MF

PHY 321 HW 7 pg 1

1) a) Eq 4: $m\ell \frac{d^2\theta}{dt^2} + \nu \frac{d\theta}{dt} + mg\sin\theta = 0$ — ν is constant parameterizing viscosity.

Eq 5: $m\ell \frac{d^2\theta}{dt^2} + \nu \frac{d\theta}{dt} + mg\sin\theta = A\sin(\omega t)$

I'll do this for equation 5, and then we can trivially find the result for equation 4.

Dividing both sides of Eq 5 by $m\ell$ gives:

$$\frac{d^2\theta}{dt^2} + \frac{\nu}{m\ell}\frac{d\theta}{dt} + \frac{g}{\ell}\sin\theta = \frac{A}{m\ell}\sin(\omega t)$$

Just like with harmonic oscillator, set coefficient of θ term to ω_0^2

So $\omega_0^2 = \frac{g}{\ell} \Rightarrow \boxed{\omega_0 = \sqrt{\frac{g}{\ell}}}$ This will be our natural frequency, and we can again choose $\boxed{\tau = \omega_0 t.}$

Again, consider $\tilde{\omega}$ as $\frac{\omega}{\omega_0}$. Do this because we have $\sin(\omega t) = \sin\left(\frac{\omega}{\omega_0}\cdot\omega_0 t\right) = \sin(\tilde{\omega}\tau)$

Plugging in ω_0 and τ gives the following:

$$\omega_0^2 \frac{d^2\theta}{d\tau} + \frac{\nu\omega_0}{m\ell}\frac{d\theta}{dt} + \frac{g}{\ell}\sin\theta = \frac{A}{m\ell}\sin(\tilde{\omega}\tau)$$

since $\omega_0^2 = \frac{g}{\ell} \Rightarrow \frac{g}{\ell}\frac{d^2\theta}{d\tau^2} + \frac{\nu\omega_0}{m\ell}\frac{d\theta}{\underset{d\tau}{\cancel{dt}}} + \frac{g}{\ell}\sin\theta = \frac{A}{m\ell}\sin(\tilde{\omega}\tau)$

dividing by $\frac{g}{\ell}$, $\frac{d^2\theta}{d\tau^2} + \frac{\nu\omega_0}{mg}\frac{d\theta}{d\tau} + \sin\theta = \frac{A}{mg}\sin(\tilde{\omega}\tau)$ — equation 5 rewritten

Thus our final equation is $\boxed{\frac{d^2\theta}{d\tau^2} + \frac{\nu\omega_0}{mg}\frac{d\theta}{d\tau} + \sin\theta = \frac{A}{mg}\sin(\tilde{\omega}\tau)}$

For equation 4, we just wouldn't have the driving force, so $\boxed{\frac{d^2\theta}{d\tau^2} + \frac{\nu\omega_0}{mg}\frac{d\theta}{d\tau} + \sin\theta = 0}$

☆ or coupled equations $\boxed{v = \frac{d\theta}{d\tau}}$ (not ν) and $\boxed{\frac{dv}{d\tau} = \frac{A}{mg}\sin(\tilde{\omega}\tau) - \frac{\nu\omega_0}{mg}v - \sin(\theta)}$

without $\frac{A}{mg}\sin(\tilde{\omega}\tau)$ for equation 4

$\frac{3}{6}+\frac{2}{6}$ $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

2) $\vec{R}_{cm}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$ $\vec{r}\equiv\vec{r}_1-\vec{r}_2$ $\vec{F}_{12}=-\vec{F}_{21}$

a) rewrite $\vec{r}_2$ and $\vec{r}_1$ as $\vec{r}_2=\vec{R}-\frac{m_1}{m_1+m_2}\vec{r}$ and $\vec{r}_1=\vec{R}+\frac{m_2}{m_1+m_2}\vec{r}$

Total linear momentum $\vec{P}=\sum_{i=1}^{N}P_i=\sum_{i=1}^{N}m_i\frac{d\vec{r}_i}{dt}$

$$M\frac{d\vec{R}}{dt}=M\dot{\vec{R}}=\sum m_i\frac{d\vec{r}_i}{dt}=\sum P_i=M\dot{\vec{R}}$$

Here we ~~can~~ are assuming we actually only have 2 particles.

