UNIVERSITÀ DEL SALENTO

# Classification

## Concepts and Techniques

# Classification: Definition

Data is a collection of records (training set)

Each **record** is by characterized by a **tuple** $(\boldsymbol{x},y)$, where $\boldsymbol{x}$ is the *attribute* set and $y$ is the *class* label

$\boldsymbol{x}$: attribute, predictor, independent variable → input

$y$: class, response, dependent variable → output

Task: Learn a model that maps each attribute set $\boldsymbol{x}$ into one of the predefined class labels

***A classification model is an abstract representation of the relationship between the attribute set and the class label***

# Examples of Classification Task

## Binary classification

problems, in which each data instance can be categorized into one of two classes

## Multiclass classification

problems, in which each data instance can be categorized into one of multiple classes

| Task | Attribute set **(x)** | Class label (y) | *Classification type* |
|---|---|---|---|
| Categorizing email messages | Features extracted from email message header and content | spam or non-spam | Binary |
| Identifying tumor cells | Features extracted from x-rays or MRI scans | malignant or benign cells | Binary |
| Cataloging galaxies | Features extracted from telescope images | Elliptical, spiral, or irregular-shaped galaxies | Multiclass |

# Classification Models

## Roles of Classification Models:

- **Predictive Model**: Classifies previously unlabeled instances.
- Descriptive Model: Identifies distinguishing characteristics between different classes.

  Example: Medical diagnosis, where justifying predictions is crucial.

## Key Requirements:

- *Accuracy*: Predict outcomes correctly.
- *Efficiency*: fast response times.

# General Framework for Classification

**Key Concepts:**

- **Classifier**: A model used to perform classification.
- **Training Set**: A set of instances with attribute values and class labels used to build the model.
- **Learning Algorithm**: The systematic approach used to create the model from the training set. This process is called **Induction** (learning/building a model).
- **Deduction**: Applying the learned model to new, unseen test instances to predict their class labels.

**Steps in Classification:**

- **Induction (training)**: Learn the model from training data.
- **Deduction (inference)**: Apply the model to classify new instances.

| ID | Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

| ID | Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|---|
| 11 | No | Married | 55K | ? |
| 12 | Yes | Divorced | 80K | ? |
| 13 | Yes | Single | 110K | ? |
| 14 | No | Single | 95K | ? |
| 15 | No | Married | 67K | ? |

**Figure 3.3.** General framework for building a classification model.

# Classification performance

**Pivotal concepts**

**Independence of Training and Test Sets**: Ensures the model can predict class labels for unseen instances.

**Generalization Performance:** A model with good generalization can accurately predict labels for new, unseen data.

**Model Performance Evaluation:** Compare **predicted labels** against **true labels** to evaluate accuracy.

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- **Confusion Matrix**

- **Accuracy**: number of correct over total
- **Error Rate**: number of wrong over total

# Classification Techniques

## Base Classifiers

- Decision Tree based Methods
- Nearest-neighbor
- Naïve Bayes and Bayesian Belief Networks
- Support Vector Machines
- Neural Networks, Deep Neural Nets

## Ensemble Classifiers

- Boosting, Bagging, Random Forests

# Decision

# Tree

# Decision Tree

**A decision tree** is a supervised machine learning algorithm used for both classification and regression tasks. It models decisions and their possible consequences in a tree-like structure, where:

**Root node:** node with no incoming link

**Nodes:** represent decisions or tests on attributes (features).

**Edges:** represent the outcome of the test, leading to either more nodes or leaf nodes.

**Leaf or terminal nodes:** represent the final outcome or class label in classification tasks.

The tree is built by splitting the data based on feature values that best separate the classes or predict the target variable.

The goal is to create a tree that accurately classifies new instances by following the decision rules in the tree structure.

# Example of a Decision Tree

| ID | Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|---|
| | | | | |
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
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| 10 | No | Single | 90K | Yes |

categorical categorical continuous class

**Training Data**

**Model: Decision Tree**

# Apply Model to Test Data

Start from the root of tree.

## Test Data

| Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|
| No | Married | 80K | ? |

# Apply Model to Test Data

## Test Data

| Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|
| No | Married | 80K | ? |

# Apply Model to Test Data

## Test Data

| Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|
| No | Married | 80K | ? |

# Apply Model to Test Data

## Test Data

| Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|
| No | Married | 80K | ? |

# Apply Model to Test Data

## Test Data

| Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|
| No | Married | 80K | ? |

# Apply Model to Test Data

## Test Data

| Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|
| No | Married | 80K | ? |

Assign Defaulted to "No"

# Another Example of Decision Tree

| ID | Home Owner | Marital Status | Annual Income | Defaulted Borrower |
|---|---|---|---|---|
| | | categorical | categorical | continuous | class |
| 1 | Yes | Single | 125K | No |
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| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

**There could be more than one tree that fits the same data!**

# Decision Tree Induction

Many Algorithms:

- Hunt's Algorithm (one of the earliest)
- CART
- ID3, C4.5
- SLIQ,SPRINT

# General Structure of Hunt's Algorithm

Let $D_t$ be the set of training records that reach a node t

General Procedure:

- If $D_t$ contains records that belong the **same class** $y_t$, then t is a leaf node labeled as $y_t$
- If $D_t$ contains records that belong to more than one class, use an attribute test to split the data into smaller subsets. Recursively apply the procedure to each subset.

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| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Hunt's Algorithm

Defaulted

**(7,3)**
(no, yes)
(a)

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# Design Issues of Decision Tree Induction

How should training records be **split**?

- Method for expressing test condition depending on *attribute types*
- Measure for **evaluating the goodness** of a test condition

How should the splitting procedure **stop**?

- Stop splitting if all the records belong to the **same class** or have **identical attribute** values
- Early termination

# Methods for Expressing Test Conditions

Depends on attribute types

- Binary
- Nominal
- Ordinal
- Continuous

# Test Condition for Nominal Attributes

## Multi-way split:

- Use as many partitions as distinct values.

## Binary split:

- Divides values into two subsets

# Test Condition for Ordinal Attributes

## Multi-way split:

- Use as many partitions as distinct values

## Binary split:

- Divides values into two subsets
- Preserve order property among attribute values

# Test Condition for Continuous Attributes

(i) Binary split

(ii) Multi-way split

# Splitting Based on Continuous Attributes

Different ways of handling

- Discretization to form an ordinal categorical attribute

  Ranges can be found by equal interval bucketing, equal frequency bucketing (percentiles), or clustering.

  - Static – discretize once at the beginning
  - Dynamic – repeat at each node

- Binary Decision: $(A < v)$ or $(A \geq v)$

  - consider all possible splits and finds the best cut
  - can be more compute intensive

# How to determine the Best Split

**Goodness of an attribute test condition:** many measures try to give preference to attribute test conditions that partition the training instances into **purer subsets** in the child nodes, which mostly have the same class labels

**Before Splitting:** 10 records of **class 0,**
10 records of **class 1**

*Which test condition is the best?*

| Customer Id | Gender | Car Type | Shirt Size | Class |
|---|---|---|---|---|
| 1 | M | Family | Small | C0 |
| 2 | M | Sports | Medium | C0 |
| 3 | M | Sports | Medium | C0 |
| 4 | M | Sports | Large | C0 |
| 5 | M | Sports | Extra Large | C0 |
| 6 | M | Sports | Extra Large | C0 |
| 7 | F | Sports | Small | C0 |
| 8 | F | Sports | Small | C0 |
| 9 | F | Sports | Medium | C0 |
| 10 | F | Luxury | Large | C0 |
| 11 | M | Family | Large | C1 |
| 12 | M | Family | Extra Large | C1 |
| 13 | M | Family | Medium | C1 |
| 14 | M | Luxury | Extra Large | C1 |
| 15 | F | Luxury | Small | C1 |
| 16 | F | Luxury | Small | C1 |
| 17 | F | Luxury | Medium | C1 |
| 18 | F | Luxury | Medium | C1 |
| 19 | F | Luxury | Medium | C1 |
| 20 | F | Luxury | Large | C1 |

**NOTE: Larger trees are less desirable as they are more susceptible to model overfitting**

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|---|---|---|---|---|
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| 2 | M | Sports | Medium | C0 |
| 3 | M | Sports | Medium | C0 |
| 4 | M | Sports | Large | C0 |
| 5 | M | Sports | Extra Large | C0 |
| 6 | M | Sports | Extra Large | C0 |
| 7 | F | Sports | Small | C0 |
| 8 | F | Sports | Small | C0 |
| 9 | F | Sports | Medium | C0 |
| 10 | F | Luxury | Large | C0 |
| 11 | M | Family | Large | C1 |
| 12 | M | Family | Extra Large | C1 |
| 13 | M | Family | Medium | C1 |
| 14 | M | Luxury | Extra Large | C1 |
| 15 | F | Luxury | Small | C1 |
| 16 | F | Luxury | Small | C1 |
| 17 | F | Luxury | Medium | C1 |
| 18 | F | Luxury | Medium | C1 |
| 19 | F | Luxury | Medium | C1 |
| 20 | F | Luxury | Large | C1 |

**NOTE: Larger trees are less desirable as they are more susceptible to model overfitting**

# How to determine the Best Split

Greedy approach: Nodes with purer class distribution are preferred

*We need a measure of node impurity !*

C0: 5
C1: 5

**High degree of impurity**

C0: 9
C1: 1

**Low degree of impurity**

Let's define the ***RELATIVE FREQUENCY*** $p_i(t)$

$C_0 = 5$ $\quad P_0(t) = \frac{5}{10}$

$C_1 = 5$ $\quad P_1(t) = \frac{5}{10}$

$C_0 = 1$ $\quad P_0(t) = \frac{1}{10}$

$C_1 = 9$ $\quad P_1(t) = \frac{9}{10}$

# Measures of **Single Node Impurity**

**Gini Index**

$$Gini\ Index = 1 - \sum_{i=0}^{c-1} p_i(t)^2$$

Where $\boldsymbol{p_i(t)}$ is the frequency of class $\boldsymbol{i}$ at node **t**, and $\boldsymbol{c}$ is the total number of classes

