Dictionary-Based Block Term Decomposition for Third-Order Tensors

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Introdcution

Block term decomposition (BTD) decomposes a third-order tensor into multiple terms (L_r, M_r, N_r), which has garnered increasing attention for high-dimensional image representation. However, BTD practically struggles to reveal these underlying different structures in the original domain, which locks the potential of the BTD. In this work, we propose a dictionary-based BTD (DBTD) for third-order tensors by revisiting the BTD from the convolutional dictionary learning perspective, which can better reveal the underlying different structures of the original tensor.

Keywords: block term decomposition, convolutional dictionary learning, high-dimensional image recovery, optimization algorithm.

Motivation

BTD practically struggles to reveal the underlying different structures of the gih-dimensional images in the original domain. We can see that when the term number R is large, the underlying different structures extracted by the BTD become insignificant, which leads to a negative impact on the recovery performance.

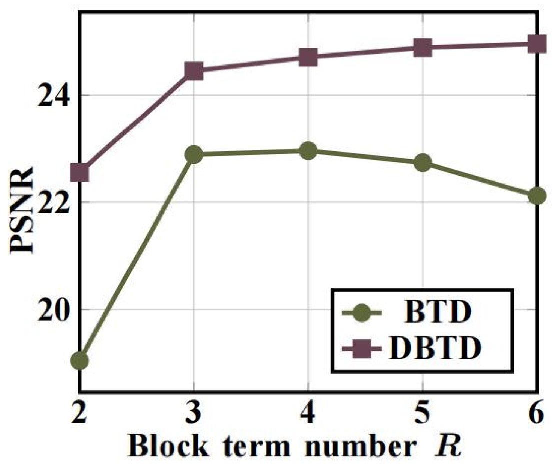


Figure 1: The PSNR of BTD and DBTD with varying block term number R.

Dictionary-Based Block Term Decomposition

To unlock the potential of BTD, we propose a dictionary-based BTD (DBTD) by revisiting BTD from the convolutional dictionary learning perspective. Each term is represented by the convolution of an adaptive dictionary and a low-rank coefficient, as shown in **Figure** 2. Herein, the adaptive dictionaries can represent distinct patterns contribute to reveal the underlying different structures of the original tensor.

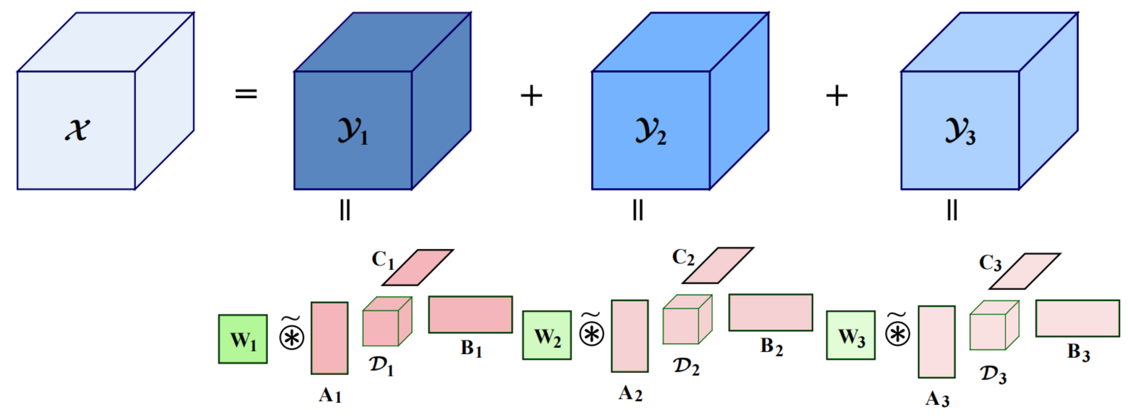


Figure 2: Visualization of DBTD.

