VMLS - Study Guide Chapter 11 - Matrix InversesName: Ben Chen

Always include this title page with your PDF. Include your name above.

- Submit your work in Gradescope as a PDF - you will identify where your "questions are."
- Identify the question number as you submit. Since we grade "blind" if the questions are NOT identified, the work WILL NOT BE GRADED and a 0 will be recorded. Always leave enough time to identify the questions when submitting.
- One section per page (if a page or less) - We prefer to grade the main solution in a single page, extra work can be included on the following page.
- Long instructions may be removed to fit on a single page.
- **Do not start a new question in the middle of a page.**
- Solutions to book questions are provided for reference.
- You may NOT submit given solutions - this includes minor modifications - as your own.
- Solutions that do not show individual engagement with the solutions will be marked as no credit and can be considered a violation of honor code.
- If you use the given solutions you must reference or explain how you used them, in particular...

For full credit, EACH book exercise in the Study Guides must use one or more of the following methods and FOR EACH QUESTION. Identify the number the method by number to ensure full credit.

Method 1 - Provide original examples which demonstrate the ideas of the exercise in addition to your solution.

Method 2 - Include and discuss the specific topics needed from the chapter and how they relate to the question.

Method 3 - Include original Python code, of reasonable length (as screenshot r text) to show how the topic or concept was explored.

Method 4 - Expand the given solution in a significant way, with additional steps and comments. All steps are justified. This is a good method for a proof for which you are only given a basic outline.

Method 5 - Attempt the exercise without looking at the solution and then the solution is used to check work. Words are used to describe the results.

Method 6 - Provide an analysis of the strategies used to understand the exercise, describing in detail what was challenging, who helped you or what resources were used. The process of understanding is described.

Method #5

1. (20 pts) Annotation/reprove/add your own explanation communicating the ideas behind the section Triangular Matrices on VMLS page 206.

Why is this result interesting and useful?
Why is this proof so much fun?

Negative matrix powers. We can now give a meaning to matrix powers with negative integer exponents. Suppose A is a square invertible matrix and k is a positive integer. Then by repeatedly applying property (11.2), we get

$$(A^k)^{-1} = (A^{-1})^k.$$

We denote this matrix as A^{-k}. For example, if A is square and invertible, then $A^{-2} = A^{-1}A^{-1} = (AA)^{-1}$. With A^0 defined as $A^0 = I$, the identity $A^{k+l} = A^k A^l$ holds for all integers k and l.

Triangular matrix. A triangular matrix with nonzero diagonal elements is invertible. We first discuss this for a lower triangular matrix. Let L be $n \times n$ and lower triangular with nonzero diagonal elements. We show that the columns are linearly independent, *i.e.*, $Lx = 0$ is only possible if $x = 0$. Expanding the matrix-vector product, we can write $Lx = 0$ as

L_{ij}

$$\begin{aligned} L_{11}x_1 &= 0 \\ L_{21}x_1 + L_{22}x_2 &= 0 \\ L_{31}x_1 + L_{32}x_2 + L_{33}x_3 &= 0 \\ &\vdots \\ L_{n1}x_1 + L_{n2}x_2 + \cdots + L_{n,n-1}x_{n-1} + L_{nn}x_n &= 0. \end{aligned}$$

Since $L_{11} \neq 0$, the first equation implies $x_1 = 0$. Using $x_1 = 0$, the second equation reduces to $L_{22}x_2 = 0$. Since $L_{22} \neq 0$, we conclude that $x_2 = 0$. Using $x_1 = x_2 = 0$, the third equation now reduces to $L_{33}x_3 = 0$, and since L_{33} is assumed to be nonzero, we have $x_3 = 0$. Continuing this argument, we find that all entries of x are zero, and this shows that the columns of L are linearly independent. It follows that L is invertible.

A similar argument can be followed to show that an upper triangular matrix with nonzero diagonal elements is invertible. One can also simply note that if R is upper triangular, then $L = R^T$ is lower triangular with the same diagonal, and use the formula $(L^T)^{-1} = (L^{-1})^T$ for the inverse of the transpose.

Inverse via QR factorization. The QR factorization gives a simple expression for the inverse of an invertible matrix. If A is square and invertible, its columns are linearly independent, so it has a QR factorization $A = QR$. The matrix Q is orthogonal and R is upper triangular with positive diagonal entries. Hence Q and R are invertible, and the formula for the inverse product gives

$$\boxed{A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T.} \tag{11.3}$$

In the following section we give an algorithm for computing R^{-1}, or more directly, the product $R^{-1}Q^T$. This gives us a method to compute the matrix inverse.

The triangular matrix proof wants to show that a matrix with nonzero diagonal elements is invertible (which means that it is undo-able), much like how dividing a number can undo the multiplication done to it.

So the strategy is to assume that Lx = 0, that simply multiplying the matrix by some vector gives the zero vector and we want to show this only happens when x = 0). You can prove that the columns are linearly independent which is the same as saying the matrix is invertible.

If you look at the left, the first equation involves just x1, and since the diagonal entry L11 is not 0, that means x1 is zero. Then you look at the next line, and see that it involves x1 and x2. But since we know x1 is zero, that means x2 must also be zero. L22x2 = 0. And so on so forth with the third equation and onward. What does this mean? It means that every step "unlocks" the next and each time, the fact that the diagonal is nonzero forces the next xi to be zero.

Why is this crazy and interesting? Because you don't need crazy techniques. This is just basic algebra and making sure the matrices are aligned in a certain way. This would also apply to inverses and transpose behavior depending on how much you want to exploit the structure of triangular matrices.

