NUMERICAL OPTIMIZATION

Sheet 3: Convexity

D: A set S is *convex* if $\forall \lambda \in [0,1]$ and $\forall x \in S, y \in S$ we have that $\lambda x + (1-\lambda)y \in S$.

D: A set C is a *cone* if for any element $v \in C$ and $\forall \alpha \geq 0$, we have that $\alpha v \in C$.

D: We say a set is a *convex cone* if it is convex and a cone.

Exercise One Prove the following equivalence:

A set C is a convex cone if and only if it satisfies the following property: $\forall a \in C, b \in C$ and $\forall \alpha \geq 0, \beta \geq 0$ it holds that $\alpha a + \beta b$ must lie in C.

Exercise Two Use the second-order convexity criterion to check whether the following function is convex, concave, or neither:

$$f(x,y,z) = 12.5x^2 + 9y^2 + 5.5z^2 + 15xy - 5xz.$$

D: A *hyperplane* is any affine space in $\mathbb{R}^d$ of dimension $d-1$. Thus, on a 2D plane, any line is a hyperplane. In the 3D space, any plane is a hyperplane, and so on.

Any hyperplane can be written as $HP = \{x \in \mathbb{R}^d | c^T x = d\}$ for some scalar d and a vector c.

A hyperplane splits the space $\mathbb{R}^d$ into two *halfspaces*. We count the hyperplane itself as a part of both halfspaces.

Any halfspace can thus be written as $HS = \{x \in \mathbb{R}^d | c^T x \leq d\}$ for some scalar d and a vector c.

T(Hyperplane separation lemma): If $A, B \subset \mathbb{R}^n$ are convex sets, $A \cap B = \emptyset$, then there exists a linear function $c^T x$ and $b \in \mathbb{R}$ such that $c^T x \leq b$ for $x \in A$ and $c^T x \geq b$ for $x \in B$. If A is open, then $c^T x < b$ for $x \in A$. If A is closed and B is compact, then we can choose c and b so that $c^T x \leq b$ for $x \in A$ and $c^T x > b$ for $x \in B$.

Exercise Three Prove the following:

A closed set $S \subseteq \mathbb{R}^n$ is convex if and only if there exists a family $\mathcal{F}$ of halfspaces in $\mathbb{R}^n$ such that $S = \bigcap_{F \in \mathcal{F}} F$.

Hint: The hard implication is that if we have a convex set S, then we can generate the family $\mathcal{F}$. This family can (and sometimes must) be of infinite size. To get $\mathcal{F}$, use the hyperplane separation lemma and separate S from "everything else".

Exercise Four Use the second-order convexity criterion to check whether the following function is convex, concave, or neither:

$f(x,y) = x^2/y^4$ where $y > 0$.

Exercise Five Write the gradient and the Hessian matrix of the function $g(x_1, x_2, \ldots, x_n) = \log(\sum_{i=1}^n e^{x_i})$.

Exercise Six The Powell's optimization problem can be described as minimizing the following function:

$$\min f(x_1,x_2,x_3,x_4) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4.$$

Your task is:

1. Without doing any complicated math, show that the global minimum of this function is at $(0,0,0,0)$.

2. Use the second-order convexity criterion to check whether the function is convex, concave or neither.