Finite Elements
Laboratory 1

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1. Chapter 1: Direct stiffness method

1.1. Truss element 2D

A truss element is a structural element that can only carry axial loads. It is assumed that the truss element is made of a linear elastic material and that the cross-sectional area is constant along its length. Each truss element is defined by two nodes and has two degrees of freedom (DOF´s) at each node: Vertical and horizontal displacements. Also, it is important to note, that for each DOF, there is a corresponding force.

$$\mathbf{f} = \begin{bmatrix} f_{x0} \\ f_{y0} \\ f_{x1} \\ f_{y1} \\ \vdots \\ f_{x_{n-1}} \\ f_{y_{n-1}} \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} k_{x0x0} & k_{x0y0} & k_{x0x1} & k_{x0y1} & \cdots \\ k_{y0x0} & k_{y0y0} & k_{y0x1} & k_{y0y1} & \cdots \\ k_{x1x0} & k_{x1y0} & k_{x1y1} & k_{x1y1} & \cdots \\ k_{y1x0} & k_{y1y0} & k_{y1y1} & k_{y1y1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_{x0} \\ u_{y0} \\ u_{x1} \\ u_{y1} \\ \vdots \\ u_{x_{n-1}} \\ u_{y_{n-1}} \end{bmatrix} \tag{1}$$

So, if we have the forces acting on the nodes of the truss element and the stiffness matrix of the element, we can calculate the displacements of the nodes. Which is the main goal of the finite element method.

$$\mathbf{f} = \mathbf{K} \cdot \mathbf{u} \tag{2}$$

2. Finite element method for elastostatic problems

Now we pass to the finite element method for elastostatic problems. The main goal of this method is to find the displacements of the nodes of the structure. To understant this method, we can use the problem of plane stress as an example.

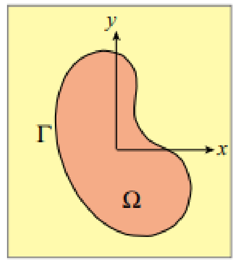

Figura 1: Plane stress problem

As we can see in the figure 1, we have a the domain geometry Γ and Specifiied interior Forces, wich are known forces that act in the interior Ω of the plate.

Also one of the most important things are de Specified Surface Forces, these are known forces that act on the boundary Γ and the displacement boundary conditions, these specify how the plate is supported.

2.1. Problem Unkowns

The unkown fields are the displacements, strains and stresses.

$$\mathbf{u}(\mathbf{x},\mathbf{y}) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}, \qquad \varepsilon(\mathbf{x},\mathbf{y}) = \begin{bmatrix} e_{xx}(x,y) \\ e_{yy}(x,y) \\ 2e_{xy}(x,y) \end{bmatrix} \qquad \sigma(\mathbf{x},\mathbf{y}) = \begin{bmatrix} \sigma_{xx}(x,y) \\ \sigma_{yy}(x,y) \\ \sigma_{xy}(x,y) \end{bmatrix} \tag{3}$$

2.2. Governing equations

The governing equations are the equilibrium equations, the strain-displacement equations and the stress-strain equations.

$$\varepsilon = Du \tag{4}$$

$$\sigma = E\varepsilon \tag{5}$$

$$D^T\sigma + b = 0 \tag{6}$$

Is important to note, that b is the body force vector, E is the 3x3 stress-strain matrix of plane stress elastic moduli D is th 3x2 symmetric-gradient operator and its s transpose the 2×3 tensor-divergence operator

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} \tag{7}$$

2.3. boundary conditions

There are two boundary conditions prescribed on Γ:

- Discplacement boundary conditions (Γ_u): These are the displacements that are prescribed in the form of $u = \hat{u}$.
- Force boundary conditions(Γ_t): These are the forces that are prescribed in the form $\sigma_n = \hat{t}$.

To see it better, we can see the following figure:

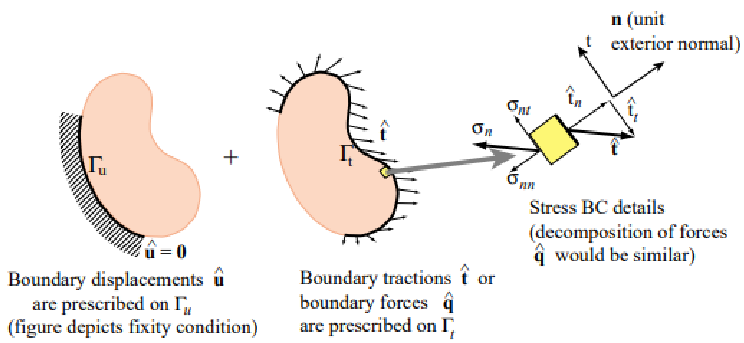

Figura 2: Boundary conditions

2.4. displacement interpolation

The displacement field $u^e(x,y)$ over the element is interpolated from the node displacements. We shall assume that the same interpolation functions are used for both displacement components