internal $\rightarrow\vec{F}_{net}=\dot{\vec{P}}=M\cdot\ddot{\vec{R}}=0.$

This is because $\vec{F}_{net}=\sum_{i\neq j}^{N}\vec{F}_{ij}=\vec{F}_{12}+\vec{F}_{21}=\vec{F}_{12}-\vec{F}_{12}=0$

With $M\cdot\ddot{\vec{R}}=0$, we have $\boxed{\ddot{\vec{R}}_{cm}=0}$

in relative motion, acceleration $\frac{d^2\vec{r}}{dt^2}=\ddot{\vec{r}}=\ddot{\vec{r}}_1-\ddot{\vec{r}}_2=\frac{\vec{F}_{12}}{m_1}-\frac{\vec{F}_{21}}{m_2}=\vec{F}_{12}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$

$$=\vec{F}_{12}\left(\frac{m_1+m_2}{m_1\cdot m_2}\right)$$

From $\ddot{\vec{r}}=\vec{F}_{12}\left(\frac{m_1+m_2}{m_1\cdot m_2}\right)\Rightarrow\boxed{\vec{F}_{12}=\ddot{\vec{r}}\left(\frac{m_1m_2}{m_1+m_2}\right)\text{ or }\mu\ddot{\vec{r}}=\vec{F}_{12}}$

using $\mu=\frac{m_1m_2}{m_1+m_2}$

b) We already showed in part a that $\boxed{\vec{P}=M\cdot\dot{\vec{R}}_{cm}}$.

We got this from $\vec{P}=\sum_{i=1}^{N}P_i=\sum_{i=1}^{N}m_i\frac{d\vec{r}_i}{dt}=\sum P_i=M\dot{\vec{R}}$

Next we want to show $q=\frac{m_2\vec{P}_1-m_1\vec{P}_2}{m_1+m_2}=\mu\dot{\vec{r}}$

$\dot{\vec{r}}=\dot{\vec{r}}_1-\dot{\vec{r}}_2$ similarly to our acceleration relationship of $\ddot{\vec{r}}=\ddot{\vec{r}}_1-\ddot{\vec{r}}_2$.

since $\vec{P}_1=m_1\dot{\vec{r}}_1$ and $\vec{P}_2=m_2\dot{\vec{r}}_2$, this can be written as $\dot{\vec{r}}=\frac{\vec{P}_1}{m_1}-\frac{\vec{P}_2}{m_2}$,

Using $\mu=\frac{m_1m_2}{m_1+m_2}$, $\mu\dot{\vec{r}}=\frac{m_1m_2}{m_1+m_2}\left(\frac{\vec{P}_1}{m_1}-\frac{\vec{P}_2}{m_2}\right)=\frac{\vec{P}_1\cdot m_2}{m_1+m_2}-\frac{\vec{P}_2\cdot m_1}{m_1+m_2}$

$$=\frac{\vec{P}_1m_2-\vec{P}_2m_1}{m_1+m_2},$$

$\boxed{\text{showing } q=\frac{m_2\vec{P}_1-m_1\vec{P}_2}{m_1+m_2}=\mu\dot{\vec{r}}}$

c) kinetic energy. $K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{dr_2}{dt}\right)^2 = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2$

$$\dot{r}_1 = \frac{d}{dt}\left(R + \frac{m_2}{m_1+m_2}r\right) = \dot{R} + \frac{m_2}{m_1+m_2}\dot{r}$$

$$\dot{r}_2 = \frac{d}{dt}\left(R - \frac{m_1}{m_1+m_2}r\right) = \dot{R} - \frac{m_1}{m_1+m_2}\dot{r}$$

$$\Rightarrow K = \frac{1}{2}m_1\left(\dot{R} + \frac{m_2}{m_1+m_2}\dot{r}\right)^2 + \frac{1}{2}m_2\left(\dot{R} - \frac{m_1}{m_1+m_2}\dot{r}\right)^2 = \frac{1}{2}m_1\left(\dot{R}^2 + 2\dot{R}\dot{r}\frac{m_2}{m_1+m_2} + \left(\frac{m_2}{m_1+m_2}\right)^2\dot{r}^2\right) + \frac{1}{2}m_2\left(\dot{R}^2 - 2\dot{R}\dot{r}\frac{m_1}{m_1+m_2} + \left(\frac{m_1}{m_1+m_2}\right)^2\dot{r}^2\right)$$

