Problem 1.

Proof. We prove by induction that for all integers $n > 10$,

$$n - 2 < \frac{n^2 - n}{12}.$$

Base Case: $n = 11$

$$11 - 2 < \frac{11^2 - 11}{12} \implies 9 < \frac{110}{12} \implies 9 < 9.166 \quad (\text{True}).$$

Inductive Step: Assume for some $k > 10$ that

$$k - 2 < \frac{k^2 - k}{12}.$$

We show for $k + 1$:

$$(k + 1) - 2 < \frac{(k+1)^2 - (k+1)}{12} \implies k - 1 < \frac{k^2 + k}{12}.$$

Proof: From the inductive hypothesis,

$$k - 2 < \frac{k^2 - k}{12}.$$

Add 1 to both sides:

$$k - 1 < \frac{k^2 - k}{12} + 1.$$

Now observe that

$$\frac{k^2 - k}{12} + 1 \leq \frac{k^2 + k}{12} \quad \text{if and only if} \quad 1 \leq \frac{2k}{12} \implies 6 \leq k.$$

Since $k > 10$, this inequality holds. Therefore,

$$k - 1 < \frac{k^2 - k}{12} + 1 \leq \frac{k^2 + k}{12},$$

which completes the inductive step.

By mathematical induction, the statement holds for all integers $n > 10$.

$\square$

Problem 2.

Proof. We prove by induction that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \sqrt{i} > \frac{2n\sqrt{n}}{3}.$$

1

Base Case: $n = 1$

$$\sum_{i=1}^{1} \sqrt{i} = \sqrt{1} = 1, \quad \frac{2 \cdot 1 \cdot \sqrt{1}}{3} = \frac{2}{3}, \quad 1 > \frac{2}{3}.$$

The base case holds.

Inductive Step: Assume for some $k \geq 1$ that

$$\sum_{i=1}^{k} \sqrt{i} > \frac{2k\sqrt{k}}{3}.$$

We show for $k + 1$:

$$\sum_{i=1}^{k+1} \sqrt{i} > \frac{2(k+1)\sqrt{k+1}}{3}.$$

Starting from the left side:

$$\sum_{i=1}^{k+1} \sqrt{i} = \sum_{i=1}^{k} \sqrt{i} + \sqrt{k+1} > \frac{2k\sqrt{k}}{3} + \sqrt{k+1}.$$

It suffices to show that:

$$\frac{2k\sqrt{k}}{3} + \sqrt{k+1} \geq \frac{2(k+1)\sqrt{k+1}}{3}.$$

Multiply both sides by 3:

$$2k\sqrt{k} + 3\sqrt{k+1} \geq 2(k+1)\sqrt{k+1}.$$

Rearrange terms:

$$2k\sqrt{k} \geq 2(k+1)\sqrt{k+1} - 3\sqrt{k+1} = (2k-1)\sqrt{k+1}.$$

Square both sides (valid since all terms are positive for $k \geq 1$):

$$(2k\sqrt{k})^2 \geq ((2k-1)\sqrt{k+1})^2,$$

$$4k^2 \cdot k \geq (2k-1)^2 \cdot (k+1),$$

$$4k^3 \geq (4k^2 - 4k + 1)(k+1).$$

Expand the right side:

$$4k^3 \geq 4k^3 + 4k^2 - 4k^2 - 4k + k + 1 = 4k^3 - 3k + 1.$$

Subtract $4k^3$ from both sides:

$$0 \geq -3k + 1 \quad \Longleftrightarrow \quad 3k \geq 1 \quad \Longleftrightarrow \quad k \geq \frac{1}{3}.$$

Since $k \geq 1$, this inequality holds. Therefore,

$$\sum_{i=1}^{k+1} \sqrt{i} > \frac{2(k+1)\sqrt{k+1}}{3},$$

which completes the inductive step.

By mathematical induction, the statement holds for all integers $n \geq 1$. $\quad \square$

Problem 3.

Proof. We prove by induction that for all integers $n \geq 0$,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Base Case ($n = 0$):

$$\sum_{i=0}^{0} i^2 = 0 \quad \text{and} \quad \frac{0(0+1)(2 \cdot 0 + 1)}{6} = 0.$$

Thus, the base case holds.

Inductive Step: Assume that for some $k \geq 0$,

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

We will show that the statement holds for $k+1$, that is,

$$\sum_{i=0}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Starting from the left-hand side:

$$\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^{k} i^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right]$$

$$= (k+1)\left[\frac{2k^2 + k + 6k + 6}{6}\right]$$

$$= (k+1)\left[\frac{2k^2 + 7k + 6}{6}\right]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}.$$

This completes the inductive step.

By mathematical induction, the statement holds for all integers $n \geq 0$. $\qquad\square$

Problem 4.

Proof. We prove by **strong induction** that every integer $n > 1$ is either prime or can be written as a product of prime numbers.

