Uniform Circular Motion

Introduction

Earlier we defined acceleration as being the change in velocity with time:

$$a = \frac{\Delta v}{t}$$

Until now we have only talked about changes in the magnitude of the acceleration: the speeding up or slowing down of objects. However, since velocity is a vector, it has both a magnitude and a direction; so another way that velocity can change is by changing its direction...even while its speed remains constant. This is the type of acceleration that will be explored in this chapter.

A very important case of changing direction while maintaining constant speed is called uniform circular motion. In that case, the object's speed remains constant, hence "uniform", while its direction is constantly changing, this is necessary to keep it moving in a circle. If there were no acceleration it would travel in a straight line.

This type of motion occurs in a number of instances: important examples being the motion of the planets around the sun or the moon around the earth. It was through an analysis of uniform circular motion that Newton was able to develop this theory of Universal Gravitation; so it's natural that that will also be explored in this chapter.

Uniform Circular Motion

From Newton's first law we know that if there is no net force acting on an object, it will travel in a straight line at constant speed. Whenever an object fails to travel in this way it is, by definition accelerating. By Newton's second law we can also conclude that there must be a net force acting on it. Circular motion is a special case of an object experiencing a constant acceleration.

In the case of circular motion, an object is constantly changing direction. Rather than traveling in a straight line, its path is always bent towards the center of the circle that defines its path. As shown below, its velocity is always tangent to the circle that describes its motion and its acceleration is always directed towards the center of the circle. By examining the velocity vector of the object over a small time difference, Δt, it can be seen that the change in velocity, the arrow connecting the tips of the earlier velocity vector to the tip of the later vector is always directed towards the center. This arrow represents the change in velocity over the time, Δt: thus it represents the direction of the acceleration.
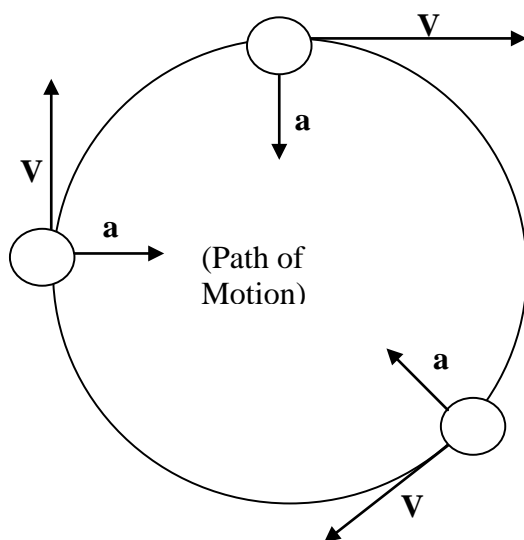
The object's velocity is always tangential to its path about the circle.

The object's acceleration is always pointing towards the center of the circle, perpendicular to the velocity vector.

Also, it can be seen that the triangle made up of v_1 and v_2 (shown below) is similar to the triangle made up of the radii of the circle and the distance traveled during time Δt, vt (shown below).
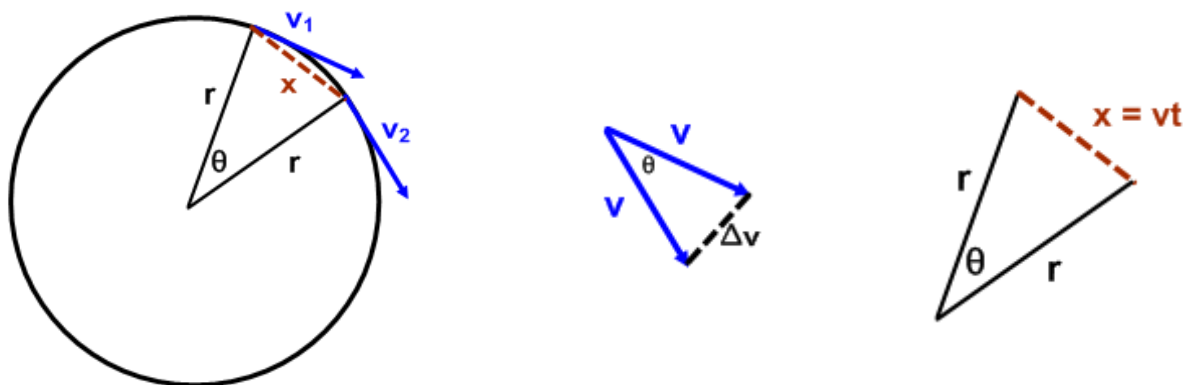
The magnitude of v_1 and v_2 are the same, since the object's speed is constant, so we can set up the following proportion:

$$\frac{\Delta v}{v} = \frac{vt}{r}$$

multiply both sides by v and divide by t

$$\frac{\Delta v}{t} = \frac{v^2}{r}$$

since $\Delta v/t = a$

$$a = \frac{v^2}{r}$$

This is a very important relationship as it gives us the magnitude of the acceleration of any object traveling along a circular path. The direction of that acceleration is towards the center of the circle.

> The acceleration of an object moving along a circular path is given by:
> $$a = \frac{v^2}{r} \text{ towards the center of the circle}$$

This is a very important result as it can be added to a few other special cases we have previously explored.

> **Special Cases of Acceleration**
> An object traveling in a straight line at constant speed:
> $a = 0$
> An unsupported object near the surface of the earth:
> $a = g = 9.8$ m/s² towards the center of the earth
> An object moving along a circular path:
> $$a = \frac{v^2}{r} \text{ towards the center of the circle}$$

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Example 1
A 4.0 kg object is traveling in uniform circular motion of radius 2.0m. The magnitude of its velocity, its speed, is 15 m/s. Determine its acceleration. Determine the net force acting on it.

Since the object is traveling in circular motion the magnitude of its acceleration is give by $a = \frac{v^2}{r}$ and the direction of its acceleration is towards the center of the circle.

$$a = \frac{v^2}{r}$$

$$a = \frac{(15\frac{m}{s})^2}{2m}$$

$$a = \frac{225\frac{m^2}{s^2}}{2m}$$

a = 112.5 m/s² towards the center of the circle

The net force acting on the object is responsible for its acceleration so:

Σ**F** = m**a**
Σ**F** = (4kg)(112.5 m/s²)
F$_{net}$ = 450N

Since the acceleration of an object is always directed in the same direction as the net force, the net force must also be directed towards the center of the circle.

F$_{net}$ = 450N towards the center of the circle

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Example 2
How much net force is required to keep a 5.0 kg object traveling in a circle of radius 6.0m with a speed of 12 m/s?

Since the object is traveling in a circle its acceleration must be v²/r, so

Σ**F** = m**a**

$$\Sigma\mathbf{F} = m\mathbf{(\frac{v^2}{r})}$$

$$\Sigma\mathbf{F} = (5kg)\frac{(12\frac{m}{s})^2}{6m}$$

$$\Sigma\mathbf{F} = (5kg)\frac{144\frac{m^2}{s^2}}{6m}$$

Σ**F** = 120 kg·m/s²
F$_{net}$ = 120 N towards the center of the circle

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Period and Frequency
There are two closely related terms that are used in describing circular motion: **period** and **frequency**. **The period of an object's motion is the time it takes it to go once around a circle.** Period is a measure of time so the standard units for period are seconds and the symbol for period is "T" (easily confused with the symbol for Tension).   If an object completes a certain number of rotations, n, in a given amount of time, t, then it follows that T = t/n, since that is the time it must be taking to complete each rotation. For example, if I go around in a circle ten times in five seconds, my period is the time it takes for each trip around the circle and is given by:

$$T = \frac{t}{n}$$

$$T = \frac{10s}{5}$$

$$T = 2s$$

Sometimes it's helpful to think about the number of times I go around per second rather than the number of seconds it takes me to go around once. This is called the frequency of rotation since it describes how frequently I complete a cycle. **The frequency of an object's motion is the number of times that it goes around a circle in a given unit of time.** The symbol for frequency is "f" (easily confused with friction). If an object completes a certain number of rotations, n, in a given amount of time, t, then it follows that $f = \frac{n}{t}$, since that is the number of times I go around in a given time. The units for frequency must reflect the number of "repetitions per unit time". If time is measured in seconds, as will typically be done in this book, the unit of frequency is 1/s or s^{-1}. This unit, s^{-1}, has also been named the Hertz (Hz).

