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1a) $\cos(\omega t) = 1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} \cdots\cdots$   Exact: $\sum_{n=0}^{\infty} (-1)^n \frac{(\omega t)^{2n}}{(2n)!}$

$\sin(\omega t) = \omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \cdots$   Exact: $\sum_{n=0}^{\infty} (-1)^n \frac{(\omega t)^{2n+1}}{(2n+1)!}$

1b) $e^{i\omega t} = 1 + i\omega t + \frac{(i\omega t)^2}{2!} + \frac{(i\omega t)^3}{3!} + \cdots$   Exact: $\sum_{n=0}^{\infty} \frac{(i\omega t)^n}{n!}$

1c) $e^{i\omega t} = 1 + i\omega t + \frac{(i\omega t)^2}{2!} + \frac{(i\omega t)^3}{3!} + \frac{(i\omega t)^4}{4!} + \frac{(i\omega t)^5}{5!}$

$= 1 + i\omega t + \frac{-1(\omega t)^2}{2!} + \frac{-i(\omega t)^3}{3!} + \frac{(\omega t)^4}{4!} + \frac{i(\omega t)^5}{5!}$

by rearranging $= \underbrace{1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!}}_{\cos(\omega t)} + i\omega t - \frac{i(\omega t)^3}{3!} + \frac{i(\omega t)^5}{5!}$

$= \cos(\omega t) + i\underbrace{\left(\omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!}\right)}_{\sin(\omega t)}$

$= \cos(\omega t) + i\sin(\omega t)$

1d) $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ from part c.

Since $\omega$ is cycles per unit time, we can choose $\omega t = \pi$.

Thus $e^{i\omega t} = e^{i\pi} = \cos(\omega t) + i\sin(\omega t) = -1$

$e^{i\pi} = -1$

By taking ln of both sides we get $i\pi = \ln(-1)$

$\ln(-1) = i\pi$

2a) We know $\vec{a}\cdot\vec{b} = |a||b|\cos\theta$ where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$

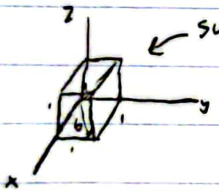
$$\vec{a}\cdot\vec{b} = \begin{pmatrix}1\\2\\4\end{pmatrix}\cdot\begin{pmatrix}4\\2\\1\end{pmatrix} = 4+4+4 = 12$$

$$|a| = \sqrt{1^2+2^2+4^2} = \sqrt{21}$$
$$|b| = \sqrt{4^2+2^2+1^2} = \sqrt{21}$$

Thus $12 = \sqrt{21}\cdot\sqrt{21}\cos\theta$

$$\cos\theta = \frac{12}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{12}{21}\right)$$
$$\boxed{\theta = 55.15^\circ} \text{ or } \boxed{0.9626 \text{ rad}}$$

2b)




$$\vec{a} = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \vec{b} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

length of body diagonal: $|a| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

length of face diagonal: $|b| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$$

$$\vec{a}\cdot\vec{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = 1+1+0 = 2$$

$$2 = \sqrt{3}\cdot\sqrt{2}\cdot\cos\theta$$
$$\cos\theta = \frac{2}{\sqrt{6}}$$
$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$$
$$\boxed{\theta = 35.26^\circ}$$

3a) Let $\vec{a}$ be $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\vec{b}$ be $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, and $\vec{c}$ be $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

$$\underline{\vec{a}\cdot(\vec{b}+\vec{c})} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1+c_1 \\ b_2+c_2 \\ \vdots \\ b_n+c_n \end{pmatrix} = a_1(b_1+c_1) + a_2(b_2+c_2) + \dots + a_n(b_n+c_n)$$

$$\Rightarrow a_1(b_1+c_1) + a_2(b_2+c_2) + \dots + a_n(b_n+c_n) = \underline{a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + \dots + a_nb_n + a_nc_n}$$

Now looking at $\underline{\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}}$, we have $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

This equals $a_1b_1 + a_2b_2 + \dots + a_nb_n + a_1c_1 + a_2c_2 + \dots + a_nc_n$

Rearranging this gives us $\underline{a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + \dots + a_nb_n + a_nc_n}$

=

Thus $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ for any vectors $\vec{a}, \vec{b}, \vec{c}$.

