# Integrable p-wave superconductivity in 2d

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University of Queensland, August 26th, 2010

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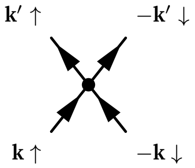
# Outline

# BCS theory [Bardeen, Cooper, Schrieffer 1957]

$2M$ electrons, $L$ energetic levels, $\epsilon_k$
an interaction between zero-momentum pairs of electrons in spin singlet (phonon interaction)

$$H_P = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} - g \sum_{\mathbf{k}\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$




the mean field BCS solution: $\Delta = g \sum_{\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle$:

$$H_{BCS} = \sum_{\mathbf{k}\alpha} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} - \sum_{\mathbf{k}} \left\{ c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \Delta + \text{h.c.} \right\}$$

can be diagonalized [Schrieffer 1957]:

quasi-particle spectrum: $E^2_{\mathbf{k}} = \xi^2_{\mathbf{k}} + \Delta^2_{\mathbf{k}}$
$\Delta$ satisfies the *gap equation*: $\Delta = g \sum_{\mathbf{k}'} \frac{\Delta}{2E_{\mathbf{k}}}$

# BCS theory (II)

- $H_{BCS}$ is not particle-conserving $[H_{BCS}, \hat{N}] \neq 0$. it acts on a Hilbert sp.: $\mathscr{H} = \oplus_N \mathscr{H}_N$
- grand-canonical ensemble: the average number of particles is fixed $\langle \hat{N} \rangle = N_0$ ($\xi_k = \epsilon_k - \mu$, $\mu$ Lagrange multiplier)
- the ground state: $|\psi\rangle = \exp \sum_{\mathbf{k}} g_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |0\rangle$
- $g(\mathbf{k}) = v(\mathbf{k})/u(\mathbf{k})$, with:

$$v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \qquad u_{\mathbf{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

## fluctuations in the particle number

superconductivity in nanograins $\sim$ 5nm [Ralph, Black, Tinkham 1997]
new phenomena: need for a description of pairing in the canonical ensemble

Anderson's conjecture [Anderson 1959]: superconductivity desappears when $\Delta < d$, level spacing

theoretical attempts to solve $H_P$ in finite-$N$:

- number-conserving BCS [Braun, von Delf 1998]
- perturbative R.G. [Berger Halperin 1998]
- Lanczos up to $L = 23$ [Mastellone *et al* 1998]
- DMRG up to $L = 400$ [Sierra, Dukelsky 1999]

the exact solution of the Pairing Hamiltonian [Richardson 1963] was then saved from oblivion
Much more efficient than Lanczos and DMRG,
it was applied to the study of superconducting small grains [Sierra *et al* 2000]

# Richardson exact solution of the Pairing Hamiltonian [Richardson 1963]

the pairing Hamiltonian:

$$H_P = \sum_{\mathbf{k},\alpha} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} - g \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

is exactly solvable. how? some preliminaries:

in terms of the hard-core boson operators:

$$\begin{cases} b_{\mathbf{k}} = c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \\ n_{\mathbf{k}} = b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \\ [b_{\mathbf{k}}, b_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \, (1 - 2n_{\mathbf{k}}) \end{cases}$$

it is (no unpaired electrons):

$$H_P = \sum_{\mathbf{k}} 2\xi_{\mathbf{k}} n_{\mathbf{k}} + g \sum_{\mathbf{k}\mathbf{k}'} b^\dagger_k b_{k'}$$

a general (unnormalized) state of $2M$ particles is: $b^\dagger_{\mathbf{k}_1} \ldots b^\dagger_{\mathbf{k}_M} |\mathrm{vac}\rangle$.

