ECEn 370

Homework #6 | Days 16, 17, 18 | Benjamin Bergeson

4.1.4 Q1

(a) $\int_{-\infty}^{\infty} f_X(u)\,du = 1 = \int_{-1}^{1} cu^2\,du = c\int_{-1}^{1} u^2 du = c\left[\frac{u^3}{3}\right]_{-1}^{1} = c\left[\frac{1}{3}+\frac{1}{3}\right] = \frac{2}{3}c = 1 \quad c = \frac{3}{2}$

(b) $EX = \int_{-\infty}^{\infty} u f_X(u)\,du = \int_{-1}^{1} u\,cu^2\,du = \frac{3}{2}\int_{-1}^{1} u^3 du = \frac{3}{2}\left[\frac{u^4}{4}\right]_{-1}^{1} = \frac{3}{2}\left[\frac{1}{4}-\frac{1}{4}\right] = 0$

$Var(X) = E[X^2] - (EX)^2 = E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)\,du = \frac{3}{2}\int_{-1}^{1} u^4 du = \frac{3}{2}\left[\frac{u^5}{5}\right]_{-1}^{1}$

$= \frac{3}{2}\left[\frac{1}{5}+\frac{1}{5}\right] = \frac{3}{2}\left[\frac{2}{5}\right] = \frac{3}{5}$

(c) $P(X \geq \frac{1}{2}) = \int_{1/2}^{\infty} f_X(u)\,du = \int_{1/2}^{1} \frac{3}{2}u^2 du = \frac{3}{2}\left[\frac{u^3}{3}\right]_{1/2}^{1} = \frac{3}{2}\left[\frac{1}{3} - \frac{1/8}{3}\right] = \frac{3}{2}\left[\frac{1}{3} - \frac{1}{24}\right]$

$= \frac{3}{2}\left[\frac{8-1}{24}\right] = \frac{3}{2}\left[\frac{7}{24}\right] = \frac{7}{16}$

4.1.4 Q3

$P(X \leq \frac{2}{3} \mid X > \frac{1}{3}) = \dfrac{P(\frac{1}{3} < X \leq \frac{2}{3})}{P(X > \frac{1}{3})}$

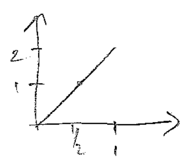
$P(\frac{1}{3} < X \leq \frac{2}{3}) = \int_{1/3}^{2/3} 4x^3 dx = \frac{5}{27}$

$P(X > \frac{1}{3}) = \int_{1/3}^{1} 4x^3 dx = \frac{80}{81}$

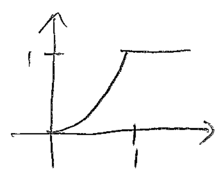
$\frac{5}{27} \cdot \frac{81}{80} = \boxed{\frac{3}{16}}$

Schaum's 2.22

(a) $\int_0^1 kx\,dx = k\left[\frac{x^2}{2}\right]_0^1 = k\left[\frac{1}{2}\right] = \frac{1}{2}k = 1 \quad k = 2$



$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$



(b) $F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$

(c) $P(\frac{1}{4} < X \leq 2) = \int_{1/4}^{1} 2x\,dx = 2\left[\frac{x^2}{2}\right]_{1/4}^{1} = 2\left[\frac{1}{2} - \frac{1/16}{2}\right] = 2\left[\frac{16}{32} - \frac{1}{32}\right] = 2\left[\frac{15}{32}\right] = \boxed{\frac{15}{16}}$

Schaum's 2.33

$$E\,X = \int_{-\infty}^{\infty} u f_X(u)\,du = 2\int_0^1 u^2\,du = 2\left[\frac{u^3}{3}\right]_0^1 = 2\left[\frac{1}{3}\right] = \frac{2}{3}$$

$$E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)\,du = \int_0^1 2u^3\,du = 2\left[\frac{u^4}{4}\right]_0^1 = 2\left[\frac{1}{4}\right] = \frac{1}{2}$$

$$Var(X) = E[X^2] - (E\,X)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

4.1.4 Q2

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}e^{-|x|}dx = \int_0^{\sqrt{y}} \frac{1}{2}e^{-x}dx = 1 - e^{-\sqrt{y}}$$

$$F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

4.1.4 Q4

$$f_X(x) = \begin{cases} x^2(2x + 3/2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad Y = \frac{2}{X} + 3$$

$$Var(Y) = Var\left(\frac{2}{X} + 3\right) = 4\,Var\left(\frac{1}{X}\right) \qquad Var\left(\frac{1}{X}\right) = E\left[\frac{1}{X^2}\right] - \left(E\left[\frac{1}{X}\right]\right)^2$$

$$E\left[\frac{1}{X}\right] = \int_0^1 x(2x + \tfrac{3}{2})\,dx = \int_0^1 2x^2 + \tfrac{3}{2}x\,dx = \frac{17}{12}$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 2x + \tfrac{3}{2}\,dx = \frac{5}{2}$$

$$Var\left(\frac{1}{X}\right) = \frac{5}{2} - \frac{289}{144} = \frac{71}{144} \qquad Var(Y) = \frac{71}{36}$$

Schaum's 4.3

(a) $Y = aX + b$ $\quad F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$ If $a > 0$

