NOTES ON INTUITIONISTIC MATHEMATICS 5

1. The Fan Theorem

1.1. **Finitary spreads.** Recall that an element σ of $\mathcal{N}$ is called a *spread-law* if and only if the following conditions are satisfied.

(i) $\sigma(\langle\ \rangle) = 0)$, and
(ii) for all s, $\sigma(s) = 0$ if and only if $\exists n[\sigma(s * \langle n\rangle) = 0]$.

A spread-law σ is called a *finitary-spread*-law or a *fan*-law if and only if:

$$\forall s[\sigma(s) = 0 \rightarrow \exists n \forall m[\sigma(s * \langle m\rangle) = 0 \rightarrow m \leq n]].$$

If σ is a fan-law, then, given any s that is admitted by σ, there are only **finitely many immediate extensions** of (the finite sequence coded by) s that are admitted by σ.

A subset $\mathcal{X}$ of $\mathcal{N}$ is called a *finitary spread* or a *fan* if and only if there exists a fan-law σ such that $\mathcal{X} = \mathcal{F}_\sigma$, the set of all infinite sequences α such that $\forall n[\sigma(\overline{\alpha}n) = 0]$.

In his early publications, Brouwer uses the expression *finite Menge* for what he later calls a *fan*.

An important and famous example of a fan-law is σ_2. It is defined by:

$$\forall s[\sigma_2(s) = 0 \leftrightarrow \forall n < \mathit{length}(s)[s(n) = 0 \ \vee \ s(n) = 1]],$$

that is, σ_2 just admits the (code numbers of the) finite binary sequences, the elements of Bin.

The set $\mathcal{F}_{\sigma_2}$ is also denoted by $\mathcal{C}$ and is called *Cantor space*.

Another useful example of a fan-law is τ_2. It is defined by:

$$\forall s[\tau_2(s) = 0 \leftrightarrow \forall i < \mathit{length}(s)[s(i) < 2 \ \wedge \ \big(i+1 < \mathit{length}(s) \rightarrow s(i) \leq s(i+1)\big)]],$$

that is, τ_2 just admits the (code numbers of the) *nondecreasing* finite binary sequences.

The set $\mathcal{F}_{\tau_2}$ is also called: $\mathcal{T}_2$.

Let us define a function f from $\mathbb{N}$ to $\mathcal{T}_2$:

> $f(0) = \underline{0}$, and for each n, $f(n+1) = \overline{\underline{0}}n * \underline{1}$

From a classical point of view, f seems to enumerate *all* elements of $\mathcal{T}_2$. Intuitionistically, however, we see that the following infinite sequence α belongs to $\mathcal{T}_2$:

> For each n, if $n < k_{99}$, then $\alpha(n) = 0$, and, if $n \geq k_{99}$, then $\alpha(n) = 1$.

Suppose that we find n such that $f(n) = \alpha$. Then either $n = 0$ and $\forall p[p < k_{99}]$, or $n > 0$ and $n = k_{99}$. The statement: $\exists n[f(n) = \alpha]$ thus is a *reckless* statement.

We now prove that the statement '$\forall \alpha \in \mathcal{T}_2 \exists n[f(n) = \alpha]$' leads to a contradiction.

Assume: $\forall \alpha \in \mathcal{T}_2 \exists n[f(n) = \alpha]$. Using Brouwer's Continuity Principle we find m, n such that $\forall \alpha \in \mathcal{T}_2[\overline{\underline{0}}m \sqsubset \alpha \rightarrow f(n) = \alpha]$. Conclude: *either* $n = 0$ and $\forall \alpha \in \mathcal{T}_2[\overline{\underline{0}}m \sqsubset \alpha \rightarrow \alpha = \underline{0}]$, and that is false, *or:* $n > 0$ and $\forall \alpha \in \mathcal{T}_2[\overline{\underline{0}}m \sqsubset \alpha \rightarrow \alpha(n) = 1]$, and that is false. We thus see: the statement: '$\forall \alpha \in \mathcal{T}_2 \exists n[f(n) = \alpha]$' leads to a contradiction, that is: $\neg \forall \alpha \in \mathcal{T}_2 \exists n[f(n) = \alpha]$.

1.2. **Brouwer's proof of the Fan Theorem.** Let $\mathcal{X}$ be a subset of $\mathcal{N}$ and let B be a subset of $\mathbb{N}$. We define: *B is a bar in $\mathcal{X}$* if and only if: $\forall\alpha \in \mathcal{X}\exists n[\overline{\alpha}n \in B]$.

For each subset $\mathcal{X}$ of $\mathcal{N}$, we define: $\mathcal{X} \cap s := \{\alpha \in \mathcal{X} | s \sqsubset \alpha\}$.