Definition 1 (DBTD). *Given a tensor* $\mathcal{X} \in \mathbb{R}^{I\times J\times K}$, *its DBTD is defined as*

$$\mathcal{X} = \sum_{r=1}^{R} \mathcal{Y}_r = \sum_{r=1}^{R} \mathbf{W}_r \tilde{\circledast} (\mathcal{D}_r \times_1 \mathbf{A}_r \times_2 \mathbf{B}_r \times_3 \mathbf{C}_r), \quad r = 1, \ldots, R. \tag{1}$$

where $\mathbf{W}_r \in \mathbb{R}^{d\times d}$ *(with* $d \ll \min\{I, J\}$*) is the* r*th dictionary,* $\mathcal{D}_r \in \mathbb{R}^{L_r\times M_r\times N_r}$ *has the multilinear rank* $\mathrm{rank}_{\boxplus}(\mathcal{D}_r) = (L_r, M_r, N_r)$.

Essential Uniqueness

Theorem 1. *Let* $\mathcal{X} \in \mathbb{R}^{I\times J\times K}$ *be a third-order tensor. Suppose there exist two DBTD-*(L, M, N) *decompositions of* $\mathcal{X}$*. If the following three conditions hold:*

(1) Each pair $(\mathbf{W}_r, \mathcal{D}_r)$ *is generic;*

(2) Both $\mathbf{W}$ *and* $\mathcal{D}$ *are generic;*

(3)

$$\begin{cases} N > L + M - 2, \\ k'_{\mathbf{A}} = R \text{ and } k'_{\mathbf{B}} + k'_{\mathbf{C}} \geqslant R + 2 \\ (\text{or } k'_{\mathbf{B}} = R \text{ and } k'_{\mathbf{A}} + k'_{\mathbf{C}} \geqslant R + 2), \end{cases} \tag{2}$$

or

$$\begin{cases} N > L + M - 2, \\ k'_{\mathbf{A}} + k'_{\mathbf{B}} + k'_{\mathbf{C}} \geqslant 2R + 2, \end{cases} \tag{3}$$

then the two DBTD-(L, M, N) *decompositions are essentially equal.*

Recovery Model

Based on the DBTD, we furthe propose a high-dimensional image denoising model. Given a noisy tensor $\mathcal{T} \in \mathbb{R}^{n_1\times n_2\times n_3}$ with the additive Gaussian noise, the recovery model is formulated as

