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● *Review*

A CRITICAL REVIEW AND UNIFORMIZED REPRESENTATION OF STATISTICAL DISTRIBUTIONS MODELING THE ULTRASOUND ECHO ENVELOPE

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Abstract—In ultrasound imaging, various statistical distributions have been proposed to model the first-order statistics of the amplitude of the echo envelope. We present an overview of these distributions based on their compound representation, which comprises three aspects: the modulated distribution (Rice or Nakagami); the modulating distribution (gamma, inverse Gaussian or even generalized inverse Gaussian); and the modulated parameters (the diffuse signal power with or without the coherent signal component or the coherent signal power). This unifying point of view makes the comparison of the various models conceptually easier. In particular, we discuss the implications of the modulated parameters on the mean intensity and the signal-to-noise ratio of the intensity in the case of a vanishing diffuse signal. We conclude that the homodyned K-distribution is the only model among the literature for which the parameters have a physical meaning that is consistent with the limiting case, although the other distributions may fit real data. (E-mail: guy.cloutier@umontreal.ca) © 2010 World Federation for Ultrasound in Medicine & Biology.

Key Words: **Echo envelope, B-mode, Homodyned K-distribution, Generalized K-distribution, Rician inverse Gaussian distribution, Nakagami distribution, Nakagami-gamma distribution, Nakagami-generalized inverse Gaussian distribution.**

INTRODUCTION AND LITERATURE

Various models have been introduced in the literature for the first-order statistics of the echo envelope of ultrasound images. One aspect of a statistical distribution is its good fitness with real data in a specific field of application. However, in problems such as tissue characterization (Shankar et al. 1993, 2001; Oelze and O'Brien 2007; Tsui et al. 2008), the estimated parameters themselves are used as classifying features. In such a framework, we believe it is desirable to also pay attention to the physical meaning of the distribution parameters. For instance, a mixture of a sufficiently large number of Gaussian distributions suffices to model the histogram of the echo envelope, but the physical meaning of the proportions, of the means and of the variances of such mixtures is unclear because a Gaussian distribution is not (directly) meaningful for the first-order statistics of the echo envelope.

In this review article, we present the various models for the first-order statistics of the echo envelope found in the literature, in a unified way based on their compound representation. The point is to compare them in view of a physical interpretation of their parameters. We do not discuss the important aspect of estimation methods for these distributions or their applications in problems such as tissue classification, or segmentation of anatomical parts. For that reason, we chose to include only the references that introduced these models for the first time in the scientific literature, and also specifically in ultrasound imaging. Thus, we have omitted numerous papers that studied or used these models, except for the few recent papers on tissue characterization mentioned before.

When the product of the wave number with the mean size of the scatterers is much smaller than the wavelength, and acoustic impedance of the scatterers is close to the impedance of the embedding medium, a high density of scatterers results in a packing organization that implies a correlation between the individual signals produced by the scatterers (Hayley et al. 1967; Twersky 1975, 1978,

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1987, 1988; Lucas and Twersky 1987; Berger et al. 1991). Apart from that case, the backscattered echo signal received at the transducer of an ultrasound device is viewed as the vector sum of the individual signals produced by the scatterers distributed in the medium (Wagner et al. 1983, 1987). In this paper, the framework leading to the physical interpretation of the parameters of the statistical distributions assumes that the individual contributions of the scatterers are independent. A high density of dependent scatterers might be characterized with the proposed models, but the physical interpretation of the parameters should be done with caution in that case.

If there exists a periodicity pattern in the position of the scatterers (Wagner et al. 1983, 1987), or if there exists strong specular reflections, then a coherent (or deterministic) component appears in the received signal, because of a long-range organization (relative to the wavelength). The power of the coherent component is called the coherent signal power. The remaining power (from the total signal power) is called the diffuse signal power and corresponds to the diffuse (or random) component, made of a diffuse collection of scatterers.

In ultrasound imaging, the Rayleigh distribution corresponds to the distribution of the gray level (also called *amplitude*) in an unfiltered B-mode image, viewed as the envelope of the radiofrequency (RF) image, in the case of a high density of random scatterers with no coherent signal component (Wagner et al. 1983). The Rayleigh distribution was first introduced in 1880 in the context of sound propagation. The Rice distribution also corresponds to a high density of random scatterers (the diffuse signal component), but combined with the presence of a coherent signal component of power ε^2 (Insana et al. 1986). Thus, the Rayleigh distribution is the special case of the Rice distribution, where $\varepsilon = 0$. The Rice distribution itself first appeared in the context of wave propagation (Nakagami 1940; Rice 1945).

The K-distribution corresponds to a variable (effective) density α of random scatterers, with no coherent signal component and was introduced in ultrasound imaging by Shankar et al. (1993). The parameter α can be viewed as the number of scatterers per resolution cell (hence, the word ‘‘density’’) multiplied by a coefficient depending on the scanning geometry and parameters, and the backscatter coefficient statistics (hence, the word ‘‘effective’’) (Jakeman and Pusey 1976; Shankar et al. 1993). The parameter α is also called the *scatterer clustering parameter* (Dutt and Greenleaf 1994). The distribution itself appeared first in Lord (1954) in the context of random walks, and was further studied by Jakeman and Pusey (1976) in the context of sea echo. Finally, the homodyned K-distribution corresponds to the general case of a variable effective density of random scatterers with or without a coherent signal component (Dutt and Greenleaf 1994). The homodyned K-distribution was first introduced and studied (Jakeman 1980; Jakeman and Tough 1987) in the context of random walks viewed as a model of weak scattering. Thus, the K-distribution is a special case of the homodyned K-distribution, and the Rayleigh and the Rice distributions are limiting cases of the two former distributions (namely, the effective density α of random scatterers is ‘‘infinite’’).

One important result (Jakeman and Tough 1987) is that the homodyned K-distribution admits a compound representation. Namely, the distribution can be viewed as the marginal distribution of a model in which the Rice distribution has its diffuse signal power $2\sigma^2$ modulated by a gamma distribution with mean and variance α (*i.e.*, the effective density of random scatterers). Namely, the model gives the joint probability of the amplitude A and the modulating variable w (distributed according to a gamma distribution), and the marginal distribution of the variable A is obtained by integrating the joint probability over the domain of w. In the same manner, the K-distribution is the marginal distribution of a model in which the Rayleigh distribution has its diffuse signal power $2\sigma^2$ that is modulated by a gamma distribution. See eqn (11) and eqn (16).

Another modeling possibility introduced in Barakat (1986) and further developed in Jakeman and Tough (1987) is equivalent to modulate both the coherent signal component ε and the diffuse signal power $2\sigma^2$ of the Rice distribution by a gamma distribution. This gives rise to the generalized K-distribution. See eqn (18). Note that this distribution has not been used in ultrasound imaging as of now. However, in Eltoft (2005), the Rician inverse Gaussian distribution (RiIG) is introduced, and we observe, using Eltoft (2005), that it corresponds to a model in which both the coherent signal component ε and the diffuse signal power $2\sigma^2$ of a Rice distribution are modulated by an inverse Gaussian (IG) distribution, instead of a gamma distribution. See eqn (20).

The homodyned K-distribution, the generalized K-distribution and the RiIG are distributions with three parameters: two parameters for the modulated Rice distribution and one parameter for the modulating (gamma or IG) distribution. A simpler model consists in modeling the gray level of the speckle pattern in a B-mode image by a Nakagami distribution (Shankar 2000). The Nakagami distribution is a two-parameter distribution first introduced in Nakagami (1943, 1960) in the context of wave propagation. It can be viewed as an approximation of the homodyned K-distribution, at least in the special cases of the Rice distribution and the K-distribution (see Theorems 6 and 7). That what essentially the point of view of Nakagami et al. (1953) and Nakagami (1960) in the context of random walks and wave propagation, although the homodyned K-distribution was not yet introduced.

Three other distributions were introduced in the context of ultrasound imaging. The first one is called the generalized Nakagami distribution (Shankar 2001) and is obtained from the Nakagami distribution by a change of variable of the form $y = A^{1/s}$, where s is a shape adjustment parameter and A is the amplitude of the signal. This distribution was also proposed independently in Raju and Srinivasan (2002) (in the equivalent form of a generalized gamma distribution). The second other distribution is called the Nakagami-gamma (NG) distribution (Shankar 2003). That distribution can be viewed as the marginal distribution of a model in which the Rice distribution is approximated by a Nakagami distribution, and in which its total signal power Ω (that would correspond to the total signal power $\varepsilon^2 + 2\sigma^2$ of the Rice distribution) is modulated by a gamma distribution. See eqn (27). Equivalently, the corresponding Rice distribution would have both its coherent signal power ε^2 and its diffuse signal power $2\sigma^2$ modulated by the gamma distribution. Note that Shankar (2001) appeared before Shankar (2003), and we gather that the latter model supersedes the former. The third distribution is called the Nakagami-generalized inverse Gaussian (NGIG) distribution (Agrawal and Karmeshu 2006), and it corresponds to a model in which the (approximating) Nakagami distribution has its total signal power Ω modulated by a generalized inverse Gaussian (GIG) distribution instead of a gamma distribution. See eqn (31). The notation of concepts that will appear frequently in this paper are presented in Table 1. In Table 2, the various compound representations presented in this paper are summarized.