Theorem 1.1 (Brouwer's Fan Theorem)**.** *Let $\mathcal{F} \subseteq \mathcal{N}$ be a finitary spread, (that is: a fan), and let B be a subset of $\mathbb{N}$ that is a bar in $\mathcal{F}$.*

Then there exists a finite subset of B that is a bar in $\mathcal{F}$, that is:

$$\exists s[\forall i < \mathit{length}(s)[s(i) \in B] \ \wedge \ \forall\alpha \in \mathcal{F}\exists n\exists i < \mathit{length}(s)[\overline{\alpha}n = s(i)]].$$

Proof. The proof is philosophical rather than mathematical and the Fan Theorem perhaps should be called an axiom rather than a theorem.

Let $\mathcal{F} \subseteq \mathcal{N}$ be a fan. Find a fan-law σ such that $\mathcal{F} = \mathcal{F}_\sigma$. Let B be a subset of $\mathbb{N}$.

We define, for every s:

s is in $\mathcal{F}_\sigma$ **safe** *with respect to B* if and only if *$\sigma(s) = 0$ and B is bar in $\mathcal{F}_\sigma \cap s$.*

Now assume: B is a bar in $\mathcal{F}_\sigma$, that is: $\langle\, \rangle$ is in $\mathcal{F}_\sigma$ safe with respect to B.

According to Brouwer, there now must exist a **canonical proof** of the fact:

$\langle\, \rangle$ is in $\mathcal{F}_\sigma$ safe with respect to B.

The *conclusion* of the canonical proof is: '$\langle\, \rangle$ is in $\mathcal{F}_\sigma$ safe with respect to B.'

The *starting points* or: *initial reasoning steps* of the canonical proof have the form:

$\sigma(s) = 0$ and $s \in B$.
Therefore: s is safe in $\mathcal{F}_\sigma$ with respect to B.

There are two kinds of reasoning steps: *forward reasoning steps* and *backward reasoning steps*. Forward means: in the direction of the conclusion, backward means: away from the conclusion.

A *forward reasoning step* has the form:

$s * \langle n_0 \rangle$ is safe in $\mathcal{F}_\sigma$ with respect to B.
$s * \langle n_1 \rangle$ is safe in $\mathcal{F}_\sigma$ with respect to B.
$\ldots$
$s * \langle n_k \rangle$ is safe in $\mathcal{F}_\sigma$ with respect to B.
$n_0, n_1, \ldots, n_k$ is the list of all n such that $\sigma(s * \langle n \rangle) = 0$.
Therefore: s is safe in $\mathcal{F}_\sigma$ with respect to B.

A *backward reasoning step* has the form:

s is safe in $\mathcal{F}_\sigma$ with respect to B.
$\sigma(s * \langle n \rangle) = 0$.
Therefore: $s * \langle n \rangle$ is safe in $\mathcal{F}_\sigma$ with respect to B.

Brouwer's crucial idea is: to use the canonical proof as a framework for making a new proof.

We define, for every s:

s is in $\mathcal{F}_\sigma$ **supersafe** *with respect to B* if and only if
$\sigma(s) = 0$ *and* $\exists u[\forall i < \mathit{length}(u)[u(i) \in B] \ \wedge \ \forall\alpha \in \mathcal{F}_\sigma \cap s\exists i\exists n[\overline{\alpha}n = u(i)]]$.

Now replace in the canonical proof everywhere the word 'safe' by the word 'supersafe'. The result will be another *valid proof.*

Why? First of all, the new initial steps are sound: if $\sigma(s) = 0$ and $s \in B$, define: $u := (s)$.

Then: the new forward reasoning steps remain sound.

For assume: $s*\langle n_0\rangle$, $s*\langle n_1\rangle$ ..., $s*\langle n_k\rangle$ are supersafe in $\mathcal{F}_\sigma$ with respect to B, and $n_0, n_1, \ldots, n_k$ is the list of all n such that $\sigma(s*\langle n\rangle) = 0$.

Find $u_0, u_1, \ldots, u_k$ such that:

$$\forall j \le k[\forall i \le \mathit{length}(u_j)[u_j(i) \in B] \wedge \forall\alpha \in \mathcal{F}_\sigma \cap s*\langle n_j\rangle \exists i < \mathit{length}(u_j)\exists n[u_j(i) = \overline{\alpha}n]].$$

Define $u := u_0 * u_1 * \ldots * u_k$ and note

$$\forall i < \mathit{length}(u)[u(i) \in B] \ \wedge\ \forall\alpha \in \mathcal{F}_\sigma \cap s\exists i < \mathit{length}(u)\exists n[u(i) = \overline{\alpha}n].$$

Conclude: s is supersafe in $\mathcal{F}_\sigma$ with respect to B.

Also: the new backward reasoning steps are sound.

For assume: s is supersafe in $\mathcal{F}_\sigma$ with respect to B and: $\sigma(s*\langle n\rangle) = 0$.

Find u such that

$\forall i < \mathit{length}(u)[u(i) \in B] \ \wedge\ \forall\alpha \in \mathcal{F}_\sigma \cap s\exists i < \mathit{length}(u)\exists n[u(i) = \overline{\alpha}n]$.

