Outline for Midterm Examination of ChE 573

February 12, 2007

The midterm examination will be mainly based on the lecture notes, which have been posted on and are printable from the relevant course homepage. **Chapters 2, 3, 4, 6** will be covered in the midterm. Basically, problems to appear in the midterm will have the similar forms to those in the given assignments. Some necessary formulas will be provided. The key in the midterm is to test students how to solve problems with the learned theory.

The exam is closed book and closed notes. Only unprogramable calculator is permitted.

The exam will be held at CME 344 from 2:00pm to 4:00pm on Friday, Feb. 16, 2007.

For the convenience of students to prepare for the midterm examination, the contents to possibly appear in the exam are summarized as follows.

Chapter 2. Understand the notations of complex variables, including correct determination of phase (angle) of a complex number in the complex plane. Use of the **Euler's rule** to solve problems, e.g. to represent any sinusoids, to derive identities and simplify operations in trigonometry. Fully understand both continuous-time (CT) and discrete-time (DT)**sinusoidal signals**, including its magnitude, frequency , fundamental period, and phase (angle). Understand the concept of aliasing in DT sinusoids. Be able to use the Euler formula to perform operations on sinusoids.

Chapter 3. Understand the process of A/D and D/A converters and know how to apply it to implement a digital process control system. Understand the phenomenon of aliasing in sampling. Be able to use the sampling theorem.

Chapter 4. Understand how to represent DT signals and systems. Be familiar with a few elementary DT signals, e.g. unit sample (impulse) function $\delta[n]$, unit step function $u[n]$. Be familiar with classification of DT systems. Be able to use the convolution sum to calculate the response of a DT system given the input signal, $x[n]$, and impulse response function, $h[n]$. Be able to analyze the stability and causality in terms of impulse response.

Chapter 6. Understand the concepts of z-transform and its ROC. Understand the properties of z-transform and know how to calculate it from its definition and using the properties. Know the relationship between ROC and causality for infinite-duration signals. Understand the concept of poles and zeros in the rational z-transform. Be able to apply z-transform and inverse z-transform to analyze DT LTI systems (transfer function, impulse response, stability and causality).

Midterm questions for ChE 573-2007

February 16, 2007.

Problem 1 (25 Marks)

An amplitude-modulated (AM) cosine wave is represented by the formula

$$x_a(t) = [10 + \cos(2\pi(20)t)]\cos(2\pi(40)t)$$

(a) Give the nonnegative frequency components of this signal in Hertz. (Hint: you may need $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$ or Euler's rule $\cos\theta = (e^{j\theta} + e^{-j\theta})/2$ (5 marks)

(b) Is this signal periodic? If yes, what is the fundamental period? (5marks)

(c) What is the Nyquist frequency in this signal? (3 marks)

(d) What is the range of the sampling rate F_s in order to avoid aliasing? (2 marks)

(e) If $x_a(t)$ is sampled at sampling rate F_s = 100Hertz, what is the resulting discrete-time(DT) signal $x[n]$? (5 marks)

(f) Is this DT signal $x[n]$ periodic? If yes, what is its fundamental period? (5 marks)

Solution: (a) It can be seen that

$$\begin{aligned} & \cos(2\pi(20)t)\cos(2\pi(40)t) \\ = & \frac{1}{2}(\cos(2\pi(20+40)t) + \cos(2\pi(20-40)t)) \\ = & \frac{1}{2}(\cos(2\pi(60)t) + \cos(2\pi(20)t)). \end{aligned}$$

Thus, $x_a(t)$ is changed to

$$x_a(t) = 10\cos(2\pi(40)t) + \frac{1}{2}(\cos(2\pi(60)t) + \cos(2\pi(20)t)),$$

and its frequency components are $F_1 = 40\text{Hertz}, F_2 = 60\text{Hertz}, F_3 = 20\text{Hertz}$

