WEEK 5

TWO-DIMENSIONAL VECTORS OF CONTINUOUS TYPE

- Let the random vector (X,Y) have the distribution function F, we say that (X,Y) is a random vector of continuous type, if the distribution function F can be put in the form:

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} \rho(s,t)\ ds\ dt,\ \forall (x,y) \in R^2,$$

and the function $\rho : R^2 \to R$ is called the probability density of the random vector (X,Y).

- If ρ is the probability density for (X,Y), and ρ_X ρ_Y the probability densities for X and Y respectively occur:

1) $\rho(x,y) \geq 0\ ,\forall (x,y) \in R^2$.

2) $\dfrac{\partial^2 F(x,y)}{\partial x \partial y} = \rho(x,y)$ A.P.T. on R^2.

3) $P((X,Y) \in D) = \iint\limits_{D} \rho(x,y)\ dx\ dy,\ D \subset R^2$.

4) $\iint_{R^2} \rho(x,y)\ dx \cdot dy = 1$.

5) $\rho_X(x) = \int_{-\infty}^{\infty} \rho(x,y)dy,\ \forall x \in R;\ \rho_Y(y) = \int_{-\infty}^{\infty} \rho(x,y)dx,\ \forall y \in R$.

- We say that random variables of continuous type X and Y are independent if F(x,y) = $F_X(x) \cdot F_Y(y)$, $\forall (x,y) \in R^2$.
- **Teorema 1.** X şi Y sunt independente $\Leftrightarrow$ $\rho(x,y) = \rho_X(x)\rho_Y(y)$.
- **Theorem 2.** Let be the random vector (X, Y) with the probability density ρ and ρ_z probability density of the random variable Z=X+Y,

then $\rho_z(x) = \int_{-\infty}^{\infty} \rho(u, x-u)du$, $\forall x \in R$.

Observation

1) If X,Y are independent, then:

$\rho_z(x) = \int_{-\infty}^{\infty} \rho_X(u) \cdot \rho_Y(x-u)du,\ \forall x \in R$.

2) If U= $X \cdot Y$, then $\rho_U(x) = \int_{-\infty}^{\infty} \rho(u, \frac{x}{u}) \frac{du}{|u|},\ \forall x \in R$.

3) If $V = \dfrac{X}{Y}$, then $\rho_V(x) = \int_{-\infty}^{\infty} \rho(xu, u) \cdot |u| du,\ \forall x \in R$.

- **Theorem 3.** Let be the random variable X with the probability density ρ_X and the monotonic function g:R→R. Then the random variable Y=g(X) has the probability density ρ_Y given by: $\rho_Y(x) = \dfrac{\rho_X(g^{-1}(x))}{\left|g'(g^{-1}(x))\right|}$.

- **Example 1.** *Let be the random vector (X,Y) with probability density*

$$\rho(x,y) = \begin{cases} kxy^2, & (x,y) \in [1,2]\times[1,3] \\ 0, & \text{în rest} \end{cases}.$$

Determine the actual parameter k, vector distribution function and marginal distribution functions.

Solution:

The function $\rho(x,y) = \begin{cases} kxy^2, & (x,y) \in [1,2]\times[1,3] \\ 0, & \text{în rest} \end{cases}$ is the probability density if $\rho(x,y) \geq 0$ and

$\iint_{R^2} \rho(x,y)dxdy = 1$ what the equation in k involves, $k\int_1^2\int_1^3 xy^2 dxdy = 1$, checked for $k = \dfrac{1}{13}$.

