CH 7: Nested and Nested Factorial Experiments

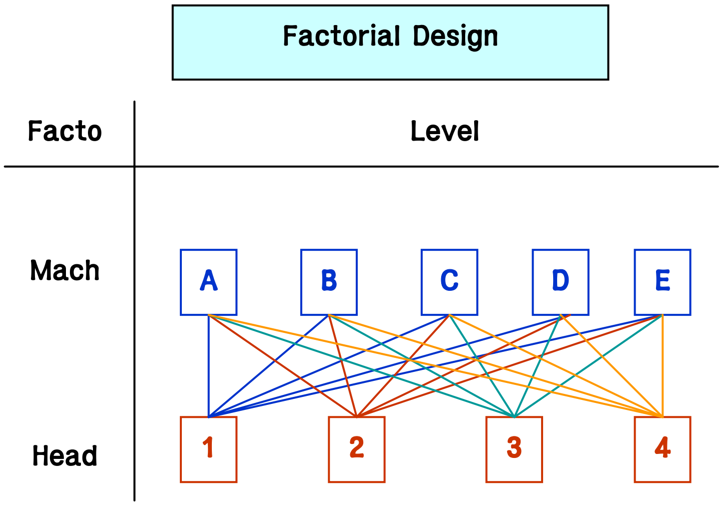
7.1 Introduction

- **Example: Study the Strain Readings of Glass Cathode Supports**
  - ← 5 different Machines
  - ← 4 Heads on each Machine
  - ← 4 replications

● **Question: What if Heads on each Machine are Different?**

← **5*4 Factorial Design???**

- **Factorial Design (Crossed)**



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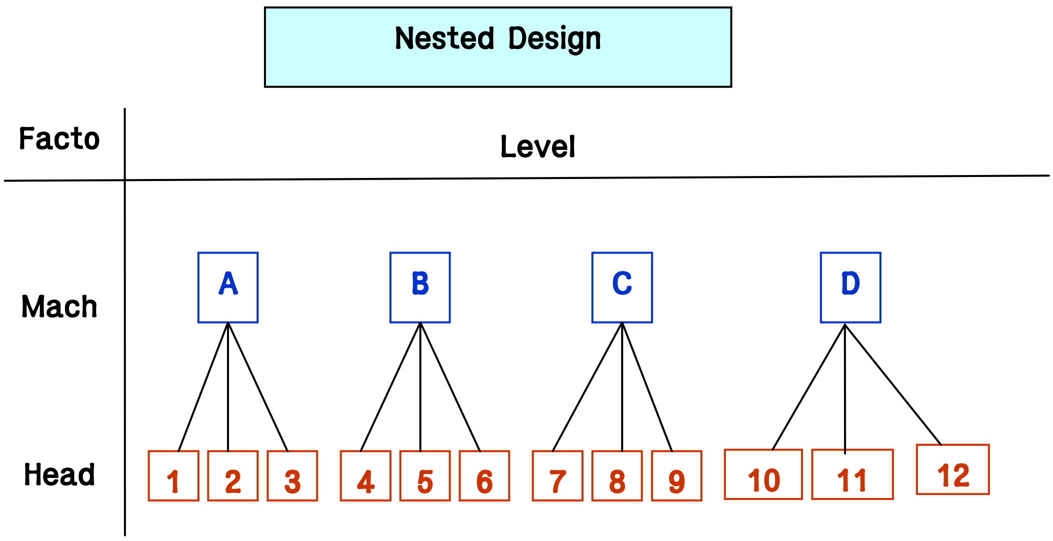
- **Nested Design (Hierarchical Experiment)**