**Entropy**

$$Entropy = -\sum_{i=0}^{c-1} p_i(t) log_2 p_i(t)$$

$2^5 = 32$

$\log_2(32) = 5$

Misclassification error

$$Classification\ error = 1 - \max[p_i(t)]$$

# Compute **Single Node Impurity**

C0: 5
C1: 5

**High degree of impurity**

$C_0 = 5$ $\quad P_0(t) = \frac{5}{10}$

$C_1 = 5$ $\quad P_1(t) = \frac{5}{10}$

$$GI = 1 - \left(\left(\frac{5}{10}\right)^2 + \left(\frac{5}{10}\right)^2\right) = 0,5$$

$$E = -\left(\frac{5}{10}\log_2\frac{5}{10} + \frac{5}{10}\log_2\frac{5}{10}\right) = 1$$

C0: 9
C1: 1

**Low degree of impurity**

$C_0 = 1$ $\quad P_0(t) = \frac{1}{10}$

$C_1 = 9$ $\quad P_1(t) = \frac{9}{10}$

$$GI = 1 - \left(\left(\frac{1}{10}\right)^2 + \left(\frac{9}{10}\right)^2\right) = 0,18$$

$$E = -\left(\frac{1}{10}\log_2\frac{1}{10} + \frac{9}{10}\log_2\frac{9}{10}\right) = 0,469$$

# Compute Collective impurity of **Child** nodes

Once a decision rule **attribute test condition** have been applied it is possible to compute the collective impurity of child nodes

Consider an **attribute test condition** that splits a node containing N training instances into k children,
{v1, v2, · · · , vk}

Let N(vj) be the number of training instances associated with a child node vj, whose impurity value is I(vj)

The **weighted impurity** measure of its **children**

$$I(\text{children}) = \sum_{j=1}^{k} \frac{N(v_j)}{N} I(v_j),$$

| ID | Home Owner | Marital Status | Annual Income | Defaulted Borrower |
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| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
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| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

$$I(\texttt{Home Owner = yes}) = -\frac{0}{3}\log_2\frac{0}{3} - \frac{3}{3}\log_2\frac{3}{3} = 0$$

$$I(\texttt{Home Owner = no}) = -\frac{3}{7}\log_2\frac{3}{7} - \frac{4}{7}\log_2\frac{4}{7} = 0.985$$

$$I(\texttt{Home Owner}) = \frac{3}{10}\times 0 + \frac{7}{10}\times 0.985 = 0.690$$

$$I(\texttt{Marital Status = Single}) = -\frac{2}{5}\log_2\frac{2}{5} - \frac{3}{5}\log_2\frac{3}{5} = 0.971$$

$$I(\texttt{Marital Status = Married}) = -\frac{0}{3}\log_2\frac{0}{3} - \frac{3}{3}\log_2\frac{3}{3} = 0$$

$$I(\texttt{Marital Status = Divorced}) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1.000$$

$$I(\texttt{Marital Status}) = \frac{5}{10}\times 0.971 + \frac{3}{10}\times 0 + \frac{2}{10}\times 1 = 0.686$$

# Finding the Best Split

1. Compute impurity measure **I(parent)** before splitting
2. Compute impurity measure **I(children)** after splitting
   - Compute impurity measure of each child node
   - **I(children)** is the weighted impurity of **child** nodes
3. Choose the attribute test condition that produces the highest gain

$$\Delta = I(\text{parent}) - I(\text{children}),$$

or equivalently, lowest impurity measure after splitting

# Finding the Best Split

**Parent**

| | |
|---|---|
| C0 | **N00** |
| C1 | **N01** |

→ **P**

**Candidate Split**

**A**

Yes → Node N1

No → Node N2

**B**

Yes → Node N1

No → Node N2

**Candidate Children**

| | |
|---|---|
| C0 | **N10** |
| C1 | **N11** |

| | |
|---|---|
| C0 | **N20** |
| C1 | **N21** |

| | |
|---|---|
| C0 | **N10** |
| C1 | **N11** |

| | |
|---|---|
| C0 | **N20** |
| C1 | **N21** |

**MA1 = Gini_Index (A, N1)**

**MA2 = Gini_Index (A, N2)**

**MB1 = Gini_Index (B, N1)**

**MB2 = Gini_Index (B, N2)**

**M A**

**M B**

**Gain A = P – M A** vs **Gain B = P – M B**

# Finding the Best Split

It can be shown that the gain is non-negative since I(parent) ≥ I(children) for any reasonable measure

The higher the gain, **the purer are the classes** in the child nodes relative to the parent node.

The splitting criterion in the decision tree learning algorithm selects the attribute test condition that shows the maximum gain (minimizing the weighted impurity measure of its children)

$$\Delta = I(\text{parent}) - I(\text{children}),$$

$$I(\text{children}) = \sum_{j=1}^{k} \frac{N(v_j)}{N} I(v_j),$$

**N(vj):** number of training instances associated with the child node vj

*Home Owner* and *Marital Status (with the highlited association)* shows the minimum weighted impurity

# Binary Attributes: Computing GINI Index

Splits into two partitions (child nodes)

Effect of Weighing partitions:

- Larger and purer partitions are sought

| | Parent |
|---|---|
| C1 | **7** |
| C2 | **5** |
| **Gini = 0.486** | |

**Gini(N1)**
**= 1 – $(5/6)^2$ – $(1/6)^2$**
**= 0.278**

**Gini(N2)**
**= 1 – $(2/6)^2$ – $(4/6)^2$**
**= 0.444**

| | N1 | N2 |
|---|---|---|
| C1 | **5** | **2** |
| C2 | **1** | **4** |
| **Gini=0.361** | | |

**Weighted Gini of N1 N2**
**= 6/12 * 0.278 +**
**6/12 * 0.444**
**= 0.361**

**Gain = 0.486 – 0.361 = 0.125**

# Categorical Attributes: Computing Gini Index

For each distinct value, gather counts for each class in the dataset

Multi-way split

| | CarType | | |
|---|---|---|---|
| | Family | Sports | Luxury |
| C1 | 1 | 8 | 1 |
| C2 | 3 | 0 | 7 |
| Gini | 0.163 | | |

Two-way split
(find best partition of values)

| | CarType | |
|---|---|---|
| | {Sports, Luxury} | {Family} |
| C1 | 9 | 1 |
| C2 | 7 | 3 |
| Gini | 0.468 | |

| | CarType | |
|---|---|---|
| | {Sports} | {Family, Luxury} |
| C1 | 8 | 2 |
| C2 | 0 | 10 |
| Gini | 0.167 | |

**Which of these is the best?**

# Continuous Attributes: Computing Gini Index

Use Binary Decisions based on one value

Several Choices for the splitting value

– Number of possible splitting values = Number of distinct values

Each splitting value has a count matrix associated with it

– Class counts in each of the partitions, A ≤ v and A > v

Simple method to choose best v

– For each v, scan the database to gather count matrix and compute its Gini index

– Computationally Inefficient! Repetition of work.

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|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
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| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
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**Annual Income ?**

| | ≤ 80 | > 80 |
|---|---|---|
| Defaulted Yes | 0 | 3 |
| Defaulted No | 3 | 4 |

# Continuous Attributes: Computing Gini Index

- For efficient computation: for each attribute,
  - Sort the attribute on values
  - Linearly scan these values, each time updating the count matrix and computing gini index
  - Choose the split position that has the least gini index

| Cheat | No | No | No | Yes | Yes | Yes | No | No | No | No |
|---|---|---|---|---|---|---|---|---|---|---|
| | Annual Income | | | | | | | | | |
| Sorted Values → | 60 | 70 | 75 | 85 | 90 | 95 | 100 | 120 | 125 | 220 |

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  - Sort the attribute on values
  - Linearly scan these values, each time updating the count matrix and computing gini index
  - Choose the split position that has the least gini index

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Cheat | No | No | No | Yes | Yes | Yes | No | No | No | No |
| | Annual Income | | | | | | | | | |
| Sorted Values → | 60 | 70 | 75 | 85 | 90 | 95 | 100 | 120 | 125 | 220 |
| Split Positions → | 55 | 65 | 72 | 80 | 87 | 92 | 97 | 110 | 122 | 172 | 230 |
| | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > |

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| Cheat | No | No | No | Yes | Yes | Yes | No | No | No | No |
|---|---|---|---|---|---|---|---|---|---|---|
| | Annual Income | | | | | | | | | |
| **Sorted Values** | 60 | 70 | 75 | 85 | 90 | 95 | 100 | 120 | 125 | 220 |

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > |
| Yes | | | | | | | 0 | 3 | | | | | | | | | | | | | | |
| No | | | | | | | 3 | 4 | | | | | | | | | | | | | | |
| Gini | | | | | | | 0.343 | | | | | | | | | | | | | | | |

# Continuous Attributes: Computing Gini Index

- For efficient computation: for each attribute,
  - Sort the attribute on values
  - Linearly scan these values, each time updating the count matrix and computing gini index
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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Cheat | No | No | No | Yes | Yes | Yes | No | No | No | No |
| | Annual Income | | | | | | | | | |
| Sorted Values → | 60 | 70 | 75 | 85 | 90 | 95 | 100 | 120 | 125 | 220 |
| Split Positions → | 55 | 65 | 72 | 80 | 87 | 92 | 97 | 110 | 122 | 172 | 230 |
| | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > | <= > |
| Yes | | | | 0 3 | 1 2 | | | | | | |
| No | | | | 3 4 | 3 4 | | | | | | |
| Gini | | | | 0.343 | 0.417 | | | | | | |

# Continuous Attributes: Computing Gini Index

- For efficient computation: for each attribute,
  - Sort the attribute on values
  - Linearly scan these values, each time updating the count matrix and computing gini index
  - Choose the split position that has the least gini index

| Cheat | No | No | No | Yes | Yes | Yes | No | No | No | No |
|---|---|---|---|---|---|---|---|---|---|---|
| | Annual Income | | | | | | | | | |
| **Sorted Values** → | 60 | 70 | 75 | 85 | 90 | 95 | 100 | 120 | 125 | 220 |

| **Split Positions** → | 55 | | 65 | | 72 | | 80 | | 87 | | 92 | | 97 | | 110 | | 122 | | 172 | | 230 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > | <= | > |
| Yes | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 3 | 1 | 2 | 2 | 1 | 3 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 3 | 0 |
| No | 0 | 7 | 1 | 6 | 2 | 5 | 3 | 4 | 3 | 4 | 3 | 4 | 3 | 4 | 4 | 3 | 5 | 2 | 6 | 1 | 7 | 0 |
| Gini | 0.420 | | 0.400 | | 0.375 | | 0.343 | | 0.417 | | 0.400 | | ***0.300*** | | 0.343 | | 0.375 | | 0.400 | | 0.420 | |

