CHE 576 ASSIGNMENT 6 SOLUTIONS

Q1. Find the range $K \in [a, b]$ such that the closed loop feedback system is stable. The process transfer function is given by:

$$G_p(s) = \frac{s+2}{s(s+5)(s^2+s+1)}$$

Closed loop transfer function:

$$G_{cl}(s) = \frac{KG_p(s)}{1+KG_p(s)} = \frac{K(s+2)}{s^4+6s^3+6s^2+(K+5)s+2K}$$

Characteristic closed loop polynomial:

$$P(s) = s^4+6s^3+6s^2+(K+5)s+2K$$

Routh-Hurwitz Criteria: Given the polynomial:

$$P(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

Define the Hurwitz matrices using the coefficients a_n where:

$$H^{(1)} = (a_1), \quad H^{(2)} = \begin{pmatrix} a_1 & 1 \\ a_3 & a_4 \end{pmatrix}, \quad H^{(3)} = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}$$

and

$$H^{(n)} = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

All of the roots of the characteristic closed loop polynomial $P(s)$ are negative or have negative real parts if and only if the determinants of all the Hurwitz matrices are positive, i.e. $\det H^j > 0$, $j = 1, 2, \ldots, n$.

General criteria:

$$\begin{aligned}
n = 1: &\quad a_1 > 0 \\
n = 2: &\quad a_1 > 0, a_2 > 0 \\
n = 3: &\quad a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0 \\
n = 4: &\quad a_1 > 0, a_3 > 0, a_4 > 0, a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0 \\
n = 5: &\quad a_j > 0 \ (j = 1, \ldots, 5),\ a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0, \\
&\quad (a_1a_4 - a^5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) - a_5(a_1a_2 - a_3)^2 - a_1a_5^2 > 0
\end{aligned}$$

For $P(s)$ of $G_{cl}(s)$:

$$\begin{aligned}
n = 1: &\quad a_1 = 6 > 0 \\
n = 2: &\quad a_1 > 0, a_2 = K + 5 > 0 \Rightarrow K > -5 \\
n = 3: &\quad \{a_1, a_3\} > 0, a_4 = 2K > 0, a_1a_2 - a_3 = 36 - (K+5) > 0 \Rightarrow K > 0, K < 31 \\
n = 4: &\quad \{a_1, a_3, a_4\} > 0, a_1a_2a_3 - a_3^2 + a_1^2a_4 > 0 \Rightarrow 155 - 46K - K^2 > 0
\end{aligned}$$

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From the last inequality:

$$(K-3.15)(K+49.15)<0 \Rightarrow K<3.15, \quad K<-49.15$$

Then the range of K such that the closed loop system is stable is:

$$K \in (0, 3.15)$$

Q2.

$$C(s)=\frac{\alpha_1 s+\alpha_0}{s+\beta_0}, \qquad G(s)=C(sI-A)^{-1}B=\frac{s-1}{s^2-2s+3}$$

Closed loop transfer function:

$$G_{cl}=\frac{G(s)}{1+G(s)C(s)}=\frac{(s+\beta_0)(s-1)}{s^3+(\alpha_1+\beta_0-2)s^2+(\alpha_0-\alpha_1-2\beta_0+3)s+(3\beta_0-\alpha_0)}$$

Characteristic polynomial:

$$P(s)=s^3+(\alpha_1+\beta_0-2)s^2+(\alpha_0-\alpha_1-2\beta_0+3)s+(3\beta_0-\alpha_0)$$

Want closed loop poles at $\sigma=\{-1,-2,-3\}$ so closed loop polynomial is:

$$P^*(s)=(s+1)(s+2)(s+3)=s^3+6s^2+11s+6$$

Get system:

$$\begin{aligned}\alpha_1+\beta_0-2 &= 6\\ \alpha_0-\alpha_1-2\beta_0+3 &= 11\\ 3\beta_0-\alpha_0 &= 6\end{aligned}$$

Solving the system to get $\alpha_1=-3, \alpha_0=27, \beta_0=11$. Controller transfer function:

$$C(s)=\frac{27-3s}{s+11}$$

Q3.

Part 1. Transfer function:

$$G(s)=C(sI-A)^{-1}B=\frac{3s-2}{s^2-3s+3}$$

Poles of $s^2-3s+3=0$ are:

$$\lambda_{1,2}=1.5\pm 0.866, \quad \mathrm{Re}(\lambda_{1,2})=1.5>0 \Longrightarrow \text{Unstable}$$

Matrix A is 2×2

$$\mathrm{Rank}(B, AB)=2=n \Rightarrow \text{Controllable}$$
$$\mathrm{Rank}(C, CA)^T=2=n \Rightarrow \text{Observable}$$

Part 2. Can determine $K=[k_1, k_2]$ by using characteristic polynomial approach similar to Question 2, i.e.

$$P(s)=(A-BK)^{-1}$$

where for $\sigma_1=-1$ and $\sigma_2=-2$, then $P^*(s)=(s+1)(s+2)$.

