CMPE250 - A Glimpse of Graph "Theory"

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Question 1

Prove or disprove: The undirected graphs with the same degree sequence are isomorphic. (Degree sequence contains degrees of each vertex in the graph in increasing order.)
Solution: Disprove by counterexample: here are two graphs on 5 vertices with degree sequence = $\{1, 1, 2, 2, 2\}$. $\square$

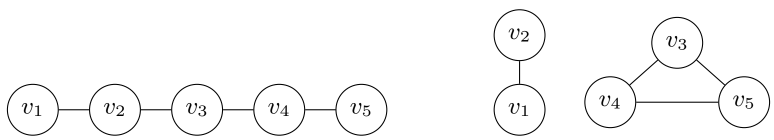

Figure 1: In the first graph, v_1 and v_5 have degree 1 whereas the rest of the vertices has degree 2. Similarly, in the second graph two vertices (v_1, v_2) have degree 1 and three vertices (v_3, v_4, v_5) have degree 2. These graphs are non-isomorphic as the first one is connected but the second one is not.

Question 2

In the good old days, people were having meetings and shaking hands fearlessly. This is a question of these old days, where there were M people in a meeting and N handshakes.

1. Prove that sum of the handshakes of all attendees is an even number.

   Solution: Let us first model the scenario with an undirected graph $G = (V, E)$ such that V represents the people in the meeting and E represents the handshakes. In this modeling, an edge $(u, v) \in E$ corresponds to a handshake between person u and person v.

   With this modeling, the question boils down to proving that the sum of degrees in G (let us call D) is an even number, since summing the handshakes of all attendees is equivalent to summing the degrees of vertices.

To do so, consider an arbitrary edge $e = (u, v) \in E$. e increments $d(u)$ and $d(v)$ by 1, and thus increments D by 2. Since e is an arbitrary edge, this holds for every edge in G, meaning that each edge increments D by 2. Therefore, for any undirected graph G, $D = 2|E|$, is an even number. $\square$

2. Prove that the number of people with odd handshakes is even. (You can use the previous result.)

   Solution: We are asked to prove that the number of people who shook the odd number of hands is even, which corresponds to proving that the number of vertices with odd degree is even.

   Let us define two sets O and $\bar{O}$ where O is the set of odd degree vertices and $\bar{O}$ is the set of even degree vertices in G. Also define, $D_o = \sum_{v \in O} d(v)$, $D_{\bar{o}} = \sum_{v \in \bar{O}} d(v)$, and $x = |O|$. With this formulation, we need to prove that x is even.

   We know by the previous result that D is even. D can be written as $D = D_o + D_{\overline{o}}$, where $D_{\overline{o}}$ is also even as it is a sum of even numbers. Therefore $D_o = D - D_{\overline{o}}$ is even too. Since D_o is a sum of x odd numbers, x must be even for D_o to be even, proving the theorem. $\square$

3. Prove that there are at least two people who shook the same number of hands.

   Solution: The question boils down to proving that there are at least two vertices with the same degree.

   Let us assume for a contradiction that each vertex has a unique degree. Since there are $|V|$ vertices, the degree sequence of such a graph must be $\{0, 1, ..., |V| - 1\}$. Therefore there is a vertex x in G with no edges ($d(x) = 0$) and a vertex y that is adjacent to every other vertex ($d(x) = |V| - 1$). Since such two vertices cannot exist simultaneously, we obtain a contradiction. Hence, our assumption is incorrect and there must be at least two vertices with the same degree in G. $\square$

4. Assume that we can divide the people into two groups such that there is only one handshake between the groups. Prove in this case that, there are at least two people in the meeting who shook the odd number of hands.

   Solution: If there are two groups with exactly one handshake between them, then there are two components in G with exactly one edge connecting them: a bridge. Thus, the question is in fact asking us to prove that when there is a bridge in G, there must be at least two odd degree vertices. We also know that there cannot be only one odd-degree vertex (Question 2.2) in G. So, it is sufficient to prove that there must be odd-degree vertices in G.

   Let us again assume for a contradiction that there is no odd-degree vertex in G. Now consider the two components C_1, C_2 and the bridge uv ($u \in$

$C_1, v \in C_2$) connecting them. They also contain no odd-degree vertices (i.e. all of the vertices in the components have even-degree).

Now let us remove the bridge. This would leave both C_1 and C_2 with one odd degree vertex, u and v, respectively. However, we know that a component of G cannot contain an odd number of odd-degree vertices and arrive at a contradiction. Therefore, our assumption is wrong and there must be some odd-degree vertices in G, if G contains a bridge. $\square$

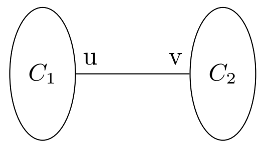

Figure 2: The scenario described in Question 2.4 with two components and a bridge