# Problem with large number of partitions

Node **impurity measures** tend to **prefer** splits that result in **large number of partitions**, each being small but pure

| Customer Id | Gender | Car Type | Shirt Size | Class |
|---|---|---|---|---|
| 1 | M | Family | Small | C0 |
| 2 | M | Sports | Medium | C0 |
| 3 | M | Sports | Medium | C0 |
| 4 | M | Sports | Large | C0 |
| 5 | M | Sports | Extra Large | C0 |
| 6 | M | Sports | Extra Large | C0 |
| 7 | F | Sports | Small | C0 |
| 8 | F | Sports | Small | C0 |
| 9 | F | Sports | Medium | C0 |
| 10 | F | Luxury | Large | C0 |
| 11 | M | Family | Large | C1 |
| 12 | M | Family | Extra Large | C1 |
| 13 | M | Family | Medium | C1 |
| 14 | M | Luxury | Extra Large | C1 |
| 15 | F | Luxury | Small | C1 |
| 16 | F | Luxury | Small | C1 |
| 17 | F | Luxury | Medium | C1 |
| 18 | F | Luxury | Medium | C1 |
| 19 | F | Luxury | Medium | C1 |
| 20 | F | Luxury | Large | C1 |

Customer ID has highest information gain because *Gini* or *Entropy* is **0** for all the children

# Gain Ratio

The number of children produced by the splitting attribute must be taken into consideration while deciding the best attribute test condition

A measure known as **Gain Ratio** is used to compensate for attributes that produce a large number of child nodes

$$\textbf{Gain Ratio} = \frac{\Delta}{Split\ Info}$$

The **Split Information** measures the entropy of splitting a node into its child nodes and evaluates if the split results in a larger number of equally-sized child nodes or not

$$Split\ Info = -\sum_{i=1}^{k} \frac{n_i}{n} log_2 \frac{n_i}{n}$$

# Gain Ratio

$$Gain\ Ratio = \frac{\Delta}{Split\ Info} \qquad Split\ Info = \sum_{i=1}^{k} \frac{n_i}{n} log_2 \frac{n_i}{n}$$

| | CarType | | |
|---|---|---|---|
| | Family | Sports | Luxury |
| C1 | 1 | 8 | 1 |
| C2 | 3 | 0 | 7 |
| Gini | 0.163 | | |

**SplitINFO = 1.52**

| | CarType | |
|---|---|---|
| | {Sports, Luxury} | {Family} |
| C1 | 9 | 1 |
| C2 | 7 | 3 |
| Gini | 0.468 | |

**SplitINFO = 0.72**

| | CarType | |
|---|---|---|
| | {Sports} | {Family, Luxury} |
| C1 | 8 | 2 |
| C2 | 0 | 10 |
| Gini | 0.167 | |

**SplitINFO = 0.97**

Parent Node, $p$ is split into $k$ partitions (children)

$n_i$ is number of records in child node $i$

# Decision Tree Based Classification

## Advantages:

- Relatively inexpensive to construct
- Extremely fast at classifying unknown records
- Easy to interpret for small-sized trees
- Robust to noise (especially when methods to avoid overfitting are employed)
- Can easily handle redundant attributes
- Can easily handle irrelevant attributes (unless the attributes are interacting)

## Disadvantages: .

- Due to the greedy nature of splitting criterion, interacting attributes (that can distinguish between classes together but not individually) may be passed over in favor of other attributed that are less discriminating.
- Each decision boundary involves only a single attribute

# Handling interactions

**Rectilinear Decision Boundaries:**

- **Single Attribute Testing**: Trees partition the attribute space based on one attribute at a time.
- **Rectilinear Boundaries**: The decision boundary is parallel to coordinate axes.
- **Limitation**: Restricts decision trees in representing complex boundaries, especially with continuous attributes.

**Oblique Decision Trees:**

- **Improvement**: Allow tests using multiple attributes (e.g., x + y < 20) for more flexible decision boundaries.
- **Result**: Can better represent complex, non-rectilinear boundaries.

# Limitations of single attribute-based decision boundaries

# Nearest-Neighbor Classifiers

# Nearest-Neighbor Classifiers

Basic idea: ***If it walks like a duck, quacks like a duck, then it's probably a duck***

A *nearest neighbor classifier* represents each example as a data point in a d-dimensional space (d the number of attributes)

Given a test instance, we compute its proximity to the training instances according to one of the proximity measures

The k-nearest neighbors of a given test instance z refer to the k training examples that are closest to z.

# Nearest-Neighbor Classifiers

Requires the following:

- A set of labeled records
- Proximity metric to compute distance/similarity between a pair of records e.g., Euclidean distance
- The value of *k*, the number of nearest neighbors to retrieve
- A method for using class labels of K nearest neighbors to determine the class label of unknown record (e.g., by taking majority vote)

# Nearest-Neighbor Classifiers

Choosing the value of k:

- If k is **too small**, sensitive to noise points (overfitting)
- If k is **too large**, neighborhood may include points from other classes
- An example of well balanced k

# Nearest-Neighbor Classifiers

## Choosing the value of k:

- If k is **too small**, sensitive to noise points (overfitting)
- If k is **too large**, neighborhood may include points from other classes
- An example of well balanced k

# Nearest-Neighbor Classifiers

How to Determine the class label of a Test Sample?

Let $k$ be the number of nearest neighbors and $D$ be the set of training examples.
**for** each test instance $z = (\mathbf{x}', y')$ **do**
    Compute $d(\mathbf{x}', \mathbf{x})$, the distance between $z$ and every example, $(\mathbf{x}, y) \in D$.
    Select $D_z \subseteq D$, the set of $k$ closest training examples to $z$.
    $y' = \underset{v}{\operatorname{argmax}} \sum_{(\mathbf{x}_i, y_i) \in D_z} I(v = y_i)$
**end for**

Take the **majority vote** of class labels among the k-nearest neighbors

<u>Weight the vote according to distance</u>

- weight factor, $w = 1/d^2$

# Nearest-Neighbor Classifiers

- $v$ is a **class label**
- $y_i$ is the **class label** for **one** of the **nearest neighbors**
- $I(\cdot)$ is an **indicator function** that returns the value 1 if its argument is true and 0 otherwise

**Majority Voting**

$$y' = \underset{v}{\operatorname{argmax}} \sum_{(\mathbf{x}_i, y_i) \in D_z} I(v = y_i),$$

**Distance-Weighted Voting**

$$y' = \underset{v}{\operatorname{argmax}} \sum_{(\mathbf{x}_i, y_i) \in D_z} w_i \times I(v = y_i).$$

**weight factor,** $w = 1/d^2$

# Nearest-Neighbor Classifiers

**Data preprocessing is often required**

Attributes may have to be **scaled** to prevent *distance measures* from being *dominated* by one of the attributes

- Example:
  - height of a person may vary from 1.5m to 1.8m
  - weight of a person may vary from 90lb to 300lb
  - income of a person may vary from $10K to $1M

- It is often useful to **standardize** data to have 0 means a standard deviation of 1

# Nearest-Neighbor Classifiers

**How to handle missing values in training and test sets?**

- Proximity computations normally require the presence of all attributes
- Some approaches use the subset of attributes present in two instances
    - This may not produce good results since it effectively uses different proximity measures for each pair of instances
    - Thus, proximities are not comparable

**How to handle Irrelevant and Redundant Attributes**

- Irrelevant attributes add noise to the proximity measure
- Redundant attributes bias the proximity measure towards certain attributes

# Nearest-Neighbor Classifiers

## Characteristics

- Nearest-Neighbor classification is part of a more general technique known as instance-based learning
- **Classifying** a test instance can be quite **expensive** because we need to compute the proximity values individually (lazy learners)
- Nearest neighbor classifiers can produce **decision boundaries** of **arbitrary shape**
- Nearest neighbor classifiers have **difficulty handling missing values** in both the training and test sets since proximity computations normally require the presence of all attributes
- The presence of irrelevant attributes can distort commonly used proximity measures, especially when the number of irrelevant attributes is large

# Bayesian Classifiers

# Bayesian Classifiers

Many classification problems involve **uncertainty**

- Observed attributes and class labels may be unreliable due to **imperfections** in the measurement process
- Set of attributes may not be fully representative of the target class, resulting in **uncertain** predictions
- Classification model learned over a finite training set may not be able to fully capture the true relationships in the overall data

In the presence of **uncertainty** we need provide not only *predictions* of class labels but a *measure of confidence*

**Probability theory** offers a systematic way for *quantifying* and *manipulating* **uncertainty** in data

Classification models that make use of *probability* are known as **probabilistic classification model**

# Bayesian Classifiers

**Introduction to probability**

Consider a variable $\boldsymbol{X}$, which can take any discrete value from the set $\{x_1, x_2, \dots, x_k\}$

$X$ has the value $x_i$ for $n_i$ data objects

Relative frequency of event $X = x_i$ is $\frac{n_i}{N}$ where $\mathrm{N} = \sum_{i=1}^{k} n_i$

The **probability** of an event e, e.g., $\boldsymbol{P(X = x_i)}$, measures how likely it is for the event $x_i$ to occur

A probability is always a number between 0 and 1

The sum of probability values of all possible events (outcomes of a variable $X$) is equal to 1

Variables that have probabilities associated with each possible outcome (values) are known as random variables

# Bayesian Classifiers

**Using Bayes Theorem for Classification**

For the purpose of classification, we are interested in computing the probability of observing a *class label* $\boldsymbol{y}$ for a data instance given its *set of attribute* $\boldsymbol{x}$

$P(y|\boldsymbol{x})$ is the posterior probability

$$P(y|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|y)P(y)}{P(\boldsymbol{x})}$$

where

$P(x|y)$ is the class-conditional probability the likelihood of observing $\boldsymbol{x}$ from the distribution of instances belonging to $y$.

