Verifying the Differential Cross Section Formula for Rutherford Scattering and the Poisson Distribution in Alpha Decay of Americium-241 by a Rutherford Scattering Experiment

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An alpha particle scattering experiment was carried out to verify the formula of the differential cross section for Rutherford scattering and to verify the Poisson distribution of alpha particle emission by Americium-241. The scattering angle dependence of the differential cross section for Rutherford scattering and the Poisson distribution of alpha decay of ^{241}Am are generally supported by the experimental result. The experimental result is not consistent with the theoretical atomic number dependence of the differential cross section for Rutherford scattering, but the comparison factors (i.e. the atomic number ratio between gold and aluminum) from the experimental results and the theory still stay on the same scale.

1 Introduction

By the end of the 19th century it was widely accepted by physicists and chemists that matter is constituted of discrete units of "atoms". However, although many evidences at that time were suggesting that "atoms" should still possess an internal structure, this internal composition remained puzzling. J. J. Thomson, after discovering electron in 1897, proposed the "plum pudding" model in 1904 to picture how charges are arranged in an atom. This model was quite influential for a time until Ernest Rutherford, who was a former research student under Thomson's supervision at the Cavendish Laboratory, discredited it with his assistants after conducting a series of alpha particle scattering experiments from 1907 to 1913.[1,2] To accommodate these experimental results, Rutherford proposed in 1911 his "planetary" model of atom with an almost sizeless central charge, either positive or negative.[1] Two years later, Rutherford was able to prove that the "nucleus" (as a term coined by himself) is positively charged based on further experimental investigations.[3] Rutherford's scattering experiments and his "planetary" model of atom were crucial for Niels Bohr, who had worked under Rutherford's supervision at the then Victoria University of Manchester in 1912, to ultimately achieve the great leap in devising his quantum model of hydrogen[1].

As undergraduate students, we carried out a scattering experiment that emulate the ones performed by Rutherford and his assistants. In this experiment, we would use gold and aluminum foils to scatter alpha particles emitted by an Americium-241 source and count the number of scattered alpha particles detected at various angles. The goal of this experiment is to verify the formula of differential cross section for Rutherford scattering and to verify the Poisson distribution of alpha particle emission by Americium-241.

2 Theory

This section describes theories that are relevant to this experiment.

2.1 Two particle scattering

As shown in Fig 1, particle m_1 is approaching particle m_2 that is at rest in the lab frame. Suppose there is some force between the two particles so that the path of m_1 will be deflected when m_1 is moving toward m_2. Suppose (throughout this article) the mass of m_2 (also denoted by "m_2") is much greater than that of

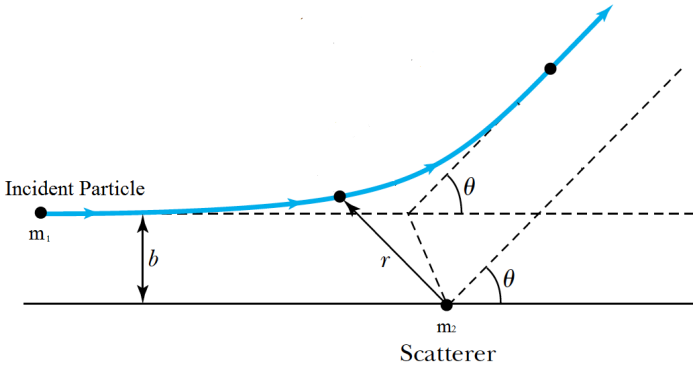

Figure 1: Schematic diagram of scattering between two particles

m_1 (also denoted by "m_1"), then we can assume that m_2 does not recoil as a result of its interaction with m_1, and thus we will call m_2 the scatterer (or deflector) and m_1 the incident particle. As shown in Fig 1, the distance b is called the **impact parameter**, which is the closest distance between the incident particle and the scatter if there were no interactions between them. Suppose the force between the two particles only depends on the distance r between the particles (central force), then the motion of the two particles should always stay on a fixed plane.[4] Then, we can define the angle between the initial velocity of the incident particle (when it was initially far away from the scatterer) and the final velocity of the incident particle (after it was deflected to a place far away from the scatterer) to be the **scattering angle** θ, as shown in Figure 1. As we can see in Fig 1, given the masses of the two particles, the kinetic energy (or speed) of the incident particle, and the force law between the incident particle and the scatterer, the scattering angle θ should solely depend on the impact parameter b. Therefore, for the two-particle scattering system with a central-force interaction, we should always be able to establish the a relationship $\theta = \theta(b)$, and equivalently,

$$b = b(\theta) \tag{1}$$

2.2 Scattering with an incident beam

Now instead of an incident particle, we consider an incident beam of particles moving toward a single scattering particle. Also, let's suppose that the incident beam has a uniform particle distribution. Then from

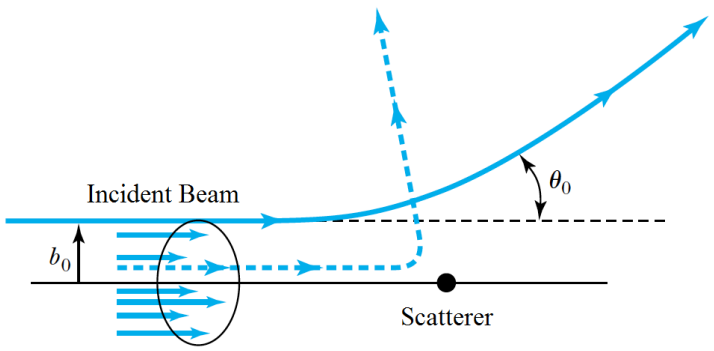

Figure 2: Scattering of a beam of incident particles with a single scattering particle

