1a) Conservative forces are defined as depending only on the position of the object on which it acts, and that the work done by the force between any two points is the same regardless of the path taken from the first point to the second. It's necessary that the curl of the force equals zero.

1b) Let some conservative force $\vec{F}$ act on a particle moving from any points $\vec{r}_1$ to $\vec{r}_2$. Let $\vec{r}_0$ be the reference point where $U(\vec{r}_0)=0$, with $U(\vec{r})$ being defined as $U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$

$$W(\vec{r}_0 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_1) + W(\vec{r}_1 \to \vec{r}_2) \quad \text{since } \vec{F} \text{ is conservative.}$$

$$W(\vec{r}_1 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_2) - W(\vec{r}_0 \to \vec{r}_1)$$

$$\Rightarrow W(\vec{r}_1 \to \vec{r}_2) = -\left(U(\vec{r}_2) - U(\vec{r}_1)\right) \quad \text{since } W(\vec{r}_0 \to \vec{r}_2) = -U(\vec{r}_2)$$
$$\text{and } W(\vec{r}_0 \to \vec{r}_1) = -U(\vec{r}_1).$$

$$-\left(U(\vec{r}_2) - U(\vec{r}_1)\right) = -\Delta U \quad \text{so} \quad W(\vec{r}_1 \to \vec{r}_2) = -\Delta U$$

The work KE-Theorem tells us $\Delta T = W(\vec{r}_1 \to \vec{r}_2)$, so

change in kinetic energy

$$\Delta T = -\Delta U$$

And $\Delta T + \Delta U = 0$

Since $E = T + U$, $\Delta E = \Delta T + \Delta U = 0$

So energy is conserved with a conservative force.

1c) N objects with only internal two-body forces. System isolated $\Rightarrow \vec{F}_{ext} = 0$

$$\vec{P} = \sum_{i=1}^{N} \vec{P}_i = \sum_{i=1}^{N} m_i \vec{v}_i$$

We know from Newton's 3rd law that $\vec{F}_{ij} = -\vec{F}_{ji}$

total force on particle:

For any particle i, $\vec{F}_i = \frac{d}{dt} m_i \vec{v}_i = \sum_{j \neq i}^{N} \vec{F}_{ij}$

For all particles $\underbrace{\frac{d}{dt} \sum_{i=0}^{N} m_i \vec{v}_i}_{\text{change in system's momentum}} = \sum_{i=0}^{N} \sum_{j \neq i}^{N} \vec{F}_{ij} = \sum_{i}^{N} \sum_{j > i} (\vec{F}_{ij} + \vec{F}_{ji}) = 0$

using Newton's 3rd law $\vec{F}_{ij} + \vec{F}_{ji} = 0$

This step is since $\vec{F} = m\vec{a} = m \frac{d\vec{v}}{dt}$
To get $\vec{F}_{tot}$ you take $\vec{F}_{ij}$ for every particle $j \neq i$.

Thus $\frac{d}{dt} \sum_{i=0}^{N} m_i \vec{v}_i = 0$, which means the net change in linear momentum for an isolated system is zero, meaning linear momentum is conserved

i.e. $P_{initial} = P_{final}$ since $\frac{dP}{dt} = 0$.

2a) For a single object with external forces only:

Angular momentum $\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$. It's the cross product of the position vector and the momentum vector for the object.

Torque $\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$. The torque is the change in angular momentum over time, which is equivalent to the cross product between the object's position vector and the net force vector on the object.

2b) Each particle has angular momentum $\ell_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$

Total Angular Momentum $\vec{L} = \sum_{\alpha=1}^{N} \ell_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha$

It's merely the sum of the angular momentum from each particle in the system.

Differentiating $\vec{L}$ we get $\dot{\vec{L}} = \sum_\alpha \dot{\vec{\ell}}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$

where $\vec{F}_\alpha = \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{ext}$ where $\vec{F}_{\alpha\beta}$ is the force on α from β.

$$\dot{\vec{L}} = \sum_\alpha \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$$

$$\Rightarrow \sum_\alpha \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = \sum_\alpha \sum_{\beta \neq \alpha} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta + \vec{F}_{\beta\alpha})$$

By Newton's 3rd Law, $F_{\alpha\beta} = -F_{\beta\alpha}$ which means the above equals

$$\sum_\alpha \sum_{\beta \neq \alpha} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta}$$

Since $(\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} = 0$, the circled double sum above $= 0$.

