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SMOOTHED PARTICLE HYDRODYNAMICS

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1. INTRODUCTION

Smoothed particle hydrodynamics (SPH) was invented to simulate nonaxisymmetric phenomena in astrophysics (Lucy 1977, Gingold & Monaghan 1977). We wanted a method that was easy to work with and could give reasonable accuracy. The SPH method satisfied these requirements. As a bonus we found the SPH was rugged, gave sensible answers in difficult situations, and could be extended to complicated physics without much trouble.

The SPH method is a particle method. Unlike the particle in cell method (PIC) (Harlow 1957, 1974, 1988), SPH does not need a grid to calculate spatial derivatives. Instead, they are found by analytical differentiation of interpolation formulae. The equations of momentum and energy become sets of ordinary differential equations which are easy to understand in mechanical and thermodynamical terms. For example, the pressure gradient becomes a force between pairs of particles. The astrophysicist can then use intuition in a way which is difficult with the original partial differential equations. This intuition, coupled with detailed analysis, has allowed SPH to be extended to a wide variety of astrophysical problems. Although very accurate finite-difference methods exist—and these are better than SPH for some problems—they cannot handle complex physics in three dimensions with the same ease.

Various aspects of SPH have been reviewed by Monaghan (1985, 1988a) and Benz (1988, 1989).

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2. FUNDAMENTALS

At the heart of SPH is an interpolation method which allows any function to be expressed in terms of its values at a set of disordered points—the particles. The ideas are given in Lucy (1977), Gingold & Monaghan (1977, 1982), and Monaghan (1985).

The integral interpolant of any function $A(\mathbf{r})$ is defined by

$$A_I(\mathbf{r}) = \int A(\mathbf{r}')\, W(\mathbf{r}-\mathbf{r}', h)\mathbf{dr}', \qquad 2.1$$

where the integration is over the entire space, and W is an interpolating kernel which has the two properties

$$\int W(\mathbf{r}-\mathbf{r}', h)\mathbf{dr}' = 1 \qquad 2.2$$

and

$$\lim_{h\to 0} W(\mathbf{r}-\mathbf{r}', h) = \delta(\mathbf{r}-\mathbf{r}'), \qquad 2.3$$

where the limit is to be interpreted as the limit of the corresponding integral interpolants. It is clear that kernel interpolation is related to the subject of singular integrals (Natanson 1960).

For numerical work the integral interpolant is approximated by a summation interpolant

$$A_S(\mathbf{r}) = \sum_b m_b \frac{A_b}{\rho_b} W(\mathbf{r}-\mathbf{r}_b, h), \qquad 2.4$$

where the summation index b denotes a particle label, and the summation is over all the particles. Particle b has mass m_b, position $\mathbf{r}_b$, density ρ_b, and velocity $\mathbf{v}_b$. The value of any quantity A at $\mathbf{r}_b$ is denoted by A_b.

The essential point is that we can construct a differentiable interpolant of a function from its values at the particles (interpolation points) by using a kernel which is differentiable. Derivatives of this interpolant can be obtained by ordinary differentiation; there is no need to use finite differences and no need for a grid. For example, if we want ∇A, we can use

$$\nabla A(\mathbf{r}) = \sum_b m_b \frac{A_b}{\rho_b} \nabla W(\mathbf{r}-\mathbf{r}_b, h), \qquad 2.5$$

though to obtain higher accuracy we would obtain the interpolant by writing

$$\rho\nabla A = \nabla(\rho A) - A\nabla\rho,$$

as shown in the examples below and, in the particular case of the pressure gradient, we would use a symmetrized form (e.g. Equation 3.3).

The original calculations of Gingold & Monaghan (1977) used a Gaussian kernel. In one dimension

$$W(x,h) = \frac{1}{h\sqrt{\pi}} e^{-(x^2/h^2)}, \qquad 2.6$$

which is the usual example of a sequence which mimics a delta function in the limit $h \to 0$. A kernel based on splines (Monaghan & Lattanzio 1986) has proven computationally efficient (other kernels will be discussed later). However, if you want to find a physical interpretation of an SPH equation, it is always best to assume the kernel is a Gaussian. This is the first golden rule of SPH.

The error in approximating Equation 2.1 by Equation 2.4 depends on the disorder of the particles (Monaghan 1982) and is normally $O(h^2)$ or better. It is important to realize that although the summations are formally over all the particles, only a small number actually contribute because W can be chosen so that it falls off rapidly for $|\mathbf{r} - \mathbf{r}_b| \geq h$.

For the rest of this review we will not distinguish between a summation interpolant and the actual function since this will be clear from the context.

The density is estimated everywhere by

$$\rho(\mathbf{r}) = \sum_b m_b W(\mathbf{r} - \mathbf{r}_b, h). \qquad 2.7$$

Another example is

$$\nabla \cdot \mathbf{v} = \sum_b m_b \mathbf{v}_b \cdot \nabla W(\mathbf{r} - \mathbf{r}_b, h), \qquad 2.8$$

but in this case it is better to remember the second golden rule of SPH which is to rewrite formulae with the density placed inside operators. For the previous case we write

$$\nabla \cdot \mathbf{v} = [\nabla \cdot (\rho \mathbf{v}) - \mathbf{v} \cdot \nabla\rho]/\rho, \qquad 2.9$$

so that the divergence of the velocity at particle a can be found from

$$\rho_a (\nabla \cdot \mathbf{v})_a = \sum_b (\mathbf{v}_b - \mathbf{v}_a) \cdot \nabla_a W_{ab}, \qquad 2.10$$

where the notation $\nabla_a W_{ab}$ denotes the gradient of $W(\mathbf{r}_a - \mathbf{r}_b, h)$ taken with respect to the coordinates of particle a. If the kernel is a Gaussian, the contribution from particle b to the divergence of the velocity at particle a is given by

$$2m_b(\mathbf{v}_a-\mathbf{v}_b)\cdot(\mathbf{r}_a-\mathbf{r}_b)W_{ab}/h^2, \qquad 2.11$$

which shows that the contribution is positive if the particles are moving away from each other, as expected. The vorticity of particle a is estimated in the same way by

$$\rho_a(\nabla\times\mathbf{v})_a=\sum_b m_b\mathbf{v}_{ab}\times\nabla_a W_{ab}. \qquad 2.12$$

Taking the kernel to be a Gaussian we find that the contribution of particle b to the vorticity of particle a is proportional to the relative angular momentum per unit mass of the two particles.

3. SIMPLE EQUATIONS OF MOTION

3.1 *The Momentum Equation*

Using the ideas outlined in the previous section, the equations of motion can be obtained easily. The pressure gradient could be estimated by using

$$\rho_a\nabla P_a=\sum_b m_b(P_b-P_a)\nabla_a W_{ab}, \qquad 3.1$$

and this has the advantage that the force vanishes exactly when the pressure is constant. However, it has the disadvantage that the linear and angular momentum are not conserved exactly (an isolated pair of particles with different pressures would bootstrap themselves to infinity), and it is difficult to construct a consistent energy equation.

In this case it is better to symmetrize the pressure gradient term by rewriting $\nabla P/\rho$ according to

$$\frac{\nabla P}{\rho}=\nabla\left(\frac{P}{\rho}\right)+\frac{P}{\rho^2}\nabla\rho. \qquad 3.2$$

The momentum equation for particle a then becomes

$$\frac{d\mathbf{v}_a}{dt}=-\sum_b m_b\left(\frac{P_b}{\rho_b^2}+\frac{P_a}{\rho_a^2}\right)\nabla_a W_{ab}, \qquad 3.3$$

where here and elsewhere d/dt denotes a derivative following the motion. The momentum equation in the form of Equation 3.3 was first derived from using a discrete form of the action principle for an adiabatic fluid.

