NUMERICAL OPTIMIZATION

Mandatory homework 2

The following is the second mandatory homework. There are two tasks. Please submit it by email or in person by Wednesday, June 19, 14:15.

If you spot any mistakes or any statement is unclear, feel free to send me an email.

EXERCISE ONE *[10 points.]* For the next two exercises, we consider the LP

$$\min_{x \in \mathbb{R}} x \text{ s.t. } x \geq 0.$$

1. Write the KKT conditions and verify the optimum solution using those conditions.
2. Determine the central path $\mathcal{C}$ and draw it.

EXERCISE TWO *[10 points.]* We continue with the same LP from the previous exercise. Assuming the complementarity condition we wish is $XSe = \sigma\mu e$ (the same as in *Framework 14.1*), write the specific formula for $\mathbf{F}$ and its Jacobian $\mathbf{J}$ for our problem. Then, compute one step of the Newton method for finding $F(x) = 0$ (a full step, i.e., with $\alpha = 1$) and show that it jumps directly to the central path from any initial point that is strictly feasible.

EXERCISE THREE *[10 points.]* Suppose we are computing one Newton step for the interior-point path-following method, meaning we solve:

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_c \\ -r_b \\ -XSe + \sigma\mu e \end{bmatrix}$$

where $\mu = x^T s/n$.

Suppose we already start from a strictly feasible point $(x, \lambda, s) \in \mathcal{F}_0$, meaning we also have $-r_c = 0$ and $-r_b = 0$ above.

Prove a simple but quite important observation that our next direction $(\Delta x, \Delta\lambda, \Delta s)$ satisfies that $(\Delta x)^T(\Delta s) = 0$, so the directions for Δx and Δs are orthogonal.

Hint: No difficult math or any extra knowledge is required. It is really a one-line proof. Just try to play with the system of equations above until you get $(\Delta x)^T(\Delta s)$ on the left.

EXERCISE FOUR *[10 points.]* This exercise illustrates the fact that the bounds $x \geq 0, s \geq 0$ are essential in relating solutions of the $F(x, \lambda, s) = 0$ (as defined in the interior-point method) to solutions of our starting linear program and its dual.

Consider the following linear program in $\mathbb{R}^2$:

$$\min x_1, \text{ subject to } x_1 + x_2 = 1, \quad (x_1, x_2) \geq 0.$$

Show that the primal-dual solution is

$$x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lambda^* = 0, \quad s^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Also verify that the system $F(x, \lambda, s) = 0$ also has the solution

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda = 1, \quad s = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

which has no relation to the solution of the linear program.

Exercise five *[10 points.]* State the definition of the neighborhood $N_{-\infty}(\gamma)$ of the central path from the lectures on primal-dual constrained optimization. In other primal-dual approaches, a different neighborhood can be important, called $N_2(\theta)$, parametrized by $\theta \in [0, 1)$:

$$N_2(\theta) = \{(x, \lambda, s) \mid (x, \lambda, s) \in \mathcal{F}_0, ||XSe - \mu e||_2 \leq \theta\mu\}.$$

Prove that for $\gamma \leq 1 - \theta$, we have $N_2(\theta) \subseteq N_{-\infty}(\gamma)$.

Some didactic notes:

1. All of the exercises are theoretical or computational – it is not allowed to solve any of these by writing a computer program and running it. You can of course use calculators.
2. Remember, the primary goal of the homework is for you to practice the understanding from the classes and to practice individual work ahead of the exam.
3. I strongly recommend to work on all tasks alone and invest time into them before looking to your colleagues or the internet for advice.
4. If you are stuck and have no idea how to proceed further, you can schedule a consultation with me and we can discuss it together.
5. Plagiarism is strictly forbidden – solutions must be in your own words, with you having full understanding of what you have written. Sharing your work is also forbidden.
6. If I suspect plagiarism or have any other doubts about the authenticity of the homework, I may discuss your solutions with you in person. Naturally, you will be expected to understand all steps in your solution.
7. If you are asking your friend for advice, make sure to use as little as possible and do not ask for the full solution – this will just tempt you to copy it, thus creating plagiarism issues for both of you.
8. Copying text directly from ChatGPT also equals plagiarism, and I can tell you as a teacher I can estimate this quite well. Be mindful of that fact.

Good luck!