Imitating the Cavendish Experiment to Determine the Universal Gravitational Constant

Gonghan Xu

An investigation that imitated the classic Cavendish experiment was carried out to determine the Newtonian gravitational constant G. A result of $G = 6.76 \times 10^{-11} m^3 kg^{-1} s^{-2}$ was obtained, which shows a good correspondence with contemporary accepted values of G.

1 Introduction

From the fall of 1797 through the spring of 1798, Henry Cavendish performed a series of experiments, now bearing his name, to determine the density of the earth.[1] He published his paper titled "Experiments to Determine the Density of the Earth" in 1798, in which he announced the mean density of the earth to be 5.48 times that of water.[2] This is the first truly accurate determination of the mean density of the earth,[1] in spite of the fact that Cavendish made a simple arithmetic mistake in addition of his results. The corrected value for the density of Earth, from his results, is 5.448 times that of water.[3]

The experiments of Cavendish, though carried out with an intent to measure the density of the earth, implicitly reveals the value of the gravitational constant G to be $6.74 \times 10^{-11} m^3 kg^{-1} s^{-2}$ (the diameter of Earth is taken to be $4.18 \times 10^7 feet$ as Cavendish knew it).[2] This would be the first truly accurate determination of the gravitational constant. In fact, Cavendish is often credited as the person who first accurately measured G.[1]

However, this did not become the standard interpretation until C. V. Boys read his paper "On the Newtonian Constant of Gravitation" at the Royal Institution in 1892, almost a century after Cavendish published his results.[4] In fact, the determination of the density of Earth was an important problem from the time of Newton to approximately 1900, whereas a measurement of a universal gravitational constant was not of much interest to investigators during that time. Newton himself raised a question about the density of Earth in Book III of his *Principia*, where he gave the relative densities of Sun, Jupiter, Saturn, and Earth. If the density of the earth were known, then that of the sun and some of the planets in the solar system could also be determined. As listed in J. H. Poynting's bibliography,[3] between 1687, when Newton published his *Principia*, and 1892, when Boys read his paper, at least 38 papers had been published dealing with measurements of gravitation. All of these papers were directed at the density of Earth, except for one that had mentioned the gravitational constant in its title.[4] George Wilson's 1851 biography of Cavendish also refers to more than 1300 experiments performed to determine the density of the earth.[1]

As a matter of fact, although Newton proposed the law of universal gravitation, he did not explicitly introduce a gravitational constant. When he applied the law in his calculations, he did not work with an equation of gravitational force but with ratios instead. The same approach was taken by Cavendish in his eponymous experiments so that he completed his analysis without ever bothering with a gravitational constant. In reality, during Cavendish's time, a unit for force had not been proposed, and Cavendish measured force in terms of its manifestation in distances and weights. Without a unit for force, an experimenter at his time was not likely to think of measuring a gravitational constant.[4]

Today, the Cavendish experiment is often replicated in undergraduate laboratories to demonstrate determination of the gravitational constant G. As an undergraduate student, I used a similar experimental setup as that used by Cavendish himself to investigate the value of G.

2 Theory

The primary theoretical concern related to this experiment is the universal gravitational law, formulated and numerically checked by Newton by 1666 and published in his *Principia* in 1687. Newton's law of universal gravitation states that *there is a universal force of attraction between two point mass particles, the magnitude of this force proportional to the product of the two masses and inversely proportional to the square of the distance between the two particles.* In mathematical form, this universal gravitational law can be written as

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\mathbf{e}_r \tag{1}$$

where m_1 and m_2 are the masses of the two particles, r is the distance between the two particles, and G is the universal gravitational constant. The unit vector $\mathbf{e}_r$ points to the particle on which this gravitational force is considered from the other particle.[5] Equation 1 can also be applied to two spherical balls with uniform density, with r being the distance between the centers of the two spheres.[6]

Besides the universal gravitational law, rotational dynamics of a rigid object is also of important concern. For planar rotational motion of a symmetric rigid body, Newton's second law can be written as

$$I\ddot{\theta} = \tau \tag{2}$$

where I is the moment of inertia of the rotating object, θ is the rotation angle, and τ is the total external torque exerted on this rotating object.[7] At rotational equilibrium, we have $\ddot{\theta} = 0$ in Equation 2, so the total external torque τ on the rotating object must also vanish.

