CS 554
Computer Vision

Homography
& Image Stitching

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Image Stitching

- We are given a bunch of photographs; how do we *stitch* them together?

- But first, we need to understand *non-linear least squares* and *homographies*

Non-linear least squares problems

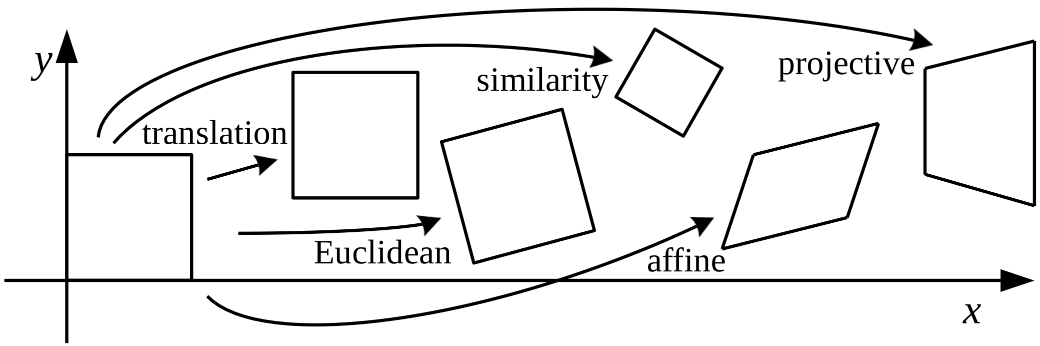
Non-linear least squares

- Fitting a panography was *linear* in the transformation parameters:

$$E = \sum_i \|f(\mathbf{x}_i; \mathbf{p}) - \mathbf{x}'_i\|^2 = \sum_i \|\mathbf{x}_i + J(\mathbf{x}_i)\mathbf{p} - \mathbf{x}'_i\|^2$$

$$= \sum_i \|J(\mathbf{x}_i)\mathbf{p} - \Delta\mathbf{x}_i\|^2$$

- For more complex motion models, the transformation is *non-linear*:

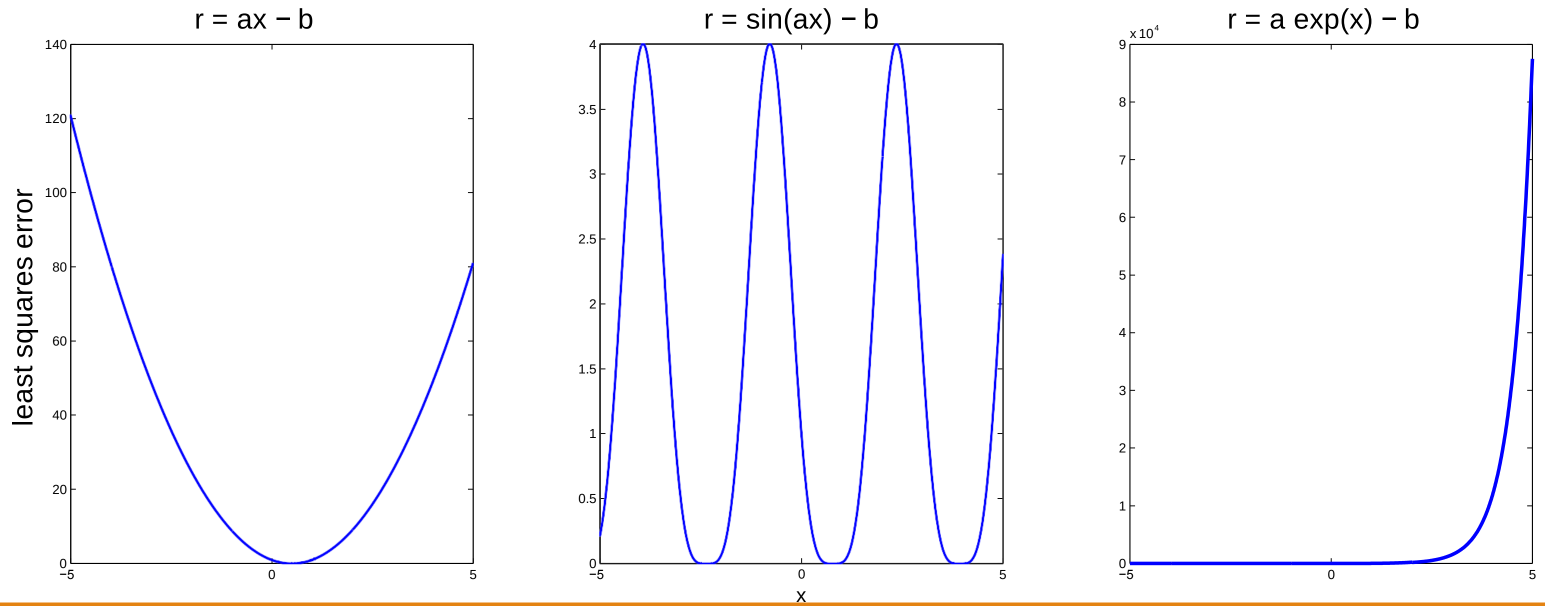


Non-linear least squares

- Consider the *non-linear least squares problem*:

$$g(\mathbf{x}) = \|f(\mathbf{x}; \mathbf{A}) - \mathbf{b}\|^2$$

- This problem is in general not *convex*; it may have multiple *local minima*:

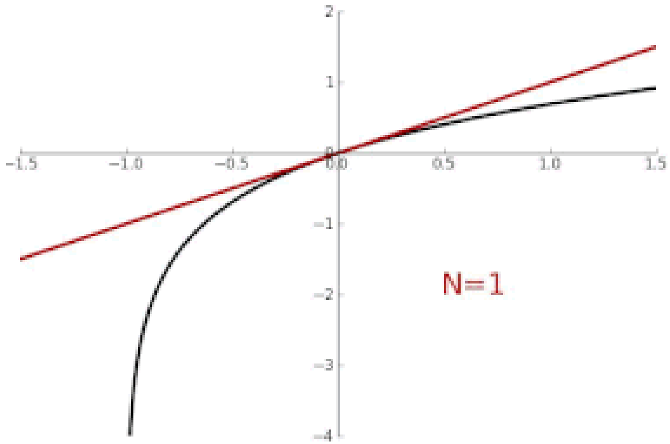


Taylor Expansion

- The *Taylor expansion* of the function $f(x)$ around a is given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- Herein, $f^{(n)}$ denotes the *n*-th derivative



Gauss-Newton Method

- Iteratively find parameter updates $\Delta\mathbf{x}$

Gauss-Newton Method

- Iteratively find parameter updates $\Delta\mathbf{x}$

- Perform a *first-order Taylor expansion* of the residual around the current $\mathbf{x}$

$$\|f(\mathbf{x}-\Delta\mathbf{x};\mathbf{A})-\mathbf{b}\|^2 \approx \|f(\mathbf{x};\mathbf{A})+J(\mathbf{x})\Delta\mathbf{x}-\mathbf{b}\|^2$$

Jacobian

parameter update objective

approximation around x

Gauss-Newton Method

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Jacobian

parameter update objective

approximation around x

- Note that the resulting residual approximation is linear in $\Delta\mathbf{x}$:
    - The parameter update $\Delta\mathbf{x}$ may be obtained via linear least squares

Gauss-Newton Method

- Writing down the linear least-squares solution for $\Delta\mathbf{x}$, we obtain:

$$\Delta\mathbf{x} = \left(J(\mathbf{x})^\top J(\mathbf{x})\right)^{-1} J(\mathbf{x})^\top r(\mathbf{x})$$

"Gauss-Newton approximation to Hessian"

- Gauss-Newton iteratively performs this update: $\mathbf{x} \leftarrow \mathbf{x} - \Delta\mathbf{x}$
- The Taylor expansion just became inaccurate! So iterate the whole process...

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"Gauss-Newton approximation to Hessian"

- Gauss-Newton iteratively performs this update: $\mathbf{x} \leftarrow \mathbf{x} - \Delta\mathbf{x}$
- The Taylor expansion just became inaccurate! So iterate the whole process...
- To implement, you only need to derive *Jacobian*: $J(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$

Newton's Method

- Perform a *second-order Taylor expansion* of $g(\mathbf{x})$ around $\mathbf{x}$:

$$g(\mathbf{x}) \approx \|r(\mathbf{x})\|^2 - 2J(\mathbf{x})r(\mathbf{x})\Delta\mathbf{x} + \left[H(\mathbf{x})r(\mathbf{x}) + J(\mathbf{x})^2\right](\Delta\mathbf{x})^2$$

with residuals: $r(\mathbf{x}) = f(\mathbf{x};\mathbf{A}) - \mathbf{b}$

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with residuals: $r(\mathbf{x}) = f(\mathbf{x};\mathbf{A}) - \mathbf{b}$

- This looks a lot like a linear least-squares problem; set gradient to zero:

$$-2J(\mathbf{x})^{\mathrm{T}}r(\mathbf{x}) + 2\left[H(\mathbf{x})r(\mathbf{x}) + J(\mathbf{x})^{\mathrm{T}}J(\mathbf{x})\right]\Delta\mathbf{x} = 0$$

$$\left[H(\mathbf{x})r(\mathbf{x}) + J(\mathbf{x})^{\mathrm{T}}J(\mathbf{x})\right]\Delta\mathbf{x} = J(\mathbf{x})^{\mathrm{T}}r(\mathbf{x})$$

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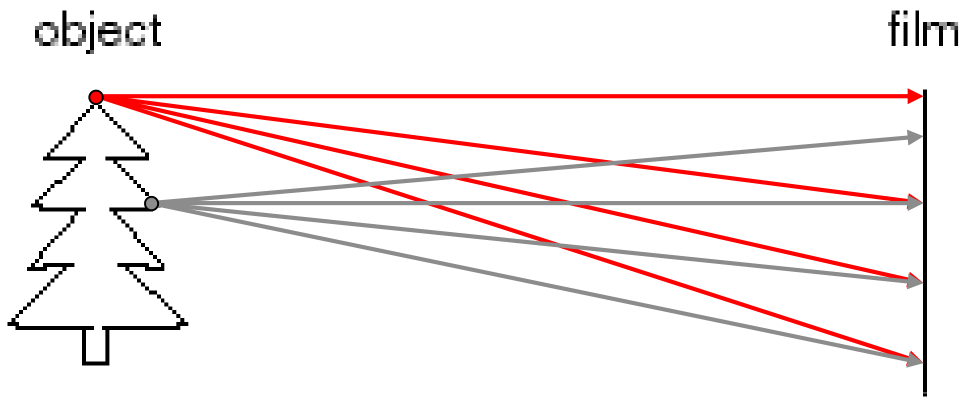
$$\left[H(\mathbf{x})r(\mathbf{x}) + J(\mathbf{x})^{\mathrm{T}} J(\mathbf{x})\right]\Delta\mathbf{x} = J(\mathbf{x})^{\mathrm{T}} r(\mathbf{x})$$