Note: $\mathcal{F}_{s*\langle n\rangle} \subseteq \mathcal{F}_\sigma$ and conclude:

$\forall i < \mathit{length}(u)[u(i) \in B] \ \wedge\ \forall\alpha \in \mathcal{F}_\sigma \cap s*\langle n\rangle\exists i < \mathit{length}(u)\exists n[u(i) = \overline{\alpha}n]$, that is: $s*\langle n\rangle$ is in $\mathcal{F}_\sigma$ supersafe with respect to B.

We must conclude: the new conclusion is true, that is:

$\langle\,\rangle$ is in $\mathcal{F}_\sigma$ supersafe with respect to B, that is:

$\exists u\forall i < \mathit{length}(u)[u(i) \in B] \ \wedge\ \forall\alpha \in \mathcal{F}_\sigma\exists i < \mathit{length}(u)\exists n[u(i) = \overline{\alpha}n]$. □

1.3. **The Uniform-Continuity Theorem.** Let f be a real function from $[0,1]$ to $\mathcal{R}$. We have seen that Brouwer's Continuity Principle proves:

$$\forall\alpha \in \mathcal{R}\forall p \in \mathbb{N}\exists m \in \mathbb{N}\forall\beta \in \mathcal{R}[|\alpha -_\mathcal{R} \beta|_\mathcal{R} <_\mathcal{R} \frac{1}{2^m} \to |f(\alpha) -_\mathcal{R} f(\beta)|_\mathcal{R} <_\mathcal{R} \frac{1}{2^p}].$$

that is, f is continuous everywhere on $[0,1]$.

We now want to prove, using the method of proof of the previous Subsection:

Theorem 1.2 (Uniform-Continuity Theorem)**.** *Let f be a real function from $[0,1]$ to $\mathcal{R}$ that is continuous everywhere on $[0,1]$.*

Then f is continuous uniformly on $[0,1]$, that is:

$$\forall p \in \mathbb{N}\exists m \in \mathbb{N}\forall\alpha \in \mathcal{R}\forall\beta \in \mathcal{R}[|\alpha -_\mathcal{R} \beta|_\mathcal{R} <_\mathcal{R} \frac{1}{2^m} \to |f(\alpha) -_\mathcal{R} f(\beta)|_\mathcal{R} <_\mathcal{R} \frac{1}{2^p}].$$

Proof. We first define a function D from Bin to $\mathbb{S}$, as follows.

(i) $D(\langle\,\rangle) = (0_\mathbb{Q}, 1_\mathbb{Q})$, and,

(ii) for each a in Bin, for each s in $\mathbb{S}$, if $D(a) = s$, then $D(a*\langle 0\rangle) = (s', \frac{1}{3}s' +_\mathbb{Q} \frac{2}{3}s'')$ and $D(a*\langle 1\rangle) = (\frac{2}{3}s' +_\mathbb{Q} \frac{1}{3}s'', s'')$.

Let n in $\mathbb{N}$ be given.

We define a subset B of Bin, as follows. Let a in Bin be given. Find s in $\mathbb{S}$ such that $D(a) = s = (s', s'')$. We define: $a \in B$ if and only if $\forall\alpha \in \mathcal{R}\forall\beta \in \mathcal{R}[\big((s')_\mathcal{R} \le_\mathcal{R} \alpha_\mathcal{R} \le (s'')_\mathcal{R} \ \wedge\ (s')_\mathcal{R} \le_\mathcal{R} \beta_\mathcal{R} \le (s'')_\mathcal{R}\big) \to |f(\alpha) -_\mathcal{R} f(\beta)| <_\mathcal{R} \frac{1}{2^n}]$.

We claim that B is a bar in Cantor space $\mathcal{C}$ and we prove this claim as follows.

Let α in $\mathcal{C}$ be given. Define β in $\mathcal{N}$ such that, for each i, $\beta(i) = D(\overline{\alpha}i)$. Note: $\beta \in \mathcal{R}$ and $0_\mathcal{R} \le_\mathcal{R} \beta \le_\mathcal{R} 1_\mathcal{R}$. Use the fact that f is continuous at β and find m such that $\forall\gamma \in [0,1][|\beta - \gamma| \le_\mathcal{R} (\frac{2}{3})^m \to |f(\beta) -_\mathcal{R} f(\gamma)| < \frac{1}{2^{n+1}}]$. Note: $\overline{\alpha}m \in B$.

We now define a subset C of Bin. Let a in Bin be given. Find s in $\mathbb{S}$ such that $D(a) = s = (s', s'')$. We define: $a \in C$ if and only if

$$\exists m\forall\alpha \in [s', s'']\forall\beta \in [s', s''][|\alpha -_\mathcal{R} \beta| <_\mathcal{R} \frac{1}{2^m} \to |f(\alpha) -_\mathcal{R} f(\beta)| <_\mathcal{R} \frac{1}{2^n}].$$

Note: $B \subseteq C$.

