ECEn 370

Homework #9 Benjamin Bergeson

$\boxed{5.3.3\ Q1}$ $\text{Cov}(XY) = E[XY] - E[X]E[Y]$

$$\int_0^1\int_0^{1-x} 2xy\,dy\,dx = \frac{1}{12} = E[XY]$$

$$f_X(x) = \begin{cases} \int_0^{1-x} 2\,dy = 2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad f_Y(y) = \begin{cases} \int_0^{1-y} 2\,dx = 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_0^1 2x(1-x)\,dx = \frac{1}{3} \quad E[Y] = \frac{1}{3}$$

$$\text{Cov}(XY) = \frac{1}{12} - \frac{1}{9} = \boxed{-\frac{1}{36}}$$

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$E[X^2] = \int_0^1 2x^2(1-x)\,dx = \frac{1}{6}$$

$$\text{Var}(Y) = \frac{1}{18}$$

$$\rho(X,Y) = \frac{-\frac{1}{36}}{\sqrt{\frac{1}{18}\cdot\frac{1}{18}}} = \boxed{-\frac{1}{2}}$$

$\boxed{\text{Schaum's 3.32}}$

$$E[XY] = \sum_x\sum_y xy\,f_{XY}(x,y) = \sum_x\sum_y xy\,f_X(x)\,f_Y(y) = \sum_x x f_X(x) \sum_y y f_Y(y) = E[X]E[Y]$$

$$E[XY] = \int_x\int_y xy\,f_X(x)f_Y(y)\,dy\,dx = \int_x\int_y xy\,f_X(x)f_Y(y)\,dy\,dx = \int_x x f_X(x)\,dx \int_y y f_Y(y)\,dy = E[X]E[Y]$$

$\boxed{\text{Schaum's 3.33}}$

$$f_X(x) = \begin{cases} \frac{1}{3} & x = 0,1,2 \\ 0 & \text{otherwise} \end{cases} \quad f_Y(y) = \begin{cases} \frac{1}{3} & y = 0 \\ \frac{2}{3} & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

$f_{XY}(x,y) \ne f_X(x)f_Y(y)$ not independent

$$E[XY] = (0)(1)(\tfrac{1}{3}) + (1)(0)(\tfrac{1}{3}) + (2)(1)(\tfrac{1}{3}) = \frac{2}{3}$$

$$E[X] = (0)(\tfrac{1}{3}) + (1)(\tfrac{1}{3}) + (2)(\tfrac{1}{3}) = \tfrac{1}{3} + \tfrac{2}{3} = 1$$

$$E[Y] = (0)(\tfrac{1}{3}) + (1)(\tfrac{2}{3}) = \tfrac{2}{3}$$

$$\text{Cov}(X,Y) = \tfrac{2}{3} - (1)(\tfrac{2}{3}) = 0$$ Uncorrelated.

$\boxed{\text{Schaum's 3.34}}$

$$f_X(x) = \int_{-\infty}^{\infty} \frac{x^2+y^2}{4\pi} e^{-(x^2+y^2)/2}\,dy =$$

Schaum's 3.36

$-1 \le \rho_{XY} \le 1$ $\quad [E(XY)]^2 \le E(X^2)E(Y^2) \Rightarrow \sigma_{XY}^2 \le \sigma_X^2 \sigma_Y^2$

$\frac{\sigma_{XY}^2}{\sigma_X^2 \sigma_Y^2} \le 1 \quad \rho_{XY}^2 \le 1 \quad |\rho_{XY}| \le$

5.3.3 Q5

$\mathrm{cov}(X+Y, X-Y) = \mathrm{cov}(X,X) - \mathrm{cov}(X,Y) + \mathrm{cov}(X,Y) - \mathrm{cov}(Y,Y)$

$= Var(X) - 0 + 0 - Var(Y) = 0$

5.3.3 Q6

$\mu_X = 0 \quad \sigma_X^2 = 1 \quad \mu_Y = -1 \quad \sigma_Y^2 = 4 \quad \rho = -\frac{1}{2}$

(a) $U = X+Y \quad EU = EX + EY = 0 - 1 = -1$

$Var(U) = Var(X) + Var(Y) + 2\,\mathrm{cov}(X,Y) = 1 + 4 + 2\sigma_X\sigma_Y\rho(X,Y)$

$= 5 - (2)(1)(2)(\frac{1}{2}) = 3$

$U \sim N(-1, 3)$

$P(U>0) = 1 - \Phi\left(\frac{0-(-1)}{\sqrt{3}}\right) = 1 - \Phi(\frac{1}{\sqrt{3}}) = 0.2819$

