CHE 576 W2012 Assignment 1
Solutions

Q1.

1. $\dfrac{Y(s)}{U(s)} = \dfrac{e^{-2s}}{s+0.01} = \dfrac{100e^{-2s}}{100s+1}$

$$\boxed{\tau = 100, \quad \tau_d = 2, \quad \tau \gg \tau_d \quad \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\tau = 10 \sim 20}$$

2. $\dfrac{Y(s)}{U(s)} = \dfrac{e^{-2s}}{s+0.1} = \dfrac{10e^{-2s}}{10s+1}$

$$\boxed{\tau = 10, \quad \tau_d = 2, \quad \tau \approx \tau_d, \quad \tau > \tau_d \quad \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\tau_d = 0.2 \sim 0.4}$$

* For 3-5:

$$\frac{Y(s)}{U(s)} = \frac{e^{-\tau_d}}{(\tau_1 s + a)(\tau_2 s + b)} = \frac{e^{-\tau_d}}{\tau_1\tau_2 s^2 + (\tau_1 b + \tau_2 a)s + ab}$$

Compare denominator with:

$$ms^2 + \gamma s + ks = 0 \qquad \text{(unforced equation of motion)}$$

$$s_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}, \qquad \xi^2 = \frac{\gamma^2}{4km}$$

Behaviour of response depends on sign of $\gamma^2 - 4km = \dfrac{\gamma^2}{4km} - 1 = \xi^2 - 1$ where $\xi = \dfrac{\gamma}{2\sqrt{km}}$

$$\begin{cases} \gamma^2 - 4km < 0 \;\Rightarrow\; \xi < 1 & \text{(underdamped, imaginary poles, oscillatory)} \\ \gamma^2 - 4km = 0 \;\Rightarrow\; \xi = 1 & \text{(critically damped)} \\ \gamma^2 - 4km > 0 \;\Rightarrow\; \xi > 1 & \text{(overdamped)} \end{cases}$$

I) If system is **underdamped** (response is oscillatory), i.e. $\gamma^2 - 4km < 0$:

$$\begin{cases} \text{Frequency of oscillation (quasi-frequency) :} & F_q = \dfrac{\sqrt{1-\xi^2}}{2m} \\ \text{Period of oscillation (quasi-period) :} & P_q = \dfrac{2\pi}{F_q} \\ \text{Cycle :} & T_c = \dfrac{1}{P_q} = \dfrac{\sqrt{1-\xi^2}}{4m\pi} \end{cases}$$

Sample at least 2 times per cycle, i.e.:

$$\Delta t < 2T_c$$

II) If system is **overdamped** convert to form:

$$\frac{Y(s)}{U(s)} = \frac{Ce^{-\tau_d}}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$\text{If} \quad \tau_1 \approx \tau_2 \quad \Rightarrow \quad \tilde{\tau} = \tau_1 + \tau_2 \quad \Rightarrow \quad \begin{cases} \tilde{\tau} \approx \tau_d & \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\min(\tilde{\tau}, \tau_d) \\ \tilde{\tau} \gg \tau_d & \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\tilde{\tau} \end{cases}$$

$$\text{If} \quad \tau_1 \gg \tau_2 \quad \Rightarrow \quad \tilde{\tau} = \tau_1 \quad \Rightarrow \quad \begin{cases} \tilde{\tau} \approx \tau_d & \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\min(\tilde{\tau}, \tau_d) \\ \tilde{\tau} \gg \tau_d & \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\tilde{\tau} \end{cases}$$

3. $\dfrac{Y(s)}{U(s)} = \dfrac{e^{-0.1s}}{0.3s^2+1.03s+0.1}$

$$\xi = 2.973 > 1 \quad \text{(overdamped)}$$

Put in form:

$$\frac{Y(s)}{U(s)} = \frac{10e^{-0.1s}}{(0.3s+1)(10s+1)}, \quad \tau_1 = 0.3, \quad \tau_2 = 10, \quad \tau_d = 0.1$$

$$\boxed{\tau_2 \gg \tau_1 \quad \Rightarrow \quad \tilde{\tau} = \tau_2 \quad \Rightarrow \quad \tilde{\tau} \gg \tau_d \quad \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\tilde{\tau} = 1 \sim 2}$$