$P(y)$ is the prior probability (prior beliefs about the distribution of class labels) [expert knowledge or distribution]

$P(x)$ can be computed as $P(x) = \sum_{i=1}^{k} P(\boldsymbol{x}|y_i)P(y_i)$

# Bayesian Classifiers

**Naïve Bayes Assumption for Classification**

**Naïve Bayes classifier** assumes that the class-conditional probability $P(\boldsymbol{x}|y)$ of all attributes x can be factored as a product of class-conditional probabilities $P(x|y)$

$$P(\boldsymbol{x}|y) = \prod_{i=1}^{d} P(x_i|y)$$

Where data instance $\boldsymbol{x}$ the set $\{x_1, x_2, \dots, x_d\}$

The basic assumption behind the previous equation is that the attribute values

$x_i$ are conditionally independent

## Naïve Bayes Assumption for Classification

The Naïve Bayes classifier computes the posterior probability for a test instance $\boldsymbol{x}$ by using the following equation

$$P(y|\boldsymbol{x}) = P(y)\prod_{i=1}^{d} P(x_i|y)$$

# Using Bayes Theorem for Classification

- Consider each attribute and class label as *random variables*
- Given a record with attributes $\{x_1, x_2, \dots, x_d\}$ the goal is to predict class $y$
- Specifically, we want to find the value of Y that maximizes $P(y|x_1, x_2, \dots, x_d)$

Can we estimate $P(y|x_1, x_2, \dots, x_d)$

directly from data?

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

$$P(y|\boldsymbol{x}) = P(y)\prod_{i=1}^{d} P(x_i|y)$$

- We need to estimate

  $P(y = yes|\boldsymbol{x})$ and $P(y = no|\boldsymbol{x})$

In the following we will replace
y = Yes by Yes, and
y = No by No

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

$$P(yes|\boldsymbol{x}) = P(yes)\prod_{i=1}^{d} P(x_i|yes)$$

$$P(no|\boldsymbol{x}) = P(no)\prod_{i=1}^{d} P(x_i|no)$$

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

$$P(yes) = 3/10$$

$$P(no) = 7/10$$

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

$$\prod_{i=1}^{d} P(x_i|yes)$$

P(X | Yes) =

P(Refund = No | Yes) x

P(Divorced | Yes) x

P(Income = 120K | Yes)

$$\prod_{i=1}^{d} P(x_i|no)$$

P(X | No) =

P(Refund = No | No) x

P(Divorced | No) x

P(Income = 120K | No)

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

$$\prod_{i=1}^{d} P(x_i|yes)$$

P(X | Yes) =

P(Refund = No | Yes) x

P(Divorced | Yes) x

P(Income = 120K | Yes)

$$\prod_{i=1}^{d} P(x_i|no)$$

P(X | No) =

P(Refund = No | No) x

P(Divorced | No) x

P(Income = 120K | No)

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

**Categorical**

**Continuous**

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

For categorical attributes:

$$P(x_i = c|yes) = n_c/n = n_c/\ n$$

where $n_c$ is number of instances having attribute value $x_i = c$ and belonging to class y

–Examples:

P(Status=Married|No) = 4/7
P(Refund=Yes|Yes) = 0

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Bayesian Classifiers: Example

For continuous attributes:

Discretization: Partition the range into bins:

Replace continuous value with bin value

Attribute changed from continuous to ordinal

Probability density estimation:

Assume attribute follows a normal distribution

Use data to estimate parameters of distribution
(e.g., mean and standard deviation)

Once probability distribution is known, use it to estimate the conditional probability $P(X_i|Y)$

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

| Tid | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

Normal distribution:

$$P(X_i \mid Y_j) = \frac{1}{\sqrt{2\pi\sigma_{ij}^2}} e^{-\frac{(X_i - \mu_{ij})^2}{2\sigma_{ij}^2}}$$

– One for each ($X_i$,$Y_i$) pair

For (Income, Class=No):

– If Class=No

sample mean = 110

sample variance = 2975

$$P(Income = 120 \mid No) = \frac{1}{\sqrt{2\pi}(54.54)} e^{-\frac{(120-110)^2}{2(2975)}} = 0.0072$$

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

**Naïve Bayes Classifier:**

P(Refund = Yes | No) = 3/7
P(Refund = No | No) = 4/7
P(Refund = Yes | Yes) = 0
P(Refund = No | Yes) = 1
P(Marital Status = Single | No) = 2/7
P(Marital Status = Divorced | No) = 1/7
P(Marital Status = Married | No) = 4/7
P(Marital Status = Single | Yes) = 2/3
P(Marital Status = Divorced | Yes) = 1/3
P(Marital Status = Married | Yes) = 0

For Taxable Income:
If class = No: sample mean = 110
sample variance = 2975
If class = Yes: sample mean = 90
sample variance = 25

- P(X | No) = P(Refund=No | No)
  $\times$ P(Divorced | No)
  $\times$ P(Income=120K | No)
  = 4/7 $\times$ 1/7 $\times$ 0.0072 = 0.0006

- P(X | Yes) = P(Refund=No | Yes)
  $\times$ P(Divorced | Yes)
  $\times$ P(Income=120K | Yes)
  = 1 $\times$ 1/3 $\times$ 1.2 $\times$ $10^{-9}$ = 4 $\times$ $10^{-10}$

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 7 | Yes | Divorced | 220K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

# Bayesian Classifiers: Example

Given a *Test Record*:

$$X = (\text{Refund} = \text{No}, \text{Divorced}, \text{Income} = 120\text{K})$$

$$P(y|\boldsymbol{x}) = P(y) \prod_{i=1}^{d} P(x_i|y)$$

$$P(no|x) = P(no)\ P(\boldsymbol{x}|no) = \mathbf{0.0004}$$

$$P(yes|x) = P(yes)\ P(\boldsymbol{x}|yes) = \mathbf{1.2\ 10^{-10}}$$

P(X | No) = P(Refund=No | No)
× P(Divorced | No)
× P(Income=120K | No)
= 4/7 × 1/7 × 0.0072 = 0.0006

$$P(no) = 7/10$$

P(X | Yes) = P(Refund=No | Yes)
× P(Divorced | Yes)
× P(Income=120K | Yes)
= 1 × 1/3 × 1.2 × $10^{-9}$ = 4 × $10^{-10}$

$$P(yes) = 3/10$$

# Bayesian Classifiers: Issue

## Naïve Bayes Classifier can make decisions with partial information about attributes in the test record

**Naïve Bayes Classifier:**

P(Refund = Yes | No) = 3/7
P(Refund = No | No) = 4/7
P(Refund = Yes | Yes) = 0
P(Refund = No | Yes) = 1
P(Marital Status = Single | No) = 2/7
P(Marital Status = Divorced | No) = 1/7
P(Marital Status = Married | No) = 4/7
P(Marital Status = Single | Yes) = 2/3
P(Marital Status = Divorced | Yes) = 1/3
P(Marital Status = Married | Yes) = 0

For Taxable Income:
If class = No: sample mean = 110
sample variance = 2975
If class = Yes: sample mean = 90
sample variance = 25

P(Yes) = 3/10 P(No) = 7/10

**If we only know that marital status is Divorced, then**:

P(Yes | Divorced) = 1/3 x 3/10 / P(Divorced)

P(No | Divorced) = 1/7 x 7/10 / P(Divorced)

**If we also know that Refund = No, then**

P(Yes | Refund = No, Divorced) = 1 x 1/3 x 3/10 P(Divorced, Refund = No)

P(No | Refund = No, Divorced) = 4/7 x 1/7 x 7/10 P(Divorced, Refund = No)

**If we also know that Taxable Income = 120, then**

P(Yes | Refund = No, Divorced, Income = 120) =
$1.2 \times 10^{-9}$ x 1 x 1/3 x 3/10 P(Divorced, Refund = No, Income = 120 )

P(No | Refund = No, Divorced Income = 120) =
0.0072 x 4/7 x 1/7 x 7/10 P(Divorced, Refund = No, Income = 120)

# Bayesian Classifiers: Issue

**Given a Test Record:** **X = (Married)**

**Naïve Bayes Classifier:**

P(Refund = Yes | No) = 3/7
P(Refund = No | No) = 4/7
P(Refund = Yes | Yes) = 0
P(Refund = No | Yes) = 1
P(Marital Status = Single | No) = 2/7
P(Marital Status = Divorced | No) = 1/7
P(Marital Status = Married | No) = 4/7
P(Marital Status = Single | Yes) = 2/3
P(Marital Status = Divorced | Yes) = 1/3
P(Marital Status = Married | Yes) = 0

For Taxable Income:
If class = No: sample mean = 110
sample variance = 2975
If class = Yes: sample mean = 90
sample variance = 25

P(Yes) = 3/10

P(No) = 7/10

P(Yes | Married) = 0 x 3/10 / P(Married)

P(No | Married) = 4/7 x 7/10 / P(Married)

# Bayesian Classifiers: Issue

Consider the table with record = 7 deleted

| *Tid* | Refund | Marital Status | Taxable Income | Evade |
|---|---|---|---|---|
| 1 | Yes | Single | 125K | No |
| 2 | No | Married | 100K | No |
| 3 | No | Single | 70K | No |
| 4 | Yes | Married | 120K | No |
| 5 | No | Divorced | 95K | Yes |
| 6 | No | Married | 60K | No |
| 8 | No | Single | 85K | Yes |
| 9 | No | Married | 75K | No |
| 10 | No | Single | 90K | Yes |

**Naïve Bayes Classifier:**

P(Refund = Yes | No) = 2/6
P(Refund = No | No) = 4/6
P(Refund = Yes | Yes) = 0
P(Refund = No | Yes) = 1
P(Marital Status = Single | No) = 2/6
P(Marital Status = Divorced | No) = 0
P(Marital Status = Married | No) = 4/6
P(Marital Status = Single | Yes) = 2/3
P(Marital Status = Divorced | Yes) = 1/3
P(Marital Status = Married | Yes) = 0/3
For Taxable Income:
If class = No: sample mean = 91
sample variance = 685
If class = No: sample mean = 90
sample variance = 25

Given X = (Refund = Yes, Divorced, 120K)

P(X | No) = 2/6 X 0 X 0.0083 = 0

P(X | Yes) = 0 X 1/3 X 1.2 X $10^{-9}$ = 0

**Naïve Bayes will not be able to classify X as Yes or No!**

# Bayesian Classifiers: Issue

If one of the conditional probabilities is zero, then the entire expression becomes zero

## Need to use other estimates of conditional probabilities than simple fractions

## Probability estimation:

$$\text{original: } P(X_i = c|y) = \frac{n_c}{n}$$

$$\text{Laplace Estimate: } P(X_i = c|y) = \frac{n_c + 1}{n + v}$$

$$\text{m} - \text{estimate: } P(X_i = c|y) = \frac{n_c + mp}{n + m}$$

$n$: number of training instances belonging to class $y$

$n_c$: number of instances with $X_i = c$ and $Y = y$

$v$: total number of attribute values that $X_i$ can take

$p$: initial estimate of ($P(X_i = c|y)$ known apriori

$m$: hyper-parameter for our confidence in $p$

# Support Vector Machines

# Support Vector Machines

A support vector machine (SVM) is a discriminative classification model that learns linear or nonlinear decision boundaries in the attribute space to separate the classes.