In this case, the distribution function will be:

$$F(x,y) = \frac{1}{13}\int_1^x\int_1^y uv^2 dudv = \begin{cases} 0 & , \textit{dacă} \ \ x<1 \ \textit{sau} \ \ y<1 \\ \frac{1}{78}(x^2-1)(y^3-1) & , \textit{dacă} \ \ (x,y)\in[1,2]\times[1,3] \\ \frac{1}{26}(y^3-1) & , \textit{dacă} \ \ y\in[1,3] \ \textit{şi} \ \ x>2 \\ \frac{1}{3}(x^2-1) & , \textit{dacă} \ \ y\in[1,2] \ \textit{şi} \ \ \mathrm{y}>3 \\ 1 & , \textit{dacă} \ \ x>2 \ \textit{şi} \ \ \mathrm{y}>3 \end{cases}$$

and we also infer that the marginal distribution functions are, respectively,

$$F_X(x) = \begin{cases} 0 & , x<1 \\ \frac{1}{3}(x^2-1) & , x\in[1,2] \\ 1 & , x>2 \end{cases} ; \quad F_Y(y) = \begin{cases} 0 & , \mathrm{y}<1 \\ \frac{1}{26}(y^3-1) & , y\in[1,3] \\ 1 & , y>3 \end{cases}$$

Example 2. *Let be the random vector (X,Y) with probability density* $\rho(x,y) = \begin{cases} k(x+y+1), x\in[0,1], y\in[0,2] \\ 0, \text{în rest} \end{cases}$

It is required:

(a) determine the constant K;

(b) determine marginal densities;

(c) investigate whether X and Y are independent or not;

(d) calculate the correlation coefficient between X and Y.

Solution

(a) of the conditions $\rho(x,y) \geq 0 \Rightarrow k \geq 0$

$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(x,y)dxdy = k\int_0^1 dx\int_0^2 (x+y+1)dy = 1 \Rightarrow k = 1/5$.

So $\rho(x,y)=\begin{cases}\frac{1}{5}(x+y+1),\ x\in[0,1],\ y\in[0,2]\\ 0,\ \text{în rest}\end{cases}$

b) $\rho_X(x)=\int_{-\infty}^{+\infty}\rho(x,y)dy=\frac{1}{5}\int_0^2(x+y+1)dy=\frac{2x+4}{5},\ x\in[0,1]$

$$\Rightarrow\ \rho_X(x)=\begin{cases}\frac{2x+4}{5},\ x\in[0,1]\\ 0,\ \text{altfel}\end{cases}$$

$$\rho_y(y)=\int_{-\infty}^{+\infty}\rho(x,y)dx=\frac{1}{5}\int_0^1(x+y+1)dx=\frac{2y+3}{10},\ y\in[0,2]$$

$$\Rightarrow\ \rho_Y(y)=\begin{cases}\frac{2y+3}{10},\ y\in[0,2]\\ 0,\ \text{altfel}\end{cases}$$

c) X and Y are not independent because: $\rho(x,y)\neq\rho_X(x)\cdot\rho_Y(y)$

d) $E(X)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x\rho(x,y)dxdy=\int_{-\infty}^{+\infty}x\rho_X(x)dx=\frac{1}{5}\int_0^1 x(2x+4)dx=\frac{8}{15}$

$$E(Y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho(x,y)dxdy=\int_{-\infty}^{+\infty}y\rho_Y(y)dy=\frac{1}{10}\int_0^2 y(2y+3)dy=\frac{17}{15}$$

$$m_2(X)=E(X^2)=\int_{-\infty}^{+\infty}x^2\rho_X(x)dx=\frac{1}{5}\int_0^1 x^2(2x+4)dx=\frac{11}{30}$$

$$m_2(Y)=E(Y^2)=\int_{-\infty}^{+\infty}y^2\rho_Y(y)dy=\frac{1}{10}\int_0^2 y^2(2y+3)dy=\frac{8}{15}$$