7.2 Nested Experiments

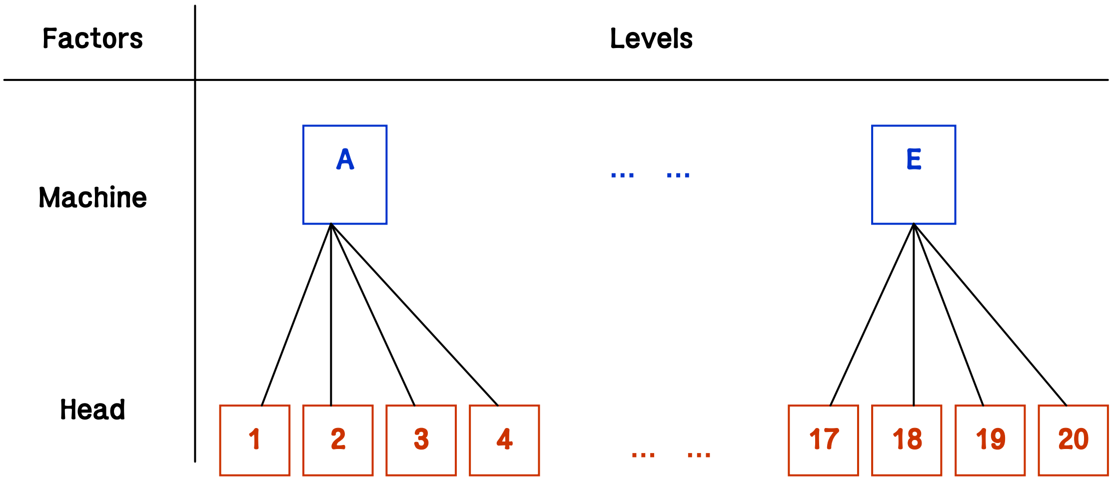
- **Model of the Example:** $Y_{ijk} = \mu + M_i + H_{j(i)} + \varepsilon_{k(ij)}$, with $\varepsilon_{ij} \sim NID(0,\ \sigma_\varepsilon^2)$

  where $i = 1, \cdots, 5 \quad j = 1, \cdots, 4 \quad k = 1, \cdots, 4$

  ← **Note: i) Model of Factorial Experiment:**

  $$Y_{ijk} = \mu + M_i + H_j + MH_{ij} + \varepsilon_{k(ij)}$$

  ii) No Interaction between Machines and Heads in Nested Design



- **Data for Strain-Reading Problem in a Nested Arrangement**

	Machine										
	A				B	C	D	E			
Head	**1**	**2**	**3**	**4**	**5-8**	**9-12**	**13-16**	**17**	**18**	**19**	**20**
	6	13	1	7				1	6	3	3
	2	3	10	4				4	7	0	7
	0	9	0	7				7	0	2	4
	8	8	6	9				9	3	2	0
Head **Totals**	16	33	17	27				21	16	7	14
Mach **Totals**	93				81	82	88	58			

- **Examples:** **Factor in the Nest**
  - i) Farms within Townships
  - ii) Classes within Schools
  - iii) Heads within Machines
  - iv) Samples within Batches

- **Model:** $Y_{ijk} = \mu + M_i + H_{j(i)} + \varepsilon_{k(ij)}$

- **EMS for Nested Experiment**

Source	5 F i	4 R j	4 R k	EMS
M_i	0	4	4	$\sigma_\varepsilon^2 + 4\sigma_H^2 + 16\phi_M$
$H_{j(i)}$	1	1	4	$\sigma_\varepsilon^2 + 4\sigma_H^2$
$\varepsilon_{k(ij)}$	1	1	1	σ_ε^2

- Minitab ANOVA for Nested Strain-Reading Problem (Table 7.4)

Factor	Type	Levels	Values
Machine	fixed	5	1, 2, 3, 4, 5
Head(Machine)	random	4	1, 2, 3, 4

Analysis of Variance for Reading

Source	DF	SS	MS	F	P
Machine	4	45.07	11.27	0.60	0.670
Head(Machine)	15	282.88	18.86	1.76	0.063
Error	60	642.00	10.70		
Total	79	969.95			

	Source	Variance component	Error term	Expected Mean Square for Each Term (using restricted model)
1	Machine		2	(3) + 4 (2) + 16 Q[1]
2	Head(Machine)	2.040	3	(3) + 4 (2)
3	Error	10.700		(3)

