VMLS - Study Guide Chapter Matrix Examples 7   Name: Ben Chen

Always include this title page with your PDF. Include your name above.

- Submit your work in Gradescope as a PDF - you will identify where your "questions are."
- Identify the question number as you submit. Since we grade "blind" if the questions are NOT identified, the work WILL NOT BE GRADED and a 0 will be recorded. Always leave enough time to identify the questions when submitting.
- One section per page (if a page or less) - We prefer to grade the main solution in a single page, extra work can be included on the following page.
- Long instructions may be removed to fit on a single page.
- **Do not start a new question in the middle of a page.**
- Solutions to book questions are provided for reference.
- You may NOT submit given solutions - this includes minor modifications - as your own.
- Solutions that do not show individual engagement with the solutions will be marked as no credit and can be considered a violation of honor code.
- If you use the given solutions you must reference or explain how you used them, in particular...

For full credit, EACH book exercise in the Study Guides must use one or more of the following methods and FOR EACH QUESTION. Identify the number the method by number to ensure full credit.

Method 1 - Provide original examples which demonstrate the ideas of the exercise in addition to your solution.

Method 2 - Include and discuss the specific topics needed from the chapter and how they relate to the question.

Method 3 - Include original Python code, of reasonable length (as screenshot r text) to show how the topic or concept was explored.

Method 4 - Expand the given solution in a significant way, with additional steps and comments. All steps are justified. This is a good method for a proof for which you are only given a basic outline.

Method 5 - Attempt the exercise without looking at the solution and then the solution is used to check work. Words are used to describe the results.

Method 6 - Provide an analysis of the strategies used to understand the exercise, describing in detail what was challenging, who helped you or what resources were used. The process of understanding is described.

Method #5

1. Annotate, summarize, and/or discuss the book example 7.4 Convolution. Be able to explain the basic ideas behind “blurring an image.” Which could be an exam question. (10 pts)

A star B is different than A * B

$C = A * B$, however, in standard mathematical notation. So we will use the notation $C = A \star B$. Hadamard Product

The same properties that we observed for 1-D convolution hold for 2-D convolution: We have $A \star B = B \star A$, $(A \star B) \star C = A \star (B \star C)$, and for fixed B, $A \star B$ is a linear function of A.

Image blurring. If the $m \times n$ matrix X represents an image, $Y = X \star B$ represents the effect of *blurring* the image by the *point spread function* (PSF) given by the entries of the matrix B. If we represent X and Y as vectors, we have $y = T(B)x$, for some $(m+p-1)(n+q-1) \times mn$-matrix $T(B)$.

As an example, with

> 1 = blur?

$$B = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}, \qquad (7.4)$$

$Y = X \star B$ is an image where each pixel value is the average of a 2×2 block of 4 adjacent pixels in X. The image Y would be perceived as the image X, with some blurring of the fine details. This is illustrated in figure 7.7 for the 8×9 matrix

$$X = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&0&0&0&0&0&1&1 \\ 1&1&1&0&1&1&0&1&1 \\ 1&1&1&0&1&1&0&1&1 \\ 1&1&1&0&1&1&0&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \end{bmatrix} \qquad (7.5)$$

and its convolution with B,

$$X \star B = \begin{bmatrix} 1/4&1/2&1/2&1/2&1/2&1/2&1/2&1/2&1/2&1/4 \\ 1/2&1&1&1&1&1&1&1&1&1/2 \\ 1/2&1&3/4&1/2&1/2&1/2&1/2&3/4&1&1/2 \\ 1/2&1&3/4&1/4&1/4&1/2&1/4&1/2&1&1/2 \\ 1/2&1&1&1/2&1/2&1&1/2&1/2&1&1/2 \\ 1/2&1&1&1/2&1/2&1&1/2&1/2&1&1/2 \\ 1/2&1&1&3/4&3/4&1&3/4&3/4&1&1/2 \\ 1/2&1&1&1&1&1&1&1&1&1/2 \\ 1/4&1/2&1/2&1/2&1/2&1/2&1/2&1/2&1/2&1/4 \end{bmatrix}.$$

1's are clear?

