# Inverse Problem of Diffusion

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We consider an inverse problem with a basis in the following differential equation

$$\frac{du(x,t)}{dt} = \frac{d^2u(x,t)}{dx^2}, \;\; u(x,0) = h_0(x), \;\; x \in (0,1), t \geq 0.$$

Data is $u(x,t) = h_t(x)$ for a given time $t > 0$. The aim of the inverse problem is $h_0(x)$.

The forward model can be written as

$$u(x,t) = h_t(x) = \frac{1}{\sqrt{4\pi t}} \int e^{-(x-y)^2/(4t)} h_0(y)dy, \;\; t \geq 0.$$

Using discretization we get

$$\mathbf{h_t} = \begin{bmatrix} h_t(x_1) \\ h_t x_2 \\ \vdots \\ h_t x_N \end{bmatrix} = \mathbf{A} \begin{bmatrix} h_0 x_1 \\ h_0 x_2 \\ \vdots \\ h_0 x_N \end{bmatrix} = \mathbf{A h_0},$$

where a regular grid of N = 100 points is used, such that $x_1 = 0$, $x_2 = 0.01$, $\ldots$, $x_M = 0.99$.

```
x = seq(from = 0, to = 1, by = 0.01)
```

The interval (0,1) is made into a circle, i.e. 1 corresponds to 0. The matrix A has elements

$$A(i,j) = \frac{0.01}{\sqrt{4\pi t}} e^{-|x_i - x_j|^2/(4t)}.$$

The distance $|x_i - x_j|$ is modular on the circle (0,1).

```
create_A <- function(x,t){
  A = diag(0.01/sqrt(4*pi*t), nrow = length(x), ncol = length(x))
  for (i in seq(1,length(x))){
    for (j in seq(i+1,length(x))){
      A[i,j] = 0.01/sqrt(4*pi*t)*exp(-(x[i]-x[j])^2/(4*t))
      A[j,i] = A[i,j]
    }
  }
  A
}
```

Measurements $\mathbf{y} = (\mathbf{y_1}, ..., \mathbf{y_N})'$ are acquired at time t = 0.001 (1ms):

$$y_i = h_t(x_i) + \epsilon_i, \;\; \epsilon_i \sim \mathcal{N}(0, 0.025^2), \;\; \text{iid.}$$

```
y = read.delim2(file = "OppgA.txt", header = F, sep = "\n", dec = ".")[[1]]
```

## Exercise a

## Exercise b

## Exercise c