Spectral Change Point Estimation via Sparse Tensor Decomposition

Based on Xinyu Zhang & Kung-Sik Chan (2024)

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Background

High-dimensional time series frequently exhibit

- nonstationarity,
- structural changes (change points),
- and complex temporal dependence.

Spectrum captures dynamic covariance structure across all lags:

- identifies periodic patterns,
- reveals structural breaks invisible in covariance alone,
- widely used in neuroscience, finance, engineering.

However, challenges remain:

- changes may be **sparse** in series dimension,
- or occur only at **specific frequencies**,
- high dimension $p \gg N$ makes detection difficult,
- existing methods cannot jointly identify **which series** and **which frequencies** are activated.

Problem Overview

We study high-dimensional multivariate time series that may experience **structural changes** over time.

Our goal:

- detect when the dependence structure changes,
- identify which components (series) are affected,
- and determine at which frequencies the changes occur.

Key idea:

- instead of looking at raw covariance,
- we analyze the **spectral structure** across time blocks.

Why Spectral Analysis?

Structural changes in high-dimensional time series may be subtle in the time domain, but become much clearer in the **frequency domain**.

Advantages of spectral analysis:

- decomposes dependence structure across frequencies,
- reveals periodic or oscillatory behavior,
- detects changes invisible to covariance-based methods,
- naturally handles high-dimensional interactions.

Key idea:

- structural breaks correspond to **changes in spectral matrices** across different time blocks.

Data Model and Block Structure

We observe a p-dimensional multivariate time series:

$$\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_N), \qquad \mathbf{X}_t \in \mathbb{R}^p.$$

To study how the dependence structure evolves over time, we divide the series into **blocks**:

$$\text{block } b = \{\mathbf{X}_t : t \in I_b\}, \qquad b = 1, \dots, B.$$

Within each block, the process is assumed to be approximately stationary. This allows us to estimate its **local spectral characteristics**.

This blockwise structure is the foundation for detecting spectral changes across time.

Blockwise Spectral Matrix Estimation

For each block b, we estimate its **local spectral density** to capture the frequency-domain dependence structure.

The result is a $p \times p$ **spectral density matrix**:

$$\hat{f}_b(\omega) \in \mathbb{C}^{p \times p}, \quad b = 1, \ldots, B.$$

Interpretation:

- Each entry $\hat{f}_b(\omega)_{ij}$ measures the correlation between series i and j at frequency ω.
- Different blocks reflect different time periods.
- Changes across b reveal **structural breaks** in the system.

Thus, $\{\hat{f}_b(\omega)\}$ provide a sequence of **frequency-specific dependence matrices** evolving over time.

From Spectral Matrices to a Tensor Structure

For each block b and frequency ω, we have a $p \times p$ spectral density matrix

$$\hat{f}_b(\omega).$$

Collecting these matrices across blocks forms a **3-way tensor representation**:

$$T(:,:,b) = \hat{f}_b(\omega), \qquad T \in \mathbb{R}^{p \times p \times B}.$$

Interpretation of tensor modes:

- Mode-1: series index i,
- Mode-2: series index j,
- Mode-3: block (time) index b.

This tensor encodes how the **frequency-domain dependence** between all pairs of series evolves over time.

Tensor CUSUM for Spectral Change-Point Detection

To detect structural breaks, we compare spectral matrices from different time blocks.

We construct a **CUSUM tensor** that measures the contrast between the first b blocks and the remaining $B - b$ blocks:

$$\mathcal{C}(b) = \sqrt{\frac{B-b}{bB}} \sum_{t \le b} T(:,:,t) - \sqrt{\frac{b}{(B-b)B}} \sum_{t > b} T(:,:,t).$$

Intuition:

- large values of $\mathcal{C}(b)$ indicate a discrepancy between the two segments,
- change points correspond to locations where this discrepancy peaks,
- each $\mathcal{C}(b)$ is still a $p \times p$ matrix: it captures which pairs of series contribute to the change.

Low-Rank Projection: Extracting Informative Directions

The CUSUM matrix $\mathcal{C}(b)$ is $p \times p$ and may be noisy in high dimensions.

To enhance the signal of change, we search for a **low-rank direction** that maximizes the CUSUM contrast:

$$\max_{\|u\|=\|v\|=1} \; u^\top \mathcal{C}(b)\, v.$$

Why low-rank?

