Handout

# Derivations of the formula's for Poisson processes

A Poisson process is a particular random process in time. Associated with it are two distributions.The first is the Poisson distribution $P_n(t)$, which is the probability that precisely $n$ events happen ffrom time 0 to time $t$. It is a **pmf** in $n$, for fixed value of time $t$. The second is the exponential distribution $P(t)$, which is a **pdf** in $t$. $P(t)$ is the pdf for an event happening after a particular time $t$. It is also the pdf for the inter-event time for events.

You should understand these derivations, but I won't ask you to reproduce them on an exam. I will test your *understanding* of this material though. This material is supplementary to Chapter 5 (6 in 2nd ed.) of the book.

Let an event $E$ have constant probability to occur per time, i.e., a Poisson process. Let's say the probability for it to occur in a small time interval $dt$ is given by $\lambda dt$. The Poisson distribution $P_n(t)$ is the probability that after the elapse of time $t$, $n$ events happened. This is a p.m.f. in $n$, but not a p.d.f. in $t$.

Let's derive

$$P_n(t) = e^{-\lambda t}(\lambda t)^n/n!$$

In particular,

$$P_0(t) = e^{-\lambda t}.$$

$P_0(t)$ is the chance that no events have taken place at time $t$, starting from $t = 0$. If we now subdivide the interval $[0\ t]$ in small segments of size $dt$, the event $E$ should not happen in each of those intervals. The chance for no event in a single interval is $1 - \lambda t$. Therefore the chance it does not happen in any interval is

$$P_0(t) = (1 - \lambda dt)^N$$

with $N = t/dt$ the number of intervals. In the limit $dt \to 0$ we have, using a well-known formula for the exponential

$$P_0(t) = \lim_{dt \to 0}(1 - \lambda dt)^{(t/dt)} = e^{-\lambda t}.$$

What is now the p.d.f. $P(t)$ for the event to happen at a particular time $t$? (Note this is a continuous distribution with $\int_{-\infty}^{\infty} P(t)dt = 1$.) $P(t)dt$ is the probability the event will happen in the infinitesimal time interval $[t\ t + dt]$. We know the chance for the event to happen in an interval $dt$, namely $\lambda dt$. For the event to happen in $[t\ t + dt]$ it should *not* happen until time $t$ *and* happen in the next time interval of size $dt$. The chance for it *not* to happen until time $t$ is just $P_0(t) = e^{-\lambda t}$. The product of the two is $e^{-\lambda t}\lambda dt$, from which it follows that

$$P(t) = \lambda e^{-\lambda t}.$$

$P(t)$ is called the exponential distribution. The inter-event time is distributed according to $P(t)$.

What remains to be derived is the rest of the Poisson distribution, $P_k(t)$ for $k > 0$. Let us derive a formula for $P_n(t + dt)$. $P_n(t + dt)$ is the chance for $n$ events to happen at time

$t + dt$. This can happen by $n$ events happening until time $t$ and no events in the interval $[t\ t + dt]$, or by $n - 1$ events to happen until time $t$ and one event in the interval $[t\ t + dt]$. This means

$$P_n(t + dt) = P_n(t)(1 - \lambda dt) + P_{n-1}(t)\lambda dt.$$

Substituting

$$P_n(t + dt) = P_n(t) + dt\frac{dP_n(t)}{dt}$$

gives

$$\frac{dP_n(t)}{dt}dt = \lambda(-P_n(t) + P_{n-1}(t))dt.$$

Dividing out $dt$ gives

$$\frac{dP_n(t)}{dt} = \lambda(P_{n-1}(t) - P_n(t)).$$

This is a recursive differential equation for $P_n(t)$. If can be seen easily by substitution the formula for the Poisson distribution that $P_n(t) = e^{-\lambda t}(\lambda t)^n/n!$ indeed satisfies this equation. Strictly speaking we should also prove uniqueness, but we'll stop here.