SVM

- strong regularization capabilities.
- good generalization performance.
- learn highly expressive models without suffering from overfitting.

**Find a linear hyperplane (decision boundary) that will separate the data**

# Support Vector Machines

**A support vector machine (SVM) is a discriminative classification model that learns linear or nonlinear decision boundaries in the attribute space to separate the classes.**

**SVM**

- **strong regularization capabilities.**
- **good generalization performance.**
- **learn highly expressive models without suffering from overfitting.**

**One Possible Solution**

# Support Vector Machines

**A support vector machine (SVM) is a discriminative classification model that learns linear or nonlinear decision boundaries in the attribute space to separate the classes.**

**SVM**

- **strong regularization capabilities.**
- **good generalization performance.**
- **learn highly expressive models without suffering from overfitting.**

**Another Possible Solution**

# Support Vector Machines

**A support vector machine (SVM) is a discriminative classification model that learns linear or nonlinear decision boundaries in the attribute space to separate the classes.**

**SVM**

- **strong regularization capabilities.**
- **good generalization performance.**
- **learn highly expressive models without suffering from overfitting.**

**Other Possible Solutions**

# Support Vector Machines

**A support vector machine (SVM) is a discriminative classification model that learns linear or nonlinear decision boundaries in the attribute space to separate the classes.**

**SVM**

- **strong regularization capabilities.**
- **good generalization performance.**
- **learn highly expressive models without suffering from overfitting.**

**Which one is better? B1 or B2?**

**How do you define better?**

# Support Vector Machines

**The point is now about the margin of a separating hyperplane and the rationale for choosing such a hyperplane with maximum margin.**

Find hyperplane maximizes the margin

B1 is better than B2

# Support Vector Machines

**The generic equation of a separating hyperplane can be written as**

$$\boldsymbol{w}^T\boldsymbol{x} + b = 0$$

**where $x$ represent the attributes and $(w, b)$ represent the hyperplane.**

**A data instance $x_i$ can belong to either side of the hyperplane depending on the sign of $w^T x + b$**

# Support Vector Machines

**The hyperplane parameters $(w, b)$ working on this condition:**

$$\boldsymbol{w}^T \boldsymbol{x_i} + b > 0 \quad if\ y_i = 1$$
$$\boldsymbol{w}^T \boldsymbol{x_i} + b < 0 \quad if\ y_i = -1$$

**That led to:**

$$\min_{w,b} \frac{\|\boldsymbol{w}\|^2}{2}$$
$$subject\ to \quad y_i(\boldsymbol{w}^T \boldsymbol{x_i} + b) > 1$$

# Support Vector Machines

**SVM to learn linear hyperplanes even in situations where the classes are not linearly separable.**

**To do this, SVM must consider the trade-off between the width of the margin and the number of training errors committed by the linear hyperplane.**

**The hyperplane parameters $(w, b)$ working on this condition:**

$$\boldsymbol{w}^T \boldsymbol{x_i} + b > 1 - \xi_i$$

**That led to:**

$$\min_{w,b,\xi_i} \frac{\|\boldsymbol{w}\|^2}{2} + C \sum_{i=1}^{n} \xi_i$$

$$\text{subject to} \quad \begin{array}{c} y_i(\boldsymbol{w}^T \boldsymbol{x_i} + b) > 1 - \xi_i \\ \xi_i > 0 \end{array}$$

**C is a hyper-parameter that makes a trade-off between *maximizing* the *margin* and *minimizing* the *training error*.**

**A large value of C pays more emphasis on minimizing the training error than maximizing the margin**

# Nonlinear Support Vector Machines

$$y(x_1, x_2) = \begin{cases} 1 & \text{if } \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2} > 0.2 \\ -1 & \text{otherwise} \end{cases}$$

*What if decision boundary is not linear?*

A nonlinear transformation $\boldsymbol{\varphi}$ is needed to map the data from its original attribute space into a new **linear separable** space

Transform the data from its original attribute space in $\boldsymbol{x}$ into a new space $\boldsymbol{\varphi}(\boldsymbol{x})$ so that a linear hyperplane

Learned hyperplane can then be projected back to the original attribute space, resulting in a nonlinear decision boundary.

# Nonlinear Support Vector Machines

$$y(x_1, x_2) = \begin{cases} 1 & \text{if } \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2} > 0.2 \\ -1 & \text{otherwise} \end{cases}$$

The transformation required is:

$$\varphi: (x_1, x_2) \rightarrow (x_1^2 - x_1, x_2^2 - x_2)$$

# Nonlinear Support Vector Machines

$$y(x_1, x_2) = \begin{cases} 1 & \text{if } \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2} > 0.2 \\ -1 & \text{otherwise} \end{cases}$$

*Learning not linear model*

The new decision boundary is:

$$\boldsymbol{w}^T \boldsymbol{\varphi}(\boldsymbol{x_i}) + b = 0$$

that results in the new learning

$$\min_{w,b,\xi_i} \frac{\|\boldsymbol{w}\|^2}{2} + C \sum_{i=1}^{n} \xi_i$$

$$\text{subject to} \quad y_i(\boldsymbol{w}^T \boldsymbol{\varphi}(\boldsymbol{x_i}) + b) > 1 - \xi_i$$

$$\xi_i > 0$$

We need only inner products of $\varphi(x)$

- What type of mapping function $\varphi$ should be used?
- How to do the computation in high dimensional space?

# Learning Nonlinear SVM

The **kernel trick** is a method for computing this similarity as a *function of the original attribute*

$$\mathrm{K}(\mathrm{u},\mathrm{v}) = \langle \boldsymbol{\varphi}(\boldsymbol{u}), \boldsymbol{\varphi}(\boldsymbol{v}) \rangle = f(u, v)$$

where f() is a function that follows certain conditions as stated by the Mercer's Theorem

$$\begin{aligned} \text{Polynomial kernel} \quad & K(\mathbf{u},\mathbf{v}) = (\mathbf{u}^T\mathbf{v}+1)^p \\ \text{Radial Basis Function kernel} \quad & K(\mathbf{u},\mathbf{v}) = e^{-\|\mathbf{u}-\mathbf{v}\|^2/(2\sigma^2)} \\ \text{Sigmoid kernel} \quad & K(\mathbf{u},\mathbf{v}) = \tanh(k\mathbf{u}^T\mathbf{v}-\delta) \end{aligned}$$

By using a kernel function, we can directly work with inner products in the transformed space without dealing with the exact forms of the nonlinear transformation function

**High-dimensional transformations**, performing calculations only in the original attribute space

# Example of Nonlinear SVM

**SVM with polynomial degree 2 kernel**

# Learning Nonlinear SVM

Advantages of using kernel:

- Don't have to know the mapping function $\boldsymbol{\varphi}$
- Computing dot product $\langle \boldsymbol{\varphi}(\boldsymbol{u}), \boldsymbol{\varphi}(\boldsymbol{v}) \rangle$ in the original space avoids curse of dimensionality

Not all functions can be kernels

- Must make sure there is a corresponding $\Phi$ in some high-dimensional space
- Mercer's theorem (see textbook)

# Characteristics of SVM

- Robust to noise
- Overfitting is handled by maximizing the margin of the decision boundary,
- SVM can handle irrelevant and redundant attributes better than many other techniques
- The user needs to provide the type of kernel function and cost function
- Difficult to handle missing values

# Ensemble Techniques

# Ensemble Methods

Construct a **set** of base classifiers learned from the training data

Predict class label of test records by **combining the predictions** made by multiple classifiers (e.g., by taking majority vote)

# General Approach of Ensemble Learning

The ensemble of classifiers can be constructed in many ways:

- By manipulating the training set
- By manipulating the input features
- By manipulating the class labels
- By manipulating the learning algorithm

The first three approaches are generic methods that are applicable to any classifier, whereas the fourth approach depends on the type of classifier used.

Using majority vote or weighted majority vote
(weighted according to their accuracy or relevance)

$$C\ (x) = f(C_1(x), C_2(x), \ldots, C_k(x))$$

# Example: Why Do Ensemble Methods Work?

- Suppose there are 25 base classifiers
  - Each classifier has error rate, $\epsilon$ = 0.35
  - Majority vote of classifiers used for classification
  - If all classifiers are identical:
    - Error rate of ensemble = $\epsilon$ (0.35)
  - If all classifiers are independent (errors are uncorrelated):
    - Error rate of ensemble = probability of having more than half of base classifiers being wrong

$$e_{\text{ensemble}} = \sum_{i=13}^{25} \binom{25}{i} \epsilon^i (1-\epsilon)^{25-i} = 0.06$$

# Necessary Conditions for Ensemble Methods

Ensemble Methods work better than a single base classifier if:

1. All base classifiers are independent of each other
2. All base classifiers perform better than random guessing (error rate < 0.5 for binary classification)

Classification error for an ensemble of 25 base classifiers, assuming their errors are uncorrelated.

