PHY 321 HW 5

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1) $\vec{F}(x) = -F_0 \sin\left(\frac{2\pi x}{b}\right) \vec{e}_x$

a) Our force depends only on position. We'll check if the curl is zero.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z} F_x \hat{j} - \frac{\partial}{\partial y} F_x \hat{k} = 0\hat{j} - 0\hat{k} = 0$$

Yes, the force is conservative. This implies that the work done by this force on the electron between 2 points is independent of the path taken, and the work done around a closed path is zero.

b) $W = \Delta KE$. I will assume $v=0$ at $x=0$.

$$\int_0^x \vec{F}\cdot d\vec{x} = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 \Rightarrow \int_0^x \vec{F}\cdot d\vec{x} = \tfrac{1}{2}mv_f^2$$

$$\int_0^x \vec{F}\cdot d\vec{x} = \int_0^x -F_0 \sin\left(\frac{2\pi x}{b}\right)\vec{e}_x \cdot d\vec{x} = -F_0 \int_0^x \sin\left(\frac{2\pi x}{b}\right)dx = -F_0\left[\frac{-b}{2\pi}\cos\left(\frac{2\pi x}{b}\right)\right]_0^x$$

$$= F_0\left[\frac{b}{2\pi}\cos\left(\frac{2\pi x}{b}\right)\right]_0^x = F_0\left(\frac{b}{2\pi}\cos\left(\frac{2\pi x}{b}\right) - \frac{b}{2\pi}\right) = \frac{F_0 b}{2\pi}\left(\cos\left(\frac{2\pi x}{b}\right) - 1\right)$$

$$\Rightarrow \tfrac{1}{2}mv^2 = \frac{F_0 b}{2\pi}\left(\cos\left(\frac{2\pi x}{b}\right) - 1\right)$$

$$mv^2 = \frac{F_0 b}{\pi}\left(\cos\left(\frac{2\pi x}{b}\right) - 1\right)$$

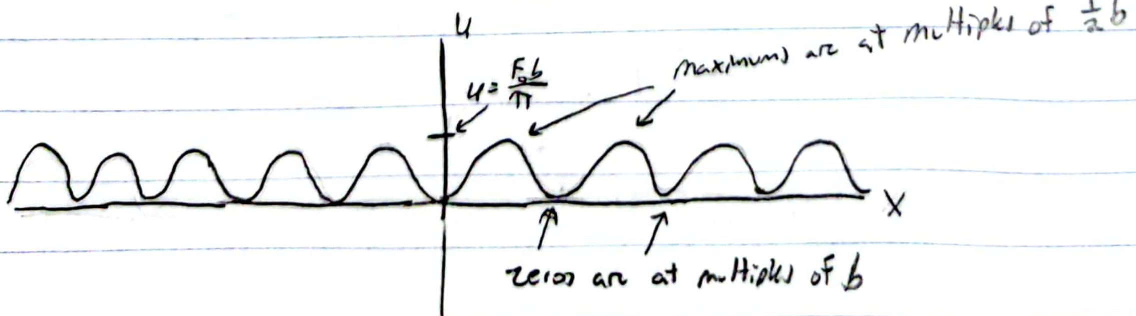
$$\boxed{v(x) = \sqrt{\frac{F_0 b}{m\pi}\left(\cos\left(\frac{2\pi x}{b}\right) - 1\right)}}$$

c) We know $\vec{F} = -\nabla U$

I'll set $U(0) = 0$ as my reference point.

$$U(r) = -\int_0^x \vec{F}(x)\cdot d\vec{x} = \boxed{\frac{-F_0 b}{2\pi}\left(\cos\left(\frac{2\pi x}{b}\right) - 1\right) = U(x)}$$

d)

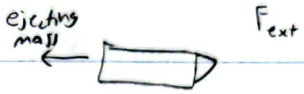


Analysis on next page!

Equilibrium points are simply where there's no force on the particle, so slope of $U(x) = 0$.
Equilibrium points are every maximum and minimum. Maxes are unstable because any movement would decrease the potential, which results in increasing kinetic energy, and further decreasing potential. Mins are stable E.P.'s because any movement results in an increase in potential energy.

