Momentum

Introduction

As was pointed out in the previous chapter, some of the most powerful tools in physics are based on **conservation principles**. The idea behind a conservation principle is that there are some properties of systems that don't change, even though other things about the system may. In this chapter we will be introducing the concept of **momentum** which, like energy, is a conserved property of any closed system. The **Conservation of Momentum principle** is a very powerful tool in physics and is, as will be shown later, a necessary consequence of Newton's Second and Third Laws.

Momentum of a Single Object

The momentum of a single object is simply equal to the product of its mass and its velocity. The symbol for momentum is "p". Since mass is a scalar and velocity is a vector, momentum is also a vector. The direction of momentum is always the same as that of the object's velocity.

$\mathbf{p} = m\mathbf{v}$

Momentum is a vector so it has a magnitude and a velocity.
Its magnitude is the product of its mass and velocity, $p = mv$.
Its direction is the same as the direction of its velocity.

Units of Momentum

The unit of momentum can be derived from the above equation.
$p = mv$

The SI units of mass are kilograms (kg) and of velocity are meters / second (m/s). Therefore, the units of momentum are kg· m/s.

There is no special name for the unit of momentum.

Example 1: A 20 kg object has a velocity of 4.5 m/s in the positive x direction. What is its momentum?
$\mathbf{p} = m\mathbf{v}$
$p = (20\text{kg})(4.5\text{ m/s})$
$p = 90\text{ kg}\cdot\text{m/s}$
$\mathbf{p} = 90\text{ kg}\cdot\text{m/s}$ *in the positive x direction*

Example 2: A 60 kg object has a velocity of 1.5 m/s in the negative x direction. What is its momentum?
$\mathbf{p} = m\mathbf{v}$
$p = (60\text{kg})(-1.5\text{ m/s})$
$= -90\text{ kg}\cdot\text{m/s}$
$\mathbf{p} = 90\text{ kg}\cdot\text{m/s}$ *in the negative x direction*

Please notice two things in the above examples. First, each answer required a magnitude and a direction. Second, since momentum is the product of mass and velocity, objects of different mass can have equal amounts of momentum. That would just require the less massive object to have greater velocity.

Momentum of a Closed System of Objects

To determine the momentum of a system that has more than one object you have to add together the momenta of all the individual objects.

$\mathbf{p}_{system} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots$
$\mathbf{p}_{system} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \ldots$

Or

$\mathbf{p}_{system} = \Sigma\mathbf{p}$
$\mathbf{p}_{system} = \Sigma m\mathbf{v}$

So if a system were comprised of the objects in Example 1 and 2, above, then the momentum of that total system would simply be the sum of those two momenta.

Example 3: Determine the momentum of a system that consists of the two objects from Example 1 and 2, above.

$\mathbf{p}_{system} = \Sigma\mathbf{p}$
$\mathbf{p}_{system} = \mathbf{p}_1 + \mathbf{p}_2$
$p_{system} = (90\ kg\cdot m/s) + (-90\ kg\cdot m/s)$
$\mathbf{p}_{system} = 0$

Example 4: Determine the momentum of a system that consists of the two objects. One object, m1, has a mass of 2 kg and a velocity of +5 m/s and the second object, m2, has a mass of 20 kg and a velocity of +3 m/s.

$\mathbf{p}_{system} = \Sigma\mathbf{p}$
$\mathbf{p}_{system} = \mathbf{p}_1 + \mathbf{p}_2$
$\mathbf{p}_{system} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$
$p_{system} = (2\ kg)\ (+5\ m/s) + (20\ kg)(+3\ m/s)$
$p_{system} = (10\ kg\cdot m/s) + (60\ kg\cdot m/s)$
$p_{system} = 70\ kg\cdot m/s$
$\mathbf{p}_{system} = 70\ kg\cdot m/s$ *in the positive x direction*

Example 5: Determine the momentum of a system that consists of the two objects. One object, m1, has a mass of 4 kg and a velocity of 17 m/s towards the east and the second object, m2, has a mass of 70 kg and a velocity of 4 m/s towards the east.

First, we need to decide how to orient our positive and negative axes. The simplest choice would be motion towards the east positive. Then, both velocities will be positive.

