Matrix Completion and Decomposition in Phase-Bounded Cones

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Problem Background

Example 1: Nuclear-Norm Matrix Completion

$$\min_{X} \ \|X\|_* \quad \text{s.t.} \quad \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M)$$

Equivalent semi-definite programming (SDP) formulation:

$$\begin{aligned} \min_{X,W,Z} \ & \tfrac{1}{2}\big(\operatorname{tr} W + \operatorname{tr} Z\big) \\ \text{s.t.} \ & \begin{bmatrix} W & X \\ X^\top & Z \end{bmatrix} \succeq 0, \ W, Z \succeq 0, \\ & \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M). \end{aligned}$$

- The feasible set lives inside the **positive semi-definite cone (PSD)**—only the *magnitudes* of singular values are enforced; *phase* information remains unrestricted.

Motivation

- In many real-world applications(e.g. *multivariable control systems*, *impedance circuit synthesis*, *robust control*) the **phase** of matrix entries must be bounded as well as their magnitude.
- We therefore introduce a more general *phase-bounded cone*（相位有界锥） $\mathrm{SS}[\alpha, \beta]$ and extend matrix completion / decomposition theory from PSD to the joint "magnitude + phase" setting.

Phase-Bounded Cone $\mathrm{SS}[\alpha, \beta]$

- **numerical range:** $W(C) = \{x^H Cx \mid \|x\|_2 = 1\} \subset \mathbb{C}$.
- **Minimum / Maximum Phase:**

  $\varphi_{\min}(C) = \min_{z \in W(C)} \arg z, \qquad \varphi_{\max}(C) = \max_{z \in W(C)} \arg z$.

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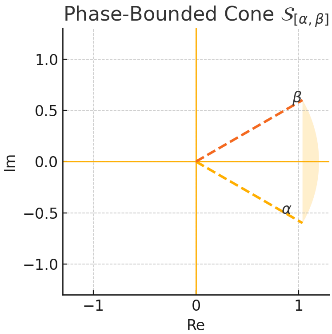
Definition

A complex matrix C is in $\mathrm{SS}[\alpha, \beta]$ if it is *semi-sectorial* and its numerical range satisfies

$$\alpha \le \varphi_{\min}(C) \le \varphi_{\max}(C) \le \beta, \qquad 0 < \beta - \alpha < \pi.$$



Methodology

Toeplitz Split

Any complex matrix C can be written as

$$C \;=\; C_H \;+\; i\,C_S, \qquad \text{with } C_H,\, C_S \in \mathbb{H}_n \text{ (Hermitian).}$$

Linear Map $R_{\alpha,\beta}$

$$R_{\alpha,\beta} \;=\; \begin{bmatrix} -\sin\alpha & \cos\alpha \\ \sin\beta & -\cos\beta \end{bmatrix} \otimes I_n.$$

It *preserves sparsity patterns* and is invertible for $0 < \beta - \alpha < \pi$.

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Key Lemma

$$C \in \mathcal{S}_{[\alpha,\beta]} \iff R_{\alpha,\beta}\begin{bmatrix} C_H \\ C_S \end{bmatrix} \in \underbrace{\mathrm{PSD} \times \mathrm{PSD}}_{\text{positive semi-definite cone（正半定锥）}}.$$

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- The lemma transfers **PSD theory** to the phase-bounded cone via an *invertible, pattern-preserving* transform.

Graph-Constrained Matrices

Definition

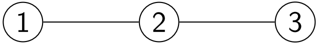
Fix an undirected graph $G = (V, E)$ on n vertices. Define

$$\mathbb{C}_G^{n\times n} := \left\{ C \in \mathbb{C}^{n\times n} \;\middle|\; C_{ij} = 0 \text{ whenever } (i,j) \notin E \cup \mathrm{diag} \right\}.$$

In other words, non-zero entries of C are allowed only on edges of G (plus the diagonal).

Toy example:

Matrix pattern respecting $E = \{(1,2),(2,3)\}$:



$$C = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} \in \mathbb{C}_G^{3\times 3}.$$

Methodology

(1) Phase-Bounded Completion

Given a *partial* matrix C whose known entries lie on E, decide whether there exists

$$K \in \mathcal{S}_{[\alpha,\beta]} \quad \text{s.t.} \quad K_{ij} = C_{ij} \ \ (\forall\, (i,j) \in E).$$

(2) Phase-Bounded Decomposition

Given a matrix $C \in \mathbb{C}_G^{n\times n}$, decide whether

$$C = \sum_{k=1}^{m} C_k, \qquad C_k \in \mathcal{S}_{[\alpha,\beta]},\ \mathrm{rank}(C_k) = 1,\ \mathrm{supp}(C_k) \subseteq E.$$

- Both tasks generalise classical PSD completion/decomposition to the *phase-bounded cone* $\mathcal{S}_{[\alpha,\beta]}$.