We claim that C is *inductive* in the following sense:

$$\forall a \in Bin[(a*\langle 0\rangle \in C \ \wedge\ a*\langle 1\rangle \in C) \to a \in C].$$

We prove this claim as follows. Let a in Bin be given such that $a*\langle 0\rangle$ and $a*\langle 1\rangle$ both belong to C. Define $s := D(a)$ and note: $D(a*\langle 0\rangle) = (s', \frac{1}{3}s' +_{\mathbb{Q}} \frac{2}{3}s'')$ and $D(a*\langle 1\rangle) = (\frac{2}{3}s' +_{\mathbb{Q}} \frac{1}{3}s'', s'')$. Find $p := \mathit{length}(s)$ and note $s'' -_{\mathbb{Q}} s' = (\frac{2}{3})^p$. Find m_0 such that, for all α, β in $[s', \frac{1}{3}s' +_{\mathbb{Q}} \frac{2}{3}s'')]$, if $|\alpha - \beta| < \frac{1}{2^{m_0}}$, then $|f(\alpha) - f(\beta)| < \frac{1}{2^n}$. Then find m_1 such that, for all α, β in $[(\frac{2}{3}s' +_{\mathbb{Q}} \frac{1}{3}s'', s'')]$, if $|\alpha - \beta| < \frac{1}{2^{m_1}}$, then $|f(\alpha) - f(\beta| < \frac{1}{2^n}$. Note: for all α, β in $[s, s'']$, if $|\alpha - \beta| < (\frac{1}{3})^p$, then either α, β both are in $[s', \frac{1}{3}s' +_{\mathbb{Q}} \frac{2}{3}s'']$ or α, β both are in $[\frac{2}{3}s' +_{\mathbb{Q}} \frac{1}{3}s'', s'']$.

Find m such that $m \geq m_0$ and $m \geq m_1$ and $\frac{1}{2^m} < (\frac{1}{3})^p$ and note: for all α, β in $[s', s'']$, if $|\alpha - \beta| < \frac{1}{2^m}$, then $|f(\alpha) - f(\beta| < \frac{1}{2^n}$. We thus see: $a \in C$.

We also claim that Q is *monotone* in the following sense:

$$\forall a \in Bin[s \in C \rightarrow (a*\langle 0\rangle \in C \ \wedge \ a*\langle 1\rangle \in C)].$$

The proof is obvious, as $\forall a \in Bin[D(a*\langle 0\rangle) \subseteq D(a) \ \wedge \ D(a*\langle 1\rangle) \subseteq D(a)]$.

Now take the canonical proof of the fact that B is a bar in $\mathcal{C}$. We have seen that this proof is an arrangement of statements of the form: 'a is safe with respect to B' that is: '$\forall \alpha \in \mathcal{C}[a \sqsubset \alpha \rightarrow \exists n[\overline{\alpha}n \in B]]$'. Replace, in this proof, every such statement by the statement: '$a \in C$'.

A *starting point* '$a \in B$' will be replaced by $a \in C$'. Note: $B \subseteq C$ and conclude: if the starting point is a true statement, also the statement replacing it is true.

A *forward reasoning step*: '$a*\langle 0\rangle$ is safe with respect to B, and $a*\langle 1\rangle$ is safe with respect to B, and, therefore: a is safe with respect to B' will be replaced by: '$a*\langle 0\rangle \in C$ and $a*\langle 1\rangle \in C$, and, therefore: $a \in C$'. Note: C is inductive and conclude: the new reasoning step is a sound step.

A *backward reasoning step*: 'a is safe with respect to B, and, therefore: $a*\langle 0\rangle$ is safe with respect to B' will be replaced by: '$a \in C$ and, therefore: $a*\langle 0\rangle \in C$'. Note: Q is monotone and conclude: the new reasoning step is a sound step.

Obviously, the same holds for a backward reasoning step of the form: 'a is safe with respect to B, and, therefore: $a*\langle 1\rangle$ is safe with respect to B'.

The conclusion of the canonical proof is: '$\langle\ \rangle$ is safe with respect to B'. It will be replaced by: '$\langle\ \rangle \in C$'. The statement '$\langle\ \rangle \in C$' now will be the conclusion of a proof that has true starting points and uses only sound, that is truth-preserving, reasoning steps. The new conclusion thus must be true too, that is:

$$\exists m \forall \alpha \in [0,1] \forall \beta \in [0,1][|\alpha -_{\mathcal{R}} \beta| <_{\mathcal{R}} \frac{1}{2^m} \rightarrow |f(\alpha) -_{\mathcal{R}} f(\beta)| <_{\mathcal{R}} \frac{1}{2^n}].$$