$\mathbf{p}_{system} = \Sigma\mathbf{p}$
$\mathbf{p}_{system} = \mathbf{p}_1 + \mathbf{p}_2$
$\mathbf{p}_{system} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$
$p_{system} = (4\ kg)\ (17\ m/s) + (70\ kg)(4\ m/s)$
$p_{system} = (68\ kg\cdot m/s) + (280\ kg\cdot m/s)$
$p_{system} = 348\ kg\cdot m/s$
$\mathbf{p}_{system} = 348\ kg\cdot m/s$ *towards the east*

Please note that in the last step we translated back to the directions that were given in the problem. The person who wrote the problem doesn't know which way we decided was positive, so we have to give our answer based on what we were give, in this case east not positive.

Example 6: Determine the momentum of a system that consists of the two objects. One object, m1, has a mass of 4 kg and a velocity of 17 m/s towards the east and the second object, m2, has a mass of 70 kg and a velocity of 14 m/s towards the west.

First, we need to decide how to orient our axes. The simplest choice would be to make motion towards the east positive. That means that motion in the opposite direction, to the west, will be negative. Then,

$\mathbf{p}_{system} = \Sigma\mathbf{p}$
$\mathbf{p}_{system} = \mathbf{p}_1 + \mathbf{p}_2$
$\mathbf{p}_{system} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$
$p_{system} = (4\ kg)\ (17\ m/s) + (70\ kg)(-4\ m/s)$
$p_{system} = (68\ kg\cdot m/s) + (-280\ kg\cdot m/s)$
$p_{system} = -212\ kg\cdot m/s$
$\mathbf{p}_{system}$ = *212 kg· m/s towards the west*

Once again, note that in our last step we interpret the negative direction to mean towards the west.

Conservation of Momentum in a closed system

We learned in the previous chapter that the energy of a closed system can only be changed by an outside force doing **Work** on it. The change in the energy of the system is equal to the work done by that outside force.

Similarly, the momentum of a closed system can only be changed by an outside force providing an **impulse** to it. The change in the momentum of the system is equal to the impulse provided by that outside force. This can be stated symbolically as:

$\mathbf{p}_0 + \mathbf{I} = \mathbf{p}_f$

where **I** is the symbol for impulse, $\mathbf{p}_0$, represents the initial momentum of the system and $\mathbf{p}_f$ represents the final momentum of that same system.

In many cases, there will be no outside forces acting on a closed system. In those cases, the momentum will not change regardless of what goes on within the system. Let's first look at those cases, where the impulse provided a system is zero.

$\mathbf{p}_0 + \mathbf{I} = \mathbf{p}_f$

but

$\mathbf{I} = 0$

so

$\mathbf{p}_0 = \mathbf{p}_f$

This means that if we measure the total momentum of a system at any point in time, its momentum will not change if it is not affected by something outside the system. The objects can collide, explode, break apart, stick together, etc. Nothing that happens within the system will change its momentum.

When we're doing the following example we're going to change how we indicate initial and final. That's because, as you'll see, the subscripts can get pretty confusing. There are just too many of them. So, the first time, we'll first write out $\mathbf{v}_{o1}$ for the initial velocity of the first object and $\mathbf{v}_{f1}$ for its final velocity. Then we're going to change to easier notation that we'll use from now on in doing momentum problems. We'll indicate the initial velocity of the first object with $\mathbf{v}_1$ and its final velocity with $\mathbf{v}'_1$. We think you'll find it's easier to keep track of everything that way. Especially as you write out your own work.

Example 7: A closed system consists of two objects. One object, m_1, has a mass of 15 kg and a velocity of 6 m/s towards the east and the second object, m_2, has a mass of 20 kg and a velocity of 3 m/s towards the west. These two objects collide and stick together. What is the velocity of the combined object?

