NUMERICAL OPTIMIZATION

Sheet 9: Constrained optimization

Exercise one A leftover from the lecture. Solve the following constrained optimization problem via *KKT enumeration*. Recall: in this method, we first select the subset of active inequalities, meaning those which hold with equality. (This means that for m inequalities, we will make 2^m choices.)

Then, once the set of active inequalities is fixed, we derive the KKT optimality conditions, and find all possible solutions of the system of equalities and inequalities. This gives us some set of candidate points C (often only one or two).

If the LICQ conditions for some points are not met, we also add them to C.

Finally, we evaluate the function value for all the candidate points in C and we we select the minimum value among them.

$$
\begin{aligned}
& \min\ 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 \\
& \text{subject to} \qquad\qquad x_1^2 + x_2^2 \leq 5, \\
& \qquad\qquad\qquad\qquad 3x_1 + x_2 \leq 6.
\end{aligned}
$$

We say a convex constrained problem is a *quadratically constrained quadratic program (QCQP)* if its objective function is quadratic, all inequality constraints are either affine or quadratic, and all equality constraints are quadratic.

We say a constrained problem is *quadratic non-convex* if some functions in its definition are not convex or if it contains quadratic equality constraints.

Exercise two Formulate the l_4-norm approximation problem as a QCQP. On input, you get a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$. The problem is:

$$\min ||Ax - b||_4 = \left(\sum_{i=1}^{m}(a_i^T x - b_i)^4\right)^{1/4}$$

Note that a_i means a row of the matrix A.

Exercise three Solve the following constrained optimization problem. If possible, use the KKT enumeration method.

$$
\begin{aligned}
& \max \qquad x_1x_2 \\
& \text{subject to } 1 - x_1^2 - x_2^2 \geq 0.
\end{aligned}
$$

Exercise four

Part 1. Why do convex programs only allow $g_i(x) = 0$ when g_i is an affine function? Of course, an easy answer is that $g_i(x) = 0$ for a convex function g_i would still lead to a non-convex feasible region.

Let us see a very simple demonstration of this fact:

1. Give a convex quadratic function $g(x)$ such that $g(x) = 0$ only has solutions $x = -1$ or $x = 1$.

2. Give a convex quadratic function $g_2(x)$ such that $g_2(x) = 0$ only has solutions $x = 0$ or $x = 1$.

Part 2. Let us formulate the problem MAX CUT. In MAX CUT, you get on input a weighted graph – a collection (V, E, w), where V is a set of vertices, E a set of edges and $w : E \to \mathbb{R}^+$ a weight function on the edges.

In MAX CUT, our goal is to partition the set of vertices V into two disjoint parts $V_1 \cup V_2$, so that the sum of edges crossing this cut – $\sum_{ij \in E | i \in V_1, j \in V_2} w_{ij}$ – is maximized.

Formulate MAX CUT as a non-convex quadratic program.

EXERCISE FIVE Solve the following constrained optimization problem. If possible, use the KKT enumeration method.

$$\begin{aligned} \min \quad & x_2^2 - x_1 \\ \text{subject to} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 2, \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2, \\ & x_1 \geq 0 \end{aligned}$$

EXERCISE SIX Solve the following constrained optimization problem. If possible, use the KKT enumeration method.

$$\begin{aligned} \min \quad & -2(x_1 - 2)^2 - x_2^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 25 \\ & x_1 \geq 0 \end{aligned}$$