Thus the system's torque $\tau_{sys} = \frac{d}{dt}\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$ = net external torque.

2c) Internal forces only.

I'm not quite sure what this question is asking. I showed in part 2b that $\frac{d}{dt}\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$ = net external torque because

$$\frac{d}{dt}\vec{L} = \sum_\alpha \sum_{b \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha b} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$$

where $\sum_\alpha \sum_{b \neq a} \vec{r}_\alpha \times \vec{F}_{\alpha b} = \sum_\alpha \sum_{\beta > \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha b} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}$

$= \sum_\alpha \sum_{\beta > \alpha} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} = 0$ since $\vec{r}_\alpha - \vec{r}_\beta$ is antiparallel to $\vec{F}_{\alpha\beta}$.

Thus $\frac{d}{dt}\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$. Since we have only internal force, $\vec{F}_\alpha^{ext} = 0$

$\Rightarrow \frac{d}{dt}\vec{L} = 0 \Rightarrow$ angular momentum conserved.

~~3 Ipex~~ The form that allows angular momentum to be conserved is that the force must be proportional to $\vec{r}_\alpha - \vec{r}_\beta$

3) Mass m. $V(x,y,z) = Ae^{-\frac{x^2+z^2}{2a^2}}$

$$\vec{F} = -\nabla V = -\left(\left(Ae^{-\frac{x^2+z^2}{2a^2}} \cdot \frac{-x}{a^2}\right)\vec{e}_1 + \left(Ae^{-\frac{x^2+z^2}{2a^2}} \cdot \frac{-z}{a^2}\right)\vec{e}_3\right)$$

$$\vec{F} = \frac{xA}{a^2} e^{-\frac{x^2+z^2}{2a^2}} \vec{e}_1 + \frac{zA}{a^2} e^{-\frac{x^2+z^2}{2a^2}} \vec{e}_3$$

3a) Yes, energy is conserved. We know this since the force only depends on the position of the object and the work around any closed path must be zero because $W(\vec{r}_1 \to \vec{r}_1) = -[U(\vec{r}_1) - U(\vec{r}_1)] = 0$. For proof, see curl calculated below!

3b) $\vec{P}_y$ is conserved but $\vec{P}_x$ and $\vec{P}_z$ are not conserved. $\vec{P}_y$ must be conserved because our force does not act in the $\hat{y}$ direction and $\frac{d}{dt}\vec{P} = \vec{F}_{ext}$. $\vec{P}_x$ and $\vec{P}_z$ are not conserved because our force acts in the directions which will cause $\vec{v}_x$ and $\vec{v}_z$ to change, which will in turn change $\vec{P}_x$ and $\vec{P}_z$ because $\vec{P}_x = m\vec{v}_x$ and $\vec{P}_z = m\vec{v}_z$. $\vec{P}_y$ is constant while $\vec{P}_x$ and $\vec{P}_z$ change.

~~3c) $\vec{L}_x$ and $\vec{L}_z$ will be conserved but $\vec{L}_y$ will not be conserved. This is because $\vec{P}_y$ is conserved. Our movement and momentum is in the x-z plane. Since our cross product follows right hand rule, $\vec{L}_y$ will vary~~

3a proof) $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \frac{\partial}{\partial y}F_z i + \frac{\partial}{\partial z}F_x j + \frac{\partial}{\partial x}F_y k - \frac{\partial}{\partial y}F_x k - \frac{\partial}{\partial z}F_y i - \frac{\partial}{\partial x}F_z j$$

Since there's no y terms, these are simply zero

$$\text{So } \vec{\nabla} \times \vec{F} = \left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right)j = 0.$$

We can clearly see that $\frac{\partial}{\partial z}F_x = \frac{\partial}{\partial x}F_z$ since F_x and F_z are identical with exchanged "x" and "z" positions, and we're also exchanging which variable we're taking the partial derivative with respect to.

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3c) We know $L = \vec{r} \times \vec{p}$ and $\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$. if $\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = 0$, then angular momentum is conserved.

