

# Reinforcement Learning

## Chapter 5: RL via Policy Gradient

Ali Bereyhi

[ali.bereyhi@utoronto.ca](mailto:ali.bereyhi@utoronto.ca)

Department of Electrical and Computer Engineering  
University of Toronto

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# Natural Policy Gradient: Main Challenges

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + \sqrt{\frac{2d_{\max}}{\nabla_k^\top \mathbf{H}_k^{-1} \nabla_k}} \mathbf{H}_k^{-1} \nabla_k$$

If we update with this rule: we could see

- ① the new point  $\boldsymbol{\theta}_{k+1}$  does not fulfill what we expect, i.e.,
  - ↳ it might do no improvement

$$\mathcal{J}(\pi_{\boldsymbol{\theta}_{k+1}}) \leq \mathcal{J}(\pi_{\boldsymbol{\theta}_k})$$

- ↳ it might violate the constraint

$$\bar{D}_{\text{KL}}(\pi_{\boldsymbol{\theta}_{k+1}} \| \pi_{\boldsymbol{\theta}_k}) > d_{\max}$$

- + But, didn't we solve the optimization problem?!
- Well! We did it approximately not exactly

# Natural Policy Gradient: Main Challenges

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + \sqrt{\frac{2d_{\max}}{\nabla_k^\top \mathbf{H}_k^{-1} \nabla_k}} \mathbf{H}_k^{-1} \nabla_k$$

If we update with this rule: *we need to*

- ② compute **Hessian** of  $\bar{D}_{\text{KL}}(\pi_{\boldsymbol{\theta}} \parallel \pi_{\boldsymbol{\theta}_k})$

↳ *we need to compute all second order derivatives, i.e., for all choices of  $i$  and  $j$*

$$\frac{\partial^2}{\partial \theta_i \partial \theta_j} \bar{D}_{\text{KL}}(\pi_{\boldsymbol{\theta}} \parallel \pi_{\boldsymbol{\theta}_k})$$

↳ *say we use ResNet-50 with  $2.6 \times 10^7$  trainable parameters*

↳ *we need to compute about  $6.6 \times 10^{14}$  derivatives*

↳ *this in principle changes the complexity in orders of magnitude*

# Natural Policy Gradient: Main Challenges

$$\theta_{k+1} = \theta_k + \sqrt{\frac{2d_{\max}}{\nabla_k^\top \mathbf{H}_k^{-1} \nabla_k}} \mathbf{H}_k^{-1} \nabla_k$$

Say we computed the Hessian: we need to

- ③ invert the **Hessian** of  $\mathbf{H}_k \in \mathbb{R}^{D \times D}$ 
  - ↳ the complexity scales as  $\mathcal{O}(D^\xi)$ 
    - ↳  $\xi = 3$  for classical Gauss-Jordan algorithm
    - ↳ it can reduce to  $\xi \approx 2.4$  if we use more optimized algorithms like CW
  - ↳ at the end, this is **computationally very expensive**

## TRPO: Backtracking Line

The first algorithmic approach proposed by Schulman et. al was

*Trust Region Policy Optimization*  $\equiv$  TRPO

*It uses two simple ideas to overcome the mentioned issues*

- *Backtracking line* challenge to get rid of the first issue
- *Conjugate gradient* to overcome the other two

Let's take a look

## TRPO: Backtracking Line

$$\theta_{k+1} = \theta_k + \sqrt{\frac{2d_{\max}}{\nabla_k^\top \mathbf{H}_k^{-1} \nabla_k}} \mathbf{H}_k^{-1} \nabla_k$$

Through analysis it turns out that: the *direction* of natural policy gradient is effective; however, the *step size* might be *overshooting*

- + Why don't we scale it back?
- Sure! We can do this *efficiently* via *backtracking line*

# TRPO: Backtracking Line

BacktrackLine():

- 1: Choose some  $\alpha < 1$ , set  $i = 0$  and start with some  $\delta > d_{\max}$
- 2: **while**  $\delta > d_{\max}$  **do**
- 3:     Replace  $\theta_{k+1}$  with

$$\theta_{k+1} \leftarrow \theta_k + \alpha^i \sqrt{\frac{2d_{\max}}{\nabla_k^\top \mathbf{H}_k^{-1} \nabla_k}} \mathbf{H}_k^{-1} \nabla_k$$

