What Makes Proper Forcing Proper?

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I wrote these notes for myself to try and better understand proper forcing. I wanted to provide an exposition of the motivation for the formulation of proper forcing through many examples, and motivate the definition of $(M, \mathbb{P})$-genericity.

We assume basic familiarity with iterated forcing.

The formulation of proper forcing began as an attempt to generalize the following fact about iterated forcing:

Theorem 1. *Any finite support iteration of ccc forcings is ccc.*

Proof. We only sketch the proof. Induct on the length of the iteration.

In the successor stage, use the fact that $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Suppose towards a contradiction that $\{(p_\xi, \dot{q}_\xi) : \xi < \omega_1\}$ is an antichain in $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Let $\dot{S} = \{(\xi, p_\xi) : \xi < \omega_1\}$, so $\dot{S}$ is a $\mathbb{P}_\alpha$ name. Clearly $\mathbb{P}_\alpha \Vdash \dot{S} \subseteq \omega_1$, and it suffices to show that $\dot{S}$ is forced to be countable. It is not too hard to see that if $\xi \neq \eta \in \dot{S}_G$ then $(\dot{q}_\xi)_G \perp (\dot{q}_\eta)_G$. Since $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha$ is ccc, $\mathbb{P}_\alpha \Vdash \dot{S}$ is countable.

In the limit case, take $\{p_\alpha : \alpha < \omega_1\} \subseteq \mathbb{P}_\alpha$ and use the delta system lemma to stabilize the (finite!) supports of the p_α's on an uncountable $B \subseteq \omega_1$. Now there is some fixed γ such that for all $\alpha \in B$, $\text{supp}(p_\alpha) \subseteq \gamma$. Then if $\{p_\alpha : \alpha \in B\}$ formed an antichain, it is not too hard to see that $\{p_\alpha \restriction \gamma : \alpha \in B\}$ forms an antichain in $\mathbb{P}_\gamma$, but $\mathbb{P}_\gamma$ is ccc. $\square$

This fact is the key in Solovay-Tenenbaum's proof of the consistency of Suslin's hypothesis (the first application of iterated forcing), as it follows that ω_1 is preserved. The idea is that it's easy to destroy one Suslin tree: the Suslin tree is itself a ccc partial order, so forcing with a Suslin tree will preserve cardinals, and the union over the generic filter will be a branch through the tree. The problem is that this necessarily adds another Suslin tree, so we can never force Suslin's hypothesis by killing one Suslin tree at a time. We need to do an iterated forcing where we use some suitable bookkeeping mechanism to ensure that we kill *every* Suslin tree at some point, thereby "catching our tail". Of course if we end up collapsing ω_1, we will not have accomplished anything. This is why the above iteration theorem for ccc forcings is essential for the construction. For completeness, we sketch the argument in a bit more detail below.

Theorem 2. *Con(ZFC) $\implies$ Con(ZFC + SH).*

Proof. Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Fix a bijection $\pi : \aleph_2 \to \aleph_2 \times \aleph_2$ such that $\pi(\xi) = (\zeta, \mu)$ with $\gamma < \alpha$. We construct a finite support iteration $\mathbb{P}$ of ccc forcings.

Since we use finite supports, we only need to specify what happens at successor stages: if δ is limit, $p \in \mathbb{P}_\delta$, and $p \Vdash \varphi$ then there is $\delta' < \delta$ such that $\mathrm{supp}(p) \subseteq \delta'$, so $p \restriction \delta' \in \mathbb{P}_{\delta'}$ and $p \restriction \delta' \Vdash \varphi$ is p is trivial past δ'.

So suppose we have defined $\mathbb{P}_\alpha$ where $\alpha < \kappa$ and $f(\alpha) = (\beta, \gamma)$. Let $1_\mathbb{P} \Vdash \dot{\lhd}$ is a well-order of the Suslin trees in the generic extension by $\mathbb{P}_\gamma$. Then let $1_\mathbb{P} \Vdash \dot{\mathbb{Q}}$ is the βth Suslin tree in generic extension by $\mathbb{P}_\gamma$, according to $\dot{\lhd}$.

