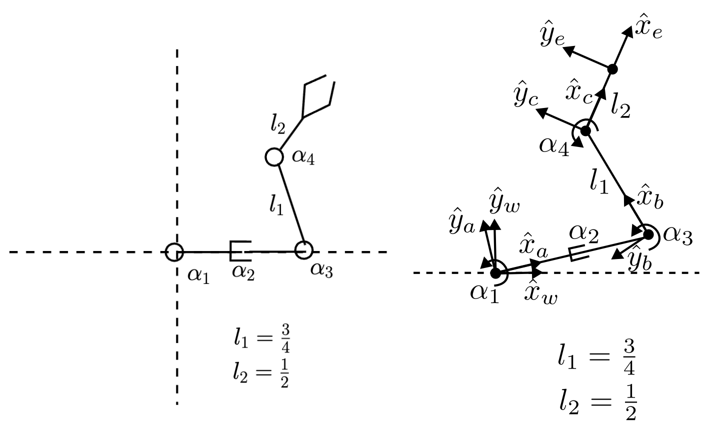
ECE 4560
Assignment 5

Due: September 26th, 11:59pm

This homework should be treated as "Practice Problems" for the upcoming Midterm. Please note that all of the points add up to *more* than 15 points, so feel free to skip questions and still get full credit. But I wanted to give you the opportunity to have more practice questions. The maximum score possible for this homework will be limited to the 15 points.

1. (1 point) The following are formulas that I expect you to have memorized. For homework this problem is a bit trivial, but you should be comfortable with the following formulas.
   (a) What is the expression for combining two vector-form transformations? I.e., write the expression for g_{ab} in vector notation for $g_{ab} = (d_1, R(\theta_1)) \cdot (d_2, R(\theta_2))$.
   (b) What is the homogenous representation of a transformation?
   (c) What is the expression for g^{-1} for both the vector and homogeneous representation of g?
   (d) What is the form of a planar rotation matrix? I.e., write the matrix expression for $R(\theta)$.
   (e) What are the expressions for $\hat{\xi}_s$ and $\hat{\xi}_b$ in terms of g_{sb} and $\dot{g}_{sb}$?

2. Consider the manipulator depicted in the Figure below, where $l_1 = \frac{3}{4}$ and $l_2 = \frac{1}{2}$. It is a serial chain manipulator with a prismatic, revolute, revolute, prismatic serial chain. At the zero configuration, all links are aligned with the horizontal x-axis. Note, for the purpose of the homework please provide numerical solutions (I recommend using MATLAB). On the exam you will not need to compute out the matrix multiplications as long as you have the formulas (with all variables plugged in or at least defined).



(a) (2 points) What is the symbolic expression for the forward kinematics, $g_e(\overrightarrow{\alpha})$, of the end-effector? Note that α_2 represents the length between the first and second joint.

(b) (2 points) What are the end-effector configurations for $\overrightarrow{\alpha}_1 = (\frac{\pi}{4}, \frac{3}{2}, \frac{\pi}{4}, \frac{\pi}{3})$ and $\overrightarrow{\alpha}_2 = (-\frac{\pi}{3}, 1, -\frac{\pi}{6}, \frac{\pi}{4})$? (I recommend using MATLAB for this problem. You would not have a question like this on the exam.)

(c) (4 points) Assume that you have two different sets of joint angles $\vec{\alpha_1}$ and $\vec{\alpha_2}$. What transformation does $g_e(\overrightarrow{\alpha_1})$ undergo to get to $g_e(\overrightarrow{\alpha_2})$? (body reference) You may either do this question symbolically or with the specific values from part b.

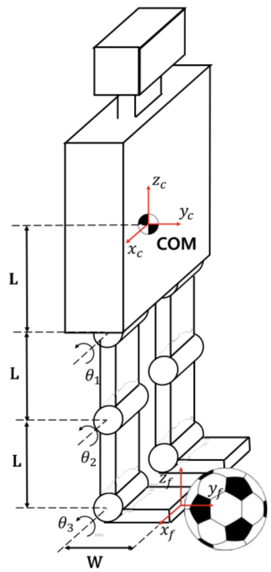
(d) (4 points) Suppose that there is a tool in the end-effector, and that the transformation from end-effector to tool is, $h = (\frac{1}{4}, \frac{1}{2}, \frac{\pi}{4})$. What transformation does the tool frame undergo as the manipulator joint-configuration goes from $\overrightarrow{\alpha_1}$ to $\overrightarrow{\alpha_2}$?

(e) (2 points) What is the Lie algebra element $\xi = (\omega, \tau)$ associated with the transformation from the initial end-effector configuration to the final end-effector configuration? (You pick the time τ, just make sure you write down what it is. Use body reference.)

(f) (2 points) What is the Lie algebra element ξ_{tool} associated to the motion of the tool? (body reference)

Note: You may want to use the equations sheet attached at the end of the assignment. This will be the same sheet that is provided during the exam.

3. The figure below shows a soccer robot in its zero configuration. The dimensions and joint variables are as indicated in the figure with $L = 1$ and $W = 0.5$.

