ECEn 370

Homework #7

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4.4 Q4

$Y = e^{-X}$ $R_Y = [\frac{1}{e}, 1]$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(e^{-X} \le y) = P(-X \le \ln y) = P(X \ge \ln y) = 1 - P(X < \ln y)$$

$$= 1 - P(X \le \ln y) + P(X = \ln y) = 1 - P(X \le \ln y) = 1 - F_X(\ln y) = 1 - \ln y$$

(a)
$$F_Y(y) = \begin{cases} 0 & y < \frac{1}{e} \\ 1 - \ln y & \frac{1}{e} \le y \le 1 \\ 1 & y > 1 \end{cases}$$

(b)
$$f_Y(y) = \begin{cases} -\frac{1}{y} & \frac{1}{e} \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $EY = E[e^{-X}] = \int_{-\infty}^{\infty} e^{-x} f_X(x)\,dx = \int_0^1 e^{-x}\,dx = 1 - \frac{1}{e}$

4.4 Q5

(a) $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_0^x \frac{5}{32} u^4\,du = \frac{x^5}{32}$ $R_Y = [0, 4]$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x^5}{32} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{32} y^{5/2} & 0 \le y \le 4 \\ 1 & y > 4 \end{cases}$$

(b)
$$f_Y(y) = \begin{cases} \frac{5}{64} y^{3/2} & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(c) $EY = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^2 x^2 \frac{x^5}{32}\,dx = 1$

4.4 Q6

$X \sim \text{Exponential}(\lambda)$ $Y = 2X$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$f_Y(y) = f_X(x_1) / |g'(x_1)|$

$f_X(x_1) = \lambda e^{-\lambda x_1}$ $g'(x_1) = 2$

$f_Y(y) = \frac{\lambda}{2} e^{-\lambda x_1} = \frac{\lambda}{2} e^{-\frac{\lambda}{2} y}$

$$f_Y(y) = \begin{cases} \frac{\lambda}{2} e^{-\frac{\lambda}{2} y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$Y \sim \text{Exponential}(\frac{\lambda}{2})$

$\boxed{4.4\ Q8}$ $X \sim N(3,9)$

(a) $P(X>0) = 1 - P(X\leq 0) = 1-\Phi\left(\frac{0-3}{3}\right) = 1-\Phi(-1)$

$= 1-(1-\Phi(1)) = \Phi(1) \approx 0.841345$

(b) $P(-3<X<8) = F_X(8) - F_X(-3) = \Phi\left(\frac{8-3}{3}\right) - \Phi\left(\frac{-3-3}{3}\right)$

$= \Phi\left(\frac{5}{3}\right) - \Phi(-2) = 0.92946$

(c) $P(X>5 \mid X>3) = \frac{P(X>5, X>3)}{P(X>3)} = \frac{P(X>5)}{P(X>3)}$

$\frac{1-F_X(5)}{1-F_X(3)} = \frac{1-\Phi\left(\frac{5-3}{3}\right)}{1-\Phi\left(\frac{3-3}{3}\right)} = \frac{0.252493}{0.5} = \boxed{0.505}$

$\boxed{4.4\ Q9}$ $X \sim N(3,9)$ $Y = 5 - X$

(a) $P(X>2) = 1-F_X(2) = 1-\Phi\left(\frac{2-3}{3}\right) = \boxed{0.631}$

(b) $P(-1<Y<3)$ $a=-1$ $b=5$

$Y \sim (-3+5, (-1)^2 9) = Y \sim (2,9)$

$P(-1<Y<3) = F_Y(3) - F_Y(-1) = \Phi\left(\frac{3-2}{3}\right) - \Phi\left(\frac{-1-2}{3}\right) = \boxed{0.472}$

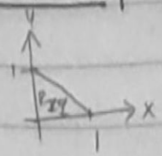
(c) $P(X>4 \mid Y<2) = P(X>4 \mid 5-X<2) = P(X>4 \mid X>3)$

$= \frac{P(X>4)}{P(X>3)} = \frac{1-F_X(4)}{1-F_X(3)} = \frac{1-\Phi\left(\frac{1}{3}\right)}{1-\Phi(0)} = \frac{0.369441}{0.5} = \boxed{0.739}$

5.2.5 Q1

$$f_{X,Y}(x,y) = \begin{cases} 3x+1 & x,y \ge 0,\ x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

1.



2. $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \int_0^{1-x} cx+1\,dy\,dx = \int_0^1 (cx+1)(1-x)\,dx$

$1 = \frac{c+3}{6} \quad c+3=6 \quad c=3$

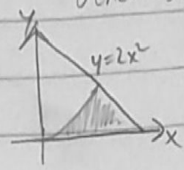
3. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{1-x} 3x+1\,dy = (3x+1)(1-x)$

$$f_X(x) = \begin{cases} (3x+1)(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 3x+1\,dx = \frac{1}{2}(1-y)(5-3y)$

$$f_Y(y) = \begin{cases} \frac{1}{2}(1-y)(5-3y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

4. $P(Y < 2X^2)$



$\int_{-\infty}^{\infty}\int_{-\infty}^{2x^2} 3x+1\,dy\,dx = \int_0^1\int_0^{\min(2x^2,1-x)} 3x+1\,dy\,dx$

$2x^2 = 1-x \quad 2x^2+x-1 \quad x = \frac{1}{2}$

$\int_0^1 \min(2x^2, 1-x)(3x+1)\,dx = \int_0^{1/2} 2x^2(3x+1)\,dx + \int_{1/2}^1 (1-x)(3x+1)\,dx$