Let $G \subseteq \mathbb{P}$ be generic. First, since we iterate ccc forcings with finite supports, the whole iteration is ccc, so all cardinals and cofinalities are preserved.

Now to check that there are no Suslin trees in $V[G]$. So suppose $T \in V[G]$ is Suslin, so there is $p \in G$ such that $p \Vdash \dot{T}$ is Suslin. Again, since we use finite supports, there is some $\gamma < \aleph_2$ such that $\mathrm{supp}(p) \subseteq \gamma$, so $p \restriction \gamma \in P_\gamma$ and $p \restriction \gamma$, so that $T \in V[G_\gamma]$, where $G_\gamma \subseteq \mathbb{P}$ is generic for $\mathbb{P}_\gamma$. Say T is a the βth Suslin tree in $V[G_\gamma]$ according to $\dot{\lhd}_{G_\gamma}$. Since π is a bijection, there is some $\alpha > \gamma$ such that $\pi(\alpha) = (\beta, \gamma)$. Then $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = T$, by our construction, so we already killed T at stage α. So T is not Suslin in $V[G]$.

Hence $V[G]$ satisfies Suslin's hypothesis. $\square$

Now we abstract the framework used above:

Goal 1. *Let φ be a property of subsets of ω_1. I want to concoct an iteration to destroy* all *subsets of ω_1 with the property φ, all while preserving ω_1.*

But ccc forcings are too restrictive, and finite support iterations of non-ccc forcings will collapse ω_1, as the next two examples show.

Proposition 1. *Suppose $V \models \diamondsuit$, $\mathbb{P}$ is ccc, and $|\mathbb{P}| \leq \aleph_1$. Then $V[G] \models \diamondsuit$.*

Proof. Let $\vec{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$ sequence. Since $|\mathbb{P}| = \aleph_1$, we may assume that $\mathbb{P}$ is a partial order on $\aleph_1$. In $V[G]$, let $B_\alpha = (\check{A}_\alpha)_G \cap \alpha$.

We will show that $\vec{B}$ is a $\diamondsuit$ sequence in $V[G]$. Let $X \subseteq \omega_1$ in $V[G]$. We may assume $|\dot{X}| = \omega_1$, so let $X^* \subseteq \omega_1$ code $\dot{X}$ in V. Let S be stationary so that $X \cap \alpha = A_\alpha$ for all $\alpha \in S$. Since $\mathbb{P}$ is ccc S remains stationary in $V[G]$, and $B_\alpha = B_\alpha = (\check{A}_\alpha)_G \cap \alpha = A_\alpha \cap \alpha = X \cap \alpha$ for all $\alpha \in S$. $\square$

This shows, for example, that we can't iterate small ccc forcings for length ω_1 and violate $\diamondsuit$. But using finite supports when iterating non-ccc forcings is too destructive.

Proposition 2. *Suppose $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ is a finite support iteration where $\mathbb{P}_n \Vdash \dot{\mathbb{Q}}_n$ is* not *ccc for all $n \in \omega$. Then in $V[G]$, ω_1 is collapsed.*

Proof. For each n, let $\mathbb{P}_n \Vdash \dot{A}_n$ is an antichain in $\dot{\mathbb{Q}}$ of size $\aleph_1$. Partition each $\dot{A}_n = \bigsqcup_{i<\omega_1} \dot{A}_n^i$ into countably many pieces. In $V[G]$, define $f : \omega \to \omega_1$ by $f(n) = i$ iff $G \restriction n \cap A_n^i \neq \emptyset$. Note that f is well defined since for $i \neq j$, $A_n^i \cap A_n^j = \emptyset$, and G must meet one A_n^i for all n by genericity. A simple genericity argument shows that f is onto, so ω_1 is collapsed. $\square$

Note that if each $\dot{\mathbb{Q}}_n$ is forced to be σ-closed, then ω_1 is preserved at every step of the iteration, and only gets collapsed at the end.

