What Makes Proper Forcing Proper?

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I wrote these notes for myself to try and better understand proper forcing. I wanted to provide an exposition of the motivation for the formulation of proper forcing through many examples, and motivate the definition of $(M, \mathbb{P})$-genericity.

We assume basic familiarity with iterated forcing.

The formulation of proper forcing began as an attempt to generalize the following fact about iterated forcing:

Theorem 1. *Any finite support iteration of ccc forcings is ccc.*

Proof. We only sketch the proof. Induct on the length of the iteration.

In the successor stage, use the fact that $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Suppose towards a contradiction that $\{(p_\xi, \dot{q}_\xi) : \xi < \omega_1\}$ is an antichain in $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Let $\dot{S} = \{(\xi, p_\xi) : \xi < \omega_1\}$, so $\dot{S}$ is a $\mathbb{P}_\alpha$ name. Clearly $\mathbb{P}_\alpha \Vdash \dot{S} \subseteq \omega_1$, and it suffices to show that $\dot{S}$ is forced to be countable. It is not too hard to see that if $\xi \neq \eta \in \dot{S}_G$ then $(\dot{q}_\xi)_G \perp (\dot{q}_\eta)_G$. Since $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha$ is ccc, $\mathbb{P}_\alpha \Vdash \dot{S}$ is countable.

In the limit case, take $\{p_\alpha : \alpha < \omega_1\} \subseteq \mathbb{P}_\alpha$ and use the delta system lemma to stabilize the (finite!) supports of the p_α's on an uncountable $B \subseteq \omega_1$. Now there is some fixed γ such that for all $\alpha \in B$, $\text{supp}(p_\alpha) \subseteq \gamma$. Then if $\{p_\alpha : \alpha \in B\}$ formed an antichain, it is not too hard to see that $\{p_\alpha \restriction \gamma : \alpha \in B\}$ forms an antichain in $\mathbb{P}_\gamma$, but $\mathbb{P}_\gamma$ is ccc. $\square$

This fact is the key in Solovay-Tenenbaum's proof of the consistency of Suslin's hypothesis (the first application of iterated forcing), as it follows that ω_1 is preserved. The idea is that it's easy to destroy one Suslin tree: the Suslin tree is itself a ccc partial order, so forcing with a Suslin tree will preserve cardinals, and the union over the generic filter will be a branch through the tree. The problem is that this necessarily adds another Suslin tree, so we can never force Suslin's hypothesis by killing one Suslin tree at a time. We need to do an iterated forcing where we use some suitable bookkeeping mechanism to ensure that we kill *every* Suslin tree at some point, thereby "catching our tail". Of course if we end up collapsing ω_1, we will not have accomplished anything. This is why the above iteration theorem for ccc forcings is essential for the construction. For completeness, we sketch the argument in a bit more detail below.

Theorem 2. *Con(ZFC) $\implies$ Con(ZFC + SH).*

Proof. Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Fix a bijection $\pi : \aleph_2 \to \aleph_2 \times \aleph_2$ such that $\pi(\xi) = (\zeta, \mu)$ with $\gamma < \alpha$. We construct a finite support iteration $\mathbb{P}$ of ccc forcings.

Since we use finite supports, we only need to specify what happens at successor stages: if δ is limit, $p \in \mathbb{P}_\delta$, and $p \Vdash \varphi$ then there is $\delta' < \delta$ such that $\operatorname{supp}(p) \subseteq \delta'$, so $p \restriction \delta' \in \mathbb{P}_{\delta'}$ and $p \restriction \delta' \Vdash \varphi$ is p is trivial past δ'.

So suppose we have defined $\mathbb{P}_\alpha$ where $\alpha < \kappa$ and $f(\alpha) = (\beta, \gamma)$. Let $1_{\mathbb{P}} \Vdash \dot{\lhd}$ is a well-order of the Suslin trees in the generic extension by $\mathbb{P}_\gamma$. Then let $1_{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is the βth Suslin tree in generic extension by $\mathbb{P}_\gamma$, according to $\dot{\lhd}$.