Moreover, the equivalence principle is assumed in this experiment, which states that *gravitational mass, as used in the calculation of the gravitational force in Equation 1, is the same as inertial mass, which is used in Newton's second law* $\boldsymbol{F} = m\boldsymbol{a}$, where $\mathbf{a}$ is the acceleration of a particle.[8]

3 Experiment

3.1 Apparatus

The main apparatus of this experiment consist of a gravitational torsion balance, a laser light source, and a measuring projection board. As shown in Figure 1, the gravitational torsion balance has a metal case with a glass cover, inside which there is a torsion pendulum.

Figure 2 gives a more detailed view of the torsion pendulum. The torsion pendulum is comprised of three components: a torsion string, a vertical pendulum bob, and a horizontal pendulum arm (or called a pendulum bar). The thin, highly sensitive, bronze torsion string, protected inside a long tube, is hung from the top of the torsion balance. The pendulum bob, onto which a mirror is attached, is suspended vertically by the torsion string. Aligned horizontally across the vertical pendulum bob is a pendulum bar, at each end of which there is a small lead ball, each with a mass $m_2 = 20g$ and a diameter of 15mm. Since the two small balls are fixed on the horizontally aligned pendulum bar and are equidistant from the torsion string, the gravitational effects on the small balls from each other and from the earth are negated. As shown in Figure 1, there is a screw on each side of the metal case, which controls locking of the pendulum bar. Once the pendulum bar is unlocked, it is allowed move. When it is rotating horizontally, the torsion string will exert a restorative torque to this rotation.

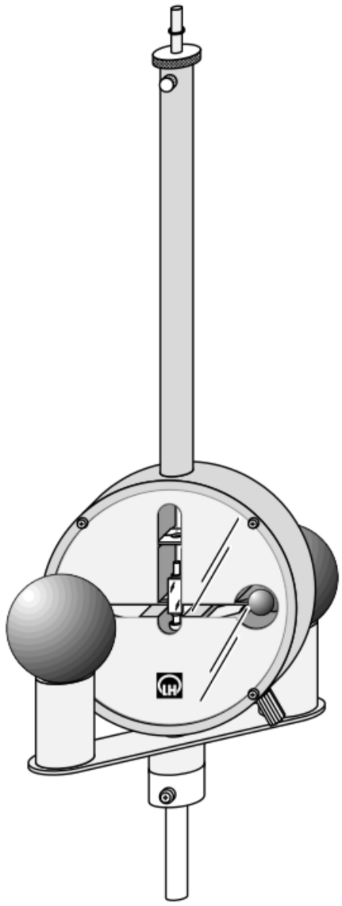

Figure 1: The gravitational torsion balance

Right below the metal case is a ball carrier that carries a pair of large, removable, lead balls, each with a mass of $m_1 = 1.5kg$ and a diameter of 64mm. The ball carrier can be rotated around the central axis of the torsion balance. When placed on the ball carrier, the large balls are horizontally aligned with the small balls in the metal case and will exert a gravitational attraction on the small balls, thus causing the pendulum arm to rotate.

As shown in Figure 3, there are two desired experimental positions of the large balls. The white locations in Figure 3 indicate Position I, while the dark locations indicate Position II. On these two positions, the large balls are very close to the metal case without touching it. Initially, the large balls are not placed on the ball carrier, and the small balls on the pendulum arm will take a resting equilibrium position. As shown in Figure 3, this is the position where the pendulum arm aligns in parallel with the surface of the metal case. When the large balls are placed on Position I, the small balls will be attracted to the large balls by gravitational force and therefore the pendulum arm will rotate and finally claim a new resting equilibrium. Similarly, when the ball carrier is rotated to Position II, the small balls will rotate and finally reach a different resting position. As shown in Figure 3, the horizontal distance between the center of a small ball to the torsion

