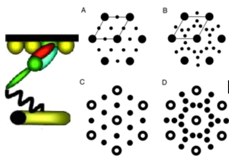
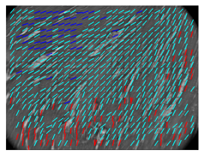
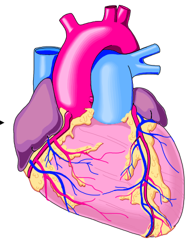
Biomechanics



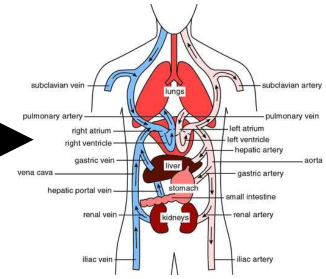
cross-bridges



3-D myocardium



ventricles



circulation

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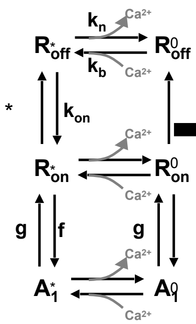


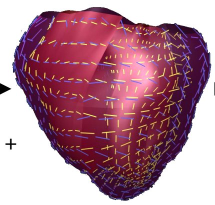
Myofilament kinetic model

$$W = \frac{1}{2} C e^{Q}$$

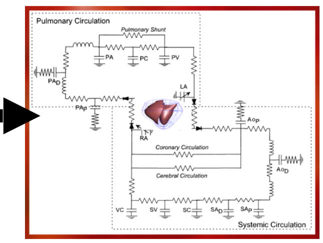
where

$$Q = b_1 E_{\Theta\Theta}^2 + b_2 E_{ZZ}^2 + b_3 E_{RR}^2 + 2b_4 E_{\Theta\Theta} E_{ZZ} + 2b_5 E_{ZZ} E_{RR} + 2b_6 E_{RR} E_{\Theta\Theta} + b_7 E_{\Theta Z}^2 + b_8 E_{ZR}^2 + b_9 E_{R\Theta}^2$$

Hyperelastic constitutive equation



3-D finite element model of ventricular stress



Boundary conditions: circulatory systems model

Continuum Mechanics Fundamentals

- The *key words* of continuum mechanics are tensors such as stress, strain, and rate-of-deformation
- The *rules* are the conservation laws of mechanics – mass, momentum and energy.
- Stress, strain, and rate of deformation vary with position and time. The relation between them is the *constitutive law.*
- The constitutive law must generally be determined by *experiment* but it is constrained by thermodynamic and other physical conditions.
- The *language* of continuum mechanics is *tensor analysis*.

Biomechanics: Mechanics$\leftrightarrow$Physiology

Continuum Mechanics	Physiology
Geometry and structure	Anatomy and morphology
Boundary conditions	Environmental influences
Conservation laws	Biological principles
• mass	• mass transport, growth
• energy	• metabolism and energetics
• momentum	• motion, flow, equilibrium
Constitutive equations	Structure-function relations

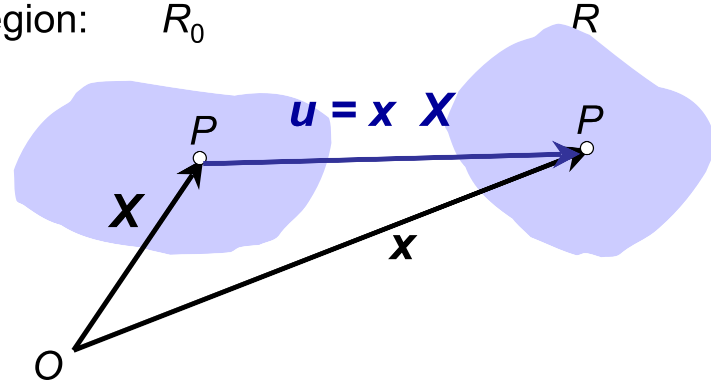
Therefore, continuum mechanics provides a mathematical framework for integrating the structure of the cell and tissue to the mechanical function of the whole organ

Kinematics

- Solid continua *deform* and fluid continua *flow*
- Flow is measured by the *rate-of-deformation tensor* which is related to *velocity gradients.*
- *Deformation* is measured by the *strain tensor*, which describes *change of shape*
- For solid biomechanics we focus on *deformation*
- In particular we derive strain tensors for *large (finite) deformations* as opposed to infinitesimal deformations

Kinematics

	Reference State ("undeformed")	Current State ("deformed")
Time:	t=0	t=t
Region:	R_0	R



$\boldsymbol{X} = X_R \boldsymbol{e}_R$	undeformed (or "material") coordinates
$\boldsymbol{x} = x_i \boldsymbol{e}_i$	current (or "spatial") coordinates

Lagrangian & Eulerian Descriptions

Lagrangian or "material" description of motion

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$$

- Motion as seen by the *convecting* material particle
- Undeformed coordinates label *material* points
- Most useful for *solid* mechanics problems

Eulerian or "spatial" description of motion

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$$

- Motion as seen by a fixed *spatial* observer
- Current *spatial* coordinates of material points change
- Most useful for *fluid flow* problems

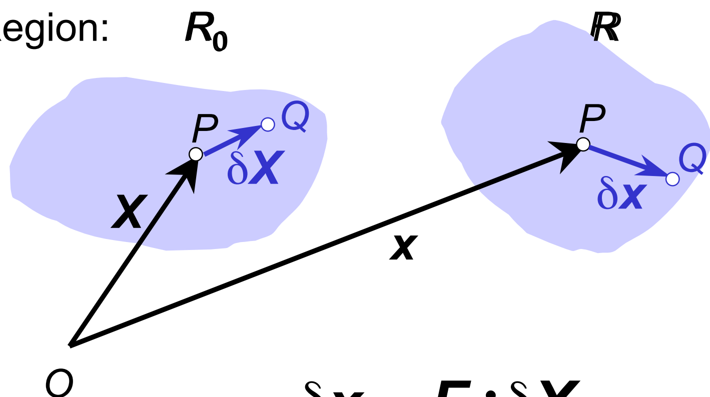
Displacement Vector

The displacement vector, $\boldsymbol{u}$

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{x}\left(\boldsymbol{X},t\right) - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}\left(\boldsymbol{x},t\right)$$

Kinematics

	Reference State ("undeformed")	Current State ("deformed")
Time:	t=0	t=t
Region:	$\boldsymbol{R_0}$	$\boldsymbol{R}$



$$\delta \boldsymbol{x} = \boldsymbol{F} \cdot \delta \boldsymbol{X}$$

Chain rule: $\delta x_i = \dfrac{\partial x_i}{\partial X_R} \delta X_R \quad \Rightarrow \quad \boxed{F_{iR} = \dfrac{\partial x_i}{\partial X_R}}$