- Note the similarity of the Newton update with the Gauss-Newton update:

$$\Delta\mathbf{x} = \left[H(\mathbf{x})r(\mathbf{x}) + J(\mathbf{x})^{\mathrm{T}} J(\mathbf{x})\right]^{-1} J(\mathbf{x})^{\mathrm{T}} r(\mathbf{x})$$

Homography

Let's design a camera

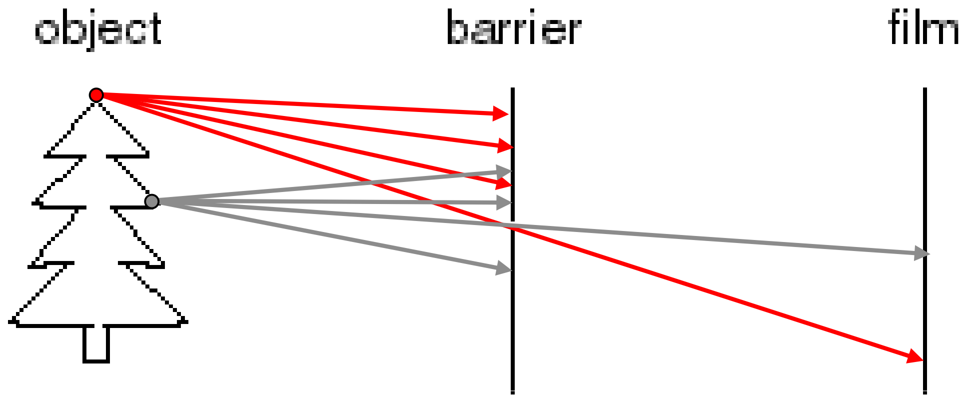


Idea 1: put a piece of film in front of an object

Do we get a reasonable image?

Slide by Steve Seitz

Let’s design a camera

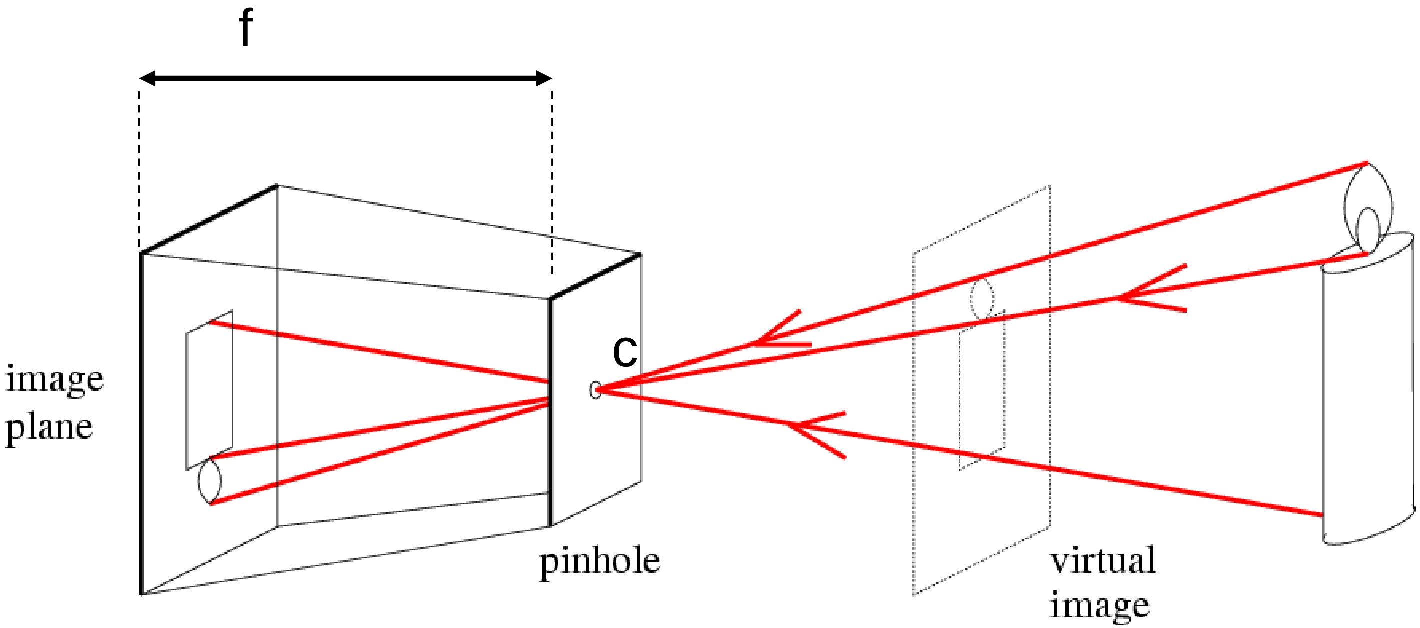


Add a barrier to block off most of the rays

- This reduces blurring
- The opening is known as the **aperture**

Slide by Steve Seitz

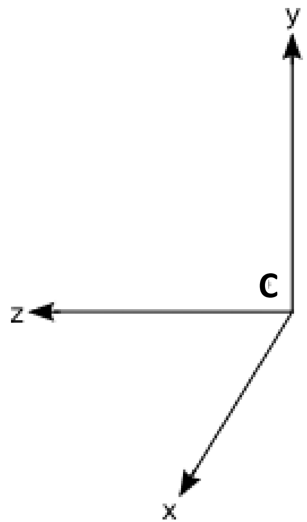
Pinhole camera



f = focal length
c = center of projection

Figure from Forsyth

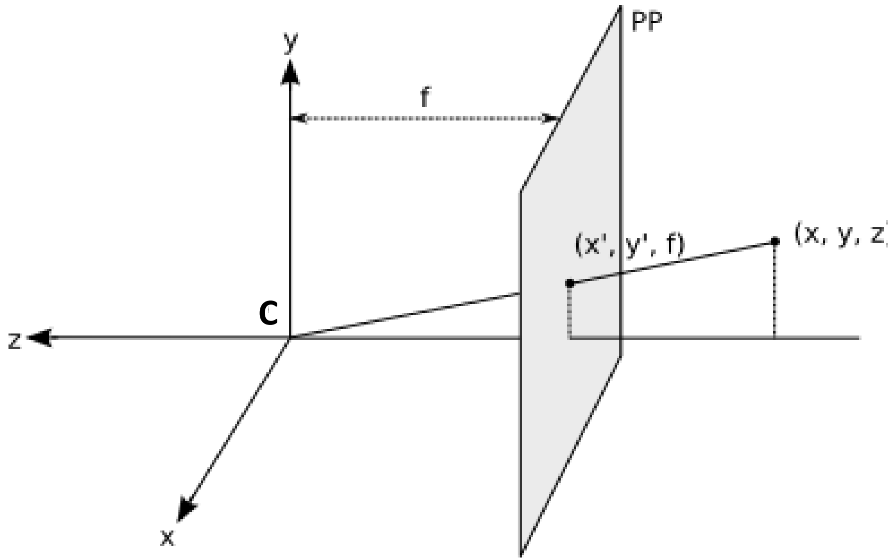
Modeling projection



The coordinate system

- We will use the pin-hole model as an approximation
- Put the optical center (**C**enter of projection) at the origin
- Where would you put the image (Projection Plane)?

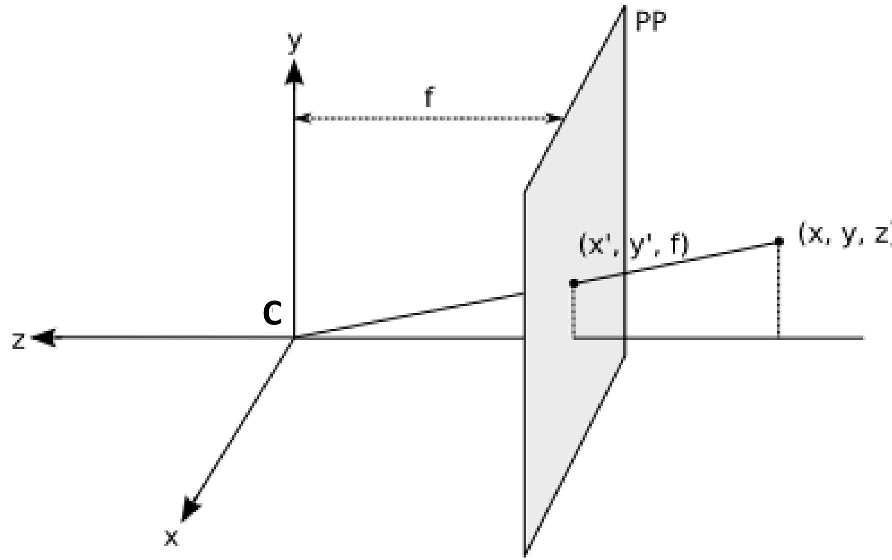
Modeling projection



The coordinate system

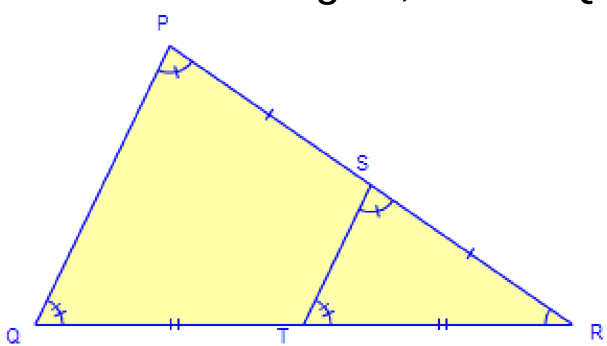
- We will use the pin-hole model as an approximation
- Put the optical center (**C**enter of projection) at the origin
- Where would you put the image (**P**rojection **P**lane)?
  - In front of the camera

Modeling projection



Projection equations

- Compute intersection with PP of ray from (x,y,z) to C in terms of *x*, *y*, *z*, and *f*
- Hint: use similar triangles; ST = PQ * (TR/QR)



$$(x, y, z) \rightarrow (x\frac{f}{z}, y\frac{f}{z}, f)$$

- How do you get the 2D projection?

$$(x, y, z) \rightarrow (x\frac{f}{z}, y\frac{f}{z})$$

Throw out the z coordinate

Modeling projection

- *Focal length* of the camera influences what is captured on the image plane:

$$(x, y, z) \rightarrow (x\frac{f}{z}, y\frac{f}{z}, f)$$

- A large focal length implies small *field of view*, and vice versa

Homogeneous coordinates

- Is this a linear transformation? $(x, y, z) \rightarrow (x\frac{f}{z}, y\frac{f}{z})$

No; division by z is non-linear

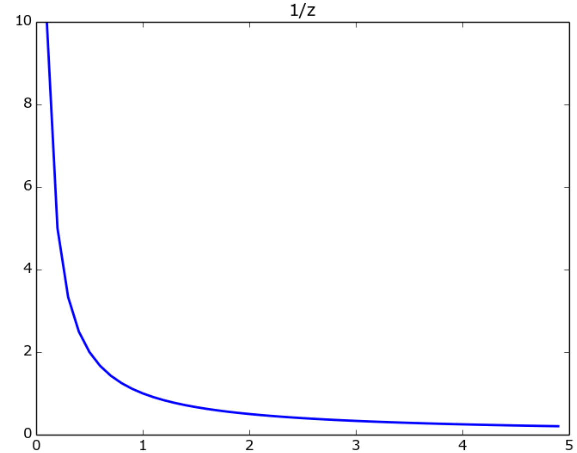
Why?

