Q3.1 Given Req

$p(x|z) = \mathcal{N}(x; Wz + \mu, \sigma^2 I_D)$ $p(x)$:

$p(z) = \mathcal{N}(z, 0, I_M)$

Solution: Substituting values into Corolary 2:

$x_a = z$, $x_b = x$, $A = W$, $\Sigma_{b/a} = \sigma^2 I_D$, $\mu_b = \mu_x$, $\Sigma_b = \Sigma_x$

$\Rightarrow \mu_x = A\mu_z + b = W \cdot 0 + \mu = \mu$

$\Rightarrow \Sigma_x = \Sigma_{x/z} + W\Sigma_z W^T = \sigma^2 I_D + W I_M W^T = \sigma^2 I_D + WW^T$

Therefore, $\underline{\underline{x = \mathcal{N}(x; \mu, \sigma^2 I_D + WW^T)}}$

Q3.2 Given Required

Refer Q3.1 $p(z/x)$

Solution

$p(z) = \mathcal{N}(z; 0, I_M)$, $p(x/z) = \mathcal{N}(x; Wz + \mu, \sigma^2 I_D)$

Using Corrolary 1: $p(z|x) = \mathcal{N}(z; \mu_{z|x}, \Sigma_{z|x})$

$$\Sigma_{z|x} = [\Sigma_z^{-1} + W^T \Sigma_{x/z}^{-1} W]^{-1}$$

$$= [I_M + W^T(\sigma^2 I_D)^{-1} W]^{-1}$$

$$= \left\{\frac{1}{\sigma^2}(\sigma^2 I_M + W^T W)\right\}^{-1}$$, using $(aA)^{-1} = 1/a\, A^{-1}$:

$$= \sigma^2 [\underbrace{\sigma^2 I_M + W^T W}_{M}] = \underline{\underline{\sigma^2 M^{-1}}}$$

$$\mu_{z|x} = \Sigma_{z/x}[\Sigma_z^{-1}\mu_z + W^T \Sigma_{z/x}^{-1}(x - \mu)]$$

$$= \sigma^2 M^{-1}[I_M^{-1} \cdot 0 + W^T(\sigma^2 I_D)^{-1}(x - \mu)]$$

$$= \sigma^2 M^{-1}\left[\frac{W^T(x-\mu)}{\sigma^2}\right]$$

$$= \underline{\underline{M^{-1} W^T (x - \mu)}}$$

Therefore: $\underline{\underline{p(z/x) = \mathcal{N}(z; M^{-1} W^T (x - \mu), \sigma^2 M^{-1})}}$

where $M = \sigma^2 I_M + W^T W$

Q3.3   Given   Required

$p_\theta(z) = \mathcal{N}(z; 0, I_M)$ $KL(q_\phi(z;x) \| p_\theta(z))$

$q_\phi(z;x) = \mathcal{N}(z; \mu_\phi(x), \mathrm{diag}(\sigma_\phi^2(x)))$

Solution: $q_\phi(z;x) = \mathcal{N}(z; \mu_\phi(x), \mathrm{diag}(\sigma_\phi^2(x)))$ - approximate post

$p_\theta(z) = \mathcal{N}(z; 0, I_M)$ - the prior

$$= \frac{1}{(2\pi)^{M/2}} \exp\left(-\tfrac{1}{2} z^T z\right)$$

$$q_\phi(z;x) = \frac{1}{(2\pi)^{M/2} \prod_{i=1}^{M} \sigma_{\phi,i}(x)} \exp\left\{-\frac{1}{2}(z-\mu_\phi(x))^T \mathrm{diag}(\sigma_\phi^{-2}(x))(z-\mu_\phi(x))\right\}$$

The KL divergence is:

$$KL(q_\phi(z;x) \| p_\theta(z)) = -\int q_\phi(z;x) \log \frac{q_\phi(z;x)}{p_\theta(z)} dz$$

$$= -\int q_\phi(z;x)\{\ln q_\phi(z;x) - \ln p_\theta(z)\} dz$$

$$= -\int q_\phi(z;x)\left[-\frac{M}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{M}\ln\sigma_{\phi,i}^2(x) - \frac{1}{2}\sum_{i=1}^{M}\frac{z_i - \mu_{\phi,i}(x)}{\sigma_{\phi,i}^2(x)} - \frac{M}{2}\ln(2\pi) - \frac{1}{2}z^Tz\right] dz$$

$$= -\int q_\phi(z;x)\left\{-\frac{1}{2}\sum_{i=1}^{M}\ln\sigma_{\phi,i}^2(x) - \frac{1}{2}\sum_{i=1}^{M}\frac{(z_i - \mu_{\phi,i}(x))^2}{\sigma_{\phi,i}^2(x)} + \frac{1}{2}z^Tz\right\} dz \quad *$$

Since $-\int q_\phi(z;x)\,dz$ is an ~~expression~~ expectation, $*$ above is also expectation. By distributing the integral and isolating the term independent of z;

a) $\int q_\phi(z,x) \sum_{i=1}^{M} \ln \sigma_{\phi,i}^2(x)\, dz = \sum_{i=1}^{M} \ln \sigma_{\phi,i}^2(x)$

Also b) $\int q_\phi(z;x) \frac{(z_i - \mu_{\phi,i}(x))^2}{\sigma_{\phi,i}^2(x)} dz = 1$ Since the expectation of $(z_i - \mu_{\phi,i}(x))^2$ is the expectation of the variance under $q_\phi(z,x)$. In other words, the expression is the pdf integrated, equalling to 1.

Including the term $\sum_{i=1}^{M} \tfrac{1}{2}(1) = \tfrac{1}{2}\sum_{i=1}^{M}(1) = M/2$

c) The expectation of z_i^2,

$$\mathbb{E}_{q_\phi}(z_i^2) = \sigma_{\phi,i}^2(x) + \mu_{\phi,i}^2(x)$$

putting terms in a, b and c together and including factors

$$KL(q_\phi(x;z) \| p_\theta(z)) = \frac{1}{2}\sum_{i=1}^{M}\left(\sigma_{\phi,i}^2(x) + \mu_{\phi,i}^2(x) - 1 - \log\sigma_{\phi,i}^2(x)\right)$$