Displacement Gradient Tensor

The displacement vector, ***u***

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \rightarrow u_i = x_i - X_i$$

$$\frac{\partial u_i}{\partial X_R} = \frac{\partial x_i}{\partial X_R} - \frac{\partial X_i}{\partial X_R}$$

$$G_{iR} = F_{iR} - \delta_{iR}$$

$$\boldsymbol{G} = \boldsymbol{F} - \boldsymbol{I}$$

Polar Decomposition Theorem

For ***F*** non-singular and square

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R}$$

where

$\boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}}, \boldsymbol{V} = \boldsymbol{V}^{\mathrm{T}}$ are the right and left stretch tensors

$\boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{R} = \boldsymbol{I}$ is the orthogonal rotation tensor

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} = (\boldsymbol{R} \cdot \boldsymbol{U})^{\mathrm{T}} \cdot (\boldsymbol{R} \cdot \boldsymbol{U}) = \boldsymbol{U}^{\mathrm{T}} \cdot \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}} \cdot \boldsymbol{U} = \boldsymbol{U}^2$$

is the right Cauchy-Green Deformation tensor (Lagrangian)

$$\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{T}} = (\boldsymbol{V} \cdot \boldsymbol{R}) \cdot (\boldsymbol{V} \cdot \boldsymbol{R})^{\mathrm{T}} = \boldsymbol{V} \cdot \boldsymbol{R} \cdot \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{V}^{\mathrm{T}} = \boldsymbol{V} \cdot \boldsymbol{V}^{\mathrm{T}} = \boldsymbol{V}^2$$

is the left Cauchy-Green Deformation tensor (Eulerian)

(Finite) Strain Tensors

Strain is a measure of *change of shape* independent of rotation. Change of shape corresponds to change of *length* (i.e. stretch)

Lagrangian Green's Strain Tensor

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right)$$

$$E_{RS} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_R}\frac{\partial x_i}{\partial X_S} - \delta_{RS}\right)$$

Eulerian Almansi's Strain Tensor

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right)$$

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_R}{\partial x_i}\frac{\partial X_R}{\partial x_j}\right)$$

Infinitesimal (Cauchy) Strain

The Green's and Alamansi strain tensors are exact measures of shape change for any finite deformation, but they are nonlinear. In terms of the displacement gradients, the finite strains are quadratic, e.g.

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} \Rightarrow E_{RS} = \frac{1}{2}\left(\frac{\partial u_R}{\partial X_S} + \frac{\partial u_S}{\partial X_R} + \frac{\partial u_i}{\partial X_R}\frac{\partial u_i}{\partial X_S}\right)$$

When the displacement gradients are small enough (<1%), we may linearize the finite strains to obtain the infinitesimal Cauchy strain tensor:

$$\varepsilon_{RS} = \frac{1}{2}\left(\frac{\partial u_R}{\partial X_S} + \frac{\partial u_S}{\partial X_R}\right) = \frac{1}{2}\left(\frac{\partial u_R}{\partial x_S} + \frac{\partial u_S}{\partial x_R}\right)$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{F} + \boldsymbol{F}^{T}\right) - \boldsymbol{I}$$

Strain is Change in Length

- The stretch and strain tensors ***U***, ***V***, ***C***, ***B***, ***E*** and ***e*** all describe how material *elements of length* change
- Consider the elements of undeformed and deformed length dL and dl respectively:

$$\mathrm{d}L^2 = \mathrm{d}\boldsymbol{X}^T \cdot \mathrm{d}\boldsymbol{X}$$
$$= \mathrm{d}X_R\,\mathrm{d}X_R = \mathrm{d}X_R\,\mathrm{d}X_S\delta_{RS}$$
$$\mathrm{d}\ell^2 = \mathrm{d}\boldsymbol{x}^T \cdot \mathrm{d}\boldsymbol{x} = \left(\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X}\right)^T \cdot \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{X}^T \cdot \boldsymbol{F}^T \cdot \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{X}^T \cdot \boldsymbol{C}\cdot\mathrm{d}\boldsymbol{X}$$
$$= \mathrm{d}x_i\,\mathrm{d}x_i = \frac{\partial x_i}{\partial X_R}\frac{\partial x_i}{\partial X_S}\mathrm{d}X_R\,\mathrm{d}X_S = C_{RS}\,\mathrm{d}X_R\,\mathrm{d}X_S$$

Therefore,
$$\boxed{\begin{aligned}&\mathrm{d}\ell^2 - \mathrm{d}L^2 = 2\mathrm{d}\boldsymbol{X}^T \cdot \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{X} = 2\mathrm{d}\boldsymbol{x}^T \cdot \boldsymbol{e}\cdot\mathrm{d}\boldsymbol{x}\\ &\mathrm{d}\ell^2 - \mathrm{d}L^2 = 2E_{RS}\,\mathrm{d}X_R\,\mathrm{d}X_S = 2e_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j\end{aligned}}$$

i.e., the finite strain tensors are measures of *change in squared lengths*

Area Changes

- If we can obtain length changes from ***F***, we must be able to derive *area changes* too.
- *Nanson's* formula relates *elements of deformed area* da (of the surface with *deformed* outward normal ***n***) to the corresponding *undeformed area element* dA (with *undeformed* outward normal ***N***)

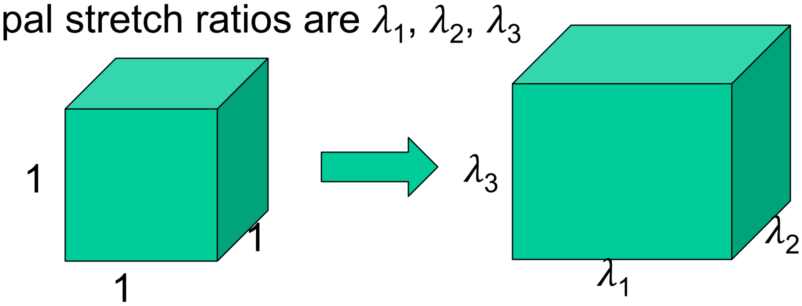
$$\boldsymbol{n}\cdot\boldsymbol{F}\frac{\mathrm{d}a}{\mathrm{d}A} = \boldsymbol{N}\det\boldsymbol{F}$$
$$\left(\frac{\mathrm{d}a}{\mathrm{d}A}\right)^2 = \frac{\boldsymbol{N}\cdot\boldsymbol{N}\left(\det\boldsymbol{F}\right)^2}{\left(\boldsymbol{n}\cdot\boldsymbol{F}\right)\cdot\left(\boldsymbol{n}\cdot\boldsymbol{F}\right)} = \frac{\left(\det\boldsymbol{F}\right)^2}{\boldsymbol{n}\cdot\boldsymbol{B}\cdot\boldsymbol{n}} = \frac{\left(\det \mathrm{F}\right)^2}{n_i n_j B_{ij}}$$