The contribution to the force on particle a from particle b when the kernel is a Gaussian is

$$\frac{2m_a m_b}{h^2}\left(\frac{P_b}{\rho_b^2}+\frac{P_a}{\rho_a^2}\right)(\mathbf{r}_a-\mathbf{r}_b)W_{ab}, \qquad 3.4$$

which shows that the pressure gradient, when translated into an SPH equation, produces a symmetric central force between pairs of particles. As a result linear and angular momenta are conserved.

Infinitely many symmetric forms of the SPH momentum equations can be constructed. For example, noting

$$\frac{\nabla P}{\rho} = \frac{P}{\rho^{\sigma}}\nabla\left(\frac{1}{\rho^{1-\sigma}}\right) + \frac{1}{\rho^{2-\sigma}}\nabla\left(\frac{P}{\rho^{\sigma-1}}\right), \tag{3.5}$$

the SPH momentum equation becomes

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b\left(\frac{P_b}{\rho_a^{2-\sigma}\rho_b^{\sigma}} + \frac{P_a}{\rho_a^{\sigma}\rho_b^{2-\sigma}}\right)\nabla_a W_{ab}, \tag{3.6}$$

which is symmetric for any value of σ. Another symmetric combination is obtained by noting

$$\nabla P = 2\sqrt{P}\nabla\sqrt{P}. \tag{3.7}$$

My preference is for the form of Equation 3.2 since it arises naturally from an action principle.

The effect of a constant external pressure P_{ext} can be approximated by replacing P everywhere by $P - P_{\text{ext}}$ (Lattanzio et al 1985a).

3.2 *The Continuity Equation*

The continuity equation can be replaced either by the interpolant

$$\rho_a = \sum_b m_b W_{ab}, \tag{3.8}$$

or by

$$\frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab}\nabla_a W_{ab}, \tag{3.9}$$

where the notation $\mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b$, has been used. Almost all SPH calculations use Equation 3.8, but there are definite advantages in using Equation 3.9. In particular, if Equation 3.8 had been used in SPH calculations of the motion of water (Monaghan 1991), the density would have dropped near the edge of the fluid (this is SPH's valiant attempt to model an edge) and the resulting pressure would have caused the edge to oscillate. With Equation 3.9 the initial density of each particle can be set, and it will only vary when particles move relative to each other. Tests on shocks show that Equation 3.8 has no advantages over Equation 3.9. There is also a computational advantage in using Equation 3.9 since the rates of change of all physical variables can be computed in one subroutine, or one pass

over a tree. The disadvantage is that exact conservation of mass is not retained.

In some accretion problems it can be useful to treat accretion onto a body as loss of mass to a sink (Anzer et al 1987). The continuity equation then becomes

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v} - f(\mathbf{r}), \tag{3.10}$$

where $f(\mathbf{r})$ is, for example, a continuous function which is zero outside a sphere surrounding the origin. If we assume h is constant in space and time (the general case is treated later) we can write the continuity equation as

$$\frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab}\cdot\nabla W_{ab} - \sum_b m_b \frac{f_b}{\rho_b} W_{ab}, \tag{3.11}$$

where we have used an interpolant form of the sink to facilitate solving the equation. If we substitute for ρ_a (Equation 3.8), and allow the mass to vary with time, we find

$$\frac{dm_b}{dt} = -m_b \frac{f_b}{\rho_b}, \tag{3.12}$$

which shows that, in SPH, a sink is interpreted as a region within which the particles lose mass. This treatment of the sink is smoother than simply eliminating particles that enter a sphere surrounding the sink hole.

3.3 *The Thermal Energy Equation*

The equation for the rate of change of thermal energy per unit mass

$$\frac{du}{dt} = -\left(\frac{P}{\rho}\right)\nabla\cdot\mathbf{v} \tag{3.13}$$

can be written for particle a in the form

$$\frac{du_a}{dt} = \left(\frac{P_a}{\rho_a^2}\right)\sum_b m_b \mathbf{v}_{ab}\cdot\nabla_a W_{ab} \tag{3.14}$$

or, by noting

$$\frac{du}{dt} = -\nabla\left(\frac{P\mathbf{v}}{\rho}\right) + \mathbf{v}\cdot\nabla\left(\frac{P}{\rho}\right), \tag{3.15}$$

the thermal energy equation for particle a can be written

$$\frac{du_a}{dt} = \sum_b m_b \left(\frac{P_b}{\rho_b^2}\right) \mathbf{v}_{ab} \cdot \nabla W_{ab}. \tag{3.16}$$

By taking the average of the two previous equations, we find

$$\frac{du_a}{dt} = \frac{1}{2} \sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2}\right) \mathbf{v}_{ab} \cdot \nabla_a W_{ab}, \tag{3.17}$$

which has the same symmetric factors as Equation 3.3.

It is characteristic of SPH that gradient terms can be written in many different ways. Similar transformations are possible for finite-difference equations.

Any of the above forms of the energy equation, when interpreted using a Gaussian kernel, show that the thermal energy of particle a increases when particle b approaches it. This is the SPH equivalent of $\nabla \cdot \mathbf{v} < 0$. Benz (1989) finds that in some cases it is better to use Equation 3.14 instead of Equation 3.17.

It has been found (Hernquist 1991, private communication) that if the thermal energy equation is integrated using any of the SPH forms above, and if the density is calculated using Equation 3.8, the total entropy is not conserved as accurately as the energy. If an entropy equation is integrated then the total energy is not conserved as accurately as the entropy. However, if the gas is ideal, the total entropy will be conserved exactly if the density is calculated using Equation 3.9 and the thermal energy is calculated using any of the forms given above. In this case the mass is not conserved exactly. It seems one cannot have everything!

3.4 *Moving the Particles*

Particles are moved using either

$$\frac{d\mathbf{r}_a}{dt} = \mathbf{v}_a, \tag{3.18}$$

or the XSPH variant (where "X" is the unknown factor) (Monaghan 1989)

$$\frac{d\mathbf{r}_a}{dt} = \hat{\mathbf{v}}_a = \mathbf{v}_a + \varepsilon \sum_b m_b \left(\frac{\mathbf{v}_{ba}}{\bar{\rho}_{ab}}\right) W_{ab}, \tag{3.19}$$

with $\bar{\rho}_{ab} = (\rho_a + \rho_b)/2$ and $\varepsilon (0 \leq \varepsilon \leq 1)$ as a constant. The XSPH variant moves a particle with a velocity that is closer to the average velocity in its neighborhood. Strict consistency then requires that if ρ is found using Equation 3.9, $\hat{\mathbf{v}}$ should be used in place of $\mathbf{v}$ in $\mathbf{v}_{ab}$.

No dissipation is introduced by XSPH but it increases the dispersion

(which can be reduced by using a different kernel in Equation 3.19). The XSPH variant has proven useful in the simulation of nearly incompressible fluids such as water, where it keeps the particles orderly in the absence of viscosity.

Another interesting feature of the XSPH variant is that if pressure and viscous forces are set to zero, it simulates the Burgers equation with very large effective Reynolds number. Experiments (Monaghan, unpublished) show that for this problem the XSPH results are as good as the best finite-difference equations.

3.5 *Equation of State*

The equation of state can be as complicated as desired. For example Benz et al (1986, 1987) use equations of state for metals and minerals, including phase changes, in their simulation of the formation of the Moon.