3.11 → 2)

a) F^{ext} acting on line of motion.



We start with $P(t+dt) = (m+dm)(v+dv) - dm(v - v_{ex}) = mv + mdv + dm v_{ex}$

without F^{ext}

with F^{ext}, $\quad dP = P(t+dt) - P(t) = mdv + dm v_{ex}$ where $dP = F^{ext}_{net}\, dt$

$\Rightarrow F^{ext} dt = m dv + dm\, v_{ex}$

dividing by dt $\Rightarrow$ $F^{ext} = \frac{m dv}{dt} + \frac{dm\ v_{ex}}{dt} = m\frac{dv}{dt} + v_{ex}\cdot\frac{dm}{dt}$

$\Rightarrow F^{ext} = m\dot{v} + \dot{m} v_{ex}$

rearranging gives $\boxed{m\dot{v} = -\dot{m} v_{ex} + F^{ext}}$

b) IF F^{ext} is gravity, then this becomes $m\dot{v} = -\dot{m} v_{ex} - mg$

$\dot{m} = -k \Rightarrow m = m_0 - kt \qquad \Rightarrow m\dot{v} = k v_{ex} - mg$

or $(m_0 - kt)\dot{v} = +k v_{ex} - (m_0 - kt) g$

$m\dot{v} = k v_{ex} - mg \Rightarrow \dot{v} = \frac{dv}{dt} = \frac{k}{m} v_{ex} - g \Rightarrow dv = \left(\frac{k}{m} v_{ex} - g\right) dt = \left(\frac{k}{m_0 - kt} v_{ex} - g\right) dt$

$v_0 = 0!$ integrate both sides: $\int_{v_0}^{v} dv = \int_0^t \left(\frac{k v_{ex}}{m_0 - kt} - g\right) dt$

$kt - m_0$

$\Rightarrow v = -v_{ex} \ln|kt - m_0| \Big|_0^t - gt = -v_{ex}\ln|-m| - \ln|m_0| - gt$

$\Rightarrow \boxed{v = v_{ex} \ln\left|\frac{m_0}{m}\right| - gt}$

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$m = m_0 - kt$

$kt = m_0 - m$

$k = \frac{m_0 - m}{t}$

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c) from 3.7 $m_0 = 2\times10^6$ kg, m after 2 min $= 1\times10^6$ kg. $v_{ex} = 3000$ m/s. $v_0 = 0$

$\frac{m_0}{m} = \frac{2\times10^6}{1\times10^6} = 2$, $v = 3000 \cdot \ln(2) - 9.81\cdot120 = \boxed{902.24 \text{ m/s}}$ ← with gravity

If there was no gravity: $v = 3000\,\ln(2) = \boxed{2079.44 \text{ m/s}}$ ← no gravity

d) If $-\dot{m}v_{ex}$ was less than $-mg$, than the force of gravity would be larger than the thrust force, and the rocket wouldn't lift off until enough mass/weight would be lost from the exhaust. Your overall $F = ma = 0$ since your contact force with the ground would equal whatever extra gravitational force is left. Eventually, reducing mass would cause mg to be smaller, and the rocket would take off.

3) 3.13) we have $v(t) = v_{ex}\ln\left|\frac{m_0}{m}\right| - gt = v_{ex}\ln\left|\frac{m_0}{m_0-kt}\right| - gt$

Integrate gives $y(t) = v_{ex}\cdot\int\ln\left|\frac{m_0}{m_0-kt}\right|dt - \frac{1}{2}gt^2$

$$\int\ln\left|\frac{m_0}{m_0-kt}\right|dt \overset{u=\frac{m_0}{m_0-kt},\ du=-k\,dt}{=} \int\ln|u|\cdot -k\,du = -k\int\ln(u)\,du = -k\left(u\cdot(\ln(u)-1)\right)$$

$$\int\ln\left|\frac{m_0}{m_0-kt}\right|dt = -\int\ln\left|\frac{m_0-kt}{m_0}\right|dt = -\left[t\left(\ln\left(\frac{m_0-kt}{m_0}\right)\cdot\frac{m_0}{-k}-1\right)\right] = t - \frac{m_0}{-k}\ln\left(\frac{m}{m_0}\right)$$

