VMLS - SG Chp. 8 - Systems Linear Equations Name: Ben Chen

Always include this title page with your PDF. Include your name above.

- Submit your work in Gradescope as a PDF - you will identify where your "questions are."
- Identify the question number as you submit. Since we grade "blind" if the questions are NOT identified, the work WILL NOT BE GRADED and a 0 will be recorded. Always leave enough time to identify the questions when submitting.
- One section per page (if a page or less) - We prefer to grade the main solution in a single page, extra work can be included on the following page.
- Long instructions may be removed to fit on a single page.
- **Do not start a new question in the middle of a page.**
- Solutions to book questions are provided for reference.
- You may NOT submit given solutions - this includes minor modifications - as your own.
- Solutions that do not show individual engagement with the solutions will be marked as no credit and can be considered a violation of honor code.
- If you use the given solutions you must reference or explain how you used them, in particular...

For full credit, EACH book exercise in the Study Guides must use one or more of the following methods and FOR EACH QUESTION. Identify the number the method by number to ensure full credit.

Method 1 - Provide original examples which demonstrate the ideas of the exercise in addition to your solution.

Method 2 - Include and discuss the specific topics needed from the chapter and how they relate to the question.

Method 3 - Include original Python code, of reasonable length (as screenshot r text) to show how the topic or concept was explored.

Method 4 - Expand the given solution in a significant way, with additional steps and comments. All steps are justified. This is a good method for a proof for which you are only given a basic outline.

Method 5 - Attempt the exercise without looking at the solution and then the solution is used to check work. Words are used to describe the results.

Method 6 - Provide an analysis of the strategies used to understand the exercise, describing in detail what was challenging, who helped you or what resources were used. The process of understanding is described.

Method #5

1. (20 pts) Solve and explain the solution to 8.1 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

8.1 *Sum of linear functions.* Suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ and $g : \mathbf{R}^n \to \mathbf{R}^m$ are linear functions. Their *sum* is the function $h : \mathbf{R}^n \to \mathbf{R}^m$, defined as $h(x) = f(x)+g(x)$ for any n-vector x. The sum function is often denoted as $h = f+g$. (This is another case of overloading the $+$ symbol, in this case to the sum of functions.) If f has matrix representation $f(x) = Fx$, and g has matrix representation $f(x) = Gx$, where F and G are $m \times n$ matrices, what is the matrix representation of the sum function $h = f + g$? Be sure to identify any $+$ symbols appearing in your justification.

f(x) = Fx, and g(x) = Gx <-- I assume the question had a typo.

F and G exist in R^(m x n) where m x n are matrices.

We are asked to find the matrix representation of their sum where h = f + g and where h(x) = f(x) + g(x).

If we substitute the matrix, h(x) = f(x) + g(x) = Fx + Gx... Fx + Gx = (F + G)x.

So h(x) = (F+G)x where F + G exist within R^(m x n).

Method #5

In the solution to 8.4 (b), is the correct image (the 9 boxes) shown for rotating clockwise?

2. (40 pts) Solve and Explain the solution to 8.4 here in your own words. (Since you are given a solution, you will be graded on your ability to explain). Use multiple pages.

8.4 *Linear functions of images.* In this problem we consider several linear functions of a monochrome image with $N \times N$ pixels. To keep the matrices small enough to work out by hand, we will consider the case with $N = 3$ (which would hardly qualify as an image). We represent a 3×3 image as a 9-vector using the ordering of pixels shown below.

1	4	7
2	5	8
3	6	9

(This ordering is called *column-major*.) Each of the operations or transformations below defines a function $y = f(x)$, where the 9-vector x represents the original image, and the 9-vector y represents the resulting or transformed image. For each of these operations, give the 9×9 matrix A for which $y = Ax$.

(a) Turn the original image x upside-down.

(b) Rotate the original image x clockwise 90°.

(c) Translate the image up by 1 pixel and to the right by 1 pixel. In the translated image, assign the value $y_i = 0$ to the pixels in the first column and the last row.

(d) Set each pixel value y_i to be the average of the neighbors of pixel i in the original image. By neighbors, we mean the pixels immediately above and below, and immediately to the left and right. The center pixel has 4 neighbors; corner pixels have 2 neighbors, and the remaining pixels have 3 neighbors.

A)

Pixel Position	Vector Index
x1 = (1,1)	1
x2 = (2,1)	2
x3 = (3,1)	3
x4 = (1,2)	4
x5 = (2,2)	5
x6 = (3,2)	6
x7 = (1,3)	7
x8 = (2,3)	8
x9 = (3,3)	9

When we flip the image vertically, the top row with the bottom row within each column is swapped. That means Row1 and Row3 are going to be reversed while Row2 is not changed.

New f(x) would be (x3, x2, x1, x6, x5, x4, x9, x8 x7) in that order. The textbook turns:

[1, 4, 7,
2, 5, 7,
3, 6, 9] to [3, 6, 9,
2, 5, 8,
1, 4, 7] and middle row does not change.

