than any L^p topology for $1 \leq p < \infty$.

Our next theorem demonstrates that the paradoxical property of CARA and CRRA Fechnerian models cannot be resolved by allowing noise structures that depend (continuously) on the menu.

Theorem 3. *For any continuous noise assignments* Φ^A, Φ^B *and distinct* CARA (CRRA) *utilities* U_A, U_B*, there exist paradoxical lotteries.*

In other words, there exist lotteries $S >_{\mathrm{c}} R$ such that

$$\Phi^A_{S,R}(U_A(S) - U_A(R)) < \Phi^B_{S,R}(U_B(S) - U_B(R)).$$

Theorem 3 is proved in Appendix D. Theorem 3 highlights that the non-monotonicity result of the menu-independent Fechnerian models (Theorem 1) is not merely a byproduct of asymptotic properties of choice probabilities when stakes are increased. Indeed, in the following example noise assignments are engineered so that choice probabilities are scale-invariant, yet monotonicity does not obtain.

Example 1. *Let* $U_A = \mathrm{CRRA}_a$ *and* $U_B = \mathrm{CRRA}_b$ *where* $a > b > 0$*, and let* F *be the CDF of the standard normal distribution. Define the noise assignments* Φ^A *and* Φ^B *by*

$$\Phi^A_{X,Y}(t) = F\left(\frac{t}{\sigma_A(X,Y)}\right)$$
$$\Phi^B_{X,Y}(t) = F\left(\frac{t}{\sigma_B(X,Y)}\right),$$

where

$$\sigma_A(X,Y) = \mathbb{E}\Big[X^{a-1} + Y^{a-1}\Big]^{-1},$$
$$\sigma_B(X,Y) = \mathbb{E}\Big[X^{b-1} + Y^{b-1}\Big]^{-1}.$$

In this example, the noise structure normal with standard deviation $\sigma_A(X,Y)$, which decreases as X and Y are scaled up, i.e., $\sigma_A(\alpha X, \alpha Y) < \sigma_A(\beta X, \beta Y)$ for $\alpha > \beta$. Thus, this noise assignment captures the behavioral property that individuals pay more attention when presented with menus of lotteries with higher stakes. Moreover, the standard deviations are chosen so that each individual's probability of choosing βX over βY does not depend on $\beta > 0$.[24] Interestingly, however, since

[24] To see this, note that $\tau_A(\beta X, \beta Y) = \beta^{a-1}\tau_A(X,Y)$ and

$$U_A(\beta X) - U_A(\beta Y) = \beta^{1-a}(U_A(X) - U_A(Y)).$$

$\sigma_A(X_n, Y_n) \to \sigma_A(X, Y)$ whenever $X_n \to X$ and $Y_n \to Y$, it follows from Theorem 3 that even these calibrated models cannot avoid the paradox. Indeed, no continuous noise assignment can resolve the paradox.

Theorem 3 demonstrates that CARA and CRRA utilities are fundamentally paradoxical when choices are noisy, even if we allow noise structures to vary across individuals and menus. Thus, if we hope to model the stochastic choice behavior of A and B, where A consistently chooses safer lotteries more frequently than B, we must forgo CARA and CRRA utilities. We take this on in the next section, where we suggest alternative vNM utilities that can accommodate this behavior under standard models of noise.

7 Monotone Alternatives to CARA and CRRA

In the previous sections, we have seen that when agents A and B have CARA or CRRA preferences and A is more risk-averse than B, there are no reasonable noise structures that can accommodate A consistently choosing safer options more often. In this section, we revisit the various stochastic choice models discussed above and investigate which vNM utility functions, beyond CARA and CRRA, have this problem. We associate agents A and B with their corresponding vNM utilities U_A and U_B and noise assignments Φ^A and Φ^B. We say that A is *stochastically more risk-averse* than B if (U_A, Φ^A) always yields higher choice probability for safer lotteries than (U_B, Φ^B).[25] As we have seen, A may not be stochastically more risk-averse than B, even if U_A is more risk averse than U_B.

We thus depart from the Arrow-Pratt order and study an alternative notion of comparative risk between vNM utilities that leads to more frequent risk-averse choices in the presence of noise. We say $U_A = \mathbb{E}[u_A]$ is *absolutely more concave* than $U_B = \mathbb{E}[u_B]$, denoted $U_A \geq_{\text{abs}} U_B$, if $u_A - u_B$ is concave.[26]

It follows that

$$\begin{aligned}\Phi^A_{\beta X,\beta Y}(U_A(\beta X) - U_A(\beta Y)) &= F(\tau_A(\beta X, \beta Y) \cdot (U_A(\beta X) - U_A(\beta Y)) \\ &= F(\tau_A(X, Y) \cdot (U_A(X) - U_A(Y)) \\ &= \Phi^A_{X,Y}(U_A(X) - U_A(Y)).\end{aligned}$$

[25]This notion is due to Wilcox (2011) who defines it for general stochastic choice functions.

[26]Recall that U_A is more risk averse than U_B if U_A is *more concave* than U_B in the sense that

The next proposition states that absolute comparative concavity aligns with stochastic comparative risk aversion under identical Fechnerian noise.