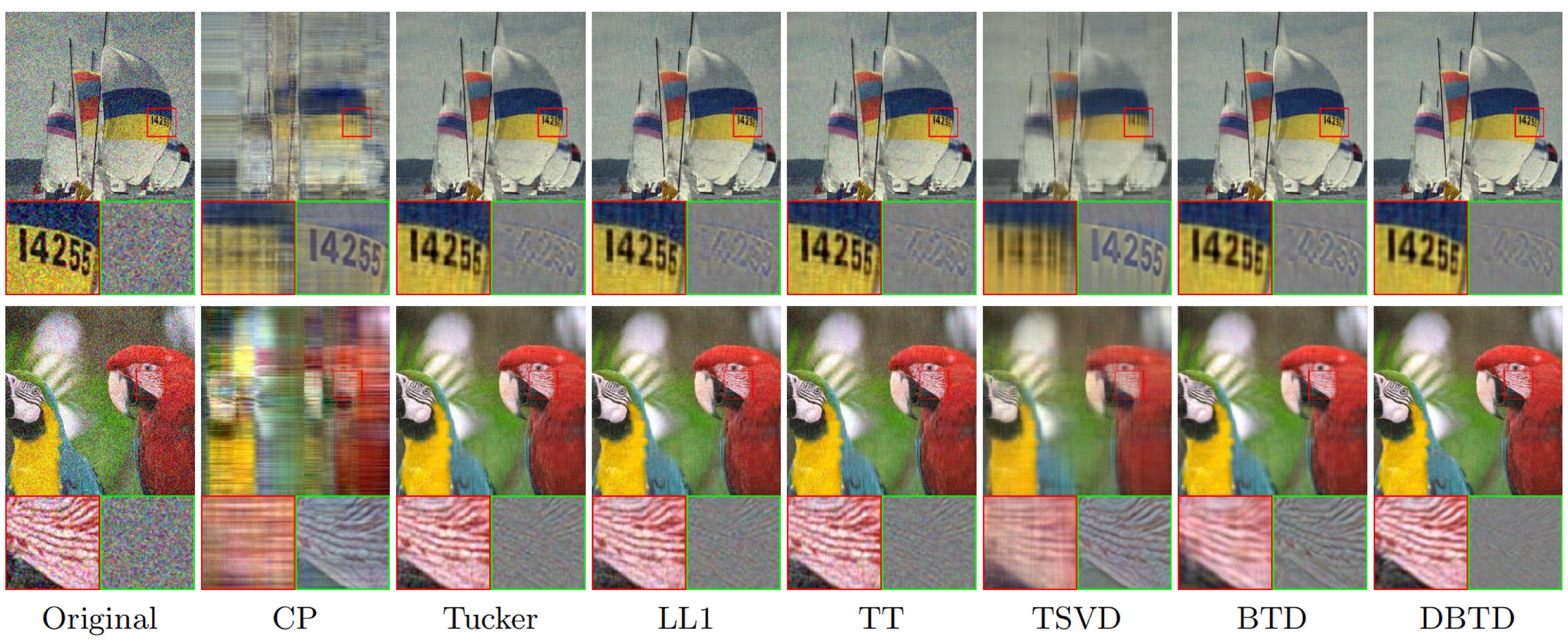
$$\min_{\substack{\mathcal{X}, \mathbf{W}_r, \mathcal{D}_r \\ \mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r}} \frac{1}{2}\|\mathcal{T} - \mathcal{X}\|_F^2 + \frac{\alpha}{2}\left\|\mathcal{X} - \sum_{r=1}^{R} \mathbf{W}_r \tilde{\circledast}(\mathcal{D}_r \times_1 \mathbf{A}_r \times_2 \mathbf{B}_r \times_3 \mathbf{C}_r)\right\|_F^2$$

$$\text{s.t.} \begin{cases} \mathbf{A}_r^\top \mathbf{A}_r = \mathbf{I}_I, \quad \mathbf{B}_r^\top \mathbf{B}_r = \mathbf{I}_J, \quad \mathbf{C}_r^\top \mathbf{C}_r = \mathbf{I}_K, \\ \|\mathbf{W}_r\|_F^2 = 1, \quad \mathbf{W}_r \geq \mathbf{0}, \quad r = 1, \ldots, R, \end{cases}$$

in which $\mathcal{X} \in \mathbb{R}^{I\times J\times K}$ is the required tensor, the first term is the fidelity term, α is a trade-off parameter, $\mathbf{W}_r \in \mathbb{R}^{d\times d}$ (with $d \ll \{I, J\}$) are the dictionaries, $\mathcal{D}_r \in \mathbb{R}^{L_r\times M_r\times N_r}$ are the core tensors. $\mathbf{A}_r \in \mathbb{R}^{I\times L_r}$, $\mathbf{B}_r \in \mathbb{R}^{J\times M_r}$, and $\mathbf{C}_r \in \mathbb{R}^{K\times N_r}$ are semi-orthogonal matrices by following the same setting in BTD. The normalized constraint $\|\mathbf{W}_r\|_F^2 = 1$ contributes to improving the stability of the proposed model. Additionally, the nonnegative constraint $\mathbf{W}_r \geq \mathbf{0}$ enhances the interpretability of the dictionaries, which is beneficial for learning significant structures.

We develop an Alternating Direction Method of Multipliers (ADMM) algorithm with the convergence guarantee to solve this model.

Recovery Results



Contact Information

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