Let $G \subseteq \mathbb{P}$ be generic. First, since we iterate ccc forcings with finite supports, the whole iteration is ccc, so all cardinals and cofinalities are preserved.

Now to check that there are no Suslin trees in $V[G]$. So suppose $T \in V[G]$ is Suslin, so there is $p \in G$ such that $p \Vdash \dot{T}$ is Suslin. Again, since we use finite supports, there is some $\gamma < \aleph_2$ such that $\operatorname{supp}(p) \subseteq \gamma$, so $p \restriction \gamma \in P_\gamma$ and $p \restriction \gamma$, so that $T \in V[G_\gamma]$, where $G_\gamma \subseteq \mathbb{P}$ is generic for $\mathbb{P}_\gamma$. Say T is a the βth Suslin tree in $V[G_\gamma]$ according to $\dot{\lhd}_{G_\gamma}$. Since π is a bijection, there is some $\alpha > \gamma$ such that $\pi(\alpha) = (\beta, \gamma)$. Then $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = T$, by our construction, so we already killed T at stage α. So T is not Suslin in $V[G]$.

Hence $V[G]$ satisfies Suslin's hypothesis. $\square$

So we are able to do iterated forcing to get a model of Suslin's hypothesis. In general, we wish to do the following:

Goal 1. *Let φ be a property of subsets of ω_1. I want to show it is consistent that there is* no *subset of ω_1 with property φ. To do this, I want to concoct an iteration to destroy* all *subsets of ω_1 with the property φ, all while crucially preserving ω_1.*

But ccc forcings are too restrictive, and finite support iterations of non-ccc forcings will collapse ω_1, as the next two examples show.

Proposition 1. *Suppose $V \models \diamondsuit$, $\mathbb{P}$ is ccc, and $|\mathbb{P}| \leq \aleph_1$. Then $V[G] \models \diamondsuit$.*

Proof. Let $\vec{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$ sequence. Since $|\mathbb{P}| = \aleph_1$, we may assume that $\mathbb{P}$ is a partial order on $\aleph_1$. In $V[G]$, let $B_\alpha = (\check{A}_\alpha)_G \cap \alpha$.

We will show that $\vec{B}$ is a $\diamondsuit$ sequence in $V[G]$. Let $X \subseteq \omega_1$ in $V[G]$. We may assume $|\dot{X}| = \omega_1$, so let $X^* \subseteq \omega_1$ code $\dot{X}$ in V. Let S be stationary so that $X \cap \alpha = A_\alpha$ for all $\alpha \in S$. Since $\mathbb{P}$ is ccc S remains stationary in $V[G]$, and $B_\alpha = B_\alpha = (\check{A}_\alpha)_G \cap \alpha = A_\alpha \cap \alpha = X \cap \alpha$ for all $\alpha \in S$. $\square$

This shows, for example, that we can't iterate small ccc forcings for length ω_1 and violate $\diamondsuit$. But using finite supports when iterating non-ccc forcings is too destructive.

Proposition 2. *Suppose $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ is a finite support iteration where $\mathbb{P}_n \Vdash \dot{\mathbb{Q}}_n$ is* not *ccc for all $n \in \omega$. Then in $V[G]$, ω_1 is collapsed.*

Proof. For each n, let $\mathbb{P}_n \Vdash \dot{A}_n$ is an antichain in $\dot{\mathbb{Q}}$ of size $\aleph_1$. Working in $V[G]$, partition each $A_n = \bigsqcup_{\alpha<\omega_1} A_n^\alpha$ into countably many pieces. Then define $f : \omega \to \omega_1$ by $f(n) = i$ iff $G \restriction n \cap A_n^i \neq \emptyset$. Note that f is well defined since for $i \neq j$, $A_n^i \cap A_n^j = \emptyset$, and G must meet one A_n^i for all n by genericity. A simple genericity argument shows that f is onto, so ω_1 is collapsed. □

Note that if each $\dot{\mathbb{Q}}_n$ is forced to be σ-closed, then ω_1 is preserved at every step of the iteration, and only gets collapsed at the end.

