Math 401 - Groupwork #11
Singular Value Decomposition

Instructions: Work through the problems below in order. Write down all of your answers on paper, to be turned in at the end of class. I do not want you to turn in any MATLAB code. Make sure that the name of every student in the group is on the paper.

1. If `A` is a real symmetric $n \times n$ matrix, then the MATLAB command `[P,D] = eig(A)` will produce the orthogonal diagonalization of `A`.

   (a) Use MATLAB to give the orthogonal diagonalization $A = PDP^T$ of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

   (b) Compute $P^T P$. What does the result confirm about P?

2. Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 1 & 5 \end{bmatrix}$. In this problem, do not use MATLAB's `svd` command.

   (a) Compute $A^T A$ and use the `eig` command to find the eigenvalues/eigenvectors of $A^T A$.

   (b) Compute AA^T and use the `eig` command to find the eigenvalues/eigenvectors of AA^T.

   (c) Give the singular values of A.

   (d) Give the right singular vectors of A.

   (e) Give the left singular vectors of A. Be sure they are compatible with the right singular vectors in the sense that $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$.

   (f) Put everything together to give the singular value decomposition $A = U\Sigma V^T$.

   (g) Use your singular value decomposition of A to give a singular value decomposition of A^T (Hint: think about what happens if you take the transpose of $A = U\Sigma V^T$.)

   (h) Use your U, Σ, V to compute the pseudoinverse A^+ of A.

   (i) Use the pseudoinverse A^+ to find the least-squares solution of $A\mathbf{x} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

   (j) The columns of A are linearly independent. Compute $(A^T A)^{-1} A^T$ in MATLAB. Is it the same as the pseudoinverse A^+?

   (k) Use your singular value decomposition for A^T to compute the pseudoinverse $(A^T)^+$. How does it relate to A^+?

   (l) Use the pseudoinverse to find a least-squares solution to $A^T\mathbf{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

   (m) The system from the previous part is actually consistent, with infinitely many solutions. Your least-squares solution is actually a solution, one of infinitely many. What is special about the one you found?

More on the back!

3. MATLAB can compute an SVD instantly using the command `[U,S,V] = svd(A)`. It will assign U, Σ, V from the SVD $A = U\Sigma V^T$ to the variables `U`, `S`, `V`. It can also compute a pseudoinverse using `pinv(A)`.

   (a) Use the `svd` command to give the SVD of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

   (b) Is it true or false that $A^2 = U\Sigma^2 V^T$?

   (c) Use your `U`, `S`, `V` to compute the pseudoinverse A^+. Then use the command `pinv(A)` to verify your pseudoinverse is correct.

4. Consider the team ranking problem below in which we have four teams such that each pair of teams played exactly once:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ -7 \\ 5 \\ 1 \\ -6 \end{bmatrix}.$$

   Note that we are not including the requirement that the sum of the rankings is 0 in the above linear system.

   (a) This linear system is inconsistent and has infinitely many least-squares solutions. Use the pseudoinverse of the coefficient matrix A to find a least-squares solution.

   (b) Do your rankings that you found using the pseudoinverse satisfy $r_1 + r_2 + r_3 + r_4 = 0$?

   (c) Write down the pseudoinverse A^+.

   (d) The pseudoinverse encodes instructions for how to take the vector containing the margins of victory and produce the team rankings. Stare at the matrix for a while to try to figure out what is going on. Then explain the relationship between a team's numerical ranking and the three margins of victory (or defeat) in the three games it played. (Note: this is all specific to the scenario described by A: four teams, each pair plays exactly once.)

   (e) Suppose that the total number of points that team 2 scored in its three games is greater than the total number of points its opponents scored in those three games (This doesn't imply that team 2 won all three games.) Explain why $r_2 > 0$.