Volume Change

- ***Elements of volume*** in the deformed dv and undeformed dV states are related by the determinant of $\boldsymbol{F}$:

$$dv = \det \boldsymbol{F}\, dV$$

- E.g. Consider a unit cube that is deformed so that its principal stretch ratios are λ_1, λ_2, λ_3



$dV = 1$

$$dv = \lambda_1\lambda_2\lambda_3 dV$$
$$= \det\boldsymbol{U}.dV = \det\boldsymbol{F}.dV$$
(since $\det\boldsymbol{R} = 1$)

Simple extension

Uniform extension

$$x_1 = \lambda_1 X_1 \qquad x_2 = \lambda_2 X_2 \qquad x_3 = \lambda_3 X_3$$

$$[F] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [B] = [C] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$[E] = \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix} \quad [e] = \frac{1}{2}\begin{bmatrix} 1 - \frac{1}{\lambda_1^2} & 0 & 0 \\ 0 & 1 - \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & 1 - \frac{1}{\lambda_3^2} \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix}$$

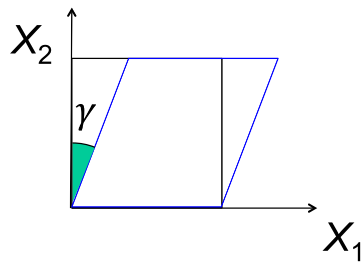
In 1-D, λ = stretch ratio $= \dfrac{\ell}{\ell_0}$

$$\varepsilon = \frac{(\ell - \ell_0)}{\ell_0} \qquad E = \frac{1}{2}\left(\frac{\ell^2}{\ell_0^2} - 1\right) = \frac{1}{2}\left(\frac{\ell^2 - \ell_0^2}{\ell^2}\right) \qquad e = \frac{1}{2}\left(1 - \frac{\ell_0^2}{\ell^2}\right) = \frac{1}{2}\left(\frac{\ell^2 - \ell_0^2}{\ell_0^2}\right)$$

Simple Shear

Simple Shear

$$x_1 = X_1 + X_2 \tan\gamma$$
$$x_2 = X_2 \quad x_3 = X_3$$



$$[F] = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow [E] = \frac{1}{2}\begin{bmatrix} 0 & \tan\gamma & 0 \\ \tan\gamma & \tan^2\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\tan^2\gamma$ vanishingly small for small strain

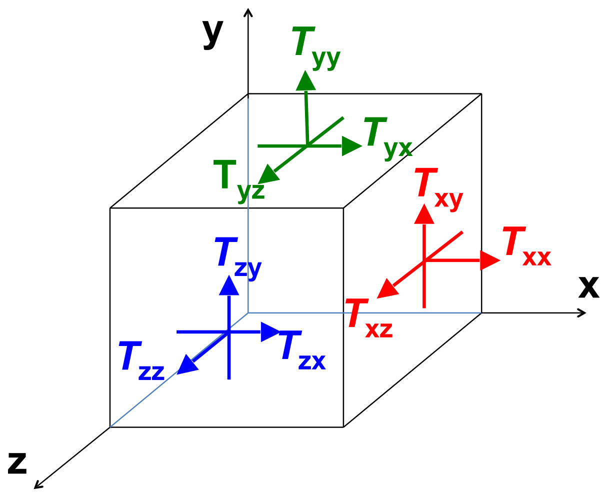
Pure Torsion of a Cylinder

Pure torsion $\quad r = R \quad \theta = \Theta + \alpha Z \quad z = Z$

$$[F] = \mathrm{Grad}\,\boldsymbol{x} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} & \dfrac{\partial r}{\partial Z} \\ r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} & r\dfrac{\partial \theta}{\partial Z} \\ \dfrac{\partial z}{\partial R} & \dfrac{1}{R}\dfrac{\partial z}{\partial \Theta} & \dfrac{\partial z}{\partial Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

α is the twist per unit length of the tube

The Cauchy Stress Tensor



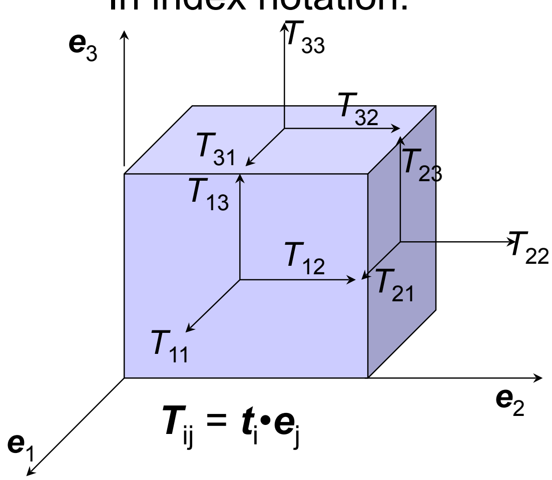
Cauchy Stress tensor ***T***

$$\boldsymbol{T} = \begin{bmatrix} T_{xx} & T_{yx} & T_{zx} \\ T_{xy} & T_{yy} & T_{zy} \\ T_{xz} & T_{yz} & T_{zz} \end{bmatrix}$$

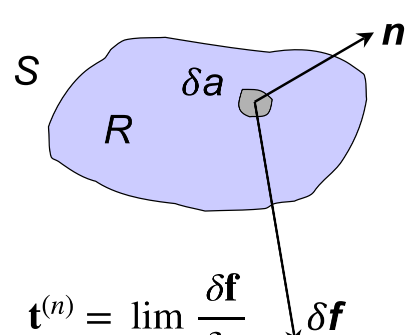
Cauchy's Formula

Cauchy's formula: $\boldsymbol{t}^{(n)} = \boldsymbol{n} \cdot \boldsymbol{T}$

In index notation: $t_j^{(n)} = n_i T_{ij}$



$T_{ij} = \boldsymbol{t}_i \cdot \boldsymbol{e}_j$



$$\mathbf{t}^{(n)} = \lim_{\delta a \to 0} \frac{\delta \mathbf{f}}{\delta a}$$

T_{ij} is the component in the x_j direction of the traction vector $\boldsymbol{t}^{(n)}$ acting on the face normal to the x_i axis in the *deformed* state of the body. The "true" stress.