- Sum of Squares of the Heads within a Machine:

Machine A: $$SS_H = \frac{(16)^2 + (33)^2 + (17)^2 + (27)^2}{4} - \frac{(93)^2}{16}$$

$$= 590.750 - 540.5624 = 50.1875$$

… … …

… … …

Machine E: $$SS_H = \frac{(21)^2 + (16)^2 + (7)^2 + (14)^2}{4} - \frac{(58)^2}{16}$$

$$= 235.500 - 210.2500 = 25.2500$$

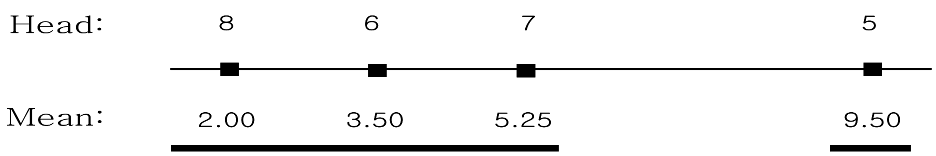
- Detailed ANOVA for Strain-Reading Data

Source	df	SS	MS	F	P
M_i	**4**	**45.075**	**11.269**	**0.60**	**0.670**
$H_{j(i)}$	**15**	**282.875**	**18.858**	**1.76**	**0.063**
$H_j(A)$	**3**	**50.188**	**16.729**	**1.56**	**0.208**
$H_j(B)$	**3**	**126.188**	**42.063**	**3.93**	**0.013**
$H_j(C)$	**3**	**74.750**	**24.917**	**2.33**	**0.083**
$H_j(D)$	**3**	**6.500**	**2.1667**	**0.20**	**0.896**
$H_j(E)$	**3**	**25.250**	**8.4167**	**0.79**	**0.504**
$\varepsilon_{k(ij)}$	**60**	**642.000**	**10.7000**		
Totals	79	969.9500			

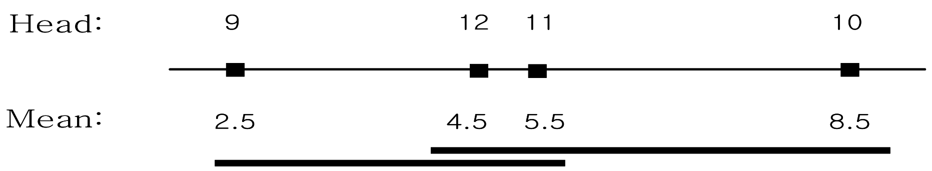
- Further Analysis of the Heads on Machine B and C

	p:	2	3	4
→ $S_{\bar{Y}} = \sqrt{10.70/4} \approx 1.636$ and	LSR:	3.866	4.841	5.418
	$(df\,60)$:	2.363	2.959	3.312

- **Newman-Keuls Test for Machine B → (Q) Any Information/Conclusion?**



- **Newman-Keuls Test for Machine C → (Q) Any Information/Conclusion?**



- General ANOVA for a Nested Experiment

Source	df	SS	MS
A_i	$a-1$	$\sum_i^a T_{i..}^2/nb - T_{...}^2/nab$	$SS_A/(a-1)$
$B_{j(i)}$	$a(b-1)$	$\sum_i^a \sum_j^b T_{ij.}^2/n - \sum_i^a T_{i..}^2/nb$	$SS_B/a(b-1)$
$\varepsilon_{k(ij)}$	$ab(n-1)$	$\sum_i^a \sum_j^b \sum_k^n Y_{ijk}^2 - \sum_i^a \sum_j^b T_{ij.}^2/n$	$SS_\varepsilon/ab(n-1)$
Totals	$abn-1$	$\sum_i^a \sum_j^b \sum_k^n Y_{ijk}^2 - T_{...}^2/nab$	

7.3 ANOVA Rationale

- **Model:** $Y_{ijk} = \mu + A_i + B_{j(i)} + \varepsilon_{k(ij)}$

- **Decomposition:**

$$Y_{ijk} - \mu \equiv (\mu_{i\cdot} - \mu) + (\mu_{ij} - \mu_{i\cdot}) + (Y_{ijk} - \mu_{ij})$$