4. VISCOSITY AND THERMAL CONDUCTION

4.1 *Viscosity*

Many forms of artificial viscosity have been proposed (Lucy 1977, Wood 1981, Monaghan & Gingold 1983, Evrard 1988, Loewenstein & Mathews 1984), but the most commonly used artificial viscosity is obtained by writing the momentum equation as

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab}, \tag{4.1}$$

where Π_{ab} is given by

$$\Pi_{ab} = \begin{cases} \dfrac{-\alpha \bar{c}_{ab} \mu_{ab} + \beta \mu_{ab}^2}{\bar{\rho}_{ab}} & \mathbf{v}_{ab} \cdot \mathbf{r}_{ab} < 0; \\ 0 & \mathbf{v}_{ab} \cdot \mathbf{r}_{ab} > 0; \end{cases}$$

and

$$\mu_{ab} = \frac{h \mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{\mathbf{r}_{ab}^2 + \eta^2}. \tag{4.2}$$

The viscosity vanishes when $\mathbf{v}_{ab} \cdot \mathbf{r}_{ab} > 0$, which is the SPH equivalent of the condition $\nabla \cdot \mathbf{v} > 0$.

The expression for Π_{ab} contains a term that is linear in the velocity differences, which produces a shear and bulk viscosity (Monaghan 1985). The quadratic term is necessary to handle high Mach number shocks, and

is roughly equivalent to the Von Neumann–Richtmyer viscosity used in finite-difference methods. This viscosity has a number of desirable features:

1. it is Galilean invariant;
2. it vanishes for rigid body rotation;
3. it conserves total linear and angular momenta.

Many tests have confirmed that, with this viscosity, shock fronts are spread over $\sim 3h$, which is sufficiently accurate for most purposes in astrophysics. The values of α and β are not critical, but they should be near $\alpha = 1$ and $\beta = 2$ for best results. The parameter η^2 in the formula for μ_{ab} prevents singularities. It should be small enough to prevent severe smoothing of the viscous term in high density regions, and this is normally achieved by taking $\eta^2 = 0.01h^2$. This choice of η^2 means that smoothing of the velocity will only take place if the particle spacing is $< 0.1h$.

In the case of accretion disks, the presence of viscosity is required in order to fit the observations, and this can be achieved by taking $\beta = 0$, and choosing appropriate values for α and h. In many other situations the physical viscosity is small, but shocks can still occur. To treat the shocks correctly requires $\alpha \sim 1$ but the shear viscosity may then be too large. In order to escape this difficulty there are several possible routes:

1. Construct a more general viscosity similar to the linear term involving α. This can be done (Monaghan 1991), but the resulting tensor viscosity does not guarantee angular momentum conservation.
2. Follow the finite-difference methods and use a switch. The idea is to monitor some quantity, for example the density, and try to predict the rapid change that comes with a shock. Experiments (Monaghan 1990) with a switch based on the density variation worked for one-dimensional shocks, but in astrophysical collapse problems the switch was always on. A switch based on the rate of change of the force was also examined. The force is a sensitive predictor of a shock, but a universal rule to control the switch has not been found.
3. Use a predictor-corrector method for the time steps, and only use the viscosity in the predictor step as in the two step Lax-Wendroff scheme (Richtmyer & Morton 1967, Gadd 1978).

A more general viscosity would be of very great value.

4.2 *Thermal Conduction*

The thermal conduction term

$$\frac{1}{\rho}\nabla \cdot (\kappa \nabla u) \qquad 4.4$$

can be approximated in a standard way, but because of the second derivatives it is found to be very sensitive to particle disorder (Brookshaw 1986, Monaghan 1988a). A better way is to base the SPH form on an integral approximation which in SPH form becomes

$$-\sum_b m_b \frac{(q_a+q_b)(u_a-u_b)(\mathbf{r}_{ab}\cdot\nabla_a W_{ab})}{\bar{\rho}_{ab}(\mathbf{r}_{ab}^2+\eta^2)}, \tag{4.5}$$

where $q = \kappa/\rho$ has the dimensions of length $\times$ velocity. If the kernel is a Gaussian, and the thermal conductivity and density are constant, the contribution of particle b to the heat conduction of particle a is given by

$$\frac{-2m_b\kappa(u_a-u_b)W_{ab}}{\rho h^2}. \tag{4.6}$$

Thermal conduction therefore takes place by the exchange of heat between pairs of particles. If the particles are in one dimension, and equispaced, it is easy to show that the SPH conduction equation mimics finite-difference equations.

The SPH conduction term (Equation 4.5) conserves total energy and, when the thermal energy increases monotonically with temperature, it is easy to show that the total entropy increases, as it should.

Equation 4.5 can be varied by replacing

$$\frac{q_a+q_b}{\bar{\rho}_{ab}} \quad \text{by} \quad \frac{\kappa_a+\kappa_b}{\rho_a\rho_b}. \tag{4.7}$$

In application to infinite strength shocks (Monaghan 1988) the heat conduction term removes the unwanted wall heating. In this case it is convenient to replace

$$q_a+q_b \quad \text{by} \quad hg(\bar{c}_{ab}+4|\mu_{ab}|), \tag{4.8}$$

where $g \sim 0.5$ is a constant.

If diffusive radiation transport occurs it can be included in the thermal energy equation using a term similar to the thermal conduction term above (Brookshaw 1985). However, this cannot be the complete story because the conduction conserves total energy, and there will be no heat loss. Heat loss requires a surface term. This can be derived by physical arguments (Brookshaw 1986) or by starting with the exact equation and transforming it by multiplying by the kernel and integrating over all space (Campbell 1988). Surface terms then appear naturally. However, as with finite-difference methods, there are still substantial obstacles to overcome in simulating radiative processes in complex astrophysical phenomena.

If there are sources of heating or cooling, these can be added to the

thermal energy equation. Several authors (Monaghan & Varnas 1988, Monaghan & Lattanzio 1991, Hernquist & Katz 1989) have included cooling from atomic and molecular processes. The cooling time is normally extremely short compared to the dynamical time scale and it is necessary to integrate the thermal energy equation implicitly. There is no difficulty in doing this.

6. SPATIALLY-VARYING RESOLUTION

The early experiments with SPH used an h that varied with time but was constant in space. The most common rule was to take

$$h \propto 1/\langle\rho\rangle^{1/\nu}, \qquad \langle\rho\rangle = \frac{1}{n}\sum_b \rho_b, \tag{6.1}$$

where ν is the number of dimensions and n is the number of particles. It is h which determines the resolution and the number of neighbors that contribute to the properties at a point. The efficiency and the accuracy would therefore be greater if h was chosen so that it depended on the local particle number density. Several authors (Hernquist & Katz 1989, Evrard 1988, Benz et al 1990) have used local resolution lengths, which has increased the resolution enormously. Typically, h_a is calculated from

$$\frac{dh_a}{dt} = -\left(\frac{h_a}{\nu\rho_a}\right)\frac{d\rho_a}{dt}. \tag{6.2}$$

When each particle has its own h, momentum can be conserved if the kernel is symmetric. A symmetric kernel can be obtained by using any of the standard kernels with h replaced by a symmetric combination of the hs for the two particles. A simple example is the arithmetric mean. An alternative is to use the average of two kernels, one with h_a and one with h_b (Hernquist & Katz 1989). This form appears when the equations are derived from an action principle.

A suitable density interpolant is

$$\rho(\mathbf{r}) = \sum_b m_b W(\mathbf{r}-\mathbf{r}_b, h_b), \tag{6.3}$$

and integration over all space shows that mass is conserved. The set of equations to be solved when the resolution length varies in space and time is

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b\left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab}\right)\nabla_a \tilde{W}_{ab}, \tag{6.4}$$

$$\frac{du_a}{dt} = \frac{1}{2}\sum_b m_b\left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab}\right)\mathbf{v}_{ab}\cdot\nabla_a\tilde{W}_{ab}, \quad 6.5$$

$$\frac{d\rho_a}{dt} = \sum_b m_b\mathbf{v}_{ab}\cdot\nabla_a\tilde{W}_{ab}, \quad 6.6$$

$$\frac{dh_a}{dt} = -\left(\frac{h_a}{\nu\rho_a}\right)\sum_b m_n\mathbf{v}_{ab}\cdot\nabla_a\tilde{W}_{ab}, \quad 6.7$$

$$\frac{d\mathbf{r}_a}{dt} = \mathbf{v}_a, \quad 6.8$$

where

$$\tilde{W}_{ab} = \frac{1}{2}[W(\mathbf{r}_{ab}, h_a) + W(\mathbf{r}_{ab}, h_b)]$$

or

$$\tilde{W}_{ab} = W(\mathbf{r}_{ab}, h_{ab})$$

and h_{ab} is an average of the h_a and h_b. The XSPH variant can be used instead of Equation 6.8 and a summation can be used instead of Equation 6.6.