Figure 2, we can see that incident particles with a larger impact parameter will have a smaller scattering angle, because incident particles further away from the scatterer should be affected less. Then, as shown

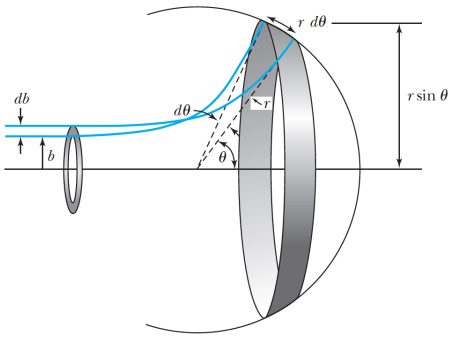

Figure 3: Scattering of incident particles with an impact parameter b to b+db into an angular region of θ to $\theta + d\theta$

in Figure 3, incident particles with an impact parameter from b to b+db will be scattered into an angular region of θ to $\theta + d\theta$, where db and $d\theta$ are opposite in sign. With this in mind, we can define an important quantity $\sigma(\theta)$, called the **differential cross section**,

$$\sigma(\theta) = \frac{dA}{d\Omega} \tag{2}$$

where $d\Omega$ is an infinitesimal angular region (or solid angle) that some incident particles are scattered into, and dA is the infinitesimal cross-sectional area of the incident beam that "contributes" the particles scattered to the solid angle $d\Omega$. For example, as per Figure 3, the ring-shaped solid angle from θ to $\theta + d\theta$ would be $d\Omega$, and the infinitesimal cross-sectional ring of the incident beam from b to db would be dA. Notice that, in Figure 2, the differential cross section is only a function of θ. This is true for a central-force interaction and a uniform incident beam (as in our case) that make the system azimuthally symmetric. In general, differential cross section $\sigma(\theta, \phi)$ is a function of both the polar angle θ and the azimuthal angle ϕ, taking the scatterer as the origin and direction of the incident particles as the +z direction. However, we will constrain our analyses to azimuthally symmetric scattering in this article. Therefore, as shown in Figure 3, the differential cross section $\sigma(\theta)$ in this case can be calculated from Equation 2 as

$$\sigma(\theta) = \frac{2\pi b\,\mathrm{db}}{2\pi \sin\theta\,|d\theta|} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{3}$$

where the absolute value comes from the opposite sign between db and $d\theta$.

Differential cross section is an important quantity because it allows us to know, in a relative sense, how many incident particles are scattered into a particular solid angle. As explained in the previous paragraph, we can interpret $\sigma(\theta)$ as the "contributing", or "effective" cross section of the incident beam in which all the incident particles are scattered into a small solid angle at polar angle θ. Therefore, the larger $\sigma(\theta)$ is, the larger the "contributing" cross section for the small solid angle at θ will be, and thus the more incident particles the small solid angle at θ will receive (since we assume the incident particles are uniformly distributed in the beam). Therefore, we can think of differential cross section $\sigma(\theta)$ as indicating the relative number of incident particles that are scattered into a small solid angle at θ.

2.3 Rutherford scattering

When the scattering process described above takes place between charged particles, it is called Rutherford (or Coulomb) scattering, where the dominant interaction between the incident particle and the scattering particle is the Coulomb force. The relationship between the impact parameter of the incident particle and its scattering angle (Equation 1) can be derived as

$$b = \frac{q_1 q_2/(4\pi\epsilon_0)}{2T}\cot(\theta/2) \tag{4}$$

where q_1 and q_2 are the respective charges of the incident and scattering particles, ϵ_0 is the permittivity of free space, and T is the initial kinetic energy of the incident particle when it is far away from the scatterer.[1]

Then, from Equation 3, we can derive the the differential cross section for Rutherford scattering of two particles as

$$\sigma(\theta) = \frac{[q_1 q_2/(4\pi\epsilon_0)]^2}{(4T)^2 \sin^4(\theta/2)} \tag{5}$$

Notice that a quantum mechanical calculation of the differential cross section for Rutherford scattering will surprisingly yield the same result.[5]

2.4 Alpha decay of Americium-241

Americium-241 (^{241}Am) is an isotope of Americium (atomic number 95). It is radioactive with a half-life of 432 years, and decays primarily through alpha decay during which an ^{241}Am atom will emit an alpha particle (i.e. Helium-4 nucleus).[6] Therefore, the equation that describes the decay of ^{241}Am is

$$N = N_0 e^{-\ln 2(\frac{t}{432yr})} \tag{6}$$

where N is the number of the ^{241}Am atoms present, N_0 is the initial number of ^{241}Am atoms, and t is time with $t = 0$ representing the initial time. Let's assume that an ^{241}Am atom is equally likely to decay at any instant of time and at any particular moment of time any two ^{241}Am atoms are equally likely to decay. Then the probability p for an ^{241}Am atom to decay in a time interval Δt can be calculated as

$$\begin{aligned} p &= \frac{\text{The number of decayed } ^{241}\text{Am atoms during } \Delta t}{\text{The number of } ^{241}\text{Am atoms present at the beginning of } \Delta t} \\ &= \frac{N_0 - N_0 e^{-\ln 2(\frac{\Delta t}{432yr})}}{N_0} \\ &= 1 - e^{-\ln 2(\frac{\Delta t}{432yr})} \\ &\approx \ln 2(\frac{\Delta t}{432yr}) \end{aligned} \tag{7}$$

when Δt is much smaller than 432 years. Notice that this is also the probability that an ^{241}Am atom will emit an alpha particle during Δt. Therefore, during a time interval Δt, the total number of alpha particles emitted is a random variable that follows a binomial distribution with parameters N_0 and p.[7] When N_0 is large and p is small, the binomial distribution can be approximated by a Poisson distribution with parameter $\lambda = N_0 p$.[7] In the case of alpha decay of Americium-241, When Δt is much smaller than the half-life of Americium-24, the probability p will be very small. Since the total number of atoms present in an ^{241}Am source is usually a very large number, we can treat the number of alpha particles emitted by the ^{241}Am source over a time interval much less than 432 years as a random variable of Poisson distribution, whose the expectation value (i.e. mean) and variance are equal.[8]