In this way, we prove:

$$\forall n \exists m \forall \alpha \in [0,1] \forall \beta \in [0,1][|\alpha -_{\mathcal{R}} \beta| <_{\mathcal{R}} \frac{1}{2^m} \rightarrow |f(\alpha) -_{\mathcal{R}} f(\beta)| <_{\mathcal{R}} \frac{1}{2^n}],$$

that is, f is continuous *uniformly* on $[0,1]$. □

1.4. **Bar Induction in Finitary Spreads.** The proof that we gave for Theorem 1.2 may be generalized and then gives the following more abstract conclusion:

Theorem 1.3 (A Principle of Bar Induction)**.** *Let σ in $\mathbb{N}$ be a fan-law such that $\sigma(\langle\ \rangle) = 0$, and let $\mathcal{F}_\sigma$ be the finitary spread determined by σ.*

Let B be a subset of $\mathbb{N}$ that is a bar in $\mathcal{F}_\sigma$. Let C be a subset of $\mathbb{N}$ that satisfies the following conditions:

(i) *$B \subseteq C$, and*
(ii) C is monotone within the set $\{s|\sigma(s) = 0\}$: *for all s such that $\sigma(s) = 0$, if $s \in C$, then for all n such that $\sigma(s*\langle n\rangle) = 0$, $s*\langle n\rangle \in C$, and,*
(iii) C is inductive within the set $\{s|\sigma(s) = 0\}$: *for all s such that $\sigma(s) = 0$, if for all n such that $\sigma(s*\langle n\rangle) = 0$, $s*\langle n\rangle \in C$, then $s \in C$.*

Then $\langle\ \rangle \in C$.

Proof. Define, for every s such that $\sigma(s) = 0$, *s is in $\mathcal{F}_\sigma$ safe with respect to B* if and only if: *B is a bar in $\mathcal{F}_\sigma \cap s$*. Build, as in the proof of Theorem 1.1, a canonical proof of the statement: '*$\langle\ \rangle$ is in $\mathcal{F}_\sigma$ safe with respect to B*'. Replace, in this proof, every statement '*s is in $\mathcal{F}_\sigma$ safe with respect to B*' by the statement: '$s \in C$'. Note that the result of these replacements is a valid proof. Conclude: $\langle\ \rangle \in C$. □

The idea to formulate the assumptions underlying Brouwer's Fan Theorem as a Principle of Bar Induction is due to S.C. Kleene and G. Kreisel.

We now easily prove another formulation of the Fan Theorem.

Theorem 1.4 (Brouwer's Fan Theorem, second formulation)**.** *Let $\mathcal{F} \subseteq \mathcal{N}$ be fan and let B be a subset of $\mathbb{N}$.*

If $\forall \alpha \in \mathcal{F} \exists n[\overline{\alpha}n \in B]$, then $\exists m \forall \alpha \in \mathcal{F} \exists n \leq m[\overline{\alpha}n \in B]$.

Proof. Assume: $\forall \alpha \in \mathcal{F} \exists n[\overline{\alpha}n \in B]$.

Let C be the set all s such that $\exists m \forall \alpha \in \mathcal{F} \cap s \exists n \leq m[\overline{\alpha}n \in B]$.

Note: $B \subseteq C$ and C is monotone and inductive.

Using Theorem 1.3, conclude: $\langle\ \rangle \in C$, that is: $\exists m \forall \alpha \in \mathcal{F} \exists n \leq m[\overline{\alpha}n \in B]$. □

1.5. **König's Lemma.** *König's Lemma* is the statement: *every infinite finitely-branching tree has an infinite branch*:

> Let σ in $\mathcal{N}$ be a fan-law such that $\sigma(\langle\ \rangle) = 0$. Let T be a subset of $\mathbb{N}$ such that $\forall s \in T[\sigma(s) = 0]$ and $\forall s \forall n[s * \langle n \rangle \in T \to s \in T]$.
>
> (A subset T of $\mathbb{N}$ satisfying this second condition is called a *tree*.)
>
> If T is infinite, that is: $\forall m \exists n > m[n \in T]$, then T has an infinite branch, that is: $\exists \alpha \forall n[\overline{\alpha}n \in T]$.

1.5.1. *The classical proof of König's Lemma.* Let σ, T satisfy the conditions of König's Lemma.

Define

$$C := \{s \in T \mid \forall m \exists t > m[n \in T \ \wedge \ s \sqsubset t]\}.$$

Note: $s \in C$ if and only if there are infinitely many t in T such that $s \sqsubseteq t$.

Note: for all s, t in T, $s \sqsubseteq t$ if and only if either $s = t$ or there exists n such that $\sigma(s * \langle n \rangle) = 0$ and $s * \langle n \rangle \sqsubseteq t$.

As, for each each s such that $\sigma(s) = 0$ there are only finitely many n such that $\sigma(s * \langle n \rangle) = 0$, conclude: $\forall s[s \in C \to \exists n[s * \langle n \rangle \in C]]$.