Looking at $\vec{r} \times \vec{F}$:
$$\begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = yF_z i + zF_x j + xF_y k - yF_x k - zF_y i - xF_z j$$

$$= (yF_z - zF_y)i + (zF_x - xF_z)j + (xF_y - yF_x)k$$

We know $F_y = 0$, $F_x = \frac{xA}{a^2} e^{-\frac{x^2+z^2}{2a^2}}$, $F_z = \frac{zA}{a^2} e^{-\frac{x^2+z^2}{2a^2}}$.

$$\Rightarrow \frac{d\vec{L}}{dt} = yF_z i + (zF_x - xF_z)j - yF_x k$$

$$= \frac{yzA}{a^2} e^{-\frac{x^2+z^2}{2a^2}} i + \left(\frac{xzA}{a^2} e^{-\frac{x^2+z^2}{2a^2}} - \frac{xzA}{a^2} e^{-\frac{x^2+z^2}{2a^2}}\right) j - \frac{xyA}{a^2} e^{-\frac{x^2+z^2}{2a^2}} k$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \frac{yzA}{a^2} e^{-(x^2+z^2)/2a^2} i - \frac{xyA}{a^2} e^{-(x^2+z^2)/2a^2} k$$

Thus angular momentum is conserved in the y-direction, but not in the x or z directions.

$\vec{L}_y$ is conserved, but $\vec{L}_x$ and $\vec{L}_z$ are not conserved.

4) At $t=0$, $\vec{r}_0 = x_0\vec{e}_x + y_0\vec{e}_y = x_0\hat{x} + y_0\hat{y}$.

Add force $\vec{F} = F\vec{e}_x = F\hat{x}$ at $t=0$.

object at rest at $t=0$.

4a) $\vec{F} = \frac{d\vec{p}}{dt}$

we know $\vec{v}(t) = v_0 + \int_{t_0}^{t} \vec{a}\, dt'$

$$\Rightarrow \vec{v}(t) = \int_0^t \frac{\vec{F}}{m} dt' \quad \text{where } m = \text{mass}$$

all in x-direction: $v(t) = \int_0^t \frac{F}{m} dt' = \frac{F}{m} t$, so $\boxed{\vec{v}(t) = \frac{F_x}{m} t\, \vec{e}_1 = \frac{F}{m} t\, \vec{e}_1}$

$\vec{p} = p_x \vec{e}_1$ since $\vec{v}_y = \vec{v}_z = 0$, so $\vec{p} = m v_x \vec{e}_1 = m \cdot \frac{F_x}{m} t\, \vec{e}_1 = F_x t\, \vec{e}_1 = F t\, \vec{e}_1$

$$\boxed{\vec{p}(t) = F_x t\, \vec{e}_1 = F t\, \vec{e}_1}$$

4b) $\vec{r}(t) = \int \vec{v}(t) = \int_0^t \frac{F}{m} t\, \hat{x}\, dt \quad \frac{F}{2m} t^2 \hat{x} + C$ where C = initial position

$$\boxed{\vec{r}(t) = \left(x_0 + \frac{F}{2m} t^2\right) \vec{e}_1 + y_0 \vec{e}_2}$$

4c) $\vec{\ell} = \vec{r} \times \vec{p} = \left(\left(x_0 + \frac{F}{2m} t^2\right) \vec{e}_1 + y_0 \vec{e}_2\right) \times F t\, \vec{e}_1$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_0 + \frac{F}{2m} t^2 & y_0 & 0 \\ Ft & 0 & 0 \end{vmatrix} = -F t y_0 \hat{z} = -F t y_0 \vec{e}_3$$

$$\boxed{\vec{\ell} = -F t y_0 \vec{e}_3}$$

$$\tau = \frac{d}{dt}\vec{L} = \frac{d}{dt}\left(-y_0 F t \vec{e}_3\right) = \boxed{-y_0 F \vec{e}_3}$$

Angular momentum is <u>not conserved</u> since $\tau \neq 0$.

5) particle mass m, $V = \frac{\alpha}{x}$ where x is the displacement.