The dream would be that some iteration of ω_1-preserving posets is ω_1-preserving. Unfortunately, no choice of supports for iteration ω_1-preserving posets will be guaranteed to preserve ω_1, as the following example shows.

Proposition 3. *There is an iteration of countable length of ω_1-preserving forcings that collapses ω_1, with any choice of supports.*

Proof. First we define the partial order. For a stationary, co-stationary $S \subseteq \omega_1$ let $\mathbb{P}_S$ be the partial order consisting of closed, bounded subsets of S. $\mathbb{P}_S$ is ordered by end extension. It is easy to see that if ω_1 is preserved then in $V[G]$, $C = \bigcup G$ is a club subset of S, so that the complement of S ceases to be stationary.

Claim 1. *$\mathbb{P}_S$ preserves ω_1 (in fact, $\mathbb{P}_S$ adds no new ω-sequences).*

Proof of claim. To see that ω_1 is preserved, let $p \Vdash \dot{f} : \omega \to Ord$. Fix a countable $M \prec \langle H_\lambda, \in, \mathbb{P}, p, \dot{f}, \lhd \rangle$ where λ is sufficiently large and $\lhd$ is a wellorder of H_λ such that $M \cap \omega_1 = \delta \notin S$. This last requirement is no issue since S is co-stationary (hence not a club). Let $\langle \delta_n : n < \omega \rangle$ be increasing and cofinal in δ. Build a chain of conditions $p_0 = p$ and p_{n+1} is the $\lhd$-least condition below p_n such that there is some β_n so that $p_{n+1} \Vdash \dot{f}(n) = \beta_n$ and $\max p_{n+1} \geq \delta_n$. By elementarity, $\langle p_n : n < \omega \rangle \in M$, so that $\max_{n<\omega} p_n \leq \delta$. Then taking $q = \bigcup_{n<\omega} n \cup \{\delta\}$ then $q \in \mathbb{P}$ and for all n, $q \Vdash \dot{f}(n) = \beta_n$. Hence $\dot{f}_G \in V$ for any generic G. ⊣

Now we use $\mathbb{P}_S$ to construct our example. Partition $\omega_1 = \bigsqcup_{n<\omega} S_n$ into countably many stationary sets. Consider the iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ where $\dot{\mathbb{Q}}_n$ names the partial order to shoot a club through S_n. Note that each $\dot{\mathbb{Q}}_n$ is ω-distributive and hence preserves ω_1. No matter what supports we take in the iteration, we will have that $\omega_1^V \setminus S_n$ is nonstationary for all n yet ω_1^V is now a countable union of nonstationary sets: $\omega_1^V = \bigcup_{n<\omega} \omega_1^V \setminus S_n$, a contradiction. □

So we can't iterate ω_1-preserving forcings, but this example leads one to consider the class of forcings that preserve stationary subsets of ω_1. We call such forcings *stationary set-preserving.* This study of this class is very subtle and the class's iteration scheme is quite intricate. The naive attempt is to iterate stationary set-preserving forcings with countable support, which fails.

Proposition 4. *There is an iteration of length ω of stationary set-preserving forcings that taking either finite or countable supports collapses ω_1.*

Proof. To simplify the proof that our example is stationary set-preserving, we assume CH.

Let f_η denote the ηth canonical function on ω_1, for $\eta < \omega_2$, i.e. $f_0(\alpha) = 0$, $f_{\eta+1}(\alpha) = f_\eta(\alpha)+1$, and for limit η, f_η is a least upper bound of every $f_\xi, \xi < \eta$ in $\leq_{NS}$.