So far, the distributions mentioned before concern the envelope of the RF signal. When a log-compression or other (nonlinear or linear) operators are applied to the envelope, the distribution of the gray levels no longer follows the distributions computed on the RF echo envelope. In the case of log-compression, the resulting distribution has been modeled in Dutt and Greenleaf (1996), assuming the K-distribution for the envelope. In Prager et al. (2003), a decompression algorithm is proposed, assuming the homodyned K-distribution for the envelope. As mentioned before, operators other than log-compression can be applied on the envelope. In Nillesen et al. (2008), a linear filter was applied to the RF data before computing the envelope. Five distributions were tested to fit the data: the Rayleigh distribution, the K-distribution, the Nakagami distribution, the inverse Gaussian distribution and the gamma distribution. The authors showed, based on empirical tests, that, overall, the gamma distribution best fit the data. In this paper, we are concerned with the statistical distributions of the amplitude of the unfiltered envelope of the RF image, and therefore we will not discuss further distributions on the filtered B-mode image. Note that one should not confuse the gamma distribution on the amplitude of the (filtered) B-mode image, with the gamma distribution on the intensity (*i.e.*, the square of the amplitude) of the (unfiltered) B-mode image, which is equivalent to the Nakagami distribution on the amplitude of the (unfiltered) B-mode image (Shankar 2000). Incidentally, in the context of this paper, we view the mean intensity (according to its statistical distribution) as the signal power (*i.e.*, the signal intensity averaged over space).

The remaining part of this paper is organized as follows. We first present the Rayleigh, the Rice, the K and the homodyned K-distributions. Then, we present the generalized K and the RiIG distributions. Afterward, we present the Nakagami, the Nakagami-gamma and the NGIG distributions. Then, we discuss the differences and common points of those distributions. Finally, we conclude with some open problems. Because the proofs of Theorems 1–4 are rather delicate, we present them in an appendix, as well as sketches of proofs of three other theorems. We refer the reader to Abramowitz and Stegun (1972) for the notions of Bessel functions and confluent hypergeometric series.

Table 1. Notation of important notions with a clear physical interpretation

Notion	Notation
Amplitude	A
Intensity	$I = A^2$
Coherent signal component of the Rice distribution	ε
Coherent signal power of the Rice distribution	ε^2
Diffuse signal power of the Rice distribution	$2\sigma^2$
Total signal power of the Rice distribution	$\varepsilon^2 + 2\sigma^2$
Scatterer clustering parameter (effective density of random scatterers) of the homodyned K-distribution	α
Total signal power of the Nakagami distribution	Ω
SNR of the intensity of the Nakagami distribution	$\sqrt{m}$
Structure parameter (ratio of the coherent signal power with the diffuse signal power)	κ

SNR = signal-to-noise.

Table 2. Compound representation of probability density distributions modeling the ultrasound echo envelope

Distribution	Modulated distribution	Modulating distribution	Modulated parameters
Homodyned K-distribution	Rice	Gamma	σ^2
Generalized K-distribution	Rice	Gamma	ε, σ^2
RiIG distribution	Rice	IG	ε, σ^2
NG distribution	Nakagami	Gamma	ε^2, σ^2
NGIG distribution	Nakagami	GIG	ε^2, σ^2

THE HOMODYNED K-DISTRIBUTION AND CLOSELY RELATED DISTRIBUTIONS

We present the homodyned K-distribution and its related distributions (Rayleigh, Rice and K-distributions) in the context of n-dimensional random walks, viewed as a model of weak scattering by randomly distributed independent particles; see, for instance, Twersky (1987) for a model that considers the packing organization of dependent particles. Recall that in an n-dimensional random walk, an object moves in the Euclidean space of dimension n by discrete independent random steps according to a specific probability distribution. The accumulation of the random scatterings can be modeled by a random walk of component phasors (Burckhardt 1978; Wagner et al. 1983). Under that point of view, the intensity of the received signal from scatterers corresponds to the square of the amplitude of the random walk (the amplitude of a random walk is the distance of the moving object to the origin of the n-dimensional Euclidean space). In ultrasound 2-D imaging, the dimension of the corresponding random walk is $n = 2$, and the amplitude corresponds to the gray level of the B-mode image (envelope of the RF signal) without applying log-compression or various filters. Here, the mean intensity according to the intensity distribution is viewed as the signal power, *i.e.*, the signal intensity averaged over space.

Rayleigh distribution

The n-dimensional Rayleigh distribution (Jakeman and Tough 1987) is defined by

$$P_{\mathrm{Ra}}\left(A\middle|\sigma^2\right)=\frac{2}{\Gamma(n/2)}\left(\frac{1}{2\sigma^2}\right)^{n/2}A^{n-1}\exp\left(-\frac{A^2}{2\sigma^2}\right), \tag{1}$$

where A represents the amplitude of the signal, $\sigma > 0$, and Γ denotes the Euler gamma function. In Jakeman and Tough (1987), the distribution is expressed in terms of the variable $\overline{a^2}=n\sigma^2$ (see Theorem 1 for the meaning of that variable). The case $n = 2$ corresponds to Rayleigh (1880). Equivalently, the intensity I, *i.e.*, the square of the amplitude A, is distributed according to an exponential distribution.

Consider an n-dimensional random walk

$$\mathbf{A}=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbf{a}_j, \tag{2}$$

where N is the number of steps. Here, the random vectors $\mathbf{a}_j$ are independent, and each one is characterized by independent phase and amplitude, together with a uniformly distributed phase.

Theorem 1. (Central Limit) Let N tend to infinity in the random walk of eqn (2). Then the distribution of the amplitude of the resulting random walk is a Rayleigh distribution with parameter

$$\sigma^2=\overline{a^2}/n, \tag{3}$$

where $\overline{a^2}$ is the mean intensity of the random step $\mathbf{a}_j$ (before scaling by the factor $1/\sqrt{N}$).

Note that $\overline{a^2}=n\sigma^2$ corresponds to the mean intensity of one scatterer, where $n = 2$ is the dimension of the random walk. After normalization of the contribution of N independent scatterers by the factor $1/\sqrt{N}$, one also obtains $\overline{a^2}$ as the mean intensity (in fact, before or after taking the limit as $N\to\infty$). In other words, the idea behind the normalization factor of $1/\sqrt{N}$ (instead of $1/N$) is to average out the intensity of the scatterers (rather than their amplitude), to preserve the mean intensity. In the case of the Rayleigh distribution, the mean intensity (*i.e.*, $2\sigma^2$) can be interpreted as the diffuse signal power, because there is no coherent component in the signal.

The Rice distribution

The n-dimensional Rice distribution (Jakeman and Tough 1987) is expressed as

$$P_{\mathrm{Ri}}\left(A\middle|\varepsilon,\sigma^2\right)=\left(\frac{\varepsilon}{\sigma^2}\right)\times\left(\frac{A}{\varepsilon}\right)^{n/2}I_{n/2-1}\left(\frac{\varepsilon}{\sigma^2}A\right)\exp\left(-\frac{\left(\varepsilon^2+A^2\right)}{2\sigma^2}\right), \tag{4}$$

where $\sigma > 0$ and $\varepsilon \geq 0$ are real numbers, n is the dimension and I_p denotes the modified Bessel function of the first kind of order p (the intensity I should not be confused with the Bessel function I_p). See Jakeman and Tough (1987, eqn (2.16)). The special case where $\varepsilon \to 0$ yields the Rayleigh distribution. The case $n = 2$ corresponds to Nakagami (1940) and Rice (1945). In Nakagami (1960, eqn (5)), the Rice distribution is called the ''n-distribution'' (Nakagami 1940).

Consider an n-dimensional random walk, obtained by adding a constant vector $\vec{\varepsilon}$ to the random walk $\mathbf{A}$ of eqn (2) (after scaling by the factor $1/\sqrt{N}$)

$$\mathbf{A}=\vec{\varepsilon}+\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathbf{a}_j. \tag{5}$$

Theorem 2. (Central Limit) Let N tend to infinity in the random walk of eqn (5). Then, the distribution of the amplitude of the resulting random walk is a Rice distribution with parameters

$$\varepsilon=\|\vec{\varepsilon}\|,\quad \sigma^2=\overline{a^2}/n, \tag{6}$$

where $\overline{a^2}$, as mentioned above, is the mean intensity of the random step $\mathbf{a}_j$ (before scaling by the factor $1/\sqrt{N}$).

For later reference, the mean intensity $E[I]$ of the Rice distribution and its signal-to-noise ratio (SNR) (*i.e.*, $E[I]/\sqrt{E[I^2]-E[I]^2}$) are as follows in the case $n = 2$

$$E[I] = \varepsilon^2+2\sigma^2; \quad SNR^2 = \frac{(\varepsilon^2+2\sigma^2)^2}{4\sigma^2(\varepsilon^2+\sigma^2)}. \tag{7}$$

The mean intensity and the SNR characterize completely the first-order statistics of the echo envelope in the case of a sufficiently large density of random independent scatterers (Insana et al. 1986). For the Rice distribution, the coherent signal component is ε, the coherent signal power is ε^2 and the diffuse signal power is $2\sigma^2$. The ratio of the coherent signal power with the diffuse signal power is called the *structure parameter* κ (Dutt and Greenleaf 1994) and is thus equal to $\varepsilon^2/(2\sigma^2)$ for the Rice distribution (it is equal to 0 for the Rayleigh distribution).

The K-distribution

The K-distribution (Lord 1954; Jakeman and Pusey 1976) is defined by

$$P_{\mathrm{K}}(A|\sigma^2,\alpha) = \frac{4A^{\alpha-1+n/2}}{(2\sigma^2)^{(\alpha+n/2)/2}\Gamma(\alpha)\Gamma(n/2)} K_{\alpha-n/2}\left(\sqrt{\frac{2}{\sigma^2}}A\right), \tag{8}$$

where $\alpha > 0$, $\sigma^2 > 0$ and K_p denotes the modified Bessel function of the second kind of order p. In Jakeman and Tough (1987, eqn (2.11)), the distribution is expressed in terms of the parameters α and $b=\sqrt{\frac{2}{\sigma^2}}$. In view of the compound representation presented later, we find the proposed parametrization more convenient.

