Math 401 - Homework #6
Discrete Dynamical Systems
due in class Monday, 6/18

Instructions: Work through the problems below. Write down your solutions on paper. You can use MATLAB for computations unless the problem says "Must do all computations by hand." For problems in which you use MATLAB to do computations, I do not want you to turn in any MATLAB code. You need to clearly present all steps of your solution so that I can follow your thought process without guessing what you are thinking (try writing in complete sentences.)

Here are some MATLAB remarks:

- `eig(A)` returns the eigenvalues of a matrix.
- `[P,D] = eig(A)` will diagonalize `A`, giving you the appropriate `P` and `D`.
- If you declare a symbolic variable, for example `syms k`, MATLAB won't be very helpful if you enter `A^k`.
- However, MATLAB will properly evaluate `D^k` for a diagonal matrix. You can then enter `P*D^k*inv(P)` to get `A^k`.
- Sometimes the output is very unreadable (with crazy fractions). It is related to the fact that the expression involves the symbolic variable `k`. You can make the output better by typing `vpa(P*D^k*inv(P),4)`

1. Let $A = \begin{bmatrix} 0.5 & 0.6 \\ -0.3 & 1.5 \end{bmatrix}$.
   (a) Give the eigenvalues and eigenvectors for A.
   (b) Give a formula for A^k. (Follow the MATLAB advice above.)
   (c) Give a formula for $\mathbf{x}_k$, the solution of the discrete dynamical system
   $$\mathbf{x}_{k+1} = A\mathbf{x}_k, \qquad \mathbf{x}_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}.$$
   (d) Viewing this dynamical system as a predator-prey system (as we had in class), do the populations of both species grow or die out?

2. We are now studying a colony of beetles. Suppose that beetles live, at most, 4 years. We divide the beetle population into four groups and keep track of their numbers year by year:
   $$\begin{aligned} A_k &= \text{number of beetles in the 0-1 age group after } k \text{ years.} \\ B_k &= \text{number of beetles in the 1-2 age group after } k \text{ years.} \\ C_k &= \text{number of beetles in the 2-3 age group after } k \text{ years.} \\ D_k &= \text{number of beetles in the 3-4 age group after } k \text{ years.} \end{aligned}$$
   Suppose that, on average, the four groups reproduce at a rate of 0, 4, 2, 1 per year (respectively). So beetles in their first year of life do not reproduce, and beetles age 1-2 produce 4 offspring per beetle, etc. Suppose also that the survival rates of the four groups are 0.7, 0.5, 0.2, 0 respectively. So 70% of beetles age 0-1 will survive to become age 1-2, etc. Note that beetles do not live past 4 years, so all beetles age 3-4 will not survive to the next year.

(a) Use the information about reproduction to express A_{k+1} in terms of A_k, B_k, C_k, D_k. (Note: newborn beetles are in the 0-1 age group!)

(b) Use the information about survival rates to find equations for each of $B_{k+1}, C_{k+1}, D_{k+1}$ in terms of the quantities A_k, B_k, C_k, D_k of the previous year.

(c) You should have four equations. Let $\mathbf{x}_k = \begin{bmatrix} A_k \\ B_k \\ C_k \\ D_k \end{bmatrix}$. Use your equations to create a discrete dynamical system $\mathbf{x}_{k+1} = L\mathbf{x}_k$ for some 4×4 matrix L (called a *Leslie matrix*).

(d) Suppose the initial beetle populations are given by $\mathbf{x}_0 = \begin{bmatrix} 30 \\ 35 \\ 25 \\ 10 \end{bmatrix}$. Determine the beetle populations after 20 years. Separate your answer into age groups.

(e) Give the eigenvalues of the Leslie matrix L. (Note that after a long time, the numbers of beetles will (approximately) grow exponentially with base λ_1, the largest eigenvalue.)

3. **(Must do all computations by hand.)** The sequence of *Lucas numbers* are defined as follows:

$$L_0 = 2, \qquad L_1 = 1, \qquad L_k = L_{k-1} + L_{k-2} \qquad (k \geq 2).$$

Note the recurrence relation is the same as for the Fibonacci numbers, but the "seeding" is different.

(a) Write the Lucas numbers L_k for $k = 0, 1, \ldots, 8$.

(b) Find a formula for L_k. You can use the fact that we already diagonalized

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \varphi & \psi \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix},$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}, \quad \psi = \dfrac{1-\sqrt{5}}{2}$.

4. **(Must do all computations by hand.)** Define a sequence G_k by

$$G_0 = 0, \qquad G_1 = 1, \qquad G_k = \frac{G_{k-1} + G_{k-2}}{2} \qquad (k \geq 2).$$

The next term in the sequence is the average of the previous two.

(a) Compute G_4.

(b) As we did for Fibonacci numbers, set up a discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ which can be used to solve for G_k.

(c) Find a formula for G_k. (Remember you must do this by hand. Everything works out nicely.)

(d) Compute $\lim_{k\to\infty} G_k$.

(e) Suppose we changed the seeding $G_0 = a$ and $G_1 = b$, but kept the recurrence relation. Express $\lim_{k\to\infty} G_k$ in terms of a and b.

5. **(Must do all computations by hand.)** Let λ be a scalar, and let $J_\lambda = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of J_λ.

(b) Explain why J_λ is not diagonalizable.

(c) Compute J_λ^k for $k = 2, 3, 4, 5$ by simply doing matrix multiplication.

(d) Based on the evidence of the previous part, give a general formula for J_λ^k. (You don't have to prove it is correct.)

(e) Use the previous part to give a closed formula for the solution $\mathbf{x}_k$ of the discrete dynamical system

$$\mathbf{x}_{k+1} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x}_k, \qquad \mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

6. Define the *Tribonacci numbers* T_k by

$$T_0 = 0, \qquad T_1 = 0, \qquad T_2 = 1, \qquad T_k = T_{k-1} + T_{k-2} + T_{k-3} \qquad (k \geq 3).$$

The next Tribonacci number is the sum of the previous three.

(a) Let $\mathbf{x}_k = \begin{bmatrix} T_k \\ T_{k+1} \\ T_{k+2} \end{bmatrix}$. Find a 3×3 matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

(b) Give the eigenvalues of A. (You should be using MATLAB here.)

(c) Two eigenvalues should be nonreal. For these λ, show that $|\lambda| < 1$ and consequently $\lambda^k \to 0$ as $k \to \infty$. (Recall the absolute value of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.)

(d) By appealing to the previous part, give an approximation for T_k in terms of the real eigenvalue of A.