A discussion of some mathematical questions concerning the relation between the SPH equations and the exact equations is deferred to a later section of this review.

7. KERNELS

The use of different kernels is the SPH analogue of the use of different difference schemes in finite difference methods. The advantage of SPH is that the kernel can be calculated in a subroutine, or a table, and it is then trivial to change a code with one kernel into a code with another. Many kernels can be devised (Natanson 1960, page 58; Monaghan 1982), but the kernel based on spline functions (Monaghan & Lattanzio 1985)

$$W(\mathbf{r}, h) = \frac{\sigma}{h^\nu}\begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 & \text{if } 0 \le \frac{r}{h} \le 1; \\ \frac{1}{4}(2-q)^3 & \text{if } 1 \le \frac{r}{h} \le 2; \\ 0 & \text{otherwise} \end{cases}$$

where ν is the number of dimensions and σ is a normalization constant with the values

$$\frac{2}{3}, \quad \frac{10}{7\pi}, \quad \frac{1}{\pi},$$

in one, two, and three dimensions respectively, has advantages. This kernel has compact support; the second derivative is continuous, and the dominant error term in the integral interpolant is $O(h^2)$. The compact support means that interactions are exactly zero for $r > 2h$; the continuity of the second derivative means that the kernel is not sensitive to disorder and the errors in approximating the integral interpolants by summation interpolants are small provided the particle disorder is not too large. The error in the integral interpolant can be determined by Taylor expansion of $A(\mathbf{r}')$ about $\mathbf{r}$ in Equation 2.1.

When the particles are equispaced, or nearly equispaced, the accuracy of kernel interpolation can be discussed using the Poisson summation formula (Monaghan 1985). This shows that the Fourier transform of good kernels falls rapidly with wave number. As an example, if particles with equal mass m are in one dimension, and equispaced with spacing λ, the density from the SPH sum is approximately

$$\rho_a = \frac{m}{s}\left[1 + F\left(\frac{2\pi h}{\lambda}\right)\right],$$

where F is the Fourier transform of the kernel. It is then easy to show that, for example, the Gaussian kernel produces negligible errors if $h > \lambda$.

More accurate kernels in one dimension can be constructed by requiring them to be normalized, to have zero first moments [i.e. $W(\mathbf{r}, h)$ an even function of r], and vanishing second moments

$$\int r^2 W(\mathbf{r}, h)\, d\mathbf{r} = 0, \tag{7.1}$$

so that by Taylor expansion of the function $A(\mathbf{r}')$ in the integral interpolant equation (2.1) the dominant error is $O(h^4)$. An example is the Super Gaussian kernel in three dimensions:

$$W(\mathbf{r}, h) = \frac{1}{\pi^{3/2} h^3}\left(\frac{5}{2} - r^2\right)\exp\left(-\frac{r^2}{h^2}\right). \tag{7.2}$$

A kernel with similar properties can be constructed by starting with any of the standard kernels and constructing a new kernel with the coefficients A and B according to

$$BW(\mathbf{r}, h)(1 - Ar^2) \qquad 7.3$$

so that the new kernel is normalized and the second moments vanish. The disadvantage of these high order kernels is that the kernel becomes negative in part of the domain, and this can have serious consequences when there is a sharp change in density. An undershoot occurs, and the density may become negative. A similar effect occurs with high order finite-difference schemes but this has been tamed by using total variation diminishing (TVD) algorithms.

The kernels give enormous flexibility to an SPH calculation. In principle different equations can have different kernels though there is no evidence that any advantage is to be gained by that. An exception may be in the XSPH variant where the kernel for the correction to the velocity could be chosen to reduce the dispersion. In the case of finite differences, Gadd (1978) shows that the two step Lax-Wendroff time integration can be improved by using a moderate accuracy $O(h^2)$ gradient in the first step with the damping term, and a high accuracy gradient in the final step. The SPH equivalent would be to insert viscosity and a standard kernel (e.g. a Gaussian) in the first step, and then use a more accurate kernel (e.g. a Super Gaussian) in the final step.

8. MAGNETIC FIELDS

The SPH formulation of magnetic forces was considered initially by Gingold & Monaghan (1977). An application was made to magnetic fields in polytropes. Further aspects of the application of SPH to magnetic field problems have been considered by Philips & Monaghan (1985), Philips (1986), and Habe (1989). The equations of MHD in SPH form have been applied by Stellingwerf (1990a) to blast waves in a magnetic medium.

8.1 *Force and Current*

The magnetic force per unit mass is

$$\frac{\mathbf{J} \times \mathbf{B}}{\rho}, \qquad 8.1$$

where the current $\mathbf{J}$ is given by

$$\mathbf{J} = \varepsilon c^2 \nabla \times \mathbf{B}, \qquad 8.2$$

where SI units have been used. The current can be estimated using

$$\rho \nabla \times \mathbf{B} = \nabla \times (\rho \mathbf{B}) - (\nabla \rho) \times \mathbf{B}, \qquad 8.3$$

so that

$$\rho_a(\nabla \times \mathbf{B})_a = \sum_b m_b(\mathbf{B}_a - \mathbf{B}_b) \times \nabla_a W_{ab}, \tag{8.4}$$

which vanishes, as it should, when **B** is constant. In general, SPH should estimate the current accurately because the current is where the matter is.

To interpret Equation 8.4, assume the kernel is a Gaussian. The contribution of particle b to the current at particle a is then

$$-\left(\frac{2\varepsilon_0 c^2 m_b W_{ab}}{h^2 \rho_b}\right)(\mathbf{B}_a - \mathbf{B}_b) \times (\mathbf{r}_a - \mathbf{r}_b). \tag{8.5}$$

This expression shows that particle b only gives information about the current perpendicular to the direction $(\mathbf{r}_a - \mathbf{r}_b)$. With this in mind, Equation 8.5 is an inverse of the Biot-Savart law.

8.2 *Time Variation of Magnetic Field*

The time variation of **B** can be taken in the form

$$\frac{d}{dt}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}\cdot\nabla}{\rho}\right)\mathbf{v} = \left(\frac{\mathbf{B}}{\rho^2}\cdot\nabla\right)\rho\mathbf{v} - \frac{\mathbf{v}}{\rho^2}(\mathbf{B}\cdot\nabla\rho). \tag{8.6}$$

If the j component of a vector field **A** at the particle b is denoted by $\mathbf{A}_{b,j}$ then the SPH form of Equation 8.6 is

$$\frac{d}{dt}\left(\frac{B_{a,j}}{\rho_a}\right) = \frac{1}{\rho_a^2}\sum_b m_b(v_{b,j} - v_{a,j})\mathbf{B}_a \cdot \nabla_a W_{ab}. \tag{8.7}$$

The effect of particle b on the rate of change of $\mathbf{B}/\rho$ of particle a can be interpreted in physical terms when the kernel is a Gaussian. The interpretation is complicated by the fact that either **B** or ρ can vary as the particles move relative to each other.