This term is commonly used in describing radio stations: when you tune your radio to 104.3MHz you are tuning it to a radio signal that repeats its cycle 104.3 million times per second. Similarly, the station at 880 kHz has a signal that repeats 880 thousand times per second. Since $T = \frac{t}{n}$ and $f = \frac{n}{t}$ it follows that:

$$\mathbf{T = \frac{1}{f}} \text{ and } \mathbf{f = \frac{1}{T}}$$

For example, if the period, T, of an object's motion is 0.2 seconds then its frequency, f, is given by:

$$f = \frac{1}{T}$$

$$f = \frac{1}{.2s}$$

$f = 5\ s^{-1}$ or 5 Hz

Similarly if an object's frequency is 20 Hz then its period is given by

$$T = \frac{1}{f}$$

$$T = \frac{1}{20Hz}$$

$$T = 0.05s$$

There is also a direct connection between period, frequency and speed. Since the distance around a circle is given by its circumference: an object must travel a distance of $2\pi r$ in order to complete one circle. That yields important relationships between speed, period and frequency.

$$s = \frac{d}{t}$$

in order to travel around one circle I must move a distance of $2\pi r$

$$s = \frac{2\pi r}{T}$$

by the definition of period, the time it takes to do that is T

$$s = \frac{2\pi r}{T}$$

since the speed is the magnitude of the instantaneous velocity of an object, this means that

$$\mathbf{v = \frac{2\pi r}{T}}$$

Also, since $f = \dfrac{1}{T}$, this can also be written as

v = 2πrf

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Relationships between period, frequency and velocity

$$T = \frac{t}{n} \text{ and } T = \frac{1}{f} \qquad\qquad f = \frac{n}{T} \text{ and } f = \frac{1}{T}$$

$$v = \frac{2\pi r}{T} \qquad\qquad v = 2\pi rf$$

The units of period, T, are seconds.

The units of frequency, f, are s⁻¹, or Hz.

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Example 3

An object is traveling in a circle of radius 4m and completes five cycles in 2s. What are its period, frequency and velocity?

$$T = \frac{t}{n}$$

$$T = \frac{2s}{5}$$

$$T = 0.4s$$

$$f = \frac{1}{T} \qquad\qquad \text{or} \qquad\qquad f = \frac{n}{t}$$

$$f = \frac{1}{.4s} \qquad\qquad\qquad\qquad f = \frac{5}{2s}$$

$$f = 2.5 \text{ Hz} \qquad\qquad\qquad\qquad f = 2.5 \text{ Hz}$$

$$v = \frac{2\pi r}{T} \qquad \text{or} \qquad\qquad v = 2\pi rf$$

$$v = \frac{2(3.14)(4m)}{.4s}$$

$$v = 2(3.14)(4m)(2.5Hz)$$

$$v = 62.8 \text{ m/s} \qquad\qquad\qquad v = 62.8 \text{ m/s}$$

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Example 4

A force of 250 N is required to keep an 8.0 kg object moving in a circle whose radius is 15m. What are the speed, period and frequency of the object?

$$\Sigma\mathbf{F} = m\mathbf{a}$$

$$F_{net} = m(\frac{v^2}{r})$$

$$F_{net}\frac{r}{m} = v^2$$

$$v = \sqrt{\frac{F_{net}r}{m}}$$

$$v = \sqrt{\frac{(250N)(15m)}{8kg}}$$

$$v = \sqrt{\frac{3750 kg \frac{m^2}{s^2}}{8kg}}$$

$$v = \sqrt{470 \frac{m^2}{s^2}}$$

v = 21.65 m/s

Since the object is moving in a circle, its velocity must be tangent to the circle

v = 21.65 m/s tangent to the circle

Next, we find the period of the object's motion.

$$v = \frac{2\pi r}{T}$$

$$T = \frac{2\pi r}{V}$$

$$T = \frac{2(3.14)(15m)}{21.65\frac{m}{s}}$$

T = 4.35s

And finally, we find its frequency.

