CHE 576 ASSIGNMENT 3 SOLUTIONS

Q1. (25 marks)

Part 1. (2 marks): The model is given as:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{A_1K_1} & 0 \\ \frac{1}{A_2K_1} & -\frac{1}{K_2A_2} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{A_1} \\ 0 \end{pmatrix} u(t)$$

$$F_2(t) = \begin{pmatrix} 0 & \frac{1}{K_2} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Part 2. (3 marks): Discretization ($\Delta t = 0.1$). Let,

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

Discrete state space model:

$$\boxed{x_{k+1} = \begin{pmatrix} 0.951 & 0 \\ 0.088 & 0.819 \end{pmatrix} x_k + \begin{pmatrix} 0.0488 \\ 0.0023 \end{pmatrix} u(k)}$$

$$y_k = \begin{pmatrix} 0 & 2 \end{pmatrix} x_k$$

Part 3. (5 marks): Overall transfer function:

$$X(z) = (zI - A)^{-1} z\, x_0 + (zI - A)^{-1} B u(z)$$

where

$$(zI - A)^{-1} = \begin{pmatrix} \frac{1}{z-0.9512} & 0 \\ \frac{0.0883}{(z-0.951)(z-0.819)} & \frac{1}{z-0.819} \end{pmatrix}$$

Then:

$$\boxed{X(z) = \begin{pmatrix} \frac{z}{(z-0.9512)} & 0 \\ \frac{0.0883z}{(z-0.951)(z-0.819)} & \frac{z}{z-0.819} \end{pmatrix} x_0 + \begin{pmatrix} \frac{0.0488}{z-0.951} \\ \frac{0.00431}{(z-0.951)(z-0.819)} + \frac{0.0023}{(z-0.819)} \end{pmatrix} u(z)}$$

Part 4. (5 marks): Zero-input state response ($u(z) = 0$): By partial fraction expansion:

$$\frac{0.0883z}{(z-0.951)(z-0.819)} = \left(\frac{A}{z-0.951} + \frac{B}{z-0.819}\right) z \Rightarrow A = \frac{2}{3}, \quad B = \frac{2}{3}$$

Then:

$$\boxed{x_{\text{zero input}}(n) = \begin{pmatrix} (0.951)^n & 0 \\ 0.667[(0.951)^n - (0.819)^n] & (0.819)^n \end{pmatrix} x_0}$$

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Part 5. (5 marks): Zero-state response for step input:

$$u(z) = \frac{z}{z-1}$$

Similar procedure to Part 4 yields:

$$\boxed{x_{\text{zero state}}(n) = \begin{pmatrix} 0.999[1-(0.951)^n] \\ 0.171(0.819)^n - 0.680(0.951)^n + 0.512 \end{pmatrix}}$$

Part 6. (5 marks): Overall system response for step input and initial condition $x(0) = (2 \quad 3)^T$:

$$\boxed{x(n) = \begin{pmatrix} 0.999 + 1.00(0.951)^n \\ 1.844(0.819)^n + 0.657(0.951)^n + 0.512 \end{pmatrix}}$$

Q2. (10 marks) Taking the z-transform one obtains,

$$\begin{aligned} z^2x(z) - z^2x(0) - zx(1) + 3zx(z) - 3zx(0) + 2x(z) &= u(z) \\ x(z)(z^2+3z+2) &= z^2x(0) + 2x(1) + 3zx(0) + u(z) \\ x(z) &= \tfrac{z^2x(0)+2x(1)+3zx(0)}{(z^2+3z+2)} + \tfrac{1}{(z^2+3z+2)}u(z) \\ x(z) &= \tfrac{z^2-z}{(z^2+3z+2)} + \tfrac{1}{(z^2+3z+2)}\tfrac{z}{z-1} \end{aligned}$$

we solve first response associated with initial conditions:

$$x(z) = \frac{z(z-1)}{z^2+3z+2} \quad \rightarrow \frac{x(z)}{z} = \frac{z-1}{z^2+3z+2} = \frac{A}{z+2} + \frac{B}{z+1}$$

we find coefficients as :

$$\begin{aligned} A &= \left[\frac{z-1}{(z+2)(z+1)}(z+2)\right]_{z=-2} = 3 \\ B &= \left[\frac{z-1}{(z+2)(z+1)}(z+1)\right]_{z=-1} = -2 \end{aligned}$$