$\frac{1}{2}m_1\cdot$ $\quad$ $\frac{1}{2}m_2\cdot$

$$= \frac{1}{2}m_1\dot{R}^2 + \dot{R}\dot{r}\frac{m_1m_2}{m_1+m_2} + \left(\frac{m_2}{m_1+m_2}\right)^2\dot{r}^2 + \frac{1}{2}m_2\dot{R}^2 - \dot{R}\dot{r}\frac{m_1m_2}{m_1+m_2} + \left(\frac{m_1}{m_1+m_2}\right)^2\dot{r}^2$$

$$\boxed{\star} = \underbrace{\frac{1}{2}\dot{R}^2(m_1+m_2)} + \underbrace{\frac{1}{2}\dot{r}^2\left(\frac{m_1\cdot m_2^2}{(m_1+m_2)^2} + \frac{m_2\cdot m_1^2}{(m_1+m_2)^2}\right)}$$

Yay!

looking at this term: using $P = M\dot{R} = (m_1+m_2)\dot{R}$, $\frac{1}{2}\dot{R}^2(m_1+m_2) = \frac{\dot{R}^2(m_1+m_2)^2}{2(m_1+m_2)} = \frac{P^2}{2(m_1+m_2)} = \underline{\frac{P^2}{2M}}$

now looking at this term, we want to show this equals $\frac{q^2}{2\mu}$.

$$\frac{1}{2}\dot{r}^2\left(\frac{m_1\cdot m_2^2}{(m_1+m_2)^2} + \frac{m_2\cdot m_1^2}{(m_1+m_2)^2}\right) = \frac{1}{2}\dot{r}^2\left(\frac{m_1\cdot m_2^2}{M^2} + \frac{m_2\cdot m_1^2}{M^2}\right) = \frac{1}{2}\dot{r}^2\cdot\frac{m_1m_2}{M^2}(m_2+m_1)$$

$$= \frac{1}{2}\dot{r}^2\frac{m_1m_2\cdot M}{M^2} = \frac{1}{2}\dot{r}^2\frac{m_1m_2}{M} = \frac{1}{2}\dot{r}^2\cdot\mu.$$

we showed in part b that $q = \mu\dot{\vec{r}}$, so $\frac{1}{2}\dot{r}^2\mu = \frac{\dot{r}^2\mu^2}{2\mu} = \frac{q^2}{2\mu}$ Yay!

We've thus shown our 2 parts of the starred equation are $\frac{P^2}{2M}$ and $\frac{q^2}{2\mu}$

$$\Rightarrow \boxed{K = \frac{P^2}{2M} + \frac{q^2}{2\mu}}$$

d) COM frame. Show $\vec{L} = \vec{r} \times \mu \dot{\vec{r}}$. In general, $\vec{\ell} = \vec{r} \times \vec{p}$, or $m(\vec{r} \times \vec{v})$

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = m_1(\vec{r}_1 \times \dot{\vec{r}}_1) + m_2(\vec{r}_2 \times \dot{\vec{r}}_2)$$

in COM frame: $\vec{R} = 0 \Rightarrow \vec{r}_1 = \frac{m_2 \vec{r}}{M}$ and $\vec{r}_2 = \frac{-m_1 \vec{r}}{M}$

$$\Rightarrow \vec{L} = m_1\left(\frac{+m_2\vec{r}}{M} \times \frac{+m_2\dot{\vec{r}}}{M}\right) + m_2\left(\frac{-m_1\vec{r}}{M} \times \frac{-m_1\dot{\vec{r}}}{M}\right)$$

$$= \frac{m_1 m_2^2}{M^2}(\vec{r} \times \dot{\vec{r}}) + \frac{(-m_1)^2 m_2}{M^2}(\vec{r} \times \dot{\vec{r}})$$

$$= \frac{m_1 m_2^2}{M^2}(\vec{r} \times \dot{\vec{r}}) + \frac{m_2 m_1^2}{M^2}(\vec{r} \times \dot{\vec{r}})$$

$$= \frac{m_1 m_2^2 + m_2 m_1^2}{m_1 + m_2}(\vec{r} \times \dot{\vec{r}}) = \frac{m_1 m_2(m_2 + m_1)}{m_1 + m_2}(\vec{r} \times \dot{\vec{r}})$$

$$= \left(\frac{m_1 m_2}{m_1 + m_2}\right)(\vec{r} \times \dot{\vec{r}}) = \mu(\vec{r} \times \dot{\vec{r}}) = \boxed{\vec{r} \times \mu \dot{\vec{r}}}$$