Information Sciences

Fuzzy Multiple Moderation and Moderated-Mediation Analysis based on and Metrics with Evolutionary Algorithms and Neural-Networks

--Manuscript Draft--

Manuscript Number:	INS-D-23-8689
Article Type:	VSI: FSMLI
Keywords:	Fuzzy multiple moderation Fuzzy multiple moderated-mediation Fuzzy least squares estimation Fuzzy least absolute deviation Evolutionary Algorithms Neural-Networks

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Abstract:	Many researchers have conducted empirical studies with multiple moderators, which offer a more comprehensive understanding of complex mechanisms of causal relations. In the real world, many phenomena cannot be accurately expressed using crisp numbers, especially when describing subjective human experiences using linguistic expressions. Therefore, representing such phenomena using soft methods, such as fuzzy theory, is more appropriate. While a few studies have explored fuzzy theory using relatively simple models, more complex models remain unexplored. This paper proposes fuzzy multiple moderation analysis and fuzzy moderated-mediation analysis with multiple moderators, collectively called fuzzy multiple moderators analysis (FMMA). To proceed with our analysis, an estimation methodology and statistical inference are defined using distance approaches termed fuzzy least squares estimation (FLSE) and fuzzy least absolute deviation (FLAD). Subsequently, we present a closed-form mathematical formula within FLSE and employ evolutionary algorithms such as genetic algorithm (GA) and harmony search (HS). An algorithm designed for applying neural networks to fuzzy models is also introduced. We also define statistical inference methodology utilizing L_2-metric, including the fuzzy T-test, fuzzy F-test, and fuzzy R^2. To conduct our analysis, data are utilized from psychology and domains closely related to everyday life, particularly solar power and bike rental counts.

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Cover Letter

Dear Guest Editor,

I and my colleagues completed the extended version of the proceeding that was presented at IFSA 2023.
So, we'd like to submit our paper to the SI of Information Science, “Fuzzy Set Theory, Machine Learning, and Their Interaction”.

The paper title is “Fuzzy Multiple Moderation and Moderated-Mediation Analysis based on and Metrics with Evolutionary Algorithms and Neural-Networks”.
And the above paper fits three topics from the list of topics of interest as follows:

Topics of interest

- Fuzzy Neural Networks and Neuro-Fuzzy Models
- Fuzzy Evolutionary Algorithms and Genetic Fuzzy Systems
- Soft Computing Methods in Machine Learning

Also, the Original Article Statements are as follows:

- This manuscript is the authors' original work and has not been published nor has it been submitted simultaneously elsewhere.
- All authors have checked the manuscript and have agreed to the submission.

Thank you for your time and support.
Looking forward to hearing from you soon.

Best regards,
Jin Hee Yoon

Information Sciences *** (2023) *****

Contents lists available at ScienceDirect

Information Sciences

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Fuzzy Multiple Moderation and Moderated-Mediation Analysis based on L_1 and L_2 Metrics with Evolutionary Algorithms and Neural-Networks

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A R T I C L E I N F O

Keywords:
Fuzzy multiple moderation
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A B S T R A C T

Many researchers have conducted empirical studies with multiple moderators, which offer a more comprehensive understanding of complex mechanisms of causal relations. In the real world, many phenomena cannot be accurately expressed using crisp numbers, especially when describing subjective human experiences using linguistic expressions. Therefore, representing such phenomena using soft methods, such as fuzzy theory, is more appropriate. While a few studies have explored fuzzy theory using relatively simple models, more complex models remain unexplored. This paper proposes fuzzy multiple moderation analysis and fuzzy moderated-mediation analysis with multiple moderators, collectively called fuzzy multiple moderators analysis (FMMA). To proceed with our analysis, an estimation methodology and statistical inference are defined using distance approaches termed fuzzy least squares estimation (FLSE) and fuzzy least absolute deviation (FLAD). Subsequently, we present a closed-form mathematical formula within FLSE and employ evolutionary algorithms such as genetic algorithm (GA) and harmony search (HS). An algorithm designed for applying neural networks to fuzzy models is also introduced. We also define statistical inference methodology utilizing L_2-metric, including the fuzzy T-test, fuzzy F-test, and fuzzy R^2. To conduct our analysis, data are utilized from psychology and domains closely related to everyday life, particularly solar power and bike rental counts.

1. Introduction

Social science is a branch of science that investigates human social behavior and social phenomena that occur in human-to-human relationships. Since social science is based on logical empiricism, it has proved the cause and effect of social phenomena, that is, causality, through empirical evidence. Mediation and moderation analyses are prevalent statistical methods to explain causality, especially in psychology [1-6]. There are two variables, a mediator and a moderator, first introduced by Baron and Kenny in 1986 [2] that play an important role in each analysis. A mediator explains the causal relationship between the independent variable (X) and the dependent variable (Y) by being located between them. Mediation analysis is a method of analyzing the indirect effect of X on Y through a mediator. On the other hand, a moderator plays a role of strengthening or weakening the effect of X on Y. Moderation analysis is a method to analyze the effect of a moderator whether it is statistically significant. Mediation

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https://doi.org/**
Received ** ** ****; Received in revised form ** ** ****; Accepted ** ** ****

Available online ** ** ****

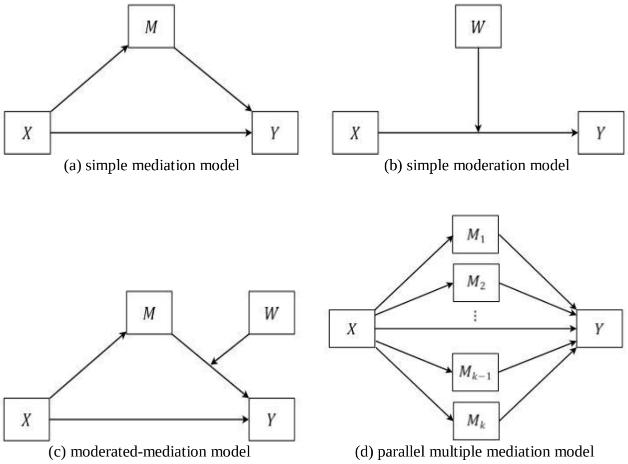


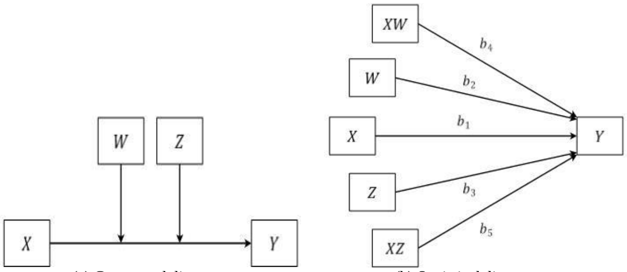
Fig. 1. Simple models, moderated-mediation model and parallel multiple mediation model

and moderation analyses have also been conducted by being integrated into a single statistical model. Several researchers mentioned the necessity of analysis of integrated model, moderated-mediation and mediated-moderation [1, 2, 7], and the following research has provided definite analytic procedures of them [8]. In addition, the moderated-mediation was conducted with various models using the concept of path analysis [9] in which a moderator controls each path of the mediation model, and also analysis of conditional indirect effects was conducted [10, 11]. The moderated-mediation model was redefined as a conditional process analysis [6], and the index of moderated-mediation, a quantification of the relationship between a moderator and an indirect effect, was continuously studied by many other authors [12, 13].

It is undeniable that it is often proper to use models with several moderators [14] due to the complexity of the real world. For instance, in psychology, as a company's job demand for an employee increases, the employee's dissatisfaction with the company will also increase. This situation can be moderated by a person's work environment such as the atmosphere of the team to which the employee belongs and the company's welfare, etc. It suggests that it is more helpful to the company itself to consider several factors at the same time, not just one factor, to reduce its employees' dissatisfaction. As in the example, if two moderators affect the causal relationship between X and Y, the model is called a multiple moderation model [6]. The multiple moderation model is divided into two types of models, the multiple additive moderation model, and moderated-moderation model, depending on the relationship between two moderators. The former's moderators moderate the effect of X on Y independently. On the other hand, in the latter case, the effect of X on Y by a primary moderator is dependent on the secondary moderator. Furthermore, the mediation model also can be moderated by multiple moderators. If two moderators independently control the indirect effect by influencing a common path, such as $X \rightarrow M$ or $M \rightarrow Y$, in mediation model, these models are called partial moderated-mediation [15]. In contrast, if the moderation of the indirect effect by the primary moderator depends on the secondary moderator, these models are named moderated moderated-mediation [15]. A few research of these models with various domains of data have been conducted so far [16-19]. These types of models were statistically analyzed using the intensified concepts such as the index of partial moderated-mediation and the index of moderated moderated-mediation [15, 20]. More active research into the analysis of these models should be conducted since it is necessary to analyze the models with interest so that we can explain complex scientific and social circumstances occurred in the real world more precisely.

Until now, most of the studies of the above models especially in humanities fields have been done on crisp numbers. However, there are many ambiguous data that cannot be precisely expressed as real numbers. For example, there are numerically ambiguous expressions such as 'about 7', 'a few', 'somewhat', and also linguistic expressions like 'small', 'moderate', 'big' that present the degree of something. In addition, there are lots of cases that we need to express human feelings with numbers in psychology, such as the degree of how much a person feels depressed. We cannot express it accurately if we use a crisp number rather than a soft number such as a fuzzy number. As mentioned above, most of analyses in humanities fields have been studied by crisp numbers which caused the loss of information. Therefore, those data which are ambiguous and vague should be expressed with a fuzzy number, which was first introduced by Zadeh [21], to increase the precision of analysis. Researches that used fuzzy numbers in mediation, moderation, and moderated-mediation model were conducted by Yoon in 2020 [22, 23]. In addition, fuzzy mediation analysis with multiple mediators was introduced by Lee et al. [24]. These researches showed that it is more reasonable to use fuzzy data in those models shown in Fig. 1 because it gives more reliable outcomes than using crisp numbers. However, fuzzy analyses that include models with multiple moderators, such as the multiple moderation model and multiple moderated-mediation model, have not been studied yet despite their necessity for explaining data.

In this paper, we present the methodology of estimation and statistical inference of fuzzy multiple moderators analysis (FMMA). By using triangular fuzzy numbers, we suggest two key estimation method; fuzzy least squares estimation (FLSE) [22, 23, 25-28] and fuzzy least absolute deviation (FLAD) based on distance approach, each of which are L_2 and L_1-metric based method. In the case of FLSE, the coefficient estimation that used closed-form formula is suggested which is the true solution and applied to data analysis in section 6. As additional methods for coefficient estimation, evolutionary algorithms are introduced to estimate coefficient with optimization, especially genetic algorithm (GA) and Harmony Search (HS). Moreover, neural-network algorithm is suggested where we use L_2-metric based objective function and utilize four optimization methods based on gradient descent to estimate the coefficients. For the methodology of statistical inference, a fuzzy T-test is defined to judge the significance of the estimated coefficients. In addition, fuzzy F-test and fuzzy R^2 are presented as model fit measurements [29]. Using closed-form



(a) Conceptual diagram (b) Statistical diagram

Fig. 2. A multiple additive moderation model

formula in FLSE and statistical inference methods, we compare the statistical outcomes of FMMA against those of CMMA to discern changes caused by missed information. In addition to the primary purpose of the mediation and moderation analysis, aimed at explaining causal relationships between the variables and estimating their effect on the dependent variables, we conduct the comparisons of the accuracy of FLSE and FLAD methods respectively with regards to fuzzy mean squared error (FMSE) and fuzzy mean absolute error (FMAE). In FLSE, we measure the performances of a range of estimation methods, including closed-form formula, evolutionary algorithms, and neural-network algorithm. On the other hand, FLAD focuses on the performance comparison solely between GA and HS by the measurements of accuracy above, as the objective function is non-differentiable.