(b) Yes, it is periodic. The period is

$$T_p = \frac{1}{20}.$$

(c) The Nyquist frequency is

$$F_{ny} = 2F_{\max} = 2 * \max\{40, 60, 20\} = 120\text{Hertz}.$$

(d) The range of F_s avoiding aliasing is

$$F_s > 2F_{ny} = 120\text{Hertz}.$$

(e) The resulting DT signal is

$$\begin{aligned}
x[n] &= x_a(nT) = x_a(\frac{n}{F_s}) \\
&= 10\cos(2\pi(40)\frac{n}{100}) + \frac{1}{2}(\cos(2\pi(60)\frac{n}{100}) + \cos(2\pi(20)\frac{100}{n})) \\
&= 10\cos(2\pi(\frac{2}{5})n) + \frac{1}{2}(\cos(2\pi(\frac{3}{5})n)) + \cos(2\pi(\frac{1}{5})n)) \\
&= 10\cos(2\pi(\frac{2}{5})n) + \frac{1}{2}\cos(2\pi n - 2\pi(\frac{3}{5})n) + \frac{1}{2}\cos(2\pi(\frac{1}{5})n) \\
&= \frac{21}{2}\cos(2\pi(\frac{2}{5})n) + \frac{1}{2}\cos(2\pi(\frac{1}{5})n)
\end{aligned}$$

(f) Yes, it is periodic. The period $N = 5$.

Problem 2 (25 marks)

Given two DT systems
$y[n] = T\{x\{n]\} = x[n] + 3u[n]$
$y[n] = T\{x[n]\} = 4x[n] - 2x[n-1] + 3x[n-2]$
where $x[n]$ and $y[n]$ are the input and output at the n^{th} sample of each system, respectively.

(a) Check if each system is linear, time-invariant, and causal. Give your explanations please (6 marks).

(b) For the LTI system, calculate its impulse response $h[n]$ and represent $h[n]$ by a finite duration sequence. (hint: use $y[n] = h[n] * x[n] = \sum_{l=-\infty}^{\infty} h[l]x[n-l]$) (5 marks).

(c) Calculate $y[n]$ of the LTI system, given $x[n] = \{2 \quad \underbrace{0.5}_{\uparrow} \quad 3\}$ (5 marks).

(d) Implement the LTI system using a block diagram (4 marks).

(e) Check the stability of the system in terms of $h[n]$. Give your explanation (5 marks).

Solution: (a) For $y[n] = T\{x\{n]\} = x[n] + 3u[n]$:
Linearity: $T\{ax_1[n] + bx_2[n]\} = (ax_1[n] + bx_2[n]) + 3u[n]$ and

$$\begin{aligned}
& aT\{x_1[n]\} + bT\{x_2[n]\} \\
&= a(x_1[n] + 3u[n]) + b(x_2[n] + 3u[n]) \\
&= (ax_1[n] + bx_2[n]) + 6u[n].
\end{aligned}$$

Hence, we can see $T\{ax_1[n] + bx_2[n] \neq aT\{x_1[n]\} + bT\{x_2[n]\}$, it is not linear.
Time-invariance: Since

$$T\{x(n-n_0)\} = x[n-n_0] + 3u[n],$$

and

$$y[n-n_0] = x[n-n_0] + 3u[n-n_0],$$

we get $y[n-n_0] \neq T\{x(n-n_0)\}$, it is not time-invariant.
Causality: It is causal.
For $y[n] = T\{x[n]\} = 4x[n] - 2x[n-1] + 3x[n-2]$:
Linearity: Since

$$\begin{aligned}
& T\{ax_1[n] + bx_2[n]\} \\
&= 4(ax_1[n] + bx_2[n]) - 2(ax_1[n-1] + bx_2[n-1]) + 3(ax_1[n-2] + bx_2[n-2])
\end{aligned}$$

and

$$
\begin{aligned}
& aT\{x_1[n]\} + bT\{x_2[n]\} \\
= \; & a(4x_1[n] - 2x_1[n-1] + 3x_1[n-2]) + b(4(x_1[n] - 2x_1[n-1] + 3x_1[n-2])) \\
= \; & 4(ax_1[n] + bx_2[n]) - 2(ax_1[n-1] + bx_2[n-1]) + 3(ax_1[n-2] + bx_2[n-2]),
\end{aligned}
$$

we get $T\{ax_1[n] + bx_2[n] = aT\{x_1[n]\} + bT\{x_2[n]\}$, and it is linear.