Lagrangian Stress Tensors

The (half) Lagrangian ***Nominal stress tensor S***

$$t_j^{(N)} = N_R S_{Rj}$$

S_{Rj} is the component in the x_j direction of the traction measured per unit *reference* area acting on the surface normal to the (undeformed) X_R axis. Useful experimentally but *not* symmetric: $\boldsymbol{S} = \det\boldsymbol{F}.\boldsymbol{F}^{-1}.\boldsymbol{T} \neq \boldsymbol{S}^{\mathrm{T}}$

The symmetric (fully) Lagrangian ***Second Piola-Kirchhoff stress tensor P***

$$\boldsymbol{P} = \boldsymbol{S}\cdot\boldsymbol{F}^{-\mathrm{T}} = (\det\boldsymbol{F})\boldsymbol{F}^{-1}\cdot\boldsymbol{T}\cdot\boldsymbol{F}^{-\mathrm{T}} = \boldsymbol{P}^{\mathrm{T}}$$

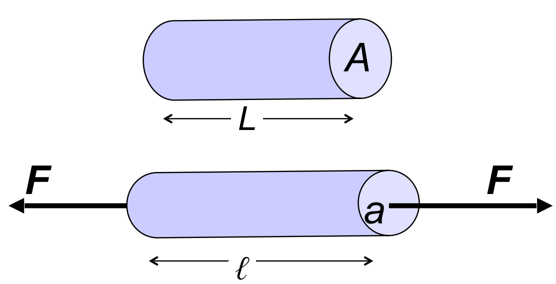
$$P_{\mathrm{RS}} = \left(\det\boldsymbol{F}\right)\frac{\partial X_R}{\partial x_i}\frac{\partial X_s}{\partial x_j}T_{\mathrm{ij}} = P_{\mathrm{SR}}$$

- Useful mathematically but no direct physical interpretation
- For small deformations differences between ***T***, ***P***, ***S*** vanish

Example: Uniaxial Stress

undeformed length = L
undeformed area = A

deformed length = ℓ
deformed area = a



Cauchy Stress

$$T = \frac{F}{a}$$

Nominal Stress

$$S = \det\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{T} = \frac{\ell a}{LA}\frac{L}{\ell}\frac{F}{a} = = \frac{F}{A}$$

Second Piola-Kirchhoff Stress

$$P = \boldsymbol{S}\cdot\boldsymbol{F}^{-1} = S\frac{L}{\ell} = \frac{F}{A}\frac{L}{\ell}$$

Governing Equations

- **Conservation Laws**
  - Conservation of Mass
  - Conservation of Momentum
    - *Linear*
    - *Angular*
  - Conservation of Energy
- **Constitutive Laws**

Conservation of Mass: Lagrangian

"The mass δm ($=\rho_0 \delta V$) of the material in the initial material volume element δV remains constant as the element deforms to volume δv with density ρ, and this must hold everywhere (i.e. for δV arbitrarily small)"

$$\iiint \rho_0 \, dV = \iiint \rho \, dv$$

$$\iiint \rho_0 \, dX_1 \, dX_2 \, dX_3 = \iiint \rho \, dx_1 \, dx_2 \, dx_3 = \iiint \rho \left| \frac{\partial x_i}{\partial X_R} \right| dX_1 \, dX_2 \, dX_3$$

Hence: $\dfrac{\rho_0}{\rho} = \dfrac{dv}{dV} = det\mathbf{F} = det\mathbf{U} = \lambda_1 \, . \, \lambda_2 \, . \, \lambda_3 = \sqrt{I_3}$

Thus, for an *incompressible* solid: $\rho = \rho_0 \Rightarrow \det \boldsymbol{F} = 1$

Conservation of Linear Momentum

"The rate of change of linear momentum of the particles that instantaneously lie within a fixed region R equals the resultant of the body forces b per unit mass acting on the particles in R plus the resultant of the surface tractions $t^{(n)}$ acting on the surface S"

$$\frac{D}{Dt}\iiint_R \rho \mathbf{v}\, dV = \iiint_R \rho b\, dV + \iint_S \mathbf{t}^{(n)}\, dS$$

$$\rightarrow \quad \boxed{\begin{gathered} \rho \frac{D\boldsymbol{v}}{Dt} = \operatorname{div}\boldsymbol{T} + \rho\boldsymbol{b} \\ \text{or} \\ \rho\frac{\partial v_i}{\partial t} + \rho v_k \frac{\partial v_i}{\partial x_k} = \frac{\partial T_{ji}}{\partial x_j} + \rho b_i \end{gathered}}$$

Conservation of Angular Momentum

"The rate of change of angular momentum of the particles that instantaneously lie within a fixed region R equals the resultant couple about the origin of the body forces b per unit mass acting on the particles in R plus the resultant couple of the surface tractions $t^{(n)}$ acting on S".

Subject to the assumption that no distributed body or surface couples act on the material in the region, this law leads simply to the symmetry of the stress tensor:

$$\boxed{\boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}}}$$

Conservation of Energy

"The rate of change of kinetic plus internal energy in the region R equals the rate at which mechanical work is done by the body forces b and surface tractions $t^{(n)}$ acting on the region plus the rate at which heat enters R across S".

$$\frac{D}{Dt}\left(\frac{1}{2}\iiint_R \rho\mathbf{v}\cdot\mathbf{v}\,dV + \iiint_R \rho e\,dV\right) = \iiint_R \rho\mathbf{b}\cdot\mathbf{v}\,dV + \iint_S \mathbf{t}^{(n)}\cdot\mathbf{v}\,dS - \iint_S \mathbf{q}\cdot\mathbf{n}\,dS$$

With some manipulation, this leads to:

$$\boxed{\rho\frac{De}{Dt} = \mathrm{tr}(\boldsymbol{T}\cdot\boldsymbol{D}) - \mathrm{div}\,\boldsymbol{q} = T_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}}$$

where: **e** is the *internal energy density*
q is the *heat flux vector*

Conservation Laws

Mass	$\frac{\rho_0}{\rho} = \frac{dv}{dV} = \det\boldsymbol{F}$	$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\boldsymbol{v}) = 0$
Momentum	$\rho\frac{D\boldsymbol{v}}{Dt} = \mathrm{div}\,\boldsymbol{T} + \rho\boldsymbol{b}$	$\boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}}$
Energy	$\rho\frac{De}{Dt} = \mathrm{tr}(\boldsymbol{T}\cdot\boldsymbol{D}) - \mathrm{div}\,\boldsymbol{q}$	

Nonlinear Biomechanics: Universal Governing Equations

Kinematics

Strain-displacement relation $\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$