Consider an n-dimensional random walk, with independent phase and amplitude, and a uniformly distributed phase, in which the number of steps is variable. Namely, assume that the number of steps N follows a negative binomial distribution $\mathrm{NegBin}(N|\alpha,p)=\frac{(N+\alpha-1)!}{N!(\alpha-1)!}p^\alpha(1-p)^N$ of mean $\overline{N}$, so that $p=1/(1+\overline{N}/\alpha)$. Let the random step be scaled by the factor $1/\sqrt{\overline{N}}$; then we obtain the random process

$$N \sim \mathrm{NegBin}(\alpha, 1/(1+\overline{N}/\alpha)) \quad \mathbf{A}|N \sim \frac{1}{\sqrt{\overline{N}}}\sum_{j=1}^{N}\mathbf{a}_j. \tag{9}$$

Theorem 3. (a) (Jakeman 1980) Let $\overline{N}$ tend to infinity in the random process of eqn (9). Then, the distribution of the amplitude of the resulting random process is a K-distribution with parameters

$$\sigma^2=\overline{a^2}/(n\alpha), \quad \alpha, \tag{10}$$

where $\overline{a^2}$ is the mean intensity of the random step $\mathbf{a}_j$ (before scaling by the factor $1/\sqrt{\overline{N}}$). (b) (Jakeman and Tough 1987) The compound representation of the K-distribution is

$$\int_0^\infty P_{\mathrm{Ra}}(A|\sigma^2 w)\mathcal{G}(w|\alpha,1)\,dw, \tag{11}$$

where P_{Ra} denotes the Rayleigh distribution, and $\mathcal{G}(w|\alpha,1)$ is the gamma distribution $w^{\alpha-1}\exp(-w)/\Gamma(\alpha)$ of mean and variance equal to α.

Here, $\overline{a^2}=n\sigma^2\alpha$ corresponds to the diffuse signal power of one scatterer. After normalization of the contribution of N independent scatterers by the factor $1/\sqrt{\overline{N}}$, one obtains the diffuse signal power $(N/\overline{N})\ \overline{a^2}$. Because N is distributed according to a negative binomial distribution of mean $\overline{N}$, the diffuse signal power is $\overline{a^2}$ (before or after taking the limit as $\overline{N} \to \infty$). Note that the diffuse signal power is $n\sigma^2$ for the Rayleigh distribution, whereas this expression is multiplied by α for the K-distribution.

The compound representation is useful to simulate the K-distribution, and in the evaluation of its value. The special case where $\alpha \to \infty$ yields the Rayleigh distribution, with parameter $\overline{a^2}/n=\lim_{\alpha\to\infty}\sigma^2\alpha$ (Jakeman and Tough 1987, eqn (2.12)).

The mean intensity of the K-distribution and its SNR in the case $n = 2$ are as follows

$$E[I] = 2\sigma^2\alpha; \quad SNR^2 = \frac{1}{1+2/\alpha}. \tag{12}$$

The homodyned K-distribution

The homodyned K-distribution (Jakeman 1980; Jakeman and Tough 1987) is defined by

$$P_{\mathrm{HK}}(A|\varepsilon,\sigma^2,\alpha) = \frac{A^{n/2}}{\varepsilon^{n/2-1}}\int uJ_{n/2-1}(u\varepsilon)J_{n/2-1}(uA)\left(1+\frac{u^2\sigma^2}{2}\right)^{-\alpha}du \tag{13}$$

where $\sigma^2 > 0$, $\alpha > 0$, $\varepsilon \geq 0$ and J_p denotes the Bessel function of the first kind of order p. In Jakeman and Tough (1987, eqn (4.13)), the homodyned K-distribution is expressed in terms of the parameters α, $\overline{a^2}=n\sigma^2\alpha$ and $a_0 = \varepsilon$.

Consider an n-dimensional random walk as in eqn (9), to which is added (after scaling by the factor $\sqrt{\overline{N}}$) a randomly phased vector $\vec{\varepsilon}$ with constant amplitude ε. So, we have the random process

$$N \sim \mathrm{NegBin}(\alpha, 1/(1+\overline{N}/\alpha)) \quad \mathbf{A}|N \sim \vec{\varepsilon}+\frac{1}{\sqrt{\overline{N}}}\sum_{j=1}^{N}\mathbf{a}_j. \tag{14}$$

Theorem 4. (a) (Jakeman 1980) If $\overline{N}$ tends to infinity, the distribution of the amplitude of the random process of eqn (14) is a homodyned K-distribution with parameters

$$\varepsilon, \quad \sigma^2 = \overline{a^2}/(n\,\alpha), \quad \alpha, \tag{15}$$

where $\overline{a^2}$ is the mean intensity of the random step $\mathbf{a}_j$ (before scaling by the factor $1/\sqrt{\overline{N}}$). (b) (Jakeman and Tough 1987) The compound representation of the homodyned K-distribution is

$$\int_0^\infty P_{\mathrm{Ri}}\left(A\middle|\varepsilon, \sigma^2 w\right)\mathcal{G}(w|\alpha, 1)d\,w, \tag{16}$$

where P_{Ri} denotes the Rice distribution and $\mathcal{G}(w|\alpha,1)$ is the gamma distribution, with mean and variance equal to α.

The compound representation is consistent with eqn (11), upon taking $\varepsilon \rightarrow 0$. The special case where $\alpha \rightarrow \infty$ is the Rice distribution (with parameters ε and $\overline{a^2}/n = \lim_{\alpha\to\infty}\sigma^2\alpha$); the case $\varepsilon \rightarrow 0$ is the K-distribution (with parameters σ^2, α); the case where $\alpha \rightarrow \infty$ and $\varepsilon \rightarrow 0$ is the Rayleigh distribution (with parameter $\overline{a^2}/n = \lim_{\alpha\to\infty}\sigma^2\alpha$). Figure 1 illustrates four representative examples of the compound representation of the homodyned K-distribution (including two examples of the K-distribution, as a special case).

Two functions of the three parameters of the homodyned K-distribution are invariant under scaling of the intensity (Dutt and Greenleaf 1994): (i) the scatterer clustering parameter α; and (ii) the structure parameter $\kappa = \varepsilon^2/(n\sigma^2\alpha)$, *i.e.*, the ratio of the coherent signal power ε^2, with the diffuse signal power $\overline{a^2} = n\sigma^2\alpha$. Again, note that the diffuse signal power is $n\sigma^2$ for the Rice distribution, whereas this expression is multiplied by α for the homodyned K-distribution.

Using the general formula for the moments of integer order of the intensity (Jakeman and Tough 1987, eqn (4.17)), we obtain (for $n = 2$) the mean intensity and the SNR of Tables 3 and 4, respectively. Note that we have not included in these two tables the Rayleigh or the K-distribution, because they are special cases of the Rice and the homodyned K-distribution, respectively.

OTHER GENERALIZATIONS OF THE K-DISTRIBUTION

We mention here two generalizations of the K-distribution other than the homodyned K-distribution.

The generalized K-distribution

The n-dimensional generalized K-distribution (Barakat 1986; Jakeman and Tough 1987) is defined by

$$P_{\mathrm{KG}}(A|\varepsilon, \sigma^2, \alpha) = \frac{2}{\sigma^2\varepsilon^{n/2-1}\left(\varepsilon^2+2\sigma^2\right)^{(\alpha-n/2)/2}\Gamma(\alpha)} \times A^\alpha I_{n/2-1}\left(\frac{\varepsilon}{\sigma^2}A\right)K_{\alpha-n/2}\left(\frac{\sqrt{\varepsilon^2+2\sigma^2}}{\sigma^2}A\right), \tag{17}$$

where $\sigma^2 > 0$, $\alpha > 0$, $\varepsilon \geq 0$ and I_p and K_p denote the modified Bessel functions of the first and second kind of order p, respectively. In Jakeman and Tough (1987, eqn (3.12)), the distribution is expressed in terms of the parameters α, $\delta = \varepsilon\alpha$, $\overline{a^2} = n\sigma^2\alpha$ and $b = \frac{\sqrt{\varepsilon^2+2\sigma^2}}{\sigma^2}$.

From Jakeman and Tough (1987, eqns (4.10 to 4.12)), the compound representation of the generalized K-distribution is

$$\int_0^\infty P_{\mathrm{Ri}}\left(A\middle|\varepsilon w, \sigma^2 w\right)\mathcal{G}(w|\alpha, 1)d\,w, \tag{18}$$

where P_{Ri} denotes the Rice distribution and $\mathcal{G}(w|\alpha,1)$ is the gamma distribution, with mean and variance equal to α. This representation is consistent with eqn (11), upon taking $\varepsilon \rightarrow 0$. Thus, as opposed to the homodyned K-distribution, both the coherent signal component ε and the diffuse signal power $2\sigma^2$ of the Rice distribution are modulated by a gamma distribution in its compound representation. In Jakeman and Tough (1987), an interpretation of the generalized K-distribution in terms of a random walk is presented.

The mean intensity of the generalized K-distribution and its SNR in the case $n = 2$ are indicated in Tables 3 and 4, respectively.

The Rician inverse Gaussian distribution

The RiIG distribution (Eltoft 2005) is expressed as

$$P_{\mathrm{RiIG}}(A|\varepsilon, \sigma^2, \lambda) = e^{\sqrt{\lambda}}\sqrt{\frac{2\lambda}{\pi}}\frac{\left(\varepsilon^2+\sigma^2\right)^{3/4}}{\sigma^2}\frac{A}{\left(A^2+\lambda\sigma^2\right)^{3/4}} \times I_0\left(\frac{\varepsilon}{\sigma^2}A\right)K_{3/2}\left(\frac{\sqrt{\left(\varepsilon^2+\sigma^2\right)\left(A^2+\lambda\sigma^2\right)}}{\sigma^2}\right), \tag{19}$$

where $\sigma^2 > 0$, $\lambda > 0$, $\varepsilon \geq 0$ and I_p and K_p denote the Bessel functions of the first and second kind of order p, respectively. In Eltoft (2005, eqn (41)), the distribution is expressed in terms of the parameters $\alpha = \sqrt{\varepsilon^2+\sigma^2}/\sigma^2$, $\beta = \varepsilon/\sigma^2$, $\delta = \sqrt{\lambda}\sigma$ and $\gamma = 1/\sigma$.