# Richardson exact solution of the Pairing Hamiltonian (II)

an ansatz for the $H_P$ eigenstates is proposed: $|\psi\rangle = B_1^\dagger \ldots B_M^\dagger |\text{vac}\rangle$ where:

$$B_j^\dagger = \sum_{\mathbf{k}} \frac{1}{2\epsilon_k - E_j}\, b_{\mathbf{k}}^\dagger$$

a superposition of a Cooper pair in all the $L$ levels, with a "wavefunction", $\frac{1}{2\epsilon_k - E}$, inspired in the single-pair problem. (in the s-p.p., the $E$'s were the pair energies. here they are chosen such that $H_P|\psi\rangle = E|\psi\rangle$)

We have [Richardson 1963]:

- $|\psi\rangle$ is an eigenstate of $H_P$ if the $E$'s satisfy the *$M$ Richardson equations* (Bethe Ansatz eqns.):

$$\frac{1}{g} - \sum_{k}^{L} \frac{1}{2\epsilon_{\mathbf{k}} - E_m} + \sum_{n \neq m} \frac{2}{E_m - E_n} = 0$$

- the eigenvalues of $H_P$ are $E(g) = \sum_m E_m(g)$

# Richardson exact solution of the Pairing Hamiltonian (II)

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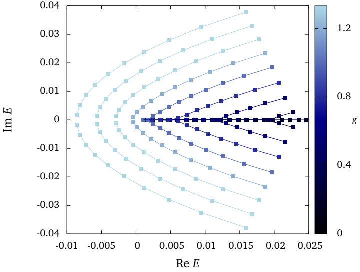
compare $|\psi\rangle$ with the (number-projected) BCS mean-field trial state:

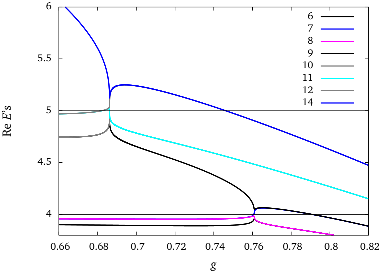
$$= \left[\sum_k g_k b_k^\dagger\right]^M |\text{vac}\rangle$$

# Richardson exact solution of the Pairing Hamiltonian (III)

about the *rapidities* $E_m$:

- $E = \sum_m^M E_m$
- $E_m \to \epsilon_m$, $m$ in a subset of $\{1, \ldots, L\}$, when $g \to 0$
- some of them become complex conjugated pairs for larger $g$



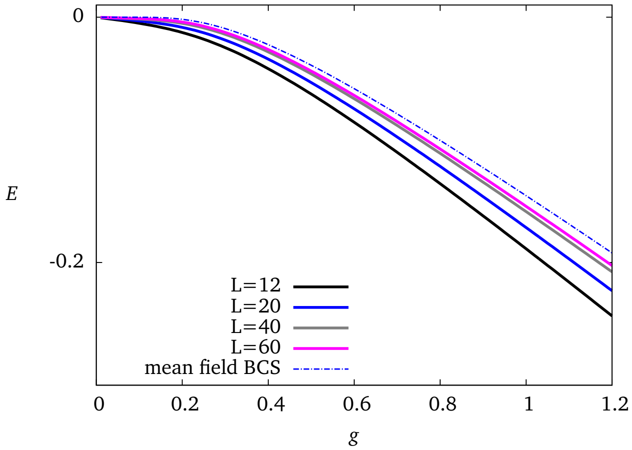

$N = 20$, $L = 40$, $\epsilon_k = k$

$M = 15$, $L = 62$, 2D $\{\mathbf{k}\}$

- for large $g$, $|E_j| >> \epsilon_k$. $\prod_j B_j^\dagger \sim \left[\sum_k b_k^\dagger\right]^M$, a bosonic condensate

# Richardson exact solution of the Pairing Hamiltonian (V)

a comparison with mean field:



# Richardson exact solution: applications

in the context of superconducting grains, Richardson exact solution accounts for the following facts:

- the parity effect in superconducting nanograins:
  unpaired electrons decouple from the interaction: the energy increases in odd-$N$ grains
- BCS predicts an abrupt superconducting/fluctuation-dominated transition at a given $d/\Delta$, even for random $\epsilon_k$ [Ambegaokar 1996]
  the exact solution predicts a smooth crossover: pairing correlation survive $\forall d/\Delta$ [Sierra *et al.* 2000]
- as a consequence, the condensation energy: $E_{\mathrm{GS}} - \langle FS|H_P|FS\rangle$ is always finite (a vanishing at a critical $d/\Delta$ is predicted by mean-field treatments)