Time-invariance: Since

$$T\{x(n-n_0)\} = 4x[n-n_0] - 2x[n-n_0-1] + 3x[n-n_0-2]],$$

and

$$y[n-n_0] = 4x[n-n_0] - 2x[n-n_0-1] + 3x[n-n_0-2]],$$

we get $y[n-n_0] = T\{x(n-n_0)\}$, it is time-invariant.

Causality: It is causal.

(b) From (a), we see that $y[n] = T\{x[n]\} = 4x[n] - 2x[n-1] + 3x[n-2]$ is LTI. $Y[n]$ can be written as

$$y[n] = \sum_{l=0}^{2} h[l]x[n-l].$$

with $h[l] = \{4, -2, 3\}$.From the definition of convolution sum, we have

$$h[n] = \{4, -2, 3\}$$

(c) We can write $h[n]$ as $h[n] = 4\delta[n] - 2\delta[n-1] + 3\delta[n-3]$, thus

$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= (4\delta[n] - 2\delta[n-1] + 3\delta[n-3]) * x[n] \\
&= 4\delta[n] * x[n] - 2\delta[n-1] * x[n] + 3\delta[n-3] * x[n] \\
&= 4x[n] - 2x[n-1] + 3x[n-2] \\
&= \{8 \;\; \underbrace{2}_{\uparrow} \;\; 12\} - \{\; \underbrace{4}_{\uparrow} \;\; 1 \;\; 6\} + \{\; \underbrace{0}_{\uparrow} \;\; 6 \;\; 1.5 \;\; 9\} \\
&= \{8 \;\; \underbrace{-2}_{\uparrow} \;\; 17 \;\; -4.5 \;\; 9\}
\end{aligned}
$$

(e) The LTI system is stable since

$$\sum_{n=-\infty}^{\infty} |h[n]| = 4 + 2 + 3 = 9$$

is finite.

Problem 3 (24 marks)

In the following figure, the ideal C-to-D converter is used to change a continuous-time (CT) signal to a discrete-time (DT) signal, and the ideal D-to-C converter is used to convert a DT signal $y[n]$ to a CT signal $y(t)$ by ideal reconstruction, that is,

$$y(t) = y[n] = y[\frac{t}{T_s}] = y[tF_s].$$

The input to the C-to-D converter is

$$x(t) = 4 + \cos(250\pi t - \frac{\pi}{4}) - 3\cos(\frac{2000\pi}{3}t).$$

The system transfer function for the LTI system is

$$H(z) = \frac{1}{3}(1 + z^{-1} + z^{-2}).$$

If $F_s = 1000$Hertz, consider the following questions.

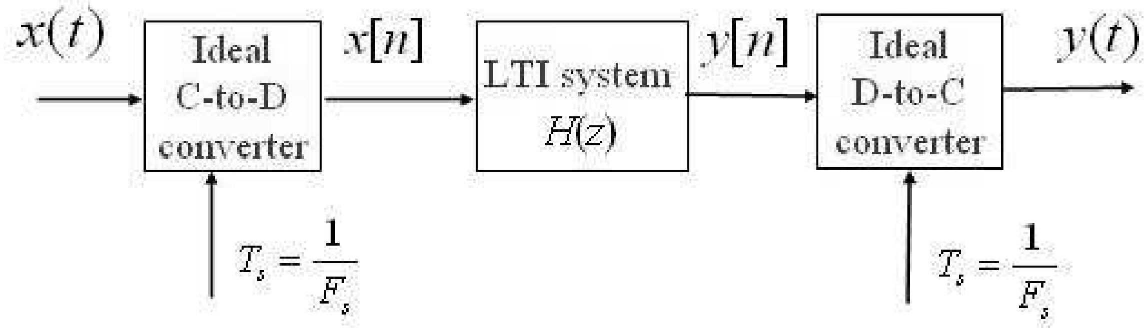
(a) What is the resulting DT signal $x[n]$ by the C-to-D converter?.(5 marks)

(b) What is the impulse response $h[n]$ for the LTI system in terms of impulse signal? Express $h[n]$ by a finite-duration sequence. (4 marks)

(c) What is the output $y[n]$ of the LTI system? (Hint: Using the convolution sum and the property $\delta[n-k] * x[n] = x[n-k]$.) (5 marks)

(d) Is the LTI system $H(z)$ causal or stable? Is the system IIR system or FIR system? Give the reasons in terms of $h[n]$.(6 marks)