If $a < 0$ $\quad F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \ge \frac{y-b}{a}\right) = 1 - P\left(X < \frac{y-b}{a}\right) = 1 - P\left(X \le \frac{y-b}{a}\right) + P\left(X = \frac{y-b}{a}\right)$

If X is continuous $P\left(X = \frac{y-b}{a}\right) = 0$

$$F_Y(y) = 1 - P\left(X \le \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

(b) $f_Y(y) = f_X(x_1) \cdot \left|\frac{dx_1}{dy}\right| \qquad x_1 = \frac{y-b}{a} \quad \frac{dx_1}{dy} = \frac{1}{a}$

$$= f_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right|$$

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Homework #6

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Schaum's 4.6

$Y = X^2 \qquad X = \pm\sqrt{Y}$

$x_1 = \pm\sqrt{y} \qquad \frac{dx_1}{dy} = \mp\frac{1}{2\sqrt{y}} \qquad \left|\frac{dx_1}{dy}\right| = \frac{1}{2\sqrt{y}}$

$$f_Y(y) = \begin{cases} \left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]\frac{1}{2\sqrt{y}} & y > 0 \\ 0 & y < 0 \end{cases}$$

4.1.4 Q6

$X \sim \text{Uniform}(-\frac{\pi}{2}, \pi) \qquad Y = \sin(X) \qquad X = \sin^{-1}(Y)$

$$f_X(x) = \begin{cases} \frac{1}{3\pi/2} & -\frac{\pi}{2} < x < \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{3\pi} & -\pi/2 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

for $y \in (-1, 0) \qquad x_1 = \sin^{-1}(y) \qquad \frac{dx_1}{dy} = \frac{1}{\sqrt{1-y^2}}$

$$f_Y(y) = \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}}$$

for $y \in (0, 1) \qquad x_1 = \sin^{-1}(y) \qquad \frac{dx_1}{dy} = \frac{1}{\sqrt{1-y^2}} \qquad x_2 = \pi - \sin^{-1}(y) \qquad \frac{dx_2}{dy} = \frac{1}{\sqrt{1-y^2}}$

$$f_Y(y) = \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}} + \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}} = \frac{4}{3\pi\sqrt{1-y^2}}$$

$$f_Y(y) = \begin{cases} \frac{2}{3\pi\sqrt{1-y^2}} & -1 < y < 0 \\ \frac{4}{3\pi\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Schaum's 4.4

$Y = aX + b \qquad X = \frac{Y-b}{a}$

$$f_X(x) = \begin{cases} \frac{1}{1-0} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$Y = b$ when $X = 0$

$Y = a + b$ when $X = 1$

$$f_Y(y) = \begin{cases} \left|\frac{1}{a}\right| \cdot 1 & y \in R_Y \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{|a|} & y \in R_Y \\ 0 & \text{otherwise} \end{cases}$$

For $a > 0 \qquad b < y < a + b$

For $a < 0 \qquad a + b < y < b$

$\boxed{\text{Schaum's } 4.5}$

$Y = N(a\mu + b;\ a^2\sigma^2)$

$Y = aX + b \qquad X = N(\mu,\ \sigma^2)$

$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \qquad x_1 = \frac{y-b}{a} \qquad \frac{dx_1}{dy} = \frac{1}{a}$

$f_Y(y) = \left|\frac{1}{a}\right| \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\frac{y-b}{a}-\mu)^2/(2\sigma^2)}$

$= \frac{1}{\sqrt{2\pi}\,\boxed{a\sigma}} e^{-(y-b-a\mu)^2/(2a^2\sigma^2)} = \frac{1}{\sqrt{2\pi}\,\boxed{a\sigma}} e^{-(y-\boxed{(a\mu+b)})^2/(\boxed{2a^2\sigma^2})}$

$Y = N(a\mu + b,\ a^2\sigma^2)$

$\boxed{\text{Schaum's } 4.8}$

$f_X(x) = \frac{1}{3} \qquad -1 < x < 2$

$Y = X^2 \qquad x_1 = \pm\sqrt{y} \qquad \frac{dx_1}{dy} = \frac{1}{2\sqrt{y}}$

$f_Y(y) = f_X(\pm\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \qquad y \in \mathbb{R}_Y$

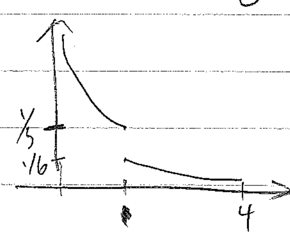
when $x < 0 \qquad y = \pm\sqrt{y}$

$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{1}{3\sqrt{y}}$

when $1 < y < 4 \qquad \sqrt{y}$

$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{3}\right) = \frac{1}{6\sqrt{y}}$

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & 0<y<1 \\ \frac{1}{6\sqrt{y}} & 1<y<4 \\ 0 & \text{otherwise} \end{cases}$$



$\boxed{\text{Schaum's } 4.9}$

$Y = e^X \qquad x_1 = \ln(y) \qquad \frac{dx_1}{dy} = \frac{1}{y}$

$f_X(x) = 1 \qquad 0 < x < 1$

$$f_Y(y) = \begin{cases} \left|\frac{1}{y}\right| & y \in \mathbb{R}_Y \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

when $x=0 \quad y=1 \quad$ when $x=1 \quad y=e$