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1) a) $\mu\ddot{r} = -\frac{dU(r)}{dr} + \mu r\dot{\phi}^2$ and $\dot{\phi} = \frac{L}{\mu r^2} \Rightarrow \mu\ddot{r} = -\frac{dU(r)}{dr} + \frac{L^2}{\mu r^3}$

Looking at $\mu\ddot{r} = -\frac{dU(r)}{dr} + \frac{L^2}{\mu r^3}$

We want to define V_{eff} such that $\mu\ddot{r} = -\frac{dV_{eff}(r)}{dr}$

So we want $V_{eff}(r) = \int -\mu\ddot{r}\,dr = \int\left(\frac{dU(r)}{dr} - \frac{L^2}{\mu r^3}\right)dr$

$$\int\left(\frac{dU(r)}{dr} - \frac{L^2}{\mu r^3}\right)dr = \underbrace{\int \frac{dU(r)}{dr}dr}_{=U(r)} - \underbrace{\int \frac{L^2}{\mu r^3}dr}_{=-\frac{1}{2}\cdot\frac{L^2}{\mu r^2}}$$

$$\int \frac{L^2 r^{-3}}{\mu}dr = -\frac{1}{2}\frac{L^2 r^{-2}}{\mu}$$

$\Rightarrow V_{eff}(r) = U(r) - -\frac{1}{2}\frac{L^2}{\mu r^2}$

So $\boxed{V_{eff} = U(r) + \frac{L^2}{2\mu r^2}}$ allows $\mu\ddot{r} = -\frac{dV_{eff}(r)}{dr}$

b) Specify $U(r) = -\frac{\alpha}{r}$

This gives $\mu\ddot{r} = -\frac{d}{dr}\left(-\frac{\alpha}{r} + \frac{L^2}{2\mu r^2}\right)$ this is what that says. $\mu\ddot{r} = -\frac{d}{dr}\left(-\frac{\alpha}{r} + \frac{L^2}{2\mu r^2}\right)$

$\Rightarrow \mu\ddot{r} = -\left[\frac{\alpha}{r^2} - \frac{L^2}{\mu r^3}\right] \Rightarrow \boxed{\mu\ddot{r} = \frac{-\alpha}{r^2} + \frac{L^2}{\mu r^3}}$

Plotting done in python, along with ~~discussion~~ discussion.

2) a) $V(r) = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2+y^2)$

As shown in 1a, $V_{eff} = V(r) + \frac{L^2}{2\mu r^2}$ where we now have new $V(r)$ and m instead of μ.

Using this, we have ~~$V_{eff} = \frac{1}{2}k(x^2+y^2) + L^2/2mr^2$~~

$$\boxed{V_{eff} = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}}$$

To find r_{min}, $\frac{dV_{eff}}{dr} = 0$ gives $kr - \frac{L^2}{mr^3} = 0$

$$kr^4 - \frac{L^2}{m} = 0$$

$$\boxed{r_{min} = \left(\frac{L^2}{km}\right)^{1/4}}$$

Using this r_{min} with equation $\boxed{\dot{\phi} = \frac{L}{mr^2}}$ gives $\dot{\phi} = \frac{L}{mr_{min}^2}$

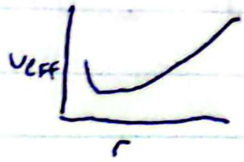
$$\Rightarrow \dot{\phi} = \frac{L}{m} \cdot \frac{(km)^{1/2}}{L} \quad \text{because } \frac{1}{r_{min}^2} = \frac{1}{\left(\frac{L^2}{km}\right)^{1/2}} = \frac{(km)^{1/2}}{L}$$

$$\Rightarrow \boxed{\dot{\phi} = \sqrt{\frac{k}{m}}}$$

r indeed constant at r_{min} given plot (in python).