Note: $\langle\ \rangle \in C$. Define α such that, for each n, $\alpha(n)$ is the least p such that $\overline{\alpha}n * \langle p \rangle \in C$. Note: $\forall n[\overline{\alpha}n \in C]$ and, therefore, $\forall n[\overline{\alpha}n \in T]$.

1.5.2. *From a classical point of view, König's Lemma is equivalent to the Fan Theorem.* Let us sketch the classical argument.

First, assume the Fan Theorem, Theorem 1.4. Let σ, T satisfy the conditions of König's Lemma. Define $B := \mathbb{N} \setminus T$. Assume: $\neg \exists \alpha \forall n[\overline{\alpha}n \in T]$. Then $\forall \alpha \in \mathcal{F}_\sigma \exists n[\overline{\alpha}n \in B]$. Find m such that: $\forall \alpha \in \mathcal{F}_\sigma \exists n \leq m[\overline{\alpha}n \in B]$. Find p such that $\forall s[(\sigma(s) = 0 \ \wedge \ \mathit{length}(s) \leq m) \to s \leq p]$. Conclude: $\neg \exists s > p[\sigma(s) = 0 \ \wedge \ s \in T]$. Contradiction. Conclude: $\neg\neg \exists \alpha \forall n[\overline{\alpha}n \in T]$ and $\exists \alpha \forall n[\overline{\alpha}n \in T]$.

Secondly, assume König's Lemma. Let σ, B satisfy the assumptions of the Fan Theorem. Let T be the set of all s such that $\sigma(s) = 0$ and $\forall n \leq \mathit{length}(s)[\overline{s}n \notin B]$. Note: T is a finitely-branching tree. Also note: $\forall \alpha \in \mathcal{F}_\sigma \exists n[\overline{\alpha}n \in B]$ and: $\neg \exists \alpha \in \mathcal{F}_\sigma \forall n[\overline{\alpha}n \in T]$. The tree T thus has no infinite branch. Conclude: T is finite. Find m such that $\forall s \in T[\mathit{length}(s) \leq m]$. Conclude: $\forall \alpha \in \mathcal{F}_\sigma \exists n \leq m + 1[\overline{\alpha}n \in B]$.

1.5.3. *König's Lemma implies* **LLPO** *and fails constructively.* Assume König's Lemma. Let T be the set of all s in Bin such that, if $length(s) > k_{99}$ and k_{99} is even, then $s(0) = 0$, and if $length(s) > k_{99}$ and k_{99} is odd, then $s(0) = 1$. Note that T is a tree and that, for each s, either $\langle 0\rangle * s \in T$ or $\langle 1\rangle * s \in T$, so T is infinite. As $T \subseteq Bin$, T is finitely-branching. Assume T has an infinite branch. Find α in $\mathcal{C}$ such that, for all n, $\overline{\alpha}n \in T$. Note: if $\alpha(n) = 0$, then $\forall n[n = k_{99} \rightarrow n \textit{ is even}]$, and, if $\alpha(n) = 1$, then $\forall n[n = k_{99} \rightarrow n \textit{ is odd}]$. In both cases, we obtain a reckless conclusion.

The argument used here may be generalized to prove:

König's Lemma implies the *Lesser Limited Principle of Omniscience* **LLPO**:

For all α such that $\forall m \forall n[(\alpha(m) \neq 0 \ \wedge\ \alpha(n) \neq 0) \rightarrow m = n]$,
(that is, α assumes at most one time a value different from 0),
either: $\forall n[\alpha(n) \neq 0 \rightarrow n \textit{ is even}]$,
or: $\forall n[\alpha(n) \neq 0 \rightarrow n \textit{ is odd}]$.

With the help of Brouwer's Continuity Principle **BCP** one may prove:

LLPO leads to a contradiction.

1.5.4. König's Lemma is due to the Hungarian mathematician D. König, the author of a famous book *Theorie der endlichen und unendlichen Graphen* (1936). The first appearance of the Lemma seems to be in 1927, and Brouwer's Fan Theorem appears for the first time in 1924.

1.6. **Kleene's Alternative to the Fan Theorem.** The Fan Theorem fails to be true in *computable analysis* where one assumes that every function α is given by a *Turing program.* Computable analysis became a possible field of research only after 1936, the famous year in which the notion of a *Turing algorithm* was discovered.

S.C. Kleene was one of the founders of the subject of *Recursion Theory* or *Computability Theory.* He visited Amsterdam in 1952 in order to learn about intuitionistic mathematics. Together with his student R.E. Vesley he wrote the book: *The Foundations of Intuitionistic Mathematics, especially in relation to the theory of recursive functions* (1965). The discovery of the failure of the Fan Theorem in the recursive context is due to him (1954).

1.6.1. *Kleene's T-predicate.* There exist a computable subset T of $\mathbb{N}$ and a computable function U from $\mathbb{N}$ to $\mathbb{N}$ such that, for every computable function φ from $\mathbb{N}$ to $\mathbb{N}$ there exists a natural number e such that:

$$\forall n[\varphi(n) = U(\mu z[T(e, n, z)])].$$

We write: '$T(e, n, z)$' where whe intend: '$\langle e, n, z\rangle \in T$'.