Deformation gradient tensor $F_{iR} = \dfrac{\partial x_i}{\partial X_R}$ $\quad \boldsymbol{F} = \mathrm{Grad}(\boldsymbol{x})$

Conservation of Momentum

Force balance $\mathrm{Div}\boldsymbol{S}+\rho\boldsymbol{b} = \mathrm{Div}(\boldsymbol{P}\bullet\boldsymbol{F}^{\mathrm{T}})+\rho\boldsymbol{b} = \boldsymbol{0}$

Moment balance $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$

Conservation of Mass

Lagrangian form (ρ is mass density) $\rho = \rho_0 \det\boldsymbol{F}$

Constitutive law

Hyperelastic relation for Lagrangian 2nd Piola-Kirchoff stress (*W* is the *strain energy* function) $P_{RS} = \dfrac{1}{2}\left(\dfrac{\partial W}{\partial E_{RS}} + \dfrac{\partial W}{\partial E_{SR}}\right)$

Eulerian Cauchy stress $\boldsymbol{T} = \dfrac{1}{\det\boldsymbol{F}}\boldsymbol{F}\bullet\boldsymbol{P}\bullet\boldsymbol{F}^{\mathrm{T}}$

The Constitutive Law

- describes the *mechanical properties* of a *material*, which depend on its *constituents*
- is a mathematical relation for *stress* as a function of *kinematic* quantities, such as *strain* or *strain-rate*
- is an *idealization* and an approximation
- the validity of the idealization depends not only on the material but on the mechanical *conditions*
- must typically be determined by *experiment*
- is *constrained* by thermodynamic and other physical conditions, e.g. conservation of mass and energy
- should be derived from considerations of material *microstructure*

Solids and Fluids

Solids

- Can support shear stress indefinitely without flowing
- Assume an unloaded *natural* shape
- Deform with minimal or substantial energy dissipation
- Usually *composites*

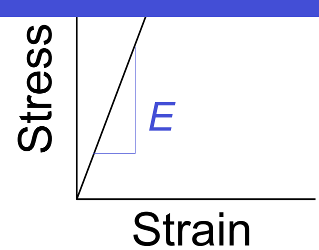
Fluids

- Liquids and gases
- Gases have lower density and higher compressibility than liquids; dependent on temperature
- Phase transition as function of temperature and pressure
- Support stress as fluid hydrostatic pressure at rest
- Can not resist a shear stress indefinitely without flowing
- No unique unloaded natural state; conform to the shape of their container
- Dissipate energy as heat when they flow
- Usually *mixtures*

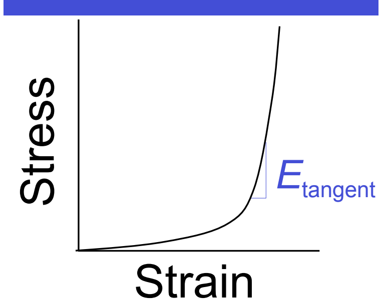
Elastic solids

- Stress depends only on strain, $T_{ij} = T_{ij}(\epsilon_{kl})$
- Return to a unique *natural* state when loads removed
- Work done during loading is stored as potential energy without dissipation (a *reversible* process)
- Example: Isotropic *Hookean* (linear) elastic solid
- Hookean solids have a constant *elastic modulus*, *E*
- In nonlinear solids, $E_{tangent}$ is dependent on strain

Linear (Hookean)

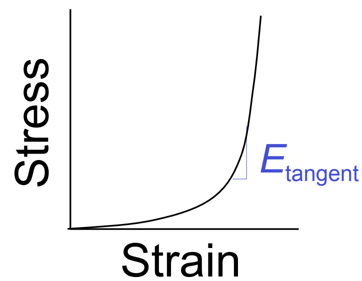


Nonlinear

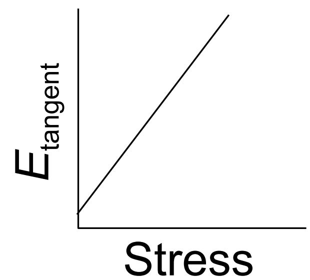


Exponential Stress-Strain Relation of Soft Tissues

Stress *vs.* Strain

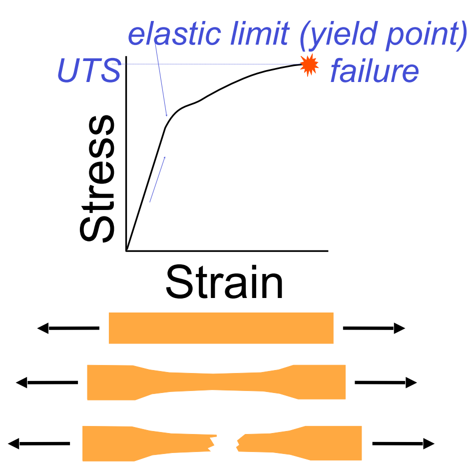


Stiffness *vs.* Stress

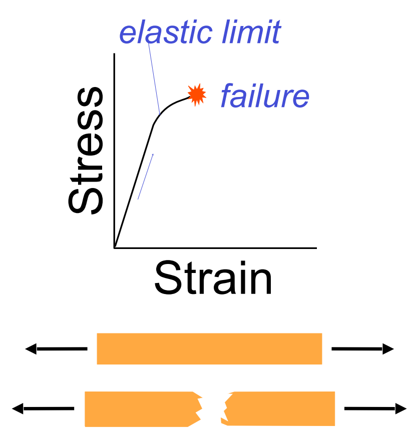


The Elastic Limit

Ductile failure



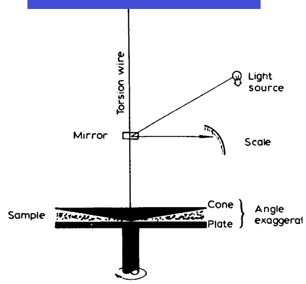
Brittle failure



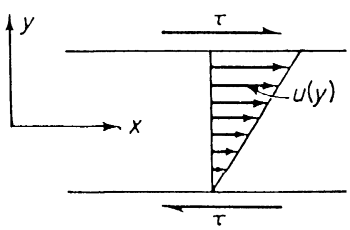
Viscous Fluids

- Shear stress depends on the *rate* of shear strain, $T_{ij} = T_{ij}(D_{kl})$
- Example: *Newtonian* viscous fluid, $T_{ij} = -p_{ij} + 2\mu D_{ij}$
- Linear viscous (Newtonian) fluids have constant *viscosity* μ
- Viscosity measures resistance to *shear*, $\tau = \mu\dot{\gamma} = \mu\dfrac{du}{dy}$
- Work done on flowing viscous fluids is dissipated as heat
- In non-Newtonian fluids, *apparent viscosity* depends on the shear rate , $\dot{\gamma}$ e.g. whole blood is *shear-thinning*