From Eltoft (2005, eqns (42) and (43)), the compound representation of the RiIG distribution is

$$\int_0^\infty P_{\mathrm{Ri}}\left(A\middle|\varepsilon w, \sigma^2 w\right)\mathrm{IG}\left(w\middle|\sqrt{\lambda}, \lambda\right)d\,w, \tag{20}$$

where P_{Ri} denotes the Rice distribution, and $\mathrm{IG}(w|\mu, \lambda)$ is the two-parameter distribution $\sqrt{\frac{\lambda}{2\pi w^3}}\exp(-\frac{\lambda(w-\mu)^2}{2\mu^2 w})$, the IG distribution, with mean μ and shape parameter λ. For

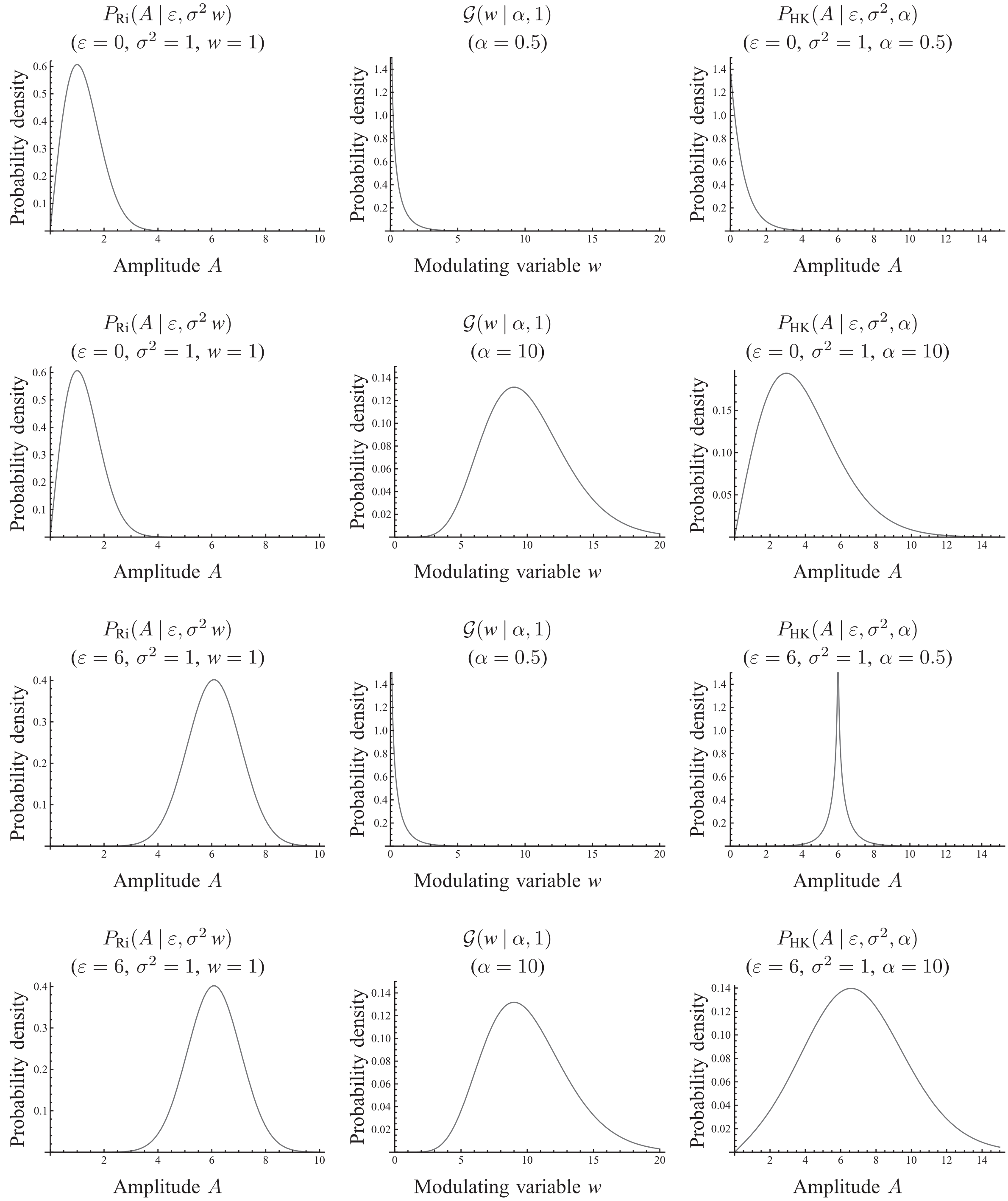

Fig. 1. Four examples of the compound representation $\int_0^\infty P_{\mathrm{Ri}}(A|\overrightarrow{\varepsilon}, \sigma^2 w)\mathcal{G}(w|\alpha, 1)\, dw$ of the homodyned K-distribution. From left to right: the modulated distribution; the modulating distribution; the resulting compound distribution. First row: $\varepsilon = 0, \sigma^2 = 1, \alpha = 0.5$. Second row: $\varepsilon = 0, \sigma^2 = 1, \alpha = 10$. Third row: $\varepsilon = 6, \sigma^2 = 1, \alpha = 0.5$. Fourth row: $\varepsilon = 6, \sigma^2 = 1, \alpha = 10$. The random variable ($A$ or w) is indicated in abscissa, and the value of the probability density function (pdf) is indicated in ordinate.

the RiIG distribution, one takes $\mu = \sqrt{\lambda}$; because the variance of the IG distribution is in general equal to μ^3/λ, the mean and the variance of $\mathrm{IG}(w|\sqrt{\lambda}, \lambda)$ are both equal to $\sqrt{\lambda}$. So, $\sqrt{\lambda}$ in $\mathrm{IG}(w|\sqrt{\lambda}, \lambda)$ plays the same role as α in $\mathcal{G}(w|\alpha, 1)$ (a gamma distribution with mean and variance equal to α). The equivalence of eqn (19) and eqn (20) with Eltoft (2005, eqn (41)), can be easily checked with the software Mathematica (Wolfram Research, Inc.,

Table 3. Mean intensity (average of the intensity in the B-mode image) in the general case (middle column) and in the limiting case of no diffuse signal power (right column). The former can be interpreted as the total signal power, whereas the latter can be interpreted as the coherent signal power

Distribution	Mean intensity	Mean intensity if no diffuse signal power
Rice distribution	$\varepsilon^2 + 2\sigma^2$	ε^2
Homodyned K-distribution	$\varepsilon^2 + 2\sigma^2\alpha$	ε^2
Generalized K-distribution	$\varepsilon^2\alpha(1+\alpha) + 2\sigma^2\alpha$	$\varepsilon^2\alpha(1+\alpha)$
RiIG distribution	$\varepsilon^2\sqrt{\lambda}(1+\sqrt{\lambda})+2\sigma^2\sqrt{\lambda}$	$\varepsilon^2\sqrt{\lambda}(1+\sqrt{\lambda})$
NG distribution	$\Omega\alpha$	$\varepsilon^2\alpha$
NGIG distribution	$\frac{\Omega\sqrt{\lambda}K_{\theta+1}(\sqrt{\lambda})}{K_\theta(\sqrt{\lambda})}$	$\frac{\varepsilon^2\sqrt{\lambda}K_{\theta+1}(\sqrt{\lambda})}{K_\theta(\sqrt{\lambda})}$

Champaign, IL, USA). The point of our notation is the similarity between the compound representation of eqn (20) with the corresponding compound representations of the homodyned and the generalized K-distributions (c.f. eqns (16) and (18)). So, as is the case for the generalized K-distribution, both the coherent signal component ε and the diffuse signal power $2\sigma^2$ are modulated by the variable ω, but that variable is distributed according to an IG distribution rather than the gamma distribution. In Eltoft (2005), an interpretation of the RiIG distribution in terms of a Brownian motion is presented.

The mean intensity of the RiIG distribution and its SNR are indicated in Tables 3 and 4, respectively. In view of the computation of the mean intensity for the generalized K-distribution and the RiIG distribution (c.f. Table 3), one may wish to compare the two distributions $\mathcal{G}(\alpha,1)$ and $\mathrm{IG}(\sqrt{\lambda},\lambda)$ with $\sqrt{\lambda}=\alpha$. We have the following approximation result for sufficiently high values of α (say, $\alpha \geq 6$).

Theorem 5. Let $\sqrt{\lambda}=\alpha$. Then, for any $\alpha \geq 6$

$$\mathcal{D}_{\mathrm{KL}}\left(\mathcal{G}(\alpha,1),\mathrm{IG}\left(\sqrt{\lambda},\lambda\right)\right)\leq 0.03, \tag{21}$$

where $\mathcal{D}_{KL}$ denotes the Kullback-Leibler divergence (Kullback and Leibler 1951) between two distributions. In fact, the function defined by eqn (21) is a decreasing function on the domain $(1, \infty)$ of α.

That result implies that for sufficiently high value of $\alpha=\sqrt{\lambda}$, the generalized K-distribution and the RiIG distribution coincide, for all practical purposes. The result can be improved slightly upon minimizing $\mathcal{D}_{\mathrm{KL}}(\mathcal{G}(\alpha,1),\mathrm{IG}(\sqrt{\lambda},\lambda))$ as a function of λ for each value of α. However, for small values of α, the resulting Kullback-Leibler divergence is still large. Thus, the two models differ significantly for small values of α (say, $0 < \alpha < 6$).