$\mu z[Q(m_0, m_1, \ldots, m_{k-1}, z)]$ stands for: *the least number z such that* $Q(m_0, m_1, \ldots, m_{k-1}, z)$. This is only meaningful if the property Q is *algorithmically decidable.* We then find z by trying first $z = 0$, then $z = 1$, and so on, until we are successful. Note that the expression still may be undefined, as there may not exist a number z such that $Q(m_0, m_1, \ldots, m_{k-1}, z)$. In the case that we consider here, where φ is supposed to be a function that is defined everywhere, this problem does not arise.

The informal meaning of $T(e, n, z)$ is: the natural number z is the code of a successful computation according to the algorithm coded by the natural number e at the input n. The function U is the the *result-extracting function*: given the code z of a successful computation, the function U reads off the *outcome* of the computation.

The number e is called an *index* of the function φ. If we start from an arbitrary number e and define a function φ by:

$$\forall n[\phi(n) \simeq U(\mu z[T(e,n,z)])],$$

then, in general, φ will not be defined everywhere in $\mathbb{N}$, but will be only a *partial* function from $\mathbb{N}$ to $\mathbb{N}$. This partial function will be denoted by φ_e.

Church's Thesis is the claim that every *informally, intuitively, computable function* from $\mathbb{N}$ to $\mathbb{N}$ has an index.

The *Halting Problem* H is the set

$$H := \{e \in \mathbb{N} | \exists z[T(e,e,z)].$$

Church's Thesis implies that the Halting Problem is *unsolvable*, that is, there does not exists an index e such that φ_e is defined everywhere and $\forall n[n \in H \leftrightarrow \varphi_e(n) = 1]$. For suppose there exists such e. Following Cantor's diagonal argument, define f from $\mathbb{N}$ to $\mathbb{N}$ such that, for each n, if $\varphi_e(n) \neq 1$, then $f(n) = 0$, and, if $\varphi_e(n) = 1$, then $f(n) = \varphi_n(n) + 1$. f is now an intuitively computable function without an index.

1.6.2. *The counterexample.* For each s, for each e, we define: *s is good for e* if and only if $\mathit{length}(s) = e + 1$ and $\forall i \le e \exists j[j = \mu z[T(e,i,z)] \wedge U(j) = s(i)]]$.

We let B be the set of all s in Bin such that, for some e, s is good for e. Note that, if s is good for e, then $\mathit{length}(s) = e + 1$, and that, for every e, there is at most one s in Bin such that s is good for e. It follows that, for each n, the number of elements of the set $\{s \in B | \mathit{length}(s) \le n\}$ is smaller than n.

We claim that B is a bar in the set of all computable elements of Cantor space $\mathcal{C}$. For assume α in $\mathcal{C}$ and α is computable. Find e such that $\alpha = \varphi_e$. Note that $\overline{\alpha}(e+1) = \overline{\varphi_e}(e+1)$ is good for e.

We also claim that every finite subset of B positively fails to be a bar in the set of all computable elements of Cantor space $\mathcal{C}$, that is, for every s, if $\forall n < \mathit{length}(s)[s(n) \in B]$, then there exists α in $\mathcal{C}$ such that α is computable and $\forall n < \mathit{length}(s)[\neg(s(n) \sqsubset \alpha)]$. For let s be given such that $\forall n < \mathit{length}(s)[s(n) \in B]$. Without loss of generality, we may assume: $\forall n < \mathit{length}(s) - 1[\mathit{length}\big(s(n)\big) \le \mathit{length}\big(s(n+1)\big)$, and, therefore, $\forall n < \mathit{length}(s)[\mathit{length}\big(s(n)\big) > n]$. Define α such that, for each n, if $n < \mathit{length}(s)$, then $\alpha(n) = 1 - \big(s(n)\big)(n)$, and, if $n \ge \mathit{length}(s)$, then $\alpha(n) = 0$. Note: α is computable and $\neg\exists n < \mathit{length}(s)[s(n) \sqsubset \alpha]$.

1.6.3. *A refinement.* The set B we defined in Subsubsection 1.6.2 is not an algorithmically decidable subset of $\mathbb{N}$ although it is an algorithmically enumerable subset of $\mathbb{N}$.

For each s, for each e, we define: *s fits e* if and only if $\mathit{length}(s) > e$ and $\forall i \le e \exists j < \mathit{length}(s)[j = \mu z[T(e,i,z)] \wedge U(j) = s(i)]]$. Note that there is algorithm by which one may decide, for all s, e, if s fits e or not. Also note: for all s in Bin, for all e, if s fits e, then $\overline{s}(e+1)$ is good for e, and, if s is good for e, then there exists m such that, for all t in Bin, if $s \sqsubseteq t$ and $\mathit{length}(t) = m$, then t fits e.