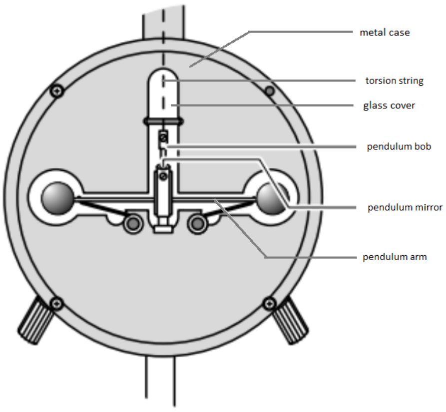

Figure 2: The torsion pendulum

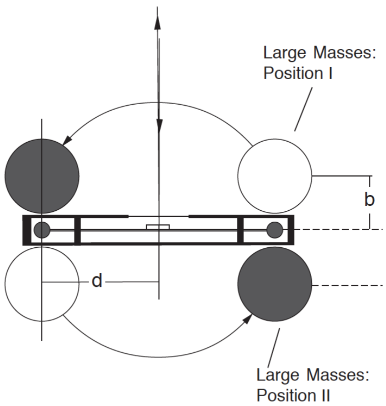

Figure 3: Desired positions of large lead balls

string is $d = 50mm$. The distance between the center of a small ball at its initial equilibrium and that of a nearby large ball when the large ball is against the surface of the metal case is $b = 47mm$.

As shown in Figure 4, a projection board and a laser pointer are set up beside the torsion balance. The projection board is aligned parallel to the metal case of the torsion balance with a separation of more than 3 meters. The laser pointer is placed horizontally between the projecting board and the torsion balance in a way such that the out-going laser can reflect from the pendulum mirror onto the projection board. A tape measure is attached on the projection board so that the horizontal position of the reflected laser point on the projection board can be measured.

3.2 Procedure

Initially, the large lead balls were placed on Position I of the ball carrier, and the reflected laser point on the projecting surface had a resting tape measure reading of 60cm. The ball carrier was then re-positioned to Position II carefully to avoid bumping the large balls with the metal case of the torsion balance. The reflected

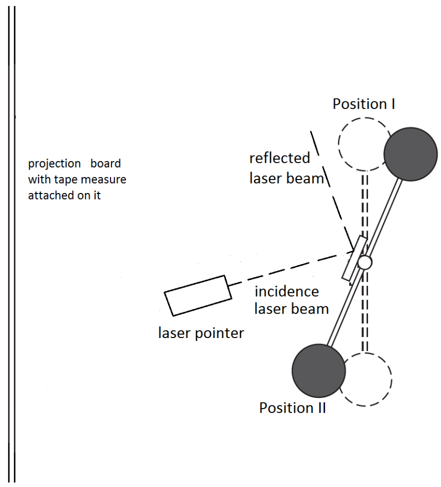

Figure 4: Schematic top view of the experimental setup (diagram not to scale)

laser point then started to oscillate horizontally on the projecting surface. The horizontal position of the oscillating point was recorded every 30 seconds, starting from the first 30 seconds after the re-positioning of the ball carrier. This recording went on for 130 minutes until the reflected point finally settled down to a well-confined horizontal length of 1mm. After this, the separation between the projection board and the metal case surface of the torsion balance was measured at a distance from the torsion balance with a tape measure as wells as the aid of a meter stick and eyes (touching the torsion balance except for the ball carrier was not allowed in the undergraduate laboratory where this experiment was carried out).

3.3 Experimental theory

Let the rotation angle of the pendulum arm at its initial equilibrium position without the gravitational influence of the large balls be $\theta = 0$, and let the positive θ direction be the counterclockwise direction on Figure 3. Then, as shown in Figure 3, when the large balls are placed at Position I, there will be a positive rotation angle θ of the pendulum arm due to the gravitational attraction between the small balls and the large balls (neglecting the mass of the pendulum arm, which is much smaller than the small lead balls). However, the torsion string, when twisted, will exert a restorative torque

$$\tau_{string} = -\kappa\theta \tag{3}$$

on the pendulum arm, where κ is the torsion constant. This linear relationship between τ_{string} and θ holds when θ is small,[9] which is exactly our case due to the relatively small gravitational influence between the small and large balls. When the small balls are at their initial equilibrium position, right after the large balls are placed on Position I, the gravitational attraction force between a large ball and its nearby small ball is $F_n = \frac{Gm_1m_2}{b^2}$, where G is the gravitational constant, m_1 is the mass of a large ball, m_2 is the mass of a small ball, and b is the distance between the center of a large ball and that of a nearby small ball. So, the total attractive torque between the large balls and their nearby small balls is $\tau_n = 2F_nd = \frac{2Gm_1m_2d}{b^2}$. However, as shown in Figure 5, there is also an attractive force between a large ball and the small ball farther from it. This attractive force is $F_f = \frac{Gm_1m_2}{b^2+(2d)^2} = \frac{Gm_1m_2}{b^2+4d^2}$. As shown in Figure 5, the total toque on the pair of small