Alternatively, can use Ackerman's formula:

$$(k_1 \quad k_2)=(0 \quad 1)(B \quad AB)^{-1}(A-\sigma_1 I)(A-\sigma_2 I)$$

to obtain the gain:

$$K=\begin{pmatrix}4 & 1\end{pmatrix}$$

Part 3. Similar to Part 2 with:

$$P_{obs}(s) = (A - LC)^{-1}, \qquad \sigma^*_{obs} = \left\{-\frac{3}{2}, -\frac{5}{2}\right\}, \qquad P^*_{obs} = \left(s + \frac{3}{2}\right)\left(s + \frac{5}{2}\right)$$

Or using Ackerman's formula:

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = (A - \sigma_1^* I)(A - \sigma_2^*) \begin{pmatrix} C \\ CA \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -7.75 \\ 14.75 \end{pmatrix}$$

Part 4. Using K and L as in Part 2,3:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= (A - BK - LC)\hat{x} + Ly \\ &= A_{cl}\hat{x} + Ly \end{aligned}$$

Taking Laplace transform:

$$\hat{X}(s) = (sI - A_{cl})^{-1} LY(s)$$

For input $u = -K\hat{x}$:

$$\frac{U(s)}{Y(s)} = K\hat{X}(s) = K\left(sI - A_{cl}\right)^{-1} L = \frac{13(5s - 21)}{2(2s^2 + 20s + 187)}$$

Part 5. The Figure 1 shows the plots of the estimated states and actual states. One can see that the estimated states converge to the actual states, and that the feedback $u = -K\hat{x}$ stabilizes the system.

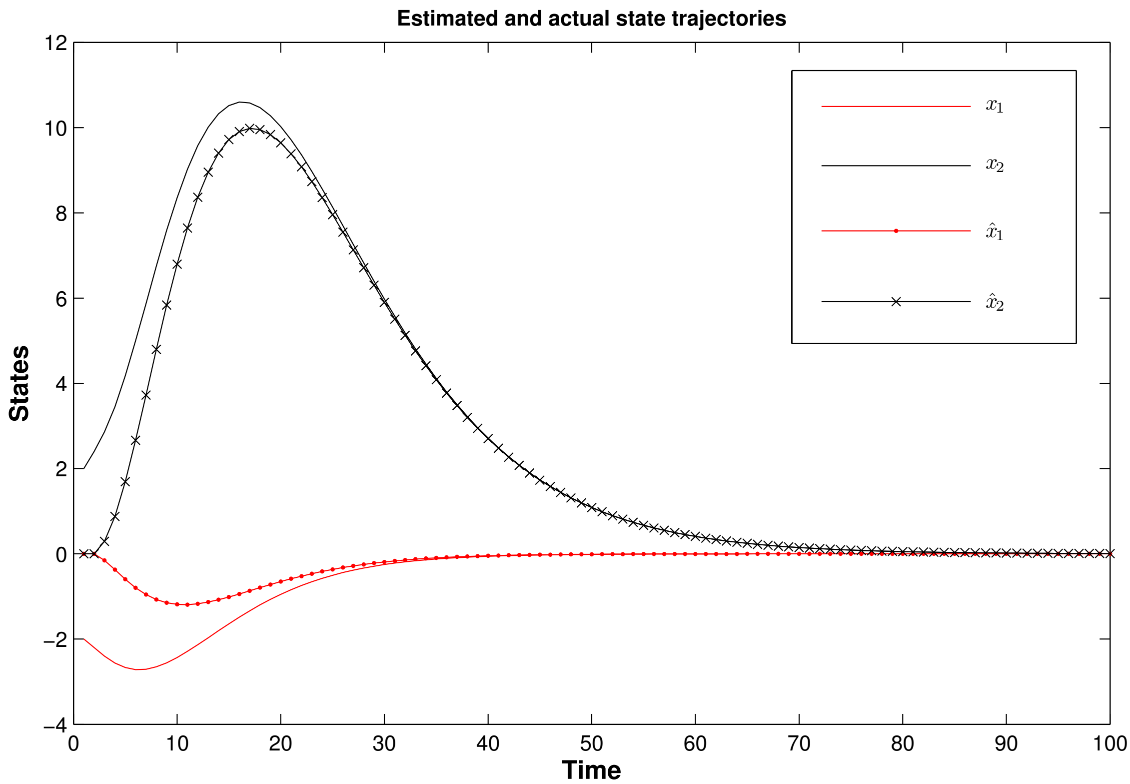

FIGURE 1. Plot of estimates states and the the actual states with state and observer gains K and L.