Let B^* be the set of all s in Bin such that, for some e, s fits e. Note that B^* is an algorithmically decidable subset of $\mathbb{N}$ and that B^* is a bar in the set of all computable elements of $\mathcal{C}$. Now let s be given such that $\forall n < \mathit{length}(s)[s(n) \in B^*]$. Find t such that $\mathit{length}(t) = \mathit{length}(s)$ and, for all $n < \mathit{length}(s)$, $t(n) \sqsubseteq s(n)$ and $t(n) \in B$. Using the result of Subsubsection 1.6.2, find α in $\mathcal{C}$ such that α is computable and $\neg\exists n < \mathit{length}(s)[t(n) \sqsubset \alpha]$, and, therefore, $\neg\exists n < \mathit{length}(s)[s(n) \sqsubset \alpha]$.

1.6.4. The following formula is sometimes called *Kleene's Alternative to the Fan Theorem*:

$$\exists\beta[\forall\alpha \in \mathcal{C}\exists n[\beta(\overline{\alpha}n) \neq 0] \ \wedge \ \forall n\exists s \in Bin[length(s) = n \ \wedge \ \forall m \leq n[\beta(\overline{s}m) = 0]].$$

This statement flatly contradicts the Fan Theorem, but it is true in the model of formal constructive analysis that we obtain by requiring every function from $\mathbb{N}$ to $\mathbb{N}$ to be Turing-computable.

Computable analysis is dramatically different from classical analysis. For instance, there exists a real function from $[0,1]$ to $\mathcal{R}$ that is everywhere continuous but positively unbounded. A survey of such results may be found in: *Archive for Mathematical Logic* 53(2014)621-693. This is also true for classical computable analysis, as Kleene's Alternative to the Fan Theorem is, from a classical point of view, an Alternative to König's Lemma.

1.7. **The Extended Fan Theorem.** Combining the Fan Theorem with Brouwer's Continuity Principle we obtain the following

Theorem 1.5 (Extended Fan Theorem)**.** *Let $\mathcal{F} \subseteq \mathcal{N}$ be a fan and let $\mathcal{T}$ be a subset of $\mathcal{N} \times \mathbb{N}$ such that $\forall\alpha \in \mathcal{F}\exists n[\alpha\mathcal{T}n]$. Then*

(i) $\exists m\forall\alpha \in \mathcal{F}\exists n\forall\beta \in \mathcal{F}[\overline{\beta}m = \overline{\alpha}m \to \beta\mathcal{T}n]$*, and*

(ii) $\exists s[\forall\alpha \in \mathcal{F}\exists n\exists i < length(s)[\overline{\alpha}n = s'(i)] \ \wedge$
$\forall i < length(s)\forall\alpha \in \mathcal{F}[s'(i) \sqsubset \alpha \to \alpha\mathcal{T}s''(i)]]$*, and*

(iii) $\exists m\forall\alpha \in \mathcal{F}\exists n \leq m[\alpha\mathcal{T}n]$.

Proof. Let σ be a fan-law such that $\mathcal{F} = \mathcal{F}_\sigma$. Let B be the set of all s such that $\sigma(s) = 0$ and $\exists n\forall\alpha \in \mathcal{F}_\sigma \cap s[\alpha\mathcal{T}n]$. According to Brouwer's Continuity Principle, B is a bar in $\mathcal{F}$.

(i) Using the second version of the Fan Theorem, Theorem 1.4, find m such that $\forall\alpha \in \mathcal{F}\exists n \leq m[\overline{\alpha}n \in B]$. Conclude: $\forall\alpha \in \mathcal{F}\exists n\forall\beta \in \mathcal{F}[\overline{\beta}m = \overline{\alpha}m \to \beta\mathcal{T}n]$.

(ii) Using the first version of the Fan Theorem, Theorem 1.1, find u such that $\forall i < length(u)[u(i) \in B]$ and $\forall\alpha \in \mathcal{F}\exists n\exists i < length(u)[u(i) = \overline{\alpha}n]$. Find t such that $length(t) = length(u)$ and $\forall i < length(u)\forall\alpha \in \mathcal{F} \cap s(i)[\alpha\mathcal{T}t(i)]$. Define s such that $length(s) = length(u)$, and, for all $i < length(u)$, $s(i) = \big(u(i), t(i)\big)$. Then: $\forall\alpha \in \mathcal{F}\exists n\exists i < length(s)[\overline{\alpha}n = s'(i)] \wedge \forall i < length(s)\forall\alpha \in \mathcal{F}[s'(i) \sqsubset \alpha \to \alpha\mathcal{T}s''(i)]$.

(iii) Let t be as defined in the proof of (ii). Define $m := \max_{i<length(t)} t(i)$. Note: $\forall\alpha \in \mathcal{F}\exists n \leq m[\alpha\mathcal{T}n]$. □