So $Var(X)=E(X^2)-[E(X)]^2=\frac{11}{30}-\frac{64}{225}=\frac{37}{450}$

$$Var(Y)=E(Y^2)-[E(Y)]^2=\frac{8}{5}-\frac{289}{225}=\frac{71}{225}$$

$$E(X\cdot Y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xy\,\rho(x,y)dxdy=\frac{1}{5}\int_0^1 dx\int_0^2 xy(x+y+1)dy=$$

$$=\frac{1}{5}\int_0^1\left(2x^2+\frac{14}{3}x\right)dx=\frac{9}{15}\Rightarrow C(X,Y)=E(XY)-E(X)E(Y)=.$$

$$=\frac{9}{15}-\frac{8}{15}\cdot\frac{12}{15}=\frac{-1}{225}$$

Is obscientious: $r(X,Y)=\frac{C(X,Y)}{\sigma_X\cdot\sigma_Y}=\frac{-\frac{1}{225}}{\sqrt{\frac{37}{450}\cdot\frac{71}{225}}}=-0{,}02758$

EXAMPLE OF LEGI DE TIP CONTINUU

- **Uniform law (rectangular)** **We** say that the random variable X follows the uniform law on the interval [a,b],if it has probability density

$$\rho(x)=\begin{cases}\dfrac{1}{b-a} & ,dacă\ \ x\in[a,b]\\ 0 & ,dacă\ \ x\notin[a,b]\end{cases}$$

- From the graph of the probability density function it is found that $\rho(x)$ indeed , thus defined, satisfies the conditions of a probability density function

1) $\rho(x)\ \ \geq 0\ ,\ \ \forall\ \ x\in R$

2) $\int\limits_{-\infty}^{+\infty}\rho(x)dx=\int\limits_{a}^{b}\frac{1}{b-a}dx=1$

- The random variable distribution function following the uniform law has the expression:

$$F(x)=\begin{cases}0 & ,x\leq a\\ \dfrac{x-a}{b-a} & ,x\in(a,b]\\ 1 & ,x>b\end{cases}$$

Really:

-if $x\leq a\ \ F(x)=\int\limits_{-\infty}^{x}\rho(t)dt=\int\limits_{-\infty}^{x}0dt=0$

-if $x\in(a,b]\ \ F(x)=\int\limits_{-\infty}^{x}\rho(t)dt=\int\limits_{-\infty}^{a}0dt+\int\limits_{a}^{x}\frac{1}{b-a}dt=\frac{x-a}{b-a}$

-if $x>b\ \ ,F(x)=\int\limits_{-\infty}^{x}\rho(t)dt=\int\limits_{-\infty}^{a}0dt+\int\limits_{a}^{b}\frac{1}{b-a}+\int\limits_{b}^{x}0dt=1$

- **Exponential law** We say that the random variable X follows the exponential parameter law $\lambda>0$, if it has probability density:

$$\rho(x)=\begin{cases}\lambda e^{-\lambda x} & ,dacă\ \ x>0\\ 0 & ,dacă\ \ x\leq 0\end{cases}$$

- Obviously, this function satisfies the conditions:

1) $\rho(x)\geq 0\ \ \forall\ \ x\in R$

2) $\int\limits_{-\infty}^{+\infty}\rho(x)dx=\int\limits_{0}^{+\infty}\lambda e^{-\lambda x}dx=-e^{-\lambda x}\Big|_{0}^{+\infty}=1$

- The distribution function for random variable X that follows the exponential parameter law $\lambda > 0$ is:

$$F(x) = \begin{cases} 0 \ , dacă \ \ x \le 0 \\ 1 - e^{-\lambda x} \ \ dacă \ \ x > 0 \end{cases}$$

- **Normal law** We say that the random variable X follows the normal law (Gauss's law)of parameters

$m \in R$ și $\sigma > 0$, $N(m,\sigma))$ if it has probability density $\rho(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$, $\forall \ x \in R$

- We obviously have:

1) $\rho(x) > 0 \ , \forall \ \ x \ \in R \quad , \quad \sigma > 0$

2) $\displaystyle\int_{-\infty}^{+\infty} \rho(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} dx = \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} e^{-t^2} dt = 1$

if the change of variable has been made $t = \dfrac{x-m}{\sigma\sqrt{2}}$, $dx = \sigma\sqrt{2dt}$.

