CHE 576 ASSIGNMENT 5 SOLUTIONS

Q1. (10 marks)

- Let: $x_1(k) = y(k), \quad x_2(k) = x_1(k+1) = y(k+1), \quad x_3(k) = x_2(k+1) = y(k+2),$
  $x_3(k+1) = y(k+3)$

  State space realization:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}$$

- Is the system controllable?

$$M_{\text{ctrl}} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{pmatrix}, \qquad n = 3$$

  $\text{rank}(M_{\text{ctrl}}) = 3 = n \Longrightarrow$ the system is *controllable.*

- Block diagram:

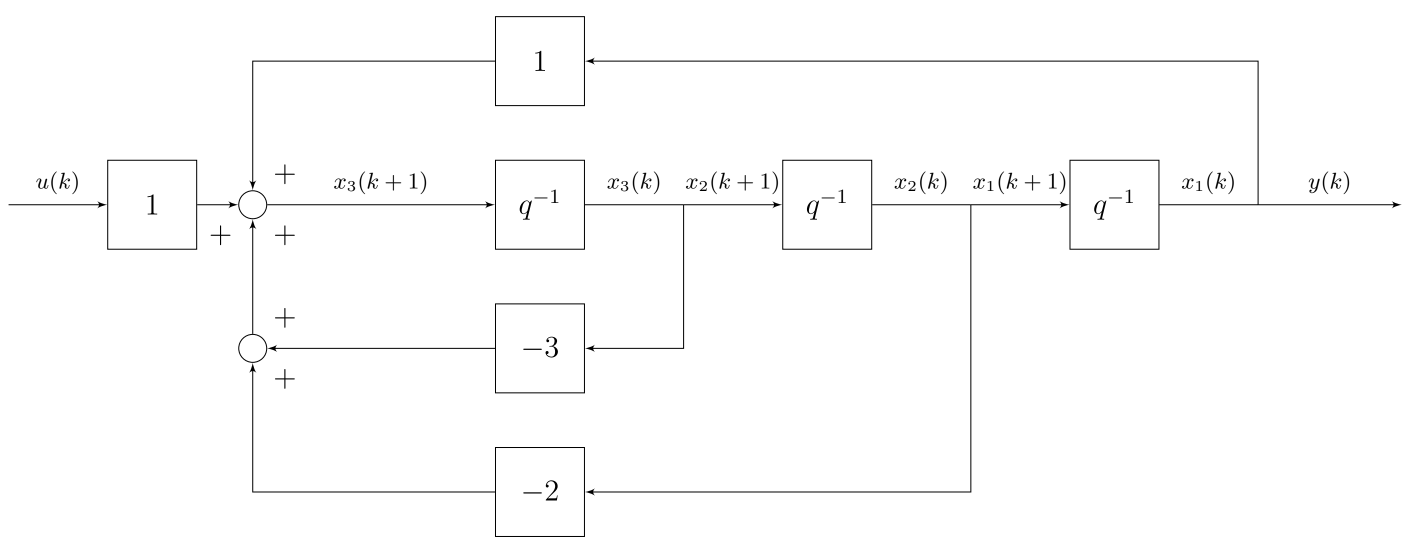

FIGURE 1. Block diagram

- Is the system observable?

$$M_{\text{obs}} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad n = 3$$

rank(M_{obs}) $= 3 = n \Longrightarrow$ the system is *observable.*

- System transfer function:

$$G(z) = C(zI - A)^{-1}B = \frac{1}{z^3 + 3z^2 + 2z - 1}$$

Q2. (8 marks)

- Is the system controllable?

$$M_{\text{ctrl}} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 1 & 11 & 43 \\ 1 & 5 & 9 \\ 1 & 1 & 1 \end{pmatrix}, \qquad n = 3$$

rank(M_{ctrl}) $= 3 = n \Longrightarrow$ the system is *controllable.*

- Is the system observable?

$$M_{\text{obs}} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 6 \\ 4 & 9 & 30 \end{pmatrix}, \qquad n = 3$$

rank(M_{obs}) $= 3 = n \Longrightarrow$ the system is *observable.*

- Determine T_c:

$$M_{\text{ctrl}}^{-1} = \begin{pmatrix} 0.0455 & -0.3636 & 1.3182 \\ -0.0909 & 0.4773 & -0.3864 \\ 0.0455 & -0.1136 & 0.0682 \end{pmatrix} \Rightarrow q = \begin{pmatrix} 0.0455 & -0.1136 & 0.0682 \end{pmatrix}$$