To change r at $r_{min,0}$ with fixed k and m, we could change L since that's the only other variable in $r_{min} = \left(\frac{L^2}{km}\right)^{1/4}$

Plot of V_{eff} done in python! Looks like



b) We have $V_{eff} = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}$

Curvature of V_{eff} is simply second derivative, so we want k_{eff} = second derivative of V_{eff} at $r = r_{min}$.

$$\frac{d^2}{dr^2}\left[V_{eff}\right] = \frac{d}{dr}\left[kr - \frac{L^2}{mr^3}\right] = \boxed{k + \frac{3L^2}{mr^4}} \quad \text{So} \quad \boxed{k_{eff} = k + \frac{3L^2}{mr_{min}^4}}$$

Plugging in r_{min} gives $k_{eff} = k + \frac{3L^2}{m} \cdot \frac{km}{L^2} = k + 3k = 4k$, so $\boxed{k_{eff} = 4k}$

$$\boxed{\omega = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}} = 2\dot{\phi}}$$ since we show $\dot{\phi} = \sqrt{\frac{k}{m}}$ in part a.

c) $\ddot{x} = \frac{-kx}{m}$, $\ddot{y} = \frac{-ky}{m}$

I'm not really sure what work is needed here. These are pretty standard, very simple second order differential equations. We know we need "oscillating", repetitive solutions, so sines or cosines or the combination thereof. This gives.

$$\boxed{\begin{aligned} x &= A\cos\omega_0 t + B\sin\omega_0 t \\ y &= C\cos\omega_0 t + D\sin\omega_0 t \end{aligned}} \quad \boxed{\text{where } \omega_0 = \sqrt{k/m}}$$

This indeed works, since $\ddot{x} = -A\omega_0^2 \cos\omega_0 t - B\omega_0^2 \sin\omega_0 t = -\omega_0^2 x = \frac{-kx}{m}$

$$\ddot{y} = -C\omega_0^2 \cos\omega_0 t - D\omega_0^2 \sin\omega_0 t = -\omega_0^2 y = \frac{-ky}{m}$$

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d) $\alpha = \frac{A^2+B^2+C^2+D^2}{2}$, $\beta = \frac{A^2-B^2+C^2-D^2}{2}$, $\gamma = AB+CD$. Show $r^2 = \alpha + (\beta^2+\gamma^2)^{1/2}\cos(2\omega_0 t - \delta)$

where $\delta = \arctan\left(\frac{\gamma}{\beta}\right)$

I don't think there's any way around a bunch of algebra :(

Have $x^2+y^2=r^2$, and equations of x and y.

$$x^2 = (A\cos\omega_0 t + B\sin\omega_0 t)^2 = A^2\cos^2(\omega_0 t) + B^2\sin^2(\omega_0 t) + 2AB\sin(\omega_0 t)\cos(\omega_0 t)$$

$$y^2 = (C\cos\omega_0 t + D\sin\omega_0 t)^2 = C^2\cos^2(\omega_0 t) + D^2\sin^2(\omega_0 t) + 2CD\sin(\omega_0 t)\cos(\omega_0 t)$$

$$\Rightarrow r^2 = x^2+y^2 = (A^2+C^2)\cos^2(\omega_0 t) + (B^2+D^2)\sin^2(\omega_0 t) + 2(AB+CD)\sin(\omega_0 t)\cos(\omega_0 t)$$

using $\sin x \cdot \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$ we know $\sin(\omega_0 t)\cos(\omega_0 t) = \frac{1}{2}\sin(2\omega_0 t) + \underbrace{\cancel{\phi}}_{\sin 0 = 0}$.

$$\Rightarrow (A^2+C^2)\cos^2(\omega_0 t) + (B^2+D^2)\sin^2(\omega_0 t) + \underbrace{(AB+CD)}_{\gamma}\sin(2\omega_0 t)$$

I need to get a $\cos(2\omega_0 t)$ in there somehow. I'll use $\cos(2x) = \cos^2 x - \sin^2 x$ I think.

or $\cos(2x) = 2\cos^2 x - 1$

I had to work backward to figure out I needed this

Think $5\cos^2 x + 3\sin^2 x = (5-3)\cos^2 x + 3$.

Applying this idea gives, $(B^2+D^2) + (A^2-B^2+C^2-D^2)\cos^2(\omega_0 t) + \gamma\sin(2\omega_0 t)$

Knew I needed this from Scott Pratt's lecture notes)

