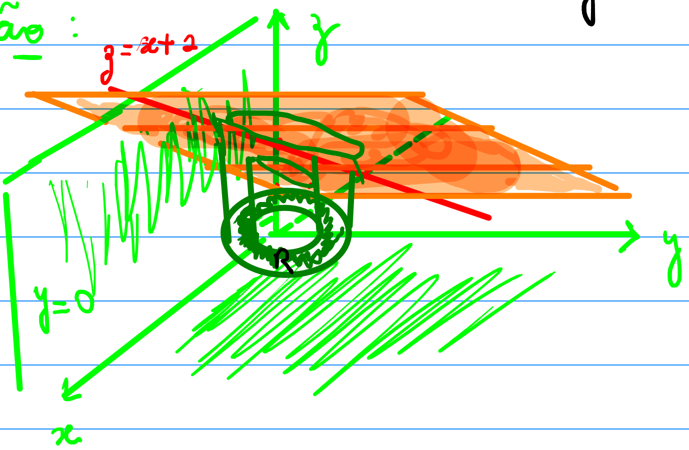
Exercício 1: Avalie $\iiint_E y\,dV$, onde E é a região abaixo do plano $z = x+2$, acima do plano Oxy, e entre os cilindros $x^2+y^2=1$ e $x^2+y^2=4$.

elemento de volume $dx\,dy\,dz$

solução:



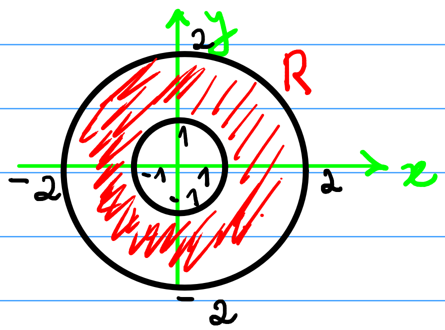
$z = x+2$

plano Oxz: $(0,0,2)$

$(-2,0,0)$

$$E = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq x+2,\ 1 \leq x^2+y^2 \leq 4\}$$

$$\iiint_E y\,dV = \iint_R \left(\int_0^{x+2} y\,dz\right) dA,$$ onde R é a projeção do sólido E sobre o plano Oxy.



coordenadas cilíndricas:

$$\begin{cases} x = \rho \cdot \cos\theta \\ y = \rho \cdot \operatorname{sen}\theta \\ z = z \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \rho$$

E: $0 \leq z \leq x+2$, $1 \leq \rho \leq 2$, $0 \leq \theta < 2\pi$

$$\underbrace{\iiint_E \underset{\rho\cdot\operatorname{sen}\theta}{y}\,dV}_{\text{coord. cartesianas}} = \underbrace{\int_0^{2\pi}\int_1^2\int_0^{\rho\cos\theta+2} \rho\cdot\operatorname{sen}\theta \cdot \underset{\text{jacobiano}}{\rho}\, dz\,d\rho\,d\theta}_{\text{coordenadas cilíndricas}} =$$

$$(i)\quad \int_0^{\rho\cdot\cos\theta+2} \rho^2 \cdot \operatorname{sen}\theta \, dz = \rho^2 \cdot \operatorname{sen}\theta \cdot \int_0^{\rho\cdot\cos\theta+2} dz$$

$$= \rho^2 \cdot \operatorname{sen}\theta \cdot (z)\Big|_{z=0}^{z=\rho\cos\theta+2}$$

$$= \rho^2 \cdot \operatorname{sen}\theta\,(\rho\cos\theta+2) = \rho^3 \operatorname{sen}\theta\cos\theta + 2\rho^2 \operatorname{sen}\theta$$

$$(ii)\quad \int_1^2 \left(\rho^3 \cdot \operatorname{sen}\theta \cdot \cos\theta + 2\rho^2 \operatorname{sen}\theta\right) d\rho$$