Proposition 3. *Let $F \in \mathcal{F}$ and let $F_A = F_B = F$. A is stochastically more risk-averse than B if and only if $U_A \geq_{\text{abs}} U_B$.*

Proposition 3 characterizes when two FNEU models with identical noise can be ordered by stochastic risk aversion. Since Fechnerian noise structures are strictly increasing, the choice probability of S is higher for A than B if and only if the utility difference is higher for A than B, i.e.,

$$U_A(S) - U_A(R) \geq U_B(S) - U_B(R). \tag{7}$$

We show in Appendix F that this inequality holds for all lotteries $S >_{\text{c}} R$ if and only if $U_A \geq_{\text{abs}} U_B$.

In the case where A and B have different noise structures, i.e., $F_A \neq F_B$, a sufficient condition for A to be stochastically more risk-averse than B is if $U_A \geq_{\text{abs}} U_B$ and F_A is *more precise than* F_B, in the sense that $F_A(t) \geq F_B(t)$ for $t \geq 0$. A more precise noise structure translates the same utility difference into a higher probability of choosing a more preferred alternative. Since A is risk averse, increasing A's precision will only increase the probability that A chooses S. In particular,

$$F_A(U_A(S) - U_A(R)) \geq F_A(U_B(S) - U_B(R)) \geq F_B(U_B(S) - U_B(R)),$$

where the first inequality follows from Proposition 3. Of course, A may experience much more noise than B, in which case A will make more mistakes and may choose riskier options more frequently than B, even with $U_A \geq_{\text{abs}} U_B$.

Proposition 3 highlights yet another property of CARA and CRRA utilities that leads to non-monotonicity. It is easy to check that distinct CARA (CRRA) utilities are never ordered by $\geq_{\text{abs}}$. Moreover, while $\geq_{\text{abs}}$ is not invariant to affine transformations, under no such transformations, T_A and T_B, are $T_A(U_A)$ and $T_B(U_B)$ ordered by $\geq_{\text{abs}}$ when U_A and U_B are distinct CARA (CRRA) utilities.[27]

$u_A = f \circ u_B$ for some increasing and concave function f. While this leads to a notion of comparative risk that is invariant to affine transformations of the underlying Bernoulli utilities, its absolute counterpart is not. This alternative comparative concavity condition is due to McElroy (1999).

[27] Indeed, (7) is equivalent to the ratio $(U_B(S) - U_B(R))/(U_A(S) - U_A(R))$ being below one. By Lemma 1, these ratios are unbounded for CARA and CRRA utilities as we vary S and R. Since the application of affine transformations only scales these ratios, they will remain unbounded.

By the following theorem, this property is exactly what determines whether stochastic comparative risk aversion may obtain under MNEU models.

Theorem 4. *Let U_A and U_B be vNM utilities. There exist continuous noise assignments Φ^A and Φ^B such that (U_A, Φ^A) is stochastically more risk-averse than (U_B, Φ^B) if and only if there exists $k > 0$ such that $kU_A \geq_{\text{abs}} U_B$.*

When $U_A = \mathbb{E}[u_A]$ and $U_B = \mathbb{E}[u_B]$, where u_A and u_B are twice differentiable strictly concave functions, the condition that there exists $k > 0$ such that

$$kU_A \geq_{\text{abs}} U_B \tag{8}$$

is equivalent to the boundedness of u''_B/u''_A.

The necessity of (8) for A to be stochastically more risk averse is proved in Appendix F.1. We prove the reverse direction here by showing that even under a simple heteroskedastic probit model, without menu dependence, this condition suffices.

Indeed, suppose that $kU_A \geq_{\text{abs}} U_B$ for some $k > 0$. Let Φ^B assign to every menu the CDF of a standard normal distribution, denoted by F, and let Φ^A always assign the CDF of a normal distribution with zero mean and standard deviation $1/k$. For any lotteries $S \geq_{\text{c}} R$,

$$\Phi^A_{S,R}(U_A(S) - U_A(R)) = F(k(U_A(S) - U_A(R))) \tag{9}$$
$$\Phi^B_{S,R}(U_B(S) - U_B(R)) = F(U_B(S) - U_B(R)). \tag{10}$$

Since $ku_A - u_B$ is concave, $kU_A(S) - U_B(S) \geq kU_A(R) - U_B(R)$. Hence, since F is strictly increasing, (9) exceeds (10), meaning that A is stochastically more risk averse.

In Appendix G, we compare (8) with the comparative risk aversion notion of Ross (1981) in the deterministic framework. We show that his notion is stronger and discuss an interpretation of it in the stochastic framework.

7.1 Parametric Families of Utility Functions

The boundedness of u''_B/u''_A is violated by many parametric families of utility functions that are used to model risk aversion. In particular, we illustrate the expo-power utility function, a two-parameter family proposed by Saha (1993) to capture increasing/decreasing absolute/relative risk aversion, suffers from the same problems as CARA and CRRA utilities.[28] The increasing and concave expo-power utility function is given by

$$u_{a,r}(x) = \frac{1 - \mathrm{e}^{-ax^{1-r}}}{a},$$

for $x \geq 0$ and positive a and $0 \leq r < 1$. Note that the absolute risk aversion is given by

$$A_{a,r}(x) = \frac{r}{x} + a(1-r)x^{-r},$$

for $x > 0$, which is decreasing, constant, or increasing depending on the parameters. Thus, unlike CARA or CRRA utility, the family of expo-power utilities, parameterized by a and r, is not totally ordered by absolute/relative risk aversion.