The dream would be that some iteration of ω_1-preserving posets is ω_1-preserving. Unfortunately, no kind of iteration ω_1-preserving posets will be guaranteed to preserve ω_1, as the following example shows.

Proposition 3. *There is an iteration of countable length of ω_1-preserving forcings that collapses ω_1, with any choice of supports.*

Proof. First we define the partial order. For a stationary, co-stationary $S \subseteq \omega_1$ let $\mathbb{P}_S$ be the partial order consisting of closed, bounded subsets of S. $\mathbb{P}_S$ is ordered by end extension. It is easy to see that if ω_1 is preserved then in $V[G]$, $C = \bigcup G$ is a club subset of S, so that the complement of S ceases to be stationary.

Claim 1. *$\mathbb{P}_S$ preserves ω_1 (in fact, $\mathbb{P}_S$ adds no new ω-sequences).*

Proof of claim. To see that ω_1 is preserved, let $p \Vdash \dot{f} : \omega \to Ord$. Fix a countable $M \prec \langle H_\lambda, \in, \mathbb{P}, p, \dot{f}, \lhd \rangle$ where λ is sufficiently large and $\lhd$ is a wellorder of H_λ such that $M \cap \omega_1 = \delta \notin S$. This last requirement is no issue since S is co-stationary (hence not a club). Let $\langle \delta_n : n < \omega \rangle$ be increasing and cofinal in δ. Build a chain of conditions $p_0 = p$ and p_{n+1} is the $\lhd$-least condition below p_n such that there is some β_n so that $p_{n+1} \Vdash \dot{f}(n) = \beta_n$ and $\max p_{n+1} \geq \delta_n$. By elementarity, $\langle p_n : n < \omega \rangle \in M$, so that $\max_{n<\omega} p_n \leq \delta$. Then taking $q = \bigcup_{n<\omega} n \cup \{\delta\}$ then $q \in \mathbb{P}$ and for all n, $q \Vdash \dot{f}(n) = \beta_n$. Hence $\dot{f}_G \in V$ for any generic G. $\dashv$

Now we use $\mathbb{P}_S$ to construct our example. Partition $\omega_1 = \bigsqcup_{n<\omega} S_n$ into countably many stationary sets. Consider the iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ where $\dot{\mathbb{Q}}_n$ names the partial order to shoot a club through S_n. Note that each $\dot{\mathbb{Q}}_n$ is ω-distributive and hence preserves ω_1. No matter what supports we take in the iteration, in $V[G]$ we will have added clubs $C_n \subseteq S_n$ whose (countable) intersection is empty. So we must have collapsed ω_1. $\square$

So we can't iterate ω_1-preserving forcings, but this example leads one to consider the class of forcings that preserve stationary subsets of ω_1. We call such forcings *stationary set-preserving.* This study of this class is very subtle and the class's iteration scheme is quite intricate. The naive attempt is to iterate stationary set-preserving forcings with countable support, which fails.

Proposition 4. *There is an iteration of length ω of stationary set-preserving forcings that taking either finite or countable supports collapses ω_1.*

Proof. To simplify the proof that our example is stationary set-preserving, we assume CH.

Let f_η denote the ηth canonical function on ω_1, for $\eta < \omega_2$, i.e. $f_0(\alpha) = 0$, $f_{\eta+1}(\alpha) = f_\eta(\alpha) + 1$, and for limit η, f_η is a least upper bound of every $f_\xi, \xi < \eta$ in $\leq_{NS}$.

