Separating Variants of LEM, LPO, and MP

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Definitions

Definition

- **The Law of Excluded Middle (LEM)**: For any proposition A, either A is true or A is false (A or $\neg A$).
- **The Weak LEM (WLEM)**: For any proposition A, either $\neg A$ or $\neg\neg A$.
- **The Limited Principle of Omniscience (LPO)**: For any binary sequence α, either $\alpha(n) = 0$ for all n or there exists n such that $\alpha(n) = 1$.
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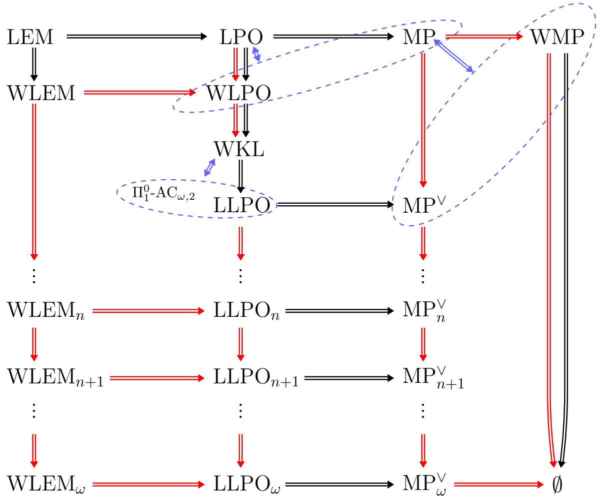
- **Markov's Principle (MP)**: If it is impossible for all terms of α to be zero, then there exists an n such that $\alpha(n) = 1$.
- **The Lesser Limited Principle of Omniscience (LLPO)**: For any binary sequence α with at most one non-zero term, either $\alpha(n) = 0$ for all even n or $\alpha(n) = 0$ for all odd n.

Implications



More Principles

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- **LLPO**$_n$: Let $(P_i)_{i<n}$ be a decidable partition of ω into blocks of size ω, and let α be a binary sequence with at most one non-zero term. Then there exists $k < n$ such that $\alpha(m) = 0$ for all $m \in P_k$.

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- **MP**$_n^\vee$: If α has at most one non-zero term and it is impossible for all terms of α to be zero, then there exists $k < n$ such that $\alpha(m) = 0$ for all $m \in P_k$.

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- **WLEM**$_n$: $\neg \bigvee_{i,j<n, i\neq j} A_i \wedge A_j \longrightarrow \bigvee_{i<n} \neg A_i$.

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Easily, $r > 0 \Rightarrow r$ is pseudo-positive $\Rightarrow r \gtrdot 0$. Mandelkern was interested in the converses.

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$i)$ $\beta \gtrdot 0$ (i.e. $\neg\neg\exists n \, (\beta(n) = 1)$) or
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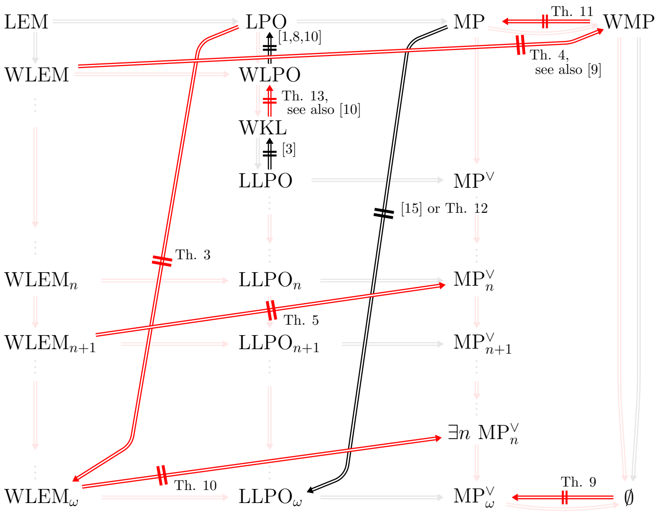
Easily, positive $\Rightarrow$ pseudo-positive $\Rightarrow$ almost positive. It is not hard to see that the assertion "if α is almost positive then α is pseudo-positive" is $\mathrm{MP}^{\vee}$.
WMP: If α is pseudo-positive then α is positive:
$\forall\beta\,(\neg\neg\exists n\;\beta(n) = 1 \vee \neg\neg\exists n(\alpha(n) = 1 \wedge \beta(n) = 0)) \to \exists n\;\alpha(n) = 1$

Non-Implications



Proof 1

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Over IZF, WLEM does not imply WMP.

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Let $f : V \to M$ be an elementary embedding of the universe of sets V into a model M with non-standard integers.