Importantly, for any pairs of coefficients (a_1, r_1) and (a_2, r_2) such that $A_{a_1,r_1}(x) \geq A_{a_2,r_2}(x)$ so that $U_1 = \mathbb{E}[u_{a_1,r_1}(X)]$ is more risk-averse than $U_2 = \mathbb{E}[u_{a_2,r_2}(X)]$, it must hold that $r_1 = r_2$ and $a_1 > a_2$. A simple calculation shows that $u''_{a_2,r_2}(x)/u''_{a_1,r_1}(x)$ tends to infinity as $x \to \infty$. Thus, we have the following corollary.

Corollary 2. *For any continuous noise assignments Φ^A, Φ^B and distinct expo-power utility functions U_A,U_B such that U_A is more risk-averse than U_B, there exist paradoxical lotteries.*

Corollary 2 highlights that the paradoxical properties arising from CARA (CRRA) utilities are pervasive, and that they do not depend on the strong assumption of constant absolute (relative) risk aversion.

We conclude by proposing parametric families under which the Arrow-Pratt notion of comparative risk and its stochastic counterpart coincide. Recall that for strictly increasing concave functions f and g, f is more concave than g if $f \circ g^{-1}$ is increasing and concave. We start with the following remark.

Remark 2. *Let f and g be strictly increasing concave functions such that f is more concave than g. Let $u_A(x) = af(x) + g(x)$ and $u_B(x) = bf(x) + g(x)$, where $a > b \geq 0$. Then $U_A = \mathbb{E}[u_A]$ is more risk-averse than $U_B = \mathbb{E}[u_B]$ and $U_A \geq_{\text{abs}} U_B$.*

These utilities are parametrized by the coefficient on the more risk-averse Bernoulli utility function f, i.e., the higher the coefficient, the greater the risk aversion.[29]

[28] Indeed, Holt and Laury (2002) uses the expo-power utility function to model increasing relative risk aversion exhibited in their data.

[29] Indeed,

$$-\frac{u''_A(x)}{u'_A(x)} \geq -\frac{u''_B(x)}{u'_B(x)} \iff (a-b)\left(-\frac{f''(x)}{f'(x)} + \frac{g''(x)}{g'(x)}\right) \geq 0.$$

Moreover, since f is concave, $u_A - u_B = (a-b)f$ is concave, i.e., $U_A \geq_{\text{abs}} U_B$. Note that while Remark 2 holds for these particular utility representations, any affine transformations of these representations necessarily lead to agreement between the traditional and stochastic notions of comparative risk, in the senses of Arrow-Pratt and (8). Thus, we have the following proposition.

Proposition 4. *Let f and g be strictly increasing concave functions such that f is more concave than g. Let $1 > a > b \geq 0$. Let $u_A(x) = af(x) + (1-a)g(x)$ and $u_B(x) = bf(x) + (1-b)g(x)$. Then the following statements hold.*

1. *$U_A = \mathbb{E}[u_A]$ is more risk-averse than $U_B = \mathbb{E}[u_B]$*
2. *There exists $k > 0$ such that $kU_A \geq_{\text{abs}} U_B$.*

Proposition 4 suggests a replacement of the CARA utility family for estimation. One specifies an upper and lower bound on absolute risk aversion, corresponding to two CARA utilities, and then estimates the weight on each utility. The same exercise can be done with bounds on relative risk aversion and CRRA utilities. If an analyst wants to deploy a stochastic model whose predictions are consistent with comparative risk preferences in the presence of noise, these novel classes of utility functions may be of interest. Importantly, in order to allow for one agent that is stochastically more risk averse than another, such models should allow for heteroskedastic errors.

8 Conclusion

This paper establishes that the paradoxes long noted for CARA and CRRA utilities under homoskedastic Fechnerian noise are not artifacts of restrictive parametric assumptions, but instead reflect a deeper incompatibility between these utility forms and noisy choice. We show that the same non-monotonic comparative statics arise under highly flexible noise specifications—including heterogeneous noise across individuals and menu-dependent noise.

We obtain a simple characterization of whether two utility functions can yield sensible predictions in the presence of noise. This condition rules out even the more general expo-power utilities, and points to new parametric families that restore

Since f is more concave than g, it follows that $a \geq b$ is equivalent to U_A being more risk-averse than U_B.

intuitive comparative statics. These families, which include interpolation between two CARA or two CRRA utilities, provide empirically tractable, well-behaved alternatives for measuring risk preferences in noisy environments. We leave to future work an empirical examination of how risk aversion and decision noise co-vary across different stake levels, as well as systematic tests of these proposed utility families under a range of noise specifications.

We conclude by noting that the continuity requirement on noise assignments may be restrictive in some settings. For example, discontinuous noise assignments may capture interesting features of binary comparisons, such as salience or complexity. Understanding which discontinuous models generate paradoxical reversals thus remains an open and potentially important direction for future research.

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A Proof of Theorem 1

Proof of Lemma 1. Let $S >_{\mathrm{c}} R$ and $a > b > 0$. For CARA, we have

$$\begin{aligned}\frac{\mathrm{CARA}_b(S+x) - \mathrm{CARA}_b(R+x)}{\mathrm{CARA}_a(S+x) - \mathrm{CARA}_a(R+x)} &= \frac{a}{b}\frac{\mathbb{E}\left[\mathrm{e}^{-b(R+x)}\right] - \mathbb{E}\left[\mathrm{e}^{-b(S+x)}\right]}{\mathbb{E}[\mathrm{e}^{-a(R+x)}] - \mathbb{E}[\mathrm{e}^{-a(S+x)}]} \\ &= \frac{a}{b}\left(\frac{\mathbb{E}\left[\mathrm{e}^{-bR}\right] - \mathbb{E}\left[\mathrm{e}^{-bS}\right]}{\mathbb{E}[\mathrm{e}^{-aR}] - \mathbb{E}[\mathrm{e}^{-aS}]}\right)\mathrm{e}^{(a-b)x} \\ &\propto \mathrm{e}^{(a-b)x}.\end{aligned}$$

