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Solutions of the Diophantine Equations $p^x + (p+1)^y + (p+2)^z = M^2$ for Primes $p \geq 2$ when $1 \leq x, y, z \leq 2$

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Dedicated to the remarkable outstanding Professor Alan Rubinow

Abstract. In this article, we investigate the solutions of the Diophantine equations $p^x + (p+1)^y + (p+2)^z = M^2$ for primes $p \geq 2$ when $1 \leq x, y, z \leq 2$. We establish : (i) When $p = 2$ and $x = y = z = 1$, the equation has a unique solution. (ii) When $p = 4N+1$ and $1 \leq x, y, z \leq 2$, the equations have no solutions. (iii) When $p = 4N+3$ and $x = y = z = 1$, the equation has infinitely many solutions. (iv) When $3 \leq p \leq 199$ and $x = 1$, $y = z = 2$, the equation has exactly one solution. (v) In all other cases $1 \leq x, y, z \leq 2$ which are not mentioned above, the equations have no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this asticle, we extend the above equation, and consider $p^x + (p+1)^y + (p+2)^z = M^2$ for primes $p \geq 2$, integers x, y, z where $1 \leq x, y, z \leq 2$. The value M is a positive integer. We employ our new method which uses the last digits of certain powers. We establish the solutions for all values x, y, z above. As in such equations, cases of infinitely many solutions, no solution cases and unique solutions are determined.

The primes $p = 2$, $p = 4N+1$ and $p = 4N+3$ are respectively discussed in Sections 2, 3 and 4. All the theorems and the cases within are self-contained.

2. All the solutions of $p^x + (p+1)^y + (p+2)^z = M^2$ when $p = 2$, $1 \leq x, y, z \leq 2$

In this section all the solutions of equation $2^x + 3^y + 4^z = M^2$ are determined.

Theorem 2.1. Let $1 \le x, y, z \le 2$. Then the equation $2^x + 3^y + 4^z = M^2$ has a unique solution when $x = y = z = 1$. In all other cases, the equation has no solutions.

Proof: When $1 \le x, y, z \le 2$, the eight cases of $2^x + 3^y + 4^z = M^2$ are listed below.

(1) $2^1 + 3^1 + 4^1 = 3^2 = M^2$.
(2) $2^1 + 3^1 + 4^2 = 21 \ne M^2$.
(3) $2^1 + 3^2 + 4^1 = 15 \ne M^2$.
(4) $2^2 + 3^1 + 4^1 = 11 \ne M^2$.
(5) $2^1 + 3^2 + 4^2 = 27 \ne M^2$.
(6) $2^2 + 3^1 + 4^2 = 23 \ne M^2$.
(7) $2^2 + 3^2 + 4^1 = 17 \ne M^2$.
(8) $2^2 + 3^2 + 4^2 = 29 \ne M^2$.

It follows that case **(1)** when $x = y = z = 1$ yields a solution for which $M = 3$, whereas in all other cases **(2) – (8)** the equation has no solutions as asserted.

This completes the proof of Theorem 2.1. □

3. All the solutions of $p^x + (p+1)^y + (p+2)^z = M^2$ when $p = 4N+1$, $1 \le x, y, z \le 2$

Here we consider $p^x + (p+1)^y + (p+2)^z = M^2$ for all primes of the form $p = 4N+1$, when $1 \le x, y, z \le 2$. We establish in Theorem 3.1 that the equations have no solutions.

Theorem 3.1. Let $1 \le x, y, z \le 2$. If $p = 4N+1$, no solutions exist for $p^x + (p+1)^y + (p+2)^z = M^2$.

Proof: When $1 \le x, y, z \le 2$ and $p = 4N+1$ is prime, eight cases exist:

(1) $(4N + 1)^1 + (4N + 2)^1 + (4N + 3)^1 = M^2$.
(2) $(4N + 1)^1 + (4N + 2)^1 + (4N + 3)^2 = M^2$.
(3) $(4N + 1)^1 + (4N + 2)^2 + (4N + 3)^1 = M^2$.
(4) $(4N + 1)^2 + (4N + 2)^1 + (4N + 3)^1 = M^2$.
(5) $(4N + 1)^1 + (4N + 2)^2 + (4N + 3)^2 = M^2$.
(6) $(4N + 1)^2 + (4N + 2)^1 + (4N + 3)^2 = M^2$.
(7) $(4N + 1)^2 + (4N + 2)^2 + (4N + 3)^1 = M^2$.
(8) $(4N + 1)^2 + (4N + 2)^2 + (4N + 3)^2 = M^2$.