2.5 Confidence interval of Poisson distribution

Given an observation k from a random variable X of Poisson distribution, the confidence interval with a confidence level of (1-α) for the expectation value of X is

$$[\frac{1}{2}\chi^2(\alpha/2, 2k), \quad \frac{1}{2}\chi^2(1-\alpha/2, 2k+2)] \tag{8}$$

where $\chi^2(x, n)$ is the quantile function of the chi-squared distribution with n degrees of freedom.[9]

Since the number of alpha particles emitted by a large number of Americium-241 atoms over a relatively short duration is approximately a random variable of Poisson distribution, if we want to estimate the expectation value of this random variable from one observation, then we can use the above formula to estimate the uncertainty of this observation.

3 Experiment

3.1 Experimental setup

As shown in Figure 4, the apparatus of the Rutherford scattering experiment mainly consists of a scattering chamber, a discriminator, an oscilloscope, a counter, and a vacuum pump. The scattering chamber

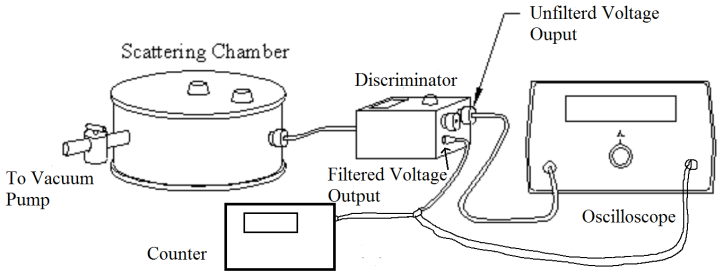

Figure 4: Experimental setup of the Rutherford scattering experiment

is where the scattering process will take place. It consists of a glass lid (transparent) and a cylindrical container. As shown in Figure 5, an alpha particle emitter and a foil plate holder are fixed on a swivel arm that

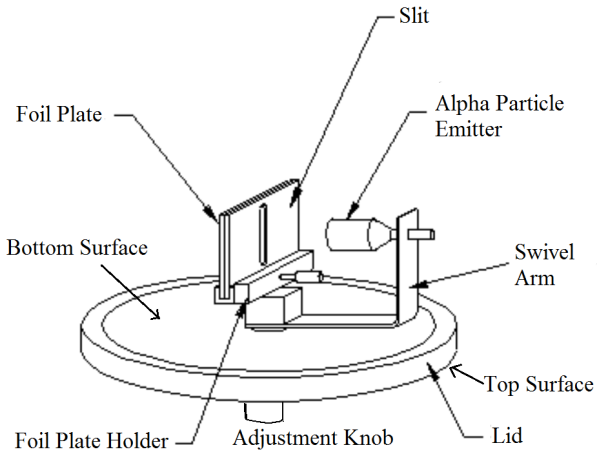

Figure 5: The lid of the scattering chamber

is attached to the bottom surface of the lid. The swivel arm can be rotated with an adjustment knob on the top surface of the lid. On the top surface of the lid, there is also an angular graduation (5 degrees between two adjacent calibrations) with reference to which the swivel arm can be rotated by a desirable angle (there is a straight hollow groove in the middle the swivel arm made to align with the angular calibrations).

The alpha particle emitter contains an Americium (^{241}Am) source, which constantly emits alpha particles as it decays. The foil plate holder is used for holding a removable foil plate which, as shown in Figure 6, consists of two taped-together plastic cards and a piece of thin metal foil (gold or aluminum, $2\mu m$ and $7\mu m$ thick respectively) in between. In the middle of one plastic card is an 1mm-wide slit, while at the center of the other plastic card is a circular hole that is considerably wider than the slit. Therefore, incident alpha particles can pass through the slit on the one side of the foil plate, hit on the foil, and then escape out of hole on the other side of the plate.

The alpha particle detector is fixed on the wall of the chamber container at a height such that it will stay on the same horizontal plane as the emitter when the lid is properly placed on the chamber container. When the swivel arm is rotated to different angles with respect to the graduation (we will call this "**calibration**

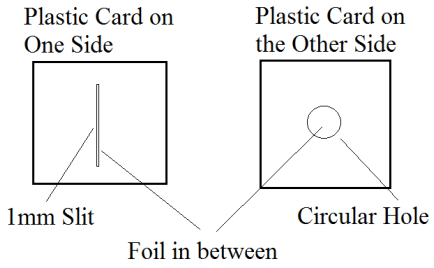

Figure 6: The foil plate

angle" throughout this article), the detector will be able to detect incident alpha particles that are deflected by different scattering angles. The 0° calibration angle should roughly corresponds to the 0° scattering angle.

Besides, the chamber container has on its wall a circular hole that is connected to a vacuum pump through a tube, and the chamber lid has a gasket on it. Therefore, when the lid closes on the scattering chamber container, the vacuum pump can be turned on to pump air out of the chamber without air leaking in from outside.