$$= t + \frac{m_0}{k}\ln\left(\frac{m}{m_0}\right) = t - \frac{m_0}{k}\ln\left(\frac{m_0}{m}\right)$$

plugging back in ⇒ $y(t) = v_{ex}\left(t - \frac{m_0}{k}\ln\left(\frac{m_0}{m}\right)\right) - \frac{1}{2}gt^2 = v_{ex}t - \frac{v_{ex}m}{k}\ln\left(\frac{m_0}{m}\right) - \frac{1}{2}gt^2$

$$\Rightarrow \boxed{y(t) = v_{ex}t - \frac{1}{2}gt^2 - \frac{m v_{ex}}{k}\ln\left(\frac{m_0}{m}\right)}$$

after 120 seconds, $\frac{m_0}{m} = 2$, $k = \frac{m_0 - m}{t} = \frac{10^6}{120} = 8333.3\bar{3}$

$y(t) = 3000\cdot120 - \frac{1}{2}\cdot9.81\cdot120^2 - \frac{10^6\cdot3000}{10^6/120}\ln(2) = 360{,}000 - 70{,}632 - 249533$

$y(120) \approx 39835$

So after 2 minutes, $\boxed{y(120) = 39{,}835 \text{ m}}$

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3.14) $\vec{F} = -b\vec{v}$ $\qquad m\dot{v} = -\dot{m}v_{ex} + F^{ext} = -\dot{m}v_{ex} - bv$

$k = -\dot{m} \Rightarrow m = m_0 - kt$

$$m\frac{dv}{dt} = +kV_{ex} - bv \quad \text{or} \quad (m_0 - kt)\frac{dv}{dt} = kV_{ex} - bv$$

$$\cancel{m\frac{dv}{dt} + bv = -kV_{ex}} \qquad \frac{m_0 - kt}{dt} = \frac{kV_{ex} - bv}{dv}$$

$$\cancel{\frac{dv}{dt} + \frac{b}{m}v = \frac{-k}{m}V_{ex}}$$

Integrate both sides $\rightarrow$ $$\frac{1}{m_0 - kt}dt = \frac{1}{kV_{ex} - bv}dv$$

$$\int_0^t \frac{1}{m_0 - kt'}dt' = \int_0^v \frac{1}{kV_{ex} - bv'}dv'$$

$$\left[-\frac{1}{k}\ln|m_0 - kt'|\right]_0^t = \frac{-1}{k}\left(\ln|m_0 - kt| - \ln|m_0|\right) = \frac{1}{k}\left(\ln|m_0| - \ln|m|\right) = \frac{1}{k}\ln\left|\frac{m_0}{m}\right|$$

$$\left[-\frac{1}{b}\ln|kV_{ex} - bv'|\right]_0^v = \frac{-1}{b}\left[\ln|kV_{ex} - bv| - \ln|kV_{ex}|\right] = \frac{1}{b}\left[\ln|kV_{ex}| - \ln|kV_{ex} - bv|\right]$$

$$= \frac{1}{b}\ln\left|\frac{kV_{ex}}{kV_{ex} - bv}\right|$$

$$\Rightarrow \frac{1}{k}\ln\left|\frac{m_0}{m}\right| = \frac{1}{b}\ln\left|\frac{kV_{ex}}{kV_{ex} - bv}\right|$$

$$\ln\left|\frac{kV_{ex}}{kV_{ex} - bv}\right| = \frac{b}{k}\ln\left|\frac{m_0}{m}\right|$$

$$\frac{kV_{ex}}{kV_{ex} - bv} = \left(\frac{m_0}{m}\right)^{b/k} \Rightarrow kV_{ex} = \left(\frac{m}{m_0}\right)^{b/k}(kV_{ex} - bv)$$

$$\Rightarrow kV_{ex} - bv = kV_{ex}\left(\frac{m}{m_0}\right)^{b/k}$$

$$v = \frac{kV_{ex}}{b} - \frac{kV_{ex}}{b}\left(\frac{m}{m_0}\right)^{b/k}$$

$$\Rightarrow \boxed{v = \frac{k}{b}V_{ex}\left(1 - \left(\frac{m}{m_0}\right)^{b/k}\right)}$$

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4) 3.20) M_1 M_2 at $\vec{R}_1$ $\vec{R}_2$. Let M_1 and M_2 be composed of n and p point particles respectively.