B) [1, 4, 7,
2, 5, 7,
3, 6, 9] turns into [3, 2, 1,
6, 5, 4,
9, 8, 7] the image in the solution is incorrect. The solution in the book is counter-clockwise rotation of the original image by 90 degrees. Not clock-wise.

C) [x1,x4,x6,
x2,x5,x8,
x3, x6, x9] translate up 1, right 1 shifts pixels up 1 row and right 1 column and pixels moving outbound are lost, hence the 0's

new matrix looks like: [0, x2, x5,
0, x3, x6,
0, 0, 0]

D) "neighbor" can be defined as above and below, but not diagonal so corner pixels has 2 neighbors, and non corners have 3, center has 4.

[1, 4, 7,
2, 5, 7,
3, 6, 9] gets normalized to: [(x2+x4)/2, (x1 + x5 + x7)/3, (x4 +x8)/2,
(x1+x3+x5)/3, (x2 + x4 + x6 + x8)/4, (x5 + x7 + x9)/3,
(x2 + x6)/2, (x3 + x5 + x9)/3, (x6 + x8)/2]

The 3x3 matrix turns into a sparse matrix because if the original image is 9x1 and the transformed image is 9x1 vector, then matrix A has to be 9x9 (81).

A =
[0, 1/2, 0, 1/2, 0, 0, 0, 0, 0,
1/3, 0, 1/3, 0, 1/3, 0, 0, 0, 0,
0, 1/2, 0, 0, 0, 1/2, 0, 0, 0,
1/3, 0, 0, 0, 1/3, 0, 1/3, 0, 0,
0, 1/4, 0, 1/4, 0, 1/4, 0, 1/4, 0,
0, 0, 1/3, 0, 1/3, 0, 0, 0, 1/3,
0, 0, 0, 1/2, 0, 0, 0, 1/2, 0,
0, 0, 0, 0, 1/3, 0, 1/3, 0, 1/3,
0, 0, 0, 0, 0, 1/2, 0, 1/2, 0]

Method #5

3. (20pts) Solve and Explain the solution to 8.7 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

8.7 *Interpolation of polynomial values and derivatives.* The 5-vector c represents the coefficients of a quartic polynomial $p(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4$. Express the conditions

$$p(0) = 0, \quad p'(0) = 0, \quad p(1) = 1, \quad p'(1) = 0,$$

as a set of linear equations of the form $Ac = b$. Is the system of equations under-determined, over-determined, or square?

We want to show Ac = b with the given conditions

1) p(0) = c1 + c2*0 + c3*0 + c4*0 + c5*0 = 0 where c1 = 0
2) p'(x) = c2 + 2c3x + 3c4x^2 + 4c5x^3 where p'(0) = c2 + 2xc3*0 +3c4*0 + 4c5*0 = c2 =0
3) p(1) = c1 + c2 + c3 + c4 + c5 = 1
4) p'(1) = c2 + 2c3 + 3c4 + 4c5 = 0

Matrix A looks like =
[1, 0, 0, 0, 0,
0, 1, 0, 0, 0,
1, 1, 1, 1, 1,
0, 1, 2, 3, 4]

b = [0, 0, 1, 0]

4. (20pts) Solve and Explain the solution to 8.9 here in your own words. (Since you are given a solution, you will be graded on your ability to explain).

8.9 *Required nutrients.* We consider a set of n basic foods (such as rice, beans, apples) and a set of m nutrients or components (such as protein, fat, sugar, vitamin C). Food j has a cost given by c_j (say, in dollars per gram), and contains an amount N_{ij} of nutrient i (per gram). (The nutrients are given in some appropriate units, which can depend on the particular nutrient.) A daily diet is represented by an n-vector d, with d_i the daily intake (in grams) of food i. Express the condition that a diet d contains the total nutrient amounts given by the m-vector n^{des}, and has a total cost B (the budget) as a set of linear equations in the variables $d_1, \ldots, d_n$. (The entries of d must be nonnegative, but we ignore this issue here.)

n = foods

m = nutrients

c = cost per gram of food (size n x 1)

Nij = nutrient i in 1 gram of food j where N is size m x n

d = daily intake of food in grams size n x 1

Nd = n^des = desired total nutrients size m x 1 = gives linear equations in d1, ..., dn

B = budget=c Transpose d (scalar) gives 1 linear equation in d1,...,dn

Matrix A = matrix with (m + 1) rows and n-columns
[N
c Transpose]

where size is (size (m + 1) x n)

b =
[n^des
B]

where size of b = (size (m + 1) x 1) and so the calculation is Ad = b (the linearity)

A size is (m + 1) x n

The calculation of the system can underdetermine if m + 1 < n, overdetermine if m + 1 > n and is just right (square) when m + 1 = b