NUMERICAL OPTIMIZATION

Mandatory homework 1

The following is the mandatory homework. There are two tasks. Please submit it by email or in person by Monday, May 6, 12:15.

If you spot any mistakes or any statement is unclear, feel free to send me an email.

EXERCISE ONE *[20 points.]* Formulate a linear program for an agricultural cooperative. It has 1200 hectares of arable land available for crop cultivation. Suitable crops for the given area include wheat, barley, rapeseed, corn, and sunflower. Rapeseed cannot be grown on more than 250 hectares of land, and sunflowers cannot be grown on more than 500 hectares of land. The table below shows the costs of cultivating 1 hectare of crops and the expected yields per hectare.

Crop	Costs (CZK/ha)	Yields (CZK/ha)
Wheat	300	2000
Barley	400	5000
Rapeseed	600	8000
Corn	500	6000
Sunflower	700	9000

At the beginning, the cooperative has a capital of 75000 CZK, which must cover the costs, and of course, it wants to maximize its profit (yields minus costs).

The cooperative also needs to incorporate the following two EU directives:

- Rapeseed must be cultivated on at least the area equal to the absolute difference between the cultivation of corn and barley.
- Additionally, the cooperative will receive a subsidy of 5000 CZK per hectare for the *minimum* area of a single crop. (Specifically, the EU subsidy for balanced farming.)

Note: Remember that we ask for a linear program. Any non-linear constraints or expressions in your solution will lead to significant point decreases. Using integer linear programming is also not allowed.

EXERCISE TWO *[30 points.]* In this exercise, we will deal with the following optimization problem:

$$\min f(x,y,z) = x^2 + 13y^2 + 4z^2 + 6xy + 2xz + 10yz + 6x - 2y + 4z + 3.$$

1. Prove or disprove whether the function f is convex.
2. Write the optimization problem using the standard matrix notation for quadratic programs, in other words, find A, b, c such that

$$\min_{x,y,z} f(x,y,z) = \min_{v \in \mathbb{R}^3} \frac{1}{2} v^T A v + b^T v + c.$$

3. Find a global minimum solution x^*. Explain in full detail how you can compute such a solution for this quadratic program (and similar) using pen, paper and calculator, without use of some computer code.
4. Perform one step of the gradient descent method with backtracking via the Armijo condition, starting at the initial point $(2, 1, 2)$. For the backtracking, start with the initial step length $\alpha = 1$ and use the parameters $c = 0.8$ (sufficient decrease parameter of Armijo) and $\rho = 0.1$ (contraction multiplier that is used to contract α).

Be thorough in your explanation – explain all computations that you performed to execute it, and do not skip any detail (for example, how to compute the gradient).

5. Perform one step of Newton's method, starting at the same initial point $(2, 1, 2)$, and using the same parameters $\alpha = 1$, $c = 0.8$ and $\rho = 0.1$.
   Again, be thorough in your explanations; if you use any linear algebraic subroutines (for example, Gaussian elimination), do not skip the computation and explain it in detail.

Some didactic notes:

1. All of the exercises are theoretical or computational – it is not allowed to solve any of these by writing a computer program and running it. You can of course use calculators.
2. Remember, the primary goal of the homework is for you to practice the understanding from the classes and to practice individual work ahead of the exam.
3. I strongly recommend to work on all tasks alone and invest time into them before looking to your colleagues or the internet for advice.
4. If you are stuck and have no idea how to proceed further, you can schedule a consultation with me and we can discuss it together.
5. Plagiarism is strictly forbidden – solutions must be in your own words, with you having full understanding of what you have written. Sharing your work is also forbidden.
6. If I suspect plagiarism or have any other doubts about the authenticity of the homework, I may discuss your solutions with you in person. Naturally, you will be expected to understand all steps in your solution.
7. If you are asking your friend for advice, make sure to use as little as possible and do not ask for the full solution – this will just tempt you to copy it, thus creating plagiarism issues for both of you.
8. I have verified that some of the exercises are solved incorrectly by some versions of ChatGPT. Copying text directly from ChatGPT also equals plagiarism, and I can tell you as a teacher I can estimate this quite well. Be mindful of that fact.

Good luck!