WEEK 6

NUMERICAL CHARACTERISTICS

Mean value (mathematical hope)

- Let be the random variable X with the distribution $X\begin{pmatrix} x_i \\ p_i \end{pmatrix}, i \in I$. It is called the average value, the numerical characteristic $M(X) = \sum_{i \in I} x_i p_i$.

Observations

1) If I is finite, the average value exists.

2) If I is infinitely countable, M(X) exists when the series defining it is absolutely convergent.

- Let be random variable X of continuous type $X\begin{pmatrix} x \\ \rho(x) \end{pmatrix}$, $x \in R$. It is called the mean value of variable X, the numerical characteristic $M(X) = \int_{-\infty}^{+\infty} x\, \rho(\mathrm{x})\mathrm{dx}$.
- The mean value exists when the improper integral that defines it converges.

Properties of average value The statements take place:

1) $M(a\,\mathrm{X} + \mathrm{b}) = \mathrm{a}\,\mathrm{M(X)} + \mathrm{b}, \forall \mathrm{a}, \mathrm{b} \in \mathrm{R}$

2) M(X + Y) = M(X) + M(Y)

3) X,Y independent $\Rightarrow$ M(X Y) = M(X) M(Y)

Dispersia

- It is called the dispersion (variance) of random variable X, the numerical characteristic $D^2(X) = M\left[(X - M(X))^2\right]$ and $\sigma(X) = \sqrt{D^2(X)}$ it is called the mean square deviation.
- Explicitly, the dispersion has the expression $D^2(X) = \sum_{i \in I} (x_i - M(X))^2 \cdot p_i$, $I \subset N$, if X is a discrete random variable or $D^2(X) = \int_R (x - M(X))^2 \rho(x)dx$, if X is a continuous random variable.
- Dispersion is a numerical indicator of how scattered (or dispersed) the values of a random variable are scattered around its mean value.

Dispersion properties

a) $D^2(X)=M(X^2)-[M(X)]^2$

b) $D^2(aX+b)=a^2D^2(X), \forall \text{a,b} \in \text{R}$

c) Independent X,Y $\Rightarrow D^2(X+Y)=D^2(X)+D^2(Y)$

Example 1. *If X is a discrete random variable*

$$X:\begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 3 \\ \frac{1}{12} & \frac{2}{12} & \frac{2}{12} & \frac{5}{12} & \frac{1}{12} & \frac{1}{12} \end{pmatrix}$$

Then we infer that:

$$M(X)=-2\frac{1}{12}-1\frac{2}{12}+0\frac{2}{12}+1\frac{5}{12}+2\frac{1}{12}+3\frac{1}{12}=\frac{1}{2}$$

$$M(X^2)=4\frac{1}{12}+1\frac{2}{12}+0\frac{2}{12}+1\frac{5}{12}+4\frac{1}{12}+9\frac{1}{12}=2$$

$$D^2(X)=M(X^2)-[M(X)]^2=2-\frac{1}{4}=\frac{7}{4}$$

$$\sigma(X)=\sqrt{D^2(X)}=\frac{\sqrt{7}}{2}$$

Example 2. *If X is a continuous random variable*

$$X:\begin{pmatrix} x \\ \rho(x) \end{pmatrix}, \ \rho(x)=\begin{cases} \frac{x}{4}, & x\in[1,3] \\ 0, & \text{în rest} \end{cases}$$

Then we infer that:

$$M(X)=\int_1^3 x\rho(x)dx=\frac{x^3}{12}\bigg|_1^3=\frac{13}{6}\ ,\ M(X^2)=\int_1^3 x^2\rho(x)dx=\frac{x^4}{16}\bigg|_1^3=5$$

$$D^2(X)=M(X^2)-[M(X)]^2=5-\frac{169}{36}=\frac{11}{36}$$

Example 3. *Consider the random variable* $\text{X}:\begin{pmatrix} -1 & 0 & 2 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{pmatrix}$.

Calculate: E(X), E(3X), E(4X-2), $Var(X)$, σ_X.

