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- Two types of traders

  Informed traders - have inside knowledge of the future value or true price V. They trade in a way that leverages their inside knowledge to exploit the market maker to gain a profit. In return, the market maker gains information about the true price

  Uninformed traders - trade for liquidity reasons. They do not have inside knowledge. They make trading decisions exogenous to the true price or market price.

- Definitions

  V - The value of the asset being traded

  H_t - Publicly available information

  J_t - Information available only to infomed traders

  B - Bid price

  A - Ask price

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Behavior of trader at time t:

$$\begin{aligned} &\text{Buy if } Z_t > A \\ &\text{Sell if } Z_t < B \end{aligned}$$

Where $Z_t = E[V]$

Let $U_t = \begin{cases} 0 & \text{if informed} \\ 1 & \text{if uninformed} \end{cases}$

$$Z_t = (1-U_t) \cdot E[V \mid H_t, J_t, A, B] + U_t \cdot E[V \mid H_t, A, B]$$

$$= \begin{cases} E[V \mid H_t, J_t, A, B] & \text{if informed} \\ E[V \mid H_t, A, B] & \text{if uninformed} \end{cases}$$

Traders maximize expected utility based on their information to date.

Let S_t = information available to the market maker at time t

The market maker's expected profit at time t:

$$EP = E\left[(A-V) \cdot I_{\{Z_t > A\}} + (V-B) \cdot I_{\{Z_t < B\}} \mid S_t \right]$$

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Where

$$I_{\{Z_t > A\}} = 1 \text{ when } Z_t > A, \text{ else } 0$$

$$I_{\{Z_t < B\}} = 1 \text{ when } Z_t < B, \text{ else } 0$$

Z_t is the expected value of V

Assume there are zero costs associated with holding inventory or short positions.

Assume market maker earns zero expected profit on each transaction.
$\therefore \; EP = 0$

Assume no transaction costs.

If S_t is the same as H_t, then:

$$A_t = \inf \{ a : a \geq E[V \mid H_t, Z_t > a] \}$$

$$B_t = \sup \{ b : b < E[V \mid H_t, Z_t < b] \}$$

inf = infimum
sup = supremum

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inf - infimum - greatest element that is less than or equal to all elements
↳ greatest lower bound

sup - supremum - least element that is greater than or equal to all elements
↳ least upper bound

- The ask price is what the revised expectation of V will be if the MM sells (trader buys)

The bid price is what the revised expectation of V will be if the MM buys (trader sells)

Once the bid/ask prices are set, we know what the possible revised expectations are of V.

- Proposition 1

$$A_t = E\left[V \mid S_t,\ Z_t > A_t\right]$$

$$B_t = E\left[V \mid S_t,\ Z_t < B_t\right]$$

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Assume $EP = 0$ conditions, then:

$$A_t \geq E[V] \geq B_t$$

↳ The expected value of V falls inside the spread

Proposition 2

Sequence of transaction prices $\{p_k\}$ forms a martingale relative to information represented by $\{S_k\}$ and $\{H_k\}$. Knowledge of past events don't help predict future values.

$$p_k = E[V \mid S_k]$$

$$E[p_{k+1} \mid S_k] = E\big[E[V \mid S_{k+1}] \mid S_k\big]$$

$$= E[V \mid S_k]$$

$$= p_k$$

$$\therefore E[p_{k+1} \mid S_k] = p_k$$

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Proposition 3

Markets with large volumes will have small spreads.

Markets with small volumes will have large spreads.

Spreads decline as more trades occur. This reflects the market's assimilation of the insiders' infomation.

Proposition 4

The expectations of the market maker and the traders converge as the number of trades increases.

Proposition 5

The spread widens when:

(i) The insider's information gets better
(ii) The ratio of informed traders to uninformed traders increases
(iii) The price elasticity of uninformed supply / demand increases.

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- The adverse selection problem gets worse when there are more informed traders and when they have better information. This forces the market maker to increase the spread. The spread decreases when there is a greater desire for the uninformed traders to trade.

