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Abstract Operator Systems over the Cone of Positive Semidefinite Matrices

Martin Berger & Tim Netzer

Operator Systems Over the Psd Cone

$$
\begin{tikzcd}
\vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
\mathrm{Sep}_{d,3} \arrow[d, dash] & \subseteq & \mathrm{Dpsd}_{\mathrm{d},3} \arrow[d, dash] & \subseteq & \mathrm{Psd}_{d,3} \arrow[d, dash] & + & \mathrm{Psd}^{\Gamma}_{d,3} \arrow[d, dash] & = & \mathrm{Decomp}_{d,3} \arrow[d, dash] & \subseteq & \mathrm{Bpsd}_{d,3} \arrow[d, dash] \\
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& & & & & & \mathrm{Psd}_{d,1} \arrow[ullllll, dash] \arrow[ullll, dash] \arrow[ull, dash] \arrow[u, dash] \arrow[urr, dash] \arrow[urrrr, dash] & & & &
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Abstract Operator Systems and Free Spectrahedra

Convex Cones

- A subset C of an $\mathbb{R}$-vector space is called a *cone* if for every nonnegative scalars α and $c \in C$ it follows
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$$c_1 + c_2 \in C\,.$$
- We call a convex cone C *salient* if $C \cap (-C) = \{0\}$.

Convex Cones - Examples

A cone with empty interior.

Convex Cones - Examples

A salient convex cone with nonempty interior.

Abstract Operator Systems

- $\mathcal{V}$ denotes a $\mathbb{C}$-vector space with involution $*$
- $\mathcal{V}_h$ the $\mathbb{R}$-subspace of its Hermitian elements
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Definition (e.g. [Pau03, Chapter 13])

An *abstract operator system* $\mathcal{C}$ on $\mathcal{V}$ consists for each $s \geq 1$ of a closed and salient convex cone $\mathcal{C}_s \subseteq \mathrm{Mat}_s(\mathcal{V})_h$ with nonempty interior such that

$$A \in \mathcal{C}_s, V \in \mathrm{Mat}_{s,t}(\mathbb{C}) \Rightarrow V^*AV \in \mathcal{C}_t.$$

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Abstract Operator Systems - Examples

- $\mathcal{V} = \mathrm{Mat}_d(\mathbb{C})$, $\mathcal{V}_h = \mathrm{Her}_d(\mathbb{C})$
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For each $s \geq 1$ define

$$\mathrm{Psd}_{d,s} := \left\{ \sum_{i=1}^{n} A_i \otimes B_i \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) \;\middle|\; n \in \mathbb{N}, \sum_{i=1}^{n} A_i \otimes B_i \geqslant 0 \right\}.$$

and call $\mathrm{Psd}_d = (\mathrm{Psd}_{d,s})_{s\in\mathbb{N}}$ the *operator system of positive semidefinite matrices*.

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and call $\mathrm{Psd}_d = (\mathrm{Psd}_{d,s})_{s \in \mathbb{N}}$ the *operator system of positive semidefinite matrices*. Similar, for each $s \geq 1$ define

$$\mathrm{Psd}^{\Gamma}_{d,s} := \left\{ \sum_{i=1}^{n} A_i \otimes B_i \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) \;\middle|\; n \in \mathbb{N}, \sum_{i=1}^{n} A_i^T \otimes B_i \geqslant 0 \right\}$$

and call $\mathrm{Psd}^{\Gamma}_d = (\mathrm{Psd}^{\Gamma}_{d,s})_{s \in \mathbb{N}}$ the *operator system of matrices with positive partial transpose*.

Abstract Operator Systems

In the following we assume that $\mathcal{V} = \mathbb{C}^d$, and thus $\mathcal{V}_h = \mathbb{R}^d$. Then

$$\mathsf{Mat}_s(\mathcal{V}) = \mathbb{C}^d \otimes_{\mathbb{C}} \mathsf{Mat}_s(\mathbb{C}) = \mathsf{Mat}_s(\mathbb{C})^d, \quad \mathsf{Mat}_s(\mathcal{V})_h = \mathsf{Her}_s(\mathbb{C})^d.$$

Hence for an operator system $\mathcal{C}$ on $\mathcal{V}$ it holds $\mathcal{C}_s \subseteq \mathsf{Her}_s(\mathbb{C})^d$.