The alpha particle detector in the scattering chamber, of course, detects incident alpha particles. When an alpha particle arrives at the detector, the detector will send a current signal through a wire to the discriminator. When receiving the current signal, the discriminator will convert the current signal to a voltage pulse. Moreover, the discriminator can set a voltage filter level which can filter out voltage pulses (noise) that are lower than the filter level. As shown in Figure 4, the discriminator has two outputs, where one delivers the unfiltered voltage signals while the other transmits the filtered ones. Both of these outputs are connected to the oscilloscope so that on the display screen of the oscilloscope, we can check whether the voltage filter level is set properly to filter in only alpha particle detection signals. The filtered output of the discriminator is also connected to the counter, which counts the number of received filtered voltage pulses and displays the number on its screen. Therefore, we can directly read from the screen the number of particles detected.

3.2 Experimental theory

3.2.1 Scattering with metal foil

In Section 2, we described characteristics of Rutherford scattering with a single scattering particle. However, in this experiment, we were using gold and aluminum foils to produce the scattering effect, so we will have a large number of scattering particles. Therefore, we need to reconsider how to describe the scattering process in this situation. First, let's calculate the inter-nuclei spacing of gold (or if you do not want to assume so quickly that a gold atom has a nucleus, you can think of it as the side of the cube that each gold atom is assigned). If we take 1 cm^3 of gold, then at room conditions its has a mass of 19.3g. Since the molar mass of gold is 197g/mol, and we know 1 mol is equal to 6.022×10^{23}, we can calculate the number of gold atoms in a 1 cm^3 cube to be $\frac{19.3g}{197g/\text{mol}} \times (6.022 \times 10^{23}/\text{mol}) = 5.90 \times 10^{22}$. Therefore, the inter-nuclei spacing of gold will be $\frac{1cm}{\sqrt[3]{5.90 \times 10^{22}}} = 2.57 \times 10^{-10}m$. To simplify our analysis, we can try to think of the gold foil, which is quite thin (2 μm thick), to "compress" into one single layer of gold. A thickness of 2 μm will correspond to $\frac{2\mu m}{2.57 \times 10^{-10}m} = 8000$ gold atoms in a row, so if the gold foil were compressed to only one square layer of gold, each side of the square would need to accommodate $\sqrt{8000} \approx 90$ times the original number of gold atoms. Therefore, the **effective inter-nuclei spacing** in a gold foil "compressed" to one single layer would have an effective inter-nuclei spacing of $\frac{2.57 \times 10^{-10}m}{90} \approx 3 \times 10^{-12}m$.

If we consider half of the effective inter-nuclei spacing of the gold foil, which is $1.5 \times 10^{-12}m$, then using Equation 4, we can calculate how big a scattering angle that an impact parameter of half the effective inter-nuclei distance of gold would lead to. Let's consider a scattering nucleus with charge Ze, where e is

the proton charge, and remember that an alpha particle is a ^{4}He nucleus with charge 2e, then according Equation 4, the scattering angle θ for the incident alpha particle is

$$b = \frac{2Z(e^2/4\pi\epsilon_0)}{2T}\cot\frac{\theta}{2} \tag{9}$$

Since alpha decay of Americium-241 will emit alpha particles with a kinetic energy around 5.5MeV,[10] we can evaluate the above equation as

$$b = (2.6 \times 10^{-16}\text{meters})Z\cot\frac{\theta}{2} \tag{10}$$

For gold (Z=79), using the above equation, we can calculate the scattering angle corresponding to an impact parameter of half the effective inter-nuclei spacing to be 1.6 degrees, which is quite a small angle. Remember that the larger the impact parameter, the smaller the scattering angle. Therefore, we can see that the effective inter-nuclei distance of the gold foil is still large enough so that only alpha particles that are close enough to a gold nucleus will be deflected by a non-trivial angle. For example, if we were to produce a scattering angle of 5 degrees for an incident alpha particle, we would need an impact parameter of $(2.6 \times 10^{-16}\text{meters}) \times 79 \times \cot\frac{5^\circ}{2} = 4.7 \times 10^{-13}\text{meters}$, which is less than one sixth of the effective inter-nuclei distance of gold. Hence, this alpha particle would be hardly affected by the neighbors of the scattering nucleus (and the rest of the gold nuclei in the foil). Therefore, for scattering with a gold foil, it is reasonable to assume that, on average, an incident alpha particle that has a non-trivial final scattering angle (say, larger or equal to 5 degrees) should only be effectively scattered once by a single gold nucleus.

The same calculations and arguments can be applied to the aluminum foil (7μm thick), too. We found that aluminum has a nucleus number density of $6.02 \times 10^{22}/cm^3$ and an aluminum nucleus would have a scattering angle of 0.5° for its half inter-nuclei distance (even less than the 1.6° for gold). Therefore, the same conclusions for the gold foil can also be applied to the aluminum foil.

Notice that we are neglecting the effects of the electrons in the foils. Since the electrons are much lighter than the alpha particles, the trajectories of the alpha particles should be hardly affected by them.

3.2.2 Total differential cross section

Since there is a grating of metal nuclei in our foil scattering experiment, we need to reconsider how many of the incident particles would be scattered into a small solid angle $d\Omega$ at polar angle θ (see Figure 3). If we had only one scattering nucleus instead of a foil, then as described in Section 2.2, the corresponding differential cross section $\sigma(\theta)$ would represent the "contributing" cross-sectional area of the incident beam for this small solid angle $d\Omega$.