In summary, section 2 provides some basic concepts of the multiple moderation model and multiple moderated-mediation model. Section 3 provides basic concepts of fuzzy theory and our proposed fuzzy model. Section 4 elaborates on the methodology of estimation, FLSE and FLAD, including closed-form formula, evolutionary algorithms, and neural-network algorithm. In section 5, we detail the methodology of statistical inference using L_2-metric. Section 6 provides data analysis based on LSE and FLSE using closed-form formula in the psychology field and also data related to our daily life, especially solar power data, applying to models proposed in section 2, 3. Section 7 offers a comparison of the performance among our methods in fuzzy model respectively, and finally, the conclusion is suggested in section 8.

2. Multiple Moderation Analysis & Moderated-Mediation Analysis with Multiple Moderators

In this section, the basic concepts of multiple moderation analysis and mediation analysis with multiple moderators are proposed, involving the regression equation and the meaning of some coefficients [6, 15].

2.1. Multiple Moderation Model

2.1.1 Multiple Additive Moderation Model

The model is illustrated in Fig. 2. Two moderators of this model control the effect of X on Y independently. Before mentioning the regression equation of the model, consider a multiple linear regression model with three antecedent variables X, W, and Z which is basic for the multiple moderation model.

$$Y = b_0 + b_1X + b_2W + b_3Z + \varepsilon_1. \tag{1}$$

We could replace b_1 with an additive linear function $f(W,Z)$ of both W and Z, as follows:

$$Y = b_0 + f(W,Z)X + b_2W + b_3Z + \varepsilon_1, \tag{2}$$
$$f(W,Z) = b_1 + b_4W + b_5Z. \tag{3}$$

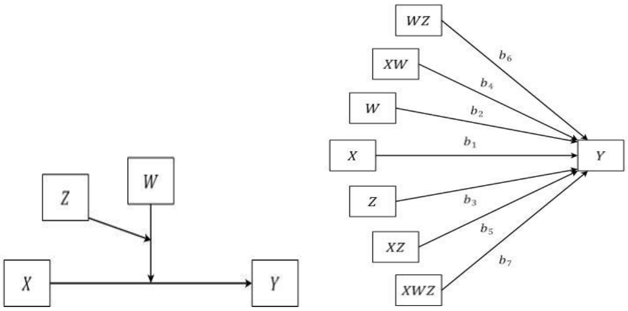
Then, we can get the regression equation (4) by substituting (3) for $f(W,Z)$ in (2), and expansion of (4) is (5).

$$Y = b_0 + (b_1 + b_4W + b_5Z\ )X + b_2W + b_3Z + \varepsilon_1, \tag{4}$$
$$Y = b_0 + b_1X + b_2W + b_3Z + b_4XW + b_5XZ\ + \varepsilon_1\,. \tag{5}$$

It can be expressed with the conditional effect of X on Y ($\theta_{X\to Y}$) as follows:

$$\theta_{X\to Y} = b_1 + b_4W + b_5Z,$$
$$Y = b_0 + \theta_{X\to Y}X + b_2W + b_3Z + \varepsilon_1. \tag{6}$$

The regression coefficients of the product of X and W (b_4) and X and Z (b_5) estimate the rate of change in the conditional effect of X on Y as W or Z changes by one unit when Z or W is constant, respectively.



(a) Conceptual diagram (b) Statistical diagram

Fig. 3. A moderated-moderation model

2.1.2 Moderated-Moderation Model

Illustrated in Fig. 3, this model is also called three-way interaction, since X, W, and Z interact. In this model, a moderator W)'s moderation of effect of X on Y depends on another moderator (Z). The product WZ and XWZ are added to (5) as following equation.

$$Y = b_0 + b_1X + b_2W + b_3Z + b_4XW + b_5XZ + b_6WZ + b_7XWZ + \varepsilon_1. \tag{7}$$

This can be rewritten as follows:

$$Y = b_0 + (b_1 + b_4W + b_5Z+b_7WZ)X + b_2W + b_3Z + b_6WZ + \varepsilon_1. \tag{8}$$

It can be expressed with the conditional effect of X on Y ($\theta_{X\to Y}$) as follows:

$$\begin{aligned}&\theta_{X\to Y} = b_1 + b_4W + b_5Z + b_7WZ,\\ &Y = b_0+\theta_{X\to Y}X + b_2W + b_3Z + b_6WZ + \varepsilon_1 .\end{aligned} \tag{9}$$

To find out how the effect of X on Y by W is moderated by Z, we can rewrite (8) again.

$$Y = b_0 + (b_1 + b_5Z)X + [(b_4 + b_7Z)W]X + b_2W + b_3Z + b_6WZ + \varepsilon_1. \tag{10}$$

Seeing $[(b_4 + b_7Z)W]X$, the moderation of the effect of X on Y by W is moderated by Z, and the conditional effect of W by Z, which can be represented as $\theta_{XW\to Y}$, is $b_4 + b_7Z$, so-called the conditional moderation of X by W. In this model, b_7 estimates three-way interaction of X, W, and Z.

2.2. Moderated-Mediation Model with Multiple Moderators

2.2.1 Partial Moderated-Mediation Model

This model is integration of multiple additive moderation and mediation, shown in Fig. 4. In this model, two moderators independently moderate either the first path ($X \to M$) or the second path ($M \to Y$).

1) A first stage dual moderated-mediation

Two moderators moderate the $X \to M$ path. The two regression equations can be expressed like below.

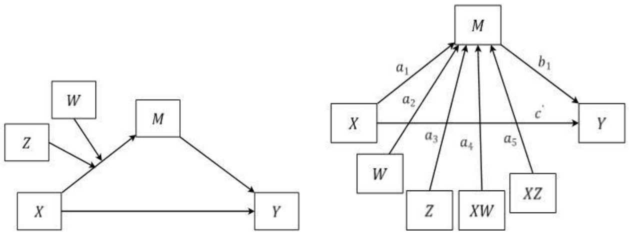
$$\begin{aligned}&M = a_0 + a_1X + a_2W + a_3Z + a_4XW + a_5XZ + \varepsilon_2,\\ &Y = b_0 + c'X + b_1M + \varepsilon_2.\end{aligned} \tag{11}$$

The conditional effect of X on M is

$$\theta_{X\to M} = a_1 + a_4W + a_5Z. \tag{12}$$

The conditional indirect effect of X is equal as the multiplication of $\theta_{X\to M}$ and b_1 ($M \to Y$) as follows:

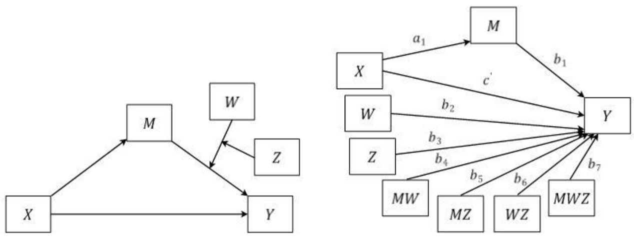
$$\omega = \theta_{X\to M}b_1 = (a_1 + a_4W + a_5Z)b_1 = a_1b_1 + a_4b_1W + a_5b_1Z. \tag{13}$$



(a) Conceptual diagram (b) Statistical diagram

(c) Conceptual diagram (d) Statistical diagram

Fig. 4. A first stage dual moderated-mediation model(a, b) and a second stage dual moderated-mediation(c, d)



(c) Conceptual diagram (d) Statistical diagram

Fig. 5. A first stage moderated moderated-mediation model(a, b) and a second stage moderated moderated-mediation(c, d)

2) A second stage dual moderated-mediation

Two moderators moderate the $M \rightarrow Y$ path, and it is illustrated in Fig. 4. The two regression equations can be expressed like below.

$$M = a_0 + a_1 X + \varepsilon_1,$$
$$Y = b_0 + c^{`} X + b_1 M + b_2 W + b_3 + b_4 MW + b_5 MZ + \varepsilon_2. \tag{14}$$

The conditional effect of X on M is

$$\theta_{M\rightarrow Y} = b_1 + b_4 W + b_5 Z. \tag{15}$$

The conditional indirect effect of $X \rightarrow M \rightarrow Y$ is equal as the multiplication of a_1, the effect of X on M, and $\theta_{M\rightarrow Y}$.

$$\omega = a_1 \theta_{M\rightarrow Y} = a_1(b_1 + b_4 W + b_5 Z) = a_1 b_1 + a_1 b_4 W + a_1 b_5 Z. \tag{16}$$

2.2.2 Moderated Moderated-Mediation Model

This model is integration of moderated-moderation and mediation, represented in Fig. 5. In this model, two moderators that a primary moderator is moderated by the other are involved in the first stage ($X \rightarrow M$) or the second stage ($M \rightarrow Y$).

1) A first stage moderated moderated-mediation model

A three-way interaction by X, W, Z is involved in $X \rightarrow M$ path in this model. Two regression equations are below.

$$M = a_0 + a_1 X + a_2 W + a_3 Z + a_4 XW + a_5 XZ + a_6 WZ + a_7 XWZ + \varepsilon_1,$$
$$Y = b_0 + c^{`} X + b_1 M + \varepsilon_2. \tag{17}$$

The conditional effect of X on M, and the conditional indirect effect of $X \rightarrow M \rightarrow Y$ are below.

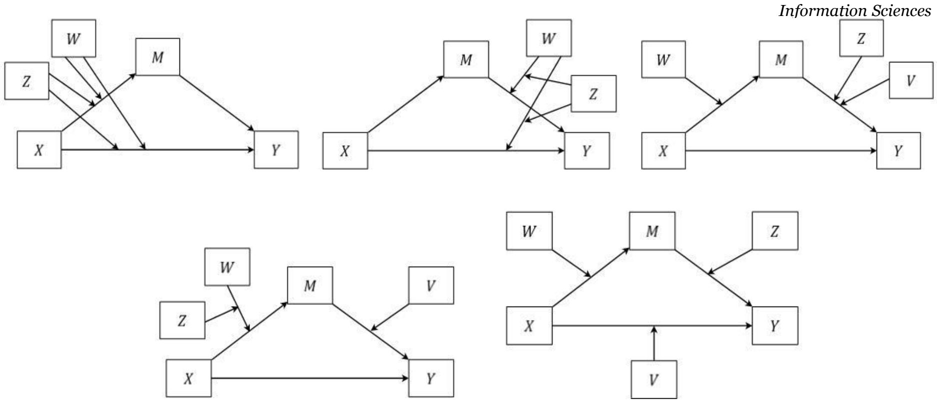


Fig. 6. More complex moderated-mediation models

$$\theta_{X\to M} = a_1 + a_4W + a_5Z + a_7WZ, \tag{18}$$
$$\omega = \theta_{X\to M}b_1 = (a_1 + a_4W + a_5Z + a_7WZ)b_1 = a_1b_1 + a_5b_1Z + (a_4b_1 + a_7b_1Z)W. \tag{19}$$

2) A second stage moderated moderated-mediation model

A three-way interaction by X, W, Z is involved in $M \to Y$ path in this model. Two regression equations are below.

$$M = a_0 + a_1X + \varepsilon_1,$$
$$Y = b_0 + c'X + b_1M + b_2W + b_3Z + b_4MW + b_5MZ + b_6WZ + b_7MWZ + \varepsilon_2. \tag{20}$$

The conditional effect of X on M, and the conditional indirect effect of $X \to M \to Y$ are below.

$$\theta_{M\to Y} = b_1 + b_4W + b_5Z + b_7WZ, \tag{21}$$
$$\omega = a_1\theta_{M\to Y} = a_1(b_1 + b_4W + b_5Z + b_7WZ) = a_1b_1 + a_1b_5Z + (a_1b_4 + a_1b_7Z)W. \tag{22}$$

More complex models that multiple moderators influence mediation are illustrated in Fig. 6.

3. Proposed Fuzzy Multiple Moderation Analysis & Fuzzy Moderated-Mediation Analysis with Multiple Moderators

In this section, some features of fuzzy numbers and some kinds of them are mentioned [21, 30]. Moreover, applying basic concepts from section 2, fuzzy multiple moderation and fuzzy multiple moderated-mediation analyses are proposed.

