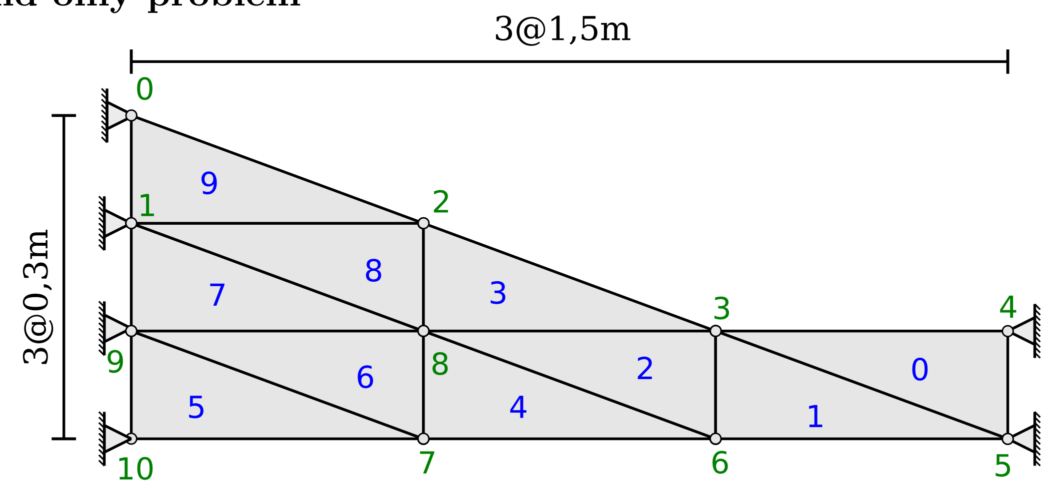
Midterm Exam # 1

First and only problem



The figure above shows a finite-element mesh, composed of CST elements, which is used to model a variable height beam with both ends fixed. The static self-weight analysis is carried out using a Young's modulus for concrete of 2000 (GPa), Poisson's ratio $\nu = 0{,}25$, concrete mass density of $\rho = 2400\ \left(\mathrm{kg/m^3}\right)$ and a gravitational acceleration constant of $g = 9{,}81\ \left(\mathrm{m/s^2}\right)$. Assume the beam has a width of 1 (m).

1. **(0.5 pt.)** Placing the origin of the system of coordinates at node 10, with the x axis pointing towards the right, write down the nodal, connectivity, nodal DOFs and element DOF tables for that mesh.

Solution

Nodes table

0	0.00	0.90
1	0.00	0.60
2	1.50	0.60
3	3.00	0.30
4	4.50	0.30
5	4.50	0.00
6	3.00	0.00
7	1.50	0.00
8	1.50	0.30
9	0.00	0.30
10	0.00	0.00

Nodal DOFs table

0	0	1
1	2	3
2	4	5
3	6	7
4	8	9
5	10	11
6	12	13
7	14	15
8	16	17
9	18	19
10	20	21

Element connectivity

0	3	4	5
1	3	5	6
2	6	3	8
3	3	2	8
4	6	8	7
5	10	7	9
6	7	8	9
7	9	8	1
8	8	2	1
9	1	2	0

Element DOFS

0	6	7	8	9	10	11
1	6	7	10	11	12	13
2	12	13	6	7	16	17
3	6	7	4	5	16	17
4	12	13	16	17	14	15
5	20	21	14	15	18	19
6	14	15	16	17	18	19
7	18	19	16	17	2	3
8	16	17	4	5	2	3
9	2	3	4	5	0	1

2. **(1.0 pt.)** Compute the element stiffness matrix for element 2. Show all work and comment on your assumptions.

Solution

$$\boldsymbol{k}^2 = t^{(2)} A^{(2)} {\boldsymbol{B}^{(2)}}^T \boldsymbol{E}_\sigma \boldsymbol{B}^{(2)}$$

$$t^{(2)} = 1{,}0$$

$$A^{(2)} = {,}22499999999999998$$

$$\boldsymbol{B}^{(2)} = \begin{bmatrix} 0. & 0. & 0{,}66666667 & 0. & -0{,}66666667 & 0. \\ 0. & -3{,}33333333 & 0. & 3{,}33333333 & 0. & 0. \\ -3{,}33333333 & 0. & 3{,}33333333 & 0{,}66666667 & 0. & -0{,}66666667 \end{bmatrix}$$

