PSI 5794 - Matrix Analysis — Final Exam — June 6, 2016

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> 1. Print the exam sheet;
> 2. Write your solutions in blank sheets (A4), number and name everything;
> 3. Solutions due Monday, June 13th, by 7:30 PM sharply. NO EXTENSIONS.

Name:____

1. An experiment was carried out in order to determine a vector of unknown parameters x, resulting in the system $Ax = b$, with $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 3 & 3 \end{bmatrix}$ and $b^T = [1 \quad 1 \quad -1]$. Retrieve x.

2. Consider the N-point Discrete Time Fourier Transform (DTFT), defined as follows

$$V(k) = \sum_n x(n)\, e^{-j2\pi kn/N}, \quad k, n = 0, \ldots, N-1 \tag{1}$$

   (a) Show that the DTFT can be posed in matrix form.
   (b) For $N = 8$, compute $\mathcal{F}x$, for $x^* = [1\ 1 \ldots 1]$ and $x^* = [1\ 0\ \ldots\ 0]$.
   (c) For $N = 4$, compute the eigenvalues and eigenvectors pairs.
   (d) Is this transformation unitary? Explain. If not, how can you fix it?

3. Solve by hand via QR decomposition the linear system $Ax = b$ in which

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ -1 & -2 & -1 \end{bmatrix} \qquad b^T = [1 \quad 2 \quad 2] \tag{2}$$

   (a) Using Gram-Schmidt.
   (b) Using Householder transforms, showing in details the intermediate matrices H_k and how the reflectors P_k were determined.

4. Consider the two matrices A and B below.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ -1 & -2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 2 & -3 \end{bmatrix} \tag{3}$$

   (a) Compute by hand, showing all the steps, their full SVD. Verify that your expansion is correct by retrieving A and B from their decompositions
   (b) Calculate by hand bases for their four fundamental subspaces

5. Let $A \in \mathbb{C}^{n\times n}$ be a hermitian matrix. Show that there exists a constant $c \in \mathbb{R}$ such that $A + cI$ is non-singular. Is this new matrix also hermitian?

6. Let $A \in \mathbb{C}^{m\times n}$ be a complex matrix and let $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ and $\Sigma \in \mathbb{C}^{m\times n}$ be matrices such that $U\Sigma V^*$ is its SVD. Denote X^+ as the Moore-Penrose pseudo-inverse of X.

(a) Compute Σ^+.

(b) Verify that $A^+ = V\Sigma^+U^*$.

(c) Let $x \in \mathbb{C}^{n\times 1}$ and $y \in \mathbb{C}^{m\times 1}$. Compute the SVD of $yx^* \in \mathbb{C}^{m\times n}$ and its pseudo-inverse.

7. Let $A \in \mathbb{C}^{n\times n}$ be a non-singular matrix and $\|\cdot\|$ the matrix norm induced by the ℓ_2 vector norm. Remember the definition of the condition number $\kappa(A) = \|A\| \cdot \|A^{-1}\|$. Show that

$$\kappa(A)^2 = \kappa(A^*A) = \kappa(AA^*) = \kappa(A^*)^2.$$

What can you say about numerically solving $A^*Ax = A^*b$ instead of solving $Ax = b$?

8. Let P_n be the space of complex polynomials of degree less than or equal to n. We choose a set of $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$ and define a form $\langle\cdot,\cdot\rangle$ as

$$\langle p, q\rangle = \sum_{i=0}^{n} p(x_i)\overline{q(x_i)}.$$

(a) Show that $\langle\cdot,\cdot\rangle$ is an inner product.

(b) Let $A\colon P_n \to P_n$ be a linear operator and $A^*\colon P_n \to P_n$ its adjoint in relation to this inner product. Let $q(x) \in P_n$. Show that the polynomial $\hat{p}(x) \in P_n$ that solves the problem

$$\min_p \|q - A(p)\|^2, \qquad \text{where } \|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle},$$

is such that $A^*A(\hat{p}) = A^*(\hat{p})$.

(c) Assume n = 3 and the set of points to be $\{-1, 0, 1, 2\}$. Starting from the canonical basis $\mathcal{C} = \{1, x, x^2, x^3\}$, apply the Gram-Schmidt process to obtain an orthonormal basis $\mathcal{O}$ of P_3 in relation to the inner product defined above.

(d) Let $D\colon P_3 \to P_3$ be the derivative operator, that is, $D(p(x)) = p'(x)$ for every $p \in P_3$. Find $[D]_{\mathcal{C}}$ and $[D]_{\mathcal{O}}$, the representations of the operator D in the bases $\mathcal{C}$ and $\mathcal{O}$. Describe the adjoint D^* in both bases.

(e) Find a solution $\hat{p}$ to the least squares problem of (b) with $A = D$ and $q(x) = x^3 + x^2 + x + 1$. Explicitly, what polynomial is $D(\hat{p}(x))$?

(f) Let $s \in P_3$ be a polynomial such that $[D]_{\mathcal{C}}[D]^*_{\mathcal{C}}[s]_{\mathcal{C}} = [D]^*_{\mathcal{C}}[s]_{\mathcal{C}}$. Is s a solution to the previous problem? If not, which least-squares problem does it solve?

The End.