- 4:     Set  $\delta \leftarrow \bar{D}_{\text{KL}}(\pi_{\theta_{k+1}} \| \pi_{\theta_k})$
- 5:     Update  $i \leftarrow i + 1$
- 6: **end while**

- + But we are **only** checking the **constraint**!?
- It turns out that *this could also guarantee policy improvement*

## TRPO: Conjugate Gradient

The next trick in TRPO is to write down the update in a form that can be computed via **conjugate gradient**: let's take a look at the update rule

$$\theta_{k+1} \leftarrow \theta_k + \alpha^i \sqrt{\frac{2d_{\max}}{\nabla_k^\top \mathbf{H}_k^{-1} \nabla_k}} \mathbf{H}_k^{-1} \nabla_k$$

We can define the vector

$$\mathbf{y}_k = \mathbf{H}_k^{-1} \nabla_k$$

It is then easy to say that

$$\begin{aligned} \nabla_k^\top \mathbf{H}_k^{-1} \nabla_k &= \nabla_k^\top \mathbf{H}_k^{-1} \mathbf{I} \nabla_k = \nabla_k^\top \mathbf{H}_k^{-1} \underbrace{\mathbf{H}_k \mathbf{H}_k^{-1}}_{\mathbf{I}} \nabla_k \\ &= \underbrace{\nabla_k^\top \mathbf{H}_k^{-1}}_{\mathbf{y}_k^\top} \mathbf{H}_k \underbrace{\mathbf{H}_k^{-1} \nabla_k}_{\mathbf{y}_k} = \mathbf{y}_k^\top \mathbf{H}_k \mathbf{y}_k \end{aligned}$$

# TRPO: Conjugate Gradient

If we have  $\mathbf{y}_k$ , we could update more easily

$$\theta_{k+1} \leftarrow \theta_k + \alpha^i \sqrt{\frac{2d_{\max}}{\mathbf{y}_k^\top \mathbf{H}_k \mathbf{y}_k}} \mathbf{y}_k$$

Let's see if there is any efficient way to find  $\mathbf{y}_k$  at least approximately

$$\mathbf{y}_k = \mathbf{H}_k^{-1} \nabla_k \rightsquigarrow \mathbf{H}_k \mathbf{y}_k = \nabla_k$$

Now, let's define  $\mathbf{g}(\theta) = \nabla \mathcal{L}_k(\theta)$ : obviously, we have

$$\nabla_k = \mathbf{g}(\theta_k)$$

$$\mathbf{H}_k = \nabla \mathbf{g}(\theta) |_{\theta=\theta_k}$$

# TRPO: Conjugate Gradient

Let's use these facts to expand our equation

$$\begin{aligned}\mathbf{y}_k &= \mathbf{H}_k^{-1} \nabla_k \rightsquigarrow \mathbf{H}_k \mathbf{y}_k = \nabla_k \\ \nabla \mathbf{g}(\boldsymbol{\theta}_k) \mathbf{y}_k &= \mathbf{g}(\boldsymbol{\theta}_k) \\ \nabla (\mathbf{g}(\boldsymbol{\theta}) \mathbf{y}_k) |_{\boldsymbol{\theta}=\boldsymbol{\theta}_k} &= \mathbf{g}(\boldsymbol{\theta}_k)\end{aligned}$$

The above functional equation can be solved for  $\mathbf{y}_k$  via conjugate gradient algorithm<sup>1</sup>, even without knowing the complete  $\mathbf{H}_k = \nabla \mathbf{g}(\boldsymbol{\theta}_k)$ !

In practice, we do the following

- Compute the gradient estimator  $\hat{\nabla}_k$
- Compute a sample Hessian  $\hat{\mathbf{H}}_k$
- Solve  $\hat{\mathbf{H}}_k \mathbf{y}_k = \hat{\nabla}_k$  via conjugate gradient

<sup>1</sup>You could check [this tutorial](#) if you are interested to know more about that