4. $\dfrac{Y(s)}{U(s)} = \dfrac{e^{-3s}}{(s+2)(s+0.25)} = \dfrac{e^{-3s}}{s^2+2.25s+0.5}$

$$\xi = 1.591 > 1 \quad \text{(overdamped)}$$

Put in form:

$$\frac{Y(s)}{U(s)} = \frac{2e^{-3s}}{(0.5s+1)(4s+1)}, \quad \tau_1 = 0.5, \quad \tau_2 = 4 \quad \tau_d = 3$$

$$\boxed{\tau_2 \gg \tau_1 \quad \Rightarrow \quad \tilde{\tau} = \tau_2 \quad \Rightarrow \quad \tilde{\tau} \sim \tau_d \quad \Rightarrow \quad \Delta t = (0.1 \sim 0.2)\tilde{\tau} = 0.3 \sim 0.6}$$

5. $\dfrac{Y(s)}{U(s)} = \dfrac{e^{-3s}}{s^2+0.5s+0.5} = \dfrac{2e^{-3s}}{2s^2+s+1} = \dfrac{2e^{-3s}}{(\sqrt{2}s+a)(\sqrt{2}s+b)}$

$$\xi = 0.354 < 1 \quad \text{(underdamped, oscillatory)}$$

$$2T_c = \frac{\sqrt{1-\xi^2}}{2m\pi} = 0.149$$

$$\boxed{\Delta t < 0.149}$$

Q2.

1. $\dfrac{dh}{dt} = -\dfrac{1}{KA}h + \dfrac{1}{A}f_{in}$
2. Assume that the tank height is measured, $y(t) = h(t)$

$$\Sigma(A,B,C,D) = \Sigma\left(-\frac{1}{KA}, \frac{1}{A}, 1, 0\right)$$

3. $u(t) = f_{in}(t), \quad y(t) = h(t)$
4.

$$\Phi = e^{-\frac{0.1}{KA}}, \quad \Gamma = -K\left(e^{-\frac{0.1}{KA}} - 1\right), \quad \theta = 1$$

$$\Sigma(A_d, B_d, C_d, D_d) = \Sigma\left(e^{-\frac{0.1}{KA}}, -K\left(e^{-\frac{0.1}{KA}} - 1\right), 1, 0\right)$$

5. Laplace transform:

$$\frac{Y(s)}{U(s)} = \frac{10}{s+1}$$

$$\boxed{\tau = 1, \quad \tau_d = 0 \Rightarrow \Delta t = (0.1 \sim 0.2)1\tau = 0.1 \sim 0.2}$$

Q3.

1. Consider:

$$\frac{dy}{dt} + ay = bu(t)$$

where a, b are constant. Look for integrating factor $f(t)$ which should satisfy:

$$\frac{df}{dt} = af$$

Solving yields $f(t) = e^{at}$ where:

$$\begin{aligned} e^{at}\frac{dy}{dt} + ae^{at}y &= e^{at}bu(t) \\ \text{or} \quad \frac{d}{dt}(e^{at}y) &= e^{at}bu(t) \end{aligned}$$

Integrating:

$$e^{at}y = b\int e^{at}u(t)dt + C$$

where C is the constant determined from the given initial condition. The general integral solution is of the form

$$y(t) = e^{-at}b\int_{t_0}^{t} e^{as}u(s)ds + Ce^{-at}$$

Substituting the corresponding values:

$$y(t) = 3e^{0.5t} + 1.5e^{0.5t}\int_0^t e^{-0.5s}u(s)ds$$

2. Exact discretization $\Delta t = 0.1$:

$$\begin{aligned} y_{k+1} &= e^{a\Delta t} y_k + A^{-1}(e^{a\Delta t} - 1) b u_k \\ &= 1.0513 y_k + 0.1538 u_k \end{aligned}$$

3. Finite difference method $\Delta t = 0.1$:

$$\begin{aligned} \frac{dy}{dt} \approx \frac{y_{k+1} - y_k}{\Delta t} &= 0.5 y_k + 1.5 u_k \\ y_{k+1} &= (1 + 0.5\Delta t) y_k + 1.5(\Delta t) u_k \\ &= 1.05 y_k + 0.15 u_k \end{aligned}$$

