Problem 1.

Proof. We prove by induction that for all integers $n > 10$,

$$n - 2 < \frac{n^2 - n}{12}.$$

Base Case: $n = 11$

$$11 - 2 < \frac{11^2 - 11}{12} \implies 9 < \frac{110}{12} \implies 9 < 9.166 \quad \text{(True)}.$$

Inductive Step: Assume for some $k > 10$ that

$$k - 2 < \frac{k^2 - k}{12}.$$

We show for $k + 1$:

$$(k+1) - 2 < \frac{(k+1)^2 - (k+1)}{12} \implies k - 1 < \frac{k^2 + k}{12}.$$

Proof: From the inductive hypothesis,

$$k - 2 < \frac{k^2 - k}{12}.$$

Add 1 to both sides:

$$k - 1 < \frac{k^2 - k}{12} + 1.$$

Now observe that

$$\frac{k^2 - k}{12} + 1 \leq \frac{k^2 + k}{12} \quad \text{if and only if} \quad 1 \leq \frac{2k}{12} \implies 6 \leq k.$$

Since $k > 10$, this inequality holds. Therefore,

$$k - 1 < \frac{k^2 - k}{12} + 1 \leq \frac{k^2 + k}{12},$$

which completes the inductive step.

By mathematical induction, the statement holds for all integers $n > 10$.

□

Problem 2.

Proof. We prove by induction that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \sqrt{i} > \frac{2n\sqrt{n}}{3}.$$

Base Case: $n = 1$

$$\sum_{i=1}^{1} \sqrt{i} = \sqrt{1} = 1, \quad \frac{2 \cdot 1 \cdot \sqrt{1}}{3} = \frac{2}{3}, \quad 1 > \frac{2}{3}.$$

The base case holds.

Inductive Step: Assume for some $k \geq 1$ that

$$\sum_{i=1}^{k} \sqrt{i} > \frac{2k\sqrt{k}}{3}.$$

We show for $k+1$:

$$\sum_{i=1}^{k+1} \sqrt{i} > \frac{2(k+1)\sqrt{k+1}}{3}.$$

Starting from the left side:

$$\sum_{i=1}^{k+1} \sqrt{i} = \sum_{i=1}^{k} \sqrt{i} + \sqrt{k+1} > \frac{2k\sqrt{k}}{3} + \sqrt{k+1}.$$

It suffices to show that:

$$\frac{2k\sqrt{k}}{3} + \sqrt{k+1} \geq \frac{2(k+1)\sqrt{k+1}}{3}.$$

Multiply both sides by 3:

$$2k\sqrt{k} + 3\sqrt{k+1} \geq 2(k+1)\sqrt{k+1}.$$

Rearrange terms:

$$2k\sqrt{k} \geq 2(k+1)\sqrt{k+1} - 3\sqrt{k+1} = (2k-1)\sqrt{k+1}.$$

Square both sides (valid since all terms are positive for $k \geq 1$):

$$(2k\sqrt{k})^2 \geq ((2k-1)\sqrt{k+1})^2,$$

$$4k^2 \cdot k \geq (2k-1)^2 \cdot (k+1),$$

$$4k^3 \geq (4k^2 - 4k + 1)(k+1).$$

Expand the right side:

$$4k^3 \geq 4k^3 + 4k^2 - 4k^2 - 4k + k + 1 = 4k^3 - 3k + 1.$$

Subtract $4k^3$ from both sides:

$$0 \geq -3k + 1 \iff 3k \geq 1 \iff k \geq \frac{1}{3}.$$

Since $k \geq 1$, this inequality holds. Therefore,

$$\sum_{i=1}^{k+1} \sqrt{i} > \frac{2(k+1)\sqrt{k+1}}{3},$$

which completes the inductive step.

By mathematical induction, the statement holds for all integers $n \geq 1$. $\square$

Problem 3.

Proof. We prove by induction that for all integers $n \geq 0$,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Base Case ($n = 0$):

$$\sum_{i=0}^{0} i^2 = 0 \quad \text{and} \quad \frac{0(0+1)(2 \cdot 0+1)}{6} = 0.$$

Thus, the base case holds.

Inductive Step: Assume that for some $k \geq 0$,

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

We will show that the statement holds for $k+1$, that is,

$$\sum_{i=0}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Starting from the left-hand side:

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \sum_{i=0}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right] \\
&= (k+1)\left[\frac{2k^2 + k + 6k + 6}{6}\right] \\
&= (k+1)\left[\frac{2k^2 + 7k + 6}{6}\right] \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

This completes the inductive step.
By mathematical induction, the statement holds for all integers $n \geq 0$. $\square$

Problem 4.