- structural changes often affect only a few linear combinations,
- reduces high-dimensional noise,
- reveals which pairs of variables contribute to the change.

This projection is the key step behind ISLET.

Frequency-Specific Projection Approach

For each frequency ω, the CUSUM tensor $T_{1,B}(\omega)$ contains information about structural changes.

Goal:

- find a projection vector $\beta(\omega) \in \mathbb{R}^p$
- such that the projected series preserves the largest change signal.

Projection:

$$F(\omega) \times_1 \beta(\omega) \times_2 \beta(\omega)$$

turns the $p \times p \times B$ tensor into a univariate series.

The optimal projection is the **leading mode-1 component** in the tensor decomposition of $T_{1,B}(\omega)$.

Special Tensor Structure

The CUSUM tensor $T_{1,B}(\omega)$ has the form

$$T_{1,B}(\omega) = g(\omega) \circ \alpha'(\omega).$$

Let the eigendecomposition of $g(\omega)$ be

$$g(\omega) = \sum_{i=1}^{r} \lambda_i(\omega)\, \gamma_i(\omega) \circ \gamma_i(\omega).$$

Normalizing $\alpha'(\omega)$ as

$$\alpha(\omega) = \frac{\alpha'(\omega)}{\|\alpha'(\omega)\|},$$

we obtain the CP form:

$$T_{1,B}(\omega) = \sum_{i=1}^{r} \lambda_i(\omega)\, \|\alpha'(\omega)\|\, \gamma_i(\omega) \circ \gamma_i(\omega) \circ \alpha(\omega).$$

This structure enables efficient extraction of the leading component $\gamma_1(\omega)$.

Extracting the Leading Component

Given the CP structure

$$T_{1,B}(\omega) = \sum_{i=1}^{r} \lambda_i(\omega) \, \|\alpha'(\omega)\| \, \gamma_i(\omega) \circ \gamma_i(\omega) \circ \alpha(\omega),$$

the goal is to recover the leading spatial component $\gamma_1(\omega)$.

Key observations:

- mode-3 is shared across all rank-1 terms,
- modes 1 and 2 are identical and symmetric,
- the tensor behaves like a "rank-1" object once projected along mode-3.

Projection along mode-3:

$$T_{1,B}(\omega) \times_3 \alpha(\omega) \;\propto\; \sum_{i=1}^{r} \lambda_i(\omega) \, \gamma_i(\omega) \circ \gamma_i(\omega).$$

This reduces the problem to finding the leading eigenvector of a $p \times p$ symmetric matrix.

ISLET: Extraction of the Leading Component

- **Key idea:** Use tensor contraction + sparse matrix power to recover the dominant spatial component $\gamma_1(\omega)$.

Algorithm 1: Component Extraction

Input: CUSUM tensor $T_{1,B}(\omega)$, matrix $g(\omega)$, sparsity s, iterations T
Output: Leading component $\gamma_1(\omega)$

Initialize: Random unit-norm vector $\gamma^{(0)}(\omega)$

for $t = 1, 2, \ldots, T$ **do**

$v \leftarrow T_{1,B}(\omega) \times_1 \gamma^{(t-1)}(\omega) \times_2 \gamma^{(t-1)}(\omega)$
$v \leftarrow g(\omega)\, v$
$\gamma^{(t)}(\omega) \leftarrow \text{Truncate}(v, s)$
$\gamma^{(t)}(\omega) \leftarrow \gamma^{(t)}(\omega) / \|\gamma^{(t)}(\omega)\|_2$

end for

Full Change-Point Detection Algorithm

Algorithm 3: Multiple spectral change points detection algorithm

Input: Time series $X_t \in \mathbb{R}^p$, block length B, sparsity s, iterations T.
Output: Estimated change points.

1. **Spectral estimation:** Compute spectral density estimates $\hat{f}_b(\omega)$ for each block b.
2. **CUSUM tensor construction:** Form the tensor $T_{1,B}(\omega) = g(\omega) \circ \alpha'(\omega)$.
3. **Leading component estimation:** For each frequency ω, compute $\gamma_1(\omega)$ using Algorithm 1.
4. **Projected CUSUM statistic:** Compute $S_b(\omega) = \gamma_1(\omega)^\top \hat{f}_b(\omega)\gamma_1(\omega)$.
5. **Aggregate over frequencies:** $S_b = \sum_\omega S_b(\omega)$.
6. **Change-point detection:** Identify peaks of S_b over $b = 1, \ldots, B$.

