NUMERICAL OPTIMIZATION

Sheet 10: Duality

Recommended review from the lecture: Nocedal-Wright, *Numerical Optimization*, Chapter 12, Section 12.9.

EXERCISE ONE We have seen at the lecture that dual of a linear program is equivalent to another linear program. Namely, we have seen that the dual of $\min c^T x, Ax = b, x \geq 0$ is equivalent to $\max b^T y, A^T y \leq c, y \in \mathbb{R}^m$.

Using the Lagrange duality rules, create equivalent programs to duals of linear programs that also contain inequalities of other types, as well as conditions such as $x_j \in \mathbb{R}$ or $x_j \leq 0$.

Then, fill in the following useful table for linear programs, and explain your results:

Original linear program:	*In the dual:*
minimum	maximum
$\min c^T x$	$\max b^T y$
m constraints n variables	variables constraints
the i-th constraint is $\leq$ the i-th constraint is $\geq$ the i-th constraint is $=$	$y_i \quad 0$ $y_i \quad 0$ $y_i \in \mathbb{R}$
$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	the j-th constraint is $\leq$ the j-th constraint is the j-th constraint is

EXERCISE TWO Dualize the linear programming relaxation of the integer program for MINIMUM VERTEX COVER for a weighted graph $G = (V, E, w)$. To be precise, the task is to dualize the following:

$$\min \sum_{v \in V} w(v) x_v$$
$$\forall e = (uv) \in E : x_u + x_v \geq 1$$
$$\forall v \in V : x_v \geq 0$$

EXERCISE THREE Prove that for linear programs, the dual of a dual is equivalent to the original linear program.

EXERCISE FOUR We are given some linear program (P) which has some optimum solution, but we do not know it yet. The program (P) is of this form:

$$\max c^T x, Ax \leq b, x \geq 0.$$

Using duality, formulate a new LP that satisfies the following:

- it has no objective function (so it is just a polytope),
- if somebody gives us any feasible solution x of the polytope, we can read from its coordinates the optimum solution of the program (P).

EXERCISE FIVE Josephine K. cheated during her final exam. To her horror, she copied a statement of the primal and an optimal solution of the dual:

$$
\begin{aligned}
\min 10x_1 - 4x_2 \\
x_1 + 0.6x_3 + 4x_4 &\geq 43 \\
x_1 - x_2 + 0.6x_3 + 10x_4 &\geq 27 \\
x_1 - x_2 - 0.4x_3 - x_4 &\geq 24 \\
x_1 - x_2 - 0.4x_3 - 2x_4 &\geq 22 \\
x_1 + 3.6x_3 - 3x_4 &\geq 56 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

The optimal dual solution is $\lambda = (3.36, 0, 0, 6.48, 0.16)$. However, the goal was to compute the optimal solution of the primal. Help Josephine figure out the optimal solution of the primal.

EXERCISE SIX Create the dual problem for

$$\text{minimize} \quad -\sum_{i=1}^{m} \log\left(b_i - a_i^T x\right)$$

with domain $\left\{x \mid a_i^T x < b_i, i = 1, \ldots, m\right\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^T x$.