$$u_x(x,y) = \sum_{i=1}^{n} N_i^e(x,y)u_{xi} \qquad u_y(x,y) = \sum_{i=1}^{n} N_i^e(x,y)u_{yi} \tag{8}$$

where $N_i^e(x,y)$ are the element shape functions. This N (with superscript e omitted to reduce clutter) is called the shape function matrix. It has dimensions 2 × 2n

$$\mathbf{N} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \cdots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \cdots & 0 & N_n^e \end{bmatrix} \tag{9}$$

Differentiating the finite element displacement field yields the strain-displacement relations:

$$\varepsilon(x,y) = DN \cdot u^e(x,y) = B \cdot u^e(x,y) \tag{10}$$

This B = D N is called the strain-displacement matrix. It is dimensioned 3 × 2n

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \cdots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \cdots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \cdots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix} \tag{11}$$

2.5. Element Stiffness Equations

With the following relations, we can calculate the element stiffness matrix and the element force vector.

$$u = Nu^e \tag{12}$$
$$\varepsilon = Bu^e \tag{13}$$
$$\sigma = E\varepsilon \tag{14}$$

The element stiffness matrix is given by the following equation:

$$\mathbf{K}^e = \int_{\Omega^e} hB^T EBd\Omega^e \tag{15}$$
$$\tag{16}$$

and the consistent element nodal force vector is:

$$\mathbf{f}^e = \int_{\Omega^e} hN^T bd\Omega^e + \int_{\Gamma^e} hN^T \hat{t} d\Gamma^e \tag{17}$$
$$\tag{18}$$

where h is the thickness of the element, b is the body force vector and Ω^e is the volume of the element.

Finally, we can describe the element stiffness matrix and the element force vector in a more compact form:

$$\int_{\Omega^e} hB^T EBd\Omega^e \cdot u = \int_{\Omega^e} hN^T bd\Omega^e + \int_{\Gamma^e} hN^T \hat{t} d\Gamma^e \quad \rightarrow \quad \mathbf{K}^e \cdot u = \mathbf{f}^e \tag{19}$$

2.6. 2D Elasticity

2.6.1. Plane stress

Plane stress is ocupied when the stress in the z direction is negligible. In other words when the plane has a very small thickness. The stress-strain relations are given by the following equations:

$$\sigma = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \varepsilon \tag{20}$$

2.6.2. Plane strain

Plane strain is ocupied when the strain in the z direction is infinite. In other words when the plane has a very big thickness. The stress-strain relations are given by the following equations:

$$\sigma = \frac{E}{(1+\nu)(1+2\nu)} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \varepsilon \tag{21}$$

3. The constant strain/stress triangle

The triangle is a 2D element with three nodes. Each node has its own coordinates (x_i, y_i) fot i= 1,2,3. But we can also use the local coordinates (ζ_i) for i = 1,2,3. The local coordinates are used to simplify the calculations. The only restriction is that:

$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$

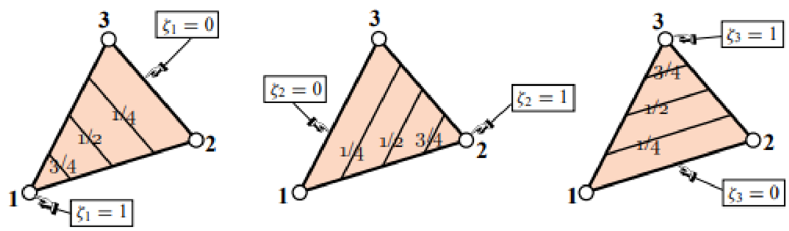

Figura 3: Constant strain triangle

To transform the global coordinates to the local coordinates, we can use the following equations:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \tag{22}$$

This changes the form to calculate the Stiffnes matrix and the force vector. The stiffness matrix is given by the following equation:

$$\mathbf{K}^e = \int_{\Omega^e} B^T E B d\Omega^e \rightarrow B^T(x,y) \cdot E_x \cdot B(x,y) \cdot A^e \cdot t^e \tag{23}$$

While the force vector is given by the following equation:

$$\mathbf{f}^e = \int_{\Omega^e} N^T b d\Omega^e \rightarrow \frac{A^e \cdot t^e}{3} \cdot \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix} \tag{24}$$