Each of these cases is considered separately, and is self-contained.

(1) The case $(4N + 1)^1 + (4N + 2)^1 + (4N + 3)^1 = M^2$.
The left side of the equation yields

$$(4N + 1) + (4N + 2) + (4N + 3) = 12N + 6 = 6(2N + 1).$$

The prime 2 in the factor 6 has an odd exponent equal to 1. Since $(2N + 1)$ is odd, it follows that $6(2N + 1)$ is not a square.

The equation $(4N + 1)^1 + (4N + 2)^1 + (4N + 3)^1 = M^2$ has no solutions.

(2) The case $(4N + 1)^1 + (4N + 2)^1 + (4N + 3)^2 = M^2$.

The left side of the equation yields

$$(4N + 1) + (4N + 2) + (16N^2 + 24N + 9) = 4(4N^2 + 8N + 3).$$

If the product $4(4N^2 + 8N + 3)$ equals a square M^2, then $(4N^2 + 8N + 3)$ must satisfy

$$4N^2 + 8N + 3 = T^2. \tag{1}$$

Consider the even square $(2N + 2)^2 = 4N^2 + 8N + 4 = Q^2$. If for some value N, there exists a value T satisfying (1), we have

$$Q^2 - T^2 = (4N^2 + 8N + 4) - (4N^2 + 8N + 3) = 1$$

which is impossible since no two squares differ by 1. Hence (1) is false.

The equation $(4N + 1)^1 + (4N + 2)^1 + (4N + 3)^2 = M^2$ has no solutions.

(3) The case $(4N + 1)^1 + (4N + 2)^2 + (4N + 3)^1 = M^2$.

The left side of the equation yields

$$(4N + 1) + (16N^2 + 16N + 4) + (4N + 3) = 8(2N^2 + 3N + 1). \tag{2}$$

We shall assume that $8(2N^2 + 3N + 1) = M^2$ and reach a contradiction. Since M^2 is even, denote $M = 2T$ where T is an integer and $M^2 = 4T^2$. From (2) we then obtain

$$2(2N^2 + 3N + 1) = T^2. \tag{3}$$

If N is even, then 2 with an odd exponent equal to 1 and $(2N^2 + 3N + 1)$ being odd imply that (3) is impossible. Therefore by our assumption N is odd. Denote $N = 2m + 1$ where m is an integer. From (3) we obtain

$$T^2 = 2\big(2N^2 + 3N + 1\big) = 2\big(2(2m + 1)^2 + 3(2m + 1) + 1\big) = 4(4m^2 + 7m + 3) \tag{4}$$

where in (4) it follows that $(4m^2 + 7m + 3) = R^2$.

Consider the following two consecutive squares $A^2 = (2m + 1)^2$ and $(A + 1)^2 = (2m + 2)^2$. The first square yields $(2m + 1)^2 = 4m^2 + 4m + 1$, whereas the second square yields $(2m + 2)^2 = 4m^2 + 8m + 4$. Then we have

$$A^2 = 4m^2 + 4m + 1 < 4m^2 + 7m + 3 < 4m^2 + 8m + 4 = (A + 1)^2 \tag{5}$$

which clearly implies that $(4m^2 + 7m + 3) \ne R^2$ since the squares on the left and right of (5) are two consecutive squares. Our assumption is therefore false.

The equation $(4N + 1)^1 + (4N + 2)^2 + (4N + 3)^1 = M^2$ has no solutions.

(4) The case $(4N + 1)^2 + (4N + 2)^1 + (4N + 3)^1 = M^2$.

The left side of the equation yields

$$(16N^2 + 8N + 1) + (4N + 2) + (4N + 3) = 2(8N^2 + 8N + 3).$$

The prime 2 has an odd exponent equal to 1. Since $(8N^2 + 8N + 3)$ is always odd, therefore $2(8N^2 + 8N + 3)$ is not a square.

The equation $(4N + 1)^2 + (4N + 2)^1 + (4N + 3)^1 = M^2$ has no solutions.

(5) The case $(4N + 1)^1 + (4N + 2)^2 + (4N + 3)^2 = M^2$.