Definition of a linear function:

$\mathrm{f}(ax) = a\ \mathrm{f}(x)$

$\mathrm{f}(x + y) = \mathrm{f}(x) + \mathrm{f}(y)$

What does 1/z look like?



Lets look at a small example for f = 1/z:

1+2=3 → f(1) + f(2) = f(1+2) → $\frac{1}{1} + \frac{2}{2} = \frac{3}{3}$ → $1 + 1 = 1$? ? ?

Trick: add one more coordinate to remember what 1/z was.

1+2=3 → $(\frac{1}{1}, 1) + (\frac{2}{2}, \frac{1}{2}) = (\frac{3}{3}, \frac{1}{3})$

How would you use this coordinate to fix the problem?

Divide by the extra coordinate: $1 + 2 = 3$

Homogeneous coordinates

- Is this a linear transformation? $(x, y, z) \rightarrow (x\frac{f}{z}, y\frac{f}{z})$

  No; division by z is non-linear

Trick: add one more coordinate:

2D:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

3D:

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

Converting *from* homogeneous coordinates

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \Rightarrow (\tilde{x}/\tilde{w}, \tilde{y}/\tilde{w})$$

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} \Rightarrow (\tilde{x}/\tilde{w}, \tilde{y}/\tilde{w}, \tilde{z}/\tilde{w})$$

Homogeneous coordinates

- We are used to describing a location in *Cartesian coordinates*:

$$\mathbf{x} = [x \quad y]^{\mathrm{T}} \qquad \mathbf{x} = [x \quad y \quad z]^{\mathrm{T}}$$

- Alternatively, we can describe locations in *homogeneous coordinates*:

$$\tilde{\mathbf{x}} = [\tilde{x} \quad \tilde{y} \quad \tilde{w}]^{\mathrm{T}} \qquad \tilde{\mathbf{x}} = [\tilde{x} \quad \tilde{y} \quad \tilde{z} \quad \tilde{w}]^{\mathrm{T}}$$

- The corresponding Cartesian coordinates are given by:

$$\mathbf{x} = [\tilde{x}/\tilde{w} \quad \tilde{y}/\tilde{w}]^{\mathrm{T}} \qquad \mathbf{x} = [\tilde{x}/\tilde{w} \quad \tilde{y}/\tilde{w} \quad \tilde{z}/\tilde{w}]^{\mathrm{T}}$$

- Essentially, you can think of $\tilde{w}$ as a way to deal with object scale (*"disparity"*)

- Homogeneous coordinates are very useful when working with *perspective transformations* (*homographies*)

Perspective Projection Matrix

Projection is a matrix multiplication using homogeneous coordinates: $(x, y, z) \rightarrow (x\frac{f}{z}, y\frac{f}{z})$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z/f \end{bmatrix} \Rightarrow (x\frac{f}{z}, y\frac{f}{z})$$

divide by the third coordinate

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In practice: split into lots of different coordinate transformations…

$$\begin{pmatrix} \text{2D} \\ \text{point} \\ \text{(3x1)} \end{pmatrix} = \begin{pmatrix} \text{Camera to} \\ \text{pixel coord.} \\ \text{trans. matrix} \\ \text{(3x3)} \end{pmatrix} \begin{pmatrix} \text{Perspective} \\ \text{projection matrix} \\ \text{(3x4)} \end{pmatrix} \begin{pmatrix} \text{World to} \\ \text{camera coord.} \\ \text{trans. matrix} \\ \text{(4x4)} \end{pmatrix} \begin{pmatrix} \text{3D} \\ \text{point} \\ \text{(4x1)} \end{pmatrix}$$

Camera Matrix

In practice: split into lots of different coordinate transformations…

$$\begin{bmatrix}\text{2D}\\\text{point}\\\text{(3x1)}\end{bmatrix} = \begin{pmatrix}\text{Camera to}\\\text{pixel coord.}\\\text{trans. matrix}\\\text{(3x3)}\end{pmatrix}\begin{pmatrix}\text{Perspective}\\\text{projection matrix}\\\text{(3x4)}\end{pmatrix}\begin{pmatrix}\text{World to}\\\text{camera coord.}\\\text{trans. matrix}\\\text{(4x4)}\end{pmatrix}\begin{bmatrix}\text{3D}\\\text{point}\\\text{(4x1)}\end{bmatrix}$$

$$\tilde{\mathbf{x}} = \mathbf{K}\left[\mathbf{R}|\mathbf{t}\right]\mathbf{p} = \mathbf{P}\mathbf{p}$$

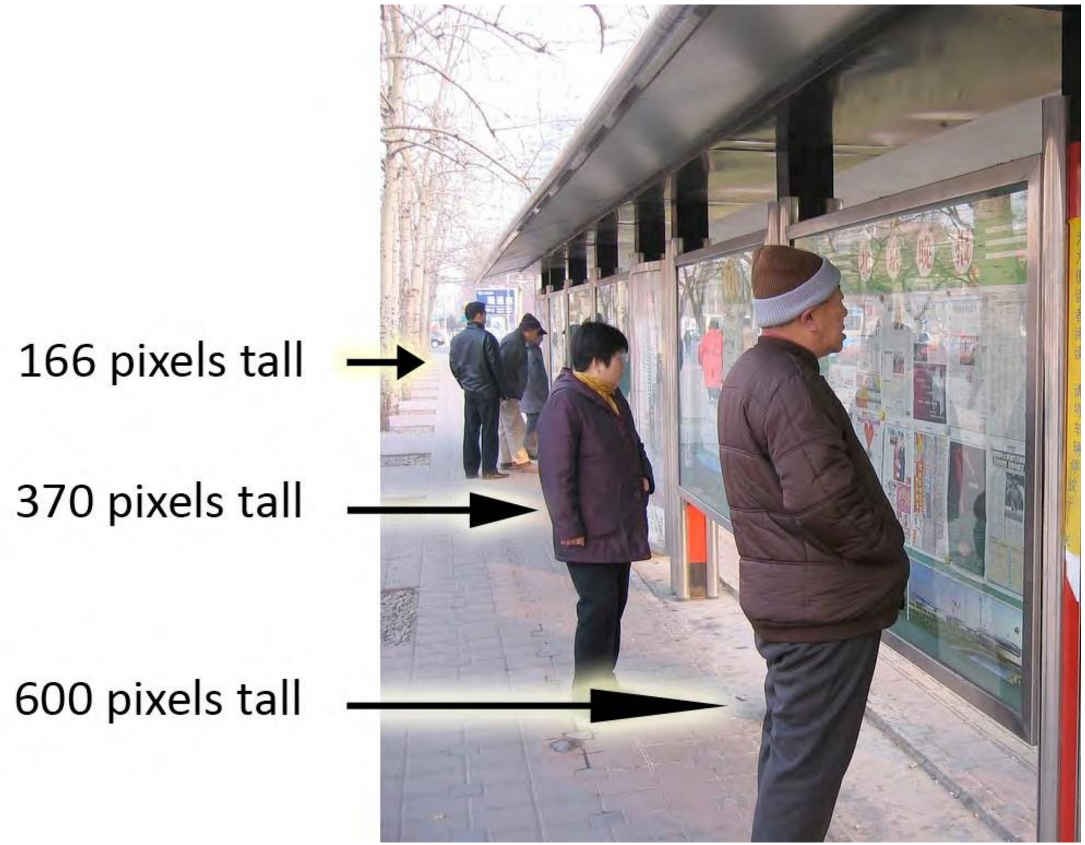
camera intrinsics
(calibration matrix)

camera extrinsics
(rotation + translation)

camera matrix

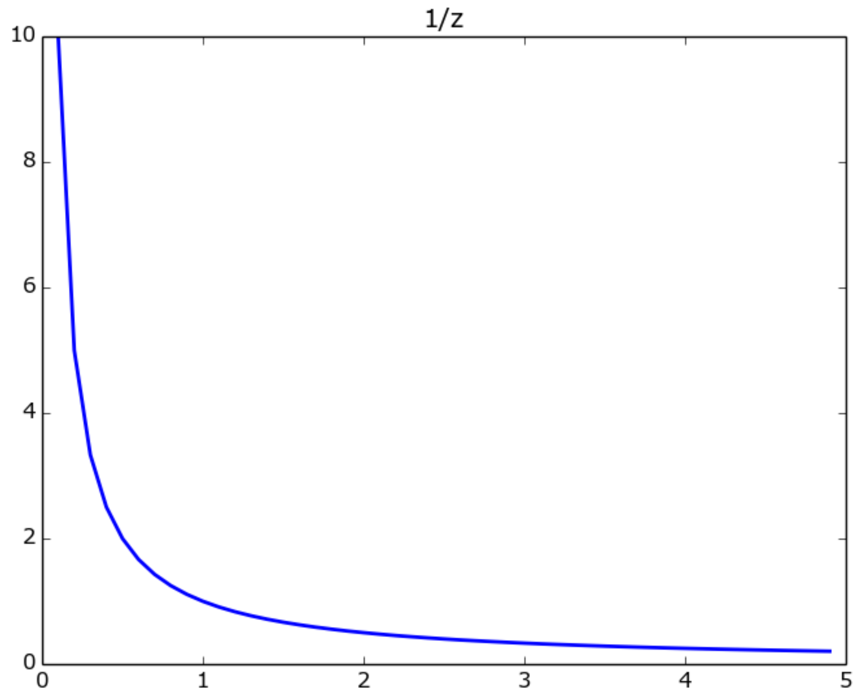
How does a camera see a 3D point?