Base Case: $(n = 2)$ The number 2 is prime by definition, as its only positive divisors are 1 and 2. A single prime is trivially considered a product of primes. Thus, the statement holds for $n = 2$.

Inductive Step: Assume the inductive hypothesis: that for some integer $k \geq 2$, every integer j with $2 \leq j \leq k$ is a product of primes (i.e., is prime itself or can be factored into primes). We must show that the integer $k + 1$ is also a product of primes.

We consider two cases:

- **Case 1:** If $k + 1$ is prime, then it is trivially a product of primes (itself), and we are done.

- **Case 2:** If $k + 1$ is composite, then by the definition of a composite number, it has positive divisors other than 1 and itself. Therefore, it can be written as:
$$k + 1 = a \cdot b$$

where a and b are integers satisfying $1 < a, b < k + 1$.

Since $2 \leq a \leq k$ and $2 \leq b \leq k$, the strong induction hypothesis applies to both a and b. Hence, both are products of primes:

$$a = p_1 p_2 \cdots p_m,$$
$$b = q_1 q_2 \cdots q_n,$$

where each p_i and q_j is a prime number.

Substituting these products, we find:

$$k + 1 = a \cdot b = (p_1 p_2 \cdots p_m)(q_1 q_2 \cdots q_n).$$

This is clearly a product of prime numbers.

In both cases, $k + 1$ is a product of primes. By the principle of strong mathematical induction, every integer $n > 1$ is a product of primes. $\qquad\square$

Problem 6.

Proof. We prove by induction that for all positive integers n,

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n},$$

where F_i denotes the ith Fibonacci number, defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Base Case: $(n = 1)$

$$F_1 = 1 \quad \text{and} \quad F_2 = 1, \quad \text{so} \quad F_1 = F_2.$$

Thus, the base case holds.

Inductive Hypothesis: Assume for some $k \geq 1$ that

$$F_1 + F_3 + \cdots + F_{2k-1} = F_{2k}.$$

Inductive Step: We now show that the statement holds for $n = k+1$, i.e.,

$$F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} = F_{2k+2}.$$

Starting from the left-hand side:

$$
\begin{aligned}
F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} &= (F_1 + F_3 + \cdots + F_{2k-1}) + F_{2k+1} \\
&= F_{2k} + F_{2k+1} \quad \text{(by the inductive hypothesis)} \\
&= F_{2k+2} \quad \text{(by the Fibonacci recurrence relation).}
\end{aligned}
$$

This completes the inductive step.

By mathematical induction, the statement holds for all positive integers n. $\qquad\square$

Problem 7.

Proof. We will prove by induction that $2^n > n^2$ for all natural numbers $n \geq 5$.

Base Case: $n = 5$

$$2^5 = 32 \quad \text{and} \quad 5^2 = 25.$$

Since $32 > 25$, the base case holds.

Inductive Step: Assume the induction hypothesis holds for some integer $k \geq 5$, that is, assume

$$2^k > k^2. \tag{IH}$$

We must now prove that the inequality holds for $k + 1$, i.e.,

$$2^{k+1} > (k+1)^2.$$

Starting with the left-hand side of the desired inequality:

$$2^{k+1} = 2 \cdot 2^k.$$

By the induction hypothesis (IH), $2^k > k^2$, so we can substitute:

$$2^{k+1} > 2 \cdot k^2. \tag{1}$$

Our goal is to show that the right-hand side of (1) is greater than $(k+1)^2$. Let us therefore examine the inequality:

$$2k^2 > (k+1)^2.$$

Expanding the right-hand side gives:

$$2k^2 > k^2 + 2k + 1.$$

Subtracting k^2 from both sides yields the equivalent inequality:

$$k^2 > 2k + 1. \tag{2}$$

We now show that inequality (2) is true for $k \geq 5$. Consider the function $f(k) = k^2 - 2k - 1$. Its derivative $f'(k) = 2k - 2$ is positive for $k > 1$, so $f(k)$ is increasing for $k \geq 5$. Since $f(5) = 25 - 10 - 1 = 14 > 0$, it follows that $k^2 > 2k + 1$ for all $k \geq 5$. Therefore, inequality (2) holds.

We can now chain the inequalities together. From (1) we have $2^{k+1} > 2k^2$, and from (2) we have $2k^2 > k^2 + 2k + 1 = (k+1)^2$. Hence,

$$2^{k+1} > 2k^2 > (k+1)^2,$$

which completes the inductive step.

By the principle of mathematical induction, $2^n > n^2$ for all natural numbers $n \geq 5$. $\qquad\square$

Problem 9.

Proof. We prove by induction that for all integers $n \geq 1$, $9^n - 2^n$ is divisible by 7.

Base Case ($n = 1$):
$$9^1 - 2^1 = 9 - 2 = 7$$

Since 7 is divisible by 7, the base case holds.