# spin triplet pairing [Balian Werthamer 1963]

generalising BCS: the potential scatters spin components ($V$, $p$-wave),

$$H_P = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} - \sum_{\mathbf{k}\mathbf{k}'} \sum_{\alpha\alpha'} V_{\mathbf{k},\mathbf{k}'}\ c^\dagger_{\mathbf{k}\alpha} c^\dagger_{-\mathbf{k}\alpha'} c_{-\mathbf{k}'\alpha'} c_{\mathbf{k}'\alpha}$$

with order parameter: $\Delta_{\mathbf{k};\alpha,\alpha'} = -\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle c_{-\mathbf{k}'\alpha} c_{\mathbf{k}'\alpha'} \rangle$
the mean-field Hamiltonian can be diagonalized. The gap equations:

$$\hat{\Delta}_{\mathbf{k}} = -\sum_{\mathbf{k}'} \frac{\hat{\Delta}_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'}}{2E_{\mathbf{k}'}}, \qquad \hat{\Delta} = \begin{pmatrix} \Delta_{\uparrow\uparrow} & \Delta_{\uparrow\downarrow} \\ \Delta_{\downarrow\uparrow} & \Delta_{\downarrow\downarrow} \end{pmatrix}$$

the order parameter can be parametrised: $\hat{\Delta}_{\mathbf{k}} = i(\boldsymbol{\sigma}\sigma_2).\mathbf{d}(\mathbf{k})$, $\boldsymbol{\sigma}$, Pauli
$\mathbf{d}$, general triplet pairing: $|\text{spin}\rangle = d_x|1,x\rangle + d_y|1,y\rangle + d_z|1,z\rangle$
$\mathbf{d}$ is such that:

- $E^2_{\mathbf{k}} = \epsilon^2_{\mathbf{k}} + \mathbf{d}\mathbf{d}^* \pm |\mathbf{d}\times\mathbf{d}^*|$
- $\mathbf{d}(\hat{R}\mathbf{k})$ as $\hat{R}\mathbf{k}$
- $\mathbf{d}$ is a $S=1$ (triplet) irrep. of $SO(3)_S$: $\hat{R}\mathbf{d}$ as $e^{i\hat{R}\boldsymbol{\sigma}}|\text{spin}\rangle$

## spin triplet pairing (II)

many possible $\mathbf{d}(\mathbf{k})$. some (local minima of $\langle H_{BCS}\rangle$) are shown [Volovik 2003]:

| $\mathbf{d}(\mathbf{k})$ | corresponding $^3$He phase | $\hat{\Delta}$ | pairing |
|---|---|---|---|
| $\mathbf{z}(k_x \pm ik_y)$ | A | $\begin{pmatrix} 0 & k_x \pm ik_y \\ k_x \pm ik_y & 0 \end{pmatrix}$ | $\lvert 1,0\rangle \perp \mathbf{k}$ |
| $\mathbf{k}$ | B | $\begin{pmatrix} -k_x + ik_y & k_z \\ k_z & k_x + ik_y \end{pmatrix}$ | isotropic $\perp \mathbf{k}$ |
| $(\mathbf{x} + i\mathbf{y})(k_x + ik_y)$ | A1 | $\begin{pmatrix} k_x + ik_y & 0 \\ 0 & 0 \end{pmatrix}$ | $\lvert \uparrow\uparrow\rangle \perp \mathbf{k}$ |

for the $\mathbf{d}(\mathbf{k}) = (\mathbf{x} + i\mathbf{y})(k_x + ik_y)$ case:

- (spinless fermions, $\uparrow\uparrow$) $\Delta(\mathbf{k}) = \Delta_0(k_x + ik_y)$
- symmetries of the normal phase: $SO(2)_L \times SO(3)_S \times U(1)_N$ phase
- and of the sym. breaking phase: $U(1)_L \times U(1)_S \times Z_2$