→ **Decomposition (Sample Version):**

$$Y_{ijk} - \overline{Y}_{\cdots} \equiv (\overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdots}) + (\overline{Y}_{ij\cdot} - \overline{Y}_{i\cdot\cdot}) + (Y_{ijk} - \overline{Y}_{ij\cdot})$$

→ **Sum of Squares of each term over (i, j, k) gives**

$$\sum_{i,j,k}^{a,b,n}(Y_{ijk} - \overline{Y}_{\cdots})^2 \equiv \sum_i^a nb(\overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdots})^2 + \sum_i^a \sum_j^b n(\overline{Y}_{ij\cdot} - \overline{Y}_{i\cdot\cdot})^2 + \sum_{i,j,k}^{a,b,n}(Y_{ijk} - \overline{Y}_{ij\cdot})^2$$

$$SS_{Total} = SS_A + SS_B + SS_{Error}$$

$$(abn-1) = (a-1) + a(b-1) + ab(n-1)$$

- **Remarks:**

$$SS_B = \sum_i^a \sum_j^b n(\overline{Y}_{ij.} - \overline{Y}_{i..})^2 = \sum_i^a \sum_j^b T_{ij.}^2 / n - \sum_i^a T_{i..}^2 / nb$$

$$= \sum_i^a \left(\sum_j^b T_{ij.}^2 / n - T_{i..}^2 / nb\right) = \sum_i^a SS_{B(i)}$$

← Sum of Squares between levels of ***B*** for each level of ***A*,** and then Pooling over all levels of ***A***

7.4 Nested-Factorial Experiments

- In some Multi-Factor Experiments, Both Factors are Crossed with others and Factors are Nested within Levels of the Others

  → i.e., Both Factorial and Nested Factors, fixed or random, appear in the same experiment: **Nested-Factorial Experiment**

- **Example 7.1: Gun-Loading Problem**

  Response: Number of Rounds per minute that could be fired from a Naval Gun

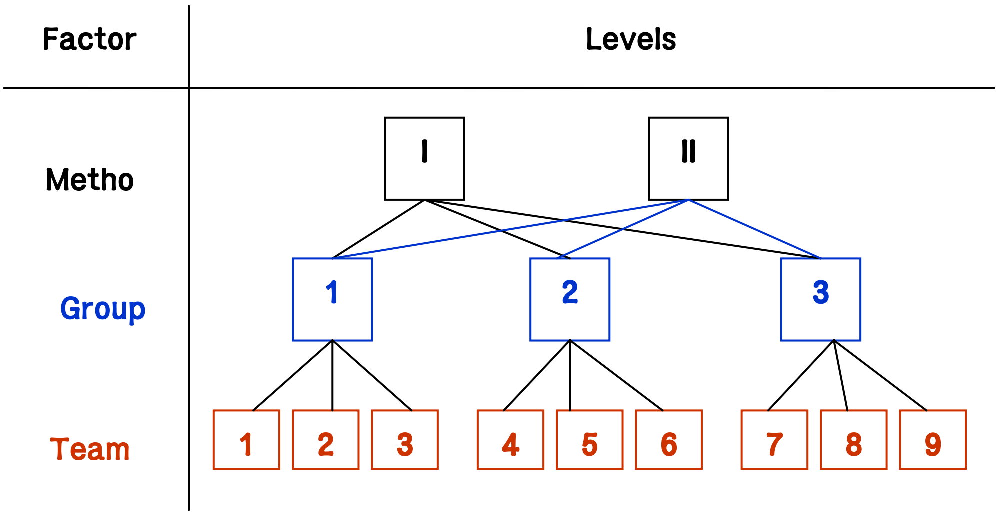
  Factors:

  - Loading Methods: M_1 (New) and M_2 (Existing)
  - Physical group: G_1 (Slight), G_2 (Average), G_3 (Heavy)
  - Teams: 3 teams Randomly Chosen for each group
  - Two replications for each team