- The camera matrix is an example of a *projective transformation*:



How does a camera see a 3D point?

What does this curve tell you about perspective effects?



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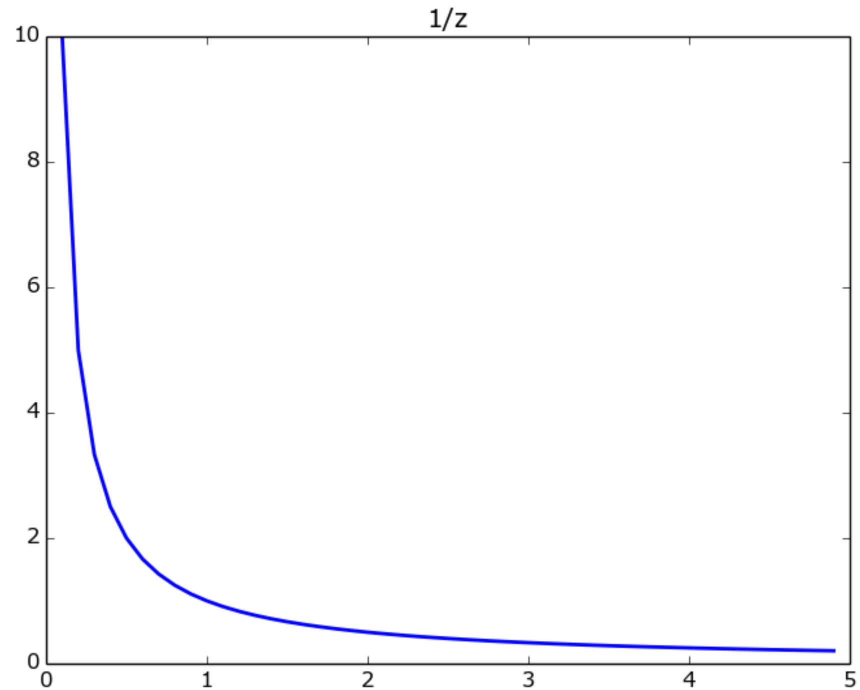

Play limited role if the object is “far away”

How does a camera see a place?

- Assume we have two cameras with projection matrices $\tilde{\mathbf{P}}_0$ and $\tilde{\mathbf{P}}_1$
- Where does camera 1 see the point that camera 0 sees at $\tilde{\mathbf{x}}_0$?

$$\tilde{\mathbf{x}}_1 \sim \tilde{\mathbf{P}}_1 \tilde{\mathbf{P}}_0^{-1} \tilde{\mathbf{x}}_0 = \mathbf{M}_{10} \tilde{\mathbf{x}}_0$$

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- *Disparity* of point is irrelevant, so we can take the 3x3 sub-matrix of $\mathbf{M}_{10}$:

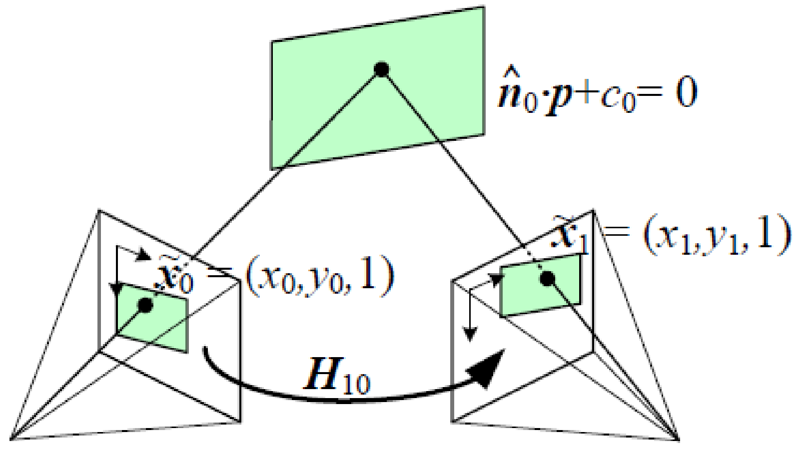
$$\tilde{\mathbf{x}}_1 \sim \tilde{\mathbf{H}}_{10} \tilde{\mathbf{x}}_0$$

- This is known as a *homography* $\tilde{\mathbf{H}}_{10}$ between the two cameras

* Notation in book is a bit sloppy here: page 56

How does a camera see a place?

- Illustration of homography between two camera (for point and plane):



$$H_{10} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & 1 \end{bmatrix}$$

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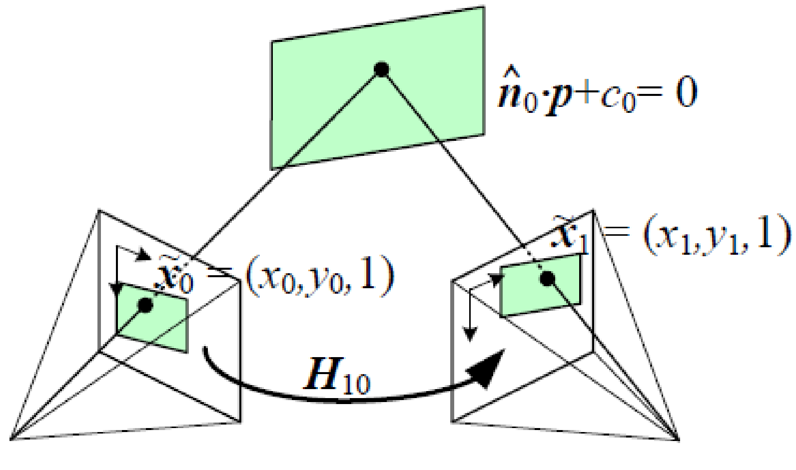


$$\boldsymbol{H}_{10} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & 1 \end{bmatrix}$$

- In Cartesian coordinates, the homography is given by:

$$x_1 = \frac{h_{00}x_0 + h_{01}y_0 + h_{02}}{h_{20}x_0 + h_{21}y_0 + 1} \qquad y_1 = \frac{h_{10}x_0 + h_{11}y_0 + h_{12}}{h_{20}x_0 + h_{21}y_0 + 1}$$

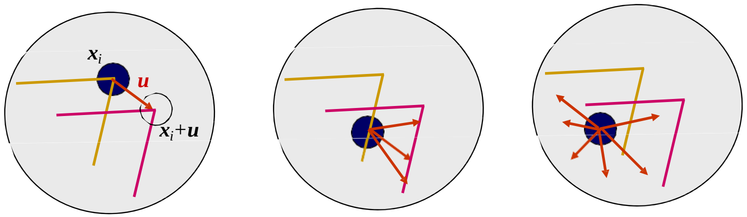
Image stitching

Image stitching

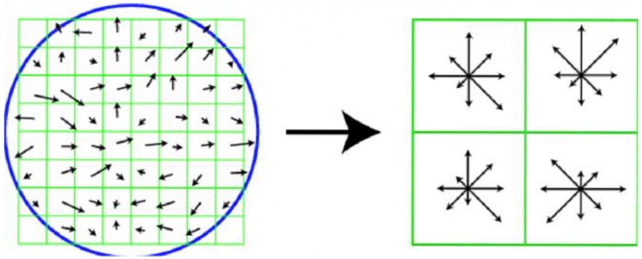
- Stitching algorithms typically have four main ingredients:
  - Method to determine *correspondences* between images
  - Model describing the set of possible *motions* between images (homography)
  - Algorithm to perform *alignment* of the images (bundle adjustment)
  - Algorithm that *composites* the images after alignment (blending; seam finding)

Determining correspondences

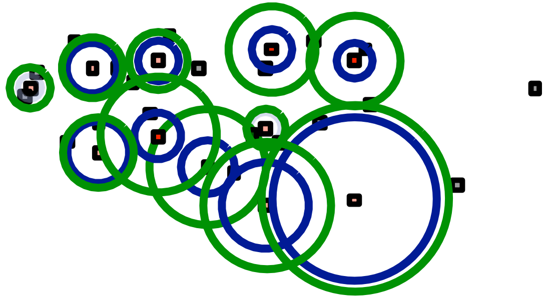
- Feature detection:



- Feature description:

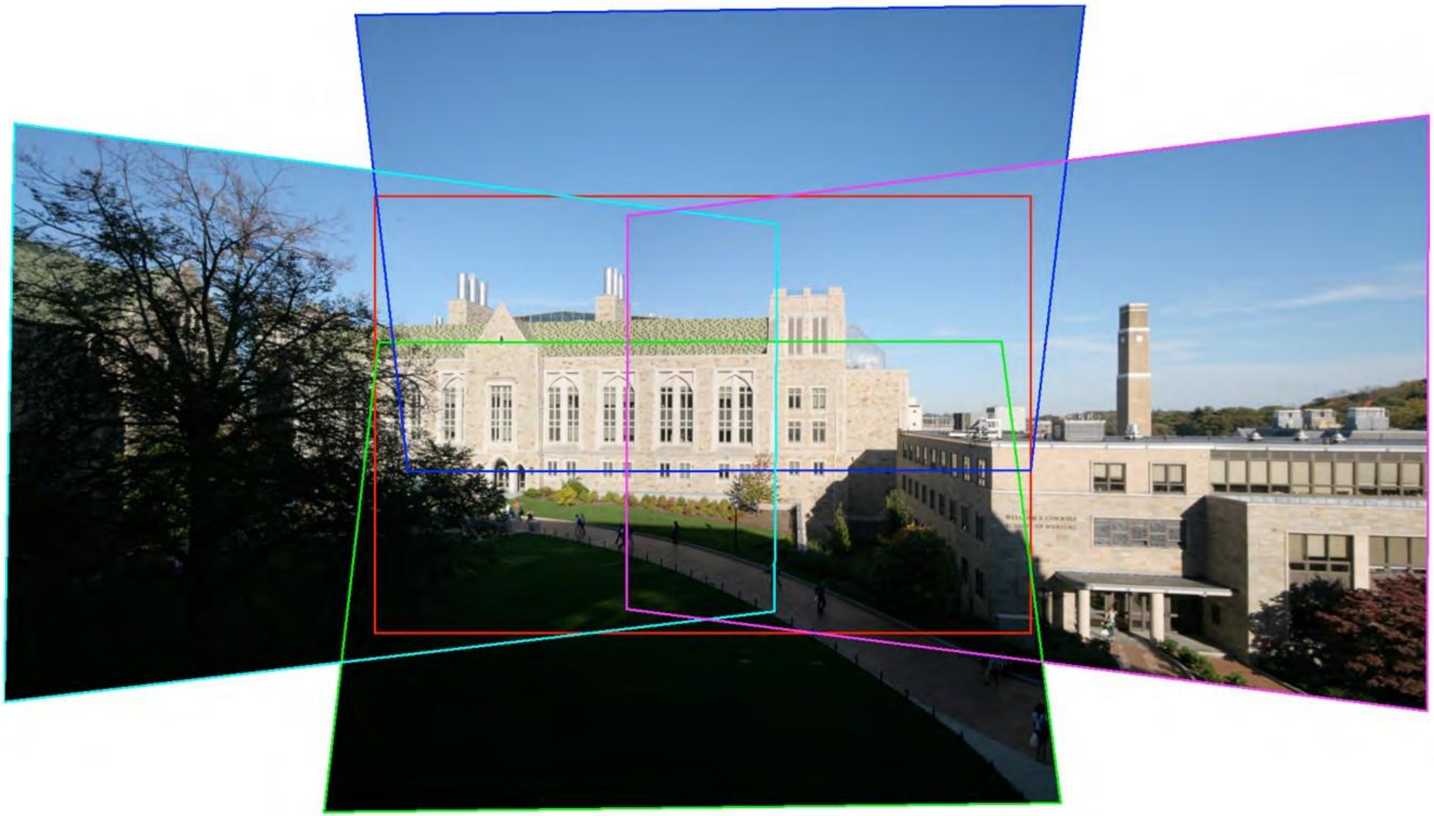


- Feature matching:



Motion model

- We will consider *homographies* because of their generality:



Fitting a homography

- Recall the definition of a homography in Cartesian coordinates:

$$x' = f(x, y) = \frac{h_{00}x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + 1} \qquad y' = g(x, y) = \frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + 1}$$

- This makes the alignment objective a non-linear least squares problem:

$$\sum_i \left\| \begin{bmatrix} f(x_i, y_i; \mathbf{H}) \\ g(x_i, y_i; \mathbf{H}) \end{bmatrix} - \begin{bmatrix} x'_i \\ y'_i \end{bmatrix} \right\|^2$$

Fitting a homography

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- Simply use Gauss-Newton's (or Newton's) method to solve this problem:
  - You may have to use RANSAC to deal with outliers!

Remark on RANSAC

- The treatment of RANSAC we saw last week was somewhat simplified
- It was not random at all!

Remark on RANSAC

- The treatment of RANSAC we saw last week was somewhat simplified
- It was not random at all! Full RANSAC actually works as follows:

1) Select random subset from 50% of “best” matches as initial inliers

2) Model is fitted to the *hypothetical inliers*

3) Data are tested against the fitted model to determine hypothetical inliers

4) Return to step 2) until sufficient points are classified as inliers
(or fixed number of times)

5) Return to step 1) a fixed number of times

Fitting a homography

- Consider the following two images and SIFT matches between them:

Fitting a homography

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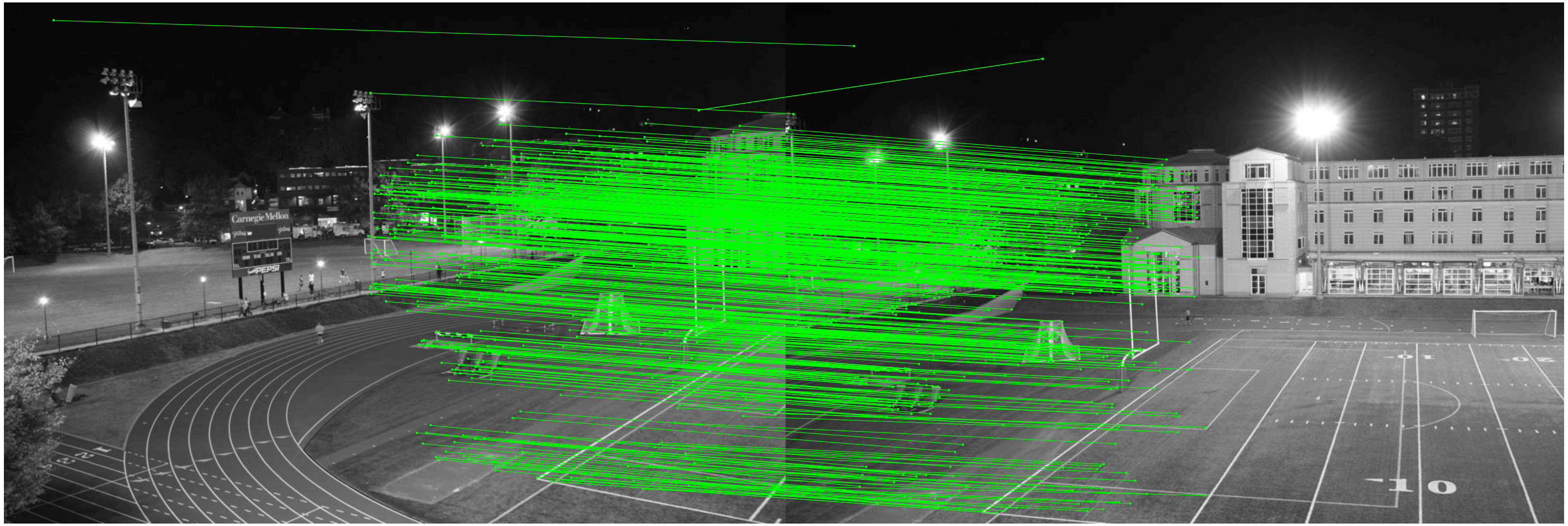
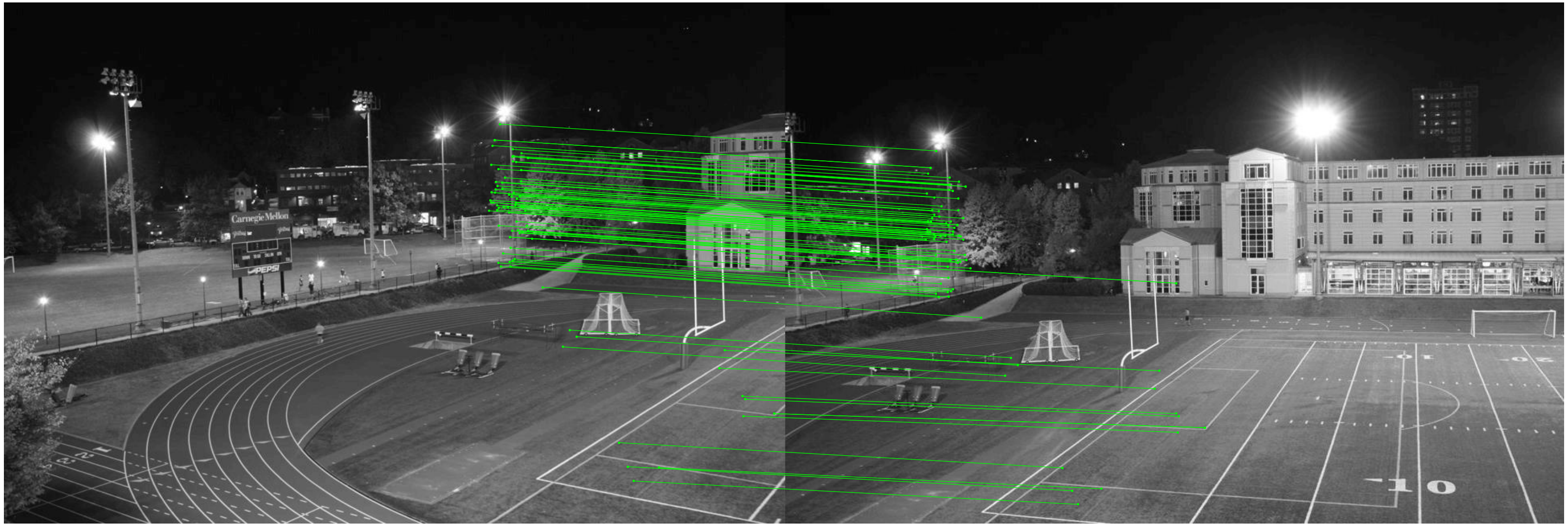


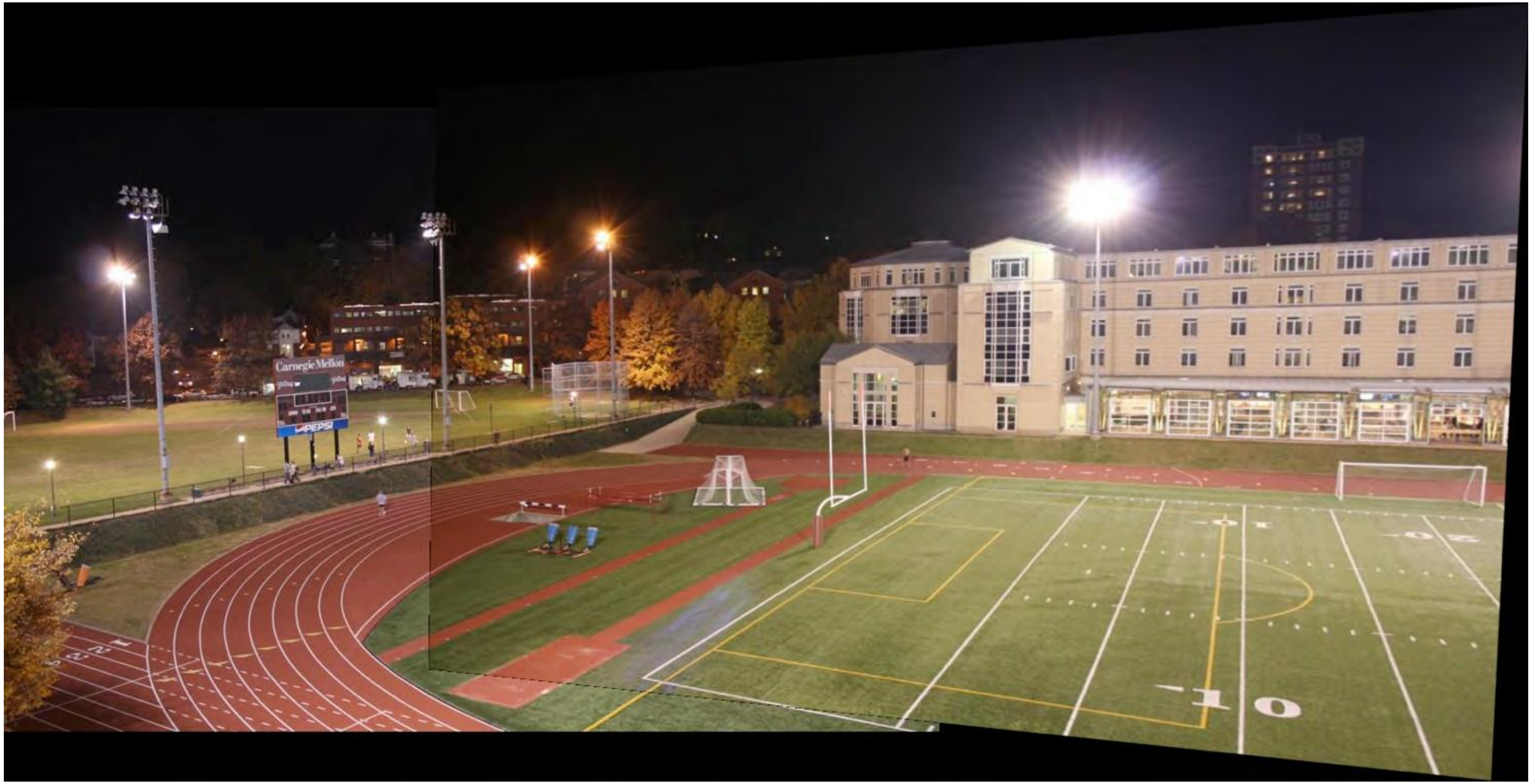
Fitting a homography

- Visualization of the inliers found by RANSAC:



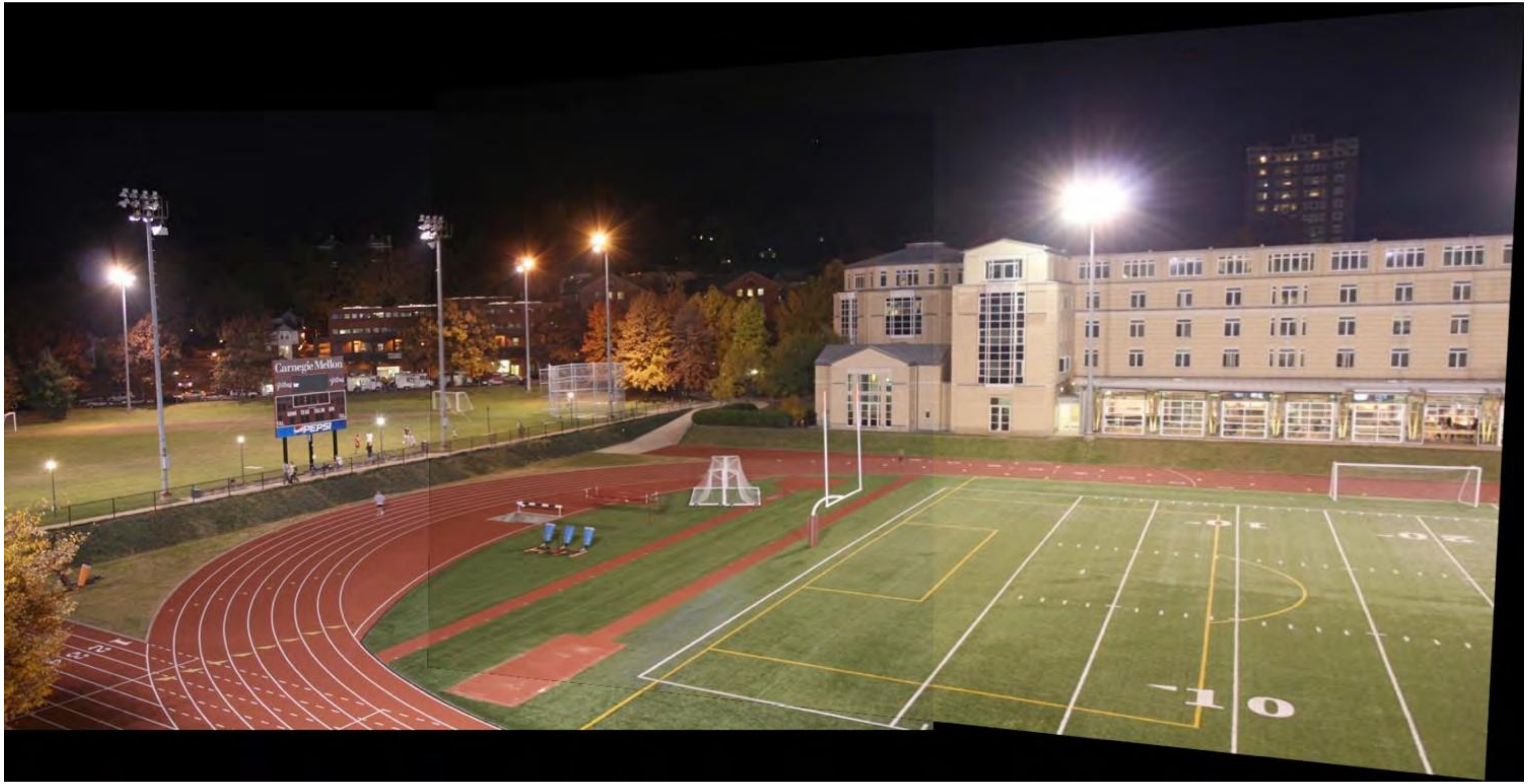
Fitting a homography

- Visualization of the homography found between the two images:



Alpha blending

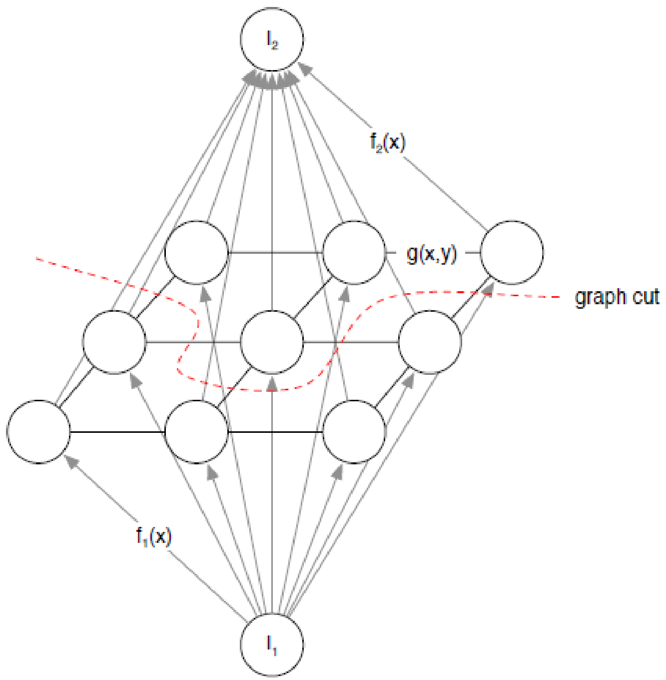
- *Alpha blending* averages the images; leads to *"ghosting"* and visible *seams*:



Finding an optimal seam

- We would like to find a *seam* that minimizes the difference between images

- This can be formulated as a *graph min-cut* problem, as follows:



- When you want pixel $\mathbf{x}$ to be taken from I_1, set $f_2(\mathbf{x}) = \infty$ (and vice versa)

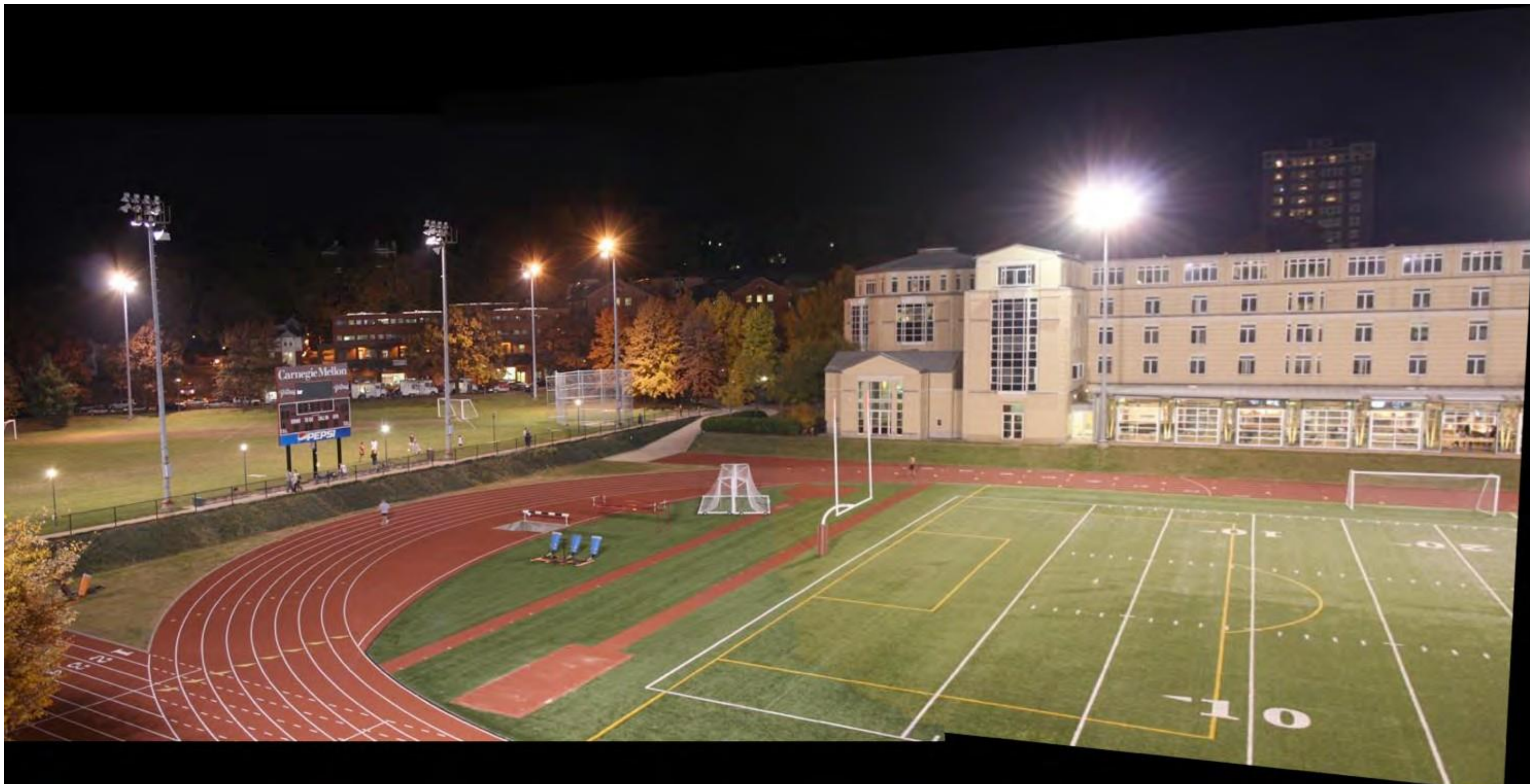
- In all other cases: $f_1(\mathbf{x}) = f_2(\mathbf{x}) = 0$

