1) Review typed in jupyter notebook.

2) a) $\vec{F} = k(x\vec{e}_1 + 2y\vec{e}_2 + 3z\vec{e}_3)$ k is a constant so I'll factor that in at the end.

technically k. this

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = \frac{\partial}{\partial y}3z\hat{i} + \frac{\partial}{\partial z}x\hat{j} + \frac{\partial}{\partial x}2y\hat{k} - \frac{\partial}{\partial y}x\hat{k} - \frac{\partial}{\partial z}2y\hat{i} - \frac{\partial}{\partial x}3z\hat{j}$$
$$= 0 + 0 + 0 - 0 - 0 - 0 = 0$$

$\vec{\nabla}\times\vec{F} = 0 \Rightarrow$ $\boxed{\text{Conservative force}}$

b) $\vec{F} = y\vec{e}_1 + x\vec{e}_2 + 0\vec{e}_3$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = 0 + \frac{\partial}{\partial z}y\hat{j} + \frac{\partial}{\partial x}x\hat{k} - \frac{\partial}{\partial y}y\hat{k} - \frac{\partial}{\partial z}x\hat{i} - 0$$
$$= 0 + 0 + 1\hat{k} - 1\hat{k} - 0 - 0 = 0$$

$\vec{\nabla}\times\vec{F} = 0 \Rightarrow$ $\boxed{\text{Conservative force}}$

c) $\vec{F} = k(-y\vec{e}_1 + x\vec{e}_2 + 0\vec{e}_3)$ again, I'll factor in the k at the end.

$$\vec{\nabla}\times\vec{F} = k\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 0 + \frac{\partial}{\partial z}\cdot y\cdot\hat{j} + \frac{\partial}{\partial x}x\hat{k} - \frac{\partial}{\partial y}\cdot -y\cdot\hat{k} - \frac{\partial}{\partial z}x\hat{i} - 0$$
$$= 0 + 0 + \hat{k} - -\hat{k} - 0 - 0 = 2\hat{k}$$

factoring back in the k, we get $\boxed{\vec{\nabla}\times\vec{F} = 0\hat{i} + 0\hat{j} + 2k\hat{k} \neq 0 \Rightarrow \underline{\text{not}} \text{ conservative force}}$

2d) We need to find V for parts a and b.

★ part a: $\vec{F} = k(x\vec{e}_1 + 2y\vec{e}_2 + 3z\vec{e}_3)$. ~~we need~~ ~~[illegible]~~

we need $F = -\nabla V$ i.e. $-\frac{\partial}{\partial x} = kx$, $-\frac{\partial}{\partial y} = k2y$ $-\frac{\partial}{\partial z} = k3z$.

In this we can just integrate.

$$\Rightarrow \boxed{V = -k\left(\tfrac{1}{2}x^2 + y^2 + \tfrac{3}{2}z^2\right)}$$

To confirm, $-\nabla V = -\left(\frac{\partial}{\partial x}V\vec{e}_1 + \frac{\partial}{\partial y}V\vec{e}_2 + \frac{\partial}{\partial z}V\vec{e}_3\right) = -\left(-k\left(x\vec{e}_1 + 2y\vec{e}_2 + 3z\vec{e}_3\right)\right)$

$= k(x\vec{e}_1 + 2y\vec{e}_2 + 3z\vec{e}_3) = \vec{F}$ ✓

★ part b: $\vec{F} = y\vec{e}_1 + x\vec{e}_2 + 0\vec{e}_3 = y\vec{e}_1 + x\vec{e}_2$.

we need $-\frac{\partial}{\partial x} = y$ and $-\frac{\partial}{\partial y} = x$.

So we get $\boxed{V = -xy}$

To confirm, $-\nabla V = -\left(\frac{\partial}{\partial x}V\vec{e}_1 + \frac{\partial}{\partial y}V\vec{e}_2 + \frac{\partial}{\partial z}V\vec{e}_3\right) = -(-ye_1 - xe_2 + 0\vec{e}_3)$

$= y\vec{e}_1 + x\vec{e}_2 + 0e_3 = \vec{F}$

✓

3) a) See jupyter notebook! Looks like

b) Equilibrium points where $F=0$. Since $F=-\nabla V$, using $V=V_0\left(\left(\frac{a}{r}\right)^{12}-\left(\frac{b}{r}\right)^{6}\right)$

Technically ignoring direction here

$$F=-\frac{d}{dr}\left[V_0\left(\left(\frac{a}{r}\right)^{12}-\left(\frac{b}{r}\right)^{6}\right)\right]=-V_0\cdot\frac{d}{dr}\left(a^{12}r^{-12}-b^6r^{-6}\right)=0$$

$$=-V_0\cdot\left(\frac{-12a^{12}}{r^{13}}+\frac{6b^6}{r^7}\right)=\frac{12V_0a^{12}}{r^{13}}-\frac{6V_0b^6}{r^7}=0$$

$$\frac{12V_0a^{12}}{r^{13}}=\frac{6V_0b^6}{r^7}\Rightarrow\frac{2a^{12}}{r^{13}}=\frac{b^6}{r^7}$$

$$\Rightarrow 2a^{12}=b^6r^6\Rightarrow r^6=\frac{2a^{12}}{b^6}\Rightarrow\boxed{r=(2)^{1/6}\cdot\frac{a^2}{b}\approx 1.122\frac{a^2}{b}}$$