FAMILY OF NAKAGAMI DISTRIBUTIONS

The Nakagami distribution

The Nakagami distribution (Nakagami 1943, 1960) is defined by

$$\mathrm{N}(A|m,\Omega)=\frac{2m^m}{\Gamma(m)\Omega^m}A^{2m-1}e^{-mA^2/\Omega}, \tag{22}$$

for $A \geq 0$, where Γ is the Euler gamma function. The real numbers $m > 0$ and $\Omega > 0$ are called the shape parameter and the scaling parameter, respectively. Equivalently, the intensity $I = A^2$ follows a gamma distribution.

We have the following expressions for the mean intensity of the Nakagami distribution and its SNR:

Table 4. Square of the SNR (ratio of the average intensity with its standard deviation) in the general case (middle column) and in the limiting case of no diffuse signal power (right column)

Distribution	SNR^2	SNR^2 if no diffuse signal power
Rice distribution	$\frac{(\varepsilon^2+2\sigma^2)^2}{4\sigma^2(\varepsilon^2+\sigma^2)}$	∞ (if $\varepsilon > 0$)
Homodyned K-distribution	$\frac{(\varepsilon^2+2\sigma^2\alpha)^2}{4\sigma^2\alpha(\varepsilon^2+\sigma^2(2+\alpha))}$	∞ (if $\varepsilon > 0$)
Generalized K-distribution	$\frac{\alpha(\varepsilon^2(1+\alpha)+2\sigma^2)^2}{2\varepsilon^4(1+\alpha)(3+2\alpha)+4\varepsilon^2(1+\alpha)(4+\alpha)\sigma^2+4(2+\alpha)\sigma^4}$	$\frac{\alpha(\varepsilon^2(1+\alpha))^2}{2\varepsilon^4(1+\alpha)(3+2\alpha)}$
RiIG distribution	$\frac{\sqrt{\lambda}(\varepsilon^2(1+\sqrt{\lambda})+2\sigma^2)^2}{\varepsilon^4(15+14\sqrt{\lambda}+4\lambda)+4\varepsilon^2(6+5\sqrt{\lambda}+\lambda)\sigma^2+4(2+\sqrt{\lambda})\sigma^4}$	$\frac{\sqrt{\lambda}(\varepsilon^2(1+\sqrt{\lambda}))^2}{\varepsilon^4(15+14\sqrt{\lambda}+4\lambda)}$
NG distribution	$\frac{m\alpha}{(1+m)(1+\alpha)-m\alpha}$	α
NGIG distribution	$\frac{mK_{\theta+1}(\sqrt{\lambda})^2}{(m+1)K_{\theta+2}(\sqrt{\lambda})K_\theta(\sqrt{\lambda})-mK_{\theta+1}(\sqrt{\lambda})^2}$	$\frac{K_{\theta+1}(\sqrt{\lambda})^2}{K_{\theta+2}(\sqrt{\lambda})K_\theta(\sqrt{\lambda})-K_{\theta+1}(\sqrt{\lambda})^2}$

$$\begin{aligned} E[I] &= \Omega; \\ SNR^2 &= m. \end{aligned} \tag{23}$$

The Nakagami distribution and the Rice distribution (case m > 1)

The following approximation result is crucial in understanding the Nakagami-gamma and the Nakagami-generalized inverse Gaussian distributions presented below.

Theorem 6. Let $m = \frac{(\varepsilon^2+2\sigma^2)^2}{4\sigma^2(\varepsilon^2+\sigma^2)}$ *and* $\Omega = \varepsilon^2 + 2\sigma^2$. *Then,*

$$\mathcal{D}_{\mathrm{KL}}\left(P_{\mathrm{Ri}}\left(\varepsilon, \sigma^2\right), \mathrm{N}(m, \Omega)\right) \leq 0.02. \tag{24}$$

The choice of m and Ω is consistent with eqn (7) and eqn (23), as well as Nakagami (1960, eqn (55)). When $\varepsilon = 0$, the approximation is actually exact (m = 1 corresponds to the Rayleigh distribution). This observation can be found in Nakagami (1960, eqns (50) and (51)).

The Nakagami distribution and the K-distribution (case m < 1)

We have the following approximation result.

Theorem 7. Let $m = \frac{\alpha}{(\alpha+1)}$ *and* $\Omega = 2\sigma^2\alpha$. *Then,*

$$\mathcal{D}_{\mathrm{KL}}\left(P_{\mathrm{K}}\left(\sigma^2, \alpha\right), \mathrm{N}(m, \Omega)\right) \leq 0.0325. \tag{25}$$

The Nakagami distribution is not included in Tables 3 and 4, because one would need the analogue of Theorems 6 and 7 for the homodyned K-distribution. Thus, we have presented in Tables 3 and 4 only the five distributions that cover the full generality of the configurations of the scatterers (namely, the distributions of Table 2), as well as the limiting case of the Rice distribution.

We now mention two distributions that have been proposed as alternatives to the family of K-distributions.

Nakagami-gamma distribution

The Nakagami-gamma (NG) distribution (Shankar 2003) is defined by

$$\mathrm{N}_{\mathrm{G}}(A|m, \Omega, \alpha) = \frac{4m^{\frac{\alpha+m}{2}}}{\Gamma(\alpha)\Gamma(m)\Omega^{\frac{\alpha+m}{2}}} A^{\alpha+m-1} K_{\alpha-m}\left(2A\sqrt{\frac{m}{\Omega}}\right), \tag{26}$$

where m, Ω and α are positive numbers, and K_p denotes the modified Bessel function of the second kind of order p (we have corrected a typo in Shankar 2003, eqn (8)).

Its compound representation (equivalent to the one given in Shankar 2003) is expressed as

$$\int_0^\infty \mathrm{N}(A|m, \Omega w)\mathcal{G}(w|\alpha, 1)d\,w, \tag{27}$$

where N denotes the Nakagami distribution and $\mathcal{G}(w|\alpha, 1)$ is the gamma distribution with mean and variance α. Now, because the total signal power Ω of the Nakagami distribution is equal to $\varepsilon^2 + 2\sigma^2$, where ε and σ^2 are the parameters of the corresponding Rice distribution (c.f. eqn (24)), we conclude that the coherent signal power ε^2 and the diffuse signal power $2\sigma^2$ are modulated by the variable w. This is consistent with the approximation in eqn (24). Namely, we have

$$\begin{aligned} &\int_0^\infty P_{\mathrm{Ri}}(A|\varepsilon\sqrt{w}, \sigma^2 w)\mathcal{G}(w|\alpha, 1)d\,w \\ &\approx \int_0^\infty \mathrm{N}\left(A|\frac{(\varepsilon^2 w+2\sigma^2 w)^2}{4\sigma^2 w(\varepsilon^2 w+\sigma^2 w)}, \varepsilon^2 w+2\sigma^2 w\right) \\ &\times\mathcal{G}(w|\alpha, 1)d\,w \\ &= \int_0^\infty \mathrm{N}(A|m, \Omega w)\mathcal{G}(w|\alpha, 1)d\,w \\ &= \mathrm{N}_{\mathrm{G}}(A|m, \Omega, \alpha). \end{aligned} \tag{28}$$

So, $\sqrt{m}$ and Ω correspond to the SNR and the total signal power, respectively, of the underlying modulated Nakagami distribution, whereas α plays at first glance the role of the effective density of random scatterers (see the Summary and Discussion). The mean intensity of the NG distribution and its SNR are as indicated in Tables 3 and 4.

Let us mention that there is also the generalized Nakagami distribution (Shankar 2001) defined by

$$\mathrm{N}_{\mathrm{Ge}}(A|m, \Omega, s) = \frac{2sm^m}{\Gamma(m)\Omega^m} A^{2ms-1} e^{-mA^{2s}/\Omega}, \tag{29}$$

where m, Ω and s are positive real numbers (s is a shape adjustment parameter). Considering the change of variable $y = A^{1/s}$, one obtains a Nakagami distribution on the variable y, *i.e.*, $\mathrm{N}(y|m, \Omega) = \frac{2m^m}{\Gamma(m)\Omega^m} y^{2m-1} e^{-my^2/\Omega}$. Thus, this distribution seems somewhat artificial, since there is no physical or statistical reason for considering such a change of variable. Also, note that Shankar (2001) appeared before Shankar (2003); hence it seems that the latter supersedes the former. Finally, the mean intensity cannot be expressed analytically, unless s is an integer, as far as we can tell.

Nakagami-generalized inverse Gaussian (NGIG) distribution

The NGIG distribution (Agrawal and Karmeshu 2006) is defined by

$$\mathrm{N}_{\mathrm{GIG}}(A|m,\Omega,\theta,\lambda) = \frac{2(m/\Omega)^m}{\lambda^{\theta/2}K_\theta(\sqrt{\lambda})\Gamma(m)}A^{2m-1} \times\left(\frac{\Omega}{2mA^2+\lambda\Omega}\right)^{\frac{m-\theta}{2}}K_{\theta-m}\left(\sqrt{\lambda+\frac{2mA^2}{\Omega}}\right), \tag{30}$$

where m, Ω, θ and λ are positive real numbers, and K is the modified Bessel function of the second kind. In Agrawal and Karmeshu (2006), that distribution is expressed in terms of the parameters "m" = m, "γ" = θ, "θ"=$\Omega\sqrt{\lambda}$, and "λ" = $\lambda\Omega$. Our choice of notation clarifies the relation between the NGIG distribution and the RiIG and NG distributions, as is discussed later.