Solution

$$E(X)=\sum_{i=1}^{3}x_ip_i=-1\cdot 0{,}2+0\cdot 0{,}3+2\cdot 0{,}5=0{,}8$$

$$E(3X)=3E(X)=3\cdot 0{,}8=3{,}4;\quad E(4X-2)=4E(X)-2=4\cdot 0{,}8-2=1{,}2$$

$Var(X) = E(X^2) - [E(X)]^2 = 2{,}2 - 0{,}64 = 1{,}56$; $E(X^2) = (-1)^2 \cdot 0{,}2 + 0^2 \cdot 0{,}3 + 2^2 \cdot 0{,}5 = 2{,}2$;

$\sigma_X = \sqrt{Var(X)} = \sqrt{1{,}56} = 1{,}24$

Example 4. *Calculate the mean value and dispersion of the random variable that has the probability density*

$$\rho(x) = \begin{cases} 1 - |1 - x|, \text{ dacă } x \in (0,2) \\ 0, \text{ altfel} \end{cases}$$

Solution

We note that: $\rho(x) = \begin{cases} x, \text{ dacă } 0 < x \le 1 \\ 2 - x, \text{ dacă } 1 < x < 2 \\ 0, \text{ altfel} \end{cases}$

Taking into account the definition we have:

$$E(X) = \int_{-\infty}^{+\infty} x\,\rho(x)dx = \int_0^1 x^2 dx + \int_1^2 x(2-x)dx = \frac{x^3}{3}\Big|_0^1 + x^2\Big|_1^2 - \frac{x^3}{3}\Big|_1^2 = 1$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 \rho(x)dx = \int_0^1 x^3 dx + \int_1^2 x^2(2-x)dx = \frac{x^4}{4}\Big|_0^1 + 2\frac{x^3}{3}\Big|_1^2 - \frac{x^4}{4}\Big|_1^2 = \frac{7}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

Theorem 1. If X is a discrete random variable following binomial law, then: M(X) = np and D2(X) = npq.

Demonstration:

From the definition of average value we have:

$$M(X) = \sum_{k=0}^{n} kP(n,k) = \sum_{k=0}^{n} kC_n^k p^k q^{n-k}$$

We consider the relationship: $(pt+q)^n = \sum_{k=0}^{n} C_n^k p^k q^{n-k} t^k$

which we derive with respect to t and get:

(*) $np(pt+q)^{n-1} = \sum_{k=0}^{n} kC_n^k p^k q^{n-k} t^{k-1}$ and for t=1

We $np = \sum_{k=0}^{n} kC_n^k p^{kq} q^{n-1} = M(X)$

To calculate the dispersion we use the calculation formula: $D^2(X) = M(X^2) - [M(X)]^2$

We derive with respect to t the relation (*) and obtain

$$n(n-1)p^2(pt+q)^{n-2}=\sum_{k=1}^{n}k(k-1)C_n^k p^k q^{n-k}t^{k-2}=\sum_{k=0}^{n}k^2C_n^k p^k\, q^{n-k}t^{k-2}-\sum_{k=0}^{n}kC_n^k p^k q^{n-k}t^{k-2}$$ iar pentru t = 1, rezultă n (n - 1) p2 = M(X2) - M(X), adică M(X2) = n2p2 - np2 + np = n2p2 + npq, iar dispersia este D2(X) = n2p2 + npq - $(np)^2$ = npq.

Theorem 2. If X is a discrete random variable following the hypergeometric law then:

$$M(X)=np \quad şi \quad D^2(X)=npq\frac{N-n}{N-1}$$

unde N=a+b , p=$\frac{a}{a+b}$, $q=\frac{b}{a+b}$ $p+q=1$.