This is a stable equilibrium point, since the force will push nearby particles towards this point.

c) I already found most of it from $-\frac{dU}{dr}=V_0\left(\frac{12a^{12}}{r^{13}}-\frac{6b^6}{r^7}\right)$

This is our magnitude. Now we just need to consider direction to make the force vector.

total vector divided by magnitude

$$\vec{F}=V_0\left(\frac{12a^{12}}{r^{13}}-\frac{6b^6}{r^7}\right)\cdot\frac{\vec{r}}{\|\vec{r}\|}$$

Assuming just 2 particles, this gives:

$$\boxed{\vec{F}=V_0\left(\frac{12a^{12}}{|r_i-r_j|^{13}}-\frac{6b^6}{|r_i-r_j|^7}\right)\cdot\frac{\vec{r}_i-\vec{r}_j}{|\vec{r}_i-\vec{r}_j|}}$$

$\boxed{\text{Yes, this is a conservative force}}$ It depends only on position, and can be written in terms of a single dimension $\vec{r}$.

Note: I'm combining 4a and 4b, since its intuitive to scale in the analysis.

4a) $V(x) = k\frac{x^2}{2}$. No driving force. Drag force $-bv$.
and 4b) ↗ This is standard for harmonic oscillator. Since $F = -\nabla V$, this gives $\boxed{F = -kx}$ ⇓ force from potential

As usual, this gives the differential equation below. Note we have the additional force $\boxed{F = -bv}$

from $F=ma$ with both forces above $\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

dividing by m gives $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \underbrace{\frac{k}{m}}_{\omega_0^2}x = 0$

4b part → Setting $\omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$, $\gamma = \frac{b}{2m\omega_0}$ and $\tau = \omega_0 \cdot t$, we can rewrite the above equation as

*scaled equation $\boxed{\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x = 0}$

~~note that we divided by ω_0 here~~

Note I skipped intermediate $\omega_0^2\frac{d^2x}{d\tau^2} + \frac{b\omega_0}{m}\frac{dx}{d\tau} + \omega_0^2 x = 0$, where you then divide by ω_0^2, giving this

We've now isolated an expression for the acceleration

This above circled equations yields solution of the form:

$$x(\tau) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = A\cos(\tau) + B\sin(\tau),$$

didn't necessarily need this { or rewritten in ~~terms~~ terms of exponentials, $x(\tau) = Ce^{i\tau} + De^{-i\tau}$

From diff eq standards, this general solution can be ~~[illegible]~~ of form

$$x(\tau) = Ae^{r\tau}$$

Plugging this back into scaled differential equation gives

$$r^2 \cdot Ae^{r\tau} + 2\gamma r Ae^{r\tau} + Ae^{r\tau} = 0$$

$$Ae^{r\tau}(r^2 + 2\gamma r + 1) = 0$$

$A=0$ is boring, same with $e^{r\tau} = 0$.

looking at $r^2+2\gamma r+1=0$ yields $\boxed{r=-\gamma\pm\sqrt{\gamma^2-1}}$.

$r_1=-\gamma+\sqrt{\gamma^2-1}$

$r_2=-\gamma-\sqrt{\gamma^2-1}$

Considering the form earlier, $x(\tau)=A_1e^{r_1\tau}+A_2e^{r_2\tau}=A_1e^{-\gamma\tau+\tau\sqrt{\gamma^2-1}}+A_2e^{-\gamma\tau-\tau\sqrt{\gamma^2-1}}$

The $e^{-\gamma\tau}$ gives exponential decay. Note $\gamma>0$.

<u>critical damping</u>: $\gamma=1$, causes the $\tau\sqrt{\gamma^2-1}$ term to equal 0, leaving us with just ~~the exponential decay~~. $x(\tau)=A_1e^{-\gamma\tau}+A_2e^{-\gamma\tau}\cdot\tau$

quickest decay!

added since otherwise we can't have 2 initial conditions. Comes back to diff eq requirements. Can show $\tau e^{-\gamma\tau}$ is also a solution.

<u>over damping</u>: $\gamma>1$, causes the $\tau\sqrt{\gamma^2-1}$ to be >0. This value is added/subtracted in the exponents. Exponential decay. More gradual than critical damping.

supposed to look exponential. My drawing skills aren't great

$A_1e^{-(\gamma+\sqrt{\gamma^2-1})\tau}+A_2e^{-(\gamma-\sqrt{\gamma^2-1})\tau}$

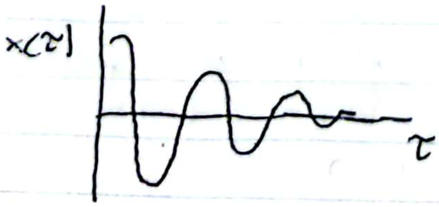
<u>under damping</u>: $\gamma<1$. $\tau\sqrt{\gamma^2-1}$ = imaginary # times τ. let $c=\sqrt{\gamma^2-1}$

$r_1=-\gamma+ic$, $r_2=-\gamma-ic$

$x(\tau)=A_1e^{-\gamma\tau}e^{ic\tau}+A_2e^{-\gamma\tau}e^{-ic\tau}$

Considering rewriting imaginary exponentials with sines and cosines, this comes out to be

$x(\tau)=B_1e^{-\gamma\tau}\cos(c\cdot\tau)+B_2e^{-\gamma\tau}\sin(c\cdot\tau)$, giving us oscillatory motion with exponential decay.