Four Fundamental Cones

Definitions

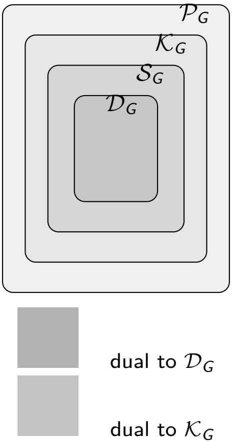
$$\mathcal{S}_G := \mathbb{C}_G^{n\times n} \cap \mathcal{S}_{[\alpha,\beta]} \quad \text{(sparse phase-bounded matrices)}$$

$$\mathcal{P}_G := \left\{ C \in \mathbb{C}_G^{n\times n} \mid C[K] \in \mathcal{S}_{[\alpha,\beta]} \ \ \forall \text{ cliques } K \subseteq G \right\}$$

$$\mathcal{K}_G := \left\{ C \in \mathbb{C}_G^{n\times n} \mid \exists\, B \in \mathbb{C}_{G^c}^{n\times n} \ C + B \in \mathcal{S}_{[\alpha,\beta]} \right\}$$

$$\mathcal{D}_G := \left\{ \sum_k C_k \;\middle|\; C_k \in \mathcal{S}_{[\alpha,\beta]},\ \mathrm{rank}(C_k) = 1,\ \mathrm{supp}(C_k) \subseteq \right.$$

- All four are **closed, pointed, convex** cones.
- $\mathcal{D}_G \subseteq \mathcal{S}_G \subseteq \mathcal{K}_G \subseteq \mathcal{P}_G$.



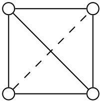
Chordal-Graph Characterisation

Main Theorem

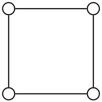
For $0 < \beta - \alpha < \pi$ and an undirected graph G, the following statements are *equivalent*:

1. G is **chordal**(every cycle of length ≥ 4 has a chord);
2. $\mathcal{D}_G = \mathcal{S}_G$;
3. $\mathcal{K}_G = \mathcal{P}_G$.

- Extends the PSD result to the *phase-bounded cone* $\mathcal{S}_{[\alpha,\beta]}$.
- Provides a **necessary & sufficient** graph criterion for phase-bounded completion and decomposition.



chordal



non-chordal

Completion Criterion for Chordal Graphs

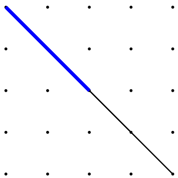
Corollary

Let $0 < \beta - \alpha < \pi$ and G be **chordal**. A partial matrix C with graph G admits a completion

$$K \in \mathcal{S}_{[\alpha,\beta]} \quad \text{such that} \quad K_{ij} = C_{ij}\ (i,j) \in E(G)$$

iff every specified clique submatrix $C[K]$ already lies in $\mathcal{S}_{[\alpha,\beta]}$.

- **Practical meaning**: only maximal cliques need to be checked —no global SDP.
- For banded or tree patterns this reduces to testing a few small principal blocks.



5×5 banded graph ($w = 2$). Blue triangle = a maximal clique to check.

Decomposition Criterion for Chordal Graphs

Corollary

Let $0 < \beta - \alpha < \pi$ and G be **chordal**. A sparse matrix $C \in \mathbb{C}_G^{n\times n}$ admits a rank-one sum

$$\boxed{C = \sum_{k=1}^{m} C_k, \quad C_k \in \mathcal{S}_{[\alpha,\beta]},\ \mathrm{rank}(C_k) = 1,\ \mathrm{supp}(C_k) \subseteq E(G)}$$

iff the matrix itself already lies in the phase-bounded cone:

$$C \in \mathcal{S}_{[\alpha,\beta]}.$$

tree pattern (chordal). Blue edge $=$ one rank-one component C_k.

Banded Graphs $\Rightarrow$ Two PSD Sub-Problems

Key idea: transform one complex problem into two real, band-preserving PSD problems, easy to solve.

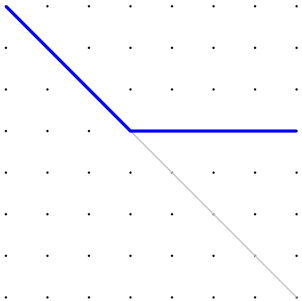
Key fact for banded graphs (Lemma 3.3 + Remark 5.1)

$$C \in \mathcal{S}_{[\alpha,\beta]} \iff \begin{cases} C_\alpha = -\sin\alpha\, C_H + \cos\alpha\, C_S \;\in\; \mathrm{PSD}, \\ C_\beta = \;\;\sin\beta\, C_H - \cos\beta\, C_S \;\in\; \mathrm{PSD}. \end{cases}$$

- **Band-preserving** (same half-bandwidth w).
- Phase-bounded completion reduces to two banded-PSD sub-problems, solvable in $\mathcal{O}(nw^2)$ time.

Staircase Algorithm

Algorithm 5.1 (staircase fill): Start at the main diagonal and successively complete principal submatrices of size $w+1,\ w+2, \ldots, n$ via Schur complements. The result H_c is the **unique** phase-bounded completion maximising $\det H$.



$n = 8,\ w = 2$ Blue = fill order.

Conclusions & Outlook

Takeaways

- Extended classical PSD completion/decomposition to **phase-bounded cones** $\mathcal{S}_{[\alpha,\beta]}$.
- Chordal graphs give *iff* criteria; banded graphs admit an $\mathcal{O}(nw^2)$ staircase algorithm and a det-max *central completion*.

Future Works

- Real-time applications in robust control and impedance circuit synthesis.
- Scalable ADMM / primal-dual solvers for large $\mathcal{S}_{[\alpha,\beta]}$-SDPs.
- Exploring matrix completion and decomposition methods for other constrained graphs, including non-chordal and non-banded graphs.