Once again, let's choose east to be positive and west to be negative. Then,

$\boldsymbol{p}_0 + \boldsymbol{I} = \boldsymbol{p}_f$

Since there are no external forces, I = 0 so

$\boldsymbol{p_0} = \boldsymbol{p_f}$

$m_1\mathbf{v}_{o1} + m_2\mathbf{v}_{o2} = m_1\mathbf{v}_{f1} + m_2\mathbf{v}_{f2}$

Simplifying the subscripts gives us

$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$

Since the objects stick together they must have the same velocity afterwards, $\mathbf{v}'_1 = \mathbf{v}'_2 = \mathbf{v}'$

$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}' + m_2\mathbf{v}'$
$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v}'$
$\mathbf{v}' = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2) / (m_1 + m_2)$
$\mathbf{v}' = ((15\ kg)(6\ m/s) + (20\ kg)(-3\ m/s)) / (15\ kg + 20\ kg)$
$\mathbf{v}' = ((90\ kg\cdot m/s) + (-60\ kg\cdot m/s)) / (35\ kg)$
$\mathbf{v}' = (30 kg\cdot m/s) / (35\ kg)$
$\mathbf{v}' = 0.86\ m/s$
$\mathbf{v}'$ *= 0.86 m/s towards the east*

Example 8: A closed system consists of a stationary object that explodes into two pieces. After the explosion, one piece, m_1, has a mass of 25 kg and a velocity of 8 m/s towards the north and the second object, m_2, has a mass of 40 kg. Determine the velocity of the second piece.

Let's choose north to be positive and south to be negative. Then,

$\boldsymbol{p_0} + \boldsymbol{I} = \boldsymbol{p_f}$

Since there are no external forces, I = 0 so

$\boldsymbol{p_0} = \boldsymbol{p_f}$

Since the object had no initial velocity, $p_0 = 0$

$0 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$

Then we solve for $\mathbf{v}'_2$

$m_2\mathbf{v}'_2 = - m_1\mathbf{v}'_1$
$\mathbf{v}'_2 = (- m_1\mathbf{v}'_1) / m_2$

And then substitute in numbers

$v' = (- (25\ kg)(8\ m/s) / (4\ kg)$
$v' = (- (25\ kg)(8\ m/s)) / (4\ kg)$
$v' = (- 200\ kg\cdot m/s) / (4\ kg)$
$v' = (- 50\ m/s)$

Then we interpret our negative velocity in terms of direction.

$\mathbf{v}'$ *= 50 m/s towards the south*

So this is how you solve all momentum problems if no outside forces are acting on the system. There are a whole class of problems, called Collision problems where this is the case.

Collisions

The collisions described in this section are all between objects within a closed system. No external forces are affecting the system, so there is no impulse acting on the system, **I** = 0.
All collisions fall into one of three categories, **Perfectly Inelastic**, **Inelastic** or **Perfectly Elastic**. In the case of perfectly inelastic collisions, the colliding objects stick together. While momentum is conserved, as it is in all these cases, **mechanical energy is not conserved**. Some of the mechanical energy goes into heat, sound, deforming the material, etc. While the total energy is conserved, it is now in forms with which we haven't yet worked. That is, the total of the gravitational potential, kinetic and elastic potential energy is not conserved in these types of collisions.

That is true of any inelastic collision. While in the case of perfectly inelastic collisions the two colliding objects stick together, in the case of inelastic collisions, while they don't stick together, some mechanical energy is still lost to other forms.

The only case where mechanical energy is conserved is during a perfectly elastic collision. In this instance, the two objects bounce off each other with no loss of mechanical energy. (A good example of this is when billiard balls collided.) In that case, and in that case alone, we can combine our equations for conservation of energy with our equations for conservation of momentum.

Problem Solving Strategies for Collisions

The first step in solving a collision problem is to determine the type of collision. You can't assume that; the problem must tell you either directly or indirectly. Obviously if they simply tell you the type of collision, that's fine. On the other hand, they could tell you indirectly through hints. For instance, if you are told that the objects stick together, then you know it was a **perfectly inelastic collision** without being told that outright. From that information you know that mechanical energy was not conserved and, most importantly, that $\mathbf{v'_1} = \mathbf{v'_2} = \mathbf{v'}$. That last fact is critical to solving the problem. An example of this type can be found in Example 7.

If you are not told the type of collision, that the objects stick together or that mechanical energy is conserved, then you can only assume this was an **elastic collision**. In that case, they must tell you some more information in order to solve the problem, for instance, the velocity and mass of one of the objects after the collision. Without that added information these problems can't be solved.