Let g be an upper bound of all the f_η's in $\leq_{NS}$. Define $\mathbb{P}_g$ to be triples $(h, c, \{c_\eta : \eta \in A\})$ where:

1. $h : \alpha + 1 \to \omega_1$ for some $\alpha < \omega_1$
2. c, c_η are closed in $\alpha + 1$
3. A is countable, and $h < g$ on c and $f_\eta < h$ on c_η

We say $(h, c, \{c_\eta : \eta \in A\}) \leq (f, b, \{b_\eta : \eta \in B\})$ iff $h \supseteq f$, c end-extends b, and $A \supseteq B$.

Claim 2. *$\mathbb{P}_g$ preserves stationary subsets of ω_1.*

Proof of claim. Let $S \in V$ be stationary and $p \in \mathbb{P}$ such that $p \Vdash \dot{C}$ is club. Let λ be sufficiently large and $M \prec \langle H_\lambda, \in, \mathbb{P}, \dot{C}, S, P\rangle$ such that $|M| = \aleph_1$ and $M^\omega \subseteq M$, which is possible since we assume CH.

We may also assume that there is an increasing, continuous chain $\langle N_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of M such that $\bigcup_{\alpha<\omega_1} N_\alpha = M$ and for all $\beta < \omega_1$, $\langle N_\alpha : \alpha < \beta\rangle \in N_\beta$.

Let $\delta = M \cap \omega_2$, and let $\langle \delta_i : i < \omega_1\rangle$ be increasing and cofinal in δ. For every $i < \omega_1$ let C_i be such that $f_{\delta_i} <_{NS} f_\delta$.

Similarly, let D_0 be a club so that $f_\delta(\alpha) < g(\alpha)$ for all $\alpha \in D_0$. Let D_1 be the club of α such that $N_\alpha \cap \delta = \langle \delta_i : i < \alpha\rangle$. Further let $D_2 = \{\alpha < \omega_1 : \forall (i < \alpha)\, i \in C_\alpha\}$ Finally take $D = D_0 \cap D_1 \cap D_2$ and note that D is club. Choose some $\xi \in D$ and let $\langle \xi_n : n < \omega\rangle$ be cofinal in ξ.

Now we define a descending sequence of conditions $\langle p_n : n < \omega\rangle \subseteq M$. Let $p_0 = p$ and $p_{n+1} \leq p_n$ such that $\xi_n \subseteq \mathrm{dom}(p_{n+1})$ and $p_{n+1} \Vdash \gamma_n \in \dot{C}$ for some $\gamma_n \geq \xi_n$.

Now let take p^* to be the union of all the p_n's while also defining $h(\xi) = f_\delta(\xi)$, where h is the function part of p^*. Now $p^* \Vdash \xi \in \dot{C}$ since $\dot{C}$ is closed and $\xi \in S$. $\dashv$

Now consider the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega\rangle$ where $\mathbb{P}_0 = \mathbb{P}_g$, and if f_n is generic function added by $\dot{\mathbb{Q}}_n$ then $\mathbb{P}_{n+1} \Vdash \dot{\mathbb{Q}}_{n+1} = \mathbb{P}_{f_n}$. Then in $V[G]$, for all $n < \omega$, there is a club C_n such that for all $\alpha \in C_n$, $f_{n+1}(\alpha) < f_n(\alpha)$. If ω_1 were preserved by the iteration, $D = \bigcap_{n<\omega} C_n$ would be club. In particular, there would be an $\alpha \in D$, so $f_0(\alpha) > f_1(\alpha) > \cdots$, a contradiction. Hence the iteration collapses ω_1. $\square$

However, Shelah found that the following (only slightly) more restricting class works with countable support:

Definition 3. A partial order $\mathbb{P}$ is **proper** if $\mathbb{P}$ preserves all stationary $S \subseteq [\lambda]^\omega$ for all uncountable λ.

Notice that ω_1 is club in $[\omega_1]^\omega$, so proper implies stationary set-preserving, and hence ω_1-preserving. In fact, it is easy to see that proper posets have the stronger property of ω-distributivity. It is also easy to see that if $\mathbb{P}$ is proper and $\mathbb{P} \Vdash \dot{\mathbb{Q}}$ is proper, the $\mathbb{P} * \dot{\mathbb{Q}}$ is proper.