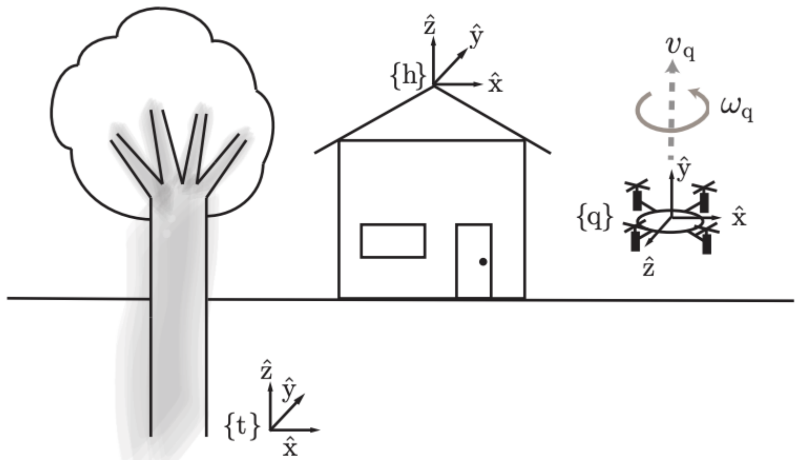


(a) (2 points) Compute the forward kinematics expression g_f for the frame located at the tip of the foot with respect to the hip joint (a fixed frame located at the joint θ_1).

(b) (2 points) Assuming the robot's kicking leg is at the initial configuration $\vec{\theta} = (\theta_1, \theta_2, \theta_3) = (-\pi/2, -\pi/2, 0)$ what is the distance between the hip joint and the foot tip frame?

(c) (2 points) Assume that the hip joint is actuated so that an angular velocity is created of $\omega = (1, 0, 0)$. All other joint angles are held constant. What is the linear velocity of the foot tip frame?

4. Consider the scene below.



The scene despicts a quadcopter (frame q) flying near a tree (frame t) and house (frame h). The quadcopter is at a position with a displacement $d_{tq} = (10, 5, 5)$ m expressed in the tree frame t, and the house is at a position with a displacement $d_{th} = (0, 10, 10)$ m expressed in the tree frame t. The quadcopter is flying upwards with a velocity of 1 m/s, and rotating with a velocity of 1 rad/s (about the y-axis).

(a) (2 points) Calculate the quadcopter's twist in frame q and frame t.

(b) (2 points) Use the adjoint map to express the twist in the house frame h.

Provided below are some useful equations.

$$v_b = R_{ab}^{\top} \dot{d}_{ab} \qquad v_s = -\dot{R}_{ab} R_{ab}^{\top} d_{ab} + \dot{d}_{ab}$$
$$\omega_b = (R_{ab}^{\top} \dot{R}_{ab})^{\vee} \qquad \omega_s = (\dot{R}_{ab} R_{ab}^{\top})^{\vee}$$

5. (0 points) **LAB COMPONENT:** Due to the upcoming Midterm, there will be no lab component this week.

Equations Sheet:

In $SE(2)$, the exponential and the logarithm for the Lie algebra element $\xi = (v, \omega) \in \mathbb{R}^3$ are:

$$\exp(\hat{\xi}\tau) = \begin{cases} \left[\begin{array}{c|c} R(\omega\,\tau) & \frac{1}{\omega}(I - R(\omega\tau))\mathbb{J}v \\ \hline 0 & 1 \end{array}\right] & \text{if } \omega \neq 0 \\ \left[\begin{array}{c|c} I & v\tau \\ \hline 0 & 1 \end{array}\right] & \text{if } \omega = 0 \end{cases}$$

$$\ln_\tau(g) = \begin{cases} \omega = \frac{1}{\tau}\text{atan}(R_{21}, R_{11}) & \\ v = -\omega\mathbb{J}\,(I - R)^{-1}\,d & \text{if } \omega \neq 0 \\ v = \frac{1}{\tau}d & \text{if } \omega = 0 \end{cases}$$

With the matrix $\mathbb{J}$ defined as:

$$\mathbb{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In $SO(3)$, the exponential and logarithm for the Lie algebra element $[\omega]_\times \in \mathbb{R}^{3\times 3}$ are:

$$\exp([\omega]_\times\tau) = \begin{cases} I + \frac{[\omega]_\times}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{[\omega]_\times^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau)) & \text{if } \omega \neq 0 \\ I & \text{if } \omega = 0. \end{cases}$$

$$\ln R = \begin{cases} \|\omega\| = \frac{1}{\tau}\cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right) & \\ [\omega]_\times = \frac{\|\omega\|}{2\sin(\|\omega\|\tau)}\left(R - R^\top\right) & \text{if } \|\omega\| \neq 0\,\cdot \\ [\omega]_\times = 0 & \text{if } \|\omega\| = 0 \end{cases}$$

In $SE(3)$, the exponential and the logarithm for the Lie algebra element $\xi = (v, \omega) \in \mathbb{R}^6$ are:

$$\exp(\hat{\xi}\tau) = \begin{cases} \left[\begin{array}{c|c} \exp([\omega]_\times\,\tau) & (I - \exp([\omega]_\times\tau))\frac{[\omega]_\times\,v}{\|\omega\|^2} + \frac{\omega\omega^\top}{\|\omega\|^2}v\tau \\ \hline 0 & 1 \end{array}\right] & \text{if } \omega \neq 0 \\ \left[\begin{array}{c|c} I & v\tau \\ \hline 0 & 1 \end{array}\right] & \text{if } \omega = 0 \end{cases}$$

$$\ln(g) = \begin{cases} \omega = \ln_\tau(R) & \\ v = \|\omega\|^2\,\left(\left((I - R)\,[\omega]_\times + \tau\omega\omega^\top\right)\right)^{-1} d & \text{if } \omega \neq 0 \\ v = \frac{1}{\tau}d & \text{if } \omega = 0. \end{cases}$$

In these equations, $[\omega]_\times \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix of $\omega \in \mathbb{R}^3$:

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}.$$

Lastly, in $SO(2)$, a rotation matrix has the form:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

In SO(3), the rotation matrices corresponding to rotations about the x, y, and z axes are:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \quad R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$