(e) What is the overall output $y(t)$ in the figure? (4 marks)



Solution: (a)

$$\begin{aligned} x[n] &= x(\frac{n}{F_s}) \\ &= 4 + \cos(250\pi\frac{n}{1000} - \frac{\pi}{4}) - 3\cos(\frac{2000\pi}{3}\frac{n}{1000}) \\ &= 4 + \cos(\frac{\pi}{4}n - \frac{\pi}{4}) - 3\cos(\frac{2\pi}{3}n). \end{aligned}$$

(b) The impulse response is

$$\begin{aligned} h[n] &= Z^{-1}\{H(z)\} \\ &= \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2]). \\ &= \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}. \end{aligned}$$

(c)

$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2] \\
&= \frac{1}{3}\Big(4 + \cos(\frac{\pi}{4}n - \frac{\pi}{4}) - 3\cos(\frac{2\pi}{3}n) + 4 + \cos(\frac{\pi}{4}(n-1) - \frac{\pi}{4}) - 3\cos(\frac{2\pi}{3}(n-1)) \\
&\quad +4 + \cos(\frac{\pi}{4}(n-2) - \frac{\pi}{4}) - 3\cos(\frac{2\pi}{3}(n-2))\Big)
\end{aligned}
$$

(d) It is causal since $h[n] = 0$ for $n < 0$. It is also stable since $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$. It is an FIR system since $h[n]$ has finite non-zero elements.

(e)

$$
\begin{aligned}
y(t) &= y[tF_s] \\
&= \frac{1}{3}\Big(4 + \cos(\frac{\pi}{4}1000t - \frac{\pi}{4}) - 3\cos(\frac{2\pi}{3}1000t) + 4 + \cos(\frac{\pi}{4}(1000t-1) - \frac{\pi}{4}) \\
&\quad -3\cos(\frac{2\pi}{3}(1000t-1)) + 4 + \cos(\frac{\pi}{4}(1000t-2) - \frac{\pi}{4}) - 3\cos(\frac{2\pi}{3}(1000t-2))\Big)
\end{aligned}
$$

Problem 4 (26 marks)

Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the different equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1]$$

(a) Determine the transfer function $H(z)$ of the LTI system. (4 marks)

(b) Determine the poles and zeros of $H(z)$.Sketch the zero-pole plot. (5 marks)

(c) If this system is BIBO stable, determine the ROC of $H(z)$. Is this system causal? Explain the reason. (4 marks)

(d) If this system is causal, determine the ROC of $H(z)$. Is this system BIBO stable? Explain the reason. (4 marks)

(e) For the system given in (d), determine the impulse response $h[n]$. (5 marks)

(f) For the $h[n]$ obtained in (e), determine $h[0]$ from the expression of $H(z)$. Verify your result from $h[n]$ obtained in (e).(4 marks)

Solution: (a) Taking the $z-$transform on the both sides of the equation, we obtain

$$Y(z) - \frac{5}{2}z^{-1}Y(z) + z^{-2}Y(z) = X(z) - z^{-1}X(z),$$

from which the TF can be computed as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

(b) $H(z)$ has poles at $p_1 = \frac{1}{2}, p_2 = 2$ and two zeros at $z_1 = 1, z_2 = 0$.

(c) We know that ROC does not include poles. Furthermore, if the system is stable, then ROC of $H(z)$ includes the unit circle. As a result, its ROC must be $\frac{1}{2} < |z| < 2$. This system is non-causal since its ROC is a ring.

(d) If the system is causal, then ROC of $H(z)$ is the exterior of a circle. Thus, ROC is $|z| > 2$. In this case, the system is unstable since the ROC does not include the unit circle.

(e) In this case, $h[n]$ is causal. Expand $H(z)$ as

$$H(z) = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - 2z^{-1}} = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

We obtain

$$A = \frac{1}{3}, B = \frac{2}{3}.$$

So we have

$$H(z) = \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1 - 2z^{-1}}$$

and

$$h[n] = \frac{1}{3}(\frac{1}{2})^n u[n] + \frac{2}{3}(2)^n u[n]$$

(f) Since $h[n]$ is causal, from the initial value theorem

$$h[0] = \lim_{z \to \infty} H(z) = 1$$