But now we have an effective layer of metal nuclei (as described in the previous section) and we know that the alpha particles deflected by non-trivial scattering angles will only be effectively scattered once. Moreover, each metal nucleus should have its own differential cross section for the same non-trivial scattering angle θ. Therefore, the total differential cross section for $d\Omega$ should be the sum of these individual differential cross sections, i.e. $\sigma(\theta)\times$(The number of metal nuclei on the part of the foil that the incident particles would pass through). Notice that in our experimental setting, the total incident cross section of the alpha particles on the metal foil is fixed by the total cross-sectional area of the incident beam and the width of the slit. Therefore, the total differential cross section for a solid angle at θ should be

$$\begin{aligned}&\sigma(\theta) \times \text{number density of the metal nuclei}\\ &\quad\times \text{thickness of the foil}\\ &\quad\times \text{the total cross-sectional area that the incident beam intersects with the slit}\end{aligned}$$

Since the total incident cross-sectional area on the slit is fixed, we have

$$\Sigma(\theta) \propto \sigma(\theta) \times n \times d \tag{11}$$

where $\Sigma(\theta)$ is the total differential cross section for a small solid angle $d\Omega$ at polar angle θ, n is the number density of the metal nuclei, and d is the thickness of the foil.

Notice that, experimentally the small solid angle is confined by the area of the detecting surface. In assuming that the detector will receive alpha particles all with the same scattering angle θ, we have taken advantage of the fact that the slit width is much smaller than the distance between the foil and the detector.

Applying Equation 5 in Equation 11, then for a fixed scattering angle θ but a variable type of the metal foil, we have

$$\Sigma(\theta) \propto Z^2 \times n \times d \tag{12}$$

where Z is the atomic number of the scattering nuclei.

Remember from Section 2.2 that the differential cross section for a scattering angle will indicate the relative number of incident particles that are deflected by that angle. Therefore, if we count the number of particles detected over a certain time interval at a particular scattering angle, then for gold and aluminum foils, the ratio of the number of detected alpha particles should follow, from Equation 12,

$$\begin{aligned} \frac{\mathrm{CR}_{Au}}{\mathrm{CR}_{Al}} &= (\frac{Z_{Au}}{Z_{Al}})^2 \times \frac{n_{Au}}{n_{Al}} \times \frac{d_{Au}}{d_{Al}} \\ &= (\frac{Z_{Au}}{Z_{Al}})^2 \times \frac{5.90 \times 10^{22}/cm^3}{6.02 \times 10^{22}/cm^3} \times \frac{2\mu m}{7\mu m} \\ &= 0.28 \times (\frac{Z_{Au}}{Z_{Al}})^2 \end{aligned} \tag{13}$$

where CR (Count Rate) means the number of alpha particles received at a fixed scattering angle for a fixed amount of time, and Au and Al denotes gold and aluminum respectively. Therefore, if we know the count rate of the alpha particles at a non-trivial angle for the gold and the aluminum foils, then we can obtain the ratio of the atomic number of gold and aluminum by

$$\frac{Z_{Au}}{Z_{Al}} = \sqrt{3.6 \times \frac{\mathrm{CR}_{Au}}{\mathrm{CR}_{Al}}} \tag{14}$$

3.2.3 Uncertainty estimation

In this experiment, we will typically measure the number count of alpha particles received at a particular scattering angle for an interval of time at a scale of at most several days. From Equation 7, we can calculate that the probability of an Americium-241 atom to decay over a day is $\ln 2 \frac{1day}{432yr} = 4 \times 10^{-6} << 1$. On the other hand, in undergraduate laboratories the amount of Americium-241 supplied for the Rutherford scattering experiment is typically characterized by a scale of 300kBq, where Bq (becquerel) means one nucleus decay per second. From Equation 7, we can calculate the probability for a singe Americium-241 atom to decay in a second is about 5×10^{-11}. Therefore, the number of Americium-241 supplied is on a scale of $\frac{300 \times 10^3}{5 \times 10^{-11}} = 6 \times 10^{15}$, which is a large number. So, according to Section 2.4, the total number of alpha particles emitted by the Americium source in this experiment during a fixed time interval of our interest should be well approximated by a random variable of Poisson distribution. Moreover, since the probability of a Americium-241 atom to decay over days is small, we can treat the total number of remaining Americium atoms as unchanged during the entire period of this experiment. Therefore, if we repeatedly counted the total number of alpha particles emitted during time intervals of equal length, then the number counts should be observations of the same Poisson random variable. Notice that according to Equation 11, over any non-trivial time interval, the ratio between the total number of alpha particles deflected by a fixed scattering angle and the total number of alpha particles emitted should be a constant given a fixed experimental setting. Therefore, the total number of alpha particles detected by a particular scattering angle θ should be $C(\theta)X$, where C is a function of θ that is always less than 1, and X is the Poisson random variable that describes the total number of alpha particles emitted during the fixed time interval. Notice that CX no longer follows a Poisson distribution, but for practical purpose we can still use Formula 8 to estimate the confidence intervals for CX. This is reasonable for the following reason. Over a given time interval, let's suppose the Americium source emits a total number of N alpha particles. Since for a Poisson random variable, the expectation value (mean) equals the square of the standard deviation, we know that the uncertainty for N with respect to the mean of X is on a scale of $\sqrt{N}$. At a particular scattering angle, we would detect CN particles, so the real uncertainty for our observation CN should be at a scale of $\mathrm{C}\sqrt{N}$. However, if we used Formula 8 to estimate

the uncertainty of CN, we would get an estimated uncertainty of about $\sqrt{CN} = \sqrt{C}\sqrt{N} > C\sqrt{N}$, since $C < 1$. Hence, in general we would not underestimate the uncertainty for an observation of the number of particles received for a particular scattering angle if we borrow Formula 8 to do our uncertainty estimation.

Therefore, in this experiment, we would use Formula 8 at a 95 percent confidence level to estimate the uncertainty of the number of alpha particles detected at different scattering angles.

