CONVEX CONE

Debartha Paul Sayantika Sengupta
Saikat Bera
Presidency University, Kolkata

September 29, 2021

Contents

List of Figures

1 Introduction

Convex cone is a part of linear algebra which helps in various branches of linear programming problem. A convex cone is regarded as the subset of a vector space over an ordered field that is closed under linear combinations with coefficients. To understand the concepts of a convex cone we need to understand the concept of a convex set, ordered field and vector space. Here we shall also discuss some properties of a convex cone and try to find out how a convex cone helps in Operations Research, especially in Linear Programming.

2 Definitions

2.1 Convex Set

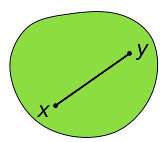


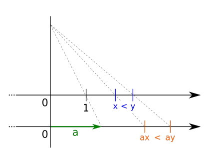
Figure 1: Convex Set

A set X is called a convex set if for any points x_1, x_2 belonging to the set X the line joining the points also belong to the set X.

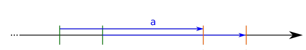
i.e.

$$\forall x_1, x_2 \in X, \quad \lambda x_1 + (1-\lambda)\, x_2 \in X \quad ; 0 \le \lambda \le 1$$

2.2 Ordered field



(a) The property $a > 0 \wedge x < y \Rightarrow ax < ay$



(b) The property $x < y \Rightarrow a + x < a + y$

Figure 2: Ordered field

An ordered field is a field containing a subset of elements closed under addition and multiplication and having the property that every element in the field is either 0, in the subset, or has its additive inverse in the subset.

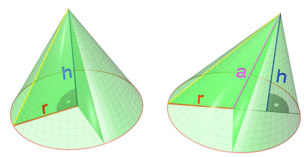


Figure 3: A right circular cone and an oblique circular cone

2.3 Cone

A subset $\mathcal{C}$ of a vector space $\mathcal{V}$ over an ordered field $\mathcal{F}$ is a cone (or sometimes called a linear cone) if for each x in $\mathcal{C}$ and positive scalar α in $\mathcal{F}$, the product αx is in $\mathcal{C}$. Mathematically, we can write that as:

$$\forall x \in \mathcal{C}, \alpha\,(>0) \in \mathcal{F}, \alpha x \in \mathcal{C}$$

2.4 Convex Cone

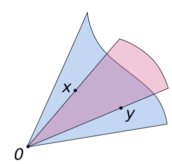


Figure 4: Convex Cone

A cone $\mathcal{C}$ is a convex cone if $\alpha x + \beta y$ belongs to $\mathcal{C}$, for any positive scalars α, β, and any x, y in $\mathcal{C}$. This, when written mathematically, turns out as:

$$\forall x, y \in \mathcal{C} \text{ and } \forall \alpha, \beta\,(>0) \in \mathcal{F}, \alpha x + \beta y \in \mathcal{C}$$

A cone $\mathcal{C}$ is convex if and only if $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$.

2.4.1 An interesting result and another definition!!

If $\mathcal{C}$ is a convex cone, then for any positive scalar α and any $x \in \mathcal{C}$, the vector $\alpha x = \frac{\alpha}{2}x + \frac{\alpha}{2}x \in C$. It follows that a convex cone $\mathcal{C}$ is a special case of a linear cone. And thus, it follows from this property that a convex cone can also be ***defined*** as a linear cone that is closed under convex combinations, or just under additions.

3 Examples of convex cones

3.1 Norm cone is a convex cone

Proof A norm cone is defined as $\mathcal{C} \equiv \left\{x, t : x \in \mathbb{R}^d, t \geq 0, \|x\| \leq t\right\} \subseteq \mathbb{R}^{d+1}$. Let $(x_1, t_1), (x_2, t_2) \in \mathcal{C}$. Then, $t_1, t_2 \geq 0$ and thus, $\alpha t_1 + (1-\alpha)\,t_2 \geq 0$ for any

$0 \leq \alpha \leq 1$. Thus, we have,

$$\begin{aligned}\|\alpha x_1 + (1-\alpha)\, x_2\| &\leq \|\alpha x_1\| + \|\,(1-\alpha)\, x_2\| \quad \text{by triangle inequality} \\ &\leq \alpha \|x_1\| + (1-\alpha)\, \|x_2\| \\ &\leq \alpha t_1 + (1-\alpha)\, t_2\end{aligned}$$

Thus, whenever, $(x_1, t_1), (x_2, t_2) \in \mathcal{C}$, $\alpha x_1 + (1-\alpha)\, x_2 \in \mathcal{C}$ for any $0 \leq \alpha \leq 1$. Thus, it is a convex cone.