The left side of the equation yields

$$(4N + 1) + (16N^2 + 16N + 4) + (16N^2 + 24N + 9) = 2(16N^2 + 22N + 7).$$

The prime 2 has an odd exponent equal to 1. Since $(16N^2 + 22N + 7)$ is always odd, hence $2(16N^2 + 22N + 7)$ is not a square.

The equation $(4N + 1)^1 + (4N + 2)^2 + (4N + 3)^2 = M^2$ has no solutions.

(6) The case $(4N + 1)^2 + (4N + 2)^1 + (4N + 3)^2 = M^2$.

The left side of the equation yields

$$(16N^2 + 8N + 1) + (4N + 2) + (16N^2 + 24N + 9) = 4(8N^2 + 9N + 3). \quad (6)$$

We shall assume that for some value N, $4(8N^2 + 9N + 3) = M^2$ and reach a contradiction. Since M^2 is even, denote $M = 2T$ where T is an integer and $M^2 = 4T^2$. Thus from (6) we have

$$8N^2 + 9N + 3 = T^2. \quad (7)$$

Suppose that N is even.

Then T^2 is odd. One could easily verify that each cycle of five consecutive even values $N = 2, 4, 6, 8, 10, \dots,$ yields five respective values T^2 which end in the digits 3, 7, 5, 7, 3. An odd square T^2 cannot end in the digits 3 and 7. Therefore we shall consider only the case in which N ends in the digit 6. Denote by $N = 10K + 6$ all the integers whose last digit is equal to 6, where $K \geq 0$ is an integer. From (7) we obtain

$$8(10K + 6)^2 + 9(10K + 6) + 3 = 5(160K^2 + 210K + 69) = T^2. \quad (8)$$

In (8), the prime 5 has an odd exponent equal to 1. Since $5 \nmid (160K^2 + 210K + 69)$, it follows that $5(160K^2 + 210K + 69) \neq T^2$ and (8) is false when N is even.

Suppose that N is odd.

Then T^2 is even. It is clearly seen that each cycle of five consecutive odd values $N = 1, 3, 5, 7, 9, \dots,$ yields five respective values T^2 which end in the digits 0, 2, 8, 8, 2. An even square T^2 cannot end in the digits 2 and 8. Hence, we shall consider only the case in which N ends in the digit 1. Denote by $N = 10K + 1$ all integers whose last digit is equal to 1, where $K \geq 0$ is an integer. From (7) we have

$$8(10K + 1)^2 + 9(10K + 1) + 3 = 10(80K^2 + 25K + 2) = T^2. \quad (9)$$

In (9) $10 = 2^1 \cdot 5^1$, where the prime 5 has an odd exponent equal to 1, and $5 \nmid (80K^2 + 25K + 2)$. Therefore, when N is odd, then $8N^2 + 9N + 3 \neq T^2$ and (9) is false.

We have shown that no value N exists which satisfies the equation $(4N + 1)^2 + (4N + 2)^1 + (4N + 3)^2 = M^2$. This contradicts our assumption.

The equation $(4N + 1)^2 + (4N + 2)^1 + (4N + 3)^2 = M^2$ has no solutions.

(7) The case $(4N + 1)^2 + (4N + 2)^2 + (4N + 3)^1 = M^2$.

The left side of the equation yields

$$(16N^2 + 8N + 1) + (16N^2 + 16N + 4) + (4N + 3) = 4(8N^2 + 7N + 2). \quad (10)$$

We shall assume that for some value N, $4(8N^2 + 7N + 2) = M^2$ and reach a contradiction. Since M^2 is even, denote $M = 2T$ where T is an integer and $M^2 = 4T^2$. Hence from (10) we obtain

$$8N^2 + 7N + 2 = T^2. \tag{11}$$

We shall consider two cases, namely N is even and N is odd.

Suppose that N is even.

Then T^2 is even. It is easily verified that each cycle of five consecutive even values $N = 2, 4, 6, 8, 10, \ldots$, yields five respective values T^2 which end in the digits 8, 8, 2, 0, 2. An even square T^2 cannot end in the digits 8 and 2. Therefore we shall consider only the case in which N ends in the digit 8. Denote by $N = 10K + 8$ all the integers whose last digit is equal to 8, where $K \geq 0$ is an integer. From (11) we have

$$8(10K+8)^2 + 7(10K+8) + 2 = 5(160K^2 + 270K + 114) = T^2. \tag{12}$$

In (12), the prime 5 has an odd exponent equal to 1. Since $5 \nmid (160K^2 + 270K + 114)$, it follows that $5(160K^2 + 270K + 114) \neq T^2$ and (12) is false when N is even.