An alternative to Equation 8.6 is to use

$$\frac{d\mathbf{B}}{dt} = -\mathbf{B}(\nabla\cdot\mathbf{v}) + (\mathbf{B}\cdot\nabla)\mathbf{v}, \tag{8.8}$$

with the SPH equivalent

$$\frac{dB_{a,j}}{dt} = \frac{1}{\rho_a}\sum_b m_b(B_{a,j}\mathbf{v}_{ab} - v_{ab,j}\mathbf{B}_a)\cdot\nabla_a W_{ab}, \tag{8.9}$$

where $v_{ab,j}$ is the jth component of $\mathbf{v}_{ab}$. The first term increases **B** by squeezing the field lines. The second term in the summation affects the j component of **B** by shearing.

Equation 8.9 can also be written

$$\frac{d\mathbf{B}_a}{dt} = \frac{1}{\rho_a}\sum_b m_b(\mathbf{v}_{ab}\times\mathbf{B}_a)\times\nabla W_{ab}. \qquad 8.10$$

8.3 *Magnetic Forces*

The simplest way to calculate the magnetic force is to substitute for $\mathbf{J}_a$ (calculated from Equation 8.4) in $\mathbf{J}\times\mathbf{B}$. This force does not conserve linear and angular momenta exactly. A force per unit mass that does conserve linear and angular momenta (Phillips & Monaghan 1985) is given by

$$\frac{1}{\rho}\frac{\partial \mathscr{M}_{ij}}{\partial x_j}, \qquad 8.11$$

where the stress tensor is defined by

$$\mathscr{M}_{ij} = \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}B^2\delta_{ij}\right).$$

The SPH equivalent is then

$$\sum_b m_b\left[\left(\frac{\mathscr{M}_{ij}}{\rho^2}\right)_a + \left(\frac{\mathscr{M}_{ij}}{\rho^2}\right)_b\right]\nabla_{a,j}W_{ab}.$$

However, when we examine a disturbance propagating along the x axis in a uniform isothermal gas with a pure B_x field, the motion is found to be unstable if the wavelengths are sufficiently short ($kh > 2$) and $2c_s^2 < B_x^2/(\rho\mu_0)$, i.e. the Alfvén speed is greater than $\sqrt{2}$ times the sound speed. This result was confirmed by numerical simulation. The reason for the instability is that the magnetic stress tensor in this case is positive, and the pair force between the particles is negative. The particles therefore attract each other and clump on the scale of h. A simple way of removing the problem is to sweep over the particles and find the maximum value of the magnetic stress tensor, then subtract this from the stress tensor in Equation 8.12. Experiments show that the algorithm is then stabilized (Phillips & Monaghan 1985), but it would be preferable to have a more elegant procedure.

8.4 *Additional Remarks*

Because the current is located where the particles are, it would be an advantage to update $\mathbf{J}$, then calculate $\mathbf{B}$ via the Biot-Savart law. An algorithmic advantage is that the tree code used for the gravitational field could be used to implement the Biot-Savart law.

The additional terms from finite conductivity can be included without

difficulty. These terms are diffusion terms and they may be estimated using expressions similar to those suggested for heat diffusion.

In star forming regions the ionization can be so low that the field can slip through the material. To simulate this process requires the inclusion of neutrals, ions, grains, and electrons. This is a simple generalization of the SPH equations used for modeling gas and dark matter in cosmology.

9. SPECIAL RELATIVITY

9.1 *Energy Momentum Tensor*

SPH equations for special relativity were given by Monaghan (1985). These equations are similar to those used by Amsden et al (1976) who used PIC to simulate high speed collisions of atomic nuclei.

We assume the fluid consists of baryons for which the energy momentum tensor is (Landau & Lifshitz 1975)

$$\mathcal{T}^{\mu\nu} = (nm_0c^2 + n\hat{e} + P)U^\mu U^\nu + Pg^{\mu\nu}, \tag{9.1}$$

where Greek indices run from 0 to 3 and the metric coefficients are defined by

$$g_{00} = -1, \qquad g_{ij} = 1.$$

In these equations n is the number density of baryons in the rest frame of the element of fluid, P is the pressure, and $\hat{e}$ is the thermal energy in the rest frame of the fluid. The details of these thermodynamic quantities are given by Chandrasekhar (1958) and the speed of sound c is given by Synge (1957). U^ν is the 4-velocity with $U_\nu U^\nu = -1$, and m_0 is the baryon rest mass.

9.2 *The Momentum Equation*

The momentum equations follow from

$$\frac{\partial \mathcal{T}^{i\mu}}{\partial x^\mu} = 0 \tag{9.2}$$

which, on writing $X = nm_0c^2 + n\hat{e}$, becomes

$$\frac{\partial M^i}{\partial t} + \frac{\partial}{\partial x^j}(M^i v^j) = 0, \tag{9.3}$$

where

$$\mathbf{M} = \frac{c^2}{\gamma^2}(P+X)\mathbf{v}, \qquad \gamma = \frac{1}{\sqrt{(1-v^2/c^2)}}. \tag{9.4}$$

M is the momentum per unit volume of the fluid as seen by the computing observer. For SPH we need quantities per particle. The momentum per baryon is denoted by

$$\mathbf{q} = \frac{\mathbf{M}}{N}, \tag{9.5}$$

where $N = \gamma n$ is the number density of the fluid seen by the computing observer. Noting the conservation law for the baryon number

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x^j}(Nv^j) = 0, \tag{9.6}$$

we can write Equation 9.3 in the form

$$\frac{d\mathbf{q}}{dt} = -\frac{1}{N}\nabla P, \tag{9.7}$$

where

$$\frac{d\mathbf{q}}{dt} = \frac{\partial \mathbf{q}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{q} \tag{9.8}$$

is the usual derivative following the motion. The special relativistic momentum equation is therefore almost identical to the nonrelativistic equation.

The SPH form is obtained by first noting that the interpolation formula is now

$$A_1(\mathbf{r}) = \sum_b v_b W(\mathbf{r}-\mathbf{r}_b, h),$$

where v_b is the number of baryons associated with particle b. This formula is not relativistically invariant, nor is it meant to be. It is the particular interpolation suited to our computational observer, and in this sense it is equivalent to that observer dividing space into cells for a finite-difference calculation. The SPH form for Equation 9.7 is

$$\frac{d\mathbf{q}_a}{dt} = -\sum_b v_b\left(\frac{P_a}{n_a^2} + \frac{P_b}{N_b^2}\right)\nabla_a W_{ab}. \tag{9.9}$$

The relativistic momentum equation has all the properties of the nonrelativistic equation. In particular it conserves relativistic linear and angular momenta:

$$\sum_a v_a \mathbf{q}_a, \qquad \sum_a v_a \mathbf{r}_a \times \mathbf{q}_a. \tag{9.10}$$

9.3 *The Energy Equation*

The energy equation is

$$\frac{\partial \mathcal{T}^{0j}}{\partial x^j} = 0 \tag{9.11}$$

which becomes (Monaghan 1985)

$$\frac{\partial}{c\partial t}[(P+X)\gamma^2 - P] + \nabla \cdot \left[\frac{\gamma^2}{c}(P+X)\mathbf{v}\right] = 0. \tag{9.12}$$

The energy per unit volume is

$$E = (P+X)\gamma^2 - P, \tag{9.13}$$

and the energy per baryon is

$$\varepsilon = \frac{E}{N}, \tag{9.14}$$

with the nonrelativistic form

$$mc^2 + \frac{1}{2}mv^2 + \hat{e}. \tag{9.15}$$

With the definition (9.14), and the continuity equation, we can write Equation 9.12 as

$$\frac{d\varepsilon}{dt} = -\frac{1}{N}\nabla \cdot (P\mathbf{v}) \tag{9.16}$$

which has the same form as the nonrelativistic equation for the total energy (kinetic plus thermal) per unit mass. An SPH form of this equation is

$$\frac{d\varepsilon_a}{dt} = -\sum_b m_b \left(\frac{P_a \mathbf{v}_a}{N_a^2} + \frac{P_b \mathbf{v}_b}{N_b^2}\right) \nabla W_{ab}. \tag{9.17}$$

These equations have been used by Lahy (1988) to model relativistic shock phenomena.

