ECEn 370

Homework #11 | 32, 33, 34 | Benjamin Bergeson

Schaum's 2.36 $X = N(\mu;\sigma^2)$ (2.76) $\mu_X = E(X) = \mu$ (2.77) $\sigma_X^2 = Var(X) = \sigma^2$

$$\mu_X = E[X] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\,dx + \frac{\mu}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y e^{-y^2/(2\sigma^2)}\,dy + \mu\int_{-\infty}^{\infty} f_X(x)\,dx = \mu$$

$$\sigma_X^2 = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}\,dx$$

$$\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \sigma\sqrt{2\pi} \qquad \int_{-\infty}^{\infty}\frac{(x-\mu)^2}{\sigma^3} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \sqrt{2\pi}$$

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \sigma^2$$

Schaum's 2.51 $E[X] = 1000$ $Var(X) = 2500$ $A: \{X < 900\} \cup \{X > 1100\}$

$$P(A) = P(X<900) + P(X>1100) = F_X(900) + [1 - F_X(1100)]$$

$$F_X(900) = \Phi\left(\frac{900-1000}{\sqrt{2500}}\right) = \Phi(-2) = 1 - \Phi(2) = 1 - 0.97725 = 0.02275$$

$$[1 - F_X(1100)] = 1 - \Phi(2) = 0.02275 \qquad 0.02275 + 0.02275 = \boxed{0.0455}$$

Schaum's 4.1 $X = N(\mu,\sigma^2)$ $Z = (X-\mu)/\sigma$

$$F_Z(z) = P(Z \le z) = P\left(\frac{X-\mu}{\sigma} \le z\right) = P(X \le z\sigma + \mu) = \int_{-\infty}^{z\sigma+\mu}\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma)^2}\,dx$$

Let $y = \frac{x-\mu}{\sigma}$ $F_Z(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$ which is the CDF for $N(0,1)$

Schaum's 4.85 $P(Y \le y) \approx \Phi\left(\frac{y-np}{\sqrt{np(1-p)}}\right)$

$$Z_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - E(X_i)}{Var(X_i)}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - p}{\sqrt{p(1-p)}}\right) \qquad P(Z_n \le x) \approx \Phi(x)$$

$$P\left[\frac{1}{\sqrt{np(1-p)}}\left(\sum_{i=1}^{n}(X_i - p)\right) \le x\right] = P\left[Y \le x\sqrt{np(1-p)} + np\right] \approx \Phi(x)$$

$$P(Y \le y) \approx \Phi\left(\frac{y-np}{\sqrt{np(1-p)}}\right)$$

Schaum's 4.86 $P(Y \le y) \approx \Phi\left(\frac{y-\lambda}{\sqrt{\lambda}}\right)$

$$Z = \frac{Y - E(Y)}{\sqrt{Var(Y)}} = \frac{Y-\lambda}{\sqrt{\lambda}} \qquad P(Z \le z) \approx \Phi(z)$$

$$P\left(\frac{Y-\lambda}{\sqrt{\lambda}} \le z\right) = P(Y \le \sqrt{\lambda}z + \lambda) \approx \Phi(z) \qquad P(Y \le y) \approx \Phi\left(\frac{y-\lambda}{\sqrt{\lambda}}\right)$$

Schaum's 4.106 $P(X>520)$

(a) $P(Y\le y) \approx \Phi\left(\frac{y-np}{\sqrt{np(1-p)}}\right)$ (4.178)

$P(Y\le y) \approx \Phi\left(\frac{y+\frac{1}{2}-np}{\sqrt{np(1-p)}}\right)$ (4.181)

$P(X>520) = 1-P(X\le 520) \approx 1-\Phi\left(\frac{520-1000(0.5)}{\sqrt{1000(0.5)(0.5)}}\right) = 1-\Phi\left(\frac{20}{\sqrt{250}}\right) = 1-0.897048 = \boxed{0.102952}$

(b) $P(X>520) = 1-P(X\le 520) \approx 1-\Phi\left(\frac{520+\frac{1}{2}-(1000)(0.5)}{\sqrt{(1000)(0.5)(0.5)}}\right) = 1-\Phi\left(\frac{20.5}{\sqrt{250}}\right) = 1-0.902604 = \boxed{0.097396}$

Schaum's 4.107 $P(Y\le y) \approx \Phi\left(\frac{y-\lambda}{\sqrt{\lambda}}\right)$ (4.182) $P(Y\le y) \approx \Phi\left(\frac{y+\frac{1}{2}-\lambda}{\sqrt{\lambda}}\right)$

(a) $Y \sim \text{Poisson}(\lambda)$ $\quad Y \sim \text{Poisson}(r,t)$

$E[Y(\lambda)] = \lambda = rt$ $\quad E[Y(r,t)] = rt$

$Var[Y(\lambda)] = \lambda = rt$ $\quad Var[Y(r,t)] = rt$

$P(Y(r,t) > 200) = 0.90$ $\quad 0.1 = P(Y(r,t) \le 200) \approx \Phi\left(\frac{200-rt}{\sqrt{rt}}\right)$

$\frac{200-rt}{\sqrt{rt}} = -1.281552$ $\quad t = \frac{218.948}{100} = \boxed{2.18948}$

(b) $0.1 = P(Y(r,t) \le 201) \approx \Phi\left(\frac{y+\frac{1}{2}-\lambda}{\sqrt{\lambda}}\right) = \Phi\left(\frac{200+0.5-rt}{\sqrt{rt}}\right)$

$\frac{200.5-rt}{\sqrt{rt}} = -1.281552$ $\quad \boxed{t = 2.1947}$

>> X = inverse_F(rand(100000,1));

>> histogram(X, 'Normalization', 'probability', 'NumBins', 100); figure(1);