Rough volatility models - Project
Due by 11:59 pm on March 31st 2023

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INSTRUCTIONS:

- **You will work in GROUPS OF 3 PEOPLE.**
- **Write your code in Python.**
- **Your WELL-COMMENTED CODE must contain:**
  - **For Exercises 1, 2.5 and 3.3, the SCRIPTS / NOTEBOOKS doing the corresponding tasks.**
  - **For Exercise 2.3, 2.4, 3.1 and 3.2 the FUNCTIONS Ch_Lifted_Heston, Call_Price_Lifted_Heston, Sim_Lifted_Bergomi and Call_Price_Lifted_Bergomi, respectively.**
- **In addition, send a PDF FILE with the numerical results and plots for Exercises 1, 2.5 and 3.3, as well as the answer to Exercises 2.1 and 2.2.**
- **Send everything by EMAIL by March 31st 2023 before 11:59 pm to**

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  In the subject of the email write: Projet_RoughVol_2023_[Name1]_[Name2]_[Name3]

  For [Name1] indicate the surname of the first member of the group, etc.

  Emails received after 11:59 pm will automatically receive a zero grade.

1 Estimation of the parameter H

Let V_t be the spot variance of a *Lifted Heston model* with n factors and initial curve $g_0(t)$ given by

$$g_0(t) = V_0 + \lambda\theta \sum_{i=1}^{n} c_i \int_0^t \mathrm{e}^{-x_i(t-s)}\,\mathrm{d}s.$$

The weights $(c_i)_{i=1}^n$ and the means reversions $(x_i)_{i=1}^n$ are parametrized as in slide 10 of Part III of the lectures with a constant r which we will denote by r_n.

Consider a Lifted Heston model with $n = 20$ factors and the following parameters

$$V_0 = 0.05; \quad \lambda = 0.3; \quad \theta = 0.05; \quad \nu = 0.1; \quad \alpha = H + 0.5 = 0.6; \quad r_{20} = 2.5.$$

1. Simulate and plot a path of the variance process $(V_{t_k})_{k=0}^m$ on the interval $[0,1]$ using a uniform partition $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots, t_m = m\Delta t$ with $m = 10^5$ (and $\Delta t = 10^{-5}$). For the simulation use the method described in slide 19 of Part III of the lectures.

2. Using the method described in the first lecture (slide 38 of Part I with W^H replaced by V) estimate the value of H for the simulation in part 1 by using moments $q = 0.5, 1, 1.5, 2$ and lags $\Delta = 1, \ldots, 10$.

3. Repeat the estimation of H from part 2 with the sampled path $V_0, V_{l\Delta t}, V_{2l\Delta t}, \ldots, V_{\lfloor m/l \rfloor l\Delta t}$ for $l \in \{1, \ldots, 10\}$. For each value of l let $H(l)$ be the corresponding estimation of the parameter H. Plot l against the values $H(l)$. What do you observe?

4. Repeat parts 1, 2 and 3 with V replaced by a classical Brownian motion W. What do you observe?

5. Repeat parts 1, 2 and 3 with V replaced by a fractional Brownian motion W^H (with $H = 0.1$). For the simulation use the Cholesky method and consider a time grid with $m = 10^3$ (and $\Delta t = 10^{-3}$). What do you observe?

2 Implied volatility in the Lifted Heston model

1. Consider a Lifted Heston model as defined in slide 12 of Part III of the lectures. Define the process

$$M_t = \exp\left(u \log(S_t) + \phi(t,T) + \sum_{i=1}^{n} c_i \psi^i(T-t) U_t^i\right) \tag{1}$$

where $u \in \mathbb{C}$, and ϕ, ψ^i are functions with $\psi^i(0) = 0$, $i = 1, \ldots, n$, that satisfy the Ordinary Differential Equations (ODEs) described in slide 14 of Part III of the lectures.

   (a) Deduce by Itô's formula that M is a *local* martingale.

   (b) Show that if M is a *true* martingale then

$$M_t = \mathbb{E}\left[\exp\left(u \log S_T\right) \middle| \mathcal{F}_t\right].$$

2. The Carr-Madan formula for the price of a European call option C_0 with strike K, maturity T, and interest rate r_{int}, is

$$C_0 = \frac{e^{-r_{int}T - \alpha_2 \log(K)}}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{\Phi_T(u - (\alpha_2+1)i)}{(\alpha_2 + iu)(\alpha_2 + 1 + iu)} e^{-i\log(K)u}\right) \mathrm{d}u \tag{2}$$

where $\Phi_T(u) = \mathbb{E}[\exp(iu\log(S_T))]$, $u \in \mathbb{R}$, is the characteristic function of the log underlying price and $\alpha_2 > 0$ is a damping factor. Deduce this formula from the remarks in slide 22 of Part II of the lectures. *Hint:* Take $\omega = \alpha_2 + 1$. Also, notice that here we consider a real value u while in the formula of the previous part u could be complex. In fact, $\phi_T(u)$ corresponds to M_0, as in (1), by replacing u by iu.