3.2.4 Procedure

This experiment consists of three trials. In the first trial we used the gold foil plate as the deflector for the incident alpha particles. First, we inserted the gold foil plate into the foil plate holder such that the alpha particle emitter was directly facing the 1mm slit on the plate. Next, we closed up the scattering chamber with the lid, and turned on the vacuum pump. We then checked that the scattering chamber had been properly sealed such that the lid could not be opened due to the air pressure outside, and we kept the vacuum pump running throughout the trial. After this, we rotated the adjustment knob on the lid so that the groove on the swivel arm aligned with the calibration of 0 degree. Then, we turned on the discriminator, the oscilloscope, and the counter. We adjusted the voltage filter level of the discriminator such that we could see from the oscilloscope's screen that only the apparently strong voltage peaks were filtered in and the more trivial voltage oscillations were filtered out. Meanwhile, we made sure that the counter was counting the filtered voltage peaks properly. After this, we aligned the swivel arm to -15 degrees, and reset the counter to restart counting for 1 minute. After 1 minute, we recorded the calibration angle (-15 degrees), the number count of detected alpha particles, and the time it took for counting (1 min). We then rotated the swivel arm to -12.5 degrees and restarted the same counting and recording process. This process was repeated, with a calibration angle increment of 2.5 degrees, up to 12.5 degrees. We then conducted the same process at calibration angles of 15 degrees (2 min), 20 degrees (15 min), 30 degrees (20 min), 40 degrees (60 min), 50 degrees (2 hr 10 min), and 60 degrees (45 hr 14 min). We did one more counting for the calibration angle of -15 degrees for 1 minute.

In the second trial, we kept the experimental configuration of the first trial and measured the number counts at a calibration angle of 0 degree over 10 seconds for 50 times. We recorded all these number counts.

The third trial was conducted for the aluminum foil plate with basically the same procedure as the first trial. The only difference is that we only measured the number counts of detected alpha particles at calibrations angles of -7.5 degrees (1 min), -5 degrees (2 min), -2.5 degrees (2 min), 0 degree (1 min), and 20 degrees (120 min). We recorded all these number counts.

4 Data analysis

4.1 Verification of Poisson distribution for alpha decay of ^{241}Am

In Trial 2, we measured the number count of alpha particles detected at a calibration angle of 0 degree over 10 seconds for 50 rounds in scattering with the gold foil. Figure 7 shows the histogram of the number

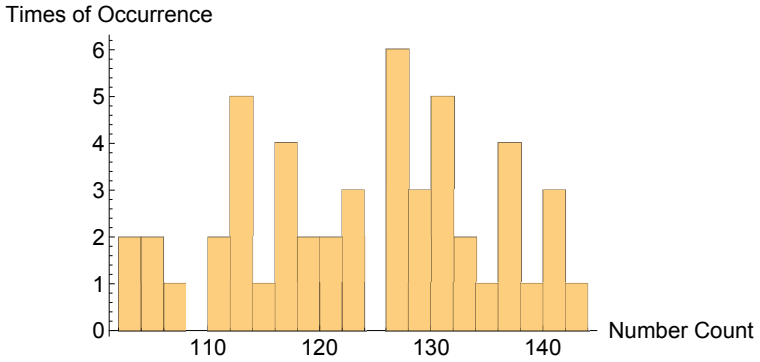

Figure 7: The histogram for the number counts of alpha particles detected at a calibration angle of 0 degree in scattering with the gold foil for 10 seconds per round over 50 rounds

counts of the alpha particles detected. The mean of the number counts over these 50 rounds is 123.58, and the unbiased (or corrected) sample variance is 117.39. They only differ by about 5 percent.

In a Poisson distribution, the expectation value (mean) of the Poisson random variable should be equal to the variance. Remember that as described in Section 3.2.3, the number of alpha particles detected at a particular scattering angle is represented by a random variable CX where C is a constant and X is a Poisson random variable. Although CX does not have a Poisson distribution, its variance should still equal its expectation value given X being a Poisson random variable.[11] Therefore, in our case, over the 50 rounds the mean of the number counts should be close to the unbiased sample variance, which is true. Therefore, our results support the claim that the number of alpha particles emitted by the Americium source over a constant amount of time should follow a Poisson distribution.

4.2 Verification of the $\frac{1}{\sin^4(\frac{1}{2}\theta)}$ dependence of the differential cross section formula for Rutherford Scattering

In Trial 1, we measured the number counts of alpha particles at various calibration angles for various lengths of time intervals. The data for the number counts and the respective calibration angles and count times are presented in Table 1.

Calibration Angle (degree)	Number Count (1)	Time Taken (min)
-15	333	2
-12.5	265	1
-10	366	1
-7.5	585	1
-5	761	1
-2.5	1509	2
0	745	1
2.5	655	1
5	464	1
7.5	318	1
10	201	1
12.5	103	1
15	145	2
20	380	15
30	90	20
40	88	60
50	100	130
60	675	2714

Table 1: Number counts of detected alpha particles for scattering with the gold foil at various calibration angles and time intervals

Notice that in this experiment, the counting time is precise to 0.01 min, based on which we will decide the significant digits. Then, using Formula 8, we can obtain the estimated uncertainties for the number counts in Table 1. Since we are only interested in comparing the number counts of the detected alpha particles of different scattering angles for the same amount of counting time in order to verify the Rutherford differential cross section formula, we will focus on the count rates at the different calibration angles instead, i.e. the number counts of alpha particles detected per minute. The calculated count rates and the corresponding lower and upper bounds obtained from the aforementioned uncertainty estimation process are presented in Table 2.