Now use $\cos^2 x = \frac{\cos(2x)+1}{2}$ $\Rightarrow$ ~~$(B^2+D^2)+(A^2-B^2+C^2-D^2)$~~

$$\Rightarrow (B^2+D^2) + (A^2-B^2+C^2-D^2)\left(\frac{\cos(2\omega_0 t)+1}{2}\right) + \gamma\sin(2\omega_0 t)$$

$$= B^2+D^2 + \beta(\cos(2\omega_0 t)+1) + \gamma\sin(2\omega_0 t)$$

$$B^2+D^2 + \beta\cos(2\omega_0 t) + \frac{A^2-B^2+C^2-D^2}{2} + \gamma\sin(2\omega_0 t)$$

$$= \frac{A^2+B^2+C^2+D^2}{2} + \beta\cos(2\omega_0 t) + \gamma\sin(2\omega_0 t)$$

$$= \alpha + \beta\cos(2\omega_0 t) + \gamma\sin(2\omega_0 t)$$

~~Now I'm stuck~~ & nevermind

~~identity~~ ~~Acos x~~ identity $a\cos x + b\sin x = \sqrt{(a^2+b^2)}\cos(x - \tan^{-1}(\frac{b}{a}))$

square root ends

$\sqrt{a^2+b^2}\cdot\cos(x - \tan^{-1}(\frac{b}{a}))$

That should pretty much do it.

when $a=\beta$, $b=\gamma$ we get

$$= \alpha + \sqrt{\beta^2+\gamma^2} \cdot \cos\left(2\omega_0 t - \arctan\left(\frac{\gamma}{\beta}\right)\right)$$

$$\boxed{r^2 = \alpha + \sqrt{\beta^2+\gamma^2} \cdot \cos(2\omega_0 t - \delta)}$$

welp.

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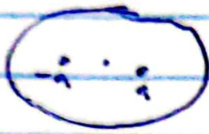
4) $\sqrt{(x-a)^2+y^2} + \sqrt{(x+a)^2+y^2} = 2D$ with foci at $-a, a$.

$\left(\sqrt{(x-a)^2+y^2}+\sqrt{(x+a)^2+y^2}\right)\left(\ldots\ldots\right)$



write as $\frac{x^2}{D^2}+\frac{y^2}{D^2-a^2}=1$

Squaring both sides

$\Rightarrow (x-a)^2+y^2+2\sqrt{(x-a)^2+y^2}\sqrt{(x+a)^2+y^2}+(x+a)^2+y^2=4D^2$

$\Rightarrow (x-a)^2+(x+a)^2+2y^2+2\sqrt{\left((x-a)^2+y^2\right)\left((x+a)^2+y^2\right)}=4D^2$

simplifying square root

$\sqrt{(x-a)^2(x+a)^2+y^2(x-a)^2+y^2(x+a)^2+y^4}$

$(x-a)^2 = x^2-2ax+a^2 \qquad (x+a)^2 = x^2+2ax+a^2$

cancel

$2x^2+2a^2+2y^2+2\sqrt{\text{all this stuff}}=4D^2$

$x^2+a^2+y^2+\sqrt{\text{stuff}}=2D^2$ same $(x-a)^2+(x+a)^2$ simplification

$x^2+a^2+y^2+\sqrt{(x-a)^2(x+a)^2+y^2(2x^2+2a^2+y^2)}=2D^2$

$\sqrt{\text{stuff}}=2D^2-x^2-a^2-y^2$

$\text{stuff}=(2D^2-x^2-a^2-y^2)^2$

$(x-a)^2(x+a)^2+y^2(2x^2+2a^2+y^2)=4D^4-4D^2a^2-4D^2x^2-4D^2y^2+\cancel{a^4}+2a^2x^2+\cancel{2a^2y^2}+\cancel{x^4}+\cancel{2x^2y^2}+\cancel{y^4}$

$\cancel{a^4}-2a^2x^2+\cancel{x^4}+\cancel{2x^2y^2}+\cancel{2a^2y^2}+\cancel{y^4}=$ same as before

$-2a^2x^2=4D^4-4D^2a^2-4D^2x^2-4D^2y^2+2a^2x^2$

$4D^4-4D^2a^2-4D^2x^2-4D^2y^2+4a^2x^2=0$

$D^4-D^2a^2-D^2x^2-D^2y^2+a^2x^2=0$

$D^2-a^2-x^2-y^2+\frac{a^2x^2}{D^2}=0$

$x^2+y^2-\frac{a^2x^2}{D^2}=D^2-a^2$

$x^2\left(1-\frac{a^2}{D^2}\right)+y^2=D^2-a^2$

$x^2\frac{\left(1-\frac{a^2}{D^2}\right)}{D^2-a^2}+\frac{y^2}{D^2-a^2}=1$

$\frac{1-\frac{a^2}{D^2}}{D^2-a^2}=\frac{\frac{D^2-a^2}{D^2}}{D^2-a^2}=\frac{1}{D^2}$

$$\boxed{\frac{x^2}{D^2}+\frac{y^2}{D^2-a^2}=1}$$ ✓