$$\boldsymbol{E}_\sigma = \begin{bmatrix} 2{,}13333333 \times 10^{10} & 5{,}33333333 \times 10^{09} & 0{,}00000000 \times 10^{00} \\ 1{,}33333333 \times 10^{09} & 2{,}13333333 \times 10^{10} & 0{,}00000000 \times 10^{00} \\ 0{,}00000000 \times 10^{00} & 0{,}00000000 \times 10^{00} & 8{,}00000000 \times 10^{09} \end{bmatrix}$$

$$\boldsymbol{k}^2 = \begin{bmatrix} 200{,}00 & 0{,}00 & -200{,}00 & -40{,}00 & 0{,}00 & 40{,}00 \\ 0{,}00 & 533{,}33 & -6{,}67 & -533{,}33 & 6{,}67 & 0{,}00 \\ -200{,}00 & -26{,}67 & 221{,}33 & 66{,}67 & -21{,}33 & -40{,}00 \\ -40{,}00 & -533{,}33 & 46{,}67 & 541{,}33 & -6{,}67 & -8{,}00 \\ 0{,}00 & 26{,}67 & -21{,}33 & -26{,}67 & 21{,}33 & 0{,}00 \\ 40{,}00 & 0{,}00 & -40{,}00 & -8{,}00 & 0{,}00 & 8{,}00 \end{bmatrix} \times 10^8$$

3. **(1.0 pt.)** Compute the diagonal components of the global (assembled) stiffness matrix $K[12,12]$ and $K[13,13]$. What node do they correspond to and what elements connect at that node?

Solution

Looking at the nodal DOFs table, DOFs 12 and 13 correspond to node 6. The element connectivity matrix shows that elements 1, 2 and 4 connect at that place.

We have the stiffness matrix for element 2, elements 4 and 1 are just rotated version of this element matrix. At node 6 all 3 corners are present so, after global assembly we'll have at node 6 the addition of the stiffnesses at all 3 corners.

Therefore,

$$K[12,12] = \boldsymbol{k}^2[0,0] + \boldsymbol{k}^2[2,2] + \boldsymbol{k}^2[4,4] = (200. + 221{,}33 + 21{,}33) \times 10^8 = (442{,}66) \times 10^8$$
$$K[13,13] = \boldsymbol{k}^2[1,1] + \boldsymbol{k}^2[3,3] + \boldsymbol{k}^2[5,5] = (533. + 541{,}33 + 8{,}0) \times 10^8 = (1082{,}33) \times 10^8$$

4. **(1.0 pt.)** Compute the full nodal load vector, $\boldsymbol{f}$, the right hand side of equation $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$, which is due to self weight. Show all work and comment on your assumptions.

Solution

First, compute $\boldsymbol{f}^e$ for any element, say $e = 2$

$$\boldsymbol{f}^2 = t^{(2)} A^{(2)} \begin{bmatrix} b_x & b_y & b_x & b_y & b_x & b_y \end{bmatrix}^T$$
$$t^{(2)} = 1{,}0$$
$$A^{(2)} = {,}22499999999999998$$
$$b_x = 0$$
$$b_y = -\rho g = -2400 \times 9{,}81 = -23544{,}00$$
$$\boldsymbol{f}^2 = -5297{,}40 \begin{bmatrix} 0{,}0 & 1{,}0 & 0{,}0 & 1{,}0 & 0{,}0 & 1{,}0 \end{bmatrix}^T$$

Now we can assemble the global force vector by inspecting how many elements meet at each node, the value of the force there at the vertical DOF will be $-5297{,}40$ times the number of elements meeting at that nodes. This gives:

$$\boldsymbol{f} = -5297{,}40 \cdot \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 3 & 0 & 6 & 0 & 3 & 0 & 1 \end{bmatrix}^T$$

5. **(1.5 pt.)** The solution vector and amplified deformed shape are shown in the next page. Using that information compute:

   (a) **(1.0 / 1.5)** All strain components ($\boldsymbol{\epsilon}$) for element 4.