Example 9: A bullet if fired at a piece of wood which is at rest on a frictionless surface. The bullet has a mass of 0.025 kg and a velocity of 300 m/s and the mass of the wood is 2.0 kg. After passing through the wood the velocity of the bullet is 200 m/s. What is the velocity of the wood?
Since they don't tell us that this is a perfectly elastic or perfectly inelastic collision, that the objects stick together or that energy is conserved, we have to assume it is an inelastic collision. But they did give us the added information of the velocity of the bullet after the collision.

$\mathbf{p_0} + \mathbf{I} = \mathbf{p_f}$

Since there are no external forces, I = 0 so

$\mathbf{p_0} = \mathbf{p_f}$
$m_b\mathbf{v}_b + m_w\mathbf{v}_w = m_b\mathbf{v'}_b + m_w\mathbf{v'}_w$
But the wood was originally at rest so $\mathbf{v}_w = 0$
$m_b\mathbf{v}_b = m_b\mathbf{v}_b' + m_w\mathbf{v}_w'$
$m_b\mathbf{v}_b - m_b\mathbf{v}_b' = m_w\mathbf{v}_w'$
$m_w\mathbf{v}_w = m_b\mathbf{v}_b - m_b\mathbf{v}_b'$
$\mathbf{v}_w = m_b(\mathbf{v}_b - \mathbf{v}_b) / m_w$
$v_w = (.025kg)(300\ m/s - 200m/s) / (2\ kg)$
$v_w = (2.5\ kg\cdot m/s) / (2\ kg)$

$\boldsymbol{v_w}$ = *(1.25 m/s) in the same direction as the bullet was traveling*

If you are told that mechanical energy is conserved, then you know it was a **perfectly elastic collision**. That allows you to use what we learned about conservation of energy in solving the problem. It turns out in this case that a very important general result can be obtained for perfectly elastic collisions. That is the difference of velocities prior to the collision is equal to the opposite of that difference after the collision. (This is derived at the end of this chapter.)
$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_2' - \mathbf{v}_1'$

Or another way of saying that is that the total of the velocities before and after the collision are the same for both objects.

$\mathbf{v}_1 + \mathbf{v}_1' = \mathbf{v}_2 + \mathbf{v}_2'$

Example 10: Two objects collide and bounce off of each other such that mechanical energy is conserved. One object, m_1, has a mass of 2.0 kg and a velocity of 8 m/s towards the east and the second object, m_2, has a mass of 4.0 kg and a velocity of 3 m/s towards the west. What are the velocities of the two objects after the collision?

Let's choose east to be positive. Then,

$\boldsymbol{p_0} + \boldsymbol{I} = \boldsymbol{p_f}$

Since there are no external forces, I = 0 so

$\boldsymbol{p_0} = \boldsymbol{p_f}$
$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$

Since the collision is elastic, $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_2' - \mathbf{v}_1'$

Solving for $\mathbf{v}_2'$ yields, $\mathbf{v}_2' = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_1'$

Then substituting that for $\mathbf{v}_2'$ yields

$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2(\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_1')$

Distributing m_2 yields

$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}_1 - m_2\mathbf{v}_2 + m_2\mathbf{v}_1'$

Leaving all the terms with $\mathbf{v}'_1$ *on the right but moving the rest to the left*

$m_1\mathbf{v}_1 - m_2\mathbf{v}_1 + m_2\mathbf{v}_2 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}_1'$

Factoring out and combining common terms

$(m_1 - m_2)\mathbf{v}_1 + 2m_2\mathbf{v}_2 = \mathbf{v}'_1(m_1 + m_2)$

Switching $\mathbf{v}'_1(m_1 + m_2)$ *to the right*

$\mathbf{v}'_1(m_1 + m_2) = (m_1 - m_2)\mathbf{v}_1 + 2m_2\mathbf{v}_2$

Solving form $\mathbf{v}'_1$ *by dividing by* $(m_1 + m_2)$

$\mathbf{v}'_1 = ((m_1 - m_2)\mathbf{v}_1 + 2m_2\mathbf{v}_2) / (m_1 + m_2)$
Substituting numbers