Demonstration:

According to the relation of definition of the average value we have:

M(X)=$\sum_{k=0}^{n}kP(n,k)=\frac{1}{C_{a+b}^n}\sum_{k=0}^{n}kC_a^kC_b^{n-k}=$

$$=\frac{a}{C_{a+b}^n}\sum_{k-1}^{n}C_{a-1}^{k-1}C_b^{n-k}=a\frac{C_{a+b-1}^{n-1}}{C_{a+b}^n}=n\frac{a}{a+b}=np$$

whether Vandermonde's relationship and the $kC_a^k=aC_{a-1}^{k-1}$

To calculate the dispersion, we use the calculation formula $D^2(X)=M(X^2)-[M(X)]^2$ where $M(X^2)=\sum_{k=0}^{n}k^2P(n,k)=\frac{1}{C_{a+b}^n}\sum_{k=0}^{n}k^2C_a^kC_b^{n-k}=$

$$=\frac{a}{C_{a+b}^n}\sum_{k=0}^{n}[k(k-1)+k]C_a^kC_b^{n-k}=\frac{1}{C_{a+b}^n}\left(\sum_{k=0}^{n}k(k-1)C_a^kC_b^{n-k}+\sum_{k=0}^{n}kC_a^kC_b^{n-k}\right)$$

dar k(k-1) $C_a^k=a(a-1)C_{a-2}^{k-2}$ iar k $C_a^k=aC_{a-1}^{k-1}$

so M $(X^2)=\frac{a}{C_{a+b}^n}\sum_{k=1}^{n}C_{a-1}^{k-1}C_b^{n-k}+\frac{a(a-1)}{C_{a+b}^n}\sum_{k=2}^{n}C_{a-2}^{k-2}C_b^{n-k}=$

$$=\frac{a}{C_{a+b}^n}C_{a+b-1}^{n-1}+\frac{a(a-1)}{C_{a+b}^n}C_{a+b-2}^{n-2}=\frac{na}{a+b}+n(n-1)\frac{a(a-1)}{(a+b)(a+b-1)}$$

if Vandermonde's relationship was considered. How to get

D $^2(X)=M(X^2)-[M(X)]^2=\frac{n(n-1)a(a-1)}{(a+b)(a+b-1)}-\frac{n^2a^2}{(a+b)^2}=$ =

$$\frac{na}{a+b}\frac{b}{a+b}\frac{a+b-n}{a+b-1}=npq\frac{N-n}{N-1}$$

Theorem 3. The mean value and dispersion of the random variable X following Poisson's law are equal to λ ,i.e. M(X)=D2(X)= λ .

Demonstration:

We

$$M(X) = \sum_{k=0}^{+\infty} kP_k(\lambda) = e^{-\lambda}\sum_{k=0}^{+\infty} k\frac{\lambda^k}{k!} = \lambda e^{-\lambda}\sum_{k=1}^{+\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

Calculate

$$M(X2)= \sum_{k=0}^{+\infty} k^2 P_k(\lambda) = \lambda e^{-\lambda}\sum_{k=1}^{+\infty} k\frac{\lambda^{k-1}}{(k-1)!} =$$

$$= \lambda e^{-\lambda}\sum_{k=1}^{+\infty}[(k-1)+1]\frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda}\left(\sum_{k=1}^{+\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{+\infty}\frac{\lambda^{k-1}}{(k-1)!}\right) =$$

$$= \lambda^2 e^{-\lambda}\sum_{k=2}^{+\infty}\frac{\lambda^{k-2}}{(k-2)!} + \lambda e^{-\lambda}\sum_{k=1}^{+\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda^2 e^{-\lambda}e^{\lambda} + \lambda e^{-\lambda}e^{\lambda} = \lambda^2 + \lambda$$

Get

$$D^2(X) = M(X^2) - [M(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Theorem 4. If the random variable X follows the geometric law then M(X)= $1/p$ and D2(X)= q/p^2 .

Demonstration:

We

$$M(X)= \sum_{k=1}^{+\infty} kP(k) = \sum_{k=1}^{+\infty} kpq^{k-1} = p\sum_{k=1}^{+\infty} kq^{k-1} =$$

$$= p(1+2q+2q^2+\ldots) = p(q+q^2+q^3+\ldots)' = p(\frac{q}{1-q})' = \frac{p}{(1-q)^2} = \frac{1}{p}.$$

Calculate

$$M(X=^2) = \sum_{k=1}^{+\infty} k^2 P(k) = \sum_{k=1}^{+\infty} k^2 pq^{k-1} = p\sum_{k=1}^{+\infty} k^2 q^{k-1}$$

$$= p(1+2^2q+3^2q^2+\ldots) = p(q+2q^2+3q^3+\ldots)' = p[q(1+2q+3q^2+\ldots)]' =$$

$$= p(q\frac{1}{(1-q)^2})' = \frac{1+q}{(1-q)^3}p = \frac{1+q}{p^2}.$$

Obžinem $D^2(X) = M(X^2) - [M(X)]^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{1+q-1}{p^2} = \frac{q}{p^2}$.