3.1. *Fuzzy Numbers*

Whether an element is included in a set can be expressed by the membership function. If the element is included in A which is a crisp subset of $\mathbb{R}$, the value of membership function is 1 also expressed as $\mu_A(x) = 1$, and if not, $\mu_A(x) = 0$. In other words, the membership function matches the element to a set $\{0, 1\}$, called mapping. However, if A is a fuzzy subset of $\mathbb{R}$, the membership function matches $\mathbb{R}$ to $[0, 1]$. Thus, the possibility of the element's belonging, or the membership function, becomes a value between 0 and 1. It means that the boundary of a fuzzy set is ambiguous. For any $\alpha \in (0, 1]$ the crisp $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ is called the α-cut or α-level set of A. If a fuzzy set defined in $\mathbb{R}$ is convex, normalized, and its membership function is continuous, this fuzzy set is called a fuzzy number. There are various types of fuzzy numbers, and they are usually expressed with the membership function. Among them, we often use a special form of fuzzy numbers, the so-called LR-fuzzy numbers, defined as follows:

$$\mu_A(x) = \begin{cases} L(\frac{m-x}{l}) & if \quad x \leq m \\ R(\frac{x-m}{r}) & if \quad x > m \end{cases} \qquad for\ x \in \mathbb{R},$$

where m means the mode of the fuzzy number A, and $l, r > 0$ mean left and right spread respectively. L and R are reference functions of X where $L, R : \mathbb{R}^+ \to [0, 1]$ are fixed left-continuous and non-increasing functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. We express the LR-fuzzy number as $A = \langle m, l, r\rangle_{LR}$. The degree of fuzziness of the number depends on the value of l and r that a fuzzy number could be either symmetric or asymmetric. If $l = r = 0$, the number has no fuzziness, so it can be regarded as a crisp number. We can get α-cuts of fuzzy numbers expressed as the intervals $A_\alpha = [m - L^{-1}(\alpha)l, m + R^{-1}(\alpha)r]$. Particularly, if $A = \langle m, l, r\rangle_{LR}$ and its reference functions are $L(x) = R(x) = [1 - x]^+$, then we regard A as a triangular fuzzy number and expressed as $A = \langle m, l, r\rangle_{LR} = \langle m - l, m, m + r\rangle_\Delta$. Let X, Y triangular fuzzy numbers. Basic operations based on extension principle [21] can be applied to X, Y as follows:

$$X \oplus Y = (l_x + l_y, x + y, r_x + r_y),$$
$$kX = \begin{cases}(kl_x, kx, kr_x) & if\ k \geq 0\\ (kr_x, kx, kl_x) & if\ k < 0\end{cases},$$

where $X = (l_x, x, r_x), Y = (l_y, y, r_y)$.

3.2. Fuzzy Multiple Moderation Model

3.2.1 Fuzzy Multiple Additive Moderation Model

Fuzzy simple moderation model was studied by Yoon previously [23]. Applied to this, if $\tilde{X}$ is a fuzzy explanatory variable and $\tilde{Y}$ is a fuzzy response variable, then $\widetilde{W}$ and $\tilde{Z}$ are fuzzy moderators which controls the effect of $\tilde{X}$ on $\tilde{Y}$. Thus, the proposed fuzzy multiple additive moderation model can be expressed with additive function $f(\widetilde{W}, \tilde{Z})$ as follows:

$$\tilde{Y} = b_0 \oplus f(\widetilde{W}, \tilde{Z}) \circledast \tilde{X} \oplus b_2\widetilde{W} \oplus b_3\tilde{Z} \oplus \tilde{\varepsilon}_1,$$
$$f(\widetilde{W}, \tilde{Z}) = b_1 \oplus b_4\widetilde{W} \oplus b_5\tilde{Z}.$$

It can be rewritten as

$$\tilde{Y} = b_0 \oplus (b_1 \oplus b_4\widetilde{W} \oplus b_5\tilde{Z}) \circledast \tilde{X} \oplus b_2\widetilde{W} \oplus b_3\tilde{Z} \oplus \tilde{\varepsilon}_1.$$

The constant term b_0 is actually $\tilde{1} \cdot b_0$, where $\tilde{1} = (1, 1, 1)$. The effect of $\tilde{X}$ on $\tilde{Y}$ depends on $\widetilde{W}$ and $\tilde{Z}$, so the conditional effect of $\tilde{X}$ on $\tilde{Y}(\theta_{\tilde{X}\to\tilde{Y}})$ is equal to $b_1 \oplus b_4\widetilde{W} \oplus b_5\tilde{Z}$.

3.2.2 Fuzzy Moderated-Moderation Model

In this model, the effect of $\tilde{X}$ on $\tilde{Y}$ by a primary fuzzy moderator ($\widetilde{W}$) depends on a secondary fuzzy moderator ($\tilde{Z}$), and it can be expressed as follows:

$$\begin{aligned}\tilde{Y} &= b_0 \oplus b_1\tilde{X} \oplus b_2\widetilde{W} \oplus b_3\tilde{Z} \oplus b_4\tilde{X} \circledast \widetilde{W} \oplus b_5\tilde{X} \circledast \tilde{Z} \oplus b_6\widetilde{W} \circledast \tilde{Z} \oplus b_7\tilde{X} \circledast \widetilde{W} \circledast \tilde{Z} \oplus \tilde{\varepsilon_1}\\ &= b_0 \oplus (b_1 \oplus b_4\widetilde{W} \oplus b_5\tilde{Z} \oplus b_7\widetilde{W} \circledast \tilde{Z}) \circledast \tilde{X} \oplus b_2\widetilde{W} \oplus b_3\tilde{Z} \oplus b_6\widetilde{W} \circledast \tilde{Z} \oplus \tilde{\varepsilon_1}\\ &= b_0 \oplus (b_1 \oplus b_5\tilde{Z}) \circledast \tilde{X} \oplus [(b_4 \oplus b_7\tilde{Z}) \circledast \widetilde{W}] \circledast \tilde{X} \oplus b_2\widetilde{W} \oplus b_3\tilde{Z} \oplus b_6\widetilde{W} \circledast \tilde{Z} \oplus \tilde{\varepsilon_1}.\end{aligned}$$

The fuzzy conditional effect (FCE) of $\tilde{X}$ on $\tilde{Y}$ is $\theta_{\tilde{X}\to\tilde{Y}}$ where $\theta_{\tilde{X}\to\tilde{Y}} = b_1 \oplus b_4\widetilde{W} \oplus b_5\tilde{Z} \oplus b_7\widetilde{W} \circledast \tilde{Z}$. The fuzzy conditional moderation effect (FCME) is $\theta_{\tilde{X}\circledast\widetilde{W}\to\tilde{Y}}$ where $\theta_{\tilde{X}\circledast\widetilde{W}\to\tilde{Y}} = b_4 \oplus b_7\tilde{Z}$, which quantifies the changes of moderation of the effect of $\tilde{X}$ on $\tilde{Y}$ through $\widetilde{W}$ by $\tilde{Z}$. If b_7 is statistically significant, we can say that $\tilde{Z}$ influences $\widetilde{W}$'s moderation of the effect of $\tilde{X}$ on $\tilde{Y}$.

3.3. Fuzzy Moderated-Mediation Model with Multiple Moderators

3.3.1 Fuzzy Partial Moderated-Mediation Model

Fuzzy moderated-mediation model was first introduced by Yoon [23].

1) A first stage dual fuzzy moderated-mediation

This model can be expressed as following equations:

$$\begin{aligned}\widetilde{M} &= a_0 \oplus a_1\tilde{X} \oplus a_2\widetilde{W} \oplus a_3\tilde{Z} \oplus a_4\tilde{X} \circledast \widetilde{W} \oplus a_5\tilde{X} \circledast \tilde{Z} \oplus \tilde{\varepsilon_1}\\ &= a_0 \oplus \theta_{\tilde{X}\to\widetilde{M}} \circledast \tilde{X} \oplus a_2\widetilde{W} \oplus a_3\tilde{Z} \oplus \tilde{\varepsilon_1},\\ \tilde{Y} &= b_0 \oplus c'\tilde{X} \oplus b_1\widetilde{M} \oplus \tilde{\varepsilon_2},\end{aligned}$$

where $\theta_{\tilde{X}\to\widetilde{M}}$ is the conditional effect of $\tilde{X}$ on $\widetilde{M}$, represented as $\theta_{\tilde{X}\to\widetilde{M}} = a_1 \oplus a_4\widetilde{W} \oplus a_5\tilde{Z}$. The fuzzy conditional indirect effect (FCIDE) [23] of $\tilde{X}$ on $\tilde{Y}$ through $\widetilde{M}$ is $b_1\theta_{\tilde{X}\to\widetilde{M}}$, which represented as $b_1\theta_{\tilde{X}\to\widetilde{M}} = b_1(a_1 \oplus a_4\widetilde{W} \oplus a_5\tilde{Z}) = a_1b_1 \oplus a_4b_1\widetilde{W} \oplus a_5b_1\tilde{Z}$.

2) A second stage dual fuzzy moderated-mediation

This model can be expressed as following equations:

$$\widetilde{M} = a_0 \oplus a_1\tilde{X} \oplus \tilde{\varepsilon_1},$$
$$\tilde{Y} = b_0 \oplus c`\tilde{X} \oplus b_1\widetilde{M} \oplus b_2\widetilde{W} \oplus b_3\tilde{Z} \oplus b_4\widetilde{M} \circledast \widetilde{W} \oplus b_5\widetilde{M} \circledast \tilde{Z} \oplus \tilde{\varepsilon_2}$$

$$= b_0 \oplus c`\tilde{X} \oplus \theta_{\tilde{M}\to\tilde{Y}} \circledast \tilde{M} \oplus b_2\tilde{W} \oplus b_3\tilde{Z} \oplus \tilde{\varepsilon_2},$$

where $\theta_{\tilde{M}\to\tilde{Y}}$ is conditional effect of $\tilde{M}$ on $\tilde{Y}$, represented as $\theta_{\tilde{M}\to\tilde{Y}} = b_1 \oplus b_4\tilde{W} \oplus b_5\tilde{Z}$. The fuzzy conditional indirect effect (FCIDE) of $\tilde{X}$ on $\tilde{Y}$ through $\tilde{M}$ is $a_1\theta_{\tilde{M}\to\tilde{Y}}$, which is represented as $a_1\theta_{\tilde{M}\to\tilde{Y}} = a_1(b_1 \oplus b_4\tilde{W} \oplus b_5\tilde{Z}) = a_1b_1 \oplus a_1b_4\tilde{W} \oplus a_1b_5\tilde{Z}$.

3.3.2 Fuzzy Moderated Moderated-Mediation Model

1) A first stage moderated moderated-mediation model

The model can be expressed by following equations:

$$\tilde{M} = a_0 \oplus a_1\tilde{X} \oplus a_2\tilde{W} \oplus a_3\tilde{Z} \oplus a_4\tilde{X} \circledast \tilde{W} \oplus a_5\tilde{X} \circledast \tilde{Z} \oplus a_6\tilde{W} \circledast \tilde{Z} \oplus a_7\tilde{X} \circledast \tilde{W} \circledast \tilde{Z} \oplus \tilde{\varepsilon_1}$$
$$= a_0 \oplus \theta_{\tilde{X}\to\tilde{M}} \circledast \tilde{X} \oplus a_2\tilde{W} \oplus a_3\tilde{Z} \oplus a_6\tilde{W} \circledast \tilde{Z} \oplus \tilde{\varepsilon_1},$$
$$\tilde{Y} = b_0 \oplus c`\tilde{X} \oplus b_1\tilde{M} \oplus \tilde{\varepsilon_2},$$

where $\theta_{\tilde{X}\to\tilde{M}}$ is conditional effect of $\tilde{X}$ on $\tilde{M}$, expressed as $\theta_{\tilde{X}\to\tilde{M}} = a_1 \oplus a_4\tilde{W} \oplus a_5\tilde{Z} \oplus a_7\tilde{W} \circledast \tilde{Z}$. The fuzzy conditional indirect effect (FCIDE) of $\tilde{X}$ on $\tilde{Y}$ through $\tilde{M}$ can be expressed as follows:

$$b_1\theta_{\tilde{X}\to\tilde{M}} = b_1(a_1 \oplus a_4\tilde{W} \oplus a_5\tilde{Z} \oplus a_7\tilde{W} \circledast \tilde{Z})$$
$$= a_1b_1 \oplus a_4b_1\tilde{W} \oplus a_5b_1\tilde{Z} \oplus a_7b_1\tilde{W} \circledast \tilde{Z}$$
$$= a_1b_1 \oplus a_5b_1\tilde{Z} \oplus \left(a_4b_1 \oplus a_7b_1\tilde{Z}\right) \circledast \tilde{W}.$$