An alternative formulation of relativistic SPH has been described by Khefets et al (1990). Their formulation retains the covariant form, and is therefore more elegant than the preceding formulation. It remains to be seen which is the more convenient for simulations.

9.4 *Dissipation*

The usual recipes for dissipation in relativistic fluid dynamics, for example those given by Landau & Lifshitz (1975) or Weinberg (1972), lead to instabilities (Hiscock & Lindblom 1985) with typical time scales for water in the bath of 10^{-34} seconds (this is not a misprint!). The nonrelativistic limit, $c \to \infty$, is singular. An alternative suggested by Carter is also known to lead to instabilities (Olson & Hiscock 1990). Hawley et al (1984) refer to Weinberg's dissipative terms, but in their numerical calculations they use an artificial pressure proportional to $(\nabla \cdot \mathbf{v})^2$ which is similar to the Von Neumann–Richtmyer viscosity. They take the practical point of view that the artificial viscosity does not have to be physical, but only has to prevent post-shock oscillations. The PIC calculations do not use explicit dissipation but, because physical quantities are averaged over a cell, dissipation is implicit. Total energy and momentum are conserved, but the contribution of dissipation to the inertia is never included. The formulation of relativistic dissipation terms is clearly in a bad state and all numerical methods face these difficulties. Experience suggests that the best form of dissipation for relativistic SPH would use velocity or momentum differences. No suitable formula has been suggested.

10. IMPLEMENTATION

An SPH calculation is initiated by specifying the mass, position, velocity, and thermal energy of each particle. In addition, if the density is calculated from Equation 3.9 the initial density is specified. Other quantities, such as the mixture of elements associated with each particle, may be needed.

10.1 *The Particle Setup*

It is often convenient to set the particles up on a regular grid. In that case a Cartesian grid with equal spacing can be used, though there are advantages in using a body centered cubic lattice in three dimensions (Monaghan & Lattanzio 1991), namely (*a*) it gives a better representation of integrals by sums than the simple Cartesian grid, (*b*) it gives a particle more nearest neighbors, and (*c*) if planar compression occurs, the particle distribution is better because the particles are closer together perpendicular to the direction of compression. Regardless of the grid, if the cell size associated with the position of particle a is ΔV_a, then m_a can be taken as $\rho \Delta V_a$.

10.2 *Data Structures*

For problems where the same h can be used for each particle, the data structure should be based on link lists. These are discussed by Monaghan

(1985) and by Hockney & Eastwood (1981). If the spline-based kernel is used, the link list cells should be $2h$ wide so that only neighboring cells can contribute to particles in a given cell. For some computers (e.g. Vax) the calculations are much faster if the labels of all the contributing particles are first gathered into a single array. Relabelling the particles so that the numerical differences of labels of particles in neighboring cells are small is worthwhile.

When each particle has its own h the calculation of the SPH sums can be made part of a tree code calculation (Appel 1985, Barnes & Hut 1986, Hernquist & Katz 1989, Benz et al. 1989) which is the natural method for calculating self-gravitational forces on a set of particles. The tree code can be vectorized (Hernquist 1990, Makino 1990).

10.3 *Time Stepping*

The numerical integration of the ordinary differential equations for the physical variables at each particle can be carried out by standard methods (e.g. leapfrog or predictor-corrector) with a time-step control that involves the Courant condition, the force terms, and the viscous diffusion term (Monaghan 1989). The time step δt can then be chosen by first calculating δt_f and δt_{cv} according to

$$\delta t_f = \min_a \left(\frac{h_a}{|\mathbf{f}_a|} \right)$$

and

$$\delta t_{cv} = \min_a \frac{h}{c_a + 0.6(\alpha c_a + \beta \max_b \mu_{ab})},$$

then $\delta t = 0.25 \min(\delta t_a, \delta t_{cv})$. Here δt_f is based on the force per unit mass $\mathbf{f}$, and δt_{cv} combines the Courant and the viscous time-step controls. There is evidence from computer experiments that the coefficient 0.25 can be replaced by 0.4 for the δt_{cv} term.

If the time step is chosen correctly, the total energy should be conserved to within 0.5% over 400 time steps. If a predictor-corrector or leapfrog method is used it is possible to ensure exact linear and angular momentum conservation. Benz (1989) reports good results with a second-order Runge Kutta integrator due to Fehlberg which can often work with a time step larger than would be expected from the Courant condition.

When rapid molecular cooling occurs the energy equation must be integrated implicitly. This requires the straightforward solution of a non-linear equation for each particle (Monaghan & Varnas 1988, Hernquist & Katz 1989, Monaghan & Lattanzio 1991).

For many astrophysical problems there is a wide range of time scales. To cope with this the particles can be grouped into sets, each with its own time step (Hernquist & Katz 1989).

11. APPLICATIONS

11.1 *Gas Dynamics*

The simplest SPH test is to use it to study linear wave phenomena. A series of tests (propagation of density and velocity perturbations and boundary oscillations) show that good agreement with theory is obtained if the wavelength is $>2\pi h$. The deviation from the correct dispersion relation depends on the Fourier transform of the kernel [an example is given by Monaghan (1990)]. It is interesting to note that in these calculations with particle spacing s, the results are significantly more accurate (in particular better propagation speeds) if $h = s$ or $h = 2s$ and least accurate with $h = 1.5s$.

In various tests (Monaghan, unpublished) SPH was applied to the problem of supersonic flow of an ideal gas over a step. The results, while satisfactory, were not as sharp as those which can be obtained with high quality TVD finite-difference schemes. A similar situation occurs in the simulation of supersonic flow over a cylinder. In both cases the shock profiles are blurred by the viscosity. In these cases there is the additional problem of how best to model the boundaries. However, recent work on nearly incompressible fluid flow with rigid boundaries (see below) suggests a convenient treatment for any boundary. It would be worth applying this treatment to supersonic flow.

The application of SPH to shocks and shock tube phenomena have been described by many authors (Monaghan & Gingold 1983, Monaghan & Pongracic 1985, Lattanzio et al 1985b, Hernquist & Katz 1989). The typical problem is the Sod (1978) shock tube for which SPH gives excellent results. In this problem it is useful to introduce a small amount of heat diffusion ($g = 0.125$) to remove a blip in the pressure at the contact discontinuity which is otherwise treated very accurately. SPH has also been applied to the plane and cylindrical infinite strength shocks of Noh (Monaghan 1988b). The calculations include thermal conduction as well as viscosity. These results are typical, and show that the SPH equations give satisfactory shock profiles and good contact discontinuities.

The development of a compressible Rayleigh-Taylor instability in an isothermal gas with the layers having a density ratio of 4:1 has been studied using SPH (Monaghan 1989). The configuration has a fixed top and bottom boundary, and periodic side conditions. A series of similar calculations confirm that the SPH equations determine the onset of the

stability in agreement with the analysis of Bernstein & Book (1983). This instability is of considerable interest in the development of a supernova, and SPH calculations (see below) agree with accurate finite-difference calculations.

Blast waves calculated with SPH (Stellingwerf 1990a) agree well with theory.

11.2 *Binary Stars and Stellar Collisions*

The interaction of two stars is ideally suited to SPH. Matter, energy, and momentum are transported accurately by the SPH particles as the stars move through space. A finite-difference calculation for the same problem would introduce errors from the advection through the grid, and would require a large number of cells to cover the space within which the stars are moving.