Transformation matrix:

$$T_c = \begin{pmatrix} q \\ qA \\ qA^2 \end{pmatrix} = \begin{pmatrix} 0.0455 & -0.1136 & 0.0682 \\ 0.0909 & 0.0227 & -0.1136 \\ 0.1818 & 0.2955 & 0.5227 \end{pmatrix}$$

Controllable state space form, $\Sigma(A_c, B_c, C_c)$:

$$A_c = T_c A T_c^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}, \quad B_c = T_c B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C_c = B_c^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

- Determine T_o:

$$M_{\text{obs}}^{-1} = \begin{pmatrix} 1.0000 & 0 & 0 \\ -1.0000 & 0.8333 & -0.1667 \\ 0.1667 & -0.2500 & 0.0833 \end{pmatrix} \Rightarrow p = \begin{pmatrix} 0 \\ -0.1667 \\ 0.0833 \end{pmatrix}$$

Transformation matrix:

$$T_o = \begin{pmatrix} p & Ap & A^2p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1.0000 \\ -0.1667 & 0.1667 & 0.5000 \\ 0.0833 & 0.0833 & 0.0833 \end{pmatrix}$$

Observable state space form, $\Sigma(A_o, B_o, C_o)$:

$$A_o = T_o^{-1} A T_o = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -5 \\ 0 & 1 & 4 \end{pmatrix}, \quad C_o = CT_o = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad B_o = C_o^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Q3. (6 marks) Discrete state space system $\Sigma_d(A, B, C)$:

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ -7 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 0 & 0 \end{pmatrix}$$

- Controllable/Observable?

$$M_{\text{ctrl}} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 1 & -1 & 10 \\ 0 & -1 & -5 \\ 1 & -6 & -1 \end{pmatrix}, \quad n = 3 = \text{rank}(M_{\text{ctrl}}) \Longrightarrow \text{controllable}$$

$$M_{\text{obsv}} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \text{rank} \begin{pmatrix} 4 & 0 & 0 \\ 4 & 4 & -8 \\ 52 & -8 & -12 \end{pmatrix}, \quad n = 3 = \text{rank}(M_{\text{obsv}}) \Longrightarrow \text{observable}$$

- Target state: $x^* = \begin{pmatrix} a \\ 2a \\ 3a \end{pmatrix}$, initial state: $x(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ State evolution:

$$x(1) = Ax(0) + Bu(0) = Bu(0)$$

$$x(2) = Ax(1) + Bu(1) = ABu(0) + Bu(1)$$

$$x(3) = Ax(2) + Bu(2) = A^2Bu(0) + ABu(1) + Bu(2) = \begin{pmatrix} B & AB & A^2B \end{pmatrix} \begin{pmatrix} u(2) \\ u(1) \\ u(0) \end{pmatrix}$$

Target state:

$$\begin{pmatrix} a \\ 2a \\ 3a \end{pmatrix} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} \begin{pmatrix} u(2) \\ u(1) \\ u(0) \end{pmatrix} = M_{\text{ctrb}} \begin{pmatrix} u(2) \\ u(1) \\ u(0) \end{pmatrix}$$

Solving for $u(0), u(1), u(2)$:

$$\begin{pmatrix} u(2) \\ u(1) \\ u(0) \end{pmatrix} = M_{\text{ctrb}}^{-1} \begin{pmatrix} a \\ 2a \\ 3a \end{pmatrix} = \begin{pmatrix} 7.571a \\ 0.857a \\ -0.571a \end{pmatrix}$$

$\Rightarrow$ Takes 3 steps to reach target state x^* from the initial state $x(0)$ with the input sequence $u(k) = \{-0.571a, 0.857a, 7.571a\}$.

- Measurement sequence $y(k) = \begin{pmatrix} 3a & 2a & a \end{pmatrix}$, $k = 0, 1, 2$:

$$\begin{aligned} &y(0) = Cx(0) \\ &y(1) = Cx(1) = C(Ax(0) + Bu(0)) = CAx(0) + CBu(0) \\ &y(2) = Cx(2) = C(A^2Bu(0) + ABu(1) + Bu(2)) = CA^2Bu(0) + CABu(1) + CBu(2) \end{aligned}$$

Rearrange to get the system:

$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} x(0) + \begin{pmatrix} 0 & 0 \\ CB & 0 \\ CAB & CB \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}$$

Solving for the initial state:

$$x(0) = M_{\text{obsv}}^{-1} \left\{ \begin{pmatrix} 3a \\ 2a \\ a \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ CB & 0 \\ CAB & CB \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \right\} = \begin{pmatrix} 0.750a \\ 3.260a \\ 1.469a \end{pmatrix}$$