Inductive Step: Assume that for some integer $k \geq 1$, the statement holds. That is, assume

$$9^k - 2^k = 7i \quad \text{for some integer } i.$$

This is our inductive hypothesis. We will show that the statement is true for $k + 1$, i.e., that $9^{k+1} - 2^{k+1}$ is divisible by 7.

We begin with the expression for $k + 1$ and manipulate it to incorporate the inductive hypothesis:

$$
\begin{aligned}
9^{k+1} - 2^{k+1} &= 9 \cdot 9^k - 2 \cdot 2^k \\
&= 9 \cdot 9^k - 9 \cdot 2^k + 9 \cdot 2^k - 2 \cdot 2^k \quad \text{(Adding and subtracting } 9 \cdot 2^k) \\
&= 9(9^k - 2^k) + 2^k(9 - 2) \\
&= 9(7i) + 2^k \cdot 7 \quad\quad\quad\quad\quad\quad \text{(By the inductive hypothesis)} \\
&= 7(9i + 2^k)
\end{aligned}
$$

Let $j = 9i + 2^k$, which is an integer. Therefore, we have shown that

$$9^{k+1} - 2^{k+1} = 7j,$$

which is divisible by 7.

By the principle of mathematical induction, $9^n - 2^n$ is divisible by 7 for all integers $n \geq 1$. $\qquad\square$

Problem 10.

Proof. We will prove by induction that $\sum_{i=1}^{n}(4i - 3) = \frac{n(4n-2)}{2}$ for all natural numbers n.

Base Case ($n = 1$):

$$\sum_{i=1}^{1}(4i - 3) = 4(1) - 3 = 1.$$

$$\frac{1 \cdot (4 \cdot 1 - 2)}{2} = \frac{1 \cdot 2}{2} = 1.$$

Since both expressions equal 1, the base case holds.

Inductive Step: Assume that for some integer $k \geq 1$, the statement holds. That is, assume the inductive hypothesis:

$$\sum_{i=1}^{k}(4i - 3) = \frac{k(4k - 2)}{2}.$$

We will show that the statement is true for $k + 1$, i.e.,

$$\sum_{i=1}^{k+1}(4i - 3) = \frac{(k + 1)(4(k + 1) - 2)}{2} = \frac{(k + 1)(4k + 2)}{2}.$$

Starting from the left-hand side of the statement for $k + 1$:

$$\sum_{i=1}^{k+1}(4i - 3) = \left(\sum_{i=1}^{k}(4i - 3) \right) + (4(k + 1) - 3)$$

$$= \frac{k(4k - 2)}{2} + (4k + 4 - 3) \qquad \text{(by the inductive hypothesis)}$$

$$= \frac{k(4k - 2)}{2} + (4k + 1)$$

$$= \frac{k(4k - 2) + 2(4k + 1)}{2} \qquad \text{(obtaining a common denominator)}$$

$$= \frac{4k^2 - 2k + 8k + 2}{2}$$

$$= \frac{4k^2 + 6k + 2}{2}.$$

Now, we factor the numerator to show it matches our target expression:

$$\frac{4k^2 + 6k + 2}{2} = \frac{2(2k^2 + 3k + 1)}{2}$$

$$= 2k^2 + 3k + 1.$$

And indeed, the target expression also simplifies to the same result:

$$\frac{(k+1)(4k+2)}{2} = \frac{(k+1) \cdot 2(2k+1)}{2}$$
$$= (k+1)(2k+1)$$
$$= 2k^2 + k + 2k + 1$$
$$= 2k^2 + 3k + 1.$$

Therefore, we have shown that

$$\sum_{i=1}^{k+1}(4i - 3) = \frac{(k+1)(4k+2)}{2},$$

which completes the inductive step.

By the principle of mathematical induction, the formula $\sum_{i=1}^{n}(4i - 3) = \frac{n(4n-2)}{2}$ holds for all natural numbers n. □

Problem 11.

Proof. We prove by induction that $2^n \geq 1 + n$ for all $n \geq 1$.

Base Case $(n = 1)$:
$$2^1 = 2 \geq 1 + 1 = 2.$$

Thus, the base case holds.

Inductive Step: Assume that for some integer $k \geq 1$, the statement holds. That is, assume
$$2^k \geq 1 + k.$$

This is our inductive hypothesis. We will show that the statement is true for $k + 1$, i.e., that $2^{k+1} \geq 1 + (k + 1)$.

$$2^{k+1} = 2^k \cdot 2$$
$$\geq (1 + k) \cdot 2 \quad \text{(by the inductive hypothesis)}$$
$$= 2 + 2k$$
$$= (k + 2) + k$$
$$\geq k + 2 \quad \text{(since } k \geq 1 > 0)$$
$$= 1 + (k + 1).$$

Thus, the statement holds for $k + 1$.

By mathematical induction, $2^n \geq 1 + n$ for all $n \geq 1$. □