$$f = \frac{1}{T}$$

$$f = \frac{1}{4.35s}$$

f = 0.23Hz

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One important aspect of the equations for uniform motion is that they apply even for the brief periods of time that an object's motion can be considered circular. For instance, as you drive around a curve in a road, that curve can be approximated as a part of a circle. You don't have to travel around a complete circle for these equations to be useful: just like you didn't have to drive 60 miles in one hour for your speed to be 60 miles per hour. **These equations can be applied anytime the motion of an object is even briefly circular in nature.**

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Example 4
An object is attached to a string which is supplying a Tension that keeps it moving in a circle of radius 0.50m as it slides along a frictionless table. The object has a mass of 2.0 kg and its motion has a period of 0.63s. What is the tension in the string?

$\Sigma \mathbf{F} = m\mathbf{a}$

$F_T = m(\frac{v^2}{r})$

but $v = \frac{2\pi r}{T}$ so $v = \frac{4\pi^2 r^2}{T^2}$ ...substituting

$F_T = m(\frac{\frac{4\pi^2 r^2}{T^2}}{R})$

$F_T = \frac{4m\pi^2 r}{T^2}$

$$F_T = \frac{4(2\text{kg})(3.14)^2(.5\text{m})}{(.63\text{s})^2}$$

F_T = 99.37 kg·m/s²

$\mathbf{F_T}$ = 99.37 N towards the center of the circle

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Multiple forces and circular motion

Often more than one force is acting on an object...including an object traveling in uniform circular motion. In that case, you treat this case the same way you did in any dynamics problem, **the sum of the forces matters...not any one force**. So for instance, if we changed the prior example by having the object moving in a vertical circle, rather than a horizontal one, we have two forces acting on the object to keep its motion circular, the weight of the object will always be down, but the tension force would always point towards the center of the circle. Since the acceleration of the object is always given by v²/r, then the sum of the forces will always equal ma or mv²/r. Thus as long as the velocity is constant the net force must be as well.

However, the tension force will sometimes be opposed by the weight of the object, at the bottom of the circle, and will sometimes be in the same direction, at the top of the circle. That means that the tension force will have to vary since the sum of the forces is constant and the weight can't vary.

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Example 5

An object is attached to a string which is supplying a Tension that helps keeps it moving in a vertical circle of radius 0.50m. The object has a mass of 2.0 kg and is traveling at a constant speed of 5.0 m/s (impractical to do, but let's use that as an assumption). What is the tension in the string in the following situations?

  a. When the object is at the top of the circle.
  b. When the object is at the bottom of the circle.
  c. When the string is horizontal.

a. At the top of the circle, both the weight and the tension point downwards. So does the acceleration of the object, since the acceleration of an object in uniform circular motion is always directed at the center of the circle. If we define down as negative:

$\Sigma\mathbf{F} = m\mathbf{a}$

-T - W = -ma   but since all the signs are negative, we can multiply by negative one on both sides and make them all positive.

T + W = ma

T = ma - W

$T = m\frac{v^2}{r}$ - mg

$T = m(\frac{v^2}{r} - g)$

$T = (2.0 \text{ kg}) (\frac{(5\frac{m}{s})^2}{.5m}) - 9.8 \text{ m/s}^2)$

$T = (2.0 \text{ kg})((\frac{25\frac{m^2}{s^2}}{.5m})) - 9.8 \text{ m/s}^2)$

T = (2.0 kg)((50 m²/s² – 9.8 m/s²)

T = 80.4 kg·m/s² = 80.4 N downwards

b. At the bottom of the circle, the weight points downward and the tension points upwards, towards the center of the circle. The acceleration of the object also point upwards towards the center of the circle, since

the acceleration of an object in uniform circular motion is always directed at the center of the circle. If we define down as negative:

$\Sigma \mathbf{F} = m\mathbf{a}$

T - W = ma

Note that in this case T and ma are positive and only W is negative.