3.2 The set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, x_2 \in \mathbb{R}\} \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}, x_2 = 0\}$ is a cone but not a convex cone.

Proof

- **It's a cone:** Let $\mathcal{C} = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, x_2 \in \mathbb{R}\big\} \cup \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}, x_2 = 0\big\}$. Then, we observe that $\mathcal{C}$ is a subset of a vector space $\mathcal{V} = \mathbb{R}^2$. Now, for any $\alpha \in \mathbb{R}$ and $x \in \mathcal{C}$, $\alpha x \in \mathcal{C}$. This proves that $\mathcal{C}$ is also a cone.

- **But not a convex cone:** Let $a = (1, 0)$ and $b = (0, 1)$. This implies that, $a, b \in \mathcal{C}$. Let $\alpha = \frac{3}{5}$ and $\beta = \frac{2}{5}$. However, $\alpha a + \beta b = \left(\frac{3}{5}, \frac{2}{5}\right) \notin \mathcal{C}$. Thus, we found out that two linear cones is not closed under convex combination. Hence, we say that the union of two convex cones may not be a convex cone.

4 Results

4.1 The intersection of two convex cones in the same vector space is again a convex cone, but their union may fail to be one

Proof

- **Intersection part:** Let $\mathcal{V}$ be a vector space and $\mathcal{F}$ be the field. Let $\mathcal{C}_1, \mathcal{C}_2$ be two convex cones in $\mathcal{V}$. Further, let $x, y \in \mathcal{C}_1 \cap \mathcal{C}_2$. Then, we have, $x, y \in \mathcal{C}_1$ and $x, y \in \mathcal{C}_2$. Further, as $\mathcal{C}_1$ is a convex cone, we have, $\forall \alpha, \beta \in \mathcal{F}$,

  $$\alpha x + \beta y \in \mathcal{C}_1$$

  Similarly, as $\mathcal{C}_2$ is also a convex cone, we have, $\forall \alpha, \beta \in \mathcal{F}$,

  $$\alpha x + \beta y \in \mathcal{C}_2$$

  Thus, we find that, $\forall \alpha, \beta \in \mathcal{F}$, if $x, y \in \mathcal{C}_1 \cap \mathcal{C}_2$, then $\alpha x + \beta y \in \mathcal{C}_1 \cap \mathcal{C}_2$. This proves that the intersection of two convex cones in the same vector space is a convex cone.

- **Union part:** We will provide a counterexample in support of our statement. Let $\mathcal{V} = \mathbb{R}^2$ be the vector space and $\mathcal{F} = \mathbb{R}$ be the field. Then, the $x - axis$ and the $y - axis$ form two subspaces of $\mathcal{V}$ and in thus, they form two convex cones (as a subspace is a convex cone). So, let $\mathcal{C}_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $\mathcal{C}_2 = \{(0, y) : y \in \mathbb{R}\}$. Let $a = (1, 0)$ and $b = (0, 1)$. This implies that, $a, b \in \mathcal{C}_1 \cup \mathcal{C}_2$. Let $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$. Then, $\alpha + \beta = 1$. However, $\alpha a + \beta b = \left(\frac{1}{3}, \frac{2}{3}\right) \notin \mathcal{C}_1 \cup \mathcal{C}_2$. Thus, we found out that two linear cones is not closed under convex combination. Hence, we say that the union of two convex cones may not be a convex cone.

