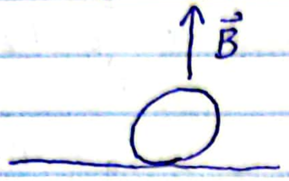
PHY 321 Final p1

1) Weather balloon released from ground.

a) $\vec{B} = B_z \hat{k}$



b) $F = ma$

$B_z \hat{k} = ma$ $\boxed{\vec{a} = \frac{B_z \hat{k}}{m}}$ where m is mass of weather balloon.

c) $v = \int a\,dt = \int \frac{B_z \hat{k}}{m}\,dt = \frac{B_z \hat{k} \cdot t}{m} + V_0 \hat{k}$

If we assume we're starting when it's first released,

$$\boxed{\vec{V}(t) = \frac{B_z \hat{k} \cdot t}{m}}$$

Note: If we wanted to, we could remove vector notation since it's all in one-dimension.

$x(t) = \int v(t)\,dt = \frac{B_z \hat{k} \cdot t^2}{2m} + X_0 \hat{k}$

If we assume we're starting at position zero (ground)

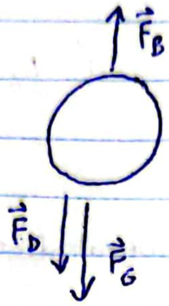
then $\boxed{\vec{x}(t) = \frac{B_z t^2 \cdot \hat{k}}{2m}}$

If we didn't assume zero initial velocity and initial position, we'd have

$$\boxed{\begin{gathered}\vec{V}(t) = \left(\frac{B_z \cdot t}{m} + V_0\right)\hat{k} \\ \vec{x}(t) = \left(\frac{B_z t^2}{2m} + V_0 t + X_0\right)\hat{k}\end{gathered}}$$

where I'm still assuming everything is in the z-direction.

d) $\vec{F}_D = -Dv\vec{v}$. Let's look at our total forces acting on the balloon



where $\vec{F}_B = B_z\hat{k}$
$\vec{F}_G = -mg\hat{k}$
$\vec{F}_D = -Dv\vec{v}$

We know $\Sigma\vec{F} = m\vec{a}$.

$$\cancel{\Sigma F \Rightarrow B_z} \quad \Sigma F = \vec{F}_B + \vec{F}_G + \vec{F}_D$$

$$= B_z\hat{k} - mg\hat{k} - Dv\vec{v} = m\vec{a}$$

$$\Rightarrow \boxed{\vec{a} = \frac{B_z}{m}\hat{k} - g\hat{k} - \frac{Dv\vec{v}}{m}}$$

Since we've defined our system as ~~exclusive to the~~ the upward z-direction, we could write this as

$$\boxed{a_z = \frac{B_z}{m} - g - \frac{Dv\vec{v}}{m}}$$

e) terminal velocity occurs when forces cancel out, resulting in $\Sigma\vec{F} = 0 \Rightarrow a = 0$.

$$\text{So } B_z - mg - Dv\vec{v} = 0$$

$$|V_{term}| \Rightarrow B_z - mg - DV_{term}^2 = 0$$

$$|V_{term}| = \sqrt{\frac{B_z - mg}{D}}$$

Assuming $B_z > mg$, we get $\boxed{V_{term} = \sqrt{\frac{B_z - mg}{D}}\,\hat{k}}$

If $B_z < mg$ and we're releasing from ground, we'd have $V_{term} = 0$, since the normal force from the ground would keep the balloon stationary.

For the sketches, I'll assume that $B_z > mg$, since that should be the case for any functioning weather balloon.

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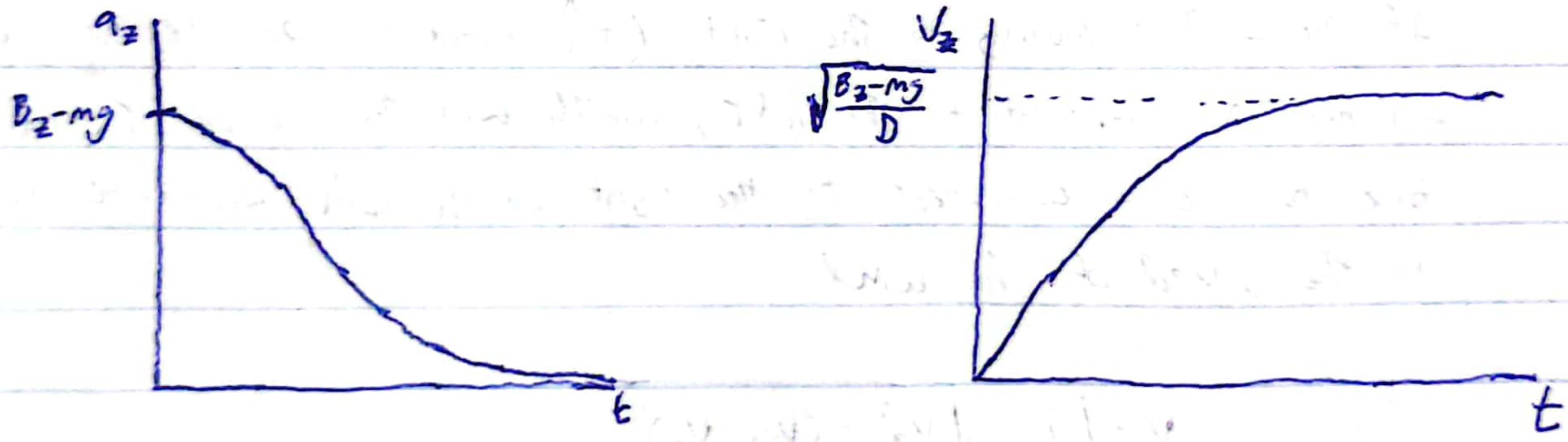

