1 Complete Elliptic Integrals of First, Second, and Third Kind

$$
\begin{aligned}
\mathbf{G}(k',p,a,b) &= \int_0^\infty \frac{a+b\xi^2}{(1+p\xi^2)\sqrt{(1+\xi^2)(1+k'^2\xi^2)}}\,d\xi \\
&= \int_0^{\pi/2} \frac{a\cos^2\phi + b\sin^2\phi}{\cos^2\phi + p\sin^2\phi}\frac{d\phi}{\sqrt{\cos^2\phi + k'^2\sin^2\phi}} \qquad (k'^2>0).
\end{aligned}
$$

For $p<0$, this integral is defined by its principal value. See **Notes** for special cases.

The functions K(k) and E(k):

$$
\begin{aligned}
\mathrm{K}(k) &= \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \qquad (|k|<1), \\
\mathrm{E}(k) &= \int_0^{\pi/2} \sqrt{1-k^2\sin^2\psi}\,d\psi \qquad (|k|\le 1).
\end{aligned}
$$

Other common definitions of the complete elliptic integrals and their relations to $\mathbf{F}_1^*$, $\mathbf{F}_2^*$, $\mathbf{F}_3^*$ are listed here for convenience ($k^2+k'^2=1$):

First kind:

$$
F(k,\pi/2) = \mathrm{K}(k) = \mathbf{F}_1^*(k') \qquad (|k|<1),
$$

$$
\widehat{F}(1,k) = \int_0^1 \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}} = \mathbf{F}_1^*(k') \qquad (|k|<1).
$$

Second kind:

$$
E(k,\pi/2) = \mathrm{E}(k) = \mathbf{F}_2^*(k',1,k'^2) \qquad (|k|\le 1),
$$

$$
\widehat{E}(1,k) = \int_0^1 \sqrt{\frac{1-k^2\eta^2}{1-\eta^2}}\,d\eta = \mathbf{F}_2^*(k',1,k'^2) \qquad (|k|\le 1).
$$

Third kind:

$$
\begin{aligned}
\Pi(\pi/2,h,k) &= \int_0^{\pi/2} \frac{d\psi}{(1+h\sin^2\psi)\sqrt{1-k^2\sin^2\psi}} = \mathbf{F}_3^*(k',h+1) \qquad (|k|<1), \\
\widehat{\Pi}(1,h,k) &= \int_0^1 \frac{d\eta}{(1+h\eta^2)\sqrt{(1-\eta^2)(1-k^2\eta^2)}} = \mathbf{F}_3^*(k',h+1) \qquad (|k|<1).
\end{aligned}
$$

`FUNCTION` subprograms
User Entry Names:

The redundant parameter AK2 in RELI3C and DELI3C permits improved accuracy when k^2 is small, i.e. $k' \approx 1$. In this case, AK2 $= k^2$ should be calculated using higher-precision arithmetic and then truncated before calling the subprogram. Special examples are

$$\begin{aligned}
\mathrm{K}(k) &= \mathbf{G}(k', 1, 1, 1), \\
\mathrm{E}(k) &= \mathbf{G}(k', 1, 1, k'^2) \\
(\mathrm{K}(k) - \mathrm{E}(k))/k^2 &= \mathbf{G}(k', 1, 0, 1), \\
(\mathrm{K}(k) - k'^2\mathrm{E}(k))/k^2 &= \mathbf{G}(k', 1, 1, 0), \\
\Pi(h, k) &= \mathbf{G}(k', h+1, 1, 1), \\
(\mathrm{K}(k) - \Pi(h, k))/h &= \mathbf{G}(k', h+1, 0, 1),
\end{aligned}$$

If $ab \geq 0$ then $\mathbf{G}$ will evaluate any linear combination of $\mathrm{K}(k)$, $\mathrm{E}(k)$, $\Pi(h,k)$ without cancellation (such as would occur, for example, if $(\mathrm{K}(k){-}\mathrm{E}(k))/k^2$ were to be computed from values of $\mathrm{K}(k)$ and $\mathrm{E}(k)$ which had been computed separately.

Other functions which can be represented by $\mathbf{G}$ are the Jacobian Zeta function $\mathbf{Z}(\Phi, k)$ and the Heuman Lambda function $\Lambda_0(\Phi, k)$ (see Ref. 5):

$$\begin{aligned}
\mathbf{Z}(\Phi, k) &= k^2 \frac{\sin\Phi\cos\Phi}{\mathrm{K}(k)} \mathbf{G}(k', q, 0, \sqrt{q}) \qquad (q = \cos^2\Phi + k'^2 \sin^2\Phi) \\
\Lambda_0(\Phi, k) &= \frac{2}{\pi}\sqrt{q}\sin\Phi \; \mathbf{G}(k', q, 1, k'^2) \qquad (q = 1 + k^2\tan^2\Phi).
\end{aligned}$$

(Quoted from Ref. 3, slightly modified).

The subprograms for $\mathbf{F}_1^*$, $\mathbf{F}_2^*$ are based on the Algol60 procedures *cel1, cel2* in Ref. 1, those for $\mathbf{F}_3^*$ on *cel3* in Ref. 2, and those for $\mathbf{G}$ on *cel* in Ref. 3.

1. R. Bulirsch, Numerical calculation of elliptic integrals and elliptic functions, Numer. Math. **7** (1965) 78–90.

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4. W.J. Cody, Chebyshev approximations for the complete elliptic integrals K and E, Math. Comp. **19** (1965) 105–112.

5. P.F. Byrd and M.D. Friedman, Handbook of elliptic integrals for engineers and scientists, 2nd ed., Springer-Verlag Berlin (1971) 33–37.