# Rationale for Ensemble Learning

Ensemble Methods work best with **unstable base classifiers**

- Classifiers that are sensitive to minor perturbations in training set, due to *high model complexity*
- Examples: Unpruned decision trees, ANNs, …

# Bias-Variance Decomposition

**Bias-variance decomposition is a formal method for analyzing the generalization error of a predictive model**

- Noise term is intrinsic to the target class
- Bias and Variance terms depend on the choice of the classification model
  - **Bias** of a model represents how close the average prediction of the model is to the average target
  - **Variance** of a model captures the stability of its predictions

For classification, the generalization error of model $m$ can be given by:

$$gen.error(m) = c_1 + bias(m) + c_2 \times variance(m)$$

# Bias-Variance Trade-off and Overfitting

Ensemble methods try to reduce the variance of complex models (with low bias) by ***aggregating*** responses of multiple base classifiers (even help in reducing the bias)

**Overfitting**

Low bias but high variance, it can become susceptible to overfitting,

**Bias-Variance trade-off** can be used to explain why ensemble learning improves the generalization performance of unstable classifiers

**Underfitting**

# Bagging (Bootstrap AGGregatING)

**Bagging** (AKA bootstrap aggregating), is a technique that repeatedly *samples* (*with replacement*) from a data set according to a uniform probability distribution

- Each bootstrap sample has the same size as the original data
- Some instances may **appear** several times in the same training set
- Probability of a training instance being selected in a bootstrap sample is:
  - *$1 - (1 - 1/n)^n$ (n: number of training instances)*
  - *~0.632 when n is large*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Original Data** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| **Bagging (Round 1)** | 7 | 8 | 10 | 8 | 2 | 5 | 10 | 10 | 5 | 9 |
| **Bagging (Round 2)** | 1 | 4 | 9 | 1 | 2 | 3 | 2 | 7 | 3 | 2 |
| **Bagging (Round 3)** | 1 | 8 | 5 | 10 | 5 | 5 | 9 | 6 | 3 | 7 |

## Build classifier on each bootstrap sample

# Bagging Algorithm

**Algorithm 4.5** Bagging algorithm.

Let $k$ be the number of bootstrap samples.
**for** $i = 1$ to $k$ **do**
    Create a bootstrap sample of size $N$, $D_i$.
    Train a base classifier $C_i$ on the bootstrap sample $D_i$.
**end for**
$C^*(x) = \underset{y}{\operatorname{argmax}} \sum_i \delta\big(C_i(x) = y\big)$.
$\{\delta(\cdot) = 1$ if its argument is true and 0 otherwise.$\}$

# Bagging Example

Consider 1-dimensional data set:

**Original Data:**

| x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

Classifier is a ***decision stump*** (decision tree of size 1)

- Decision rule: $x \leq k$ versus $x > k$
- Split point k is chosen based on entropy

# Bagging Example

Bagging Round 1:

| x | 0.1 | 0.2 | 0.2 | 0.3 | 0.4 | 0.4 | 0.5 | 0.6 | 0.9 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |

x <= 0.35 ➔ y = 1
x > 0.35 ➔ y = -1

Bagging Round 2:

| x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.5 | 0.9 | 1 | 1 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 |

x <= 0.7 ➔ y = 1
x > 0.7 ➔ y = 1

Bagging Round 3:

| x | 0.1 | 0.2 | 0.3 | 0.4 | 0.4 | 0.5 | 0.7 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 |

x <= 0.35 ➔ y = 1
x > 0.35 ➔ y = -1

Bagging Round 4:

| x | 0.1 | 0.1 | 0.2 | 0.4 | 0.4 | 0.5 | 0.5 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 |

x <= 0.3 ➔ y = 1
x > 0.3 ➔ y = -1

Bagging Round 5:

| x | 0.1 | 0.1 | 0.2 | 0.5 | 0.6 | 0.6 | 0.6 | 1 | 1 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

x <= 0.35 ➔ y = 1
x > 0.35 ➔ y = -1

Bagging Round 6:

| x | 0.2 | 0.4 | 0.5 | 0.6 | 0.7 | 0.7 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

x <= 0.75 ➔ y = -1
x > 0.75 ➔ y = 1

Bagging Round 7:

| x | 0.1 | 0.4 | 0.4 | 0.6 | 0.7 | 0.8 | 0.9 | 0.9 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 |

x <= 0.75 ➔ y = -1
x > 0.75 ➔ y = 1

Bagging Round 8:

| x | 0.1 | 0.2 | 0.5 | 0.5 | 0.5 | 0.7 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

x <= 0.75 ➔ y = -1
x > 0.75 ➔ y = 1

Bagging Round 9:

| x | 0.1 | 0.3 | 0.4 | 0.4 | 0.6 | 0.7 | 0.7 | 0.8 | 1 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

x <= 0.75 ➔ y = -1
x > 0.75 ➔ y = 1

Bagging Round 10:

| x | 0.1 | 0.1 | 0.1 | 0.1 | 0.3 | 0.3 | 0.8 | 0.8 | 0.9 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

x <= 0.05 ➔ y = 1
x > 0.05 ➔ y = 1

# Bagging Example

Summary of Trained Decision Stumps:

| Round | Split Point | Left Class | Right Class |
|---|---|---|---|
| 1 | 0.35 | 1 | -1 |
| 2 | 0.7 | 1 | 1 |
| 3 | 0.35 | 1 | -1 |
| 4 | 0.3 | 1 | -1 |
| 5 | 0.35 | 1 | -1 |
| 6 | 0.75 | -1 | 1 |
| 7 | 0.75 | -1 | 1 |
| 8 | 0.75 | -1 | 1 |
| 9 | 0.75 | -1 | 1 |
| 10 | 0.05 | 1 | 1 |

# Bagging Example

Use majority vote (sign of sum of predictions) to determine class of ensemble classifier

| Round | x=0.1 | x=0.2 | x=0.3 | x=0.4 | x=0.5 | x=0.6 | x=0.7 | x=0.8 | x=0.9 | x=1.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 4 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 5 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 6 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 7 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 8 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 9 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Sum | 2 | 2 | 2 | -6 | -6 | -6 | -6 | 2 | 2 | 2 |
| Sign | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

**Predicted Class**

Bagging can also increase the complexity (representation capacity) of simple classifiers such as decision stumps

# Bagging Example

Use majority vote (sign of sum of predictions) to determine class of ensemble classifier

| Round | x=0.1 | x=0.2 | x=0.3 | x=0.4 | x=0.5 | x=0.6 | x=0.7 | x=0.8 | x=0.9 | x=1.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 4 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 5 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 6 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 7 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 8 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 9 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Sum | 2 | 2 | 2 | -6 | -6 | -6 | -6 | 2 | 2 | 2 |
| Sign | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

**Predicted Class**

Bagging can also increase the complexity (representation capacity) of simple classifiers such as decision stumps

# Bagging Example

Use majority vote (sign of sum of predictions) to determine class of ensemble classifier

| Round | x=0.1 | x=0.2 | x=0.3 | x=0.4 | x=0.5 | x=0.6 | x=0.7 | x=0.8 | x=0.9 | x=1.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 4 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 5 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| 6 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 7 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 8 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 9 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Sum | 2 | 2 | 2 | -6 | -6 | -6 | -6 | 2 | 2 | 2 |
| Sign | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

**Predicted Class**

| x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

**Predicted Class**

Bagging can also increase the complexity (representation capacity) of simple classifiers such as decision stumps

# Boosting

Boosting is an iterative procedure used to adaptively change the distribution of training examples for learning base classifiers so that they increasingly focus on examples that are hard to classify

Boosting assigns a weight to each training example and may adaptively change the weight at the end of each boosting round

1. They can be used to inform the sampling distribution used to draw a set of bootstrap samples from the original data.

2. They can be used to learn a model that is biased toward examples with higher weight.

Examples that are classified incorrectly will have their weights increased, while those that are classified correctly will have their weights decreased

This forces the classifier to focus on examples that are difficult to classify in subsequent iterations

# AdaBoost

Consider a dataset $\{(x_i, y_i) | j = 1,2, \dots, N\}$

In **Adaboost** the ***importance*** of a classifier $C_i$ *depends* on the error rate :

$$\epsilon_i = \frac{1}{N} \sum_{j=1}^{N} w_j^{(i)} \, \delta(C_i(x_j) \neq y_j)$$

The ***importance*** of a base classifier is **given** by

$$\alpha_i = \frac{1}{2} \ln\left(\frac{1-\varepsilon_i}{\varepsilon_i}\right)$$

# AdaBoost Algorithm

***Weight update:***

$$w_j^{(i+1)} = \frac{w_j^{(i)}}{Z_i} \times \begin{cases} e^{-\alpha_i} & \text{if } C_i(x_j) = y_j \\ e^{\alpha_i} & \text{if } C_i(x_j) \neq y_j \end{cases}$$

Where $Z_i$ is the normalization factor

If any intermediate rounds produce error rate higher than 50%, the weights are reverted back to 1/n and the resampling procedure is repeated

$$C^*(x) = \arg\max_y \sum_{i=1}^{T} \alpha_i \delta(C_i(x) = y)$$

# AdaBoost Algorithm

**Algorithm 4.6** AdaBoost algorithm.

1: $\mathbf{w} = \{w_j = 1/N \mid j = 1, 2, \ldots, N\}$. {Initialize the weights for all $N$ examples.}
2: Let $k$ be the number of boosting rounds.
3: **for** $i = 1$ to $k$ **do**
4: Create training set $D_i$ by sampling (with replacement) from $D$ according to $\mathbf{w}$.
5: Train a base classifier $C_i$ on $D_i$.
6: Apply $C_i$ to all examples in the original training set, $D$.
7: $\epsilon_i = \frac{1}{N}\left[\sum_j w_j \, \delta\big(C_i(x_j) \neq y_j\big)\right]$ {Calculate the weighted error.}
8: **if** $\epsilon_i > 0.5$ **then**
9: $\mathbf{w} = \{w_j = 1/N \mid j = 1, 2, \ldots, N\}$. {Reset the weights for all $N$ examples.}
10: Go back to Step 4.
11: **end if**
12: $\alpha_i = \frac{1}{2} \ln \frac{1-\epsilon_i}{\epsilon_i}$.
13: Update the weight of each example according to Equation 4.103.
14: **end for**
15: $C^*(\mathbf{x}) = \underset{y}{\operatorname{argmax}} \sum_{j=1}^{T} \alpha_j \delta(C_j(\mathbf{x}) = y))$.