My initial conditions are $x=0$, $v=0$, $a = B_z - mg$ with $B_z > mg$.

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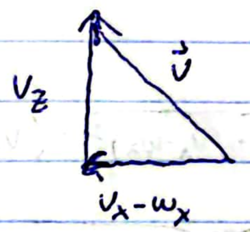
f) Wind velocity $\vec{\omega} = \omega_x \hat{i}$ along x-axis.
This will exert a force in the $\hat{i}$ direction until the balloon accelerates up to $\omega_x \hat{i}$, at which point it will no longer feel the wind.

$\vec{F}_D$ still follows equation $-Dv\vec{v}$, but now $\vec{v}$ is a combination of our vertical velocity and the difference between our horizontal velocity and the wind velocity.

Let's say our balloon is moving with $V_z \hat{k}$ and $V_x \hat{i}$.
Relative to the air which is moving $\omega_x \hat{i}$, its relative horizontal velocity is $(V_x - \omega_x)\hat{i}$

Our total relative velocity then looks like:

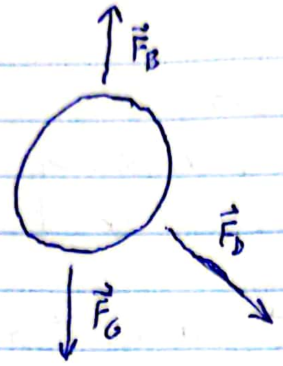


If the wind is blowing to the right ($+\hat{i}$ direction) our relative velocity will have a component to the left, which will then cause the air resistance force to have a component to the right, which will accelerate our balloon towards the speed of the wind.

$$v = |\vec{v}| = \sqrt{V_z^2 + (V_x - \omega_x)^2}$$
$$\vec{v} = V_z \hat{k} + (V_x - \omega_x)\hat{i}$$

Still have $\vec{F}_D = -Dv\vec{v}$ so now this is $-D \cdot \sqrt{V_z^2 + (V_x - \omega_x)^2} \cdot \left(V_z \hat{k} + (V_x - \omega_x)\hat{i}\right)$

g)



$\vec{F}_B$ is buoyancy force
$\vec{F}_G$ is gravity force
$\vec{F}_D$ is air resistance force

This picture is assuming $\vec{F}_B > \vec{F}_G$ and wind is moving to the right. Otherwise $\vec{F}_D$ could be oriented differently.
Relative to the ground, the balloon is moving up and to the right, but relative to the air it's moving up and to the left.

h) Still have $\Sigma \vec{F} = m\vec{a}$

$$\Sigma \vec{F} = \vec{F}_B + \vec{F}_G + \vec{F}_D$$
$$= B_z \hat{k} - mg\hat{k} - Dv\vec{v}$$
$$= B_z \hat{k} - mg\hat{k} - D\sqrt{V_z^2 + (V_x - w_x)^2} \cdot \left(V_z \hat{k} + (V_x - w_x)\hat{i}\right)$$

$$ma = \Sigma \vec{F} = B_z \hat{k} - mg\hat{k} - DV_z\sqrt{V_z^2 + (V_x - w_x)^2}\,\hat{k} - D(V_x - w_x)\sqrt{V_z^2 + (V_x - w_x)^2}\,\hat{i}$$

$$\boxed{\vec{a} = \left(\frac{B_z}{m} - g - \frac{DV_z}{m}\sqrt{V_z^2 + (V_x - w_x)^2}\right)\hat{k} - \frac{D(V_x - w_x)}{m}\sqrt{V_z^2 + (V_x - w_x)^2}\,\hat{i}}$$

Initial condition is still $\vec{F}_B > \vec{F}_G$ so that we don't have to worry about a normal force from the balloon resting on the ground.

i) Motion in the z and x directions are coupled because the acceleration in the z direction depends on the velocity in the x direction, and acceleration in the x direction depends on the velocity in the z direction. We can see this from part h where the $\hat{k}$ term has not only V_z but also V_x and the $\hat{i}$ term has both V_x and V_z dependence. Because of this coupling, we can't determine the motion analytically and will use numerical methods.

j & k) Done in Jupyter notebook.