- The curve representing the probability density graph ρ is called the Gauss curve (or Gauss bell).

Note: If the random variable X follows the normal law $N(m,\sigma)$, then the distribution function F is given by the relation

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}} dt \qquad \forall \in R$$

If the variable change is made, it $(t-m)/\sigma = u$ is obtained

$$\Rightarrow F(x) = \frac{1}{2} + \Phi(\frac{x-m}{\sigma}) \qquad unde \qquad \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{t^2}{2}} dt \quad , \quad \forall x \in R$$

is Laplace's function.

- Laplace's function has tabulated values for positive arguments.
- If x<0 then $\Phi(x) = -\Phi(-x)$.
- If parameters m and σ have values 0 and 1 respectively, then we say that X follows the normal and normed law or reduced normal law.
- **Gamma law** We say that the random variable X follows the gamma law with parameters a,b>0 if it has probability density

$$\rho(x) = \begin{cases} \dfrac{1}{\Gamma(a)b^a} x^{a-1} e^{-\frac{x}{b}}, x > 0 \\ 0, x \le 0 \end{cases}$$

where $\Gamma(a)$ is Euler's function of the second kind, i.e $\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx\ldots$

- Obviously, the conditions are met:

1) $\rho(x)\geq 0 \quad \forall \quad x\in R$

2) $\int_{-\infty}^{+\infty}\rho(x)dx=1$

- In the particular case $a=1, b=1/\lambda$, exponential law is obtained.
- The distribution function of a random variable following the gamma law has the expression

$$F(x)=\frac{1}{\Gamma(a)b^a}\int_0^x y^{a-1}e^{-\frac{y}{b}}dy \quad , \quad x>0$$

known as incomplete gamma function.

- **Beta law** We say that the random variable X follows the beta law with parameters a,b>0, if it has probability density

$$\rho(x)=\begin{cases}\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1} \quad , \quad dacă \;\; x\in[0,1] \\ 0 \quad ,dacă \;\; x\notin[0,1]\end{cases}$$

where B(a,b) is Euler's function of the first kind, i.e. $B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$

known as incomplete beta function.

- Obviously, the conditions are met:

1) $\rho(x)\geq 0, \forall \;\; x\in R$

2) $\int_{-\infty}^{+\infty}\rho(x)dx=\frac{1}{B(a,b)}\int_0^1 x^{a-1}(1-x)^{b-1}dx=\frac{B(a,b)}{B(a,b)}=1$

- The distribution function associated with a variable following the beta law has the expression:

$$F(x)=\begin{cases} 0 \;\; ,x<0 \\ \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt, \quad x\in[0,1] \\ 1 \;\; ,x>1\end{cases}$$

- **Law χ^2 (hi-squared)** We say that random variable X follows the law χ^2 (hi-squared), if it has probability density:

$$\rho(x)=\begin{cases}\frac{1}{\Gamma(\frac{n}{2})\sigma^n 2^{\frac{n}{2}}}\int_0^{+\infty} x^{\frac{n}{2}-1}e^{-\frac{x}{2\sigma^2}} \quad ,dacă \;\; x>0 \\ 0 \;\; ,dacă \;\; x\leq 0\end{cases}$$

where $\sigma>0$,iar $n\in N^*$ and is called the number of degrees of freedom.

- We obviously have:

1) $\rho(x) \geq 0 \ \forall \ x \in R$

2) $\int_{-\infty}^{+\infty} \rho(x)dx = \frac{1}{\Gamma(\frac{n}{2})\sigma^n 2^{\frac{n}{2}}} \int_0^{+\infty} x^{\frac{n}{2}-1} e^{-\frac{x}{2\sigma^2}} dx = \frac{2^{\frac{n}{2}}\sigma^n}{\Gamma(\frac{n}{2})\sigma^n 2^{\frac{n}{2}}} \int_0^{+\infty} t^{\frac{n}{2}-1} e^{-t} dt = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2})} = 1$ whether the variable change has been made $t = x/2\sigma^2$.