For CRRA, we have

$$\begin{aligned}\frac{\mathrm{CRRA}_b(k\cdot S) - \mathrm{CRRA}_b(k\cdot R)}{\mathrm{CRRA}_a(k\cdot S) - \mathrm{CRRA}_a(k\cdot R)} &= \frac{1-a}{1-b}\left(\frac{\mathbb{E}\left[S^{1-b}\right] - \mathbb{E}\left[R^{1-b}\right]}{\mathbb{E}[S^{1-a}] - \mathbb{E}[R^{1-a}]}\right)\frac{k^{1-b}}{k^{1-a}} \\ &\propto k^{a-b}.\end{aligned}$$

$\square$

Lemma 1 provides sequences of lottery pairs (S_n, R_n) along which the ratio of expected-utility differences $(U_B(S_n) - U_B(R_n))/(U_A(S_n) - U_A(R_n))$ diverges, for CARA and CRRA utilities. Lemma 2 establishes that when such a sequence exists there will be paradoxical lotteries. An important subtlety of Lemma 2 is that the paradoxical lotteries may not be part of this sequence. Indeed, Lemma 2 relies on the following lemma, which establishes that under the additional condition that utility differences tend to zero, there will be paradoxical lotteries in the sequence.

Lemma 3. *Let $F_A, F_B \in \mathcal{F}$ and let U_A and U_B be risk-averse vNM utility functions with U_A more risk averse. There exist $M \in \mathbb{R}$ and $\delta > 0$ such that all lotteries $S >_{\mathrm{c}} R$ satisfying*

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M \quad \textit{and} \quad U_i(S) - U_i(R) < \delta \tag{11}$$

for $i = A, B$ are paradoxical.

Proof of Lemma 3. Let $S >_{\mathrm{c}} R$. Let $f_A(t) = \frac{d}{dt}F_A(t)$ and $f_B(t) = \frac{d}{dt}F_B(t)$. Since F_A is continuously differentiable, f_A is continuous at 0 so that for each $\varepsilon_A > f_A(0)$ there is $\delta_A > 0$ such that for each $t \in (0, \delta_A)$, $f_A(t) < \varepsilon_A$ and $F_A(t) < \frac{1}{2} + t\varepsilon_A$. Since F_B is continuously differentiable and $f_B > 0$, for each $0 < \varepsilon_B < f_B(0)$, there exists $\delta_B > 0$ such that for any $t \in (0, \delta_B)$, $f_B(t) > \varepsilon_B$ and $F_B(t) > \frac{1}{2} + t\varepsilon_B$. Let $\delta = \min\{\delta_A, \delta_B\}$ and let $M = \varepsilon_A/\varepsilon_B$.

Let S and R as in the statement of the lemma. Then

$$\begin{aligned} F_B(U_B(S) - U_B(R)) &> \frac{1}{2} + (U_B(S) - U_B(R))\varepsilon_B \\ &\geq \frac{1}{2} + (U_A(S) - U_A(R))\varepsilon_A \\ &> F_A(U_A(S) - U_A(R)). \end{aligned}$$

□

In light of Lemma 3, Lemma 2 follows from the observation that for any lotteries $S >_{\mathrm{c}} R$ and $\varepsilon > 0$, there exist lotteries $S' >_{\mathrm{c}} R'$ with the same utility difference such that the denominator is less than ε.

Proof of Lemma 2. Let $F_A, F_B \in \mathcal{F}$ and let U_A and U_B be risk-averse vNM utility functions with U_A more risk averse. By Lemma 3, there exist $M \in \mathbb{R}$ and $\delta > 0$ such that all lotteries $S >_{\mathrm{c}} R$ satisfying

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M \quad \text{and} \quad U_i(S) - U_i(R) < \delta \tag{12}$$

for $i = A, B$ are paradoxical. By hypothesis, there exist lotteries $S >_{\mathrm{c}} R$ satisfying the first inequality.

For $\lambda \in (0, 1)$, let $R_\lambda S$ denote a lottery distributed as a compound lottery that yields R with probability λ and yields S with probability $1 - \lambda$. Then $S >_{\mathrm{c}} R_\lambda S$ and

$$\frac{U_B(S) - U_B(R_\lambda S)}{U_A(S) - U_A(R_\lambda S)} = \frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)},$$

by the linearity of expected utility, i.e., $U(R_\lambda S) = \lambda U(R) + (1 - \lambda)U(S)$ for any vNM utility U. Moreover, as λ tends to zero, $U_i(S) - U_i(R_\lambda S)$ tends to zero. Hence, S and $R_\lambda S$ are paradoxical lotteries for all λ small enough.

$\square$

B Proof of Proposition 2

Proposition 2 shows how to generate the paradoxical lotteries of Theorem 1 by generalizing Proposition 1 to arbitrarily different noise structures. The CARA case relies on the following lemma about diminishing utility differences as background wealth increases.

Lemma 4. *Let $S \geq_{\mathrm{c}} R$. Then*

$$\lim_{x \to \infty} \mathrm{CARA}_a(S + x) - \mathrm{CARA}_a(R + x) = 0$$

for all $a > 0$.