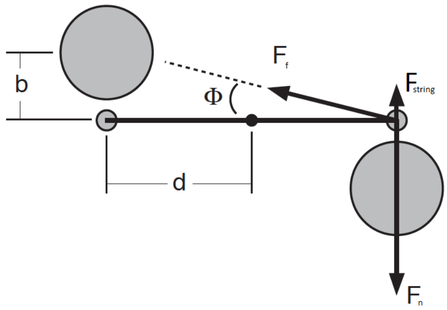

Figure 5: Gravitational force between a large ball and the farther small ball

balls due to the farther large balls is

$$\tau_f = -2F_f d \sin\phi = -2F_f d\frac{b}{\sqrt{b^2+(2d)^2}} = -\frac{2Gm_1m_2d}{b^2}\frac{b^3}{(b^2+4d^2)^{\frac{3}{2}}} = -\tau_n\beta$$

where $\beta = \frac{b^3}{(b^2+4d^2)^{\frac{3}{2}}}$ and the minus sign is due to the restorative nature of this torque. So, when the small balls are at their initial equilibrium position and the large balls are just put at Position I, the total torque on the small balls around the torsion string due to the large balls is

$$\tau = \tau_n + \tau_f = \tau_n(1-\beta) = \frac{2Gm_1m_2d}{b^2}(1-\beta) \tag{4}$$

When the small balls are rotating to reach their final equilibrium, we assume the torque on the small balls due to the large balls remain unchanged, since the rotation angle of the pendulum bar will be very small owing to the very weak gravitational interaction between the small and large balls. So, when the small balls finally reach their equilibrium position with the large balls at Position I, we have $\tau_{eq} = \tau_{string} + \tau = 0$. From Equation 3 and Equation 4, we have $\kappa\theta = \frac{2Gm_1m_2d}{b^2}(1-\beta)$. So,

$$G = \kappa\theta\frac{b^2}{2m_1m_2d}\frac{1}{1-\beta} \tag{5}$$

Notice that only κ and θ on the right hand side of Equation 5 are unknown in determining the gravitational constant G.

From Figure 4, we can see rotation of the pendulum bar will cause the reflected laser point to deflect from its original position on the projection board. Suppose there were an initial reflected laser point at rest on the projection board, named P_0. We also name the reflected laser point at rest on the projection board when the large balls are at Position I P_1 and that when the large balls are at Position II P_2. Name the laser incidence point on the pendulum mirror P_i. Then due to the law of reflection,[10] $\angle P_0P_iP_1 = 2\theta$, where θ is the equilibrium rotation angle of the pendulum bar when the large balls are at Position I. Since Position I and Position II of the large balls are symmetric with respect to the initial equilibrium position of the pendulum arm, the equilibrium rotation angle of the pendulum arm when the large balls are at Position II will be $-\theta$. So, $\angle P_0P_iP_2 = 2\theta$. Since P_1 and P_2 should stay on different sides of P_0, $\angle P_1P_iP_2 = 4\theta$. So, let the horizontal separation of P_1 and P_2 be ΔS, and the distance between the projecting board and the pendulum mirror at its initial equilibrium position be L, then we have $4\theta = \angle P_1P_iP_2 = \frac{\Delta S}{L}$, since the equilibrium angle θ should be very small. So, we have

$$\theta = \frac{\Delta S}{4L} \tag{6}$$

Since ΔS and L in Equation 6 are both measurable quantities, θ can be determined from the experiment.

Now we need to determine the torsion constant κ. During the rotation of the pendulum bar from its equilibrium position with the large balls at Position I to a new equilibrium position with the large balls at Position II, the small balls will experience three torques: the motivational torque due to the large balls, the restorative torque due to the torsion string, and a damping torque due to the air in the surrounding of the pendulum. At low angular velocities, the damping torque $\tau_{damping} = -c\dot{\theta}$, where c is a constant damping coefficient and $\dot{\theta}$ denotes the time derivative of θ.[11] From Equation 3, we have $\tau_{string} = -\kappa\theta$, so the equation of motion of the small balls rotating from their equilibrium position with the large balls at Position I to a new equilibrium position with the large balls at Position II is

$$I\ddot{\theta} = -\tau + \tau_{string} + \tau_{damping} = -\tau - \kappa\theta - c\dot{\theta} \tag{7}$$

$$\therefore \ddot{\theta} + \frac{c}{I}\dot{\theta} + \frac{\kappa}{I}\theta = -\frac{\tau}{I} \tag{8}$$

where I is the moment of inertia of the pair of small balls with respect to the torsion string as the rotation axis, and -τ is the torque on the small balls due to the large balls at Position II. Remember that we assume τ to be a constant during the rotation of the small balls.