  ← Check the randomization and experimental procedures (page 198-9)

- **Data and ANOVA for Gun-Loading Problem (Table 7.8)**

	Group								
	I			II			III		
Team	1	2	3	4	5	6	7	8	9
Methd I	20.2	26.2	23.8	22.0	22.6	22.9	23.1	22.9	21.8
	24.1	26.9	24.9	23.5	24.6	25.0	22.9	23.7	23.5
Methd II	14.2	18.0	12.5	14.1	14.0	13.7	14.1	12.2	12.7
	16.2	19.1	15.4	16.1	18.1	16.0	16.1	13.8	15.1



- **Model:** $Y_{ijkm} = \mu + M_i + G_j + MG_{ij} + T_{k(j)} + MT_{ik(j)} + \varepsilon_{m(ijk)}$

- **EMS for Gun-Loading Problem**

Source	모수	2 F i	3 F j	3 R k	2 R m	EMS
M_i	ϕ_M	0	3	3	2	$\sigma_\varepsilon^2 + 2\sigma_{MT}^2 + 18\phi_M$
G_j	ϕ_G	2	0	3	2	$\sigma_\varepsilon^2 + 4\sigma_T^2 + 12\phi_G$
MG_{ij}	ϕ_{MG}	0	0	3	2	$\sigma_\varepsilon^2 + 2\sigma_{MT}^2 + 6\phi_{MG}$
$T_{k(j)}$	σ_T^2	2	1	1	2	$\sigma_\varepsilon^2 + 4\sigma_T^2$
$MT_{ik(j)}$	σ_{MT}^2	0	1	1	2	$\sigma_\varepsilon^2 + 2\sigma_{MT}^2$
$\varepsilon_{m(ijk)}$	σ_ε^2	1	1	1	1	σ_ε^2

← **EMS 계산시, 계수의 곱을 합할 때 첨자의 괄호 포함 여부는 무관 (참고)**

e.g., $T_{k(j)}$ **항에 대한 EMS 계산**

- **Minitab ANOVA for Gun-Loading Problem (Table 7.9)**

Factor	Type	Levels	Values
Method	fixed	2	1, 2
Group	fixed	3	1, 2, 3
Team(Group)	random	3	1, 2, 3

Analysis of Variance for Speed

Source	DF	SS	MS	F	P
Method	1	651.951	651.95	364.84	0.000
Group	2	16.052	8.026	1.23	0.358
Method*Group	2	1.187	0.594	0.33	0.730
Team(Group)	6	39.258	6.543	2.83	0.040
Method*Team(Group)	6	10.722	1.787	0.77	0.601
Error	18	41.590	2.311		
Total	35	760.760			

	Source	Variance component	Error term	Expected Mean Square for Each Term (using restricted model)
1	Method		5	(6) + 2 (5) + 18 Q[1]
2	Group		4	(6) + 4 (4) + 12 Q[2]
3	Method*Group		5	(6) + 2 (5) + 6 Q[3]
4	Team(Group)	1.0581	6	(6) + 4 (4)
5	Method*Team(Group)	-0.2618	6	(6) + 2 (5)
6	Error	2.3106		(6)

- **After ANOVA: → (Q) Any Information/Conclusion?**

Note that <u>Method</u> and <u>Team(Group)</u> are Significant!

→ i) Further Analysis of Team Effects for each Group (see Table 7-10~12)

$$SS_{Team(G_1)} = 35.7350, \quad SS_{Team(G_2)} = 1.6217, \quad SS_{Team(G_3)} = 1.9017$$

ii) Group 1 has significant different effect than Group 2 and 3

→ Further Analysis between Teams within Group 1:

→ Team 2 is exceptionally faster than the other two teams !!!