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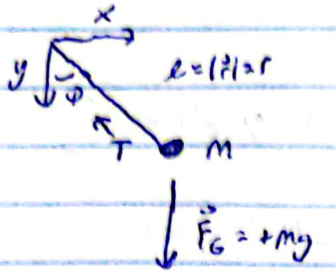
2)



we're defining +y as going down.

constant length

$\vec{r} = l\sin\phi\,\hat{i} + l\cos\phi\,\hat{j}$

a) This seems very simple, so I hope I'm not oversimplifying it.
As shown in the diagram above, our object has 2 forces on it, the force from gravity and the tension force from the rod.

$\vec{F}_G$ $\qquad\qquad$ $\vec{T}$

By Newton's second law $\Sigma\vec{F} = m\vec{a}$.
Our position is represented by $\vec{r}$, so acceleration is simply $\ddot{\vec{r}}$

This gives $\Sigma\vec{F} = \vec{F}_G + \vec{T} = m\ddot{\vec{r}}$, so $\boxed{m\ddot{\vec{r}} = \vec{F}_G + \vec{T}}$

b) Given $\ddot{\phi}(t) = -\omega_0^2\sin(\phi(t))$
Using small angle approximation for ~~...~~ $\sin\phi$ gives $\ddot{\phi}(t) = -\omega_0^2\phi(t)$
This is a classic form that we've encountered ~~...~~ tons of times.
Since $\ddot{\phi}(t) = \text{constant}\cdot\phi(t)$, our $\phi(t)$ needs to stay after 2 derivatives, which is true of sines and cosines.

For a differential equation of the form $\frac{d^2x}{dt^2} = -\omega_0^2 x$
the general solution is $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.

For us, this would be $\boxed{\phi(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)}$

often times $-\theta$ is written, but that's more of a convention thing

We can instead write $\boxed{\phi(t) = A\cos(\omega_0 t + \theta)}$ where θ is the phase angle.

If we assume we ~~...~~ have $t=0$ when the mass is released, this would simply be $\phi(t) = A\cos(\omega_0 t)$ where A is initial angle.
I'll rename A to ϕ_0, giving:

$$\boxed{\phi(t) = \phi_0\cos(\omega_0 t)}$$

c) $\frac{d\dot{\phi}}{dt} = -\omega_0^2 \sin(\phi)$ and $\frac{d\phi}{dt} = \dot{\phi}$

Scale in terms of $\hat{t} = \omega_0 t$. For clarity I'll use $\underline{\underline{\tau}} = \omega_0 t$ instead of $\hat{t}$.

For $\frac{d\dot{\phi}}{dt} = -\omega_0^2 \sin\phi$, replacing t with τ gives:

$$\cancel{\omega_0 \frac{d\dot{\phi}}{d\tau} = -\omega_0^2 \sin\phi} \qquad \omega_0 \frac{d\dot{\phi}}{d\tau} = -\omega_0^2 \sin\phi$$

dividing by ω_0 gives: $\boxed{\frac{d\dot{\phi}}{d\tau} = -\omega_0 \sin\phi}$

for $\frac{d\phi}{dt} = \dot{\phi}$, replacing t with τ gives

$$\omega_0 \frac{d\phi}{d\tau} = \dot{\phi}$$

dividing by ω_0 gives: $\boxed{\frac{d\phi}{d\tau} = \frac{1}{\omega_0}\dot{\phi}}$

For my algorithm I'm going to use Velocity-Verlet, since our system conserves energy. Euler-Cromer and Velocity-Verlet are both often used for energy conserving systems, although Velocity-Verlet should do a slightly better job. Since $\frac{d\dot{\phi}}{d\tau}$ depends only on ϕ, Velocity-Verlet is a great candidate.

No dependence on ϕ'

Algorithm: This algorith goes in the order of calculating $\phi_{i+1} \rightarrow \phi''_{i+1} \rightarrow \phi'_{i+1}$

where primes represent the number of derivatives with respect to τ.