$$= \operatorname{sen}\theta \cdot \cos\theta \cdot \int_1^2 \rho^3\, d\rho + 2\operatorname{sen}\theta \cdot \int_1^2 \rho^2\, d\rho$$

$$= \operatorname{sen}\theta \cdot \cos\theta \cdot \left(\frac{\rho^4}{4}\right)\Big|_1^2 + 2\operatorname{sen}\theta \cdot \left(\frac{\rho^3}{3}\right)\Big|_1^2$$

$$= \frac{\operatorname{sen}\theta \cdot \cos\theta}{4} \cdot (16-1) + \frac{2\operatorname{sen}\theta}{3} \cdot (8-1)$$

$$= \frac{15}{4} \cdot \operatorname{sen}\theta \cdot \cos\theta + \frac{14}{3}\operatorname{sen}\theta$$

$$(iii)\quad \int_0^{2\pi} \left(\frac{15}{4}\operatorname{sen}\theta \cdot \cos\theta + \frac{14}{3}\operatorname{sen}\theta\right) d\theta$$

$$= \frac{15}{4} \cdot \int_0^{2\pi} \underbrace{\operatorname{sen}\theta}_{u} \cdot \underbrace{\cos\theta\, d\theta}_{du} + \frac{14}{3} \cdot \int_0^{2\pi} \operatorname{sen}\theta\, d\theta$$

$$= \frac{15}{4}\left(\frac{\operatorname{sen}^2\theta}{2}\right)\Big|_0^{2\pi} + \frac{14}{3} \cdot (-\cos\theta)\Big|_0^{2\pi}$$

$$= \frac{15}{8} \cdot (0-0) - \frac{14}{3} \cdot (1-1) = 0$$

Professor, e se eu, uma pessoa cheia de coragem e ousadia, quisesse resolver a integral em coordenadas cartesianas?

$$\iiint_E y\,dV =$$

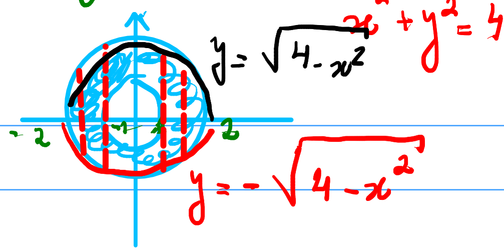
$$\int_{-2}^{-1}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x+2} y\,dz\,dy\,dx \;+$$

$$+\int_{1}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x+2} y\,dz\,dy\,dx \;+$$

$$+\int_{-1}^{1}\int_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}}\int_0^{x+2} y\,dz\,dy\,dx$$

$$+\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}\int_0^{x+2} y\,dz\,dy\,dx$$

$0 \leq z \leq x+2$



Exercício 2: Converta a integral tripla

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy$$

para coordenadas esféricas.

solução: $0 \leq \underbrace{\sqrt{x^2+y^2} \leq z}\; \underbrace{\leq \sqrt{18-x^2-y^2}}$, $\;0 \leq x \leq \sqrt{9-y^2}$, $\;0 \leq y \leq 3$