Let g be an upper bound of all the f_η's in $\leq_{NS}$. Define $\mathbb{P}_g$ to be triples $(h, c, \{c_\eta : \eta \in A\})$ where:

1. $h : \alpha + 1 \to \omega_1$ for some $\alpha < \omega_1$
2. c, c_η are closed in $\alpha + 1$
3. A is countable, and $h < g$ on c and $f_\eta < h$ on c_η

We say $(h, c, \{c_\eta : \eta \in A\}) \leq (f, b, \{b_\eta : \eta \in B\})$ iff $h \supseteq f$, c end-extends b, and $A \supseteq B$.

Claim 2. *$\mathbb{P}_g$ preserves stationary subsets of ω_1.*

Proof of claim. Let $S \in V$ be stationary and $p \in \mathbb{P}$ such that $p \Vdash \dot{C}$ is club. Let λ be sufficiently large and $M \prec \langle H_\lambda, \in, \mathbb{P}, \dot{C}, S, P\rangle$ such that $|M| = \aleph_1$ and $M^\omega \subseteq M$, which is possible since we assume CH.

We may also assume that there is an increasing, continuous chain $\langle N_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of M such that $\bigcup_{\alpha<\omega_1} N_\alpha = M$ and for all $\beta < \omega_1$, $\langle N_\alpha : \alpha < \beta\rangle \in N_\beta$.

Let $\delta = M \cap \omega_2$, and let $\langle \delta_i : i < \omega_1\rangle$ be increasing and cofinal in δ. For every $i < \omega_1$ let C_i be such that $f_{\delta_i} <_{NS} f_\delta$.

Similarly, let D_0 be a club so that $f_\delta(\alpha) < g(\alpha)$ for all $\alpha \in D_0$. Let D_1 be the club of α such that $N_\alpha \cap \delta = \langle \delta_i : i < \alpha\rangle$. Further let $D_2 = \{\alpha < \omega_1 : \forall (i < \alpha)\, i \in C_\alpha\}$ Finally take $D = D_0 \cap D_1 \cap D_2$ and note that D is club. Choose some $\xi \in D$ and let $\langle \xi_n : n < \omega\rangle$ be cofinal in ξ.

Now we define a descending sequence of conditions $\langle p_n : n < \omega\rangle \subseteq M$. Let $p_0 = p$ and $p_{n+1} \leq p_n$ such that $\xi_n \subseteq \mathrm{dom}(p_{n+1})$ and $p_{n+1} \Vdash \gamma_n \in \dot{C}$ for some $\gamma_n \geq \xi_n$.

Now let take p^* to be the union of all the p_n's while also defining $h(\xi) = f_\delta(\xi)$, where h is the function part of p^*. Now $p^* \Vdash \xi \in \dot{C}$ since $\dot{C}$ is closed and $\xi \in S$. $\dashv$

Now consider the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega\rangle$ where $\mathbb{P}_0 = \mathbb{P}_g$, and if f_n is generic function added by $\dot{\mathbb{Q}}_n$ then $\mathbb{P}_{n+1} \Vdash \dot{\mathbb{Q}}_{n+1} = \mathbb{P}_{f_n}$. Then in $V[G]$, for all $n < \omega$, there is a club C_n such that for all $\alpha \in C_n$, $f_{n+1}(\alpha) < f_n(\alpha)$. If ω_1 were preserved by the iteration, $D = \bigcap_{n<\omega} C_n$ would be club. In particular, there would be an $\alpha \in D$, so $f_0(\alpha) > f_1(\alpha) > \cdots$, a contradiction. Hence the iteration collapses ω_1. $\square$

However, Shelah found that the following more restricting class works with countable support:

Definition 3. A forcing $\mathbb{P}$ is **proper** if $\mathbb{P}$ preserves all stationary $S \subseteq [\lambda]^\omega$ for all uncountable λ.