- Example from Market Microstructure Theory

Let $V = 1$ or $V = 0$

$V = 0$ with prob δ
$V = 1$ with prob $1 - \delta$

When $\delta = 0.5$:

$P(V = 0) = 0.5$
$P(V = 1) = 0.5$

Suppose:

$P(\text{informed trader}) = 0.5$
$P(\text{uninformed trader}) = 0.5$

$Q_1 = S$ – trade at $t=1$ is a sell

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$$P(V=0 \mid S) = \frac{P(V=0)\cdot P(S \mid V=0)}{P(V=0)\cdot P(S \mid V=0) + P(V=1)\cdot P(S \mid V=1)}$$

Let:

$P(\text{informed}) = 0.5$
$P(\text{uninformed}) = 0.5$

$P(\text{informed } S \mid V=0) = 1.0$
$P(\text{uninformed } S \mid V=0) = 0.5$
$P(\text{informed } S \mid V=1) = 0.0$
$P(\text{uninformed } S \mid V=1) = 0.5$

$P(\text{informed } B \mid V=0) = 0.0$
$P(\text{uninformed } B \mid V=0) = 0.5$
$P(\text{informed } B \mid V=1) = 1.0$
$P(\text{uninformed } B \mid V=1) = 0.5$

$$P(S \mid V=0) = P(\text{informed})\cdot P(\text{informed } S \mid V=0) + P(\text{uninfomed})\cdot P(\text{uninfomed } S \mid V=0)$$

$$= (0.5)(1.0) + (0.5)(0.5) = 0.75$$

$$P(S \mid V=1) = P(\text{informed})\cdot P(\text{informed } S \mid V=1) + P(\text{uninformed})\cdot P(\text{uninformed } S \mid V=1)$$

$$= (0.5)(0.0) + (0.5)(0.5) = 0.25$$

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$$P(B \mid V=0) = P(\text{informed}) \cdot P(\text{informed } B \mid V=0) + P(\text{uninformed}) \cdot P(\text{uninformed } B \mid V=0)$$

$$= (0.5)(0.0) + (0.5)(0.5) = 0.25$$

$$P(B \mid V=1) = P(\text{informed}) \cdot P(\text{informed } B \mid V=1) + P(\text{uninformed}) \cdot P(\text{uninformed } B \mid V=1)$$

$$= (0.5)(1.0) + (0.5)(0.5) = 0.75$$

Solving:

$$P(V=0 \mid S) = \frac{P(V=0) \cdot P(S \mid V=0)}{P(V=0) \cdot P(S \mid V=0) + P(V=1) \cdot P(S \mid V=1)}$$

$$= (0.5)(0.75) \,/\, (0.5)(0.75) + (0.5)(0.25)$$
$$= 0.75$$

$$P(V=0 \mid B) = \frac{P(V=0) \cdot P(B \mid V=0)}{P(V=0) \cdot P(B \mid V=0) + P(V=1) \cdot P(B \mid V=1)}$$

$$= (0.5)(0.25) \,/\, (0.5)(0.25) + (0.5)(0.75)$$
$$= 0.25$$

$$P(V=1 \mid S) = 1 - P(V=0 \mid S) = 0.25$$

$$P(V=1 \mid B) = 1 - P(V=0 \mid B) = 0.75$$

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If the MM sees a buy, then $P(V=0) = 0.25$ and $P(V=1) = 0.75$ at $t=2$.

If the MM sees a sell, then $P(V=0) = 0.75$ and $P(V=1) = 0.25$ at $t=2$.

$$
\begin{aligned}
\text{Bid} &= E(V \mid S) \\
&= (V=1)\,P(V=1 \mid S) + (V=0)\,P(V=0 \mid S) \\
&= (1)(0.25) + (0)(0.75) \\
&= 0.25
\end{aligned}
$$

$$
\begin{aligned}
\text{Ask} &= E(V \mid B) \\
&= (V=1)\,P(V=1 \mid B) + (V=0)\,P(V=0 \mid B) \\
&= (1)(0.75) + (0)(0.25) \\
&= 0.75
\end{aligned}
$$

Suppose there is a buy i.e. the trader buys the ask at 0.75. Compute the following:

$P(V=0 \mid B, B)$

$P(V=0 \mid B, S)$

$E(V \mid B, B)$

$E(V \mid B, S)$

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$$P(V=0 \mid B,B) = \frac{P(V=0 \mid B) \cdot P(B \mid V=0)}{P(V=0 \mid B) \cdot P(B \mid V=0) + P(V=1 \mid B)\, P(B \mid V=1)}$$

$$= (0.25)(0.25) \,/\, (0.25)(0.25) + (0.75)(0.75)$$

$$= 0.0625 \,/\, 0.0625 + 0.5625$$

$$= 0.1$$

$$P(V=0 \mid B,S) = \frac{P(V=0 \mid B) \cdot P(S \mid V=0)}{P(V=0 \mid B) \cdot P(S \mid V=0) + P(V=1 \mid B) \cdot P(S \mid V=1)}$$