T = ma + W

$T = m\dfrac{v^2}{r} + mg$

$T = m(\dfrac{v^2}{r} + g)$

$T = (2.0 \text{ kg}) \left(\dfrac{(5\frac{m}{s})^2}{.5m}\right) + 9.8 \text{ m/s}^2)$

$T = (2.0 \text{ kg})\left(\dfrac{25\frac{m^2}{s^2}}{.5m} + 9.8 \text{ m/s}^2\right)$

$T = (2.0 \text{ kg})((50 \text{ m}^2/\text{s}^2 + 9.8 \text{ m/s}^2)$

$T = 119.6 \text{ kg·m/s}^2$

T = 119.6 N upwards

c. When the string is horizontal, the weight points downward and the tension points sideways, towards the center of the circle. The acceleration of the object, due to its circular motion, also point sideways towards the center of the circle, since the acceleration of an object in uniform circular motion is always directed at the center of the circle. In this case, the weight does not contribute to keeping the object moving in a circle...it will serve to speed the object up. But since we are only considering the forces causing circular motion, the weight will not affect the Tension in the string. This then becomes the same as Example 4, shown above...W will not play a role in maintaining the circular motion of the object.

$\Sigma \mathbf{F} = m\mathbf{a}$

$T = m(\dfrac{v^2}{r})$

$T = (2.0 \text{ kg})\left(\dfrac{(5\frac{m}{s})^2}{.5m}\right)$

$T = (2.0 \text{ kg})\left(\dfrac{25\frac{m^2}{s^2}}{.5m}\right)$

$T = \left(\dfrac{50kg\frac{m^2}{s^2}}{.5m}\right)$

$T = 100 \text{ kg·m/s}^2$

T = 100 N towards the center of the circle

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In example 5a, it can be seen that there will be a minimum speed that the object can travel in a vertical circle. At that minimum speed, the tension in the string becomes zero at the top of the circle and only the weight of the object supplies the force necessary to maintain circular motion. If an object moves any slower than that, it will depart from circular motion as it falls towards the center of the circle. In Example 5, the relationship that gives us this speed is seen in the equation: $T = m(\dfrac{v^2}{r} - g)$. From this it can be seen that when $\dfrac{v^2}{r} - g = 0$, the tension in the string will be zero. Since the tension in a string cannot be negative, this is the lowest possible velocity for circular motion; and this velocity is independent of the mass of the object. Solving for "v" gives us:

$\dfrac{v^2}{r} - g = 0$

$$\frac{v^2}{r} = g$$

$$v^2 = gr$$

$$v = \sqrt{gr}$$

So in example 5, the lowest possible velocity for the object to maintain circular motion would be:

$$v_{min} = \sqrt{gr}$$

$$v_{min} = \sqrt{\left(9.8\,\frac{m}{s^2}\right)(.5m)}$$

$$v_{min} = \sqrt{4.9\,\frac{m^2}{s^2}}$$

$$v_{min} = 2.2 \text{ m/s}$$

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Example 6
A bucket of water is spun in a vertical circle such that the bucket is upside down with the water in it at the top of the circle. The person water in the bucket is spinning in a circle of radius 0.80 m.

    a. *What is the minimum velocity that the water must maintain to stay in the bucket and not drench the person below?*
    b. *If the water has a mass of 0.25kg and the velocity of the bucket is the same at the bottom as it is at the top of the circle, what normal force must the bucket provide at the bottom of the circle?*

The water experiences two forces as it travels through along its circular path: its weight down and the normal force of the bottom of the bucket directed towards the center of the circle. The sum of these two forces must always equal ma, or $m\frac{v^2}{r}$.
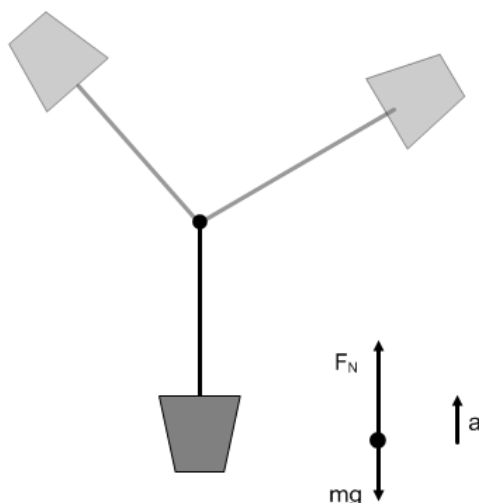
a. At the top of the circle, the weight, normal force and acceleration all point down. So,

$\Sigma\mathbf{F} = m\mathbf{a}$
-F$_N$ - W = -ma   but since all the signs are negative, we can multiply by negative one on both sides and make them all positive.