The proofs that ccc and σ-closed forcings preserve stationary subsets of ω_1 easily generalize to show that they are proper.

Proposition 5. *Both ccc and σ-closed forcings are proper.*

Proof. First we prove that ccc forcings are proper. Let $p \Vdash \dot{C} \subseteq [\lambda]^\omega$ is club. We use the fact that there $\dot{F} : \lambda^{<\omega} \to \omega$ such that $p \Vdash \dot{C}$ contains the closure points of $\dot{F}$. In V, let $f : \lambda^{<\omega} \to [\lambda]^\omega$ where

$$f(e) = \{\alpha < \lambda : (\exists q \leq p)\, \dot{F}(e) = \alpha$$

By the ccc, $|f(e)|$ is countable for all e. Let D be the club of closure points of f. Then $p \Vdash \forall e\, \dot{F}(e) \in f(e)$ so $p \Vdash D \subseteq \dot{C}$. Since every club in the extension contains a ground model club, all stationary sets are preserved.

Now we shall prove that σ-closed forcings are proper. Let S be stationary, and $p \Vdash \dot{F} : \lambda^{<\omega} \to \lambda$. We find an extension of p that forces S to contain a closure point of $\dot{F}$.

There is some $M \prec H_\lambda$ countable such that $N \cap \lambda \in S$. Let $N \cap \lambda = x$. Enumerate $x^{<\omega} = \{e_n : n < \omega\}$. We construct a descending sequence of conditions $p_{n+1} \leq p_n$ such that $p_0 = p$ and for every n there is an α_n such that $p_{n+1} \Vdash \dot{F}(e_n) = \alpha_n$. Using closure of the poset, let q be a lower bound of the p_n's. Then $q \Vdash \dot{F}\,``x^{<\omega} \subseteq x$. $\square$

Note that the proof of properness of σ-closed forcings can be generalized to $< \omega_1$ strategically-closed forcings. Although properness is easily preserved by two-step iteration, properness is not necessarily preserved under *products.*

Proposition 6. *There are proper forcings $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not proper, and in fact collapses ω_1.*

Proof. Let $\mathbb{P} = \omega_2^{<\omega_1}$, so $\mathbb{P}$ adds a surjection from ω_1 to ω_2. It is easy to see that $\mathbb{P}$ is σ-closed, so $\mathbb{P}$ is proper.

Let $\mathbb{Q}_0 = \mathrm{Add}(\omega, \omega_2) \times \mathrm{Col}(\omega_1, 2^{\omega_1})$. After forcing with $\mathbb{Q}_0$, the cofinality of ω_2^V becomes ω_1 so $\mathbb{P}$ has no branch with supremum ω_2. By an argument of Silver, no new ω_1-branches are added to $\mathbb{P}$ so $\mathbb{P}$ has at most ω_1 many cofinal branches. So let $\mathbb{Q}_1$ be the Baumgartner's forcing to seal the branches of $\mathbb{P}$. Note also that $\mathbb{Q}_0$ is of the form ccc $*\,\sigma$-closed, so that $\mathbb{Q}_0$ is proper. Also, $\mathbb{Q}_1$ is ccc and hence proper. Then $\mathbb{Q} = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$ is proper.

However, $\mathbb{P} \times \mathbb{Q}$ is not proper, as it collapses ω_1. $\square$

Proper forcings end up being the right class for a nice iteration theory, as Shelah proved that any *countable* support iteration of proper forcings remain proper. As far as I know, there is no proof of this theorem using the above formulation of properness. However, Shelah was able to generalize an analogue of the following model-theoretic characterization of ccc:

Lemma 1. *Let $\mathbb{P}$ be any notion of forcing. The following are equivalent:*

1. *$\mathbb{P}$ is ccc.*

2. *For all sufficiently large λ and countable $M \prec H_\lambda$, if $\dot{G}$ names the generic filter then $1_{\mathbb{P}} \Vdash \dot{G} \cap M$ is M-generic.*