# spin triplet pairing (III)

possible realizations of the $\Delta(\mathbf{k}) = \Delta_0(k_x + ik_y)$ state

- Stronthium Ruthenate $Sr_2RuO_4$ [Mackenzie *et al* 2003]
- $^3$He-$A_1$ phase
  "external magnetic field immediatly below the transition from the normal state"
  [Volovik 2003]
- superfluids of cold atoms in optical traps [Gurarie *et al* 2005]

# 2D $p_x + ip_y$ pairing [Read, Green 2000]

mean-field 2D-Hamiltonian: $2M$ spinless electrons in $L$ levels

$$H_{mf} = \sum_{\mathbf{k}}^{L} \xi_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} - \frac{1}{4} \sum_{\mathbf{k}}^{L} \left\{ c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger \Delta_{\mathbf{k}} + \text{h.c.} \right\}$$

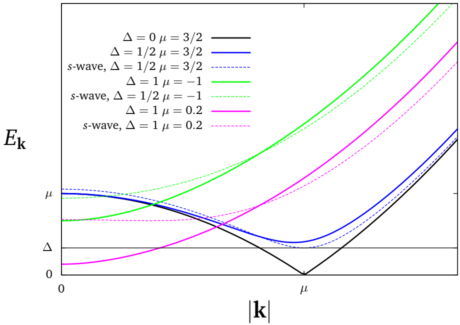
such that the order parameter is:

$$\Delta_{\mathbf{k}} = (k_x - ik_y)\Delta_0, \qquad \Delta_0 = G \sum_{\mathbf{k}} (k_x + ik_y) \langle c_{-\mathbf{k}} c_{\mathbf{k}} \rangle$$

# 2D $p_x + ip_y$ pairing (II)

$H_{mf}$ can be diagonalized
(*s*-wave/*p*-wave comparison):

quasi-particle spectrum:
$E_{\mathbf{k}}^2 = (\mathbf{k}^2 - \mu)^2 + \mathbf{k}^2\ |\hat{\Delta}|^2$




$$\frac{1}{G} = \sum_{|\mathbf{k}|} \frac{\mathbf{k}^2}{2E_{\mathbf{k}}} \qquad \text{(gap eqn.)}$$

$$2M - L + G^{-1} = \mu \sum_{|\mathbf{k}|} \frac{1}{2E_{\mathbf{k}}} \qquad \text{(chem. pot. eqn.)}$$

## a phase transition...

$$E_{\mathbf{k}} = \sqrt{(\mathbf{k}^2 - \mu)^2 + \mathbf{k}^2 \, |\Delta_0|^2}$$

$$|\psi\rangle = \left[ \sum_{|\mathbf{k}|>0} g(\mathbf{k}) c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} \right]^M |\text{vac}\rangle, \qquad g(\mathbf{k}) = \frac{2(E_{\mathbf{k}} - \mathbf{k}^2 + \mu)}{(k_x + i k_y)\Delta_0^*}$$

$$\text{when } |\mathbf{k}| \to 0 \qquad g(\mathbf{k}) \sim \begin{cases} k_x - i k_y, & \mu < 0 \\ 1/(k_x + i k_y), & \mu > 0 \end{cases}$$

the different behaviour of $g$ implies a topological (non-Landau) phase transition at $\mu = 0$ [Read, Green 2000]

$\mu = 0 \Rightarrow 2M = L - 1/G$ or $2x = 1 - 1/g$: the *Read-Green line* of the phase diagram $(x = M/L,\ g = GL)$

## ...which is topological [Volovik 1998]



$$g(\mathbf{k}) = \frac{2(E_{\mathbf{k}} - \mathbf{k}^2 + \mu)}{(k_x + ik_y)\Delta_0^*}$$

$g(\mathbf{k})$ can be viewd as a $S^2 \to S^2$ map

the winding number $w = \frac{1}{\pi}\int\int d\mathrm{Re}[g]\, d\mathrm{Im}[g] \frac{1}{(1+|g|^2)^2}$
defines the homotopy class $\boldsymbol{\pi}(S^2) = \mathbb{Z}$

it turns out that:

| wavefunction | $w$ |
|---|---|
| $g(\mathbf{k})$, $\mu < 0$ | 0 |
| $s$-wave $g(\mathbf{k})$ | 0 |
| $g(\mathbf{k})$, $\mu > 0$ | 1 |