- **Computation of Sum of Squares:** $SS_{M\times T(G)}$

- **Data on Gun-Loading Problem for Group I (Table 7.10)**

	Team			Method
	1	2	3	Totals
Method I	20.2	26.2	23.8	
	24.1	26.9	24.9	
	44.3	53.1	48.7	146.1
Method II	14.2	18.0	12.5	
	16.2	19.1	15.4	
	30.4	37.1	27.9	95.4
Team totals	74.7	90.2	76.6	241.5

→ **Computation: Factorial Experiment (see, Page 147)**

$$SS_{Cell} = \frac{44.3^2 + 53.1^2 + \cdots + 27.9^2}{2} - \frac{241.5^2}{12} = 256.1975$$

$$SS_{Method} = \frac{146.1^2 + 95.4^2}{6} - \frac{241.5^2}{12} = 214.2075$$

$$SS_{Team} = \frac{74.7^2 + 90.2^2 + 76.6^2}{4} - \frac{241.5^2}{12} = 35.7350$$

→ $SS_{M\times T} = SS_{Cell} - SS_{Method} - SS_{Team} = 256.1975 - 214.2075 - 35.7350 = 6.2550$

- **Data on Gun-Loading Problem for Group II**

$SS_{Cell} = 199.9567$, $SS_{Method} = 196.8300$, $SS_{Team} = 1.6217$

→ $SS_{M\times T} = 199.9567 - 196.8300 - 1.6217 = 1.5050$

- **Data on Gun-Loading Problem for Group III**

$SS_{Cell} = 246.9642$, $SS_{Method} = 242.1009$, $SS_{Team} = 1.9017$

→ $SS_{M\times T} = 246.9642 - 242.1009 - 1.9017 = 2.9616$

→ $$SS_{M\times T(G)} = SS_{M\times T(G_1)} + SS_{M\times T(G_2)} + SS_{M\times T(G_3)} = 6.2550 + 1.5050 + 2.9616 \approx 10.722$$ **: By Pooling for All Groups**

7.5 Repeated-Measures Design Nested-Factorial Experiments

- **Repeated Measure** = A Special Case of Factorial and Nested-Factorial Design
  ← Same Subjects are measured **repeatedly**

  - **Example 7.2 (Pre- and Post- Effects):** Evaluate Physical Strength Before and After a Specified Training Period (Factorial Experiment Type)

Subjects	Pretest	Post-Test
1	100	115
2	110	125
3	90	105
4	110	130
5	125	140
6	130	140
7	105	125

← Between Subjects

← Within Subjects

← **Model:** $Y_{ij} = \mu + S_i + \varepsilon_{j(i)}$

(DF 6 7)

ANOVA (Repeated Measure Design)

Source	DF	SS	MS	F
Between Sub (S_i)	**6**	**2084.72**	**347.45**	
Within Subjects	**7**	**902.00**	**---**	
Test (T_j)	**1**	**864.29**	**864.29**	**145**
Residuals	**6**	**35.71**	**5.96**	
Totals	**13**	**2985.71**		

Model: $Y_{ij} = \mu + S_i + T_j + ST_{ij}$ **(Factorial Design with No Replication)**

(DF 6 1 6) ← **Subjects (Random), Test (Fixed)**

ANOVA (Repeated Measure Design)

Source	DF	7 R i	2 F j	1 R k	EMS
S_i	**6**	**1**	**2**	**1**	$\sigma_\varepsilon^2 + 2\sigma_S^2$
T_j	**1**	**7**	**0**	**1**	$\sigma_\varepsilon^2 + \sigma_{ST}^2 + 7\phi_T$ ← MS_S / MS_{ST}
ST_{ij}	**6**	**1**	**0**	**1**	$\sigma_\varepsilon^2 + \sigma_{ST}^2$
$\varepsilon_{k(ij)}$	**0**	**1**	**1**	**1**	σ_ε^2 **(not retrievable)**