Define $\omega_0^2 = \kappa/I$ and $2\gamma = c/I$, then the equation of motion (Eq. 8) becomes

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta = -\frac{\tau}{I} \tag{9}$$

The general solution[12] for this second order differential equation (Eq. 9) is

$$\theta = Ae^{-\gamma t}\cos(\omega t + \psi) - \frac{\tau}{\kappa} \tag{10}$$

where A and ψ are constants that depend on the initial condition, and $\omega = \sqrt{\omega_0^2 - \gamma^2}$. So, the pendulum arm will be oscillating with an exponentially attenuated amplitude. Recall from Section 3.2 that the horizontal position of the oscillating reflected laser point on the projection board was recorded every 30 seconds until the reflected point was virtually at rest. If we modify Equation 6 slightly, we can obtain the instantaneous rotation angle of the pendulum arm

$$\theta(t) = \frac{\Delta S}{4L} - \frac{\Delta S(t)}{2L} \tag{11}$$

where $\theta(t)$ is the instantaneous rotation angle of the pendulum arm and $\Delta S(t)$ is the separation between the instantaneous reflected point and P_1. Notice that we have in this case $\theta(t) = \theta$ when the reflected point is at P_1 and $\theta(t) = -\theta$ when the reflected point is at P_2, following our previous definition of the sign of θ strictly. Since $\Delta S(t)$ can be easily obtained from the recorded data, we can obtain the value of $\theta(t)$ for every 30 seconds. According to Equation 10, the local minimums of $\theta(t)$ obtained from the experiment can be fitted with an exponential decaying (in magnitude) line by computer to obtain the damping parameter γ. We can also determine the period of oscillation T from the average of time intervals between the aforementioned local minimums of $\theta(t)$. Then, ω in Equation 10 is equal to $\frac{2\pi}{T}$. Recall that $\frac{\kappa}{I} = \omega_0^2 = \omega^2 + \gamma^2$. So with ω and γ determinable from the experimental data, we can determine the torsion constant

$$\kappa = I(\omega^2 + \gamma^2) \tag{12}$$

From Figure 3, using the moment of inertia formula for a homogeneous ball[13] and the parallel axis theorem[14], we can determine the moment of inertia for the two small balls around the torsion string to be $2(m_2d^2 + \frac{2}{5}m_2r^2) = 2m_2(d^2 + \frac{2}{5}r^2)$, where r is the radius of the small ball. With this and $\omega = \frac{2\pi}{T}$, κ from Equation 12 can be further written as

$$\kappa = 2m_2(d^2 + \frac{2}{5}r^2)\left((\frac{2\pi}{T})^2 + \gamma^2\right) \tag{13}$$

So, from Equations 5, 6, and 13, we can determine the gravitational constant G from our experimental data

$$G = 2m_2(d^2 + \frac{2}{5}r^2)\left((\frac{2\pi}{T})^2 + \gamma^2\right)\frac{\Delta S}{4L}\frac{b^2}{2m_1m_2d}\frac{1}{1-\beta} \tag{14}$$

Simplify and rearrange Equation 14, we have

$$G = \left[\left((\frac{2\pi}{T})^2 + \gamma^2\right)\frac{\Delta S}{L}\right]\left(\frac{1}{4}(d^2 + \frac{2}{5}r^2)\frac{b^2}{m_1d}\frac{1}{1-\beta}\right) \tag{15}$$

Notice that the expression outside the square bracket on the right hand side of Equation 15 can be determined from the given parameters of the experimental apparatus. The expression inside the square bracket can be determined from the experimental data and data analysis.