- Set, e.g., $g(x, y) = |I_1(\mathbf{x}) - I_2(\mathbf{y})| + |I_2(\mathbf{x}) - I_1(\mathbf{y})|$

- Efficient graph-cut algorithms exist

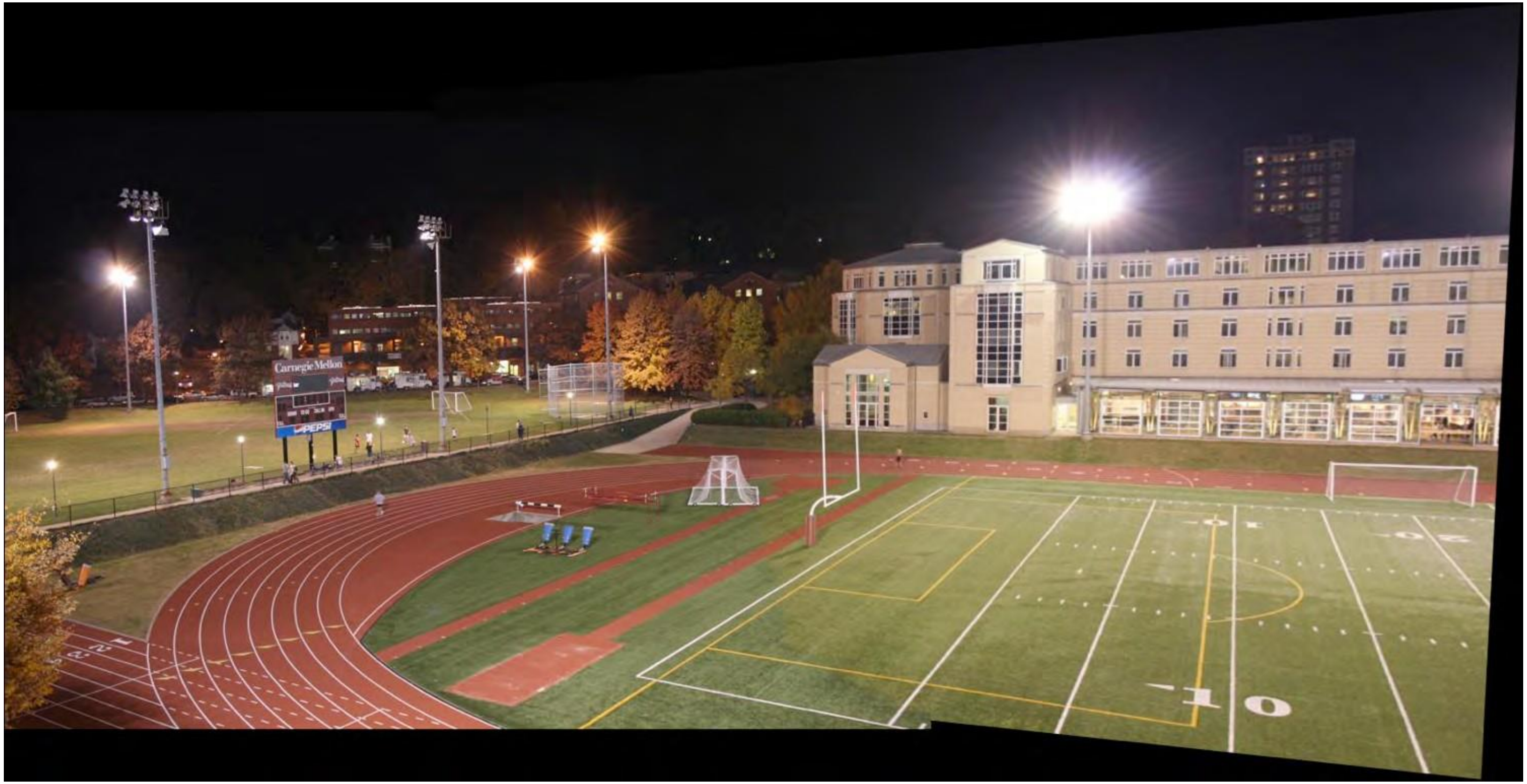
Finding an optimal seam

- Ghosting can be prevented by finding a *seam* at which to switch images:



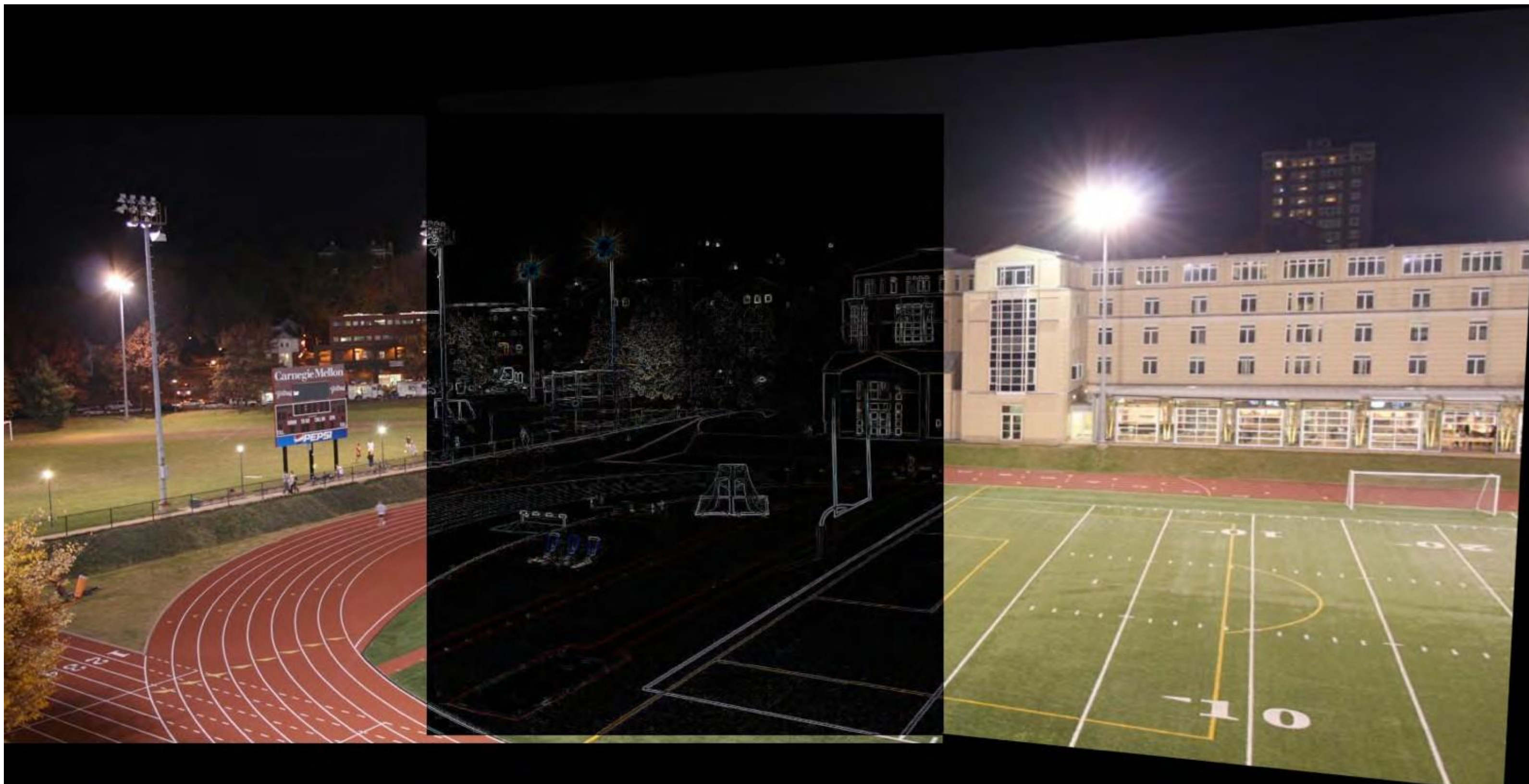
Feathering

- Visibility of seam may be reduced by *“blurring”* weights of both images:



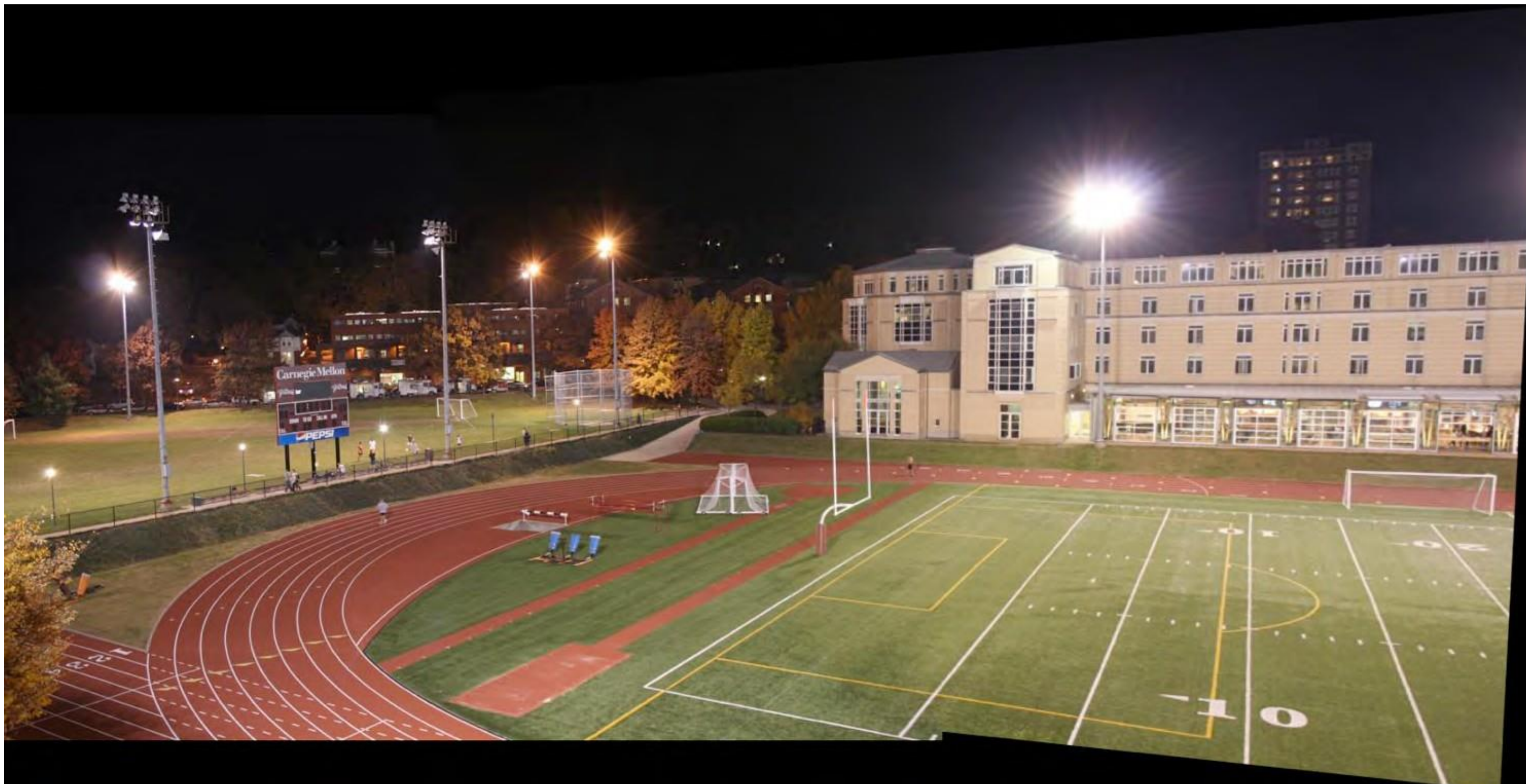
Correcting for exposure differences

- Instead of using seam to select image, we use it to select the *image gradient*:



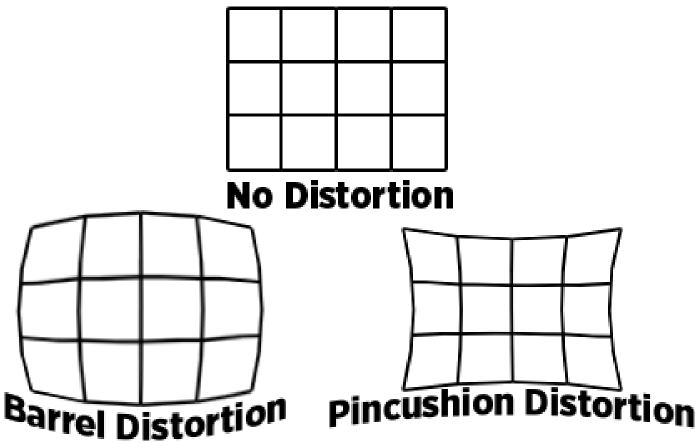
Correcting for exposure differences

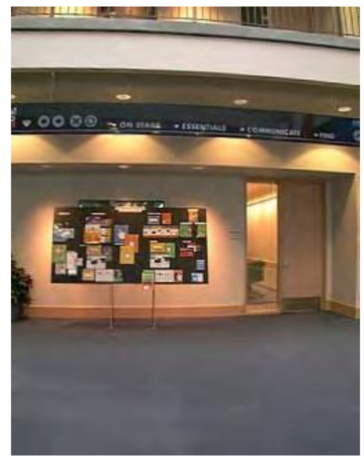
- The stitched image can then be recovered from this gradient:



Lens distortion

- Homography model may be too simple when lens distortion is present:





- A common way of dealing with this is by incorporating a distortion model in the objective, and also minimizing w.r.t. the parameters of that model

Other applications of homographies

Other applications of homographies

Other applications: Image Warping

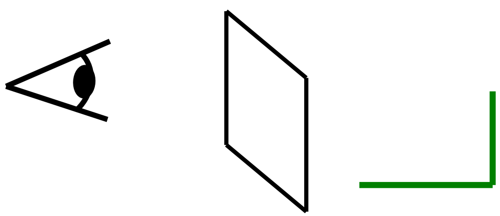
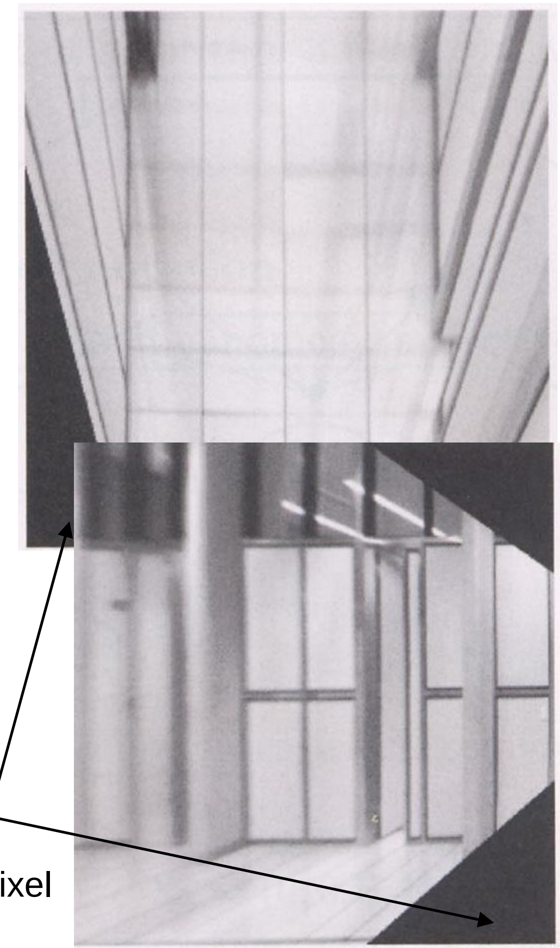


image plane in front

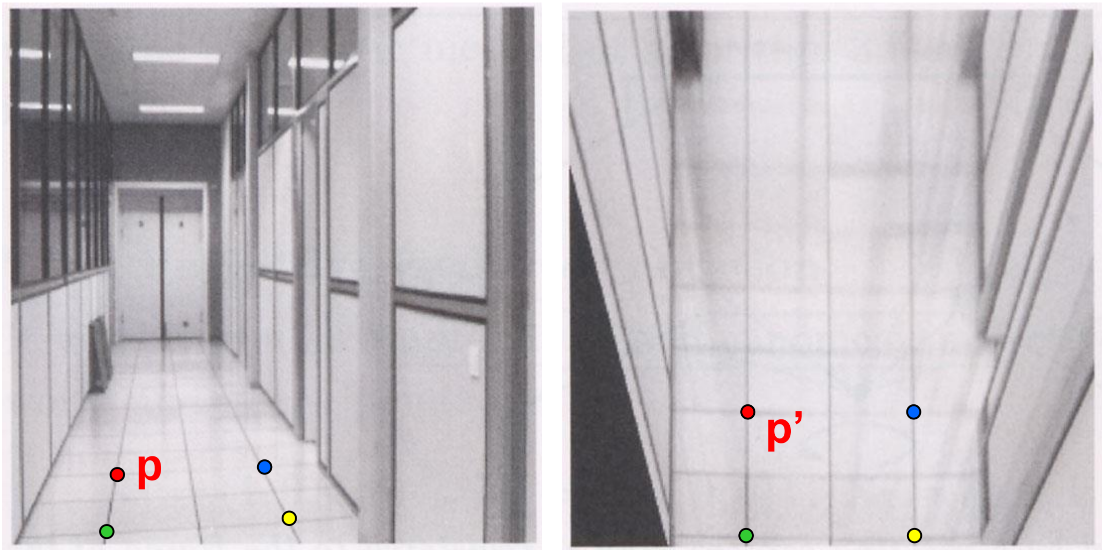


black area
where no pixel
maps to

How would you do this?

Source: Steve Seitz

Other applications: Image rectification



Other applications: Analyzing patterns/shapes

What is the shape of the b/w floor pattern?

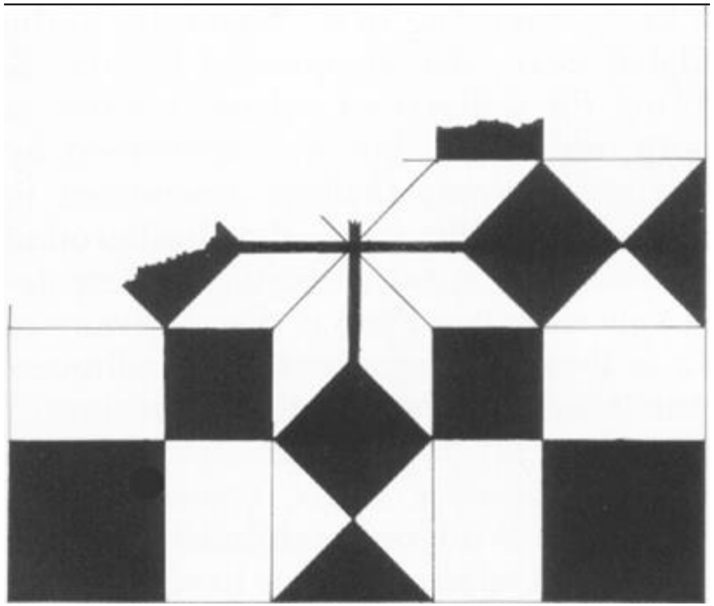
The floor (enlarged)

Automatically rectified floor

Slide: Criminisi

Other applications: Analyzing patterns/shapes

Automatic rectification



From Martin Kemp *The Science of Art*
(manual reconstruction)

Slide: Criminisi

Other applications: Analyzing patterns/shapes

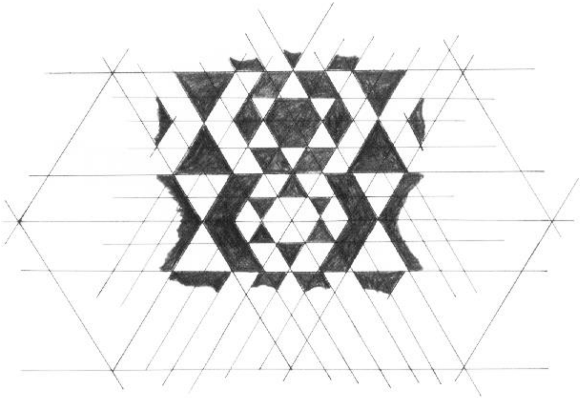
St. Lucy Altarpiece, D. Veneziano

Automatically rectified floor

Slide: Criminisi

Other applications: Analyzing patterns/shapes

Automatic rectification



From Martin Kemp, *The Science of Art* *(manual reconstruction)*

Slide: Criminisi

Reading material: Section 2 and 9 of Szeliski