Its compound representation, equivalent to the one given in Agrawal and Karmeshu (2006), is expressed as

$$\int_0^\infty \mathrm{N}(A|m,\Omega w)\mathrm{GIG}\left(w\middle|\theta,\sqrt{\lambda},\lambda\right)d\,w, \tag{31}$$

where N denotes the Nakagami distribution and $\mathrm{GIG}(w|\theta,\mu,\lambda)=\frac{1}{2\mu^\theta K_\theta(\lambda/\mu)}w^{\theta-1}\times\exp(-\frac{1}{2}(\frac{\lambda}{w}+\frac{\lambda}{\mu^2}w))$ is the GIG distribution. The IG distribution is obtained from the GIG distribution upon setting $\theta = -1/2$. The equivalence of eqn (30) and eqn (31) with Agrawal and Karmeshu (2006, eqns 6 and 5), can be checked with Mathematica. Again, this is consistent with the approximation in eqn (24). Namely, we have

$$\begin{aligned}&\int_0^\infty P_{\mathrm{Ri}}(A|\varepsilon\sqrt{w},\sigma^2w)\mathrm{GIG}\left(w|\theta,\sqrt{\lambda},\lambda\right)d\,w\\ &\approx\int_0^\infty \mathrm{N}\left(A|\frac{\left(\varepsilon^2w+2\sigma^2w\right)^2}{4\sigma^2w(\varepsilon^2w+\sigma^2w)},\varepsilon^2w+2\sigma^2w\right)\\ &\quad\times\mathrm{GIG}\left(w\middle|\theta,\sqrt{\lambda},\lambda\right)d\,w\\ &=\int_0^\infty \mathrm{N}(A|m,\Omega w)\mathrm{GIG}\left(w\middle|\theta,\sqrt{\lambda},\lambda\right)d\,w\\ &=\mathrm{N}_{\mathrm{GIG}}(A|m,\Omega,\theta,\lambda).\end{aligned} \tag{32}$$

So, $\sqrt{m}$ and Ω correspond to the SNR and the total signal power, respectively, of the underlying modulated Nakagami distribution, whereas $\sqrt{\lambda}$ plays at first glance the role of the effective density of random scatterers (again, see the Summary and Discussion). As for θ, its physical meaning remains to be explained. The mean intensity of the NGIG distribution and its SNR are indicated in Tables 3 and 4.

Obviously, at this point, one could introduce the Rician generalized inverse Gaussian (RiGIG) distribution $P_{\mathrm{RiGIG}}(r|\varepsilon, \sigma^2, \theta, \lambda)$ defined by

$$\int_0^\infty P_{\mathrm{Ri}}\left(A\middle|\varepsilon w,\sigma^2 w\right)\mathrm{GIG}\left(w\middle|\theta,\sqrt{\lambda},\lambda\right)d\,w. \tag{33}$$

See eqn (20). This distribution has four parameters and has not been used in ultrasound imaging yet. But, of course, one would have to explain the physical meaning of the extra parameter θ.

SUMMARY AND DISCUSSION

In summary, there are three aspects in the compound representation of the distributions discussed before: (i) the modulated distribution, (ii) the modulating distribution and (iii) the modulated parameters. Table 2 summarizes the various compound representations presented in this paper. Figure 2 gives five examples of those compound representations.

For the first aspect, one uses the Rice distribution (homodyned K-distribution, generalized K-distribution, RiIG distribution) or its approximation based on Theorem 6 (NG distribution, NGIG distribution). Considering the good quality of the approximation, this aspect is not crucial and seems to play a role only in view of obtaining an explicit form of the distribution.

For the second aspect, the modulation distribution is the gamma distribution (homodyned K-distribution, generalized K-distribution, NG) or the IG distribution (RiIG distribution), and even the more general GIG distribution (NGIG distribution). The IG is obtained from the GIG distribution by setting $\theta = -1/2$. It remains to have a physical interpretation of the parameter θ of the GIG distribution. For a wide range of values, the IG distribution is a good approximation of the gamma distribution, upon taking $\lambda = \alpha^2$ (say $\alpha \geq 6$ as in Theorem 5). The difference between the two models for small values of α remains to be studied.

The third aspect seems to be the most important one, in our opinion. The modulated parameters can be the diffuse signal power $2\sigma^2$ (homodyned K-distribution), the coherent signal component ε and the diffuse signal power (generalized K-distribution and RiIG distribution) or the coherent signal power ε^2 and the diffuse signal power (NG and NGIG distributions).

As mentioned in Jakeman and Tough (1987), an unsatisfactory feature of the generalized K-distribution is that it predicts that the mean intensity of the scattered field will fluctuate because of variations in the number of scatterers, even if the mean intensity of each scatterer tends to 0. On the other hand, the homodyned K-distribution model predicts that the mean intensity of the scattered field will tend to the mean intensity of the coherent signal component, under the same condition.

One can actually see this phenomenon upon considering the mean intensity of the various distributions presented here before. The analytical expressions of the mean intensities are presented in Table 3. In the case of the first three distributions (homodyned K-distribution, generalized K-distribution, RiIG distribution), let $\sigma^2 \to 0$, corresponding to a vanishing inhomogeneity of the medium, *i.e.*, the vanishing of the diffuse component. In

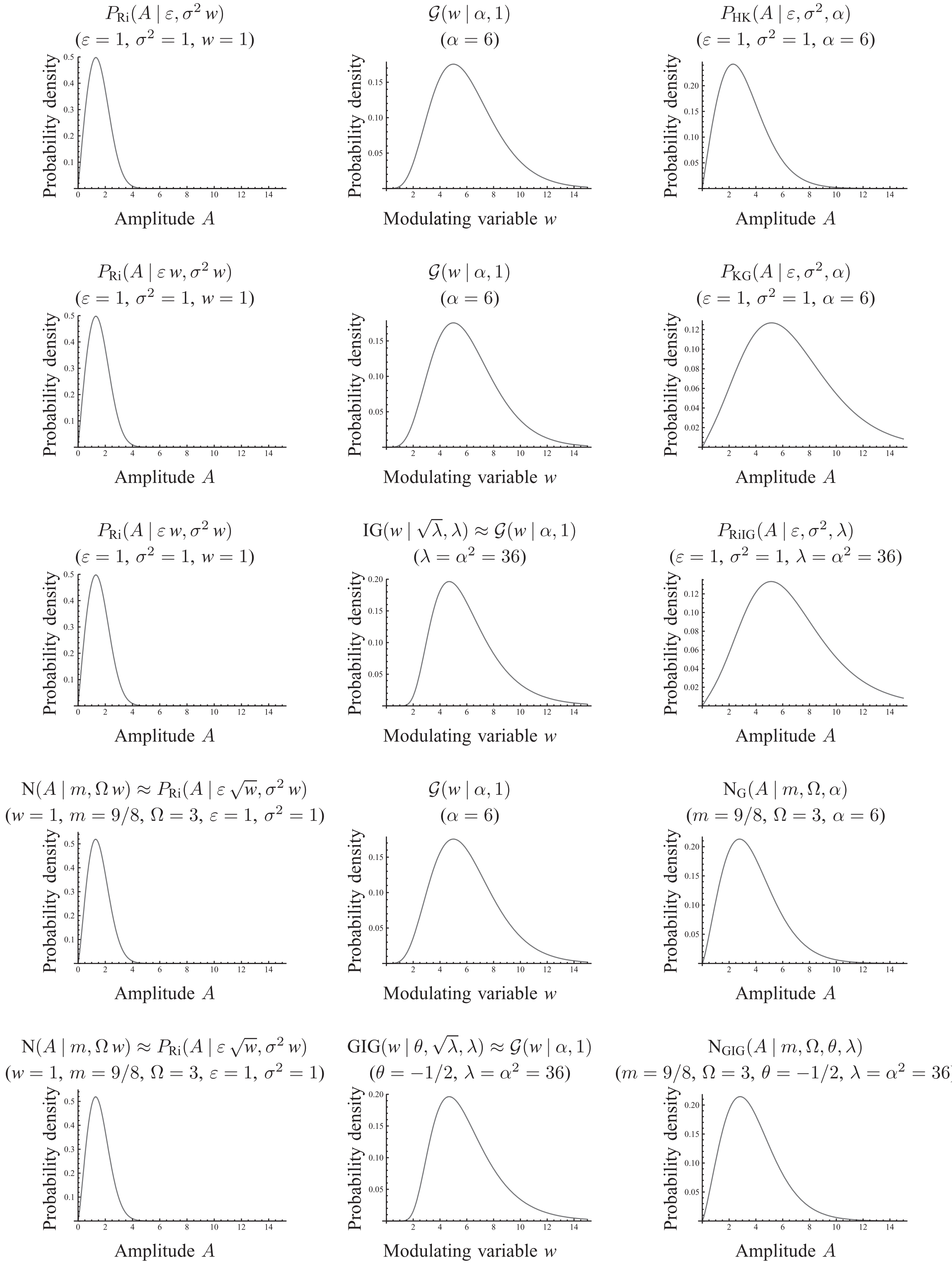

Fig. 2. Five examples of compound representations. From left to right: the modulated distribution; the modulating distribution; the resulting compound distribution. First row: the homodyned K-distribution. Second row: the generalized K-distribution. Third row: the RiIG distribution. Fourth row: the NG distribution. Fifth row: the NGIG distribution. The random variable (A or w) is indicated in abscissa, and the value of the probability density function (pdf) is indicated in ordinate. For the third and fifth rows, we use the approximation $\mathcal{G}(w|\alpha,1) \approx \mathrm{IG}(w|\sqrt{\lambda},\lambda)$, with $\lambda = \alpha^2$ (c.f. Theorem 5). For the fourth and fifth rows, we use the approximation $P_{\mathrm{Ri}}(A|\varepsilon\sqrt{w},\sigma^2 w) \approx \mathrm{N}(A|m,\Omega w)$, with $m = \frac{(\varepsilon^2+2\sigma^2)^2}{4\sigma^2(\varepsilon^2+\sigma^2)}$ and $\Omega = \varepsilon^2 + 2\sigma^2$ (c.f. Theorem 6).

the case of the two other distributions (NG distribution, NGIG distribution), let $\Omega \rightarrow \varepsilon^2$, corresponding to the same limiting situation (because $\Omega = \varepsilon^2 + 2\sigma^2$ in Theorem 6). Then the limiting moments are indicated in the second column of Table 3. As one can see, only the homodyned K-distribution offers the feature of a limiting mean intensity that does not depend further on the density parameters α, or θ and λ.