2) A second stage moderated moderated-mediation model

The model can be expressed by following equations:

$$\tilde{M} = a_0 \oplus a_1\tilde{X} \oplus \tilde{\varepsilon_1},$$
$$\tilde{Y} = b_0 \oplus c`\tilde{X} \oplus b_1\tilde{M} \oplus b_2\tilde{W} \oplus b_3\tilde{Z} \oplus b_4\tilde{M} \circledast \tilde{W} \oplus b_5\tilde{M} \circledast \tilde{Z} \oplus b_6\tilde{W} \circledast \tilde{Z} \oplus b_7\tilde{M} \circledast \tilde{W} \circledast \tilde{Z} \oplus \tilde{\varepsilon_2}$$
$$= b_0 \oplus c`\tilde{X} \oplus \theta_{\tilde{M}\to\tilde{Y}} \circledast \tilde{M} \oplus b_2\tilde{W} \oplus b_3\tilde{Z} \oplus b_6\tilde{W} \circledast \tilde{Z} \oplus \tilde{\varepsilon_2},$$

where $\theta_{\tilde{M}\to\tilde{Y}}$ is conditional effect of $\tilde{M}$ on $\tilde{Y}$, expressed as $\theta_{\tilde{M}\to\tilde{Y}} = b_1 \oplus b_4\tilde{W} \oplus b_5\tilde{Z} \oplus b_7\tilde{W} \circledast \tilde{Z}$. The fuzzy conditional indirect effect (FCIDE) of $\tilde{X}$ on $\tilde{Y}$ through $\tilde{M}$ can be expressed as follows:

$$a_1\theta_{\tilde{M}\to\tilde{Y}} = a_1(b_1 \oplus b_4\tilde{W} \oplus b_5\tilde{Z} \oplus b_7\tilde{W} \circledast \tilde{Z})$$
$$= a_1b_1 \oplus a_1b_4\tilde{W} \oplus a_1b_5\tilde{Z} \oplus a_1b_7\tilde{W} \circledast \tilde{Z}$$
$$= a_1b_1 \oplus a_1b_5\tilde{Z} \oplus \left(a_1b_4 \oplus a_1b_7\tilde{Z}\right) \circledast \tilde{W}.$$

4. Methodology: Estimation

4.1. Estimation of the Proposed Model with Distance Approach

In order to analyze the fuzzy multiple moderation models based on regression, it is first necessary to clearly define the fuzzy subtraction operation to express the difference between two fuzzy numbers. The reason is that since the subtraction operation of the fuzzy number is not identically defined, different values are derived depending on the calculation method even if it is the same equation. Therefore, by defining this operation as a distance approach using the L_2-metric, the coefficient estimation in fuzzy model was previously studied in [25-28]. We denote the estimation method using L_2-metric as FLSE. In addition, we also suggest the method of estimating coefficients, FLAD, by using L_1-metric as described in section 4.1.2.

4.1.1 Estimation by FLSE using Closed-Form Mathematical Formula

On the spaces of crisp sets, we normally use the least squares estimation to estimate the regression coefficients. There also exist several metrics that are suitable for fuzzy sets applied for the least squares estimation [31, 32, 33]. Generally, the distance between two fuzzy numbers is based on the distance between their α-cuts. A metric which is useful can be defined through support functions. The support function of any compact convex set $A \in \mathbb{R}^d$ is defined as a function $s_A: S^{d-1} \to \mathbb{R}$ given by for all $r \in S^{d-1}$.

$$\mathrm{s_A}(r) = \sup_{a\in A}\langle r, a\rangle,$$

where S^{d-1} is the unit sphere of $(d-1)$ dimension in $\mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ represents the scalar product on $\mathbb{R}^d$. Note that the support function s_A is uniquely determined for convex and compact $A \in \mathbb{R}^d$. An L_2-metric on a fuzzy number set is defined on the space of Lebesgue integrable as below, and

$$\delta_2(A, B) = \left[d \cdot \int_0^1 \int_{S^0} |s_A(\alpha, r) - s_B(\alpha, r)|^2 \mu(dr) d\alpha\right]^{1/2}.$$

Based on this equation, L_2-metric for triangular fuzzy numbers can be defined as follows:

$$d_2(\tilde{X},\tilde{Y}) = \left[\left(l_x - l_y\right)^2 + (x-y)^2 + \left(r_x - r_y\right)^2\right]^{1/2}, \tag{23}$$

where $\tilde{X} = (l_x, x, r_x), \tilde{Y} = (l_y, y, r_y)$.

A fuzzy regression model introduced previously [25, 26] is suggested as follows:

$$\tilde{Y}_i = \beta_0 \oplus \beta_1\tilde{X}_{i1} \oplus \beta_2\tilde{X}_{i2} \oplus \cdots \oplus \beta_p\tilde{X}_{ip} \oplus \tilde{E}_i, \tag{24}$$

where $\tilde{X}_{ij} = (l_{x_{ij}}, x_{ij}, r_{x_{ij}})$, $\tilde{Y}_i = (l_{y_i}, y_i, r_{y_i})$ $(i = 1,\cdots,n;\ j = 1,\cdots,p)$, and $\tilde{E}_i$ are presumed as fuzzy error terms that indicate fuzziness.

All cases can be encompassed by

$$l_{x_{ij}} = \begin{cases} x_{ij} - \xi_{lij}, if\ \beta_j \geq 0, \\ x_{ij} + \xi_{rij}, if\ \beta_j < 0, \end{cases}$$
$$r_{x_{ij}} = \begin{cases} x_{ij} + \xi_{rij}, if\ \beta_j \geq 0, \\ x_{ij} - \xi_{lij}, if\ \beta_j < 0, \end{cases}$$

where $\xi_{l_{ij}}$ and $\xi_{r_{ij}}$ are the left and right spreads of $\tilde{X}_{ij}$, respectively. Now we can get estimators by minimizing following function:

$$\begin{aligned} Q(\beta_{k0}, \beta_{k1}, \cdots, \beta_{kp}) &= \textstyle\sum_{i=1}^n d_2^2(\tilde{Y}_i, \sum_{j=0}^p \beta_{kj}\tilde{X}_{ij}) \\ &= \textstyle\sum_{i=1}^n \left(\left(l_{y_i} - \sum_{j=0}^p \beta_{kj} l_{x_{ij}}\right)^2 + \left(y_i - \sum_{j=0}^p \beta_{kj} x_{ij}\right)^2 + \left(r_{y_i} - \sum_{j=0}^p \beta_{kj} r_{x_{ij}}\right)^2\right), \end{aligned} \tag{25}$$

for $k = 1,2,\cdots,h$, where h is the number of the regression model in this fuzzy analysis. This function can be acquired based on the L_2-metric (23).

To minimize (25), we obtain the normal equation applying $\frac{\partial Q}{\partial \beta_{kl}} = 0$ for each $k = 1,2,\cdots,h$ and for $l = 0,1,\cdots,p$ as follows:

$$\frac{\partial Q}{\partial \beta_{kl}} = 2\textstyle\sum_{i=1}^n \left(l_{y_i} - \sum_{j=0}^p \beta_{kj} l_{x_{ij}}\right) l_{x_{il}} + 2\sum_{i=1}^n \left(y_i - \sum_{j=0}^p \beta_{kj} x_{ij}\right) x_{il} + 2\sum_{i=1}^n \left(r_{y_i} - \sum_{j=0}^p \beta_{kj} r_{x_{ij}}\right) r_{x_{il}}, \tag{26}$$

and for each $k = 1,2,\cdots,h$ the normal equation which has $\hat{\beta}_{kl}$ as solutions can be obtained as follows:

$$\textstyle\sum_{j=0}^p \hat{\beta}_{kj} \sum_{i=1}^n \left(l_{x_{il}} l_{x_{ij}} + x_{il}x_{ij} + r_{x_{il}} r_{x_{ij}}\right) = \sum_{i=1}^n \left(l_{x_{il}} l_{y_i} + x_{il} y_i + r_{x_{il}} r_{y_i}\right). \tag{27}$$

To find the solution vector, a triangular fuzzy matrix (t.f.m.) is defined as follows:

$$\tilde{X} = \begin{bmatrix} (1,1,1) & (l_{x_{11}}, x_{11}, r_{11}) & \cdots & (l_{x_{1p}}, x_{1p}, r_{1p}) \\ \vdots & & \ddots & \vdots \\ (1,1,1) & (l_{x_{n1}}, x_{n1}, r_{n1}) & \cdots & (l_{x_{np}}, x_{np}, r_{np}) \end{bmatrix}.$$

This matrix is denoted as $\tilde{X} = [\tilde{X}_{ij}]_{n\times(p+1)}$ simply, where $\tilde{X}_{ij}$ is a triangular fuzzy number for $i = 1,\cdots,n;\ j = 1,\cdots,p$. Also, a tringular fuzzy vector $\tilde{Y} = [\tilde{Y}_i]^t$ is defined.

To minimize the above objective function (25), fuzzy operations previously studied [25, 28] have been applied.

$$\tilde{X} \circ \tilde{Y} = l_x l_y + xy + r_x r_y,$$
$$\tilde{X} \circledast \tilde{Y} = (l_x l_y, xy, r_x r_y).$$

For given two $n \times n$ t.f.m's, $\tilde{\Gamma} = [\tilde{X}_{ij}]$, $\tilde{\Lambda} = [\tilde{Y}_{ij}]$, and a crisp matrix $\tilde{A} = [a_{ij}]$, the operations are defined as follows:

$$\tilde{\Gamma} \circ \tilde{\Lambda} = [\textstyle\sum_{k=1}^n \tilde{X}_{ik} \circ \tilde{Y}_{kj}],\ \tilde{\Gamma} \circledast \tilde{\Lambda} = [\bigoplus_{k=1}^n \tilde{X}_{ik} \circ \tilde{Y}_{kj}],$$
$$\tilde{A}\tilde{\Gamma} = [\textstyle\bigoplus_{k=1}^n a_{ik}\tilde{X}_{kj}],\ \ k\tilde{\Gamma} = [\bigoplus_{k=1}^n a_{ik}\tilde{X}_{ij}],$$
$$\tilde{X}\tilde{A} = [a_{ij}X],\ \ \tilde{X} \circ \tilde{\Gamma} = [X \circ X_{ij}],$$
$$\tilde{X} \circledast \tilde{\Gamma} = [X \circledast X_{ij}].$$

Using above operations and algebraic properties, we can obtain solutions of normal equation fuzzy estimators for each $k = 1,2,\ldots,h$ by

$$\widehat{\beta_k} = \left(\tilde{X}^t \diamond \tilde{X}\right)^{-1} \tilde{X}^t \diamond \tilde{y}\,, \tag{28}$$

where

$$\tilde{X}^t \diamond \tilde{X} = \left[\sum_{i=1}^{n}(l_{x_{il}} l_{x_{ij}} + x_{il}x_{ij} + r_{x_{il}} r_{x_{ij}})\right]_{(p+1)\times(p+1)},$$
$$\tilde{X}^t \diamond \tilde{y} = \left[\sum_{i=1}^{n}(l_{x_{il}} l_{y_i} + x_{il}y_i + r_{x_{il}} r_{y_i})\right]_{(p+1)\times 1},$$

for $l = 0,1,\cdots,p$. Note that (28) exists if $det(\tilde{X}^t \diamond \tilde{X}) \neq 0$. The closed-form formula (28) is true solution of FLSE method.

4.1.2 Estimation by FLAD

Least absolute deviation (LAD) is a statistical optimality measure and statistical optimization technique based on minimizing an absolute deviation. Unlike the more commonly used LSE, LAD offers a more robust solution in the presence of outliers, which can significantly influence the outcome. LAD does not have a closed-form formula for estimating coefficients due to the non-differentiability of absolute values. Therefore, we can only attain approximate solutions through optimization methods.