Free Spectrahedra

Definition

Let $B_1, \ldots, B_d \in \mathrm{Her}_r(\mathbb{C})$ denote Hermitian matrices. Then a *(classical) spectrahedron* is a set of the form

$$\left\{a \in \mathbb{R}^d \;\middle|\; a_1 B_1 + \cdots + a_d B_d \geqslant 0\right\}.$$

For any $s \geq 1$, we define

$$\mathcal{S}_s(B_1, \ldots, B_d) := \left\{(A_1, \ldots, A_d) \in \mathrm{Her}_s(\mathbb{C})^d \;\middle|\; B_1 \otimes A_1 + \cdots + B_d \otimes A_d \geqslant 0\right\}.$$

The family of cones $\mathcal{S}(B_1, \ldots, B_d) = (\mathcal{S}_s(B_1, \ldots, B_d))_{s \geq 1}$ is called the *free spectrahedron* defined by $B_1, \ldots, B_d$.

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Remark: For every free spectrahedron each level is a classical spectrahedron in $\mathbb{R}^{s^2 d}$.

Finite-Dimensional Realizable Operator Systems

Definition

We call an abstract operator system $\mathcal{C}$ *finite-dimensional realizable* if it constitutes a free spectrahedron, i. e. there exist hermitian matrices $B_1, \ldots, B_d$ such that

$$\mathcal{C}_s = \mathcal{S}_s(B_1, \ldots, B_d) = \left\{(A_1, \ldots, A_d) \in \mathrm{Her}_s(\mathbb{C})^d \;\middle|\; B_1 \otimes A_1 + \cdots + B_d \otimes A_d \geqslant 0\right\}$$

holds for all $s \geq 1$.

Finite-Dimensional Realizable – Example

Let $E_{ij} \in \mathbb{C}^{d\times d}$ denote the ij-th standard matrix, then

$$E_{ii}, \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}), \frac{i}{\sqrt{2}}(E_{ij} - E_{ji})$$

is an orthonormal basis of $\mathrm{Her}_d(\mathbb{C})$ and it holds

$$\mathrm{Psd}_d = \mathcal{S}\left(E_{ii}, \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}), \frac{i}{\sqrt{2}}(E_{ij} - E_{ji})\right) .$$

Similar for Psd_d^Γ.

Free Duality

Duality of Cones

Let $\mathcal{C}$ be an operator system such that $\mathcal{C}_s \subseteq \mathrm{Her}_s(\mathbb{C})^d$.

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for $A = (A_1, \ldots, A_d), B = (B_1, \ldots, B_d) \in \text{Her}_s(\mathbb{C})^d$, where tr denotes the trace of a matrix.

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$$\mathcal{C}_s^{\vee} = \left\{ (A_1, \ldots, A_d) \in \mathrm{Her}_s(\mathbb{C})^d \;\middle|\; \sum_{i=1}^{d} \mathrm{tr}(B_i A_i) \geq 0 \text{ for every } (B_1, \ldots, B_d) \in \mathcal{C}_s \right\}.$$

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In contrast, we define the *free dual* $\mathcal{C}^{\vee \mathrm{fr}} = (\mathcal{C}_s^{\vee \mathrm{fr}})_{s \geq 1}$ of $\mathcal{C}$ by

$$\mathcal{C}_s^{\vee \mathrm{fr}} := \left\{ (A_1, \ldots, A_d) \in \mathrm{Her}_s(\mathbb{C})^d \;\middle|\; \sum_{i=1}^{d} B_i^T \otimes A_i \geqslant 0 \text{ for every } (B_1, \ldots, B_d) \in \mathcal{C} \right\}.$$

Free Duality

Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ be operator systems with $\mathcal{C}_s, \mathcal{D}_s \subseteq \mathrm{Her}_s(\mathbb{C})^d$ for every $s \geq 1$. Then the following holds:

1. $\mathcal{C}_s^{\vee\mathrm{fr}} \subseteq \mathcal{C}_s^{\vee}$ *for every* s*,*
2. $\mathcal{C}^{\vee\mathrm{fr}}$ *is an operator system.*
3. $(\mathcal{C}^{\vee\mathrm{fr}})^{\vee\mathrm{fr}} = \mathcal{C}$*.*
4. *Let* $\mathcal{C} + \mathcal{D}$ *and* $\mathcal{C} \cap \mathcal{D}$ *denote the (level-wise) Minkowski sum and intersection respectively. Then*
$$(\mathcal{C} + \mathcal{D})^{\vee\mathrm{fr}} = \mathcal{C}^{\vee\mathrm{fr}} \cap \mathcal{D}^{\vee\mathrm{fr}}.$$

Finitely Generated OS

Definition

An operator system $\mathcal{C}$ is *finitely generated*, if there is an element $A \in \mathcal{C}$ such that each element from $\mathcal{C}$ is of the form

$$\sum_i V_i^* A V_i$$

for some finitely many complex matrices V_i.