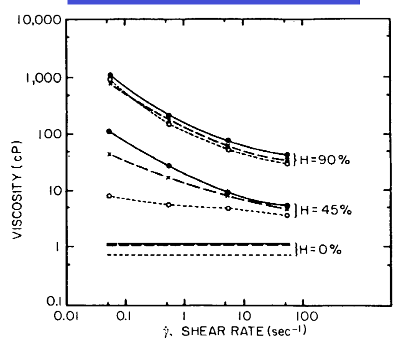
Viscometer



Couette Flow



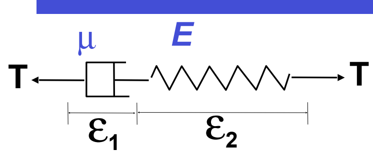
Blood



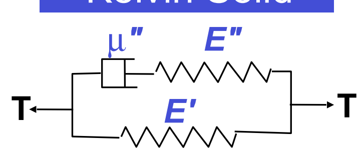
Viscoelastic Solids and Fluids

- Stress depends on strain *and* strain-rate, $T_{ij} = T_{ij}(\varepsilon_{kl}, D_{kl})$
- *Creep* at constant stress
- *Stress relaxation* at constant strain
- *Hysteresis*, energy dissipation during loading and unloading

Maxwell Fluid



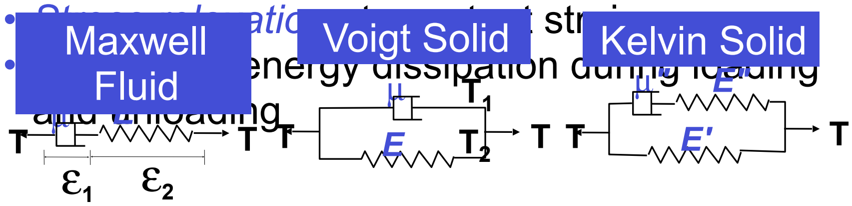
Kelvin Solid



- Elastic stress depends on strain (spring)
- Viscous stress depends on strain-rate (dashpot)
- Strains add in series, stresses are the same
- Stresses add in parallel, strains are the same

Viscoelastic Solids and Fluids

- Stress depends on strain *and* strain-rate, $T_{ij} = T_{ij}(\varepsilon_{kl}, D_{kl})$
- *Creep* at constant stress
- *Stress relaxation* at constant strain
- Energy dissipation during loading and unloading



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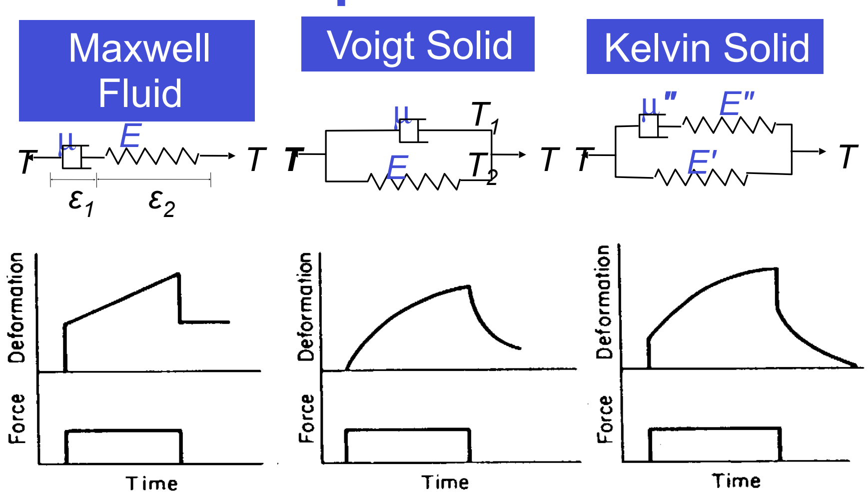
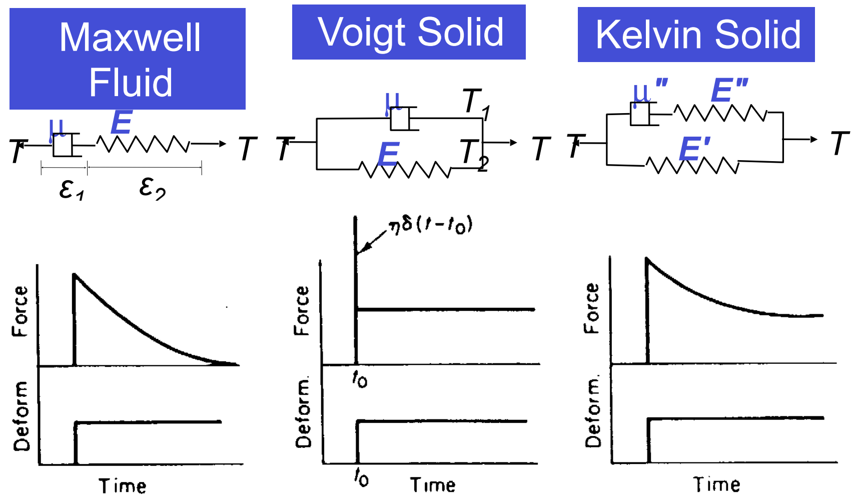
Linear Viscoelastic Models: Creep Functions