Based on the LAD, we propose fuzzy least absolute deviation (FLAD) as another coefficient estimation method, and L_1-metric for fuzzy numbers is defined in this paper as follows:

$$d_1(\tilde{X},\tilde{Y}) = \left|l_x - l_y\right| + |x - y| + \left|r_x - r_y\right|, \tag{29}$$

where $\tilde{X} = (l_x, x, r_x), \tilde{Y} = \left(l_y, y, r_y\right)$.

If we set a fuzzy regression model represented as (24) where $\tilde{X}_{ij} = (l_{x_{ij}}, x_{ij}, r_{x_{ij}})$, $\tilde{Y}_i = (l_{y_i}, y_i, r_{y_i})$ $(i = 1,\cdots,n;\ j = 1,\cdots,p)$, we can get an objective function by applying the L_1-metric above as follows:

$$\begin{aligned} Q\left(\beta_{k0},\beta_{k1},\cdots,\beta_{kp}\right) &= \sum_{i=1}^{n} d_1(\tilde{Y}_i, \sum_{j=0}^{p}\beta_{kj}\tilde{X}_{ij}) \\ &= \sum_{i=1}^{n}\left(\left|l_{y_i} - \sum_{j=0}^{p}\beta_{kj} l_{x_{ij}}\right| + \left|y_i - \sum_{j=0}^{p}\beta_{kj}x_{ij}\right| + \left|r_{y_i} - \sum_{j=0}^{p}\beta_{kj} r_{x_{ij}}\right|\right), \end{aligned} \tag{30}$$

for $k = 1,2,\cdots,h$, where h is the number of the regression model in this fuzzy analysis.

To minimize (30), we use GA and HS for the optimization, which are mentioned in section 4.2, and we repeatedly update coefficients to the direction of minimizing the objective function (30).

4.2. Estimation of the Proposed Model with Evolutionary Algorithms

In this study, two evolutionary algorithms, namely GA and HS, are employed. The flowcharts of each algorithm are visually presented in Fig. 7 [34, 35]. These two methods were used to estimate the coefficients of a fuzzy model, and this process is elaborated in detail in Algorithm 1. As described in Algorithm 1, the objective function corresponding to FLAD or FLSE is set, and evolutionary algorithms are utilized to minimize this function.

4.2.1 Genetic Algorithm (GA)

Genetic Algorithm (GA) is a well-known optimization algorithm among meta-heuristic method, which is inspired from natural selection. GA was first proposed by J.H. Holland in 1992 [36]. GA is highly effective because there is no fixed optimization algorithm, which enables the potential for a wide range of diverse outcomes and is applicable to various forms of problems. By exploring various solutions, the likelihood of getting stuck in local optima is low, making it useful for finding global optimum solutions that other optimization algorithms can miss. The implementation of the GA typically starts with a population of chromosomes, often generated randomly. New populations are generated through the repeated application of genetic operators to individual chromosome in the population. The genetic operators are selection, crossover, and mutation. In selection, chromosomes are chosen based on their fitness score for subsequent processing. The fitness score of each chromosome is obtained by fitness function which corresponds to the objective function for the problem. By the iterative application of these genetic operators, optimal solutions with higher fitness can be obtained.

4.2.2 Harmony Search (HS)

Harmony Search (HS) is a meta-heuristic optimization algorithm proposed by Geem in 2001 [37], inspired by the process of finding musical harmony in natural phenomena. HS is applicable to a wide range of optimization problems and is characterized by its exceptional

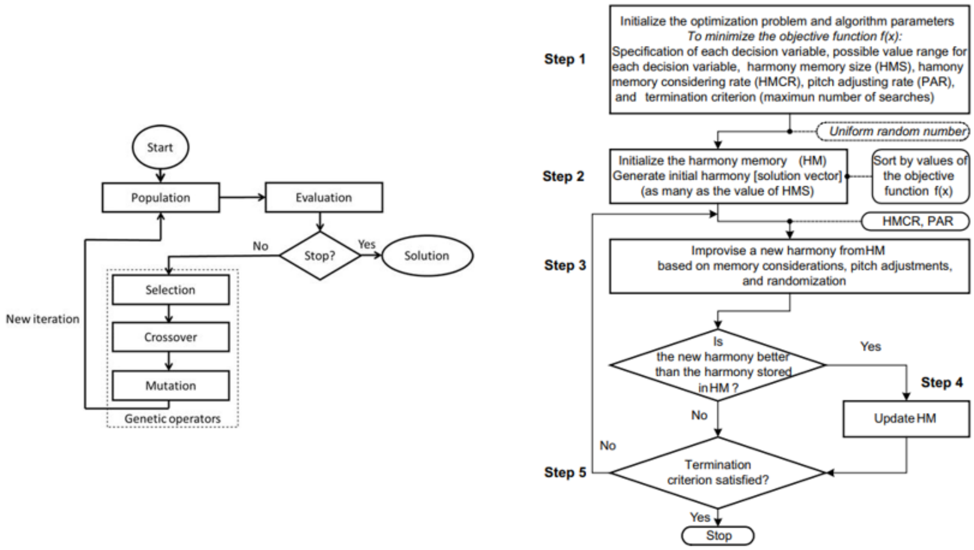
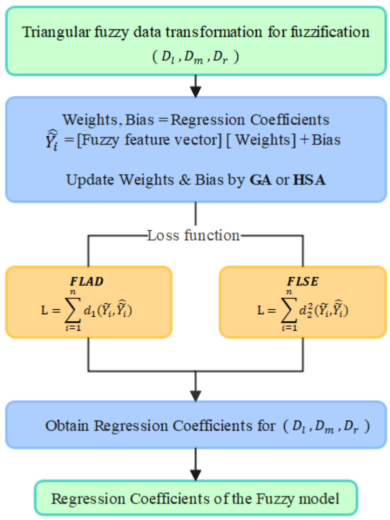


Fig. 7. Genetic algorithm flowchart (left), Harmony Search flowchart (right)



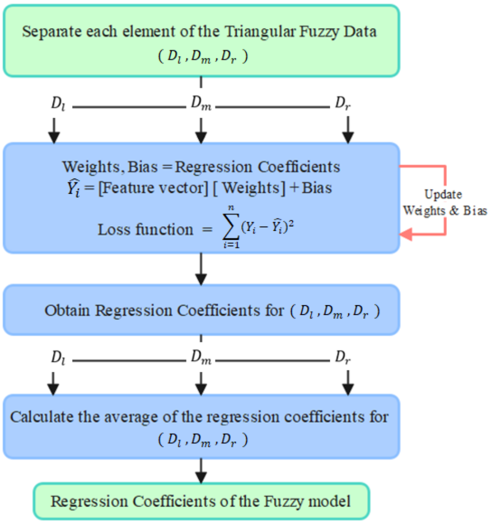
Algorithm 1. Application of Evolutionary Algorithms for fuzzy model analysis

flexibility and rapid convergence. Its simplicity in implementation and ease of parameter tuning makes it highly effective in practical applications. Moreover, it offers a natural-inspired approach to exploring creative solutions and has demonstrated strong optimization performance across various application domains. The algorithm of HS begins by creating an initial set of candidate solutions, known as the harmony memory. Each of these candidates is evaluated based on the problem's objective function to determine their fitness. Next, it goes through a series of iterations, including memory consideration, pitch adjustment, and randomization. This iteration process continues until a predefined termination criterion is met. Common termination criteria include reaching a fixed number of iterations, exceeding a specified time limit, or observing convergence in fitness values.

4.3. Estimation of the Proposed Model with Neural-Network Algorithm

In recent times, there has been an increasing adoption of models based on neural-network algorithm, which have proven effective in addressing a diverse range of problems. Motivated by this, we aim to contribute to the development of an algorithm that incorporates neural-network in moderation and mediation analyses. We propose Algorithm 2 for conducting FMMA by applying neural-network, and we use gradient descent-based optimization methods to minimize the loss function. As described in Algorithm 2, we separate three elements of the triangular fuzzy number, and estimate regression coefficients for each element. Finally, we compute the average of the coefficients of the three elements, and that average is estimated coefficients for triangular fuzzy data.

We employ the closed-form formula (28), evolutionary algorithms, and neural-network algorithm to the estimation in FLSE. However, when



Algorithm 2. Application of neural-network for fuzzy model analysis

dealing with L_1-metric, closed-form formula and neural-network algorithm are not applicable due to the impossibility of the differentiation. Thus, evolutionary algorithms are only used for the estimation in FLAD.

5. Methodology: Statistical Inference

5.1. *Sum of Squares for the Proposed Fuzzy Model Analysis*

In the fuzzy regression model, a fuzzy total sum of squares (FTSS), a fuzzy residual sum of squares (FSSE), and a fuzzy regression sum of squares (FSSR) using L_2-metric are defined [29] as follows:

$$
\begin{aligned}
FTSS &= \textstyle\sum_{i=1}^{n} d_2^2(\tilde{Y}_i, \bar{\tilde{Y}}) = \sum_{i=1}^{n}\{(l_{y_i} - \overline{l_{y_i}})^2 + (y_i - \bar{y}_i)^2 + (r_{y_i} - \overline{r_{y_i}})^2\},\\
FSSE &= \textstyle\sum_{i=1}^{n} d_2^2(\tilde{Y}_i, \hat{\tilde{Y}}_i) = \sum_{i=1}^{n}\{(l_{y_i} - \widehat{l_{y_i}})^2 + (y_i - \hat{y}_i)^2 + (r_{y_i} - \widehat{r_{y_i}})^2\},\\
FSSR &= \textstyle\sum_{i=1}^{n} d_2^2(\hat{\tilde{Y}}_i, \bar{\tilde{Y}}) = \sum_{i=1}^{n}\{(\widehat{l_{y_i}} - \overline{l_{y_i}})^2 + (\hat{y}_i - \bar{y}_i)^2 + (\widehat{r_{y_i}} - \overline{r_{y_i}})^2\}.
\end{aligned}
\tag{31}
$$

Unlike the crisp regression model, the formula (32) is not always guaranteed because the fuzzy sum of squares also requires additional consideration of the squares of the difference between the left and right spread of the fuzzy number.

$$FTSS \neq FSSR + FSSE. \tag{32}$$

Fuzzy R^2, which measures how well the object is represented by a regression model, is expressed using FTSS and FSSE.

$$\text{Fuzzy } R^2 = 1 - \frac{FSSE}{FTSS}. \tag{33}$$

5.2. *Proposed Fuzzy F-test*

After regression analysis, the regression model can be evaluated with various indicators, one of which is R^2 introduced above. The other is an F-test that checks the significance of the estimated regression model [29]. First, a fuzzy F-statistic is obtained, and then the null hypothesis is established and verified to determine significance. The hypothesis used to confirm the significance of the model as follows:

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_p = 0 \quad v.s. \quad H_1: H_0 \text{ is not true}.$$

The null hypothesis means that since all regression coefficients are zero, the necessity of doing regression analysis is gone. In other words, it means that the estimated model is not significant.
For this test, we obtain a fuzzy F-statistic approximately following the F distribution as follows:

$$Fuzzy\ F = \frac{FMSR}{FMSE} = \frac{FSSR/p}{FSSE/(n-p-1)} \sim f(p, n-p-1). \tag{34}$$

Then, the p-value is obtained using the calculated fuzzy F-statistic. Finally, comparing this with the significance level, we can determine whether the null hypothesis can be rejected.

5.3. Proposed Fuzzy T-test

R^2 and the F-test evaluate the estimated overall regression model. We can also conduct a T-test to determine the degree to which a linear relationship exists between each independent variable and dependent variable. Unlike the F-test which evaluates several coefficients at the same time, the T-test has the advantage of being able to evaluate only one regression coefficient. The hypothesis is used to determine the significance of each regression coefficient as follows:

$$H_0: \beta_k = 0, k = 1, 2, \dots, p \quad v.s. \quad H_1: H_0 \; is \; not \; true.$$

The null hypothesis means that the corresponding regression coefficient is zero. In other words, it means that there is no relationship between the independent variable and the dependent variable.