4d) With the driving force, our diff eq now looks like

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t)$$

We can again scale this using $\omega_0 = \sqrt{\frac{k}{m}}$ and $\tau = \omega_0 t$.

Note that $F_0\cos(\omega t)$ can be written as $F_0\cos(\frac{\omega}{\omega_0}\tau)$ since $\tau = \omega_0 t$.

Using scaling the same as in part a with the exception of the driving force written above, we get

$$m\omega_0^2\frac{d^2x}{d\tau^2} + b\omega_0\frac{dx}{d\tau} + kx = F_0\cos(\frac{\omega}{\omega_0}\tau)$$

divide by $\omega_0^2 m$

$$\Rightarrow \frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x = \frac{F_0}{m\omega_0^2}\cos(\frac{\omega}{\omega_0}\tau)$$ where we have the same $\gamma = \frac{b}{2m\omega_0}$

Define $\tilde{\omega} = \frac{\omega}{\omega_0}$ and $\tilde{F}_0 = \frac{F_0}{m\omega_0^2}$ to write as

$$\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x = \tilde{F}_0\cos(\tilde{\omega}\tau)$$

where r_1 and r_2 are the same as before

At this point we guess solution of the form $\boxed{x(\tau) = \underbrace{D\cos(\tilde{\omega}\tau - \delta)}_{x_p(\tau)} + \underbrace{c_1e^{r_1\tau} + c_2e^{r_2\tau}}_{\text{general solution}}}$

making guess on this

Plug $x_p(\tau)$ back into diff eq gives: $D[-\omega^2\cos(\tilde{\omega}\tau - \delta) - 2\gamma\tilde{\omega}\sin(\tilde{\omega}\tau - \delta) + \cos(\tilde{\omega}\tau - \delta)] = \tilde{F}_0\cos(\tilde{\omega}\tau)$

rewrite as $D[\underbrace{(-\tilde{\omega}^2\cos\delta + 2\gamma\tilde{\omega}\sin\delta + \cos\delta)}_{\text{must equal } \tilde{F}_0}\cos(\tilde{\omega}\tau) + \underbrace{(-\tilde{\omega}^2\sin\delta - 2\gamma\tilde{\omega}\cos\delta + \sin\delta)}_{\text{must equal 0, since no sine}}\cdot\sin(\tilde{\omega}\tau)]$
$= \tilde{F}_0\cos(\tilde{\omega}\tau)$

Match sine and cosine part. On RHS we have only cosine

Using orthogonality, ★ $D[-\tilde{\omega}^2\cos\delta + 2\gamma\tilde{\omega}\sin\delta + \cos\delta] = \tilde{F}_0$

$$-\tilde{\omega}^2\sin\delta - 2\gamma\tilde{\omega}\cos\delta + \sin\delta = 0$$

$$-\tilde{\omega}^2\tan\delta - 2\gamma\tilde{\omega} = -\tan\delta \Rightarrow \boxed{\tan\delta = \frac{2\gamma\tilde{\omega}}{1-\tilde{\omega}^2}}$$

rewrite → $\sin\delta = \frac{2\gamma\tilde{\omega}}{\sqrt{4\gamma^2\tilde{\omega}^2 + (1-\tilde{\omega}^2)^2}}$ using trig equivalencies.

$$\cos\delta = \frac{(1-\tilde{\omega}^2)^2}{\sqrt{4\gamma^2\tilde{\omega}^2+(1-\tilde{\omega}^2)^2}}$$

Put $\sin\delta$ and $\cos\delta$ back into ☆ equation. $\Rightarrow$ $\boxed{D = \frac{\tilde{F}_0}{\sqrt{(1-\tilde{\omega}^2)^2+4\tilde{\omega}^2\gamma^2}}}$

So finally we have our solution $x_p(\tau) = D\cos(\tilde{\omega}\tau-\delta)$

when $\delta = \tan^{-1}\left(\frac{2\gamma\tilde{\omega}}{1-\tilde{\omega}^2}\right)$ and $D = \frac{\tilde{F}_0}{\sqrt{(1-\tilde{\omega}^2)^2+4\tilde{\omega}^2\gamma^2}}$

What we expect to see this this particular solution dominating after sufficient time has passed, since the general solution $C_1 e^{r_1\tau} + C_2 e^{r_2\tau}$ will exhibit exponential decay (recall $r_1, r_2 = -\gamma \pm \sqrt{\gamma^2-1}$). since $\gamma > 0$, both r_1 & $r_2 < 0$, so exponential decay.

We'll check for this in our numerical solution.