Simulation Setup (DGP1–DGP3)

Goal: Evaluate spectral change-point detection under different high-dimensional time series models.

Single Change Point (DGP1)

- Model: High-dimensional factor model $X_t = \Lambda F_t + \varepsilon_t, \quad t = 1, \ldots, N.$
- Purpose: Assess performance under sparse and weak spectral shifts.

Multiple Change Points (DGP2, DGP3)

- **DGP2:** VMA(2) model $X_t = \sum_{k=0}^{2} B_{q,k}\, \varepsilon_{t-k}, \quad t \in (\tau_{q-1}, \tau_q].$
- **DGP3:** VAR(2) model $X_t = A_{q,1} X_{t-1} + A_{q,2} X_{t-2} + \varepsilon_t.$
- Purpose: Evaluate robustness in complex multi-change-point scenarios.

Simulation Data Results

Table 2: Simulation results for DGP2 with $N = 6000$

N	k_0	$\hat{q} < q$	$\hat{q} = q$	$\hat{q} > q$	ARI	$\hat{q} < q$	$\hat{q} = q$	$\hat{q} > q$	ARI	$\hat{q} < q$	$\hat{q} = q$	$\hat{q} > q$	ARI
		SCP				SBS-LSW				Series-by-series			
	3	0.02	0.98	0.00	0.98	0	0.82	0.18	0.98	0	0.19	0.81	0.90
	8	0.01	0.99	0.00	0.99	0	0.76	0.24	0.98	0	0.21	0.79	0.90
	40	0.00	1.00	0.00	1.00	0	0.31	0.69	0.95	0	0.42	0.58	0.91
	80	0.00	1.00	0.00	1.00	0	0.11	0.89	0.92	0	0.92	0.08	0.95
		FVAR-c				FVAR-i				FreSpeD			
6000	3	1.00	0.00	0.00	0.00	0	1.00	0.00	0.98	0	0.08	0.92	0.89
	8	0.93	0.07	0.00	0.33	0	1.00	0.00	0.98	0	0.01	0.99	0.80
	40	0.00	0.92	0.08	0.97	0	1.00	0.00	0.98	0	0.00	1.00	0.64
	80	0.00	0.72	0.28	0.95	0	1.00	0.00	0.98	0	0.00	1.00	0.66

Real Data Experiment Setup

Dataset: S&P 100 Constituent Returns

- Daily log-returns of S&P100 constituent stocks.
- Time period: 2000–2021.
- Dimension: $p = 79$ stocks (after filtering missing series).
- Sample size: $N = 5520$ trading days.

Preprocessing

- Returns standardized to zero mean and unit variance.
- Heavy-tailed behavior addressed using the **normal quantile transform** (NQT).
- Spectral density estimated over blocks of size $L = 60$ (approx. 3 months per block).

Real Data Results

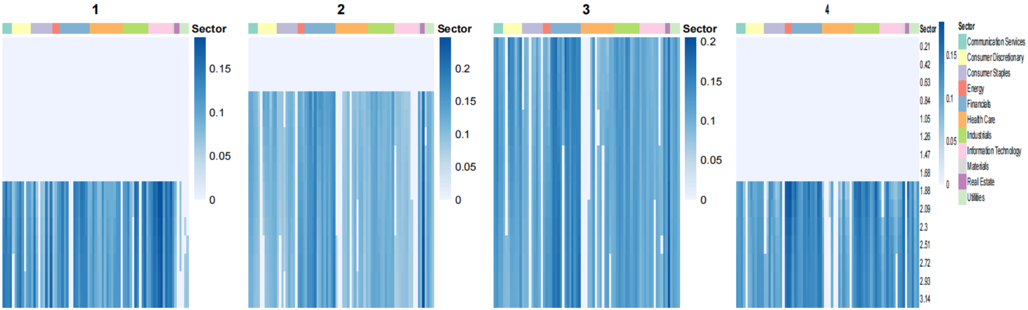

Figure 6.1: Projection per frequency (each row) and series (each column) for the four change points.

Thank you!