With the point spread function

$$D^{\mathrm{hor}} = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

the pixel values in the image $Y = X \star D^{\mathrm{hor}}$ are the horizontal first order differences of those in X:

$$Y_{ij} = X_{ij} - X_{i,j-1}, \quad i = 1, \ldots, m, \quad j = 2, \ldots, n$$

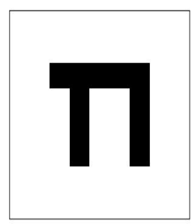
(and $Y_{i1} = X_{i1}$, $X_{i,n+1} = -X_{in}$ for $i = 1, \ldots, m$). With the point spread function

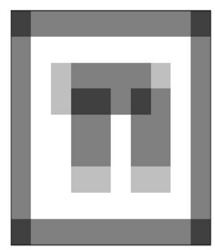
$$D^{\mathrm{ver}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

Image blurring in this context is using a 2D convolution between image X and matrix B called a point spread function. X represented by a matrix where each entry responds to a pixel's intensity. B is the PSF to determine how the pixels are averaged to produce the burred effect.

In this case, pixel value of 1 = white, 3/4 is bright gray (light gray), 1/2 = medium gray, 1/4 = dark gray, 0 is fully black.

The range is 1 <-> -1





Method #5

2. (20 pts) Solve and explain the solution to 7.2 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

7.2 *3-D rotation.* Let x and y be 3-vectors representing positions in 3-D. Suppose that the vector y is obtained by rotating the vector x about the vertical axis (*i.e.*, e_3) by 45° (counterclockwise, *i.e.*, from e_1 toward e_2). Find the 3×3 matrix A for which $y = Ax$. *Hint.* Determine the three columns of A by finding the result of the transformation on the unit vectors e_1, e_2, e_3.

The rotation about the vertical axis e3 by 45 degrees means the rotation is happening in the plane formed by e1 and e2 while the vertical axis e3 stays fixed. The 3D rotation matrix for rotation about the e3 axis (z-axis) by angle theta.

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\theta = 45°$

$$R_z(45°) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$e_1 = (1, 0, 0) \rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$

$e_2 = (0, 1, 0) \rightarrow \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$

$e_3 = (0, 0, 1) \rightarrow (0, 0, 1)$

So Matrix A

Method #5

3. (20 pts) Solve and Explain the solution to 7.3 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

7.3 *Trimming a vector.* Find a matrix A for which $Ax = (x_2, \ldots, x_{n-1})$, where x is an n-vector. (Be sure to specify the size of A, and describe all its entries.)

$x = n \times 1$ vector $= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ almost same

Ax is $(n-2) \times 1$ vector $= \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix}$

Matrix A must be
$(n-2)$ rows
n columns
so $(n-2) \times n$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad (n-2) \times n$$

$$A_{ij} \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}, \quad 1 \le i \le n-2, \quad 1 \le j \le n$$

Matrix A selects all components of x except the first and last, so each row of A has exactly 1 entry of value 1, positioned in a diagonal pattern to pick out the correct element of x, and all the other entries are going to be 0.

Method #5

4. (10pts) Solve and Explain the solution to 7.7 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

7.7 *Incidence matrix of reversed graph.* (See exercise 6.5.) Suppose A is the incidence matrix of a graph. The reversed graph is obtained by reversing the directions of all the edges of the original graph. What is the incidence matrix of the reversed graph? (Express your answer in terms of A.)

Since columns are edges and rows are nodes, we can represent each column to have exactly A +1 at the head (the node the edge enters), A -1 at the tail (where the node leaves), and assume 0 is everywhere else.

When we reverse the direction of all the nodes in the graph, then the directions are flipped, and the node previously receiving the edge will turn into one that is starting the node. So all the +1 will turn into -1 and vice versa.

So each element in the matrix that represents the reversed graph is just the negative of the original matrix A. So the answer should be -A.