4.2 For a vector space $\mathcal{V}$, the empty set, the space $\mathcal{V}$, and any linear subspace of $\mathcal{V}$ are convex cones

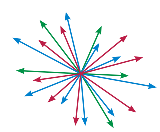


Figure 5: Vectors

Proof

- Let $x \in \mathcal{V}$. Then, according to the definition of a vector space, for any $\alpha \in \mathcal{F}$, $\alpha x \in \mathcal{V}$. Thus, we can say that $\mathcal{V}$ is a cone. Also, we know that a vector space is closed under addition. This implies that, for any $\alpha, \beta \in \mathcal{F}$,

$$\alpha x + \beta y \in \mathcal{V} \quad \forall x, y \in \mathcal{V}$$

  This proves that the vector space $\mathcal{V}$ is a convex cone.

- A linear subspace $\mathcal{W}$ of $\mathcal{V}$ is itself a vector space. This implies that if $\mathcal{V}$ is a convex cone, so is $\mathcal{W}$. Hence the proof.

- The empty set ϕ is a subspace of $\mathcal{W}$. This implies that ϕ is a convex cone.

5 Properties

5.1 Half Spaces

A (linear) hyperplane is a set in the form $\{x \in \mathcal{V} \mid f(x) = c\}$ where f is a linear function on the vector space $\mathcal{V}$. A closed half-space is a set in the form $\{x \in V \mid f(x) \leq c\}$ or $\{x \in V \mid f(x) \geq c\}$, and likewise an open half-space uses strict inequality. Half-spaces (open or closed) are affine convex cones. Moreover (in finite dimensions), any convex cone $\mathcal{C}$ that is not the whole space $\mathcal{V}$ must

(a) Half Space

(b) Hyperplane

Figure 6: Hyperplanes and Half spaces

be contained in some closed half-space $\mathcal{H}$ of $\mathcal{V}$; this is a special case of Farkas' lemma.

Generalized Farkas' lemma can be interpreted geometrically as follows: either a vector is in a given closed convex cone, or there exists a hyperplane separating the vector from the cone; there are no other possibilities.

6 Uses in LPP

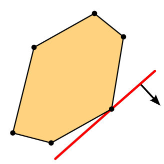


Figure 7: Linear Programming (Illustration)

6.1 Conic Optimisation

In order to do complex optimization we often use the simplex method (e.g. surplus variable etc). But sometimes we face the problem of duality where decreasing one criteria can often lead to increasing another one and vise versa. For this situation we can't use simplex method. So we use convex optimization (i.e. conic optimization). To solve the problem a vector space and a convex real-valued function co-exist in a cone, i.e., the convex cone. So considering the dual nature of the problem use of convex cone is required. It turns the problem into a simple linear programming problem which is much easier to solve.

Given a real vector space $\mathcal{X}$, a convex, real-valued function $f : \mathcal{C} \to \mathbb{R}$ defined on a convex cone $\mathcal{C} \subset \mathcal{X}$, and an affine subspace $\mathcal{H}$ defined by a set of affine constraints $h_i(x) = 0$, a conic optimization problem is to find the point x in $C \cap \mathcal{H}$ for which the number $f(x)$ is smallest.

Conclusion

This paper overviews the powerful idea behind the convex cone and it's application and uses for solving different types of problems. We have seen the relation between ordered fields, convex set, cones, convex cones and half-spaces. We also observed different properties and results concerning the convex cones. Convex cones have been found very useful in diverse applications including Multi-objective integer programming and evolutionary multi-objective optimisation. We conclude with brief state-of-the-art recommendation on how to use convex cone in duality problems.

Acknowledgement

We would like to cordially thank our guide Prof. Krishna Chaitanya Yerroju for giving us this opportunity to do this presentation. We would also like to thank our classmates and peers for their kind help and support. Last, but not the least, we would like to extend our gratitude to our parents who encouraged us during our work.

References

1. Convex Cone Wikipedia
2. Conic Optimization Wikipedia
3. Duality in Conic Programming by Thomas Anderson and Nathan D'Addio
4. Optimization Lecture 4 by Quanquan Gu
5. Linear Programming by Hadley