$\mathbf{v'_1}$ = *((2kg- 4kg)**(8m/s)** + **2**(4kg)(-**3m/s)** / **(**2 kg + 4 kg)*
v'_1 = *((-16 kg· m/s) + -24 kg· m/s) / (6 kg)*
v'_1 = *-6.7 m/s*

Then substituting this back into the equation we got from conservation of energy $\mathbf{v_2' = v_1 - v_2 + v_1'}$
v_2' = 8 m/s – (-3 m/s) + (-6.7 m/s)
v_2' = 8 m/s + 3 m/s -6.7 m/s
v_2' = 4.3 m/s
$\mathbf{v'_1}$ = *6.7 m/s towards the west*
$\mathbf{v_2'}$ = 4.3 m/s towards the east

Change of Momentum and Impulse

If no external force acts on a closed system, its momentum will not change. However, if an external force does act on a system, the systems momentum will be altered changed by the Impulse applied by that external force. So the initial amount of momentum of a system can be altered by an amount equal to the Impulse applied. The result is that the final momentum of the system has changed.

$\mathbf{p_0} + \mathbf{I} = \mathbf{p_f}$
$\mathbf{I} = \mathbf{p_f} - \mathbf{p_0}$

Now let's use Newton's Second Law to get an equation for Impulse (**I**).

We know that p = mv, so let's substitute that in for p

$\mathbf{I} = m\mathbf{v_f} - m\mathbf{v_0}$

Now factor out m

$\mathbf{I} = m\,(\mathbf{v_f} - \mathbf{v_0})$
$\mathbf{I} = m\,\Delta\mathbf{v}$

But acceleration is defined as $\mathbf{a} = \Delta\mathbf{v} / \Delta t$
So we can replace $\Delta\mathbf{v}$ by $\mathbf{a}\,\Delta t$

$\mathbf{I} = m\mathbf{a}\,\Delta t$

But if there is one force acting on a system then

$\mathbf{F} = m\mathbf{a}$ so we can replace m**a** by **F**
$\mathbf{I} = \mathbf{F}\,\Delta t$

So the impulse on a system, or an object, due to an external force is just equal to the force that's applied to it times by the amount of time that that force is applied. Since this is also equal to the change in the momentum of the system, or object, we can also write this as

$\mathbf{I} = \Delta\mathbf{p} = \mathbf{F}\,\Delta t$

It can be seen from this equation that the units of impulse must also be equal to he units of momentum, *kg· m/s.*

Example 11: A stationary ball whose mass is 4.5 kg is subject to an impulse of 36 kg· m/s towards the north. What will its velocity be after this? What will the change in its momentum be?

$\mathbf{p_0} + \mathbf{I} = \mathbf{p_f}$
$\mathbf{p_f} = \mathbf{p_0} + \mathbf{I}$

mv´ = m**v** + **I**

But the ball was initially at rest so v = 0

mv´= I
v´= **I**/m
v´= *36 kg· m/s* / 4.5 kg
v´= *36 kg· m/s* / 4.5 kg
v´= *8 m/s*
v´= *8 m/s* towards the north
I = Δ**p**
So Δ**p** = *36 kg· m/s towards the north*

Example 12: If the force on the ball in example 11 had been applied for 1.8s, what was its average magnitude and direction?

I = **F** Δt
F = **I** / Δt
F = (*36 kg· m/s(/ (1.8s)*
***F** = 20 kg· m/s*^{2} towards the north
F = *20 N* towards the north

The force and impulse are always in the same direction since Δt is a scalar.

Example 13: A 5.0 kg ball is traveling at a velocity of 25 m/s to the east when it bounces off of a wall and rebounds with a velocity of 20 m/s to the west. (a) What was its change in momentum? (b)What impulse did the wall deliver to the ball? (c)If the ball was in contact with the wall for 0.25s, what was the average force acting on the ball? (d)What was the average force acting on the wall? (e)What was the impulse delivered by the wall by ball? (Remember to answer all questions about vectors with both magnitudes and directions.)

Let's take east to be the positive direction.