One can also consider the limiting behavior of the SNR (Table 4). In the case of the first three distributions (homodyned K-distribution, generalized K-distribution, RiIG distribution), let $\sigma^2 \rightarrow 0$ (*i.e.*, vanishing of the diffuse component). In the case of the two other distributions (NG and NGIG), let $m \rightarrow \infty$ (see the expression of m in Theorem 6). Then, the limiting moments are indicated in the bottom rows of Table 4. As one can see, the behavior of the homodyned K-distribution differs drastically from the other distributions. Namely, only the homodyned K-distribution offers the property of a limiting infinite SNR, which corresponds to a vanishing variance of the signal in the case of a vanishing inhomogeneity of the medium.

But now, both features of the homodyned K-distribution seem to be desirable properties, because one would expect the coherent signal power ε^2 to be mainly the result of a difference of impedance between two adjacent tissues, and not on the number of diffuse scatterers when their power vanishes. In particular, the statistical parameters of the four other distributions lose the physical meaning of their counterparts in the homodyned K-distribution. For instance, in the case of the homodyned K-distribution, the coherent signal power and the diffuse signal power correspond to ε^2 and $2\sigma^2\alpha$, respectively. On the other hand, for the generalized K-distribution, Table 3 suggests that one should take $\varepsilon^2\alpha(1 + \alpha)$ and $2\sigma^2\alpha$, respectively. Thus, it seems that the interpretation of the parameters ε and α are more difficult in the case of the generalized K-distribution. A similar remark holds for the three other distributions. In particular, the parameter α or $\sqrt{\lambda}$ might not represent the effective density of random scatterers for other distributions than the homodyned K-distribution, even if formally they seem to do so, as was mentioned after eqn (28) and eqn (32). This fact does not come as a surprise; after all, the models are different and the coherent signal component or its square are modulated simultaneously with the diffuse signal power in all cases, except for the homodyned K-distribution. It shows that one should be careful in interpreting physically and clinically the parameters of the various distributions presented in this paper. For that matter, we believe that the homodyned K-distribution is the most suitable distribution among the ones that we have presented in the context of tissue characterization. On the other hand, the Nakagami distribution is a good approximation of the homodyned K-distribution, but with only two parameters, that is also consistent with the limiting case of a vanishing diffuse signal power, at least in the case where $m > 1$. Moreover, the Rayleigh, Rice and K-distributions are also consistent with the limiting case of a vanishing diffuse signal power, because they are limiting or special cases of the homodyned K-distribution.

CONCLUSION

We have presented a unified overview of the main distributions used to model the amplitude (gray level) of the envelope of the RF image, based on their compound representation. Based on the computation of the mean intensity and the SNR in the limit case of a vanishing diffuse signal power, we have argued in favor of the homodyned K-distribution. In the case of the other distributions, the parameters lose their physical meaning, although the distributions may very well fit real data. It remains to prove our argument experimentally, with simulations or real data. Thus, an open problem in ultrasound imaging is to clarify the choice of a model based on the modulated parameters. Determining the most appropriate model is important, for instance, in the context of tissue characterization. In particular, it would be desirable to use a same model to favor the comparison of results. As it seems, new statistical distributions are introduced in the literature, not so much because the homodyned K-distribution does not have good modeling properties, but simply because it does not admit an explicit analytical expression. Nevertheless, we do not believe that this is a major drawback for using the homodyned K-distribution. For instance, the fractional moments of the intensity admit an analytical expression based on Prager et al. (2002).

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APPENDIX: PROOFS OF THE RESULTS

Proof of Theorem 1

See Jakeman and Tough (1987, eqn (2.13)). From Lord (1954, eqn (9)), the characteristic function of $\mathbf{a}$ is equal to $\phi(\mathbf{u})=\Phi(u)=\overline{\Lambda_{n/2-1}(ua)}$, where $u=\|\mathbf{u}\|$ and $\Lambda_\alpha(z)=\Gamma(\alpha+1)(\frac{z}{2})^{-\alpha}J_\alpha(z)=1-\frac{1}{2(2\alpha+2)}z^2+\mathcal{O}(z^4)$. Thus, after scaling by the factor $1/\sqrt{N}$, we obtain $(1-\frac{u^2\overline{a^2}}{2n}N^{-1}+\mathcal{O}(N^{-2}))^N$ as characteristic function of $\mathbf{A}$. Letting N tend to infinity gives $\exp(-\frac{u^2\overline{a^2}}{2n})$, which is the characteristic function of an n-dimensional Gaussian distribution with mean $\overrightarrow{0}$ and variance equal to the diagonal matrix with identical entries of $\sigma^2=\overline{a^2}/n$. The distribution of the amplitude is thus the Rayleigh distribution. Note that from the inversion formula (Lord 1954, eqn (10)), the distribution $P(A)$ of the amplitude of the random process can also be expressed as $\frac{1}{2^{n/2-1}\Gamma(n/2)}\int_0^\infty (uA)^{n/2}J_{n/2-1}(uA)\exp(-\frac{u^2\overline{a^2}}{2n})\,du$.

Proof of Theorem 2

By definition (see Jakeman and Tough 1987, eqn (2.15)), the characteristic function of the random process $\overrightarrow{\varepsilon}+\frac{1}{\sqrt{N}}\sum_{j=1}^N \mathbf{a}_j$ of eqn (5) as $N\to\infty$ is equal to $\exp(i\mathbf{u}\cdot\overrightarrow{\varepsilon})\Phi(u)$, where $\Phi(u)$ is the characteristic function of the random process $\frac{1}{\sqrt{N}}\sum_{j=1}^N \mathbf{a}_j$ of eqn (2) as $N\to\infty$, *i.e.*, $\exp(-\frac{u^2\overline{a^2}}{2n})$ from the proof of Theorem 1. So we obtain the characteristic function of an n-dimensional Gaussian distribution with mean $\overrightarrow{\varepsilon}$ and variance, the diagonal matrix with identical entries, equals to $\sigma^2=\overline{a^2}/n$. The distribution of the amplitude is thus the Rice distribution.

In (Jakeman and Tough 1987, eqn (2.16)), an alternative proof is presented based on the evaluation of the integral $\frac{A^{n/2}}{\varepsilon^{n/2-1}}\int uJ_{n/2-1}(uA)J_{n/2-1}(u\varepsilon)\exp(-\frac{u^2\overline{a^2}}{2n\alpha})\,du$. We derive that integral representation of the Rice distribution as follows. From the inversion formula (Lord 1954, eqn (3)), the distribution $p(\mathbf{A})$ of the random process underlying the Rice distribution is equal to

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int \exp(-i\mathbf{u}\cdot\mathbf{A})\exp(i\mathbf{u}\cdot\overrightarrow{\varepsilon})\exp\left(-\frac{u^2\overline{a^2}}{2n}\right)d\mathbf{u}\\ &=\int\frac{u^{n-1}}{(2\pi)^n}\left\{\int\exp(-i\mathbf{u}\cdot\mathbf{A})\exp(i\mathbf{u}\cdot\overrightarrow{\varepsilon})\,d\hat{\mathbf{u}}\right\}\\ &\times\exp\left(-\frac{u^2\overline{a^2}}{2n}\right)du,\end{aligned}$$

where u and $\hat{\mathbf{u}}$ denote the norm and the direction of $\mathbf{u}$, respectively. From (Lord 1954), we deduce that the amplitude has distribution $P(A)$ equal to

$$\begin{aligned}&A^{n-1}\int p(\mathbf{A})\,d\hat{\mathbf{A}}\\ &=A^{n-1}\int\frac{u^{n-1}}{(2\pi)^n}\exp\left(-\frac{u^2\overline{a^2}}{2n}\right)\\ &\times\left\{\int\left(\int\exp(-i\mathbf{u}\cdot\mathbf{A})\,d\hat{\mathbf{A}}\right)\exp(i\mathbf{u}\cdot\overrightarrow{\varepsilon})\,d\hat{\mathbf{u}}\right\}du.\end{aligned}$$

But from (Lord 1954), the integral $\int\exp(-i\mathbf{u}\cdot\mathbf{A})\,d\hat{\mathbf{A}}$ is equal to the expression $(2\pi)^{n/2}(uA)^{-n/2+1}J_{n/2-1}(uA)$. Thus, after simplifications, $P(A)$ is equal to

$$\begin{aligned}&A^{n/2}\int\frac{u^{n/2}}{(2\pi)^{n/2}}J_{n/2-1}(uA)\left\{\int\exp(i\mathbf{u}\cdot\overrightarrow{\varepsilon})\,d\hat{\mathbf{u}}\right\}\\ &\times\exp\left(-\frac{u^2\overline{a^2}}{2n}\right)du.\end{aligned}$$

And by the same argument this is equal to

$$\frac{A^{n/2}}{\varepsilon^{n/2-1}}\int uJ_{n/2-1}(uA)J_{n/2-1}(u\varepsilon)\exp\left(-\frac{u^2\overline{a^2}}{2n}\right)du.$$

Finally, we have shown in the previous paragraph that this must be the Rice distribution.