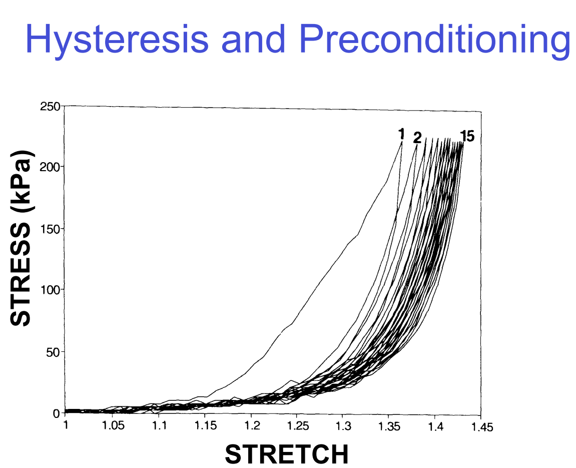


Linear Viscoelastic Models: Relaxation Functions

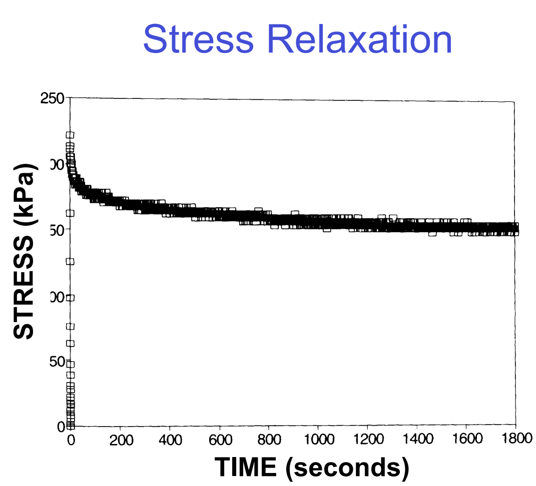


Viscoelastic Properties of Soft Tissues

Bovine Coronary Artery



data from Humphrey JD, Salunke N, Tippett B, 1996



Other Material Properties

Viscoplastic

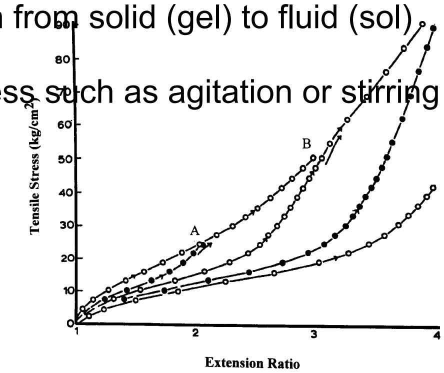
- behaves like a viscous fluid after shear stress exceeds a finite *yield stress* (e.g. whole blood)

Thixotropic

- sol-gel transformation from solid (gel) to fluid (sol) properties
- induced by shear stress such as agitation or stirring (e.g. actin)

Strain softening

- Also known as the Mullins effect
- progressive, irreversible reduction in elastic stiffness induced by increased *maximum* previous strain
- e.g. elastomers and small intestine



Considerations in Biomechanics

MODELING CONSIDERATION	EXPERIMENTAL CONSIDERATION	TISSUE EXAMPLE
linear/nonlinear e.g. Hookean/non-Hookean e.g. Newtonian/Non-Newtonian	small or large strain testing hard vs. soft tissues test over a varied shear rates	ligament bone vs. skin plasma vs. whole blood
symmetry: isotropic/anisotropic	microstructure, histology	skin, muscle, lung
homogeneous/nonhomogeneous	morphology, architecture	trabecular and cortical bone blood vessel wall layers
composites micromechanics, homogenization	cellular, extracellular and vascular components	lung alveolar structure
biphasic/poroelastic theories	tissue hydration and fluid movement	cartilage and synovial fluid
environment	in-vivo vs. in-vitro, temperature	isolated perfused heart
variability	species, subject	numerous collagen subtypes
irreversible properties	injury, repair	scarring of skin
changes in natural shape and material properties	growth and remodeling	muscle hypertrophy, bone remodeling
active stress and strain	contraction, locomotion	muscle, white blood cells
contact properties	adhesion, cell and tissue interfaces	platelets , joints

Introduction to Tissue Mechanics: Summary of Key Points

- *Biomechanics* is mechanics applied to biology; our specific focus is solid mechanics applied to physiology.
- Biomechanics involves the interplay of *experimental measurement* in living tissues and *theoretical analysis* based on physical foundations
- Biomechanics has numerous *applications* in biomedical engineering, biophysics, medicine, and other fields.
- The constitutive law describes the mechanical properties of a particular material. A major objective of biomechanics is identifying the constitutive law for biological cells and tissues.
- In an *elastic solid*, the stress depends only on the *strain*; it returns to its undeformed natural state when unloaded.
- In an *viscous fluid*, the stress depends only the *strain-rate*.
- Stress depends on *strain and strain rate* in *viscoelastic* materials; they exhibit creep, relaxation, hysteresis.

Topic 2: Summary of Key Points

- The constitutive law describes the *mechanical properties* of a *material,* which depend on its *constituents*
- In an elastic solid, the *stress* depends only the *strain;* it returns to its undeformed natural state when unloaded.
- Stress depends on strain *and* strain rate in viscoelastic materials; they exhibit *creep, relaxation, hysteresis*.

Two Definitions of Elasticity

In words

*In an **elastic material** the stress depends only on the strain.*

*The work done by the stress producing strain in a **hyper-elastic material** is stored as potential energy in a thermo-dynamically reversible process.*

Mathematically

$$\boldsymbol{T} = \boldsymbol{T}(\varepsilon) \qquad \mathrm{d}W = \boldsymbol{T}\,\mathrm{d}\varepsilon \;\rightarrow\; \boldsymbol{T} = \frac{\partial W}{\partial \varepsilon}$$

$$T_{ij} = T_{ij}(\varepsilon_{kl}) \qquad T_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

W (Work per unit volume) is also called the *strain energy density*

Strain Energy in Reversible Processes

Rate of change of Internal Energy	=Rate of Work Done by Stresses	+Rate of Heat Absorbed

$$\rho\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}t} + \rho\frac{\mathrm{d}Q}{\mathrm{d}t}$$

For a reversible process, the change in total entropy:

$$\mathrm{d}S = \frac{\mathrm{d}Q}{\theta} \rightarrow \mathrm{d}W = \rho(\mathrm{d}I - \mathrm{d}Q) = \rho(\mathrm{d}I - \theta\,\mathrm{d}S) = T_{ij}\,\mathrm{d}\varepsilon_{ij}$$

where θ is temperature

- Stress is the derivative of *strain energy* with respect to strain.
- Stress arises from an *increase in internal energy I* or a *decrease in entropy S* with strain; strain energy is stored as either or both of these:
- *Crystalline* materials (e.g. collagen) derive stress from an increase in the internal energy between their bonds, ($W \equiv \rho I$)
- *Rubbery* materials (e.g. elastin) derive stress from a *decrease* in entropy, ($W \equiv \rho F = \rho(I - \theta S)$, the Helmholtz Free Energy

Hyperelastic Constitutive Law for Finite Deformations

Second Piola-Kirchhoff Stress

$$P_{RS} = \frac{\partial W}{\partial C_{RS}} + \frac{\partial W}{\partial C_{SR}} = \frac{1}{2}\left(\frac{\partial W}{\partial E_{RS}} + \frac{\partial W}{\partial E_{SR}}\right) = \frac{\partial W}{\partial E_{RS}}$$

Cauchy Stress

$$T_{ij} = \frac{\rho}{\rho_o} \frac{\partial x_i}{\partial X_R} \frac{\partial x_j}{\partial X_S} \frac{\partial W}{\partial E_{RS}}$$