The early SPH calculations for binary stars were concerned with the Roche and Darwin problems for polytropes (Gingold & Monaghan 1978). Only a small number of particles (less than 400) were used, but the results were in good agreement with stability theory. In a later calculation (Gingold & Monaghan 1979, 1980) the Roche problem for central orbits was studied. In this work the SPH calculations were compared with results from the integration of equations describing the coupling between the linear oscillations of the polytrope and the nonlinear orbital motion. Good agreement was obtained. For stronger interactions the SPH simulation gave the first direct results for tidal disruption and capture. This work has since been superseded by the more accurate work of Benz & Hills (1987) and Goodman & Hernquist (1991) on binary polytropes. The interaction of three polytropes was studied by Cleary & Monaghan (1990) for an ensemble of configurations using ~ 100 particles per polytrope. This quite small number of particles is near the lower limit for a reasonable calculation, but it appears to be adequate to indicate deviations from the classical 3-body system. The collision of white dwarf stars has been considered by Benz & Thielemann (1990) and Benz et al (1990). In the first of these papers allowance was made for radiation pressure and energy generation using a reduced nuclear network. In the second paper the question of mass exchange between interacting white dwarfs and the formation of a disk was considered in detail. This paper contains a very useful and detailed description of their tree code.

Rasio & Shapiro (1991) have applied SPH to the collision of a giant star and a compact object.

11.3 *Formation of the Moon and Impact Problems*

The collision theory of the origin of the Moon has been studied in detail by Benz et al (1986, 1987, 1989a) using SPH. This work shows the power

of SPH. The configuration consists of a planetesimal that collides with an embryonic Earth. The equation of state (originally the Tillotson, but later the Char D ANEOS) is appropriate for rocky materials and iron and therefore complicated, but it can be included in an SPH code without difficulty.

A problem of considerable interest is the cratering produced by the impact of an asteroid or comet on a planetary surface. In particular the impact on the Earth is of interest for the Cretaceous extinction event. This problem has been examined by Pongracic (1989) who modeled the impact in two dimensions using a variety of equations of state. A variety of configurations were considered, including impact on water over a typical rocky sea bottom, impact on model mountains, and impacts at oblique angles. A related problem concerns the delivery of material to the Earth by comets. This problem has been examined by Chyba et al (1990) using an SPH code. None of these cometary or asteroid impact problems have been taken beyond the initial stage of crater formation, though this is often enough to estimate the amount of material flung to large distances and the maximum temperatures reached.

Stellingwerf & Campbell (1990) have examined the hypervelocity impact of metals. Laser ablation has been studied by Stellingwerf (1990b).

11.4 *Fragmentation and Cloud Collisions*

Lattanzio et al (1985a), Lattanzio & Henriksen (1988), and Keto & Lattanzio (1989) applied SPH to the study of interacting isothermal clouds as a model for the processes conjectured for molecular cloud complexes. The SPH simulation allowed a systematic study of the way complex structures formed in the interactions between initially spherical clouds. The authors used a fourth-order multigrid Poisson solver with the source terms mapped to the grid and the grid forces mapped back to the particles. The smoothing length h was the same for all particles. A similar calculation (Nagasawa et al 1988) used a spatially-varying h. In a later calculation (Monaghan & Varnas 1988) an attempt was made to simulate an entire cloud complex with the isothermal assumption replaced by a cooling formula suitable for the interstellar medium.

SPH has been used to study the fragmentation in isothermal rotating clouds (Gingold & Monaghan 1981, 1983, Miyama et al 1984) and in clouds and disks (Monaghan & Lattanzio 1991) using an accurate molecular cooling formula. The results from a molecular cooling sequence have been applied to an observed molecular fragmentation structure (Keto & Lattanzio 1989, Keto et al 1991). Most of these calculations used $\sim$30,000 particles, and the same h was used for each particle.

An interesting study of instabilities in a rotating cloud was made by

Durisen et al (1986) who compared an SPH simulation with a finite-difference simulation and found the SPH simulation to be much easier and more efficient. It is useful to note that the SPH calculation was completed in a few days whereas the finite-difference calculation took more than a year to implement satisfactorily.

11.5 *Cosmological and Galactic Problems*

The typical cosmological problem requires the simulation of the growth of fluctuations in two fluids (gas and dark matter) in an expanding universe. SPH is ideal for this type of problem. The two fluids can be easily distinguished by using a tag for each particle. A link list can then be constructed for the gas particles to compute thermodynamic quantities, and the gravitational field can be obtained using all the particles. More fluids could be easily incorporated.

Applications of SPH to problems arising in cosmology have been considered by Evrard (1988) and Hernquist & Katz (1989) who laid the foundations for an effective combination of a tree code with SPH which has subsequently been used to investigate a variety of problems in cosmology and galaxy interactions (Barnes & Hernquist 1991). Bond et al (1989) have studied cosmological problems using a fourth-order multigrid Poisson solver in place of the tree code.

11.6 *Disks and Rings*

Zurek & Benz (1986) used SPH to simulate the nonlinear development of an unstable thick barytrope disk around a star. Their work showed that the disk evolved to a structure with a rotation law predicted by linear theory. In a related calculation Monaghan (1990) explored the stability of the isothermal rings postulated by a theory of the solar system. Artymowicz et al (1991) used SPH to study the effect of a disk on the elements of a central binary.

11.7 *Radio Jets*

SPH simulations of jet models (Coleman & Bicknell 1985, 1988; see also Bicknell et al 1990) making use of axial symmetry, and special kernels which take into account boundaries by changing the normalization, have been used to determine observational properties of jets. In related work Balsara et al (1991) have used SPH to model supersonic shear layers and have noted that SPH can automatically model subgrid turbulence.

11.8 *Motion Near Black Holes*

A comprehensive use of SPH for fluid dynamics near a black hole awaits the development of a general relativistic version of SPH (see below).

Bicknell & Gingold (1983) examined the effect of a polytrope moving in the gravitational field of a massive body. Evans & Kochanek (1989) applied SPH to the disruption of a polytrope in the neighborhood of a black hole, but they confined the simulation to the first passage and only included the zeroth-order gravitational field. This problem is of special interest in active galactic nuclei where the formation of an accretion disk and its subsequent development as a source of fuel for the black hole is the object of the study. It is an intrinsically difficult problem because the material of the disrupted star becomes spread out over very great distances, and to simultaneously achieve good resolution and model the collision and viscous forces requires a very large number of particles.

11.9 *Supernovae*

The mixing due to Rayleigh-Taylor instabilities during a supernova explosion has been the subject of many computer simulations. The first SPH simulations of supernova explosions were carried out for polytropes (Nagasawa et al 1988). The results disagreed with finite-difference calculations. Benz & Thielmann (1990) pointed out that Nagasawa et al had used a δ function source of energy at the center. With a smoothed source the SPH formulation gave results in agreement with finite-difference calculations. Herant & Benz (1991) have performed further SPH calculations with realistic initial states.

11.10 *Special and General Relativity*

Only a small number of SPH relativistic calculations have been carried out. Lahy (1989) applied SPH to shock tube and nuclear collision problems. The results were poor by comparison with the nonrelativistic shock results because a good artificial viscosity for SPH hasn't been found.

The SPH equations for post-Newtonian fluid dynamics were set up by Thompson (1984) and applied to the collapse of rotating neutron stars. The SPH equations for fluid dynamics in a known stationary metric were described by Monaghan & Lahy (1989).

11.11 *Magnetic Phenomena*

SPH has been used to study the collapse of magnetic gas clouds (Phillips 1986, Habe 1989), the propagation of Alfvén waves (Monaghan & Phillips 1985), and the structure of static magnetic polytropes (Gingold & Monaghan 1977). The development of a blast wave in a magnetic cloud has been considered by Stellingwerf & Peterkin (1990). There is now abundant evidence that magnetic fields must be taken into account for star formation and the time would seem to be ripe for the development of a robust SPH

algorithm for both ideal MHD and for MHD in the presence of plasma drift.