$$= (0.25)(0.75) \,/\, (0.25)(0.75) + (0.25)(0.75)$$

$$= 0.1875 \,/\, 0.1875 + 0.1875$$

$$= 0.5$$

$$P(V=1 \mid B,B) = 1 - P(V=0 \mid B,B) = 0.9$$
$$P(V=1 \mid B,S) = 1 - P(V=0 \mid B,S) = 0.5$$

$$\text{Bid} = E(V \mid B,S)$$
$$= (V=1)\,P(V=1 \mid B,S) + (V=0)\,P(V=0 \mid B,S) = 0.5$$

$$\text{Ask} = E(V \mid B,B)$$
$$= (V=1)\,P(V=1 \mid B,B) + (V=0)\,P(V=0 \mid B,B) = 0.9$$

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Note that at $t=2$:

$$P(V=0 \mid B,B) = \frac{P(V=0)\left[P(B \mid V=0)\right]^2}{P(V=0)\left[P(B \mid V=0)\right]^2 + P(V=1)\left[P(B \mid V=1)\right]^2}$$

$$= (0.5)(0.25)^2 \Big/ (0.5)(0.25)^2 + (0.5)(0.75)^2$$

$$= 0.1$$

General formula:

b = # buys
s = # sells

$$q = P(B \mid V=0) = 1 - P(S \mid V=0)$$

$$p = P(B \mid V=1) = 1 - P(S \mid V=1)$$

$$P(V=0 \mid b,s) = \frac{P(V=0)\, q^b (1-q)^s}{P(V=0)\, q^b (1-q)^s + P(V=1)\, p^b (1-p)^s}$$

$$P(V=1 \mid b,s) = \frac{P(V=1)\, p^b (1-p)^s}{P(V=0)\, q^b (1-q)^s + P(V=1)\, p^b (1-p)^s}$$

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- Realize that the following values are fixed as long as the ratio of informed / uninformed is fixed:

$$P(S \mid V=0) = 0.75$$
$$P(S \mid V=1) = 0.25$$
$$P(B \mid V=0) = 0.25$$
$$P(B \mid V=1) = 0.75$$

As information becomes more dispersed over time, then the fraction of informed traders increases, giving the MM another parameter to learn. This is not considered in the Glosten Milgrom model.

$EP = 0$ – The expected profit set to zero in the Glosten Milgrom model.

- Example from "Better Explained" – cancer screening example that demonstrates Bayes theorem:

1% of population has cancer
80% of cancer is detected by test
9.6% of test report false positives

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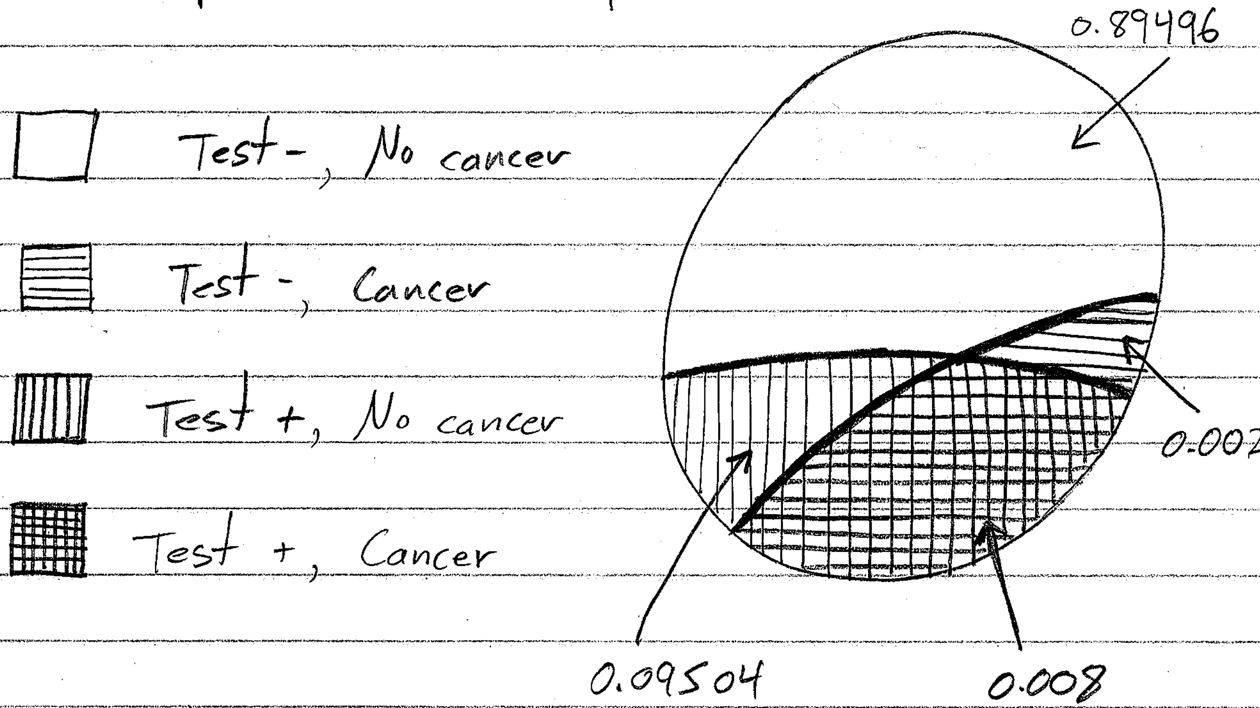
	Cancer (1%)	No Cancer (99%)
Test +	80%	9.6%
Test −	20%	90.4%