(a) $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_o$
$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_o$
$\Delta\mathbf{p} = m\mathbf{v}_f - m\mathbf{v}_o$
$\Delta\mathbf{p} = m(\mathbf{v}_f - \mathbf{v}_o)$
Δp = 5.0 kg(-20 m/s -25 m/s)
Δp = 5.0 kg(-45 m/s)
***Δp** = 220 kg· m/s to the west*
(b) **I** = Δ**p** = *220 kg· m/s to the west*
(c) **I = F** Δt
F = **I** / Δt
F = *220 kg· m/s / .25s*
F = 880 N towards the west

(d) By Newton's third law we know that the force acting on the wall from the ball must be equal and opposite to the force acting on the ball due to the wall. Therefore, The force on the wall must be 880 N towards the east.

(e) **I = F** Δt

Since the force acting on the wall is equal and opposite to that acting on the ball, and Δt is the same, that means that the impulse delivered to the wall by the ball must be equal and opposite the impulse given the ball by the wall. So the impulse is *220 kg· m/s to the east.*

It is always true that if two objects collide that the impulse one delivers to the other is equal and opposite to the impulse it receives.

Derivation of relationship of velocities before and after an elastic collision.

$\mathbf{p}_0 + \mathbf{I} = \mathbf{p}_f$

There are no outside forces so I = 0

$\mathbf{p}_0 = \mathbf{p}_f$

$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$

Reorganize this to put all the m_1 terms on one side and all the m_2 terms on the other

$m_1\mathbf{v}_1 - m_1\mathbf{v}_1' = m_2\mathbf{v}_2' - m_2\mathbf{v}_2$

Factor out the masses

$m_1(\mathbf{v}_1 - \mathbf{v}_1') = m_2(\mathbf{v}_2' - \mathbf{v}_2)$

Leave that result for later. Now let's use mechanical energy conservation for perfectly elastic collisions to get a second equation

$\mathbf{E}_0 + \mathbf{W} = \mathbf{E}_f$

But there are no outside forces so W = 0

$\mathbf{E}_0 = \mathbf{E}_f$

There are no height changes or springs involved so all the energy is kinetic

$KE_0 = KE_f$

$\frac{1}{2} m_1\mathbf{v}_1^2 + \frac{1}{2} m_2\mathbf{v}_2^2 = \frac{1}{2} m_1\mathbf{v}_1'^2 + \frac{1}{2} m_2\mathbf{v}_2'^2$

Multiplying by 2 simplifies this

$m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2 = m_1\mathbf{v}_1'^2 + m_2\mathbf{v}_2'^2$

Now reorganize this to put all the m_1 terms on one side and all the m_2 terms on the other

$m_1\mathbf{v}_1^2 - m_1\mathbf{v}_1'^2 = m_2\mathbf{v}_2'^2 - m_2\mathbf{v}^2$

Factor out the masses

$m_1(\mathbf{v}_1^2 - \mathbf{v}_1'^2) = m_2(\mathbf{v}_2'^2 - \mathbf{v}_2^2)$

Now factor using difference of squares

$m_1(\mathbf{v}_1 - \mathbf{v}_1')(\mathbf{v}_1 + \mathbf{v}_1') = m_2(\mathbf{v}_2' - \mathbf{v}_2)(\mathbf{v}_2' + \mathbf{v}_2)$

Now divide this equation, from energy conservation by the one we got from momentum conservation.

$$\frac{m_1(\mathbf{v_1} - \mathbf{v_1'})(\mathbf{v_1} + \mathbf{v_1'}) = m_2(\mathbf{v_2} - \mathbf{v_2'})(\mathbf{v_2} + \mathbf{v_2'})}{m_1(\mathbf{v_1} - \mathbf{v_1'}) = m_2(\mathbf{v_2} - \mathbf{v_2'})}$$

$\mathbf{v}_1 + \mathbf{v}_1' = \mathbf{v}_2' + \mathbf{v}_2$

Which is fine, or you can say that

$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v_2}' - \mathbf{v_1}'$

Which indicates that the difference of velocities is reversed during a collision.

Most importantly, notice that this result is true regardless of the masses. They can be the same or different, it has no affect on this result.