4 Data analysis

From Equation 15, we can see that we need the oscillation period T of the reflected laser point, the damping parameter γ of the air, the horizontal separation ΔS between the two reflected laser points at rest on the projection board when the large balls were at Position I and II respectively, and the distance L between the projecting board and the pendulum mirror at its initial equilibrium position. L was measured to be 353cm. The initial resting position of the reflected laser point P_1 when the large balls were at Position I had a tape measure reading of 60cm. The final resting position of the laser point P_2 when the large balls were at Position II was stabilizing between a tape measure reading of 70.8cm and 70.9cm. We will take 70.85cm as the final resting position of P_2, then the separation $\Delta S = 70.85cm - 60cm = 10.85cm$.

As mentioned in Section 3.2, the position of the reflected laser point on the projection board was recorded every 30 seconds until it eventually stabilized. Using Equation 11, we can obtain the values of $\theta(t)$ for every 30 seconds during the oscillation of the reflected laser point. As mentioned in Section 3.3, we can determine the period of oscillation T from the average of time intervals between the local minimums of $\theta(t)$. The

t (s)	270	840	1470	2070	2670	3270	3900
$\theta(t)$ (rad)	-0.106905	-0.101664	-0.0982649	-0.0964235	-0.0951487	-0.0942989	-0.0937323
t (s)	4500	5100	5730	6330	6960	7500	
$\theta(t)$ (rad)	-0.0933074	-0.0931657	-0.0930241	-0.0928824	-0.0927408	-0.0927408	

Table 1: Local minimums of rotation angle and respective time

local minimums of $\theta(t)$ and their respective time t are given in Table 1. The average of the individual time intervals between local minimums of $\theta(t)$ is 602.5s. So, $T = 602.5s$.

Now, as mentioned in Section 3.3, we need to fit the local minimums of the $\theta(t)$ obtained from the experiment by an exponential decaying line to obtain the damping parameter γ. As shown in Table 1, the local minimums of $\theta(t)$ are much smaller than their respective t. So, to facilitate the fitting process, we used hours as the unit for time t. According to Table 1, local minimums of $\theta(t)$ converge to around -0.0927408 rad, so we used $\theta(t) = Ae^{-\gamma t} - 0.0927408$ to fit the points of local minimums of $\theta(t)$ vs. t. The fitting

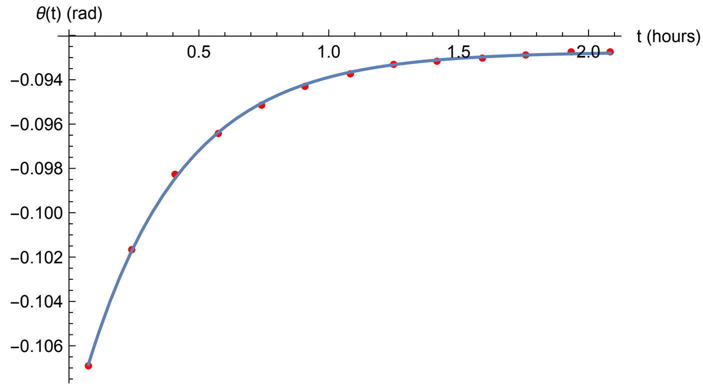

Figure 6: Local minimums of $\theta(t)$ vs. t fitted by exponential decaying curve

process was completed with Mathematica, and as shown in Figure 6, the computed fitting function is $\theta(t) = -0.0172825e^{-2.71241t} - 0.0927408$. So, the damping parameter $\gamma = 2.71241 hour^{-1} = 7.53447 \times 10^{-4} s^{-1}$.

So far, we have obtained all the parameters that need to be determined experimentally. The distance between the projecting board and the pendulum bar is $L = 353cm$. The separation between the reflected laser points at rest is $\Delta S = 10.85cm$. The oscillating period is $T = 602.5s$. And the damping parameter is $\gamma = 7.53447 \times 10^{-4} s^{-1}$. Replicating Equation 15, we have

$$G = \left[\left((\frac{2\pi}{T})^2 + \gamma^2 \right) \frac{\Delta S}{L} \right] \left(\frac{1}{4}(d^2 + \frac{2}{5}r^2) \frac{b^2}{m_1 d} \frac{1}{1-\beta} \right) \tag{16}$$