so that $x(z) = A\frac{z}{z+2} + B\frac{z}{z+1}$ which leads to $x_{ic}(n) = 3(-2)^n - 2(-1)^n$.
The response to the step input is given by:

$$x(z) = \frac{z}{(z^2+3z+2)(z-1)} \rightarrow \frac{x(z)}{z} = \frac{1}{(z^2+3z+2)(z-1)} \tag{1}$$

and by the same procedure we obtain coefficients in partial fraction expansion:

$$\begin{aligned} A &= \left[\frac{1}{(z+2)(z+1)(z-1)}(z+2)\right]_{z=-2} = 1/3 \\ B &= \left[\frac{1}{(z+2)(z+1)(z-1)}(z+1)\right]_{z=-1} = -1/2 \\ C &= \left[\frac{1}{(z+2)(z+1)(z-1)}(z-1)\right]_{z=1} = 1/6 \end{aligned}$$

finally one obtains $x_{input}(n) = 1/3(-2)^n - 1/2(-1)^n + 1/6$. The final solution is $x(n) = x_{input}(n) + x_{ic}(n) = 3(-2)^n - 2(-1)^n + 1/3(-2)^n - 1/2(-1)^n + 1/6 = 10/3(-2)^n - 5/2(-1)^n + 1/6$.

Q3. (10 marks) To obtain solution by power series method we must transform expression in the following form:

$$x(z) = \frac{z(z+1)}{(z-1)^2} \rightarrow \quad x(z) = \frac{1+z^{-1}}{1-2z^{-1}+z^{-2}} \tag{2}$$

therefore we obtain:

$$\begin{array}{l}
1+z^{-1} : 1-2z^{-1}+z^{-2} = 1+3z^{-1}+5z^{-2}+7z^{-3}+9z^{-4}+\cdots \\
-\underline{(1-2z^{-1}+z^{-2})} \\
\quad 3z^{-1}-z^{-2} \\
\quad -\underline{(3z^{-1}-6z^{-2}+3z^{-3})} \\
\qquad 5z^{-2}-3z^{-3} \\
\qquad -\underline{(5z^{-2}-10z^{-3}+5z^{-4})} \\
\qquad\quad 7z^{-3}-5z^{4} \\
\qquad\quad -\underline{(7z^{-3}+14z^{-4}+7z^{-5})} \\
\qquad\qquad 9z^{-4}-5z^{-5} \\
\qquad\qquad \cdots
\end{array}$$

therefore the output sequence is $x(n) = \{1, 3, 5, 7, 9, \cdots\}$. By applying the partial fraction one obtains:

$$\frac{x(z)}{z} = \frac{z+1}{(z-1)^2} \rightarrow \frac{x(z)}{z} = \frac{A}{(z-1)^2} + \frac{B}{z-1}$$

coefficients are obtained:

$$\begin{aligned}
A &= \left[\frac{z+1}{(z-1)^2}(z-1)^2\right]_{z=1} = 2 \\
B &= \left[\frac{d}{dz}\left[\frac{z+1}{(z-1)^2}(z-1)^2\right]\right]_{z=1} = 1
\end{aligned}$$

so that the final expression becomes, $x(z) = 2\frac{z}{(z-1)^2} + \frac{z}{z-1}$ which leads to $x(n) = 2n+1$.

Q4. (25 marks) For the tank system model given in the Figure below by the following transfer function $G(z) = \frac{1}{z+0.1}$, find the response of the system to the step input signal. The closed loop transfer function is $G_{cl}(z) = \frac{G(z)}{1+G(z)} = \frac{1}{z+1.1}$. The step input to $G_{cl}(z)$ is given as

$$y(z) = \frac{1}{z+1.1}\frac{z}{z-1} \rightarrow y(z)/z = \frac{A}{z+1.1} + \frac{B}{z-1}$$

$$\begin{aligned}
A &= \left[\frac{1}{(z-1)(z+1.1)}(z+1.1)\right]_{z=-1.1} = -1/2.1 \\
B &= \left[\frac{1}{(z-1)(z+1.1)}(z-1)\right]_{z=1} = 1/2.1
\end{aligned}$$

which becomes $y(n) = -\frac{1}{2.1}(-1.1)^n + \frac{1}{2.1}$. As $n \rightarrow \infty$ response diverges.