$g$ cannot be deformed continuously from $w = 0$ to $w = 1$: a discontinuity must occur at $\mu = 0$

# the ground state phase diagram [Ibáñez *et al*, 2009]

for $\mu > 0$ and $|\mathbf{k}| \to 0$, $|\psi\rangle$ approaches the Moore-Read state of the FQHE [Read, Green 2000]:

$$|MR\rangle = \left[\sum_{|\mathbf{k}|>0} \frac{1}{k_x + ik_y} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}\right]^M |0\rangle$$

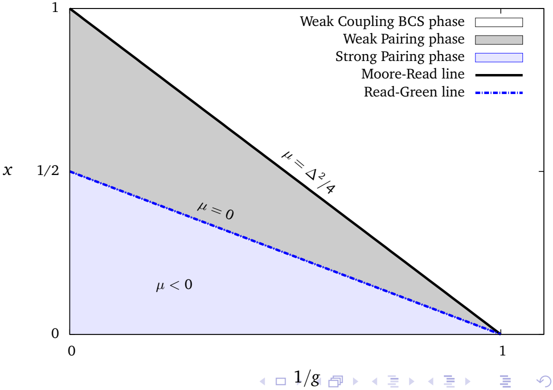
morover, $|\psi\rangle = |MR\rangle \ \forall\ \mathbf{k}$ if $\Delta_0^2 = 4\mu$

$\Delta_0^2 = 4\mu \Rightarrow x = 1 - 1/g$, the *Moore-Read line* of the phase diagram

for this state $\langle H_{mf}\rangle = 0$ !

$w[g]$ in the W. C.-BCS is 0!

the whole p. d. is independent on $\{\mathbf{k}\}, \epsilon_{\mathbf{k}}$ !!



## duality [Ibáñez *et al*, 2009]

consider two states with filling fractions $x_S, \; x_W$, both at $g$, such that

$$x_W + x_S = 1 - 1/g$$

they satisfy:

$$\begin{aligned} \mu_S &= -\mu_W \\ \langle H_{mf} \rangle_S &= \langle H_{mf} \rangle_W \\ \Delta_S^2 &= \Delta_W^2 - 4\mu_W \end{aligned}$$

- the Moore-Read line is dual to the vaccum
- the Read-Green line is self-dual

# vortices and degenerate ground state

a vortex is a solution of the BdG equation

$$\begin{pmatrix} -\mu(\mathbf{r}) & \frac{i}{2}\{\Delta(\mathbf{r}), \partial_x + i\partial_y\} \\ \frac{i}{2}\{\Delta^*(\mathbf{r}), \partial_x - i\partial_y\} & \mu(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix}$$

such that $\Delta(\mathbf{r}) = ie^{i\ell\varphi}|\Delta(r)|$, $\Delta(r)$ vanishing at $r \to 0$

a Bogolubov transformation: $\gamma_n = \int d\mathbf{r}(u_n^*(\mathbf{r})c_n(\mathbf{r}) + v_n^*(\mathbf{r})c_n^\dagger(\mathbf{r}))$, $H_{mf} = \sum_E \gamma_E^\dagger \gamma_E$
the symmetry of $H_{mf} = -\sigma_1 H_{mf}^* \sigma_1$ is such that solutions are paired: $\gamma_E^\dagger = \gamma_{-E}$

- there always exist one $E = 0$, $\Delta(\mathbf{r})$ bounded solution if $\mu > 0$, $\ell$ odd [Read, Green 2000]
- it is $\gamma = \gamma^\dagger$ (a Majorana fermion)
- since #(0-modes) changes $\pm 2$, a single, isolated, 0-mode is topologically protected
- if several vortices at $\mathbf{r}_j$, $r_{jk} >> m/\Delta$, several 0-modes, $\{\gamma_j, \gamma_k\} = \delta_{j,k}$ exist, localized at $\mathbf{r}_j$