To do the T-test, the standard error of $\hat{\beta}_k$ is obtained, and the T-statistic is obtained. , we have

$$C = (X^t X)^{-1},$$
$$S_{\widehat{\beta}_k} = \sqrt{\frac{SSE}{n-p-1} C_{kk}}, \; k = 1,2, \dots, p,$$
$$T = \frac{\widehat{\beta}_k}{S_{\widehat{\beta}_k}} \sim t(n-p-1), \; k = 1,2, \dots, p. \tag{35}$$

After calculating the p-value through this statistic, it is compared with the significance level to determine whether to reject the null hypothesis. Now, we define the standard error of $\hat{\beta}_k$ in fuzzy model using FSSE, the fuzzy residual sum of squares, and $\diamond$, the fuzzy operation defined above, and we newly define the fuzzy T-statistic in fuzzy model. The fuzzy T-statistic below roughly follows the t-distribution for the same reason as the fuzzy F-statistic.

$$\tilde{C} = \left(\tilde{X}^t \diamond \tilde{X}\right)^{-1},$$
$$S_{\widehat{\beta}_k} = \sqrt{\frac{FSSE}{n-p-1} \tilde{C}_{kk}}, \; k = 1,2, \dots, p,$$
$$Fuzzy \;\; T = \frac{\widehat{\beta}_k}{S_{\widehat{\beta}_k}} \sim t(n-p-1), \; k = 1,2, \dots, p. \tag{36}$$

After that, similarly, the p-value is obtained, and it is possible to check whether the null hypothesis is rejected by comparing it with the significance level.

6. Data Analysis based on Least Squares Estimation

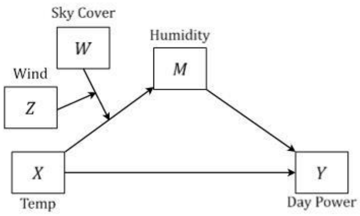
In this section, we compare CMMA and FMMA using the estimation method of LSE and FLSE with close-form mathematical formula respectively. Furthermore, we conduct statistical inference following the estimation, a methodology introduced in section 5.

6.1. Fuzzy Moderated Moderated-Mediation Analysis for SOLAR POWER Data

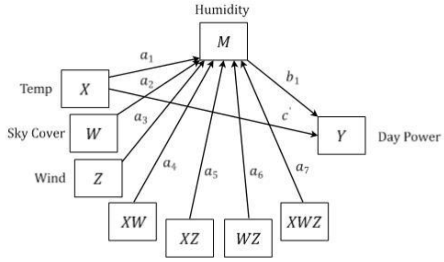
This data was originally created by Rami Mashkouk, who combined Daily Power Production of Solar Panels dataset from [38] and dataset of weather status in Antwerp, Belgium from 'Timeanddate' website [39]. It contains several weather conditions and solar power data. We used data from January 1, 2016 to November 19, 2019 and about 20 missing values were replaced by cubic spline interpolation. Consequently, we proceeded with the analysis using 1332 data. The variables 'Temp' (X), 'Wind' (Z), and 'Humidity' (M) represent the daily average of temperature, wind speed, and humidity. 'Day Power' (Y) indicates the daily power production obtained by solar panels. 'Sky Cover' (W) literally indicates the sky cover at 3 p.m. in a day when solar energy comes with a peak, and it is classified into eight categories based on sky cover percent by 'WeatherSTEM' as follows: 'Sunny', 'Partly sunny', 'Passing clouds', 'Scattered clouds', 'Broken clouds', 'fog', 'Ice fog or Haze', 'Overcast'. These were coded from 'Sunny' as 1 to 'Overcast' as 8, and the higher the value, the greater the amount of clouds or fog covering the sky. Temperature, wind, and humidity are not fixed but continuous values so loss of information can be occurred if we use crisp data intact. Therefore, we transformed the crisp values of the above three variables into triangular fuzzy numbers with spreads defined as half of the difference between values of two consecutive days. For instance, if the temperature in a certain period is 10(℃) and the next period is 20(℃), the triangular fuzzy number of 10 becomes $(5, 10, 15)$. In the case of 'Sky Cover', the categories are in linguistic expressions and their boundaries are ambiguous. For example, the criteria for classifying 'Passing clouds' and 'Scattered clouds' varies from an observer. Thus, it is better to express them as fuzzy numbers so we fuzzified them with spread 1. All variables except 'Sky cover' are normalized from 0 to 1. The model of this data is shown in Fig. 8.

The terminology CMMA that we defined earlier means an analyses of models with multiple moderators on crisp numbers. Based on CMMA, the regression equations of this model applied above data are estimated as follows:

$$\widehat{M} = 0.9370 - 0.9212X - 0.0035W - 0.9191Z + 0.0777XW + 2.0611XZ + 0.1339WZ - 0.3276XWZ,$$
$$\hat{Y} = 0.6099 + 0.4950X - 0.8307M.$$



(a) Conceptual diagram



(b) Statistical diagram

Fig. 8. Fuzzy moderated moderated-mediation model for SOLAR POWER data

Table 1
Estimated coefficients and p-values from moderated moderated-mediation model using SOLAR POWER data

Method	Estimated coefficients								
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	c′	b_1
CMMA	−0.9212	−0.0035	−0.9191	0.0777	2.0611	0.1339	−0.3276	0.4950	−0.8307
FMMA	−0.8038	0.0116	−0.7326	0.0566	1.6342	0.0974	−0.2280	0.5308	−0.7549
	p-values								
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	c′	b_1
CMMA	< 0.001 ***	0.793	< 0.001 ***	0.011 *	< 0.001 ***	0.003 **	0.002 **	< 0.001 ***	< 0.001 ***
FMMA	< 0.001 ***	0.306	< 0.001 ***	0.019 *	< 0.001 ***	0.007 **	0.005 **	< 0.001 ***	< 0.001 ***

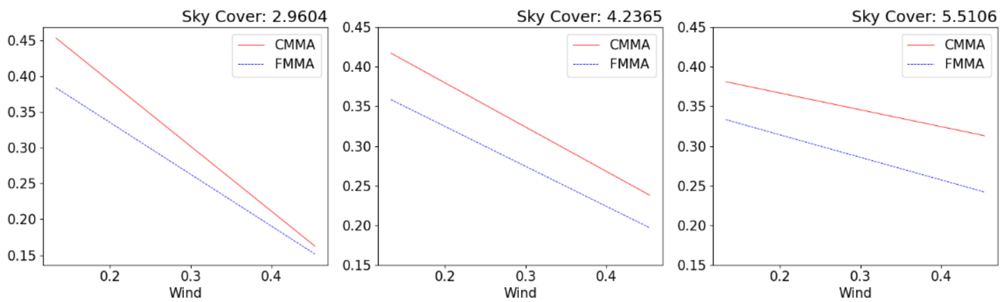


Fig. 9. Conditional effects of X on Y

We suggest fuzzy multiple moderators analysis (FMMA). In FMMA, the regression coefficients are estimated using fuzzy least squares estimation with each spread mentioned above.

$$\begin{aligned}\hat{\widetilde{M}} &= 0.8503 \oplus (-0.8038)\tilde{X} \oplus 0.0116\widetilde{W} \oplus (-0.7326)\tilde{Z} \\ &\quad \oplus 0.0566\tilde{X} \circledast \widetilde{W} \oplus 1.6342\tilde{X} \circledast \tilde{Z} \oplus 0.0974\widetilde{W} \circledast \tilde{Z} \oplus (-0.2280)\tilde{X} \circledast \widetilde{W} \circledast \tilde{Z}, \\ \hat{\tilde{Y}} &= 0.5437 \oplus 0.5308\tilde{X} \oplus (-0.7549)\widetilde{M}.\end{aligned}$$

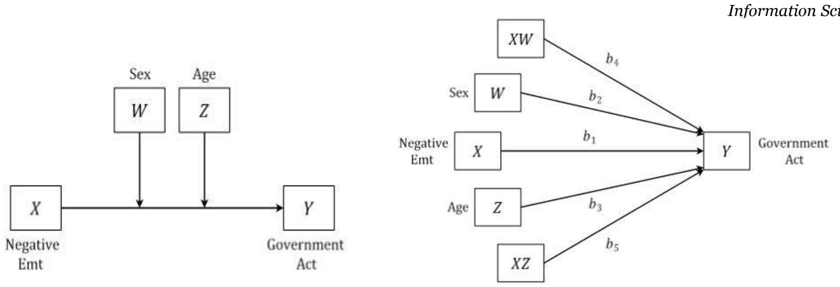
The estimated coefficients and each p-value both on crisp and fuzzy numbers are shown in Table 1. The noticeable change has shown that the sign of a_2, which estimates the effect of W on M, was changed to positive. This represents that humidity actually increases if the sky becomes cloudy, but it was misinterpreted in CMMA. In the meantime, the p-value of a_2 becomes much smaller from 0.793 to 0.306 while other is similar or slightly increases in CMMA and FMMA. These results occurred because using crisp numbers on ambiguous data caused the loss of information.

The conditional indirect effect (CIDE) of X on Y in CMMA and the fuzzy conditional indirect effect (FCIDE) of $\tilde{X}$ on $\tilde{Y}$ in FMMA can be expressed as below:

$$\begin{aligned}\omega &= \theta_{X\to M} b_1 = a_1 b_1 + a_5 b_1 Z + (a_4 b_1 + a_7 b_1 Z)W, \\ \tilde{\omega} &= \theta_{\tilde{X}\to\widetilde{M}} b_1 = (a_1 \oplus a_4\widetilde{W} \oplus a_5\tilde{Z} \oplus a_7\widetilde{W} \circledast \tilde{Z})\, b_1 \\ &= a_1 b_1 \oplus a_4 b_1 \widetilde{W} \oplus a_5 b_1 \tilde{Z} \oplus a_7 b_1 \widetilde{W} \circledast \tilde{Z} \\ &= a_1 b_1 \oplus a_5 b_1 \tilde{Z} \oplus \left(a_4 b_1 \oplus a_7 b_1 \tilde{Z}\right) \circledast \widetilde{W}.\end{aligned}$$

They are estimated as $\omega = 0.7652 - 0.0645W - 1.7122Z + 0.2721WZ$, $\tilde{\omega} = 0.6068 \oplus (-0.0427\widetilde{W}) \oplus (-1.2337)\tilde{Z} \oplus 0.1721\widetilde{W} \circledast \tilde{Z}$, respectively. Shown in Fig. 9, the graph is divided according to the value of 'Sky Cover' (mean-1sd, mean, mean+1sd). As Sky Cover get bigger, the amount of change in slope of FMMA is smaller than CMMA. In other words, FCIDE is less sensitive than CIDE in this data.

The validity of the model can be determined by R^2 and F-statistic in a F-test. Using these statistical methods, the necessity of transforming



(a) Conceptual diagram (b) Statistical diagram

Fig. 10. Fuzzy multiple additive moderation model for CLIMATE CHANGE data

Table 2
Estimated coefficients and p-values from multiple additive moderation model using CLIMATE CHANGE data

Method	Estimated coefficients				
	b_1	b_2	b_3	b_4	b_5
CMMA	0.1154	−0.8449	−0.0241	0.2208	0.0054
FMMA	0.3571	−0.6532	−0.0196	0.1761	0.0043
	p-values				
	b_1	b_2	b_3	b_4	b_5
CMMA	0.173	< 0.001 ***	< 0.001 ***	< 0.001 ***	0.001 **
FMMA	< 0.001 ***	0.002 **	0.002 **	0.001 **	0.010 **

ambiguous values to fuzzy numbers can be proven by comparing original methods with fuzzy R^2 and fuzzy F-statistic which are proposed in previous section. In this model, firstly, fuzzy R^2 (0.645) is slightly less than R^2 (0.662). However, this result never means that FMMA is less convincing than CMMA. We should not approach in a quantitative way, but consider that it can be estimated exaggeratingly without considering fuzziness of data. Fuzzy F-statistic (1206.08) was estimated less than F-statistic in CMMA (1299.55), suggesting that the significance of the model is slightly magnified, but they were both considered significant as a result.