Suppose that N is odd.

Then T^2 is odd. One can easily see that each cycle of five consecutive odd values $N = 1, 3, 5, 7, 9, \ldots$, yields five respective values T^2 which end in the digits 7, 5, 7, 3, 3. An odd square T^2 does not end in the digits 7 and 3. Hence, we shall consider only the case in which N ends in the digit 3. Denote by $N = 10K + 3$ all the integers whose last digit is equal to 3, where $K \geq 0$ is an integer. From (11) we then obtain

$$8(10K+3)^2 + 7(10K+3) + 2 = 5(160K^2 + 110K + 19) = T^2. \tag{13}$$

In (13), the prime 5 has an odd exponent equal to 1. Since $5 \nmid (160K^2 + 110K + 19)$, it follows that $5(160K^2 + 110K + 19) \neq T^2$ and (13) is impossible when N is odd.

We have shown that no value N exists which satisfies the equation $(4N+1)^2 + (4N+2)^2 + (4N+3)^1 = M^2$. This contradicts our assumption.

The equation $(4N+1)^2 + (4N+2)^2 + (4N+3)^1 = M^2$ has no solutions.

(8) The case $(4N+1)^2 + (4N+2)^2 + (4N+3)^2 = M^2$.

The left side of the equation yields

$$(16N^2 + 8N + 1) + (16N^2 + 16N + 4) + (16N^2 + 24N + 9) = 2(24N^2 + 24N + 7).$$

The prime 2 has an odd exponent equal to 1. Since $(24N^2 + 24N + 7)$ is always odd, it follows that $2(24N^2 + 24N + 7)$ is not a square.

The equation $(4N+1)^2 + (4N+2)^2 + (4N+3)^2 = M^2$ has no solutions.

The proof of Theorem 3.1 is complete. □

Remark 3.1. It is worthy of remark that $p = 4N + 1$ is prime was not used at all in the proofs of the eight cases. Therefore, the results obtained in Theorem 3.1 are valid for all primes of the form $4N + 1$ as well as for all composites of this form.

4. Solutions of $p^x + (p+1)^y + (p+2)^z = M^2$ when $p = 4N+3$, $1 \leq x, y, z \leq 2$

In this section we consider $p^x + (p+1)^y + (p+2)^z = M^2$ when $1 \leq x, y, z \leq 2$, and the primes p are of the form $p = 4N + 3$.

Theorem 4.1. Let $1 \le x, y, z \le 2$. Then $p^x + (p+1)^y + (p+2)^z = M^2$ has:
(i) Infinitely many solutions when $x = y = z = 1$ with primes $p = 4N + 3$. (ii) Exactly one solution when $3 \le p \le 199$ and $x = 1$, $y = z = 2$. (iii) No solutions for all other possibilities.

Proof: When $1 \le x, y, z \le 2$ and $p = 4N + 3$ is prime, eight cases exist:

(1) $(4N + 3)^1 + (4N + 4)^1 + (4N + 5)^1 = M^2$.
(2) $(4N + 3)^1 + (4N + 4)^1 + (4N + 5)^2 = M^2$.
(3) $(4N + 3)^1 + (4N + 4)^2 + (4N + 5)^1 = M^2$.
(4) $(4N + 3)^2 + (4N + 4)^1 + (4N + 5)^1 = M^2$.
(5) $(4N + 3)^1 + (4N + 4)^2 + (4N + 5)^2 = M^2$.
(6) $(4N + 3)^2 + (4N + 4)^1 + (4N + 5)^2 = M^2$.
(7) $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^1 = M^2$.
(8) $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^2 = M^2$.

Each case is considered separately, and is self-contained.