3. **In what follows we will assume that $r_{int} = 0$.** In the Lifted Heston model, the function Φ_T comes from a system of Riccati ODEs (see slide 14 of Part III of the lectures). Using the scheme described in slide 19 of Part III of the lectures write a function

$$\mathrm{Ch\_Lifted\_Heston(u, S0, T, rho, lamb, theta, nu, V0, n, rn, alpha, M)}$$

that approximates the characteristic function in the Lifted Heston model. Here

$$(\text{rho}, \text{lamb}, \text{theta}, \text{nu}, \text{V0}, \text{alpha}) = (\rho, \lambda, \theta, \nu, V_0, \alpha),$$

n is the number of factors and $r_n = r$ as in Exercise 1. The parameter M is the number of time discretization steps used in the scheme described in slide 19 of Part III of the lectures.

4. Using the function from the previous part and the Carr-Madan formula (2) write a function

$$\text{Call\_Price\_Lifted\_Heston}(\text{S0}, \text{K}, \text{T}, \text{rho}, \text{lamb}, \text{theta}, \text{nu}, \text{V0}, \text{n}, \text{rn}, \text{alpha}, \text{M}, \text{alpha2}, \text{L})$$

that approximates the call option price in the Lifted Heston model. The new inputs of the function are the strike K, the damping factor in the Carr-Madan formula α_2, and the truncation level L to approximate the integral in (2).

5. Consider the following parameters

$$S_0 = 1; \quad \rho = -0.7; \quad \lambda = 0.3; \quad \theta = 0.02; \quad \nu = 0.3; \quad V_0 = 0.02;$$
$$\alpha = H + 0.5 = 0.6; \quad \alpha_2 = 1; \quad M = 100.$$

Let $r_n = 1 + 10n^{-0.9}$ be the constant used in order to parametrize the weights $(c_i)_{i=1}^n$ and the means reversions $(x_i)_{i=1}^n$.

(a) For $T = 1$ choose the truncation level $L = 100$. Consider 20 equidistant log strikes between -1.2 and 0.2. Plot the implied volatility smiles for $n = 5, 10, 20, 50$ factors. What do you observe?

(b) For $T = 1/26$ choose the truncation level $L = 1000$. Consider 20 equidistant log strikes between -0.15 and 0.05. Plot the implied volatility smiles for $n = 5, 10, 20, 50$ factors. What do you observe?

3 Implied volatility in the Lifted/Rough Bergomi model

In this section we will assume that riskless interest rates are zero. We first recall that in the rough Bergomi model the spot variance has the form

$$V_t = \xi_0(t) \exp\left(\eta\sqrt{2H} \int_0^t (t-s)^{H-\frac{1}{2}} \, \mathrm{d}W_s - \frac{1}{2}\eta^2 t^{2H}\right) \tag{3}$$

where W is a Brownian motion, and the spot underlying (discounted) prices are driven by an equation of the form

$$\mathrm{d}S_t = S_t \sqrt{V_t} \, \mathrm{d}B_t \tag{4}$$

with B a Brownian motion such that $d\langle B, W\rangle_t = \rho \, \mathrm{d}t$.

Using a *multi-factor approximation* (in the same line of what we did in the Lifted Heston model) we can approximate

$$\frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^t (t-s)^{H-\frac{1}{2}} \, \mathrm{d}W_s \approx \int_0^t K^n(t-s) \, \mathrm{d}W_s = \sum_{i=1}^n c_i Y_t^i$$

where $dY_t^i = -x_i Y_t^i \, \mathrm{d}t + \mathrm{d}W_t$, $Y_0^i = 0$ for all $1 \leq i \leq n$, and $K^n(t) = \sum_{i=1}^n c_i \mathrm{e}^{-x_i t} \approx \frac{t^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}$.

This suggests to define the following model, that we call the *Lifted Bergomi model*

$$\begin{aligned} V_t^n &= \xi_0(t) \exp\left(\eta\sqrt{2H}\Gamma\left(H + \frac{1}{2}\right) \int_0^t K^n(t-s) \, \mathrm{d}W_s - H\eta^2\Gamma^2\left(H + \frac{1}{2}\right) \int_0^t (K^n(s))^2 \, \mathrm{d}s\right), \\ \mathrm{d}S_t^n &= S_t^n \sqrt{V_t^n} \, \mathrm{d}B_t. \end{aligned} \tag{5}$$

1. Taking inspiration from the work already done in the Lifted Heston model, implement a function Sim_Lifted_Bergomi to simulate trajectories of the Lifted Bergomi model (V_t^n, S_t^n) over an arbitrary interval $[0, T]$.

2. Using the function from the previous part, write a function Call_Price_Lifted_Bergomi to price – via Monte Carlo – a Call option (written on S) with strike K and maturity T in the Lifted Bergomi model. *Remark:* The number of simulations and time-discretization step that you use are left to your discretion.

3. Consider the following parameters

$$H = 0.1, \quad \eta = 1.9, \quad \rho = -0.9, \quad S_0 = 1, \quad \xi_0(t) \equiv 0.02. \tag{6}$$

   For a maturity and an interval of strikes of your choice, and using the function from the previous part, plot a implied volatility smile in the Lifted Bergomi model for $n = 5, 10, 20$ (you can take $r_n = 1 + 10n^{-0.9}$ in the kernel approximation as in the previous section).