(b) $\mathrm{Cov}(aX+Y, X+2Y) = 0$

$\mathrm{Cov}(aX+Y, X+2Y) = a\,\mathrm{Cov}(X,X) + 2a\,\mathrm{cov}(X,Y) + \mathrm{cov}(X,Y) + 2\,\mathrm{cov}(Y,Y)$

$= a(1) + 2a(-\frac{1}{2}) + -\frac{1}{2} + 2(4) = -a + 7 \quad a = 7$

(c) $U = X+Y \quad V = 2X - Y$

$EU = -1 \quad EV = 2EX - EY = 2(0) + 1 = 1$

$Var(U) = 3 \quad Var(V) = 2Var(X) - Var(Y) \cancel{+} 2\,\mathrm{cov}(X,Y) = 12$

$Var(2X-Y) = E[(2X-Y)^2] = E[4X^2 - 4XY + Y^2] = E[4X^2] - 4E[XY] + E[Y^2]$

$= 4Var(X) - 4\,\mathrm{cov}(XY) + Var(Y) = 4(1) - 4\sigma_X\sigma_Y\rho(X,Y) + 4$

$= 8 - (4)(1)(2)(-\frac{1}{2}) = 8 + 4 = 12$

$\mathrm{cov}(U,V) = \mathrm{cov}(X+Y, 2X-Y) = 2\,\mathrm{cov}(X,X) - \mathrm{cov}(X,Y) + 2\,\mathrm{cov}(X,Y) - \mathrm{cov}(Y,Y)$

$= 2Var(X) + \mathrm{cov}(X,Y) - Var(Y) = 2(1) - 4 + (-\frac{1}{2})(1)(2) = -2 - 1 = -3$

$\rho(U,V) = \frac{-3}{\sqrt{(3)(12)}} = \frac{-3}{6} = -\frac{1}{2}$

$E[U|V=0] = \mu_U + \rho(U,V)\sigma_U \frac{0-\mu_V}{\mu_V} = -\frac{3}{4} \quad Var(U|V=0) = (1-\rho_{UV}^2)\sigma_U^2 = \frac{9}{4}$

$P(X+Y > 0 \mid 2X - Y = 0) = P(U>0 \mid V=0) = 1 - \Phi\left(\frac{0-(-3/4)}{3/2}\right)$

$= 1 - \Phi(\frac{1}{2}) = 0.3085$

$\boxed{\text{Schaum's 3.49}}$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2}q(x,y)\right] \qquad q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[\frac{-(x-\mu_x)^2}{2\sigma^2 x}\right]$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[\frac{-(y-\mu_y)^2}{2\sigma_y^2}\right]$$

(b) $$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}q(x,y)\right] = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$

$f_{XY}(x,y) = f_X(x)\, f_Y(y)$ *Independent*

$\boxed{\text{Schaum's 3.51}}$ $E(Y \mid x)$

$$E(Y \mid x) = \int_{-\infty}^{\infty} y\, f_{Y|x}(y \mid x)\,dy \qquad f_{Y|x}(y \mid x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_{Y|x}(y \mid x) = \frac{1}{\sqrt{2\pi}\sigma_y(1-\rho^2)^{1/2}} \exp\left\{\frac{1}{2\sigma_y^2(1-\rho^2)}\left[y - \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x) - \mu_y\right]^2\right\}$$

$$E(Y \mid x) = \mu_Y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)$$

$\boxed{\text{6.2.6 Q1}}$

$0.2 + 0.2 + 0.2 + 0.2 = 0.8$ 80%

$\boxed{\text{Schaum's 3.54}}$

$$K = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix} \qquad |\det K|^{1/2} = \sigma_1\sigma_2\cdots\sigma_n = \prod_{i=1}^{n}\sigma_i \qquad K^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_n^2} \end{bmatrix}$$

$$(x-\mu)^T K^{-1}(x-\mu) = \sum_{i=1}^{n}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2 \qquad f_{x_1\cdots x_n}(x_1\cdots x_n) = \frac{1}{(2\pi)^{n/2}\left(\prod_{i=1}^{n}\sigma_i\right)} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right]$$

$$f_{x_1\cdots x_n}(x_1\cdots x_n) = \prod_{i=1}^{n} f_{x_i}(x_i) \qquad f_{x_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[\frac{-(x_i-\mu_x)^2}{2\sigma_i^2}\right]$$