(1) The case $(4N + 3)^1 + (4N + 4)^1 + (4N + 5)^1 = M^2$.
The left side of the equation yields

$$(4N + 3) + (4N + 4) + (4N + 5) = 12(N + 1). \tag{14}$$

In (14), the equality $12(N + 1) = M^2$ is true provided $N + 1 = 3^a$ or $N + 1 = 3^a \cdot G$ where $a \ge 1$ is an odd integer and G is a product of squares only. For instance, when $a = 1, 3, 5, 7$, then $N + 1 = 3^a$ yields the respective primes $p = 11, 107, 971, 8747$, and the respective values $M = 6, 18, 54, 162$. The values $a = 1$ and $G = 2^2$, $a = 1$ and $G = 4^2$, $a = 3$ and $G = 5^2$ yield the respective primes $p = 47, 191, 2699$, and the respective values $M = 12, 24, 90$. Evidently then, infinitely many solutions of the equation exist in which $4N + 3$ is prime.

The equation $(4N + 3)^1 + (4N + 4)^1 + (4N + 5)^1 = M^2$ in which $4N + 3$ is prime has infinitely many solutions.

(2) The case $(4N + 3)^1 + (4N + 4)^1 + (4N + 5)^2 = M^2$.
The left side of the equation yields

$$(4N + 3) + (4N + 4) + (16N^2 + 40N + 25) = 16(N^2 + 3N + 2). \tag{15}$$

In (15) the factor $(N^2 + 3N + 2)$ must be a square C^2 in order for a solution to exist. Consider the following two consecutive squares $(N + 1)^2$ and $(N + 2)^2$. The first square yields $(N + 1)^2 = N^2 + 2N + 1$, whereas the second square yields $(N + 2)^2 = N^2 + 4N + 4$. Then, we have

$$N^2 + 2N + 1 < N^2 + 3N + 2 < N^2 + 4N + 4 \tag{16}$$

which implies that $N^2 + 3N + 2 \ne C^2$, since the squares on the left and right of (16) are consecutive squares.

The equation $(4N + 3)^1 + (4N + 4)^1 + (4N + 5)^2 = M^2$ has no solutions.

(3) The case $(4N + 3)^1 + (4N + 4)^2 + (4N + 5)^1 = M^2$.

The left side of the equation yields

$$(4N+3) + (16N^2 + 32N + 16) + (4N + 5) = 8(2N^2 + 5N + 3). \quad (17)$$

In (17) the number $8 = 2^3$ has an odd exponent equal to 3. If N is even, then the factor $(2N^2 + 5N + 3)$ is odd, and hence the equation has no solutions. Therefore, if the equation has a solution, then N must be odd. Denote $N = 2m+1$ where m is an integer. Then $(2N^2 + 5N + 3) = 2(2m + 1)^2 + 5(2m + 1) + 3 = 2(4m^2 + 9m + 5)$ implying that $(4m^2 + 9m + 5)$ must be a square A^2 for a solution to exist.

Consider the following two consecutive squares $(2m + 2)^2$ and $(2m + 3)^2$. The first square yields $(2m + 2)^2 = 4m^2 + 8m + 4$, whereas the second square yields $(2m + 3)^2 = 4m^2 + 12m + 9$. Then we have

$$4m^2 + 8m + 4 < 4m^2 + 9m + 5 < 4m^2 + 12m + 9 \quad (18)$$

implying that $(4m^2 + 9m + 5) \neq A^2$, since the two squares in (18) are consecutive squares. Thus N is not odd.

It follows that in (17) no value N exists for which $8(2N^2 + 5N + 3)$ is a square.
The equation $(4N+3)^1 + (4N + 4)^2 + (4N + 5)^1 = M^2$ has no solutions.

(4) The case $(4N + 3)^2 + (4N + 4)^1 + (4N + 5)^1 = M^2$.
The left side of the equation yields

$$(16N^2 + 24N + 9) + (4N + 4) + (4N + 5) = 2(8N^2 + 16N + 9). \quad (19)$$

In (19), the prime 2 has an odd exponent equal to 1. The factor $(8N^2 + 16N + 9)$ is odd for all values N. It therefore follows that $2(8N^2 + 16N + 9) \neq M^2$.
The equation $(4N + 3)^2 + (4N + 4)^1 + (4N + 5)^1 = M^2$ has no solutions.

(5) The case $(4N+3)^1 + (4N + 4)^2 + (4N + 5)^2 = M^2$.
The left side of the equation yields

$$(4N + 3) + (16N^2 + 32N + 16) + (16N^2 + 40N + 25) = 4(8N^2 + 19N + 11).$$

When $N = 0, 1$, the equation has no solutions. When $N = 2$, then $p = 11$ and $M = 18$. The first solution of the equation has been achieved. For any other solution if such exists, it follows that $(8N^2 + 19N + 11) = T^2$ where T is an integer, and $N \geq 3$. All values $3 \leq N \leq 50$ have been examined, and $(8N^2 + 19N + 11) \neq T^2$.
The equation $(4N + 3)^1 + (4N + 4)^2 + (4N + 5)^2 = M^2$ has exactly one solution ($N = 2$) when $0 \leq N \leq 50$. For all primes $3 \leq p \leq 199$, $p = 11$ is the only solution.

