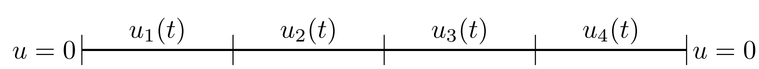
Math 401 - Groupwork #9
Heat Diffusion and Systems of Differential Equations

Instructions: Work through the problems below in order. Write down all of your answers on paper, to be turned in at the end of class. I do not want you to turn in any MATLAB code. Make sure that the name of every student in the group is on the paper.

Here are some MATLAB remarks:

- The command to compute the matrix exponential e^A is `expm(A)`.
- DO NOT accidentally use the command `exp(A)`. This command DOES NOT give the matrix exponential e^A. Instead, it just exponentiates each entry of the matrix (this is very different from a matrix exponential.)
- If you declare a symbolic variable `syms t`, MATLAB can usually handle evaluating `expm(t*A)` and presenting it as a function of `t`.
- Sometimes, MATLAB tries too hard to give exact symbolic expressions and it becomes unreadable. Sometimes it is best to use `vpa(expm(t*A),4)`, which gives decimal approximations.
- Given an initial condition `x0` for $\mathbf{x}'(t) = A\mathbf{x}(t)$, you can enter `expm(t*A)*x0` to get the solution of the differential equation (`t` is symbolic, as above.)
- If you just want to know the values of the solution at a given time, say $\mathbf{x}(1.236)$, you can enter `expm(1.236*A)*x0`.

1. Consider a thin metal rod divided into four portions.



   As in class, we assume that heat can escape to the left of region 1 and to the right of region 4. Heat transfers across any boundary at a rate equal to the difference of the temperatures on both sides. Suppose that initially we have

$$u_1(0) = 0, \qquad u_2(0) = 100, \qquad u_3(0) = 0, \qquad u_4(0) = 0.$$

   (a) Write down the system of differential equations for $u_1(t), u_2(t), u_3(t), u_4(t)$, as in class. Then present the initial value problem in the form $\mathbf{u}'(t) = A\mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0$.

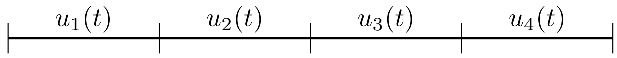
   (b) Compute $e^A\mathbf{u}_0$ using MATLAB and write the result down. What does this vector represent?

   (c) Approximately find the smallest time t beyond which all four temperatures are below 1. (Give your answer to the nearest tenth.)

   (d) Have MATLAB produce the solution $\mathbf{u}(t)$ of this problem, written symbolically in terms of t. (Don't write it down, yet.) Appreciate the fact that it is a little messy, and then rerun your command wrapped in a `vpa`, as described above. Write down the third function $u_3(t)$ of the solution.

   (e) Suppose that we change the initial condition so that $u_1(0) = u_2(0) = u_3(0) = u_4(0) = 100$. Do you think that all four portions will continue to have the same temperature as time goes on? Do something to figure out whether or not this is the case.

2. Consider a thin metal rod divided into four portions as before, but now we assume that heat cannot escape out of the rod.



So for example, the only heat that goes in/out of the first region comes/goes across the boundary with the second region. Consequently, $u_1'(t) = u_2(t) - u_1(t)$.

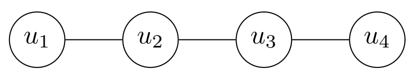
(a) Write the system $\mathbf{u}'(t) = A\mathbf{u}(t)$ of differential equations for this adjusted scenario.

(b) Suppose that $\mathbf{u}(0) = \begin{bmatrix} 0 & 100 & 0 & 0 \end{bmatrix}^T$, like in the previous problem. What do you expect to happen to $\mathbf{u}(t)$ after a long time?

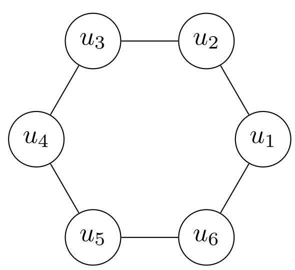
(c) Compute the solution $\mathbf{u}(t)$ to the initial value problem (with $\mathbf{u}_0$ from the previous part) and write it down (use `vpa`). Then use your solution to explain what happens to $\mathbf{u}(t)$ after a long time.

(d) Write down the eigenvalues of A. Then explain, in terms of the eigenvalues of A, why the solution $\mathbf{u}(t)$ did not just decay to $\mathbf{0}$.

3. Another way to think about the situation in the previous problem is as the graph



Each circle (also called vertex or node) represents one of the regions of the metal rod. A line (called an edge) connecting two vertices indicates that heat can flow between those regions. As in problem 2, we are assuming that heat cannot escape the system. So we can describe heat diffusion across differently shaped objects by specifying different graphs. For example, consider the hexagon below.



(a) Write down the diffusion equations $\mathbf{u}'(t) = A\mathbf{u}(t)$ for this hexagon.

(b) Write down the solution $\mathbf{u}(t)$ that has initial condition $\mathbf{u}_0 = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.

(c) Write down the eigenvalues of A. Consider your solution from the previous part. Which of the eigenvalues seems to be playing the largest role in determining the speed at which equilibrium is approached?

(d) Consider instead the initial condition $\mathbf{u}_0 = \begin{bmatrix} 2 & 0 & 2 & 0 & 2 & 0 \end{bmatrix}^T$. Write down the solution $\mathbf{u}(t)$ that corresponds to this initial condition. Comment on the speed at which it approaches equilibrium, contrasting it with the solution for the previous initial condition.

(e) Try to find another initial condition $\mathbf{u}_0$ which falls somewhere in between the previous two examples, in terms of speed in approaching equilibrium. Choose your $\mathbf{u}_0$ so that the sum of the initial temperatures is 6.

(f) Suppose we add edges connecting opposite vertices (u_1 to u_4, u_2 to u_5, u_3 to u_6) so that heat can also flow directly between these opposite regions. In general (meaning for a general initial condition), how will this change the speed at which equilibrium is approached? Explain in terms of eigenvalues. Experiment with different initial conditions.