Proof. Since S and R have equal means, we suppose, without loss of generality, that $\mathbb{E}[S] = \mathbb{E}[R] = 0$. Let $u_a(x) = \frac{1 - \mathrm{e}^{-ax}}{a}$ denote the CARA Bernoulli utility under coefficient a, and let m denote the essential infimum of S.[30] By monotonicity, $u_a(x) - u_a(m + x) \geq u_a(x) - \mathrm{CARA}_a(S + x)$ and by concavity, $u_a(x) - \mathrm{CARA}_a(S + x) \geq 0$. Since $\mathbb{E}[S] = 0$, $m \leq 0$, by concavity,

$$\frac{d}{dx} u_a(m + x) \cdot |m| \geq u_a(x) - u_a(m + x).$$

[30]This is the largest value that S exceeds with probability 1.

Note that $\frac{d}{dx}u_a(m+x) = \mathrm{e}^{-a(m+x)}$. Thus

$$\mathrm{e}^{-a(m+x)} \cdot |m| \geq u_a(x) - u_a(m+x) \geq u_a(x) - \mathrm{CARA}_a(S+x) \geq 0.$$

Taking the limit as $x \to \infty$, we see that $u_a(x) - \mathrm{CARA}_a(S+x) \to 0$. Since S was an arbitrary mean-zero lottery and $\mathbb{E}[R] = 0$, we have

$$\lim_{x\to\infty} u_a(x) - \mathrm{CARA}_a(R+x) = 0$$

as well, concluding the proof. □

Proof of Proposition 2. For the case of CARA utilities, Lemma 1 establishes that for any pair of lotteries $S >_{\mathrm{c}} R$, the ratio

$$\frac{U_B(S+x) - U_B(R+x)}{U_A(S+x) - U_A(R+x)}$$

tends to infinity with x, while Lemma 4 establishes that the numerator and denominator go to zero. Hence, the result follows from Lemma 3.

For the CRRA case, Lemma 1 establishes that for any pair of lotteries $S >_{\mathrm{c}} R$, the ratio

$$\frac{U_B(k \cdot S) - U_B(k \cdot R)}{U_A(k \cdot S) - U_A(k \cdot R)}$$

tends to infinity with k. As in the proof of Lemma 2,

$$\frac{U_B(k \cdot S) - U_B(k \cdot R_\lambda S)}{U_A(k \cdot S) - U_A(k \cdot R_\lambda S)} = \frac{U_B(k \cdot S) - U_B(k \cdot R)}{U_A(k \cdot S) - U_A(k \cdot R)},$$

and as λ tends to zero, $U_i(k \cdot S) - U_i(k \cdot R_\lambda S)$ tends to zero for $i = A, B$, and the result follows from Lemma 3. □

C Uniform Convergence of Location-Scale Families

Let $g : \mathbb{R} \to [0, \infty)$ be a continuous probability density function. Then $\mu \in \mathbb{R}$ and $\sigma > 0$ parameterize the location-scale family with densities

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma}\, g\left(\frac{x-\mu}{\sigma}\right).$$

Proposition 5. *If $(\mu_n, \sigma_n) \to (\mu, \sigma)$, then $\|f_{\mu_n,\sigma_n} - f_{\mu,\sigma}\|_\infty \longrightarrow 0$.*

Proof of Proposition 5. Because g is continuous and integrable on $\mathbb{R}$, it necessarily satisfies $\lim_{|y|\to\infty} g(y) = 0$; hence g is bounded and uniformly continuous on $\mathbb{R}$. Define $a_n = \frac{\sigma}{\sigma_n}$ and $b_n = \frac{\mu-\mu_n}{\sigma_n}$. Note that $a_n \to 1$ and $b_n \to 0$. Changing variables $y = \frac{x-\mu}{\sigma}$, we have

$$\begin{aligned}\|f_{\mu_n,\sigma_n} - f_{\mu,\sigma}\|_\infty &= \sup_{y\in\mathbb{R}} \left| \frac{1}{\sigma}\Big(a_n\, g(a_n y + b_n) - g(y)\Big) \right| \\ &\leq \frac{1}{\sigma}\Big(|a_n - 1|\, \|g\|_\infty + \sup_{y\in\mathbb{R}} |g(a_n y + b_n) - g(y)|\Big).\end{aligned}$$

The first term $|a_n - 1|\, \|g\|_\infty \to 0$ since $a_n \to 1$. For the second term, $\sup_{y\in\mathbb{R}} |g(a_n y + b_n) - g(y)|$, fix $\varepsilon > 0$. By uniform continuity of g, there exists $\delta > 0$ such that $|u - v| < \delta$ implies $|g(u) - g(v)| < \varepsilon$. For large n, since $a_n \to 1$ and $b_n \to 0$, we have $\sup_y |(a_n - 1)y + b_n| < \delta$ on any compact set, and the tail contribution is negligible because $g(y) \to 0$ uniformly as $|y| \to \infty$. Hence $\sup_{y\in\mathbb{R}} |g(a_n y + b_n) - g(y)| < \varepsilon$ for n large. $\square$

D Proof of Theorem 3

The proof of Theorem 3 makes use of the following lemma, which is a strengthening of Lemma 2.

Lemma 5. *Let $\Phi^A, \Phi^B : \mathcal{L} \times \mathcal{L} \to \mathcal{F}$ be continuous noise assignments. Let U_A and U_B be vNM utility functions with the property for each $M \in \mathbb{R}$ there exist lotteries $S >_{\mathrm{c}} R$ such that*

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M.$$

Then there exist lotteries $S >_{\mathrm{c}} R$ such that S is chosen more frequently under (U_B, Φ^B) than (U_A, Φ^A).