Method #5

2. (20 pts) Solve and Explain the solution to 11.2 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

Provide examples.

Yes, x does have left inverse and does not have right inverse.

Easiest way to solve this problem is to fall back to the fundamentals on what makes right and left inverse

11.2 *Left and right inverses of a vector.* Suppose that x is a nonzero n-vector with $n > 1$.

(a) Does x have a left inverse?

(b) Does x have a right inverse?

In each case, if the answer is yes, give a left or right inverse; if the answer is no, give a specific nonzero vector and show that it is not left- or right-invertible.

left inverse: $x \in \mathbb{R}^n$ is row vector $z^T \in \mathbb{R}^{1\times n}$ s.t. $z^T x = 1$

right inverse: $x \in \mathbb{R}^n$ is column vector $y \in \mathbb{R}^n$ s.t. $xy^T = I$ (not possible for scalar/vector)

since $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, $xy^T \in \mathbb{R}^{n \times n}$

and product is rank 1, so it cannot be identity if $n > 1$

A) yes, has left inverse

Example: $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ we want $z^T x = 1$

let $z = [a \quad b]$, then $a(1) + b(2) = 1 \Rightarrow a + 2b = 1$

↳ infinitely many solutions, thus left inverse

B) no, x has no right inverse

$x \in \mathbb{R}^n$, we try to find $y \in \mathbb{R}^n$

s.t. $xy^T = I_n$

compare ⇒

xy^T is outer product which means rank-1 matrix (product of 2 vectors)

I_n has rank n, xy^T cannot = identity matrix

3. (UNGRADED) Solve and Explain the solution to 11.3 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

Give an examples.

Method #5

4. (20pts) Solve and Explain the solution to 11.4 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

11.4 *Transpose of orthogonal matrix.* Let U be an orthogonal $n \times n$ matrix. Show that its transpose U^T is also orthogonal.

Orthogonal $U^TU = I$

$U^{-1} = U^T$

Show U^T is also orthogonal $(U^T)^TU^T = I$

$U^TU = I$ $\quad (U^TU)^T = U^{TT}U^T = UU^T$

so

$UU^T = I$

$(U^T)^TU^T = I$

$\therefore$ if U is orthogonal, then U^T is also orthogonal

Method #5

We are going to use Schur's complement to solve this problem.

https://www.statlect.com/matrix-algebra/Schur-complement

5. (20 points) Solve and Explain the solution to 11.5 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

11.5 *Inverse of a block matrix.* Consider the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} I & a \\ a^T & 0 \end{bmatrix},$$

where a is an n-vector.

(a) When is A invertible? Give your answer in terms of a. Justify your answer.

(b) Assuming the condition you found in part (a) holds, give an expression for the inverse matrix A^{-1}.

a) Invertibility using Schur's complement

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

if B is invertible then matrix is invertible iff Schur comp. $S = E - DB^{-1}C$ is invertible

$B = I$, so it's invertible

$C = a$, $D = a^T$, $E = 0$

$S = 0 - a^T I^{-1} a = -a^T a = -\|a\|^2$

so A is invertible iff $\|a\|^2$ is not 0

so $a \neq 0$ vector

b) $A^{-1} = \begin{bmatrix} I & a \\ a^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} I + \frac{1}{\|a\|^2} aa^T & -\frac{1}{\|a\|^2} a \\ -\frac{1}{\|a\|^2} a^T & \frac{1}{\|a\|^2} \end{bmatrix}$ ← block formula and Schur complement

Formula: $\begin{bmatrix} D^{-1} + B^{-1}CS^{-1}DB^{-1} & -B^{-1}CS^{-1} \\ -S^{-1}DB^{-1} & S^{-1} \end{bmatrix}$

where $S = E - DB^{-1}C$

$A = \begin{bmatrix} I & a \\ a^T & 0 \end{bmatrix}$

$B = I$

$C = a$

$D = a^T$

$E = 0$

$S = E - DB^{-1}C = 0 - a^T I a = -a^T a = -\|a\|^2$

$A \cdot A^{-1} = I_{n+1}$ to verify

6. (20 points) Solve and Explain the solution to 11.16 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

Be sure to add your own interpretation and explanation.

What is a "running sum" matrix? Why is it called this?

11.16 *Inverse of running sum matrix.* Find the inverse of the $n \times n$ running sum matrix,

$$S = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Does your answer make sense?

if $1/4$ Inverted to $=1$ is $4/1$

$4/1 \cdot 1/4 = 1$

then yes it makes sense in matrix form

$S^{-1}S = I$, $SS^{-1} = I$

$S \in \mathbb{R}^{n \times n}$

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \qquad y_i = \sum_{j=1}^{i} x_j$$

Cumulative Sum matrix notation

maps vector x to vector y whose entries are running sums of x

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

to recover x from y we need to find the first difference

$x_1 = y_1$

$x_2 = y_2 - y_1$

etc. etc.

row 1 y_1 as is $\rightarrow x_1 = y_1$

row 2 $y_2 - y_1 \rightarrow x_2$

row 3 $y_3 - y_2 \rightarrow x_3$

row n $y_n - y_{n-1} \rightarrow x_n$

$y = Sx$ $S^{-1}y$ is the difference between entries in y and reconstructs x

$S^{-1}(Sx) = x$

S^{-1} is the "undo-button"

A running sum matrix S maps a vector to its cumulative sums

The inverse S^-1 is the first-difference matrix, minusing the previous entries

So if the inverse exists, the multiplying of the two gives back the identity matrix

So, the answer makes sense.