According to Section 2.2, 2.3, and Section 3.2.2, the count rate at a particular scattering angle θ should be proportional to $\frac{1}{\sin^4(\frac{1}{2}\theta)}$ for non-trivial scattering angles. Therefore, in general, the smaller the scattering angle a calibration angle corresponds to, the more alpha particles the detector will detect. From Table 2, we can see from the trend of the count rates that the theoretical zero scattering angle should lie between the calibration angles -2.5 degrees and 0 degree. Moreover, it should lie more inclined to -2.5 degree. Therefore, we will use the average of -2.5 degrees and -1.25 degrees, i.e. -1.875 degrees, as the calibration angle that corresponds to the zero scattering angle of the incident alpha particles. Let's denote the calibration angle by θ_c and -1.875° by θ_0, then the count rates listed in Table 2 should ideally follow

$$\mathrm{CR}(\theta_c) = \frac{k}{\sin^4(\frac{1}{2}(\theta_c - \theta_0))} \tag{15}$$

Calibration Angle (degree)	Count Rate (1/min)	Count Rate Lower Bound (1/min)	Count Rate Upper Bound (1/min)
-15	167	149	185
-12.5	265	234	299
-10	366	329	405
-7.5	585	539	634
-5	761	708	817
-2.5	754	717	794
0	745	692	800
2.5	655	606	707
5	464	423	508
7.5	318	284	355
10	201	174	231
12.5	103	84.1	125
15	72.5	61.2	85.3
20	25.33	22.85	28.01
30	4.500	3.619	5.531
40	1.467	1.176	1.807
50	0.76923	0.62588	0.93559
60	0.248710	0.230299	0.268202

Table 2: Observed count rates, and the respective lower and upper bounds for their expectation values at various calibration angles in the gold scattering experiment.

where CR denotes count rate (number counts/min) and k is a positive proportionality constant. Equation 15 can be re-written as

$$\ln \mathrm{CR}(\theta_c) = \ln k + \ln \frac{1}{\sin^4(\frac{1}{2}(\theta_c - \theta_0))} \tag{16}$$

where ln is the natural logarithm. Therefore, we can see that $\ln \mathrm{CR}(\theta_c)$ should be a linear function of $\ln \frac{1}{\sin^4[\frac{1}{2}(\theta_c - \theta_0)]}$ with slope 1. A Plot of $\ln \mathrm{CR}(\theta_c)$ (with error bars from the aforementioned uncertainty estimation process) vs. $\ln \frac{1}{\sin^4[\frac{1}{2}(\theta_c - \theta_0)]}$, fitted with a line of slope 1, is shown in Figure 8. The least-squares

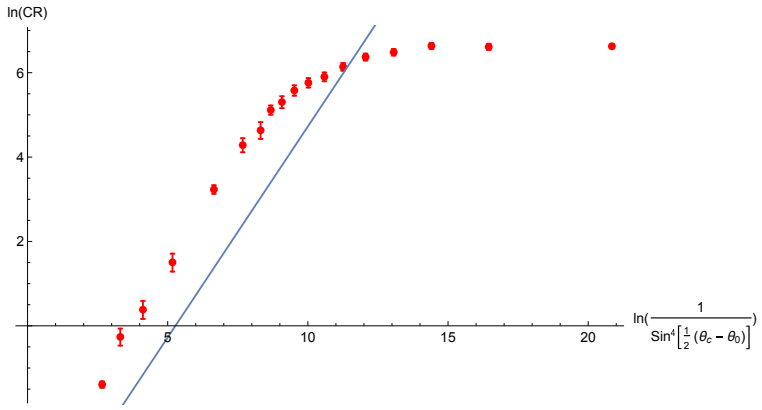

Figure 8: $\ln \mathrm{CR}(\theta_c)$ vs. $\ln \frac{1}{\sin^4[\frac{1}{2}(\theta_c - \theta_0)]}$ for the gold foil scattering experiment ($\theta_0 = -1.875°$).

fitting line in Figure 8 has a slope of 1 and a vertical-axis interception of -5.27. From Figure 8, we can clearly see that the value of $\ln \mathrm{CR}(\theta_c)$ increases as $\ln \frac{1}{\sin^4[\frac{1}{2}(\theta_c - \theta_0)]}$ increases, which means that the smaller the scattering angle that our detector was aligned with, the fewer the alpha particles would be detected. This trend generally agrees with Equation 5. However, in Figure 8, it seems that the data points corresponding to smaller scattering angles (i.e. larger values of $\ln \frac{1}{\sin^4[\frac{1}{2}(\theta_c - \theta_0)]}$) have a flattening out slope, while those corresponding to larger scattering angels display a slope close to 1. To examine this feature more closely, we plot only the data points with a scattering angle $|\theta - \theta_0| >= 10°$, as shown in Figure 9.

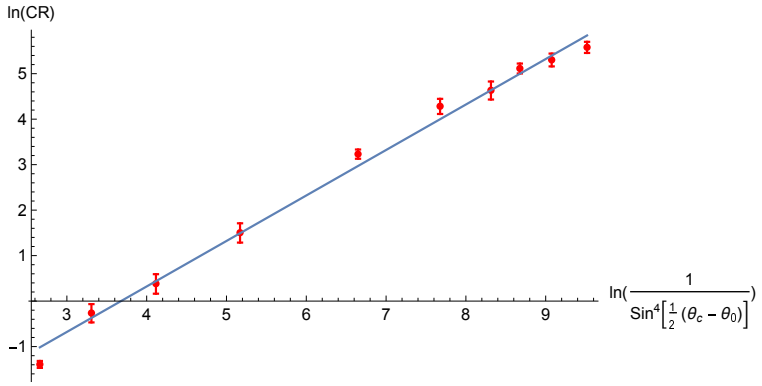

Figure 9: $\ln \mathrm{CR}(\theta_c)$ vs. $\ln \frac{1}{\sin^4[\frac{1}{2}(\theta_c - \theta_0)]}$ for the gold foil scattering experiment with scattering angles >= 10° ($\theta_0 = -1.875°$).