$P(\text{Cancer}) = 0.01$

$P(\text{No cancer}) = 0.99$

$P(\text{Test} +) = (0.01)(0.80) + (0.99)(0.096) = 0.10304$

$P(\text{Test} -) = 1 - 0.10304 = 0.89696$

$$P(\text{Cancer} \mid \text{Test} +) = \frac{(0.01)(0.80)}{(0.01)(0.80) + (0.99)(0.096)}$$

$$= \frac{0.008}{0.008 + 0.09504}$$

$$= 0.0776$$

$$= 7.76\ \%$$

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Example from Sanmay Das paper:

$$P(V=V_i \mid Sell) = \frac{P(Sell \mid V=V_i) \cdot P(V=V_i)}{P(Sell)}$$

$$P(V=V_i \mid Take) = \frac{P(Take \mid V=V_i) \cdot P(V=V_i)}{P(Take)}$$

$$P(V=V_i \mid None) = \frac{P(None \mid V=V_i) \cdot P(V=V_i)}{P(None)}$$

$$P(Sell \mid V=V_i,\ V_i < bid) = (1-\alpha)\eta + \alpha$$

$$P(Sell \mid V=V_i,\ V_i \geq bid) = (1-\alpha)\eta$$

$$P(Take \mid V=V_i,\ V_i \leq ask) = (1-\alpha)\eta$$

$$P(Take \mid V=V_i,\ V_i > ask) = (1-\alpha)\eta + \alpha$$

$$P(None \mid V=V_i,\ V_i < bid) = (1-\alpha)(1-2\eta)$$

$$P(None \mid V=V_i,\ ask \leq V_i \leq bid) = (1-\alpha)(1-2\eta) + \alpha$$

$$P(None \mid V=V_i,\ V_i > ask) = (1-\alpha)(1-2\eta)$$

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$$P(Sell) = \sum_{V_i=min}^{V_i=max} P(Sell \mid V=V_i) \cdot P(V=V_i)$$

$$= \sum_{V_i=min}^{V_i=max} (1-\alpha)\eta \cdot P(V=V_i) \qquad (unf)$$

$$+ \sum_{V_i=min}^{V_i=bid-1} \alpha \cdot P(V=V_i) \qquad (inf)$$

$$P(Take) = \sum_{V_i=min}^{V_i=max} P(Take \mid V=V_i) \cdot P(V=V_i)$$

$$= \sum_{V_i=min}^{V_i=max} (1-\alpha)\eta \cdot P(V=V_i) \qquad (unf)$$

$$+ \sum_{V_i=ask+1}^{V_i=max} \alpha \cdot P(V=V_i) \qquad (inf)$$

$$P(None) = \sum_{V_i=min}^{V_i=max} (1-\alpha)(1-2\eta) \cdot P(V=V_i) \qquad (unf)$$

$$+ \sum_{V_i=min}^{V_i=bid-1} 0$$

$$+ \sum_{V_i=bid}^{V_i=ask} \alpha \cdot P(V=V_i) \qquad (inf)$$

$$+ \sum_{V_i=ask+1}^{V_i=max} 0$$