And remember, from Section 3.1 that $d = 50mm = 0.05m$, $b = 47mm = 0.047m$, $r = 15mm/2 = 7.5mm = 0.0075m$, and $m_1 = 1.5kg$. Also recall from Section 3.3 that $\beta = \frac{b^3}{(b^2+4d^2)^{\frac{3}{2}}}$. So, we have assembled all the necessary components to calculate the gravitational constant G. Plugging all the relevant parameters into Equation 16, we obtain

$$G = 6.76 \times 10^{-11} m^3 kg^{-1} s^{-2} \tag{17}$$

5 Conclusion

The value of the gravitational constant $G = 6.76 \times 10^{-11} m^3 kg^{-1} s^{-2}$ determined from this experiment differs from the value $G = 6.674 \times 10^{-11} m^3 kg^{-1} s^{-2}$ recommended by Committee on Data for Science and Technology by only 1.3 percent.[15] So, the value of the gravitational constant determined from this experiment is reasonably accurate. One major source of error likely comes from measuring the separation between the projecting board and the pendulum mirror. Because this experiment was done in an undergraduate laboratory where touching the pendulum balance except the ball carrier is prevented, the separation between the torsion pendulum and the projecting board was measured, in considerable part, with the aid of eyes. Supposing the error of measuring this separation L is ΔL, then from Equation 16, we can see the measured value of G increase by a factor of $\frac{1}{1+\frac{\Delta L}{L}} \approx 1 - \frac{\Delta L}{L}$ from the true value, corresponding to a fractional error of $\frac{\Delta L}{L}$. Therefore, an error on measuring the separation L will lead to the same fractional error on the calculated value of G. Besides the error on L, the reflected laser point on the projection board was determined to ultimately stabilize between a tape measure reading of 70.8cm and 70.9cm, corresponding to a ΔS value between 10.8cm and 10.9cm. Since we took $\Delta S = 10.85cm$ in our final calculation of G, there is a 0.05cm uncertainty on this value, corresponding to an uncertainty of around 0.5 percent. Like L, we can see from Equation 16 that an error on the value of ΔS used for calculation will result in the same fractional error on G, so ΔS is another major source of error. Apart from these errors, the experimentally determined values of the oscillation period T, the damping parameter γ, and the given parameters of the experimental apparatus all possess their intrinsic uncertainties, and the squaring of these parameters in Equation 16 can further amplify any errors on these values. In the end, the 1.3 percent error on our calculated value of G is reasonable. To improve the calculated value of G, more accurate measurement of L and ΔS is needed. ΔS could be measured by waiting longer for the reflected laser point to finally settle at rest on the projection board. L could be measured in a closer vicinity of the pendulum balance to minimize the use of eyes to align the tape measure with the two ends of the separation L.

References

[1] S. P. Lally, The Physics Teacher **37**, 34 (1999).

[2] H. Cavendish, Philosophical Transactions of the Royal Society of London **88**, 469–526 (1798).

[3] J. H. Poynting, *The mean density of the earth: an essay to which the adams prize was adjudged in 1893 in the university of cambridge* (C. Griffin, Limited, 1894).

[4] B. E. Clotfelter, Am. J. Phys. **55**, 210 (1987).

[5] S. Thornton and J. Marion, *Classical dynamics of particles and systems*, 5th ed. (Brooks/Cole, 2004), p. 182.

[6] S. Thornton and J. Marion, *Classical dynamics of particles and systems*, 5th ed. (Brooks/Cole, 2004), pp. 184–189.

[7] J. Cutnell and K. Johnson, *Physics*, 9th ed. (Wiley, 2012), pp. 252–255.

[8] S. Thornton and J. Marion, *Classical dynamics of particles and systems*, 5th ed. (Brooks/Cole, 2004), p. 52.

[9] F. Beer, E. Jr. Johnston, J. DeWolf, and D. Mazurek, *Mechanics of materials*, 6th ed. (McGraw-Hill Education, 2011), p. 159.

[10] D. Griffiths, *Introduction to electrodynamics*, 4th ed. (Pearson Education, 2014), p. 407.

[11] S. Thornton and J. Marion, *Classical dynamics of particles and systems*, 5th ed. (Brooks/Cole, 2004), pp. 58–59.

[12] S. Thornton and J. Marion, *Classical dynamics of particles and systems*, 5th ed. (Brooks/Cole, 2004), pp. 108–117.

[13] J. Cutnell and K. Johnson, *Physics*, 9th ed. (Wiley, 2012), p. 254.

[14] J. Cutnell and K. Johnson, *Physics*, 9th ed. (Wiley, 2012), pp. 272–273.

[15] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **77**, 1–107 (2005).