We know $\vec{R}_1 = \frac{1}{M_1}\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha$ and $\vec{R}_2 = \frac{1}{M_2}\sum_{\beta=1}^{p} m_\beta \vec{r}_\beta$. α and β count through separate particles.

where $M_1 = \sum_{\alpha=1}^{n} m_\alpha$ and $M_2 = \sum_{\beta=1}^{p} m_\beta$

We know the total center of mass $\vec{R} = \left[\sum_{a=1}^{n+p}(m_a)\right]^{-1} \cdot \sum_{a=1}^{n+p} m_a \vec{r}_a$

$$\Rightarrow \; = \left[\sum_{a=1}^{n} m_a + \sum_{a=n+1}^{n+p} m_a\right]^{-1} \cdot \left[\underbrace{\sum_{a=1}^{n} m_a \vec{r}_a}_{M_1 R_1} + \underbrace{\sum_{a=n+1}^{n+p} m_a \vec{r}_a}_{M_2 R_2}\right]$$

$$= \left[M_1 + M_2\right]^{-1} \cdot \left[M_1 \vec{R}_1 + M_2 \vec{R}_2\right]$$

$$\Rightarrow \boxed{\vec{R} = \frac{M_1 \vec{R}_1 + M_2 \vec{R}_2}{M_1 + M_2}}$$

5) a) $1\ AU = 1.5\times10^{11}\ m$. $M_\odot = 2\times10^{30}\ kg$. $M_{Earth} = 6\times10^{24}$

initialize $x = 1\ AU$. $y = 0$. Assume almost circular motion.

$$F_G = \frac{-GM_\odot M_E}{r^2} \qquad \frac{d^2x}{dt^2} = \frac{-F_{G,x}}{M_{Earth}} \qquad \frac{d^2y}{dt^2} = \frac{-F_{G,y}}{M_{Earth}}$$

since circular: $F_G = \frac{M_{Earth} v^2}{r} = \frac{GM_\odot M_{Earth}}{r^2}$

use $r = \sqrt{x^2+y^2}$ to rewrite.

$r\cos\theta$

$$F_x = \frac{-GM_\odot M_E}{r^2}\cos\theta = -\frac{GM_\odot M_E}{r^3}x$$

$$F_y = \frac{-GM_\odot M_E}{r^2}\sin\theta = -\frac{GM_\odot M_E}{r^3}y$$

4 first-order coupled differential equations:

$$\boxed{\frac{dv_x}{dt} = \frac{-GM_\odot}{r^3}x \qquad \frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y \qquad \frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y}$$

Now we need to ~~set~~ make these dimensionless.

using centripetal force: $\frac{M_E v^2}{r} = F = \frac{GM_\odot M_E}{r^2} \Rightarrow GM_\odot = v^2 r$

assuming circular orbit, $v = \frac{2\pi r}{year} = \frac{2\pi \cdot 1AU}{year} \Rightarrow \boxed{GM_\odot = v^2 r = 4\pi^2 \frac{(AU)^3}{yr^2}}$

b) using euler's method, this discretizes as, using step length h.

$$\boxed{\begin{aligned} v_{x,i+1} &= v_{x,i} - h\frac{4\pi^2}{r_i^3}x_i \\ x_{i+1} &= x_i + h v_{x,i} \\ v_{y,i+1} &= v_{y,i} - h\frac{4\pi^2}{r_i^3}y_i \\ y_{i+1} &= y_i + h v_{y,i} \end{aligned}}$$

or $v_{x,i+1}$ and $v_{y,i+1}$ if euler-cromer method

using velocity-verlet method:

$$x_{i+1} = x_i + \Delta t\, v_i + \frac{\Delta t^2}{2} a_i$$

$$v_{i+1} = v_i + \frac{\Delta t}{2}(a_{i+1} + a_i)$$

where a_{i+1} depends only on x_{i+1}

Note, these same equations work for y or z direction, just replace for example x with y, v_x with v_y, and a_x with a_y.

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