3b) Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and let $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$$\underline{\frac{d}{dt}(\vec{a}\cdot\vec{b})} = \frac{d}{dt}\left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}\right) = \frac{d}{dt}(a_1b_1 + a_2b_2 + \dots + a_nb_n)$$

$$= \frac{d}{dt}(a_1b_1) + \frac{d}{dt}(a_2b_2) + \dots + \frac{d}{dt}(a_nb_n)$$

$$= \underline{a_1\frac{d}{dt}(b_1) + b_1\frac{d}{dt}(a_1) + a_2\frac{d}{dt}(b_2) + b_2\frac{d}{dt}(a_2) + \dots + a_n\frac{d}{dt}(b_n) + b_n\frac{d}{dt}(a_n)}$$

Now looking at $\underline{\vec{a}\frac{d\vec{b}}{dt} + \vec{b}\frac{d\vec{a}}{dt}}$, we have $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \frac{d}{dt}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \cdot \frac{d}{dt}\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

this equals $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} db_1/dt \\ db_2/dt \\ \vdots \\ db_n/dt \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \cdot \begin{pmatrix} da_1/dt \\ da_2/dt \\ \vdots \\ da_n/dt \end{pmatrix}$

$$= a_1\frac{d}{dt}(b_1) + a_2\frac{d}{dt}(b_2) + \dots + a_n\frac{d}{dt}(b_n) + b_1\frac{d}{dt}(a_1) + b_2\frac{d}{dt}(a_2) + \dots + b_n\frac{d}{dt}(a_n)$$

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Rearranging this gives us $a_1 \frac{d}{dt}(b_1) + b_1 \frac{d}{dt}(a_1) + a_2 \frac{d}{dt}(b_2) + b_2 \frac{d}{dt}(a_2) + \dots + a_n \frac{d}{dt}(b_n) + b_n \frac{d}{dt}(a_n)$

Thus $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\frac{d\vec{b}}{dt} + \vec{b}\frac{d\vec{a}}{dt}$

4a) Switching to 3D vectors for simplicity.

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

Using $\underline{\vec{a}\times(\vec{b}+\vec{c})}$, $\vec{b}+\vec{c} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1+c_1 \\ b_2+c_2 \\ b_3+c_3 \end{pmatrix}$

$$\Rightarrow \vec{a}\times(\vec{b}+\vec{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1+c_1 \\ b_2+c_2 \\ b_3+c_3 \end{pmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix} = a_2(b_3+c_3)\vec{i} + a_3(b_1+c_1)\vec{j} + a_1(b_2+c_2)\vec{k} - a_2(b_1+c_1)\vec{k} - a_3(b_2+c_2)\vec{i} - a_1(b_3+c_3)\vec{j}$$

$$= \underline{\begin{pmatrix} a_2(b_3+c_3) - a_3(b_2+c_2) \\ a_3(b_1+c_1) - a_1(b_3+c_3) \\ a_1(b_2+c_2) - a_2(b_1+c_1) \end{pmatrix}}$$

Now using $\underline{\vec{a}\times\vec{b} + \vec{a}\times\vec{c}}$, we have $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

✓

$$= a_2b_3\vec{i} + a_3b_1\vec{j} + a_1b_2\vec{k} - a_2b_1\vec{k} - a_3b_2\vec{i} - a_1b_3\vec{j} + a_2c_3\vec{i} + a_3c_1\vec{j} + a_1c_2\vec{k} - a_2c_1\vec{k} - a_3c_2\vec{i} - a_1c_3\vec{j}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 + a_2c_3 - a_3c_2 \\ a_3b_1 - a_1b_3 + a_3c_1 - a_1c_3 \\ a_1b_2 - a_2b_1 + a_1c_2 - a_2c_1 \end{pmatrix} = \underline{\begin{pmatrix} a_2(b_3+c_3) - a_3(b_2+c_2) \\ a_3(b_1+c_1) - a_1(b_3+c_3) \\ a_1(b_2+c_2) - a_2(b_1+c_1) \end{pmatrix}}$$

Thus cross products are distributive.

4b) Use definitions of $\vec{a}$ and $\vec{b}$ from part a.

Looking at $\underline{\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)}$, $\quad \vec{a}\times\vec{b} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

$$= a_2b_3\vec{i} + a_3b_1\vec{j} + a_1b_2\vec{k} - a_2b_1\vec{k} - a_3b_2\vec{i} - a_1b_3\vec{j}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}a_2b_3 - a_3b_2\\ a_3b_1 - a_1b_3\\ a_1b_2 - a_2b_1\end{pmatrix} = \begin{pmatrix}\frac{d}{dt}(a_2b_3) - \frac{d}{dt}(a_3b_2)\\ \frac{d}{dt}(a_3b_1) - \frac{d}{dt}(a_1b_3)\\ \frac{d}{dt}(a_1b_2) - \frac{d}{dt}(a_2b_1)\end{pmatrix}$$

$$= \begin{pmatrix}a_2\frac{db_3}{dt} + b_3\frac{da_2}{dt} - a_3\frac{db_2}{dt} - b_2\frac{da_3}{dt}\\ a_3\frac{db_1}{dt} + b_1\frac{da_3}{dt} - a_1\frac{db_3}{dt} - b_3\frac{da_1}{dt}\\ a_1\frac{db_2}{dt} + b_2\frac{da_1}{dt} - a_2\frac{db_1}{dt} - b_1\frac{da_2}{dt}\end{pmatrix} = \underbrace{\begin{pmatrix}a_2\frac{db_3}{dt} - a_3\frac{db_2}{dt} + b_3\frac{da_2}{dt} - b_2\frac{da_3}{dt}\\ a_3\frac{db_1}{dt} - a_1\frac{db_3}{dt} + b_1\frac{da_3}{dt} - b_3\frac{da_1}{dt}\\ a_1\frac{db_2}{dt} - a_2\frac{db_1}{dt} + b_2\frac{da_1}{dt} - b_1\frac{da_2}{dt}\end{pmatrix}}$$