(6) The case $(4N+3)^2 + (4N + 4)^1 + (4N + 5)^2 = M^2$.
The left side of the equation yields

$$(16N^2 + 24N + 9) + (4N + 4) + (16N^2 + 40N + 25) = 2(16N^2 + 34N + 19). \quad (20)$$

The prime 2 has an odd exponent equal to 1, and the factor $(16N^2 + 34N + 19)$ is odd for all values N. Thus, the right side of (20) is not equal to a square.
The equation $(4N + 3)^2 + (4N + 4)^1 + (4N + 5)^2 = M^2$ has no solutions.

(7) The case $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^1 = M^2$.
The left side of the equation yields

$$(16N^2 + 24N + 9) + (16N^2 + 32N + 16) + (4N + 5) = 2(16N^2 + 30N + 15). \quad (21)$$

In (21), the prime 2 has an odd exponent equal to 1, and the factor $(16N^2 + 30N + 15)$ is odd for all values N. Hence, the right side of (21) is not equal to a square.
The equation $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^1 = M^2$ has no solutions.

(8) The case $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^2 = M^2$.
The left side of the equation yields

$$(16N^2 + 24N + 9) + (16N^2 + 32N + 16) + (16N^2 + 40N + 25) = 2(24N^2 + 48N + 25). \quad (22)$$

In (22), the prime 2 has an odd exponent equal to 1, and the factor $(24N^2 + 48N + 25)$ is odd for all values N. Therefore, the right side of (22) is not equal to a square.
The equation $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^2 = M^2$ has no solutions.

This concludes the proof of Theorem 4.1. □

Based on our findings for case (5), we state the following conjecture.

Conjecture 1. The equation $(4N + 3) + (4N + 4)^2 + (4N + 5)^2 = M^2$ has no solutions for all values $N > 50$.

5. Conclusion

The famous equation $p^x + q^y = z^2$ mentioned earliar was considered by many authors. The equations $p^x + (p + 1)^y + (p + 2)^z = M^2$ when $p \geq 2$ is prime and $1 \leq x, y, z \leq 2$ form an extension of the previous equation. We have shown: (a) A unique solution exists for $p = 2$ and $x = y = z = 1$. (b) No solutions exist for all primes $p = 4N + 1$ when $1 \leq x, y, z \leq 2$. (c) When $x = y = z = 1$, infinitely many primes $p = 4N + 3$ exist for which the equation has a solution. (d) For $x = 1$, $y = z = 2$, the equation has exactly one solution when $3 \leq p \leq 199$. (e) No solutions exist for all other unmentioned cases $1 \leq x, y, z \leq 2$. The results were achieved in an elementary manner which includes our new method that uses the last digits of certain powers.

This is a pioneering and preliminary article in the extended direction, since to the best of our knowledge other authors have not considered equations such as $p^x + (p + 1)^y + (p + 2)^z = M^2$ for primes $p \geq 2$ when $1 \leq x, y, z \leq 2$. It is therefore obvious, that there are no references on such equations.

REFERENCES

1. N. Burshtein, On solutions of the diophantine equation $8^x + 9^y = z^2$ when x, y, z are positive integers, *Annals of Pure and Applied Mathematics*, 20 (2) (2019) 79-83.
2. N. Burshtein, A note on the diophantine equation $p^x + (p + 1)^y = z^2$, *Annals of Pure and Applied Mathematics*, 19 (1) (2019) 19-20.
3. N. Burshtein, A note on the diophantine equation $p + q + r = M^2$ and the Goldbach Conjectures, *Annals of Pure and Applied Mathematics*, 13 (2) (2017) 293-296.

4. B. Poonen, Some diophantine equations of the form $x^n + y^n = z^m$, *Acta. Arith.*, 86 (1998) 193-205.
5. B. Sroysang, On the diophantine equation $5^x + 7^y = z^2$, *Int. J. Pure Appl. Math.*, 89 (2013) 115-118.