## TRPO: Comments on Estimating Hessian

As long as we need only *an estimate*, we can estimate Hessian by sampling: if we extend our derivative in Assignment 3, we will see

$$\begin{aligned}
 \mathbf{H}_k &= \nabla^2 \bar{D}_{\text{KL}} (\pi_{\theta} \| \pi_{\theta_k}) |_{\theta_k} \\
 &= \int \int \underbrace{d_{\theta_k}(s) \nabla \pi_{\theta_k}(a|s)}_{s \ a} \nabla \log \pi_{\theta_k}(a|s)^T \\
 &= \int \int \underbrace{d_{\theta_k}(s) \pi_{\theta_k}(a|s)}_{\text{distribution}} \underbrace{\nabla \log \pi_{\theta_k}(a|s) \nabla \log \pi_{\theta_k}(a|s)^T}_{\text{sample outer product}} \\
 &= \mathbb{E}_{\pi_{\theta_k}} \left\{ \nabla \log \pi_{\theta_k}(A|S) \nabla \log \pi_{\theta_k}(A|S)^T \right\}
 \end{aligned}$$

This is the *Fisher information matrix* and can be estimated by *sampling*

# TRPO

`TRPO()`:

- 1: *Initiate with  $\theta$  and dampening factor  $\alpha < 1$*
- 2: **while** interacting **do**
- 3:   **for** mini-batch  $b = 1 : B$  **do**
- 4:     Sample  $S_0, A_0 \xrightarrow{R_1} \dots \xrightarrow{R_{T-1}} S_{T-1}, A_{T-1} \xrightarrow{R_T} S_T$  by policy  $\pi_\theta$
- 5:     Compute sample  $U_t = R_{t+1} + \gamma v_{\pi_\theta}(S_{t+1}) - v_{\pi_\theta}(S_t)$  **for**  $t = 0 : T - 1$
- 6:     Compute sample gradient as  $\hat{\nabla}_b = \sum_t U_t \nabla \log \pi_\theta(A_t | S_t)$
- 7:   **end for**
- 8:   Compute estimator as  $\hat{\nabla} = \text{mean}(\hat{\nabla}_1, \dots, \hat{\nabla}_B)$  and a Hessian estimator  $\hat{\mathbf{H}}$
- 9:   Solve  $\hat{\mathbf{H}}\mathbf{y} = \hat{\nabla}$  for  $\mathbf{y}$  via conjugate gradient with multiple iterations
- 10:   Backtrack on a line: find minimum integer  $i$  such that

$$\theta' \leftarrow \theta + \alpha^i \sqrt{\frac{2d_{\max}}{\mathbf{y}^\top \hat{\mathbf{H}} \mathbf{y}}} \mathbf{y}$$

satisfies  $\bar{D}_{\text{KL}}(\pi_{\theta'} \| \pi_\theta) \leq d_{\max}$

- 11:   Update  $\theta \leftarrow \theta'$
- 12: **end while**

# Back to Trust Region PGM

AdvantageGD() :

- 1: Start with some initial  $\theta_0$
- 2: **for**  $k = 1 : K$  **do**
- 3:   Compute the **surrogate function**  $\mathcal{L}_k(\pi_\theta)$
- 4:   Update the parameters as

$$\theta_{k+1} = \underset{\theta}{\operatorname{argmax}} \mathcal{L}_k(\pi_\theta) \text{ subject to } \pi_\theta \text{ and } \pi_{\theta_k} \text{ are close}$$

- 5: **end for**

- + Was this whole “closeness” metric worth it?
- Well! Maybe not!

## Back to Trust Region PGM: Alternative Formulation

Let's check back what was our concern: we wanted to maximize

$$\mathcal{L}_k(\pi_{\theta}) = \mathbb{E}_{\pi_{\theta_k}} \left\{ u_{\pi_{\theta_k}}(S, A) \frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)} \right\}$$

while making sure that

$$\text{Var} \left\{ \hat{\mathcal{L}}_k(\pi_{\theta}) \right\} \propto \frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)}$$

does not explode!

- + Why don't we check the ratio of policies for "closeness"?
- Sounds like a good idea!