Looking at $\underline{\vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}}$

$$\vec{a}\times\frac{d\vec{b}}{dt} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}db_1/dt\\db_2/dt\\db_3/dt\end{pmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ \frac{db_1}{dt} & \frac{db_2}{dt} & \frac{db_3}{dt}\end{vmatrix} = a_2\frac{db_3}{dt}\vec{i} + a_3\frac{db_1}{dt}\vec{j} + a_1\frac{db_2}{dt}\vec{k} - a_3\frac{db_2}{dt}\vec{i} - a_1\frac{db_3}{dt}\vec{j} - a_2\frac{db_1}{dt}\vec{k}$$

$$\frac{d\vec{a}}{dt}\times\vec{b} = \begin{pmatrix}da_1/dt\\da_2/dt\\da_3/dt\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{da_1}{dt} & \frac{da_2}{dt} & \frac{da_3}{dt}\\ b_1 & b_2 & b_3\end{vmatrix} = b_3\frac{da_2}{dt}\vec{i} + b_1\frac{da_3}{dt}\vec{j} + b_2\frac{da_1}{dt}\vec{k} - b_2\frac{da_3}{dt}\vec{i} - b_3\frac{da_1}{dt}\vec{j} - b_1\frac{da_2}{dt}\vec{k}$$

$$\Rightarrow \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b} = \underbrace{\begin{pmatrix}a_2\frac{db_3}{dt} - a_3\frac{db_2}{dt} + b_3\frac{da_2}{dt} - b_2\frac{da_3}{dt}\\ a_3\frac{db_1}{dt} - a_1\frac{db_3}{dt} + b_1\frac{da_3}{dt} - b_3\frac{da_1}{dt}\\ a_1\frac{db_2}{dt} - a_2\frac{db_1}{dt} + b_2\frac{da_1}{dt} - b_1\frac{da_2}{dt}\end{pmatrix}}$$

Thus $\frac{d}{dt}\left(\vec{a}\times\vec{b}\right) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$

5a) Exer 1.18a). By definition of the cross product, we know the magnitude of the cross product is the area of the parallelogram with those 2 vectors as adjacent sides.

In our formula, the magnitude value of the cross products takes care of any potential negative, and the triangle with any 2 vectors as sides has half the area of the corresponding parallelogram with those 2 vectors as adjacent sides.

Thus $\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{b}\times\vec{c}| = \frac{1}{2}|\vec{c}\times\vec{a}|$ = area of the triangle formed by these 3 vectors as sides.

5a cont) Alternatively, think choosing as the origin and an adjacent side as the x-axis. Taking cross product of both sides adjacent to origin point. So long as you order them properly, you'll get the x-part of the "x-axis" and the y-component of the other, which is the height. $\frac{1}{2}$ this then completes triangle formula. ie. $\frac{1}{2}|\vec{b}\times\vec{c}| = \frac{1}{2}(\underset{\text{base}}{b_x}\underset{\text{height}}{c_y} + 0)$. or $\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\underset{\text{base}}{a_x}\underset{\text{height}}{b_y} + 0|$

or $\frac{1}{2}|\vec{c}\times\vec{a}| = \frac{1}{2}|\underset{\text{base}}{c_x} + \underset{\text{height}}{a_y} + 0|$

5b) Exer 1.18 b).

$$\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{b}\times\vec{c}| = \frac{1}{2}|\vec{c}\times\vec{a}| \Rightarrow |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$$

Since $|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\theta$ where $\theta$ is the angle between $\vec{v},\vec{w}$, we can make our expression

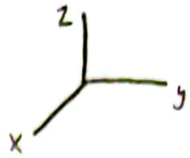
$$\Rightarrow \quad ab\sin\gamma = bc\sin\alpha = ca\sin\beta$$

dividing by $abc$ $\Rightarrow \dfrac{\sin\gamma}{c} = \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

taking the reciprocal gives $\dfrac{c}{\sin\gamma} = \dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$$\boxed{\text{So } \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}$$

NOT NEEDED?




6a) $\vec{R}^T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6b) $\vec{R}^T\vec{R} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \cos\phi\sin\phi & 0 \\ \cos\phi\sin\phi - \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow$ Identity matrix So $R^T R = I$

Because of this, $\vec{R}$ and $\vec{R}^T$ are invertible matrices and merely represent rotations. As such, dot products are invariant under these transformations.

6c) We want some vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to be transformed to $\begin{pmatrix} x \\ -z \\ y \end{pmatrix}$

I'm doing a $90°$ counterclockwise rotation about the x-axis.

This rotation is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}$$

because

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -z \\ y \end{pmatrix}$$