Proof of Lemma 5. Let U_A and U_B be as in the statement of the lemma. Let $Z, Z', \Lambda \in \mathcal{L}$ such that Z and Z' have the same distribution, and Λ is uniformly distributed on $[0, 1]$ and independent of Z and Z'. Let $F^A = \Phi^A(Z, Z')$ and $F^B = \Phi^B(Z, Z')$ and let $f^A(t) = \frac{d}{dt}F^A(t)$ and $f^B(t) = \frac{d}{dt}F^B(t)$. Let $\varepsilon_A > f^A(0)$ and $\varepsilon_B < f^B(0)$. Since f^A and f^B are positive and continuous at 0, there is $\delta > 0$ such that for all $0 < t < \delta$, $f^A(t) < \varepsilon_A$ and $f^B(t) > \varepsilon_B$.

By assumption, there are lotteries $X >_{\mathrm{c}} Y$ such that

$$\frac{U_B(X) - U_B(Y)}{U_A(X) - U_A(Y)} > \frac{\varepsilon_A}{\varepsilon_B}.$$

Since the inequality only depends on the distributions of X and Y, we may choose X and Y to be independent of Λ. We define the random variables S_λ and R_λ by

$$S_\lambda(\omega) = \begin{cases} X(\omega) & \Lambda(\omega) \leq \lambda \\ Z(\omega) & \Lambda(\omega) > \lambda \end{cases}$$

and

$$R_\lambda(\omega) = \begin{cases} Y(\omega) & \Lambda(\omega) \leq \lambda \\ Z'(\omega) & \Lambda(\omega) > \lambda. \end{cases}$$

Note that for $\lambda \in (0, 1)$, $S_\lambda >_{\mathrm{c}} R_\lambda$ and

$$U_A(S_\lambda) - U_A(R_\lambda) = \lambda(U_A(X) - U_A(Y)),$$

since $U_A(Z) = U_A(Z')$. Hence,

$$\frac{U_B(S_\lambda) - U_B(R_\lambda)}{U_A(S_\lambda) - U_A(R_\lambda)} = \frac{U_B(X) - U_B(Y)}{U_A(X) - U_A(Y)} > \frac{\varepsilon_A}{\varepsilon_B}, \tag{13}$$

and $U_A(S_\lambda) - U_A(R_\lambda), U_B(S_\lambda) - U_B(R_\lambda) > 0$.

Let $F_\lambda^A = \Phi^A(S_\lambda, R_\lambda)$, $F_\lambda^B = \Phi^B(S_\lambda, R_\lambda)$ and let $f_\lambda^A(t) = \frac{d}{dt} F_\lambda^A(t)$ and $f_\lambda^B(t) = \frac{d}{dt} F_\lambda^B(t)$. Since $S_\lambda \to Z$ and $R_\lambda \to Z'$ as $\lambda \to 0$, continuity of Φ^A and Φ^B imply that $f_\lambda^A \to f^A$ and $f_\lambda^B \to f^B$ compactly. Thus, there is $\delta' > 0$ and λ_A such that for all $\lambda \in (0, \lambda_A)$ and all $t \in [0, \delta']$, $f_\lambda^A(t) < \varepsilon_A$ and $F_\lambda^A(t) < \frac{1}{2} + t\varepsilon_A$. Likewise, there is λ_B such that for all $\lambda \in (0, \lambda_B)$ and all $t \in [0, \delta']$, $F_\lambda^B(t) > \frac{1}{2} + t\varepsilon_B$.

For λ small enough,

$$U_A(S_\lambda) - U_A(R_\lambda) < \delta'$$

and

$$U_B(S_\lambda) - U_B(R_\lambda) < \delta'.$$

Thus, for λ small enough, we have

$$\begin{aligned} F_\lambda^B(U_B(S_\lambda) - U_B(R_\lambda)) &> \frac{1}{2} + (U_B(S_\lambda) - U_B(R_\lambda))\varepsilon_B \\ &> \frac{1}{2} + (U_A(S_\lambda) - U_A(R_\lambda))\varepsilon_A \\ &> F_\lambda^A(U_A(S_\lambda) - U_A(R_\lambda)). \end{aligned}$$

The second inequality follows from (13).

$\square$

By Lemma 1, any distinct CARA utilities U_A and U_B, where U_A is more risk-averse than U_B, satisfy the hypothesis of Lemma 5, and likewise for CRRA utilities. Hence, Theorem 3 follows from Lemma 5.

E NEU Models

Recall that $\mathcal{H}$ is the set of all continuously differentiable noise structures H whose partial derivatives are nowhere zero.