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Let $f : V \to M$ be an elementary embedding of the universe of sets V into a model M with non-standard integers. Consider the Kripke model:

$$\begin{array}{c} \bullet\ M \\ | \\ \bullet\ V. \end{array}$$

□

Proof 2

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To get WMP to be false, iterate:

$$\begin{tikzcd}[row sep=small]
\vdots \arrow[d, no head] \\
\bullet\, M_2 \arrow[d, no head] \\
\bullet\, M_1 \arrow[d, no head] \\
\bullet\, V.
\end{tikzcd}$$

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Same picture:

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| \\
\bullet\ M_2 \\
| \\
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| \\
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\end{array}
$$

Only this time, sets are allowed to change once. They have to settle down by the node after the one where they first appear.

Proof 4

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Proof.

To get DC to hold, use a topological model: Let X be $\omega \cup \{*\}$ and $\mathcal{U}$ be a non-principal ultrafilter on ω. Take the discrete topology on ω, and a neighborhood of $*$ to be $\{*\} \cup u$ where $u \in \mathcal{U}$.

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WLEM: Consider $X_0 := \{k \in \omega : \{k\} \Vdash \neg A\}$ and
$X_1 := \{k \in \omega : \{k\} \Vdash A\}$.

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WMP: Consider α such that $\{k\} \Vdash \alpha(n) = 1$ iff $k = n$. No neighborhood of $*$ forces α to be positive. To see that α is pseudo-positive, given β, consider $X_0 := \{k \in \omega : \{k\} \Vdash \exists n\, \beta(n) = 1\}$ and $X_1 := \{k \in \omega : \{k\} \Vdash \forall n\, \beta(n) = 0\}$. $\square$

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Proof.

To get DC and ¬WMP, iterate. Let T be $\omega^{<\omega}$, the nodes of the countably branching tree. A basic open set $\mathcal{O}$ contains a unique shortest node $\sigma_{\mathcal{O}}$, and, for all $\sigma \in \mathcal{O}, \{n \mid \sigma^\frown n \in \mathcal{O}\} \in \mathcal{U}$. □

WMP and MP

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Take the topological model over the rationals $\mathbb{Q}$.
For $\neg$MP, let $\mathcal{O}_n$ be a nested sequence of irrational intervals with intersection $\{r\}$. Let $[\![\alpha(n) = 0]\!] = \mathcal{O}_n$.

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For $r \in \mathcal{O}$, let $\mathcal{O}_n$ decide $\alpha(n)$ $(r \in \mathcal{O})$. Worst case:
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For $r \in \mathcal{O}$, let $\mathcal{O}_n$ decide $\alpha(n)$ $(r \in \mathcal{O})$. Worst case: $\mathcal{O}_n \Vdash \alpha(n) = 0$. WLOG r is the left endpoint of $\bigcap \mathcal{O}_n$. Let $[\![\beta(n) = 0]\!] = \mathcal{O}_n \cup (r, \infty)$. Then β contradicts the hypothesis on $\mathcal{O}$. $\square$

WKL and WLPO

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Over IZF, WKL does not imply WLPO.

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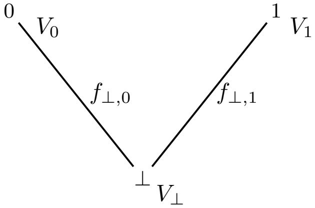

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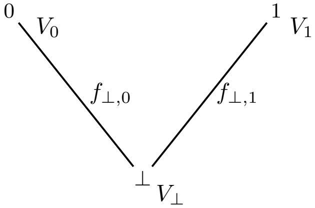

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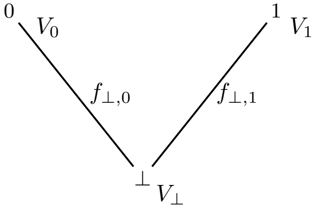

V_0 the standard V, V_1 ω-non-standard. For WKL, if at $\perp$ Tr is an infinite tree, consider a non-standard branch. For $\neg$WLPO, let $\alpha(n)$ be 0 for all standard n and 1 for some non-standard n. $\square$

References

- Matt Hendtlass and Robert Lubarsky, 'Separating Fragments of WLEM, LPO, and MP', to appear (hopefully), available at http://math.fau.edu/lubarsky/pubs.html.
- Hajime Ishihara, 'Constructive reverse mathematics: compactness properties', in L. Crosilla and P. Schuster, eds., *From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics*, Oxford University Press, p. 245–267, 2005.
- Ulrich Kohlenbach, *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*, Springer, 2008.
- Mark Mandelkern, 'Constructively complete finite sets', Zeitschr. f. math. Logic und Grundlagen d. Math. 34, p. 97-103, 1988.
- Fred Richman, 'Polynomials and linear transformations', Linear Algebra Appl. 131, p. 131–137, 1990.