11.12 *Nearly Incompressible Flow*

SPH was designed for compressible flow problems, but it can be extended to nearly incompressible flow (Monaghan & Humble 1991). The essential point is that an artificial equation of state can be constructed so that compressibility effects are at or below the 1% level. All that is required is that the Mach number M of the flow should be ~ 0.1, since compressibility effects are $O(\mathrm{M}^2)$. Because terrestrial flows can be characterized by a typical velocity (for example a bursting dam of height H produces typical velocities $\sim\sqrt{2gH}$, where g is the acceleration due to gravity), it is possible to design an equation of state so that $\mathrm{M} \sim 0.1$. The SPH treatment is then straightforward, and free-surface problems (bursting dams, tidal bores, waterfalls, etc.) can be treated easily. The boundaries can be replaced by chains of fixed particles which interact with the water particles by forces which are modeled on molecular forces.

The disadvantage is that time steps are a factor ten shorter than normal, but since all standard finite-difference techniques for free-surface problems include an iteration cycle of several steps, this disadvantage is minimal. Thermal convection and fluids with different densities can be treated without difficulty.

12. THEORETICAL POINTS CONCERNING SPH

12.1 *The Derivation of SPH when* h = h(t)

One convenient way to derive the SPH equations is to start with the original equations, multiply through by a kernel and, by integrations by parts or Gauss's theorem, determine an equation which the SPH integral interpolants must satisfy. This idea has been used to examine the SPH equations when the resolution length varies with space and time.

For the present we consider the case in which h varies only with time. Start with the exact continuity equation in one dimension:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x'}(\rho v) = 0. \qquad 12.1$$

If Equation 12.1 is multiplied by $W(x-x', h)$ and integrated over x' assuming that in the integration by parts the integrand vanishes at the limits, we get

$$\frac{\partial \rho_{\mathrm{I}}}{\partial t} - \dot{h}\frac{\partial \rho_{\mathrm{I}}}{\partial h} + \frac{\partial}{\partial x}(\rho v)_{\mathrm{I}} = 0. \qquad 12.2$$

The term involving $\dot{h}$ exists because changes in ρ can occur when h changes even when the particles are held fixed. This is equivalent to the change in physical quantities in a finite-difference calculation which occurs if the grid is altered and physical quantities are mapped from the old grid to the new grid.

If we approximate ρ_I by the summation interpolant

$$\rho(x)_I = \sum_b m_b W(x - x_b, h)$$

then

$$\frac{\partial \rho_I}{\partial t} = \sum_b m_b \left[\left(\frac{\partial W}{\partial x} \right) (-\dot{x}_b) + \dot{h} \frac{\partial W}{\partial h} \right] \quad . \tag{12.3}$$

and

$$\frac{\partial}{\partial x}(\rho v)_I = \sum_b m_b v_b \frac{\partial W}{\partial x}. \tag{12.4}$$

These equations satisfy the interpolant form of the continuity equation provided $\dot{x}_b = v_b$, the usual assumption in SPH.

These results show that the summation interpolant is an exact solution of the integral interpolant form of the exact equations. Of course the integral and summation interpolants differ, and this difference is the error in the calculation of the density.

If we use Equation 3.9 we are approximating the rate of change of the true density, not the rate of change of the integral interpolant. There is therefore no $\dot{h}$ term.

The momentum equation can be discussed in the same way by working with the equation

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x'}(\rho v^2) = -\frac{\partial P}{\partial x'}. \tag{12.5}$$

Proceeding as before we find

$$\frac{\partial}{\partial t}(\rho v)_I + \frac{\partial}{\partial x'}(\rho v^2)_I - \dot{h}\frac{\partial}{\partial h}(\rho v)_I = -\left(\frac{\partial P}{\partial x}\right)_I. \tag{12.6}$$

If we substitute the following SPH summation interpolants for the integral interpolants

$$(\rho v)_I = \sum_b m_b v_b W(x - x_b, h) \tag{12.7}$$

and

$$\left(\frac{\partial P}{\partial x}\right)_{\mathrm{I}} = \sum_b m_b \frac{1}{\rho_b}\left(\frac{\partial P}{\partial x}\right)_b W(x-x_b, h), \tag{12.8}$$

Equation 12.6 is satisfied if

$$\dot{v}_b = -\frac{1}{\rho_b}\left(\frac{\partial P}{\partial x}\right)_b \tag{12.9}$$

which is the usual SPH equation. The thermal energy equation can be derived in the same way.

12.2 *Deriving the SPH Equations when* $\mathrm{h} = \mathrm{h}(\mathbf{r}, \mathrm{t})$

The previous arguments have been extended by Bicknell (1991) to the case of h varying with space and time. If, for example, we assume h depends on the coordinate of the contributing particle, we multiply the continuity equation by $W[x-x', h(x', t)]$, and integrate as before. We find

$$\frac{\partial \rho_{\mathrm{I}}}{\partial t} + \frac{\partial}{\partial x}(\rho v)_{\mathrm{I}} - \int \rho \dot{h} \frac{\partial W}{\partial h} dx' = 0, \tag{12.10}$$

where $\dot{h}$ is the derivative of h following the motion. Equation 12.10 is satisfied exactly by the summation interpolant for the density and for ρv. The summation interpolant conserves mass because

$$\int \rho_{\mathrm{S}}\, dx = \sum_b m_b. \tag{12.11}$$

If we apply the same procedure to the momentum equation we recover Equation 12.9.

A point worth noting is that if we use interpolants with h dependent upon the contributing particle, then the spatial gradients do not involve derivatives of h. For example

$$\frac{\partial}{\partial x}(\rho v)_{\mathrm{S}} = \sum_b m_b v_b \frac{\partial}{\partial x} W(x-x_b, h_b). \tag{12.12}$$

On the other hand if we use an h which depends on the position where we want the estimate, then the spatial derivative of the summation interpolant involves the spatial derivative of h. This is the case for the interpolation which uses an average h, or an average of the kernels, as in Section 6. The errors are of $O(h^2)$ provided the scale of variation of h is comparable to that of other physical quantities (Hernquist & Katz 1989). In this case total mass is not conserved exactly since

$$\int W[x-x_b, h(x)]\,dx \neq 0. \qquad 12.13$$

The conclusion that we draw from this is: It is possible to interpolate with hs that vary in space as well as time but local errors [typically $O(h^2)$] are inescapable. These difficulties occur in adaptive grid finite-difference methods, but SPH is incomparably easier to work with.

12.3 *Remarks Concerning Errors*

As we have seen, the errors in the integral interpolants can be estimated analytically. It is more difficult to estimate the errors in the summation interpolants unless the particle positions are orderly. The original discussion of SPH by Lucy (1977), and by Gingold & Monaghan (1977), assumed that the particles were randomly distributed and the summations were Monte Carlo estimates of the integral interpolants. The results were therefore expected to show large fluctuations and correspondingly large errors. These large fluctuations were not found, even in fission calculations, because the particles were not distributed according to a random number generator. They are distributed by the dynamics which is an altogether different matter. In the cases where some moderate disorder occurs, as in the collapse of a rotating cloud, the best estimate of the errors is probably that of Niedereiter (1978) who estimated the errors as being $\propto n^{-1}\log n^{\nu-1}$, where n is the number of points. A related result due to Wozniakowski (1991) gives the average error as $\propto n^{-1}\log n^{(\nu-1)/2}$. (This remarkable result was produced by a challenge with a payoff of sixty-four dollars!)

In complicated dynamical problems, large variations in the physical properties occur naturally, and this is clearly true for astronomical objects like star forming regions and interacting galaxies. Similar large variations are expected in the numerical simulations, and it is a mistake to assume that these large variations are numerical artifacts produced by the simulation. For this reason the shock calculations of Rasio & Shapiro (1991), where the initial properties are given random variations, are not necessarily a measure of the errors that occur in practice. The ultimate test of accuracy is how well the numerical method reproduces known results. The applications described in this paper show that SPH provides a robust accurate tool for the study of astrophysical fluid dynamics.

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