Proof of Theorem 3

From the proof of Theorem 1, the characteristic function of $\mathbf{A}$ conditional to N is equal to $\Phi(u|N)=(1-\frac{u^2\overline{a^2}}{2n}\overline{N}^{-1}+\mathcal{O}(\overline{N}^{-2}))^N=\Phi(u|1)^N$, after scaling by the factor $1/\sqrt{\overline{N}}$. Thus, the characteristic function of $\mathbf{A}$ is equal to $\Phi(u)=\sum_{N=0}^\infty \mathrm{NegBin}(N|\alpha,1/(1+\overline{N}/\alpha))\Phi(u|N)$. And this is equal to $\sum_{N=0}^\infty\frac{(N+\alpha-1)!}{N!(\alpha-1)!}\frac{(\overline{N}/\alpha)^N}{(1+\overline{N}/\alpha)^{N+\alpha}}\Phi(u|1)^N=(1+\frac{\overline{N}}{\alpha}(1-\Phi(u|1)))^{-\alpha}$. Letting $\overline{N}$ tend to infinity yields $(1+\frac{u^2\overline{a^2}}{2n\alpha})^{-\alpha}$. From the inversion formula (Lord 1954, eqn (10)), the distribution of the amplitude of the random process is equal to $P(A)=\frac{1}{2^{n/2-1}\Gamma(n/2)}\int_0^\infty(uA)^{n/2}J_{n/2-1}(uA)(1+\frac{u^2\overline{a^2}}{2n\alpha})^{-\alpha}\,du$. From Jakeman and Tough (1987, eqns (2.10) and (2.11)), this is the K-distribution. As an alternative proof, we will use Jakeman and Tough (1987, eqn (4.1) to (4.6)) and prove part (b) of the Theorem at the same time.

So, let us observe that $(1+\frac{u^2\overline{a^2}}{2n\alpha})^{-\alpha}$ is equal to $\int_0^\infty\exp(-\frac{u^2\overline{a^2}w}{2n\alpha})\mathcal{G}(w|\alpha,1)\,dw$, where $\mathcal{G}(w|\alpha,1)$ is as in the statement of the Theorem (see Jakeman and Tough 1987, eqn (4.2)) and (Erdélyi 1954, vol. I, p. 312, eqn (1)). Changing the order of integration in the equation of the previous paragraph, we obtain that $P(A)$ is equal to

$$\begin{aligned}&\int_0^\infty\left\{\frac{1}{2^{n/2-1}\Gamma(n/2)}\right.\\ &\left.\times\int_0^\infty(uA)^{n/2}J_{n/2-1}(uA)\exp\left(-\frac{u^2\overline{a^2}w}{2n\alpha}\right)du\right\}\\ &\times\mathcal{G}(w|\alpha,1)\,dw.\end{aligned}$$

From the proof of Theorem 1, the inner integral is equal to the Rayleigh distribution $P_{\mathrm{Ra}}(A|\frac{\overline{a^2}w}{2n\alpha})$. Thus, the resulting distribution is $\int_0^\infty P_{\mathrm{Ra}}(A|\sigma^2w)\mathcal{G}(w|\alpha,1)\,dw$. Finally, from Erdélyi (1954, vol. I, p. 146, eqn (29)), one obtains the distribution of eqn (8).

Proof of Theorem 4

As in Jakeman and Tough (1987), the characteristic function of the random process $\mathbf{A}$ of eqn (14) as $\overline{N}\to\infty$ is equal to $\Psi(u)\Phi(u)$, where $\Psi(u)$ is the characteristic function of the randomly phased vector $\overrightarrow{\varepsilon}$ of constant amplitude ε, and $\Phi(u)$ is the characteristic function of the random process of eqn (9) as $\overline{N}\to\infty$, *i.e.*, $(1+\frac{u^2\overline{a^2}}{2n\alpha})^{-\alpha}$ from the proof of Theorem 3. From Lord (1954), we have $\Psi(u)=\frac{(2\pi)^{n/2}}{(u\varepsilon)^{n/2-1}}J_{n/2-1}(u\varepsilon)\times\frac{\Gamma(n/2)}{2\pi^{n/2}}=\frac{2^{n/2-1}\Gamma(n/2)}{(u\varepsilon)^{n/2-1}}J_{n/2-1}(u\varepsilon)$.

From the inversion formula (Lord 1954, eqn (10)), we deduce that the distribution $P(A)$ of the amplitude of the random process underlying the homodyned K-distribution is equal to

$$\begin{aligned}&\frac{1}{2^{n/2-1}\Gamma(n/2)}\int(uA)^{n/2}J_{n/2-1}(uA)\frac{2^{n/2-1}\Gamma(n/2)}{(u\varepsilon)^{n/2-1}}J_{n/2-1}(u\varepsilon)\left(1+\frac{u^2\overline{a^2}}{2n\alpha}\right)^{-\alpha}du\\ &=\frac{A^{n/2}}{\varepsilon^{n/2-1}}\int uJ_{n/2-1}(u\varepsilon)J_{n/2-1}(uA)\left(1+\frac{u^2\overline{a^2}}{2n\alpha}\right)^{-\alpha}du.\end{aligned}$$

Finally, this is the definition of the homodyned K-distribution. See Jakeman and Tough (1987, eqn (4.13)).

Next, as in the proof of Theorem 3, we can rewrite the homodyned K-distribution as

$$\begin{aligned}&\int_0^\infty\mathcal{G}(w|\alpha,1)\left\{\frac{A^{n/2}}{\varepsilon^{n/2-1}}\right.\\ &\left.\times\int uJ_{n/2-1}(uA)J_{n/2-1}(u\varepsilon)\exp\left(-\frac{u^2\overline{a^2}w}{2n\alpha}\right)du\right\}dw.\end{aligned}$$

Now, from the proof of Theorem 2, we know that the inner integral is the Rice distribution $P_{\mathrm{Ri}}(A|\varepsilon,\sigma^2w)$. The compound representation of the homodyned K-distribution follows from there.

Remark on Theorems 2 and 4

Let $\mathbf{B}$ be a random process with characteristic function of the form $\phi(\overrightarrow{u})=\Phi(u)$ (where $u=\|\overrightarrow{u}\|$), such as is the case for the process underlying the Rayleigh distribution or the K-distribution. Let $\overrightarrow{\varepsilon}_1$ be a random process consisting of a constant vector of amplitude ε. Then, the argument in the proof of Theorem 2 shows that the random process $\overrightarrow{\varepsilon}_1+\mathbf{B}$ (where the two terms are viewed as independent random processes) has an amplitude with distribution $P(A)$ equal to $\frac{A^{n/2}}{\varepsilon^{n/2-1}}\int uJ_{n/2-1}(u\varepsilon)J_{n/2-1}(uA)\Phi(u)\,du$. On the other hand, let $\overrightarrow{\varepsilon}_2$ be a randomly phased vector of constant amplitude ε. Then, the argument in the proof of Theorem 4 shows that the random process $\overrightarrow{\varepsilon}_2+\mathbf{B}$ (where the two terms are viewed as independent random processes) has also an amplitude with distribution $P(A)$ equal to $\frac{A^{n/2}}{\varepsilon^{n/2-1}}\int uJ_{n/2-1}(u\varepsilon)J_{n/2-1}(uA)\Phi(u)\,du$. Therefore, the two different processes $\overrightarrow{\varepsilon}_1+\mathbf{B}$ and $\overrightarrow{\varepsilon}_2+\mathbf{B}$ yield the same distribution of their amplitude. Thus, in eqn (5) and eqn (14), one may assume that $\overrightarrow{\varepsilon}$ is either a constant vector of amplitude ε or a randomly phased vector of constant amplitude ε, as far as the distribution of the amplitude is concerned.

Sketch of a proof of Theorem 5

Recall that $\mathcal{D}_{\mathrm{KL}}(f,g)=\int f(x)\,\log\frac{f(x)}{g(x)}\,dx$. Using Mathematica, it is straightforward to compute

$$\mathcal{D}_{\mathrm{KL}}(\mathcal{G}(\alpha,1),\mathrm{IG}(\alpha,\alpha^2))$$

$$=\tfrac{1}{2}\left(\tfrac{\alpha}{\alpha-1}-2\alpha+\log(2\pi)\right.$$

$$\left.-2\log\Gamma(\alpha+1)+(1+2\alpha)\psi(\alpha)\right)$$

From there, the statement of the Theorem follows conjecturally after inspection of the graph of that function of α.

Sketch of a proof of Theorem 6

We first observe that for any $\rho>0$, $\mathrm{N}(A|m,\rho\Omega)=\rho^{-1}\mathrm{N}(\rho^{-1}A|m,\Omega)$ and $P_{\mathrm{Ri}}(A|\sqrt{\rho}\varepsilon,\rho\sigma^2)=\rho^{-1}P_{\mathrm{Ri}}(\rho^{-1}A|\varepsilon,\sigma^2)$. Therefore, $\mathcal{D}_{\mathrm{KL}}(P_{\mathrm{Ri}}(\sqrt{\rho}\varepsilon,\rho\sigma^2),\mathrm{N}(m,\rho\Omega))$ is invariant in ρ. So, it is enough to consider the case $\sigma^2=1$. The result now follows conjecturally after inspection of the graph of the Kullback-Leibler divergence as a function of ε.

Sketch of a proof of Theorem 7

Again, because $P_{\mathrm{K}}(A|\rho\sigma^2,\alpha)=\rho^{-1}P_{\mathrm{K}}(\rho^{-1}A|\sigma^2,\alpha)$, for any $\rho>0$, it follows that $\mathcal{D}_{\mathrm{KL}}(P_{\mathrm{K}}(\rho\sigma^2,\alpha),\mathrm{N}(m,\rho\Omega))$ is invariant in ρ. So, it is enough to consider the case $\sigma^2=1$. The result now follows conjecturally after inspection of the graph of the Kullback-Leibler divergence as a function of α.