## Trust Region PGM: Ratio-Limited Policy Optimization

Let's assume  $\mathcal{C}(\cdot)$  is a function that limits its argument into a restricted interval of variation: then we can define

$$\tilde{\mathcal{L}}_k(\pi_{\theta}) = \mathbb{E}_{\pi_{\theta_k}} \left\{ u_{\pi_{\theta_k}}(S, A) \mathcal{C}\left(\frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)}\right) \right\}$$

If we optimize this *surrogate function*, we *proximally* satisfy what we want

LimitedRatioAdvantageGD() :

- 1: Start with some initial  $\theta_0$
- 2: **for**  $k = 1 : K$  **do**
- 3:   Compute the *surrogate function*  $\tilde{\mathcal{L}}_k(\pi_{\theta})$
- 4:   Update the parameters as

$$\theta_{k+1} = \operatorname{argmax}_{\theta} \tilde{\mathcal{L}}_k(\pi_{\theta})$$

- 5: **end for**

# Proximal Policy Optimization

A common form of this approach is used in

$$\text{Proximal Policy Optimization} \equiv \text{PPO}$$

In this algorithm, we set

$$\tilde{\mathcal{L}}_k(\pi_{\theta}) = \mathbb{E}_{\pi_{\theta_k}} \left\{ \mathcal{L}_k^{\text{Clip}}(S, A, \theta) \right\}$$

where  $\mathcal{L}_k^{\text{Clip}}(S, A, \theta)$  is importance sample of advantage with clipped ratio, i.e.,

$$\mathcal{L}_k^{\text{Clip}}(S, A, \theta) = \min \left\{ u_{\pi_{\theta_k}}(S, A) \frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)}, \ell_{\varepsilon} \left( u_{\pi_{\theta_k}}(S, A) \right) \right\}$$

for the clipping function

$$\ell_{\varepsilon}(x) = \begin{cases} (1 + \varepsilon)x & x > 0 \\ (1 - \varepsilon)x & x \leq 0 \end{cases}$$

# Proximal Policy Optimization

- + This clipping looks quite **complicated**! How does it restrict the domain of variation?
- It is indeed **complicated**, but we may understand it by a simple example

Say we have only one sample trajectory with single **state  $S$**  and **action  $A$** : we hence estimate the restricted **surrogate** as

$$\tilde{\mathcal{L}}_k(\pi_{\theta}) \approx \mathcal{L}_k^{\text{Clip}}(S, A, \theta)$$

Now, say that this sample pair gives **sample advantage**  $u_{\pi_{\theta_k}}(S, A)$ : this can be

- a **positive** advantage
- a **negative** advantage

Let's see output of our restricted surrogate in each case

# Proximal Policy Optimization

- + What happens when we have a **positive** sample advantage?
- In this case, we have

$$\mathcal{L}_k^{\text{Clip}}(S, A, \theta) = u_{\pi_{\theta_k}}(S, A) \min \left\{ \frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)}, 1 + \varepsilon \right\}$$

Since the **advantage is positive**, surrogate is optimized by  $\theta$  that **increases the ratio**: the clipping operator lets us do it only up to some  $\theta$  that

$$\frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)} \leq 1 + \varepsilon$$

if the **ratio** happens to be more, it clips it by  $1 + \varepsilon$

# Proximal Policy Optimization

- + What happens when we have a **negative sample advantage**?
- In this case, we have

$$\mathcal{L}_k^{\text{Clip}}(S, A, \theta) = u_{\pi_{\theta_k}}(S, A) \max \left\{ \frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)}, 1 - \varepsilon \right\}$$

Since the **advantage is negative**, surrogate is maximized by  $\theta$  that **reduces the ratio**: the clipping operator lets us do it only up to some  $\theta$  that

$$\frac{\pi_{\theta}(A|S)}{\pi_{\theta_k}(A|S)} \geq 1 - \varepsilon$$

if the **ratio** happens to lie below, it clips it by  $1 - \varepsilon$

# Proximal Policy Optimization

## Moral of Story

Clipping will keep the maximizer of the *restricted surrogate* such that the new policy described by the maximizer of the *surrogate* has *controlled* variation as compared to  $\pi_{\theta_k}$ . This controlled variation is tuned by  $\varepsilon$

Doing so we are still keeping our new policy within a *trust region*; however,

- We *don't* need to check KL-divergence
- We *don't* need to estimate *Hessian*
- We *don't* need to implement *conjugate gradient* algorithm
- We *don't* need *backtracking line*

Or in a nutshell: the life becomes much easier 😊

# PPO Algorithm

`PPO()`:

- 1: *Initiate with  $\theta$  and learning  $\alpha < 1$*
- 2: **while interacting do**
- 3:   **for mini-batch  $b = 1 : B$  do**
- 4:     *Sample  $S_0, A_0 \xrightarrow{R_1} \dots \xrightarrow{R_{T-1}} S_{T-1}, A_{T-1} \xrightarrow{R_T} S_T$  by policy  $\pi_\theta$*
- 5:     *Compute sample  $U_t = R_{t+1} + \gamma v_{\pi_\theta}(S_{t+1}) - v_{\pi_\theta}(S_t)$  for  $t = 0 : T - 1$*
- 6:   **end for**
- 7:   *Compute the restricted surrogate*

$$\tilde{\mathcal{L}}(\pi_x) = \text{mean}_b \left[ \sum_{t=1}^T \min \left\{ U_t \frac{\pi_x(A_t | S_t)}{\pi_\theta(A_t | S_t)}, \ell_\varepsilon(U_t) \right\} \right]$$

- 8:   **for  $i = 1 : I$  potentially  $I = 1$  do**
- 9:     *Update  $\theta \leftarrow \theta + \alpha \nabla \tilde{\mathcal{L}}(\pi_x) |_{x=\theta}$*
- 10:   **end for**
- 11: **end while**

## Sample Reuse with TRPO and PPO

- + Very nice! You did a great job; however, you did not mention anything about **sample efficiency**!
  - ↳ With TRPO and PPO, we can make sure that our updated policy will be within the vicinity of previous policy
  - ↳ But, we still **sample** a **mini-batch**, apply SGD and drop it!
- Well! As long as we are using TRPO and PPO, we can reuse our **previous samples** for some time! This can help us with **sample efficiency**

In practice, we can use **experience buffer** here as well

- We collect multiple sample trajectory and save them into into a **buffer**
- We treat the **buffer** as a **dataset** and break it into **mini-batches**
- We go **multiple epochs** over this **dataset**
- ★ We **remove** old trajectories **periodically** as our policy is getting far gradually

## Sample Reuse with TRPO and PPO

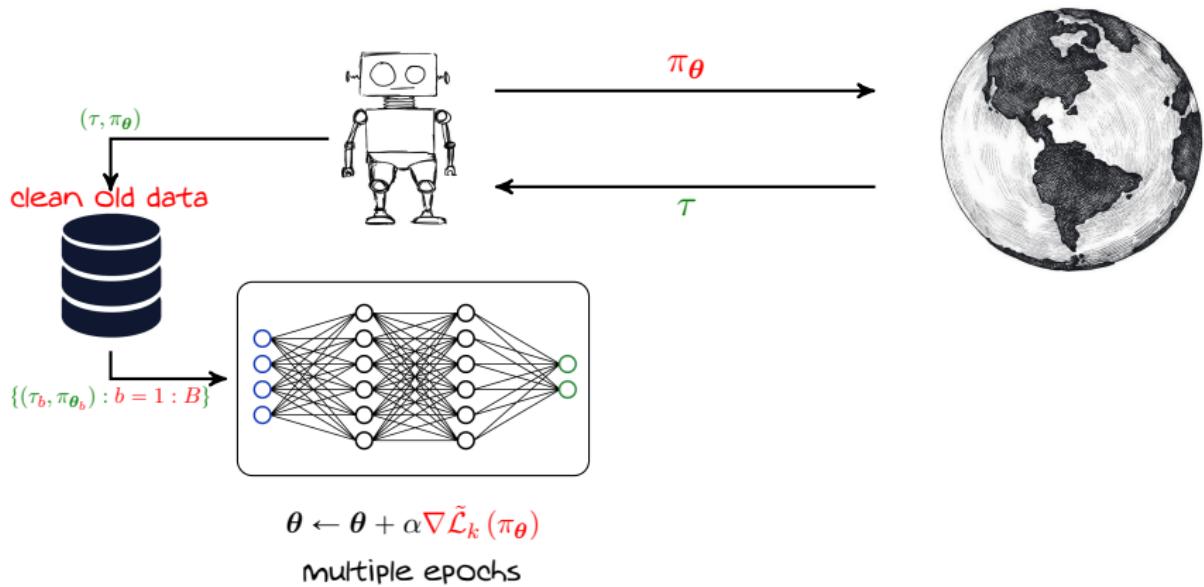
There is a **tiny change** that we need to consider in this case: when we compute the **surrogate** function, we should do the importance sampling with the **policy** that we sampled the trajectory with