   Solution

   Element 4 is just a rotated version of element 2, for which we already have its strain-displacement matrix $\boldsymbol{B}$, therefore:

$$\boldsymbol{B}^{(4)} = \begin{bmatrix} 0{,}66666667 & 0. & 0. & 0. & -0{,}66666667 & 0. \\ 0. & 0. & 0. & 3{,}33333333 & 0. & -3{,}33333333 \\ 0. & 0{,}66666667 & 3{,}33333333 & 0. & -3{,}33333333 & -0{,}66666667 \end{bmatrix}$$

   And, $\boldsymbol{\epsilon}^{(4)} = \boldsymbol{B}^{(4)} \boldsymbol{u}^{(4)}$

$$\boldsymbol{u}^{(4)} = \begin{bmatrix} 1{,}7 \times 10^{07} \\ -9{,}0 \times 10^{06} \\ -9{,}8 \times 10^{08} \\ -7{,}1 \times 10^{06} \\ -8{,}9 \times 10^{07} \\ -7{,}1 \times 10^{06} \end{bmatrix} \quad \boldsymbol{\epsilon}^{(4)} = \begin{bmatrix} 7{,}1 \times 10^{-7} \\ -8{,}3 \times 10^{-8} \\ 1{,}4 \times 10^{-6} \end{bmatrix}$$

   (b) **(0.5 / 1.5)** all stress components ($\boldsymbol{\sigma}$) for element 4. **Solution**

$$\boldsymbol{\sigma}^{(4)} = \boldsymbol{E}_\sigma \boldsymbol{\epsilon}^{(4)} = \begin{bmatrix} 714701{,}9 \\ -821{,}6 \\ 11069{,}5 \end{bmatrix}$$

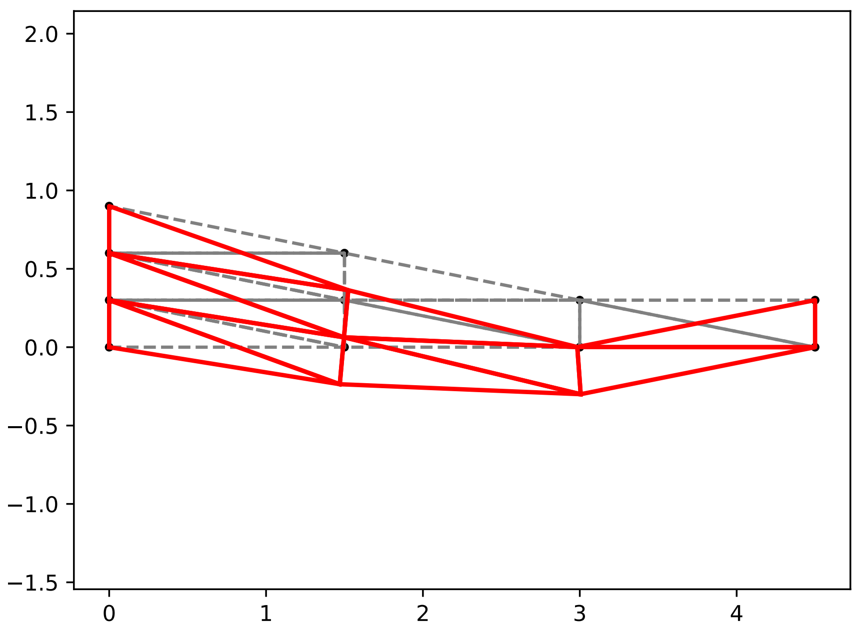
6. **(1.0 pt.)** List all assumptions you can think of behind the finite-element model presented. Is it a better model than a line-element (mechanics of material) model using Bernoulli-Euler formulation? In what sense is it better or worse?

Solution

Some assumptions:

a) Isotropic material
b) Elastic material
c) Uncracked concrete properties
d) Small deformations
e) Small displacements
f) Linear interpolation of displacements inside each element
g) Perfect fixity at ends (zero displacement conditions)
h) 3-D effects can be ignored (plane-stress analysis)
i) Un-stressed initial condition (no initial stresses or strains due to concrete creep, thermal effects, etc.)
j) Uniform gravitational field

Bernoulli-Euler theory is valid for high slenderness elements, because it does not consider shear deformations, and of constant cross section. The FEM model is better in the sense that it does not make these assumptions, it considers shear and bending deformations as well as a varying cross section. On the other hand, its is very likely that the chosen mesh is insufficient to properly resolve the stress and strain fields. More elements would be required to be able to trust the FEM model results.



$$\boldsymbol{u} = \begin{bmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 7{,}1 \\ -70{,}6 \\ -4{,}8 \\ -89{,}3 \\ 0. \\ 0. \\ 0. \\ 0. \\ 1{,}7 \\ -89{,}6 \\ -8{,}9 \\ -70{,}6 \\ -1. \\ -70{,}8 \\ 0. \\ 0. \\ 0. \\ 0. \end{bmatrix} \times 10^{-7} \qquad (m)$$