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Theorem ([BN21],[HKM17])

An operator system $\mathcal{C}$ is generated by $A = (A_1, \ldots, A_d)$ if and only if $\mathcal{C}^{\vee \mathrm{fr}} = \mathcal{S}(A_1^T, \ldots, A_d^T)$. In particular, $\mathcal{C}$ is finitely generated if and only if $\mathcal{C}^{\vee \mathrm{fr}}$ is finite-dimensional realizable, and vice versa.

Finitely Generated OS - Example

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Proposition

We have

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Since

$$\mathrm{Psd}_d = \mathrm{Psd}_d^{\vee_{\mathrm{fr}}} = \mathcal{S}\left(E_{ii}, \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}), \frac{i}{\sqrt{2}}(E_{ij} - E_{ji})\right)$$

Psd_d is generated by the corresponding tuple of transposed matrices.

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Psd_d is generated by the corresponding tuple of transposed matrices. Similar for Psd_d^{Γ}.

Operator Systems Over the Psd Cone

Operator Systems Over the Psd Cone

- Interested in operator systems $\mathcal{C}$ with the cone of psd matrices at level one
- Let $\mathcal{V} = \mathrm{Mat}_d(\mathbb{C})$ with the usual involution $*$
- $\mathcal{V}_h = \mathrm{Her}_d(\mathbb{C})$
- $\mathrm{Mat}_s(\mathcal{V}) = \mathrm{Mat}_d(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Mat}_s(\mathbb{C}) = \mathrm{Mat}_d\big(\mathrm{Mat}_s(\mathbb{C})\big)$
- $\mathcal{C}_s \subseteq \mathrm{Her}_d(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Her}_s(\mathbb{C})$
- $\mathcal{C}_1 = \mathrm{Psd}_{d,1}$

Operator Systems Over the Psd Cone

The smallest such operator system is denoted by $\mathrm{Sep}_d = (\mathrm{Sep}_{d,s})_{s\in\mathbb{N}}$ where

$$\mathrm{Sep}_{d,s} := \left\{ \sum_{i=1}^{n} A_i \otimes B_i \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) \;\middle|\; n \in \mathbb{N}, A_i \geqslant 0, B_i \geqslant 0 \right\},$$

and is called the operator system of *separable matrices*.

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and is called the operator system of *separable matrices*.

The largest one is the operator system of *block positive matrices* $\mathrm{Bpsd}_d = (\mathrm{Bpsd}_{d,s})_{s\in\mathbb{N}}$, where

$$\mathrm{Bpsd}_{d,s} := \left\{ \sum_{i=1}^{n} A_i \otimes B_i \;\middle|\; n \in \mathbb{N}, (x\otimes y)^* \left(\sum_{i=1}^{n} A_i \otimes B_i \right) (x \otimes y) \geq 0 \ \forall x \in \mathbb{C}^d, y \in \mathbb{C}^s \right\}.$$

Operator Systems Over the Psd Cone

In between lie the operator systems Psd_d where

$$\mathrm{Psd}_{d,s} = \left\{ \sum_{i=1}^{n} A_i \otimes B_i \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) \;\middle|\; n \in \mathbb{N}, \sum_{i=1}^{n} A_i \otimes B_i \geqslant 0 \right\}$$

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and the operator system Psd_d^Γ where

$$\mathrm{Psd}_{d,s}^\Gamma = \left\{ \sum_{i=1}^{n} A_i \otimes B_i \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) \;\middle|\; n \in \mathbb{N}, \sum_{i=1}^{n} A_i^T \otimes B_i \geqslant 0 \right\}.$$

Operator Systems Over the Psd Cone

We call the level-wise intersection

$$\mathrm{Dpsd}_{d,s} := \mathrm{Psd}_{d,s} \cap \mathrm{Psd}_{d,s}^{\Gamma}$$

the operator system of *doubly positive matrices* and write $\mathrm{Dpsd}_d = (\mathrm{Dpsd}_{d,s})_{s\in\mathbb{N}}$.