Notice that ω_1 is club in $[\omega_1]^\omega$, so proper implies stationary set-preserving, and hence ω_1-preserving. In fact, one can show the following:

Proposition 5. *If $\mathbb{P}$ is proper then every countable set of ordinals in $V[G]$ is contained in a countable set in V (i.e. proper forcings have the ω_1-cover property).*

Proof. Let X be a countable set of ordinals in $V[G]$ and λ regular uncountable in V such that $X \subseteq \lambda$. Then $([\lambda]^\omega)^V$ remains stationary in $V[G]$ and hence meets the club $\{A \subseteq [\lambda]^\omega : A \supseteq X\}$ in $V[G]$. Hence there is some $A \in ([\lambda]^\omega)^V$ such that $X \subseteq A$. □

So if $\mathbb{P}$ is proper and does not add reals then $\mathbb{P}$ adds no new sequences of ordinals.

The proofs that ccc and σ-closed forcings preserve stationary subsets of ω_1 easily generalize to show that they are proper.

Proposition 6. *Both ccc and σ-closed forcings are proper.*

Proof. First we prove that ccc forcings are proper. Let $p \Vdash \dot{C} \subseteq [\lambda]^\omega$ is club. We use the fact that there $\dot{F} : \lambda^{<\omega} \to \omega$ such that $p \Vdash \dot{C}$ contains the closure points of $\dot{F}$. In V, let $f : \lambda^{<\omega} \to [\lambda]^\omega$ where

$$f(e) = \{\alpha < \lambda : (\exists q \leq p)\, \dot{F}(e) = \alpha$$

By the ccc, $|f(e)|$ is countable for all e. Let D be the club of closure points of f. Then $p \Vdash \forall e\, \dot{F}(e) \in f(e)$ so $p \Vdash D \subseteq \dot{C}$. Since every club in the extension contains a ground model club, all stationary sets are preserved.

Now we shall prove that σ-closed forcings are proper. Let S be stationary, and $p \Vdash \dot{F} : \lambda^{<\omega} \to \lambda$. We find an extension of p that forces S to contain a closure point of $\dot{F}$.

There is some $M \prec H_\lambda$ countable such that $N \cap \lambda \in S$. Let $N \cap \lambda = x$. Enumerate $x^{<\omega} = \{e_n : n < \omega\}$. We construct a descending sequence of conditions $p_{n+1} \leq p_n$ such that $p_0 = p$ and for every n there is an α_n such that $p_{n+1} \Vdash \dot{F}(e_n) = \alpha_n$. Using closure of the poset, let q be a lower bound of the p_n's. Then $q \Vdash \dot{F}\text{"}x^{<\omega} \subseteq x$. □

Note that the proof of properness of σ-closed forcings can be generalized to $< \omega_1$ strategically-closed forcings.

It is also easy to see that if $\mathbb{P}$ is proper and $\mathbb{P} \Vdash \dot{\mathbb{Q}}$ is proper, then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper. Although properness is easily preserved by two-step iteration, properness is not necessarily preserved under *products*.

Proposition 7. *There are proper forcings $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not proper, and in fact collapses ω_1.*

Proof. Let $\mathbb{P} = \omega_2^{<\omega_1}$, so $\mathbb{P}$ adds a surjection from ω_1 to ω_2. It is easy to see that $\mathbb{P}$ is σ-closed, so $\mathbb{P}$ is proper.

Let $\mathbb{Q}_0 = \mathrm{Add}(\omega, \omega_2) \times \mathrm{Col}(\omega_1, 2^{\omega_1})$. After forcing with $\mathbb{Q}_0$, the cofinality of ω_2^V becomes ω_1 so $\mathbb{P}$ has no branch with supremum ω_2. By an argument of Silver, no new ω_1-branches are added to $\mathbb{P}$ so $\mathbb{P}$ has at most ω_1 many cofinal branches. So let $\mathbb{Q}_1$ be the Baumgartner's forcing to seal the branches of $\mathbb{P}$. Note also that $\mathbb{Q}_0$ is of the form ccc $* \sigma$-closed, so that $\mathbb{Q}_0$ is proper. Also, $\mathbb{Q}_1$ is ccc and hence proper. Then $\mathbb{Q} = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$ is proper.