Proof of Theorem 2. First, we show that NEU models are a subclass of menu-dependent Fechnerian models. Indeed, let $H \in \mathcal{H}$ and let U be a CARA or CRRA vNM utility. Define $\Phi_{X,Y}(t) = H(\bar{U} + \frac{t}{2}, \bar{U} - \frac{t}{2})$ where $\bar{U} = \frac{U(X)+U(Y)}{2}$. Then

$$\Phi_{X,Y}(U(X) - U(Y)) = H(U(X), U(Y)).$$

Note that $\Phi_{X,Y}(t) + \Phi_{X,Y}(-t) = 1$ since $H(s,t) + H(t,s) = 1$. Let $\phi_{X,Y}(t) = \frac{d}{dt}\Phi_{X,Y}(t)$. Then

$$2\phi_{X,Y}(t) = H_1\left(\bar{U} + \frac{t}{2}, \bar{U} - \frac{t}{2}\right) - H_2\left(\bar{U} + \frac{t}{2}, \bar{U} - \frac{t}{2}\right).$$

Since H_1 is continuous and positive and H_2 is continuous and negative, it follows that $\phi_{X,Y}(t)$ is continuous and positive. Thus, $\Phi_{X,Y} \in \mathcal{F}$ for all $(X,Y) \in \mathcal{L} \times \mathcal{L}$.

It remains to be shown that Φ is a continuous assignment, i.e., when $X_n \to X$ and $Y_n \to Y$, it holds that $\phi_{X_n,Y_n} \to \phi_{X,Y}$ uniformly on each compact set. Letting $\bar{U}_n = \frac{U(X_n)+U(Y_n)}{2}$, we have

$$2\phi_{X_n,Y_n}(t) = H_1\left(\bar{U}_n + \frac{t}{2}, \bar{U}_n - \frac{t}{2}\right) - H_2\left(\bar{U}_n + \frac{t}{2}, \bar{U}_n - \frac{t}{2}\right).$$

Since U is a vNM utility function, it is continuous, and $\bar{U}_n \to \bar{U}$. Since H_1 and H_2 are continuous, they are uniformly continuous on compact sets. Thus,

$$\lim_n \sup_{-1 \le t \le 1} |\phi_{X_n,Y_n}(t) - \phi_{X,Y}(t)| = 0.$$

The result follows from Theorem 3. $\square$

F Monotonicity Results

Recall that U_A is absolutely more concave than U_B if the difference of their Bernoulli utilities is concave. Proposition 3 states that absolute comparative concavity is equivalent to stochastic comparative risk aversion under identical noise.

Proof of Proposition 3. Since A and B have the same noise structure F which is a strictly increasing function, for A to be stochastically more risk-averse than B, we require that for all $S \geq_{\mathrm{c}} R$,

$$U_A(S) - U_A(R) \geq U_B(S) - U_B(R),$$

or equivalently,

$$U_A(S) - U_B(S) \geq U_A(R) - U_B(R). \tag{14}$$

We must show that (14) is equivalent to the concavity of $u_A - u_B$. Let $h = u_A - u_B$. If h is concave, then (14) is satisfied by the definition of the concave order. Conversely, for the sake of contradiction, suppose that h is not concave. Then there are $x, y \in \mathbb{R}$ and $\lambda \in (0, 1)$ such that

$$h(\lambda x + (1 - \lambda)y) < \lambda h(x) + (1 - \lambda)h(y).$$

Let S denote a degenerate lottery that always pays $\lambda x + (1 - \lambda)y$, and let R denote a lottery that pays x with probability λ and y with complementary probability. Then

$$U_A(S) - U_B(S) = h(\lambda x + (1 - \lambda)y) < \lambda h(x) + (1 - \lambda)h(y) = U_A(R) - U_B(R),$$

contradicting (14). Thus $u_A - u_B$ is concave, meaning $U_A \geq_{\mathrm{abs}} U_B$. $\square$

F.1 Proof of Theorem 4

Theorem 4 provides a characterization of vNM utilities that lead to consistent predictions under some noise structures.

Proof of Theorem 4. In Section 7, we showed that $kU_A \geq_{\mathrm{abs}} U_B$ suffices for A to be stochastically more risk averse than B under some noise structures. Here, we show that it is necessary.

Contrapositively, suppose that, for all $k > 0$, it is not the case that $kU_A \geq_{\text{abs}} U_B$. Then, by Proposition 3, for each $k > 0$, there are lotteries $S >_{\text{c}} R$ such that

$$k(U_A(S) - U_A(R)) < U_B(S) - U_B(R).$$

Hence, the result follows from Lemma 5. □

G Comparison with Standard Comparative Risk Notions

Note that U_A is more risk-averse than U_B in the Arrow-Pratt sense if and only if, for all x,

$$\frac{u''_B(x)}{u''_A(x)} \leq \frac{u'_B(x)}{u'_A(x)}. \tag{15}$$

While stochastic comparative risk depends on bounding the ratio of second derivatives by a constant, the traditional Arrow-Pratt notion of comparative risk depends on bounding this ratio by the ratio of marginal utilities. Thus, when $u'_B(x)/u'_A(x)$ is bounded, the traditional notion of comparative risk is sufficient for the stochastic notion. On the other hand, Ross' (1981) stronger comparative notion requires

$$\frac{u''_B(x)}{u''_A(x)} \leq k \leq \frac{u'_B(x)}{u'_A(x)}, \tag{16}$$

for some $k > 0$ and all x, implying the boundedness of u''_B/u''_A. Moreover, Ross (1981) shows (16) is equivalent to the existence of k such that $ku_A - u_B$ is concave and decreasing, while our condition does not require it to be decreasing.

The next proposition states that, in the presence of noise related by a scale factor, Ross' (1981) notion captures stochastic comparative risk aversion stemming from preferences rather than greater choice precision. To formalize this, we introduce the following notion of relative choice precision.

We say that A has *greater choice precision* than B if, whenever X first-order dominates Y, A chooses X more than B.