In terms of a generic x, v, and a it's $x_{i+1} \rightarrow a_{i+1} \rightarrow v_{i+1}$. Error order $O(\Delta\tau^3)$

Could say step zero is finding $\phi''_i = -\omega_0^2 \sin(\phi_i)$

1) $\phi_{i+1} = \phi_i + \phi'_i \cdot \Delta\tau + \phi''_i \cdot \frac{(\Delta\tau)^2}{2}$

2) $\phi''_{i+1} = -\omega_0^2 \cdot \sin(\phi_{i+1})$

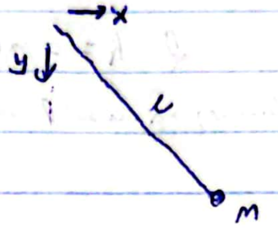
3) $\phi'_{i+1} = \phi'_i + \frac{\Delta\tau}{2}(\phi''_{i+1} + \phi''_i)$

where ϕ_i, ϕ'_i, ϕ''_i are all from previous iteration, and ϕ''_{i+1} in step 3 uses the value found in step 2.

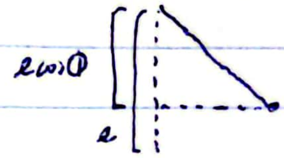
d) Done in jupyter notebook.

e) K and V in term of r and Φ.

we set the problem up like



we'll define the zero point for potential energy at $(0, l)$, so when the mass is hanging straight down.



height $= l - l\cos\Phi$. $V = mgh$

Potential energy

$$\Rightarrow V = m \cdot g \cdot (l - l\cos\Phi) = \boxed{mgl(1-\cos\Phi)}$$

$l = r$

or $\boxed{mgr(1-\cos\Phi)}$

Kinetic Energy $K = \frac{1}{2}mv^2$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\Phi}^2) = \frac{1}{2}m(0 + l^2\dot{\Phi}^2) = \boxed{\frac{1}{2}ml^2\dot{\Phi}^2 = \frac{1}{2}mr^2\dot{\Phi}^2}$$

Kinetic Energy

Note: I've been using $\omega_0 = \sqrt{\frac{g}{l}}$ throughout.

I'll choose $g = 1$ and therefore $l = \frac{1}{\omega_0^2}$.

Plots on jupyter notebook.

f) So far I've been using $V = mg\ell(1-\cos\phi)$ and $K = \frac{1}{2}m\ell^2\dot{\phi}^2$.
I've already ~~[illegible]~~ incorporated the constraint of $r=\ell$ (constant) into the equation for kinetic energy, so I'm going to take a step back so that I can do the full lagrangian formalism with constraint.

In cartesian: $K = \frac{1}{2}m(v_x^2 + v_y^2)$ which in polar becomes $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$

In cartesian: $V = mgy$ which in polar coordinates becomes $-mgr\cos\phi$.
negative

Note before I ~~have~~ had $V = mgr - mgr\cos\phi$, but this is just down to ~~a~~ a constant from where $V=0$ is defined, which is irrelevant.

Using these, I can ~~f~~ find $\mathcal{L} = K - V + \lambda \cdot \text{constraint}$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mgr\cos\phi + \lambda \cdot \underbrace{\text{constraint}}_{(\ell - r) = 0}$$

Note: Could write $\mathcal{L}'$ for with constraint, but I just used $\mathcal{L}$.

Now using Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 + mg\cos\phi - \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r}$$

Plugging these in gives $mr\dot{\phi}^2 + mg\cos\phi - \lambda - m\ddot{r} = 0$

$m\ddot{r} = mr\dot{\phi}^2 + mg\cos\phi - \lambda = 0$ since $\ddot{r} = 0$

Solving for λ: $\boxed{\lambda = mr\dot{\phi}^2 + mg\cos\phi}$ $\boxed{\lambda = m\ell\dot{\phi}^2 + mg\cos\phi}$

since $r=\ell$

λ is a force! It can be ~~[illegible]~~ interpreted as the tension force!

Now using $\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$: $\frac{\partial \mathcal{L}}{\partial \phi} = -mgr\sin\phi$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\ddot{\phi}$$

Plugging these values in gives $-mgr\sin\phi - mr^2\ddot{\phi} = 0$

$$mr^2\ddot{\phi} = -mgr\sin\phi$$

$$\boxed{\ddot{\phi} = \frac{-g}{r}\sin\phi}$$

~~[illegible]~~

Since $r = \ell$, $\boxed{\ddot{\phi} = \frac{-g}{\ell}\sin\phi}$

with $\omega_0^2 = \frac{g}{\ell}$, $\boxed{\ddot{\phi} = -\omega_0^2\sin\phi}$

So we've found everything we wanted!! ☺

We got $\ddot{\phi} = -\omega_0^2\sin\phi$ with $\omega_0^2 = \frac{g}{\ell}$

$\lambda = m\ell\dot{\phi}^2 + mg\cos\phi$ where λ can be interpreted as the tension force.