$\underbrace{z = \sqrt{x^2+y^2}}_{\text{cone invertido}}$

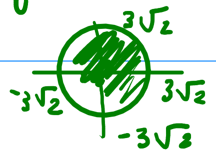
$z = \sqrt{18-x^2-y^2} \rightarrow$

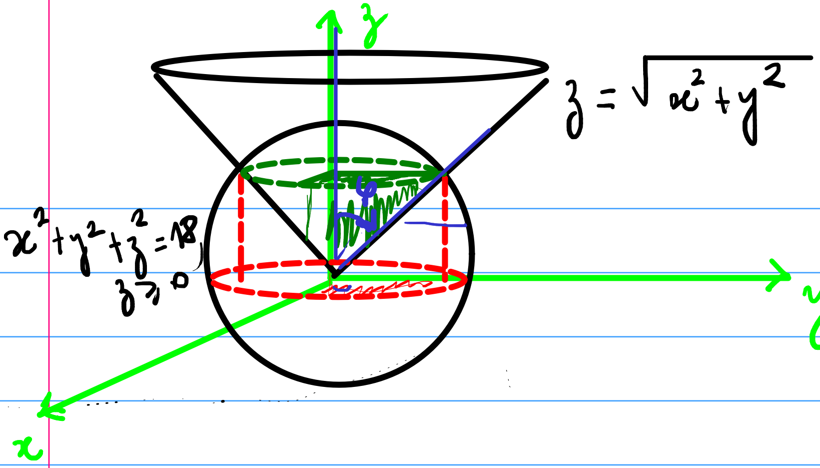
$$18 - x^2 - y^2 \geq 0$$
$$18 \geq x^2 + y^2$$
$$x^2 + y^2 \leq 18$$

$z^2 = 18 - x^2 - y^2$

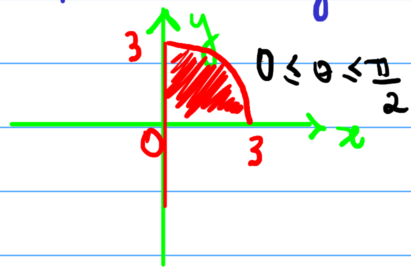
$\underbrace{x^2 + y^2 + z^2 = 18}$

esfera de centro $(0,0,0)$ e raio $\sqrt{18} \approx 3\sqrt{2}$





região R (projeção do sólido sobre o plano Oxy



$0 \leq \theta \leq \frac{\pi}{2}$

- $0 \leq \theta \leq \pi/2$

$$\underbrace{x^2+y^2+z^2}=18$$

$$\rho^2=18$$

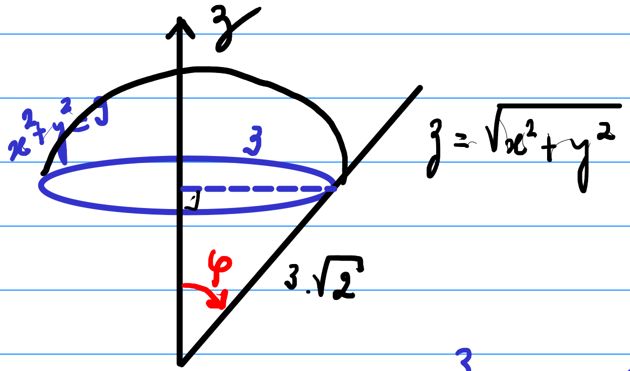
- $0 \leq \rho \leq 3\sqrt{2}$

$0 \leq x \leq \sqrt{9-y^2}$, $0 \leq y \leq 3$

$$x=\sqrt{9-y^2}$$
$$x^2=9-y^2$$
$$x^2+y^2=9$$

Intersecção entre o cone $z=\sqrt{x^2+y^2}$ e a esfera $x^2+y^2+z^2=18$

$$x^2+y^2+(x^2+y^2)=18$$
$$2(x^2+y^2)=18$$
$$x^2+y^2=9$$



$$\operatorname{sen}\varphi=\frac{3}{3\sqrt{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$$\Rightarrow \varphi=\pi/4$$

$$0 \leq \varphi \leq \frac{\pi}{4}$$

coordenadas esféricas:

$$\begin{cases} x=\rho \cdot \operatorname{sen}\varphi \cdot \cos\theta \\ y=\rho \cdot \operatorname{sen}\varphi \cdot \operatorname{sen}\theta \\ z=\rho \cdot \cos\varphi \end{cases}$$

$$x^2+y^2+z^2=\rho^2$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}=\rho^2 \cdot \operatorname{sen}\varphi$$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{18-x^2-y^2}} \underbrace{(x^2+y^2+z^2)}\, dz\, dx\, dy =$$