# AdaBoost Example

Consider 1-dimensional data set:

**Original Data:**

| x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

Classifier is a ***decision stump*** (decision tree of size 1)

- Decision rule: $x \leq k$ versus $x > k$
- Split point k is chosen based on entropy

# AdaBoost Example

## Training sets for the first 3 boosting rounds:

Boosting Round 1:

| x | 0.1 | 0.4 | 0.5 | 0.6 | 0.6 | 0.7 | 0.7 | 0.7 | 0.8 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 |

Boosting Round 2:

| x | 0.1 | 0.1 | 0.2 | 0.2 | 0.2 | 0.2 | 0.3 | 0.3 | 0.3 | 0.3 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Boosting Round 3:

| x | 0.2 | 0.2 | 0.4 | 0.4 | 0.4 | 0.4 | 0.5 | 0.6 | 0.6 | 0.7 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |

**Weights**

| Round | x=0.1 | x=0.2 | x=0.3 | x=0.4 | x=0.5 | x=0.6 | x=0.7 | x=0.8 | x=0.9 | x=1.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| 2 | 0.311 | 0.311 | 0.311 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| 3 | 0.029 | 0.029 | 0.029 | 0.228 | 0.228 | 0.228 | 0.228 | 0.009 | 0.009 | 0.009 |

# AdaBoost Example

## Alpha

| Round | Split Point | Left Class | Right Class | alpha |
|---|---|---|---|---|
| 1 | 0.75 | -1 | 1 | 1.738 |
| 2 | 0.05 | 1 | 1 | 2.7784 |
| 3 | 0.3 | 1 | -1 | 4.1195 |

## Classification

| Round | x=0.1 | x=0.2 | x=0.3 | x=0.4 | x=0.5 | x=0.6 | x=0.7 | x=0.8 | x=0.9 | x=1.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| Sum | 5.16 | 5.16 | 5.16 | -3.08 | -3.08 | -3.08 | -3.08 | 0.397 | 0.397 | 0.397 |
| Sign | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 |

**Predicted Class**

# Random Forest Algorithm

Construct an ensemble of decision trees by manipulating training set as well as features

- Use bootstrap sample to train every decision tree (similar to Bagging)
- Use the following tree induction algorithm:
  - At every internal node of decision tree, randomly sample p attributes for selecting split criterion
  - Repeat this procedure until all leaves are pure (unpruned tree)

# Random Forest Algorithm

Construct an ensemble of decision trees by manipulating training set as well as features

- Base classifier are unpruned trees and hence are unstable classifier
- Base classifier are uncorrelated (due to randomization in training and features)
- Random forest reduce variance of unstable classifier without impacting the bias

# Model Overfitting

# Overfitting

If a model fits well over the training data, it can still show poor generalization performance, a phenomenon known as Model **Overfitting**

**Reasons for Model Overfitting**

Model overfitting occurs when, in the pursuit of minimizing the training error rate, an overly complex model is selected that captures specific patterns in the training data but fails to learn the true nature of relationships between attributes and class labels in the overall data.

The opposite is Model **Underfitting**: the learned model is too simplistic, and thus, incapable of fully representing the true relationship between the attributes and the class labels.

# Classification Errors

**Training errors**: Errors committed on the training set

**Test errors**: Errors committed on the test set

**Generalization errors**: Expected error of a model over random selection of records from same distribution

| Tid | Attrib1 | Attrib2 | Attrib3 | Class |
|---|---|---|---|---|
| 1 | Yes | Large | 125K | No |
| 2 | No | Medium | 100K | No |
| 3 | No | Small | 70K | No |
| 4 | Yes | Medium | 120K | No |
| 5 | No | Large | 95K | Yes |
| 6 | No | Medium | 60K | No |
| 7 | Yes | Large | 220K | No |
| 8 | No | Small | 85K | Yes |
| 9 | No | Medium | 75K | No |
| 10 | No | Small | 90K | Yes |

| Tid | Attrib1 | Attrib2 | Attrib3 | Class |
|---|---|---|---|---|
| 11 | No | Small | 55K | ? |
| 12 | Yes | Medium | 80K | ? |
| 13 | Yes | Large | 110K | ? |
| 14 | No | Small | 95K | ? |
| 15 | No | Large | 67K | ? |

# Example Data Set: two class problem

+ : 5400 instances

- 5000 instances generated from a Gaussian centered at (10,10)
- 400 noisy instances added

o : 5400 instances

- Generated from a uniform distribution

**10 % of the data used for training and 90% of the data used for testing**

# Decision Tree with 4 nodes

A decision tree made up of **few levels** led to simple decision boundaries

**Decision boundaries on Training data (4 level)**

# Decision Tree with 50 nodes

A decision tree made up of **many levels** led to complex decision boundaries

**Decision boundaries on Training data (50 level)**

# Which tree is better?

**The complex solution involving several layers is better than the simple one?**

**Decision Tree with 4 nodes**

**Decision Tree with 50 nodes**

# Model Underfitting and Overfitting

**Let's involve the test set.**

As the model becomes more and **more complex**, **test errors** can start **increasing** even though **training error** may be **decreasing**

**Underfitting: when model is too simple, both training and test errors are large**

**Overfitting: when model is too complex, training error is small but test error is large**

# Model Overfitting – Impact of Training Data Size

Increasing the size of training data (10% to 20%) reduces the difference between training and testing errors at a given size of model

**Training set 10%**

**Training set 20%**

# Reasons for Model Overfitting

**Limited Training Size**

A training set consisting of a finite number of instances can only provide a limited representation of the overall data.

It is possible that the patterns learned from a training set do not fully represent the true patterns in the overall data

**High Model Complexity**

A more complex model can better represent complex patterns in the data.

An **overly complex model** also tends to **learn specific patterns** in the training set that do not generalize well over unseen instances

With a limited training set and a complex model, there is a high chance of picking a spurious combination of parameters that maximizes the evaluation metric just by random chance.

# Model Evaluation

(and Selection)

# Model Selection

The process of *selecting a model* with the right level of complexity, which is expected to generalize well over unseen test instances, is known as **model selection**

We can estimate the **generalization error rate** of a model by using "out-of-sample" estimates on **VALIDATION SET**

The validation error rate is a good indicator of generalization performance since the validation set has not been used for training the model

Small Training Set → Poor Classification model

Small Val Set → Unreliable result for selecting the model

# Model Selection

**Occam's razor** principle (principle of parsimony) suggests that given **two models** with the **same errors**, the **simpler model** is preferred over the more complex model.

We represent ***gen.error(m)*** as a function of not just the **training error** rate but also the **model complexity**

$$gen.error(m) = train.error(m, D.train) + \alpha \times complexity(M)$$

# Model Evaluation

**Model evaluation** estimates its generalization performance, i.e., its performance on unseen instances outside of train set

Holdout Method → Split labelled dataset in training set ***D.train*** and the test set ***D.test***

A correct approach for **model evaluation** would be to *assess the performance* of a learned model on a labelled test set has not been used at any stage of **model selection**

<u>Small Tr</u> Set → Poor Classification model

<u>Small Test</u> Set → Unreliable result for selecting the model

# Model Evaluation: Cross Validation

**Cross-validation** is a model evaluation method that aims to make effective use of all labeled instances in labeled dataset D for both **training** and **testing**.

**Cross-Validation** is also referred as **k-fold** Cross Validation

Let's partition the Labeled Dataset into k equal sized subset S1, S2 and S3.

Run 1 → Train (S1, S2), Test (S3)

Run 2 → Train (S1, S3), Test (S2)

Run 3 → Train (S2, S3), Test (S1)

Example with k=3

# Model Evaluation: k-fold

Procedure:

- Segment the labeled data D (of size N) into k equal-sized partitions (or folds)
- During the i-th run
  - The i-th partitions of D is chosen as D.test(i)
  - The rest of the partitions are used as D.train(i)

The total test error rate, $err_{test}$, is then computed as

$$err_{test} = \frac{\sum_{i=1}^{k} err_{test}(i)}{k}$$

# Model Evaluation: k-fold

**Small** value of k will result in a smaller training set and larger test set at every run

**Large** value of k will result in a larger training set and small test set at every run

A special case is the **Leave-one-out** approach (k=N) → Computationally Expensive

This approach of estimating the generalization error rate and its variance is known as the complete cross-validation approach

A more practical solution is to repeat the cross-validation approach multiple times, using a different random partitioning

Usually, k is chosen equal to 5 or 10

**Stratified cross-validation** → Guarantee the equality of positive and negative instances in every partition of the data

# Model Evaluation: k-fold

k-fold cross-validation → A different model at every run → $err_{test}(i)$

$err_{test}$ does not reflect the generalization error rate of any of the k models

$err_{test}$ represent the expected generalization error rate of the model selection approach

Applied on a training set of the same size as one of the training folds (N (k – 1)/k)

When the size of the training folds is closer to the size of the overall data (when k is large),

$err_{test}$ resembles the expected performance of a model learned over a data set of the same size as **D**

# Hyper-parameter Selection

***Hyper-parameters*** are parameters of learning algorithms that need to be determined before learning the classification model

Hyper-parameters do not appear in the final model

Regular model parameters, such as the test conditions in the decision tree, appear in the final model

The **hyper-parameter selection** must be performed during model selection and considered in model evaluation

# Hyper-parameter Selection

**Algorithm 3.2** Procedure `model-select`($k$, $\mathcal{P}$, $D.train$)

$N_{train} = |D.train|$ {Size of $D.train$.}
Divide $D.train$ into $k$ partitions, $D.train_1$ to $D.train_k$.
**for** each run $i = 1$ to $k$ **do**
  $D.val(i) = D.train_i$. {Partition used for validation.}
  $D.tr(i) = D.train \setminus D.train_i$. {Partitions used for training.}
  **for** each parameter $p \in \mathcal{P}$ **do**
    $m =$ `model-train`($p$, $D.tr(i)$). {Train model}
    $err_{sum}(p, i) =$ `model-test`($m$, $D.val(i)$). {Sum of validation errors.}
  **end for**
**end for**
$err_{val}(p) = \sum_i^k err_{sum}(p, i)/N_{train}$. {Compute validation error rate.}
$p^* = \text{argmin}_p \, err_{val}(p)$. {Select best hyper-parameter value.}
$m^* =$ `model-train`($p^*$, $D.train$). {Learn final model on $D.train$}
return ($p^*$, $m^*$).

# Nested Cross-Validation

Algorithm 3.2 returns the final classification model m* (learned with parameter p*) but not an estimate of its generalization performance $err_{test}$

The validation error rates used in Algorithm 3.2 cannot be used as estimates of generalization performance

## Nested cross-validation or double cross-validation

**Algorithm 3.3** The nested cross-validation approach for computing $err_{test}$.

1: Divide $D$ into $k$ partitions, $D_1$ to $D_k$.
2: **for** each outer run $i = 1$ to $k$ **do**
3: $D.test(i) = D_i$. {Partition used for testing.}
4: $D.train(i) = D \setminus D_i$. {Partitions used for model selection.}
5: $(p^*(i), m^*(i)) =$ `model-select`$(k, \mathcal{P}, D.train(i))$. {Inner cross-validation.}
6: $err_{sum}(i) =$ `model-test`$(m^*(i), D.test(i))$. {Sum of test errors.}
7: **end for**
8: $err_{test} = \sum_i^k err_{sum}(i)/N$. {Compute test error rate.}

# Class Imbalance Problem

In many datasets (**skewed** or **imbalanced**), there are a disproportionate number of instances that belong to different classes

This is frequent, especially in critical applications (health care, fraud)

The degree of imbalance varies across different applications or different datasets of the same application.

Accuracy is not well suited to evaluate models in the presence of class imbalance

We need of:

- methods for **building classifiers** when there is class imbalance in the training set
- methods for **adapting classification** decisions in the presence of a skewed test set.
- methods for **evaluating classification** in the presence of a skewed test set.