However, $\mathbb{P} \times \mathbb{Q}$ is not proper, as it collapses ω_1. $\square$

Proper forcings end up being the right class for a nice iteration theory, as Shelah proved that any *countable* support iteration of proper forcings remain proper. As far as I know, there is no proof of this theorem using the above formulation of properness. However, Shelah was able to generalize an analogue of the following model-theoretic characterization of ccc:

Lemma 1. *Let $\mathbb{P}$ be any notion of forcing. The following are equivalent:*

1. *$\mathbb{P}$ is ccc.*

2. *For all sufficiently large λ and countable $M \prec H_\lambda$, if $\dot{G}$ names the generic filter then $1_{\mathbb{P}} \Vdash \dot{G} \cap M$ is M-generic.*

Proof. ($1 \Rightarrow 2$). Let $M \prec H_\lambda$ be countable, $D \in M$ dense. If $A \in M$ is a maximal antichain, then A must be countable by the ccc. Consider $D = \{p \in \mathbb{P} : \exists q \in A$ such that $q \leq p\}$. Then $D \in M$ and D is dense. So $1_{\mathbb{P}} \Vdash \exists q \in \dot{G} \cap D$, and since $\dot{G}$ is upwards closed, $1_{\mathbb{P}} \Vdash \exists p \in \dot{G} \cap A$. So $1_{\mathbb{P}}$ forces $\dot{G}$ to meet every maximal antichain of M.

($2 \Rightarrow 1$). Suppose $\mathbb{P}$ is not ccc and A is a maximal antichain, so A is uncountable. Let $M \prec H_\lambda$ be countable with $A \in M$. Choose some $p \in A \setminus M$. Since any generic G contains exactly one element of A, if $p \in G$ then $G \cap A \cap M = \emptyset$ so $p \Vdash \dot{G} \cap M$ is not M-generic. $\square$

The condition in 2. suggests the following definition:

Definition 4. Let $\mathbb{P}$ be a partial order and $p \in \mathbb{P}$. For λ sufficiently large and $M \prec H_\lambda$ countable, we say p is $(M, \mathbb{P})$-generic if $p \Vdash \dot{G} \cap M$ is M-generic, where $\dot{G}$ names the generic filter.

So $\mathbb{P}$ is ccc iff $1_{\mathbb{P}}$ is $(M, \mathbb{P})$-generic for every countable M. The proof of the last lemma easily generalizes to show the following:

Lemma 2. *Let $\mathbb{P}$ be any forcing, $p \in \mathbb{P}$, and λ sufficiently large, and $M \prec H_\lambda$ countable. The following are equivalent:*

1. *p is (M, P)-generic (i.e. $p \Vdash \dot{G} \cap M$ is M-generic)*

2. *For all maximal antichains $A \subseteq \mathbb{P}$, $A \cap M$ is predense below p, i.e. for all $q \leq p$ there is $r \in A \cap M$ such that $r \not\perp q$.*

Then one can prove the following model-theoretic characterization of properness which is used in proving the iteration theorem, and is generally easier to verify.

Lemma 3. *Let $\mathbb{P}$ be any forcing. The following are equivalent:*

1. *$\mathbb{P}$ is proper.*
2. *For all sufficiently large λ there is a club of countable $M \prec H_\lambda$ such that for all $p \in M$ there is a $q \leq p$ that is (M, P)-generic.*

Proof. ($1 \Rightarrow 2$). Suppose towards a contradiction that there is a stationary $S \subseteq [H_\lambda]^\omega$ such that for all $M \in S$, there is a $p \in M$ with no $q \leq p$ that is $(M, \mathbb{P})$-generic. We may assume that there is one fixed p for all $M \in S$: if $f : S \to H_\lambda$ by $f(M) = p$ where $p \in M$ is such that there is no $(M, \mathbb{P})$-generic $q \leq p$, we see f is regressive and by Fodor constant on a stationary set.