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The operator system of *decomposable matrices* $\mathrm{Decomp}_d = (\mathrm{Decomp}_{d,s})_{s\in\mathbb{N}}$ is defined as the level-wise Minkowski sum of the systems Psd_d and Psd_d^{Γ}, i.e. by

$$\mathrm{Decomp}_{d,s} := \left\{ X + Y \middle| X \in \mathrm{Psd}_{d,s}, Y \in \mathrm{Psd}_{d,s}^{\Gamma} \right\} .$$

Operator Systems Over the Psd Cone

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\begin{tikzcd}
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\end{tikzcd}
$$

Figure: Schematic representation of the relationships between the discussed cones.

Operator Systems Over the Psd Cone

Proposition

- *We have* $\mathrm{Sep}_d^{\vee\mathrm{fr}} = \mathrm{Bpsd}_d$ *and* $\mathrm{Bpsd}_d^{\vee\mathrm{fr}} = \mathrm{Sep}_d$. *Both systems are neither finitely generated nor finite-dimensional realizable.*
- $\mathrm{Psd}_d^{\vee\mathrm{fr}} = \mathrm{Psd}_d$, $(\mathrm{Psd}_d^{\Gamma})^{\vee\mathrm{fr}} = \mathrm{Psd}_d^{\Gamma}$, *and both systems are both finitely generated and finite-dimensional realizable.*
- $\mathrm{Dpsd}_d^{\vee\mathrm{fr}} = \mathrm{Decomp}_d$ *and* $\mathrm{Decomp}_d^{\vee\mathrm{fr}} = \mathrm{Dpsd}_d$. *Furthermore the system* Dpsd_d *is finite-dimensional realizable, the system* Decomp_d *is finitely generated.*

Operator Systems Over the Psd Cone - Main Result

Theorem (Berger & Netzer 2021 [BN21])

For $d \geq 2$, the operator system Decomp_d of decomposable matrices does not admit a finite-dimensional realization, and the operator system Dpsd_d of doubly positive matrices is not finitely generated.

Operator Systems Over the Psd Cone - Main Result

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Remark:

- Intersection of finitely generated operator systems $\not\Rightarrow$ finitely generated
- Sum of finite-dimensional realizable operator systems $\not\Rightarrow$ finite-dimensional realizable

Sketch of the Proof

1. Show Decomp_2 is not a free spectrahedron.
   - Recall that each level of a free spectrahedron is a classical one.
   - We exhibited a two-dimensional subspace on which $\mathrm{Decomp}_{2,2}$ is not a classical spectrahedron.
2. By embedding it follows that Decomp_d is not a free spectrahedron for every $d \geq 2$. Thus it is not a finite-dimensional realizable abstract operator system.
3. The result for Dpsd_d follows by free duality.

Sketch of the Proof

For $a, b \in \mathbb{R}$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \frac{a}{4} \\ 0 & \frac{1}{4} & \frac{a}{4} & 0 \\ 0 & \frac{a}{4} & b & 0 \\ \frac{a}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Mat}_2(\mathbb{C}) \tag{1}$$

is *decomposable* if and only if

$$(a, b) \in S_1 \cup S_2$$

where

$$\begin{aligned} S_1 &:= \{(a, b) \in \mathbb{R}^2 \mid (b + 1 - a^2)^2 - 4b \leq 0\}, \\ S_2 &:= \{(a, b) \in \mathbb{R}^2 \mid (b + 1 - a^2)^2 - 4b \geq 0, b \geq 0, a^2 - b - 1 \leq 0\}. \end{aligned} \tag{2}$$

Sketch of the Proof

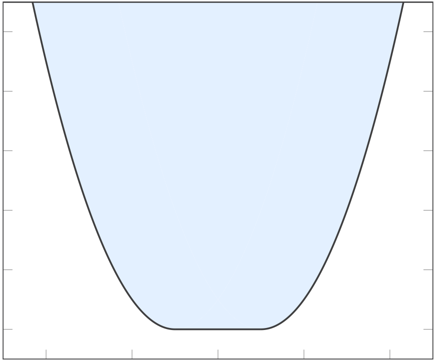
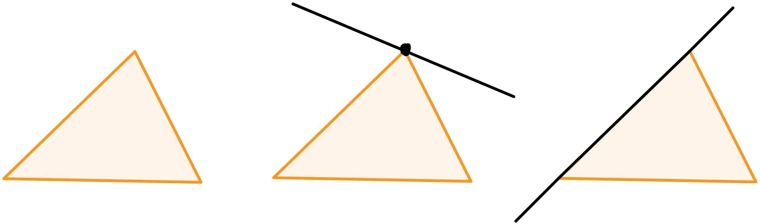


Figure: Section of the area $S_1 \cup S_2$.