- The law, χ^2 also called the Helmert–Pearson law, is a particular case of the gamma law, namely, for $a = n/2$ *şi* $b = 2\sigma^2$.
- The distribution function associated with a random variable X that follows the law χ^2 (hi-squared) is of the form

$$F(x) = P(X < x) = P(\chi^2 < \chi_q^2) = F(\chi_q^2) =$$
$$\frac{1}{\Gamma(\frac{n}{2})\sigma^n 2^{\frac{n}{2}}} \int_0^{x_q^2} x^{\frac{n}{2}-1} e^{-\frac{x}{2\sigma^2}} dx = 1 - P(\chi^2 > \chi_q^2) = 1 - q$$

where q are probabilities of the form $q = P(\chi^2 > \chi_q^2) = \frac{1}{\Gamma(\frac{n}{2})\sigma^n 2^{\frac{n}{2}}} \int_{x_q^2}^{+\infty} x^{\frac{n}{2}-1} e^{-\frac{x}{2\sigma^2}} dx$

- Karl Pearson edited tables for the distribution function corresponding to a random variable that follows the distribution χ^2.
- **Say** that random variable X follows Student's law with n degrees of freedom, if it has probability density

$$\rho(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}(1+\frac{x^2}{n})^{-\frac{n+1}{2}} \quad , pentru \quad x \in R$$

- The conditions of a probability density are checked, i.e.

1) $\rho(x) \geq 0 \forall \ x \in R$ obviously

2) $\int_{-\infty}^{+\infty} \rho(x)dx = \int_{-\infty}^{+\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma(\frac{n}{2})}(1+\frac{x^2}{n})^{-\frac{n+1}{2}} dx = 1$

whether the variable change has been made $y = 1/(1+x^2/n)$ and relationships have been used

$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

- If n=1, then the probability density takes the form $\rho(x) = \frac{1}{\pi(1+x^2)}$, $x \in R$, i.e. the probability density corresponding to the standard Cauchy distribution is obtained.

- **Fisher–Snedecor law** (Fs distribution **)** We say that the random variable X follows the Fisher–Snedecor law if it has probability density

$$\rho(x)=\begin{cases}\dfrac{\Gamma(\frac{r+s}{2})}{\Gamma(\frac{r}{2})\Gamma(\frac{s}{2})}r^{\frac{r}{2}}s^{\frac{s}{2}}\dfrac{x^{\frac{r}{2}-1}}{(rx+s)^{\frac{r+s}{2}}} & ,\ \textit{dacă}\ x\geq 0\\ 0 & ,\ \textit{dacă}\ \ x<0\end{cases}$$

unde s,r>0.

- The conditions of a probability density are checked, i.e.

1) $\rho(x)\geq 0 \quad \forall x\in R$ obviously

2) $\int\limits_{-\infty}^{+\infty}\rho(x)dx=\int\limits_{0}^{+\infty}\dfrac{\Gamma(\frac{r+s}{2})}{\Gamma(\frac{r}{2})\Gamma(\frac{s}{2})}r^{\frac{r}{2}}s^{\frac{s}{2}}\dfrac{x^{\frac{r}{2}-1}}{(rx+s)^{\frac{r+s}{2}}}dx=\dfrac{\Gamma(\frac{r+s}{2})}{\Gamma(\frac{r}{2})\Gamma(\frac{s}{2})}\int\limits_{0}^{1}y^{\frac{r}{2}-1}(1-y)^{\frac{s}{2}-1}dy=\dfrac{B(\frac{r}{2},\frac{s}{2})}{B(\frac{r}{2},\frac{s}{2})}=1$

where the variable change was made $y=rx/(rx+s)$.