6.2. *Fuzzy Multiple Additive Moderation Analysis for CLIMATE CHANGE Data*

Climate change data are introduced in [6]. These data were collected in online surveys from 815 residents of the United States including 417 females and 398 males. Participants of surveys responded to a question asking how often they feel each of three negative emotions ("worried", "alarmed", "concerned") when they think about global warming (X : Negative Emt). The response options of frequency were numerically coded 1 ("not at all") to 6 ("a great deal"), which means that the higher score, the more often they feel the emotions respectively. The sum of scores of all three emotions were averaged which is a measure of X. Participants were also asked five questions about how much he or she supports several policies or actions by the government to alleviate the threat of climate change (Y : Government Act). The questions' responses were measured 1 ("Strongly opposed") to 7 ("Strongly support), and the average of the questions was used. Prior to the research, each respondent answered his or her sex (W) and age (Z). As you can see, the response options in the online surveys are in linguistic forms such as "not at all", "a great deal", "Strongly support", etc. When the linguistic responses are changed into numbers, the loss of information is inevitable. Thus, it is reasonable to analyze ambiguous data like linguistic expressions with fuzzy numbers. The variables X and Y in the data are fuzzified with spread 1. The model of this data is illustrated in Fig. 10.

Based on CMMA, the regression equation of this model applied above data is estimated as follows:

$$\hat{Y} = 4.4748 + 0.1154X - 0.8449W - 0.0241Z + 0.2208XW + 0.0054XZ.$$

In FMMA, the regression coefficients are estimated using fuzzy least squares estimation with spread 1 mentioned above.

$$\hat{\tilde{Y}} = 3.5682 \oplus 0.3571\tilde{X} \oplus (-0.6532)\widetilde{W} \oplus (-0.0196)\tilde{Z} \oplus 0.1761\tilde{X} \circledast \widetilde{W} \oplus 0.0043\tilde{X} \circledast \tilde{Z}.$$

The estimated coefficients and each p-value both on crisp and fuzzy numbers are shown in Table 2. Shown in table, the absolute value of coefficients slightly decreased except for X. In other words, the influence of X on Y has increased. In CMMA, p-value of coefficient of X (b_1) is 0.173 which is not significant, but in a linear regression model without moderators, the coefficient of X is estimated as 0.5142 and is actually significant with very small p-value, which means X is an important variable. However, in FMMA, b_1 was significant that it involves the fact that X is an meaningful variable in the analysis. In other words, FMMA reflects more information of ambiguous data.

Here, the conditional effects of X on Y in this model in CMMA and FMMA are expressed as $\theta_{X\to Y} = b_1 + b_4W + b_5Z$, $\theta_{\tilde{X}\to\tilde{Y}} = b_1 \oplus b_4\widetilde{W} \oplus b_5\tilde{Z}$. Those in CMMA and FMMA are $\theta_{X\to Y} = 0.1154 + 0.2208W + 0.0054Z$, $\theta_{\tilde{X}\to\tilde{Y}} = 0.3571 \oplus 0.1761\widetilde{W} \oplus 0.0043\tilde{Z}$, respectively. Fig. 11 shows both conditional effects of X on Y depending on age and sex in CMMA and FMMA. b_4 indicates the moderation effect of sex (W), and b_5 indicates the moderation effect of age (Z). Each coefficient is estimated when the other is constant. In Fig. 11, the slopes of lines represent b_5. The slopes of lines of FMMA (0.0043) are more gradual than those of CMMA (0.0054) in both graphs, which means that the conditional effect of

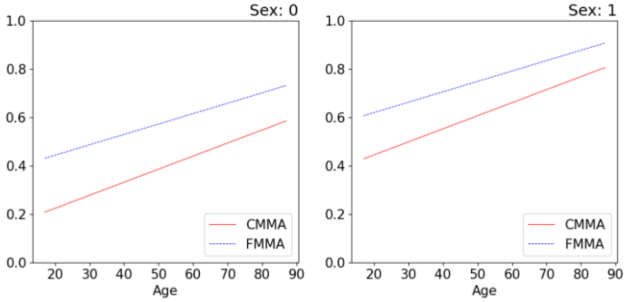


Fig. 11. Conditional effect of X on Y

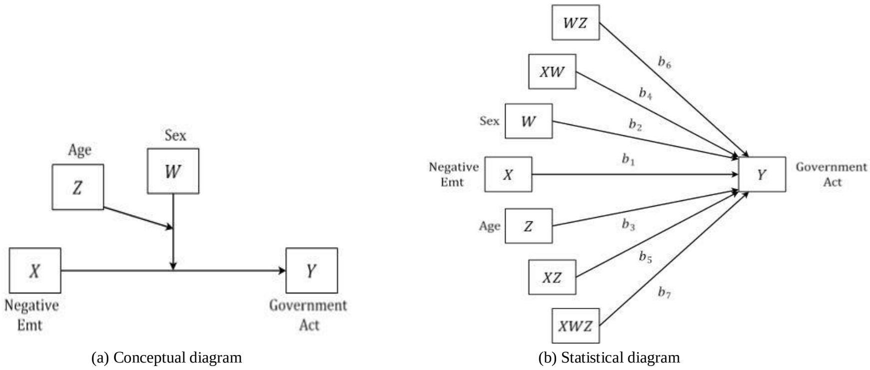


Fig. 12. Fuzzy moderated-moderation model for CLIMATE CHANGE data

Table 3
Estimated coefficients and p-values from moderated-moderation model using CLIMATE CHANGE data

Method	Estimated coefficients						
	b_1	b_2	b_3	b_4	b_5	b_6	b_7
CMMA	0.3120	0.5985	−0.0071	−0.1413	0.0012	−0.0287	0.0072
FMMA	0.5045	0.4595	−0.0064	−0.0928	0.0011	−0.0223	0.0054
	p-values						
	b_1	b_2	b_3	b_4	b_5	b_6	b_7
CMMA	0.011 *	0.371	0.469	0.415	0.623	0.024 *	0.029 *
FMMA	< 0.001 ***	0.503	0.526	0.598	0.665	0.089 **	0.109

FMMA is less sensitive than of CMMA. In the case of sex (M), when age is constant, the difference in the conditional effect when sex is 0 and 1 (b_4) of FMMA, 0.1761 is less than that of CMMA, 0.2208, which also means the moderation effect by sex is slightly exaggerated when we use crisp numbers for expressing ambiguous information.

Now we mention R^2 and F-test of both crisp and fuzzy models, and compare them to judge the validity of the models. In this model, fuzzy R^2 (0.460) in FMMA is significantly higher than R^2 (0.368) in CMMA, which implies the regression model explains given data reflecting fuzziness better. Moreover, F-statistic of the model in FMMA (138.78) is much higher than CMMA's (94.40). But each p-value is very small, so the model is considered significant in both CMMA and FMMA.

6.3. *Fuzzy Moderated-Moderation Analysis for CLIMATE CHANGE Data*

The climate change data is also used to analyze moderated-moderation model. The model for this data is shown in Fig. 12. In this model, the moderation of the effect of X on Y by sex (W) depends on age (Z). Both in CMMA and FMMA, estimated regression equations are expressed respectively as follows:

$$\hat{Y} = 3.6733 + 0.3120X + 0.5985W - 0.0071Z - 0.1413XW + 0.0012XZ - 0.0287WZ + 0.0072XWZ,$$
$$\begin{aligned}\hat{\tilde{Y}} &= 2.9476 \oplus 0.5045\tilde{X} \oplus 0.4595\tilde{W} \oplus (-0.0064)\tilde{Z}\\ &\oplus (-0.0928)\tilde{X} \circledast \tilde{W} \oplus 0.0011\tilde{X} \circledast \tilde{Z} \oplus (-0.0223)\tilde{W} \circledast \tilde{Z} \oplus 0.0054\tilde{X} \circledast \tilde{W} \circledast \tilde{Z}.\end{aligned}$$

The estimated coefficients and each p-value both in CMMA and FMMA are presented in Table 3. The absolute value of coefficients except for X were all gotten a bit smaller. In addition, p-values except for b_1 slightly declined, which suggests that the significances of coefficients were a little bit exaggerated with using crisp numbers. Especially, b_7 quantifies the moderated-moderation of X's effect on Y by W and Z, which is

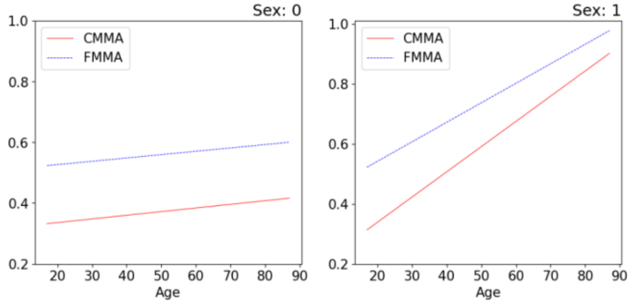
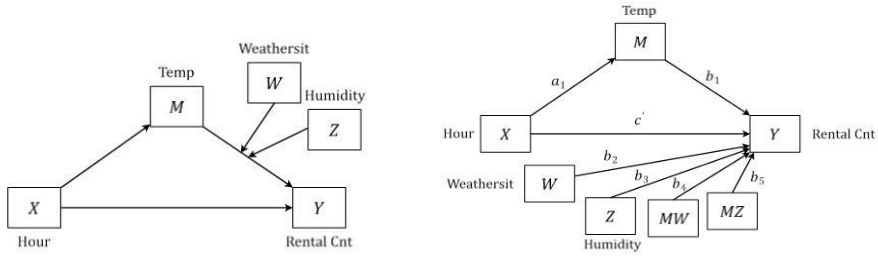


Fig. 13. Conditional effect of X on Y



(a) Conceptual diagram (b) Statistical diagram

Fig. 14. Fuzzy partial moderated-mediation model for BIKE data

estimated as 0.0072 in CMMA and 0.0054 in FMMA. Although the moderated-moderation effect of FMMA is less significant than CMMA in Table 3, this absolutely does not mean that considering fuzziness is not reasonable. We can say that the effect is estimated in an exaggerated way due to the ignorance of fuzziness.

The conditional effect of X on Y in this model is expressed as $\theta_{X\to Y} = b_1 + b_4W + b_5Z + b_7WZ$, and $\theta_{\tilde{X}\to\tilde{Y}} = b_1 \oplus b_4\tilde{W} \oplus b_5\tilde{Z} \oplus b_7\tilde{W} \circledast \tilde{Z}$ in CMMA and FMMA respectively. Using estimated coefficients, those effects can be written as follows:

$$\theta_{X\to Y} = 0.3120 - 0.1413W + 0.0012Z + 0.0072WZ,$$
$$\theta_{\tilde{X}\to\tilde{Y}} = 0.5045 \oplus (-0.0928)\tilde{W} \oplus 0.0011\tilde{Z} \oplus 0.0054\tilde{W} \circledast \tilde{Z}.$$

The conditional effects of X on Y both in CMMA and FMMA are shown in Fig. 13. When sex is 0, the slopes of lines of CMMA and FMMA are parallel, but it increases more in CMMA than FMMA when sex is 1. In other words, since the difference in the conditional effects of X on Y by sex is bigger in CMMA than FMMA when age is, for example, 30 and 70, we can notice that the moderation effect of the conditional effect depending on sex when age increases becomes greater in CMMA than FMMA. This represents that the conditional effects are less sensitive when data is fuzzified.

As in the previous models using the same data, fuzzy R^2 (0.462) is also higher than that (0.372) of in CMMA, which highlights the fact that changing ambiguous data to fuzzified values can be more helpful in explaining data with the model. Furthermore, the fuzzy F-statistic of the model in FMMA (98.96) is fairly higher than F-statistic (68.11) in CMMA, but they are both considered significant with p-values near zero.