4. Using MATLAB:

$$y_{k+1} = 1.0513 y_k + 0.1538 u_k$$

5. Plots for finite-difference and exact discretization

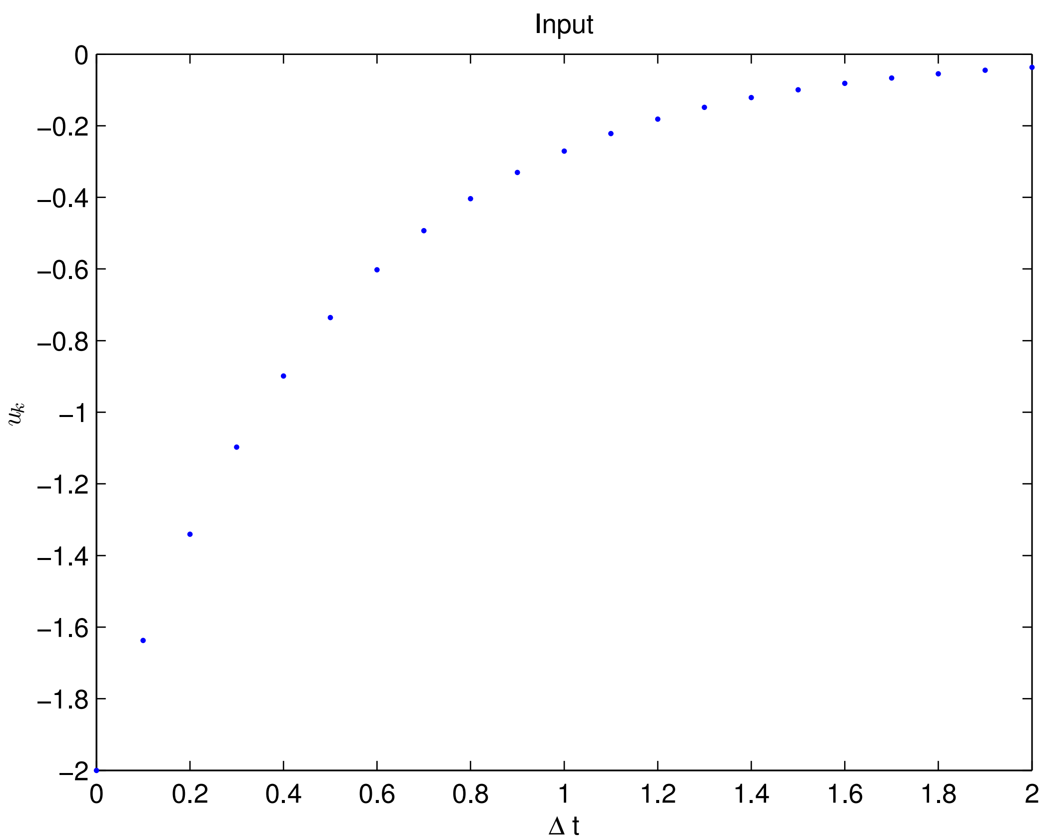

Figure 1: Input plot

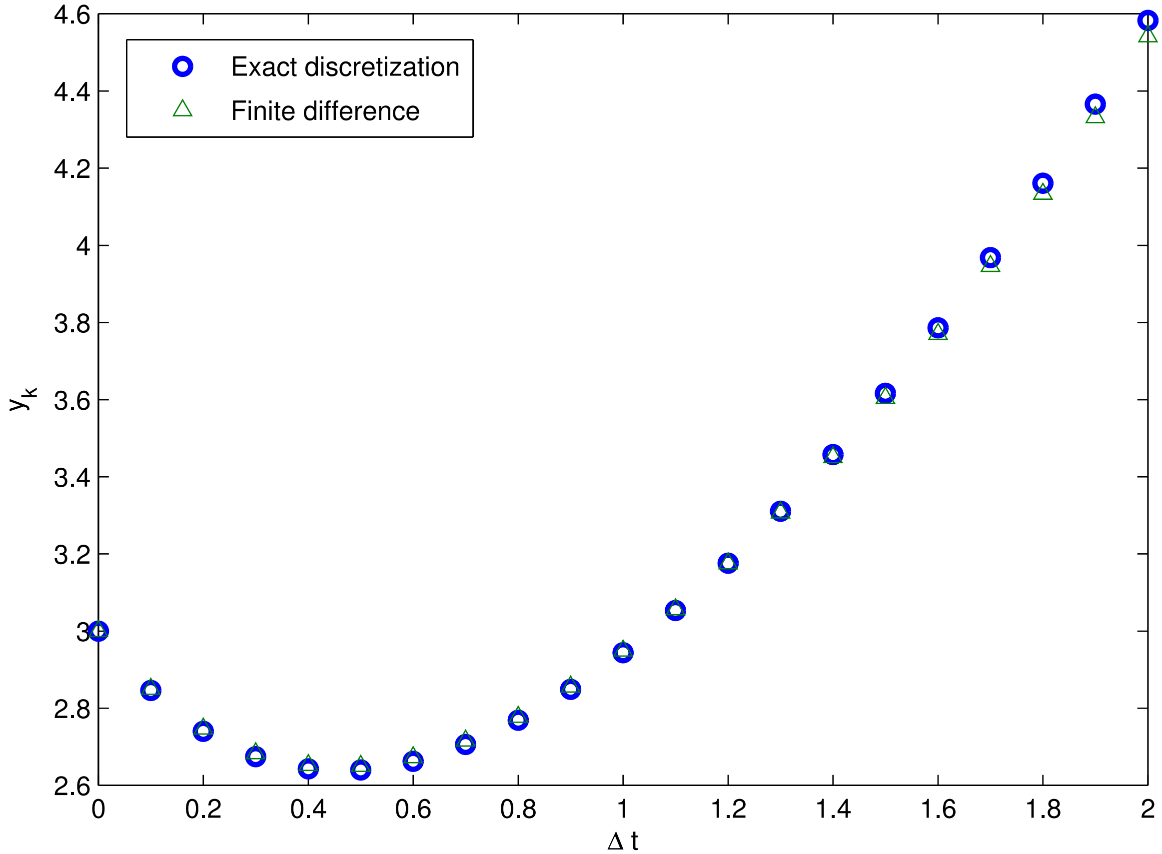

Figure 2: Output y_k

Q4.

$$
\begin{aligned}
M_1\ddot{x}_1 + C\dot{x}_1 + k_1x_1 - k_2x_2 &= f(t) \\
M_2\ddot{x}_2 + k_2x_2 &= 0
\end{aligned}
$$

Let $x_1 = x_0, \dot{x}_1 = x_{10}, x_2 = x_{20}, \dot{x}_2 = x_{30}$

$$
\begin{pmatrix} \dot{x}_0 \\ \dot{x}_{10} \\ \dot{x}_{20} \\ \dot{x}_{30} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_1/M_1 & -C/M_1 & k_2/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_2/M_2 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{10} \\ x_{20} \\ x_{30} \end{pmatrix} + \begin{pmatrix} 0 \\ 1/M_1 \\ 0 \\ 0 \end{pmatrix} f(t)
$$

Assuming the measurements are $y_1 = x_1$ and $y_2 = x_2$:

$$
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{10} \\ x_{20} \\ x_{30} \end{pmatrix}
$$

For $\Delta t = 0.1$, $M_1 = 1$, $M_2 = 2$, $k_1 = 0.5$, $k_2 = 0.2$, $C = 1$:

$$
\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.5 & -0.1 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/M_1 \\ 0 \\ 0 \end{pmatrix} f(t)
$$

Exact discretization using MATLAB with $\Delta t = 0.1$ yields:

$$
\Phi = \begin{pmatrix} 0.9975 & 0.0994 & 0.0010 & 0.0000 \\ -0.0497 & 0.9876 & 0.0199 & 0.0010 \\ 0 & 0 & 0.9990 & 0.1000 \\ 0 & 0 & -0.0200 & 0.9990 \end{pmatrix}, \qquad \Gamma_0 = \begin{pmatrix} 0.0050 \\ 0.0994 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D = 0
$$