$= \frac{17}{96} + \frac{3}{8} = \boxed{\frac{53}{96}}$

Schaum's 3.20

$$f_{XY}(x,y) = \begin{cases} k & 0<y\le x<1 \\ 0 & \text{otherwise} \end{cases}$$

(a) $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$

$= \int_0^1\int_0^x k\,dy\,dx = \int_0^1 kx\,dx = \frac{k}{2}$

$\frac{k}{2} = 1 \quad \boxed{k=2}$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^x 2\,dy = 2x$

$$f_X(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^1 2\,dx = 2(1-y)$

$$f_Y(y) = \begin{cases} 2(1-y) & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $P(0<X<\frac{1}{2}, 0<Y<\frac{1}{2}) = P(0<X<\frac{1}{2}, 0<Y<X)$

$$\int_0^{1/2}\int_0^{x} 2\,dy\,dx = \int_0^{1/2} 2x\,dx = \boxed{1/4}$$

$\boxed{\text{Schaum's } 3.21}$ $f_{XY}(x,y) = \begin{cases} k & x^2+y^2 \le R^2 \\ 0 & x^2+y^2 > R^2 \end{cases}$

(a)

$$\int_{-R^2}^{R^2}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} k\,dy\,dx = \int_{-R^2}^{R^2} 2k\sqrt{R^2-x^2}\,dx = k(\pi R^2) = 1$$

$$\boxed{k = \frac{1}{\pi R^2}}$$

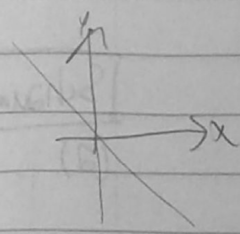
(b) $$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2}$$

$$f_X(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-x^2} & |x| \le R \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-y^2} & |y| \le R \\ 0 & \text{otherwise} \end{cases}$$

(c) $P(R^2 \le a) = P(X^2+Y^2 \le a) = \int$

$\boxed{\text{Schaum's } 3.17}$ $f_{X,Y}(x,y) = \begin{cases} k(x+y) & 0<x<2, 0<y<2 \\ 0 & \text{otherwise} \end{cases}$



(a) $$\int_0^2\int_0^2 k(x+y)\,dy\,dx = 8k \quad 8k=1 \quad \boxed{k=\frac{1}{8}}$$

(b) $$\int_0^2 \frac{1}{8}(x+y)\,dy = \frac{x+1}{4} \qquad f_X(x) = \begin{cases} \frac{x+1}{4} & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^2 \frac{1}{8}(x+y)\,dx = \frac{y+1}{4} \qquad f_Y(y) = \begin{cases} \frac{y+1}{4} & 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

(c) $f_{X,Y}(x,y) \ne f_X(x)f_Y(y)$ X and Y are not independent.

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Schaum's 3.18 $f_{X,Y}(x,y) = \begin{cases} kxy & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

(a) $\int_0^1\int_0^1 kxy\,dy\,dx = 1 = \frac{k}{4}$ $k=4$

(b) $f_X(x) = \int_0^1 4xy\,dy = 2x$ $f_X(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(y) = \int_0^1 4xy\,dx = 2y$ $f_Y(y) = \begin{cases} 2y & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

$f_X(x)f_Y(y) = 4xy$ $0<x<1,\ 0<y<1$ X and Y are independent

(c) $P(X+Y<1) = \int_0^1\int_0^{1-x} 4xy\,dy\,dx = \boxed{\frac{1}{6}}$

Schaum's 3.19 $f_{XY}(x,y) = \begin{cases} kxy & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$

(a) $\int_0^1\int_0^y kxy\,dx\,dy = 1 = \frac{k}{8}$ $\boxed{k=8}$

(b) $f_X(x) = \int_x^1 8xy\,dy = 4x(1-x^2)$ $f_X(x) = \begin{cases} 4x(1-x^2) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(y) = \int_0^y 8xy\,dx = 4y^3$ $f_Y(y) = \begin{cases} 4y^3 & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

$f_{XY}(x,y) \neq f_X(x)f_Y(y)$ So X and Y are not independent

Schaum's 3.28 $f_{X,Y}(x,y) = \begin{cases} k(x+y) & 0<x<2,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases}$

(a) $f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{(x+y)4}{8(x+1)} = \frac{1}{2}\frac{(x+y)}{x+1}$

$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{(x+y)4}{8(y+1)} = \frac{1}{2}\frac{(x+y)}{y+1}$

(b) $P(0<Y<\frac{1}{2} \mid X=1) = \int_0^{1/2} f_{Y|X}(y|x=1)\,dy = \int_0^{1/2} \frac{1}{2}\frac{1+y}{2}\,dy = \frac{5}{32}$

$\boxed{\text{Schaum's 3.29}}$ $f_{X,Y}(x,y) = \begin{cases} 4xy & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{4xy}{2x} = 2y \qquad 0<y<1,\ 0<x<1$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{4xy}{2y} = 2x \qquad 0<y<1,\ 0<x<1$$

$\boxed{\text{Schaum's 3.30}}$ $f_{XY}(x,y) = \begin{cases} 2 & 0<y\le x<1 \\ 0 & \text{otherwise} \end{cases}$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x} \qquad \begin{matrix} 0<x<1 \\ y\le x<1 \end{matrix}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y} \qquad \begin{matrix} 0<x<1 \\ y\le x<1 \end{matrix}$$