In Figure 9, the least-squares fitting line has a slope of 1 and a vertical interception of -3.68. The selected data points that correspond to a scattering angle of no less than 10 degrees are those with a calibration angle <= -12.5° or >= 10°. We can see that these data points are well fitted by a line of slope 1, so these points adhere to Equation 5 much better than those of smaller scattering angles.

4.3 Finding the ratio between gold and aluminum's atomic numbers

From Section 3.2.2, we have discussed that we can use the ratio between the scattering count rates of the gold and the aluminum foils at the same scattering angle to find out the ratio between their atomic numbers. From Trial 3, we have measured the scattering number counts for aluminum at several different calibration angles. The data from this trial is presented in Table 3.

Calibration Angle (degree)	Number Count (1)	Time Taken (min)	Count Rate (1/min)
-7.5	1060	1	1060
-5	2912	2	1456
-2.5	2709	2	1354.5
0	1018	1	1018
20	48	120	0.4

Table 3: Number counts and the corresponding count rates of detected alpha particles for scattering with the aluminum foil at various calibration angles and time intervals

We want to use the count rate at the calibration angle of 20° as a reference angle to compare the atomic numbers using Equation 14, because as discussed in the previous section, gold scatterings at larger calibration angles seem to follow the Rutherford differential cross section formula better. In Table 3, we can clearly see that a scattering angle of 0° will correspond to a calibration angle between -5° and -2.5°. Therefore, the mid-point calibration angle between -5° and -2.5°, i.e. -3.75°, could be used as a good approximation of the zero scattering angle. So, the 20° calibration angle in aluminum scattering should correspond to a scattering angle of about 24°. This should correspond to a scattering angle in the gold scattering experiment of (-1.875+24)°=22°. Then we have $\ln \frac{1}{\sin^4[\frac{1}{2}(22°)]} = 6.6$. Remember that the least squares fitting line for the larger scattering angles in Figure 9 has a slope of 1 and an y-axis intersection of -3.68. Therefore, the corresponding count rate for gold scattering is $e^{6.6-3.68} = 18$. Then, using Equation 14, we can calculate the atomic number ratio between gold and aluminum to be $\sqrt{\frac{18}{0.4} \times 3.6} = 13$.

5 Conclusion

In this experiment, we used gold and aluminum foils to scatter incident alpha particles that are emitted by an ^{241}Am source. We recorded the number counts of alpha particles detected at different scattering angles over various time intervals in order to verify (1) the Poisson distribution of alpha particle emission by ^{241}Am, (2) the $\frac{1}{\sin^4(\frac{1}{2}\theta)}$ dependence of the Rutherford scattering differential cross section formula, and (3) the Z^2 dependence of the scattering particle's atomic number in the same formula.

In Trial 2, we recorded the number of alpha particles detected at the zero calibration angle over 50 rounds for 10 seconds per round, using the gold foil as the scatterer. The average of these 50 number counts is 123.58 and the unbiased (corrected) variance is 117.39. Since these two numbers only differ from each other by about 5 percent, our data support the claim that the number of alpha particles emitted over a fixed time interval by ^{241}Am should be a random variable of Poisson distribution.

In Trial 1, we counted and recorded the number of alpha particles detected at different calibration angles over various time intervals using the gold foil as the scatterer. As analyzed in Section 4.2, the count rates of the alpha particles with a scattering angle >= 10° correspond to the $\frac{1}{\sin^4(\frac{1}{2}\theta)}$ dependence of Equation 5 very well, but those scattering angles less than 10° would yield count rates that are smaller than expected from Equation 5. As discussed in Section 3.2.1, the scattering process of the incident particles with smaller scattering angles tend to be more complicated because these incident particles could be effectively influenced by more than one metal nuclei. To better understand these experimental results, the internal structure of gold needs to be studied more carefully. However, our experimental result does suggest that gold is internally structured in such a way that will prevent deflecting incident alpha particles by small scattering angles.

In Trial 3, we counted and recorded the number of alpha particles detected at several calibration angles over various time intervals using the aluminum foil as the scatterer. In Section 4.3, the atomic number ratio between gold and aluminum is calculated to be about 13, but the true ratio should be 6.08. Now that the count rate for gold scattering (CR_{Au}) as in Equation 14 is obtained from linear fitting for the gold scattering data points at scattering angles no less than 10°, from Table 2, we can find out the average uncertainty for the gold scattering count rates at scattering angles no less than 10° to be ± 15 percent. For the aluminum scattering at the calibration angle of 20°, the uncertainty of the count rate is obtained from Equation 8 as +33 percent and -26 percent. Therefore, according to Equation 14, the lower-bound uncertainty of our calculated atomic ratio of 13 is $1 - \sqrt{\frac{1-0.15}{1+0.33}} = 20$ percent, which would correspond to a lower bound of $13 \times (1 - 0.20) = 10$ for our determined atomic ratio between gold and aluminum. This is still larger than the true value of 6.08 by 64 percent. According to Equation 14, the source of this mismatch should only come from the count rates for gold at scattering angles no less than 10° and the count rate for aluminum at the the calibration angle of 20°. It seems that these count rates for gold are more likely to be accurate since they together display a good consistency with the $\frac{1}{\sin^4(\frac{1}{2}\theta)}$ dependency. Therefore, the dominant error is more likely to reside in the number count for aluminum scattering at 20°, which seems to be smaller than what it should have been. In fact, the count rate should have been 1.75/min instead of the measured 0.4/min to produce the true atomic number ratio. The cause of this deviation is not very clear at this stage. To further investigate the cause, the aluminum scattering experiment should be performed more thoroughly with alpha particle number counts recorded for multiple large scattering angles that are larger than 10°. We could then calculate the gold-aluminum atomic number ratios one by one with respect to these large angles. We would investigate the variation trend of these calculated ratios for more hints about the aforementioned error.

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