Isotropic Strain Energy Functions

Let, $W = W(I_1, I_2, I_3)$

where, I_1, I_2, I_3 are the *principal invariants* of C_{RS}

$$I_1 = \operatorname{tr}\boldsymbol{C} = C_{11} + C_{22} + C_{33}$$

$$I_2 = \tfrac{1}{2}\left(\left(\operatorname{tr}\boldsymbol{C}\right)^2 - \operatorname{tr}\boldsymbol{C}^2\right) = C_{11}C_{22} + C_{11}C_{33} + C_{33}C_{22}$$

$$I_3 = \det\boldsymbol{C}$$

$$\frac{\partial W}{\partial C_{RS}} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial C_{RS}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial C_{RS}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial C_{RS}}$$

Chain rule

$$= \frac{\partial W}{\partial I_1}\delta_{RS} + \frac{\partial W}{\partial I_2}\left(I_1\delta_{RS} - C_{RS}\right) + \frac{\partial W}{\partial I_3}\left(I_1\delta_{RS} - I_1 C_{RS} + C_{RP}C_{SP}\right)$$

Isotropic Strain Energy Functions: Examples

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) = C_1(\mathrm{tr}\,\boldsymbol{C} - 3) + \tfrac{1}{2}C_2\left((\mathrm{tr}\,\boldsymbol{C})^2 - \mathrm{tr}\,\boldsymbol{C}^2 - 3\right)$$

(Mooney-Rivlin model)

$$W = C_1\left(e^{\alpha(I_1-3)} - 1\right) + C_2(I_2 - 3)$$

(Veronda-Westman model) $\alpha \approx 5$ for skin and heart muscle

$$W = \tfrac{1}{2}\lambda\delta_{ij}\delta_{kl}\varepsilon_{ij}\varepsilon_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\varepsilon_{ij}\varepsilon_{kl}$$

(Isotropic Hooke's Law)

Strain Energy Functions for Myocardium

Transversely Isotropic Exponential (Humphrey & Yin, 1989)

$$W = 0.21\left(e^{9.4(I_1-3)} - 1\right) + 0.35\left(e^{66(\lambda_F - 1)^2} - 1\right)$$

Transversely Isotropic Exponential (Guccione et al, 1990)

$$W = 0.6\,(e^{Q} - 1)\,,$$

where, in the dog

$$Q = 26.7E_{11}^2 + 2.0(E_{22}^2 + E_{33}^2 + E_{23}^2 + E_{32}^2) + 14.7(E_{12}^2 + E_{21}^2 + E_{13}^2 + E_{31}^2)\,,$$

and, in the rat

$$Q = 9.2E_{11}^2 + 2.0(E_{22}^2 + E_{33}^2 + E_{23}^2 + E_{32}^2) + 3.7(E_{12}^2 + E_{21}^2 + E_{13}^2 + E_{31}^2)\,.$$

Orthotropic Exponential (Holzapfl and Ogden, 2009)

$$\Psi = \frac{a}{2b}\exp[b(I_1 - 3)] + \sum_{i=\mathrm{f,s}} \frac{a_i}{2b_i}\{\exp[b_i(I_{4i} - 1)^2] - 1\} + \frac{a_{\mathrm{fs}}}{2b_{\mathrm{fs}}}[\exp(b_{\mathrm{fs}} I_{8\mathrm{fs}}^2) - 1],$$

Orthotropic Power Law (Hunter et al)

$$W = 0.36\left(\frac{\lambda_1^{32}}{32} + \frac{\lambda_2^{30}}{30} + \frac{\lambda_3^{31}}{31} - 3\right)$$

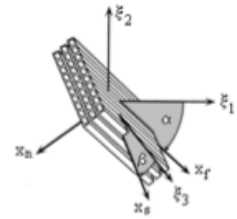
Orthotropic Exponential (Usyk et al, *J Elast*, 2000)

$$W = \tfrac{c}{2}e^{Q}$$

$$Q = b_1E_{FF}^2 + b_2E_{SS}^2 + +b_3E_{NN}^2 + b_4\left(E_{FS}^2 + E_{SF}^2\right) + b_5\left(E_{FN}^2 + E_{NF}^2\right) + b_6\left(E_{SN}^2 + E_{NS}^2\right)$$

Slightly Compressible Orthotropic Constitutive Model

Constitutive law:



$$W = \frac{1}{2} C\,(e^{Q} - 1) + C_{compr}(\,J \ln J - J + 1);$$

$$Q = b_{ff}\, E_{ff}^2 + b_{ss}\, E_{ss}^2 + b_{nn}\, E_{nn}^2 + b_{fs}\,(E_{fs}^2 + E_{sf}^2\,)$$

$$+\, b_{fn}\,(E_{fn}^2 + E_{nf}^2\,) + b_{ns}\,(E_{ns}^2 + E_{sn}^2\,);$$

where E_{IJ} are components of Green's strain tensor ***E*** in an orthogonal coordinate system having fiber, sheet and sheet-normal (*f, s, n*) axes respectively; *J* is the determinant of the stretch tensor U. The following material constants: C = 0.88 kPa; b_{ff} = 18.5; b_{ss} = 3.58; b_{nn} = 3.58; b_{fs} = 2.8; b_{fn} = 2.8; b_{ns} = 2.8; C_{compr} =100

T. Usyk et al., J Elast, 2000

Incompressible Materials

Stress is not completely determined by the strain because a *hydrostatic pressure* can be added to T_{ij} without changing C_{RS}. The extra condition is the *kinematic incompressibility constraint*

$$I_3 = \left(\det F\right)^2 = \left(\lambda_1\lambda_2\lambda_3\right)^2 = 1$$

To avoid derivative of W tending to ∞

$$W = W\left(I_1, I_2\right) - \frac{1}{2} p\left(I_3 - 1\right)$$

p is a Lagrange multiplier (a pressure) that is an additional unknown that must be solved for, now that we have added the additional equation ($I_3=1$)

$$P_{RS} = 2\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial C_{RS}} + 2\frac{\partial W}{\partial C_{RS}}\frac{\partial I_2}{\partial C_{RS}} - p\underbrace{\frac{\partial X_R}{\partial x_i}\frac{\partial X_S}{\partial x_i}}_{C_{RS}^{-1}}$$

Slightly Compressible Materials

Although most tissues are approximately incompressible, in reality vascular, lymphatic or interstitial fluid can be squeezed out by stress thus resulting in small volume changes.

Rather than adding an explicit *constraint* on the volume changes, we add a term to W that is a function of I_3, with a (relatively large) material parameter K that controls the degree of compressibility:

$$W = W\left(I_1, I_2\right) + K\left(J - 1\right)\ln\left(J\right)$$

where

$$J = \det(\boldsymbol{F}) = \sqrt{I_3}$$