Problem 1.

Proof. We prove by induction that for all integers $n > 10$,

$$n - 2 < \frac{n^2 - n}{12}.$$

Base Case: $n = 11$

$$11 - 2 < \frac{11^2 - 11}{12} \implies 9 < \frac{110}{12} \implies 9 < 9.166 \quad \text{(True)}.$$

Inductive Step: Assume for some $k > 10$ that

$$k - 2 < \frac{k^2 - k}{12}.$$

We show for $k + 1$:

$$(k+1) - 2 < \frac{(k+1)^2 - (k+1)}{12} \implies k - 1 < \frac{k^2 + k}{12}.$$

Proof: From the inductive hypothesis,

$$k - 2 < \frac{k^2 - k}{12}.$$

Add 1 to both sides:

$$k - 1 < \frac{k^2 - k}{12} + 1.$$

Now observe that

$$\frac{k^2 - k}{12} + 1 \leq \frac{k^2 + k}{12} \quad \text{if and only if} \quad 1 \leq \frac{2k}{12} \implies 6 \leq k.$$

Since $k > 10$, this inequality holds. Therefore,

$$k - 1 < \frac{k^2 - k}{12} + 1 \leq \frac{k^2 + k}{12},$$

which completes the inductive step.

By mathematical induction, the statement holds for all integers $n > 10$.

□

problem 2.

Proof. We prove by induction that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \sqrt{i} > \frac{2n\sqrt{n}}{3}.$$

Base Case: $n = 1$

$$\sum_{i=1}^{1} \sqrt{i} = \sqrt{1} = 1, \quad \frac{2 \cdot 1 \cdot \sqrt{1}}{3} = \frac{2}{3}, \quad 1 > \frac{2}{3}.$$

The base case holds.

Inductive Step: Assume for some $k \geq 1$ that

$$\sum_{i=1}^{k} \sqrt{i} > \frac{2k\sqrt{k}}{3}.$$

We show for $k+1$:

$$\sum_{i=1}^{k+1} \sqrt{i} > \frac{2(k+1)\sqrt{k+1}}{3}.$$

Starting from the left side:

$$\sum_{i=1}^{k+1} \sqrt{i} = \sum_{i=1}^{k} \sqrt{i} + \sqrt{k+1} > \frac{2k\sqrt{k}}{3} + \sqrt{k+1}.$$

It suffices to show that:

$$\frac{2k\sqrt{k}}{3} + \sqrt{k+1} \geq \frac{2(k+1)\sqrt{k+1}}{3}.$$

Multiply both sides by 3:

$$2k\sqrt{k} + 3\sqrt{k+1} \geq 2(k+1)\sqrt{k+1}.$$

Rearrange terms:

$$2k\sqrt{k} \geq 2(k+1)\sqrt{k+1} - 3\sqrt{k+1} = (2k-1)\sqrt{k+1}.$$

Square both sides (valid since all terms are positive for $k \geq 1$):

$$\begin{gathered}(2k\sqrt{k})^2 \geq ((2k-1)\sqrt{k+1})^2, \\ 4k^2 \cdot k \geq (2k-1)^2 \cdot (k+1), \\ 4k^3 \geq (4k^2 - 4k + 1)(k+1).\end{gathered}$$

Expand the right side:

$$4k^3 \geq 4k^3 + 4k^2 - 4k^2 - 4k + k + 1 = 4k^3 - 3k + 1.$$

Subtract $4k^3$ from both sides:

$$0 \geq -3k + 1 \quad \Longleftrightarrow \quad 3k \geq 1 \quad \Longleftrightarrow \quad k \geq \frac{1}{3}.$$

Since $k \geq 1$, this inequality holds. Therefore,

$$\sum_{i=1}^{k+1} \sqrt{i} > \frac{2(k+1)\sqrt{k+1}}{3},$$

which completes the inductive step.

By mathematical induction, the statement holds for all integers $n \geq 1$. $\square$