F$_N$ + W = ma

The minimum speed will be when $F_N = 0$. At that point, the water will appear to be weightless...in free fall.

$W = ma$

$mg = ma$

$a = g$

$\dfrac{v^2}{r} = g$

$v^2 = gr$

$v = \sqrt{gr}$

$v_{min} = \sqrt{\left(9.8\,\frac{m}{s^2}\right)(.8m)}$

$v_{min} = \sqrt{7.84\,\frac{m^2}{s^2}}$

$v_{min} = 2.8$ m/s

Note that the mass of the water didn't matter, only the radius of the circle.

b. At the bottom of the circle, the bucket and acceleration are pointed upwards, while the weight of the water is pointed down. So,

$\Sigma \mathbf{F} = m\mathbf{a}$

$F_N - W = ma$ but since all the signs are negative, we can multiply by negative one on both sides and make them all positive.

$F_N = W + ma$

$F_N = mg + ma$

$F_N = m(g + a)$

$F_N = m(g + \dfrac{v^2}{r})$

We then use the velocity we calculated for the top of the circle

$F_N = (0.25$ kg$)(9.8$ m/s$^2 + \dfrac{(2.8\frac{m}{s})^2}{.8m})$

$F_N = (0.25$ kg$)(9.8$ m/s$^2 + 9.8$ m/s$^2)$

$F_N = (0.25$ kg$)(19.6$ m/s$^2)$

$F_N = 4.9$ N

Note that this normal force is double what it would be if that were holding the water is stationary. The apparent weight of the water is double at the bottom and is zero at the top of the circle.

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Sometimes the force keeping an object in circular motion is due to friction. In that case, there are really three dimensions involved in solving the problem: the two dimensions in which the circular motion is defined plus the normal force, which is perpendicular to the plane of the circular motion. It will be seen that in these cases the mass of the object does not affect the outcome.

Example 7
A car is rounding a curve with a speed of 20 m/s. At that location, the curve can be approximated by a circle of radius 150m. What is the minimum coefficient of static friction that will allow the car to make the curve without sliding off the road?

From the top, a sketch of this problem would show the car traveling in a circle. However, from that perspective it's not possible to draw a free body diagram showing all the forces necessary to solve this

problem. So it's also important to make a sketch from the perspective of someone standing on the road with the car driving away.

Top Down Free Body Diagram   Side view Free Body Diagram
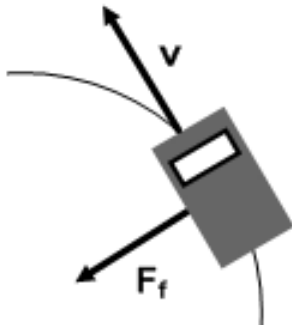
radial-direction

$\Sigma\mathbf{F} = m\mathbf{a}$

$F_{sf} = ma$

$\mu_s F_N = m(\frac{v^2}{r})$

$\mu_s mg = m\frac{v^2}{r}$

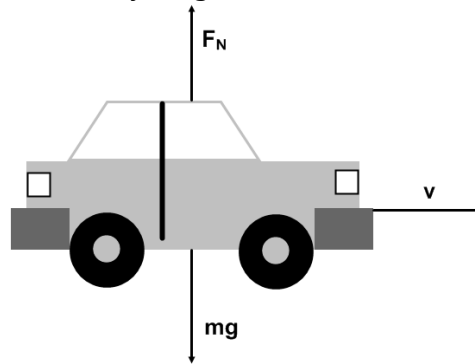$\mu_s = \frac{v^2}{gr}$

$\mu_s = \frac{(20\frac{m}{s})^2}{\left(9.8\frac{m}{s^2}\right)(150m)}$

$\mu_s = 0.27$

vertical-direction

$\Sigma\mathbf{F} = m\mathbf{a}$

$F_N - mg = 0$

$F_N = mg$

Note that the mass of the car doesn't matter. So when a road is being designed, it's not important to know the mass of the cars that will be using it when determining the maximum curvature of the road.

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