Proposition 6. *Let $U_A = \mathbb{E}[u_A]$ and $U_B = \mathbb{E}[u_B]$ be vNM utilities. Define $F_A(t) = F(t/\sigma_A)$ and $F_B(t) = F(t/\sigma_B)$ for some $\sigma_A, \sigma_B > 0$ and $F \in \mathcal{F}$. Then A is stochastically more risk-averse than B while B has greater choice precision if and only if $u_A/\sigma_A - u_B/\sigma_B$ is decreasing and concave.*

Proof of Proposition 6. For any $X, Y \in \mathcal{L}$, A chooses X over Y more than B iff

$$U_A(X)/\sigma_A - U_B(X)/\sigma_B \geq U_A(Y)/\sigma_A - U_B(Y)/\sigma_B. \tag{17}$$

Thus, if $u_A/\sigma_A - u_B/\sigma_B$ is decreasing and concave, then A chooses X less often when X first-order dominates Y and more often when X concave-order dominates Y.

Conversely, suppose that A is stochastically more risk-averse than B while B has greater choice precision. It follows from Proposition 3 that $u_A/\sigma_A - u_B/\sigma_B$ is concave. Moreover, for any $x > y$, a degenerate lottery X paying x first-order dominates Y paying y. Since B has greater choice precision, the right-hand side of (17) is larger, and we conclude that $u_A/\sigma_A - u_B/\sigma_B$ is decreasing.

□

H Axiomatic Foundation of NEU

In this section, we formulate axioms on stochastic choice rules that underpin NEU models. Formally, a stochastic choice rule $\rho : \mathcal{L} \times \mathcal{L} \to [0, 1]$ is any function satisfying

$$\rho(X, Y) + \rho(Y, X) = 1,$$

where $\rho(X, Y)$ is interpreted as the probability that X is chosen over Y. We say that ρ is monotone in a utility $U : \mathcal{L} \to \mathbb{R}$ if for all X, Y, Z, $U(X) \geq U(Y)$ implies that $\rho(X, Z) \geq \rho(Y, Z)$.[31] As the following lemma states, ρ must be a function of utilities.

Lemma 6. *If ρ is monotone in a utility U, then there is a function $H : \mathbb{R}^2 \to [0, 1]$ such that for all $X, Y \in \mathcal{L}$,*

$$\rho(X, Y) = H(U(X), U(Y)).$$

Moreover, H is increasing in the first argument, decreasing in the second argument, and satisfies $H(s, t) + H(t, s) = 1$.

Proof of Lemma 6. For all $s, t \in Im(U)$, define $\tilde{H}(s, t) = \rho(X, Y)$ for some X and Y with $U(X) = s, U(Y) = t$. To see that $\tilde{H}$ is well-defined, suppose that $U(X') = s, U(Y') = t$. Then, since ρ is monotone in U,

$$\rho(X', Y') = \rho(X, Y') = 1 - \rho(Y', X) = 1 - \rho(Y, X) = \rho(X, Y).$$

[31] This property is related to *strong stochastic transitivity*. See Strzalecki (2025).

Clearly,

$$\tilde{H}(s,t)+\tilde{H}(t,s)=\rho(X,Y)+\rho(Y,X)=1.$$

Moreover, $\tilde{H}$ must additionally satisfy:

$$s_1 \leq s_2, t_1 \geq t_2 \implies \tilde{H}(s_1,t_1) \leq \tilde{H}(s_2,t_2),$$

for all $s_1, s_2, t_1, t_2 \in Im(U)$. Indeed, $\tilde{H}(s_1,t_1) \leq \tilde{H}(s_2,t_1) = 1-\tilde{H}(t_1,s_2) \leq 1-\tilde{H}(t_2,s_2) = \tilde{H}(s_2,t_2)$.

Define the lower and upper extensions to $\mathbb{R}^2$:

$$H_L(x,y)=\sup\{\tilde{H}(s,t) \mid s,t \in Im(U), s \leq x, t \geq y\}, \quad \text{setting} \quad \sup\emptyset = 0,$$
$$H_U(x,y)=\inf\{\tilde{H}(s,t) \mid s,t \in Im(U), s \geq x, t \leq y\}, \quad \text{setting} \quad \inf\emptyset = 1.$$

Note that $H_L \leq H_U$. Define $H(x,y)=\frac{1}{2}\left(H_L(x,y)+H_U(x,y)\right)$. Clearly, H extends $\tilde{H}$, as if $x,y \in Im(U)$, then $H_L(x,y)=H_U(x,y)=\tilde{H}(x,y)$.

Moreover, H_L and H_U are monotone in their first argument; hence, so is their average, H. Finally, note that

$$\begin{aligned}
H_L(y,x) &= \sup\{\tilde{H}(s,t) \mid s,t \in Im(U), s \leq y, t \geq x\} \\
&= \sup\{1-\tilde{H}(t,s) \mid s,t \in Im(U), s \leq y, t \geq x\} \\
&= 1-\inf\{\tilde{H}(t,s) \mid t,s \in Im(U), t \geq x, s \leq y\} \\
&= 1-H_U(x,y).
\end{aligned}$$

Hence, $H(x,y)+H(y,x)=1$, and it follows that H is decreasing in its second argument. □

Since monotonicity in a utility is an ordinal requirement, we could rather impose that ρ is monotone in a weak order $\succsim$ on $\mathcal{L}$. If we additionally impose that $\succsim$ satisfy the expected-utility axioms: independence and continuity, and moreover that $X \sim Y$ whenever they have the same distribution, then the conclusion of Lemma 6 holds for some vNM utility U.