6.4. Fuzzy Partial Moderated-Mediation Analysis for BIKE Data

This data is combination of bicycle rental data of Capital bike-share system from 2011 to 2012 [40] and the weather and season data [41], provided by Roberto Frias, totaling 10,948 data. We conducted the analysis of this data to show the usage of mediation and moderation in daily life and to emphasize the importance of using fuzzy numbers in this type of data. All variables of the raw data except 'Rental cnt', meaning the number of bicycle rented, were scaled from 0 to 1, so we scaled 'Rental cnt' from 0 to 1, too. Moreover, we sorted data in chronological order to transform it into time series data using variables 'Dteday' and 'Hour'. 'Hour' (X) means the time data was collected. 'Temp' (M) means the temperature of the time. 'Weathersit' (W) indicates the conditions of the weather which was measured by Freemeteo. The weather conditions are divided into 4 categories and coded 1 to 4 (1 : 'Clear', 'Few clouds', etc, 2 : 'Mist+Cloudy', etc, 3 : 'Mist+Broken clouds', etc, 4 : 'Snow+Fog', etc). In the case of 'Temp' and 'Humidity', they are collected as fixed values but actually always changes. Thus, it is more reasonable to express them with numbers fuzzified rather than crisp numbers. We transformed crisp data of above two variables into triangular fuzzy numbers with spread defined as a half of difference between values of two consecutive periods. 'Weathersit' variable was expressed as linguistic forms and coded with crisp numbers, so we fuzzified with spread 1. The model of this data is shown in Fig. 14.

In this model, two moderators influence $M \to Y$ independently, so we can get estimated regression equation both in CMMA and FMMA as follows:

Table 4
Estimated coefficients and p-values from partial moderated-mediation model using BIKE data

Method	Estimated coefficients						
	a_1	c'	b_1	b_2	b_3	b_4	b_5
CMMA	0.0040	0.0075	0.6419	−0.0311	0.1165	0.0528	−0.6545
FMMA	0.0040	0.0075	0.6202	−0.0220	0.0891	0.0411	−0.6026
	p-values						
	a_1	c'	b_1	b_2	b_3	b_4	b_5
CMMA	< 0.001 ***	< 0.001 ***	< 0.001 ***	< 0.001 ***	< 0.001 ***	< 0.001 ***	< 0.001 ***
FMMA	< 0.001 ***	< 0.001 ***	< 0.001 ***	0.029 *	< 0.001 ***	< 0.001 ***	< 0.001 ***

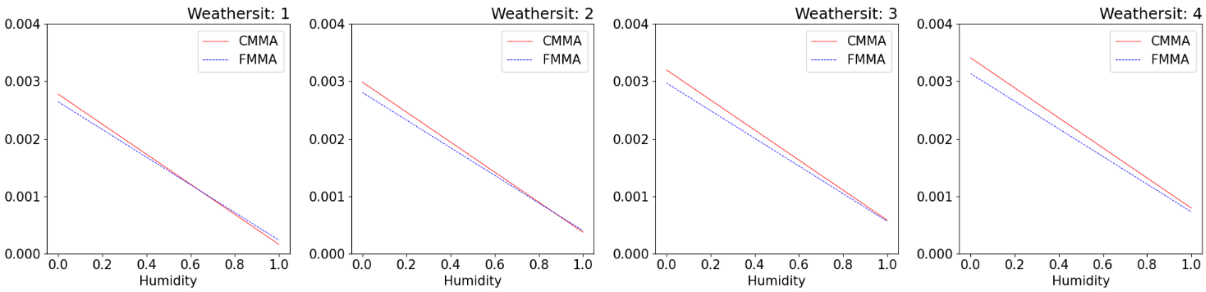


Fig. 15. Conditional effect of X on Y

$$\hat{M} = 0.4520 + 0040X,$$
$$\hat{Y} = -0.0745 + 0.0075X + 0.6419M - 0.0311W + 0.1165Z + 0.0528MW - 0.6545MZ,$$

$$\hat{\tilde{M}} = 0.4520 + 0040\tilde{X},$$
$$\hat{\tilde{Y}} = -0.0669 \oplus 0.0075\tilde{X} \oplus 0.6202\tilde{M} \oplus (-0.0220)\tilde{W} \oplus 0.0891\tilde{Z} \oplus 0.0411\tilde{M} \circledast \tilde{W} \oplus (-0.6026)\tilde{M} \circledast \tilde{Z}.$$

In Table 4, estimated coefficients and their p-values are shown. All of the absolute value of estimated coefficients in FMMA are almost equal or slightly decreased than those of in CMMA. It represents that the influence of the products is actually exaggerated since it ignored the fuzziness data has. Also, some p-values increases slightly but are all significant with very small value.

The conditional indirect effect (CIDE) of X on Y in CMMA and the fuzzy conditional indirect effect (FCIDE) of $\tilde{X}$ on $\tilde{Y}$ in FMMA can be expressed as below:

$$\omega = a_1\theta_{M\to Y} = a_1(b_1 + b_4W + b_5Z) = a_1b_1 + a_1b_4W + a_1b_5Z,$$
$$\tilde{\omega} = a_1\theta_{\tilde{M}\to\tilde{Y}} = a_1(b_1 \oplus b_4\tilde{W} \oplus b_5\tilde{Z}) = a_1b_1 \oplus a_1b_4\tilde{W} \oplus a_1b_5\tilde{Z}.$$

The estimation of the conditional effects above are $\omega = 0.0026 + 0.0002W - 0.0026Z$, $\tilde{\omega} = 0.0025 \oplus 0.0002\tilde{W} \oplus (-0.0024)Z$, respectively. Fig. 15 represents the graph of the conditional indirect effects of changes in ‘Weathersit’ (W) and ‘Humidity’ (Z). The gradient of the lines are all same as a_1b_5, the partial moderated-mediation effect of ‘Humidity’ (Z) in each analysis, which are -0.0026 in CMMA and -0.0024 in FMMA. We can notice that partial moderated-mediation effect is estimated in an exaggeratedly way.

Fuzzy R^2 (0.331) is estimated to be slightly lower than that (0.332) of in CMMA, but it does not mean that FMMA is less effective analysis, as we mentioned in analysis of solar power data. Regarding F-test, F-statistic (906.95) and fuzzy F-statistic (902.22) are estimated, and these are considered significant with very small p-values.

7. Comparison on Performance among FLSE and FLAD Methods

In section 6, we estimated regression coefficients and analyzed causal relationships among variables using closed-form formula (28) in FLSE. As mentioned earlier, we additionally compare the accuracy of the models under FLSE and FLAD methods, depending on L_1 or L_2-metric based objective functions using solar power data in section 6.1.

The proposed accuracy measurements, FMSE and FMAE, are defined as follows:

$$FMSE = \frac{1}{n}\sum_{i=1}^{n} d_2^2\left(\tilde{Y}, \hat{\tilde{Y}}\right) = \frac{1}{n}\sum_{i=1}^{n}\left\{\left(l_{y_i} - \widehat{l_{y_i}}\right)^2 + (y_i - \hat{y}_i)^2 + \left(r_{y_i} - \widehat{r_{y_i}}\right)^2\right\},$$
$$FMAE = \frac{1}{n}\sum_{i=1}^{n} d_1\left(\tilde{Y}, \hat{\tilde{Y}}\right) = \frac{1}{n}\sum_{i=1}^{n}\left\{\left|l_{y_i} - \widehat{l_{y_i}}\right| + |y_i - \hat{y}_i| + \left|r_{y_i} - \widehat{r_{y_i}}\right|\right\}.$$

7.1. L_2-metric based method (FLSE)

Using the L_2-metric based function (25) as the objective function, we apply four gradient descent-based optimization methods in neural-network, SGD, Momentum, Adagrad, Adam, and two evolutionary algorithms, GA and HS, to estimate coefficients and calculate the accuracy.

Prior to

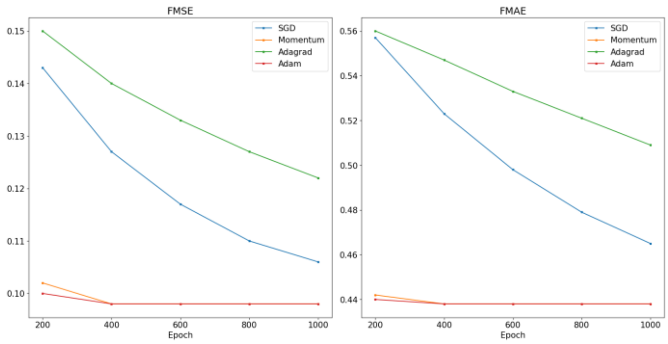


Fig. 16. Variation of FMSE and FMAE for neural-network algorithm according to epoch

Table 5

Measurements of the methods based on L_2-metric (FLSE) and L_1-metric (FLAD)

Measurements	FLSE							FLAD	
	Closed-Form Solution (CS)	Evolutionary Algorithms		Neural-Network				Evolutionary Algorithms	
		GA	HS	SGD	Momentum	Adagrad	Adam	GA	HS
FMSE	0.098	0.107	0.098	0.106	0.098	0.122	0.098	0.121	0.098
FMAE	0.437	0.456	0.438	0.465	0.438	0.509	0.438	0.492	0.433

comparing accuracy, we examine the variation of FMSE and FMAE of gradient descent-based optimizations to confirm the convergence of solutions when the number of epochs increases, as illustrated in Fig. 16. In Table 5, the FMSE and FMAE values for the FLSE methods are presented, with the optimization methods based on gradient descent observed at 1000 epochs, when the measurements are top-notch.

Since the objective function is convex, there exists only one global minimum. Thus, the coefficients estimated by the closed-form formula is the true solution, resulting in the lowest FMSE (0.098). Similarly, CS shows superior performance in terms of FMAE (0.437), which is the convergence value. Therefore, using the formula (28), we can achieve better performance without the need for additional computational work. The other optimization methods aim to approximate the regression coefficients of this global minimum. In this data, it is confirmed that HS, Momentum, and Adam converge most closely to the global minimum.

It is worth noting that when performing coefficient estimation using optimization, the solutions can vary due to various factors. Despite the presence of such variability, it is essential to apply multiple optimization algorithms to address diverse problems, including non-linear functions, large-scale optimization problems, and situations where differentiation of the objective function is not possible, as discussed in the following section.

7.2. L_1-metric based method (FLAD)

In this section, we utilize L_1-metric based function (30) as the objective function for coefficient estimation and calculate accuracy. FLAD approximates the objective function since it is not differentiable in its original form. Therefore, we employ the evolutionary algorithms, namely GA and HS, as optimization methods for coefficient estimation. As shown in Table 5, we confirm that HS excels over GA in both FMSE and FMAE when minimizing the objective function with L_1-metric applied.

8. Conclusion

In this paper, we introduce fuzzy multiple moderators analysis based on FLSE and FLAD. In data analysis, we conducted estimation based on LSE and FLSE using closed-form formula. By employing fuzzy data in various domains, FMMA aims to address the limitations associated with using crisp data in previous research. According to the results of the data analysis, most of the independent variables' influences were slightly decreased, and the p-value increased. However, some of them showed remarkable changes such as increases in coefficients, shifts in sign or significance. This is because crisp data does not reflect fuzziness and ambiguity of certain phenomena, so it cannot fully represent the relationship between the independent variables and the dependent variable. In addition, the conditional effects of X on Y by moderators were less sensitive in FMMA than in CMMA, and the moderation effects by moderators were estimated slightly excessively in CMMA.

In terms of fitness of the model, we compared the result of F-test and R^2 with fuzzy F-test and fuzzy R^2. A slight difference in F-statistic between CMMA and FMMA has occurred, but there was no change in significance. In the case of R^2, both results that R^2 increased or decreased have appeared. Even though fuzzy R^2 is lower than R^2, it does not mean that prediction of the fuzzy model is poor than the crisp model, and this is also a distorted result caused by loss of information.

Furthermore, we employed our evolutionary algorithms and neural-network algorithms to apply fuzzy model analysis. Using these algorithms, we compared the performance of those FLSE and FLAD methods respectively, with FMSE and FMAE. For the result of the comparison, model that used closed-form formula demonstrated superior performance among FLSE methods, and HS exhibited superior performance within FLAD.

For the future work, our neural-network algorithm can be extended to the application of the logic of fuzzy mediation and moderation analyses into the intricate structures of deep neural network (DNN). By doing so, we can develop models that exhibit both strong predictive capabilities and high interpretability, which are fundamental aspects of XAI. Consequently, this contribution can enhance the field of XAI by offering improved model explainability methods using our extended models that incorporate fuzzy analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Publicly available data was used.

Acknowledgements

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1A2C1A01011131; No. 2019M3F2A1073179).

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Declaration of competing interest

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