Proof. We prove by **strong induction** that every integer $n > 1$ is either prime or can be written as a product of prime numbers.

Base Case: ($n = 2$) The number 2 is prime by definition, as its only positive divisors are 1 and 2. A single prime is trivially considered a product of primes. Thus, the statement holds for $n = 2$.

Inductive Step: Assume the inductive hypothesis: that for some integer $k \geq 2$, every integer j with $2 \leq j \leq k$ is a product of primes (i.e., is prime itself or can be factored into primes). We must show that the integer $k + 1$ is also a product of primes.

We consider two cases:

- **Case 1:** If $k + 1$ is prime, then it is trivially a product of primes (itself), and we are done.
- **Case 2:** If $k + 1$ is composite, then by the definition of a composite number, it has positive divisors other than 1 and itself. Therefore, it can be written as:

  $$k + 1 = a \cdot b$$

  where a and b are integers satisfying $1 < a, b < k + 1$.

  Since $2 \leq a \leq k$ and $2 \leq b \leq k$, the strong induction hypothesis applies to both a and b. Hence, both are products of primes:

  $$a = p_1 p_2 \cdots p_m,$$
  $$b = q_1 q_2 \cdots q_n,$$

  where each p_i and q_j is a prime number.

  Substituting these products, we find:

  $$k + 1 = a \cdot b = (p_1 p_2 \cdots p_m)(q_1 q_2 \cdots q_n).$$

  This is clearly a product of prime numbers.

In both cases, $k + 1$ is a product of primes. By the principle of strong mathematical induction, every integer $n > 1$ is a product of primes. $\square$

Problem 6.

Proof. We prove by induction that for all positive integers n,

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n},$$

where F_i denotes the i^{th} Fibonacci number, defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Base Case: $(n = 1)$

$$F_1 = 1 \quad \text{and} \quad F_2 = 1, \quad \text{so} \quad F_1 = F_2.$$

Thus, the base case holds.

Inductive Hypothesis: Assume for some $k \geq 1$ that

$$F_1 + F_3 + \cdots + F_{2k-1} = F_{2k}.$$

Inductive Step: We now show that the statement holds for $n = k + 1$, i.e.,

$$F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} = F_{2k+2}.$$

Starting from the left-hand side:

$$\begin{aligned} F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} &= (F_1 + F_3 + \cdots + F_{2k-1}) + F_{2k+1} \\ &= F_{2k} + F_{2k+1} \quad \text{(by the inductive hypothesis)} \\ &= F_{2k+2} \quad \text{(by the Fibonacci recurrence relation).} \end{aligned}$$

This completes the inductive step.

By mathematical induction, the statement holds for all positive integers n. ☐

Problem 7.

Proof. We will prove by induction that $2^n > n^2$ for all natural numbers $n \geq 5$.

Base Case: $n = 5$

$$2^5 = 32 \quad \text{and} \quad 5^2 = 25.$$

Since $32 > 25$, the base case holds.

Inductive Step: Assume the induction hypothesis holds for some integer $k \geq 5$, that is, assume

$$2^k > k^2. \tag{IH}$$

We must now prove that the inequality holds for $k + 1$, i.e.,

$$2^{k+1} > (k+1)^2.$$

Starting with the left-hand side of the desired inequality:

$$2^{k+1} = 2 \cdot 2^k.$$

By the induction hypothesis (IH), $2^k > k^2$, so we can substitute:

$$2^{k+1} > 2 \cdot k^2. \tag{1}$$

Our goal is to show that the right-hand side of (1) is greater than $(k+1)^2$. Let us therefore examine the inequality:

$$2k^2 > (k+1)^2.$$

Expanding the right-hand side gives:

$$2k^2 > k^2 + 2k + 1.$$

Subtracting k^2 from both sides yields the equivalent inequality:

$$k^2 > 2k + 1. \tag{2}$$

We now show that inequality (2) is true for $k \geq 5$. Consider the function $f(k) = k^2 - 2k - 1$. Its derivative $f'(k) = 2k - 2$ is positive for $k > 1$, so $f(k)$ is increasing for $k \geq 5$. Since $f(5) = 25 - 10 - 1 = 14 > 0$, it follows that $k^2 > 2k + 1$ for all $k \geq 5$. Therefore, inequality (2) holds.

We can now chain the inequalities together. From (1) we have $2^{k+1} > 2k^2$, and from (2) we have $2k^2 > k^2 + 2k + 1 = (k+1)^2$. Hence,

$$2^{k+1} > 2k^2 > (k+1)^2,$$

which completes the inductive step.

By the principle of mathematical induction, $2^n > n^2$ for all natural numbers $n \geq 5$. $\square$