# anyons, roughly speaking

"[triplet pairing] is the most elementary way in which a non-Abelian state can emerge as the ground-state of a many-body system" [Nayak *et al* 2008]

main ingredients for this:

- with $2n$ vortices, $n$ fermions $c_n = \gamma_{2n-1} + i\gamma_{2n}$ can be created (a $2^n$ degenerated ground state)
- brading $j$-th vortex around $k$-th vortex is equivalent to a $2\pi$ rotation of $\Delta \to e^{i2\pi}\Delta$
- since $\Delta \to e^{i\varphi}\Delta$ as $c \to e^{i\varphi/2}c$ and $c^\dagger \to e^{-i\varphi/2}c^\dagger$, the exchanging of $j$ by $k$ is equivalent to the change of the sign of one of them:

$$\gamma_k \to \gamma_j$$
$$\gamma_j \to -\gamma_k$$

  this is realized in the $2^n$-dimensional g.s. space by an operator $U_{jk}$ (non-Abelian representation of the braiding)

$\rightarrow$ topologically protected: the realization of the braiding is possible whenever there are zero modes (symmetry of the Hamiltonian) and

$\rightarrow$ when $\gamma$ changes phase under a braiding ($Z_2$ symmetry of $\mathbf{d}$)

# integrable $p_x + ip_y$ [Ibáñez, Links, Sierra, Zhao 2009]

$$H_P = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} - \frac{G}{4} \sum_{\mathbf{k}\mathbf{k}'} (k_x - ik_y)(k'_x + ik'_y) c^\dagger_{\mathbf{k}\alpha} c^\dagger_{-\mathbf{k}\alpha'} c_{-\mathbf{k}'\alpha'} c_{\mathbf{k}'\alpha}$$

the exact solution:

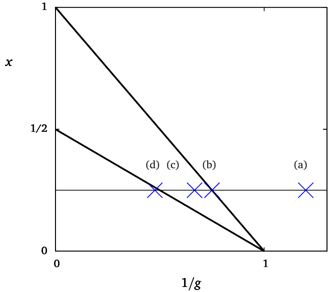
$$|\psi\rangle = \prod_{j=1}^{M} C(y_j)|0\rangle, \quad C(y) = \sum_{|\mathbf{k}|>0} \frac{k_x - ik_y}{\mathbf{k}^2 - y} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}$$

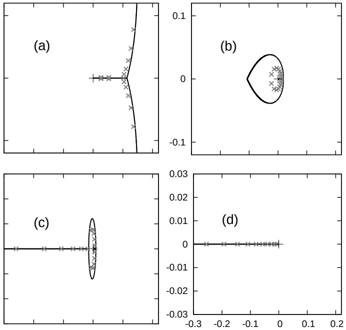
where $y_m$ satisfy Bethe ansatz equations ($2q = 1/g - 1 + 2x - 1/2$):

$$-\frac{1}{2} \sum_{k=1}^{L} \frac{1}{y_m - \epsilon_k^2} - \frac{q}{y_m} + \sum_{j \neq m}^{M} \frac{1}{y_m - y_j} = 0, \qquad m = 1, \ldots, M$$

and $E = (1 + G) \sum_m^M y_m$

# Bethe roots



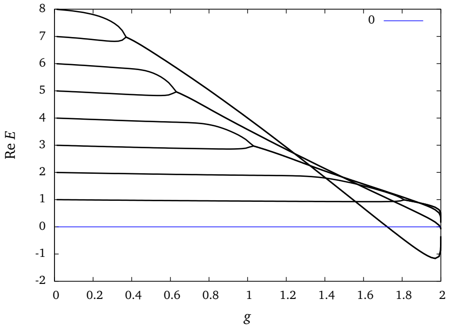

$L = 64$, $M = 16$, 2D $\{\mathbf{k}\}$ $(\rightarrow)$

continuous lines are Gaudin arcs $\Gamma$ such that $(L, M \to \infty, x < \infty)$:

$$\int_\Omega d\varepsilon \frac{\rho(\varepsilon)}{\varepsilon - y} - \frac{q}{y} - P\int_\Gamma |dy'| \frac{r(y')}{y' - y} = 0, \quad \forall y \in \Gamma$$

where $2\int_{\Omega\subset\mathbb{R}} d\varepsilon\, \rho(\varepsilon) = L$ and $\int_\Gamma |dy|\, r(y) = M$

