Math 401 - Homework #7
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due in class Wednesday, 6/20

Instructions: Work through the problems below. Write down your solutions on paper. You can use MATLAB for computations unless the problem says "Must do all computations by hand." For problems in which you use MATLAB to do computations, I do not want you to turn in any MATLAB code. You need to clearly present all steps of your solution so that I can follow your thought process without guessing what you are thinking (try writing in complete sentences.)

1. Do Exercise 7.7 from the text.

2. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.

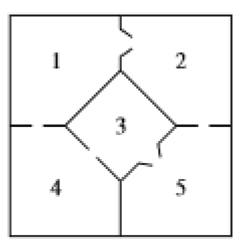
   (a) Write down the transition matrix T for this situation.

   (b) If a student is healthy today, what is the probability that he or she will be healthy three days from now?

   (c) On any given day, what percentage of students do you expect to be ill?

   (d) Suppose that 30% of all students are ill after a recent outbreak. How many days need to pass until the percentage of ill students is within a 1% difference of what we expect to have under normal circumstances?

3. Suppose a mouse is in the maze below, which has some "one way doors".

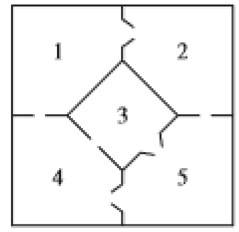


   Doors like the one connecting room 3 to room 5 are one way. That door can be traversed from room 3 to room 5, but not from room 5 to room 3. Doors like the one connecting rooms 1 and 4 can be traversed in either direction. Suppose that a mouse moves to a new room each time step. Assume that the mouse is equally likely to take any one of the exits out of the room it is in.

   (a) Give the transition matrix T for this situation.

   (b) Is T regular? Explain your answer.

   (c) Suppose we leave the mouse in the maze for a long time. After returning, we check to see what room the mouse is in. For each room, give the probability that we find the mouse there.

4. Repeat the previous question with the following maze: 

5. Suppose a very simple version of English consists of five parts of speech: Noun (cat, dog, etc.), Verb (eat, sleep, etc.), Preposition (to, with, etc.), Article (a, the, etc.) and Conjunction (and, or). In any given sentence each of these has a probability of being followed by one of the others according to the following chart. To clarify, for example, the value 0.65 means that a Verb has a 0.65 probability of being followed by an Article.

	N	V	P	A	C
N	0	0.20	0.05	1.00	0.25
V	0.70	0	0.15	0	0.30
P	0.10	0	0	0	0.10
A	0	0.65	0.80	0	0.35
C	0.20	0.15	0	0	0

(a) The sentence fragment “the cat eats the cheese” has the form ANVAN. Starting with A, calculate the associated probability of ANVAN appearing.

(b) The sentence fragment “dog to eat and with” has the form NPVCP. Starting with N, calculate the associated probability of NPVCP appearing.

(c) Compare the previous two values. What does their difference suggest?

(d) What is the probability that a Noun is followed two words later by an Article?
Note: This means something like “Noun ***** Article”.

(e) What is the probability that an Article is followed four words later by a Noun?

(f) The steady state vector can be interpreted to give the probabilties of the parts of speech appearing in the language. Calculate these probabilities.

6. **(Must do all computations by hand.)** Suppose that $T = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$ is a regular transition matrix with $0 < a, b < 1$.

(a) Show that $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for T and find the eigenvalue λ. (Hint: Multiply $T\mathbf{w}$.)

(b) Explain why the eigenvalue λ from the previous part satisfies $-1 < \lambda < 1$.

(c) If we were to draw all of the unit vectors in $\mathbb{R}^2$, we would get the unit circle. Probability vectors are different than unit vectors. Draw a sketch of all probability vectors in $\mathbb{R}^2$.

(d) Use your sketch the explain why the difference of any two probabilities vectors in $\mathbb{R}^2$ is a scalar multiple of $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(e) Let $\mathbf{v}$ denote a steady-state vector (a probability eigenvector for $\lambda = 1$) for the 2×2 transition matrix T given above. If $\mathbf{x}_0$ is any other probability vector, then we can write $\mathbf{x}_0 = \mathbf{v} + c\mathbf{w}$ for some scalar c, by the previous part. Use this and the fact that $\mathbf{v}, \mathbf{w}$ are eigenvectors to show directly that $\lim_{k\to\infty} T^k \mathbf{x}_0 = \mathbf{v}$. (You should not be citing the theorem from class. This is justifying why the theorem works in the 2×2 case.)