Method #5

5. (20pts) Solve and Explain the solution to 7.9 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

7.9 *Social network graph.* Consider a group of n people or users, and some symmetric social relation among them. This means that some pairs of users are *connected*, or *friends* (say). We can create a directed graph by associating a node with each user, and an edge between each pair of friends, arbitrarily choosing the direction of the edge. Now consider an n-vector v, where v_i is some quantity for user i, for example, age or education level (say, given in years). Let $\mathcal{D}(v)$ denote the Dirichlet energy associated with the graph and v, thought of as a potential on the nodes.

(a) Explain why the number $\mathcal{D}(v)$ does not depend on the choice of directions for the edges of the graph.

(b) Would you guess that $\mathcal{D}(v)$ is small or large? This is an open-ended, vague question; there is no right answer. Just make a guess as to what you might expect, and give a short English justification of your guess.

A)

$$D(v) = \sum_{\text{edges}(i,j)} (v_i - v_j)^2$$

$$(v_i - v_j)^2 = (v_j - v_i)^2$$

D(v) depends on the differences between node attributess across edges in the summation form on the left.

Since this involves squared differences, this means that we're calculating absolute value (and the distances between vi and vj are mirrored with respect to i and j). Hence how you can get the second equation on the left.

Reversing the direction of any edge doesn't cahnge the calculated Dirichlet energy, the energy remains the same because D(v) cares more about the absolute distance, and not in a specific direction.

B)

I would guess D(v) is small assuming the attributes are measured is related to the individuals. If I think of Facebook, I think of a person's profile as their age, school, their friends, their likes. So people who are connected socially often share same characteristics, like how car enthusiasts care about car pages on Facebook.

Method #5

6. (20pts) Solve and Explain the solution to 7.10 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

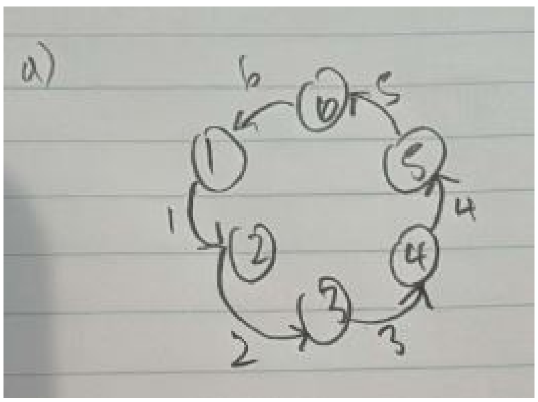
7.10 *Circle graph.* A *circle graph* (also called a *cycle graph*) has n vertices, with edges pointing from vertex 1 to vertex 2, from vertex 2 to vertex 3, $\ldots$, from vertex $n-1$ to vertex n, and finally, from vertex n to vertex 1. (This last edge completes the circle.)

(a) Draw a diagram of a circle graph, and give its incidence matrix A.

(b) Suppose that x is a circulation for a circle graph. What can you say about x?

(c) Suppose the n-vector v is a potential on a circle graph. What is the Dirichlet energy $\mathcal{D}(v) = \|A^T v\|^2$?

Remark. The circle graph arises when an n-vector v represents a periodic time series. For example, v_1 could be the value of some quantity on Monday, v_2 its value on Tuesday, and v_7 its value on Sunday. The Dirichlet energy is a measure of the roughness of such an n-vector v.



b) If x is a ciruclation, then it would mean that Ax = 0, since the circle graph has a cycle structure, Ax = 0 implies that each node has equal incoming and outgoing edges. All components of x must be equal, so x1 = x2 = ... = xn. It's a circle/cycle graph.

c) $A^T v$ is n-vector where each entry corresponds to diff. potential edges

$$A^T v = \begin{bmatrix} v_2 - v_1 \\ v_3 - v_2 \\ \vdots \\ v_n - v_{n-1} \\ v_1 - v_n \end{bmatrix}$$

so $D(v) = \|A^T v\|^2 = (v_2 - v_1)^2 + (v_3 - v_2)^2 + \ldots + (v_n - v_{n-1})^2 + (v_1 - v_n)^2$

The energy here is corresponding to the varation of the potentials around the cycle. If all vi are similar, then the energy is small, and if there is high variability, energy is large.