Theorem 5. If X is a random variable evenly distributed over the interval [a,b]

Then: $M(X)=(a+b)/2$ *şi* $D^2(X)=(b+a)^2/12$.

Demonstration:

We $M(X)=\int_{-\infty}^{+\infty} x\rho(x)dx=\frac{1}{b-a}\int_{-\infty}^{+\infty} xdx=\frac{1}{b-a}\frac{b^2-a^2}{2}=\frac{a+b}{2}$

We calculate the dispersion using the calculation formula

$$D^2(X)=M(X^2)-[M(X)]^2$$

We

$$M(X^2)=\int_{-\infty}^{+\infty} x^2\rho(x)dx=\frac{1}{b-a}\int_{-\infty}^{+\infty} x^2dx==\frac{a^2+ab+b^2}{3}$$

$$\textit{rezultă}\ \ D^2(X)=\frac{a^2+ab+b^2}{3}-(\frac{a+b}{2})^2=\frac{(b-a)^2}{12}$$

$M(X)=(a+b)/2$, we have the median value equal to the mean value .

Theorem 6. If random variable X follows the exponential parameter law $\lambda>0$ then: $M(X)=1/\lambda$ *şi* $D^2(X)=1/\lambda^2$.

Demonstration:

We

$$M(X)=\int_{-\infty}^{+\infty} x\rho(x)dx=\lambda\int_{0}^{+\infty} xe^{-\lambda x}dx=\frac{1}{\lambda}.$$

Calculate:

$$M(X^2)=\int_{-\infty}^{+\infty} x^2\rho(x)dx=\int_{0}^{+\infty} x^2\lambda e^{-\lambda x}=$$

$$=\lambda\int_{0}^{+\infty} x^2(-\frac{1}{\lambda}e^{-\lambda x})'dx=-x^2e^{-\lambda x}\Big|_{0}^{+\infty}+2\int_{0}^{+\infty} xe^{-\lambda x}dx=\frac{2}{\lambda}\int_{0}^{+\infty} x\lambda e^{-\lambda x}dx=\frac{2}{\lambda}\frac{1}{\lambda}=\frac{2}{\lambda^2}$$

We get $D^2(X)=M(X^2)-[M(X)]^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$.

Theorem 7. If X is a random variable following the normal law of parameters $m\in R$ and $\sigma>0$ *atunci* $M(X)=m$ *şi* $D^2(X)=\sigma^2$.

Demonstration:

We have $M(X)=\int_{-\infty}^{+\infty} x\rho(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} xe^{-\frac{(x-m)^2}{2\sigma^2}}dx$.

The variable change is made $t=\dfrac{x-m}{\sigma\sqrt{2}}$, $dx=\sigma\sqrt{2}dt$. Results

$$M(X)=\frac{1}{\sigma\sqrt{2\pi}}\sigma\sqrt{2}\int_{-\infty}^{+\infty}(m+\sigma\sqrt{2}t)e^{-t^2}dt=\frac{2m}{\sqrt{\pi}}\int_{0}^{+\infty}e^{-t^2}dt+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}te^{-t^2}dt=m$$

$$D^2(X)=\int_{-\infty}^{+\infty}(x-m)^2\rho(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}(x-m)^2e^{-\frac{(x-m)^2}{2\sigma^2}}dx=$$

$$=\frac{4\sigma^2}{\sqrt{\pi}}\int_{0}^{+\infty}t^2e^{-t^2}dt=-\frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{+\infty}t^2(e^{-t^2})'dt=-\frac{2\sigma^2}{\sqrt{\pi}}te^{-t^2}\Big|_{0}^{+\infty}+\frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{+\infty}e^{-t^2}dt=\sigma^2$$