Now let G be generic with $p \in G$ and work in $V[G]$. For every maximal antichain A below p, let $q_A \in G \cap A$. Note that q_A is unique. Let

$$C = \{M \prec H_\lambda^V : \text{if } A \in M \text{ then } q_A \in M\}$$

It is easy to see that C is club. Since S remains stationary, in $V[G]$ there is an $M \in S \cap C$. For each $A \in M$, since $q_A \in M$ there is an upper bound q_A^M of $A \cap M$ in G. By genericity, there is a lower bound q^* of all the q_A^M's. Then $q^* \leq p$ is $(M, \mathbb{P})$-generic, but $M \in S$, a contradiction.

($2 \Rightarrow 1$). Let $S \subseteq [\lambda]^\omega$ be stationary and let $p \Vdash \dot{F} : \lambda^{<\omega} \to \lambda$. We want to show that there is a $q \leq p$ that forces S to contain a closure point of $\dot{F}$. Let $\mu \geq \lambda$ be sufficiently large and $C \subseteq [H_\mu]^\omega$ club so that for all $M \in C$ and $p \in M$, there is an $(M, \mathbb{P})$-generic $q \leq p$. Note that $\{M \cap \lambda : M \in C\}$ contains a club in $[\lambda]^\omega$, so there is some $M \in C$ with $M \cap \lambda \in S$ and $p \in M$.

Let $q \leq p$ be $(M, \mathbb{P})$-generic. We shall show that $q \Vdash M \cap \lambda$ is closed under $\dot{F}$. To this end let $e \in (M \cap \lambda)^{<\omega}$. Let

$$A = \{r \in M : r \leq p \text{ and } \exists \alpha\ r \Vdash \dot{F}(e) = \alpha\}$$

So that $A \in M$ and A is a maximal antichain below p. Suppose that $r \leq q$ and $r \Vdash \dot{F}(e) = \alpha$. Since $A \cap M$ is predense below q (by $(M, \mathbb{P})$-genericity), there is an $r' \in A$ compatible with r. By the definability lemma, $\alpha \in M$. $\square$

This characterization of properness is generally easier to check. For example, we can now easily prove that Axiom A forcings are proper:

Proposition 8. *Axiom A forcings are proper.*

Proof. Suppose $\mathbb{P}$ is Axiom A. We recall that since $\mathbb{P}$ is Axiom A, there is a sequence of partial orderings $\{\leq_n : n < \omega\}$ on $\mathbb{P}$ such that $p \leq_0 q$ implies $p \leq q$ and for all n, $p \leq_{n+1} q$ implies $p \leq q$ and

1. If $\langle p_n : n < \omega \rangle$ is a sequence such that $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 p_3 \geq \cdots$ then there is a q such that $q \leq_n p_n$ for all n.
2. For all $p \in \mathbb{P}$ and n and every ordinal name $\dot{\alpha}$ there is $q \leq_n p$ and a countable set B such that $q \Vdash \dot{\alpha} \in \check{B}$.

Now let M be countable and $p \in M$. Enumerate the maximal antichains in M as $\langle A_n : n < \omega \rangle$. We define a sequence $p_{n+1} \leq_n p_n$ with $p_0 = p$. Take p_{n+1} such that $\{a \in A_n : a \perp p_{n+1}\}$ is countable. Now let $q \leq_n p_n$ for all n.

We claim that q is $(M, \mathbb{P})$-generic: For every n, since A_n is predense, also $\{a \in A_n : a \perp q\} \subseteq M$ is predense. $\square$

This characterization is also key in verifying that a countable support iteration of proper forcings is proper. For a proof of the properness iteration theorem, we refer the reader to chapter 3 of [She17], chapter 31 of [Jec03], or Uri Abraham's chapter in the Handbook of Set Theory [Abr10].

Theorem 5. *Any countable support iteration of proper forcings is proper.*

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