For instance, say we sample  $B$  trajectories from the **buffer**

- It might be that each trajectory has been sampled by one policy  $\pi_{\theta_b}$
- ⚠ They are all **close policies** as we clean **buffer** periodically
- If we use PPO, we could compute the **surrogate** as

$$\tilde{\mathcal{L}}(\pi_x) = \text{mean}_b \left[ \sum_{t=1}^T \min \left\{ U_t \frac{\pi_x(A_t | S_t)}{\pi_{\theta_b}(A_t | S_t)}, \ell_\varepsilon(U_t) \right\} \right]$$

# Sample Reuse with TRPO and PPO: Visualization



# PPO Algorithm: Sample Efficient Example

PPO() :

- 1: Initiate with  $\theta$ , learning  $\alpha < 1$ , and an experience buffer with limited size
- 2: while interacting do
- 3:    Sample  $\tau : S_0, A_0 \xrightarrow{R_1} \dots \xrightarrow{R_{T-1}} S_{T-1}, A_{T-1} \xrightarrow{R_T} S_T$  by policy  $\pi_\theta$
- 4:    if experience buffer is full then
- 5:     Remove oldest sample
- 6:    end if
- 7:    Save sample  $(\tau, \pi_\theta)$  into experience buffer as most recent
- 8:    for  $i = 1 : I$  potentially for multiple epochs of buffer do
- 9:     Sample a mini-batch with  $B$  trajectories from experience buffer
- 10:    Compute the restricted surrogate

$$\tilde{\mathcal{L}}(\pi_x) = \text{mean}_b \left[ \sum_{t=1}^T \min \left\{ \frac{\pi_x(A_t|S_t)}{\pi_{\theta_b}(A_t|S_t)}, \ell_\varepsilon(U_t) \right\} \right]$$

- 11:       Update  $\theta \leftarrow \theta + \alpha \nabla \tilde{\mathcal{L}}(\pi_x) |_{x=\theta}$
- 12:    end for
- 13: end while

# Sample Reuse with TRPO and PPO: Final Notes

Though we use *experience reply* as in *DQL*, we should note

- In *DQL*, we are not very restricted with *memory update*
  - ↳ We could keep all samples and reuse them
  - ↳ This was because we could go **totally off-policy** with *DQL*
- In *policy optimization*, we are *strictly* restricted with *memory update*
  - ↳ We could **not** use very old samples *efficiently*
    - ↳ If we use them, we will have *large variance*
  - ↳ We can only *mildly* go *off-policy*
    - ↳ We *keep a sample* and squeeze the its *most possible juice*

## Important Conclusion

In terms of *sample efficiency*, we always have

$$\text{Policy Gradient Methods} \ll \text{DQL}$$

But they could become *more stable* than DQL as they directly control the *policy*

## Last Stop: Actor-Critic Approaches

We are finished with PGMs

- ✓ We know how to train **efficiently** a policy network
  - ↳ We can use TRPO and PPO pretty much in any problem
- ✗ But we assumed that we have **access** to the **value function**
  - ↳ This is **not** really the case in practice!

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We now go for the last chapter, where we learn to

approximate the **value function** via a **value network**

This will complete our box of tools and we are ready to solve any RL problem!

## Efficient Implementation: TorchRL



*In larger RL projects, you might find it easier to have access to some pre-implemented modules: TorchRL does that for you*

- It's a **library** implemented in PyTorch
- It contains lots of **useful modules**, e.g., to implement **experience replay**
- It **does not** give you implemented algorithms
  - ↳ Instead, it gives you modules that you need to implement the algorithm
- It's **compatible** with **Gymnasium**

*Since we often use **PyTorch** for **DL** implementations and **Gymnasium** to implement **environment**, TorchRL is a very efficient toolbox*

## Torch RL: Sample Modules



*Some sample lines of code*

```
from torchrl.collectors import SyncDataCollector
from torchrl.data.replay_buffers import ReplayBuffer
from torchrl.data.replay_buffers.samplers import SamplerWithoutReplacement
from torchrl.data.replay_buffers.storages import LazyTensorStorage
```

## Some Resources

- Take a look at the [introductory presentation](#) by Vincent Moens
- Go over its documentation at [TorchRL page](#)