Proof. ($1 \Rightarrow 2$). Let $M \prec H_\lambda$ be countable, $D \in M$ dense. If $A \in M$ is a maximal antichain, then A must be countable by the ccc. Consider $D = \{p \in \mathbb{P} : \exists q \in A$ such that $q \leq p\}$. Then $D \in M$ and D is dense. So $1_{\mathbb{P}} \Vdash \exists q \in \dot{G} \cap D$, and since $\dot{G}$ is upwards closed, $1_{\mathbb{P}} \Vdash \exists p \in \dot{G} \cap A$. So $1_{\mathbb{P}}$ forces $\dot{G}$ to meet every maximal antichain of M.

($2 \Rightarrow 1$). Suppose $\mathbb{P}$ is not ccc and A is a maximal antichain, so A is uncountable. Let $M \prec H_\lambda$ be countable with $A \in M$. Choose some $p \in A \setminus M$. Then it is easily checked that $p \Vdash \dot{G}$ is not M-generic. □

The condition in 2. suggests the following definition:

Definition 4. Let $\mathbb{P}$ be a partial order and $p \in \mathbb{P}$. For λ sufficiently large and $M \prec H_\lambda$ countable, we say p is $(M, \mathbb{P})$-generic if $p \Vdash \dot{G} \cap M$ is M-generic, where $\dot{G}$ names the generic filter.

So $\mathbb{P}$ is ccc iff $1_{\mathbb{P}}$ is $(M, \mathbb{P})$-generic for every countable M. The proof of the last lemma easily generalizes to show the following:

Lemma 2. *Let $\mathbb{P}$ be a partial order, $p \in \mathbb{P}$, and λ sufficiently large, and $M \prec H_\lambda$ countable. Then p is (M, P)-generic iff for all dense $D \subseteq \mathbb{P}$, $D \cap M$ is predense below p.*

Then one can prove the following model-theoretic characterization of properness, which is used in proving the iteration theorem.

Lemma 3. *$\mathbb{P}$ is proper iff for all sufficiently large λ there is a club of countable $M \prec H_\lambda$ such that for all $p \in M$ there is a $q \leq p$ that is (M, P)-generic.*

Proof. □

This characterization of properness is often easier to check, as one on is only considering objects in the ground model instead of considering clubs in the extension. For example, we can now easily prove that Axiom A forcings are proper:

Proposition 7. *Axiom A forcings are proper.*

Proof. Suppose $\mathbb{P}$ is Axiom A. We recall that since $\mathbb{P}$ is Axiom A, there is a sequence of partial orderings $\{\leq_n : n < \omega\}$ on $\mathbb{P}$ such that $p \leq_0 q$ implies $p \leq q$ and for all n, $p \leq_{n+1} q$ implies $p \leq q$ and

1. If $\langle p_n : n < \omega \rangle$ is a sequence such that $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 p_3 \geq \cdots$ then there is a q such that $q \leq_n p_n$ for all n.

2. For all $p \in \mathbb{P}$ and n and every ordinal name $\dot{\alpha}$ there is $q \leq_n p$ and a countable set B such that $q \Vdash \dot{\alpha} \in \check{B}$.

Now let M be countable and $p \in M$. Enumerate the maximal antichains in M as $\langle A_n : n < \omega \rangle$. We define a sequence $p_{n+1} \leq_n p_n$ with $p_0 = p$. Take p_{n+1} such that $\{a \in A_n : a \perp p_{n+1}\}$ is countable. Now let $q \leq_n p_n$ for all n.

We claim that q is $(M, \mathbb{P})$-generic: For every n, since A_n is predense, also $\{a \in A_n : a \perp q\} \subseteq M$ is predense. $\square$

For a proof of the properness iteration theorem, we refer the reader to chapter 3 of [She17].

Theorem 5. *Any countable support iteration of proper forcings is proper.*

References

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