# Class Imbalance Problem: Undersampling/Oversampling

**Undersampling:** the frequency of the majority class is **reduced** to match the frequency of the minority class

- N random samples of the majority class are selected (where N is the number of instances of the minority class)
- Some useful examples could be lost
- Reducing the training set could result in low model capabilities

**Oversampling**: artificial examples of the minority class are **created** to make them equal to the number of negative instances

- Duplicate every positive instance $n_{majority}/n_{minority}$
- Generate synthetic positive instances in the neighborhood of existing positive instances
- Duplicating a positive instance is analogous to doubling its weight during the training stage
- It does not help improve the representation of the positive class outside the boundary of existing positive instances.

# Class Imbalance Problem: Assigning Scores to Test Instances

If a classifier returns an ordinal score s(x) for a test instance x:

- Higher score → greater likelihood of x belonging to the positive class
- Lower score → greater likelihood of x belonging to the negative class

Moving a threshold $\boldsymbol{s_T}$ we can create a new binary classifier where a test instance **x** is classified **positive** only if s(x) > **sT**

This way, we can compensate bias introduced by the poor number of instances in the minority class.

This approach works if, in general, $s(x_1) > s(x_2) \quad if. \quad P(y = 1|x_1) > P(y = 1|x_2)$

# Confusion Matrix

**True positive (TP)** or $f_{++}$, which corresponds to the number of positive examples correctly predicted by the classifier.

**False positive (FP)** or $f_{-+}$ (also known as Type I error), which corresponds to the number of negative examples wrongly predicted as positive by the classifier.

**False negative (FN)** or $f_{+-}$ (also known as Type II error), which corresponds to the number of positive examples wrongly predicted as negative by the classifier.

**True negative (TN)** or $f_{--}$, which corresponds to the number of negative examples correctly predicted by the classifier.

| | PREDICTED CLASS | | |
|---|---|---|---|
| GT | | Class=**Yes** | Class=**No** |
| | Class=**Yes** | TP | FN |
| | Class=**No** | FP | TN |

$$Accuracy = \frac{TP + TN}{TP + TN + FP + FN}$$

# Confusion Matrix

## Consider a 2-class problem

- Number of Class NO examples = 990
- Number of Class YES examples = 10

If a model predicts everything to be class NO, accuracy is 990/1000 = 99 %

- This is misleading because this trivial model does not detect any class YES example
- Detecting the rare class is usually more interesting (e.g., frauds, intrusions, defects, etc)

| | | PREDICTED CLASS | |
|---|---|---|---|
| | | Class=**Yes** | Class=**No** |
| GT | Class=**Yes** | TP=0 | FN=10 |
| | Class=**No** | FP=0 | TN=990 |

$$Accuracy = \frac{TP+TN}{TP+TN+FP+FN} = \mathbf{99\%}$$

# Confusion Matrix

- **True positive rate (TPR), Recall, Sensitivity:** fraction of positive test instances correctly predicted
- **True negative rate (TPR), Specificity:** fraction of negative test instances correctly predicted
- **False positive rate (FPR):** fraction of negative test instances wrongly predicted as positive
- **False negative rate (FBR):** fraction of positive test instances wrongly predicted as negative

$$TPR = \frac{TP}{TP + FN} \qquad TNR = \frac{TN}{FP + TN}$$

$$FPR = \frac{FP}{FP + TN} \qquad FNR = \frac{FN}{FN + TP}$$

| | PREDICTED CLASS | | |
|---|---|---|---|
| GT | | Class=**Yes** | Class=**No** |
| | Class=**Yes** | TP=0 | FN=10 |
| | Class=**No** | FP=0 | TN=990 |

$$Accuracy = \frac{TP+TN}{TP+TN+FP+FN} = 0.99$$

$$\boxed{Balanced\ Accuracy = \frac{TPR+TNN}{2} = \frac{0+1}{2} = 0.5}$$

# Which model is better?

| | PREDICTED CLASS | | |
|---|---|---|---|
| GT | | Class=**Yes** | Class=**No** |
| | Class=**Yes** | TP=0 | FN=10 |
| | Class=**No** | FP=0 | TN=990 |

| | PREDICTED CLASS | | |
|---|---|---|---|
| GT | | Class=**Yes** | Class=**No** |
| | Class=**Yes** | TP=10 | FN=0 |
| | Class=**No** | FP=500 | TN=490 |

# Which model is better?

| | PREDICTED CLASS | | |
|---|---|---|---|
| GT | | Class=**Yes** | Class=**No** |
| | Class=**Yes** | TP=0 | FN=10 |
| | Class=**No** | FP=0 | TN=990 |

$$\boxed{Balanced\ Accuracy = \frac{TPR+TNN}{2} = \frac{0+1}{2} = 0.5}$$

| | PREDICTED CLASS | | |
|---|---|---|---|
| GT | | Class=**Yes** | Class=**No** |
| | Class=**Yes** | TP=10 | FN=0 |
| | Class=**No** | FP=500 | TN=490 |

$$\boxed{Balanced\ Accuracy = \frac{TPR+TNN}{2} = \frac{1+0.5}{2} = 0.75}$$

# Alternative Measures

**Precision** (also called positive predictive value) A classifier that has a high precision is likely to have most of its positive predictions correct.

**Recall** is the fraction of positive test instances correctly predicted

Low Precision → Many **false positive**

Low Recall → Many **false negative**

| | | PREDICTED CLASS | |
|---|---|---|---|
| | | Class=**Yes** | Class=**No** |
| GT | Class=**Yes** | TP | FN |
| | Class=**No** | FP | TN |

**Precision VS Recall**: may be valuable to prioritize one metric over the other in cases where the outcome of a false positive or false negative is costly

In medical diagnosis, the **cost of a false negative** is high, and **recall** may be a valuable metric

In fraud detection, the **cost of a false positive** is high and **precision** may be a more valuable metric

$$Precision(p) = \frac{TP}{TP + \boldsymbol{FP}}$$

$$Recall\ (r) = \frac{TP}{TP + \boldsymbol{FN}} = TPR$$

$$F1\ score = \frac{2rp}{r + p} = \frac{2\ TP}{2\ TP + FP + FN}$$

# ROC (Receiver Operating Characteristic)

A graphical approach for displaying **trade-off** between **TPR** (detection rate) and **FPR** (false alarm rate)

Developed in 1950s for signal detection theory to analyze noisy signals

ROC curve plots TPR against FPR

**Let's say we have a classifier providing a score s where:**

$$s(x_1) > s(x_2) \quad if. \quad P(y = 1|x_1) > P(y = 1|x_2)$$

- **Higher score → greater <u>likelihood</u> of x belonging to the positive class**
- **Lower score → greater <u>likelihood</u> of x belonging to the negative class**

**A threshold $th$ can divide positive and negative instances**

- **Higher $th$ → greater risk of misclassify positive as negative**
- **Lower $th$ → greater risk of misclassify negative as positive**

# ROC (Receiver Operating Characteristic)

ROC curve plots TPR against FPR

By using **different thresholds** on this value, we can create different **variations of the classifier** with TPR/FPR tradeoffs

**Let's say we have a classifier providing a score s where:**

$$s(x_1) > s(x_2) \quad if. \quad P(y = 1|x_1) > P(y = 1|x_2)$$

- **Higher score → greater <u>likelihood</u> of x belonging to the positive class**
- **Lower score → greater <u>likelihood</u> of x belonging to the negative class**

**A threshold $th$ can divide positive and negative instances**

- **Higher $th$ → greater risk of misclassify positive as negative**
- **Lower $th$ → greater risk of misclassify negative as positive**

# ROC (Receiver Operating Characteristic)

ROC curve plots TPR against FPR

(0,0): declare everything to be negative class

(1,1): declare everything to be positive class

(1,0): ideal

**Let's say we have a classifier providing a score s where:**

$$s(x_1) > s(x_2) \quad if. \quad P(y = 1|x_1) > P(y = 1|x_2)$$

- **Higher score → greater <u>likelihood</u> of x belonging to the positive class**
- **Lower score → greater <u>likelihood</u> of x belonging to the negative class**

**A threshold $th$ can divide positive and negative instances**

- **Higher $th$ → greater risk of misclassify positive as negative**
- **Lower $th$ → greater risk of misclassify negative as positive**

# ROC Curve Example

**1-dimensional data set containing 2 classes (positive and negative)**

**Any points located at x > t is classified as positive**

**At threshold t: TPR=0.5, FNR=0.5, FPR=0.12, TNR=0.88**

# How to Construct an ROC curve

| Instance | Score | True Class |
|---|---|---|
| 1 | 0.95 | + |
| 2 | 0.93 | + |
| 3 | 0.87 | - |
| 4 | 0.85 | - |
| 5 | 0.85 | - |
| 6 | 0.85 | + |
| 7 | 0.76 | - |
| 8 | 0.53 | + |
| 9 | 0.43 | - |
| 10 | 0.25 | + |

- Use a classifier that produces a continuous-valued score for each instance
  - The more likely it is for the instance to be in the + class, the higher the score
- Sort the instances in decreasing order according to the score
- Apply a threshold at each unique value of the score
- Count the number of TP, FP, TN, FN at each threshold
  - TPR = TP/(TP+FN)
  - FPR = FP/(FP + TN)

# How to construct an ROC curve

| Class | + | - | + | - | - | - | + | - | + | + | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.25 | 0.43 | 0.53 | 0.76 | 0.85 | 0.85 | 0.85 | 0.87 | 0.93 | 0.95 | 1.00 |
| TP | 5 | 4 | 4 | 3 | 3 | 3 | 3 | 2 | 2 | 1 | 0 |
| FP | 5 | 5 | 4 | 4 | 3 | 2 | 1 | 1 | 0 | 0 | 0 |
| TN | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 5 | 5 |
| FN | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 4 | 5 |
| TPR | 1 | 0.8 | 0.8 | 0.6 | 0.6 | 0.6 | 0.6 | 0.4 | 0.4 | 0.2 | 0 |
| FPR | 1 | 1 | 0.8 | 0.8 | 0.6 | 0.4 | 0.2 | 0.2 | 0 | 0 | 0 |

**ROC Curve:**

# Using ROC for Model Comparison

- No model consistently outperforms the other
  - $M_1$ is better for small FPR
  - $M_2$ is better for large FPR
- Area Under the ROC curve (AUC)
  - Ideal:
    - Area = 1
  - Random guess:
    - Area = 0.5

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