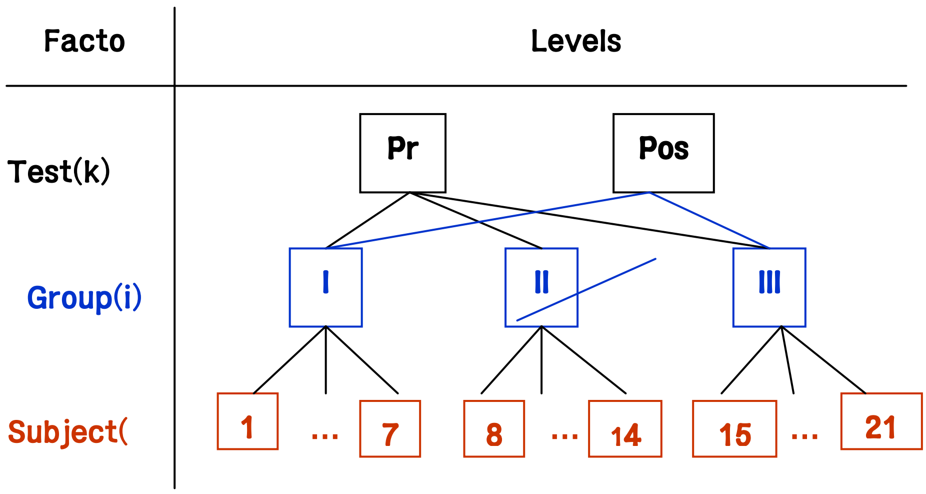
← **Same as a Randomized Block Design**

← **Same as Paired Comparison (t-Test)**

- **Example 7.3 (Pre- and Post- Effects):** Evaluate Physical Strength Before and After a Specified Training Period (Nested-Factorial Experiment Type),

Group	Subjects	Pretest	Post-Test
I	1	26.25	29.50
	...	...	...
	7	29.33	31.15
II	8	27.47	28.74
	...	...	...
	14	28.09	28.99
III	15	22.27	22.52
	...	...	...
	21	27.55	27.86

→ A Nested-Factorial Design: Subjects are nested within Groups and then Tests are Factorial on both Groups and Subjects



← Model: $Y_{ijkm} = \mu + G_i + S_{j(i)} + T_k + GT_{ik} + TS_{kj(i)} + \varepsilon_{m(ijk)}$

(DF 2 18 1 2 18 0)

3*(7-1) (3-1)*(2-1) 1*18

ANOVA

Source	DF	SS	MS	EMS	F
Between Sub	**20**	**271.05**	**--**		
Group(G_i)	**2**	**28.14**	**14.07**	$\sigma_\varepsilon^2 + 2\sigma_S^2 + 14\phi_G$	**1.04**
Sub in G ($S_{j(i)}$)	**18**	**242.91**	**13.50**	$\sigma_\varepsilon^2 + 2\sigma_S^2$	
Within Sub	**21**	**35.73**	**--**		
Test (T_k)	**1**	**21.26**	**21.26**	$\sigma_\varepsilon^2 + \sigma_{TS}^2 + 21\phi_T$	**183**
$G \times T$	**2**	**12.38**	**6.19**	$\sigma_\varepsilon^2 + \sigma_{TS}^2 + 7\phi_{GT}$	**53**
$T \times S_{kj(i)}$	**18**	**2.09**	**0.116**	$\sigma_\varepsilon^2 + \sigma_{TS}^2$	

- **Derivation of EMS: See Table 7.18 (Page 207)**

☐ **Two Cases of Three-Factor Experiments with Repeated Measure**

- **Case I: Two of Factors (Fixed) crossing (or Repeated) All Subjects**

$$\begin{aligned} Y_{ijkm} &= \mu + A_i + S_{j(i)} \\ &\quad + B_k + AB_{ik} + BS_{kj(i)} + C_m + AC_{im} + CS_{mj(i)} + BC_{km} + ABC_{ikm} + BCS_{mkj(i)} \end{aligned}$$

- **Case II: Subjects are Nested within Two Factors (Fixed) and the Third Factors Crossing (or is Repeated on) All Subjects**

$$\begin{aligned} Y_{ijkm} &= \mu + A_i + B_j + AB_{ij} + S_{k(ij)} \\ &\quad + C_m + AC_{im} + BC_{jm} + ABC_{ijm} + CS_{mk(ij)} \end{aligned}$$