# the Moore-Read line revisited



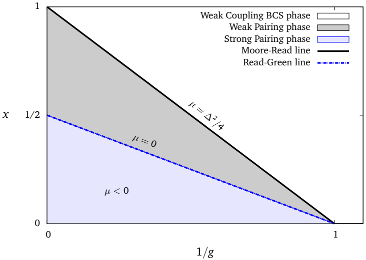

in the Moore-Read line, $y_m = 0 \;\forall m$

$$|\psi\rangle = |MR\rangle = \left[\sum_{|\mathbf{k}|>0} \frac{1}{k_x + ik_y} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}\right]^M |\mathrm{vac}\rangle$$

equivalence of mean-field finite-$M$ and exact solution descriptions in the Moore-Read line

$$|\psi_{mf}\rangle = \left[\sum_{|\mathbf{k}|>0} g_{MR}(\mathbf{k}) c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}\right]^M |\mathrm{vac}\rangle$$

## duality revisited




for two states $|S\rangle$, $|W\rangle$ in the S.P. and W.P. rspctvly, such that:

$$M_W + M_S = L - 1/G, \text{ or:}$$

$$x_W + x_S = 1 - 1/g$$

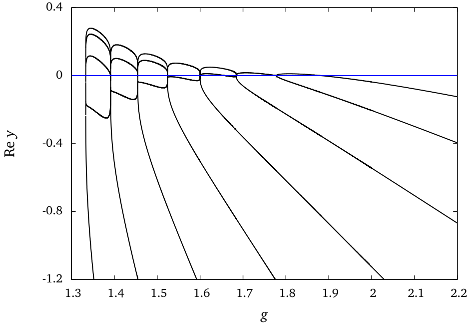
(the m.f. duality relation)

it is:

- $M_W - M_S$ of the $M_W$ roots of $|W\rangle$ @$(x_W, g)$ are zero (Moore-Read pairs)
- the remaining $M_S$ satisfy the same B. A. equations of the $M_S$ roots of $|S\rangle$ @$(x_S, g)$
- this means (*dressing*) $|W\rangle = \left[\sum_{|\mathbf{k}|>0} g_{MR}(\mathbf{k}) c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}\right]^{M_W - M_S} |S\rangle$
- $|S\rangle$ and $|W\rangle$ have the same energy: the duality observed in mean-field approximation

## duality revisited (II)

the $M_W - M_S$ pairs with zero energy are @ the *dressing points* $(x_W, g)$
$g$ rational!!
non well-defined continuum limit $M, L \to \infty$




$L = 24$, $M = 8$, 2D $\{\mathbf{k}\}$, S. P. & W. P. phases
see the animation!

# winding numbers revisited

the exact wavefunctions for 1 and $M$ pairs :

$$g_1(\mathbf{k}, y) = \frac{k_x + ik_y}{y - E},$$

$$g_M(\mathbf{k}_1, \ldots, \mathbf{k}_M; E_1, \ldots, E_M) = \sum_{\pi} \prod_{j}^{M} g_1(\mathbf{k}_{\pi(j)}, E_j)$$

to construct an $S^2 \to S^2$ map from $g_1$ we define [Ibáñez *et al* 2009]:
$\psi_M(\mathbf{k}; E_1, \ldots, E_M) = g_M(\mathbf{k} + \mathbf{c}_1, \ldots, \mathbf{k} + \mathbf{c}_M; E_1, \ldots, E_M)$,
where $\mathbf{c}_j$ are constants

we observe that, for this $S^2 \to S^2$ map:

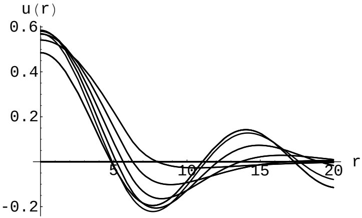
$$w = P$$

$P$ being the number of zero energies of the state

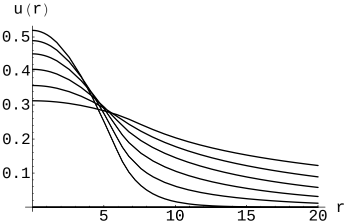
dressing points in the W. P. phase & the Moore-Read line are the only topologically non-trivial points in the phase diagram

# phase diagram dependence of vortices

the Bogolubov-DeGennes equations ($H_{mf}$ Hamiltonian) are solved for a vortex which vanishes inside a core $r > 0$. $u(r)$ is plotted in the W. C.-BCS, W. P. phases [Ibáñez *et al* 2009]:




$x = 1/4$, $g = 0.3$ to $0.13$




$x = 1/4$, $g = 1.4$ to $1.9$

different qalitative behaviour in different phases
the Moore-Read boundary line plays a role also in this context

# perspective: quenching

integrability approach to quantum dynamics after a quench [Faribault *et al* 2009]

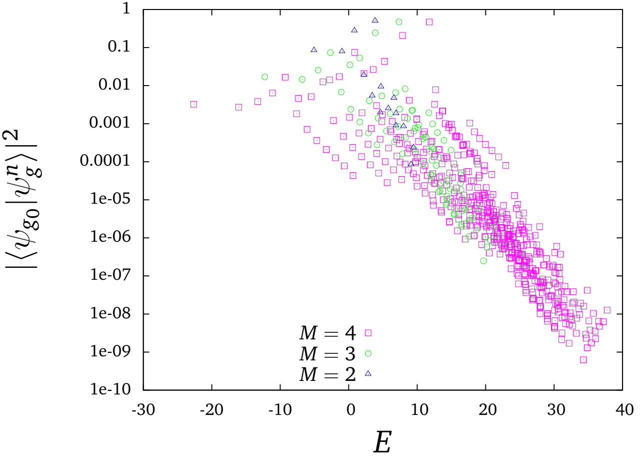
**Bethe Ansatz approach to quench dynamics in the Richardson model**

Alexandre Faribault, Pasquale Calabrese, and Jean-Sébastien Caux

ABSTRACT. By instantaneously changing a global parameter in an extended quantum system, an initially equilibrated state will afterwards undergo a complex non-equilibrium unitary evolution whose description is extremely challenging. A non-perturbative method giving a controlled error in the long time limit remained highly desirable to understand general features of the quench induced quantum dynamics. In this paper we show how integrability (via the

- quench from $g_0$ to $g$. evolution: $|\psi(t)\rangle = e^{itH_g}|\psi^{\mu}_{g_0}\rangle$ hence:
  $|\psi(t)\rangle = \sum_{\nu} e^{itE^{\nu}_g}\langle\psi^{\nu}_g|\psi^{\mu}_{g_0}\rangle|\psi^{\nu}_g\rangle$
  hence: $\langle O(t)\rangle = \sum_{n,n'} e^{it(\omega_{n'}-\omega_n)}\langle\psi^{n}_g|\psi^{\mu}_{g_0}\rangle\langle\psi^{\mu}_{g_0}|\psi^{n'}_g\rangle\langle\psi^{n'}_g|O|\psi^{n}_g\rangle$
- the Bethe Ansatz approach allows for the computation of scalar products [Slanov 1989]. expectations values of operators are computed through the QISM [Kitanine *et al* 1999], [Ibáñez *et al* 2009]
- effective Hilbert space truncations

# quenchig: a work in progress




$x = 1/4$, $g_0$ in the W. C.-BCS phase, $g$ in the S. P. phase

# conclusions

- novel phenomenology: the topologically non-trivial Moore-Read boundary line separates the W. P. from the topologically trivial W. C.-BCS phase
- the dressing points present in the W. P. phase are the only topologically non-trivial points in the phase diagram
- the different topological properties of both "phases" reflects in the different vortex behavior exhibited by each of them