5a) we know $\vec{p} = mv = \frac{m\alpha}{x}$

$\vec{F} = \frac{d\vec{p}}{dt}$. Assume energy conserved. at t_0 we have $E_0 = \frac{1}{2}mv^2 + V(x)$.

$$E_0 = \frac{1}{2}m\frac{\alpha^2}{x^2} + V(x)$$

$$\frac{dE_0}{dx} = 0 = -\frac{m\alpha^2}{x^3} + \frac{dV}{dx}$$

$$\frac{dV}{dx} = -F(x) = \frac{m\alpha^2}{x^3}$$

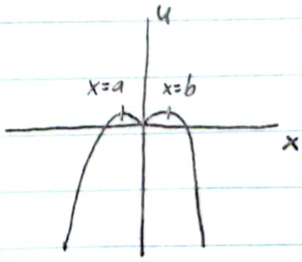
$$\boxed{F(x) = -\frac{m\alpha^2}{x^3}}$$

5b) $F = -kx + \frac{kx^3}{\alpha^2}$ $\quad k > 0$.

$$U(x) = -\int F(x)\,dx = -\int\left(-kx + \frac{kx^3}{\alpha^2}\right)dx = \int\left(kx - \frac{kx^3}{\alpha^2}\right)dx = \frac{kx^2}{2} - \frac{kx^4}{4\alpha^2}$$

$$\boxed{U(x) = \frac{kx^2}{2} - \frac{kx^4}{4\alpha^2}}$$

$U(x)$ looks like



If total energy greater than $U(a) = U(b)$, then it will leave the energy well!

The system will seek out a relative minimum potential energy, so if $x<a$, it will be pulled towards $-\infty$, if $a<x<b$, it will oscillate around $x=0$, and if $x>b$, it will be pulled towards $+\infty$. using a and b as marked on the graph. Assuming no initial kinetic energy.

5c) The max potential comes when $F=0$ and $x \neq 0$, as $x=0$ is a relative min not a relative max. ($F=0$ are critical points for $U(x)$ because $F(x) = U'(x)$.

$$F=0 = -kx + \frac{kx^3}{\alpha^2} = 0$$

$$-x + \frac{x^3}{\alpha^2} = 0 \Rightarrow -1 + \frac{x^2}{\alpha^2} = 0 \qquad \frac{x^2}{\alpha^2} = 1$$

$$x^2 = \alpha^2$$

$$x = \pm\alpha$$

Thus max U is $U(\pm\alpha)$. $U(-\alpha) = U(+\alpha)$

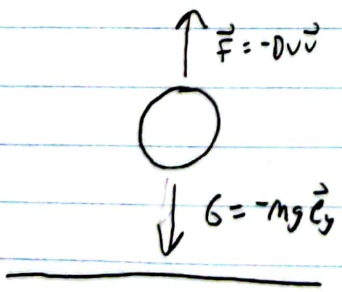
$$U(\alpha) = \frac{k\alpha^2}{2} - \frac{k\alpha^4}{4\alpha^2} = \frac{k\alpha^2}{2} - \frac{k\alpha^2}{4} = \frac{k\alpha^2}{4}$$

and $x = \pm\alpha$ i.e. potential is $\frac{1}{4}k\alpha^2$ and kinetic is 0.

Thus if the energy of the particle is $\frac{1}{4}k\alpha^2$, the particle is at a relative and absolute maximum of potential energy. At this point the particle has 0 force acting on it, but is unstable and if it gets knocked off of the perfect $x=-\alpha$ or $x=\alpha$ it will have a force acting on it to convert its potential energy to kinetic energy.

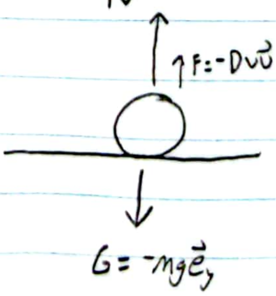
If $-\alpha < x < \alpha$ and total energy $> \frac{k\alpha^2}{4}$, the particle will escape the potential energy well.

6a) Ball has gravity, air resistance, and normal forces acting on it. Gravity is at all times, air resistance only when $V \neq 0$ and normal force only when $y(t) < R$



My code needs $\vec{N}$ to be positive, since I've defined up as the positive direction.

$$\vec{N} = +1000\,(.1 - y(t))\,\vec{e}_y$$



air resistance opposite motion of ball. This diagram is when ball is still compressing down.

$\boxed{\text{if } y(t) < R}$

$$F = ma \Rightarrow a = \frac{F}{m} \qquad \boxed{\vec{a} = \frac{-mg\vec{e}_y - Dv\vec{v} + 1000(.1 - y(t))\,\vec{e}_y}{m}}$$

$\boxed{\text{if } y(t) \geq R}$

or $\boxed{\vec{a} = \frac{-mg\vec{e}_y - Dv\vec{v}}{m}}$