$$= \underbrace{\int_0^{\pi/2} \int_0^{\frac{\pi}{4}} \int_0^{3\sqrt{2}}}_{\text{paralelepípedo}} \rho^2 \cdot \underbrace{\rho^2 \cdot \operatorname{sen}\varphi}_{\text{jacobiano}} \cdot d\rho\, d\varphi\, d\theta$$

$$= \left(\int_0^{\pi/2} d\theta\right) \cdot \left(\int_0^{\pi/4} \operatorname{sen}\varphi\, d\varphi\right) \cdot \left(\int_0^{3\sqrt{2}} \rho^4\, d\rho\right)$$

$$= (\theta)\Big|_0^{\pi/2} \cdot (-\cos\varphi)\Big|_0^{\pi/4} \cdot \left(\frac{\rho^5}{5}\right)\Big|_0^{3\sqrt{2}}$$

$$= \frac{\pi}{\cancel{2}} \cdot \left(-\frac{\sqrt{2}}{\cancel{2}} + \frac{\cancel{2}\,1}{\cancel{2}}\right) \cdot \frac{1}{5} \cdot 243 \cdot \cancel{4}^{\,2}\,\sqrt{2}$$

$$= \frac{243\pi \cdot \sqrt{2}}{5}\left(-\sqrt{2} + 2\right) = \frac{243\pi}{5}\left(2\sqrt{2} - 2\right)$$

$$= \frac{486\pi}{5}\left(\sqrt{2} - 1\right)$$

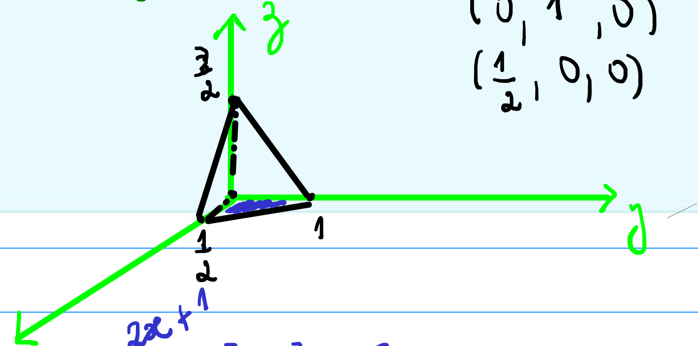
Encontre o volume da região limitada pelos planos $\underbrace{x = 0, y = 0, z = 0}_{\text{planos coordenados}}$ e $\underbrace{x + \frac{y}{2} + \frac{z}{3} = \frac{1}{2}}$. $\rightarrow$ $\frac{z}{3} = \frac{1}{2} - x - \frac{y}{2}$

$z = \frac{3}{2} - 3x - \frac{3y}{2}$

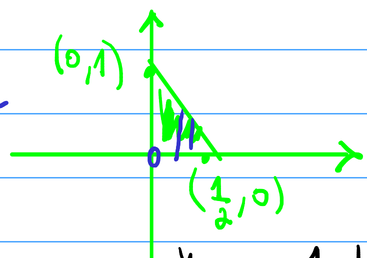
Escolha uma opção:

- a. $\frac{3}{4}$
- b. $\frac{1}{8}$
- c. $\frac{1}{4}$
- d. $\frac{1}{12}$
- e. $\frac{1}{24}$

$(0, 0, \frac{3}{2})$
$(0, 1, 0)$
$(\frac{1}{2}, 0, 0)$



$$\text{volume} = \int_0^{\frac{1}{2}} \int_0^{-2x+1} \int_0^{\frac{3}{2} - 3x - \frac{3y}{2}} dz\, dy\, dx$$



$y - 0 = \frac{1}{-\frac{1}{2}}\left(x - \frac{1}{2}\right)$

$y = -2\left(x - \frac{1}{2}\right)$
$-2x + 1$

Calcule o volume do sólido E que está entre as esferas $x^2+y^2+z^2=4$ e $x^2+y^2+z^2=16$, e que está acima do cone $z^2=\frac{1}{3}(x^2+y^2)$