Sketch of the proof - Exposed Faces

Definition

A proper face F of a convex set C is called exposed, if there exists a supporting hyperplane which touches C precisely at F.



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Theorem ([RG95])

Every face of a spectrahedron is exposed.

Sketch of the Proof

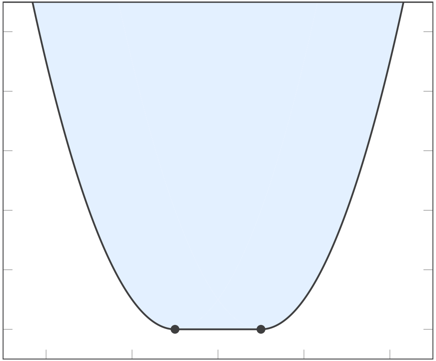

Figure: Section of the area $S_1 \cup S_2$.

Sketch of the Proof

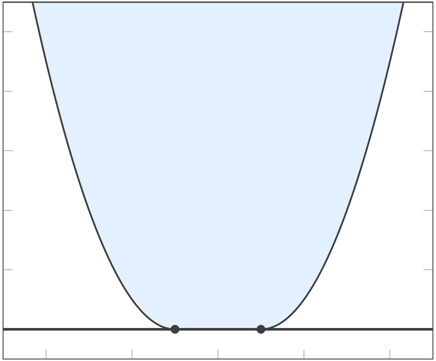

Figure: Section of the area $S_1 \cup S_2$.

Thank you!

Appendix - Algebraic Boundary

Definition

Let $S \subseteq \mathbb{R}^d$ be a semialgebraic set.

(*i*) The *algebraic boundary* $\partial_a S$ of S is the Zariski closure in $\mathbb{A}^n$ of its boundary ∂S in the Euclidean topology.

(*ii*) S is called *regular*, if it is contained in the closure of its interior (w.r.t. the Euclidean topology).

Proposition ([Sin11])

Let $S \subseteq \mathbb{R}^d$ be a nonempty regular semialgebraic set, and suppose that its complement $\mathbb{R}^d \setminus S$ is also regular and nonempty. If the interior of S intersects the algebraic boundary of S in a regular point, then S is not basic closed semialgebraic.

Appendix - Algebraic Boundary

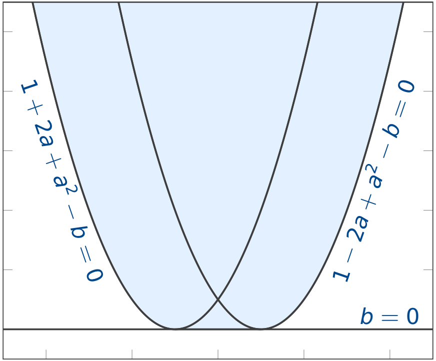

Figure: Section of the area $S_1 \cup S_2$ with its algebraic boundary.

Appendix - Subspace

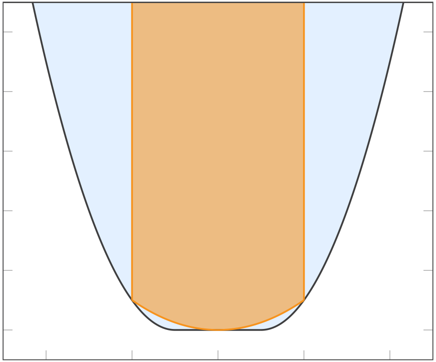

Figure: The sets $\mathsf{Sep}_{2,2} = \mathsf{Dpsd}_{2,2} = \mathsf{Psd}_{2,2} = \mathsf{Psd}^{\Gamma}_{2,2}$ (orange) and $\mathsf{Decomp}_{2,2}$ (blue) in our two-dimensional subspace.

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